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Master Thesis

Explaining the Structure of Quark and Lepton Mass Matrices via Flavour non-Universal Gauge Groups

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Introduction

The Standard Model (SM) is so far the best theory we have to describe the Universe. Three out of four of the fundamental interactions that govern Nature have been well explained and tested at very high precision. Nevertheless, we know that SM is not the final theory. Even neglecting gravitational interactions, there exist some questions that SM is not able to answer. One of them is about the observed fermion masses. In the SM these masses are strictly linked to the strength of the couplings between the fermions and the Higgs field. These couplings are called Yukawa couplings. In total there are 12 different fermions, divided into four categories of three generations each: the up- and down-type quarks and the charged and neutral leptons. So far the SM predicts universality among the three generations: namely, the three different fermions of the same type share the same quantum numbers and interaction strengths with all the SM particles, but the Higgs field. What we know is that the three generations of each of the four types of fermions have very different masses. The first generation is very light while the third one could be even 10^5 times bigger. This leads to a very hierarchical structure in the Yukawa couplings or, better saying, in the Yukawa matrices. In addition, the values of those couplings are just set *ad hoc* to explain the observed different fermion masses. This leads to one of the open questions in Particle Physics, usually referred to as the *flavour puzzle* (or *flavour problem*): why do the Yukawa matrices have such a hierarchical structure?

Since the interaction strength between Higgs field and SM fermions is not universal, maybe this is true also for the other interactions. In fact there is not really a fundamental reason that implies the universality of the three generations. Throughout the past decades a lot of effort has been made to detect some flavour universality violation, but nothing has been detected yet. This leads to the possibility that flavour non-universality effects could be a feature that manifests at an energy scale higher than the one we have already probed, namely at the TeV scale. An interesting attempt to address the flavour problem is offered by the hypothesis of flavour non-universal Gauge Groups. More precisely, we could think that the ultraviolet (UV) completion of the SM features a deconstruction in flavour space of the flavour universal Gauge Symmetries observed up to the Electro-Weak (EW) scale. The idea of addressing the origin of the flavour puzzle via flavour non-universal Gauge Groups is not new, see for instance [1, 2]. A common feature is that this flavour non-universality couples most strongly to the third generation. This is due to the *a posteriori* justification of an approximate $U(2)^5$ flavour Symmetry acting on the light fermion generations, which is known to provide a good description of the SM spectrum plus an efficient suppression of flavour-violating effects for low-scale new physics (NP) [3, 4]. By invoking flavour non-universal Gauge interactions, the $U(2)^5$ Symmetry emerges as an accidental Symmetry of the Gauge sector.

Another open question that has not been answered yet by the SM is the Charge quantization. It is commonly known that the electric charge seems to be quantized from what we observe. Namely, the electron has a charge e and all the other possible charges are multiple of that quantity. In the SM this is closely related to the Hypercharge of the SM particles, which is the generator of a U(1) abelian Gauge Group. In fact we know that the Charge associated to non-abelian Gauge Groups must be quantized, but there is no such condition for the abelian ones (see for instance [5]). This means that, if the SM Gauge Group is the final one, there is no reason whatsoever that the SM Hypercharges are rational numbers. A possible explanation to the Charge quantization could be assuming that the UV embedding of the SM is provided by a semisimple Gauge Group, which means that there are no abelian U(1) components. Probably the most famous example is provided by the SU(5) UV completion of the SM, firstly introduced by Georgi and Glashow in 1974 [6]. However, the issue of such kind of models is that they all predict proton decay. Hence, by looking at the constraints we have on proton lifetime, the energy scale at which the SM is embedded in such kind of models is pushed above 10^{16} GeV.

A very interesting possibility is offered by the SU(4) UV completion of the SM, firstly introduced by Pati and Salam in 1974 [7]. The main idea is that leptons in the UV are not something different from quarks, but rather the three coloured quarks and the lepton form the same fermion which is a vector charged under SU(4). In such kind of models the $B-L$ component of the Hypercharge (where

B and L are the Baryon and Lepton Number respectively) together with SU(3)-Colour Gauge Group of the SM are embedded in SU(4). This also means that $B - L$ is gauged, which is an exact global Symmetry of the SM. This is possible since the 15^{th} generator of SU(4) is a 4×4 matrix proportional to a diagonal one with eigenvalues $(1/3, 1/3, 1/3, -1)$. This resembles very much the Baryon Number of the quarks and the Lepton Number of the leptons. In such kind of models Baryon and Lepton Numbers are conserved classically. This means that, like in the SM, proton decay is kept under control since it is generated only by anomalies, which are very suppressed. For this reason they are very appealing if one is looking for NP at the TeV scale.

Recently has been done a classification of the possible flavour non-universal deconstruction of the SM Gauge Group with a Pati-Salam-like UV embedding [8]. In particular, the authors studied possible models where the flavour non-universality of the third generation of fermions already happens at the TeV scale. In addition, under the assumptions that the Higgs field is an elementary particle, they assumed that it couples mainly with the third-generation fermions while the rest of the Yukawa couplings are generated by some higher dimensional or, better saying, Effective Field Theory (EFT) operators. This offers a possible explanation for the hierarchical structure of the Yukawa matrices. In particular the authors found a very interesting class of models that have not been studied yet in the literature. The peculiarity is that the flavour non-universality happens only in the $\text{SU}(3) \times \text{U}(1)$ Gauge sector of the SM Gauge Group, while the SU(2)-Left sector is kept universal. This leads *naturally* to a hierarchical structure of the Yukawa matrices such that they are almost superior-triangular matrices. Thus, when one goes from the flavour- to the mass-basis, the mixing matrices of the right-handed fermions are very close to the identity, showing very little mixing. Phenomenologically, this is a highly desirable feature for flavoured NP near the TeV scale.

In this work we try to give an explanation to the flavour puzzle focusing on models where the non-universality of the third generation is manifest in the $\text{SU}(3) \times \text{U}(1)$ Gauge sector of the SM. Also we are going to assume that it finds a Pati-Salam-like UV embedding that occurs already at the TeV scale. The fact that we want to analyse models where new degrees of freedom are close to that energy scales is not only *pragmatic*. Clearly, if new particles appear at higher energy scales there is no chance to probe these models in the next years. Nevertheless, there is a strong motivation to assume the existence of NP at that scale which is linked to the *hierarchy problem*. It arises from the fact that the Higgs mass, since it is a relevant coupling, is very sensible to quantum corrections. Hence, if NP happens well above the TeV scale, it would imply strong fine tuning between the bare quantity and its quantum corrections to explain the observed Higgs mass which is approximately 125 GeV.

Another important problem of the SM is that it predicts massless neutrinos. However neutrino oscillations have been observed. This brings as direct consequence that (at least) two of the neutrino species are massive since two different mass splittings have been measured. In addition, we know from such observations that the three generations of neutrinos have comparable masses of $\mathcal{O}(10^{-1})$ eV. Possible minimal extensions of the SM to solve this problem assume the existence of sterile right-handed neutrinos, either with a very suppressed neutrino Yukawa coupling or with a very high Majorana mass. In the latter case, the observed very small neutrino masses are explained by a Direct See-Saw mechanism (for a review see for instance [9]).

Such Pati-Salam-like models predict the existence of right-handed neutrinos and a neutrino Yukawa matrix comparable (if not equal) to the one of the up-type quarks. Thus, the most efficient way to explain the observed neutrino masses is by means of an Inverse See-Saw mechanism [2, 10]. This implies the existence of three generations of sterile fermions with small Majorana masses. Such low value is well explained by the fact that it is the only explicit source of Lepton Number violation. Nevertheless, there is an issue concerning such models: in order to reproduce the anarchic small neutrino mass-spectrum, the Majorana masses of these sterile fermions must be extremely hierarchical to compensate the hierarchy present in the neutrino Yukawa matrix. In addition, See-Saw realizations usually predict the existence of some heavy neutral leptons with Dirac or Majorana masses. Phenomenologically this implies enhanced lepton flavour violation (LFV) effects with respect to SM predictions. Such kind of processes are common features in beyond SM (BSM) theories and in the past decades there has been a big effort in the detection of such effects, both from a theoretical and experimental point of view. For instance look [11, 12, 13] and references therein.

In this work we try to face all the theoretical issues listed above which currently represent an obstruction to the desired *naturalness* that such kind of models are expected to manifest. We anticipate here that we will be able to provide a natural explanation to the neutral-fermion masses. Furthermore, this model will predict the existence of three neutral (almost) Dirac-type (almost) sterile fermions with hierarchical masses. This is a very peculiar feature and it is worth to study its phenomenological implications. Thus, we are going to study in detail the phenomenology induced by such neutral fermions with focus on LFV observables and related constraints.

To summarize, the model we are going to study will be able to solve the flavour puzzle, to explain the Charge quantization, to reproduce the observed small neutrino masses assuming NP at TeV scale, hence providing a possible solution to the hierarchy problem. All this while keeping the model very *natural*, without the requirement of any strong assumption or fine tuning among the parameters of the UV theory.

This work in organized as follows. In Chapter 1 we study in detail the SM UV embedding of the model considered, explaining the particles in the UV theory and the Symmetry breaking mechanism to go down to the SM Gauge Group. In Chapter 2 we derive the Yukawa matrices that our model is able to reproduce from an EFT description, together with a possible UV origin for such higher-dimensional operators. In Chapter 3 we describe different possibilities to provide the observed small neutrino masses by means of See-Saw realizations, reproducing the results already present in the literature. In Chapter 4 we provide a new way to explain the observed neutrino masses solving all the issues that concern *naturalness* that are present in all the models studied in the literature so far. In Chapter 5 we focus on all the phenomenological implications of the heavy neutral leptons that arise from the model we built.

The Appendix contains all the relevant calculations as well as further in-depth analysis or model-building options not discussed in the main Chapters because less relevant or too technical. In addition, there are listed many mathematical results that have been used as well as the notation and conventions adopted throughout this work.

1 Model Building

In this Chapter we are going to study a possible UV embedding of the SM with a flavour non-universal deconstruction in the Colour and Hypercharge sectors. Then we are going to provide several possible spontaneous Symmetry breaking (SSB) mechanisms to go down from the UV to the SM, comparing the common and different features for each of them.

1.1 UV Theory

Consider the SM Gauge Group

$$H_{\mathrm{SM}} = \mathrm{SU}(2)_L \times \mathrm{SU}(3)_C \times \mathrm{U}(1)_Y \tag{1.1.1}$$

where the Gauge couplings and fields[1] associated are (with the $\mathrm{SU}(N)$ indices $a = 1, ..., 8$, $i = 1, 2, 3$)

$$(g_L, g_s, g_Y) \ , \qquad \left(W_L^i, G^a, B\right) \ . \tag{1.1.2}$$

We want to study the following UV embedding that breaks the universality of the third generation

$$H_{\mathrm{UV}} = \mathrm{SU}(2)_L \times \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{SU}(2)_R^{[3]} \times \mathrm{U}(1)_X^{[12]} \tag{1.1.3}$$

where the Gauge couplings and fields associated are (with the $\mathrm{SU}(N)$ indices $a = 1, ..., 15$, $b = 1, ..., 8$, $i, j = 1, 2, 3$)

$$(g_L, g_4, g_3, g_R, g_X) \ , \qquad \left(W_L^i, H^a, C^b, W_R^j, B_{12}\right) \ . \tag{1.1.4}$$

Such a UV completion has been firstly introduced by [8] where the authors pointed out the promising features of such a UV theory in explaining the hierarchical structure of the Yukawa matrices. The importance of keeping $\mathrm{SU}(2)_L$ universal will be clear in the next Chapter. The choice to embed the Gauge sector associated to the third generation in a semisimple Gauge Group arises from the aim to give an explanation to Charge quantization. Finally, we have chosen a Pati-Salam-like UV embedding [7] because it ensures proton stability, allowing to have a NP scale already at the TeV.

The fermionic particle content in the UV theory is listed in Table 1 with corresponding Charges under the UV Gauge Group H_{UV} (1.1.3). In our notation, quark and lepton doublets are represented respectively as follows

$$Q = \begin{pmatrix} u \\ d \end{pmatrix} \ , \qquad L = \begin{pmatrix} \nu \\ e \end{pmatrix} \ . \tag{1.1.5}$$

The fermions charged under SU(4) are represented as 4-vectors with a lepton in the last entry while the quarks with their red, blue and green colours are placed in the first three. Explicitly they read as follows

$$\chi = \begin{pmatrix} Q^r & Q^b & Q^g & L \end{pmatrix}^T \ . \tag{1.1.6}$$

Since the Yukawa couplings of the first and second generations of fermions are suppressed with respect to the third one, it is sensible to assume that the Higgs field in the UV theory is charged only under the third-generation Gauge sector. Thus we define the Higgs field in the UV theory as follows with corresponding Charges under H_{UV} (1.1.3)

$$\mathcal{H} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} H^c & H \end{pmatrix} \sim (\mathbf{2}, \mathbf{1}, \mathbf{1}, \overline{\mathbf{2}}, 0) \tag{1.1.7}$$

where H is the SM Higgs field while $H^c \equiv i\sigma_L^2 H^*$ is its Charge-Conjugate with σ_L^2 being the second Pauli matrix (G.1.12) that acts in the $\mathrm{SU}(2)_L$ space.

[1] To avoid using too heavy notation we omit the Lorentz index in all the Gauge fields when unnecessary.

ψ	$\mathrm{SU}(2)_L$	$\mathrm{SU}(4)^{[3]}$	$\mathrm{SU}(3)^{[12]}$	$\mathrm{SU}(2)_R^{[3]}$	$\mathrm{U}(1)_X^{[12]}$
χ_L^3	**2**	**4**	**1**	**1**	0
Q_L^i	**2**	**1**	**3**	**1**	1/6
L_L^i	**2**	**1**	**1**	**1**	-1/2
χ_R^3	**1**	**4**	**1**	**2**	0
u_R^i	**1**	**1**	**3**	**1**	2/3
d_R^i	**1**	**1**	**3**	**1**	-1/3
e_R^i	**1**	**1**	**1**	**1**	-1
ν_R^i	**1**	**1**	**1**	**1**	0

Table 1: Fermionic particle content in the UV theory. The index $i = 1, 2$ labels the first and second generations of fermions.

1.2 Spontaneous Symmetry Breaking Mechanisms

To go from the UV theory down to the SM, we have to break the UV Gauge Symmetry. This is made through a SSB mechanism and it requires the introduction of some scalar fields that have to eventually acquire a vacuum expectation value (VEV).

We could assume that the SSB mechanism of H_{UV} (1.1.3) down to the SM Gauge Group (1.1.1) happens in two consecutive steps. There are in total three possible ways to do this[2]:

$$\text{Model A} \quad \begin{array}{ll} 1^{\text{st}} \text{ Step:} & \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_X^{[12]} \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_{X'} \\ 2^{\text{nd}} \text{ Step:} & \mathrm{SU}(2)_R^{[3]} \times \mathrm{U}(1)_{X'} \to \mathrm{U}(1)_Y \end{array} \tag{1.2.1a}$$

$$\text{Model B} \quad \begin{array}{ll} 1^{\text{st}} \text{ Step:} & \mathrm{SU}(2)_R^{[3]} \times \mathrm{U}(1)_X^{[12]} \to \mathrm{U}(1)_{X'} \\ 2^{\text{nd}} \text{ Step:} & \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_{X'} \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_Y \end{array} \tag{1.2.1b}$$

$$\text{Model C} \quad \begin{array}{ll} 1^{\text{st}} \text{ Step:} & \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(2)_R^{[3]} \to \mathrm{SU}(3)^{[3]} \times \mathrm{U}(1)_X^{[3]} \\ 2^{\text{nd}} \text{ Step:} & \mathrm{SU}(3)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_X^{[3]} \times \mathrm{U}(1)_X^{[12]} \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_Y \end{array} \tag{1.2.1c}$$

We can even write them in a more schematic, yet clearer, way as follows:

$$\text{Model A:} \qquad 4321 \to 321 \to 31 \tag{1.2.2a}$$

$$\text{Model B:} \qquad 4321 \to 431 \to 31 \tag{1.2.2b}$$

$$\text{Model C:} \qquad 4321 \to 3311 \to 31 \tag{1.2.2c}$$

Another possibility is to consider a slightly modified version of them which consists in doing a SSB in three steps, where the first one is always the same and it is given by the breaking of

$$\mathrm{SU}(2)_R^{[3]} \to \mathrm{U}(1)_R^{[3]} \,. \tag{1.2.3}$$

This is important to get a little generalization of the models above producing more *naturally* the structure of the Yukawa matrices. Specifically the new models have the following SSB steps:

$$\text{Model D} \quad \begin{array}{ll} 2^{\text{nd}} \text{ Step:} & \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_X^{[12]} \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_{X'} \\ 3^{\text{rd}} \text{ Step:} & \mathrm{U}(1)_R^{[3]} \times \mathrm{U}(1)_{X'} \to \mathrm{U}(1)_Y \end{array} \tag{1.2.4a}$$

$$\text{Model E} \quad \begin{array}{ll} 2^{\text{nd}} \text{ Step:} & \mathrm{U}(1)_R^{[3]} \times \mathrm{U}(1)_X \to \mathrm{U}(1)_{X'} \\ 3^{\text{rd}} \text{ Step:} & \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_{X'} \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_Y \end{array} \tag{1.2.4b}$$

$$\text{Model F} \quad \begin{array}{ll} 2^{\text{nd}} \text{ Step:} & \mathrm{SU}(4)^{[3]} \times \mathrm{U}(1)_R^{[3]} \to \mathrm{SU}(3)^{[3]} \times \mathrm{U}(1)_X^{[3]} \\ 3^{\text{rd}} \text{ Step:} & \mathrm{SU}(3)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_X^{[3]} \times \mathrm{U}(1)_X^{[12]} \to \mathrm{SU}(3)_C \times \mathrm{U}(1)_Y \end{array} \tag{1.2.4c}$$

[2]Since $\mathrm{SU}(2)_L$ must be kept unbroken, it is just a spectator in the SSB mechanisms and is omitted.

and schematically they read as follows:

$$\begin{aligned}
&\text{Model D:} \qquad 4321 \to 4311 \to 311 \to 31 && (1.2.5a)\\
&\text{Model E:} \qquad 4321 \to 4311 \to 431 \to 31 && (1.2.5b)\\
&\text{Model F:} \qquad 4321 \to 4311 \to 3311 \to 31 && (1.2.5c)
\end{aligned}$$

What are left to be defined are the scalar fields responsible to trigger such SSB mechanisms. We define the following ones with corresponding Charges under H_{UV} (1.1.3)

$$\begin{aligned}
&\text{Model A,B,D,E} \quad \Omega_1 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, -1/2)\,, \quad \Omega_3 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{3}, \mathbf{1}, 1/6)\,, \quad \Sigma_R \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, 1/2)\,, && (1.2.6a)\\
&\text{Model C,F} \quad \Omega_1 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, -1/2)\,, \quad \Omega_3 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{3}, \mathbf{1}, 1/6)\,, \quad \Delta_3 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{2}, 0)\,. && (1.2.6b)
\end{aligned}$$

As it will be clear in the following Sections, the $\mathrm{U}(1)_X^{[12]}$ Charges are chosen in such a way that in the broken phase are reproduced exactly the SM Hypercharges of all the fields[3]. Then, for Models D, E and F, we need a further scalar field to trigger the SSB step (1.2.3). This must be done only by a field that transforms under the adjoint representation of $\mathrm{SU}(2)_R^{[3]}$. Hence we define the following scalar field with corresponding Charges under H_{UV} (1.1.3)

$$\Sigma_3 \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}, 0)\,. \tag{1.2.7}$$

To conclude, we also define the $\mathrm{SU}(2)_R^{[3]}$ Charge-Conjugate fields of Σ_R and Δ_3 as follows

$$\Sigma_R^c \equiv i\sigma_R^2 \Sigma_R^* \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, -1/2)\,, \qquad \Delta_3^c \equiv i\sigma_R^2 \Delta_3^* \sim (\mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{2}, 0) \tag{1.2.8}$$

with σ_R^2 being the second Pauli matrix (G.1.12) that acts in the $\mathrm{SU}(2)_R^{[3]}$ space.

In the following Sections we go through all the SSB steps for each of the models considered. At last, in Section 1.9 we list all the resulting Gauge boson fields in the broken phase together with their masses and the expression of the covariant derivative.

1.3 Model A

The first step is triggered by Ω_1 and Ω_3 and we choose their VEVs to be

$$\langle \Omega_1 \rangle = \begin{pmatrix} 0 & 0 & 0 & \omega_1 \end{pmatrix}\,, \qquad \langle \Omega_3 \rangle = \begin{pmatrix} \omega_3 & 0 & 0 & 0 \\ 0 & \omega_3 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \end{pmatrix}\,. \tag{1.3.1}$$

The SSB is evaluated in Section A.1 in the Appendix and we find that the residual Gauge Group is

$$H_A = \mathrm{SU}(2)_L \times \mathrm{SU}(3)_C \times \mathrm{SU}(2)_R^{[3]} \times \mathrm{U}(1)_{X'} \tag{1.3.2}$$

with Gauge couplings and fields denoted by

$$(g_L, g_s, g_R, g_{X'})\,, \qquad \left(W_L^i, G^a, W_R^j, B'\right) \tag{1.3.3}$$

where $a = 1, ..., 8$ and $i, j = 1, 2, 3$ and (where $\hat{g}_4 = \sqrt{6}\, g_4/2$)

$$g_s = \frac{g_4 g_3}{\sqrt{g_4^2 + g_3^2}}\,, \qquad g_{X'} = \frac{\hat{g}_4 g_X}{\sqrt{\hat{g}_4^2 + g_X^2}}\,. \tag{1.3.4}$$

[3]Actually, in full generality for Models C and F we could assume more general $\mathrm{U}(1)_X^{[12]}$ Charges, namely for $\Omega_3 \sim \eta/6$ while for $\Omega_1 \sim -\eta/2$ where η is a free parameter. To get the SM Hypercharges we just need to rescale all the Charges of the particle content as $X^{[12]} \to \eta X^{[12]}$. However this seems to be a minor and not well justified generalization.

The $\mathrm{U}(1)_{X'}$ generator is given by

$$X' = X^{[12]} + \frac{\sqrt{6}}{3}\hat{T}^{15} \tag{1.3.5}$$

and the diagonalized Gauge boson basis reads (where $a = 1, ..., 8$)

$$G^a = s_{43}H^a + c_{43}C^a \qquad m_G^2 = 0\,, \tag{1.3.6a}$$

$$G'^a = c_{43}H^a - s_{43}C^a \qquad m_{G'}^2 = \omega_3^2\left(g_3^2 + g_4^2\right)\,, \tag{1.3.6b}$$

$$\begin{aligned} U_r^{\pm} &= \frac{1}{\sqrt{2}}\left(H^9 \mp iH^{10}\right) \\ U_b^{\pm} &= \frac{1}{\sqrt{2}}\left(H^{11} \mp iH^{12}\right) \qquad m_U^2 = \frac{g_4^2}{2}\left(\omega_1^2 + \omega_3^2\right)\,, \\ U_g^{\pm} &= \frac{1}{\sqrt{2}}\left(H^{13} \mp iH^{14}\right) \end{aligned} \tag{1.3.6c}$$

$$B' = s_{4X}H^{15} + c_{4X}B_{12} \qquad m_{B'}^2 = 0\,, \tag{1.3.6d}$$

$$Z' = c_{4X}H^{15} - s_{4X}B_{12} \qquad m_{Z'}^2 = \frac{1}{2}\left(\omega_1^2 + \frac{\omega_3^2}{3}\right)\left(\hat{g}_4^2 + g_X^2\right) \tag{1.3.6e}$$

where we have defined

$$c_{43} = \frac{g_4}{\sqrt{g_4^2 + g_3^2}}\,, \quad s_{43} = \frac{g_3}{\sqrt{g_4^2 + g_3^2}}\,, \quad c_{4X} = \frac{\hat{g}_4}{\sqrt{\hat{g}_4^2 + g_X^2}}\,, \quad s_{4X} = \frac{g_X}{\sqrt{\hat{g}_4^2 + g_X^2}}\,. \tag{1.3.7}$$

The massive boson fields have Charges $U_c^{\pm} \sim (\mathbf{1}, \mathbf{3}, \mathbf{1}, \pm 2/3)$ (where $c = r, b, g$), $G'^a \sim (\mathbf{1}, \mathbf{8}, \mathbf{1}, 0)$ and $Z' \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0)$ under H_A. The covariant derivative reads as follows

$$\begin{aligned} D_\mu = {}& \partial_\mu - ig_L T_L^i W_L^i - ig_s T^a G^a - ig_R T_R^j W_R^j - ig_{X'} X' B' \\ & - ig_4 \sum_{c=r,b,g}\left(\hat{T}_c^+ U_c^+ + \hat{T}_c^- U_c^-\right) - i\frac{g_4}{2c_{43}}\left(c_{43}^2 - s_{43}^2\right) T^a G'^a - i\frac{\hat{g}_4}{c_{4X}}\left[\frac{\sqrt{6}}{3}\hat{T}^{15} - s_{4X}^2 X'\right] Z' \end{aligned} \tag{1.3.8}$$

where we have defined

$$\hat{T}_r^{\pm} = \frac{1}{\sqrt{2}}\left(\hat{T}^9 \pm i\hat{T}^{10}\right)\,, \quad \hat{T}_b^{\pm} = \frac{1}{\sqrt{2}}\left(\hat{T}^{11} \pm i\hat{T}^{12}\right)\,, \quad \hat{T}_g^{\pm} = \frac{1}{\sqrt{2}}\left(\hat{T}^{13} \pm i\hat{T}^{14}\right)\,. \tag{1.3.9}$$

The second step is triggered by $\Sigma_R \sim (\mathbf{1}, \mathbf{1}, \mathbf{2}, 1/2)$ (whose Charges are referred to H_A) and its VEV is chosen to be

$$\langle \Sigma_R \rangle = \begin{pmatrix} 0 \\ v_R \end{pmatrix}\,. \tag{1.3.10}$$

The SSB is evaluated in Section A.2 in the Appendix and we find H_{SM} as residual Gauge Group with Hypercharge

$$Y = T_R^3 + X' = T_R^3 + X^{[12]} + \frac{\sqrt{6}}{3}\hat{T}^{15} \tag{1.3.11}$$

and coupling

$$g_Y = \frac{g_R g_{X'}}{\sqrt{g_R^2 + g_{X'}^2}}\,. \tag{1.3.12}$$

To find the diagonalized Gauge boson basis we need to notice that we have an interaction between Σ_R and Z' which changes slightly the mixing evaluated in Section A.2. In particular the covariant derivative gets the following contribution

$$D_\mu \Sigma_R \supset -\frac{i}{2}\frac{g_X^2}{\sqrt{g_X^2 + \hat{g}_4^2}} Z' \Sigma_R\,. \tag{1.3.13}$$

When Σ_R acquires its VEV, it modifies the mass-matrix as follows

$$\mathcal{L}_M \supset \frac{1}{4}\left(\omega_1^2 + \frac{\omega_3^2}{3}\right)\left(\hat{g}_4 H^{15} - g_X B_{12}\right)^2 + \frac{1}{4} v_R^2 \left(g_R W_R^3 - g_X B_{12}\right)^2 . \tag{1.3.14}$$

1.4 Model B

The first step is triggered by Σ_R and we choose its VEV to be

$$\langle \Sigma_R \rangle = \begin{pmatrix} 0 \\ v_R \end{pmatrix} . \tag{1.4.1}$$

The SSB is evaluated in Section A.2 in the Appendix and we find that the residual Gauge Group is

$$H_B = \mathrm{SU}(2)_L \times \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_{X'} \tag{1.4.2}$$

with Gauge couplings and fields denoted by

$$(g_L, g_4, g_3, g_{X'}) , \qquad \left(W_L^i, H^a, C^b, B'\right) \tag{1.4.3}$$

where $a = 1, ..., 15$, $b = 1, ..., 8$ and $i = 1, 2, 3$ and

$$g_{X'} = \frac{g_X g_R}{\sqrt{g_R^2 + 4X_R^2 g_X^2}} . \tag{1.4.4}$$

The $\mathrm{U}(1)_{X'}$ generator is given by

$$X' = T_R^3 + X^{[12]} \tag{1.4.5}$$

and the diagonalized Gauge boson basis reads

$$W_R^\pm = \frac{1}{\sqrt{2}}\left(W_R^1 \mp i W_R^2\right) \qquad m_{W_R}^2 = \frac{g_R^2}{2} v_R^2 , \tag{1.4.6a}$$

$$B' = c_{RX} B_{12} + s_{RX} W_R^3 \qquad m_{B'}^2 = 0 , \tag{1.4.6b}$$

$$Z' = -s_{RX} B_{12} + c_{RX} W_R^3 \qquad m_{Z'}^2 = \frac{g_R^2}{2} v_R^2 \left(1 + 4\frac{g_X^2}{g_R^2} X_R^2\right) \tag{1.4.6c}$$

where we have defined

$$c_{RX} = \frac{g_R}{\sqrt{g_R^2 + 4g_X^2 X_R^2}} , \quad s_{RX} = \frac{2 g_X X_R}{\sqrt{g_R^2 + 4g_X^2 X_R^2}} . \tag{1.4.7}$$

The massive boson fields have Charges $W_R^\pm \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \pm 1)$ and $Z' \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0)$ under H_B (1.4.2). The covariant derivative reads as follows

$$\begin{aligned} D_\mu = \partial_\mu &- i g_L T_L^i W_L^i - i g_4 \hat{T}^a H^a - i g_3 T^b C^b - i g_{X'} X' B' \\ &- i g_R \left(T_R^+ W_R^+ + T_R^- W_R^-\right) - i \frac{g_R}{c_{RX}} \left[T_R^3 - s_{RX}^2 X'\right] Z' \end{aligned} \tag{1.4.8}$$

where we have defined

$$T_R^\pm = \frac{1}{\sqrt{2}}\left(T_R^1 \pm i T_R^2\right) . \tag{1.4.9}$$

The second step is triggered by $\Omega_1 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, -1/2)$ and $\Omega_3 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{3}, 1/6)$ (whose Charges are referred to H_B) and their VEVs are chosen to be

$$\langle \Omega_1 \rangle = \begin{pmatrix} 0 & 0 & 0 & \omega_1 \end{pmatrix} , \qquad \langle \Omega_3 \rangle = \begin{pmatrix} \omega_3 & 0 & 0 & 0 \\ 0 & \omega_3 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \end{pmatrix} . \tag{1.4.10}$$

The SSB is evaluated in Section A.1 in the Appendix and we find $H_{\rm SM}$ as residual Gauge Group with Hypercharge

$$Y = \frac{\sqrt{6}}{3}\hat{T}^{15} + X' = \frac{\sqrt{6}}{3}\hat{T}^{15} + X^{[12]} + T^3_R \tag{1.4.11}$$

and couplings

$$g_s = \frac{g_4 g_3}{\sqrt{g_4^2 + g_3^2}}\,, \qquad g_Y = \frac{\hat{g}_4 g_{X'}}{\sqrt{\hat{g}_4^2 + 4\, g_{X'}^2 X_\Omega^2}}\,. \tag{1.4.12}$$

To find the diagonalized Gauge boson basis we need to notice that we have an interaction between Ω_1, Ω_3 and Z' which changes slightly the mixing evaluated in Section A.1. In particular the covariant derivatives get the contributions

$$D_\mu\Omega_1 \supset \frac{i}{2}\frac{g_X^2}{\sqrt{g_R^2 + g_X^2}} Z'\,\Omega_1\,, \qquad D_\mu\Omega_3 \supset -\frac{i}{6}\frac{g_X^2}{\sqrt{g_R^2 + g_X^2}} Z'\,\Omega_3\,. \tag{1.4.13}$$

When Ω_1 and Ω_3 acquire their VEVs, it modifies the mass-matrix as follows

$$\mathcal{L}_M \supset \frac{1}{4}v_R^2\left(g_R W_R^3 - g_X B_{12}\right)^2 + \frac{1}{4}\left(\omega_1^2 + \frac{\omega_3^2}{3}\right)\left(\hat{g}_4 H^{15} - g_X B_{12}\right)^2\,. \tag{1.4.14}$$

1.5 Model C

The first step is triggered by Δ_3 and we choose its VEV to be

$$\langle\Delta\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w \end{pmatrix}\,. \tag{1.5.1}$$

The SSB is evaluated in Section A.3 in the Appendix and we find that the residual Gauge Group is

$$H_C = {\rm SU}(2)_L \times {\rm SU}(3)^{[3]} \times {\rm SU}(3)^{[12]} \times {\rm U}(1)_X^{[3]} \times {\rm U}(1)_X^{[12]} \tag{1.5.2}$$

with Gauge couplings and fields denoted by

$$\left(g_L, g_s^h, g_s^\ell, g_X^h, g_X^\ell\right)\,, \qquad \left(W_L^i, H^a, C^b, B_3, B_{12}\right) \tag{1.5.3}$$

where $a, b = 1, ..., 8$ and $i = 1, 2, 3$ and

$$g_s^h = g_4\,, \qquad g_s^\ell = g_3\,, \qquad g_X^h = \frac{\hat{g}_4 g_R}{\sqrt{\hat{g}_4^2 + g_R^2}}\,, \qquad g_X^\ell = g_X\,. \tag{1.5.4}$$

The ${\rm U}(1)_X^{[3]}$ generator is given by

$$X^{[3]} = T_R^3 + \frac{\sqrt{6}}{3}\hat{T}^{15} \tag{1.5.5}$$

and the diagonalized Gauge boson basis reads (we omit H^a for $a = 1, ..., 8$ that are still present)

$$\begin{aligned} U_r^\pm &= \frac{1}{\sqrt{2}}\left(H^9 \mp iH^{10}\right) \\ U_b^\pm &= \frac{1}{\sqrt{2}}\left(H^{11} \mp iH^{12}\right) \qquad m_U^2 = \frac{g_4^2}{2}w^2\,, \\ U_g^\pm &= \frac{1}{\sqrt{2}}\left(H^{13} \mp iH^{14}\right) \end{aligned} \tag{1.5.6a}$$

$$W_R^\pm = \frac{1}{\sqrt{2}}\left(W_R^1 \mp iW_R^2\right) \qquad m_{W_R}^2 = \frac{g_R^2}{2}w^2\,, \tag{1.5.6b}$$

$$B_3 = c_{4R}W_R^3 + s_{4R}H^{15} \qquad m_{B_3}^2 = 0\,, \tag{1.5.6c}$$

$$Z' = -s_{4R}W_R^3 + c_{4R}H^{15} \qquad m_{Z'}^2 = \frac{1}{2}w^2\left(\hat{g}_4^2 + g_R^2\right) \tag{1.5.6d}$$

where we have defined

$$c_{4R} = \frac{\hat{g}_4}{\sqrt{\hat{g}_4^2 + g_R^2}}\,, \quad s_{4R} = \frac{g_R}{\sqrt{\hat{g}_4^2 + g_R^2}}\,. \tag{1.5.7}$$

The massive boson fields have Charges $U_c^{\pm} \sim (\mathbf{1}, \mathbf{3}, \mathbf{1}, \pm 2/3, 0)$ (where $c = r, b, g$), $W_R^{\pm} \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \pm 1, 0)$ and $Z' \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0)$ under H_C. The covariant derivative reads as follows

$$\begin{aligned} D_\mu = \partial_\mu &- i g_L T_L^i W_L^i - i g_s^h T^a H^a - i g_s^\ell T^b C^b - i g_X^h X^{[3]} B_3 - i g_X^\ell X^{[12]} B_{12} \\ &- i g_R \left(T_R^+ W_R^+ + T_R^- W_R^-\right) - i g_4 \sum_{c=r,b,g} \left(\hat{T}_c^+ U_c^+ + \hat{T}_c^- U_c^-\right) - i \frac{\hat{g}_4}{c_{4R}} \left[\frac{\sqrt{6}}{3} \hat{T}^{15} - s_{4R}^2 X^{[3]}\right] Z'\,. \end{aligned} \tag{1.5.8}$$

The second step is triggered by Ω_1 and Ω_3 which at this stage each one is split into two fields, one of which will acquire a VEV. Namely, under H_{UV} they can be written in the following forms

$$\Omega_1 = \begin{pmatrix} \Omega_1' & \tilde{\Omega}_1 \end{pmatrix}\,, \qquad \Omega_3 = \begin{pmatrix} \Omega_3' & \tilde{\Omega}_3 \end{pmatrix} \tag{1.5.9}$$

and under H_C each of them have Charges

$$\begin{aligned} &\Omega_1' \sim (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1}, -1/6, -1/2)\,, \qquad \Omega_3'(\mathbf{1}, \mathbf{1}, \mathbf{3}, 1/2, 1/6)\,, \\ &\tilde{\Omega}_1 \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, 1/2, -1/2)\,, \qquad \tilde{\Omega}_3 \sim (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{3}, -1/6, 1/6)\,. \end{aligned} \tag{1.5.10}$$

We make only the last two to acquire a VEV that we choose to be

$$\langle \tilde{\Omega}_1 \rangle = \omega_1\,, \qquad \langle \tilde{\Omega}_3 \rangle = \begin{pmatrix} \omega_3 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}\,. \tag{1.5.11}$$

The SSB is evaluated in Section A.5 in the Appendix and we find H_{SM} as residual Gauge Group with Hypercharge

$$Y = X^{[3]} + X^{[12]} = T_R^3 + \frac{\sqrt{6}}{3} \hat{T}^{15} + X^{[12]} \tag{1.5.12}$$

and couplings

$$g_s = \frac{g_s^h g_s^\ell}{\sqrt{(g_s^h)^2 + (g_s^\ell)^2}}\,, \qquad g_Y = \frac{g_X^h g_X^\ell}{\sqrt{(g_X^h)^2 + (g_X^\ell)^2}}\,. \tag{1.5.13}$$

To find the diagonalized Gauge boson basis we need to notice that we have an interaction between Ω_1, Ω_3 and $Z', U_c^{\pm}$ which changes slightly the mixing evaluated in Section A.5. In particular the covariant derivatives get the contributions

$$D_\mu \Omega_1 \supset i g_4 \sum_{c=r,b,g} \left(\Omega_1 \hat{T}_c^- U_c^-\right) + \frac{i}{2} \frac{g_R^2}{\sqrt{g_R^2 + \hat{g}_4^2}} Z' \,\Omega_1\,, \tag{1.5.14a}$$

$$D_\mu \Omega_3 \supset i g_4 \sum_{c=r,b,g} \left(\Omega_3 \hat{T}_c^+ U_c^+\right) - \frac{i}{6} \frac{g_R^2}{\sqrt{g_R^2 + \hat{g}_4^2}} Z' \,\Omega_3\,. \tag{1.5.14b}$$

When Ω_1 and Ω_3 acquire their VEVs, it modifies the mass-matrix as follows

$$\mathcal{L}_M \supset \frac{g_4^2}{2}(\omega_1^2 + \omega_3^2) \sum_{c=r,b,g} U_c^+ U_c^- + \frac{1}{4} w^2 \left(g_R W_R^3 - \hat{g}_4 H^{15}\right)^2 + \frac{1}{4}\left(\omega_1^2 + \frac{\omega_3^2}{3}\right)\left(\hat{g}_4 H^{15} - g_X B_{12}\right)^2\,. \tag{1.5.15}$$

1.6 Model D

The first step is triggered by Σ_3 and we choose its VEV to be

$$\langle\Sigma_3\rangle = \frac{v_3}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = v_3\, T_R^3\,. \tag{1.6.1}$$

The SSB is evaluated in Section A.6 in the Appendix and we find that the residual Gauge Group is

$$H_D = \mathrm{SU}(2)_L \times \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_R^{[3]} \times \mathrm{U}(1)_X^{[12]} \tag{1.6.2}$$

with Gauge fields and couplings denoted by

$$(g_L, g_4, g_3, g_R, g_X)\;, \qquad \left(W_L^i, H^a, C^b, B_3, B_{12}\right) \tag{1.6.3}$$

where $a = 1, ..., 15$, $b = 1, ..., 8$ and $i = 1, 2, 3$. The $\mathrm{U}(1)_R^{[3]}$ generator is given by

$$X_R^{[3]} = T_R^3 \tag{1.6.4}$$

and the diagonalized Gauge boson basis (regarding the broken sector) is given by

$$W_R^{\pm} = \frac{1}{\sqrt{2}}\left(W_R^1 \mp i W_R^2\right) \qquad m_{W_R}^2 = g_R^2\, v_3^2\,, \tag{1.6.5a}$$

$$B_3 = W_R^3 \qquad m_{B_3}^2 = 0\,. \tag{1.6.5b}$$

The new massive Gauge bosons have Charges $W_R^{\pm} \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \pm 1, 0)$ under H_D. The covariant derivative reads as follows

$$D_\mu = \partial_\mu - i g_L T_L^i W_L^i - i g_4 \hat{T}^a H^a - i g_3 T^b C^b - i g_X X^{[12]} B_{12} - i g_R X_R^{[3]} B_3 - i g_R \left(T_R^+ W_R^+ + T_R^- W_R^-\right)\,. \tag{1.6.6}$$

The second step is triggered by $\Omega_1 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, 0, -1/2)$ and $\Omega_3 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{3}, 0, 1/6)$ (whose Charges are referred to H_D) and their VEVs are chosen to be

$$\langle\Omega_1\rangle = \begin{pmatrix}0 & 0 & 0 & \omega_1\end{pmatrix}\,, \qquad \langle\Omega_3\rangle = \begin{pmatrix}\omega_3 & 0 & 0 & 0\\ 0 & \omega_3 & 0 & 0\\ 0 & 0 & \omega_3 & 0\end{pmatrix}\,. \tag{1.6.7}$$

The SSB is evaluated in Section A.1 in the Appendix and we find that the residual Gauge Group is

$$H_D' = \mathrm{SU}(2)_L \times \mathrm{SU}(3)_C \times \mathrm{U}(1)_R^{[3]} \times \mathrm{U}(1)_{X'} \tag{1.6.8}$$

with Gauge couplings and fields denoted by

$$(g_L, g_s, g_R, g_{X'})\;, \qquad \left(W_L^i, G^a, B_3, B'\right) \tag{1.6.9}$$

where $a = 1, ..., 8$ and $i = 1, 2, 3$ and

$$g_s = \frac{g_4 g_3}{\sqrt{g_4^2 + g_3^2}}\,, \qquad g_{X'} = \frac{\hat{g}_4 g_X}{\sqrt{\hat{g}_4^2 + g_X^2}}\,. \tag{1.6.10}$$

The $\mathrm{U}(1)_{X'}$ generator is given by

$$X' = X^{[12]} + \frac{\sqrt{6}}{3}\hat{T}^{15} \tag{1.6.11}$$

and the diagonalized Gauge boson basis reads the same found in (1.3.6) (with even same notation) and the new massive boson fields have Charges $U_c^\pm \sim (\mathbf{1},\mathbf{3},0,\pm 2/3)$ (where $c = r, b, g$), $G'^a \sim (\mathbf{1},\mathbf{8},0,0)$ and $Z' \sim (\mathbf{1},\mathbf{1},0,0)$ under H'_D. The covariant derivative reads as follows

$$
\begin{aligned}
D_\mu = \partial_\mu &- i g_L T_L^i W_L^i - i g_s T^a G^a - i g_R X_R^{[3]} B_3 - i g_{X'} X' B' - i g_R \left(T_R^+ W_R^+ + T_R^- W_R^- \right) \\
&- i g_4 \sum_{c=r,b,g} \left(\hat{T}_c^+ U_c^+ + \hat{T}_c^- U_c^- \right) - i \frac{g_4}{2 c_{43}} \left(c_{43}^2 - s_{43}^2 \right) T^a G'^a - i \frac{\hat{g}_4}{c_{4X}} \left[\frac{\sqrt{6}}{3} \hat{T}^{15} - s_{4X}^2 X' \right] Z' \,.
\end{aligned} \tag{1.6.12}
$$

The third step is triggered by Σ_R which at this stage is split into two fields, one of which will acquire a VEV. Namely under H_{UV} it can be written in the following form

$$
\Sigma_R = \begin{pmatrix} \Sigma_R^+ \\ \Sigma_R^0 \end{pmatrix} \tag{1.6.13}
$$

and under H'_D each of them have Charges

$$
\Sigma_R^0 \sim (\mathbf{1},\mathbf{1},-1/2,1/2)\,, \qquad \Sigma_R^+ \sim (\mathbf{1},\mathbf{1},1/2,1/2)\,. \tag{1.6.14}
$$

We make only the first one to acquire a VEV that we choose to be

$$
\langle \Sigma_R^0 \rangle = v_R \,. \tag{1.6.15}
$$

The SSB is evaluated in Section A.7 in the Appendix and we find H_{SM} as residual Gauge Group with Hypercharge

$$
Y = X_R^{[3]} + X' = X^{[12]} + \frac{\sqrt{6}}{3} \hat{T}^{15} + T_R^3 \tag{1.6.16}
$$

and coupling

$$
g_Y = \frac{g_R g_{X'}}{\sqrt{g_R^2 + g_{X'}^2}} \,. \tag{1.6.17}
$$

To find the diagonalized Gauge boson basis we need to notice that we have an interaction between Σ_R^0 and $Z', W_R^\pm$ which changes slightly the mixing evaluated in Section A.7. In particular the covariant derivative gets the contributions

$$
D_\mu \Sigma_R^0 \supset -\frac{i}{2} \frac{g_X^2}{\sqrt{g_X^2 + \hat{g}_4^2}} Z' \Sigma_R^0 - i g_R T_R^+ W_R^+ \Sigma_R^0 \,. \tag{1.6.18}
$$

When Σ_R^0 acquires its VEV, it modifies the mass-matrix as follows

$$
\mathcal{L}_M \supset \frac{1}{4} \left(\omega_1^2 + \frac{\omega_3^2}{3} \right) \left(\hat{g}_4 H^{15} - g_X B_{12} \right)^2 + \frac{1}{4} v_R^2 \left(g_R W_R^3 - g_X B_{12} \right)^2 + g_R^2 v_3^2 \left(1 + \frac{v_R^2}{2 v_3^2} \right) W_R^+ W_R^- \,. \tag{1.6.19}
$$

1.7 Model E

The first step is completely equivalent to the one of Model D. The second step is triggered by Σ_R which is a doublet of two scalar fields as in (1.6.13) with Charges under $H_E = H_D$ (1.6.2)

$$
\Sigma_R^0 \sim (\mathbf{1},\mathbf{1},\mathbf{1},-1/2,1/2)\,, \qquad \Sigma_R^+ \sim (\mathbf{1},\mathbf{1},\mathbf{1},1/2,1/2)\,. \tag{1.7.1}
$$

We make only the first one to acquire a VEV that we choose to be

$$
\langle \Sigma_R^0 \rangle = v_R \,. \tag{1.7.2}
$$

The SSB is evaluated in Section A.7 in the Appendix and we find that the residual Gauge Group is

$$H'_E = \mathrm{SU}(2)_L \times \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_{X'} \tag{1.7.3}$$

with Gauge couplings and fields denoted by

$$(g_L, g_4, g_3, g_{X'}) \ , \qquad \left(W_L^i, H^a, C^b, B'\right) \tag{1.7.4}$$

where $a = 1, ..., 15$, $b = 1, ..., 8$ and $i = 1, 2, 3$ and

$$g_{X'} = \frac{g_X g_R}{\sqrt{g_X^2 + g_R^2}} \, . \tag{1.7.5}$$

The $\mathrm{U}(1)_{X'}$ generator is given by

$$X' = X^{[12]} + T_R^3 \, . \tag{1.7.6}$$

Since Σ_R is a $\mathrm{SU}(2)_R^{[3]}$ doublet, we have an interaction between Σ_R^0 and $W_R^\pm$ in the covariant derivative which changes slightly the masses of $W_R^\pm$. In particular the covariant derivative gets the contribution

$$D_\mu \Sigma_R^0 \supset -i g_R T_R^+ W_R^+ \Sigma_R^0 \, . \tag{1.7.7}$$

When Σ_R^0 acquires its VEV, it modifies the mass-matrix as follows

$$\mathcal{L}_M \supset g_R^2 v_3^2 \left(1 + \frac{v_R^2}{2 v_3^2}\right) W_R^+ W_R^- = m_{W_R}^2 W_R^+ W_R^- \, . \tag{1.7.8}$$

The diagonalized Gauge boson basis reads

$$B' = s_{XR} B_{12} + c_{XR} B_3 \qquad m_{B'}^2 = 0 \, , \tag{1.7.9a}$$

$$Z' = c_{XR} B_{12} - s_{XR} B_3 \qquad m_{Z'}^2 = \frac{1}{2} v_R^2 \left(g_X^2 + g_R^2\right) \tag{1.7.9b}$$

where we have defined

$$c_{XR} = \frac{g_X}{\sqrt{g_X^2 + g_R^2}} \, , \quad s_{XR} = \frac{g_R}{\sqrt{g_X^2 + g_R^2}} \, . \tag{1.7.10}$$

The massive boson field has Charges $Z' \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0)$ under H'_E. The covariant derivative reads as follows

$$\begin{aligned} D_\mu = \partial_\mu - i g_L T_L^i W_L^i - i g_4 \hat{T}^a H^a - i g_3 T^b C^b - i g_{X'} X' B' \\ - i g_R \left(T_R^+ W_R^+ + T_R^- W_R^-\right) - i \frac{g_X}{c_{XR}} \left(X^{[12]} - s_{XR}^2 X'\right) Z' \, . \end{aligned} \tag{1.7.11}$$

The third step is triggered by $\Omega_1 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, -1/2)$ and $\Omega_3 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{3}, 1/6)$ (whose Charges are referred to H'_E) and their VEVs are chosen to be

$$\langle \Omega_1 \rangle = \begin{pmatrix} 0 & 0 & 0 & \omega_1 \end{pmatrix} \, , \qquad \langle \Omega_3 \rangle = \begin{pmatrix} \omega_3 & 0 & 0 & 0 \\ 0 & \omega_3 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \end{pmatrix} \, . \tag{1.7.12}$$

The SSB is evaluated in Section A.1 in the Appendix and we find H_{SM} as residual Gauge Group with Hypercharge

$$Y = \frac{\sqrt{6}}{3} \hat{T}^{15} + X' = \frac{\sqrt{6}}{3} \hat{T}^{15} + X^{[12]} + T_R^3 \tag{1.7.13}$$

and coupling

$$g_Y = \frac{\hat{g}_4 g_{X'}}{\sqrt{\hat{g}_4 + g_{X'}^2}} \, . \tag{1.7.14}$$

To find the diagonalized Gauge boson basis we need to notice that we have an interaction between Ω_1, Ω_3 and Z' which changes slightly the mixing evaluated in Section A.1. In particular the covariant derivatives get the contributions

$$D_\mu \Omega_1 \supset \frac{i}{2} \frac{g_X^2}{\sqrt{g_R^2 + g_X^2}} Z' \, \Omega_1 \,, \qquad D_\mu \Omega_3 \supset -\frac{i}{6} \frac{g_X^2}{\sqrt{g_R^2 + g_X^2}} Z' \, \Omega_3 \,. \tag{1.7.15}$$

When Ω_1 and Ω_3 acquire their VEVs, it modifies the mass-matrix as follows

$$\mathcal{L}_M \supset \frac{1}{4} v_R^2 \left(g_R W_R^3 - g_X B_{12} \right)^2 + \frac{1}{4} \left(\omega_1^2 + \frac{\omega_3^2}{3} \right) \left(\hat{g}_4 H^{15} - g_X B_{12} \right)^2 \,. \tag{1.7.16}$$

1.8 Model F

The first step is completely equivalent to the one of Model D. The second step is triggered by Δ_3 which at this stage is split into two fields, one of which will acquire a VEV. Namely it can be written in the following form

$$\Delta_3 = \begin{pmatrix} \Delta_3^+ \\ \Delta_3^0 \end{pmatrix} \tag{1.8.1}$$

and under $H_F = H_D$ (1.6.2) each of them have Charges

$$\Delta_3^0 \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, -1/2, 0) \,, \qquad \Delta_3^+ \sim (\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, 1/2, 0) \,. \tag{1.8.2}$$

We make only the first one to acquire a VEV that we choose to be

$$\langle \Delta_3 \rangle = \begin{pmatrix} 0 & 0 & 0 & w \end{pmatrix} \,. \tag{1.8.3}$$

The SSB is evaluated in Section A.4 in the Appendix and we find that the residual Gauge Group is

$$H_F' = \mathrm{SU}(2)_L \times \mathrm{SU}(3)^{[3]} \times \mathrm{SU}(3)^{[12]} \times \mathrm{U}(1)_X^{[3]} \times \mathrm{U}(1)_X^{[12]} \tag{1.8.4}$$

with Gauge couplings and fields denoted by

$$\left(g_L, g_s^h, g_s^\ell, g_X^h, g_X^\ell \right) \,, \qquad \left(W_L^i, H^a, C^b, B_3, B_{12} \right) \tag{1.8.5}$$

where $a, b = 1, ..., 8$ and $i = 1, 2, 3$ and

$$g_s^h = g_4 \,, \qquad g_s^\ell = g_3 \,, \qquad g_X^h = \frac{\hat{g}_4 g_R}{\sqrt{\hat{g}_4^2 + g_R^2}} \,, \qquad g_X^\ell = g_X \,. \tag{1.8.6}$$

The $\mathrm{U}(1)_X^{[3]}$ generator is given by

$$X^{[3]} = T_R^3 + \frac{\sqrt{6}}{3} \hat{T}^{15} \,. \tag{1.8.7}$$

Since Δ_3 is a $\mathrm{SU}(2)_R^{[3]}$ doublet, we have an interaction between Δ_3^0 and $W_R^\pm$ in the covariant derivative which changes slightly the masses of $W_R^\pm$. In particular the covariant derivative gets the contribution

$$D_\mu \Delta_3^0 \supset -i g_R T_R^+ W_R^+ \Delta_3^0 \,. \tag{1.8.8}$$

When Σ_R^0 acquires its VEV, it modifies the mass-matrix as follows

$$\mathcal{L}_M \supset g_R^2 v_3^2 \left(1 + \frac{w^2}{2 v_3^2} \right) W_R^+ W_R^- = m_{W_R}^2 W_R^+ W_R^- \,. \tag{1.8.9}$$

The diagonalized Gauge boson basis reads (we omit H^a for $a = 1, ..., 8$ that are still present)

$$
\begin{aligned}
U_r^{\pm} &= \frac{1}{\sqrt{2}}\left(H^9 \mp iH^{10}\right) \\
U_b^{\pm} &= \frac{1}{\sqrt{2}}\left(H^{11} \mp iH^{12}\right) \qquad m_U^2 = \frac{g_4^2}{2}w^2\,, \\
U_g^{\pm} &= \frac{1}{\sqrt{2}}\left(H^{13} \mp iH^{14}\right)
\end{aligned}
\tag{1.8.10a}
$$

$$
B_3 = c_{4R}W_R^3 + s_{4R}H^{15} \qquad m_{B_3}^2 = 0\,, \tag{1.8.10b}
$$

$$
Z' = -s_{4R}W_R^3 + c_{4R}H^{15} \qquad m_{Z'}^2 = \frac{1}{2}w^2\left(\hat{g}_4^2 + g_R^2\right) \tag{1.8.10c}
$$

where we have defined

$$
c_{4R} = \frac{\hat{g}_4}{\sqrt{\hat{g}_4^2 + g_R^2}}\,, \qquad s_{4R} = \frac{g_R}{\sqrt{\hat{g}_4^2 + g_R^2}}\,. \tag{1.8.11}
$$

The massive boson fields have Charges $U_c^{\pm} \sim (\mathbf{1}, \mathbf{3}, \mathbf{1}, \pm 2/3, 0)$ (where $c = r, b, g$), $W_R^{\pm} \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, \pm 1, 0)$ and $Z' \sim (\mathbf{1}, \mathbf{1}, \mathbf{1}, 0)$ under H'_F. The covariant derivative reads as follows

$$
\begin{aligned}
D_\mu = \partial_\mu &- ig_L T_L^i W_L^i - ig_s^h T^a H^a - ig_s^\ell T^b C^b - ig_X^h X^{[3]} B_3 - ig_X^\ell X^{[12]} B_{12} \\
&- ig_R\left(T_R^+ W_R^+ + T_R^- W_R^-\right) - ig_4 \sum_{c=r,b,g}\left(\hat{T}_c^+ U_c^+ + \hat{T}_c^- U_c^-\right) - i\frac{\hat{g}_4}{c_{4R}}\left[\frac{\sqrt{6}}{3}\hat{T}^{15} - s_{4R}^2 X^{[3]}\right] Z'\,.
\end{aligned}
\tag{1.8.12}
$$

The third step is completely equivalent to the second step of Model C with even same notation.

1.9 Gauge Spectrum in the Broken Phase

In this Section we list and compare the Gauge mass-spectrum in the UV broken phase (thus at the SM Symmetry Group) for each of the different SSB mechanisms considered.

From the breaking of $\mathrm{SU}(4)^{[3]} \to \mathrm{SU}(3)^{[3]}$ we have six massive leptoquarks $U_c^{\pm}$ (where $c = r, b, g$). From the breaking of $\mathrm{SU}(3)^{[3]} \times \mathrm{SU}(3)^{[12]} \to \mathrm{SU}(3)_C$ we have eight massive Gluons G'^a (where $a = 1, ..., 8$). From the breaking of $\mathrm{SU}(2)_R^{[3]} \to \mathrm{U}(1)_R^{[3]}$ we have two massive W-type bosons $W_R^{\pm}$. At last, from the breaking of the diagonal subgroups of $\mathrm{SU}(4)^{[3]}$ and $\mathrm{SU}(2)_R^{[3]}$, we have two neutral massive Z-type bosons Z' and Z''. Explicitly we have that in all the models the massive gluons, leptoquarks and W-type vector bosons read as follows with corresponding Charges under H_{SM} (1.1.1)

$$
G'^a = \frac{g_4}{\sqrt{g_4^2 + g_3^2}}H^a - \frac{g_3}{\sqrt{g_4^2 + g_3^2}}C^a \qquad \sim (\mathbf{1}, \mathbf{8}, 0)\,, \tag{1.9.1a}
$$

$$
\left.
\begin{aligned}
U_r^{\pm} &= \frac{1}{\sqrt{2}}\left(H^9 \mp iH^{10}\right) \\
U_b^{\pm} &= \frac{1}{\sqrt{2}}\left(H^{11} \mp iH^{12}\right) \\
U_g^{\pm} &= \frac{1}{\sqrt{2}}\left(H^{13} \mp iH^{14}\right)
\end{aligned}
\right\} \sim (\mathbf{1}, \mathbf{3}, \pm 2/3)\,, \tag{1.9.1b}
$$

$$
W_R^{\pm} = \frac{1}{\sqrt{2}}\left(W_R^1 \mp iW_R^2\right) \qquad \sim (\mathbf{1}, \mathbf{1}, \pm 1)\,. \tag{1.9.1c}
$$

Their masses depend on the model considered and explicitly they read as follows

$$
\text{Model A,B} \qquad m_{G'}^2 = \omega_3^2\left(g_3^2 + g_4^2\right)\,, \quad m_U^2 = \frac{1}{2}g_4^2\left(\omega_1^2 + \omega_3^2\right)\,, \quad m_{W_R}^2 = \frac{1}{2}g_R^2 v_R^2\,, \tag{1.9.2a}
$$

$$
\text{Model C} \qquad m_{G'}^2 = \omega_3^2\left(g_3^2 + g_4^2\right)\,, \quad m_U^2 = \frac{1}{2}g_4^2 w^2\left(1 + \frac{\omega_1^2 + \omega_3^2}{w^2}\right)\,, \quad m_{W_R}^2 = \frac{1}{2}g_R^2 w^2\,, \tag{1.9.2b}
$$

Model D,E $\quad m_{G'}^2 = \omega_3^2\left(g_3^2+g_4^2\right)\,,\quad m_U^2 = \frac{1}{2}g_4^2\left(\omega_1^2+\omega_3^2\right)\,,\quad m_{W_R}^2 = g_R^2 v_3^2\left(1+\frac{v_R^2}{2v_3^2}\right)\,, \quad (1.9.2c)$

Model F $\quad m_{G'}^2 = \omega_3^2\left(g_3^2+g_4^2\right)\,,\quad m_U^2 = \frac{1}{2}g_4^2 w^2\left(1+\frac{\omega_1^2+\omega_3^2}{w^2}\right)\,,\quad m_{W_R}^2 = g_R^2 v_3^2\left(1+\frac{w^2}{2v_3^2}\right)\,.$ (1.9.2d)

For all the models the eight massless gluons (Gauge bosons associated to $\mathrm{SU}(3)_C$) are given by (where $a = 1,...,8$)

$$G^a = \frac{g_3}{\sqrt{g_4^2+g_3^2}}H^a + \frac{g_4}{\sqrt{g_4^2+g_3^2}}C^a\,. \tag{1.9.3}$$

The Hypercharge (generator of $\mathrm{U}(1)_Y$) is given by

$$Y = T_R^3 + \frac{\sqrt{6}}{3}\hat{T}^{15} + X^{[12]}\,. \tag{1.9.4}$$

Also the couplings associated to H_{SM} are the same for all the models (as expected) and explicitly they are given by

$$g_s = \frac{g_3 g_4}{\sqrt{g_3^2+g_4^2}} = \left(\frac{1}{g_3^2}+\frac{1}{g_4^2}\right)^{-1/2}\,,\quad g_Y = \frac{\hat{g}_4 g_R g_X}{\sqrt{\hat{g}_4^2 g_R^2+\hat{g}_4^2 g_X^2+g_R^2 g_X^2}} = \left(\frac{1}{\hat{g}_4^2}+\frac{1}{g_R^2}+\frac{1}{g_X^2}\right)^{-1/2} \tag{1.9.5}$$

where $\hat{g}_4 = \sqrt{6}\,g_4/2$.

