ALGANT Master Thesis in Mathematics

Modular Representations of GL_2 over a Local Field

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Preface

The main object of study in number theory is the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as it encodes the data of all number fields. A first step to understand this group is to look at its abelianization, which is the Galois group of the maximal abelian extension of $\mathbb{Q}$. This subgroup is well understood via class field theory. One fruitful approach in class field theory is to study first the abelian extensions of $\mathbb{Q}_p$ for each prime p and then use some local-global principles to obtain an analogous result for the global case. Thus, one usually establishes first the local version of class field theory and then obtains from there a global version.

For a local field F the well known local Artin reciprocity map can be seen as giving a bijection between some characters of $F^\times$ and some characters of the Galois group $\mathrm{Gal}(F^{\mathrm{ab}}/F)$. One can ask whether by allowing higher dimensional representations we can obtain a similar bijection between some p-adic representations of $\mathrm{GL}_n(F)$ and some n-dimensional p-adic representations of the absolute Galois group. This is the main object of study of the p-adic Langlands program.

In this work, we will study the "automorphic" side, i.e. we will study the side concerning representations of general linear groups of a local field. We shall study the natural case that comes right after local class field theory: the two dimensional case. Before studying p-adic representations, it is convenient to establish the theory of modular representations, i.e. representations over a field of prime characteristic. Since a local field can be seen as an extension of $\mathbb{Q}_p$ for some prime p, we shall distinguish two cases. The first one is when the characteristic of the field is different from p, in this case the classical methods work and its study is similar to that of complex representations [Vig89]. In this work, we study the case where the characteristic is exactly p. The aim of this work is to classify the irreducible modular representations of $\mathrm{GL}_2(F)$.

In Chapter 1, we shall introduce some preliminaries and the tree of SL_2, which is the main tool along with Hecke algebras to classify these representations. This tree allows us to use more geometrical arguments when dealing with the Hecke algebras, which makes proofs more intuitive an easier to handle than dealing directly with the Hecke algebras. We shall also see how the group $\mathrm{GL}_2(F)$ and some of its subgroups act on this tree and prove some important group decompositions which will be useful later.

In Chapter 2 we introduce the concept of Hecke algebras and we give the E-algebra structure in two specific cases. Later, we study some modules over these two Hecke algebras and we give existence of irreducible $\mathrm{GL}_2(F)$-modules for each value $\lambda \in E$ where λ will be a parameter linked to a special Hecke operator. We conclude the chapter proving two important technical results.

In Chapter 3 we construct some special representations, which are constructed from a representation of $\mathrm{GL}_2(\mathcal{O}_F)$ and lifted to a representation of $\mathrm{GL}_2(F)$ via compact induction. We shall see that almost all irreducible two dimensional representations arise in this manner. We prove these representations are irreducible for all values $\lambda \neq 0, \pm 1$.

In Chapter 4 we deal with the special case when $\lambda = \pm 1$. In this case, the aforementioned representation is not irreducible, but it admits an irreducible quotient called the special series. We prove this special series is irreducible. We note that we make the proofs only for the case $\lambda = 1$ as the other case can be reduced to this one by twisting.

In Chapter 5 we give the classification of the irreducible representations in the case that the vector space we are acting on contains a vector fixed by the subgroup $\mathrm{GL}_2(\mathcal{O}_F)$. For the general case we just outline how to adapt what we already did to get a classification in this case. Lastly, we give some short comments on the supersingular representations for completeness.

Lastly let us introduce some notation. The notation ":=" will be used to define notation that will be fixed for the rest of the work. Left cosets will be denoted by G/H, right cosets by $H\backslash G$, and double cosets by $H\backslash G/K$. Composition will be written from left to right unless otherwise stated, i.e. fg means apply first f and then g.

Chapter 1

Preliminaries and the tree of SL_2

In this chapter one will be introduced to the elemental definitions and results needed to classify the irreducible modular representations of $\mathrm{GL}_2(F)$. The main tool is the so called *tree of* SL_2, which will make the proofs more geometric and tangible for the reader. We will follow [BL95] for sections 1-3, to introduce definitions, notations and well known classical results. Sections 4-7 will be based on [Cas14] and aim to provide the basic tools for the next chapters. Lastly, for section 8, we follow [BL95] with a little twist to introduce some technical results that are needed later at some point.

1.1 First definitions

Let G be a topological group. G is said to be *locally profinite* if it is Hausdorff, totally disconnected and locally compact. An equivalent definition of a locally profinite group is to ask for the topological group G to be Hausdorff and that each neighborhood of the identity contains an open compact subgroup. In particular, if G is not only locally compact but compact one gets the topological definition of a profinite group. For the remaining of the section, G will denote a locally profinite group, H a closed subgroup of G, E a field of positive characteristic p and V a vector space over E. Some results may be true in a more general setting, but this setting is general enough for the subsequent sections.

Let $\rho : G \to V$ be a linear representation. It is said to be *smooth* if for every $v \in V$, $\mathrm{Stab}\,(v)$ is an open subgroup of G, or, equivalently, if v is fixed by an open subgroup. It is said to be *admissible* if, additionally, for any compact open subgroup $C \leq G$, the C-fixed subspace V^C is finite dimensional over E. From now on, all G-representations will be assumed to be smooth.

There are different ways to look at a G-representation $\rho : G \to V$. It can be seen as a G-action $G \times V \to V$, as a group homomorphism $G \to \mathrm{GL}(V)$ or as a functor $\mathbf{G} \to \mathbf{Vect}$ where $\mathbf{G}$ is the one object category associated to the group G. This last way of looking at representations shows they form a category **G-Rep** with G-representations as the objects and the G-equivariant maps as the morphisms. Recall that for two G-representations $\rho : G \to V$ and $\tau : G \to W$, a G-equivariant map is no more than a linear map $f : V \to W$ such that

$$\begin{array}{ccccc} (*) & & V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle g} & \longrightarrow & {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \tau(g)} \\ (*) & & V & \xrightarrow{f} & W. \end{array}$$

is commutative for all $g \in G$. In the following, one may refer to a representation in any of the above ways.

Given a G-representation $\rho : G \to V$ and a subgroup $H \leq G$, one may define the H-representation $\rho_{|H} : H \to V$ called the *restriction* of ρ to H, by just considering the action of the elements of H. Now, let us consider a H-representation $\sigma : H \to V$. The *space of locally constant functions*, $\mathcal{F}(G, H, \sigma)$, is the E-vector space of locally constant functions $f : G \to V$ with compact support modulo H such that $f(hg) = \sigma(h)f(g)$ for all $h \in H$ and $g \in G$. Note such an f is totally determined by its image on each $g_i \in H\backslash G$. G acts on $\mathcal{F}(G, H)$ via right translation $gf(x) := f(xg)$. This action is called the *induced representation* $\mathrm{ind}_H^G\sigma$. Note the induced representation $\mathrm{ind}_H^G\sigma$ lifts the H-representation σ to a G-representation, and one may consider this lifting as a kind of inverse to the restriction of G-representations to H. Formally, this is an adjuntion of functors and in this context it will be called the *Frobenius reciprocity.*

1.2 Classical results

We will start by stating the so called *Frobenius reciprocity.*

PROPOSITION 1.1 (FROBENIUS RECIPROCITY). *The restriction and the induced functors are adjoint, i.e. for $\rho : G \to V$ and $\sigma : H \to W$ representations and H open in G there is a natural isomorphism*

$$\mathrm{Hom}_G(\mathrm{ind}_H^G\sigma, \rho) \cong \mathrm{Hom}_H(\sigma, \rho_{|H})$$

PROOF. Let $\{g_i\} = H\backslash G$. One first defines a map

$$A : \mathrm{Hom}_G(\mathrm{ind}_H^G\sigma, \rho) \to \mathrm{Hom}_H(\sigma, \rho_{|H}).$$

For $\phi \in \mathrm{Hom}_G(\mathrm{ind}_H^G \sigma, \rho)$, and $w \in W$, one wants to define $A(\phi)(w)$. First, define

$$f_w(g) = \begin{cases} w, & g = 1, \\ 0, & 1 \neq g \in H \backslash G. \end{cases}$$

It is clear that $f_w \in \mathcal{F}(G, H, \sigma)$ and

$$\begin{array}{ccccccc}
(*) & & \overset{w}{W} & \longrightarrow & \overset{f_w}{\mathcal{F}(G,H,\sigma)} & \longrightarrow & \overset{\phi(f_w)}{V} \\
\big\downarrow h & \longrightarrow & \sigma(h)\big\downarrow & & \big\downarrow \mathrm{ind}_H^G \sigma(h) & & \big\downarrow \rho(h) \\
(*) & & \underset{\sigma(h)(w)}{W} & \longrightarrow & \underset{f_{\sigma(h)(w)}}{\mathcal{F}(G,H,\sigma)} & \longrightarrow & \underset{\phi(f_{\sigma(h)(v)})}{V}
\end{array}$$

where the first square commutes by the definition of f_w and the second one commutes for ϕ being a G-equivariant map. Thus, the large square commutes and one may define

$$A(\phi)(w) = \phi(f_w),$$

which makes $A(\phi)$ into a H-equivariant map independent of the choice of the representatives $H \backslash G$, as only H is considered.

Now, one defines the inverse morphism to A. Let us define

$$B : \mathrm{Hom}_H(\sigma, \rho_{|H}) \to \mathrm{Hom}_G(\mathrm{ind}_H^G \sigma, \rho),$$

for $\psi \in \mathrm{Hom}_H(\sigma, \rho_{|H})$ and $f \in \mathcal{F}(G, H, \sigma)$ as $B(\psi)(f) := \sum_i \rho(g_i^{-1})\psi(f(g_i))$. This is a G-equivariant map since for any $g \in G$ we may write $g_i g = h g_j$ and

$$\begin{aligned}
gB(\psi)(f) &= g \sum_j \rho(g_j^{-1})\psi(f(g_j)) = \sum_i \rho(g_i^{-1}h)\psi(f(g_j)) \\
&= \sum_i \rho(g_i^{-1})\psi(f(g_i g)) = \sum_i \rho(g_i^{-1})\psi(g f(g_i)) \\
&= B(\psi)(gf).
\end{aligned}$$

Also, note

$$B(A(\phi))(f) = \sum_i \rho(g_i^{-1}) A(\phi)(f(g_i)) = \sum_i \rho(g_i^{-1})\phi(f_{f(g_i)}) = \phi(f),$$

where the last equality comes from noting the left hand side is exactly $f(g_i)$ on each $g_i \in H \backslash G$. Also,

$$(A(B(\psi))(w) = B(\psi)(f_w) = \psi(w).$$

Hence, A and B are inverses to each other concluding the proof. □

Remark. The explicit isomorphism B of the proof is going to be used later.

THEOREM 1.2 (SCHUR'S LEMMA). *Let $\rho : G \to V$ be an irreducible smooth G-representation and assume the coefficient field of V, E, is algebraically closed. Suppose either that ρ is also admissible or that G is a countable union of compacts and E is uncountable. Then, an endomorphism of V commuting with G is a scalar.*

PROOF. The reader is referred to [IB76] for a proof over the complex numbers, but their proof works in the generality of the statement too. □

In other words, if one restricts to admissible representations, as we shall do, one can always assume their centers act as scalars, i.e. they have a central character.

LEMMA 1.3. *Let $\rho : U \to V$ be a U-representation with U a pro-p-group, over a field of characteristic p. If $V \neq \{0\}$, then $V^U \neq \{0\}$.*

PROOF. Let $v \in V$ be non zero. Since ρ is smooth, $\text{Stab}(v)$ is open and since U is a pro-p-group it is compact; thus, one has $|U : \text{Stab}(v)| = p^n$ for a non negative integer n. If $n = 0$ then $v \in V^U$ and one is done. Then, assume $n \geq 1$ and consider the action of the quotient $U/\text{Stab}(v)$, which is a finite p group of order p^n, over the orbit of v, let us call it X. Note X has cardinality p^n. Then, $X \setminus X^{U/\text{Stab}(v)}$ is a finite union of orbits of cardinality a positive power of p so $0 \equiv |X| \equiv |X^{U/\text{Stab}(v)}| \bmod p$. But $0 \in X^{U/\text{Stab}(v)}$ so it contains at least a non trivial vector w. But w is fixed by all U since any $u \in U$ may be written as $u = rs$ with $r \in U\backslash\text{Stab}(v)$ and $s \in \text{Stab}(v)$ and both r and s fix w. Thus, $w \in X^U \subset V^U \neq \{0\}$. □

1.3 Structure of locally constant function spaces

Now, one shall see the structure of $\mathcal{F}(G, H, 1)$ and $\mathcal{F}(G, H, 1)^{\tilde{H}}$.

Structure of $\mathcal{F}(G, H, 1)$

Let triv be the irreducible trivial G-representation, i.e. triv: $G \to E$ such that $G \to \text{GL}(E)$ is the trivial group homomorphism mapping all G to the identity in $\text{GL}(E)$. One may also write 1 for triv for simplicity of notation. Then, $\mathcal{F}(G, H, 1)$ are the locally constant functions to the field of coefficients E with compact support modulo H such that $f(hg) = f(g)$, i.e. they are constant on the right cosets of H. Recall the characteristic function of H, $\mathbf{1}_H$, is the locally constant function mapping H to 1 and everything else to 0. Then, it is clear that the G-translates of $\mathbf{1}_H$ form an E-basis of $\mathcal{F}(G, H, 1)$. In other words, $\{\mathbf{1}_{Hg}\}$ where $g \in H\backslash G$ is an E-basis for $\mathcal{F}(G, H, 1)$. This simple observation will be used throughout the chapter.

Structure of $\mathcal{F}(G, H, 1)^{\tilde{H}}$

One knows that the $\{\mathbf{1}_{Hg}\}$ form an E-basis for $\mathcal{F}(G, H, 1)$. However, if now one considers the subspace fixed by a subgroup $\tilde{H} \leq G$, $\mathcal{F}(G, H, 1)^{\tilde{H}}$, one gets less functions in there. If one considers the $\tilde{H}$-action induced on the cosets of H, one sees that by construction the elements of $\mathcal{F}(G, H, 1)^{\tilde{H}}$ must be constant on the $\tilde{H}$-orbits of each coset Hg and not only on the coset Hg itself; thus, one can obtain an E-basis of $\mathcal{F}(G, H, 1)^{\tilde{H}}$ via double cosets as $\{\mathbf{1}_{Hg\tilde{H}}\}$, for $g \in H\backslash G/\tilde{H}$.

1.4 Lattices and group decompositions

L is said to be a *lattice* in F if L is a finitely generated $\mathcal{O}_F$ module and L spans F^2 as an F-vector space. This is equivalent to L being a free $\mathcal{O}_F$-module of rank 2. To see this equivalence just use elementary operations in columns, i.e. multiplications by elements of $\mathrm{GL}_m(\mathcal{O}_F)$ on the left, to get two generators just in the first two columns proving $m = 2$, with m being the number of generators of L. Thus, we shall usually denote $L = [u, v]$ where u and v are the generators of L as an $\mathcal{O}_F$-module. In the same lines, using also elementary row operations i.e. multiplications on the right, one can prove that for $g \in \mathrm{GL}_2(F)$, there exists $D = \mathrm{diag}(\pi^m, \pi^n)$ with $m \leq n$ and $k_1, k_2 \in \mathrm{GL}_2(\mathcal{O}_F)$ such that $g = k_1 D k_2$. That is to say we get the following group decomposition:

$$\mathrm{GL}_2(F) = \mathrm{GL}_2(\mathcal{O}_F) Z A \mathrm{GL}_2(\mathcal{O}_F), \quad A = \left\{ \alpha^n : \alpha := \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \right\},$$

where Z denotes the center of $\mathrm{GL}_2(F)$. This is called the *Cartan decomposition.* The following result is now immediate.

Theorem 1.4 (Principal Divisor Theorem). *Let L, M be two lattices over F. Then, there exist $e, f \in F$ and m, n integers such that $m \leq n$ and $L = [e, f]$ and $M = [\pi^m e, \pi^n f]$.*

Let us fix some more notation. For the rest of the chapter, let $G := \mathrm{GL}_2(F)$ and $K := \mathrm{GL}_2(\mathcal{O}_F)$. Let Z be the center of G as before. Let B the Borel subgroup of G, i.e. the subgroup of upper triangular matrices in G. Then, one defines the subgroup $N \leq B$ as

$$N := \left\{ n(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in B : a \in F \right\},$$

i.e. the unipotent radical of B. It is straightforward to prove the so called *Bruhat decomposition* of G

$$G = B \sqcup BwN, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Just assume $M \in G \setminus B$. Then,

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad/c - b & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix},$$

where one can safely assume $c \neq 0$ since $M \notin B$.