The main differences among the models are given by the three neutral vector bosons. They are always given by two neutral massive vector bosons with Charges $Z' \sim (\mathbf{1},\mathbf{1},0)$, $Z'' \sim (\mathbf{1},\mathbf{1},0)$ under H_{SM} (1.1.1) and one massless which is identified with B (the Gauge boson associated to $\mathrm{U}(1)_Y$). In terms of UV-theory quantities, they are generated by a linear combination of the vector bosons associated with the diagonal subgroups of the UV Gauge Group. The mixing matrices are found by diagonalizing the neutral mass-Lagrangian provided by the models. Also, to get somehow nicer expressions, we have worked up to first order in ε which, after having defined

$$\Omega^2 \equiv \omega_1^2 + \frac{\omega_3^2}{3}\,, \tag{1.9.6}$$

it is given by $\varepsilon = v_R^2/\Omega^2$ for Models A and D, $\varepsilon = \Omega^2/v_R^2$ for Models B and E while $\varepsilon = \Omega^2/w^2$ for Models C and F. We assume ε to be small thanks to the present hierarchy between the VEVs of the scalar fields which are acquired at different energy scales. The mixing matrices read as follows

Model A,D

$$\begin{pmatrix} B \\ Z'' \\ Z' \end{pmatrix} = \frac{1}{\sigma}\begin{pmatrix} g_R g_X & \hat{g}_4 g_X & \hat{g}_4 g_R \\ -\hat{g}_4 g_X^2 \rho_{4X}\left(1+\frac{v_R^2}{\Omega^2}\rho_{4X}^4\sigma^2\right) & g_R\rho_{4X}^{-1} & -\hat{g}_4^2 g_X\rho_{4X}\left(1-\frac{v_R^2}{\Omega^2}\frac{g_X^2}{\hat{g}_4^2}\rho_{4X}^4\sigma^2\right) \\ \hat{g}_4\rho_{4X}\sigma\left(1-\frac{v_R^2}{\Omega^2}g_X^4\rho_{4X}^4\right) & g_R g_X^2\rho_{4X}^3\sigma\frac{v_R^2}{\Omega^2} & -g_X\rho_{4X}\sigma\left(1+\frac{v_R^2}{\Omega^2}\hat{g}_4^2 g_X^2\rho_{4X}^4\right) \end{pmatrix}\begin{pmatrix} H^{15} \\ W_R^3 \\ B_{12} \end{pmatrix} \tag{1.9.7a}$$

Model B,E

$$\begin{pmatrix} B \\ Z'' \\ Z' \end{pmatrix} = \frac{1}{\sigma}\begin{pmatrix} g_R g_X & \hat{g}_4 g_X & \hat{g}_4 g_R \\ \hat{g}_4\rho_{RX}^{-1} & -g_R g_X^2\rho_{RX}\left(1+\frac{\Omega^2}{v_R^2}\rho_{RX}^4\sigma^2\right) & -g_R^2 g_X\rho_{RX}\left(1-\frac{\Omega^2}{v_R^2}\frac{g_X^2}{g_R^2}\rho_{RX}^4\sigma^2\right) \\ \hat{g}_4 g_X^2\rho_{RX}^3\sigma\frac{\Omega^2}{v_R^2} & g_R\rho_{RX}\sigma\left(1-\frac{\Omega^2}{v_R^2}g_X^4\rho_{RX}^4\right) & -g_X\rho_{RX}\sigma\left(1+\frac{\Omega^2}{v_R^2}g_R^2 g_X^2\rho_{RX}^4\right) \end{pmatrix}\begin{pmatrix} H^{15} \\ W_R^3 \\ B_{12} \end{pmatrix} \tag{1.9.7b}$$

Model C,F

$$\begin{pmatrix} B \\ Z'' \\ Z' \end{pmatrix} = \frac{1}{\sigma} \begin{pmatrix} g_R g_X & \hat{g}_4 g_X & \hat{g}_4 g_R \\ -\hat{g}_4 g_R^2 \rho_{4R} \left(1 - \dfrac{\Omega^2}{w^2} \dfrac{\hat{g}_4^2}{g_R^2} \rho_{4R}^4 \sigma^2\right) & -\hat{g}_4^2 g_R \rho_{4R} \left(1 + \dfrac{\Omega^2}{w^2} \rho_{4R}^4 \sigma^2\right) & g_X \rho_{4R}^{-1} \\ -\hat{g}_4 \rho_{4R} \sigma \left(1 + \dfrac{\Omega^2}{w^2} \hat{g}_4^2 g_R^2 \rho_{4R}^4\right) & g_R \rho_{4R} \sigma \left(1 - \dfrac{\Omega^2}{w^2} \hat{g}_4^4 \rho_{4R}^4\right) & g_X \hat{g}_4^2 \rho_{4R}^3 \sigma \dfrac{\Omega^2}{w^2} \end{pmatrix} \begin{pmatrix} H^{15} \\ W_R^3 \\ B_{12} \end{pmatrix} \tag{1.9.7c}$$

where we have defined

$$\rho_{4X}^{-1} = \sqrt{\hat{g}_4^2 + g_X^2}\,, \quad \rho_{4R}^{-1} = \sqrt{\hat{g}_4^2 + g_R^2}\,, \quad \rho_{RX}^{-1} = \sqrt{g_R^2 + g_X^2}\,, \quad \sigma = \sqrt{\hat{g}_4^2 g_R^2 + \hat{g}_4^2 g_X^2 + g_R^2 g_X^2}\,. \tag{1.9.8}$$

Notice that the massless vector boson B is the same in all the models as expected. Also we can check that all the mixing matrices are an element of SO(3). The masses of the neutral vector bosons depend on the model considered and explicitly they read as follows

$$\text{Model A,D} \qquad m_B^2 = 0\,, \quad m_{Z'}^2 = \frac{\Omega^2}{2} \rho_{4X}^{-2} \left(1 + \frac{v_R^2}{\Omega^2} g_X^4 \rho_{4X}^4\right)\,, \quad m_{Z''}^2 = \frac{v_R^2}{2} \sigma^2 \rho_{4X}^2\,, \tag{1.9.9a}$$

$$\text{Model B,E} \qquad m_B^2 = 0\,, \quad m_{Z'}^2 = \frac{v_R^2}{2} \rho_{RX}^{-2} \left(1 + \frac{\Omega^2}{v_R^2} g_X^4 \rho_{RX}^4\right)\,, \quad m_{Z''}^2 = \frac{\Omega^2}{2} \sigma^2 \rho_{RX}^2\,, \tag{1.9.9b}$$

$$\text{Model C,F} \qquad m_B^2 = 0\,, \quad m_{Z'}^2 = \frac{w^2}{2} \rho_{4R}^{-2} \left(1 + \frac{\Omega^2}{w^2} \hat{g}_4^4 \rho_{4R}^4\right)\,, \quad m_{Z''}^2 = \frac{\Omega^2}{2} \sigma^2 \rho_{4R}^2\,. \tag{1.9.9c}$$

The last quantity to write down is the covariant derivative in terms of the broken vector bosons. In particular the SM Gauge contributions are common to all the models as expected. The contributions provided by the massive gluons, leptoquarks and W-type vector bosons seem to be independent on the SSB mechanisms exploited. The only differences are in the contributions provided by the neutral Z-type vector bosons. Explicitly we find that

$$\begin{aligned} D_\mu = \partial_\mu - i g_L T_L^i W_L^i - i g_s T^a G^a - i g_Y Y B \\ -i g_R \left(T_R^+ W_R^+ + T_R^- W_R^-\right) - i g_4 \sum_{c=r,b,g} \left(\hat{T}_c^+ U_c^+ + \hat{T}_c^- U_c^-\right) - \frac{i}{2} \frac{g_4^2 - g_3^2}{\sqrt{g_4^2 + g_3^2}} T^a G'^a + D_\mu^Z \end{aligned} \tag{1.9.10}$$

where we have defined

$$T_R^\pm = \frac{1}{\sqrt{2}} \left(T^1 \pm i T^2\right)\,, \tag{1.9.11a}$$

$$\hat{T}_r^\pm = \frac{1}{\sqrt{2}} \left(\hat{T}^9 \pm i \hat{T}^{10}\right)\,, \quad \hat{T}_b^\pm = \frac{1}{\sqrt{2}} \left(\hat{T}^{11} \pm i \hat{T}^{12}\right)\,, \quad \hat{T}_g^\pm = \frac{1}{\sqrt{2}} \left(\hat{T}^{13} \pm i \hat{T}^{14}\right)\,. \tag{1.9.11b}$$

The Z-type contribution to the covariant derivative D_μ^Z for each of the models considered reads as follows

Model A,D

$$\begin{aligned} i D_\mu^Z = &\left[-g_X^2 \hat{g}_4^2 \frac{\rho_{4X}}{\sigma} \left(1 + \frac{v_R^2}{\Omega^2} \rho_{4X}^4 \sigma^2\right) (B-L)^{[3]} - g_X^2 \hat{g}_4^2 \frac{\rho_{4X}}{\sigma} \left(1 - \frac{v_R^2}{\Omega^2} \frac{\rho_{4X}^4 \sigma^2}{\hat{g}_4^2}\right) Y^{[12]} + \frac{g_R^2}{\rho_{4X} \sigma} T_R^{[3]}\right] Z' \\ &+ \left[\hat{g}_4^2 \rho_{4X} \left(1 - \frac{v_R^2}{\Omega^2} g_X^4 \rho_{4X}^4\right) (B-L)^{[3]} - g_X^2 \rho_{4X} \left(1 - \frac{v_R^2}{\Omega^2} g_X^2 \hat{g}_4^2 \rho_{4X}^4\right) Y^{[12]} + \frac{v_R^2}{\Omega^2} g_X^2 g_R^2 \rho_{4X}^3 T_R^{[3]}\right] Z'' \end{aligned} \tag{1.9.12a}$$

Model B,E

$$iD_\mu^Z = \left[-g_X^2 g_R^2 \frac{\rho_{RX}}{\sigma}\left(1+\frac{\Omega^2}{v_R^2}\rho_{RX}^4\sigma^2\right)T_R^{[3]} - g_X^2 g_R^2\frac{\rho_{RX}}{\sigma}\left(1-\frac{\Omega^2}{v_R^2}\frac{\rho_{RX}^4\sigma^2}{g_R^2}\right)Y^{[12]} + \frac{\hat{g}_4^2}{\rho_{RX}\sigma}(B-L)^{[3]}\right]Z'$$
$$+\left[g_R^2\rho_{RX}\left(1-\frac{\Omega^2}{v_R^2}g_X^4\rho_{RX}^4\right)T_R^{[3]} - g_X^2\rho_{RX}\left(1-\frac{\Omega^2}{v_R^2}g_X^2 g_R^2\rho_{RX}^4\right)Y^{[12]} + \frac{\Omega^2}{v_R^2}g_X^2\hat{g}_4^2\rho_{RX}^3(B-L)^{[3]}\right]Z'' \tag{1.9.12b}$$

Model C,F

$$iD_\mu^Z = \left[-\hat{g}_4^2 g_R^2 \frac{\rho_{4R}}{\sigma}\left(1+\frac{\Omega^2}{w^2}\rho_{4R}^4\sigma^2\right)T_R^{[3]} - \hat{g}_4^2 g_R^2\frac{\rho_{4R}}{\sigma}\left(1-\frac{\Omega^2}{w^2}\frac{\rho_{4R}^4\sigma^2}{g_R^2}\right)(B-L)^{[3]} + \frac{g_X^2}{\rho_{4R}\sigma}Y^{[12]}\right]Z'$$
$$+\left[g_R^2\rho_{4R}\left(1-\frac{\Omega^2}{w^2}\hat{g}_4^4\rho_{4R}^4\right)T_R^{[3]} - \hat{g}_4^2\rho_{4R}\left(1-\frac{\Omega^2}{w^2}\hat{g}_4^2 g_R^2\rho_{4R}^4\right)(B-L)^{[3]} + \frac{\Omega^2}{w^2}\hat{g}_4^2 g_X^2\rho_{4R}^3 Y^{[12]}\right]Z'' \tag{1.9.12c}$$

where we have defined

$$(B-L)^{[3]} \equiv \frac{\sqrt{6}}{3}\hat{T}^{15}\,, \qquad T_R^{[3]} \equiv T_R^3\,, \qquad Y^{[12]} \equiv X^{[12]}\,. \tag{1.9.13}$$

At last, we observe that in some models are present some scalar fields which did not acquired a VEV and are remnants of the scalar bosons that triggered the SSB. In particular for Models A and B we do not have any. For Model C we have that Ω_1 and Ω_3 split when Δ_3 acquires its VEV, and one of those components each is still present and are given by (notice that one is the antiparticle of the other)

$$\Omega_1' \sim (\mathbf{1},\overline{\mathbf{3}},-2/3)\,, \qquad \Omega_3' \sim (\mathbf{1},\mathbf{3},2/3)\,. \tag{1.9.14}$$

For Models D and E we have that Σ_R splits when Σ_3 acquires its VEV and one of those components is still present and it is given by

$$\Sigma_R^+ \sim (\mathbf{1},\mathbf{1},1)\,. \tag{1.9.15}$$

For Model F we have that are present the two components of Ω_1 and Ω_3 in (1.9.14) and two components of Δ_3 (plus their antiparticles) which are given by

$$\Delta_1^+ \sim (\mathbf{1},\mathbf{1},1)\,, \qquad \Delta_3^- \sim (\mathbf{1},\mathbf{3},-1/3)\,. \tag{1.9.16}$$

Clearly we expect all of them to be heavy enough to not have been detected yet. Their masses will depend on the details of the potentials of the scalar fields, which is something we have not taken into consideration.

2 Yukawa Matrices

In this Chapter we are going to study in detail the structure of the Yukawa matrices that are generated by the models we considered in this work. In particular our aim is to explain the hierarchical structure of the Yukawa matrices of the SM. To do so, we are going to assume that the suppressed Yukawa couplings are such because they are generated from some EFT operators which are *naturally* suppressed with respect to the non-suppressed ones. Then we are going to provide a possible UV origin for such higher-dimensional operators.

2.1 General Structure of the Yukawa Matrices

In the SM Yukawa matrices represent the couplings between the Higgs field and the SM fermions. They are four in total, one for the up-type quarks Y^U, one for the down-type quarks Y^D, one for the neutral leptons Y^N and one for the charged leptons Y^E. All of them are 3×3 completely general complex matrices. Not all their entries are physical, but there are some constraints that are fixed by experimental observations. The most prominent ones are given by the observed fermion masses. They are intimately related to their eigenvalues. In fact, once they have been diagonalized with a singular-valued decomposition (for more information see Section D.2 in the Appendix), we have that the three eigenvalues y_i are related to the fermion masses m_i of a given fermion type as follows

$$m_i = \frac{y_i}{\sqrt{2}} v \tag{2.1.1}$$

where $v \approx 246\,\text{GeV}$ is the Higgs' VEV. We know the masses of all the quarks and charged leptons. For Y^U, Y^D and Y^E the eigenvalues (at the $M_Z \approx 91\,\text{GeV}$ scale) that we would like to explain are respectively [14]

$$y_t \approx 0.99\,, \qquad y_c \approx 3.6 \cdot 10^{-3}\,, \qquad y_u \approx 7.3 \cdot 10^{-6}\,; \tag{2.1.2a}$$
$$y_b \approx 1.7 \cdot 10^{-2}\,, \qquad y_s \approx 3.2 \cdot 10^{-4}\,, \qquad y_d \approx 1.7 \cdot 10^{-5}\,; \tag{2.1.2b}$$
$$y_\tau \approx 1.0 \cdot 10^{-2}\,, \qquad y_\mu \approx 5.9 \cdot 10^{-4}\,, \qquad y_e \approx 2.8 \cdot 10^{-6}\,. \tag{2.1.2c}$$

One can appreciate the clear hierarchy between the three generations. In this Chapter we are going to explain the hierarchy between the third and second generations, assuming that the one between the second and first will be described by a further UV embedding of the model we considered at an energy scale well above the TeV. The masses of the neutral leptons (the neutrinos) are not known yet, but we expect them to be all comparable and of $\mathcal{O}(10^{-1})$ eV. For this reason we postpone the explanation of neutrino masses in the next Chapter.

Regarding the Yukawa matrices of the quarks, there are further constraints that we need to satisfy. Phenomenologically, a highly desirable feature for flavoured NP at the TeV scale is provided by suppressed right-handed mixing between light and heavy fields [8]. This condition reads generically as follows

$$|Y^F_{3i}| \ll |Y^F_{i3}| \qquad \text{where} \qquad F = U, D \tag{2.1.3}$$

and it is *naturally* reproduced by our model thanks to the choice of leaving $\mathrm{SU}(2)_L$ universal. In addition, the way we diagonalize the Yukawa matrices is linked to the CKM matrix that plays an important role in Charged-Current Weak Interactions. In particular, we can perturbatively diagonalize the Yukawa matrices with such a hierarchical structure to find the mixing matrices and hence the CKM matrix. This has been done in Section C.1 in the Appendix, finding that

$$V_{cb} \approx Y^D_{23}/Y^D_{33} - Y^U_{23}/Y^U_{33}\,. \tag{2.1.4}$$

To find this result we assumed that $Y_{i3} \ll Y_{33}$, but this turns out to be a necessary condition since we have that $|V_{cb}| \approx 0.04$ (unless we allow fine tuning between the two subtracted quantities). This condition puts other constraints on the Yukawa matrices that we want to reproduce, namely

$$|Y^U_{i3}| \le |V_{cb}|\, y_t \sim 10^{-2}\,, \qquad |Y^D_{i3}| \le |V_{cb}|\, y_b \sim 10^{-3}\,. \tag{2.1.5}$$

Then, if we also require (2.1.3), we need the further constraints

$$|Y^U_{3i}| \lesssim 10^{-3}\,, \qquad |Y^D_{3i}| \lesssim 10^{-4}\,. \tag{2.1.6}$$

Another result that have been derived in Section C.1 is that the third- and second-generation eigenvalues of the Yukawa matrices are (not surprisingly) given by (with $F = U, D, N, E$)

$$y^F_2 \approx |Y^F_{22}|\,, \qquad y^F_3 \approx |Y^F_{33}|\,. \tag{2.1.7}$$

2.2 Yukawa Matrices as EFT

In this Section we list all the possible EFT operators that could be used to generate the Yukawa matrices. To do so, the only possibility is to use some scalar fields that will eventually acquire a VEV at low energies. The suppression behaviors that we want to reproduce in the Yukawa matrices are ensured by the fact that higher-dimensional operators are suppressed with respect to the 4-dimensional ones.

Since the Higgs field is assumed to be charged only under the third-generation sector of the UV Gauge Group, we can explain this hierarchical structure by assuming that the Higgs field couples only with the third-generation fermions and the other couplings are generated by some EFT operators. By construction we can easily see that the Yukawa coupling between the Higgs and the third generation is given by

$$\mathcal{L} \supset \mathcal{L}_{Y,33} = -c_{33}\, \overline{\chi}^3_L \mathcal{H} \chi^3_R + h.c. \tag{2.2.1}$$

All the other Yukawa couplings can only be generated by some EFT operators. Notice that those operators are independent on the precise way it happens the SSB from the UV theory down to the SM. The only thing that matters are the scalar fields that we assume to be present in the theory. Hence we can divide the models studied in the previous Chapter in two categories[4]: Models A, B, D and E are Type I Models, while C and F are Type II.

The easiest to write down are the couplings of type c_{i3} (with $i = 1, 2$) and they are given by the following 5-dimensional operators (common for both of the Models)

$$\mathcal{L} \supset \mathcal{L}_{Y,i3} \sim \frac{c_{i3}}{\Lambda} \left(\overline{Q}^i_L \mathcal{H} \Omega_3 \chi^3_R + \overline{L}^i_L \mathcal{H} \Omega_1 \chi^3_R \right) \tag{2.2.2}$$

where Λ is the scale at which these operators get a UV completion. The next ones are the couplings of type c_{ij} (with $i, j = 1, 2$). For Model I a possibility is given by the following 5-dimensional operators

$$\mathcal{L} \supset \mathcal{L}_{Y,ij} \sim \frac{c_{ij}}{\Lambda} \left[\overline{Q}^i_L \mathcal{H} \left(\Sigma_R d^j_R + \Sigma^c_R u^j_R \right) + \overline{L}^i_L \mathcal{H} \left(\Sigma_R e^j_R + \Sigma^c_R \nu^j_R \right) \right] \tag{2.2.3}$$

while for Model II we can use the following 6-dimensional operators

$$\mathcal{L} \supset \mathcal{L}_{Y,ij} \sim \frac{c_{ij}}{\Lambda^2} \left[\overline{Q}^i_L \mathcal{H} \left(\Delta_3 \Omega^\dagger_1 d^j_R + \Omega_1 \Delta^c_3\, u^j_R \right) + \overline{L}^i_L \mathcal{H} \left(\Delta_3 \Omega^\dagger_1 e^j_R + \Omega_1 \Delta^c_3\, \nu^j_R \right) \right]. \tag{2.2.4}$$

For the couplings of type c_{3i} (with $i = 1, 2$) we must exploit higher-dimensional operators than the previous ones. For Model I a possibility is given by the following 6-dimensional operators

$$\mathcal{L} \supset \mathcal{L}_{Y,3i} \sim \frac{c_{3i}}{\Lambda^2} \left[\overline{\chi}^3_L \mathcal{H} \Omega^\dagger_3 \left(\Sigma_R d^j_R + \Sigma^c_R u^j_R \right) + \overline{\chi}^3_L \mathcal{H} \Omega^\dagger_1 \left(\Sigma_R e^j_R + \Sigma^c_R \nu^j_R \right) \right] \tag{2.2.5}$$

[4] The presence or absence of the scalar field Σ_3 is just a minor difference in the classification of EFT operators as it will be clear in the following.

while for Model II we can use the following 7-dimensional operators

$$\mathcal{L} \supset \mathcal{L}_{Y,3i} \sim \frac{c_{3i}}{\Lambda^3}\left[\overline{\chi}_L^3 \mathcal{H}\Omega_3^\dagger\left(\Delta_3\Omega_1^\dagger d_R^j + \Omega_1\Delta_3^c\, u_R^j\right) + \overline{\chi}_L^3 \mathcal{H}\Omega_1^\dagger\left(\Delta_3\Omega_1^\dagger e_R^j + \Omega_1\Delta_3^c\, \nu_R^j\right)\right]. \tag{2.2.6}$$

Notice that up to now we are not able to explain the mass splittings between the different third-generation fermions. In particular the ones between bottom-tau and top-bottom. The mass splitting between b and τ can be achieved by exploiting the fields that break $\mathrm{SU}(4)^{[3]}$. This can be done by using one of the following 6-dimensional operators

$$\mathcal{L} \supset \frac{c_{b\tau}}{\Lambda^2}\overline{\chi}_L^3\mathcal{H}\left(\Omega_3^\dagger\Omega_3\right)\chi_R^3\,, \quad \mathcal{L} \supset \frac{c_{b\tau}}{\Lambda^2}\overline{\chi}_L^3\mathcal{H}\left(\Omega_1^\dagger\Omega_1\right)\chi_R^3\,, \quad \mathcal{L} \supset \frac{c_{b\tau}}{\Lambda^2}\overline{\chi}_L^3\mathcal{H}\left(\Delta_3^\dagger\Delta_3\right)\chi_R^3 \tag{2.2.7}$$

where the last one is at work only in Model II, while the others in both of the Models. The mass splitting between t and b (or more generically between the up and down components of an $\mathrm{SU}(2)_R^{[3]}$-doublet) can be achieved by exploiting the fields that break $\mathrm{SU}(2)_R^{[3]}$. Namely we can write down the following 6-dimensional operators

$$\mathcal{L} \supset \frac{c_{tb}}{\Lambda^2}\overline{\chi}_L^3\mathcal{H}\left(\Sigma_R\Sigma_R^\dagger\right)\chi_R^3\,, \qquad \mathcal{L} \supset \frac{c_{tb}}{\Lambda^2}\overline{\chi}_L^3\mathcal{H}\left(\Delta_3\Delta_3^\dagger\right)\chi_R^3 \tag{2.2.8}$$

where the first one is at work in Model I while the second one in Model II. Furthermore, if Σ_3 is present, we can write down the following 5-dimensional operator

$$\mathcal{L} \supset \frac{c_{tb}}{\Lambda}\overline{\chi}_L^3\mathcal{H}\Sigma_3\chi_R^3\,. \tag{2.2.9}$$

Notice that we could even add a Σ_3 insertion in all the EFT operators written so far (at the price of increase by 1 their dimensions), and this could provide a further way to reproduce the desirable Yukawa structure.

In fact there is another possibility to provide different masses between the up and down components of an $\mathrm{SU}(2)_R^{[3]}$-doublet that does not involve an EFT description. This is given by assuming that the Higgs field in the UV theory weights differently H and H^c. Namely we could define the Higgs field as follows

$$\mathcal{H} = \frac{1}{\sqrt{1+\xi^2}}\begin{pmatrix} H^c & \xi H\end{pmatrix} \tag{2.2.10}$$

where ξ is a generic real parameter and when $\xi = 1$ we recover the definition made in Section 1.1. This possibility has been discussed in Section C.2 in the Appendix and we will come back to the consequences of such a choice at the end of this Chapter.

2.3 UV Completion for Model I

Once we have listed all the possible EFT operators that could be utilize to generate the Yukawa matrices, we can provide a possible UV origin to them. In this Section we focus on Model I while in the next one on Model II.

For Model I we notice that the operators of type c_{i3} and c_{ij} are both 5-dimensional, which means that their UV completion can be done by using a single VLF. The most efficient way to do this is to use the following VLF with corresponding Charges under H_{UV} (1.1.3)

$$\rho = \rho_3 \oplus \rho_1 \sim (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{2},1/6) \oplus (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},-1/2)\,. \tag{2.3.1}$$

It allows the following UV completion

$$\begin{aligned} -\mathcal{L}_\rho = & M_\rho\left(\overline{\rho}_1\rho_1 + \overline{\rho}_3\rho_3\right) - c_\rho^3\overline{\rho}_3\Sigma_3\rho_3 - c_\rho^1\overline{\rho}_1\Sigma_3\rho_1 + \Big(- c_{\rho\mathcal{H}}^3\overline{Q}_L^i\mathcal{H}\rho_3 - c_{\rho\mathcal{H}}^1\overline{L}_L^i\mathcal{H}\rho_1 \\ & + c_{\rho\Omega}^3\overline{\rho}_3\Omega_3\chi_R^3 + c_{\rho\Omega}^1\overline{\rho}_1\Omega_1\chi_R^3 + c_{\rho R}^3\overline{\rho}_3\Sigma_R d_R^j + \overline{c}_{\rho R}^3\overline{\rho}_3\Sigma_R^c u_R^j + c_{\rho R}^1\overline{\rho}_1\Sigma_R e_R^j + \overline{c}_{\rho R}^1\overline{\rho}_1\Sigma_R^c\nu_R^j + h.c.\Big)\,. \end{aligned} \tag{2.3.2}$$

We also assumed that ρ_3 and ρ_1 find an embedding in a unique VLF ρ and that they share the same mass. For this reason we expect the following coefficients to be similar

$$c^3_{\rho\mathcal{H}} \approx c^1_{\rho\mathcal{H}}\,, \qquad c^3_{\rho\Omega} \approx c^1_{\rho\Omega}\,, \qquad c^3_{\rho R} \approx \bar{c}^3_{\rho R} \approx c^1_{\rho R} \approx \bar{c}^1_{\rho R}\,, \qquad c^3_{\rho} \approx c^1_{\rho}\,. \tag{2.3.3}$$

Regarding the operators to type c_{i3}, a further possibility could be using the following VLF with corresponding Charges under $H_{\rm UV}$ (1.1.3)

$$\lambda \sim (\mathbf{2},\mathbf{4},\mathbf{1},\mathbf{1},0)\,. \tag{2.3.4}$$

It allows the following UV completion

$$-\mathcal{L}_\lambda = M_\lambda \overline{\lambda}\lambda + \left(-c_{\lambda\mathcal{H}}\,\overline{\lambda}\mathcal{H}\chi^3_R + c^3_{\lambda\Omega}\overline{Q}^i_L\Omega_3\lambda + c^1_{\lambda\Omega}\overline{L}^i_L\Omega_1\lambda + m\,\overline{\chi}^3_L\lambda + h.c.\right)\,. \tag{2.3.5}$$

Notice that the last term accounts for possible mass-mixing terms between χ^3_L and λ since they share the same quantum numbers.

For the operators of type c_{3i} we could either add an additional VLF to provide a UV origin at tree-level, or assume that they are generated directly at loop-level in the EFT. The same holds also for the operators needed to generate the mass splittings.

In case Σ_3 is present, we still have to generate the 5-dimensional operator (2.2.9). To generate it we need to use the following VLF with corresponding Charges under $H_{\rm UV}$ (1.1.3)

$$\eta \sim (\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{2},0)\,. \tag{2.3.6}$$

It allows the following UV completion

$$-\mathcal{L}_\eta = M_\eta\overline{\eta}\eta - c_\eta\overline{\eta}\Sigma_3\eta + \left(-c_{\eta\mathcal{H}}\,\overline{\chi}^3_L\mathcal{H}\eta + c_{\eta\Sigma}\overline{\eta}\Sigma_3\chi^3_R + m\,\overline{\eta}\chi^3_R + h.c.\right)\,. \tag{2.3.7}$$

As before, notice the last term accounts for possible mass-mixing terms between χ^3_R and η since they share the same quantum numbers.

Furthermore, the addition of η allows by free a possible UV origin for the EFT operators of type c_{3i} and the $b\tau$ mass-splitting operators (2.2.7) just by adding to the renormalizable Lagrangian the following terms

$$\mathcal{L}_{\rho\eta} = c^3_{\eta\rho}\overline{\eta}\Omega^\dagger_3\rho_3 + c^1_{\eta\rho}\overline{\eta}\Omega^\dagger_1\rho_1 + h.c. \tag{2.3.8}$$

Again we would like to assume that

$$c^3_{\eta\rho} \approx c^1_{\eta\rho}\,. \tag{2.3.9}$$

Finally, in case Σ_3 is not present, we need to generate the tb mass-splitting operators (2.2.8). To do so we can introduce the following VLF with corresponding Charges under $H_{\rm UV}$ (1.1.3)

$$\eta' \sim (\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{1},-1/2)\,. \tag{2.3.10}$$

It allows the following UV completion

$$-\mathcal{L}_{\eta'} = M_{\eta'}\overline{\eta}'\eta' + \left(-c_{\eta R}\,\overline{\eta}\Sigma_R\eta' + \bar{c}_{\eta R}\,\overline{\eta}'\Sigma^\dagger_R\chi^3_R + h.c.\right)\,. \tag{2.3.11}$$

Notice that we need in any case the introduction of the VLF η to generate $b\tau$ of tb mass splittings. Thus unavoidably we generate at tree-level the operators of type c_{3i}.

An important remark to make follows from the fact that VLFs λ and η share the same quantum numbers of χ^3_L and χ^3_R respectively. This allows a possible mixing between the two which could be used as a source of mass splittings between the fermions of the third generation. This possibility is discussed in Section C.3 in the Appendix. Nevertheless it seems that it could generate only small splittings. Thus we will not consider this possibility in the following.

2.4 UV Completion for Model II

For Model II we notice that the operators of type c_{i3} are the same of Model I. Thus we can provide a UV origin in the same way we did in the previous Section by using the single VLF ρ (2.3.1). It allows the following UV completion

$$\begin{aligned}-\mathcal{L}_\rho =& M_\rho\left(\overline{\rho}_1\rho_1+\overline{\rho}_3\rho_3\right)-c_\rho^3\overline{\rho}_3\Sigma_3\rho_3-c_\rho^1\overline{\rho}_1\Sigma_3\rho_1\\&+\left(-c^3_{\rho\mathcal{H}}\overline{Q}_L^i\mathcal{H}\rho_3-c^1_{\rho\mathcal{H}}\overline{L}^i_L\mathcal{H}\rho_1+c^3_{\rho\Omega}\overline{\rho}_3\Omega_3\chi_R^3+c^1_{\rho\Omega}\overline{\rho}_1\Omega_1\chi^3_R+h.c.\right)\,.\end{aligned}\tag{2.4.1}$$

The operators of type c_{ij} are 6-dimensional. Thus we need at least a further VLF to find their UV completion. In fact, due to the complexity of the operators needed, we need the following two VLFs with corresponding Charges under $H_{\rm UV}$ (1.1.3)

$$\eta_3\oplus\eta_1\sim(\mathbf{1},\mathbf{4},\mathbf{3},\mathbf{1},1/6)\oplus(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{1},-1/2)\,,\tag{2.4.2a}$$

$$\eta_3'\oplus\eta_1'\sim(\mathbf{1},\mathbf{4},\mathbf{3},\mathbf{2},2/3)\oplus(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{2},0)\,.\tag{2.4.2b}$$

The drawback is that we need η_3' which is heavily charged under the Gauge Group[5]. In addition we get for free a UV origin at tree-level of the Yukawa operators of type c_{3i}, the $b\tau$ mass splitting and (if Σ_3 is present) the 5-dimensional operator (2.2.9) which generates the tb mass splitting. They allow the following UV completion

$$\begin{aligned}-\mathcal{L}_\eta =& M_\eta\left(\overline{\eta}_1\eta_1+\overline{\eta}_3\eta_3\right)+\left(-c^3_{\eta\Delta}\overline{\rho}_3\Delta_3\eta_3-c^1_{\eta\Delta}\overline{\rho}_1\Delta_3\eta_1+c^3_{\eta\Omega}\overline{\eta}_3\Omega_1^\dagger d_R^j+c^1_{\eta\Omega}\overline{\eta}_1\Omega_1^\dagger e_R^j+h.c.\right)\\&+M_{\eta'}\left(\overline{\eta}_1'\eta_1'+\overline{\eta}_3'\eta_3'\right)+\left(-\overline{c}^3_{\eta\Omega}\overline{\rho}_3\Omega_1\eta_3'-\overline{c}^1_{\eta\Omega}\overline{\rho}_1\Omega_1\eta_1'+\overline{c}^3_{\eta\Delta}\overline{\eta}_3'\Delta_3^c\,u_R^j+\overline{c}^1_{\eta\Delta}\overline{\eta}_1'\Delta_3^c\,\nu_R^j+h.c.\right)\\&+\left(-c_{\eta\mathcal{H}}\overline{\chi}_L^3\mathcal{H}\eta_1'+c_{\eta\Sigma}\overline{\eta}_1'\Sigma_3\chi_R^3+m\overline{\eta}_1'\chi_R^3-\overline{c}^{13}_{\eta\Omega}\overline{\rho}_3\Omega_3\eta_1'+h.c.\right)-c^3_\eta\overline{\eta}_3'\Sigma_3\eta_3'-c^1_\eta\overline{\eta}_1'\Sigma_3\eta_1'\end{aligned}\tag{2.4.3}$$

where the first two lines generate the Yukawa operators of type c_{ij} while the last one the Yukawa operator of type c_{3i} and the mass splittings. As we did for ρ, we assume that η_1 and η_3 will find an embedding in a unique VLF and that they share the same mass. The same is assumed for η_1' and η_3'. Thus we expect the following coefficients to be similar

$$c^3_{\eta\Delta}\approx c^1_{\eta\Delta}\,,\quad c^3_{\eta\Omega}\approx c^1_{\eta\Omega}\,,\quad \overline{c}^3_{\eta\Delta}\approx\overline{c}^1_{\eta\Delta}\,,\quad \overline{c}^3_{\eta\Omega}\approx\overline{c}^1_{\eta\Omega}\approx\overline{c}^{13}_{\eta\Omega}\,.\tag{2.4.4}$$

At last, to generate the mass-splitting operators with Δ_3 in (2.2.7) and (2.2.8), we need to add further VLFs since we have that

$$\Delta_3^\dagger\chi_R^3\sim(\mathbf{4}\times\mathbf{4},\mathbf{1})\in{\rm SU}(4)^{[3]}\times{\rm SU}(2)_R^{[3]}\,,\qquad \Delta_3\chi_R^3\sim(\mathbf{1},\mathbf{2}\times\mathbf{2})\in{\rm SU}(4)^{[3]}\times{\rm SU}(2)_R^{[3]}\,.\tag{2.4.5}$$

In particular this means that a tb mass splitting is at work only if Σ_3 is present if we do not add new VLFs to the theory (or we do not make other assumptions like (2.2.10)).

2.5 Yukawa Matrices in Model I

We have provided a UV origin to the Yukawa matrices, seen from an EFT description. The next step is to integrate out in the UV theory the heavy degrees of freedom (DOFs) to find the explicit realization of the Yukawa matrices. Also, to make this calculation as systematic as possible, we use the Matchete package for Wolfram Mathematica developed by [15]. In this Section we focus on Model I while in the next one on Model II.

[5]If we do not want to employ VLFs that transform heavily under the Gauge Group like η_3', we could modify Model II to allow the presence of the scalar field Σ_R in the theory. In this way we can generate the Yukawa operators in the same way as done for Model I. This has the only consequence to partially change the final Gauge mass-spectrum of the theory.

For Model I we assume to work with just ρ and η VLFs which entail the heavy DOFs that we want to integrate out. The UV-completed Lagrangian reads as follows

$$\mathcal{L}_Y^{\mathrm{UV}} = -\left(c_{33}\overline{\chi}_L^3\mathcal{H}\chi_R^3 + h.c.\right) + \mathcal{L}_\rho + \mathcal{L}_\eta + \mathcal{L}_{\rho\eta} \tag{2.5.1}$$

where the $\mathcal{L}_\rho$, $\mathcal{L}_\eta$ and $\mathcal{L}_{\rho\eta}$ are reported in (2.3.2), (2.3.7) and (2.3.8). By integrating out ρ and η, working in the static limit and doing tree-level matching up to 6-dimensional operators, we find the following effective Lagrangian[6]

$$\begin{aligned}
-\mathcal{L}_Y^{\mathrm{IR}} =& c_{33}\overline{\chi}_L^3\mathcal{H}\chi_R^3 + \frac{1}{M_\rho}c^3_{\rho\mathcal{H}}\overline{Q}^i_L\mathcal{H}\sigma^3_\rho J^3_\rho + \frac{1}{M_\rho}c^1_{\rho\mathcal{H}}\overline{L}^i_L\mathcal{H}\sigma^1_\rho J^1_\rho \\
&+ \frac{1}{M_\eta}c_{\eta\mathcal{H}}\overline{\chi}_L^3\mathcal{H}\sigma_\eta J_\eta + \frac{1}{M_\rho M_\eta}c_{\eta\mathcal{H}}\overline{\chi}_L^3\mathcal{H}\sigma_\eta\left(c^3_{\eta\rho}\sigma^3_\rho\Omega_3^\dagger J^3_\rho + c^1_{\eta\rho}\sigma^1_\rho\Omega_1^\dagger J^1_\rho\right) \\
&+ \frac{1}{M_\rho M_\eta}c^3_{\rho\mathcal{H}}(c^3_{\eta\rho})^*\overline{Q}^i_L\mathcal{H}\sigma^3_\rho\sigma_\eta\Omega_3 J_\eta + \frac{1}{M_\rho M_\eta}c^1_{\rho\mathcal{H}}(c^1_{\eta\rho})^*\overline{L}^i_L\mathcal{H}\sigma^1_\rho\sigma_\eta\Omega_1 J_\eta + h.c.
\end{aligned} \tag{2.5.2}$$

where we have defined

$$\sigma^3_\rho = \left(1 - c^3_\rho\frac{\Sigma_3}{M_\rho}\right)^{-1}, \quad \sigma^1_\rho = \left(1 - c^1_\rho\frac{\Sigma_3}{M_\rho}\right)^{-1}, \quad \sigma_\eta = \left(1 - c_\eta\frac{\Sigma_3}{M_\eta}\right)^{-1}, \tag{2.5.3a}$$

$$J^3_\rho = c^3_{\rho\Omega}\Omega_3\chi^3_R + c^3_{\rho R}\Sigma_R d^j_R + \overline{c}^3_{\rho R}\Sigma^c_R u^j_R\,, \tag{2.5.3b}$$

$$J^1_\rho = c^1_{\rho\Omega}\Omega_1\chi^3_R + c^1_{\rho R}\Sigma_R d^j_R + \overline{c}^1_{\rho R}\Sigma^c_R \nu^j_R\,, \tag{2.5.3c}$$

$$J_\eta = c_{\eta\Sigma}\Sigma_3\chi^3_R + m\chi^3_R\,. \tag{2.5.3d}$$

When all the fields (but the Higgs) acquire their VEVs, we find that (where just for this equation $i, j = 1, 2, 3$)

$$-\mathcal{L}_Y^{\mathrm{IR}} \supset \overline{Q}^i_L H^c\, Y^U_{ij}\, u^j_R + \overline{Q}^i_L H\, Y^D_{ij}\, d^j_R + \overline{L}^i_L H^c\, Y^N_{ij}\, \nu^j_R + \overline{L}^i_L H\, Y^E_{ij}\, e^j_R + h.c. \tag{2.5.4}$$

Explicitly the Yukawa matrix of the up-type quarks reads as follows

$$\begin{aligned}
Y^U_{33} =& \frac{c_{33}}{\sqrt{2}} + \frac{c_{\eta\mathcal{H}}}{\sqrt{2}}\frac{m}{M_\eta}\left(1 - c_\eta\frac{v_3}{2M_\eta}\right)^{-1} + \frac{v_3}{2M_\eta}\frac{c_{\eta\mathcal{H}}}{\sqrt{2}}c_{\eta\Sigma}\left(1 - c_\eta\frac{v_3}{2M_\eta}\right)^{-1} \\
&+ \frac{\omega_3^2}{M_\rho M_\eta}\frac{c_{\eta\mathcal{H}}}{\sqrt{2}}c^3_{\eta\rho}c^3_{\rho\Omega}\left(1 - c_\eta\frac{v_3}{2M_\eta}\right)^{-1}\left(1 - c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1},
\end{aligned} \tag{2.5.5a}$$

$$\begin{aligned}
Y^U_{i3} =& \frac{\omega_3}{M_\rho}\frac{c^3_{\rho\mathcal{H}}}{\sqrt{2}}c^3_{\rho\Omega}\left(1 - c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1} \\
&+ \frac{\omega_3 v_3}{2\,M_\rho M_\eta}\frac{c^3_{\rho\mathcal{H}}}{\sqrt{2}}(c^3_{\eta\rho})^*\left(c_{\eta\Sigma} + \frac{2m}{v_3}\right)\left(1 - c_\eta\frac{v_3}{2M_\eta}\right)^{-1}\left(1 - c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1},
\end{aligned} \tag{2.5.5b}$$

$$Y^U_{ij} = \frac{v_R}{M_\rho}\frac{c^3_{\rho\mathcal{H}}}{\sqrt{2}}\overline{c}^3_{\rho R}\left(1 - c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1}, \tag{2.5.5c}$$

$$Y^U_{3j} = \frac{v_R\omega_3}{M_\rho M_\eta}\frac{c_{\eta\mathcal{H}}}{\sqrt{2}}c^3_{\rho\Omega}\overline{c}^3_{\rho R}\left(1 - c_\eta\frac{v_3}{2M_\eta}\right)^{-1}\left(1 - c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1}. \tag{2.5.5d}$$

The Yukawa matrix for the down-type quarks is given by Y^U_{ij} but replacing $v_3 \to -v_3$ and $\overline{c}^3_{\rho R} \to c^3_{\rho R}$. The Yukawa matrices for the neutral and charged leptons are given by Y^U_{ij} and Y^D_{ij} but replacing $\omega_3 \to \omega_1$, $c^3_{\eta\rho} \to c^1_{\eta\rho}$, $c^3_{\rho\Omega} \to c^1_{\rho\Omega}$, $c^3_{\rho\mathcal{H}} \to c^1_{\rho\mathcal{H}}$, $c^3_{\rho R} \to c^1_{\rho R}$, $\overline{c}^3_{\rho R} \to \overline{c}^1_{\rho R}$ and $c^3_\rho \to c^1_\rho$ respectively.

[6] In case Σ_3 is absent one must set $c^3_\rho = c^1_\rho = c_\eta = 0$.

Notice that in the evaluation of the EFT we have integrated out Σ_3 exactly (yet at tree-level and in the static limit approximation) assuming that it has a VEV comparable to the heavy DOFs. This could have been done easily since it is a scalar field. Then, when Σ_3 acquires its VEV, we have that the terms containing this scalar field in the denominator must be interpreted as a series. Thus, to provide the final expression of the Yukawa matrices we have exploited the following fact: given a matrix $M = \mathrm{diag}(a, -a)$, we have that

$$(1-M)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \begin{pmatrix} 1^n & 0 \\ 0 & (-1)^n \end{pmatrix} = \begin{pmatrix} (1-a)^{-1} & 0 \\ 0 & (1+a)^{-1} \end{pmatrix} . \tag{2.5.6}$$

2.6 Yukawa Matrices in Model II

For Model II we assume to work with ρ, η and η' VLFs which entail the heavy DOFs that we want to integrate out. The UV-completed Lagrangian reads as follows

$$\mathcal{L}_Y^{\mathrm{UV}} = -\left(c_{33}\overline{\chi}_L^3 \mathcal{H} \chi_R^3 + h.c.\right) + \mathcal{L}_\rho + \mathcal{L}_\eta \tag{2.6.1}$$

where the $\mathcal{L}_\rho$ and $\mathcal{L}_\eta$ are reported in (2.4.1) and (2.4.3). By integrating out ρ, η and η', working in the static limit and doing tree-level matching up to 6-dimensional operators, we find the following effective Lagrangian[7]

$$\begin{aligned}
-\mathcal{L}_Y^{\mathrm{IR}} =& c_{33}\overline{\chi}_L^3 \mathcal{H}\chi_R^3 + \frac{1}{M_{\eta'}} c_{\eta\mathcal{H}}\overline{\chi}_L^3 \mathcal{H}\sigma_\eta^1 \left(c_{\eta\Sigma}\Sigma_3\chi_R^3 + m\chi_R^3 + \overline{c}_{\eta\Delta}^1 \Delta_3^c\, \nu_R^j\right) \\
&+ \frac{1}{M_\rho} c_{\rho\mathcal{H}}^3 c_{\rho\Omega}^3 \overline{Q}_L^i \mathcal{H}\sigma_\rho^3 \Omega_3 \chi_R^3 + \frac{1}{M_\rho} c_{\rho\mathcal{H}}^1 c_{\rho\Omega}^1 \overline{L}_L^i \mathcal{H}\sigma_\rho^1 \Omega_1 \chi_R^3 + \frac{1}{M_\rho M_{\eta'}} c_{\rho\mathcal{H}}^3 \overline{c}_{\eta\Delta}^1 \overline{c}_{\eta\Omega}^{13} \overline{Q}_L^i \mathcal{H}\sigma_\rho^3 \sigma_\eta^1 \Omega_3 \Delta_3^c\, \nu_R^j \\
&+ \frac{1}{M_\rho M_{\eta'}} c_{\eta\mathcal{H}}\overline{\chi}_L^3 \mathcal{H}\sigma_\eta^1 \left(\sigma_\rho^3 (\overline{c}_{\eta\Omega}^{13})^* c_{\rho\Omega}^3 \Omega_3^\dagger \Omega_3 + \sigma_\rho^1 (\overline{c}_{\eta\Omega}^1)^* c_{\rho\Omega}^1 \Omega_1^\dagger \Omega_1\right) \chi_R^3 \\
&+ \frac{1}{M_\rho M_{\eta'}} \left(c_{\rho\mathcal{H}}^3 \overline{c}_{\eta\Omega}^{13} \overline{Q}_L^i \mathcal{H}\sigma_\rho^3 \Omega_3 + c_{\rho\mathcal{H}}^1 \overline{c}_{\eta\Omega}^1 \overline{L}_L^i \mathcal{H}\sigma_\rho^1 \Omega_1\right) \sigma_\eta^1 \left(c_{\eta\Sigma}\Sigma_3\chi_R^3 + m\chi_R^3\right) \\
&+ \frac{1}{M_\rho M_\eta} c_{\rho\mathcal{H}}^3 c_{\eta\Omega}^3 c_{\eta\Delta}^3 \overline{Q}_L^i \mathcal{H}\sigma_\rho^3 \Delta_3 \Omega_1^\dagger d_R^j + \frac{1}{M_\rho M_{\eta'}} c_{\rho\mathcal{H}}^3 \overline{c}_{\eta\Omega}^3 \overline{c}_{\eta\Delta}^3 \overline{Q}_L^i \mathcal{H}\sigma_\rho^3 \sigma_\eta^3 \Omega_1 \Delta_3^c\, u_R^j \\
&+ \frac{1}{M_\rho M_\eta} c_{\rho\mathcal{H}}^1 c_{\eta\Omega}^1 c_{\eta\Delta}^1 \overline{L}_L^i \mathcal{H}\sigma_\rho^1 \Delta_3 \Omega_1^\dagger e_R^j + \frac{1}{M_\rho M_{\eta'}} c_{\rho\mathcal{H}}^1 \overline{c}_{\eta\Omega}^1 \overline{c}_{\eta\Delta}^1 \overline{L}_L^i \mathcal{H}\sigma_\rho^1 \sigma_\eta^1 \Omega_1 \Delta_3^c\, \nu_R^j + h.c.
\end{aligned} \tag{2.6.2}$$

where we have defined

$$\sigma_\rho^3 = \left(1 - c_\rho^3 \frac{\Sigma_3}{M_\rho}\right)^{-1}, \quad \sigma_\rho^1 = \left(1 - c_\rho^1 \frac{\Sigma_3}{M_\rho}\right)^{-1}, \quad \sigma_\eta^3 = \left(1 - c_\eta^3 \frac{\Sigma_3}{M_{\eta'}}\right)^{-1}, \quad \sigma_\eta^1 = \left(1 - c_\eta^1 \frac{\Sigma_3}{M_{\eta'}}\right)^{-1}. \tag{2.6.3}$$

When all the fields (but the Higgs) acquire their VEVs, we find that (where just for this equation $i, j = 1, 2, 3$)

$$-\mathcal{L}_Y^{\mathrm{IR}} \supset \overline{Q}_L^i H^c\, Y_{ij}^U u_R^j + \overline{Q}_L^i H\, Y_{ij}^D d_R^j + \overline{L}_L^i H^c\, Y_{ij}^N \nu_R^j + \overline{L}_L^i H\, Y_{ij}^E e_R^j + h.c. \tag{2.6.4}$$

Explicitly the Yukawa matrix for the up-type quarks reads as follows

$$\begin{aligned}
Y_{33}^U = \frac{c_{33}}{\sqrt{2}} + \frac{c_{\eta\mathcal{H}}}{\sqrt{2}} \frac{m}{M_{\eta'}} \left(1 - c_\eta^1 \frac{v_3}{2M_{\eta'}}\right)^{-1} + \frac{v_3}{2M_{\eta'}} \frac{c_{\eta\mathcal{H}}}{\sqrt{2}} c_{\eta\Sigma} \left(1 - c_\eta^1 \frac{v_3}{2M_{\eta'}}\right)^{-1} \\
+ \frac{\omega_3^2}{M_\rho M_{\eta'}} \frac{c_{\eta\mathcal{H}}}{\sqrt{2}} (\overline{c}_{\eta\Omega}^{13})^* c_{\rho\Omega}^3 \left(1 - c_\eta^1 \frac{v_3}{2M_{\eta'}}\right)^{-1} \left(1 - c_\rho^3 \frac{v_3}{2M_\rho}\right)^{-1},
\end{aligned} \tag{2.6.5a}$$

[7] In case Σ_3 is absent one must set $c_\rho^3 = c_\rho^1 = c_\eta^1 = c_\eta^3 = 0$.

$$\begin{aligned} Y^U_{i3} =& \frac{\omega_3}{M_\rho}\frac{c^3_{\rho\mathcal{H}}}{\sqrt{2}}c^3_{\rho\Omega}\left(1-c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1} \\ &+\frac{v_3\omega_3}{2M_\rho M_{\eta'}}\frac{c^3_{\rho\mathcal{H}}}{\sqrt{2}}\overline{c}^{13}_{\eta\Omega}\left(c_{\eta\Sigma}+\frac{2m}{v_3}\right)\left(1-c^1_\eta\frac{v_3}{2M_{\eta'}}\right)^{-1}\left(1-c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1}, \end{aligned} \tag{2.6.5b}$$

$$Y^U_{ij} = \frac{w\,\omega_1}{M_\rho M_{\eta'}}\frac{c^3_{\rho\mathcal{H}}}{\sqrt{2}}\overline{c}^3_{\eta\Omega}\overline{c}^3_{\eta\Delta}\left(1-c^3_\eta\frac{v_3}{2M_{\eta'}}\right)^{-1}\left(1-c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1}, \tag{2.6.5c}$$

$$Y^U_{3j} = \mathcal{O}(M^{-3})\,. \tag{2.6.5d}$$

Instead the Yukawa matrix for the down-type quarks is given by

$$Y^N_{ij} = \frac{w\,\omega_1}{M_\rho M_\eta}\frac{c^3_{\rho\mathcal{H}}}{\sqrt{2}}c^3_{\eta\Delta}c^3_{\eta\Omega}\left(1+c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1}, \qquad Y^U_{3i} = \mathcal{O}(M^{-3}) \tag{2.6.6}$$

while components Y^D_{33} and Y^D_{i3} are given by Y^U_{33} and Y^U_{i3} but replacing $v_3 \to -v_3$. The Yukawa matrices for the neutral and charged leptons are given by Y^U_{ij} and Y^D_{ij} but replacing $\omega_3 \to \omega_1$, $c^3_{\rho\Omega} \to c^1_{\rho\Omega}$, $\overline{c}^{13}_{\eta\Omega} \to \overline{c}^1_{\eta\Omega}$, $c^3_{\rho\mathcal{H}} \to c^1_{\rho\mathcal{H}}$, $c^3_{\eta\Omega} \to c^1_{\eta\Omega}$, $\overline{c}^3_{\eta\Omega} \to \overline{c}^1_{\eta\Omega}$, $c^3_{\eta\Delta} \to c^1_{\eta\Delta}$, $\overline{c}^3_{\eta\Delta} \to \overline{c}^1_{\eta\Delta}$, $c^3_\eta \to c^1_\eta$ and $c^3_\rho \to c^1_\rho$ respectively[8]. Also for Y^N_{ij} only the Y^N_{3j} component is not that much suppressed, and its leading contribution reads as follows

$$Y^N_{3j} = \frac{w}{M_{\eta'}}\frac{c_{\eta\mathcal{H}}}{\sqrt{2}}\overline{c}^1_{\eta\Delta}\left(1-c^3_\eta\frac{v_3}{2M_{\eta'}}\right)^{-1}. \tag{2.6.7}$$

Notice that also in this case in the evaluation of the EFT we have integrated out Σ_3 exactly (yet at tree-level and in the static limit approximation) assuming that it has a VEV comparable to the heavy DOFs.

2.7 Complete Mass-Matrix Diagonalization

What we did in the last two Sections was to reproduce the Yukawa matrices from an EFT description. Nevertheless, once we have written the UV-completed Lagrangian, there is no need to integrate out the heavy DOFs to find the Yukawa matrices that, after they have been diagonalized, provide the predictions for the fermion masses. This method is useful to clearly see the whole Yukawa structure predicted by the model, but brings some errors due to the truncation that is done at a given order in the EFT description. In addition, if the scalar field's VEVs are comparable to the VLFs' masses, the EFT approximation could be not so precised. For this reason it is worth to try to diagonalize directly with a singular-valued decomposition the total mass-matrices generated by the model, comprehensive of the SM fermions as well as the NP VLFs, once all the fields have acquired their VEVs (Higgs field included). This is what we do in this Section in the framework of Model I. For Model II we are not going to provide such an analysis since, as discussed with great detail in next Chapters, it is not very appealing once we try to explain also the neutrino masses.

For Model I the sector of the UV-completed Lagrangian that generates the fermion masses is reported in (2.5.1). When we make all the scalar fields to acquire their VEVs we go from the Gauge Group of the UV theory down to the broken phase where the residual Gauge Group is just $\mathrm{U}(1)_Q$. In this phase the mass-Lagrangian reads as follows

$$-\mathcal{L}_M = \overline{U}_L\,\mathbb{M}^U U_R + \overline{D}_L\,\mathbb{M}^D D_R + \overline{N}_L\,\mathbb{M}^N N_R + \overline{E}_L\,\mathbb{M}^E E_R + h.c. \tag{2.7.1}$$

[8] Notice that this is just a replacement, not an exchange: it does not work backwards. This because of the asymmetry between η'_1 and η'_3.

where (recall that $i,j=1,2$, but for simplicity here we work as if they are equivalent)

$$U=\begin{pmatrix}u^i\\u^3\\u^\rho\\u^\eta\end{pmatrix}\,,\qquad D=\begin{pmatrix}d^i\\d^3\\d^\rho\\d^\eta\end{pmatrix}\,,\qquad N=\begin{pmatrix}\nu^i\\\nu^3\\\nu^\rho\\\nu^\eta\end{pmatrix}\,,\qquad E=\begin{pmatrix}e^i\\e^3\\e^\rho\\e^\eta\end{pmatrix} \tag{2.7.2}$$

and the mass-matrix for the up-type quarks reads as follows

$$\mathbb{M}^U=\begin{pmatrix}0 & -c^3_{\rho\mathcal{H}}v/2 & 0 & 0\\ 0 & c_{33}v/2 & 0 & -c_{\eta\mathcal{H}}v/2\\ \overline{c}^3_{\rho R}v_R & c^3_{\rho\Omega}\omega_3 & \overline{M}_\rho & -(c^3_{\eta\rho})^*\omega_3\\ 0 & \overline{m} & -c^3_{\eta\rho}\omega_3 & \overline{M}_\eta\end{pmatrix} \tag{2.7.3}$$

where we have defined

$$\overline{M}_\rho\equiv M_\rho-c^3_\rho\frac{v_3}{2}\,,\qquad \overline{m}\equiv m+c_{\eta\Sigma}\frac{v_3}{2}\,,\qquad \overline{M}_\eta\equiv M_\eta-c_\eta\frac{v_3}{2}\,. \tag{2.7.4}$$

For the other mass-matrices we have that they are given by $\mathbb{M}^U$ but applying the same prescriptions applied for the Yukawa matrices. Thus $\mathbb{M}^D$ is given by $\mathbb{M}^U$ but replacing $v_3\to -v_3$ and $\overline{c}^3_{\rho R}\to c^3_{\rho R}$. $\mathbb{M}^{\mathbb{N}}$ and $\mathbb{M}^E$ are given by $\mathbb{M}^U$ and $\mathbb{M}^D$ but replacing $\omega_3\to\omega_1$, $c^3_{\eta\rho}\to c^1_{\eta\rho}$, $c^3_{\rho\Omega}\to c^1_{\rho\Omega}$, $c^3_{\rho\mathcal{H}}\to c^1_{\rho\mathcal{H}}$, $c^3_{\rho R}\to c^1_{\rho R}$, $\overline{c}^3_{\rho R}\to\overline{c}^1_{\rho R}$ and $c^3_\rho\to c^1_\rho$ respectively. In the following we are going to diagonalize only $\mathbb{M}^U$ leaving as understood that to get the others one may need to apply the same prescriptions.