Now, one proves the *Iwasawa decomposition*, $G = BK$. For $d \neq 0$ and $n = \mathrm{val}(d)$

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pi^{-n} \begin{pmatrix} a - bc/d & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \pi^n & 0 \\ c\pi^n/d & \pi^n \end{pmatrix} \in BK,$$

and if $d = 0$ just note $M\omega \in B$ and $w^{-1} = -w \in K$.

1.5 The definition of the Bruhat-Tits tree

Now, one is in position to define the Bruhat-Tits tree of $\mathrm{SL}_2(F)$ which will be denoted by Δ. The vertices of Δ will be the equivalence classes of lattices modulo similarity, i.e. L and M will be equivalent if there exists $n \in \mathbb{Z}$ such that $M = \pi^n L$ and $v_L = \{\pi^n L\} \in \mathrm{V}(\Delta)$ will denote the vertex associated to the equivalence class of L in Δ. The equivalence class of L will be denoted as $[\![L]\!]$ and the associated vertex in Δ as v_L. Given two lattices L and M, the Principal Divisor Theorem tells us that we can write $L = [e, f]$ and $M = [\pi^m e, \pi^n f]$ with $m \leq n$, so that the difference $n - m \geq 0$ is an invariant of the similiarity classes, i.e. $\mathrm{inv}(v_L : v_M) = n - m$ is well defined. It will be called the *lattice pair invariant*. L and M will be called *neighbors* if $\mathrm{inv}(v_L : v_M) = 1$. In this case they will be joined by an edge in Δ. All the edges in Δ arise in this manner, i.e. there is an edge in Δ joining v_L and v_M if and only if L and M are neighbors.

Remark. The invariant $\mathrm{inv}(v : w)$ is exactly 1 if and only if v and w have representatives L and M respectively, such that $L/M \cong (\mathcal{O}_F/\mathfrak{m})^2 \cong \mathbb{F}_q^2$. In other words, $\pi L \subset M \subset L$. Thus, the neighbors of L correspond to the lines in $\mathbb{F}_q^2$ and there are exactly $(q^2 - 1)/(q - 1) = q + 1$. of them. They can also be seen as elements of the projective space $\mathbb{P}^1(\mathbb{F}_q)$.

A finite or half-infinite sequence of nodes linked by edges will be called a *chain* in Δ and it will be denoted by $(v_i) = (v_0, v_1, \dots, v_n, \dots)$. We can choose a representative L_n for each v_n such that

$$L_0 \supset L_1 \supset \cdots \supset L_n \supset \cdots$$

with $L_n \supset L_{n+1} \supset \pi L_n$. This chain will also be represented as (L_i). A chain will be said to be *simple* if it is not a cycle in Δ, i.e. it is neither a finite closed path or an infinite repetition of a finite closed path in Δ. An infinite simple chain starting at $v \in V(\Delta)$, will be called a *branch from* v. The union of two branches from v with no common edges will be called an *apartment*.

1.6 Group actions on Δ

Let $L = [u, v]$ be a lattice. First, note the intersection of two, and thus of a finite number, lattices is again a lattice. Let us define the natural G-action in the set of lattices $\mathcal{L}$ as $gL = [gu, gv]$ for $g \in G$. It is known G acts transitively in the set of F-basis of F^2; thus, it acts transitively in $\mathcal{L}$. It is also clear that $\mathrm{Stab}\left(\mathcal{O}_F^2\right)$ is exactly K, so one can identify $\mathcal{L}$ with the quotient group G/K. Recall Z is the subgroup of scalar nonzero matrices. Then, $V(\Delta)$ may be identified with G/KZ by the definition of the similarity classes. One notes that K is an open and compact subgroup of G. Since the G-action is transitive one can write any lattice as $L = g\mathcal{O}_F^2$ for some $g \in G$, so $\mathrm{Stab}\,(L) = \mathrm{Stab}\left(g\mathcal{O}_F^2\right) = \mathrm{Stab}\left(\mathcal{O}_F^2\right)^{g^{-1}}$ is also a compact and open subgroup since conjugation is an homeomorphism. Conversely, it is easy to see any compact open subgroup of $C \in G$ stabilizes some lattice in $\mathcal{L}$. If one considers $H := C \cap K$, H has finite index in C as it is open in a compact group, and one may write it as a finite disjoint union of cosets $C = \coprod c_i H$. Then, since finite intersection of lattices is a lattice, $L = \bigcap c_i \mathcal{O}_F^2$ is a lattice, and it is stabilized by C. To see this, just fix $c_i, c_k \in H \backslash C$ and let c_j be the unique element in $H \backslash C$ such that $c_i c_j H = c_k H$; thus,

$$c_i(c_j \mathcal{O}_F^2) = c_k \mathcal{O}_F^2,$$

and one gets $C(L) = L$ from the definition of L.

Now, G preserves the equivalence of lattices defined before since one has $g\pi^n L = \pi^n gL$. It also preserves the lattice pair invariant. To see this just note that if $L = [e, f]$ and $M = [e, \pi^n f]$, $gL = [ge, gf]$ and $gM = [ge, \pi^n gf]$. Therefore, it preserves edges of Δ and it acts on Δ transitively.

There is a very natural chain in Δ deserving a special attention. Let us fix the canonical F-basis, $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and for each integer n let $v_n = [\![e_1, \pi^n e_2]\!]$. In particular, $v_0 = [\![\mathcal{O}_F^2]\!]$. Then, it is clear from its definition that v_n has v_{n-1} and v_{n+1} as neighbors; thus, it is very natural to build a chain of the form, possibly finite, $(v_0, v_{-1}, \dots)$. Such a chain will be called a *standard chain.* The *standard branch* $\mathcal{B}_0 = (v_0, v_{-1}, \dots)$ together with its opposite branch $B_\infty := (v_0, v_1, \dots)$ define the *standard apartment* denoted as $\mathcal{A} := \mathcal{B}_0 \cup \mathcal{B}_\infty$. Standard chains play a special role as the next proposition shows.

Proposition 1.5. *Every finite simple chain in Δ may be transformed to a standard one by an element of G. For F complete, the result still holds for half-infinite simple chains too.*

As a consequence of this proposition, one shows that the distance between two nodes v and w in the tree Δ, i.e. the number of edges of the shortest path joining them, is given by the lattice pair invariant $\mathrm{inv}(v : w)$. Lastly, one can use the Principal Divisor Theorem to show Δ is connected by constructing an explicit chain of lattices for any pair of lattices L and M. This last proposition shows this chain cannot have any loops since a standard one has no loops, proving Δ is a tree, finally justifying its name.

Now, let us prove Proposition 1.5. First, one may work over an associated chain of lattices

$$L_0 \supset L_1 \supset \cdots \supset L_n,$$

such that $L_i \supset L_{i+1} \supset \pi L_i$. Now, one needs to find $g_0, g_1, \dots, g_n$ such that for each $i \leq n$, g_i maps L_i to a representative of v_{-i} and fixes the chosen representative of v_{-j} for $j \leq i$. Then, $\prod_{i=0}^n g_i \in G$ is the transformation mapping the given chain to a standard one. Note, for the half infinite case this product is infinite so the extra assumption of F being complete will ensure convergence of the product so that $\prod g_i \in G$. Since G acts transitively on $\mathcal{L}$, there is some g_0 mapping L_0 to $\mathcal{O}_F^2$. To define g_1, one considers the image of L_1 on the residue field $\mathcal{O}_F/\pi\mathcal{O} \cong \mathbb{F}_q$, which is just a line and it can be transformed to the line passing through $(0,1)$ by an element $g \in \mathrm{GL}_2(\mathbb{F}_q)$. Taking g_1 any lift of g to K, g_1 fixes $\mathcal{O}_F$ and it maps L_1 to $[\pi, 1]$.

To construct g_i for $i > 1$, one proceeds by induction on i, looking at the image of L_i on $L_{i-1}/\pi L_{i-1}$ which is a line different from πL_{i-2} by assumption. One may assume $L_{i-1} = [\pi^{i-1}, 1]$ by induction. Since $\pi L_{i-2} = [\pi^{i-1}, \pi]$ projects to the line through $(1,0)$ in $L_{i-1}/\pi L_{i-1}$, the image of L_i must be a line through a point $(y, 1)$ which can be transformed to the point $(0,1)$ by

$$M_{-y} = \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q).$$

Then, for x a lift of y to $\mathcal{O}_F$, one may take $g_i := M_{\pi^{i-1}x}$ which will satisfy the wanted conditions. Just note $v_{-j} = [\![\pi^j, 1]\!]$ for $j < i-1$ and compute the action of g_i on the generators to obtain $g[\pi^j, 1] = [\pi^j + \pi^{i-1}x, 1] = [\pi^j \epsilon, 1] = [\pi^j, 1]$ since $\epsilon \in \mathcal{O}_F^\times$ by the assumption $j < i-1$ and elemental properties of discrete valuations. Also, note that the infinite product of these matrices will converge by construction if F is complete. This concludes the proof.

The following immediate corollary will be very useful later on.

Corollary 1.6. *In particular, vertices at distance exactly n from v_0 are exactly the vertices K-equivalent to v_{-n}. There are exactly $(q+1)q^{n-1}$ of them.*

1.7 Iwahori subgroups

Let $I \leq K$ denote the *Iwahori* subgroup, the subgroup formed by the lifts of upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_q)$, i.e.

$$I := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : \text{with } \nu(c) \geq 1 \right\}.$$

The natural action of I in Δ clearly fixes ν_0 and v_1, so it preserves the edge e_{01} from v_0 to v_1. Then, the stabilizer of e_{01} is exactly IZ and since the G-action on the edges is transitive, one can identify $E(\Delta)$ with G/IZ.

Remark. Δ is considered as an oriented tree. One may also consider Δ as a non oriented tree, where in particular $e_{01} = e_{10}$ but then the stabilizer of this edge is strictly bigger since one is allowed to consider transformations such that $v_0 \leftrightarrow v_1$ see [Cas14]. But the key in this work is to identify the set of edges with the quotient group G/IZ; thus, one needs to consider oriented edges to this end.

One also defines the subgroup $I_1 \leq I$ as

$$I_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I : \text{with } a \equiv d \equiv 1 \bmod \pi\mathcal{O}_F \right\}.$$

One shall see I_1 is a pro-p-group. Let $\varphi : K \to \mathrm{GL}_2(\mathbb{F}_q)$ denote the usual projection. Clearly, $\varphi(I_1) \cong \mathbb{F}_q$, so it is a p-group. Then, since $\ker\varphi \leq I_1$ it is enough to prove that this kernel is a pro-p-group. Since $K = \varprojlim \mathrm{GL}_2(\mathcal{O}_F/\pi^n)$, $\ker\varphi = \varprojlim \ker\varphi_n$ for $\varphi_n : \mathrm{GL}_2(\mathcal{O}_F/\pi^n) \to \mathrm{GL}_2(\mathbb{F}_q)$. But each $\ker\varphi_n$ is clearly a p-group so $\ker\varphi$ is a pro-p-group as wanted.

Both identifications $V(\Delta) \cong G/KZ$ and $E(\Delta) \cong G/IZ$ are isomorphisms in the category of G-sets, i.e. isomorphic via invertible G-equivariant maps.

1.8 Heights

For a vertex $v \in V(\Delta)$, one defines its *height* as $h(v) := k - d(v, v_k)$ for the first $v_k \in \mathcal{A}$ in the unique path from v to v_0. In particular, $h(v_k) = k$. One defines the *antecedent* of v, $a(v)$, as the unique vertex of height $h(a(v)) = h(v) + 1$ adjacent to v. More explictly, it is the first vertex in the unique path from v to v_0 for $v \notin \mathcal{A}$ and $a(v_k) := v_{k+1}$.

One shall denote $n(X) \leq N$ to the subgroup of N formed by the elements $n(x)$ with $x \in X \subset F$ whenever this makes sense. Let us define a decreasing filtration of N as

$$\{1\} \subset \cdots \subset N_k \subset N_{k-1} \subset \cdots \subset N, \quad N_k := n(\pi^k \mathcal{O}_F), \quad k \in \mathbb{Z}.$$

It is clear that any $n \in N$ must lie in some N_k with k minimal, i.e. $n \notin N_{k-1}$, and that N_{-r} fixes v_s with $s \geq r$. Then, N_k stabilizes v_r if and only if $k \leq -r$, but all such N_k are contained in N_{-r} proving the stabilizer of v_r in N is exactly N_{-r}. Let $n \in N$, $h(nv_r) = j - d(nv_r, v_j)$ with v_j as in the definition of the height and let us choose $k \geq 0$ big enough such that $n \in N_{-j-k}$ and $r \leq j + k$. Then,

$$\begin{aligned} h(nv_r) &= j - d(nv_r, v_j) = j + k - d(nv_r, v_{j+k}) \\ &= j + k - d(nv_r, nv_{j+k}) = j + k - d(v_r, v_{j+k}) \\ &= j + k - j - k + r = r = h(v_r), \end{aligned}$$

i.e. N preserves the height of v_r. Since N_{-j-k} fixes v_s for $s \geq j + k$ one gets

$$\begin{aligned} |N_{-j-k} v_r| = |N_{-j-k} : N_{-r}| = q^{j+k-r} = 1 + \sum_{i=0}^{j+k-r-1} (q-1) q^i \\ = |\{v \in V(\Delta) : h(v) = r + i - d(v, v_{r+i}) = r, \ 0 \leq i \leq j + k - r\}|, \end{aligned}$$

proving the N_{-j-k}-orbit of v_r is exactly the last set appearing in the equality. The following result is now obvious.

Lemma 1.7. *For $r \in \mathbb{Z}$, N_{-r} fixes v_r and*

$$(N/N_{-r}) v_r = \{v \in V(\Delta) : h(v) = r\}.$$

In particular, N preserves the height and adjacency so $a(nv_r) = nv_{r+1}$ for any $n \in N$.

Chapter 2

Hecke algebras

In this chapter, the reader will be introduced to the concept of *Hecke algebras*, as they will be crucial to classify irreducible modular representations. The goal is to find an appropiate Hecke operator T, such that if G is acting on V and $V^K \neq \{0\}$, there exists an eigenvector $v \in V^K$ for T. This chapter will follow [BL95]. In section 1, we introduce the first definitions and properties of these Hecke algebras. In section 2 and section 3 we determine the E-algebra structure of the Hecke algebras corresponding to the subgroups KZ and IZ respectively. Sections 4 and 5 study the actions of these algebras on the space of locally constant functions, and finally section 6 proves the existence of the above mentioned eigenvector in the unramified case, i.e. when $V^K \neq \{0\}$.

2.1 Definition and first properties

Let $H \leq G$ be an open subgroup of G. The *Hecke algebra of G relative to H* is defined as

$$\mathcal{H}(G,H) := \mathrm{Hom}_G(\mathrm{ind}_H^G 1, \mathrm{ind}_H^G 1),$$

with coefficients over any commutative ring with identity. Note that for two elements $T, S \in \mathcal{H}(G,H)$, $T(\mathbf{1}_H) = S(\mathbf{1}_H)$ already implies $T = S$ since they are G-equivariant by definition and the G-translates of $\mathbf{1}_H$ give an E-basis of $\mathcal{F}(G,H,1)$ as seen in the preliminaries. One shall call this argument *by G-equivariance*, and it will be extensively used later.