We need to diagonalize $\mathbb{M}^U$ with a singular-valued decomposition (for more information look at Section D.2 in the Appendix). In particular we need to find two unitary matrices L_U and R_U such that

$$L_U^\dagger\,\mathbb{M}^U R_U=\hat{\mathbb{M}}^U \tag{2.7.5}$$

where $\hat{\mathbb{M}}^U$ is diagonal and positive-definite. Unfortunately the calculation is very demanding computationally. Nevertheless, since the Higgs' VEV is assumed to be much lighter than all the others mass-scales in $\mathbb{M}^U$, we notice that the mass-matrix can be written in the following form

$$\mathbb{M}^U=\begin{pmatrix}m^U\\M^U\end{pmatrix} \tag{2.7.6}$$

where m^U and M^U are the sub-matrices of the first and last two rows respectively and we have the hierarchy $m^U\ll M^U$. Hence we can perturbatively block-diagonalize this matrix using the results derived in Section D.5 in the Appendix. In particular we can find a unitary matrix $\mathbb{L}^U$ such that

$$\mathbb{L}_U^\dagger\mathbb{M}^U(\mathbb{M}^U)^\dagger\mathbb{L}_U=\begin{pmatrix}(m^U_\ell)^2 & \mathbb{O}\\ \mathbb{O} & (M^U_h)^2\end{pmatrix} \tag{2.7.7}$$

where $(m^U_\ell)^2$ and $(M^U_h)^2$ are 2×2[9] positive-definite matrices and

$$\mathbb{L}_U=\begin{pmatrix}\mathbb{I}-\frac{1}{2}FF^\dagger & F\\ -F^\dagger & \mathbb{I}-\frac{1}{2}F^\dagger F\end{pmatrix}+\mathcal{O}(F^3)\,. \tag{2.7.8}$$

We find that

$$F\approx\frac{v/2}{|\overline{M}_\rho\overline{m}|^2+|\overline{M}_\rho\overline{M}_\eta|^2}\begin{pmatrix}F_{2\rho} & F_{2\eta}\\ F_{3\rho} & F_{3\eta}\end{pmatrix} \tag{2.7.9}$$

[9] In principle $(m^U_\ell)^2$ is a 3×3, but explicitly it is treated as a 2×2 because we do not distinguish between the first and second generations of SM fermions.

where

$$F_{2\rho} = c^3_{\rho\mathcal{H}}\omega_3\overline{M}^*_\rho\left(c^3_{\rho\Omega}\overline{m}^* - (c^3_{\eta\rho})^*\overline{M}^*_\eta\right) , \tag{2.7.10a}$$

$$F_{2\eta} = c^3_{\rho\mathcal{H}}\overline{M}^*_\rho\left(|\overline{m}|^2 + |\overline{M}_\eta|^2\right) , \tag{2.7.10b}$$

$$F_{3\rho} = |\overline{M}_\rho|^2\left(c_{33}\overline{m}^* - c_{\eta\mathcal{H}}\overline{M}^*_\eta\right) , \tag{2.7.10c}$$

$$F_{3\eta} = c^3_{\eta\rho}\omega_3\overline{M}^*_\rho\left(c_{33}\overline{m}^* - c_{\eta\mathcal{H}}\overline{M}^*_\eta\right) + \omega_3\left(c_{\eta\mathcal{H}}\overline{m} + c_{33}\overline{M}_\eta\right)\left(c^3_{\eta\rho}\overline{m}^* + (c^3_{\rho\Omega})^*\overline{M}^*_\eta\right) . \tag{2.7.10d}$$

The heavy mass-matrix reads as follows

$$(M^U_h)^2 \approx \begin{pmatrix} |\overline{m}|^2 + |\overline{M}_\eta|^2 + |c^3_{\eta\rho}|^2\omega_3^2 & (c^3_{\rho\Omega})^*\omega_3\overline{m} - c^3_{\eta\rho}\omega_3\left(\overline{M}_\eta + \overline{M}^*_\rho\right) \\ c^3_{\rho\Omega}\omega_3\overline{m}^* - (c^3_{\eta\rho})^*\omega_3\left(\overline{M}^*_\eta + \overline{M}_\rho\right) & |\overline{M}_\rho|^2 + \left(|c^3_{\eta\rho}|^2 + |c^3_{\rho\Omega}|^2\right)\omega_3^2 + |\overline{c}^3_{\rho R}|^2 v_R^2 \end{pmatrix} \tag{2.7.11}$$

and in first approximation the two heavy states have masses

$$M_1^2 \sim |\overline{m}|^2 + |\overline{M}_\eta|^2 , \qquad M_2^2 \sim |\overline{M}_\rho|^2 . \tag{2.7.12}$$

The light mass-matrix is given by

$$(m^U_\ell)^2 \approx \frac{v^2}{4N}\begin{pmatrix} (m^2_\ell)_{22} & (m^2_\ell)_{23} \\ (m^2_\ell)_{32} & (m^2_\ell)_{33} \end{pmatrix} \tag{2.7.13}$$

where

$$(m^2_\ell)_{22} = |c^3_{\rho\mathcal{H}}|^2\left[\omega_3^2\left|c^3_{\rho\Omega}\overline{M}_\eta + (c^3_{\eta\rho})^*\overline{m}\right|^2 + |\overline{c}^3_{\rho R}|^2 v_R^2\left(|\overline{m}|^2 + |\overline{M}_\eta|^2\right)\right] , \tag{2.7.14a}$$

$$\begin{aligned}(m^2_\ell)_{32} =& (c^3_{\rho\mathcal{H}})^*\omega_3\Big[c^3_{\eta\rho}|\overline{c}^3_{\rho R}|^2 v_R^2\left(c_{\eta\mathcal{H}}\overline{M}^*_\eta - c_{33}\overline{m}^*\right) \\ &+ \left(c_{\eta\mathcal{H}}\overline{m}\overline{M}_\rho + c_{33}\overline{M}_\eta\overline{M}_\rho + c^3_{\eta\rho}c_{\eta\mathcal{H}}c^3_{\rho\Omega}\omega_3^2 - c_{33}|c^3_{\eta\rho}|^2\omega_3^2\right)\left(c^3_{\eta\rho}\overline{m}^* + (c^3_{\rho\Omega})^*\overline{M}^*_\eta\right)\Big] ,\end{aligned} \tag{2.7.14b}$$

$$(m^2_\ell)_{23} = (m^2_\ell)^*_{3j} , \tag{2.7.14c}$$

$$\begin{aligned}(m^2_\ell)_{33} =& \left|c_{33}\overline{M}_\eta\overline{M}_\rho + c_{\eta\mathcal{H}}\overline{m}\overline{M}_\rho + c_{\eta\mathcal{H}}c^3_{\rho\Omega}c^3_{\eta\rho}\omega_3^2 - c_{33}|c^3_{\eta\rho}|^2\omega_3^2\right|^2 \\ &+ |\overline{c}^3_{\rho R}|^2 v_R^2\left[\left|c_{\eta\mathcal{H}}\overline{m} + c_{33}\overline{M}_\eta\right|^2 - |c_{33}|^2|\overline{M}_\eta|^2 + |c^3_{\eta\rho}|^2\omega_3^2\left(|c_{\eta\mathcal{H}}|^2 + |c_{33}|^2\right)\right] ,\end{aligned} \tag{2.7.14d}$$

$$\begin{aligned}N =& \omega_3^2\left|(c^3_{\eta\rho})^*\overline{m} + c^3_{\rho\Omega}\overline{M}_\eta\right|^2 + \left|\omega_3^2|c^3_{\eta\rho}|^2 - \overline{M}_\eta\overline{M}_\rho\right|^2 \\ &+ \left|c^3_{\rho\Omega}c^3_{\eta\rho}\omega_3^2 + \overline{m}\overline{M}_\rho\right|^2 + |\overline{c}^3_{\rho R}|^2 v_R^2\left(|\overline{m}|^2 + |\overline{M}_\eta|^2 + |c^3_{\eta\rho}|^2\omega_3^2\right) .\end{aligned} \tag{2.7.14e}$$

We can find the squared-masses of the second and third generations of SM fermions by diagonalizing $(m^U_\ell)^2$. In particular, under the assumptions that the scale of the VLF masses is greater than the ones of the scalar field's VEVs, we have that the mass of the third-generation fermion of a given type F (with $F = U, D, E$) is given by

$$(m^F_3)^2 \approx \left((m^F_\ell)^2\right)_{33} . \tag{2.7.15}$$

2.8 UV Parameter Space

What is left to understand is where all the UV parameters introduced lie in the parameter space to reproduce the desired Yukawa matrices. In particular the aim is to see if the couplings could be all of $\mathcal{O}(1)$ and if some fine tuning is needed.

The major constraint to satisfy is the tb mass splitting which at the M_Z scale is given by

$$m_t/m_b \approx 58\,. \tag{2.8.1}$$

Such large splitting is difficult to reproduce when it is generated by an EFT description. This implies unavoidably some fine tuning among the UV parameters. Because of that the scalar field Σ_3 plays a crucial role to avoid very large fine tuning since the mass splitting provided by this field comes from a 5-dimensional operator (otherwise it would have been 6-dimensional). Not only, but we do not need that the mass scale of the VEV of Σ_3 is below the mass scale of the VLFs. This because the hierarchical structure of the Yukawa matrices is already ensured by a hierarchy among the mass scales of the VEVs of Ω_1, Ω_3, Σ_R and the masses of the VLFs. To study this fine tuning it is worth to use the results derived in Section 2.7. If we assume that the leading terms are given by the mass scales $\overline{M}_\rho$, $\overline{M}_\eta$, $\overline{m}$ and v_3, we have that

$$m_t \approx \frac{v}{2}\left(|\overline{M}_\eta|^2 + |\overline{m}|^2\right)^{-1/2}\left|c_{33}M_\eta + c_{\eta\mathcal{H}}m + \frac{v_3}{2}\left(c_{\eta\mathcal{H}}c_{\eta\Sigma} - c_\eta c_{33}\right)\right|\,, \qquad m_b \approx m_t|_{v_3\to -v_3}\,. \tag{2.8.2}$$

Thus we need the following fine tuning among the parameters to reproduce the observed tb mass splitting

$$\left(1 + c_\eta\frac{v_3}{2M_\eta}\right)\frac{c_{33}}{c_{\eta\mathcal{H}}} - \left(c_{\eta\Sigma} - \frac{2m}{v_3}\right)\frac{v_3}{2M_\eta} \sim 10^{-2} \div 10^{-1}\,. \tag{2.8.3}$$

In addition, the tb mass splitting brings other consequences due to the constraints imposed on V_{cb}. In fact this is given by (2.1.4) and it implies that Y^D_{23} must be suppressed with respect to Y^D_{33}, which on its own is suppressed by fine tuning among the UV parameters. Hence we suspect that there could be some further fine tuning to suppress Y^D_{23} with respect to Y^U_{23}. In fact this is the case. Explicitly we have that using the results of Section 2.5

$$Y^U_{23} = \frac{\omega_3}{M_\rho}c^3_{\rho\mathcal{H}}\left(1 - c^3_\rho\frac{v_3}{2M_\rho}\right)^{-1}\left[c^3_{\rho\Omega} + \frac{v_3}{2M_\eta}(c^3_{\eta\rho})^*\left(c_{\eta\Sigma} + \frac{2m}{v_3}\right)\left(1 - c_\eta\frac{v_3}{2M_\eta}\right)^{-1}\right]\,, \tag{2.8.4a}$$

$$Y^D_{23} = Y^U_{23}\Big|_{v_3\to -v_3}\,. \tag{2.8.4b}$$

The leading part provides the suppression factor with respect to Y_{33} under the assumption that there is a hierarchy between ω_3 and M_ρ. Then, because Y^D_{33} is suppressed, we need the further following fine tuning among the parameters to reproduce the observed CKM component V_{cb}

$$\left(1 + c_\eta\frac{v_3}{2M_\eta}\right)\frac{c^3_{\rho\Omega}}{(c^3_{\eta\rho})^*} - \left(c_{\eta\Sigma} - \frac{2m}{v_3}\right)\frac{v_3}{2M_\eta} \sim 10^{-2} \div 10^{-1}\,. \tag{2.8.5}$$

Interestingly, notice that this second fine tuning is directly satisfied assuming the first one and the condition among the UV couplings

$$c_{33}/c_{\eta\mathcal{H}} \approx c^3_{\rho\Omega}/(c^3_{\eta\rho})^*\,. \tag{2.8.6}$$

A possible explanation to this could be that η and χ^3_R share the same quantum numbers, so the couplings to operators with same field content but χ^3_R or η should be very similar. If this is true, the latter condition follows immediately.

Once we have pointed out all the possible sources of fine tuning among the UV parameters, it is worth to identify which UV parameters are related to the other constraints that we would like to impose.

The first one is given by the $b\tau$ mass splitting. However, since the masses of b and τ are both suppressed by some fine tuning among the UV parameters, it is not so easy to provide the leading contribution to them. Nevertheless, by using the results of Section 2.7 and the fine tuning condition

$$\left|c_{\eta\mathcal{H}}\overline{m} + c_{33}\overline{M}_\eta\right| \ll \omega_3, \omega_1, v_R \tag{2.8.7}$$

which comes from the tb mass splitting, we find that

$$m_b^2 \approx \frac{v^2}{4} \frac{\omega_3^4 |c^3_{\eta\rho}|^2 \left| c_{\eta\mathcal{H}} c^3_{\rho\Omega} - c_{33} (c^3_{\eta\rho})^* \right|^2 + |c^3_{\rho R}|^2 v_R^2 \left(|c^3_{\eta\rho}|^2 \omega_3^2 \left(|c_{\eta\mathcal{H}}|^2 + |c_{33}|^2 \right) - |c_{33}|^2 |\overline{M}_\eta|^2 \right)}{|\overline{M}_\rho|^2 \left(|\overline{m}|^2 + |\overline{M}_\eta|^2 \right)} \tag{2.8.8}$$

while m_τ^2 is given by m_b^2 after having done the proper substitutions. If we further assume condition (2.8.6), we can provide the following leading contribution that generates $b\tau$ mass splitting

$$\frac{m_b^2}{m_\tau^2} \approx \left| \frac{c^3_{\rho R}}{c^1_{\rho R}} \right|^2 \left[\frac{|c^3_{\eta\rho}|^2 \omega_3^2 \left(|c_{\eta\mathcal{H}}|^2 + |c_{33}|^2 \right) - |c_{33}|^2 |\overline{M}_\eta|^2}{|c^1_{\eta\rho}|^2 \omega_1^2 \left(|c_{\eta\mathcal{H}}|^2 + |c_{33}|^2 \right) - |c_{33}|^2 |\overline{M}_\eta|^2} \right] \tag{2.8.9}$$

and it can easily reproduce the observed $b\tau$ mass splitting in some regions of the UV parameters space.

Last condition that must be satisfied is the one reported in (2.1.3). Explicitly we have that using the results of Section 2.5

$$\frac{Y^U_{3j}}{Y^U_{i3}} \approx \frac{v_R}{M_\eta} c_{\eta\mathcal{H}} c^3_{\rho\Omega} \bar{c}^3_{\rho R} \left(1 - c_\eta \frac{v_3}{2M_\eta} \right)^{-1} \left[c^3_{\rho\Omega} + \frac{v_3}{2M_\eta} (c^3_{\eta\rho})^* \left(c_{\eta\Sigma} + \frac{2m}{v_3} \right) \left(1 - c_\eta \frac{v_3}{2M_\eta} \right)^{-1} \right]^{-1}, \tag{2.8.10a}$$

$$\frac{Y^D_{3j}}{Y^D_{i3}} \approx \frac{v_R}{M_\eta} c_{\eta\mathcal{H}} c^3_{\rho\Omega} c^3_{\rho R} \left(1 + c_\eta \frac{v_3}{2M_\eta} \right)^{-1} \left[c^3_{\rho\Omega} - \frac{v_3}{2M_\eta} (c^3_{\eta\rho})^* \left(c_{\eta\Sigma} - \frac{2m}{v_3} \right) \left(1 + c_\eta \frac{v_3}{2M_\eta} \right)^{-1} \right]^{-1}. \tag{2.8.10b}$$

In particular we have that for the down-type quarks the expression in the parenthesis is suppressed because of the fine tuning imposed by the constraints on V_{cb}. Thus we need that

$$\frac{v_R}{M_\eta} c^3_{\rho R} \lesssim 10^{-2}\,. \tag{2.8.11}$$

This is achievable without any fine tuning, but just with a suitable choice of the UV parameters and mass scales. Notice that in this region of the UV parameter space we have that

$$Y^D_{3j} \sim Y^D_{ij} \sim 10^{-4} \tag{2.8.12}$$

and such value corresponds to the correct order of magnitude for y_s. Nevertheless, it is not important to tune precisely the Yukawa couplings of the second generations. This because we could think of a possible further UV embedding in which is violated the universality of the first and second generations. This should be the correct framework where to study precisely the structure of the Y_{ij} sector in the Yukawa matrices. However this is not going to be analysed in this work.

Keeping in mind all the constraints that limit the region of the UV parameter space, we are able to reproduce all the Yukawa matrices by assuming $\mathcal{O}(1)$ couplings (more precisely from the interval $[0.1, 1]$), $\mathcal{O}(5)$ TeV VEVs (expect for the VEV of Σ_3 which we allow to be even higher, of $\mathcal{O}(20)$ TeV) and $\mathcal{O}(20)$ TeV VLF masses. Also this happens in both of the models, although we have studied in detail the constraints on the UV parameter space only for Model I. We recall that this choice will be clarified in the following, more precisely at the end of Chapter 4. We want also to enlighten the fact that this model appears to be very robust in the sense that, under a small change of the UV parameters, also the Yukawa matrices change slightly (apart from the constraints imposed by the little fine tuning already discussed). In addition, thanks to Σ_3, we have been able to explain the structure of the all Yukawa matrices in a very *natural* way where the only fine tuning is exploited to reproduce the observed tb mass splitting and it requires just an $\mathcal{O}(10)$ cancellation among the UV parameters.

A possible choice for the values of the UV parameters with corresponding Yukawa matrices will be provided in Section 4.7, together with the predictions on neutrino masses.

As last remark, we have already pointed out from the beginning of this Chapter that we could avoid such fine tuning among the UV parameters or even the need of Σ_3 to generate the tb mass splittings at the price of assuming an Higgs field where the $\mathrm{SU}(2)_R^{[3]}$-doublet components are weights differently as shown in (2.2.10). The issue with this approach is that, as discussed in Section C.2 in the Appendix, we need $\xi = \mathcal{O}(10^{-2})$ to explain the tb mass splitting. Thus we should explain why there are such different weights between the two components of the $\mathrm{SU}(2)_R^{[3]}$ doublet, which is again a fine tuning problem.

3 Neutrino Masses

In the previous Chapter we have shown that the models we considered in this work are able to reproduce the desired Yukawa matrices for the up- and down-type quarks as well as for the charged leptons. What must still be understood is how to explain the observed neutrino masses. In this Chapter we are going to study this issue in great detail with focus on how to implement the See-Saw mechanism in the framework of the models considered. In the first part we are going to study several possible See-Saw mechanisms that could be utilize to explain the observed active-neutrino masses. In the second part we are going to study which of the parameters entering in the See-Saw mechanisms could be generated in the models we considered, together with the hierarchies among them.

3.1 The See-Saw Mechanism

An unavoidable feature of Pati-Salam-like models is a degeneracy of the up-type quarks and neutral leptons interactions since they are grouped together in a fermionic SU(4)-vector. Even though we have introduced sources of splittings among the different types of fermions, the up-type quark and neutral lepton Yukawa matrices are unavoidably comparable. The issue with neutrino masses arises from neutrino oscillation measurements. We know that the neutrino masses are anarchic and in the range $10^{-2}\,\mathrm{eV} \lesssim m_\nu \lesssim 10^{-1}\,\mathrm{eV}$ (unless you allow fine tuning) with well-measured squared-mass splittings

$$\Delta m^2_{32} = 2.56 \cdot 10^{-3}\,\mathrm{eV}^2\,, \qquad \Delta m^2_{21} = 7.36 \cdot 10^{-5}\,\mathrm{eV}^2\,. \tag{3.1.1}$$

Thus we need to introduce a mechanism to suppress the latter masses with respect to the ones of the up-type quarks.

The See-Saw mechanism is probably the most efficient way to explain the tiny neutrino masses giving Yukawa couplings of $\mathcal{O}(1)$. The only constraint is that it requires the existence of Majorana fermions. A proper definition of a Majorana field is provided in Section E.1 in the Appendix. There exist two different kinds of See-Saw mechanisms: the Direct (DSS) and Inverse (ISS). For a review see for instance [9]. DSS works assuming the existence of (at least) one sterile right-handed (RH) neutrino with a Majorana mass. Working in a single-flavour model with only one left-handed (LH) neutrino ν_L and one RH ν_R, the mass-Lagrangian reads as follows

$$-\mathcal{L} = \frac{1}{2} M_R \overline{\nu}^c_R \nu_R + m_D \overline{\nu}_L \nu_R + h.c. \tag{3.1.2}$$

where m_D is the so-called Dirac mass for the LH neutrino. Using the fact that

$$\overline{\nu}_L \nu_R = \frac{1}{2}\left(\overline{\nu}_L \nu_R + \overline{\nu}^c_R \nu^c_L\right) \tag{3.1.3}$$

we can rewrite the Lagrangian as follows

$$-\mathcal{L} = \frac{1}{2} \overline{N}_L M N^c_L + h.c. \tag{3.1.4}$$

where $N_L = (\nu_L, \nu^c_R)^T$ and the neutrino mass-matrix reads

$$M = \begin{pmatrix} 0 & m_D \\ m_D & M_R \end{pmatrix}\,. \tag{3.1.5}$$

To find the mass-eigenstates we have to do a unitary transformation of N_L to diagonalize M. In addition, working in the limit $m_D \ll M_R$, we can do this perturbatively as shown in Section D.4 in the Appendix. We find one active (or better saying very light) neutrino mass-eigenstate and one very heavy. Their masses read as follows respectively

$$m_\nu \approx \frac{m_D^2}{M_R}\,, \qquad M_h \approx M_R\,. \tag{3.1.6}$$

The eigenstates are almost given by ν_L and ν_R^c their-selves since the mixing angle between the two species is of $\mathcal{O}(m_D/M_R)$ and it is extremely suppressed.

ISS works assuming the existence of (at least) three neutrino species with (at least) one with a small Majorana mass. Working in a single-flavour model with two LH neutrinos ν_L and s_L and one RH ν_R, the Lagrangian reads as follows

$$-\mathcal{L} = \frac{1}{2}\overline{N}_L M N_L^c + h.c. \tag{3.1.7}$$

where $N_L = (\nu_L, \nu_R^c, s_L)^T$ and the neutrino mass-matrix reads

$$M = \begin{pmatrix} 0 & D & 0 \\ D & m & N \\ 0 & N & \mu \end{pmatrix} . \tag{3.1.8}$$

As before, we need to do a unitary transformation of N_L to diagonalize M. This time we assume the hierarchy $m, \mu \ll D, N$. Thus M is made of a leading part plus a perturbation

$$M = M_0 + \Delta M = \begin{pmatrix} 0 & D & 0 \\ D & 0 & N \\ 0 & N & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{pmatrix} . \tag{3.1.9}$$

Such kind of matrix has been perturbatively diagonalized in Section C.4 in the Appendix. We find that there are one active and two heavy neutrino mass-eigenstates. Their squared-masses read as follows respectively

$$m_\nu^2 \approx \frac{|\mu|^2|D|^4}{(|D|^2+|N|^2)^2} , \qquad M_R^2 \approx |D|^2 + |N|^2 . \tag{3.1.10}$$

The mixing matrix between the three neutrinos is given by the eigenvectors. In particular, the active eigenstate is a mixing (almost) between ν_L and s_L with mixing angle given by $\tan\theta \sim D/N$.

In the following Sections we apply those mechanisms in the framework of the models we are considering in this work to see which could be utilize to explain the observed neutrino masses.

3.2 Direct See-Saw

Up to now in the models considered we have the following neutrinos[10]: ν_L^3, ν_L^i, ν_R^3 and ν_R^j. The most general Lagrangian allows Yukawa couplings among ν_L and ν_R neutrinos; then we can have a Majorana mass-matrix for ν_R^j and ν_R^3. Therefore the most general mass-Lagrangian reads as follows

$$-\mathcal{L} = \frac{1}{2}\overline{N}_L M N_L^c + h.c. \tag{3.2.1}$$

where $N_L = (\nu_L^i, \nu_L^3, (\nu_R^j)^c, (\nu_R^3)^c)^T$ and

$$M = \begin{pmatrix} 0 & D \\ D^T & \mu_R \end{pmatrix} . \tag{3.2.2}$$

D and μ_R are two 3×3 matrices and explicitly[11]

$$D = \frac{v}{\sqrt{2}} Y^N = \frac{v}{\sqrt{2}} \begin{pmatrix} Y_{ij}^N & Y_{i3}^N \\ Y_{3j}^N & Y_{33}^N \end{pmatrix} . \tag{3.2.3}$$

[10] In principle $i = 1, 2$, but we treat them as they are just one generation to simplify the computations. Conceptually it does not change anything.

[11] We write in compact notation the entries associated to the first and second generations.

We assume the hierarchy $D \ll \mu_R$. Thus we can block-diagonalize M using a unitary matrix W as done in Section D.3 in the Appendix to get

$$W^T M W = \begin{pmatrix} m_\nu & 0 \\ 0 & M_h \end{pmatrix} . \tag{3.2.4}$$

The mixing angle between LH and RH is of $\mathcal{O}(v/M_R)$ where

$$M_R \equiv \left[\det\left(\mu_R^\dagger \mu_R\right)\right]^{\frac{1}{4}} \tag{3.2.5}$$

and it is assumed to be very little. Thus the eigen-subspaces are given by one active-neutrino subspace generated almost only by ν_L^i and ν_L^3 and one heavy-neutrino subspace generated almost only by $(\nu_R^i)^c$ and $(\nu_R^3)^c$. Explicitly their mass-matrices read respectively as follows

$$m_\nu \approx -\frac{v^2}{2} Y^N R^{-1} (Y^N)^T , \qquad M_h \approx \mu_R . \tag{3.2.6}$$

This is equivalent to a DSS and we have that the leading contribution to m_ν parametrically goes like

$$m_\nu \sim \frac{v^2}{M_R} Y^2 . \tag{3.2.7}$$

With suitable choices of the parameters we can have active-neutrino masses of the same order of magnitude. However keep in mind that the mass scale of M_R should be around 10^{14} GeV like happens in any DSS scenarios. We will come back to this point later in this Chapter.

3.3 Inverse See-Saw

In this Section we study in detail the ISS mechanism in the framework of the models considered. Before starting with the explicit realization, let us make an observation: the sterile fermions that we have to add in the ISS must be LH. Otherwise it would be like adding new sterile RH neutrinos, which means it would be like increasing the maximum value of the index j that accounts for the generations of ν_R^j. In addition, there is no constraint that implies the existence of the first and second generations of RH neutrinos which are sterile particles in our model. Hence we are going to be completely general in accounting the neutrino species that contributes to the ISS mechanism. Thus we have in full generality that the particle content is made of at least three LH neutrinos ν_L^3 and ν_L^i with $i = 1, 2$ and one third-generation RH neutrino ν_R^3. Then we add $n_R - 1$ species of RH sterile neutrinos ν_R^j with $j = 1, ..., n_R - 1$ and n_s species of sterile LH fermions s_L^k with $k = 1, ..., n_s$. Here n_R and n_s are completely general positive integer numbers.

We assume interactions between LH and RH fermions and we provide small Majorana mass-matrices for the RH neutrinos and the LH sterile fermions s_L^k. In addition we assume that ν_L^i and s_L^k do not interact at tree-level in any way and we do not assign Majorana masses to ν_L^i. The most general mass-Lagrangian reads as follows

$$-\mathcal{L} = \frac{1}{2} \overline{N}_L M N_L^c + h.c. \tag{3.3.1}$$

where $N_L = (\nu_L^i, \nu_L^3, (\nu_R^j)^c, (\nu_R^3)^c, s_L^k)^T$ and

$$M = \begin{pmatrix} 0 & D & 0 \\ D^T & \mu_R & N^T \\ 0 & N & \mu \end{pmatrix} . \tag{3.3.2}$$

D is a $n_L \times n_R$ complex matrix (calling $n_L = 3$ the number of LH neutrinos), N is a $n_s \times n_R$ complex matrix, μ_R is a $n_R \times n_R$ symmetric complex matrix and μ is a $n_s \times n_s$ symmetric complex matrix. We assume the hierarchy $\mu_R, \mu \ll D, N$. Thus we can perturbatively diagonalize this matrix which has the following form

$$M = M_0 + \Delta M = \begin{pmatrix} 0 & D & 0 \\ D^T & 0 & N \\ 0 & N^T & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_R & 0 \\ 0 & 0 & \mu \end{pmatrix} . \tag{3.3.3}$$

From the rank of M_0 we can see that at zero-th order in perturbation theory we have $n_L + n_s - n_R$ massless eigenstates, while the others have masses of $\mathcal{O}(N, D)$. This is very important because, since we observe (at least) three active-neutrino mass-eigenstates, we must satisfy the condition $n_s \geq n_R$. The minimal realization would be in the case of three active mass-eigenstates. Hence in the following we will always assume to work with

$$n_s = n_R \, . \tag{3.3.4}$$

Notice that this result was already present in the literature, for instance see [9].

To summarize, to make an ISS we need (apart from the three LH neutrino generations) $n_s \geq 1$ sterile LH fermions s_L^k with $k = 1, ..., n_s$, then $n_s - 1$ sterile RH neutrinos ν_R^j with $j = 1, ..., n_s - 1$ and finally the third-generation RH neutrino ν_R^3. In this framework the most general mass-Lagrangian reads as follows

$$-\mathcal{L} = \frac{1}{2} \overline{N}_L M N_L^c + h.c. \tag{3.3.5}$$

where $N_L = (\nu_L^i, \nu_L^3, (\nu_R^j)^c, (\nu_R^3)^c, s_L^k)^T$ and

$$M = \begin{pmatrix} 0 & D \\ D^T & R \end{pmatrix} . \tag{3.3.6}$$

D and R are two matrices of dimension $3 \times 2n_s$ and $2n_s \times 2n_s$ respectively and explicitly

$$D = \frac{v}{\sqrt{2}} \begin{pmatrix} Y_{1j}^N & Y_{13}^N & \\ Y_{2j}^N & Y_{23}^N & \mathbb{O}_{3 \times n_s} \\ Y_{3j}^N & Y_{33}^N & \end{pmatrix} , \qquad R = \begin{pmatrix} \mu_R & m_R^T \\ m_R & \mu \end{pmatrix} \tag{3.3.7}$$

where m_R, μ_R and μ are $n_s \times n_s$ matrices. We also assume the hierarchy $\mu, \mu_R \ll m_R$ and $D \ll R$[12].

Thus we can block-diagonalize this matrix as shown in Section D.3 in the Appendix in a similar way as done in the DSS scenario. We find that the mixing angle between LH active neutrinos and the other states is of $\mathcal{O}(v/M_R)$ where

$$M_R \equiv \left[\det\left(m_R^\dagger m_R \right) \right]^{1/2n_s} \tag{3.3.8}$$

and it is assumed to be little. Thus the eigen-subspaces are given by one active-neutrino subspace generated almost only by ν_L^i and ν_L^3 and one heavy-neutrino subspace generated almost only by $(\nu_R^j)^c$, $(\nu_R^3)^c$ and s_L^k. The heavy mass-matrix is given by $M_h \approx R$ which means that the heavy mass-eigenstates have masses of order

$$M_h \sim M_R \tag{3.3.9}$$

plus small corrections of $\mathcal{O}(\mu, \mu_R)$. The light mass-matrix instead is given by

$$m_\nu \approx -D R^{-1} D^T \, . \tag{3.3.10}$$

To invert R we use the results shown in Section D.6 in the Appendix finding that

$$m_\nu \approx \frac{v^2}{2} Y^N \left(m_R^T \mu^{-1} m_R \right)^{-1} (Y^N)^T \, . \tag{3.3.11}$$

[12]In principle this assumption is needed just to provide a rather easy approximated solution and it is not necessary to get the final mass hierarchy among the active-neutrino masses. Nevertheless it will be important to not spoil the unitarity constraints on the PMNS matrix as we are going to discuss in the following.

Notice that as long as $\mu_R \ll m_R^T \mu^{-1} m_R$ (as we assume to be), a possible Majorana mass-matrix for the RH neutrinos is completely irrelevant. Also, as expected, we have that the active-neutrino mass-matrix goes parametrically like an usual ISS.

Regarding the number of sterile LH fermions n_s, if $n_s \leq 2$ we have that $\det[m_\nu] = 0$ which implies that there is at least one exactly massless eigenstate. In particular for $n_s = 2$ there is only one while for $n_s = 1$ they are two and this is not compatible with what we observe[13]. If $n_s = 3$ we have three massive eigenstates. In any case it is possible to tune the parameters to reproduce the observed neutrino mass-spectrum. Nevertheless, it seems very *natural* to expect that the first- and second-generation fermions will embed in a Gauge Group of type $\mathrm{SU}(4) \times \mathrm{SU}(2)_R$ as we did for the third generation at a higher energy scale. Thus the case $n_s = 2$ seems quite *unnatural* and we assume in the following to work with $n_s = 3$ in the ISS scenario.

3.4 Mixed See Saw

There is one last possibility that we have not studied yet. We can do a mixed See-Saw (MSS) mechanism using very heavy Majorana masses for the light-generation RH neutrinos and one new sterile LH fermion with a little Majorana mass. Thus we assume to have (apart from the three LH neutrinos) two generations of sterile RH neutrinos ν_R^j with $j = 1, 2$, the usual third-generation RH neutrino ν_R^3 and one sterile LH fermions s_L. The most general mass-Lagrangian reads as follows

$$-\mathcal{L} = \frac{1}{2} \overline{N}_L M N_L^c + h.c. \tag{3.4.1}$$

where $N_L = (\nu_L^i, \nu_L^3, (\nu_R^j)^c, (\nu_R^3)^c, s_L)^T$ and

$$M = \begin{pmatrix} 0 & D \\ D^T & R \end{pmatrix} . \tag{3.4.2}$$

D and R are two matrices of dimension 3×4 and 4×4 respectively and explicitly

$$D = \frac{v}{\sqrt{2}} \begin{pmatrix} & 0 \\ Y^N & 0 \\ & 0 \end{pmatrix} , \qquad R = \begin{pmatrix} M_R^j & (m_R^{j3})^T & (m_R^j)^T \\ m_R^{j3} & \mu_R^3 & M_R^3 \\ m_R^j & M_R^3 & \mu \end{pmatrix} \tag{3.4.3}$$

where M_R^j is a 2×2 matrix while m_R^{j3} and m_R^j are 2-dimensional row-vectors. We also assume the hierarchy $D \ll R$ required by the DSS, but also the hierarchies $\mu, m_R^{j3}, \mu_R^3 \ll M_R^j$ and $m_R^j, \mu_R^3 \ll M_R^3$ required by the ISS.

This allows us to block-diagonalize M as we did before and we find that the mixing angle between LH neutrinos and the other states is of $\mathcal{O}(v/M_R)$ where

$$M_R \equiv \left[\det\left(R^\dagger R\right)\right]^{\frac{1}{8}} \sim \left(M_R^j M_R^3\right)^{\frac{1}{2}} \tag{3.4.4}$$

and it is assumed to be little. Thus the eigen-subspaces are given by one active-neutrino mass-subspace generated almost only by ν_L^i and ν_L^3 and one heavy-neutrino mass-subspace generated almost only by $(\nu_R^j)^c$, $(\nu_R^3)^c$ and s_L. The heavy mass-matrix is given by

$$M_h \approx R \approx \begin{pmatrix} M_R^j & 0 & 0 \\ 0 & 0 & M_R^3 \\ 0 & M_R^3 & 0 \end{pmatrix} + \mathcal{O}(\mu, \mu_R^3, m_R^j, m_R^{j3}) \tag{3.4.5}$$

[13]In fact we do observe two different mass splittings among the active neutrinos in the SM. Hence at least two of them must be massive.

which means that the heavy mass-eigenstates have masses of order

$$M_h \sim M_R \tag{3.4.6}$$

plus small corrections. The active-neutrino mass-matrix is given by

$$m_\nu \approx -DR^{-1}D^T\,. \tag{3.4.7}$$

We can now invert R as shown in Section D.6 in the Appendix finding that (in the approximations $(m_R^j)^T\,\mu^{-1}m_R^j \ll M_R^j$, $m_R^{j3} \ll M_R^3\,\mu^{-1}m_R^j$ and $\mu_R^3 \ll M_R^3\,\mu^{-1}M_R^3$)

$$m_\nu \approx -\frac{v^2}{2}Y^N \begin{pmatrix} M_R^j & (m_R^j)^T\,\mu^{-1}M_R^3 \\ M_R^3\,\mu^{-1}m_R^j & M_R^3\,\mu^{-1}M_R^3 \end{pmatrix}^{-1} (Y^N)^T\,. \tag{3.4.8}$$

We observe that parametrically the active-neutrino masses are given by

$$m_\nu^j \sim \frac{v^2}{M_R^j}Y^2\,, \qquad m_\nu^3 \sim \mu\frac{v^2}{(M_R^3)^2}Y^2\,. \tag{3.4.9}$$

Thus they are a mixture between DSS and ISS as expected. Also we can tune the parameters to find the observed neutrino mass-spectrum. Nevertheless, the MSS requires somehow a fine tuning between the high Majorana masses of the first- and second-generation RH neutrinos and the mass scales involved with s_L to produce anarchic active-neutrino masses. This seems very *unnatural* and hence we discard this option.

3.5 Non-Unitarity of the PMNS Matrix

Like there is the CKM matrix in the SM for the quarks, there is also its leptonic counterpart which is called the PMNS matrix. It is defined as follows

$$N \equiv U_\ell^\dagger U_\nu \tag{3.5.1}$$

where U_ℓ and U_ν are two unitary matrices that rotate the LH charged and neutral leptons respectively from the flavour-basis to the mass-basis. By construction it is different from the identity once non-vanishing masses for the neutrinos are assumed. In addition, as it happens for the CKM matrix, the PMNS matrix is predicted to be unitary in the SM. We can parameterize its non-unitarity by defining the following matrix

$$|\eta| \equiv |\mathbb{I} - NN^\dagger|\,. \tag{3.5.2}$$

A possible detection of non-unitarity of N would imply BSM physics. Furthermore, there exist some experimental upper bounds on this parameter which puts some constraints on the possible PMNS unitarity violation induced by some NP models, like the one we are considering. Currently those bounds are given by [16]

$$|\eta| < \begin{pmatrix} 2.1\cdot 10^{-3} & 1.0\cdot 10^{-5} & 2.1\cdot 10^{-3} \\ 1.0\cdot 10^{-5} & 4.0\cdot 10^{-4} & 8.0\cdot 10^{-4} \\ 2.1\cdot 10^{-3} & 8.0\cdot 10^{-4} & 5.3\cdot 10^{-3} \end{pmatrix}\,. \tag{3.5.3}$$

A possible source of non-unitarity is provided by the addition of new sterile fermionic states that mix together with the LH SM neutrinos. This is the case at work in the models we considered. In particular PMNS unitarity gets broken because the matrix that rotates the neutrinos into their mass-eigenstates is given by

$$\overline{U}_\nu \approx \left(\mathbb{I} - \frac{1}{2}BB^\dagger\right) U_\nu \tag{3.5.4}$$

where U_ν is the unitary matrix that diagonalizes the active-neutrino mass-matrix m_ν and (in the notation used in the previous Sections[14])

$$B^\dagger \approx R^{-1} D^T \,. \tag{3.5.5}$$

In particular the non-unitarity part of the mixing matrix comes from the upper-left block of the unitary matrix W which block-diagonalizes the neutrino mass-matrix isolating the active-neutrino mass-eigenstates in (3.2.4) (and it is a feature present in all the See-Saw mechanisms considered). Thus the PMNS matrix in this framework reads as follows

$$N \equiv U_\ell^\dagger \overline{U}_\nu \approx U_\ell^\dagger \left(\mathbb{I} - \frac{1}{2} B B^\dagger \right) U_\nu \tag{3.5.6}$$

and it is not unitary anymore. Explicitly the non-unitarity parameter reads as follows

$$|\eta| \approx |B B^\dagger| \approx |D^* (R^{-1})^\dagger R^{-1} D^T| \tag{3.5.7}$$

and parametrically it goes like

$$|\eta| \sim \frac{v^2}{M_R^2} Y^2 \,. \tag{3.5.8}$$

For the DSS scenario this parameter is very low and well below the current bounds by several orders of magnitude[15]. In the ISS this should be taken into account. In our case we have that using (D.6.4) and under the assumption that $\mu_R \ll m_R^T \mu^{-1} m_R$

$$B^\dagger \approx \frac{v^2}{2} \begin{pmatrix} (m_R^T \mu^{-1} m_R)^{-1} (Y^N)^T \\ \mu^{-1} m_R (m_R^T \mu^{-1} m_R)^{-1} (Y^N)^T \end{pmatrix} \tag{3.5.9}$$

which implies that

$$|\eta| = \frac{v^2}{2} \left(Y^N m_R^{-1} \right)^* \left(Y^N m_R^{-1} \right)^T + \mathcal{O}\left(\frac{v^4}{M_R^4} \right) \,. \tag{3.5.10}$$

3.6 See-Saw Mass-Terms as EFT

In this Section we provide a list of all the possible mass-terms that enter in the See-Saw mechanisms described before and that could be generated in the framework of the models we considered in this work. We start from generating the Majorana mass-matrix for RH neutrinos and we continue generating all the mass-terms where the LH sterile fermions s_L^k are involved.

We want to generate in the framework of the models considered in this work a Majorana mass-matrix for the RH neutrinos. Explicitly we want to generate the following Lagrangian (where only for the equation below $i, j = 1, 2, 3$)

$$\mathcal{L} \supset -\frac{1}{2} (\mu_R)_{ij} (\nu_R^i)^T \mathcal{C}\, \nu_R^j + h.c. \tag{3.6.1}$$

where μ_R is a symmetric matrix. Notice that this can be done because under the SM Gauge Group (1.1.1) those are all sterile fermions. Instead for the LH neutrinos this is not possible since they are still charged. The first- and second-generation RH neutrinos are sterile. Hence we can safely write the following Majorana mass-matrix (where $i, j = 1, 2$)

$$\mathcal{L} \supset -\frac{1}{2} (\mu_R)_{ij} (\nu_R^i)^T \mathcal{C}\, \nu_R^j + h.c. \tag{3.6.2}$$

[14] In the case of DSS $R = \mu_R$.

[15] In general in a DSS scenario is required a Majorana mass $M_R = \mathcal{O}(10^{14})$ GeV and our case is not an exception.

Instead the third-generation RH neutrino is charged under the Gauge Group (1.1.3). Thus it is more complicated to generate a Majorana mass-term for it. In particular we have that

$$(\chi_R^3)_\alpha(\chi_R^3)_\beta \quad \sim \quad (\mathbf{4}\times\mathbf{4},\mathbf{2}\times\mathbf{2}) \in \mathrm{SU}(4)^{[3]}\times\mathrm{SU}(2)_R^{[3]} \tag{3.6.3}$$

where α,β represent the spinor (or Lorentz) indices of the fermionic fields. To generate a Majorana mass-term, those indices has to be contracted in the following way

$$(\chi_R^3)_\alpha \mathcal{C}_{\alpha\beta}(\chi_R^3)_\beta\,. \tag{3.6.4}$$

This means that this term must be symmetric in all its Gauge indices. Namely it has to transform under the $\mathbf{10}_S$ representation of $\mathrm{SU}(4)^{[3]}$ and the $\mathbf{3}_S$ representation of $\mathrm{SU}(2)_R^{[3]}$. This because we have that from Group Theory (for a reference see for instance [17])

$$\mathbf{4}\times\mathbf{4}=\mathbf{10}_S+\mathbf{6}_A\,,\qquad \mathbf{2}\times\mathbf{2}=\mathbf{3}_S+\mathbf{1}_A\,. \tag{3.6.5}$$

Without adding new fields to the models, we could get a Majorana mass for the third-generation neutrino by means of some EFT operators built using some of the scalar fields that will eventually acquire a VEV. For Model I we can write the following 7-dimensional operator

$$\mathcal{L}\supset -\frac{1}{2}\frac{c_7}{\Lambda^3}\left((\Sigma_R^c)^\dagger\Omega_1\chi_R^3\right)^T\mathcal{C}\left((\Sigma_R^c)^\dagger\Omega_1\chi_R^3\right)+h.c. \tag{3.6.6}$$

where Λ is the energy scale at which this operator finds its UV completion. For Model II we can write the following 5-dimensional operator

$$\mathcal{L}\supset -\frac{1}{2}\frac{c_5}{2\Lambda}\left((\Delta_3^c)^\dagger\chi_R^3\right)^T\mathcal{C}\left((\Delta_3^c)^\dagger\chi_R^3\right)+h.c. \tag{3.6.7}$$

When those fields acquire their VEVs, we can generate the following Majorana mass-terms for RH neutrinos

$$\text{Model I:}\quad (\mu_R)_{33}=\frac{c_7}{\Lambda^3}v_R^2\omega_1^2\,,\qquad\qquad \text{Model II:}\quad (\mu_R)_{33}=\frac{c_5}{\Lambda}w^2\,. \tag{3.6.8}$$

Since both are EFT operators, we expect those masses to be suppressed, or in any case not above the TeV scale. In fact the only way to get a renormalizable Majorana mass-term for ν_R^3 (and hence a mass-term which *theoretically* could be very large) is to assume the existence of a scalar boson

$$X\sim(\overline{\mathbf{10}}_S,\overline{\mathbf{3}}_S)\in\mathrm{SU}(4)^{[3]}\times\mathrm{SU}(2)_R^{[3]} \tag{3.6.9}$$

which will eventually acquire a VEV. This possibility is studied in detail in Section C.7 in the Appendix.

Lastly we need to generate the mass-mixing terms between first-, second- and third-generation RH neutrinos. Again, without adding new fields to the models, for Model I the only possibility is to use some EFT operators built using the scalar fields that will eventually acquire a VEV. In particular we can use the following 5-dimensional operator

$$-\mathcal{L}\supset\frac{c_{3j}}{\Lambda}\overline{\chi}_R^3\Omega_1^\dagger\Sigma_R^c(\nu_R^j)^c+h.c. \tag{3.6.10}$$

For Model II instead we can write the following renormalizable operator

$$-\mathcal{L}\supset c_{3j}\overline{\chi}_R^3\Delta_3^c(\nu_R^j)^c+h.c. \tag{3.6.11}$$

When those fields acquire their VEVs, we can generate the following Majorana mass-terms for RH neutrinos (where $j=1,2$)

$$\text{Model I:}\quad (\mu_R)_{3j}=\frac{c_{3j}}{\Lambda}v_R\omega_1\,,\qquad\qquad \text{Model II:}\quad (\mu_R)_{3j}=c_{3j}w\,. \tag{3.6.12}$$

In the ISS we have to assume the existence of a certain number n_s (which we assume to be $n_s = 3$) of sterile fermions s_L^k (where $k = 1, ..., n_s$). Now we want to generate in the framework of the models considered in this work all the mass-terms in which they are involved. Since they are sterile, we can safely generate for s_L^k the following Majorana mass-matrix

$$\mathcal{L} \supset -\frac{1}{2}\overline{s}_L^k\,\mu_{k\ell}(s_L^\ell)^c + h.c. \tag{3.6.13}$$

Then we need to generate the mass-mixing terms between those sterile fermions and the RH neutrinos. Namely we want to generate the mass-terms (where only for the equation below $j = 1, 2, 3$)

$$\mathcal{L} \supset -\overline{s}_L^k(m_R)_{kj}\,\nu_R^j + h.c. \tag{3.6.14}$$

The term with the first- and second-generation RH neutrinos can be always generated since all the fields are sterile and it reads as follows

$$-\mathcal{L} \supset c_{kj}\Lambda'\,\overline{s}_L^k\nu_R^j + h.c. \tag{3.6.15}$$

Λ' is a completely general energy scale which in principle could be very high (and different from the scale Λ at which all the other operators find their UV origin), even well above the TeV scale. Thus we have that (where $j = 1, 2$)

$$(m_R)_{kj} = c_{kj}\Lambda'\,. \tag{3.6.16}$$

The mass-mixing terms between the sterile fermions and the third-generation RH neutrino are less trivial to generate. Without adding new fields to the models, for Model I the only possibility is to use some EFT operators built using the scalar fields that will eventually acquire a VEV. In particular we can use the following 5-dimensional operator

$$\mathcal{L} \supset -\frac{c_{k3}}{\Lambda}\overline{s}_L^k(\Sigma_R^c)^\dagger\Omega_1\chi_R^3 + h.c. \tag{3.6.17}$$

For Model II instead we can write the following renormalizable operator

$$\mathcal{L} \supset -c_{k3}\overline{s}_L^k(\Delta_3^c)^\dagger\chi_R^3 + h.c. \tag{3.6.18}$$

When those fields acquire their VEVs, we can generate the following mass-mixing terms between sterile fermions and third-generation RH neutrino

$$\text{Model I:}\quad (m_R)_{k3} = \frac{c_{k3}}{\Lambda}v_R\omega_1\,, \qquad\qquad \text{Model II:}\quad (m_R)_{k3} = c_{k3}w\,. \tag{3.6.19}$$

3.7 UV Completion and Parameter Space

In this Section we provide a possible UV origin to the mass-terms that are required to do the See-Saw mechanism and are generated only from an EFT description. Then we analyse where the possible values lie in the UV parameter space to reproduce the observed active-neutrino masses.

In the DSS scenario, to get active-neutrino masses of $\mathcal{O}(10^{-1})$ eV with $Y^N = \mathcal{O}(10^{-2})$ and the Higgs' VEV $v \sim 10^2$ GeV, we need a Majorana mass-matrix for the RH neutrinos

$$\mu_R = \mathcal{O}(10^{12})\,\text{TeV}\,. \tag{3.7.1}$$

This is a very high scale, too high to be testable at colliders. In addition, even though it is in principle *natural* to have a very heavy Majorana mass-matrix for the first- and second-generation RH neutrinos, the same is not true for the third-generation one. From Section 3.6, it is clear that we can *naturally* assume $(\mu_R)_{3j}$ and $(\mu_R)_{33}$ up to the order of few TeV. We cannot go far above this scale if we want to keep the assumption that the UV embedding of the SM happens at the TeV scale. Even in case of the existence of the X boson (3.6.9), although theoretically possible, it is not very sensible to assign a VEV of $\mathcal{O}(10^{12})$ TeV for the same reasons. Therefore DSS seems *unnatural* in the framework of the models considered in this work.

Regarding the ISS scenario, we could safely assume that the RH neutrinos have no Majorana mass-matrix at all. What we still have to do is to provide a UV origin to the mass-terms in which the sterile fermions s_L^k are involved, listed in Section 3.6. In particular in Model I, without adding any new further VLF to the theory, we could use ρ_1 (2.3.1) to write down the following term in the UV Lagrangian (where $k = 1, 2, 3$)

$$\mathcal{L}_\rho \supset c_{s\rho} \overline{s}_L^k (\Sigma_R^c)^\dagger \rho_1 + h.c. \tag{3.7.2}$$

This adds the following EFT operators to (2.5.2)

$$-\mathcal{L}_Y^{\text{IR}} \supset \frac{1}{M_\rho} c_{s\rho} \overline{s}_L^k (\Sigma_R^c)^\dagger \sigma_\rho^1 \left(c_{\rho\Omega}^1 \Omega_1 \chi_R^3 + c_{\rho R}^1 \Sigma_R e_R^j + \overline{c}_{\rho R}^1 \Sigma_R^c \nu_R^j \right) + h.c. \tag{3.7.3}$$

When all the fields acquire their VEVs, we find that (with $k = 1, 2, 3$ and only for this equation $j = 1, 2, 3$)

$$-\mathcal{L}_Y^{\text{IR}} \supset \overline{s}_L^k \, (m_R)_{kj} \nu_R^j + h.c. \tag{3.7.4}$$

where[16]

$$(m_R)_{kj} = c_{kj}\Lambda' + \frac{v_R^2}{M_\rho} \overline{c}_{\rho R}^1 c_{s\rho} \,, \qquad (m_R)_{k3} = \frac{\omega_1 v_R}{M_\rho} c_{\rho\Omega}^1 c_{s\rho} \,. \tag{3.7.5}$$

In Model II we have that the mass-mixing terms are generated using renormalizable operators and for completeness we find that (with $k = 1, 2, 3$ and just for this equation $j = 1, 2, 3$)

$$-\mathcal{L}_Y^{\text{IR}} \supset \overline{s}_L^k \, (m_R)_{kj} \nu_R^j + h.c. \tag{3.7.6}$$

where

$$(m_R)_{kj} = c_{kj}\Lambda' \,, \qquad (m_R)_{k3} = c_{k3} w \,. \tag{3.7.7}$$

What is left to do is to identify the allowed region in the UV parameter space to reproduce the observed active-neutrino masses while respecting all the constraints imposed, which usually are given only by the bounds on PMNS non-unitarity. If we keep the UV parameters used to generate the Yukawa matrices in the region of the UV parameter space discussed in Section 2.8, we see that we can explain the observed active-neutrino mass-spectrum keeping $\mathcal{O}(1)$ couplings between s_L^k and the RH neutrinos. Nevertheless to do so we need to assume a very hierarchical structure for the Majorana mass-matrix μ of the sterile fermions s_L^k. In fact we need the eigenvalues of μ to be

$$\mu_1 = \mathcal{O}(10^{10})\,\text{eV} \,, \quad \mu_2 = \mathcal{O}(10^5)\,\text{eV} \,, \quad \mu_1 = \mathcal{O}(1)\,\text{eV} \,. \tag{3.7.8}$$

This is due to the fact that the neutral lepton Yukawa matrix has a hierarchical structure that must be compensate by an even more pronounced hierarchical structure in μ. Recall that this happens under the assumption to keep all the couplings to be of $\mathcal{O}(1)$. We are going to discuss better this point in the following Chapter. In addition, we would like to emphasize that if we allow Λ' to be much above the TeV scale, then this hierarchical structure of μ is even more pronounced. This because we have to compensate the hierarchical structures present in both Y^N and m_R. As last check, the PMNS unitarity constraints are in general satisfied. The only care must be put to ensure that the 33-component of $|\eta|$ (3.5.10) lies inside the current bounds, and this could be achieved if

$$(m_R)_{k3} \gtrsim 3\,\text{TeV} \,. \tag{3.7.9}$$

[16] Recall that the mass-mixing term $(m_R)_{kj}$ is already present at the renormalizable level and gets just a small (maybe negligible) correction from the EFT description.

The other entries are usually ensured to be inside the bounds thanks to the hierarchical structure of the Yukawa matrices.

To conclude this Chapter, let us observe that these results are not new in Pati-Salam-like models, for instance see [2, 10]. The issue is that such a strong hierarchy among the sterile fermion masses seems quite *unnatural*, or at least not really well justified. Therefore this explanation to the observed active-neutrino masses spoils the *naturalness* assumption that these models aim (and have the potential) to manifest.

4 Natural Model for Neutrinos

We saw that be best mechanism to explain the observed anarchic active-neutrino masses is provided by the ISS. However we need a very hierarchical structure in the Majorana mass-matrix of the NP sterile fermions. This somehow ruins the *naturalness* of the model we are considering. In this Chapter we are going to propose a solution to this issue which is new in the literature as far as we know. The main idea is to assume a suitable hierarchical structure in the mass-mixing terms between RH neutrinos and NP sterile fermions. In the first part we are going to study in detail this hypothesis in a model-independent way, then we provide an explicit realization in the framework of the models we are considering in this work.

4.1 Hierarchical Mass-Matrix

The main conclusion of the last Chapter was that, to explain the observed active-neutrino masses, we need a very hierarchical structure for the Majorana mass-matrix of the three generations of sterile fermions s_L^k. This is due to the fact that parametrically

$$m_\nu^i \sim \mu_i \frac{y_i^2 v^2}{(m_R^i)^2} \,. \tag{4.1.1}$$

So, considering the eigenvalues of the matrices entering in the ISS realization, if $y_2/y_3 \sim 10^{-2}$ and $m_R^2 \sim m_R^3$, we need that $\mu_2/\mu_3 \sim 10^{-4}$. Nevertheless, if we assume a suitable hierarchical structure in m_R such that $m_R^2/m_R^3 \sim y_2/y_3$, maybe we are able to generate anarchic m_ν^i with anarchic μ_i.

In analogy with what we did in Section 3.3, recall that in an ISS scenario we have the following mass-matrix in the space $\{\nu_L^i, (\nu_R^j)^c, s_L^k\}$ (where $i, j, k = 1, 2, 3$)

$$M = \begin{pmatrix} 0 & \frac{v}{\sqrt{2}} Y^N & 0 \\ \frac{v}{\sqrt{2}} (Y^N)^T & 0 & m_R^T \\ 0 & m_R & \mu \end{pmatrix} \tag{4.1.2}$$

where Y^N, m_R are 3×3 complex matrices and μ is a 3×3 symmetric complex matrix. Nevertheless, to simplify the calculations at least in the first attempt, let us put in a restricted framework where we have only two flavours. Hence $i, j, k = 2, 3$ and all the matrices are 2×2. As we will see, this simplification will not spoil conceptually the main results and the extension to the 3-flavour case will be straightforward. As stated above, the idea is to assume a hierarchical structure in Y^N and m_R such that parametrically they go as follows

$$Y^N \sim \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon^2 & 1 \end{pmatrix} , \qquad m_R \sim m_R^3 \begin{pmatrix} \varepsilon & 1 \\ \varepsilon & 1 \end{pmatrix} \tag{4.1.3}$$

where ε is assumed to be a small parameter (typically $\varepsilon \sim y_2/y_3 \sim 10^{-2}$) and m_R^3 is the biggest eigenvalue of m_R. In particular we want that the mass-mixing terms between s_L^k and ν_R^2 are suppressed with respect to the ones between s_L^k and ν_R^3.

The problems with this scenario comes out when we try to diagonalize M. In fact, in this framework it is not obvious anymore that the block-diagonalization approximation used previously works, so we should proceed with care. To further simplify the calculation we observe that we have the freedom to diagonalize Y^N with a singular-valued decomposition as shown in Section D.2 in the Appendix. In particular we have that there exist two unitary matrices U_L and U_R such that, under the unitary transformations

$$\nu_L^i \quad \rightarrow \quad (U_L)_{i\ell}\, \nu_L^\ell \,, \qquad \nu_R^j \quad \rightarrow \quad (U_R)_{j\ell}\, \nu_R^\ell \,, \tag{4.1.4}$$

the Yukawa matrix gets diagonalized as follows

$$Y^N \quad \to \quad U_L^\dagger Y^N U_R = \hat{Y}^N = \begin{pmatrix} y_2 & 0 \\ 0 & y_3 \end{pmatrix} \tag{4.1.5}$$

where y_2 and y_3 are positive real numbers. Those unitary matrices are evaluated in Section C.1 in the Appendix in terms of Yukawa entries and explicitly we have that

$$U_L \approx \begin{pmatrix} 1 & \varepsilon_L \\ -\varepsilon_L^* & 1 \end{pmatrix}, \qquad U_R \approx \begin{pmatrix} 1 & \varepsilon_R \\ -\varepsilon_R^* & 1 \end{pmatrix} \tag{4.1.6}$$

where $\varepsilon_L \sim \varepsilon$ and $\varepsilon_R \sim \varepsilon^2$ (which means that $U_R \approx \mathbb{I}$). Also we do not change the hierarchy $y_2/y_3 \sim \varepsilon$. Then, as shown in Section D.3 in the Appendix, since μ is symmetric there always exists a unitary matrix U_S such that, under the unitary transformation

$$s_L^k \quad \to \quad (U_S)_{k\ell}\, s_L^\ell\,, \tag{4.1.7}$$

the Majorana mass-matrix gets diagonalized as follows

$$\mu \quad \to \quad U_S^\dagger\, \mu\, U_S^* = \hat{\mu} = \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_3 \end{pmatrix} \tag{4.1.8}$$

where μ_2 and μ_3 are positive real numbers.

Now we can check how m_R gets modified by those transformations. In particular we observe that

$$m_R \quad \to \quad U_S^\dagger m_R U_R \tag{4.1.9}$$

and parametrically it goes as follows

$$U_S^\dagger m_R U_R \sim m_R^3 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & 1 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon^2 \\ \varepsilon^2 & 1 \end{pmatrix} \sim m_R^3 \begin{pmatrix} \varepsilon & 1 \\ \varepsilon & 1 \end{pmatrix} \tag{4.1.10}$$

so those transformations do not change its hierarchical structure[17]. After all those simplifications the mass-matrix reads as follows

$$M = \begin{pmatrix} 0 & 0 & y_2\, v/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & y_3\, v/\sqrt{2} & 0 & 0 \\ y_2\, v/\sqrt{2} & 0 & 0 & 0 & m_R^{22} & m_R^{32} \\ 0 & y_3\, v/\sqrt{2} & 0 & 0 & m_R^{23} & m_R^{33} \\ 0 & 0 & m_R^{22} & m_R^{23} & \mu_1 & 0 \\ 0 & 0 & m_R^{32} & m_R^{33} & 0 & \mu_2 \end{pmatrix} \tag{4.1.11}$$

where y_2, y_3, μ_2, μ_3 are real and positive. Also we have the hierarchies $m_R^{k2} \ll m_R^{k3}$ and $y_2 \ll y_3$.