Now, for any G-representation $\rho : G \to V$, a H-equivariant map f from $1 : H \to E$ to $\rho_{|H} : H \to V$ is completely determined by $f(1) = v \in V^H$. Thus, there is an isomorphism as E-vector spaces $V^H \cong \mathrm{Hom}_H(1, \rho_{|H})$. This isomorphism combined with the one of the Frobenius reciprocity gives

$$V^H \cong \mathrm{Hom}_H(1, \rho_{|H}) \cong \mathrm{Hom}_G(\mathrm{ind}_H^G 1, \rho).$$

In particular,

$$\mathcal{F}(G,H,1)^H \cong \mathrm{Hom}_H(1, \mathrm{ind}_H^G 1_{|H}) \cong \mathcal{H}(G,H).$$

$\mathcal{H}(G,H)$ acts in the H invariants of a G-representation as expected. For $v \in V^H$, let us denote by ϕ_v^H the corresponding element in $\mathrm{Hom}_G(\mathrm{ind}_H^G 1, \rho)$ caracterized by $\phi_v^H(\mathbf{1}_H) = v$. Then, $T \in \mathcal{H}(G,H)$ acts on v via precomposition, i.e.

$$v|T := (\mathbf{1}_H)T\phi_v^H.$$

Note this is a right action. In particular, for $\mathrm{ind}_H^G 1 : G \to \mathcal{F}(G,H,1)$, this right action corresponds under the above isomorphism to the right multiplication on itself. Also, for $v \in V^H$, the G-equivariant map $\phi_v^H : \mathcal{F}(G,H,1) \to V^H$ restricts to an E-linear morphism $\phi_v^H : \mathcal{F}(G,H,1)^H \to V^H$ but satisfying the addtional condition that for $T \in \mathcal{H}(G,H)$

$$\phi_v^H(f|T) = \phi_v^H(f)|T, \quad \text{for } f \in \mathcal{F}(G,H,1)^H.$$

To see this, note $\phi_v^H(f|T) = ((\mathbf{1}_H)T)\phi_f^H\phi_v^H$ and $\phi_v^H(f)|T = ((\mathbf{1}_H)T)\phi^H_{\phi_v^H(f)}$, so it is enough to show $\phi_f^H\phi_v^H = \phi^H_{\phi_v^H(f)}$. In particular, one shall show both morphisms take the same value at $\mathbf{1}_H$ by G-equivariance. But both take the value $\phi_f^H(f)$ at $\mathbf{1}_H$ so the equality holds. This summarizes as follows.

Lemma 2.1. *The restriction $\phi_v^H : \mathcal{F}(G,H,1)^H \to V^H$ is $\mathcal{H}(G,H,1)$-linear.*

Making use of the explicit isomorphism in the proof of the Frobenius reciprocity, one may define the action of the Hecke algebra on V^H via double cosets. By Frobenius reciprocity, one has

$$\mathcal{H}(G,H) \cong \mathcal{F}(G,H,1)^H,$$

and the latter has an E-basis represented by double cosets as $\{\mathbf{1}_{HgH}\}$. Thus, it is enough to specify how the element in $\mathcal{H}(G,H)$ corresponding to $\mathbf{1}_{HgH}$ acts on V^H to have a complete description of the action of the Hecke algebra on the H-invariants of V. Recall that if $HgH = \bigcup Hg_i$ and $T \in \mathcal{H}(\mathcal{G},\mathcal{H})$ corresponds to $\mathbf{1}_{HgH}$, by the explicit isomorphism of the Frobenius reciprocity, one has that for each $v \in V^H$,

$$v|T = \sum \rho(g_i^{-1})v.$$

It is also possible to define a left $\mathcal{H}(G,H)$-action too on $\mathcal{F}(G,H,1)^H$. For that, let $f \in \mathcal{F}(G,H,1)^H$ and ϕ_f^H the corresponding element in $\mathcal{H}(G,H)$, i.e. $\phi_f^H(\mathbf{1}_H) = f$. Then, for $T \in \mathcal{H}(G,H)$, one sets

$$Tf := (\mathbf{1}_H)\phi_f^H T$$

which corresponds under the above isomorphism to the left multiplication on itself.

Now, let us fix $\mathcal{H}_K := \mathcal{H}(G, KZ)$ and $\mathcal{H}_I := \mathcal{H}(G, IZ)$.

2.2 E-algebra structure of $\mathcal{H}_K$

Recall the isomorphism of G-sets $V(\Delta) \cong G/KZ$. Recall also the functions $\{\mathbf{1}_{KZg}\}$ indexed by $g \in KZ\backslash G$ form an E-basis for $\mathcal{F}(K) := \mathcal{F}(G, KZ, 1)$. Then, we shall consider the correspondence $\mathbf{1}_{KZg} \leftrightarrow g^{-1}v_0$, which is G-equivariant since $\mathbf{1}_{KZ}$ corresponds to v_0 and $g^{-1}\mathbf{1}_{KZ} = \mathbf{1}_{KZg}$ corresponds to $g^{-1}v_0$ as wanted. Then, we may identify $\mathcal{F}(K)$ with $C_0(\Delta)$, the space of 0-chains in Δ over E. In other words, we identify the elements in $\mathcal{F}(K)$ with finite E-linear combination of vertices of Δ. Now, one has $\{\mathbf{1}_{KZgK}\}$ is an E-basis for $\mathcal{F}(K)^K$, but by the Cartan decomposition of G, the set $KZ\backslash G/K$ can be chosen to be

$$KZ\backslash G/K = \{\alpha^n : n \in \mathbb{Z}\},$$

and one gets the E-basis consisting of the elements $\mathbf{1}_{KZ\alpha^n K}$. One notes $\mathbf{1}_{KZ\alpha^n K} = \sum \mathbf{1}_{KZg_i}$ where the sum is taken over the $g_i \in KZ\backslash KZ\alpha^n K$, which are all distinct and exactly one is 1. Thus, $g_i^{-1}v_0$ are all the distinct vertices in Δ which are K-equivalent to v_{-n} plus v_0. To see this, just note $g_i^{-1}v_0 = k\alpha^{-n}v_0 = kv_{-n}$ for $k \in K$. Hence, $\mathbf{1}_{KZ\alpha^n K}$ corresponds, geometrically, to the sum of all distinct vertices at distance n from v_0.

Let $T_n \in \mathcal{H}_K$ correspond to $\mathbf{1}_{KZ\alpha^n K}$ under the Frobenius reciprocity isomorphism. Then, $T_1 T_{2-1} = T_2 + 1$. This is easy to see when ones thinks geometrically. One identifies $\mathbf{1}_{KZ\alpha^2 K}$ with the finite sum of all distinct vertices at distance 2 from v_0. Recall each vertex has exactly $q+1$ neighbors, and that $q = 0$ in E. Now, $T_1 T_1$ corresponds to the finite sum composed of all the distinct vertices that are at at distance 1 from a vertex at distance 1 from v_0, i.e. the ones at distance exactly 2 from v_0 plus $q + 1 = 1$ repetitions of v_0; thus, the equality follows. For $n = 3$ one gets $T_2 T_1 = T_3$. From the previous equality, $T_2 T_1 = (T_1^2 - 1)T_1 = T_1^3 - T_1$. Note T_1^3 can be identified with the vertices at distance 1 from a vertex at distance 1 from a vertex at distance 1 from v_0. So it can be identified with the sum of vertices at exactly distance 3 from v_0 plus $(q+1)^2 + q = 1$ times the vertices at distance 1 from v_0. All in all, removing once the vertices at distance 1 from v_0, they are the sum of all the distinct vertices at distance 3 from v_0 proving the equality. In general, $T_n = T_{n-1}T_1$ since by induction on n, $T_n = T_{n-1}T_1 = T_1^n - T_1^{n-1}$ and same reasoning gives the equality.

Now, one notes the T_n-s form an E-basis of the Hecke algebra $\mathcal{H}_K$, and the previous paragraph shows $T_n = T_1^n - T_1^{n-1}$, so as an E-algebra,

$$\mathcal{H}_K \cong E[T_1].$$

One usually denotes $T := T_1$ and the above isomorphism as $\mathcal{H}_K \cong E[T]$. In particular, note $\mathcal{H}_K$ is commutative.

2.3 E-algebra structure of $\mathcal{H}_I$

Deducing the E-algebra structure of $\mathcal{H}_I$ takes a little more effort than the one of $\mathcal{H}_K$, but one should follow the same guidelines. First, one finds an appropiate E-basis by thinking geometrically on Δ and then one finds the relations of these generators to give generators as an E-algebra, and, in this case, some relations of these generators too. As usual, one notes the elements of the form $\mathbf{1}_{IZg}$ with $g \in IZ\backslash G$ give an E-basis of $\mathcal{F}(I) := \mathcal{F}(G, IZ, 1)$. This set corresponds to G/IZ so it corresponds to $E(\Delta)$, which gives a correspondance between the whole space $\mathcal{F}(I)$ and the space $C_1(\Delta)$ of 1-chains over E.

By the Frobenius reciprocity, $\mathcal{H}_I \cong \mathcal{F}(I)^I$, the latter having an E-basis given by the elements $\mathbf{1}_{IZgI}$ with $g \in IZ\backslash G/I$. Now, one needs an analogous of the Cartan decomposition of G to give a complete set of representatives $IZ\backslash G/I$. One achieves this by working on Δ. Since $IZ\backslash G \cong E(\Delta)$, one gets that $IZ\backslash G/I \cong E(\Delta)/I$, where modulo I means modulo I-equivariance. One notes that by Proposition 1.5, for any edge e, there is a unique edge e' in the standard apartment $\mathcal{A}$ such that (e_{01}, e) and (e_{01}, e') are G-equivalent, i.e. there is $g \in G$ such that $ge_{01} = e_{01}$ and $ge = e'$. Since g estabilizes e_{01}, $g \in I$. Thus, the edges in the standard apartment form a complete set of representatives of $E(\Delta)/I$, i.e. $\{e_{n+1,n}, e_{n,n+1}\}$ for all n integers. But if one denotes

$$\beta := \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix},$$

one has $e_{n+1,n} = \alpha^n e_{10} = \alpha^n \beta e_{01}$ and $e_{n,n+1} = \alpha^n e_{01}$, so they correspond to $\mathbf{1}_{IZ\beta\alpha^{-n}}$ and $\mathbf{1}_{IZ\alpha^{-n}}$ respectively since $\beta^2 = 1$. We shall denote the elements in $\mathcal{H}_I$ corresponding to $\mathbf{1}_{IZ\beta\alpha^{-n}I}$ and $\mathbf{1}_{IZ\alpha^{-n}I}$ by the Frobenius reciprocity as $T_{n+1,n}$ and $T_{n,n+1}$ respectively. These elements generate $\mathcal{H}_I$ as an E-vector space. One notes that if $\phi_{i,j}$ denotes the sum of all edges I-equivalent to e_{ij}, then one has the correspondances

$$T_{n+1,n} \leftrightarrow \phi_{n+1,n}, \quad T_{n,n+1} \leftrightarrow \phi_{n,n+1}.$$

In other words, $T_{n+1,n}(e_{01}) = \phi_{n+1,n}$ and $T_{n,n+1}(e_{01}) = \phi_{n,n+1}$. In particular, ϕ_{01} corresponds just to the edge e_{01}

Now, one should find the relations to describe the E-algebra structure of $\mathcal{H}_I$. One would like to relate all $T_{n,n+1}$ and $T_{n+1,n}$ to just T_{10} and T_{12}, which, by G-equivariance, is enough to do it on their images on e_{01}. One starts by studying the T_{10} and T_{12}. Note T_{10} acts on e_{01} by reversing it, i.e. $T_{10}(e_{01}) = e_{10}$; thus, by G-equivariance it acts on $E(\Delta)$ by reversing the edges, mapping each edge to its opposite. In particular,

$$T_{10}(\phi_{n+1,n}) = \phi_{n,n+1}, \quad T_{10}(\phi_{n,n+1}) = \phi_{n+1,n}. \tag{2.1}$$

It is also immediate to see that $T_{12}(e_{01})$ is just the sum of distinct edges with origin at v_1 but e_{10}, since these are the ones I-equivalent to e_{12} as v_1 is fixed by I. Then, again by G-equivalence, T_{12} acts on any edge $e \in E(\Delta)$ as $T_{12}(e) = \sum e_i$ where the sum is taken over all the distinct edges e_i with origin at the terminal vertex of e but the opposite of e. Then, it follows for $n \geq 0$

$$T_{12}(\phi_{n,n+1}) = \phi_{n+1,n+2}, \quad T_{12}(\phi_{-n+1,-n}) = \phi_{-n,-n-1}, \tag{2.2}$$

as one gets all the I-equivalent edges to $e_{n+1,n+2}$ exactly once and same for the second one. Also, one has

$$T_{12}(\phi_{21}) = (q-1)\phi_{12} = -\phi_{12}, \tag{2.3}$$

since the edges I-equivalent to e_{21} are those with terminal vertex in v_1 distinct from e_{01}, so there are exactly q such edges. Now, applying T_{12} to each of them we get exactly $q-1$ of the q edges with initial vertex v_1 distinct from e_{10}. Thus, one gets each such an edge exactly $q-1$ times, and these are exactly the vertices I-equivalent to e_{12}.

Now, one is in position to give the E-algebra structure of $\mathcal{H}_I$.

PROPOSITION 2.2. *$\mathcal{H}_I$ is a non commutative algebra finitely presented as*

$$\mathcal{H}_I \cong \frac{E[T_{10}, T_{12}]}{(T_{10}^2 - 1, T_{12}T_{10}T_{12} + T_{12})}.$$

PROOF. First, note that any $T_{n,n+1}$ and $T_{n+1,n}$ can be written in terms of the T_{10} and T_{12}. Let $n \geq 0$. One only needs to check their images on e_{01} by G-equivariance. Then,

$$T_{n,n+1}(e_{01}) = \phi_{n,n+1} = T_{12}(\phi_{n-1,n}) = \cdots = T_{12}^n(e_{01}),$$

easily from a multiple application of (2.2). Also, for any integer k

$$T_{k+1,k}(e_{01}) = \phi_{k+1,k} = T_{10}(\phi_{k,k+1}) = T_{10}T_{k,k+1}(e_{01}),$$

by (2.1). In particular,

$$T_{n+1,n}(e_{01}) = T_{10}T_{n,n+1}(e_{01}) = T_{10}T_{12}^n(e_{01}),$$

using both equalities above. Lastly, one deduces

$$T_{-n+1,-n}(e_{01}) = \phi_{-n+1,-n} = T_{12}(\phi_{-n+2,-n+1}) = \cdots = T_{12}^n(\phi_{1,0}) = T_{12}^n T_{10}(e_{01}),$$

from (2.1) and (2.2), and

$$T_{-n,-n+1}(e_{01}) = T_{10}T_{-n+1,-n}(e_{01}) = T_{10}T_{12}^n T_{10}(e_{01}).$$

This shows T_{10} and T_{12} generate $\mathcal{H}_I$ as an E-algebra. Now one finds the relations these two generators satisfy. The first one is straightforward

$$T_{10}^2(e_{01}) = T_{10}(\phi_{10}) = \phi_{01} = 1(e_{01}),$$

again from (2.1). One also gets a second relation,

$$T_{12}T_{10}T_{12}(e_{01}) = T_{12}T_{10}(\phi_{1,2}) = T_{12}(\phi_{2,1}) = -\phi_{1,2} = -T_{12}(e_{01}),$$

from (2.3). Lastly, note the monomials $T_{10}^{\epsilon}T_{12}^n T_{10}^{\delta}$ map to different ϕ_{ij} by the previous equalities, ensuring there is no further relation. Thus, one gets the isomorphism of the statement. □

One may note β normalizes I, i.e. $I^\beta = I$. In particular, $\beta I = I\beta^{-1}$. Thus, T_{10} corresponds to $\mathbf{1}_{ZI\beta I} = \mathbf{1}_{ZI\beta^{-1}}$. This gives the following Lemma.

Lemma 2.3. *Let $\rho : G \to V$ be a G-representation with trivial central character and let $v \in V^I$. Then, $v|T_{10} = \beta v$.*

2.4 $\mathcal{H}_K$-module structure of $\mathcal{F}(K)$

Recall the identification $\mathcal{F}(K) \cong C_0(\Delta)$. There is an obvious increasing filtration for this infinite dimensional E-vector space

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \subset C_0(\Delta),$$

with V_n having as E-basis the vertices at distance at most n from v_0. Then, it is clear that V_n has dimension $(q+1)q^{n-1}$ over E by Corollary 1.6. Note that to get the vertices at distance n from v_0 it is enough to take the adjacent ones for each vertex at distance $n-1$, but the one which is at distance $n-2$. All in all, for each vertex at distance $n-1$ one takes q distinct neighbors, and these are all the vertices at distance n from v_0 without repetition. Since $v|T$ is the sum of all neighbors of v, one may choose inductively v_0, and q neighbors of v_0; $q-1$ neighbors of each neighbor of v_0 and so on until one reaches a distance n from v_0. Then, one gets the full V_n, since by applying the operator

T one gets the vertex it is left in V_1, the ones left in V_2 and so on. This way, one shows $\mathcal{F}(K)$ is free over $\mathcal{H}_K$, with the inductively constructed basis

$$\beta_K = \{v_0\} \cup \{v_1^1, \dots, v_1^q \text{ distinct neighbors of } v_0\} \cup \bigcup_{n=2}^{\infty} \beta_n,$$

where $\beta_n := \{v_n^1, \dots, v_n^{q(q-1)^{n-1}}$ distinct neighbors of vertices in $\beta_{n-1}\}$, since applying T to this β_K we obtain all $V(\Delta)$, i.e. an E-basis of $\mathcal{F}(K)$, concluding β_K is a $\mathcal{H}_K$-basis for $\mathcal{F}(K)$.