Notice that this discussion can be trivially generalized to the 3-flavour case. In this framework the mass-matrix reads as follows

$$M = \begin{pmatrix} 0 & \dfrac{v}{\sqrt{2}}\hat{Y}^N & 0 \\ \dfrac{v}{\sqrt{2}}\hat{Y}^N & 0 & m_R^T \\ 0 & m_R & \hat{\mu} \end{pmatrix} \tag{4.1.12}$$

where

$$\hat{Y}^N = \mathrm{diag}(y_1, y_2, y_3)\,, \qquad \hat{\mu} = \mathrm{diag}(\mu_1, \mu_2, \mu_3)\,. \tag{4.1.13}$$

Also are assumed the following hierarchies between the parameters

$$m_R^{k1} \ll m_R^{k2} \ll m_R^{k3}\,, \qquad y_1 \ll y_2 \ll y_3\,. \tag{4.1.14}$$

[17]In fact in the following we will be sloppy in stressing the difference between m_R and $U_S^\dagger m_R U_R$. We will use the same symbol m_R neglecting the fact that it has been rotated. Nevertheless this will not change the final results as we will see in the following.

4.2 Mass-Matrix Diagonalization

Once we have simplified the mass-matrix M, we have to diagonalize it. However, even in the 2-flavour case it remains very hard to do analytically. Nevertheless there is something we can observe from the single-flavour ISS scenario studied in Section C.4 in the Appendix: we see that when D and N are comparable, we have a non-trivial mixing between ν_L and s_L. Thus we could expect that $v\,\hat{Y}^N$ and m_R cannot be much comparable, otherwise we would violate the non-unitarity constraints on the PMNS matrix. In addition, this mixing can be evaluated in the limit where $\hat{\mu} \to 0$. Hence we can try to evaluate the eigen-subspace associated to the null eigenvalue of the matrix $M^\dagger M$ and try to see how large m_R must be compared to $v\,\hat{Y}^N$.

For the moment we work in the 2-flavour case to make the calculation easier. We want to find the null eigen-subspace of the matrix $M^\dagger M$. This is easy to compute and we find that it is generated by the following two vectors

$$v_1 = \begin{pmatrix} \sqrt{2}m_R^{22}/y_2\,v & \sqrt{2}m_R^{23}/y_3\,v & 0 & 0 & -1 & 0 \end{pmatrix}^T, \tag{4.2.1a}$$

$$v_2 = \begin{pmatrix} \sqrt{2}m_R^{32}/y_2\,v & \sqrt{2}m_R^{33}/y_3\,v & 0 & 0 & 0 & -1 \end{pmatrix}^T. \tag{4.2.1b}$$

Then, to find a basis, we have to orthonormalize them, but already at this stage it is clear that, if $m_R^{kj} \approx y_j\,v$, there is a comparable mixing between ν_L^i and s_L^k. In fact, if we go to the orthonormal basis $\{v_1, v_2\} \to \{w_1, w_2\}$, we have that the unitary matrix that diagonalizes M is given by

$$U = \begin{pmatrix} w_1 & w_2 & ... \end{pmatrix} + \mathcal{O}(\hat{\mu}) \tag{4.2.2}$$

and observe that we need to compute explicitly just the first and second columns since are the ones that enter in the PMNS matrix. The unitary rotation that brings ν_L^i into their mass-basis is given by

$$\overline{U}_\nu = U_L \mathcal{W} \tag{4.2.3}$$

where $\mathcal{W}$ is the upper left 2×2 block of U and its entries are given by the first two components of w_1 and w_2. Then, since U_L is unitary, the non-unitarity of the PMNS matrix is well approximated by

$$|\eta| \approx |\mathcal{W}^\dagger \mathcal{W}|\,. \tag{4.2.4}$$

The explicit form of this matrix is rather long and useless. The only important feature is that it depends only on the ratios m_R^{kj}/y_j with $j = 2, 3$. Thus, if we have that those two ratios are comparable, $|\eta|$ turns out to be an anarchic matrix whose entries parametrically go as follows

$$|\eta| \sim \frac{y_3^2 v^2}{(m_R^3)^2}\,. \tag{4.2.5}$$

Thus the upper bounds on $|\eta|$ (3.5.3) put a lower bound on m_R^3 which turns out to be

$$m_R^3 \gtrsim 10\,\mathrm{TeV}\,. \tag{4.2.6}$$

At this point we could try to block-diagonalize M perturbatively using the technique shown in Section D.4 in the Appendix. Keeping the same notation used in that Section, we need to require that the sub-matrix B is *small*. From that computation we see that at leading order

$$B \approx D^*(R^{-1})^\dagger = \frac{v}{\sqrt{2}} \begin{pmatrix} \hat{Y}^N (m_R^*)^{-1} & \mathcal{O}(\hat{\mu}) \end{pmatrix} \tag{4.2.7}$$

and parametrically

$$v\,\hat{Y}^N (m_R^*)^{-1} \sim \frac{v}{m_R^3} \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon^{-1} & \varepsilon^{-1} \\ 1 & 1 \end{pmatrix} \sim \frac{v}{m_R^3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.2.8}$$

Thus we have that B is an anarchic matrix and it is suppressed by the ratio v/m_R^3. Therefore the assumption of having B *small* works and the block-diagonalization can be safely done perturbatively in powers of v/m_R^3. In addition we find that numerically this block-diagonalization technique works with an error of less then 5% if $m_R^3 \approx 10\,\text{TeV}$. This could be seen as an empirical validation of our claim, also considering that $v/m_R^3 \approx 2\%$. Therefore we can use the block-diagonalization method to get the analytic behaviour of the active-neutrino masses.

After block-diagonalizing M in (4.1.11) we find that the active-neutrino mass-matrix reads as follows

$$m_\nu \approx \frac{y_2^2 y_3^2 v^4}{2\left(m_R^{23} m_R^{32} - m_R^{22} m_R^{33}\right)^2} \begin{pmatrix} \mu_2 \dfrac{(m_R^{33})^2}{y_3^2\, v^2} + \mu_3 \dfrac{(m_R^{23})^2}{y_3^2\, v^2} & -\mu_2 \dfrac{m_R^{32} m_R^{33}}{y_2 y_3\, v^2} - \mu_3 \dfrac{m_R^{22} m_R^{23}}{y_2 y_3\, v^2} \\ -\mu_2 \dfrac{m_R^{32} m_R^{33}}{y_2 y_3\, v^2} - \mu_3 \dfrac{m_R^{22} m_R^{23}}{y_2 y_3\, v^2} & \mu_2 \dfrac{(m_R^{32})^2}{y_2^2\, v^2} + \mu_3 \dfrac{(m_R^{22})^2}{y_2^2\, v^2} \end{pmatrix}. \tag{4.2.9}$$

From m_ν it is very clear that to get anarchic active-neutrino masses while keeping $\hat{\mu}$ anarchic we need that

$$\frac{m_R^{k2}}{y_2} \sim \frac{m_R^{k3}}{y_3}\,. \tag{4.2.10}$$

The PMNS non-unitarity parameter $|\eta|$ is given by (3.5.10) and explicitly it reads as follows

$$|\eta| \approx \frac{y_2^2 y_3^2 v^4}{2\left|m_R^{23} m_R^{32} - m_R^{22} m_R^{33}\right|^2} \begin{pmatrix} \dfrac{|m_R^{33}|^2}{y_3^2\, v^2} + \dfrac{|m_R^{23}|^2}{y_3^2\, v^2} & -\dfrac{m_R^{32} (m_R^{33})^*}{y_2 y_3\, v^2} - \dfrac{m_R^{22} (m_R^{23})^*}{y_2 y_3\, v^2} \\ -\dfrac{(m_R^{32})^* m_R^{33}}{y_2 y_3\, v^2} - \dfrac{(m_R^{22})^* m_R^{23}}{y_2 y_3\, v^2} & \dfrac{|m_R^{32}|^2}{y_2^2\, v^2} + \dfrac{|m_R^{22}|^2}{y_2^2\, v^2} \end{pmatrix} \tag{4.2.11}$$

and if the condition (4.2.10) holds, it is anarchic as well and to respect the unitarity bounds (3.5.3) we need that $m_R^3 \gtrsim 10$ TeV. Notice that this agrees with the results of the discussion made above.

These discussions and results regarding the 2-flavour case can be trivially extended to the 3-flavour case. The block-diagonalization approximation works under the same conditions. However this time m_ν is much more complicated and we do not report the explicit expression here, but we just recall its matrix form which reads as follows

$$m_\nu \approx \frac{v^2}{2} \left(\hat{Y}^N m_R^{-1}\right) \hat{\mu} \left(\hat{Y}^N m_R^{-1}\right)^T\,. \tag{4.2.12}$$

What is important to mention is that it is anarchic if

$$\frac{m_R^{k1}}{y_1} \sim \frac{m_R^{k2}}{y_2} \sim \frac{m_R^{k3}}{y_3}\,. \tag{4.2.13}$$

Under this condition we also have an anarchic structure of $|\eta|$ and the PMNS unitarity bounds are satisfied as long as $m_R^3 \gtrsim 10$ TeV.

4.3 Heavy Mass-Eigenstates

So far we have discussed only about the light states of the ISS mechanism. In this Section we study in detail the heavy states. As we discussed in the previous Section, we can block-diagonalize the mass-matrix M (4.1.12) by means of a unitary matrix W such that

$$W^T M W = \begin{pmatrix} m_\nu & \mathbb{O} \\ \mathbb{O} & M_h \end{pmatrix} \tag{4.3.1}$$

where m_ν and M_h are 3×3 and 6×6 matrices respectively (in the 3-flavour case). In good approximation we have that

$$W \approx \begin{pmatrix} \mathbb{I} - \frac{1}{2}BB^\dagger & B \\ -B^\dagger & \mathbb{I} - \frac{1}{2}B^\dagger B \end{pmatrix} \tag{4.3.2}$$

where

$$B^\dagger \approx \frac{v}{\sqrt{2}} \begin{pmatrix} m_R^{-1}\hat{\mu}(m_R^T)^{-1}\hat{Y}^N \\ (m_R^T)^{-1}\hat{Y}^N \end{pmatrix} = \frac{v}{\sqrt{2}} \begin{pmatrix} \mathbb{O} \\ (m_R^T)^{-1}\hat{Y}^N \end{pmatrix} + \mathcal{O}(\hat{\mu})\,. \tag{4.3.3}$$

Notice that, regarding the mixing matrix, all the $\mathcal{O}(\hat{\mu})$ corrections can be safely neglected since the first non-trivial corrections are of $\mathcal{O}((v/m_R^3)^3)$. Thus we find that explicitly

$$W \approx \begin{pmatrix} \mathbb{I} - \frac{v^2}{4}\hat{Y}^N\left(m_R^T m_R^*\right)^{-1}\hat{Y}^N & \mathbb{O} & \frac{v}{\sqrt{2}}\hat{Y}^N(m_R^*)^{-1} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} \\ -\frac{v}{\sqrt{2}}(m_R^T)^{-1}\hat{Y}^N & \mathbb{O} & \mathbb{I} - \frac{v^2}{4}(m_R^T)^{-1}(\hat{Y}^N)^2(m_R^*)^{-1} \end{pmatrix}. \tag{4.3.4}$$

The Heavy mass-matrix is given by (as shown in Section D.4 in the Appendix)

$$M_h \approx \begin{pmatrix} \mathbb{O} & m_R^T \\ m_R & \mathbb{O} \end{pmatrix}. \tag{4.3.5}$$

Thus, as long as we neglect all the $\mathcal{O}(\hat{\mu})$ corrections, we have that the mass-Lagrangian of the heavy mass-eigenstates reads as follows (with $k, j = 1, 2, 3$)

$$\mathcal{L} \supset -\bar{s}_L'^k\, m_R^{kj}\, \nu_R^j + h.c. \tag{4.3.6}$$

and $s_L'^k$ is a linear combination of ν_L^i and s_L^k that comes out after a W transformation[18].

Now we need to diagonalize this mass-matrix with a singular-valued decomposition to find the heavy mass-eigenstates. As shown in Section D.2 in the Appendix, there exist two unitary matrices V_S and V_R such that, under the unitary transformations

$$s_L'^k \quad \to \quad (V_S)_{k\ell}\, s_L'^\ell\,, \qquad \nu_R^j \quad \to \quad (V_R)_{j\ell}\, \nu_R^\ell\,, \tag{4.3.7}$$

the mass-matrix gets diagonalized as follows

$$m_R \quad \to \quad V_S^\dagger m_R V_R = \hat{M} = \mathrm{diag}(M_1, M_2, M_3) \tag{4.3.8}$$

where M_1, M_2 and M_3 are positive real numbers. The diagonalization has been explicitly done in Section C.6 in the Appendix finding that the masses are given by

$$M_3 = \sqrt{|m_R^{13}|^2 + |m_R^{23}|^2 + |m_R^{33}|^2}\,, \tag{4.3.9a}$$

$$M_2 = M_3^{-1}\sqrt{|m_R^{13}m_R^{22} - m_R^{23}m_R^{12}|^2 + |m_R^{13}m_R^{32} - m_R^{33}m_R^{12}|^2 + |m_R^{23}m_R^{32} - m_R^{33}m_R^{22}|^2}\,, \tag{4.3.9b}$$

$$\begin{aligned} M_1 = (M_2M_3)^{-1}\Big| & m_R^{11}m_R^{22}m_R^{33} - m_R^{11}m_R^{23}m_R^{32} + m_R^{12}m_R^{23}m_R^{31} \\ & - m_R^{12}m_R^{21}m_R^{33} + m_R^{13}m_R^{21}m_R^{32} - m_R^{13}m_R^{22}m_R^{31}\Big| \end{aligned} \tag{4.3.9c}$$

while the mixing matrices are given by up to corrections of $\mathcal{O}(m_R^2/m_R^3)$ (where m_R^2 is the second biggest eigenvalue of m_R)

$$V_S = \begin{pmatrix} v_1^S & v_2^S & v_3^S \end{pmatrix}, \qquad V_R = \begin{pmatrix} v_1^R & v_2^R & v_3^R \end{pmatrix} \tag{4.3.10}$$

[18]The explicit form of these linear combinations are provided in Section 5.1.

where v^S_j and v^R_j (where $j = 1, 2, 3$) are reported in (C.6.15) and (C.6.7) respectively. Also notice that we have $V_R \approx \mathbb{I}$, which means that there is little mixing among the RH neutrinos.

As a side comment, notice that, as showed in Section C.5 in the Appendix, the three mass-eigenstates correspond to three Dirac-type fermions. If one considers the $\mathcal{O}(\hat{\mu})$ contributions, each of the Dirac fermion splits into two Weyl fermions with Majorana masses. However, this goes well above the level of precision we are considering.

4.4 New Symmetry for the Sterile Fermions

In the first part of this Chapter we have shown that it is possible to generate an anarchic active-neutrino mass-matrix with an anarchic structure among the three generations of the sterile fermions s^k_L. The only requirement is that it must be satisfied the condition (4.2.13). However there is a problem with this condition in the framework of the models we are considering in this work. Namely, since ν^j_R (with $j = 1, 2$) and s^k_L are both sterile, the mass-mixing terms between them m^{kj}_R are not suppressed in any way with respect to m^{k3}_R. Not only, but it is also enhanced in principle. Nevertheless a possible solution can be to introduce a Symmetry which makes such terms not allowed anymore at the renormalizable level, so that they have to be generated as some EFT higher-dimensional operators. For this reason we assume that there exists a further Symmetry such that all the fields already introduced are singlets under it but the sterile fermions s^k_L. In principle there is no need to specify anything more about this Symmetry, it could be either continue or discrete, global or local. Just for the sake to be quantitative, we will assume for the rest of this work that this is a $\mathrm{U}(1)_F$ Symmetry under which the sterile fermions have Charge

$$s^k_L \sim 1 \in \mathrm{U}(1)_F\,. \tag{4.4.1}$$

Then we assume that there is a scalar field

$$\Phi_{k\ell} \sim -2 \in \mathrm{U}(1)_F \tag{4.4.2}$$

and it is a singlet under everything else. Also $k, \ell = 1, 2, 3$ and represent its flavour indices in the flavour space of the sterile fermions. We assume that it will eventually acquire a VEV at an arbitrary high energy scale generating the Majorana mass-matrix of s^k_L. Namely we have that

$$\langle \Phi \rangle = \mu\,. \tag{4.4.3}$$

Since μ is a very small mass scale, the slight breaking of this new Symmetry we have introduced is responsible for the Fermion Number violation (this is why we choose F as subscript of this new Symmetry Group). In this framework the mass-Lagrangian of the sterile fermions before the SSB of this new Symmetry reads as follows

$$\mathcal{L} \supset -\frac{1}{2}\Phi_{k\ell}(s^k_L)^T \mathcal{C}\, s^\ell_L + h.c. \tag{4.4.4}$$

The next step is to generate the mass-mixing terms between the sterile fermions and the RH neutrinos which recall that they read as follows (with $k = 1, 2, 3$ and only for the equation below $j = 1, 2, 3$)

$$\mathcal{L} \supset -\bar{s}^k_L\, m^{kj}_R\, \nu^j_R + h.c. \tag{4.4.5}$$

In particular we need to generate a non-suppressed (hence renormalizable) operator that couples s^k_L with ν^3_R. Because of this new Symmetry, we must add to the model a new scalar field with suitable charges under the whole Symmetry Group. In addition this new field must eventually acquire a suitable VEV to give a mass-mixing term only with ν^3_R. The only possibility is provided by the following scalar field

$$\Delta \sim (\mathbf{4}, \mathbf{2}, -1) \in \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(2)^{[3]}_R \times \mathrm{U}(1)_F \tag{4.4.6}$$

with the following VEV

$$\langle \Delta_{aj} \rangle = v_F \, \delta_{a4} \delta_{j1} \tag{4.4.7}$$

where $a = 1, ..., 4$ and $i = 1, 2$ are the $\mathrm{SU}(4)^{[3]}$ and $\mathrm{SU}(2)_R^{[3]}$ indices respectively. Notice that this VEV defines the mass scale governing the ISS realization discussed above. Namely we have that m_R^3 (which we recall it is the biggest eigenvalue of m_R) is identified with v_F (apart from $\mathcal{O}(1)$ numbers). As a side comment, the addition of a new scalar field charged under the UV Gauge Group (1.1.3) and that acquires a VEV implies a modification in the final mass-spectrum of the broken Gauge bosons. This modification has been computed in Section C.8 in the Appendix.

Now the mass-mixing terms m_R^{kj} with $j = 1, 2$ cannot be anymore generate through some renormalizable operators. Thus they must be generated through some EFT operators and hence are suppressed with respect to m_R^{k3}. In addition we must impose that these suppression factors are such that condition (4.2.13) holds. In the following Section we are going to discuss this in detail.

4.5 Mass-Mixing Terms as EFT

In this Section we write down all the possible mass-mixing terms that are allowed in the framework of the models we are considering. Namely we want to generate the mass-terms (where only for the equation below $j = 1, 2, 3$)

$$\mathcal{L} \supset -\bar{s}_L^k (m_R)_{kj} \, \nu_R^j + h.c. \tag{4.5.1}$$

Keep in mind that, in contrast to Section 3.6, this time we have a new Symmetry that changes the operators allowed.

For Model I we can write down the following EFT operators

$$\mathcal{L}_{sR} \sim c_{k3} \bar{s}_L^k \Delta^\dagger \chi_R^3 + \frac{c_{kj}}{\Lambda^2} \bar{s}_L^k \Delta^\dagger \Omega_1^\dagger \Sigma_R^c \nu_R^j \,. \tag{4.5.2}$$

Thus we have that

$$m_R^{k3} \sim v_F \,, \qquad m_R^{kj} \sim \frac{\omega_1 v_R}{\Lambda^2} v_F \,. \tag{4.5.3}$$

We can make a parametric comparison between these terms and the Yukawa eigenvalues (which are given by (2.1.7) using the results of Section 2.5) focusing on the suppression behaviours. We have that parametrically

$$\frac{y_2}{y_3} \sim \frac{v_R}{\Lambda} \,, \qquad \frac{m_R^2}{m_R^3} \sim \frac{\omega_1 v_R}{\Lambda^2} \,. \tag{4.5.4}$$

Unfortunately they do not show the same suppression factors. Nevertheless, if $\omega_1 \sim \Lambda$, they could be made comparable as we would like. Also this requirement does not spoil the *naturalness* of the Model[19].

For Model II we can write the following EFT operators

$$\mathcal{L}_{sR} \sim c_{k3} \bar{s}_L^k \Delta^\dagger \chi_R^3 + \frac{c_{kj}}{\Lambda} \bar{s}_L^k \Delta^\dagger \Delta_3^c \nu_R^j \,. \tag{4.5.5}$$

Thus we have that

$$m_R^{k3} \sim v_F \,, \qquad m_R^{kj} \sim \frac{w}{\Lambda} v_F \,. \tag{4.5.6}$$

As before, we can compare these terms and the Yukawa eigenvalues (which are given by (2.1.7) using the results of Section 2.6). We have that parametrically

$$\frac{y_2}{y_3} \sim \frac{\omega_1 w}{\Lambda^2} \,, \qquad \frac{m_R^2}{m_R^3} \sim \frac{w}{\Lambda} \,. \tag{4.5.7}$$

[19]Actually one should explain why these two scales are comparable, but this could be just seen as *accidental*.

Again, we cannot reproduce exactly the same parametric suppression unless we require that $\omega_1 \sim \Lambda$. However notice that for Model I m_R^2/m_R^3 is too suppressed with respect to y_2/y_3, instead for Model II it is not as much suppressed as we would like it to be.

To summarize, we have seen that (in the 2-flavour scenario) condition (4.2.13) can be achieved if we require that $\omega_1 \sim \Lambda$ in both the models. Nevertheless we have to be careful since this requirement could spoil the hierarchical structure of the Yukawa matrices. In fact we saw that we need (at least) two VLFs to UV-complete the Yukawa matrices and they can have in principle different masses. Hence there are actually two energy scales Λ and Λ' which, for instance, could differ by one order of magnitude. Thus we must be careful in evaluating the energy scales that suppress the EFT operators. The details of the UV origin will be discussed in the following Section, but here we anticipate just some results to better refine the discussion made above.

For Model I we could use the VLF η (2.3.6) with the following UV Lagrangian

$$\mathcal{L} \supset c_{s\eta}\bar{s}_L^k \Delta^\dagger \eta + c^1_{\eta\rho}\bar{\eta}\Omega_1^\dagger \rho_1 - \bar{c}^1_{\rho R}\bar{\rho}_1 \Sigma_R^c \nu_R^j + h.c. \tag{4.5.8}$$

to generate a mass-mixing term with the following suppression factor

$$\frac{m_R^{kj}}{v_F} \sim \frac{\omega_1 v_R}{M_\eta M_\rho} \,. \tag{4.5.9}$$

This has to be compared with the following Yukawa parameters which have been derived in Section 2.5

$$Y^N_{ij} \sim \frac{v_R}{M_\rho} \,, \qquad Y^N_{i3} \sim \frac{\omega_1}{M_\rho} \,, \qquad Y^N_{3i} \sim \frac{\omega_1 v_R}{M_\rho M_\eta} \,. \tag{4.5.10}$$

Thus to satisfy condition (4.2.13) we need

$$\frac{v_R}{M_\rho} \sim \mathcal{O}(10^{-2}) \,, \qquad \frac{\omega_1}{M_\rho} \sim \mathcal{O}(10^{-1}) \,, \qquad \frac{\omega_1}{M_\eta} \sim \mathcal{O}(1) \tag{4.5.11}$$

and this is something that actually can be achieved. In this way we can get a Yukawa matrix where $Y^N_{ij} \sim Y^N_{3i} \lesssim Y^N_{i3}$ without requiring any fine tuning, but just masses of different orders of magnitude for the VLFs η and ρ.

For Model II we could use the VLF η' (2.4.2) with the following UV Lagrangian

$$\mathcal{L} \supset c_{s\eta}\bar{s}_L^k \Delta^\dagger \eta_1' - \bar{c}^1_{\eta\Delta}\bar{\eta}_1' \Delta_3^c \nu_R^j + h.c. \tag{4.5.12}$$

to generate a mass-mixing term with the following suppression factor

$$\frac{m_R^{kj}}{v_F} \sim \frac{w}{M_{\eta'}} \,. \tag{4.5.13}$$

This has to be compared to the following Yukawa parameters which have been derived in Section 2.6

$$Y^N_{ij} \sim \frac{\omega_1 w}{M_\rho M_{\eta'}} \,, \qquad Y^N_{i3} \sim \frac{\omega_1}{M_\rho} \,. \tag{4.5.14}$$

Thus to satisfy condition (4.2.13) we need

$$\frac{\omega_1}{M_\rho} \sim \mathcal{O}(1) \,. \tag{4.5.15}$$

However this implies that $Y^N_{i3} \sim \mathcal{O}(1)$. Thus this Model is incompatible with all the conditions we would like to impose[20]. Therefore in the following we are going to consider only Model I.

[20]In principle we could think that the coupling $\bar{c}^1_{\eta\Delta}$ is suppressed in a suitable way to explain condition (4.2.13). However this would spoil the *naturalness* of the model.

Actually there could be another possibility to satisfy condition (4.2.13). This requires the addition of a further scalar field

$$\Sigma \sim (\mathbf{2}, -1/2, -1) \in \mathrm{SU}(2)_R^{[3]} \times \mathrm{U}(1)^{[12]} \times \mathrm{U}(1)_F \tag{4.5.16}$$

that acquires a VEV such that

$$\langle \Sigma \rangle = \begin{pmatrix} v'_F \\ 0 \end{pmatrix} . \tag{4.5.17}$$

In this way we have that the mass-mixing terms can be generated by the following EFT operators in Model I

$$\mathcal{L}_{sR} \sim c_{k3} \overline{s}_L^k \Delta^\dagger \chi_R^3 + \frac{c_{kj}}{\Lambda} \overline{s}_L^k \Sigma^\dagger \Sigma_R^c \nu_R^j . \tag{4.5.18}$$

This implies that

$$m_R^{k3} \sim v_F , \qquad m_R^{kj} \sim \frac{v_R}{\Lambda} v'_F \tag{4.5.19}$$

which means that $m_R^2/m_R^3 \sim y_2/y_3$ if $v_F \sim v'_F$. In fact this could be achieved, but it assumes a fine tuning between the two VEVs to be both at the TeV scale. Thus we will not consider this hypothesis any further. Also notice that this possibility does not work in Model II since the ratio m_R^2/m_R^3 is less suppressed with respect to y_2/y_3, but we cannot reduce the suppression of y_2/y_3.

4.6 UV Completion

As we have seen in the last Section, there is room to have a theory that reproduces *naturally* condition (4.2.13) while keeping the hierarchical structure of the Yukawa matrices. In addition, Model I reveals to be more appealing than Model II. In this Section we provide a possible UV origin to the mass-mixing terms generated by an EFT description in the framework of Model I.

It is possible to provide a UV origin to the EFT operators generating m_R^{kj} without the addition of any new further VLF. In particular we can use the VLF η (2.3.6) and write down the following terms in the UV Lagrangian (where $k = 1, 2, 3$)

$$\mathcal{L}_\eta \supset c_{s\eta} \overline{s}_L^k \Delta^\dagger \eta + h.c. \tag{4.6.1}$$

This adds the following EFT operators to (2.5.2) (where $k = 1, 2, 3$ and $j = 1, 2$)

$$-\mathcal{L}_Y^{\mathrm{IR}} \supset \frac{1}{M_\eta} c_{s\eta} \overline{s}_L^k \Delta^\dagger \sigma_\eta J_\eta + \frac{1}{M_\rho M_\eta} c_{s\eta} \overline{s}_L^k \Delta^\dagger \sigma_\eta \left(c_{\eta\rho}^3 \sigma_\rho^3 \Omega_3^\dagger J_\rho^3 + c_{\eta\rho}^1 \sigma_\rho^1 \Omega_1^\dagger J_\rho^1 \right) + h.c. \tag{4.6.2}$$

When all the fields acquire their VEVs, we find that (with $k = 1, 2, 3$ and only for this equation $j = 1, 2, 3$)

$$-\mathcal{L}_Y^{\mathrm{IR}} \supset \overline{s}_L^k \, m_R^{kj} \nu_R^j + h.c. \tag{4.6.3}$$

where

$$\begin{aligned} \frac{m_R^{k3}}{v_F} = c_{k3} + c_{s\eta} \frac{m}{M_\eta} \left(1 - c_\eta \frac{v_3}{2M_\eta}\right)^{-1} + \frac{v_3}{2M_\eta} c_{s\eta} c_{\eta\Sigma} \left(1 - c_\eta \frac{v_3}{2M_\eta}\right)^{-1} \\ + \frac{\omega_1^2}{M_\rho M_\eta} c_{s\eta} c_{\eta\rho}^1 c_{\rho\Omega}^1 \left(1 - c_\eta \frac{v_3}{2M_\eta}\right)^{-1} \left(1 - c_\rho^1 \frac{v_3}{2M_\rho}\right)^{-1} , \end{aligned} \tag{4.6.4a}$$

$$\frac{m_R^{kj}}{v_F} = \frac{v_R \omega_1}{M_\rho M_\eta} c_{s\eta} c_{\rho\Omega}^1 \overline{c}_{\rho R}^1 \left(1 - c_\eta \frac{v_3}{2M_\eta}\right)^{-1} \left(1 - c_\rho^1 \frac{v_3}{2M_\rho}\right)^{-1} . \tag{4.6.4b}$$

Not surprisingly, we can observe from (2.5.5) that, if $c_{k3}/c_{s\eta} \approx c_{33}/c_{\eta\mathcal{H}}$, we have that

$$\frac{m_R^{kj}}{m_R^{k3}} \approx \frac{Y_{3j}^N}{Y_{33}^N} . \tag{4.6.5}$$

This is almost what we would like to reproduce. In fact if we assume values for the mass scales in the region of the UV parameter space space identified in (4.5.11), we have that $Y^N_{ij} \approx Y^N_{3j}$ and hence we can reproduce condition (4.2.13) without any fine tuning while keeping $\mathcal{O}(1)$ couplings. Thus we can explain *naturally* the structure of Yukawa matrices and the observed active-neutrino masses.

As last remark, observe that conditions (4.2.13) and (2.1.3) together imply the following constraints among UV couplings

$$c_{33}/c_{\eta\mathcal{H}} \approx c^3_{\rho\Omega}/(c^3_{\eta\rho})^* \approx c_{k3}/c_{s\eta}\,. \tag{4.6.6}$$

This is not surprising if we recall the observation made in Section 2.8. In fact this constraint follows directly if one assumes that η and χ^3_R (which share the same quantum numbers) are coupled with very similar couplings to the rest of the particle content in the UV theory.

4.7 Final Results

At this point we can argue that the model is complete. We have been able to generate the observed hierarchical structure of the Yukawa matrices and the observed active-neutrino masses with a UV theory with $\mathcal{O}(1)$ couplings among its UV particle content. Furthermore, the only source of fine tuning is required to explain the observed mass splitting between the top and bottom masses, but it is just an $\mathcal{O}(10)$ cancellation among the UV parameters as shown in (2.8.3). In this Section we provide a possible choice for the values of the UV parameters and the corresponding numerical values of the Yukawa matrices and active-neutrino masses (in the 2-flavour case). We also declare since the beginning that such values are not the precised ones in agreement with the current measurements. Our purpose is just to show that this model is able to reproduce what we observe, not to provide the exact values predicted by the theory. We recall that the analytic expressions of the relevant quantities can be found in the following Sections: vector-boson masses in Section 1.9, Yukawa matrices in Section 2.5, active- and heavy-neutrino matrices in Sections 4.2 and 4.3.

We assume the following values for the UV couplings, masses of VLFs and VEVs of scalar fields

$$\begin{gathered} c_{33} \approx 0.56\,, \quad c_{\eta\mathcal{H}} \approx 1.4\,, \quad c_{\rho\mathcal{H}} \approx 0.7\,, \quad c_{\eta\Sigma} \approx 0.36\,, \quad c_{s\eta} \approx 2.5\,, \quad c_{k3} \approx 1.0\,, \\ c_{\eta} \approx 0.1\,, \quad c_{\rho} \approx 0.5\,, \quad c_{\eta\rho} \approx 0.5\,, \quad c_{\rho\Omega} \approx 0.2\,, \quad c_{\rho R} \approx 0.1\,, \end{gathered} \tag{4.7.1a}$$

$$M_\eta \approx 5\,\mathrm{TeV}\,, \qquad M_\rho \approx 30\,\mathrm{TeV}\,, \qquad m \approx 0.6\,\mathrm{TeV}\,, \tag{4.7.1b}$$

$$\omega_1 \approx 2.0\,\mathrm{TeV}\,, \quad \omega_3 \approx 4.0\,\mathrm{TeV}\,, \quad v_3 \approx 15\,\mathrm{TeV}\,, \quad v_R \approx 1\,\mathrm{TeV}\,, \quad v_F \approx 10\,\mathrm{TeV}\,. \tag{4.7.1c}$$

We find the following numerical values for SM couplings, vector-boson masses and Yukawa matrices

$$g_Y \approx 0.37\,, \qquad g_s \approx 1.00\,, \tag{4.7.2a}$$

$$m_{G'} \approx 9.2\,\mathrm{TeV}\,, \quad m_U \approx 15.5\,\mathrm{TeV}\,, \quad m_{W_R} \approx 6.6\,\mathrm{TeV}\,, \quad m_{Z'} \approx 18.3\,\mathrm{TeV}\,, \quad m_{Z''} \approx 2.5\,\mathrm{TeV}\,, \tag{4.7.2b}$$

$$\begin{gathered} Y^U \approx \begin{pmatrix} 0.0019 & 0.0444 \\ 0.0007 & 1.1788 \end{pmatrix}, \qquad Y^D \approx \begin{pmatrix} 0.0015 & 0.0010 \\ 0.0004 & 0.0426 \end{pmatrix}, \\ Y^N \approx \begin{pmatrix} 0.0019 & 0.0222 \\ 0.0004 & 1.1682 \end{pmatrix}, \qquad Y^E \approx \begin{pmatrix} 0.0015 & 0.0005 \\ 0.0002 & 0.0365 \end{pmatrix}. \end{gathered} \tag{4.7.2c}$$

In particular, using (2.1.7) one can see that we can reproduce the observed values of the Yukawa couplings (2.1.2). In addition, there are clear top-bottom and bottom-tau mass splittings and CKM entry $|V_{cb}| \approx 0.014$. Therefore all the observational constraints are satisfied upon a suitable choice of the UV parameter values. Instead the Yukawa couplings of the second generations are very general since we expect that the structure that defines their values involves $\mathcal{O}(1)$ contributions from some UV fields that break explicitly the universality among the first and second generations.

Let us make some comments on these results. The vector-boson masses are all of the order of few TeV, allowing to a possible detection in the near future. The heaviest ones are the leptoquarks and one of the neutral Z-type boson since their masses depend on the VEV of Δ. Also be aware that those results are quite sensible to the choices of the numerical values of the UV couplings. Furthermore it is important to stress that in this numerical analysis we assume the same values for some UV couplings that in principle could differ by some $\mathcal{O}(1)$ factors, helping to reproduce better the desired Yukawa matrices. To be precised, we are referring to the expected similarities between the couplings shown in (2.3.3) and (2.3.9). Finally, it is worth to mention that, with such UV parameters, the mixing matrix between heavy VLFs and SM fermions F, defined in (2.7.9), is sufficiently little, namely $F \sim \mathcal{O}(10^{-2})$. This is important since it is related to non-unitarity of the CKM matrix which is given by $FF^\dagger \sim \mathcal{O}(10^{-4})$ and it is below the constraints we have from the observations. This was expected, but important to check.

Regarding the neutral leptons, we assume a Majorana mass-matrix for the sterile fermions μ with eigenvalues

$$\mu_3 \approx 0.5\,\mathrm{eV}\,, \qquad \mu_2 \approx 0.5\,\mathrm{eV}\,. \tag{4.7.3}$$

We find the following active-neutrino mass-matrix and PMNS non-unitarity parameter

$$m_\nu \approx 10^{-2}\,\mathrm{eV}\begin{pmatrix} 9.27 & -1.40 \\ -1.40 & 4.01 \end{pmatrix}\,, \qquad |\eta| \approx 10^{-4}\begin{pmatrix} 1.85 & 0.27 \\ 0.27 & 0.80 \end{pmatrix}\,. \tag{4.7.4}$$

In particular m_ν is anarchic and $|\eta|$ is within the experimental constraints (3.5.3). The active-neutrino masses are therefore anarchic and of $\mathcal{O}(10^{-2})$ eV. For our specific choice of UV parameters we find

$$m_\nu^3 \approx 9.62 \cdot 10^{-2}\,\mathrm{eV}\,, \qquad m_\nu^2 \approx 3.66 \cdot 10^{-2}\,\mathrm{eV}\,. \tag{4.7.5}$$

The precised values of the masses of the three Dirac-type heavy neutrinos are not important since there is no constraint to impose on them. We just state that we expect them to be of the following order

$$M_3 \sim y_t v_F\,, \qquad M_2 \sim y_c v_F\,, \qquad M_1 \sim y_u\, v_F\,. \tag{4.7.6}$$

In particular they are hierarchical with ratios similar to the ones among the up-type quarks since the neutral lepton Yukawa matrix is similar to the one of the up-type quarks. With the chosen value of v_F we expect the following masses

$$M_3 \sim 10\,\mathrm{TeV}\,, \qquad M_2 \sim 50\,\mathrm{GeV}\,, \qquad M_1 \sim 100\,\mathrm{MeV}\,. \tag{4.7.7}$$

Nevertheless we can safely change those numbers by factors of even $\mathcal{O}(10)$ by assuming some splittings between Y^N and Y^U, especially for the lightest mass-eigenstate which is the most constrained one by experimental bounds, as we will see in the next Chapter.

As final remark we want to make a comment about how *natural* is to assume such (different) energy scales among the VLFs and scalar fields' VEVs. In particular we would not like to push v_F too high since Δ breaks the $\mathrm{SU}(4)^{[3]}$ Gauge Group. Instead v_3 can be pushed very high since we like that $\mathrm{SU}(2)_R^{[3]}$ breaks at a relative higher scale. Moreover having a mass-mixing term m between η and χ_R^3 not very high could be sensible, but actually it turns out that even larger values do not spoil the results. Instead could seem *unnatural* having $M_\eta/M_\rho \sim 10$. Nevertheless η is charged only under the third-generation Gauge Group, while ρ is charged under $\mathrm{SU}(3)^{[12]}$. Thus it seems sensible if we assume that the first- and second-generation non-universality happens at higher energy scales. Therefore we believe that such ranges for the energy scales in the theory are *natural*.

5 Phenomenology of Exotic Neutrinos

In this Chapter we are going to study in detail some phenomenological implications of the model we have built in this work. In particular the phenomenology associated to the NP massive vector bosons have been already studied in the recent literature. Therefore we are going to focus on the Dirac-type neutral fermions that are predicted by the model, which we call *exotic neutrinos*. They represent the peculiar new feature introduced in this work. As we are going to discuss, the experimental bounds on LFV observables will put some constraints on the masses and couplings of those NP fermions, but they will not spoil the *naturalness* of the model. Instead they provide some NP effects beyond the SM that could hopefully be detected in the near future[21].

5.1 Neutrino Mass-Basis

The model predicts the existence of three generations of sterile Dirac-type fermions n^i (where $i = 1, 2, 3$). In this Section we make explicit the unitary rotation that enable us to pass from flavour- to mass-basis in the neutrino sector to find the couplings between n^i and SM fields.

In Chapter 4, to diagonalize the neutrino mass-matrix M (4.1.2) in the space $\{\nu_L^i, (\nu_R^j)^c, s_L^k\}$ (where $i, j, k = 1, 2, 3$), we exploited several rotations to find the mass-eigenstates of the theory. Also, to simplify the notation, we define

$$N_L \equiv \begin{pmatrix} \nu_L & \nu_R^c & s_L \end{pmatrix}^T \tag{5.1.1}$$

where ν_L, ν_R^c and s_L are 3-dimensional vectors. At first we diagonalized Y^N and μ. This was something not really necessary, but rather a way to simplify all the further calculations. In fact the results can be written (and we are going to do it) in terms of the *original* (hence non-diagonalized) matrices. Then we block-diagonalized M using a unitary matrix W as shown in (4.3.1). This means that[22]

$$-\mathcal{L} = \frac{1}{2}\overline{N}_L M N_L^c + h.c. = \frac{1}{2}\overline{N}'_L \begin{pmatrix} U_L^\dagger m_\nu U_L^* & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & U_R^T m_R^T U_S^* \\ \mathbb{O} & U_S^\dagger m_R U_R & \mathbb{O} \end{pmatrix} N_L'^c + h.c. \tag{5.1.2}$$

where

$$N'_L \equiv \begin{pmatrix} \nu'_L \\ \nu_R'^c \\ s'_L \end{pmatrix} = W^T \begin{pmatrix} U_L^\dagger \nu_L \\ U_R^T \nu_R^c \\ U_S^\dagger s_L \end{pmatrix} \tag{5.1.3}$$

and the active-neutrino mass-matrix m_ν reads

$$m_\nu = \frac{v^2}{2}\left(Y^N m_R^{-1}\right)\mu\left(Y^N m_R^{-1}\right)^T . \tag{5.1.4}$$

Then, we diagonalized m_ν and the heavy-neutrino mass-matrix m_R as follows

$$-\mathcal{L} = \frac{1}{2}\overline{\nu}'_L U_L^\dagger m_\nu U_L^* \nu_L'^c + \overline{s}'_L U_S^\dagger m_R U_R \nu'_R + h.c. = \frac{1}{2}\overline{\nu}'_L U_L^\dagger V_L \hat{m}_\nu V_L^T U_L^* \nu_L'^c + \overline{s}'_L U_S^\dagger V_S \hat{M} V_R^\dagger U_R \nu'_R + h.c. \tag{5.1.5}$$

where

$$\hat{m}_\nu = \mathrm{diag}\left(m_\nu^1, m_\nu^2, m_\nu^3\right) , \qquad \hat{M} = \mathrm{diag}\left(M_1, M_2, M_3\right) \tag{5.1.6}$$

are the diagonal active- and heavy-neutrino mass-matrices respectively, while M_i, V_S and V_R are explicitly derived[23] in (4.3.9) and (4.3.10).

[21] For the notation of this Chapter we refer to Appendix F. Notice that this is slightly different from the one used in Chapter 1.

[22] Notice that we do not have m_ν and m_R, but rather $U_L^\dagger m_\nu U_L^*$ and $U_S^\dagger m_R U_R$ because we have to take into account the rotation that we did at the beginning. In fact this time we are writing all the results in terms of the *original* matrices.

[23] Be careful that those results where derived considering purely m_R and not $U_S^\dagger m_R V_R$. However parametrically it does not change anything and those results can be taken without any change.

By taking into account all these transformations and using the explicit form of W (4.3.4), we find that the neutrino states rotate as follows[24]

$$
\begin{aligned}
\nu_L &\to \mathcal{U}\,\nu_L + \mathcal{V}\,s_L && (5.1.7a)\\
s_L &\to -\mathcal{V}'\nu_L + \mathcal{U}'s_L && (5.1.7b)\\
\nu_R &\to V_R\,\nu_R && (5.1.7c)
\end{aligned}
$$

where

$$
\mathcal{U} = \left[\mathbb{I} - \frac{v^2}{4}\left(Y^N m_R^{-1}\right)\left(Y^N m_R^{-1}\right)^\dagger\right] V_L\,, \qquad \mathcal{V} = \frac{v}{\sqrt{2}}\left(Y^N m_R^{-1}\right) V_S\,, \tag{5.1.8a}
$$

$$
\mathcal{U}' = \left[\mathbb{I} - \frac{v^2}{4}\left(Y^N m_R^{-1}\right)^\dagger\left(Y^N m_R^{-1}\right)\right] V_S\,, \qquad \mathcal{V}' = \frac{v}{\sqrt{2}}\left(Y^N m_R^{-1}\right)^\dagger V_L\,. \tag{5.1.8b}
$$

In this equation we have written all the matrices in their *original* form (hence the ones generated directly from the EFT operators and not diagonalized). Notice that, as expected, the dependence on the matrices that diagonalized Y^N and μ (which are U_L, U_R and U_S) completely disappeared. Thanks to the negligible mixing with ν_R, ν_L and s_L mix through an almost unitary matrix since

$$
\begin{pmatrix} \mathcal{U} & \mathcal{V} \\ -\mathcal{V}' & \mathcal{U}' \end{pmatrix}^\dagger \begin{pmatrix} \mathcal{U} & \mathcal{V} \\ -\mathcal{V}' & \mathcal{U}' \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix} + \mathcal{O}\left(\frac{v^4}{v_F^4}\right)\,. \tag{5.1.9}
$$

In fact this is not surprising since it is the non-trivial part of W, modulo a change of phase.

Finally recall that actually s_L and ν_R are just the chiral components of a Dirac-type sterile fermion

$$
n \equiv s_L + \nu_R\,. \tag{5.1.10}
$$

Consistently to our notation, this is a 3-dimensional vector that represents all the three generations of sterile Dirac-type fermions which are our NP exotic neutrinos. Recall also that they have three hierarchical masses M_i (where $i = 1, 2, 3$).

5.2 Interactions of the Exotic Neutrinos

In this Section we study how the rotation of neutrino states from flavour- to mass-basis affects the SM Lagrangian and, in turn, the interactions between the exotic neutrinos n and SM particles.

In the unbroken EW phase neutrinos appear in the following terms of the SM Lagrangian

$$
\mathcal{L}_{\rm SM} \supset \overline{L}_L i\not{D} L_L + \overline{\nu}_R i\not{D}\nu_R + \overline{s}_L i\not{D} s_L - \left(\overline{L}_L H Y^E e_R + \overline{L}_L H^c Y^N \nu_R + \overline{s}_L m_R\,\nu_R + h.c.\right) \tag{5.2.1}
$$

where the covariant derivative reads

$$
D_\mu = \partial_\mu - igT^i W_\mu^i - ig'YB_\mu\,. \tag{5.2.2}
$$

In the broken EW phase, instead, they are given by

$$
\begin{aligned}
\mathcal{L}_{\rm SM} \supset\; & \overline{N}_L i\not{\partial} N_L - \left(\frac{1}{2}\overline{N}_L M N_L^c + h.c.\right) + \frac{g}{2c_W}\overline{\nu}_L \not{Z}\nu_L + \left(\frac{g}{\sqrt{2}}\overline{e}_L \not{W}^-\nu_L + h.c.\right) \\
& + \left(\overline{e}_L Y^N \nu_R\,\phi^- - \overline{\nu}_L Y^E e_R\,\phi^+ - \frac{1}{\sqrt{2}}\overline{\nu}_L Y^N \nu_R\, h + \frac{i}{\sqrt{2}}\overline{\nu}_L Y^N \nu_R\,\phi^0 + h.c.\right)\,.
\end{aligned} \tag{5.2.3}
$$

[24] Notice that the matrix that rotates ν_R is almost diagonal.

When we go to the mass-basis, we find that they read

$$
\begin{aligned}
\mathcal{L}_{\rm SM} \supset\, & \overline{\nu}_L i\partial\!\!\!/ \nu_L - \left(\frac{1}{2}\overline{\nu}_L \hat{m}_\nu \nu_L^c + h.c.\right) + \overline{n}\left(i\partial\!\!\!/ - \hat{M}\right) n \\
& + \left(\frac{g}{\sqrt{2}}\overline{e}_L W\!\!\!\!/^{-} U \nu_L + \frac{g}{\sqrt{2}}\overline{e}_L W\!\!\!\!/^{-} V n_L + h.c.\right) \\
& + \frac{g}{2c_W}\overline{\nu}_L Z\!\!\!/ \left(U^\dagger U\right)\nu_L + \frac{g}{2c_W}\overline{n}_L Z\!\!\!/ \left(V^\dagger V\right) n_L + \left(\frac{g}{2c_W}\overline{\nu}_L Z\!\!\!/ \left(U^\dagger V\right) n_L + h.c.\right) \\
& + \left[\frac{g}{\sqrt{2}}\phi^-\left(\overline{e}_L V \frac{\hat{M}}{M_W} n_R - \overline{e}_R \frac{\hat{m}_e}{M_W} V\, n_L\right) - \frac{g}{\sqrt{2}}\phi^- \overline{e}_R \frac{\hat{m}_e}{M_W} U\, \nu_L + h.c.\right] \\
& - \left[\frac{g}{2}\left(\overline{\nu}_L (U^\dagger V)\frac{\hat{M}}{M_W}\, n_R + \overline{n}_L (V^\dagger V)\frac{\hat{M}}{M_W}\, n_R\right)(h - i\phi^0) + h.c.\right]
\end{aligned}
\tag{5.2.4}
$$

where $n = s_L + \nu_R$ and we have defined

$$
U_{\alpha i} \equiv \left(L_E^\dagger \mathcal{U}\right)_{\alpha i}\,, \qquad V_{\alpha i} \equiv \left(L_E^\dagger \mathcal{V}\right)_{\alpha i}\,. \tag{5.2.5}
$$

L_e is the unitary matrix that diagonalizes the charged lepton Yukawa matrix Y^E such that

$$
\frac{v}{\sqrt{2}} L_E^\dagger Y^E R_E = \frac{v}{\sqrt{2}}\hat{Y}^E \equiv \hat{m}_e = \mathrm{diag}(m_e, m_\mu, m_\tau)\,. \tag{5.2.6}
$$

In the mixing matrices U and V we have labelled with Greek and Latin indices $\alpha = e, \mu, \tau$ (or $\alpha = 1, 2, 3$) and $i = 1, 2, 3$ the charged- and neutral-lepton components respectively, and we will continue to use this notation for the rest of this Chapter[25]. Neutral Current (NC) and Charged Current (CC) Weak Interactions of the SM are modified by NP interactions with exotic neutrinos and they are suppressed by a factor of $\mathcal{O}(v/v_F)$. In addition, there are interesting interactions between the Higgs field and the neutrino states.

The rest of this Chapter is devoted to the analysis of the phenomenological implications of the above NP interactions. In particular, the first thing to consider is the lifetime of these new exotic neutrinos n to see whether they are long-lived or not. To do so, we need to compute their decay rates. In addition we are going to divide them into two categories. If $M_n > M_Z$, they are called *heavy*, otherwise they are called *light*. This classification is important since, as we are going to discuss in the following, their most relevant decay modes will be completely different. From (4.7.7), we can argue that the lightest and heaviest states are expected to be respectively light and heavy, while the one in the middle could be both.

5.3 Lifetime of Heavy Exotic Neutrinos

In this Section we want to compute the lifetimes of the exotic neutrinos with masses $M_n > M_h, M_Z, M_W$. In this framework the dominating processes accounting for their decay rates are $n \to h\nu$, $n \to Z\nu$ and $n \to W\ell$.

If we neglect the lepton masses, the decay rates of n^3 and (if kinematically allowed) n^2 read

$$
\Gamma\left(n^i \to W \ell^\alpha\right) = \frac{G_F |V_{\alpha i}|^2 M_i^3}{8\pi\sqrt{2}}\left(1 - \frac{M_W^2}{M_i^2}\right)^2\left(1 + 2\frac{M_W^2}{M_i^2}\right)\,, \tag{5.3.1a}
$$

$$
\Gamma\left(n^i \to Z\, \nu^j\right) = \frac{G_F |(U^\dagger V)_{ji}|^2 M_i^3}{16\pi\sqrt{2}}\left(1 - \frac{M_Z^2}{M_i^2}\right)^2\left(1 + 2\frac{M_Z^2}{M_i^2}\right)\,, \tag{5.3.1b}
$$

$$
\Gamma\left(n^i \to h\, \nu^j\right) = \frac{G_F |(U^\dagger V)_{ji}|^2 M_i^3}{16\pi\sqrt{2}}\left(1 - \frac{M_h^2}{M_i^2}\right)^2 \tag{5.3.1c}
$$

[25]Notice that the CKM matrix is indicated with V as well. Thus, to not confuse it with the mixing matrix between charged leptons and exotic neutrinos, in the following we are going to indicate the first as $V_{qq'}$ where $q = u, c, t$ and $q' = d, s, b$.

where $i = 2, 3$ and their explicit expressions have been computed in Section B.1 in the Appendix. Thus, if we sum over $j, \alpha = 1, 2, 3$ and use the fact that $UU^\dagger \approx \mathbb{I}$, we get that the total decay width reads[26]

$$\Gamma_{n^i} \approx \frac{G_F M_i^3}{4\pi\sqrt{2}} (V^\dagger V)_{ii} \,. \tag{5.3.2}$$

This result is worth to be compared with the decay rate of the top quark since it is mediated by a very similar decay channel. In particular, as evaluated in Section B.1 in the Appendix, the top decay width reads as follows

$$\Gamma_t = \frac{G_F |V_{tb}|^2 m_t^3}{8\pi\sqrt{2}} \left(1 - \frac{M_W^2}{m_t^2}\right)^2 \left(1 + 2\frac{M_W^2}{m_t^2}\right) . \tag{5.3.3}$$

By using the parametric behaviour of the exotic neutrino masses (4.7.6) together with the fact that $V_{\alpha j} \sim v/v_F$, $|V_{tb}| \approx 1$ and $m_t \sim v$, we find that parametrically

$$\frac{\Gamma_{n^2}}{\Gamma_t} \sim \frac{v_F}{v} \left(\frac{m_c}{m_t}\right)^3 \sim 10^{-3} \,, \qquad \frac{\Gamma_{n^3}}{\Gamma_t} \sim \frac{v_F}{v} \sim 10^2 \,. \tag{5.3.4}$$

Therefore, as expected, n^2 and n^3 are very short-lived particles. In such a case there is an important check to make to ensure that perturbative unitarity is not violated. Namely, as shown in [18, 13], we need to satisfy the following constraint

$$\frac{\Gamma_{n^i}}{M_i} < \frac{1}{2} \,. \tag{5.3.5}$$

After recalling that $\sqrt{2} G_F = v^{-2}$, for n^2 we find that

$$\frac{\Gamma_{n^2}}{M_2} \sim \left(\frac{m_c}{v}\right)^2 \ll 1 \tag{5.3.6}$$

where the dependence on v/v_F disappears. For n^3 instead we should do this with care since $m_t/v \sim 1$. We have that

$$\frac{\Gamma_{n^3}}{M_3} \approx \frac{M_3^2}{4\pi\, v^2} (V^\dagger V)_{ii} \sim \frac{1}{4\pi} < \frac{1}{2} \,. \tag{5.3.7}$$

Therefore perturbative unitarity holds. Another way to verify this is given by the following fact. We could argue that

$$\frac{\Gamma_{n^3}}{M_3} \approx \frac{\Gamma_t}{m_t} \tag{5.3.8}$$

since the decay channels are very similar. Therefore, if the top quark does not break perturbative unitarity (as he does), also n^3 will not.

5.4 Lifetime of Light Exotic Neutrinos

In this Section we compute the lifetimes of the exotic neutrinos with masses $M_n < M_h, M_Z, M_W$. In this framework, we can rely on an energy regime where Fermi Theory holds[27] and the dominating decay channel is given by $n \to \ell\ell'\nu$. Then, if kinematically allowed, there can be also contributions from hadronic decays.

For n^1 we have the following leptonic decay rate

$$\Gamma\left(n^1 \to \ell^\alpha \bar{\ell}^\beta \nu^i\right) = \frac{G_F^2 |V_{\alpha 1} U_{\beta i}^*|^2 M_1^5}{192\pi^3} \,\Phi \tag{5.4.1}$$

[26]We neglect the decay channels $n^i \to n^j Z(h)$ since they are suppressed by a further factor of $V^\dagger V \sim v^2/v_F^2$.

[27]This is a good approximation for n^1 while, for n^2, this is not guaranteed. However, as we will see, even if $M_2 < M_Z$, its lifetime will be too short. Thus it is not very interesting and we do not need to compute it carefully.

which has been computed explicitly in Section B.2 in the Appendix and the factor Φ is an $\mathcal{O}(1)$ number (and it is $\Phi = 1$ in the limit where the lepton masses are negligible with respect to M_1). One can appreciate that this is pretty similar to the muon decay rate which is given by

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}\,. \tag{5.4.2}$$

In addition, if $M_1 \gtrsim m_\pi$, there are a lot of hadronic decay processes that start to be allowed. For instance

$$\Gamma\left(n^1 \to \ell^\alpha h_P^+\right) = \frac{G_F^2}{16\pi} f_h^2 |V_{ud}|^2 |V_{\alpha 1}|^2 M_1^3 \left[\left(1 - \frac{m_\alpha^2}{M_1^2}\right)^2 - \frac{m_h^2}{M_1^2}\left(1 + \frac{m_\alpha^2}{M_1^2}\right)\right] \lambda^{\frac{1}{2}}\left(1, \frac{m_\alpha^2}{M_1^2}, \frac{m_h^2}{M_1^2}\right)\,, \tag{5.4.3a}$$

$$\Gamma\left(n^1 \to \nu^j h_P^0\right) = \frac{G_F^2}{64\pi} f_h^2 |(U^\dagger V)_{j1}|^2 M_1^3 \left(1 - \frac{m_h^2}{M_1^2}\right)^2 \tag{5.4.3b}$$

where h_P is a pseudoscalar meson with mass m_h and hadronic structure constant f_h. The above decay rates have been computed explicitly in Sections B.4 and B.5 in the Appendix.

In general, we can appreciate that those decay rates are very similar to the hadronic decay channels of the tau lepton. In particular, in Section B.4, we have computed some hadronic decay rates of τ and the results are similar to the hadronic decay rates of n^1, but without the suppression factor $|V_{j1}|^2$. In addition, the τ hadronic decay channels are comparable to its purely leptonic ones. Because of this analogy, which is at work if $M_1 \gtrsim m_\tau$, we can argue that the total decay width of n^1 is given by its leptonic decay rate multiplied by the factor k_n where

$$k_n \equiv 1 + \Gamma\left(n^1 \to \text{hadrons}\right) / \Gamma\left(n^1 \to \text{leptons}\right) = \mathcal{O}(1)\,. \tag{5.4.4}$$

To give a rather easy expression for the total decay width of n^1, we can use the following fact

$$\sum_{\alpha,\beta,i=1}^{3} |V_{\alpha 1} U_{\beta i}^*|^2 = \mathrm{Tr}\Big[U^\dagger U\Big] (V^\dagger V)_{11} \approx 3 (V^\dagger V)_{11}\,. \tag{5.4.5}$$

Thus we have that

$$\Gamma_{n^1} \approx k_n \frac{G_F^2 (V^\dagger V)_{11} M_1^5}{64\pi^3} \tag{5.4.6}$$

where to compute the total leptonic decay width we have summed over all the possible leptonic configurations in the final state with $\Phi = 1$ (this is a good approximation up to $\mathcal{O}(1)$ numbers, even if some configurations are not kinematically allowed).

We can compare now this decay rate to that of the muon. Parametrically, using (4.7.6), we find

$$\frac{\Gamma_{n^1}}{\Gamma_\mu} \sim k_n \left(\frac{v}{v_F}\right)^2 \left(\frac{M_1}{m_\mu}\right)^5 \sim k_n \left(\frac{v_F}{v}\right)^3 \left(\frac{m_u}{m_\mu}\right)^5\,. \tag{5.4.7}$$

This ratio is very sensible to M_1, so it is not clear the relation between the two decay rates. Nevertheless, there is space to n^1 to be long-lived. We anticipate that, at the end of this Chapter, we are going to be more quantitative, once we put some bounds on the involved quantities. In this case, condition (5.3.5) is easily satisfied since n^1 will turn out to be a long-lived particle. For completeness we can explicitly check that, even if $v_F/v \sim 10^5$, we have that

$$\frac{\Gamma_{n^1}}{M_1} \sim 10^{-3} \left(\frac{m_u}{v}\right)^4 \left(\frac{v_F}{v}\right)^2 \ll 1\,. \tag{5.4.8}$$

As an important remark, by looking naively at this equation, one could argue that, for v_F/v arbitrarily large, perturbative unitarity fails. In fact the ratio $v_F/v \sim 10^5$ is not chosen randomly, but it is the scale at which $M_1 \gtrsim M_Z$ and hence Γ_{n^1} changes its parametric dependence, resulting in similar ratios than the ones for n^2 and n^3 which are independent on v_F/v (and do not break perturbative unitarity).

If one considers n^2 with mass $M_2 < M_W$, we have that it decays with similar processes to n^1, although the energy scale does not allow to work within the effective Fermi Theory. Nevertheless, we expect this particle to be short-lived (and hence non-interesting like n^1) since, by looking at (5.4.7), we expect that

$$\frac{\Gamma_{n^2}}{\Gamma_\mu} \sim \left(\frac{v_F}{v}\right)^3 \left(\frac{m_c}{m_\mu}\right)^5 \gg 10^5 \,. \tag{5.4.9}$$

5.5 Lepton Flavour Violation in $\mu \to e\gamma$

Heavy neutral leptons are predicted by several BSM scenarios and they typically generate enhanced LFV processes with respect to the SM predictions. In the past decades there has been an outstanding experimental effort to detect LFV effects. However, no convincing evidences have emerged so far. One of the most prominent consequence of BSM heavy neutral leptons is a huge enhancement in the muon decay channel $\mu \to e\gamma$. In this Section we study this process in detail.