Now, one would like to build an irreducible G-module in which T has a given $\lambda \in E$ as an eigenvalue. For that, one may define the E-quotient space

$$\mathcal{F}_\lambda := \mathcal{F}(K)/(T - \lambda),$$

for any $\lambda \in E$. Let us denote W a submodule such that the quotient $\mathcal{F}_\lambda/W$ is irreducible. Then, $v|T = \lambda v$ for all $v \in \mathcal{F}_\lambda/W$ and clearly it is a G-module and v is fixed by K. Summarizing one has proved the following important result.

Proposition 2.4. *For every $\lambda \in E$ there is an irreducible G-module of the form $\mathcal{F}_\lambda/W$ with trivial central character and a nonzero K-fixed vector v such that $v|T = \lambda v$.*

2.5 $\mathcal{H}_I$-module structure of $\mathcal{F}(K)^I$

Now, one shall study the action of $\mathcal{H}_I$ on $\mathcal{F}(K)^I \cong C_0(\Delta)^I$. The elements in $C_0(\Delta)^I$ are those finite sums with the same coefficients for vertices in the same I-orbit. Note there is exactly one v_n in each I-orbit. This is clear for v_0 and v_1 as they are fixed by I, and for any other v_m and v_r one may assume $0 \leq m < r$, and note that if there would exist $i \in I$ such that $v_r = iv_m$, then

$$v_0 = \alpha^{-r} v_r = \alpha^{-r} i v_m = \alpha^{-r} i \alpha^m v_0,$$

so $\alpha^{-r} i \alpha^m \in KZ$, but as $i\alpha^m \in KZ$, this forces $\alpha^{-r} \in KZ$, which is not true. Let us denote

$$\psi_n = \sum_{v \in I v_n} v,$$

the sum of vertices in the I-orbit of v_n for each $n \in \mathbb{Z}$. Since the G action preserves distances and IZ is the stabilizer of e_{01}, the vertices in this sum are precisely those at distance $|n|$ from v_0 and at the same time at distance $|n-1|$ from v_1 for $n \neq 0, 1$. For $n = 0, 1$ it is just $\psi_n = v_n$. Now, one notes $\{\psi_n\}$

form an E-basis for $C_0(\Delta)^I$ as the elements are precisely those with constant coefficients in I-orbits. By the Frobenius reciprocity ψ_n corresponds to

$$\phi_{\psi_n}^{IZ} \in \mathrm{Hom}_G(\mathrm{ind}_{IZ}^G 1, \mathrm{ind}_{KZ}^G 1) \cong \mathrm{Hom}_G(E(\Delta), V(\Delta)).$$

If one wants to study the action of $\mathcal{H}_I$ on $\mathcal{F}(K)^I$, it is clear that it is enough to study the action of the generators of $\mathcal{H}_I$ in an E-basis of $\mathcal{F}(K)^I$, i.e. the action of T_{10} and T_{12} on each ψ_n. One shall start studying the action of $T_{n,n-1}$ and $T_{n,n+1}$ on ψ_0 which enables one to study the action of the generators on any ψ_n. First, note

$$\phi_{\psi_0}^{IZ}(\mathbf{1}_{IZ}) = \phi_{\psi_0}^{IZ}(e_{01}) = \psi_0 = v_0,$$

so by G-equivariance, $\phi_{\psi_0}^{IZ}$ maps each edge in Δ to its origin vertex. Analogously,

$$\phi_{\psi_1}^{IZ}(\mathbf{1}_{IZ}) = \phi_{\psi_1}^{IZ}(e_{01}) = \psi_1 = v_1,$$

and by G-equivariance again, $\phi_{\psi_1}^{IZ}$ maps each edge in Δ to its terminal vertex.

For $n \neq 0, 1$ one needs a little remark on the ψ_n. Recall $\phi_{n,n-1}$ ="sum of edges I-equivalent to $e_{n,n-1}$". Assume n to be positive. Then, it is clear that

$$\psi_n = \phi_{\psi_0}^{IZ}(\phi_{n,n-1}),$$

and recalling the action of the operator $T_{n,n-1}$,

$$\phi_{\psi_n}^{IZ}(e_{01}) = \psi_n = \phi_{\psi_0}^{IZ}(\phi_{n,n-1}) = \phi_{\psi_0}^{IZ}(T_{n,n-1}(e_{01})),$$

and by G-equivariance, $\phi_{\psi_n}^{IZ} = T_{n,n-1}\phi_{\psi_0}^{IZ}$. In particular,

$$\psi_0 | T_{n,n-1} = \psi_n.$$

Same reasoning gives

$$\psi_0 | T_{n,n+1} = \psi_n$$

for $n \leq 0$. Now, one can proceed to study the action of T_{12} and T_{10} on ψ_n.

LEMMA 2.5. *The action of T_{12} on ψ_n is given by:*

1. $\psi_n | T_{12} = \psi_{n+1}$ *for* $n > 0$,
2. $\psi_n | T_{12} = -\psi_{-n+1}$ *for* $n < 0$,
3. $\psi_0 | T_{12} = 0$ *for* $n = 0$.

The action of T_{10} on ψ_n is given by:

$$\psi_n | T_{10} = \psi_{1-n}.$$

Proof. For the proof one shall just recall the E-algebra structure of $\mathcal{H}_I$ and how its elements are represented in terms of T_{12} and T_{10}.

For $n > 0$

$$\begin{aligned}\psi_n|T_{12} &= \psi_0|T_{n,n-1}T_{12} = \psi_0|T_{10}T_{12}^{n-1}T_{12} = \psi_0|T_{n+1,n}\\ &= \psi_{n+1}.\end{aligned}$$

For $n < 0$

$$\begin{aligned}\psi_n|T_{12} &= \psi_0|T_{n,n+1}T_{12} = \psi_0|T_{10}T_{12}^{-n}T_{10}T_{12} = \psi_0| - T_{10}T_{12}^{-n-1}T_{12}\\ &= \psi_0| - T_{10}T_{12}^{-n}T_{10}T_{10} = \psi_0| - T_{-n,-n+1}T_{10} = \psi_0| - T_{-n+1,-n}\\ &= -\psi_{-n+1}.\end{aligned}$$

For $n = 0$

$$\begin{aligned}\psi_0|T_{12} &= \phi_{\psi_0}^{IZ}(\phi_{01})|T_{12} = \phi_{\psi_0}^{IZ}(\phi_{12})\\ &= qv_1 = 0,\end{aligned}$$

since there are exactly q edges I-equivalent to e_{12} one for each neighbor of v_1 but v_0, since v_0 is fixed by I. Lastly, by Lemma 2.3, T_{10} acts as β does, i.e. as a reflection interchanging $v_0 \leftrightarrow v_1$; thus, T_{10} interchanges the I-orbit of v_n with the I-orbit of v_{1-n}. □

Now, one is in position to prove the following key proposition.