In the SM $\mu \to e\gamma$ is very suppressed thanks to the unitarity of the PMNS matrix and the very small neutrino masses. The current experimental bound is given by MEG II [19] and it reads as follows

$$\mathrm{Br}(\mu \to e\gamma) < 3.1 \cdot 10^{-13} \,. \tag{5.5.1}$$

This is well above the SM prediction which is around 10^{-50}. Hence, the emergence of this process would represent an unambiguous signal of NP. In our model NP exotic fermions break PMNS unitarity. Hence, we expect an enhanced branching ratio with respect to the SM predictions. However, we have to be sure that it lies below the current experimental bounds. This turns out to put some bounds on the UV parameters of the model. We have explicitly evaluated this branching ratio in Section B.6 in the Appendix. The result is given by

$$\mathrm{Br}(\mu \to e\gamma) = \frac{3\alpha}{32\pi} |\delta_\nu|^2 \tag{5.5.2}$$

where

$$\delta_\nu = 2 \sum_j U_{ej} U^*_{\mu j}\, g\left(\frac{(m^j_\nu)^2}{M_W^2}\right) + 2 \sum_k V_{ek} V^*_{\mu k}\, g\left(\frac{M_k^2}{M_W^2}\right) \tag{5.5.3}$$

and

$$g(x) = \frac{2}{3} + \frac{11}{2(1-x)} - \frac{15}{2(1-x)^2} + \frac{3}{(1-x)^3} - \frac{3\,x^3}{(1-x)^4} \ln x \,. \tag{5.5.4}$$

To better study this result, it is useful to consider the following limits of $g(x)$

$$g(x) = \frac{5}{3} - \frac{x}{2} + \mathcal{O}(x^2) \qquad \text{if} \qquad x \ll 1 \,, \tag{5.5.5}$$

$$g(x) = \frac{2}{3} + \frac{3 \ln x}{x} + \mathcal{O}\left(\frac{1}{x}\right) \qquad \text{if} \qquad x \gg 1 \,. \tag{5.5.6}$$

Also it is important to observe that $g(x) = \mathcal{O}(1)$ for all the values $x > 0$ and in particular $g(1) = 17/12$. If we apply this result in the SM framework we have that

$$\delta_\nu \approx 2 \sum_j U_{ej} U^*_{\mu j} \left(\frac{5}{3} - \frac{(m^j_\nu)^2}{2\,M_W^2}\right) = \frac{1}{M_W^2} \sum_j U_{ej} U^*_{\mu j} (m^j_\nu)^2 \tag{5.5.7}$$

where U is the PMNS matrix and we use the fact that it is unitary to kill the leading contribution. Thus, if we assume $m_\nu^j \sim 0.1$ eV, we find that SM predicts

$$\mathrm{Br}(\mu \to e\gamma)_{\mathrm{SM}} \approx 10^{-52} \,. \tag{5.5.8}$$

In our model, instead, we have three further exotic neutrinos n^i contributing at the virtual level to this decay. Notice that, since $g(x)$ is limited in all its domain, it is not important how massive n^i are. Also, by using (5.1.9), we have that

$$\sum_j U_{ej} U^*_{\mu j} + \sum_k V_{ek} V^*_{\mu k} = \left(L_E L_E^\dagger\right)_{e\mu} + \mathcal{O}\left(\frac{v^4}{v_F^4}\right) = \mathcal{O}\left(\frac{v^4}{v_F^4}\right) . \tag{5.5.9}$$

Thus, since we expect $m_\nu^j, M_1 \ll M_W$ and $M_3 \gg M_W$, we can write the loop factor as follows

$$\delta_\nu = 2\, V_{e2} V^*_{\mu 2} \left[g\left(\frac{M_2^2}{M_W^2}\right) - \frac{5}{3}\right] + 2\, V_{e3} V^*_{\mu 3} \left[\frac{2}{3} - \frac{5}{3}\right] + \mathcal{O}\left(\frac{v^4}{v_F^4}\right) \approx 4\kappa^2 \frac{v^2}{v_F^2} \,. \tag{5.5.10}$$

κ is a numerical factor that quantifies the numerical values of $V_{\alpha i}$, once we have factorized out the suppression factor v/v_F. In fact, since they are components of a unitary matrix, we expect that $\kappa \approx 0.3 \div 0.7$. It is important to keep track of this small contribution since the final observable depends on an important power of κ. Hence, our model predicts the following branching ratio

$$\mathrm{Br}(\mu \to e\gamma) \approx \frac{3\alpha}{2\pi} \kappa^4 \frac{v^4}{v_F^4} \,. \tag{5.5.11}$$

By comparing this with the experimental bound (5.5.1), we need that

$$\frac{v_F}{v} \gtrsim 300\, \kappa \,. \tag{5.5.12}$$

Therefore this observable requires that[28] $v_F \gtrsim 8 \div 15\,\mathrm{TeV}$. This also means that, an improvement of the experimental upper bound on this process could lead to a possible BSM physics detection which could be explained by our model. In fact in the near future we expect to improve the experimental bounds on this decay channel. Namely MEG II [19] aims in 2026 to reach a sensitivity of $\mathrm{Br}(\mu \to e\gamma) \approx 6 \cdot 10^{-14}$. This could lead to a possible detection of BSM physics.

5.6 Other Lepton-Flavour-Violating Observables

There are several LFV observables that are affected by some NP heavy neutral leptons, like the exotic neutrinos in our model. Those are usually less prominent than $\mu \to e\gamma$, but it is important to study them as well. All those kind of processes have been already computed and analyzed in the literature. As good references, one can look at [11, 13]. In particular [13] has a very nice Table with all the updated bounds on LFV observables and future prospects.

First process to consider is the decay rate of $Z \to \ell\bar{\ell}'$ where $\ell \neq \ell'$. In particular the most precised one is given by

$$\mathrm{Br}\left(Z \to \mu e\right) = \frac{G_F^3 M_W^4 M_Z^2}{12\sqrt{2}\pi^5} \frac{M_Z}{\Gamma_Z} |F_Z^{\mu e}|^2 \approx 2 \cdot 10^{-6} |F_Z^{\mu e}|^2 \tag{5.6.1}$$

where $\Gamma_Z \approx 2.49\,\mathrm{GeV}$ is the Z-boson total decay width and $F_Z^{\mu e}$ is a loop form-factor explicitly evaluated in [11]. Since

$$F_Z^{\mu e} \sim \mathcal{O}(1) V_{\alpha j} V_{\beta k} \sim \frac{v^2}{v_F^2} \tag{5.6.2}$$

[28] Notice that v_F/v is just an approximate scaling ratio. In fact the real ratio is given by $\sqrt{2} c_{s\eta} v_F/v$. Therefore if we keep $c_{s\eta} \approx 2$, we can tune v_F to not exceed 10 TeV without violating the experimental bounds.

and the current SM bounds given by ATLAS read

$$\mathrm{Br}\,(Z \to \mu e) < 4.2 \cdot 10^{-7}\,, \tag{5.6.3}$$

we can safely say that the effects of our exotic neutrinos are well below the current possibilities to detect something. With FCC-ee we expect to improve the bounds on this branching ratio at $\mathcal{O}(10^{-10})$. Even though at this level $Z \to \mu e$ would become an interesting channel to look for LFV, $\mu \to e\gamma$ will remain the best probe of our scenario with exotic neutrinos.

The next class of LFV processes is given by the NP contribution to the invisible decay widths of Z and h. Such decay rates have been evaluated explicitly in Section B.3 in the Appendix and they are given by (apart from minor corrections)

$$\Delta_{\mathrm{NP}}\Gamma\,(Z \to \mathrm{inv}) \approx \frac{G_F M_Z^3}{6\sqrt{2}\pi}\sum_k (V^\dagger V)_{kk}\,, \tag{5.6.4a}$$

$$\Gamma\,(h \to \mathrm{inv}) \approx \frac{M_h}{8\pi\, v^2}\sum_k M_k^2 (V^\dagger V)_{kk}\left(1-\frac{M_k^2}{M_h^2}\right)^2\,. \tag{5.6.4b}$$

The sums are taken over all the possible neutrino states that are kinematically allowed. In the case of the Z boson, BSM contributions provide just a small correction to the SM invisible decay rate. Instead, for the h boson case, BSM effects are by far dominant as the invisible decay rate of h in the SM is extremely suppressed by the tiny neutrino masses. Exploiting the numerical analysis of [13] (which entails also the 1-loop contributions), we find that the invisible decay width of the Z is well in agreement with the SM already with $v_F/v \gtrsim 30$, which is not so relevant. For h instead the situation is different. We have not been able yet to measure its total decay width. The SM predicts this quantity to be $\Gamma_h \approx 4$ MeV with a negligible invisible decay width since neutrino masses are negligible. Our model predicts instead the following contribution if $M_2 < M_h$

$$\Gamma(h \to \mathrm{inv}) \approx 1.3\,\mathrm{GeV}(V^\dagger V)_{22}\frac{M_2^2}{M_h^2}\left(1-\frac{M_2^2}{M_h^2}\right)^2 \sim \frac{M_2^2}{M_h^2}\frac{v^2}{v_F^2}\,\mathrm{GeV}\,. \tag{5.6.5}$$

Therefore, even with $M_2 \lesssim M_h$ and $v_F/v \gtrsim 70$, we have that $\Gamma(h \to \mathrm{inv})/\Gamma_h \lesssim 10^{-2}$, implying that this observable is not able to unveil NP effects in the framework of our model.

Another very important LFV process that must be considered is given by the decay channel $\mu \to eee$. Currently, the experimental bound is given by SINDRUM [20]

$$\mathrm{Br}\,(\mu \to eee) < 1.0 \cdot 10^{-12}\,. \tag{5.6.6}$$

The theoretical prediction of the SM is well below the current bounds and it receives very important contributions from heavy neutral leptons which are given by

$$\mathrm{Br}\,(\mu \to eee) = \frac{G_F^2 M_W^4}{32\pi^4}\mathcal{F} \approx 2 \cdot 10^{-6}|\mathcal{F}|^2 \tag{5.6.7}$$

where $\mathcal{F}$ is a loop form-factor which has been computed explicitly in [11]. Before providing a full numerical analysis, we like to analyse the general structure of $\mathcal{F}$ which is given by

$$\mathcal{F} \sim V_{\alpha j}^2 + V_{\alpha j}^4\frac{M_n^2}{M_Z^2} + V_{\alpha j}^2 \ln\frac{M_n^2}{M_Z^2}\,. \tag{5.6.8}$$

Since M_3 is the biggest mass and

$$\frac{M_3^2}{M_Z^2} \sim \frac{v_F^2}{v^2}\frac{m_t^2}{M_Z^2} = \mathcal{O}(1)\frac{v_F^2}{v^2}\,, \tag{5.6.9}$$

we have that at most the form-factor is expected to be

$$\mathcal{F} \sim \mathcal{O}(10)\kappa^2 \frac{v^2}{v_F^2} \,. \tag{5.6.10}$$

Therefore, by comparing the theoretical prediction with the experimental bounds (5.6.6), we find

$$\frac{v_F}{v} \gtrsim 100\, \kappa \,, \tag{5.6.11}$$

which is weaker than the bound set by $\mu \to e\gamma$. Nevertheless, there are excellent future prospects to improve the current bound. In particular, Mu3e experiment [21] aims to reach a sensitivity of $\mathrm{Br}\,(\mu \to eee) = \mathcal{O}(10^{-16})$, that is four orders of magnitude better than the current bound. At this level $\mu \to eee$ will become competitive with $\mu \to e\gamma$ to probe LFV effects.

5.7 Bounds on Light Neutral Leptons

The model assumes the existence of some light NP exotic neutrinos (at least one, but they could be also two). These small masses bring with them other important bounds. In particular, if some NP neutral leptons that mix with SM charged fermions are sufficiently light, they can be produced on-shell by some decay processes changing the kinematic spectrum of the process. The most stringent bounds are set by (semi)leptonic hadronic decays, but not only. The updated bounds on heavy neutral leptons from hadron physics are listed in [12]. In particular, the relevant bounds on the squared-mixing matrix between SM charged leptons and NP heavy neutral leptons depending on the mass of the the latter are listed in Table 2, together with the name of the experiments that set them. As one can appreciate, if those NP neutral leptons are very light, there are very stringent bounds set by D-meson decay processes (namely by CHARM and BEBC experiments). Otherwise there are other bounds set by experiments at high-energy colliders (namely by ATLAS, CMS and DELPHI).

$\lvert V\rvert^2$	M_n	Experiments
$\lesssim 2 \cdot 10^{-8}$	$M_n \lesssim 2\,\mathrm{GeV}$	CHARM, BEBC
$\lesssim 2 \cdot 10^{-5}$	$2\,\mathrm{GeV} \lesssim M_n \lesssim 3\,\mathrm{GeV}$	DELPHI
$\lesssim 10^{-6} \div 10^{-7}$	$3\,\mathrm{GeV} \lesssim M_n \lesssim 10\,\mathrm{GeV}$	ATLAS, CMS
$\lesssim 2 \cdot 10^{-5}$	$10\,\mathrm{GeV} \lesssim M_n \lesssim 100\,\mathrm{GeV}$	ATLAS, CMS, DELPHI

Table 2: Bounds on the mixing matrix V between SM charged leptons and NP heavy neutral leptons with mass M_n. The last column lists the experiments that have set those bounds.

In our model, the most stringent constraints are on the lightest exotic neutrino n^1, but also partially on n^2. Namely, if $M_1 \lesssim 2$ GeV, we need the require that $v_F/v \gtrsim 10^4$, spoiling completely the *naturalness* of the model we have built[29]. Nevertheless we can assume that the lightest eigenstate of Y^N is not of the order of $\mathcal{O}(1)y_u$, but rather $\mathcal{O}(10)y_u$. In this scenario, if $v_F/v \sim 100$, we have that

$$M_1 \sim 10\, m_u \frac{v_F}{v} \sim 2\,\mathrm{GeV}\,, \qquad \lvert V\rvert^2 \sim \kappa^2 \frac{v^2}{v_F^2} \sim 10^{-5}\,. \tag{5.7.1}$$

Thus we are really at the corner of the current bounds, but still in agreement[30]. Furthermore, if this is really the case, there could be room for a possible detection of BSM physics. Notice also that in this scenario we really have stringent bounds on M_1 since, if $M_1 \gtrsim 3$ GeV, we are forced to push $v_F/v \gtrsim 10^3$.

[29] In fact, since $M_1 \sim m_u v_F/v$, if $v_F/v \gtrsim 10^3$ the bounds are satisfied because we are in a regime where $\lvert V\rvert^2 \sim 10^{-6}$ and $M_1 \gtrsim 2$ GeV.

[30] These bounds are very close to the ones set by $\mu \to e\gamma$ where we needed (5.5.12).

For n^2, which is expected to have a mass $M_2 \sim 50$ GeV, there are some bounds but, as long as $v_F/v \sim 100$, we should be at the corner of them. This is very promising since a small improvement in the experimental resolution could lead to a possible detection of BSM physics. Nevertheless, even if the bounds are more stringent, we just need to increase by an $\mathcal{O}(1)$ factor the eigenvalue of Y^N to push $M_2 \gtrsim 100\,\text{GeV}$ without pushing v_F/v too high.

5.8 Summary of Bounds and Future Prospects

In this Chapter, we have studied in detail the most prominent phenomenological implications of such exotic neutrinos that are predicted by the model we have built. In this Section, we summarize all the relevant results.

The model predicts the existence of three exotic neutrinos with hierarchical masses. The model is mainly dependent on a single scale, which is given by v_F, and we want to keep this around the TeV. Current bounds from LFV processes imply the following constraint

$$v_F \gtrsim 8 \div 15\,\text{TeV} \tag{5.8.1}$$

where the bound is set from the measurement of the decay channel $\mu \to e\gamma$. D-meson decays put the most stringent bounds on the mass of the lightest exotic neutrino. In fact, if we require to keep $v_F \sim 10$ TeV, we need

$$2\,\text{GeV} \lesssim M_1 \lesssim 3\,\text{GeV}\,. \tag{5.8.2}$$

Moreover this state could be long-lived as we discussed in Section 5.4. We observe that, if the mass of n^1 is within the above range, it is very comparable to the τ mass. Furthermore, this sterile state has decay channels very similar to the τ, the only difference is that they are suppressed by $(V^\dagger V)_{11}$. Therefore, assuming a value for v_F near the bounds set by $\mu \to e\gamma$, we could expect that

$$\frac{\Gamma_{n^1}}{\Gamma_\mu} = \frac{\Gamma_{n^1}}{\Gamma_\tau} \cdot \frac{\Gamma_\tau}{\Gamma_\mu} \sim \left(\kappa \frac{v}{v_F}\right)^2 \left(\frac{M_1}{m_\tau}\right)^5 \cdot 10^7 \sim 10^2\,. \tag{5.8.3}$$

As a result, it is possible that n^1 has a mean lifetime of the order

$$\tau_1 \sim 10^{-8}\,\text{s}\,. \tag{5.8.4}$$

In addition, the model offers a lot of possible ways to detect BSM physics, if this model turns out to be correct. In fact, with $v_F \sim 10$ TeV, the theoretical predictions of the model are really at the corner of the current bounds in many of the interesting processes that could detect the existence of some heavy neutral leptons. A possibility is offered by looking at some neutral leptons weakly coupled with the SM in the mass range from GeV up to 100 GeV, improving by $\mathcal{O}(10)$ the current bounds. In this way, it could be possible to detect n^1 but also n^2 if $M_2 \lesssim 100$ GeV. Furthermore, n^1 could have a sufficiently long lifetime to be interesting in some experiments. Another possibility is to improve the current sensitivity on rare muon decay channels, in particular $\mu \to e\gamma$ and $\mu \to eee$. This has been already taken into consideration by the experimental community and we expect to have a significant improvement in the next decade. If $v_F \sim 10$ TeV and the model is correct, we should likely detect some BSM signals in those experiments.

On the other side, if we are not going to see anything in rare muon decay channels once we reach the expected sensitivity, this would imply that

$$v_F \gtrsim 80 \div 150\,\text{TeV}\,. \tag{5.8.5}$$

Furthermore, if no neutral lepton will show up with mass around the GeV, depending on the new bounds we could need to increase v_F by a factor of 3 or even more. In any case such a big value for v_F would spoil partially the *naturalness* of the model we built. This is because Δ (the scalar boson whose VEV is v_F) breaks SU(4)[3] and we hope that this happens at the TeV scale not to destabilize too much the Higgs boson mass.

Conclusion

In this work we have studied in detail a possible UV theory where the flavour universality among the SM fermions is broken already at the TeV scale by embedding the SM Gauge Group into a larger one that breaks the universality of the third generation of fermions with respect to the other two. In particular, we assumed that the flavour non-universality happens only in the $\mathrm{SU}(3)_C \times \mathrm{U}(1)_Y$ Gauge sector of the SM Gauge Group, while the $\mathrm{SU}(2)_L$ sector is kept universal. Moreover, we have embedded the Gauge sector of the third generation in a Pati-Salam-like Gauge Group, namely $\mathrm{SU}(4) \times \mathrm{SU}(2)_R$. In this way the UV theory conserves classically the Baryon Number allowing to have the energy scale of the UV theory already at the TeV, unlike many other Grand-Unification theories where the bounds on proton decay push the energy scale around or above 10^{16} GeV. Once the UV theory has been fixed, we have studied several ways to break the UV Gauge Group down to the SM one by means of a SSB mechanism. To do so, we assumed that in the UV theory are present several scalar fields which eventually acquire a suitable VEV with energy scale around few TeV. One of the main consequences of SSB is that the broken generators of the UV theory recombine at low energies in several BSM massive vector bosons with masses in the range of few TeV.

In this framework we have assumed that the Higgs field, still considered an elementary particle, is charged only under the third-generation sector of the Gauge Group. This is well motivated by the fact that the Higgs field in the SM is mainly coupled to the fermions of the third generation. In other words, by looking at the hierarchical structure of the fermion masses, we observe that only the Yukawa couplings of the third generation are non-suppressed. In the SM such hierarchy is realized by choosing *ad hoc* the values of those couplings. Instead, our model has been able to provide a possible explanation to this. Using the scalar fields that acquire a VEV at low energies, we have been able to generate all the Yukawa matrices from an EFT approach. Therefore, the desired hierarchical structure is *naturally* given by construction since the suppressed Yukawa couplings are realized as higher-dimensional (hence suppressed) operators. Furthermore, we have provided a possible UV origin to those EFT operators. Namely, we have assumed the existence of just two heavy VLFs with masses of $\mathcal{O}(10)$ TeV which are coupled to SM fermions with $\mathcal{O}(1)$ couplings. This was enough to generate all the Yukawa matrices with their desired hierarchical structure, at least between the second and third generations.

The last, yet extremely non-trivial, ingredient left to be explained were the tiny and anarchic neutrino masses. The issue arises from the fact that Pati-Salam-like models predict naively very similar up-type quarks and neutral lepton masses. The most efficient way to provide an explanation to the latter is by means of a See-Saw mechanism. In addition, since the RH neutrinos are already present in the UV theory and charged under the UV Gauge Group (at least for the one of the third generation), the only meaningful possibility is offered by the Inverse See-Saw. Thus, we have assumed that in the UV theory are present also three generations of LH sterile fermions with a small Majorana mass-matrix. Without any further assumption, this Majorana mass-matrix turned out to be extremely hierarchical to compensate the hierarchical structure of the neutral lepton Yukawa matrix generated by our model: a rather *unnatural* (and apparently unexplained) feature.

As we have shown, it is possible to provide a solution to this latter problem. The key ingredient is the existence of a non-better-defined Symmetry under which are charged only the sterile fermions and a new scalar field. There is a large freedom on the nature of this Symmetry: it could be local or global, continues of discrete. We simply assume it is broken at an arbitrarily high scale and its breaking can be associated to Fermion Number violation. In this framework the hierarchical structure in the Yukawa matrices is compensated by a suitable hierarchical structure in the mass-mixing terms between sterile fermions and RH neutrinos. These latter terms can be generated similarly to the Yukawa matrices. Namely, we required that this further scalar field is charged under the third-generation sector of the UV Gauge Group and that it contributes to the breaking of it by acquiring a suitable VEV. Thus, the mass-mixing terms with the third-generation RH neutrino is generated by a renormalizable operator while the other terms are generated by means of some EFT operators and hence are suppressed. The leading energy scale is given by the VEV of this scalar field which we called v_F. By means of these ingredients we have been able to explain the anarchic neutrino masses without imposing any kind of hierarchical structure in the Majorana mass-matrix of the sterile fermions.

A consequence of the Inverse See-Saw mechanism is that it predicts the existence of three heavy (almost) Dirac-type fermions which are weakly coupled to the SM. This feature is present also in the model we have built. In addition, the peculiarity of those heavy states is that their masses are hierarchical with suppression factors very similar to the ones among the up-type quark masses. Namely, the heaviest one has a mass of the order of v_F while the others are suppressed by factors of order y_c/y_t and y_u/y_t. These states, which we called *exotic neutrinos*, offer a wide range of interesting phenomenological implications that we have studied in detail.

The model we have built is able to provide an explanation to several open questions in fundamental physics. It explains Charge quantization since the UV embedding of the SM Gauge Group is semisimple. More precisely, the UV Gauge Group (at least in the third-generation sector) is a combination of non-abelian groups and in particular the SM Hypercharge is realized as a linear combination of SU(N) generators. This implies that the Hypercharges of the SM particles are given by a linear combination of rational numbers. It solves the flavour puzzle since the hierarchical structure in the Yukawa matrices is *naturally* realized from an EFT description. It explains the observed small neutrino masses. Finally, it can provide a possible solution to the hierarchy problem since in this model there is no constraint that prevents assuming the UV energy scale already at the TeV. Therefore the quantum corrections to the Higgs mass are kept under control without any strong fine tuning.

In the literature there are a lot of UV models that solve all these problems. Nevertheless, compared to the others, this one is very *minimal* and *natural*. Minimal because there are not many new DOFs added to the UV theory and *natural* because the SM is reproduced without any source of fine tuning among the UV parameters. More precisely, there is no sizable hierarchy among the mass scale in the UV theory and all the dimensionless UV couplings are in the range $0.1 \div 2.0$. We recall that the only possible source of fine tuning is required to explain the observed top-bottom mass splitting, but it requires just an $\mathcal{O}(10)$ cancellation among the parameters.

Furthermore, there is a rich phenomenology coming out from this model that can be studied at colliders or precision experiments since the NP scale is around the TeV. We focused on the phenomenology of the exotic neutrinos since they are really the new peculiar feature of this model which is not yet present in the literature to our knowledge. They offer a wide range of processes that could provide possible detection of BSM physics in the near future. In particular, they predict branching ratios for the rare muon decay channels $\mu \to e\gamma$ and $\mu \to eee$ that lie just below the current experimental bounds and can be detected by the experiments MEG II and Mu3e, which are expected to run in the next few years. In addition, the lightest two exotic neutrinos have a concrete chance to be detected at LHC in the next decade.

Appendix

A Spontaneous Symmetry Breaking - Calculations

In this Chapter we do all the calculations to find the breaking spectrum of some SSB mechanisms from several possible Gauge Groups down to smaller ones. All of them have been used during this work. For simplicity, in the Section's titles every SU(N) Gauge Group is indicated with just its number N.

A.1 431 into 31 Model

We want to study the SSB pattern

$$\mathrm{SU}(4) \times \mathrm{SU}(3) \times \mathrm{U}(1)_X \quad \to \quad \mathrm{SU}(3) \times \mathrm{U}(1)_{X'} \tag{A.1.1}$$

and we define the Gauge couplings and fields associated to the UV and IR theories (with the SU(N) indices $a = 1, ..., 15$, $b = 1, ..., 8$)

$$(g_4, g_3, g_X)\ , \quad \Big(H^a, C^b, B\Big) \qquad \to \qquad (g_3', g_X')\ , \quad \Big(G^b, B'\Big)\ . \tag{A.1.2}$$

The covariant derivative in the unbroken phase reads as follows

$$D_\mu = \partial_\mu - ig_4 \hat{T}^a H^a - ig_3 T^b C^b - ig_X X B\,. \tag{A.1.3}$$

We assume the following fields charged under the full Symmetry Group

$$\Omega_1 \sim (\overline{\mathbf{4}}, \mathbf{1}, -X_\Omega)\,, \qquad \Omega_3 \sim (\overline{\mathbf{4}}, \mathbf{3}, X_\Omega/3)\,. \tag{A.1.4}$$

Notice that the Charges under U(1) have to be related one to the other to get a residual U(1) Symmetry after the SSB occurs. Their VEVs are chosen to be

$$\langle\Omega_1\rangle = \begin{pmatrix} 0 & 0 & 0 & \omega_1 \end{pmatrix}\,, \qquad \langle\Omega_3\rangle = \begin{pmatrix} \omega_3 & 0 & 0 & 0 \\ 0 & \omega_3 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 \end{pmatrix}\,. \tag{A.1.5}$$

Now we need to find the residual Gauge Group. To leave the two VEVs invariant we need to satisfy

$$\theta^a \langle\Omega_1\rangle \hat{T}^a + \beta X_\Omega \langle\Omega_1\rangle = 0\,, \qquad \theta^a \langle\Omega_3\rangle \hat{T}^a - \eta^b T^b \langle\Omega_3\rangle - \beta \frac{1}{3} X_\Omega \langle\Omega_3\rangle = 0\,. \tag{A.1.6}$$

Thus we have that

$$\langle\Omega_3\rangle \hat{T}^a = T^a \langle\Omega_3\rangle \quad \text{and} \quad \langle\Omega_1\rangle \hat{T}^a = 0 \quad \text{for} \quad a = 1, ..., 8\,; \tag{A.1.7a}$$

$$\langle\Omega_3\rangle \hat{T}^{15} = \frac{1}{2\sqrt{6}} \langle\Omega_3\rangle \quad \text{and} \quad \langle\Omega_1\rangle \hat{T}^{15} = -\frac{3}{2\sqrt{6}} \langle\Omega_1\rangle \tag{A.1.7b}$$

which represents a Gauge transformation where we keep free η^a, β and we fix

$$\theta^a = \eta^a \quad \text{for} \quad a = 1, ..., 8\,; \qquad \theta^a = 0 \quad \text{for} \quad a = 9, ..., 14\,; \qquad \theta^{15} = \frac{2\sqrt{6}}{3} X_\Omega\, \beta\,. \tag{A.1.8}$$

Therefore the residual Gauge Group is given by SU(3) × U(1) and the generator of U(1) is given by

$$X' = X + \frac{2\sqrt{6}}{3} X_\Omega\, \hat{T}^{15}\,. \tag{A.1.9}$$

To find the new Gauge boson basis and their masses we need to write down the kinetic terms of Ω_1 and Ω_3 evaluated on the VEVs. To do so we need their covariant derivatives which read as follows

$$D_\mu\langle\Omega_1\rangle = i\frac{\omega_1}{2}g_4\left(H^9+iH^{10}\,,\quad H^{11}+iH^{12}\,,\quad H^{13}+iH^{14}\,,\quad -\frac{\sqrt{6}}{2}H^{15}+2\frac{g_X}{g_4}X_\Omega\,B\right)\,, \tag{A.1.10a}$$

$$D_\mu\langle\Omega_3\rangle = i\frac{\omega_3}{2}g_4\left(\begin{array}{c|c} 2\left(H^a-\frac{g_3}{g_4}C^a\right)T^a+\frac{1}{3}\left(\frac{\sqrt{6}}{2}H^{15}-2\frac{g_X}{g_4}X_\Omega\,B\right)\mathbb{I}_{3\times3} & \begin{array}{c} H^9-iH^{10}\\ H^{11}-iH^{12}\\ H^{13}-iH^{14}\end{array}\end{array}\right)\,. \tag{A.1.10b}$$

Now we need to compute the mass-Lagrangian which is given by

$$\mathcal{L}_M \equiv \mathrm{Tr}\Big[(D_\mu\langle\Omega_1\rangle)^\dagger D_\mu\langle\Omega_1\rangle\Big]+\mathrm{Tr}\Big[(D_\mu\langle\Omega_3\rangle)^\dagger D_\mu\langle\Omega_3\rangle\Big]\,. \tag{A.1.11}$$

Using that $\mathrm{Tr}\Big[T^aT^b\Big]=\delta^{ab}/2$, $\mathrm{Tr}[\mathbb{I}_{3\times3}]=3$ and $\mathrm{Tr}[T^a]=0$ we find

$$\mathcal{L}_M = \frac{g_4^2}{2}\omega_3^2\sum_{a=1}^{8}\left(H^a-\frac{g_3}{g_4}C^a\right)^2+\frac{g_4^2}{4}\left(\omega_1^2+\omega_3^2\right)\sum_{a=9}^{14}(H^a)^2+\frac{g_4^2}{4}\left(\omega_1^2+\frac{\omega_3^2}{3}\right)\left(\frac{\sqrt{6}}{2}H^{15}-2\frac{g_X}{g_4}X_\Omega\,B\right)^2\,. \tag{A.1.12}$$

To see which is the mass-basis we need to diagonalize the X' operator. In particular we need to study the transformation properties of H^a. We know that

$$H\to H-i\theta^{15}\,H^a[\hat{T}^{15},\hat{T}^a]\qquad\Longrightarrow\qquad H^a\to H^a+\frac{2\sqrt{6}}{3}X_\Omega\,f^{a\,b\,15}H^b\,. \tag{A.1.13}$$

Using this relation it turns out that under a $\mathrm{U}(1)_{X'}$ transformation $H\to(\mathbb{I}-i\beta\,\mathbb{X}')\,H$ where

$$\mathbb{X}'=\mathrm{diag}\left(\mathbb{O}_{8\times8},\mathbb{X}_r,\mathbb{X}_b,\mathbb{X}_g,0\right)\qquad\text{and}\qquad\mathbb{X}_r=\mathbb{X}_b=\mathbb{X}_g=\frac{4}{3}X_\Omega\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\,. \tag{A.1.14}$$

Hence after diagonalizing it we find that the Gauge boson basis reads (where $a=1,...,8$)

$$G^a = s_{43}H^a+c_{43}C^a\qquad m_G^2=0\,, \tag{A.1.15a}$$

$$G'^a = c_{43}H^a-s_{43}C^a\qquad m_{G'}^2=\omega_3^2\left(g_3^2+g_4^2\right)\,, \tag{A.1.15b}$$

$$\begin{aligned} U_r^\pm &= \frac{1}{\sqrt{2}}\left(H^9\mp iH^{10}\right)\\ U_b^\pm &= \frac{1}{\sqrt{2}}\left(H^{11}\mp iH^{12}\right)\qquad m_U^2=\frac{g_4^2}{2}\left(\omega_1^2+\omega_3^2\right)\,,\\ U_g^\pm &= \frac{1}{\sqrt{2}}\left(H^{13}\mp iH^{14}\right)\end{aligned} \tag{A.1.15c}$$

$$B' = s_{4X}H^{15}+c_{4X}B\qquad m_{B'}^2=0\,, \tag{A.1.15d}$$

$$Z' = c_{4X}H^{15}-s_{4X}B\qquad m_{Z'}^2=\frac{1}{2}\left(\omega_1^2+\frac{\omega_3^2}{3}\right)\left(\hat{g}_4^2+4\,g_X^2X_\Omega^2\right) \tag{A.1.15e}$$

where we have defined using $\hat{g}_4=\sqrt{6}/2\,g_4$

$$c_{43}=\frac{g_4}{\sqrt{g_4^2+g_3^2}}\,,\quad s_{43}=\frac{g_3}{\sqrt{g_4^2+g_3^2}}\,,\quad c_{4X}=\frac{\hat{g}_4}{\sqrt{\hat{g}_4^2+4\,g_X^2X_\Omega^2}}\,,\quad s_{4X}=\frac{2\,g_X X_\Omega}{\sqrt{\hat{g}_4^2+4\,g_X^2X_\Omega^2}} \tag{A.1.16}$$

and (A.1.12) becomes

$$\mathcal{L}_M=\frac{1}{2}m_{G'}^2\sum_{a=1}^{8}(G'^a)^2+m_U^2\sum_{c=r,b,g}U_c^+U_c^-+\frac{1}{2}m_{Z'}^2(Z')^2\,. \tag{A.1.17}$$

The massive boson fields have Charges $U_c^{\pm} \sim (\mathbf{3}, \pm 4/3\, X_\Omega)$ (where $c = r, b, g$), $G'^a \sim (\mathbf{8}, 0)$ and $Z' \sim (\mathbf{1}, 0)$ under the residual Gauge Group.

By computing the covariant derivative in the broken phase we find the IR gauge couplings written in terms of UV couplings which are given by

$$g_3' = \frac{g_4 g_3}{\sqrt{g_4^2 + g_3^2}}\,, \qquad g_X' = \frac{2\,\hat{g}_4 g_X X_\Omega}{\sqrt{\hat{g}_4^2 + 4\, g_X^2 X_\Omega^2}}\,. \tag{A.1.18}$$

The covariant derivative in the broken phase reads as follows

$$\begin{aligned} D_\mu = \partial_\mu - i g_3' T^a G^a - i g_X' X' B' - i g_4 \sum_{c=r,b,g} \left(\hat{T}_c^+ U_c^+ + \hat{T}_c^- U_c^- \right) \\ - i \frac{g_4}{2 c_{43}} \left(c_{43}^2 - s_{43}^2 \right) T^a G'^a - i \frac{\hat{g}_4}{c_{4X}} \left[\frac{\sqrt{6}}{3} \hat{T}^{15} - s_{4X}^2 X' \right] Z' \end{aligned} \tag{A.1.19}$$

where we have defined

$$\hat{T}_r^{\pm} = \frac{1}{\sqrt{2}} \left(\hat{T}^9 \pm i \hat{T}^{10} \right)\,, \quad \hat{T}_b^{\pm} = \frac{1}{\sqrt{2}} \left(\hat{T}^{11} \pm i \hat{T}^{12} \right)\,, \quad \hat{T}_g^{\pm} = \frac{1}{\sqrt{2}} \left(\hat{T}^{13} \pm i \hat{T}^{14} \right)\,. \tag{A.1.20}$$

A.2 21 into 1 Model

We want to study the SSB pattern

$$\mathrm{SU}(2) \times \mathrm{U}(1)_X \quad \to \quad \mathrm{U}(1)_{X'} \tag{A.2.1}$$

and we define the Gauge couplings and fields associated to the UV and IR theories (with the SU(2) indices $i = 1, 2, 3$)

$$(g_2, g_X)\,, \quad \left(W^i, B \right) \qquad \to \qquad g_X'\,, \quad B'\,. \tag{A.2.2}$$

The covariant derivative in the unbroken phase reads as follows

$$D_\mu = \partial_\mu - i g_2 T^i W^i - i g_X X B\,. \tag{A.2.3}$$

We assume the following field charged under the full Symmetry Group

$$\Sigma \sim (\mathbf{2}, X_\Sigma)\,. \tag{A.2.4}$$

Its VEV is chosen to be

$$\langle \Sigma \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}\,. \tag{A.2.5}$$

Now we need to find the residual Gauge Group. To leave the VEV invariant we need to satisfy

$$\left(\alpha^i T^i + \beta X_\Sigma \right) \langle \Sigma \rangle = 0\,. \tag{A.2.6}$$

Thus we need a Gauge transformation where we keep free β and we fix

$$\alpha^1 = \alpha^2 = 0\,; \qquad \alpha^3 = 2 X_\Sigma \beta\,. \tag{A.2.7}$$

Therefore the residual Gauge Group is given by $\mathrm{U}(1)_{X'}$ where the generator is given by

$$X' = X + 2 X_\Sigma\, T^3\,. \tag{A.2.8}$$

To find the new Gauge boson basis and their masses we need to write down the kinetic term of Σ evaluated on the VEV. To do so we need its covariant derivatives which reads as follows

$$D_\mu\langle\Sigma\rangle = -i\frac{v}{2}\begin{pmatrix} g_2\left(W^1 - iW^2\right) \\ 2g_X X_\Sigma B - g_2 W^3\end{pmatrix} . \tag{A.2.9}$$

Now we need to compute the mass-Lagrangian which is given by

$$\mathcal{L}_M \equiv \mathrm{Tr}\Big[(D_\mu\langle\Sigma\rangle)^\dagger D_\mu\langle\Sigma\rangle\Big] = \frac{g_2^2}{4}v^2\left[(W^1)^2 + (W^2)^2\right] + \frac{g_2^2}{4}v^2\left(W^3 - 2\frac{g_X}{g_2}X_\Sigma B\right)^2 . \tag{A.2.10}$$

To see which is the mass-basis we need to diagonalize the X' operator. It turns out that under a $\mathrm{U}(1)_{X'}$ transformation $W \to (\mathbb{I} - i\beta\mathbb{X}')W$ where

$$\mathbb{X}' = \mathrm{diag}\left(\mathbb{X}_{12}, 0\right) \qquad \text{and} \qquad \mathbb{X}_{12} = 2X_\Sigma\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} . \tag{A.2.11}$$

Hence after diagonalizing it we find that the Gauge boson basis reads

$$W^\pm = \frac{1}{\sqrt{2}}\left(W^1 \mp iW^2\right) \qquad m_W^2 = \frac{g_2^2}{2}v^2 , \tag{A.2.12a}$$

$$B' = c_{2X}B + s_{2X}W^3 \qquad m_{B'}^2 = 0 , \tag{A.2.12b}$$

$$Z' = -s_{2X}B + c_{2X}W^3 \qquad m_{Z'}^2 = \frac{1}{2}v^2\left(g_2^2 + 4\,g_X'^2X_\Sigma^2\right) \tag{A.2.12c}$$

where we have defined

$$c_{2X} = \frac{g_2}{\sqrt{g_2^2 + 4g_X^2X_\Sigma^2}} , \quad s_{2X} = \frac{2g_X X_\Sigma}{\sqrt{g_2^2 + 4g_X^2X_\Sigma^2}} \tag{A.2.13}$$

and (A.2.10) becomes

$$\mathcal{L}_M = m_W^2 W^+W^- + \frac{1}{2}m_{Z'}^2(Z')^2 . \tag{A.2.14}$$

The massive boson fields have Charges $W^\pm \sim \pm 2X_\Sigma$ and $Z' \sim 0$ under the residual Gauge Group.

By computing the covariant derivative in the broken phase we find the IR gauge coupling written in terms of UV couplings which is given by

$$g_X' = \frac{2\,g_2 g_X X_\Sigma}{\sqrt{g_2^2 + 4\,g_X^2X_\Sigma^2}} . \tag{A.2.15}$$

The covariant derivative in the broken phase reads as follows

$$D_\mu = \partial_\mu - ig_X'X'B' - ig_2\left(T^+W^+ + T^-W^-\right) - i\frac{g_2}{c_{2X}}\left[T^3 - s_{2X}^2X'\right]Z' \tag{A.2.16}$$

where we have defined

$$T^\pm = \frac{1}{\sqrt{2}}\left(T^1 \pm iT^2\right) . \tag{A.2.17}$$

A.3 42 into 31 Model

We want to study the SSB pattern

$$\mathrm{SU}(4)\times\mathrm{SU}(2) \quad \to \quad \mathrm{SU}(3)\times\mathrm{U}(1)_X \tag{A.3.1}$$

and we define the Gauge couplings and fields associated to the UV and IR theories (with the $\mathrm{SU}(N)$ indices $a = 1, ..., 15$, $b = 1, ..., 8$, $i = 1, 2, 3$)

$$(g_4, g_2) , \quad \left(H^a, W^i\right) \qquad \to \qquad (g_3, g_X) , \quad \left(G^b, B\right) . \tag{A.3.2}$$

The covariant derivative in the unbroken phase reads as follows

$$D_\mu = \partial_\mu - ig_4 \hat{T}^a H^a - ig_2 T^i W^i \,. \tag{A.3.3}$$

We assume the following field charged under the full Symmetry Group

$$\Delta \sim (\overline{\mathbf{4}}, \mathbf{2}) \,. \tag{A.3.4}$$

Its VEV is chosen to be

$$\langle \Delta \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v \end{pmatrix} \,. \tag{A.3.5}$$

Now we need to find the residual Gauge Group. To leave the VEV invariant we need to satisfy

$$\theta^a \langle \Delta \rangle \hat{T}^a - \alpha^i T^i \langle \Delta \rangle = 0 \,. \tag{A.3.6}$$

We have that

$$\langle \Delta \rangle \hat{T}^a = 0 \quad \text{for} \quad a = 1, ..., 8 \quad \text{and} \quad \langle \Delta \rangle = -\frac{2\sqrt{6}}{3} \langle \Delta \rangle \hat{T}^{15} = -2\, T^3 \langle \Delta \rangle \,. \tag{A.3.7}$$

Thus it represents a Gauge transformation where we keep free α^3 and θ^a for $a = 1, ..., 8$ and we fix

$$\theta^{15} = \frac{\sqrt{6}}{3} \alpha^3 \,; \qquad \theta^a = 0 \quad \text{for} \quad a = 9, ..., 14 \,; \qquad \alpha^1 = \alpha^2 = 0 \,. \tag{A.3.8}$$

Therefore the residual Gauge Group is given by $\mathrm{SU}(3) \times \mathrm{U}(1)_X$ and the generator of U(1) is given by

$$X = T^3 + \frac{\sqrt{6}}{3} \hat{T}^{15} \,. \tag{A.3.9}$$

To find the new Gauge boson basis and their masses we need to write down the kinetic term of Δ evaluated on the VEV. To do so we need its covariant derivative which reads as follows

$$D_\mu \langle \Delta \rangle = i \frac{g_4}{2} v \begin{pmatrix} 0 & 0 & 0 & -\frac{g_2}{g_4}\left(W^1 - iW^2\right) \\ H^9 + iH^{10} & H^{11} + iH^{12} & H^{13} + iH^{14} & -\frac{\sqrt{6}}{2} H^{15} + \frac{g_2}{g_4} W^3 \end{pmatrix} \,. \tag{A.3.10}$$

Now we need to compute the mass-Lagrangian which is given by

$$\mathcal{L}_M \equiv \mathrm{Tr}\Big[(D_\mu \langle \Delta \rangle)^\dagger D_\mu \langle \Delta \rangle \Big] = \frac{g_4^2}{4} v^2 \sum_{a=9}^{14} (H^a)^2 + \frac{g_2^2}{4} v^2 \left[(W^1)^2 + (W^2)^2 \right] + \frac{g_4^2}{4} v^2 \left(\frac{\sqrt{6}}{2} H^{15} - \frac{g_2}{g_4} W^3 \right)^2 \,. \tag{A.3.11}$$

In the same way as done in Sections A.1 and A.2, we need to diagonalize X to find the new Gauge boson basis.

After diagonalizing it we find that the Gauge boson basis reads (where $a = 1, ..., 8$)

$$G^a = H^a \qquad m_G^2 = 0 \,, \tag{A.3.12a}$$

$$\begin{aligned} U_r^\pm &= \frac{1}{\sqrt{2}} \left(H^9 \mp iH^{10} \right) \\ U_b^\pm &= \frac{1}{\sqrt{2}} \left(H^{11} \mp iH^{12} \right) \qquad m_U^2 = \frac{g_4^2}{2} v^2 \,, \\ U_g^\pm &= \frac{1}{\sqrt{2}} \left(H^{13} \mp iH^{14} \right) \end{aligned} \tag{A.3.12b}$$

$$W^\pm = \frac{1}{\sqrt{2}} \left(W^1 \mp iW^2 \right) \qquad m_W^2 = \frac{g_2^2}{2} v^2 \,, \tag{A.3.12c}$$

$$B = c_{42} W^3 + s_{42} H^{15} \qquad m_B^2 = 0 \,, \tag{A.3.12d}$$

$$Z' = -s_{42} W^3 + c_{42} H^{15} \qquad m_{Z'}^2 = \frac{1}{2} v^2 \left(\hat{g}_4^2 + g_2^2 \right) \tag{A.3.12e}$$

where we have defined using $\hat{g}_4 = \sqrt{6}/2\, g_4$

$$c_{42} = \frac{\hat{g}_4}{\sqrt{\hat{g}_4^2 + g_2^2}}\,, \quad s_{42} = \frac{g_2}{\sqrt{\hat{g}_4^2 + g_2^2}} \tag{A.3.13}$$

and (A.3.11) becomes

$$\mathcal{L}_M = m_U^2 \sum_{c=r,b,g} U_c^+ U_c^- + m_W^2 W^+ W^- + \frac{1}{2} m_{Z'}^2 (Z')^2\,. \tag{A.3.14}$$

The massive boson fields have Charges $U_c^\pm \sim (\mathbf{3}, \pm 2/3)$ (where $c = r, b, g$), $W^\pm \sim (\mathbf{1}, \pm 1)$ and $Z' \sim (\mathbf{1}, 0)$ under the residual Gauge Group.

By computing the covariant derivative in the broken phase we find the IR Gauge couplings written in terms of UV couplings which are given by

$$g_3 = g_4\,, \qquad g_X = \frac{\hat{g}_4 g_2}{\sqrt{\hat{g}_4^2 + g_2^2}}\,. \tag{A.3.15}$$

The covariant derivative in the broken phase reads as follows

$$\begin{aligned} D_\mu = \partial_\mu - i g_3 T^b G^b - i g_X X B - i g_2 \left(T^+ W^+ + T^- W^- \right) \\ - i g_4 \sum_{c=r,b,g} \left(\hat{T}_c^+ U_c^+ + \hat{T}_c^- U_c^- \right) - i \frac{\hat{g}_4}{c_{42}} \left[\frac{\sqrt{6}}{3} \hat{T}^{15} - s_{42}^2 X \right] Z'\,. \end{aligned} \tag{A.3.16}$$

A.4 41 into 31 Model

We want to study the SSB pattern

$$\mathrm{SU}(4) \times \mathrm{U}(1)_X \quad \to \quad \mathrm{SU}(3) \times \mathrm{U}(1)_{X'} \tag{A.4.1}$$

and we define the Gauge couplings and fields associated to the UV and IR theories (with the SU(N) indices $a = 1, ..., 15$, $b = 1, ..., 8$)

$$(g_4, g_X)\,, \quad (H^a, B) \qquad \to \qquad (g_3, g'_X)\,, \quad \left(G^b, B' \right)\,. \tag{A.4.2}$$

The covariant derivative in the unbroken phase reads as follows

$$D_\mu = \partial_\mu - i g_4 \hat{T}^a H^a - i g_X X B\,. \tag{A.4.3}$$

We assume the following field charged under the full Symmetry Group

$$\Delta \sim (\overline{\mathbf{4}}, -X_\Delta)\,. \tag{A.4.4}$$

Its VEV is chosen to be

$$\langle \Delta \rangle = \begin{pmatrix} 0 & 0 & 0 & v \end{pmatrix}\,. \tag{A.4.5}$$

Now we need to find the residual Gauge Group. To leave the VEV invariant we need to satisfy

$$\theta^a \langle \Delta \rangle \hat{T}^a + \beta X_\Delta \langle \Delta \rangle = 0\,. \tag{A.4.6}$$

We have that

$$\langle \Delta \rangle \hat{T}^a = 0 \quad \text{for} \quad a = 1, ..., 8 \quad \text{and} \quad \langle \Delta \rangle = -\frac{2\sqrt{6}}{3} \langle \Delta \rangle \hat{T}^{15}\,. \tag{A.4.7}$$

Thus it represents a Gauge transformation where we keep free β and θ^a for $a = 1, ..., 8$ and we fix

$$\theta^{15} = \frac{2\sqrt{6}}{3} X_\Delta \beta \, ; \qquad \theta^a = 0 \quad \text{for} \quad a = 9, ..., 14 \, . \tag{A.4.8}$$

Therefore the residual Gauge Group is given by $\mathrm{SU}(3) \times \mathrm{U}(1)_X$ and the generator of U(1) is given by

$$X' = X + \frac{2\sqrt{6}}{3} X_\Delta \hat{T}^{15} \, . \tag{A.4.9}$$

To find the new Gauge boson basis and their masses we need to write down the kinetic term of Δ evaluated on the VEV. To do so we need its covariant derivative which reads as follows

$$D_\mu \langle \Delta \rangle = i \frac{g_4}{2} v \begin{pmatrix} H^9 + iH^{10} & H^{11} + iH^{12} & H^{13} + iH^{14} & -\frac{\sqrt{6}}{2} H^{15} + 2X_\Delta \frac{g_X}{g_4} B \end{pmatrix} \, . \tag{A.4.10}$$

Now we need to compute the mass-Lagrangian which is given by

$$\mathcal{L}_M \equiv \mathrm{Tr}\Big[(D_\mu \langle \Delta \rangle)^\dagger D_\mu \langle \Delta \rangle \Big] = \frac{g_4^2}{4} v^2 \sum_{a=9}^{14} (H^a)^2 + \frac{g_2^2}{4} v^2 \left[(W^1)^2 + (W^2)^2 \right] + \frac{g_4^2}{4} v^2 \left(\frac{\sqrt{6}}{2} H^{15} - 2X_\Delta \frac{g_X}{g_4} B \right)^2 . \tag{A.4.11}$$

In the same way as done in Sections A.1 and A.2, we need to diagonalize X to find the new Gauge boson basis.

After diagonalizing it we find that the Gauge boson basis reads (where $a = 1, ..., 8$)

$$G^a = H^a \qquad m_G^2 = 0 \, , \tag{A.4.12a}$$

$$\begin{aligned} U_r^\pm &= \frac{1}{\sqrt{2}} \left(H^9 \mp iH^{10} \right) \\ U_b^\pm &= \frac{1}{\sqrt{2}} \left(H^{11} \mp iH^{12} \right) \qquad m_U^2 = \frac{g_4^2}{2} v^2 \, , \\ U_g^\pm &= \frac{1}{\sqrt{2}} \left(H^{13} \mp iH^{14} \right) \end{aligned} \tag{A.4.12b}$$

$$W^\pm = \frac{1}{\sqrt{2}} \left(W^1 \mp iW^2 \right) \qquad m_W^2 = \frac{g_2^2}{2} v^2 \, , \tag{A.4.12c}$$

$$B = c_{42} W^3 + s_{42} H^{15} \qquad m_B^2 = 0 \, , \tag{A.4.12d}$$

$$Z' = -s_{42} W^3 + c_{42} H^{15} \qquad m_{Z'}^2 = \frac{1}{2} v^2 \left(\hat{g}_4^2 + 4X_\Delta^2 g_X^2 \right) \tag{A.4.12e}$$

where we have defined using $\hat{g}_4 = \sqrt{6}/2 \, g_4$

$$c_{42} = \frac{\hat{g}_4}{\sqrt{\hat{g}_4^2 + 4X_\Delta^2 g_X^2}} \, , \qquad s_{42} = \frac{2X_\Delta g_X}{\sqrt{\hat{g}_4^2 + 4X_\Delta^2 g_X^2}} \tag{A.4.13}$$

and (A.4.11) becomes

$$\mathcal{L}_M = m_U^2 \sum_{c=r,b,g} U_c^+ U_c^- + m_W^2 W^+ W^- + \frac{1}{2} m_{Z'}^2 (Z')^2 \, . \tag{A.4.14}$$

The massive boson fields have Charges $U_c^\pm \sim (\mathbf{3}, \pm 2/3)$ (where $c = r, b, g$), $W^\pm \sim (\mathbf{1}, \pm 1)$ and $Z' \sim (\mathbf{1}, 0)$ under the residual Gauge Group.

By computing the covariant derivative in the broken phase we find the IR Gauge couplings written in terms of UV couplings which are given by

$$g_3 = g_4 \, , \qquad g_X' = \frac{2X_\Delta \hat{g}_4 g_X}{\sqrt{\hat{g}_4^2 + 4X_\Delta^2 g_X^2}} \, . \tag{A.4.15}$$

The covariant derivative in the broken phase reads as follows

$$D_\mu = \partial_\mu - ig_3 T^b G^b - ig_X' X' B' - ig_4 \sum_{c=r,b,g} \left(\hat{T}_c^+ U_c^+ + \hat{T}_c^- U_c^- \right) - i \frac{\hat{g}_4}{c_{42}} \left[\frac{2\sqrt{6}}{3} X_\Delta \hat{T}^{15} - s_{42}^2 X' \right] Z' \, . \tag{A.4.16}$$

A.5 3311 into 31 Model

We want to study the SSB pattern

$$\mathrm{SU}(3)_h \times \mathrm{SU}(3)_\ell \times \mathrm{U}(1)_{X_h} \times \mathrm{U}(1)_{X_\ell} \quad \to \quad \mathrm{SU}(3) \times \mathrm{U}(1)_X \tag{A.5.1}$$

and we define the Gauge couplings and fields associated to the UV and IR theories (with SU(3) indices $a = 1, ..., 8$))

$$\left(g_3^h, g_3^\ell, g_X^h, g_X^\ell\right) , \quad (C_h^a, C_\ell^a, B_h, B_\ell) \qquad \to \qquad (g_3, g_X) , \quad (G^a, B) . \tag{A.5.2}$$

The covariant derivative in the unbroken phase reads as follows

$$D_\mu = \partial_\mu - ig_3^h T_h^a C_h^a - ig_3^\ell T_\ell^a C_\ell^a - ig_X^h X_h B_h - ig_X^\ell X_\ell B_\ell . \tag{A.5.3}$$

We assume the following fields charged under the full Symmetric Group[31]

$$\Omega_1 \sim (\mathbf{1}, \mathbf{1}, X_1, -\eta X_1) , \qquad \Omega_3 \sim (\overline{\mathbf{3}}, \mathbf{3}, X_3, -\eta X_3) . \tag{A.5.4}$$

Notice that the Charges under U(1) have to be related one to the other to get a residual U(1) Symmetry after the SSB occurs. Their VEVs are chosen to be

$$\langle \Omega_1 \rangle = \omega_1 , \qquad \langle \Omega_3 \rangle = \begin{pmatrix} \omega_3 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} . \tag{A.5.5}$$

Now we need to find the residual Gauge Group. To leave the two VEVs invariant we need to satisfy

$$-\beta_h X_1 \langle \Omega_1 \rangle + \beta_\ell \eta X_1 \langle \Omega_1 \rangle = 0 , \qquad \eta_h^a \langle \Omega_3 \rangle T_h^a - \eta_\ell^b T_\ell^b \langle \Omega_3 \rangle - \beta_h X_3 \langle \Omega_3 \rangle + \beta_\ell \eta X_3 \langle \Omega_3 \rangle = 0 . \tag{A.5.6}$$

Thus we need a Gauge transformation where we keep free β_h and η_h^a for $a = 1, ..., 8$ and we fix

$$\eta_\ell^a = \eta_h^a \quad \text{for} \quad a = 1, ..., 8 ; \qquad \beta_\ell = \beta_h / \eta . \tag{A.5.7}$$

Therefore the residual Gauge Group is given by $\mathrm{SU}(3) \times \mathrm{U}(1)_X$ where the U(1) generator is given by

$$X = X_h + \frac{1}{\eta} X_\ell . \tag{A.5.8}$$

To find the new Gauge boson basis and their masses we need to write down the kinetic terms of Ω_1 and Ω_3 evaluated on the VEV. To do so we need their covariant derivatives which read as follows

$$D_\mu \langle \Omega_1 \rangle - i\omega_1 g_X^h X_1 \left(B_h - \eta \frac{g_X^\ell}{g_X^h} B_\ell \right) , \tag{A.5.9a}$$

$$D_\mu \langle \Omega_3 \rangle = i\omega_3 g_3^h \left(C_h^a - \frac{g_3^\ell}{g_3^h} C_\ell^a \right) T^a - i\omega_3 g_X^h X_3 \left(B_h - \eta \frac{g_X^\ell}{g_X^h} B_\ell \right) \mathbb{I}_{3\times 3} . \tag{A.5.9b}$$

Now we need to compute the mass-Lagrangian which is given by

$$\mathcal{L}_M \equiv \mathrm{Tr}\Big[(D_\mu \langle \Omega_1 \rangle)^\dagger D_\mu \langle \Omega_1 \rangle \Big] + \mathrm{Tr}\Big[(D_\mu \langle \Omega_3 \rangle)^\dagger D_\mu \langle \Omega_3 \rangle \Big] . \tag{A.5.10}$$

Using that $\mathrm{Tr}\Big[T^a T^b\Big] = \delta^{ab}/2$, $\mathrm{Tr}[\mathbb{I}_{3\times 3}] = 3$ and $\mathrm{Tr}[T^a] = 0$ we find that

$$\mathcal{L}_M = \frac{(g_3^h)^2}{2} \omega_3^2 \sum_{a=1}^{8} \left(C_h^a - \frac{g_3^\ell}{g_3^h} C_\ell^a \right)^2 + (g_X^h)^2 \left(\omega_1^2 X_1^2 + 3\, \omega_3^2 X_3^2 \right) \left(B_h - \eta \frac{g_X^\ell}{g_X^h} B_\ell \right)^2 . \tag{A.5.11}$$

[31] In principle it is enough just Ω_3; however we include also Ω_1.

Hence after diagonalizing it we find that the new Gauge boson basis reads (where $a = 1, ..., 8$)

$$G^a = s_3 H^a + c_3 C^a \qquad m_G^2 = 0\,, \tag{A.5.12a}$$

$$G'^a = c_3 H^a - s_3 C^a \qquad m_{G'}^2 = \omega_3^2 \left[(g_3^h)^2 + (g_3^\ell)^2\right], \tag{A.5.12b}$$

$$B = s_X B_h + c_X B_\ell \qquad m_B^2 = 0\,, \tag{A.5.12c}$$

$$Z' = c_X B_h - s_X B_\ell \qquad m_{Z'}^2 = 2\left(\omega_1^2 X_1^2 + 3\,\omega_3^2 X_3^2\right)\left[(g_X^h)^2 + \eta^2 (g_X^\ell)^2\right] \tag{A.5.12d}$$

where we have defined

$$c_3 = \frac{g_3^h}{\sqrt{(g_3^h)^2 + (g_3^\ell)^2}}\,, \quad s_3 = \frac{g_3^\ell}{\sqrt{(g_3^h)^2 + (g_3^\ell)^2}}\,, \quad c_X = \frac{g_X^h}{\sqrt{(g_X^h)^2 + \eta^2 (g_X^\ell)^2}}\,, \quad s_X = \frac{\eta g_X^\ell}{\sqrt{(g_X^h)^2 + \eta^2 (g_X^\ell)^2}} \tag{A.5.13}$$

and (A.5.11) reads

$$\mathcal{L}_M = \frac{1}{2} m_{G'}^2 \sum_{a=1}^{8} (G'^a)^2 + \frac{1}{2} m_{Z'}^2 (Z')^2\,. \tag{A.5.14}$$

The massive boson fields have Charges $G'^a \sim (\mathbf{8}, 0)$ and $Z' \sim (\mathbf{1}, 0)$ under the residual Gauge Group.

By computing the covariant derivative in the broken phase we find the IR Gauge couplings written in terms of UV couplings which are given by

$$g_3 = \frac{g_3^h g_3^\ell}{\sqrt{(g_3^h)^2 + (g_3^\ell)^2}}\,, \qquad g_X = \frac{\eta\, g_X^h g_X^\ell}{\sqrt{(g_X^h)^2 + \eta^2 (g_X^\ell)^2}}\,. \tag{A.5.15}$$

The covariant derivative in the broken phase reads as follows

$$D_\mu = \partial_\mu - i g_3 T^a G^a - i g_X X B - i \frac{g_3^h}{2c_3}\left(c_3^2 - s_3^2\right) T^a G'^a - i \frac{g_X^h}{c_X}\left[X_h - s_X^2 X\right] Z'\,. \tag{A.5.16}$$

A.6 2 into 1 Model

We want to study the SSB pattern

$$\mathrm{SU}(2) \quad \to \quad \mathrm{U}(1)_X \tag{A.6.1}$$

and we define the Gauge couplings and fields associated to the UV and IR theories (with SU(2) indices $i = 1, 2, 3$)

$$g_2\,, \quad W^i \qquad \to \qquad g_X\,, \quad B\,. \tag{A.6.2}$$

The covariant derivative in the unbroken phase reads as follows

$$D_\mu = \partial_\mu - i g_2 T^i W^i\,. \tag{A.6.3}$$

We assume the following field charged under the full Symmetric Group

$$\Sigma \sim 3\,. \tag{A.6.4}$$

such that it transforms under the adjoint representation of the Group. Its VEV is chosen to be

$$\langle \Sigma \rangle = \frac{v}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = v\, T^3\,. \tag{A.6.5}$$

Since by construction the only vanishing commutator between the VEV and the generators is

$$[\langle \Sigma \rangle, T^3] = 0 \tag{A.6.6}$$

we have that the residual Gauge Group is given by U(1) with generator

$$X = T^3 \,. \tag{A.6.7}$$

To find the new Gauge boson basis and their masses we need to write down the kinetic term of Σ evaluated on the VEV. To do so we need its covariant derivative which reads as follows

$$D_\mu \langle \Sigma^i \rangle = -iv g_2 \varepsilon^{ij3} W^j \,. \tag{A.6.8}$$

Now we need to compute the mass-Lagrangian which is given by

$$\mathcal{L} = \mathrm{Tr}\Big[(D_\mu \langle \Sigma^i \rangle)^\dagger (D_\mu \langle \Sigma^i \rangle)\Big] = \frac{1}{2} v^2 g_2^2 \left[(W^1)^2 + (W^2)^2\right] \,. \tag{A.6.9}$$

To see which is the mass-basis we need to diagonalize the X operator. It turns out that under a U(1) transformation $W \to (\mathbb{I} - i\beta\mathbb{X})W$ where

$$\mathbb{X} = \mathrm{diag}\,(\mathbb{X}_{12}, 0) \qquad \text{and} \qquad \mathbb{X}_{12} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.6.10}$$

Hence after diagonalizing it we find that the Gauge boson basis reads

$$W^\pm = \frac{1}{\sqrt{2}} \left(W^1 \mp i W^2\right) \qquad m_W^2 = g_2^2\, v^2 \,, \tag{A.6.11a}$$

$$B = W^3 \qquad m_B^2 = 0 \tag{A.6.11b}$$

and (A.6.9) becomes

$$\mathcal{L}_M = m_W^2 W^+ W^- \,. \tag{A.6.12}$$

The massive boson fields have Charges $W^\pm \sim \pm 1$ under the residual Gauge Group.

By computing the covariant derivative in the broken phase we find the IR Gauge coupling written in terms of UV couplings which is given by

$$g_X = g_2 \,. \tag{A.6.13}$$

The covariant derivative in the broken phase reads as follows

$$D_\mu = \partial_\mu - i g_X X B - i g_2 \left(T^+ W^+ + T^- W^-\right) \,. \tag{A.6.14}$$

A.7 11 into 1 Model

We want to study the SSB pattern

$$\mathrm{U}(1)_X \times \mathrm{U}(1)_Y \quad \to \quad \mathrm{U}(1)_Z \tag{A.7.1}$$

and we define the Gauge couplings and fields associated to the UV and IR theories

$$(g_X, g_Y) \,, \quad (B_X, B_Y) \qquad \to \qquad g_Z \,, \quad B_Z \,. \tag{A.7.2}$$

The covariant derivative in the unbroken phase reads as follows

$$D_\mu = \partial_\mu - i g_X X B_X - i g_Y Y B_Y \,. \tag{A.7.3}$$

We assume the following field charged under the full Symmetric Group

$$\Delta \sim (X_\Delta, -\eta X_\Delta) \,. \tag{A.7.4}$$

Its VEV is chosen to be

$$\langle\Delta\rangle = v\,. \tag{A.7.5}$$

Now we need to find the residual Gauge Group. To leave the VEV invariant we need to satisfy

$$-\alpha X_\Delta\langle\Delta\rangle + \beta\,\eta X_\Delta\langle\Delta\rangle = 0\,. \tag{A.7.6}$$

Thus we need a Gauge transformation where we keep free α and we fix $\beta = \alpha/\eta$. Therefore the residual Gauge Group is given by $\mathrm{U}(1)_Z$ with generator

$$Z = X + \frac{1}{\eta}Y\,. \tag{A.7.7}$$

To find the new Gauge boson basis and their masses we need to write down the kinetic term of Δ evaluated on the VEV. To do so we need its covariant derivative which reads as follows

$$D_\mu\langle\Delta\rangle = -iv\,g_X X_\Delta\left(B_X - \eta\frac{g_Y}{g_X}B_Y\right)\,. \tag{A.7.8}$$

Now we need to compute the mass-Lagrangian which is given by

$$\mathcal{L} = \mathrm{Tr}\Big[(D_\mu\langle\Delta\rangle)^\dagger(D_\mu\langle\Delta\rangle)\Big] = v^2 g_X^2 X_\Delta^2\left(B_X - \eta\frac{g_Y}{g_X}B_Y\right)^2\,. \tag{A.7.9}$$

Hence after diagonalizing it we find that the new Gauge boson basis reads

$$B_Z = s_{XY}B_X + c_{XY}B_Y \qquad m_{B_Z}^2 = 0\,, \tag{A.7.10a}$$

$$Z' = c_{XY}B_X - s_{XY}B_Y \qquad m_{Z'}^2 = 2\,v^2 X_\Delta^2\left(g_X^2 + \eta^2 g_Y^2\right) \tag{A.7.10b}$$

where we have defined

$$c_{XY} = \frac{g_X}{\sqrt{g_X^2+\eta^2 g_Y^2}}\,, \quad s_{XY} = \frac{\eta\,g_Y}{\sqrt{g_X^2+\eta^2 g_Y^2}} \tag{A.7.11}$$

and (A.7.9) becomes

$$\mathcal{L}_M = \frac{1}{2}m_{Z'}^2 Z'^2\,. \tag{A.7.12}$$

The massive boson field have Charge $Z' \sim 0$ under the residual Gauge Group.

By computing the covariant derivative in the broken phase we find the IR Gauge coupling written in terms of UV couplings which is given by

$$g_Z = \frac{\eta\,g_X g_Y}{\sqrt{g_X^2+\eta^2 g_Y^2}}\,. \tag{A.7.13}$$

The covariant derivative in the broken phase reads as follows

$$D_\mu = \partial_\mu - ig_Z Z B_Z - i\frac{g_X}{c_{XY}}\left(X - s_{XY}^2 Z\right)Z'\,. \tag{A.7.14}$$

B Scattering - Calculations

In this Chapter we compute some of the relevant processes that we considered in Chapter 5 in the framework of the model we considered. Consistently to what has been done in that Chapter, for the notation we refer to Appendix F. In particular all the relevant Feynman rules are reported in Section F.4. We also recall that, to distinguish the CKM matrix from the mixing matrix between SM charged leptons and exotic neutrinos, the first will always care indices $V_{qq'}$ where $q = u, c, t$ and $q' = d, s, b$ while the second $V_{\alpha i}$ where $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$.

B.1 Top-like Decay

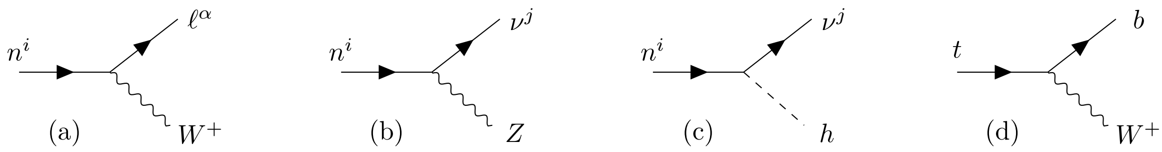


Figure 1: Feynman diagrams corresponding to n^i (a), (b), (c) and t (d) decays.

We want to compute the decay rate $n^i(p) \to W^+(k)\ell^\alpha(q)$ assuming that it is kinematically allowed. The only Feynman diagram contributing to the process is pictured in Figure 1(a). The scattering amplitude reads as follows

$$i\mathcal{M}_a = \frac{ig}{2\sqrt{2}} V_{\alpha i} \overline{u}_\alpha(q)\gamma^\mu(1-\gamma^5)u_n(p)\varepsilon^*_\mu(k)\,. \tag{B.1.1}$$

Since n^i is a Dirac-type fermion, the unpolarized squared-amplitude reads in the limit where $m_\alpha \to 0$

$$|\overline{\mathcal{M}}_a|^2 = \frac{1}{2}\sum_{\text{spin}} |\mathcal{M}_a|^2 = 2\sqrt{2}\, G_F |V_{\alpha i}|^2 \Big[M_W^2 (p\cdot q) + 2(k\cdot q)(k\cdot p) \Big]\,. \tag{B.1.2}$$

The decay rate reads from (G.3.5)

$$\Gamma = \frac{1}{2M_i}\int \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}(2\pi)^4\delta^4(p-q-k)|\overline{\mathcal{M}}_a|^2\,. \tag{B.1.3}$$

By using momentum conservation we find that

$$2\,k\cdot q = 2\,p\cdot q = M_i^2 - M_W^2\,, \qquad 2\,p\cdot k = M_i^2 + M_W^2\,. \tag{B.1.4}$$

Thus, as expected, the scattering amplitude factorizes out from the integration. Also the phase space of a decay process of type $A \to BC$ has been computed in Section E.3. Therefore we find that

$$\Gamma\left(n^i \to W\ell^\alpha\right) = \frac{G_F |V_{\alpha i}|^2 M_i^3}{8\pi\sqrt{2}}\left(1 - \frac{M_W^2}{M_i^2}\right)^2\left(1 + 2\frac{M_W^2}{M_i^2}\right) \tag{B.1.5}$$

and recall that this holds in the limit where $m_\alpha \to 0$ and when $M_i > M_W$.

Thanks to the calculation done above, we observe that we got for free the decay rate of the process $n^i(p) \to Z(k)\nu^j(q)$, represented in Figure 1(b). In fact the scattering amplitude reads as follows

$$i\mathcal{M}_b = \frac{ig}{4\,c_W}(U^\dagger V)_{ji}\overline{u}_\nu(q)\gamma^\mu(1-\gamma^5)u_n(p)\varepsilon^*_\mu(k) \tag{B.1.6}$$

which is exactly $\mathcal{M}_a$ but sending

$$V_{\alpha i} \quad \to \quad \frac{1}{\sqrt{2}\, c_W}(U^\dagger V)_{ji}\,. \tag{B.1.7}$$

Hence we find that

$$\Gamma\left(n^i \to Z\,\nu^j\right) = \frac{G_F|(U^\dagger V)_{ji}|^2 M_i^3}{16\pi\sqrt{2}}\left(1-\frac{M_Z^2}{M_i^2}\right)^2\left(1+2\frac{M_Z^2}{M_i^2}\right)\,. \tag{B.1.8}$$

The same happens for the process $t(p)\to W^+(k)b(q)$, represented in Figure 1(d). This also is the dominating decay process for the top quark, hence we will be able to compute its total decay width. The scattering amplitude reads as follows

$$i\mathcal{M}_d = \frac{ig}{2\sqrt{2}}V_{tb}^*\,\overline{u}_b(q)\gamma^\mu(1-\gamma^5)u_t(p)\varepsilon_\mu^*(k) \tag{B.1.9}$$

which is exactly $\mathcal{M}_a$ but sending

$$V_{\alpha i} \quad \to \quad V_{tb}\,. \tag{B.1.10}$$

Hence we find that

$$\Gamma_t = \frac{G_F|V_{tb}|^2 m_t^3}{8\pi\sqrt{2}}\left(1-\frac{M_W^2}{m_t^2}\right)^2\left(1+2\frac{M_W^2}{m_t^2}\right)\,. \tag{B.1.11}$$

A similar discussion can be done for the process $n^i(p)\to h(k)\nu^j(q)$, represented in Figure 1(c). The scattering amplitude reads as follows

$$i\mathcal{M}_c = -i\frac{M_i}{2v}(U^\dagger V)_{ji}\overline{u}_\nu(q)(1+\gamma^5)u_n(p)\,. \tag{B.1.12}$$

Since n^i is a Dirac-type fermion, the unpolarized squared-amplitude reads as follows

$$|\overline{\mathcal{M}}_c|^2 = \frac{1}{2}\sum_{\text{spin}}|\mathcal{M}_c|^2 = \frac{M_i^4}{2v^2}|(U^\dagger V)_{ji}|\left(1-\frac{M_h^2}{M_i^2}\right)\,. \tag{B.1.13}$$

Since the phase space is similar to the ones computed before, we end up finding that

$$\Gamma\left(n^i \to h\,\nu^j\right) = \frac{G_F|(U^\dagger V)_{ji}|^2 M_i^3}{16\pi\sqrt{2}}\left(1-\frac{M_h^2}{M_i^2}\right)^2\,. \tag{B.1.14}$$

B.2 Muon-like Decay

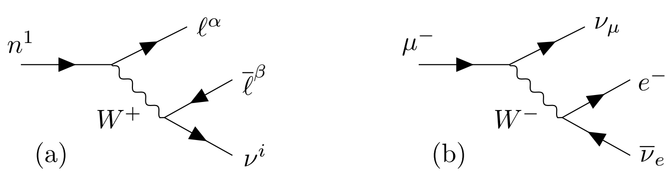


Figure 2: Feynman diagrams corresponding to n^1 (a) and μ (b) decays.

We want to compute the decay rate $n^1(k)\to\ell^\alpha(p)\overline{\ell}^\beta(q)\nu^i(r)$. The only Feynman diagram contributing to the process is pictured in Figure 2(a). The scattering amplitude reads as follows in the limit where $M_W \gg M_1$

$$i\mathcal{M}_a = -\frac{ig^2}{8M_W^2}V_{\alpha 1}U_{\beta i}^*\overline{u}_\alpha(p)\gamma_\mu(1-\gamma^5)u_n(k)\overline{u}_\nu(r)\gamma^\mu(1-\gamma^5)v_\beta(q)\,. \tag{B.2.1}$$

Since n^1 is a Dirac-type fermion, the unpolarized squared-amplitude reads as follows

$$|\overline{\mathcal{M}}_a|^2 = \frac{1}{2}\sum_{\text{spin}} |\mathcal{M}_a|^2 = 64\, G_F^2 |V_{\alpha 1} U^*_{\beta i}|^2 \Big[(p\cdot q)(r\cdot k) - m_\alpha M_1 (q\cdot r)\Big]\,. \tag{B.2.2}$$

The decay rate reads from (G.3.5)

$$\Gamma = \frac{1}{2M_1}\int \frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^4(k-p-q-r)|\overline{\mathcal{M}}_a|^2\,. \tag{B.2.3}$$

By defining

$$x_1 \equiv \frac{2\,p\cdot k}{k^2}\,,\quad x_2 \equiv \frac{2\,q\cdot k}{k^2}\,,\qquad \varepsilon_1 \equiv \frac{m_\alpha^2}{M_1^2}\,,\quad \varepsilon_2 = \frac{m_\beta^2}{M_1^2} \tag{B.2.4}$$

and using momentum conservation, we find that

$$|\overline{\mathcal{M}}_a|^2 = \frac{M_1^4}{4} f(x_1,x_2) \tag{B.2.5}$$

where

$$f(x_1,x_2) = (x_1+x_2-1-\varepsilon_1-\varepsilon_2)\,(2-x_1-x_2) + 2\sqrt{\varepsilon_1}(1-x_1+\varepsilon_1-\varepsilon_2)\,. \tag{B.2.6}$$

The phase space of a 3-body decay has been computed in Section E.4. Therefore we find that

$$\Gamma\left(n^1 \to \ell^\alpha \overline{\ell}^\beta \nu^i\right) = \frac{G_F^2 |V_{\alpha 1}U^*_{\beta i}|^2 M_1^5}{192\pi^3}\Phi \tag{B.2.7}$$

where

$$\Phi = 12\int_{x_1^{min}}^{x_1^{max}} dx_1 \int_{x_2^{min}}^{x_2^{max}} dx_2\, f(x_1,x_2) \tag{B.2.8}$$

and the extremes of integration are given by (E.4.16) and (E.4.18) with $\varepsilon_3 = 0$. We observe that in the limit where $\varepsilon_1 = \varepsilon_2 = 0$ we find that $\Phi = 1$.

Thanks to the calculation done above, we observe that we got for free the total decay width of the muon. This because the decay rate is completely dominated by the process $\mu^-(k) \to e^-(r)\nu_\mu(p)\overline{\nu}_e(q)$, represented in Figure 2(b). In fact the scattering amplitude reads as follows in the limit where $M_W \gg m_\mu$

$$i\mathcal{M}_b = -\frac{ig^2}{8M_W^2}\overline{u}_\nu(p)\gamma_\mu(1-\gamma^5)u_\mu(k)\overline{u}_e(r)\gamma^\mu(1-\gamma^5)v_\nu(q)\,. \tag{B.2.9}$$

Hence, in the limit where $m_e \to 0$, we have that the unpolarized squared-amplitude is given by (B.2.2) when $m_\alpha = 0$. Hence we find that

$$|\overline{\mathcal{M}}_b|^2 = \frac{1}{2}\sum_{\text{spin}} |\mathcal{M}_b|^2 = 64\, G_F^2 (p\cdot q)(r\cdot k)\,. \tag{B.2.10}$$

Since the phase space integration is the same as before but with $m_\alpha = m_\beta = 0$ and by sending $M_1 \to m_\mu$, the total decay width of the muon reads as follows

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{B.2.11}$$

since in this massless limit $\Phi = 1$.

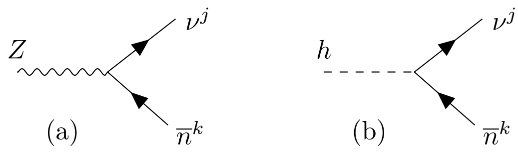


Figure 3: Feynman diagrams corresponding to Z (a) and h (b) invisible decays.

B.3 Invisible Z and Higgs Decay Widths

In this Section we compute the invisible decay widths of Z and h bosons predicted by our model. We want to compute the invisible decay rate of Z. In the SM this is given only by the process $Z \to \nu\overline{\nu}$. In the framework of the model we are considering, this gets a NP correction that we parameterize as follows

$$\Delta_{\rm NP}\Gamma(Z \to \text{inv}) \equiv \Gamma(Z \to \text{inv}) - \Gamma_{\rm SM}(Z \to \text{inv})\,. \tag{B.3.1}$$

This correction gets two different contributions: one from the decay channel $Z \to \nu\overline{n} + \overline{\nu}n$ and one from the modification in the coupling constant between Z and ν since it gains the factor $U^\dagger U \neq \mathbb{I}$. Both are of the same order since from naive dimensional analysis

$$\frac{\Gamma(Z \to \nu\overline{n} + \overline{\nu}n)}{\Gamma(Z \to \nu\overline{\nu})} \sim \frac{v^2}{v_F^2}\,, \qquad \mathbb{I} - (U^\dagger U)^2 \sim \frac{v^2}{v_F^2}\,. \tag{B.3.2}$$

However, for the simple treatment we want to do, we will need just an order of magnitude of this NP contributions. Thus the second effect will contribute only for an $\mathcal{O}(1)$ factor of correction with respect to the first one and it will not modify the main results made in Section 5.6.

We want to compute the decay rate of the channel $Z(k) \to \nu^j(q)\overline{n}^k(p)$ assuming that it is kinematically allowed. The only Feynman diagram contributing to the process is pictured in Figure 3(a). The scattering amplitude reads as follows

$$i\mathcal{M}_a = \frac{ig}{4c_W}(U^\dagger V)_{jk}\overline{u}_\nu(q)\gamma^\mu(1-\gamma^5)v_n(p)\varepsilon^*_\mu(k)\,. \tag{B.3.3}$$

Since Z is a massive vector boson, the unpolarized squared-amplitude reads as follows

$$|\overline{\mathcal{M}}_a|^2 = \frac{1}{3}\sum_{\text{spin}}|\mathcal{M}_a|^2 = \frac{2\sqrt{2}}{3}G_F|(U^\dagger V)_{jk}|^2\Big[M_Z^2(p\cdot q) + 2(k\cdot q)(k\cdot p)\Big]\,. \tag{B.3.4}$$

The decay rate reads from (G.3.5)

$$\Gamma = \frac{1}{2M_Z}\int \frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3q}{(2\pi)^3 2E_q}(2\pi)^4\delta^4(p+q-k)|\overline{\mathcal{M}}_a|^2\,. \tag{B.3.5}$$

By using momentum conservation we find that

$$2\,k\cdot q = 2\,p\cdot q = M_Z^2 - M_k^2\,, \qquad 2\,p\cdot k = M_Z^2 + M_k^2\,. \tag{B.3.6}$$

Thus, as expected, the scattering amplitude factorizes out from the integration. Also the phase space of a decay process of type $A \to BC$ has been computed in Section E.3. Therefore we find that

$$\Gamma\left(Z \to \nu^j\overline{n}^k\right) = \frac{G_F|(U^\dagger V)_{jk}|^2 M_Z^3}{12\pi\sqrt{2}}\left(1 - \frac{M_k^2}{M_Z^2}\right)^2\left(1 + \frac{M_k^2}{2\,M_Z^2}\right) \tag{B.3.7}$$

and recall that this holds when $M_Z > M_k$. Moreover this decay rate is the same also for the process $Z \to \overline{\nu}^j n^k$.

Since $UU^\dagger \approx \mathbb{I}$, we can observe that

$$\sum_{j=1}^{3} |(U^\dagger V)_{jk}|^2 = \sum_{j,\alpha=1}^{3} |U_{\alpha j}|^2 |V_{\alpha k}|^2 = \sum_{\alpha=1}^{3} (UU^\dagger)_{\alpha\alpha} |V_{\alpha k}|^2 \approx (V^\dagger V)_{kk} \tag{B.3.8}$$

Therefore we expect that (apart from minor corrections)

$$\Delta_{\rm NP}\Gamma(Z \to \text{inv}) \approx \frac{G_F M_Z^3}{6\sqrt{2}\pi} \sum_k (V^\dagger V)_{kk} \tag{B.3.9}$$

where the sum is taken over all possible kinematically allowed configurations.

We want to compute the invisible decay rate of h. In the SM this is given only by the process $h \to \nu\overline{\nu}$. However this is extremely suppressed by the small neutrino masses. Hence, in the framework of the model we are considering, we can say that the NP contribution is given by

$$\Delta_{\rm NP}\Gamma(h \to \text{inv}) \equiv \Gamma(h \to \text{inv}) - \Gamma_{\rm SM}(h \to \text{inv}) \approx \Gamma(h \to \text{inv})\,. \tag{B.3.10}$$

Also in this model the invisible decay rate of h is dominated by the decay channel $h \to \nu\overline{n} + \overline{\nu}n$.

We want to compute the decay rate of the channel $h(k) \to \nu^j(q)\overline{n}^k(p)$ assuming that it is kinematically allowed. The only Feynman diagram contributing to the process is pictured in Figure 3(b). The scattering amplitude reads as follows

$$i\mathcal{M}_b = \frac{-iM_k}{2v}(U^\dagger V)_{jk}\overline{u}_\nu(q)(1+\gamma^5)v_n(p)\,. \tag{B.3.11}$$

Since h is a real scalar boson, the unpolarized squared-amplitude reads as follows

$$|\overline{\mathcal{M}_b}|^2 = \sum_{\rm spin} |\mathcal{M}_b|^2 = \frac{2\,M_k^2}{v^2}|(U^\dagger V)_{jk}|^2 (p\cdot q)\,. \tag{B.3.12}$$

The decay rate reads from (G.3.5)

$$\Gamma = \frac{1}{2M_h}\int \frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3q}{(2\pi)^3 2E_q}(2\pi)^4\delta^4(p+q-k)|\overline{\mathcal{M}_b}|^2\,. \tag{B.3.13}$$

By using momentum conservation we find that

$$2\,p\cdot q = M_h^2 - M_k^2\,. \tag{B.3.14}$$

Thus, as expected, the scattering amplitude factorizes out from the integration. Also the phase space of a decay process of type $A \to BC$ has been computed in Section E.3. Therefore we find that

$$\Gamma\left(h \to \nu^j \overline{n}^k\right) = \frac{|(U^\dagger V)_{jk}|^2 M_k^2 M_h}{16\pi\, v^2}\left(1 - \frac{M_k^2}{M_h^2}\right)^2 \tag{B.3.15}$$

and recall that this holds when $M_h > M_k$. Moreover this decay rate is the same also for the process $h \to \overline{\nu}^j n^k$.

Therefore, using (B.3.8), we expect that (apart from minor corrections)

$$\Gamma\left(h \to \text{inv}\right) \approx \frac{M_h}{8\pi\, v^2}\sum_k M_k^2 (V^\dagger V)_{kk}\left(1 - \frac{M_k^2}{M_h^2}\right)^2 \tag{B.3.16}$$

where the sum is taken over all possible kinematically allowed configurations.

B.4 Charged Hadronic Decay

We want to compute the decay rate $n^1(k) \to \ell^\alpha(q) h_P^+$ where $h_P^+(p)$ is a positively-charged pseudoscalar meson (like π^+ or K^+). Since we have a hadron in the scattering process, we need to deal with it in the way described in Section E.2. Thus we have that the scattering amplitude reads as follows[32]

$$i\mathcal{M}_a = -\frac{G_F}{\sqrt{2}} f_h V_{\alpha 1} V_{ud}\, \overline{u}_\alpha(q) \not{p}(1-\gamma^5) u_n(k)\,. \tag{B.4.1}$$

Using momentum conservation and spinor properties we can further simplify the fermionic bilinear as follows

$$\overline{u}_\alpha(q) \not{p}(1-\gamma^5) u_n(k) = \overline{u}_\alpha(q) \left[M_1 - m_\alpha + \gamma^5 (M_1 + m_\alpha) \right] u_n(k)\,. \tag{B.4.2}$$

Since n^1 is a Dirac-type fermion, the unpolarized squared-amplitude reads as follows

$$|\overline{\mathcal{M}}_a|^2 = \frac{1}{2}\sum_{\text{spin}} |\mathcal{M}_a|^2 = G_F^2 f_h^2 |V_{ud}|^2 |V_{\alpha 1}|^2 M_1^4 \left[\left(1 - \frac{m_\alpha^2}{M_1^2}\right)^2 - \frac{m_h^2}{M_1^2}\left(1 + \frac{m_\alpha^2}{M_1^2}\right)\right] \tag{B.4.3}$$

where we have used that

$$2\, k\cdot q = M_1^2 + m_\alpha^2 - m_h^2\,. \tag{B.4.4}$$

The decay rate reads from (G.3.5)

$$\Gamma = \frac{1}{2M_1} \int \frac{d^3 q}{(2\pi)^3 2E_q} \frac{d^3 p}{(2\pi)^3 2E_p} (2\pi)^4 \delta^4(k-p-q) |\overline{\mathcal{M}}_a|^2\,. \tag{B.4.5}$$

As expected, the scattering amplitude factorizes out from the integration. Also the phase space of a decay process of type $A \to BC$ has been computed Section E.3. Therefore we find that

$$\Gamma\left(n^1 \to \ell^\alpha h_P^+\right) = \frac{G_F^2}{16\pi} f_h^2 |V_{ud}|^2 |V_{\alpha 1}|^2 M_1^3 \left[\left(1 - \frac{m_\alpha^2}{M_1^2}\right)^2 - \frac{m_h^2}{M_1^2}\left(1 + \frac{m_\alpha^2}{M_1^2}\right)\right] \lambda^{\frac{1}{2}}\left(1, \frac{m_\alpha^2}{M_1^2}, \frac{m_h^2}{M_1^2}\right) \tag{B.4.6}$$

where $\lambda(a,b,c)$ is the Källén function (E.3.11).

A similar discussion holds for the related process $h_P^-(p) \to n^1(k)\ell^\alpha(q)$. In fact the scattering amplitude reads formally the same

$$i\mathcal{M}_b = -\frac{G_F}{\sqrt{2}} f_h V_{\alpha 1} V_{ud}\, \overline{u}_\alpha(q) \not{p}(1-\gamma^5) u_n(k)\,. \tag{B.4.7}$$

The only differences are that this time

$$\overline{u}_\alpha(q) \not{p}(1-\gamma^5) u_n(k) = \overline{u}_\alpha(q) \left[M_1 + m_\alpha + \gamma^5 (M_1 - m_\alpha) \right] u_n(k) \tag{B.4.8}$$

and

$$2\, k\cdot q = m_h^2 - M_1^2 - m_\alpha^2\,. \tag{B.4.9}$$

In the end we find that

$$\Gamma\left(h_P^- \to n^1 \ell^\alpha\right) = \frac{G_F^2}{8\pi} f_h^2 |V_{ud}|^2 |V_{\alpha 1}|^2 m_h \left[M_1^2 + m_\alpha^2 - \frac{1}{m_h^2}\left(M_1^2 - m_\alpha^2\right)^2 \right] \lambda^{\frac{1}{2}}\left(1, \frac{m_\alpha^2}{m_h^2}, \frac{M_1^2}{m_h^2}\right)\,. \tag{B.4.10}$$

The same is true for the process $\tau^-(q) \to \nu^j(k)\pi^-(p)$. The scattering amplitude reads as follows

$$i\mathcal{M}_c = -\frac{G_F}{\sqrt{2}} f_\pi U_{\tau j}^* V_{ud}^*\, \overline{u}_\nu(k) \not{p}(1-\gamma^5) u_\tau\,. \tag{B.4.11}$$

[32]Here and in the following processes the CKM component V_{ud} stands for generic quark flavours that depend on $h_P^\pm$.

This time we have that

$$\overline{u}_\nu(k)\not{p}(1-\gamma^5)u_\tau(q) = m_\tau\,\overline{u}_\nu(k)(1+\gamma^5)u_\tau(q)\,. \tag{B.4.12}$$

By using momentum conservation we find

$$2\,k\cdot q = m_\tau^2 - m_\pi^2\,. \tag{B.4.13}$$

Thus by proceeding as before we end up with

$$\Gamma\left(\tau^- \to \nu^j\pi^-\right) = \frac{G_F^2}{8\pi}|V_{ud}|^2|U_{\tau j}|^2 f_\pi^2 m_\tau^3\left(1-\frac{m_\pi^2}{m_\tau^2}\right)^2\,. \tag{B.4.14}$$

B.5 Neutral Hadronic Decay

We want to compute the decay rate $n^1(k)\to\nu^j(q)h_P^0(p)$ where h_P^0 is a neutral pseudoscalar meson (like π^0 or K^0). Since we have a hadron in the scattering process, we need to deal with it in the way described in Section E.2. Thus we have that the scattering amplitude reads as follows

$$i\mathcal{M}_a = \frac{G_F}{2}f_h(U^\dagger V)_{j1}\,\overline{u}_\nu(q)\not{p}(1-\gamma^5)u_n(k)\,. \tag{B.5.1}$$

Using momentum conservation and spinor properties we can further simplify the fermionic bilinear as follows

$$\overline{u}_\alpha(q)\not{p}(1-\gamma^5)u_n(k) = M_1\,\overline{u}_\alpha(q)(1+\gamma^5)u_n(k)\,. \tag{B.5.2}$$

Since n^1 is a Dirac-type fermion, the unpolarized squared-amplitude reads

$$|\overline{\mathcal{M}}_a|^2 = \frac{1}{2}\sum_{\text{spin}}|\mathcal{M}_a|^2 = \frac{1}{4}G_F^2 f_h^2|(U^\dagger V)_{j1}|^2 M_1^4\left(1-\frac{m_h^2}{M_1^2}\right) \tag{B.5.3}$$

where we have used that

$$2\,k\cdot q = M_1^2 - m_h^2\,. \tag{B.5.4}$$

The decay rate reads from (G.3.5)

$$\Gamma = \frac{1}{2M_1}\int\frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3p}{(2\pi)^3 2E_p}(2\pi)^4\delta^4(k-p-q)|\overline{\mathcal{M}}_a|^2\,. \tag{B.5.5}$$

As expected, the scattering amplitude factorizes out from the integration. Also the phase space of a decay process of type $A\to BC$ has been computed in Section E.3. Therefore we find that

$$\Gamma\left(n^1\to\nu^j h_P^0\right) = \frac{G_F^2}{64\pi}f_h^2|(U^\dagger V)_{j1}|^2 M_1^3\left(1-\frac{m_h^2}{M_1^2}\right)^2\,. \tag{B.5.6}$$

Thanks to the calculation done above, we observe that we got for free the decay rate of the process $h_P^0(p)\to n^1(k)\nu^j(q)$. In fact the scattering amplitude reads exactly as before. The only difference is that this time

$$2\,k\cdot q = m_h^2 - M_1^2 \tag{B.5.7}$$

and the unpolarized squared-amplitude does not have the factor 1/2 to account for the spin. In the end we find that

$$\Gamma\left(h_P^0\to n^1\nu^j\right) = \frac{G_F^2}{32\pi}f_h^2|(U^\dagger V)_{j1}|^2 M_1^2 m_h\left(1-\frac{M_1^2}{m_h^2}\right)^2\,. \tag{B.5.8}$$

B.6 Decay Rate of $\mu \to e\gamma$

We want to compute the decay rate of the process $\mu(p) \to e(p')\gamma(q)$ under the assumptions of having a generic number of neutrino states ν_j with masses m_j which could be even very large in principle. A reference to this calculation can be found for instance in [17].

We observe that from Lorentz invariance the most general scattering amplitude takes the form (where $A, ..., F$ are generic scalar coefficients)

$$i\mathcal{M} = \overline{u}_e(p') \Big[(A + B\gamma^5)p^\mu + (C + D\gamma^5)\gamma^\mu + (E + F\gamma^5)p'^\mu\Big]\, u_\mu(p)\, \varepsilon^*_\mu(q) \tag{B.6.1}$$

and in particular there is no q^μ coefficient thanks to momentum conservation $p = p' + q$. Now we can use the Gordon Identity (G.2.12) to rewrite the scattering amplitude in the following form (eventually after relabelling the scalar coefficients)

$$i\mathcal{M} = \overline{u}_e(p-q) \Big[(A + B\gamma^5)i\sigma^{\mu\nu}q_\nu + (C + D\gamma^5)\gamma^\mu + (E + F\gamma^5)q^\mu\Big]\, u_\mu(p)\, \varepsilon^*_\mu(q)\,. \tag{B.6.2}$$

By imposing Ward Identity and using the on-shellness conditions of the spinors (G.2.9) we find that

$$C(m_\mu - m_e) - D\gamma^5(m_\mu + m_e) + q^2(E + F\gamma^5) = 0\,. \tag{B.6.3}$$

Since the photon is on-shell ($q^2 = 0$), we find that $C = D = 0$. Also the coefficients E and F are irrelevant because for physical polarizations we have that $q^\mu \varepsilon_\mu(q) = 0$. Thus we are left with

$$i\mathcal{M} = \overline{u}_e(p-q) \Big[(A + B\gamma^5)i\sigma^{\mu\nu}q_\nu\Big]\, u_\mu(p)\, \varepsilon^*_\mu(q)\,. \tag{B.6.4}$$

We are interested to study this process in the limit where $m_e \to 0$. Hence we have that the electron is purely LH (since the process is mediated by W exchange only). This means that we have $\overline{u}_e(p-q)(1+\gamma^5)$ at the beginning of the fermionic bilinear. This implies that $A = B$. Finally we can exploit the Gordon Identity once again to rewrite the $\sigma^{\mu\nu}q_\nu$ term as follows, leading to the final form of the scattering amplitude

$$i\mathcal{M} = \mathcal{A}\,\overline{u}_e(p-q)(1+\gamma^5)\Big[2\,p\cdot\varepsilon^*(q) - m_\mu \not{\varepsilon}^*(q)\Big]u_\mu(p) \tag{B.6.5}$$

and $\mathcal{A}$ is a scalar form-factor which is momentum-independent. This is because, thanks to momentum conservation $p = q + p'$ and the fact that all momenta are on-shell, the only independent scalar product between the momenta is

$$2\,p\cdot q = p^2 + q^2 - p'^2 = m_\mu^2 - m_e^2\,. \tag{B.6.6}$$

The precise calculation of $\mathcal{A}$ is done in the following Section.

Now we can evaluate the decay rate associated to the process considered. We have that the unpolarized squared-amplitude reads in the limit where $m_e \to 0$

$$|\overline{\mathcal{M}}|^2 = \frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 = 8m_\mu^2|\mathcal{A}|^2\Big(m_\mu^2 - p\cdot q\Big) = 4\,m_\mu^4|\mathcal{A}|^2\,. \tag{B.6.7}$$

The decay rate reads from (G.3.5)

$$\Gamma = \frac{1}{2m_\mu}\int \frac{d^3p'}{(2\pi)^3 2E_{p'}}\frac{d^3q}{(2\pi)^3 2E_q}(2\pi)^4\delta^4(p-q-p')|\overline{\mathcal{M}}|^2\,. \tag{B.6.8}$$

The phase space of a decay process of type $A \to BC$ has been computed in Section E.3. Therefore we find that

$$\Gamma\,(\mu \to e\gamma) = \frac{m_\mu^3}{4\pi}|\mathcal{A}|^2\,. \tag{B.6.9}$$

The 1-loop form-factor $\mathcal{A}$ is given by (B.7.23) while the total decay rate of the muon is given by (B.2.11). Therefore we can compute explicitly the branching ratio associated to the decay channel $\mu \to e\gamma$. It is given by

$$\mathrm{Br}(\mu \to e\gamma) \equiv \frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to e\nu\overline{\nu})} = \frac{3\alpha}{32\pi}|\delta_\nu|^2 \tag{B.6.10}$$

where

$$\delta_\nu = 2\sum_j U_{ej}U^*_{\mu j}\, g\left(\frac{m_j^2}{M_W^2}\right) \tag{B.6.11}$$

and $g(x)$ is given by (B.7.24).

B.7 1-Loop Form-Factor of $\mu \to e\gamma$

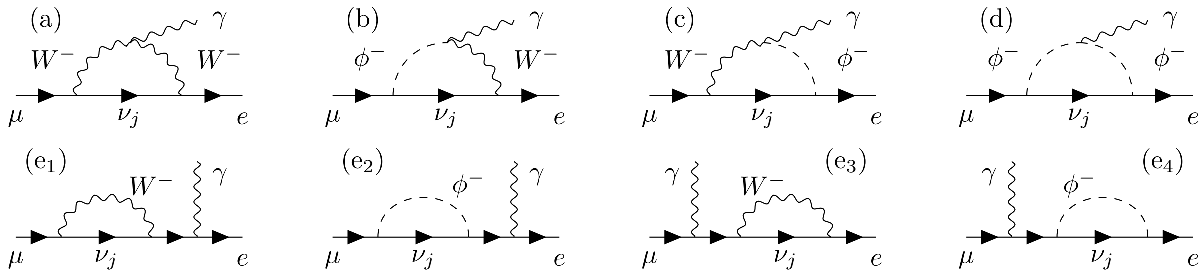


Figure 4: Feynman diagrams corresponding to the process $\mu \to e\gamma$ at 1-loop.

We recall that in the previous Section we have shown that the scattering amplitude related to the process $\mu(p) \to e(p')\gamma(q)$ reads as follows

$$i\mathcal{M} = \mathcal{A}\,\overline{u}_e(p-q)(1+\gamma^5)\Big[2\,p\cdot\varepsilon^*(q) - m_\mu \not{\varepsilon}^*(q)\Big]u_\mu(p) \tag{B.7.1}$$

and we still need to compute the scalar coefficient $\mathcal{A}$. We want to compute it at 1-loop level and we assume to work with a generic number of neutrino states ν_j with generic masses m_j. We call $U_{\alpha j}$ the generic mixing matrix between the charged lepton of flavour $\alpha = e, \mu, \tau$ and the j-th neutrino state. Also we expect that the final value will be finite since there are no counter-terms in the SM Lagrangian to possibly reabsorb any divergent behaviour.

The Feynman diagrams related to the process at 1-loop are listed in Figure 4. To find $\mathcal{A}$, we observe that we can focus only on the terms with the following Lorentz structure

$$\Gamma \equiv 2\,p\cdot\varepsilon^*(q)\,\overline{u}_e(p-q)(1+\gamma^5)u_\mu(p)\,. \tag{B.7.2}$$

This means that we can avoid calculating all the e-type diagrams since they are in the form $\overline{u}_e \not{\varepsilon}^* u_\mu$. Hence we can say that

$$\mathcal{A} = \mathcal{A}_a + \mathcal{A}_b + \mathcal{A}_c + \mathcal{A}_d \tag{B.7.3}$$

where we split the contribution coming from each of the diagrams in the first line of Figure 4. In the following we assign the internal momentum k running in the loop such that the virtual neutrino state ν_j brings momentum $p+k$. We also work in the 't Hooft Gauge $\xi = 1$. This is very helpful because in this Gauge the propagators of W^- do not have the $k^\mu k^\nu$ term in the numerator which is the only possible source of divergences in the loop integrals. Thus in this Gauge we can safely work since the beginning in $d = 4$ dimensions.

Diagram (a) gives the following scattering amplitude

$$i\mathcal{M}_a = \frac{g^2 e}{4}\sum_j U_{ej}U^*_{\mu j}\int \frac{d^4k}{(2\pi)^4}\frac{N^{\mu\nu}V_{\nu\mu}}{D} \tag{B.7.4}$$

where (recall that $q^2 = 0$ and $q \cdot \varepsilon^*(q) = 0$)

$$N^{\mu\nu} = \overline{u}_e(p-q)(1+\gamma^5)\gamma^\mu \left(\not{p} + \not{k}\right) \gamma^\nu u_\mu(p)\,, \tag{B.7.5a}$$

$$V_{\alpha\beta} = (q-k)_\beta \varepsilon^*_\alpha(q) + 2g_{\alpha\beta}\, k \cdot \varepsilon^*(q) - (2q+k)_\alpha \varepsilon^*_\beta(q)\,, \tag{B.7.5b}$$

$$D = \left[(p+k)^2 - m_j^2 + i\varepsilon\right] \left[(q+k)^2 - M_W^2 + i\varepsilon\right] \left[k^2 - M_W^2 + i\varepsilon\right]. \tag{B.7.5c}$$

To solve the loop integral, one has to introduce new parameters x, y and z such that, after using (G.3.9) we can rewrite the product of propagators as follows

$$\frac{1}{D} = 2\int_{[0,1]^3} dxdydz\, \frac{\delta(x+y+z-1)}{\mathcal{D}^3} \tag{B.7.6}$$

where

$$\mathcal{D} = (k + x\,p + y\,q)^2 - \left[(1-x)M_W^2 + x\,m_j^2\right]. \tag{B.7.7}$$

Then we perform the shift $k \to k - x\,p - y\,q$ and we notice that the only terms proportional to (B.7.2) are given by

$$N^{\mu\nu} V_{\nu\mu} \quad \to \quad -m_\mu \Gamma \left[2(1-x)^2 + y(2x-1)\right]. \tag{B.7.8}$$

Thus, using the Master Integral (G.3.6)

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - C]^3} = \frac{i}{32\pi^2}\frac{1}{C}\,, \tag{B.7.9}$$

we can perform the integration over k. Finally we can do the integrations over z and y finding that

$$\mathcal{A}_a = \frac{G_F e\, m_\mu}{16\sqrt{2}\,\pi^2} \sum_j U_{ej} U^*_{\mu j} \int_0^1 dx \frac{(1-x)^2(2x-3)}{1 - x + x\, m_j^2/M_W^2}\,. \tag{B.7.10}$$

Diagram (b) gives the following scattering amplitude

$$i\mathcal{M}_b = \frac{g^2 e}{4} \sum_j U_{ej} U^*_{\mu j} \int \frac{d^4k}{(2\pi)^4} \frac{N_b}{D} \tag{B.7.11}$$

where D is (B.7.5c) and

$$N_b = \overline{u}_e(p-q)(1+\gamma^5)\not{\varepsilon}^*(q) \left[m_j^2 - (\not{k} + m_\mu) m_\mu\right] u_\mu(p)\,. \tag{B.7.12}$$

Again we introduce parameters x, y and z and we do exactly the same steps done before. This time, after having shifted k, we have that

$$N_b \quad \to \quad y\, m_\mu \Gamma\,. \tag{B.7.13}$$

Thus we find that

$$\mathcal{A}_b = \frac{G_F e\, m_\mu}{16\sqrt{2}\,\pi^2} \sum_j U_{ej} U^*_{\mu j} \int_0^1 dx \frac{(1-x)^2}{1 - x + x\, m_j^2/M_W^2}\,. \tag{B.7.14}$$

Diagram (c) gives the following scattering amplitude

$$i\mathcal{M}_c = \frac{g^2 e}{4} \sum_j U_{ej} U^*_{\mu j} \int \frac{d^4k}{(2\pi)^4} \frac{N_c}{D} \tag{B.7.15}$$

where D is (B.7.5c) and

$$N_c = -m_j^2 \overline{u}_e(p-q)(1+\gamma^5)\not{\varepsilon}^*(q) u_\mu(p)\,. \tag{B.7.16}$$

This clearly does not produce any Lorentz structure in the form of (B.7.2) which means that

$$\mathcal{A}_c = 0\,. \tag{B.7.17}$$

Diagram (d) gives the following scattering amplitude

$$i\mathcal{M}_d = \frac{g^2 e}{4}\sum_j U_{ej}U^*_{\mu j}\int \frac{d^4k}{(2\pi)^4}\frac{N_d}{D} \tag{B.7.18}$$

where D is (B.7.5c) and

$$N_d = 2\,k\cdot\varepsilon^*(q)\frac{m_j^2}{M_W^2}\overline{u}_e(p-q)(1+\gamma^5)\not{k}u_\mu(p)\,. \tag{B.7.19}$$

Thus, after having shifted k, we have that

$$N_d \quad \to \quad m_\mu\Gamma\left[x(x+y)\frac{m_j^2}{M_W^2}\right]\,. \tag{B.7.20}$$

Thus we find that

$$\mathcal{A}_d = \frac{G_F e\, m_\mu}{16\sqrt{2}\,\pi^2}\sum_j U_{ej}U^*_{\mu j}\int_0^1 dx\frac{x(1-x)(1+x)m_j^2/M_W^2}{1-x+x\,m_j^2/M_W^2}\,. \tag{B.7.21}$$

Now we can sum all the contributions finding that the form-factor reads as follows

$$\mathcal{A} = -\frac{G_F e\, m_\mu}{16\sqrt{2}\,\pi^2}\sum_j U_{ej}U^*_{\mu j}\int_0^1 dx\frac{1-x}{1-x+x\,m_j^2/M_W^2}\left[2(1-x)(2-x)+x(1+x)\frac{m_j^2}{M_W^2}\right]\,. \tag{B.7.22}$$

If we do the integration over x we find that

$$\mathcal{A} = -\frac{G_F e\, m_\mu}{16\sqrt{2}\pi^2}\sum_j U_{ej}U^*_{\mu j}\, g\left(\frac{m_j^2}{M_W^2}\right) \tag{B.7.23}$$

where

$$g(x) = \frac{2}{3}+\frac{11}{2(1-x)}-\frac{15}{2(1-x)^2}+\frac{3}{(1-x)^3}-\frac{3\,x^3}{(1-x)^4}\ln x\,. \tag{B.7.24}$$

C Additional and Technical Analysis

In this Chapter we are going to discuss several insights that we have just mention in the main Chapters of this work. These insights could be either some technical computations, or even further model-building possibilities that turns out to be less appealing than the ones discussed, but still worth to be mentioned.

C.1 Yukawa Diagonalization

In this Section we want to diagonalize a general Yukawa matrix Y with a singular-valued decomposition. We assume for simplicity that it is a 2×2 matrix. In addition, we assume that parametrically it goes as follows where $\varepsilon \ll 1$

$$Y = \begin{pmatrix} Y_{22} & Y_{23} \\ Y_{32} & Y_{33} \end{pmatrix} \sim \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon^2 & 1 \end{pmatrix} . \tag{C.1.1}$$

This allows us to diagonalize Y perturbatively.

To do a singular-valued decomposition, we need to find two unitary matrices U_L and U_R such that

$$U_L^\dagger Y U_R = \hat{Y} = \begin{pmatrix} y_2 & 0 \\ 0 & y_3 \end{pmatrix} . \tag{C.1.2}$$

Following Section D.2, we need to diagonalize the following matrices to find U_L and U_R

$$YY^\dagger = \begin{pmatrix} |Y_{22}|^2 + |Y_{23}|^2 & Y_{22}Y_{32}^* + Y_{23}Y_{33}^* \\ Y_{32}Y_{22}^* + Y_{33}Y_{23}^* & |Y_{32}|^2 + |Y_{33}|^2 \end{pmatrix} , \quad Y^\dagger Y = \begin{pmatrix} |Y_{22}|^2 + |Y_{32}|^2 & Y_{23}Y_{22}^* + Y_{33}Y_{32}^* \\ Y_{22}Y_{23}^* + Y_{32}Y_{33}^* & |Y_{23}|^2 + |Y_{33}|^2 \end{pmatrix} . \tag{C.1.3}$$

Explicitly we have that

$$\mathrm{Tr}\Big[YY^\dagger\Big] = \mathrm{Tr}\Big[Y^\dagger Y\Big] = |Y_{22}|^2 + |Y_{23}|^2 + |Y_{32}|^2 + |Y_{33}|^2 , \tag{C.1.4a}$$

$$\det\Big[YY^\dagger\Big] = \det\Big[Y^\dagger Y\Big] = |Y_{22}|^2|Y_{33}|^2 + |Y_{23}|^2|Y_{32}|^2 - 2\,\mathrm{Re}[Y_{22}Y_{33}Y_{23}^*Y_{32}^*] . \tag{C.1.4b}$$

Given the parametric dependence (C.1.1), we have that $\det\Big[YY^\dagger\Big] \ll \mathrm{Tr}^2[YY^\dagger]$. Thus we are in the case (D.7.1) shown in Section D.7. Thus we find that the eigenvalues of $YY^\dagger$ and $Y^\dagger Y$ (which are equal) are given by

$$y_2^2 = |Y_{22}|^2 + \mathcal{O}(\varepsilon^4) , \quad y_3^2 = |Y_{33}|^2 + |Y_{23}|^2 + \mathcal{O}(\varepsilon^4) . \tag{C.1.5}$$

U_L is the matrix that diagonalizes $YY^\dagger$. Thus we find that

$$U_L = \frac{1}{|Y_{33}|^2} \begin{pmatrix} |Y_{33}|^2 - \frac{1}{2}|Y_{23}|^2 & Y_{23}Y_{33}^* \\ -Y_{33}Y_{23}^* & |Y_{33}|^2 - \frac{1}{2}|Y_{23}|^2 \end{pmatrix} + \mathcal{O}(\varepsilon^3) . \tag{C.1.6}$$

Instead U_R is the matrix that diagonalizes $Y^\dagger Y$. Thus we find that

$$U_R = \frac{1}{|Y_{33}|^2} \begin{pmatrix} |Y_{33}|^2 & Y_{23}Y_{22}^* + Y_{32}Y_{33}^* \\ -Y_{22}Y_{23}^* - Y_{33}Y_{32}^* & |Y_{33}|^2 \end{pmatrix} + \mathcal{O}(\varepsilon^3) . \tag{C.1.7}$$

Notice that parametrically we have that

$$y_2 \sim \varepsilon , \quad y_3 \sim 1 , \qquad U_L \sim \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} , \quad U_R \sim \begin{pmatrix} 1 & \varepsilon^2 \\ -\varepsilon^2 & 1 \end{pmatrix} . \tag{C.1.8}$$

Thus y_2 is suppressed with respect to y_3 and also the right mixing is suppressed with respect to the left one.

From those results we can also find the expression of V_{cb} since the CKM matrix is defined as follows

$$V \equiv (U_L^{(u)})^\dagger U_L^{(d)} \tag{C.1.9}$$

where the upper index u or d refers to the left mixing of the up- or down-type quarks respectively. After an easy computation we find that

$$V_{cb} = Y_{23}^{(d)}/Y_{33}^{(d)} - Y_{23}^{(u)}/Y_{33}^{(u)} + \mathcal{O}(\varepsilon^3)\,. \tag{C.1.10}$$

C.2 Non-Equivalently Weighted Higgs Field

In this work we have assumed that the Higgs field in the UV theory is an $\mathrm{SU}(2)_R^{[3]}$-doublet with two equally weighted components. Nevertheless nothing implies such a condition. In fact, to generate a mass splitting between top and bottom masses, it is possible to assume that the Higgs field in the UV theory is given by

$$\mathcal{H} = \frac{1}{\sqrt{1+\xi^2}}\begin{pmatrix} H^c & \xi H \end{pmatrix} \tag{C.2.1}$$

where ξ is a generic real parameter and when $\xi = 1$ we get the Higgs field definition used in this work. The normalization is chosen such that in the broken phase the kinetic term of the Higgs field is well normalized. In fact we have that

$$D_\mu \mathcal{H} = \frac{1}{\sqrt{1+\xi^2}}\begin{pmatrix} D_\mu H^c & \xi D_\mu H \end{pmatrix} \tag{C.2.2}$$

where, by looking at the covariant derivative in the broken phase (1.9.10),

$$D_\mu H = \partial_\mu H - i g_L T_L^i W_L^i H - i\frac{g_Y}{2} B\, H + \frac{i}{\xi} g_R T^+ W_R^+ H^c + D_\mu^Z H\,, \tag{C.2.3a}$$

$$D_\mu H^c = \partial_\mu H^c - i g_L T_L^i W_L^i H^c + i\frac{g_Y}{2} B\, H^c + i\xi\, g_R T^- W_R^- H + D_\mu^Z H^c\,. \tag{C.2.3b}$$

Hence the kinetic term of $\mathcal{H}$ reads as follows

$$\mathrm{Tr}\Big[(D_\mu \mathcal{H})^\dagger (D^\mu \mathcal{H})\Big] = \frac{1}{\sqrt{1+\xi^2}}\Big[(D_\mu H^c)^\dagger (D^\mu H^c) + \xi^2 (D_\mu H)^\dagger (D^\mu H)\Big]\,. \tag{C.2.4}$$

Now we observe that using properties of Pauli matrices we have that

$$(D_\mu H^c)^* = i\sigma_L^2\left[\partial_\mu H - i g_L T_L^i W_L^i H - i\frac{g_Y}{2} B\, H + i\xi\, g_R (i\sigma_L^2 T^-) W_R^+ H^c + D_\mu^Z H\right]\,. \tag{C.2.5}$$

Using this result we eventually find that, after some algebra,

$$\mathrm{Tr}\Big[(D_\mu \mathcal{H})^\dagger (D^\mu \mathcal{H})\Big] \supset (D_\mu^{\mathrm{SM}} H)^\dagger (D^{\mathrm{SM}\mu} H) \tag{C.2.6}$$

where

$$D_\mu^{\mathrm{SM}} H = \partial_\mu H - i g_L T_L^i W_L^i H - i\frac{g_Y}{2} B\, H \tag{C.2.7}$$

is the covariant derivative of the Higgs field in the SM. Hence the normalization assumed is essential to guarantee that the kinetic term of the Higgs field is properly normalized. Then ξ-dependent terms arise from the interactions between H and $W_R^\pm$ as one could expect.

With such a Higgs field in the UV theory, we have that the Yukawa matrices gets the following factors with respect to the ones derived in Sections 2.5 and 2.6

$$Y^U, Y^N \to \sqrt{\frac{2}{1+\xi^2}}\, Y^U, Y^N\,; \qquad Y^D, Y^E \to \xi\sqrt{\frac{2}{1+\xi^2}}\, Y^D, Y^E\,. \tag{C.2.8}$$

Hence we have that $Y^D/Y^U \sim \xi$, leading to a tb mass splitting while keeping at same time the ratios Y^D_{23}/Y^D_{33} and Y^U_{23}/Y^U_{33} of the same order. Thus we avoid the need of any fine tuning among the parameters that generate the Yukawa matrices (for a better discussion, see Section 2.8). We do not even need anymore the scalar field Σ_3. Nevertheless, notice that we still need some fine tuning because now

$$\frac{m_b}{m_t} \sim \xi \qquad \Longrightarrow \qquad \xi = \mathcal{O}(10^{-2})\,. \tag{C.2.9}$$

Thus we should explain why there are such different weights between the two components of the $\mathrm{SU}(2)_R^{[3]}$ doublet, which is again a fine tuning problem. Furthermore, by looking at the Yukawa matrices in this framework, such mass splitting is *naturally* present also for the light generations, but the mass ratios of m_s/m_c and m_d/m_u are not as much suppressed as the one of the third generation. Hence we would need to introduce further fine tuning to explain the non-suppressed down mass with respect the up one. For all those reasons we prefer the fine tuning among the UV parameters rather than assuming a very little ξ.

Nevertheless there is also the possibility to use a mixture of the two, so $\xi = \mathcal{O}(10^{-1})$ and a less strong fine tuning among the UV parameters. In any case the hierarchical structure of the Yukawa matrices can be explained without requiring any strong fine tuning.

C.3 Mass Splitting via Fermion Mixing

To provide a UV origin to the Yukawa matrices we have introduced some heavy VLFs which share the same quantum numbers of the third-generation fermions. This allows a further way to address the issue of mass splittings among the third-generation fermions.

For instance consider the case of the VLF η (2.3.6) that shares the same quantum numbers of χ^3_R. We can assume that a source of $\mathrm{SU}(4)^{[3]}$-breaking is induced by the following 5-dimensional operator[33]

$$-\mathcal{L} \supset -\mathcal{L}_{\eta\Omega} = M_\eta \overline{\eta}\left(1 + \frac{c^3_\Omega}{\Lambda^2}\Omega^\dagger_3\Omega_3 + \frac{c^1_\Omega}{\Lambda^2}\Omega^\dagger_1\Omega_1\right)\eta + \left(m\,\overline{\eta}\chi^3_R + h.c.\right)\,. \tag{C.3.1}$$

A possible UV origin can be easily achieved by considering a VLF with same quantum numbers of ρ (2.3.1), but much heavier than η to make sense of the EFT description. In fact in this model one must require that $\omega_{1(3)}, M_\eta \ll \Lambda$. When Ω_1 and Ω_3 acquire their VEVs, we have that the Lagrangian becomes where $\eta = (q, \ell)^T$

$$-\mathcal{L}_{\eta\Omega} \to M_\eta \overline{q}_L\left[\left(1 + c^3_\Omega\frac{\omega_3^2}{\Lambda^2}\right)q_R + \frac{m}{M_\eta}Q^3_R\right] + M_\eta\overline{\ell}_L\left[\left(1 + c^1_\Omega\frac{\omega_1^2}{\Lambda^2}\right)\ell_R + \frac{m}{M_\eta}L^3_R\right] + h.c. \tag{C.3.2}$$

The mass-matrices for the quark- and lepton-type fermions are of the type (with a and b real numbers)

$$M = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\,. \tag{C.3.3}$$

Using case (D.7.1) in Section D.7, we have the following eigen-spectrum[34]

$$\lambda_1 = 0\,, \quad v_1 = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} b \\ -a \end{pmatrix}\,; \qquad \lambda_2 = a\,, \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{C.3.4}$$

[33] Notice that the last term entails for a mass-mixing term between η and χ^3_R and it is present since they share the same quantum numbers.

[34] In principle one should diagonalize M with a singular-valued decomposition, but for the sake of this analysis this simplification works as well.

Thus we have that the true right-handed third-generation massless fermions are given by

$$Q'^3_R = -\frac{m}{\sqrt{M_\eta^2(1+c_\Omega^3\omega_3^2/\Lambda^2)^2+m^2}}q + \frac{M_\eta(1+c_\Omega^3\omega_3^2/\Lambda^2)}{\sqrt{M_\eta^2(1+c_\Omega^3\omega_3^2/\Lambda^2)^2+m^2}}Q^3_R\,, \tag{C.3.5a}$$

$$L'^3_R = -\frac{m}{\sqrt{M_\eta^2(1+c_\Omega^1\omega_1^2/\Lambda^2)^2+m^2}}\ell + \frac{M_\eta(1+c_\Omega^1\omega_1^2/\Lambda^2)}{\sqrt{M_\eta^2(1+c_\Omega^1\omega_1^2/\Lambda^2)^2+m^2}}L^3_R\,. \tag{C.3.5b}$$

The Yukawa coupling of the third generation (2.2.1) gets split as follows

$$\mathcal{L}_{Y,33} \supset c_{33}\frac{\sqrt{M_\eta^2(1+c_\Omega^3\omega_3^2/\Lambda^2)^2+m^2}}{M_\eta(1+c_\Omega^3\omega_3^2/\Lambda^2)}\overline{Q}^3_L\mathcal{H}Q'^3_R + c_{33}\frac{\sqrt{M_\eta^2(1+c_\Omega^1\omega_1^2/\Lambda^2)^2+m^2}}{M_\eta(1+c_\Omega^1\omega_1^2/\Lambda^2)}\overline{L}^3_L\mathcal{H}L'^3_R + h.c. \tag{C.3.6}$$

Therefore such $\mathrm{SU}(4)^{[3]}$-breaking operator brings a splitting in the bottom and tau Yukawa couplings. Explicitly we have that, working in the limit where $\omega_{1(3)} \ll \Lambda$,

$$\frac{y_b}{y_\tau} = \frac{m_b}{m_\tau} \approx 1 + \frac{m^2}{m^2+M_\eta^2}\frac{c_\Omega^1\omega_1^2 - c_\Omega^3\omega_3^2}{\Lambda^2},. \tag{C.3.7}$$

A similar mechanism could be exploited to explain a top-bottom mass splitting using an $\mathrm{SU}(2)_R^{[3]}$-breaking operator. In particular, if we allow the presence of the VLF λ (2.3.4), this could be done by mixing this latter VLF with χ^3_L. Nevertheless these mechanisms are not very efficient since they generate only small splittings among the Yukawa couplings of the third-generation fermions. This because the EFT description implies that Λ is a higher scale compared to the other scales present in the model. Since we need to generate mass splittings of $\mathcal{O}(1)$ (if not more), this possibility has been not considered during this work.

C.4 Single-Flavour Inverse See-Saw

In this Section we compute the mass-eigenstates that come out from a single-flavour ISS mechanism. This calculation is very standard and a possible review can be found in [9]. The mass-Lagrangian reads as follows

$$-\mathcal{L} = \frac{1}{2}\overline{N}_L M N^c_L + h.c. \tag{C.4.1}$$

where $N_L = (\nu_L, \nu^c_R, s_L)^T$ represents the three neutrino states in the flavour-basis and the mass-matrix reads as follows

$$M = \begin{pmatrix} 0 & D & 0 \\ D & m & N \\ 0 & N & \mu \end{pmatrix}\,. \tag{C.4.2}$$

To go into the mass-basis we need to diagonalize $M^\dagger M$ by means of a unitary matrix U as shown in Section D.3. In addition we assume the hierarchy $m, \mu \ll D, N$ typical of the ISS. Thus M can be seen as made of a leading part plus a small perturbation, allowing us to perform the diagonalization perturbatively. Explicitly we have that $M^\dagger M$ reads as follows

$$\begin{aligned} M^\dagger M &= M_0^2 + \Delta_I M^2 + \Delta_{II} M^2 \\ &= \begin{pmatrix} |D|^2 & 0 & D^*N \\ 0 & |D|^2+|N|^2 & 0 \\ N^*D & 0 & |N|^2 \end{pmatrix} + \begin{pmatrix} 0 & D^*m & 0 \\ m^*D & 0 & m^*N+\mu N^* \\ 0 & N^*m+\mu^*N & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & |m|^2 & 0 \\ 0 & 0 & |\mu|^2 \end{pmatrix}\,. \end{aligned} \tag{C.4.3}$$

Notice that $\Delta_I M^2$ is at first order in perturbation theory, while $\Delta_{II} M^2$ is already at second order.

The diagonalization of a matrix in perturbation theory has been discussed in Section D.1. At zero-th order we need to diagonalize M_0^2. We find as eigenvalues

$$(m_0^2)^{(0)} = 0\,, \qquad (m_1^2)^{(0)} = |D|^2 + |N|^2\,, \qquad (m_2^2)^{(0)} = |D|^2 + |N|^2 \tag{C.4.4}$$

with corresponding eigenvectors

$$v_0^{(0)} = \frac{1}{\sqrt{|D|^2+|N|^2}}\begin{pmatrix} N \\ 0 \\ -D \end{pmatrix}\,, \qquad v_1^{(0)} = \frac{1}{\sqrt{|D|^2+|N|^2}}\begin{pmatrix} D^* \\ 0 \\ N^* \end{pmatrix}\,, \qquad v_2^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\,. \tag{C.4.5}$$

The last two eigenvalues are very large and we do not need to evaluate their perturbative corrections since they will be negligible. Instead we need to find the correction to the first one which is vanishing at zero-th order. Using (D.1.14a), the correction at first order in perturbative expansion is given by

$$\Delta_I m_0^2 = \langle v_0^{(0)} | \Delta_I M^2 | v_0^{(0)} \rangle = 0\,. \tag{C.4.6}$$

Hence we need to go at second order and we find

$$\Delta_{II} m_0^2 = \langle v_0^{(0)} | \Delta_{II} M^2 | v_0^{(0)} \rangle - \frac{1}{(m_1^2)^{(0)}} \left| \langle v_1^{(0)} | \Delta_I M^2 | v_0^{(0)} \rangle \right|^2 - \frac{1}{(m_2^2)^{(0)}} \left| \langle v_2^{(0)} | \Delta_I M^2 | v_0^{(0)} \rangle \right|^2\,. \tag{C.4.7}$$

After some computation we find that

$$(m_0^2)^{(2)} = \frac{|\mu|^2 |D|^4}{(|D|^2+|N|^2)^2}\,. \tag{C.4.8}$$

Using (D.1.14b), the correction to the eigenvector associated to the null eigenvalue at first order in perturbative expansion is given by

$$\Delta_I v_0 = -\frac{\langle v_1^{(0)} | \Delta_I M^2 | v_0^{(0)} \rangle}{(m_1^2)^{(0)}} v_1^{(0)} - \frac{\langle v_2^{(0)} | \Delta_I M^2 | v_0^{(0)} \rangle}{(m_2^2)^{(0)}} v_2^{(0)}\,. \tag{C.4.9}$$

After some computation we find that

$$v_0^{(1)} = \frac{1}{\sqrt{|D|^2+|N|^2}}\begin{pmatrix} N \\ \mu\dfrac{N^* D}{|D|^2+|N|^2} \\ -D \end{pmatrix}\,. \tag{C.4.10}$$

Notice that this implies a non-negligible mixing between ν_L and s_L in case N and D are comparable. Also there is in any case almost no mixing with ν_R^c (as long as $\mu \ll N, D$).

To do this computation we could use also the block-diagonalization method shown in Section D.4 if we assumed the additional hierarchy $D \ll N$. In this case the light mass-matrix reads as follows

$$M_\ell \approx -\begin{pmatrix} D & 0 \end{pmatrix} \begin{pmatrix} m & N \\ N & \mu \end{pmatrix}^{-1} \begin{pmatrix} D \\ 0 \end{pmatrix} \approx \mu \frac{D^2}{N^2} \tag{C.4.11}$$

and it is in agreement with the expression of the almost vanishing eigenvalue found before. However to do so we needed to require a further approximation that it is not necessary in principle. The mixing matrix between ν_L and the other states is given by

$$B^\dagger \approx \begin{pmatrix} m & N \\ N & \mu \end{pmatrix}^{-1} \begin{pmatrix} D \\ 0 \end{pmatrix} \approx \frac{1}{N^2}\begin{pmatrix} -\mu D \\ DN \end{pmatrix} \tag{C.4.12}$$

and again it is in agreement with the one found before since $-B^\dagger$ corresponds to the last two entries of (C.4.10).

C.5 Heavy States in Single-Flavour Inverse See-Saw

Consider again the single-flavour ISS discussed in Section C.4, but with the further assumption that $m = 0$ (this has the only purpose to simplify the calculations). In this Section we focus on the heavy mass-eigenstates.

From the results of Section C.4, the two heavy mass-eigenstates at zero-th order in perturbative expansion are degenerate. This corresponds to the limit where $\mu \to 0$ and in this regime we cannot remove this degeneracy. Thus we have that, after using the unitary matrix U (whose columns are given by the mass-eigenvectors (C.4.5)) to diagonalize the mass-matrix M (C.4.2) (with $m = 0$) in the limit where $\mu \to 0$, we find that

$$U^T M U = \hat{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{|D|^2 + |N|^2} \\ 0 & \sqrt{|D|^2 + |N|^2} & 0 \end{pmatrix} . \tag{C.5.1}$$

Thus in the mass-basis with states $N'_L \equiv (\nu'_L, \nu'^c_R\, s'_L)^T$, the mass-Lagrangian reads as follows

$$-\mathcal{L} = \sqrt{|D|^2 + |N|^2}\, \overline{s}'_L \nu'^c_R + h.c. \tag{C.5.2}$$

This corresponds to one heavy fermionic state ψ with a Dirac-type mass and whose chiral components are given by

$$\psi = \psi_L + \psi_R \qquad \text{with} \qquad \psi_L \equiv s'_L\,, \quad \psi_R \equiv \nu'_R\,. \tag{C.5.3}$$

When $\mu \neq 0$, the two chiral components split and become two Weyl fermions with Majorana masses. Also, since the two heavy states at zero-th order are degenerate, to remove this degeneracy we need to go at first order in degenerate perturbation theory. To do so, by looking at (D.1.15), we need to diagonalize the following matrix in the $\{v_1^{(0)}, v_2^{(0)}\}$ space

$$W = \frac{1}{\sqrt{|D|^2 + |N|^2}} \begin{pmatrix} 0 & \mu^* N^2 \\ \mu (N^*)^2 & 0 \end{pmatrix} . \tag{C.5.4}$$

Thus we are in the case (D.7.7) shown in Section D.7. Thus we find that the new zero-th order eigenvectors read as follows

$$\tilde{v}_1^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta} \dfrac{D^* N^*}{N\sqrt{|D|^2 + |N|^2}} \\ -1 \\ e^{-i\theta} \dfrac{N}{\sqrt{|D|^2 + |N|^2}} \end{pmatrix} , \qquad \tilde{v}_2^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta} \dfrac{D^* N^*}{N\sqrt{|D|^2 + |N|^2}} \\ 1 \\ e^{-i\theta} \dfrac{N}{\sqrt{|D|^2 + |N|^2}} \end{pmatrix} \tag{C.5.5}$$

with corresponding eigenvalues at first order in perturbative expansion

$$(m_1^2)^{(1)} = |D|^2 + |N|^2 - \frac{|\mu||N|^2}{\sqrt{|D|^2 + |N|^2}} , \qquad (m_2^2)^{(1)} = |D|^2 + |N|^2 + \frac{|\mu||N|^2}{\sqrt{|D|^2 + |N|^2}} . \tag{C.5.6}$$

Hence we have been able to remove the degeneracy among the two heavy mass-eigenstates.

To do this computation we could use the other method shown in Section D.4 if we assumed the additional hierarchy $D \ll N$. In this case the heavy mass-matrix reads as follows (again we set $m = 0$)

$$M_h \approx \begin{pmatrix} 0 & N \\ N & \mu \end{pmatrix} . \tag{C.5.7}$$

In the limit where $\mu \to 0$ we find that this corresponds to a Dirac-type mass-term. Hence we get the same results gotten before.

If we keep $\mu \neq 0$, we have to unitary diagonalize M_h as shown in Section D.3. Also we perform this calculation in perturbation theory. Namely we want to diagonalize

$$M_h^\dagger M_h = \begin{pmatrix} |N|^2 & 0 \\ 0 & |N|^2 \end{pmatrix} + \begin{pmatrix} 0 & \mu N^* \\ \mu^* N & 0 \end{pmatrix} + \mathcal{O}(|\mu|^2) \,. \tag{C.5.8}$$

This is exactly the same diagonalization problem we solved before, but with the additional assumption that $D \ll N$. Hence we find as expected the same results.

C.6 Mixing Matrices of the Heavy-Neutrino States

In Section 4.3 we considered the mass-Lagrangian of the heavy mass-eigenstates that comes out from the ISS mechanism applied to the models we considered. In this Section we diagonalize the mass-matrix m_R with a singular-valued decomposition. In particular we want to find two unitary matrices V_S and V_R such that

$$V_S^\dagger m_R V_R = \hat{M} = \mathrm{diag}(M_1, M_2, M_3) \tag{C.6.1}$$

where M_1, M_2 and M_3 are positive real numbers. As showed in Section D.2, to find V_S and V_R we need to diagonalize $m_R m_R^\dagger$ and $m_R^\dagger m_R$ respectively. Since we are going to do this perturbatively, to keep easily track of the order of the corrections that we are making we use the short notation $\mathcal{O}(\varepsilon^n)$ where each power of ε corresponds to neglecting corrections of order

$$\varepsilon \sim \frac{m_R^2}{m_R^3} \sim \frac{m_R^1}{m_R^2} \tag{C.6.2}$$

where m_R^j (where $j = 1, 2, 3$) represents the j-th eigenvalue of the mass-matrix m_R in increasing order from the smallest one. Also it is understood that, whenever the expressions have a mass-dimension, one has to put in front of ε^n the correct power of m_R^3.

We start from the 2-flavour case since it is computationally easier. Explicitly we have that

$$m_R^\dagger m_R = \begin{pmatrix} |m_R^{22}|^2 + |m_R^{32}|^2 & m_R^{23}(m_R^{22})^* + m_R^{33}(m_R^{32})^* \\ m_R^{22}(m_R^{23})^* + m_R^{32}(m_R^{33})^* & |m_R^{23}|^2 + |m_R^{33}|^2 \end{pmatrix} , \tag{C.6.3a}$$

$$m_R m_R^\dagger = \begin{pmatrix} |m_R^{22}|^2 + |m_R^{23}|^2 & m_R^{22}(m_R^{32})^* + m_R^{23}(m_R^{33})^* \\ m_R^{32}(m_R^{22})^* + m_R^{33}(m_R^{23})^* & |m_R^{32}|^2 + |m_R^{33}|^2 \end{pmatrix} . \tag{C.6.3b}$$

We notice that both matrices are in the case (D.7.1) shown in Section D.7 with the condition that $\det\left[m_R^\dagger m_R\right] \ll \mathrm{Tr}^2[m_R^\dagger m_R]$. Also they share the same eigenvalues. Thus we find that the eigenvalues are given by

$$\lambda_3 = |m_R^{23}|^2 + |m_R^{33}|^2 + \mathcal{O}(\varepsilon^2) \,, \qquad \lambda_2 = \frac{|m_R^{23} m_R^{32} - m_R^{33} m_R^{22}|^2}{|m_R^{23}|^2 + |m_R^{33}|^2} + \mathcal{O}(\varepsilon^4) \tag{C.6.4}$$

with corresponding eigenvectors

$$v_3^R = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}(\varepsilon) \,, \qquad v_3^S = \frac{1}{\sqrt{|m_R^{33}|^2 + |m_R^{23}|^2}} \begin{pmatrix} m_R^{23} \\ m_R^{33} \end{pmatrix} + \mathcal{O}(\varepsilon) \,, \tag{C.6.5a}$$

$$v_2^R = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(\varepsilon) \,, \qquad v_2^S = \frac{1}{\sqrt{|m_R^{33}|^2 + |m_R^{23}|^2}} \begin{pmatrix} (m_R^{33})^* \\ -(m_R^{23})^* \end{pmatrix} + \mathcal{O}(\varepsilon) \,. \tag{C.6.5b}$$

The 3-flavour case is more complicated, but only computationally. The result will be a straightforward generalization of the 2-flavour case. We start from observing that parametrically $m_R^\dagger m_R$ has the same hierarchical structure of the matrix that has been perturbatively diagonalized in Section D.8. Thus we find that the eigenvalues are given by

$$\lambda_3 = |m_R^{13}|^2 + |m_R^{23}|^2 + |m_R^{33}|^2 + \mathcal{O}(\varepsilon^2)\,, \tag{C.6.6a}$$

$$\lambda_2 = |m_R^{12}|^2 + |m_R^{22}|^2 + |m_R^{32}|^2 - \frac{|m_R^{13}(m_R^{12})^* + m_R^{23}(m_R^{22})^* + m_R^{33}(m_R^{32})^*|^2}{|m_R^{13}|^2 + |m_R^{23}|^2 + |m_R^{33}|^2} + \mathcal{O}(\varepsilon^4)\,, \tag{C.6.6b}$$

$$\lambda_1 = \frac{|\det[m_R]|^2}{\lambda_2\lambda_3} + \mathcal{O}(\varepsilon^6) \tag{C.6.6c}$$

with corresponding eigenvectors

$$v_3^R = \begin{pmatrix}0\\0\\1\end{pmatrix} + \mathcal{O}(\varepsilon)\,, \qquad v_2^R = \begin{pmatrix}0\\1\\0\end{pmatrix} + \mathcal{O}(\varepsilon)\,, \qquad v_1^R = \begin{pmatrix}1\\0\\0\end{pmatrix} + \mathcal{O}(\varepsilon)\,. \tag{C.6.7}$$

Instead diagonalizing $m_R m_R^\dagger$ is very difficult since it does not show any hierarchy inside. Nevertheless we can split the different hierarchical orders as follows

$$m_R m_R^\dagger = \begin{pmatrix} |m_R^{13}|^2 & m_R^{13}(m_R^{23})^* & m_R^{13}(m_R^{33})^* \\ m_R^{23}(m_R^{33})^* & |m_R^{23}|^2 & m_R^{23}(m_R^{33})^* \\ m_R^{33}(m_R^{13})^* & m_R^{33}(m_R^{23})^* & |m_R^{33}|^2 \end{pmatrix} + \begin{pmatrix} |m_R^{12}|^2 & m_R^{12}(m_R^{22})^* & m_R^{12}(m_R^{32})^* \\ m_R^{22}(m_R^{12})^* & |m_R^{22}|^2 & m_R^{22}(m_R^{32})^* \\ m_R^{32}(m_R^{12})^* & m_R^{32}(m_R^{22})^* & |m_R^{32}|^2 \end{pmatrix} + \mathcal{O}(\varepsilon^4)\,. \tag{C.6.8}$$

The first leading contribution has a 1-dimensional eigen-subspace with eigenvalue given by λ_3 in (C.6.6a) with corresponding eigenvector

$$v_3^{(0)} = \frac{1}{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2 + |m_R^{33}|^2}} \begin{pmatrix} m_R^{13}\\ m_R^{23}\\ m_R^{33}\end{pmatrix} \tag{C.6.9}$$

and a 2-dimensional eigen-subspace with null eigenvalue. Since the latter sub-space must be orthogonal to the first, it is easy to find a basis for this subspace. A choice (already orthonormalized) is given by

$$v_1^{(0)} = \begin{pmatrix} -s_{12}^* \\ c_{12}^* \\ 0 \end{pmatrix}\,, \qquad v_2^{(0)} = \begin{pmatrix} -s_3^* c_{12} \\ -s_3^* s_{12} \\ c_3^* \end{pmatrix} \tag{C.6.10}$$

where we have defined

$$c_{12} \equiv \frac{m_R^{13}}{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2}}\,, \qquad s_{12} \equiv \frac{m_R^{23}}{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2}}\,, \tag{C.6.11a}$$

$$c_3 \equiv \frac{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2}}{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2 + |m_R^{33}|^2}}\,, \qquad s_3 \equiv \frac{m_R^{33}}{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2 + |m_R^{33}|^2}}\,. \tag{C.6.11b}$$

As discussed in Section D.1, to remove the degeneracy in the null eigen-subspace we need to go at $\mathcal{O}(\varepsilon^2)$ in (C.6.8) and in the $\{v_1^{(0)}, v_2^{(0)}\}$ basis we have to diagonalize the matrix

$$W = \begin{pmatrix} |a|^2 & a^* b \\ a b^* & |b|^2 \end{pmatrix} \tag{C.6.12}$$

where

$$a^* = \frac{m_R^{13} m_R^{22} - m_R^{23} m_R^{12}}{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2}}\,, \qquad b^* = \frac{c_{12}^*\left(m_R^{13} m_R^{32} - m_R^{33} m_R^{12}\right) + s_{12}^*\left(m_R^{23} m_R^{32} - m_R^{33} m_R^{22}\right)}{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2 + |m_R^{33}|^2}}\,. \tag{C.6.13}$$

This matrix is in the case (D.7.4) shown in Section D.7. Thus we can diagonalize it and remove the degeneracy. To this purpose it is useful to define the following quantities

$$c \equiv \frac{a}{\sqrt{|a|^2 + |b|^2}}\,, \qquad s \equiv \frac{b}{\sqrt{|a|^2 + |b|^2}}\,. \tag{C.6.14}$$

Also notice that one eigenvalue becomes positive and can be explicitly checked that it is given by (C.6.6b) as expected. In the end we have that $m_R m_R^\dagger$ is diagonalized with eigenvalues (C.6.6) and corresponding eigenvectors

$$\begin{gathered} v_3^S = \frac{1}{\sqrt{|m_R^{13}|^2 + |m_R^{23}|^2 + |m_R^{33}|^2}} \begin{pmatrix} m_R^{13} \\ m_R^{23} \\ m_R^{33} \end{pmatrix} + \mathcal{O}(\varepsilon)\,, \\ v_2^S = \begin{pmatrix} c\, s_{12}^* + s\, s_3^* c_{12} \\ -c\, c_{12}^* + s\, s_3^* s_{12} \\ -s\, c_3^* \end{pmatrix} + \mathcal{O}(\varepsilon)\,, \qquad v_1^S = \begin{pmatrix} -s^* s_{12}^* + c^* s_3^* c_{12} \\ s^* c_{12}^* + c^* s_3^* s_{12} \\ -c^* c_3^* \end{pmatrix} + \mathcal{O}(\varepsilon)\,. \end{gathered} \tag{C.6.15}$$

C.7 Majorana Mass for the Charged RH Neutrino

In the DSS scenario, to explain the observed active-neutrino masses, we need to provide an extremely heavy Majorana mass for the third-generation neutrino ν_R^3. A possible way to do so is to introduce a new scalar field which eventually acquires a VEV. However, we would like to specify since the beginning that this option revealed to be not physically relevant. Nevertheless it was a good exercise on SSB and Group Theory.

As discussed in Section 3.6, since the two-fermion operator

$$(\chi_R^3)_\alpha \mathcal{C}_{\alpha\beta} (\chi_R^3)_\beta \tag{C.7.1}$$

has mass-dimension +3, the only way to get a renormalizable operator that produces a Majorana mass-term consists in assuming the existence of a scalar field

$$X \sim (\overline{\mathbf{10}}_S, \overline{\mathbf{3}}_S) \in \mathrm{SU}(4)^{[3]} \times \mathrm{SU}(2)_R^{[3]} \tag{C.7.2}$$

which eventually acquires a suitable VEV. We represent its Gauge indices as $X_{mn,ij}$ where $m, n = 1, 2, 3, 4$ and $i, j = 1, 2$ and we require its VEV to be given by

$$\langle X_{mn,ij} \rangle = v_X \delta_{m4} \delta_{n4} \delta_{i1} \delta_{j1}\,. \tag{C.7.3}$$

Also by construction we have that

$$X_{mn,ij} = X_{nm,ij}\,, \qquad X_{mn,ij} = X_{mn,ji}\,. \tag{C.7.4}$$

We recall that the importance of having a renormalizable operator is due to the fact that we need a very heavy Majorana mass-term for the third-generation RH neutrino. Thanks to this we can write the following renormalizable operator

$$\mathcal{L} \supset -\frac{1}{2} c_\nu X_{mn,ij} (\chi_R^3)^T_{m,i} \mathcal{C}\, (\chi_R^3)_{n,j} + h.c. \tag{C.7.5}$$

which in the broken phase it reduces to

$$\mathcal{L} \supset -\frac{1}{2}c_\nu v_X (\nu_R^3)^T \mathcal{C}\, \nu_R^3 + h.c. \equiv -\frac{1}{2}(\mu_R)_{33}(\nu_R^3)^T \mathcal{C}\, \nu_R^3 + h.c. \tag{C.7.6}$$

Thus we can safely explain the observed active-neutrino masses through a DSS mechanism by fixing v_X sufficiently large.

Now we want to compute the broken Gauge spectrum of the SSB mechanisms induced by the scalar field X. Under a Gauge transformation we have that (up to first order in the Gauge parameters)

$$X_{mn,ij} \to X_{mn,ij} + i\theta^a X_{rn,ij}\hat{T}^a_{rm} + i\theta^a X_{mr,ij}\hat{T}^a_{rn} + i\alpha^a X_{mn,\ell j}T^a_{R\,\ell i} + i\alpha^a X_{mn,i\ell}T^a_{R\,\ell j}\,. \tag{C.7.7}$$

In particular its VEV transforms as follows under an $\mathrm{SU}(4)^{[3]}$ Gauge transformation

$$\langle X_{mn,ij}\rangle \to \langle X_{mn,ij}\rangle \qquad \text{if} \quad m,n \neq 4\,, \tag{C.7.8a}$$

$$\langle X_{m4,ij}\rangle \to \langle X_{m4,ij}\rangle + i\langle X_{44,ij}\rangle\,\theta^a\hat{T}^a_{4m} \qquad \text{if} \quad m \neq 4\,, \tag{C.7.8b}$$

$$\langle X_{44,ij}\rangle \to \langle X_{44,ij}\rangle \left(1 - \frac{3i}{\sqrt{6}}\theta^{15}\right)\,. \tag{C.7.8c}$$

Instead under an $\mathrm{SU}(2)_R^{[3]}$ Gauge transformation we find that

$$\langle X_{mn,22}\rangle \to \langle X_{mn,22}\rangle\,, \tag{C.7.9a}$$

$$\langle X_{mn,12}\rangle \to \langle X_{mn,12}\rangle + \frac{i}{2}\langle X_{mn,11}\rangle\,(\alpha_1 - i\alpha_2)\,, \tag{C.7.9b}$$

$$\langle X_{mn,11}\rangle \to \langle X_{mn,11}\rangle \left(1 + i\alpha^3\right)\,. \tag{C.7.9c}$$

Thus, to leave the VEV invariant, we keep free α^3 and we fix

$$\theta^a = 0 \quad \text{for} \quad a = 9, ..., 14\,; \qquad \alpha^1 = \alpha^2 = 0\,; \qquad \theta^{15} = \frac{\sqrt{6}}{3}\alpha^3\,. \tag{C.7.10}$$

Therefore the residual Gauge Group is the same left unbroken by Δ_3 in Model C with same generators and Gauge boson basis, listed in Section 1.5. The only difference is that this modifies the masses of the broken Gauge fields.

To compute the mass-spectrum we need the covariant derivative of X. It reads as follows

$$D_\mu X_{mn,ij} = \partial_\mu X_{mn,ij} + ig_4 H^a X_{rn,ij}\hat{T}^a_{rm} + ig_4 H^a X_{mr,ij}\hat{T}^a_{rn} + ig_R W^a_R X_{mn,\ell j}T^a_{R\,\ell i} + ig_R W^a_R X_{mn,i\ell}T^a_{R\,\ell j}\,. \tag{C.7.11}$$

In matrix notation we have that on the VEV it reduces to

$$D_\mu\langle X\rangle = v_X \begin{pmatrix} \mathbb{O}_{3\times3} & U & \mathbb{O}_{3\times3} & \bar{0} \\ U^T & D & \bar{0}^T & W \\ \mathbb{O}_{3\times3} & \bar{0} & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbb{O}_{4\times4}$}} \\ \bar{0}^T & W & & \end{pmatrix} \tag{C.7.12}$$

where we have defined

$$U = i\frac{g_4}{2}\begin{pmatrix} H^9 - iH^{10} \\ H^{11} - iH^{12} \\ H^{13} - iH^{14}\end{pmatrix}\,, \qquad W = i\frac{g_R}{2}(W_R^1 + iW_R^2)\,, \qquad D = i\left(g_R W_R^3 - \frac{\sqrt{6}}{2}g_4 H^{15}\right)\,. \tag{C.7.13}$$

Therefore the mass-matrix of the broken Gauge fields gets the following contributions

$$\mathcal{L}_M \supset \mathrm{Tr}\Big[(D_\mu X)^\dagger D^\mu X\Big] = \frac{g_4^2}{2}v_X^2\sum_{a=9}^{14}(H^a)^2 + \frac{g_R^2}{4}v_X^2\left[(W_R^1)^2 + (W_R^2)^2\right] + \frac{g_4^2}{4}v_X^2\left(\frac{\sqrt{6}}{2}H^{15} - \frac{g_R}{g_4}W_R^3\right)^2. \tag{C.7.14}$$

In particular a very large VEV v_X would imply very heavy leptoquarks, W-type and one neutral Z-type vector bosons.

C.8 Gauge Spectrum induced by Δ

In Section 4.4 we have introduced a new Symmetry and an additional scalar field charged under the UV Gauge Group (1.1.3). We also require that this field eventually acquires a VEV. Thus it modifies the mass-spectrum of the broken Gauge bosons. In this Section we compute how the mass-spectrum gets modified.

The new scalar field introduced reads as follows with corresponding Charges under the UV Gauge Group (1.1.3)[35]

$$\Delta \sim (\mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{2}, 0)\,. \tag{C.8.1}$$

In particular it is charged only under $\mathrm{SU}(4)^{[3]} \times \mathrm{SU}(2)_R^{[3]}$ and we represent its Gauge indices as $\Delta_{a,j}$ where $a = 1, 2, 3, 4$ and $j = 1, 2$. Its VEV is given by

$$\langle \Delta_{aj} \rangle = v_F\, \delta_{a4} \delta_{j1}\,. \tag{C.8.2}$$

Notice that this scalar field is closely related to Δ_3, the scalar field that acquires a VEV in the framework of Model C. In fact we expect to find very similar results compared to the ones found after the first SSB step in Section 1.5.

To compute the broken mass-spectrum of the SSB mechanism induced by Δ, we need to find the residual Gauge Group. To leave its VEV invariant we need to satisfy

$$\theta^a \hat{T}^a \langle \Delta \rangle + \alpha^i T^i \langle \Delta \rangle = 0\,. \tag{C.8.3}$$

We have that

$$\hat{T}^a \langle \Delta \rangle = 0 \quad \text{for} \quad a = 1, ..., 8 \quad \text{and} \quad \langle \Delta \rangle = -\frac{2\sqrt{6}}{3} \hat{T}^{15} \langle \Delta \rangle = 2\, T^3 \langle \Delta \rangle\,. \tag{C.8.4}$$

This represents a Gauge transformation where we keep free α^3 and θ^a for $a = 1, ..., 8$ and we fix

$$\theta^{15} = \frac{\sqrt{6}}{3} \alpha^3\,; \qquad \theta^a = 0 \quad \text{for} \quad a = 9, ..., 14\,; \qquad \alpha^1 = \alpha^2 = 0\,. \tag{C.8.5}$$

Therefore the residual Gauge Group is given by $\mathrm{SU}(3)^{[3]} \times \mathrm{U}(1)_X^{[3]}$. Hence its VEV does not spoil the SSB mechanisms induced by our models since all the generators that we must preserve remain unbroken.

To find the new Gauge boson basis and their masses we need to write down the kinetic term of Δ evaluated on the VEV. To do so we need its covariant derivative which reads as follows

$$D_\mu \langle \Delta \rangle = -i \frac{g_4}{2} v_F \Big((D_\mu \langle \Delta \rangle)_1 \quad (D_\mu \langle \Delta \rangle)_2 \quad (D_\mu \langle \Delta \rangle)_3 \quad (D_\mu \langle \Delta \rangle)_4 \Big)^T \tag{C.8.6}$$

where

$$\begin{aligned} (D_\mu \langle \Delta \rangle)_1 &= \begin{pmatrix} H^9 - iH^{10} \\ 0 \end{pmatrix}, \quad (D_\mu \langle \Delta \rangle)_2 = \begin{pmatrix} H^{11} - iH^{12} \\ 0 \end{pmatrix}, \\ (D_\mu \langle \Delta \rangle)_3 &= \begin{pmatrix} H^{13} - iH^{14} \\ 0 \end{pmatrix}, \quad (D_\mu \langle \Delta \rangle)_4 = \begin{pmatrix} -\frac{\sqrt{6}}{2} H^{15} + \frac{g_R}{g_4} W_R^3 \\ \frac{g_R}{g_4} \left(W_R^1 + iW_R^2 \right) \end{pmatrix}. \end{aligned} \tag{C.8.7}$$

Now we need to compute the mass-Lagrangian which is given by

$$\mathcal{L}_M \equiv \mathrm{Tr}\Big[(D_\mu \langle \Delta \rangle)^\dagger D_\mu \langle \Delta \rangle \Big] = \frac{g_4^2}{4} v_F^2 \sum_{a=9}^{14} (H^a)^2 + \frac{g_R^2}{4} v^2 \left[(W_R^1)^2 + (W_R^2)^2 \right] + \frac{g_4^2}{4} v_F^2 \left(\frac{\sqrt{6}}{2} H^{15} - \frac{g_R}{g_4} W_R^3 \right)^2. \tag{C.8.8}$$

[35]Recall that it also must be charged under the unknown new Symmetry that could be discrete or not, but this is not going to affect our analysis.

This is exactly the same mass-Lagrangian found in Section 1.5 after the first step, and it produces the same broken vector-boson mass-basis.

As discussed in Chapter 4, the introduction of a new Symmetry to explain *naturally* the observed active-neutrino masses makes sense only in framework of Model I. Thus we can restrict the analysis considering only Models A, B, D and F. From the mass-Lagrangian (C.8.8), we have that the masses of the leptoquarks $U_c^\pm$ and W-type bosons $W_R^\pm$ get modified as follows

$$\Delta m_U^2 = \frac{g_4^2}{2} v_F^2 \, , \qquad \Delta m_{W_R}^2 = \frac{g_R^2}{2} v_F^2 \, . \tag{C.8.9}$$

Instead the mass-Lagrangian of the neutral vector bosons becomes

$$\mathcal{L}_M \supset \frac{\Omega^2}{4} \left(\hat{g}_4 H^{15} - g_X B_{12} \right)^2 + \frac{v_R^2}{4} \left(g_R W_R^3 - g_X B_{12} \right)^2 + \frac{v_F^2}{4} \left(\hat{g}_4 H^{15} - g_R W_R^3 \right)^2 \, . \tag{C.8.10}$$

This changes consistently the mixing matrices for the neutral vector bosons and hence the masses and interactions with Z' and Z''. However, since the expressions are rather long and complicated, here we just state that the null mass-eigenstate which generates the Gauge boson B associated to $\mathrm{U}(1)_Y$ is the same found before as expected, while the masses of the two neutral vector bosons are given by (if we assume that $v_F \gtrsim \Omega, v_R$)

$$m_{Z'}^2 \approx \frac{v_F^2}{2} \rho_{4R}^{-2} \left[1 + \frac{\Omega^2}{v_F^2} \hat{g}_4^4 \rho_{4R}^4 + \frac{v_R^2}{v_F^2} g_R^4 \rho_{4R}^4 \right] \, , \qquad m_{Z''}^2 \approx \frac{1}{2} \left(\Omega^2 + v_R^2 \right) \sigma^2 \rho_{4R}^2 \tag{C.8.11}$$

where the notation is the same used in Section 1.9. Notice that they are very similar to the ones found in Models C and F since Δ and Δ_3 are very similar.

The interactions with the fermions and the neutral massive vector bosons in the covariant derivative are rather long and complicated and we will not write them here. Nevertheless, given the mass-Lagrangian (C.8.10), this derivation is straightforward.

D Matrix Diagonalization in Perturbation Theory

In this Chapter we are going to derive and provide several results that we exploited in this work when we needed to diagonalize a matrix under certain conditions.

D.1 Perturbative Diagonalization

Suppose to have an hermitian matrix in the form

$$M = M_0 + \Delta M \tag{D.1.1}$$

where $\Delta M = \varepsilon \hat{M}$ is a little perturbation of M_0, namely $\Delta M \ll M_0$ or[36] $\varepsilon \ll 1$. Suppose that M_0 is diagonalizable and the spectrum is given by a set of eigenvalues with the corresponding normalized eigenvectors $\{\lambda_a^{(0)}, v_a^{(0)}\}$. We want to know the spectrum of the full matrix M $\{\lambda_a, v_a\}$ at a given perturbative order. In this Section we define the scalar products between two vectors v, w and the scalar product between two vectors and a matrix A as follows

$$\langle v|w\rangle \equiv v^\dagger w\,, \qquad \langle v|A|w\rangle \equiv v^\dagger A w\,. \tag{D.1.2}$$

We make a Taylor expansion in the small parameter ε as follows

$$\lambda_a = \sum_{n=0}^{\infty} \varepsilon^n \lambda_a^{(n)}\,, \qquad v_a = \sum_{n=0}^{\infty} \varepsilon^n v_a^{(n)} \tag{D.1.3}$$

and we assume that the vectors are well normalized, namely we require that $\forall N \in \mathbb{N}$

$$\langle v_a|v_a\rangle = \sum_{n+m<N} \varepsilon^{n+m} \langle v_a^{(n)}|v_a^{(m)}\rangle = 1 + \mathcal{O}(\varepsilon^N)\,. \tag{D.1.4}$$

We also require (without loss of generality) that $\forall n \in \mathbb{N}$

$$\langle v_a^{(n)}|v_a^{(0)}\rangle \in \mathbb{R}\,. \tag{D.1.5}$$

Notice that, directly from (D.1.4) and (D.1.5), it follows that

$$\langle v_a^{(0)}|v_a^{(0)}\rangle = 1\,, \quad \langle v_a^{(1)}|v_a^{(0)}\rangle = 0\,, \quad \langle v_a^{(2)}|v_a^{(0)}\rangle = \langle v_a^{(0)}|v_a^{(2)}\rangle = \frac{1}{2}\langle v_a^{(1)}|v_a^{(1)}\rangle\,. \tag{D.1.6}$$

We write the equation that defines the spectrum of M as follows

$$\left(M_0 + \varepsilon \hat{M}\right) \sum_n \varepsilon^n v_a^{(n)} = \sum_m \varepsilon^m \lambda_a^{(m)} \sum_n \varepsilon^n v_a^{(n)}\,. \tag{D.1.7}$$

By expanding up to $\mathcal{O}(\varepsilon^2)$ we find the following relations

$$\mathcal{O}(\varepsilon^0): \quad \left(M_0 - \lambda_a^{(0)}\right) v_a^{(0)} = 0\,, \tag{D.1.8a}$$

$$\mathcal{O}(\varepsilon^1): \quad \left(M_0 - \lambda_a^{(0)}\right) v_a^{(1)} + \left(\hat{M} - \lambda_a^{(1)}\right) v_a^{(0)} = 0\,, \tag{D.1.8b}$$

$$\mathcal{O}(\varepsilon^2): \quad \left(M_0 - \lambda_a^{(0)}\right) v_a^{(2)} + \left(\hat{M} - \lambda_a^{(1)}\right) v_a^{(1)} - \lambda_a^{(2)} v_a^{(0)} = 0\,. \tag{D.1.8c}$$

We notice that (D.1.8a) is the defining equation of the spectrum of M_0 which can be explicitly solved to find $\lambda_a^{(0)}$ and $v_a^{(0)}$.

[36]In this treatment ε is an additional parameter which could be dropped at the end, but it is useful to keep during the intermediate steps.

Now assume that the eigen-subspace of $\lambda_a^{(0)}$ is non-degenerate. Then, by projecting (D.1.8b) onto $(v_a^{(0)})^\dagger$ and using (D.1.6) we find that

$$\lambda_a^{(1)} = \langle v_a^{(0)}|\hat{M}|v_a^{(0)}\rangle\,. \tag{D.1.9}$$

By projecting (D.1.8b) onto $(v_b^{(0)})^\dagger$ with $b \neq a$ and using the fact that eigen-subspaces of different eigenvalues are orthogonal between each others we find that

$$\left[\lambda_a^{(0)} - \lambda_b^{(0)}\right]\langle v_b^{(0)}|v_a^{(1)}\rangle = \langle v_b^{(0)}|\hat{M}|v_a^{(0)}\rangle\,. \tag{D.1.10}$$

Thus, by using the completeness relation

$$v_a^{(1)} = \sum_b \langle v_b^{(0)}|v_a^{(1)}\rangle v_b^{(0)} = \sum_{b\neq a} \langle v_b^{(0)}|v_a^{(1)}\rangle v_b^{(0)} \tag{D.1.11}$$

we find that

$$v_a^{(1)} = \sum_{b\neq a} \frac{\langle v_b^{(0)}|\hat{M}|v_a^{(0)}\rangle}{\lambda_a^{(0)} - \lambda_b^{(0)}} v_b^{(0)}\,. \tag{D.1.12}$$

By projecting (D.1.8c) onto $(v_a^{(0)})^\dagger$ and using (D.1.6) and (D.1.12) we find that

$$\lambda_a^{(2)} = \sum_{b\neq a} \frac{1}{\lambda_a^{(0)} - \lambda_b^{(0)}} \left|\langle v_b^{(0)}|\hat{M}|v_a^{(0)}\rangle\right|^2\,. \tag{D.1.13}$$

Therefore we have found that for non-degenerate eigen-subspace

$$\lambda_a = \lambda_a^{(0)} + \langle v_a^{(0)}|\Delta M|v_a^{(0)}\rangle + \sum_{b\neq a} \frac{1}{\lambda_a^{(0)} - \lambda_b^{(0)}} \left|\langle v_b^{(0)}|\Delta M|v_a^{(0)}\rangle\right|^2 + \mathcal{O}(3)\,, \tag{D.1.14a}$$

$$v_a = v_a^{(0)} + \sum_{b\neq a} \frac{\langle v_b^{(0)}|\Delta M|v_a^{(0)}\rangle}{\lambda_a^{(0)} - \lambda_b^{(0)}} v_b^{(0)} + \mathcal{O}(2)\,. \tag{D.1.14b}$$

Instead now assume that the eigen-subspace of $\lambda_a^{(0)}$ is degenerate with a multiplicity of m. Then we need to project (D.1.8b) onto a base of the subspace of $\lambda_a^{(0)}$ which we denote by $v_{a,r}^{(0)}$ with $r = 1, ..., m$ finding

$$\lambda_{a,r}^{(1)} w_r = W_{rs} w_s \tag{D.1.15}$$

where we have defined

$$W_{rs} \equiv \langle v_{a,r}^{(0)}|\hat{M}|v_{a,s}^{(0)}\rangle \tag{D.1.16}$$

and w is an m-dimensional vector whose coordinates w_r represent the weights in the basis of the eigen-subspace considered. Namely a vector in the eigen-subspace is represented by

$$v_a^{(0)} = \sum_{r=1}^{m} w_r v_{a,r}^{(0)}\,. \tag{D.1.17}$$

To find the splitting among the eigenvectors at zero-th order with the corresponding eigenvalues at first order we need to diagonalize W, namely we need to solve (D.1.15). Then, once we have found m eigenvectors $w^{(r)}$ (with hopefully m different eigenvalues $\lambda_{a,r}^{(1)}$), we have that we replace the zero-th order eigenvectors $v_{a,r}^{(0)}$ with

$$v_{a,r}^{(0)} \quad \to \quad \tilde{v}_{a,r}^{(0)} \equiv \sum_{s=1}^{m} w_s^{(r)} v_{a,s}^{(0)} \qquad \text{with} \quad r = 1, ..., m\,. \tag{D.1.18}$$

To find the correction to the eigenvectors at first order we proceed as we did in the non-degenerate case and we find that

$$\tilde{v}_{a,r}^{(1)} = \sum_{b\neq a} \frac{\langle v_b^{(0)}|\hat{M}|\tilde{v}_{a,r}^{(0)}\rangle}{\lambda_a^{(0)} - \lambda_b^{(0)}} v_b^{(0)} \,. \tag{D.1.19}$$

To find the second order correction to the eigenvalues we proceed as before. Thus, by projecting (D.1.8c) onto $(\tilde{v}_{a,r}^{(0)})^\dagger$ we find that

$$\lambda_{a,r}^{(2)} = \sum_{b\neq a} \frac{1}{\lambda_a^{(0)} - \lambda_b^{(0)}} \left|\langle v_b^{(0)}|\hat{M}|\tilde{v}_{a,r}^{(0)}\rangle\right|^2 \,. \tag{D.1.20}$$

Therefore we have found that for degenerate eigen-subspace

$$\lambda_{a,r} = \lambda_a^{(0)} + \varepsilon\lambda_{a,r}^{(1)} + \sum_{b\neq a} \frac{1}{\lambda_a^{(0)} - \lambda_b^{(0)}} \left|\langle v_b^{(0)}|\Delta M|\tilde{v}_{a,r}^{(0)}\rangle\right|^2 + \mathcal{O}(3) \,, \tag{D.1.21a}$$

$$v_{a,r} = \tilde{v}_{a,r}^{(0)} + \sum_{b\neq a} \frac{\langle v_b^{(0)}|\Delta M|\tilde{v}_{a,r}^{(0)}\rangle}{\lambda_a^{(0)} - \lambda_b^{(0)}} v_b^{(0)} + \mathcal{O}(2) \tag{D.1.21b}$$

where $\lambda_{a,r}^{(1)}$ and $\tilde{v}_{a,r}$ are obtained by solving (D.1.15).

D.2 Singular-Valued Decomposition

Given a matrix M, we want to find two unitary matrices U, V such that

$$U^\dagger M V = \hat{M} \tag{D.2.1}$$

where $\hat{M}$ is a diagonal matrix with non-negative eigenvalues.

To do so we look at the matrix $MM^\dagger$. It is hermitian, hence it is unitary diagonalizable with non-negative eigenvalues. Therefore we can find a unitary matrix U such that

$$U^\dagger(MM^\dagger)U = D^2 \tag{D.2.2}$$

where D^2 is positive-definite diagonal matrix (and so will be D). Now let us define the matrix

$$K \equiv M^\dagger U D^{-1} \,. \tag{D.2.3}$$

It is unitary since

$$K^\dagger K = (D^{-1\dagger}U^\dagger M)(M^\dagger U D^{-1}) = D^{-1}D^2D^{-1} = \mathbb{I} \,. \tag{D.2.4}$$

Also we can check that

$$U^\dagger M K = U^\dagger M(M^\dagger U D^{-1}) = D \,. \tag{D.2.5}$$

So we have that we can set $V = K$ and $\hat{M} = D$ and we are done.

To find V it is actually more useful to try to unitary diagonalize the matrix $M^\dagger M$. In fact we can define V such that

$$V^\dagger(M^\dagger M)V = D^2 \tag{D.2.6}$$

and notice that the diagonal matrix must be the same of before since the characteristic polynomials associated to $MM^\dagger$ and $M^\dagger M$ are the same, namely

$$\det\Big[MM^\dagger - \lambda\mathbb{I}\Big] = \det\Big[M(M^\dagger - \lambda M^{-1})\Big] = \det\Big[(M^\dagger - \lambda M^{-1})M\Big] = \det\Big[M^\dagger M - \lambda\mathbb{I}\Big] \,. \tag{D.2.7}$$

Now if we do a similarity transformation to $M^\dagger M$ using K we find that

$$K^\dagger(M^\dagger M)K = (D^{-1\dagger}U^\dagger M)(M^\dagger M)(M^\dagger U D^{-1}) = D^{-1}D^2D^2D^{-1} = D^2 \,. \tag{D.2.8}$$

Therefore $K = V$ since the diagonalization matrix is unique.

Thus to find U we need to diagonalize $MM^\dagger$, to find V we need to diagonalize $M^\dagger M$ and to find $\hat{M}$ we need to take the squared-root of the diagonal matrix that one finds after diagonalizing $MM^\dagger$.

D.3 Diagonalization of a Symmetric Mass-Matrix

In this work we needed several times to diagonalize a neutrino mass-matrix which is a complex symmetric $(n_\ell + n_h) \times (n_\ell + n_h)$ matrix of the following block-form

$$M = \begin{pmatrix} L & D \\ D^T & R \end{pmatrix} \tag{D.3.1}$$

where n_ℓ and n_h are the number of the would-be light and heavy neutrino species respectively. L, D and R are $n_\ell \times n_\ell$, $n_\ell \times n_h$ and $n_h \times n_h$ complex symmetric matrices respectively. To diagonalize it we need to find a unitary matrix U such that

$$U^T M U = \mathrm{diag}\left(m_\ell^{(1)}, ..., m_\ell^{(n_\ell)}, M_h^{(1)}, ..., M_h^{(n_h)}\right) \tag{D.3.2}$$

where $m_\ell^{(\ell)}$ and $M_h^{(h)}$ are all positive real numbers and are associated to the masses of the mass-eigenstates.

The condition $M = M^T$ is necessary to ensure that such a unitary matrix always exists. In general, given a generic matrix, we can diagonalize it with a singular-valued decomposition as shown in Section D.2. Thus we have that there always exist two unitary matrices V and U such that

$$V^\dagger M U = \hat{M} \tag{D.3.3}$$

where $\hat{M}$ is diagonal and positive-definite. To find V and U we need to diagonalize $MM^\dagger$ and $M^\dagger M$ respectively. Nevertheless, since $M = M^T$, the two equations can be simplified as follows

$$V^\dagger M M^* V = \hat{M}^2\,, \qquad U^\dagger M^* M U = \hat{M}^2\,. \tag{D.3.4}$$

Thus, upon taking the complex conjugation on the first equation, we see that $V^* = U$. Therefore we can make use of the same unitary matrix to diagonalize a symmetric matrix as done in (D.3.2).

As we have shown above, to find U we diagonalize the hermitian matrix $M^\dagger M$ and we know that

$$U^\dagger (M^\dagger M) U = \mathrm{diag}\left((m_\ell^{(1)})^2, ..., (m_\ell^{(n_\ell)})^2, (M_h^{(1)})^2, ..., (M_h^{(n_h)})^2\right)\,. \tag{D.3.5}$$

Thus we can find the eigenvalues with corresponding eigenstates of a neutrino mass-matrix by doing this procedure. Moreover, when the calculation seems too hard analytically, we can diagonalize $M^\dagger M$ perturbatively if there is a clear hierarchy in the mass-matrix. In fact we usually have that $L, D \ll R$. In the following Section we are going to study this case in detail.

D.4 Block-Diagonalization of a Symmetric Mass-Matrix

When we want to diagonalize a symmetric mass-matrix in the form of (D.3.1), sometimes it is useful to disentangle the light and heavy states and then diagonalize each sub-matrix independently. This is because, most of the times, if more than one generation is involved (namely $n_\ell, n_h > 1$), we should use degenerate perturbation theory to diagonalize the mass-matrix. Thus what we could do is to exploit a unitary matrix W such that

$$W^T M W = \begin{pmatrix} M_\ell & \mathbb{O} \\ \mathbb{O} & M_h \end{pmatrix} \tag{D.4.1}$$

where M_ℓ and M_h are $n_\ell \times n_\ell$ and $n_h \times n_h$ complex symmetric matrices respectively. Then one can easily diagonalize independently M_ℓ and M_h using the same method used for M but with reduced matrices. A reference of the calculation we are about to make can be found in [22].

To find M_ℓ and M_h we can parameterize W as follows

$$W = \begin{pmatrix} \sqrt{1-BB^\dagger} & B \\ -B^\dagger & \sqrt{1-B^\dagger B} \end{pmatrix} \tag{D.4.2}$$

where B is a generic $n_\ell \times n_h$ matrix and it is understood that the squared-root should be treated as a power series, namely

$$\sqrt{1-BB^\dagger} \equiv 1 - \frac{1}{2}BB^\dagger - \frac{1}{8}(BB^\dagger)^2 + ... \tag{D.4.3}$$

Notice that this parameterization is the generalization of an orthogonal 2×2 matrix which can be always written in the following form

$$U = \begin{pmatrix} \sqrt{1-\sin^2\theta} & \sin\theta \\ -\sin\theta & \sqrt{1-\sin^2\theta} \end{pmatrix} . \tag{D.4.4}$$

Also it is clear from this that B represents the mixing angle between light and heavy eigenstates. Moreover we want this quantity to be little.

After making explicit the parameterization of W in (D.4.1), we find that

$$M_\ell = \sqrt{1-B^*B^T}L\sqrt{1-BB^\dagger} - B^*D^T\sqrt{1-BB^\dagger} - \sqrt{1-B^*B^T}DB^\dagger + B^*RB^\dagger\,, \tag{D.4.5a}$$

$$M_h = BLB + \sqrt{1-B^TB^*}D^TB + BD\sqrt{1-B^\dagger B} + \sqrt{1-B^TB^*}R\sqrt{1-B^\dagger B}\,, \tag{D.4.5b}$$

$$\mathbb{O} = BL\sqrt{1-BB^\dagger} + \sqrt{1-B^TB^*}D^T\sqrt{1-BB^\dagger} - BDB^\dagger - \sqrt{1-B^TB^*}RB^\dagger\,. \tag{D.4.5c}$$

In particular we can solve the last equation to find B. We proceed perturbatively in powers of DR^{-1} under the assumption that $L, D \ll R$. In addition, we expect that

$$B \sim DR^{-1} + \mathcal{O}(D^2R^{-2}) \tag{D.4.6}$$

since B is supposed to be little. We find that (D.4.5c) reads as follows

$$\mathbb{O} = D^T - RB^\dagger + \mathcal{O}(DR^{-1}) \qquad \Longrightarrow \qquad B^\dagger = R^{-1}D^T + \mathcal{O}(D^2R^{-2})\,. \tag{D.4.7}$$

By substituting this condition inside the other equations we find that at lowest order

$$M_\ell \approx L - DR^{-1}D^T\,, \qquad M_h \approx R\,, \qquad B \approx D^*(R^{-1})^\dagger\,. \tag{D.4.8}$$

D.5 Block-Diagonalization of a Hierarchical Mass-Matrix

The method discussed above can be used also to diagonalize with a singular-valued decomposition a generic matrix that has a suitable hierarchical structure. In particular we can extend the method above to diagonalize perturbatively the mass-matrices of some fermionic fields. A reference of the calculation we are about to make can be found in [22].

Assume to have a mass-Lagrangian in the following form

$$\mathcal{L} = -\begin{pmatrix} \overline{\psi}_L & \overline{\chi}_L \end{pmatrix} \mathbb{M} \begin{pmatrix} \psi_R \\ \chi_R \end{pmatrix} + h.c. \tag{D.5.1}$$

where ψ and χ are n- and n'-dimensional vectors that contain the light and heavy fermionic DOFs respectively. The mass-matrix is a $(n+n')\times(n+n')$ general complex matrix and we assume that it has the following form

$$\mathbb{M} = \begin{pmatrix} m \\ M \end{pmatrix} \tag{D.5.2}$$

where m is $n \times (n+n')$ while M is $n' \times (n+n')$. We also assume the hierarchy $m \ll M$. To go from the flavour- to mass-basis we need to do diagonalize $\mathbb{M}$ with a singular-valued decomposition as shown in Section D.2. In particular to find the LH mixing matrix and the light and heavy mass-eigenstates we need to unitary diagonalize

$$\mathbb{M}\mathbb{M}^\dagger = \begin{pmatrix} mm^\dagger & mM^\dagger \\ Mm^\dagger & MM^\dagger \end{pmatrix} . \tag{D.5.3}$$

As we did for the neutrino mass-matrix in Section D.4, we can block-diagonalize this matrix such that

$$U^\dagger \mathbb{M}\mathbb{M}^\dagger U = \begin{pmatrix} M_\ell^2 & \mathbb{O} \\ \mathbb{O} & M_h^2 \end{pmatrix} \tag{D.5.4}$$

where M_ℓ^2 and M_h^2 are $n \times n$ and $n' \times n'$ hermitian, positive-definite matrices and we can assume that the unitary matrix U is in the following form

$$U = \begin{pmatrix} \sqrt{1-FF^\dagger} & F \\ -F^\dagger & \sqrt{1-F^\dagger F} \end{pmatrix} \tag{D.5.5}$$

where F is a generic $n \times n'$ matrix and it is understood that the squared-root should be treated as a power series. In analogy with before, we have that F represents the mixing angle between the light and heavy DOFs and we assume this quantity to be little.

After making explicit the parameterization of W in (D.5.4), we find that

$$M_\ell^2 = \sqrt{1-FF^\dagger}(mm^\dagger)\sqrt{1-FF^\dagger} - F(Mm^\dagger)\sqrt{1-FF^\dagger} - \sqrt{1-FF^\dagger}(mM^\dagger)F^\dagger + F(MM^\dagger)F^\dagger \, , \tag{D.5.6a}$$

$$M_h^2 = F(mm^\dagger)F + \sqrt{1-F^\dagger F}(Mm^\dagger)F + F(mM^\dagger)\sqrt{1-F^\dagger F} + \sqrt{1-F^\dagger F}(MM^\dagger)\sqrt{1-F^\dagger F} \, , \tag{D.5.6b}$$

$$\mathbb{O} = F(mm^\dagger)\sqrt{1-FF^\dagger} + \sqrt{1-F^\dagger F}(Mm^\dagger)\sqrt{1-FF^\dagger} - F(mM^\dagger)F^\dagger - \sqrt{1-F^\dagger F}(MM^\dagger)F^\dagger \, . \tag{D.5.6c}$$

In particular we can solve the last equation to find F. We proceed perturbatively in powers of $(MM^\dagger)^{-1}$. In addition we expect that[37]

$$F \sim (MM^\dagger)^{-1} + \mathcal{O}((MM^\dagger)^{-2}) \tag{D.5.7}$$

We find that (D.5.6c) reads as follows

$$\mathbb{O} = Mm^\dagger - (MM^\dagger)F^\dagger + \mathcal{O}((MM^\dagger)^{-1}) \quad \Longrightarrow \quad F^\dagger = (MM^\dagger)^{-1}Mm^\dagger + \mathcal{O}((MM^\dagger)^{-2}) \, . \tag{D.5.8}$$

By substituting this condition inside the other equations we find that at lowest order

$$M_\ell^2 \approx m\left[\mathbb{I} - M^\dagger(MM^\dagger)^{-1}M\right]m^\dagger \, , \qquad M_h^2 \approx MM^\dagger \, , \qquad F \approx mM^\dagger(MM^\dagger)^{-1} \, . \tag{D.5.9}$$

In this way we have disentangled the light and heavy DOFs and we can further diagonalize M_ℓ^2 to find the masses of the light mass-eigenstates with the corresponding LH mass-eigenstates.

Unfortunately this method cannot be used to find the RH mixing matrix since we have that

$$\mathbb{M}^\dagger\mathbb{M} = m^\dagger m + M^\dagger M \, . \tag{D.5.10}$$

Sometimes this is not even required. Nevertheless it can be obtained by exploiting the proof done in Section D.2. In particular we want to find two unitary matrices $\mathbb{U}$ and $\mathbb{V}$ such that

$$\mathbb{U}^\dagger\mathbb{M}\mathbb{V} = \hat{\mathbb{M}} \tag{D.5.11}$$

[37]It is left understood that F is adimensional.

where $\hat{\mathbb{M}}$ is diagonal. As shown above, to find $\mathbb{U}$ and $\hat{\mathbb{M}}$ we need to block-diagonalize $\mathbb{M}\mathbb{M}^\dagger$ as in (D.5.4) and then diagonalize M_ℓ^2 and M_h^2. To do so we need to find two unitary matrices U_ℓ and U_h such that

$$U_\ell^\dagger M_\ell^2 U_\ell = \hat{M}_\ell^2\,, \qquad U_h^\dagger M_h^2 U_h = \hat{M}_h^2 \tag{D.5.12}$$

where $\hat{M}_\ell^2$ and $\hat{M}_h^2$ are two positive-definite diagonal matrices. For $\mathbb{V}$ instead we know that it holds the following relation

$$\mathbb{V} = \mathbb{M}^\dagger \mathbb{U} \hat{\mathbb{M}}^{-1}\,. \tag{D.5.13}$$

We can verify that

$$\mathbb{U} = U \begin{pmatrix} U_\ell & \mathbb{O} \\ \mathbb{O} & U_h \end{pmatrix}\,, \quad \hat{\mathbb{M}} = \begin{pmatrix} \hat{M}_\ell & \mathbb{O} \\ \mathbb{O} & \hat{M}_h \end{pmatrix}\,, \quad \mathbb{V} = \begin{pmatrix} m^\dagger & M^\dagger \end{pmatrix} U \begin{pmatrix} U_\ell \hat{M}_\ell^{-1} & \mathbb{O} \\ \mathbb{O} & U_h \hat{M}_h^{-1} \end{pmatrix}\,. \tag{D.5.14}$$

Therefore at lowest order we find that

$$\mathbb{V} \approx \begin{pmatrix} \left[\mathbb{I} - M^\dagger (M M^\dagger)^{-1} M\right] m^\dagger\, U_\ell \hat{M}_\ell^{-1} & M^\dagger U_h \hat{M}_h^{-1} \end{pmatrix}\,. \tag{D.5.15}$$

D.6 Inverse of a Block-Matrix

Consider a generic 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,. \tag{D.6.1}$$

Its inverse is easy to find and it reads as follows

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\,. \tag{D.6.2}$$

Now consider a 2×2 block-matrix. It can be written in the following form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{D.6.3}$$

where A, B, C and D are generic matrices of dimensions $p \times p$, $p \times q$, $q \times p$ and $q \times q$ respectively. If D and $A - BD^{-1}C$ are invertible, then we have that

$$M^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1} BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}\,. \tag{D.6.4}$$

In the subcase where $p = q$, $A = \mathbb{O}$ and B, C and D are invertible, the latter result simplifies to

$$M^{-1} = \begin{pmatrix} C^{-1}DB^{-1} & C^{-1} \\ B^{-1} & \mathbb{O} \end{pmatrix}\,. \tag{D.6.5}$$

Now consider the case where both A and D are vanishing. We are left with a matrix in the form

$$M = \begin{pmatrix} \mathbb{O} & B \\ C & \mathbb{O} \end{pmatrix}\,. \tag{D.6.6}$$

In this case the inverse is given by

$$M^{-1} = \begin{pmatrix} \mathbb{O} & C^{-1} \\ B^{-1} & \mathbb{O} \end{pmatrix}\,. \tag{D.6.7}$$

D.7 Diagonalization of a 2×2 Matrix

Consider a generic 2×2 complex matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{D.7.1}$$

Its eigenvalues and corresponding eigenvectors (to be normalized) are given by

$$\lambda_\pm = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right) , \tag{D.7.2a}$$

$$v_- = \begin{pmatrix} d - a + \sqrt{T^2 - 4D} \\ -2c \end{pmatrix} , \qquad v_+ = \begin{pmatrix} 2b \\ d - a + \sqrt{T^2 - 4D} \end{pmatrix} \tag{D.7.2b}$$

where $T \equiv \mathrm{Tr}[M]$ and $D \equiv \det[M]$. In the limit where $D \ll T^2$ they simplify to

$$\lambda_- = \frac{D}{T}\left[1 + \mathcal{O}\left(\frac{D}{T^2}\right)\right] , \qquad \lambda_+ = T\left[1 - \frac{D}{T^2} + \mathcal{O}\left(\frac{D^2}{T^4}\right)\right] , \tag{D.7.3a}$$

$$v_- = \begin{pmatrix} d - D/T\left[1 + \mathcal{O}(D/T^2)\right] \\ -2c \end{pmatrix} , \qquad v_+ = \begin{pmatrix} 2b \\ d - D/T\left[1 + \mathcal{O}(D/T^2)\right] \end{pmatrix} . \tag{D.7.3b}$$

Consider the subcase where M is in the following form

$$M = \begin{pmatrix} |a|^2 & a^* b \\ a b^* & |b|^2 \end{pmatrix} . \tag{D.7.4}$$

The eigenvalues are given by

$$\lambda_- = 0 , \qquad \lambda_+ = |a|^2 + |b|^2 \tag{D.7.5}$$

with corresponding eigenvectors (already normalized)

$$v_- = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{pmatrix} -b \\ a \end{pmatrix} , \qquad v_+ = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{pmatrix} a^* \\ b^* \end{pmatrix} . \tag{D.7.6}$$

Consider the subcase where M is in the following form

$$M = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} . \tag{D.7.7}$$

The eigenvalues are given by

$$\lambda_\pm = \pm|a| \tag{D.7.8}$$

with corresponding eigenvectors (already normalized)

$$v_\pm = \frac{1}{\sqrt{2}|a|} \begin{pmatrix} \pm|a| \\ a \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ e^{i\theta} \end{pmatrix} \tag{D.7.9}$$

where we have defined

$$e^{i\theta} \equiv \frac{a}{|a|} \qquad \Longleftrightarrow \qquad \theta = \arg[a] . \tag{D.7.10}$$

D.8 Diagonalization of a 3×3 Matrix

Consider a generic 3×3 hermitian matrix

$$M = \begin{pmatrix} a & b & c \\ b^* & d & e \\ c^* & e^* & f \end{pmatrix} . \tag{D.8.1}$$

We assume that holds the following hierarchy among the entries where ε is a small real parameter that we assume to be $\varepsilon \ll 1$

$$a \sim \varepsilon^4 , \quad b \sim \varepsilon^3 \quad c, d \sim \varepsilon^2 , \quad e \sim \varepsilon , \quad f \sim \varepsilon^0 . \tag{D.8.2}$$

We want to diagonalize M perturbatively. To this purpose we represent the order in perturbative expansion of all the parameters adding explicitly the ε-dependence as follows

$$M = \begin{pmatrix} \varepsilon^4 a & \varepsilon^3 b & \varepsilon^2 c \\ \varepsilon^3 b^* & \varepsilon^2 d & \varepsilon\, e \\ \varepsilon^2 c^* & \varepsilon\, e & f \end{pmatrix} . \tag{D.8.3}$$

In this way it will be very easy to diagonalize it perturbatively.

To find the eigenvalues we need to derive the characteristic polynomial associated to M. Explicitly it is given by

$$\det[M - \lambda \mathbb{I}] = -\lambda^3 + \lambda^2 (a + d + f) + \lambda \left(|e|^2 + |c|^2 + |b|^2 - ad - af - df \right) + \det[M] \tag{D.8.4}$$

where

$$\det[M] = adf + bc^* e + b^* c e^* - a|e|^2 - d|c|^2 - f|b|^2 . \tag{D.8.5}$$

We want to find the roots of this polynomial in perturbation theory in powers of ε. To this purpose we write the roots as follows

$$\lambda = \lambda_0 + \varepsilon^2 \lambda_1 + \varepsilon^4 \lambda_2 + \varepsilon^6 \lambda_3 + \mathcal{O}(\varepsilon^8) \tag{D.8.6}$$

where we used that fact that all the odd powers in ε vanish because the characteristic polynomial is a power series in ε^2. By making explicit the dependence in ε, we find that (D.8.4) reads as follows

$$\begin{aligned} \det[M - \lambda \mathbb{I}] &= -\lambda_0^3 + f\lambda_0^2 + \varepsilon^2 \left(d\lambda_0^2 - df\lambda_0 + 2f\lambda_0\lambda_1 - 3\lambda_0^2\lambda_1 + |e|^2\lambda_0 \right) \\ &+ \varepsilon^4 \left(-af\lambda_0 + a\lambda_0^2 - df\lambda_1 + 2d\lambda_0\lambda_1 + f\lambda_1^2 - 3\lambda_0\lambda_1^2 + 2f\lambda_0\lambda_2 - 3\lambda_0^2\lambda_2 + |c|^2\lambda_0 + |e|\lambda_1 \right) \\ &+ \varepsilon^6 \Big(\det[M] - ad\lambda_0 - af\lambda_1 + 2a\lambda_0\lambda_1 + d\lambda_1^2 - \lambda_1^3 - df\lambda_2 + 2d\lambda_0\lambda_2 \\ &\qquad + 2f\lambda_1\lambda_2 - 6\lambda_0\lambda_1\lambda_2 + 2f\lambda_0\lambda_3 - 3\lambda_0^2\lambda_3 \Big) + \mathcal{O}(\varepsilon^8) . \end{aligned} \tag{D.8.7}$$

We can now solve it for each power of ε^2 and find the following eigenvalues

$$\Lambda_3 = f + \mathcal{O}(\varepsilon^2) , \qquad \Lambda_2 = d - \frac{|e|^2}{f} + \mathcal{O}(\varepsilon^4) , \qquad \Lambda_1 = \frac{\det[M]}{df - |e|^2} + \mathcal{O}(\varepsilon^6) . \tag{D.8.8}$$

The corresponding eigenvectors can be found by solving the equation

$$(M - \Lambda_i \mathbb{I})\, v_i = \overline{0} . \tag{D.8.9}$$

We find that the three corresponding eigenvectors (to be normalized) are given by

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mathcal{O}(\varepsilon) , \quad v_2 = \frac{1}{f} \begin{pmatrix} 0 \\ f \\ -e^* \end{pmatrix} + \mathcal{O}(\varepsilon^2) , \quad v_1 = \frac{1}{df - |e|^2} \begin{pmatrix} df - |e|^2 \\ -fb^* - ec^* \\ dc^* - b^* e^* \end{pmatrix} + \mathcal{O}(\varepsilon^3) . \tag{D.8.10}$$

E Scattering Tools

In this Chapter we are going to study in detail some important tools that we have used in this work.

E.1 Majorana Field

In this Section we study in detail how is described a fermionic field with a Majorana mass. A reference can be found for instance in [5].

Given a fermion ψ with Right- or Left-chirality, we define its Charge-Conjugate as follows

$$\psi^c \equiv \mathcal{C}\,\overline{\psi}^T = -i\gamma^2\psi^* \tag{E.1.1}$$

where

$$\mathcal{C} \equiv i\gamma^0\gamma^2 = \begin{pmatrix} -i\sigma^2 & \mathbb{O} \\ \mathbb{O} & i\sigma^2 \end{pmatrix} \tag{E.1.2}$$

is the Charge-Conjugation operator and it is a 4×4 matrix in spinor space. It satisfies the following properties

$$\mathcal{C}^{-1} = \mathcal{C}^\dagger = \mathcal{C}^T = -\mathcal{C}\,, \tag{E.1.3a}$$

$$\mathcal{C}(\gamma^\mu)^T\mathcal{C}^{-1} = -\gamma^\mu\,, \qquad \mathcal{C}\gamma^5\mathcal{C}^{-1} = \gamma^5\,. \tag{E.1.3b}$$

Using the equations above it is possible to show that

$$\overline{\psi}^c = \psi^T\mathcal{C}\,. \tag{E.1.4}$$

A Majorana fermion is described by a self-conjugate field, hence

$$\psi^c = \psi\,. \tag{E.1.5}$$

This condition implies that it has half of the DOFs of a Dirac spinor since it must be in the following form (compared to the one described in Section G.2)

$$\psi = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix}\,. \tag{E.1.6}$$

A Majorana mass m for this field can be provided only by the following term in the Lagrangian (remember that ψ has a precise fixed chirality)

$$\mathcal{L} \supset -\frac{1}{2}m\,\overline{\psi}^c\psi + h.c. = -\frac{1}{2}m\,\psi^T\mathcal{C}\,\psi + h.c. \tag{E.1.7}$$

One of the main features of this mass-term is that it breaks explicitly some Symmetries that instead are present in a Dirac-type mass-term. To see this, consider a unitary transformation on ψ which in general reads as follows

$$\psi \quad\rightarrow\quad U\psi\,. \tag{E.1.8}$$

We have that the Charge-Conjugate field transforms as follows

$$\psi^c \quad\rightarrow\quad U^*\psi^c\,. \tag{E.1.9}$$

Therefore the Majorana-type mass-term in the Lagrangian breaks explicitly any U(1) transformation of ψ. This means that a Majorana fermion implies Fermion Number violation.

When we deal with Majorana fermions, the Feynman rule of the propagator is slightly different than the Dirac-type one due to the explicit Fermion Number violation. To compute the 2-point function, we consider the free Lagrangian which reads as follows

$$\mathcal{L} = \overline{\psi} i\partial\!\!\!/ \psi - \frac{1}{2} m\, \psi^T \mathcal{C}\, \psi - \frac{1}{2} m\, \overline{\psi}\, \mathcal{C}\, \overline{\psi}^T \equiv \frac{1}{2} \overline{\Psi} M \Psi \tag{E.1.10}$$

where

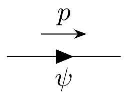
$$\Psi = \begin{pmatrix} \psi \\ \overline{\psi}^T \end{pmatrix}, \qquad M = \begin{pmatrix} i\partial\!\!\!/ & -m\,\mathcal{C} \\ -m\,\mathcal{C} & i\mathcal{C}^{-1}\partial\!\!\!/\,\mathcal{C} \end{pmatrix}. \tag{E.1.11}$$

Using the Feynman prescription to find the propagator, we need to invert M. Going to momentum space $\partial_\mu \to -ip_\mu$ we find that

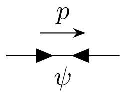
$$M^{-1} = \frac{1}{p^2 - m^2} \begin{pmatrix} \mathcal{C}^{-1} p\!\!\!/ \mathcal{C} & m\,\mathcal{C} \\ m\,\mathcal{C} & p\!\!\!/ \end{pmatrix}. \tag{E.1.12}$$

This implies that we have two possible 2-point functions. One that preserves the fermionic current and one that breaks it by two units. They explicitly read as follows with the corresponding Feynman diagrams

$$iS(p) \equiv \langle \psi \overline{\psi} \rangle = \frac{i p\!\!\!/}{p^2 - m^2} \tag{E.1.13a}$$



$$iS'(p) \equiv \langle \psi \psi^T \rangle = \frac{i m\,\mathcal{C}}{p^2 - m^2} \tag{E.1.13b}$$



Also notice that to get the antiparticle propagators one needs to invert the arrows in the Feynman diagrams, but this is equivalent to work with the propagators of the particles with opposite momentum, hence $iS(-p)$ and $iS'(-p)$.

E.2 Hadronic Scattering

Hadrons are bound states of quarks and gluons. They are the manifestation of QCD confinement. Since they are not elementary particles, we have to know how to deal with them when we want to compute scattering processes where hadrons are involved. This is what we study in this Section. A reference can be found for instance in [23].

We know that processes involving leptons and quarks are mediated by Weak Interactions (WI). Since when we deal with hadrons we can assume to be at low energy, we can use Fermi theory. The Fermi Lagrangian reads as follows

$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}} \left(J^+_\mu J^{-\mu} + J^0_\mu J^{0\,\mu} \right) \tag{E.2.1}$$

where we have defined the Charged and Neutral Currents (CC and NC)

$$J^+_\mu = \sum_{u,d} \overline{d} \gamma_\mu (1 - \gamma^5) V^*_{ud} u + \sum_{j,\ell} \overline{\ell} \gamma_\mu (1 - \gamma^5) U_{\ell j} \nu_j\,, \tag{E.2.2a}$$

$$J^0_\mu = \sum_{f = u,d,\ell,\nu} \overline{f} \gamma^\mu \left(g^f_V - g^f_A \gamma^5 \right) f\,. \tag{E.2.2b}$$

V and U are the CKM and PMNS matrices respectively, $J^-_\mu = (J^+_\mu)^\dagger$ and

$$g^f_V = T^3_f - 2 s_W \left(T^3_f + Y_f \right), \qquad g^f_A = T^3_f \tag{E.2.3}$$

where T^3 and Y are the third generator of $\mathrm{SU}(2)_L$ and the generator of $\mathrm{U}(1)_Y$ respectively.

Hadrons are specific combinations of quarks. For instance pions are given by

$$|\pi^+\rangle = |u\overline{d}\rangle\,, \qquad |\pi^-\rangle = |d\overline{u}\rangle\,, \qquad |\pi^0\rangle = \frac{1}{\sqrt{2}}|u\overline{u} - d\overline{d}\rangle \tag{E.2.4}$$

while kaons are

$$|K^+\rangle = |u\overline{s}\rangle\,, \qquad |K^-\rangle = |s\overline{u}\rangle\,, \qquad |K^0\rangle = |d\overline{s}\rangle\,, \qquad |\overline{K}^0\rangle = |s\overline{d}\rangle\,. \tag{E.2.5}$$

Then there are many others, like D mesons which are like pions and kaons but with quark c instead of u. To compute scattering amplitudes where hadrons are involved, one has to take the matrix element of the Lagrangian on initial and final states. The lepton part factorizes as usual. Instead the hadronic part is unknown. For instance, in case of an incoming pseudoscalar meson h_P in a process mediated by CC, we need to compute the following matrix element

$$\langle 0|J^+_\mu|h^-_P\rangle\,. \tag{E.2.6}$$

By dimensional analysis we observe that it has mass-dimension +2. Also the only Lorentz tensors on which it can depend on are the metric tensor $g_{\mu\nu}$ and the momentum of h_P p_μ. Hence we can write that matrix element as follows

$$\langle 0|J^+_\mu|h^-_P\rangle \equiv i f_h p_\mu \tag{E.2.7}$$

where f_h is a constant with mass-dimension $[f_h] = +1$. It contains all what we do not know from non-perturbative effects that are responsible for the formation of the quark bound state. From Symmetry considerations we can prove that matrix elements with hadrons and CC or NC are related. In particular we have that in the Isospin-preserving limit

$$\langle 0|J^0_\mu|h^0_P\rangle \equiv -\frac{i}{\sqrt{2}} f_h p_\mu\,. \tag{E.2.8}$$

An explanation on why this is true can be found in [23]. The unknown constants f_h are extracted from measurements and, in our notation, their values are listed in [23].

E.3 Phase Space of a $1 \to 2$ Decay Process

In this Section we compute the phase space of a decay process of type $A \to BC$. In full generality from (G.3.1) we have that the phase space integral reads as follows

$$\int d\Phi = \int \frac{d^3\vec{p}_B}{(2\pi)^3 2E_B}\frac{d^3\vec{p}_C}{(2\pi)^3 2E_C}(2\pi)^4\delta^4(p_A - p_B - p_C)\,. \tag{E.3.1}$$

Using momentum conservation we find that in this process the only independent scalar product among the momenta is given by

$$2\,p_A\cdot p_B = m_A^2 + m_B^2 - m_C^2\,. \tag{E.3.2}$$

Therefore the squared-amplitude is momentum independent and it factorizes out of the integration when we consider the decay rate of the process. This allow us to perform all the integrations over the momenta finding in the end just an expression to multiply to the rest to find the total decay rate.

At first we perform the integration over $d^3\vec{p}_C$ using the δ-function finding

$$\int d\Phi = \frac{1}{16\pi^2}\int \frac{d^3\vec{p}_B}{E_B E_C}\delta(E_A - E_B - E_C)\,. \tag{E.3.3}$$

Since the expression is Lorentz invariant, we can choose to work is any reference frame we want. We choose to work in the rest frame of A. This implies that explicitly

$$E_A^2 = m_A^2\,, \qquad E_B^2 = m_B^2 + |\vec{p}_B|^2\,, \qquad E_C^2 = m_C^2 + |\vec{p}_B|^2\,. \tag{E.3.4}$$

For the integration over $d^3\vec{p}_B$, we go into spherical coordinates

$$d^3\vec{p}_B = |\vec{p}_B|^2 d|\vec{p}_B| d\Omega_B = 4\pi |\vec{p}_B|^2 d|\vec{p}_B| \,. \tag{E.3.5}$$

Calling $p \equiv |\vec{p}_B|$, we have that the phase space reads as follows

$$\int d\Phi = \frac{1}{4\pi} \int dp \, \frac{p^2}{\sqrt{m_B^2 + p^2}\sqrt{m_C^2 + p^2}} \delta\left(m_A - \sqrt{m_B^2 + p^2} - \sqrt{m_C^2 + p^2}\right) . \tag{E.3.6}$$

We still have to perform the integration over p. To do so we use the last δ-function. We observe that

$$\frac{\partial}{\partial p}\left(\sqrt{m_B^2 + p^2} + \sqrt{m_C^2 + p^2} - m_A\right) = p\frac{\sqrt{m_B^2 + p^2} + \sqrt{m_C^2 + p^2}}{\sqrt{m_B^2 + p^2}\sqrt{m_C^2 + p^2}} \,. \tag{E.3.7}$$

Thus, after performing the last integration, we find that

$$\int d\Phi = \frac{1}{4\pi} \frac{\hat{p}}{\sqrt{m_B^2 + \hat{p}^2} + \sqrt{m_C^2 + \hat{p}^2}} \tag{E.3.8}$$

where $\hat{p}$ is defined such that it satisfies

$$m_A - \sqrt{m_B^2 + \hat{p}^2} - \sqrt{m_C^2 + \hat{p}^2} = 0 \,. \tag{E.3.9}$$

By solving this equation we find that

$$\hat{p}^2 = \frac{1}{4m_A^2} \lambda(m_A^2, m_B^2, m_C^2) \tag{E.3.10}$$

where we have defined the Källén function

$$\lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \tag{E.3.11}$$

Therefore the phase space of a $A \to BC$ process reads as follows

$$\Phi = \frac{1}{8\pi} \frac{1}{m_A^2} \lambda^{\frac{1}{2}}(m_A^2, m_B^2, m_C^2) \,. \tag{E.3.12}$$

E.4 3-Body Phase Space

In this Section we compute the phase space of a generic process with three particles in the final state. We define the momenta of the particles in the final state as follows (where $i = 1, 2, 3$)

$$p_i = (E_i, \vec{p}_i) \tag{E.4.1}$$

and they all satisfy their on-shellness conditions $p_i^2 = m_i^2$. We define also the transferred momentum

$$q = p_1 + p_2 + p_3 \,. \tag{E.4.2}$$

In full generality from (G.3.1) we have that the phase space integral reads as follows

$$\int d\Phi = \int \frac{d^3\vec{p}_1}{(2\pi)^3\, 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3\, 2E_2} \frac{d^3\vec{p}_3}{(2\pi)^3\, 2E_3} (2\pi)^4 \, \delta^4 \left(q - p_1 - p_2 - p_3\right) . \tag{E.4.3}$$

Since the expression is Lorentz invariant, we can work in any reference frame that we want. We choose to work in the Centre of Mass (CoM) frame. This implies that

$$q = (\sqrt{s}, \vec{0}) \tag{E.4.4}$$

where $s = q^2$ is the first Mandelstam variable. We also define the so-called Bjorken dimensionless variables as follows

$$x_i \equiv \frac{2\, p_i \cdot q}{q^2} = \frac{2E_i}{\sqrt{s}}\,, \qquad \varepsilon_i \equiv \frac{m_i^2}{s} \tag{E.4.5}$$

and we observe that from momentum conservation (E.4.2) they satisfy the following condition

$$x_1 + x_2 + x_3 = 2\,. \tag{E.4.6}$$

We want to simplify the integration over the momenta as far as possible. At first we perform the integration over $d^3\vec{p}_3$ using the δ-function finding

$$\int d\Phi = \frac{1}{8(2\pi)^5} \int \frac{d^3\vec{p}_1\, d^3\vec{p}_2}{E_1 E_2 E_3}\, \delta(\sqrt{s} - E_1 - E_2 - E_3)\,. \tag{E.4.7}$$

Now we can go into spherical coordinates and we find that the remained integration measure can be cast in the following way

$$d^3\vec{p}_1\, d^3\vec{p}_2 = |\vec{p}_1|^2 |\vec{p}_2|^2 d|\vec{p}_1| d|\vec{p}_2|\, d\Omega_1 d\Omega_{12} = 8\pi^2\, |\vec{p}_1|^2 |\vec{p}_2|^2 d|\vec{p}_1| d|\vec{p}_2|\, d\cos\theta_{12} \tag{E.4.8}$$

where $d\Omega_1$ is the spherical integral measure associated to the solid angle of $\vec{p}_1$ while $d\Omega_{12}$ is associated to the solid angle between $\vec{p}_1$ and $\vec{p}_2$. Moreover $d\Omega_1$ is trivial and it is integrated out to leave a factor of 4π, while $d\Omega_{12}$ is trivial in its azimuthal part and it leaves a factor of 2π and a non-trivial integration over $d\cos\theta_{12}$. We can see that only E_3 shows an explicit dependence on $\cos\theta_{12}$ since

$$E_3 = \sqrt{m_3^2 + |\vec{p}_1 + \vec{p}_2|^2} = \sqrt{m_3^2 + |\vec{p}_1|^2 + |\vec{p}_2|^2 + 2|\vec{p}_1||\vec{p}_2|\cos\theta_{12}}\,. \tag{E.4.9}$$

Thus we can get rid of the integration over this last angle using the last δ-function, leading to the following expression

$$\int d\cos\theta_{12}\, \delta(\sqrt{s} - E_1 - E_2 - E_3) = \left| \frac{\partial E_3}{\partial \cos\theta_{12}} \right|^{-1} = \frac{E_3}{|\vec{p}_1||\vec{p}_2|}\,. \tag{E.4.10}$$

Therefore, using the following change of integration variables

$$|\vec{p}_i| d|\vec{p}_i| = E_i dE_i\,, \qquad dE_i = \frac{\sqrt{s}}{2} dx_i \tag{E.4.11}$$

and grouping all the expressions together, we find that

$$\int d\Phi = \frac{1}{32\pi^3} \int dE_1 dE_2 = \frac{q^2}{128\pi^3} \int_{x_1^{min}}^{x_1^{max}} dx_1 \int_{x_2^{min}}^{x_2^{max}} dx_2\,. \tag{E.4.12}$$

What is left is to find are the integration bounds. We start from looking at the bounds on x_1. Its minimum is trivially when $E_1 = m_1$, while for its maximum we need to find when E_1 reaches its maximum, which is equivalent to find the minimum of $E_2 + E_3$. Furthermore the configuration in which E_1 is maximal is for sure when $\vec{p}_1$ is antiparallel to $\vec{p}_2$ and $\vec{p}_3$, which means that we need to minimize with respect to $|\vec{p}_2|$

$$E_2 + E_3 = \sqrt{m_2^2 + |\vec{p}_2|^2} + \sqrt{m_3^2 + (|\vec{p}_2| - |\vec{p}_1|)^2}\,. \tag{E.4.13}$$

This happens when

$$|\vec{p}_2| = \frac{m_2}{m_2 + m_3} |\vec{p}_1|\,. \tag{E.4.14}$$

By inserting this condition inside the energy conservation equation

$$\sqrt{s} = E_1 + E_2 + E_3\,, \tag{E.4.15}$$

we find the maximum of x_1. In the end we have that

$$x_1^{max} = 1 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - 2\sqrt{\varepsilon_2}\sqrt{\varepsilon_3}\,, \tag{E.4.16a}$$

$$x_1^{min} = 2\sqrt{\varepsilon_1}\,. \tag{E.4.16b}$$

At fixed x_1 now we need to find the bounds on x_2. To do so we need to exploit the condition that fixes $\cos\theta_{12}$. From energy conservation (E.4.15) we find the following relation written in terms of Bjorken variables

$$x_1x_2 - 2x_1 - 2x_2 + 2 + 2\varepsilon_1 + 2\varepsilon_2 - 2\varepsilon_3 = \sqrt{x_1^2 - 4\varepsilon_1}\sqrt{x_2^2 - 4\varepsilon_2}\cos\theta_{12}\,. \tag{E.4.17}$$

Since $-1 \leq \cos\theta_{12} \leq 1$, we can find the integration bounds on x_2 written in terms of x_1 by solving for x_2 the equation above when $\cos\theta_{12} = \pm 1$. We find that

$$\begin{aligned} x_2^{max} = \frac{1}{2(1+\varepsilon_1 - x_1)}\Big[&(2-x_1)(1+\varepsilon_1+\varepsilon_2-\varepsilon_3-x_1) \\ &+\sqrt{x_1^2-4\varepsilon_1}\sqrt{(\varepsilon_1-\varepsilon_2)^2+2(1-x_1)(\varepsilon_1-\varepsilon_2)+(1-\varepsilon_3-x_1)^2-2\varepsilon_3(\varepsilon_1+\varepsilon_2)}\Big]\,, \end{aligned} \tag{E.4.18a}$$

$$\begin{aligned} x_2^{min} = \frac{1}{2(1+\varepsilon_1 - x_1)}\Big[&(2-x_1)(1+\varepsilon_1+\varepsilon_2-\varepsilon_3-x_1) \\ &-\sqrt{x_1^2-4\varepsilon_1}\sqrt{(\varepsilon_1-\varepsilon_2)^2+2(1-x_1)(\varepsilon_1-\varepsilon_2)+(1-\varepsilon_3-x_1)^2-2\varepsilon_3(\varepsilon_1+\varepsilon_2)}\Big]\,. \end{aligned} \tag{E.4.18b}$$

Also it is easy to check that, in the range of x_1, all the values of x_2 in the range above are well-defined.

In the end we found an integration measure that is still Lorentz invariant and depends only on two of the Bjorken variables. It is also possible to show that any squared-amplitude associated to a process with three particles in the final state depends on only two of the Bjorken variables. This is due to the fact that, from Lorentz invariance, this quantity depends only on scalar products between initial- and final-state momenta. Moreover, since the initial-state momenta are all related and they depend only on q, the only independent scalar products are between q, p_1, p_2 and p_3. Now we can show that all these combinations can be written in terms of x_1, x_2, s and the masses of the particles. We start from replacing all p_3 with the other three momenta exploiting momentum conservation. Then $q \cdot p_1$ and $q \cdot p_2$ are easily related to x_1 and x_2. The only non-trivial scalar product is $p_1 \cdot p_2$, but exploiting momentum conservation and in particular (E.4.6) we can show that

$$2\,p_1 \cdot p_2 = m_3^2 - m_1^2 - m_2^2 + s(x_1 + x_2 - 1)\,. \tag{E.4.19}$$

By following these prescriptions one can always write any possible integrand of the 3-body phase space as function of x_1 and x_2 only (plus s and all the masses involved).

F Standard Model Lagrangian

In this Chapter we are going to write down explicitly the whole SM Lagrangian, both in the unbroken and broken EW phases. This is done to fix all the possible conventions related. Then we are going to provide the relevant Feynman rules that we have used to perform all the calculations done in this work. The conventions used are the ones in [24] with $\eta, \eta', \eta_s, \eta_e = -1$ and $\eta_Y, \eta_Z, \eta_\theta = 1$.

F.1 Unbroken Electro-Weak Phase

We consider the SM Gauge Group in the unbroken EW phase

$$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \tag{F.1.1}$$

and the Gauge couplings and fields associated[38] are (with the SU(3) and SU(2) indices $a = 1, ..., 8$, $i = 1, 2, 3$ respectively)

$$(g_s, g, g')\ , \qquad \Big(G^a_\mu, W^i_\mu, B_\mu\Big)\ . \tag{F.1.2}$$

The field-strength tensors are given by

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu\,, \tag{F.1.3a}$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\,\varepsilon^{ijk} W^j_\mu W^k_\nu\,, \tag{F.1.3b}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{F.1.3c}$$

where f^{abc} and ε^{ijk} are the structure constants of SU(3) and SU(2) respectively and they are listed in Section G.4 together with the corresponding generators that we call $\hat{T}^a$ and T^i respectively. The fermionic particle content is listed in Table 3 and they are all Weyl massless fermions. Finally we have the Higgs field $H \sim (\mathbf{1}, \mathbf{2}, 1/2)$.

ψ	$\mathrm{SU}(3)_C$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$
Q^i_L	**3**	**2**	1/6
L^i_L	**1**	**2**	-1/2
u^i_R	**3**	**1**	2/3
d^i_R	**3**	**1**	-1/3
e^i_R	**1**	**1**	-1
ν^i_R	**1**	**1**	0

Table 3: SM fermionic particle content in the unbroken EW phase. $i = 1, 2, 3$ represent the flavour index.

In the unbroken phase the SM Lagrangian reads as follows

$$\begin{aligned}\mathcal{L}_{\mathrm{SM}} = &-\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \sum_\psi \overline{\psi} i \not{D} \psi + (D_\mu H)^\dagger (D^\mu H) \\ &- \Big(\overline{Q}^i_L H Y^D_{ij} d^j_R + \overline{Q}^i_L H^c Y^U_{ij} u^j_R + \overline{L}^i_L H Y^E_{ij} e^j_R + \overline{L}^i_L H^c Y^N_{ij} \nu^j_R + h.c.\Big) - \mu^2 H^\dagger H - \lambda (H^\dagger H)^2\end{aligned} \tag{F.1.4}$$

where Y^U, Y^D, Y^N and Y^E are the Yukawa matrices, $H^c \equiv i\sigma^2 H^*$ is the Charge-Conjugate of the Higgs field and the covariant derivative reads as follows

$$D_\mu = \partial_\mu - i g_s \hat{T}^a G^a_\mu - i g T^i W^i_\mu - i g' Y B_\mu\,. \tag{F.1.5}$$

[38] Notice that in this Section (as well as in Chapter 5) we are using different notation for Gauge fields and couplings with respect to Chapter 1.

F.2 Broken Electro-Weak Phase

When the Higgs acquires a VEV, the SM Gauge Group reduces to

$$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \quad \rightarrow \quad \mathrm{SU}(3)_C \times \mathrm{U}(1)_Q \,. \tag{F.2.1}$$

This happens because the squared-mass of the Higgs doublet μ^2 becomes negative and the Higgs potential develops a new global minimum in

$$\langle H \rangle = \frac{v}{\sqrt{2}} \qquad \text{where} \qquad v^2 = -\frac{\mu^2}{\lambda} \,. \tag{F.2.2}$$

The Higgs doublet gets split as follows (as well as its Charge-Conjugate)

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\,\phi^+ \\ v + h + i\phi^0 \end{pmatrix} , \qquad H^c = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h - i\phi^0 \\ -\sqrt{2}\,\phi^- \end{pmatrix} \tag{F.2.3}$$

where h is the Higgs field in the broken EW phase with mass

$$M_h^2 = 2\lambda v^2 \tag{F.2.4}$$

while $\phi^\pm$ and ϕ^0 are the massless Goldstone bosons (GBs) associated to the broken generators of $\mathrm{SU}(2)_L$.

From the kinetic term of the Higgs doublet we get the mass-spectrum of the vector fields. We find that the mass-basis is given by

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp i W_\mu^2 \right) \qquad M_W^2 = \frac{1}{4} g^2 v^2 \,, \tag{F.2.5a}$$

$$Z_\mu = c_W W_\mu^3 - s_W B_\mu \qquad M_Z^2 = \frac{1}{4} \left(g^2 + g'^2 \right) v^2 \,, \tag{F.2.5b}$$

$$A_\mu = s_W W_\mu^3 + c_W B_\mu \qquad M_A^2 = 0 \tag{F.2.5c}$$

where we have defined

$$c_W = \frac{g}{\sqrt{g^2 + g'^2}} \,, \qquad s_W = \frac{g'}{\sqrt{g^2 + g'^2}} \,. \tag{F.2.6}$$

We have that h, ϕ^0 and Z_μ are singlets under the residual Gauge Group while $\phi^\pm, W_\mu^\pm \sim \pm 1 \in U(1)_Q$. A_μ is the Gauge field of the residual $\mathrm{U}(1)_Q$ Symmetry and the Charge associated is given by

$$Q = Y + T^3 \tag{F.2.7}$$

with coupling

$$e = \frac{g g'}{\sqrt{g^2 + g'^2}} \,. \tag{F.2.8}$$

The covariant derivative in the broken phase reads as follows

$$D_\mu = \partial_\mu - i g_s \hat{T}^a A_\mu^a - i g \left(T^+ W_\mu^+ + T^- W_\mu^- \right) - i e Q A_\mu - i \frac{g}{c_W} \left(T^3 - s_W^2 Q \right) Z_\mu \tag{F.2.9}$$

where we have defined

$$T^\pm = \frac{1}{\sqrt{2}} \left(T^1 \pm i T^2 \right) \,. \tag{F.2.10}$$

From the Yukawa interactions with the Higgs doublet in the unbroken phase are generated the fermion masses. The fermionic particle content in the broken phase is listed in Table 4 and are all Dirac fermions.

This time the SM Lagrangian is very long and we are not going to write it explicitly in this work, but it can be found in [24]. Also it is worth to define the fine structure constant and the Fermi constant

$$\alpha \equiv \frac{e^2}{4\pi} \,, \qquad G_F \equiv \sqrt{2} \frac{g^2}{8\, M_W^2} = \frac{1}{\sqrt{2}\, v^2} \,. \tag{F.2.11}$$

ψ	SU(3)$_C$	U(1)$_Q$
u^i	**3**	2/3
d^i	**3**	-1/3
e^i	**1**	-1
ν^i	**1**	0

Table 4: SM fermionic particle content in the broken EW phase. $i = 1, 2, 3$ represent the flavour index.

F.3 Gauge Fixing

Since SM is a non-abelian Gauge theory, we have some redundancy in the definition of the Gauge bosons. This is a problem because, to find the propagator of a field φ, one has to consider the free (hence quadratic) part of the Lagrangian in momentum space

$$\mathcal{L}_\varphi = \varphi^\dagger M \varphi \tag{F.3.1}$$

and invert the bilinear tensor M. However, if we have some redundant DOFs as it happens for the Gauge bosons, this is not invertible. Thus we need to add a Gauge-fixing term to *fix the Gauge* and remove this redundancy.

In the SM case in the broken phase we add the following Gauge-fixing terms[39]

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi}\left(F_G^2 + F_A^2 + F_Z^2 + 2F_+F_-\right) \tag{F.3.2}$$

where

$$F_G^a = \partial^\mu G_\mu^a\,, \tag{F.3.3a}$$
$$F_A = \partial^\mu A_\mu\,, \tag{F.3.3b}$$
$$F_Z = \partial^\mu Z_\mu - \xi M_Z \phi^0\,, \tag{F.3.3c}$$
$$F_\pm = \partial^\mu W_\mu^\pm \mp i\xi M_W \phi^\pm\,. \tag{F.3.3d}$$

This Gauge-fixing condition is called R_ξ Gauge. The parameter ξ is just a Lagrangian multiplier and it is physically irrelevant. Any observable must be ξ-independent. It can be left free during the computation of fixed to simplify it. Also it is possible to introduce different ξ-parameters for each of the Gauge-fixing terms. The Gauge-fixing terms for the massive vector bosons contain a dependence on the GBs to remove an unwanted mixing term in the free Lagrangian. In fact, when we consider the free part of SM Lagrangian in the broken EW phase, we find that

$$(D_\mu H)^\dagger(D^\mu H) \supset \frac{1}{2}M_Z^2 Z_\mu Z^\mu + M_W^2 W_\mu^+ W^{-\,\mu} + M_Z Z_\mu(\partial^\mu\phi^0) + iM_W\left[W_\mu^-(\partial^\mu\phi^+) - W_\mu^+(\partial^\mu\phi^-)\right] \tag{F.3.4}$$

and the last two (unwanted) terms are removed perfectly (up to a total derivative) since explicitly

$$\begin{aligned}\mathcal{L}_{\text{GF}} = &-\frac{1}{2\xi}\left[(\partial^\mu G_\mu^a)(\partial^\nu G_\nu^a) + (\partial^\mu A_\mu)^2 + (\partial^\mu Z_\mu)^2 + 2(\partial^\mu W_\mu^+)(\partial^\nu W_\nu^-)\right]\\ &-\frac{1}{2}\xi M_Z^2(\phi^0)^2 - \xi M_W^2\phi^+\phi^- + M_Z(\partial^\mu Z_\mu)\phi^0 + iM_W\left[(\partial^\mu W_\mu^-)\phi^+) - (\partial^\mu W_\mu^+)\phi^-\right]\,.\end{aligned} \tag{F.3.5}$$

Observe that, due to this Gauge-fixing terms, the GBs acquire a squared-mass which is given by ξ times the squared-mass of the massive vector boson associated.

[39] In principle those terms are added in the unbroken phase, but it is possible to show that, under a suitable choice of Gauge-fixing terms in the unbroken phase, one reproduces those Gauge-fixing conditions in the broken phase.

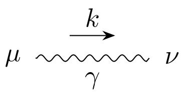
There are several possible Gauge-fixing choices for ξ. When $\xi \to \infty$ we are in the Unitary Gauge where no GBs are present in the theory. When $\xi = 0$ we are in the Landau Gauge. When $\xi = 1$ we are in the 't Hooft Gauge and it is particularly useful while dealing with loop calculations since the vector boson propagators do not show the term $k^\mu k^\nu$ in the numerator (for a better discussion see for instance [24]) which brings further divergent contributions in the loop integrals.

To conclude, let us mention that in the case of non-abelian Gauge Group it is important to add also to the Lagrangian the contributions of the ghost fields. However, since in this work we have not considered any process where they are involved, we are not going to discuss them here, but we only cite as a good reference [24] for a complete treatment.

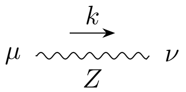
F.4 Feynman Rules

In this Section we list all the relevant Feynman rules in R_ξ Gauge that we needed to use to compute the processes considered in this work. They are derived from the SM Lagrangian in the notation presented above. For all the cases recall that $W^-_\mu = (W^+_\mu)^\dagger$ as well as $\phi^- = (\phi^+)^\dagger$. Hence by crossing symmetry one can exchange the two particles such that there is Charge conservation in the vertex. Also conventionally we do not put the arrows in the legs related to $W^\pm_\mu$ and $\phi^\pm$, but strictly speaking they should be drawn.

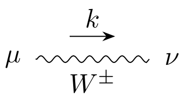
Relevant propagators. ψ is a generic fermion field with generic Gauge indices a, b.



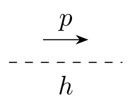
$$\frac{-i}{k^2 + i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]$$



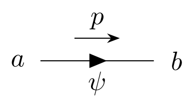
$$\frac{-i}{k^2 - M_Z^2 + i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi M_Z^2}\right]$$



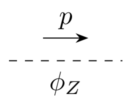
$$\frac{-i}{k^2 - M_W^2 + i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi M_W^2}\right]$$



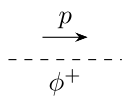
$$\frac{i}{p^2 - m_h^2 + i\varepsilon}$$



$$\frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\varepsilon}\delta^{ab}$$

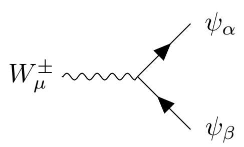


$$\frac{i}{p^2 - \xi M_Z^2 + i\varepsilon}$$

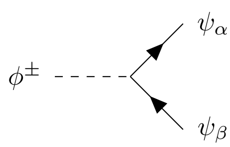


$$\frac{i}{p^2 - \xi M_W^2 + i\varepsilon}$$

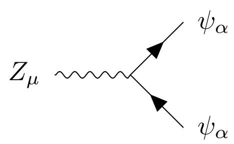
Relevant interaction vertices among vector bosons, GBs and fermions. The fermions are represented as ψ_α and ψ_β where α and β are some flavour indices that are contracted by a general matrix $V_{\alpha\beta}$ in the operator.



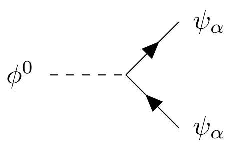
$$i\frac{g}{\sqrt{2}}\gamma^\mu P_L V_{\alpha\beta}$$



$$\pm\, i\frac{g}{\sqrt{2}}\left(\frac{m_\alpha}{M_W}P_L - \frac{m_\beta}{M_W}P_R\right)V_{\alpha\beta}$$

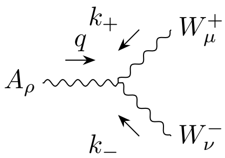


$$i\frac{g}{2c_W}\gamma^\mu\left[\left(T^3_\alpha - 2s_W^2 Q_\alpha\right) - T^3_\alpha\gamma^5\right]$$

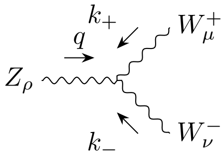


$$-\,gT^3_\alpha\frac{m_\alpha}{M_W}\gamma^5$$

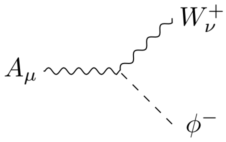
Relevant interaction vertices among vector bosons and GBs.



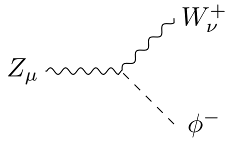
$$ie\Big[g^{\mu\rho}(k_+ - q)^\nu + g^{\mu\nu}(k_- - k_+)^\rho + g^{\nu\rho}(q - k_-)^\mu\Big]$$



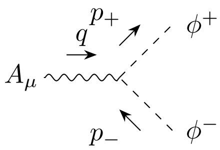
$$ie\frac{c_W}{s_W}\Big[g^{\mu\rho}(k_+ - q)^\nu + g^{\mu\nu}(k_- - k_+)^\rho + g^{\nu\rho}(q - k_-)^\mu\Big]$$



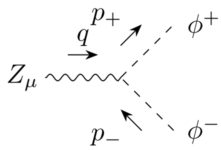
$$ieM_W g^{\mu\nu}$$



$$-\,ies_W M_Z g^{\mu\nu}$$



$$-\,ie(p_+ + p_-)^\mu$$



$$-\,ie\frac{c_W^2 - s_W^2}{2c_W s_W}(p_+ + p_-)^\mu$$

G Notation and Conventions

In this Chapter we are going to list all the notation and conventions that we used in this work. A reference that explains and derive most of the results listed here can be found for instance in [5].

G.1 Mathematical Notation and Conventions

In this Section we list all the notation and conventions used for mathematical quantities.

We represent the sets of natural, integer, rational, real and complex numbers as follows: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. Also we assume that $0 \in \mathbb{N}$.

Given a complex number $z \in \mathbb{C}$, its complex conjugate is indicated as z^*. Its argument is defined in the following two equivalent ways

$$\arg[z] \equiv \mathrm{Im}[\ln[z]] \,, \qquad e^{i \arg[z]} \equiv \frac{z}{|z|} \,. \tag{G.1.1}$$

Given a generic $n \times n$ matrix A, we call its determinant and the trace as $\det[A]$ and $\mathrm{Tr}[A]$ respectively. Given two generic matrices A and B, with the notation $A \ll B$ we mean that the eigenvalues of B are much bigger than all the entries of A. This is useful to make explicit the hierarchies between two matrices of even different dimensions. A diagonal matrix is indicated as follows

$$\mathrm{diag}(\lambda_1, ..., \lambda_n) \equiv \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \,. \tag{G.1.2}$$

The identity matrix is indicated as follows

$$\mathbb{I} \equiv \mathrm{diag}(1, ..., 1) \tag{G.1.3}$$

and it is left understood its dimension. The null matrix is indicated as $\mathbb{O}$ and it is a generic $n \times m$ matrix with all null entries. The null vector is indicated as $\overline{0}$ and it is a generic n-dimensional vector with all null entries.

The Euler δ-function is indicated as $\delta(x)$ with $x \in \mathbb{R}$ and it is defined by the following two properties

$$\int_{\mathbb{R}} dx \, \delta(x) = 1 \,, \qquad \delta(x) = 0 \quad \forall x \neq 0 \,. \tag{G.1.4}$$

To represent multi-dimensional δ-functions we use the notation

$$\delta^n(x) \equiv \prod_{i=1}^{n} \delta(x_i) \tag{G.1.5}$$

where the argument $x = (x_1, ..., x_n)$ is an n-dimensional vector.

The Euler Γ-function is defined as follows with complex argument $z \in \mathbb{C} \setminus \{0\}$

$$\Gamma(z) \equiv \int_0^\infty dt \, t^{z-1} e^{-t} \,. \tag{G.1.6}$$

It satisfies the following property

$$\Gamma(z+1) = z \, \Gamma(z) \tag{G.1.7}$$

and it has the following series expansion around $z = 1$

$$\Gamma(1+\varepsilon) = 1 - \gamma_E \, \varepsilon + \mathcal{O}(\varepsilon^2) \tag{G.1.8}$$

where $\gamma_E \approx 0.577$ is the Euler-Mascheroni constant.

The metric tensor has mostly-negative signature and it is given by

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)\,. \tag{G.1.9}$$

The Levi-Civita tensor is given by $\varepsilon_{\mu\nu\rho\sigma}$ and it is completely antisymmetric with $\varepsilon_{0123} = +1$. The scalar product between two Lorentz vectors p_μ and q_μ is indicated as follows

$$p \cdot q \equiv p_\mu q^\mu = g_{\mu\nu} p^\mu q^\nu\,. \tag{G.1.10}$$

In particular the scalar product with γ-matrices (G.2.3) is indicated as follows

$$\not{p} \equiv p_\mu \gamma^\mu\,. \tag{G.1.11}$$

The Pauli matrices are three 2×2 complex matrices and explicitly they read as follows

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{G.1.12}$$

G.2 Dirac Algebra

In this Section we list all the relevant features when dealing with fermionic fields. We work in the Weyl representation.

The Lie algebra of the Lorentz Group is represented as follows

$$\mathfrak{so}(1,3) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)\,. \tag{G.2.1}$$

Under this group a spinor of Left-chirality ψ_L transforms as $(1/2, 0)$ while one of Right-chirality ψ_R as $(0, 1/2)$. In our notation $\psi_{L/R}$ are 2-dimensional vectors in the spinor space. We can define the following 4-vectors of 2×2 matrices where σ^i are the Pauli matrices (G.1.12)

$$\sigma^\mu \equiv (\mathbb{I}, \vec{\sigma})\,, \qquad \overline{\sigma}^\mu \equiv (\mathbb{I}, -\vec{\sigma})\,. \tag{G.2.2}$$

The γ-matrices in the Weyl representation are defined as follows

$$\gamma^\mu \equiv \begin{pmatrix} \mathbb{O} & \sigma^\mu \\ \overline{\sigma}^\mu & \mathbb{O} \end{pmatrix}. \tag{G.2.3}$$

They are 4×4 matrices in the spinor space. Then we can define the last of the γ-matrices as follows

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} -\mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix}. \tag{G.2.4}$$

They satisfy the following properties (where $i = 1, 2, 3$)

$$(\gamma^0)^2 = (\gamma^5)^2 = \mathbb{I}\,, \qquad (\gamma^i)^2 = -\mathbb{I}\,, \tag{G.2.5a}$$
$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,, \qquad (\gamma^5)^\dagger = \gamma^5\,, \tag{G.2.5b}$$
$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}\,, \qquad \{\gamma^\mu, \gamma^5\} = 0\,. \tag{G.2.5c}$$

A Dirac spinor is defined as follows

$$\psi \equiv \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix} \tag{G.2.6}$$

and it is a 4-vector in the spinor space. We can also define the adjoint spinor

$$\overline{\psi} \equiv \psi^\dagger\gamma^0 = \begin{pmatrix} \chi_R^\dagger & \psi_L^\dagger \end{pmatrix}. \tag{G.2.7}$$

To project it onto its Left- and Right-chirality components we use the following Left- and Right-projectors

$$P_L \equiv \frac{1}{2}\left(\mathbb{I} - \gamma^5\right) = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \qquad P_R \equiv \frac{1}{2}\left(\mathbb{I} + \gamma^5\right) = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix} \tag{G.2.8}$$

and they satisfy the usual properties of projector operators.

When we go from coordinate to momentum space, the spinor is expanded in plane waves and the spinor polarizations are denoted by $u_s(p)$ and $v_s(p)$ for the particle and antiparticle respectively. p is the momentum of the spinor while $s = 1, 2$ accounts for the helicity state. When a spinor is on-shell, the polarizations satisfy the following equations (where m is the mass of the spinor)

$$(\not{p} - m)u_s(p) = 0\,, \qquad (\not{p} + m)v_s(p) = 0 \tag{G.2.9}$$

and they are normalized as follows

$$\overline{u}_s(p)u_{s'}(p) = 2m\,\delta_{ss'}\,, \qquad \overline{v}_s(p)v_{s'}(p) = -2m\,\delta_{ss'}\,. \tag{G.2.10}$$

Also they satisfy the following completeness relations

$$\sum_s u_s(p)\overline{u}_s(p) = \not{p} + m\,, \qquad \sum_s v_s(p)\overline{v}_s(p) = \not{p} - m\,. \tag{G.2.11}$$

Some of the most important identities that hold for on-shell spinors are given by the so-called Gordon Identities. They read as follows (where m and m' are the masses of the two spinor polarizations)

$$\overline{u}(p')\,(p+p')^\mu\, u(p) = \overline{u}(p')\Big[(m'+m)\gamma^\mu + i\sigma^{\mu\nu}(p-p')_\nu\Big]u(p)\,, \tag{G.2.12a}$$

$$\overline{u}(p')\,(p+p')^\mu\, \gamma^5 u(p) = \overline{u}(p')\Big[(m'-m)\gamma^\mu + i\sigma^{\mu\nu}(p-p')_\nu\Big]\gamma^5 u(p)\,. \tag{G.2.12b}$$

G.3 Scattering Quantities

In this Section we list many relevant quantities that are useful while computing scattering processes.

Consider a scattering process with N_f particles in the final state with momenta $p_f = (E_f, \vec{p}_f)$. The infinitesimal phase-space is defined as follows

$$d\Phi \equiv (2\pi)^4\delta^4\left(\sum_{f=1}^{N_f} p_f - q\right) S \prod_{f=1}^{N_f} \frac{d^3\vec{p}_f}{(2\pi)^3 2E_f} \tag{G.3.1}$$

where q is the total transferred momentum of the initial state and S is a Symmetry factor that accounts for the presence of identical particles. Namely, if there are N_p species of identical particles with multiplicity n_p, we have that

$$S = \prod_{p=1}^{N_p} \frac{1}{n_p!}\,. \tag{G.3.2}$$

Also it is possible to prove that this quantity is invariant under proper Lorentz transformations.

Calling $|\overline{\mathcal{M}}|^2$ the unpolarized squared-amplitude of a given process, the cross-section of a $2 \to N_f$ process reads as follows

$$\sigma\Big(2 \to N_f\Big) = \int \prod_{f=1}^{N_f} \frac{d^3\vec{p}_f}{(2\pi)^3 2E_f}(2\pi)^4\delta^4\left(\sum_{f=1}^{N_f} p_f - q_1 - q_2\right) \frac{S}{4E_{q_1}E_{q_2}v_r}|\overline{\mathcal{M}}|^2 \tag{G.3.3}$$

where q_1 and q_2 are the momenta of the initial-state particles with corresponding energies E_{q_1} and E_{q_2} while v_r is the relative velocity between them. Namely

$$v_r \equiv |\vec{v}_1 - \vec{v}_2| = \frac{1}{E_{q_1} E_{q_2}} \sqrt{(q_1 \cdot q_2)^2 - m_1^2 m_2^2}\,. \tag{G.3.4}$$

The decay rate of a particle into N_f particles, namely a $1 \to N_f$ process, reads as follows

$$\Gamma\Big(1 \to N_f\Big) = \int \prod_{f=1}^{N_f} \frac{d^3\vec{p}_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta^4 \left(\sum_{f=1}^{N_f} p_f - q \right) \frac{S}{2E_q} |\overline{\mathcal{M}}|^2 \tag{G.3.5}$$

where q is the momentum of the particle that decays.

In the framework of loop integrals, it is important to define a basis of integrals, called Master Integrals. They read as follows in $d = 4 - \varepsilon$ dimensions

$$I_{r,m} \equiv \int \frac{d^dk}{(2\pi)^d} \frac{(k^2)^r}{[k^2 - C + i\varepsilon]^m} = \frac{i(-1)^{r+m}}{(4\pi)^2} \left(\frac{4\pi}{C}\right)^{\varepsilon/2} C^{2+r-m} \frac{\Gamma\left(2 + r - \frac{\varepsilon}{2}\right)\Gamma\left(m - r - 2 + \frac{\varepsilon}{2}\right)}{\Gamma\left(2 - \frac{\varepsilon}{2}\right)\Gamma(m)} \tag{G.3.6}$$

where $r, m \in \mathbb{N}$ and $C \neq 0$ is a constant with respect to the integration variable. We also provide the following identities, easy to prove

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{[k^2 - C + i\varepsilon]^m} = 0\,, \tag{G.3.7a}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2 - C + i\varepsilon]^m} = \frac{g^{\mu\nu}}{d} I_{1,m}\,. \tag{G.3.7b}$$

In generic d dimensions Dirac Algebra takes the name of Clifford Algebra and in this framework the metric tensor is a $d \times d$ diagonal matrix with mostly-minus signature. It is possible to prove the following properties of the extended γ-matrices (from the first identity all the others follow)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \tag{G.3.8a}$$

$$\gamma_\mu \gamma^\mu = d\,, \tag{G.3.8b}$$

$$\gamma_\mu \gamma^\alpha \gamma^\mu = -(d-2)\gamma^\alpha\,, \tag{G.3.8c}$$

$$\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = (d-4)\gamma^\alpha \gamma^\beta + 4g^{\alpha\beta}\,, \tag{G.3.8d}$$

$$\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\mu = 2\gamma^\beta \gamma^\gamma \gamma^\alpha - 2\gamma^\alpha \gamma^\gamma \gamma^\beta - (d-2)\gamma^\alpha \gamma^\beta \gamma^\gamma\,, \tag{G.3.8e}$$

$$\mathrm{Tr}[\mathbb{I}] = d\,, \tag{G.3.8f}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = d g^{\mu\nu}\,, \tag{G.3.8g}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = d\left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right)\,, \tag{G.3.8h}$$

$$\mathrm{Tr}[\gamma^{\mu_1} ... \gamma^{\mu_{2k+1}}] = 0\,. \tag{G.3.8i}$$

Another important identity to utilize while computing loop integrals is the following

$$\frac{1}{\prod_{i=1}^n A_i^{\alpha_i}} = \frac{\Gamma\left(\sum_{i=1}^n \alpha_i\right)}{\prod_{i=1}^n \Gamma(\alpha_i)} \prod_{i=1}^n \int du_i\, \delta\left(1 - \sum_{i=1}^n u_i\right) \frac{\prod_{i=1}^n u_i^{\alpha_i - 1}}{\left(\sum_{i=1}^n u_i A_i\right)^{\sum_{i=1}^n \alpha_i}}\,. \tag{G.3.9}$$

G.4 SU(N) Group

In this Section we list the most relevant features of SU(N) groups. In general the group has dimension N^2-1 and it corresponds to the numbers of generators T^a with $a = 1, ..., N^2-1$. They are normalized such that

$$\mathrm{Tr}\Big[T^a T^b\Big] = \frac{1}{2}\delta^{ab} \tag{G.4.1}$$

and satisfy the commutator relation (in matrix notation)

$$[T^a, T^b] \equiv T^a T^b - T^b T^a = i f^{abc} T^c\,. \tag{G.4.2}$$

f^{abc} are the structure constants and are completely antisymmetric under any permutation of their indices. An element of the group can be represented as follows

$$U = \exp\left\{-i\,\alpha^a T^a\right\} \in \mathrm{SU}(N)\,. \tag{G.4.3}$$

For SU(2) a possible representation of the generators is given by (where $a = 1, 2, 3$)

$$T^a = \frac{1}{2}\sigma^a \tag{G.4.4}$$

where σ^a are the Pauli matrices (G.1.12). There is only one independent non-vanishing structure constant and it is given by

$$f^{123} = 1\,. \tag{G.4.5}$$

For SU(3) a possible representation of the generators is given by (where $a = 1, ..., 8$)

$$T^a = \frac{1}{2}\lambda^a \tag{G.4.6}$$

where λ^a are the Gellmann matrices and they read as follows

$$\begin{aligned}
&\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
&\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{G.4.7}$$

The independent non-vanishing structure constants are given by

$$f^{123} = 1\,, \qquad f^{147} = f^{246} = f^{257} = f^{345} = \frac{1}{2}\,, \qquad f^{156} = f^{367} = -\frac{1}{2}\,, \qquad f^{458} = f^{678} = \frac{\sqrt{3}}{2}\,. \tag{G.4.8}$$

For SU(4) a possible representation of the generators is given by (where $a = 1, ..., 15$)

$$T^a = \frac{1}{2}\lambda^a \tag{G.4.9}$$

where λ^a read as follows

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
\lambda^5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,
$$

$$
\lambda^9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \quad \lambda^{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} , \quad \lambda^{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad \lambda^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} ,
$$

$$
\lambda^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \qquad \lambda^{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} , \qquad \lambda^{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} . \tag{G.4.10}
$$

The independent non-vanishing structure constants are given by

$$
\begin{aligned}
& f^{1\,2\,3} = 1 , \qquad f^{1\,4\,7} = f^{2\,4\,6} = f^{2\,5\,7} = f^{3\,4\,5} = \frac{1}{2} , \qquad f^{1\,5\,6} = f^{3\,6\,7} = -\frac{1}{2} , \qquad f^{4\,5\,8} = f^{6\,7\,8} = \frac{\sqrt{3}}{2} , \\
& f^{1\,9\,12} = f^{2\,9\,11} = f^{2\,10\,12} = f^{3\,9\,10} = f^{4\,9\,14} = f^{5\,9\,13} = f^{5\,10\,14} = f^{6\,11\,14} = f^{7\,11\,13} = f^{7\,12\,14} = \frac{1}{2} , \\
& f^{1\,10\,11} = f^{3\,11\,12} = f^{4\,10\,13} = f^{6\,12\,13} = -\frac{1}{2} , \qquad f^{8\,9\,10} = f^{8\,11\,12} = \frac{1}{2\sqrt{3}} , \qquad f^{8\,13\,14} = -\frac{1}{\sqrt{3}} , \\
& f^{9\,10\,15} = f^{11\,12\,15} = f^{13\,14\,15} = \frac{\sqrt{2}}{\sqrt{3}} .
\end{aligned} \tag{G.4.11}
$$

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