Proposition 2.6. *Every non zero $\mathcal{H}_I$-submodule of $\mathcal{F}(K)^I$ is of finite codimension.*

Proof. Let ψ be a non zero element of $0 \neq W \leq \mathcal{F}(K)^I$. Then, it can be written in the basis $\{\psi_n\}$ as

$$\psi = \sum_{n=A}^{B} a_n\psi_n,$$

where $A \leq B$. Also, one may assume $-A < B$. Just note that otherwise one may consider $\psi|T_{10} \in W$. Both inequalities together give $-B < A \leq B$. One may also assume $a_B \neq 0$. Let us define the $\mathcal{H}_I$-subspace W' as the one generated by W and $\{\psi_{-(B-1)}, \dots, \psi_{B-1}\}$ and let us see it is the whole space $\mathcal{F}(K)^I$. One shall see every ψ_n is in W'. One starts with ψ_B

$$\psi_B = a_B^{-1}\psi - \sum_{n=A}^{B-1} a_B^{-1}a_n\psi_n \in W'.$$

Now, if B is positive, $\psi_{B+k} = \psi_B | T_{12}^k \in W'$ for any positive k. If B is not positive, then $\psi_{B+1} = -\psi_B | T_{12}$ and one may just take care of the alternating minus signs until $B+k$ gets positive. In any case, $\psi_{B+k} \in W'$ for all positive k.

Now, one is only left with ψ_{-B-k} for $k \geq 0$. But,

$$\psi_{-B-k} = \psi_{B+k+1} T_{10} \in W',$$

concluding the assertion. □

2.6 Existence of eigenvectors for $\mathcal{H}_K$. The unramified case

For a 0-chain in $C_0(\Delta)$, i.e. a finite sum of the form $\sum a_i w_i$ with $a_i \in E$ and $w_i \in V(\Delta)$, one defines its *degree* as the sum of its coefficients, i.e. $\deg \sum a_i w_i = \sum a_i$.

Theorem 2.7. *Let $\rho : G \to V$ be a smooth irreducible representation with trivial central character such that $V^K \neq 0$. Assume E is algebraically closed. Then, there exists in V^K an eigenvector for $\mathcal{H}_K$.*

Proof. Let $v \in V^K$ nonzero. Then, the morphism $\phi_v^{KZ} : \mathcal{F}(K) \to V$ induces a morphism of $\mathcal{H}_K$-modules on the KZ-invariants by Lemma 2.1, and since ρ has central character, in particular in the K-invariants

$$\phi_v^{KZ} : \mathcal{F}(K)^K \to V^K.$$

Note that $\mathrm{ind}_{KZ}^G 1$ is not irreducible. Just recall $\mathcal{F}(K) \cong C_0(\Delta)$ and note the 0-chains of degree 0 form a proper G-subspace. Then, since V is an irreducible G-module, $\ker \phi_v^{KZ} \neq 0$, otherwise V would not be irreducible as it would contain $\mathcal{F}(K)$ as a G-subspace. Now, since I_1 is a pro-p-group, one can apply Lemma 1.3 to get $(\ker \phi_v^{KZ})^{I_1} \neq 0$. Since $I \subset K$ and the ker is contained in the K-invariants, I/I_1 acts trivially on $(\ker \phi_v^{KZ})^{I_1}$, and one gets

$$(\ker \phi_v^{KZ})^I = (\ker \phi_v^{KZ})^{I_1} \neq 0.$$

By Proposition 2.6, it has finite codimension in $\mathcal{F}(K)^I$ over $\mathcal{H}_I$. By the usual isomorphism theorem, the dimension of the image $\phi_v^{KZ}(\mathcal{F}(K)^I)$ over $\mathcal{H}_I$ is finite. In particular, since $\mathcal{F}(K)^K \subset \mathcal{F}(K)^I$, $\phi_v^{KZ}(\mathcal{F}(K)^K)$ is also finite dimensional over $\mathcal{H}_I$, and non zero, since in particular it is a ring homomorphism so it cannot be zero. Since $\mathcal{H}_K \cong E[T]$ is commutative and E is assumed to be algebraically closed, the finite $\mathcal{H}_K$-submodule $\phi_v^{KZ}(\mathcal{F}(K)^K) \subset V^K$ contains an eigenvector for T and thus, for $\mathcal{H}_K$ □

Chapter 3

The universal unramified principal series

As we saw in the previous chapter, one may find an eigenvector for the Hecke operator T, and when the corresponding eigenvalue $\lambda \neq 0, \pm 1$ one shall see that the irreducible representations has a very familiar form: it is a principal series representations. These representations come from lifting a character from $F^\times$ to the Borel subgroup B and then to G via induction. In section 1, we give the first definitions and some technical results. In section 2, we study the unramified principal series and its properties. Lastly, section 3 is devoted to proving irreducibility of these principal series representations.

3.1 Preliminary results

Let Λ be a variable and let R be the localization at Λ of the polynomial ring $E[\Lambda]$, i.e. $R := E[\Lambda, \Lambda^{-1}]$. Let su denote by $\chi : F^\times \to R^\times$ the unramified character given by $\chi(a) = \Lambda^{-\mathrm{val}(a)}$. One can construct a character of the Borel subgroup, $\chi \otimes \chi^{-1} : B \to R^\times$, as

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi(a)\chi^{-1}(d) = \Lambda^{\mathrm{val}(d/a)}.$$

One can now lift this character to G via induction

$$\mathrm{ind}_B^G(\chi \otimes \chi^{-1}) : \mathcal{F}(\chi, \chi^{-1}) \to R^\times,$$

where $\mathcal{F}(\chi, \chi^{-1}) := \mathcal{F}(G, B, \chi \otimes \chi^{-1})$. Let us recall the Bruhat decomposition $G = B \sqcup BwN$, so any element $f \in \mathcal{F}(\chi, \chi^{-1})$ is completely determined by its image on 1_G and wN. Since $wN = \bigcup_{x \in F} wn(x)$ one shall study its values on each $wn(x)$. Let us define $\varphi : F \to R$ via $x \mapsto f(wn(x))$, which is locally constant for f being locally constant. Let

$$\alpha' := \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}.$$

LEMMA 3.1. *The map φ satisfies the following properties:*

1. $n(y)\varphi : x \mapsto \varphi(x+y)$, *for all* $y \in F$.
2. $\alpha\varphi : x \mapsto \Lambda^{-1}\varphi(\pi x)$ *for all* $x \in F$.
3. $\alpha'\varphi : x \mapsto \Lambda\varphi(\pi^{-1}x)$ *for all* $x \in F$.

PROOF. For the first one just note

$$n(y)\varphi(x) = n(y)f(wn(x)) = f(wn(x)n(y)) = f(wn(x+y)) = \varphi(x+y).$$

For the second one

$$\alpha\varphi(x) = \alpha f(wn(x)) = f(wn(x)\alpha) = f(\alpha' wn(\pi x)) = (\chi \otimes \chi^{-1})(\alpha')\varphi(\pi x)$$
$$= \Lambda^{-1}\varphi(\pi x).$$

The third one is done similarly

$$\alpha'\varphi(x) = \alpha' f(wn(x)) = f(wn(x)\alpha') = f(\alpha wn(\pi^{-1}x)) = (\chi \otimes \chi^{-1})(\alpha)\varphi(\pi^{-1}x)$$
$$= \Lambda\varphi(\pi^{-1}x),$$

concluding the proof. □

Now, one would like to compute the values $f(wn(x))$ explicitly. First, note

$$wn(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

Then, by the definition of f,

$$f(wn(x)) = (\chi \otimes \chi^{-1})\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} f\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} = \Lambda^{2\mathrm{val}(x)} f\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix},$$

and since f is locally constant, for x large enough this last matrix will be near 1_G, and one gets

$$f(wn(x)) = \Lambda^{2\mathrm{val}(x)} f(1_G).$$

Then, $\varphi(x) = C\Lambda^{2\mathrm{val}(x)}$ with $C = f(1_G)$ for x large enough. This means, f is totally determined by φ, since one may compute the remaining value $f(1_G)$ from φ. Then, one may work with φ or f interchangably, but carefully. Let f_0 be the unique K-invariant element in $\mathcal{F}(\chi, \chi^{-1})$ such that $f_0(1_G) = 1$. Then, since it is K-invariant, $kf_0(g) = f_0(gk) = f_0(g)$ for any $g \in G$ and any $k \in K$. In particular, for $x \in \mathcal{O}_F$, $wn(x) \in K$ so

$$f_0(wn(x)) = f_0(1_G wn(x)) = f_0(1_G) = 1,$$

and for $x \notin \mathcal{O}_F$, i.e. $\mathrm{val}(x) < 0$,

$$f_0(wn(x)) = \Lambda^{2\mathrm{val}(x)},$$

since this is true for x large enough and f_0 is K-invariant.

Let us denote $\varphi_0(x) := f_0(wn(x))$.

PROPOSITION 3.2. *The map φ_0 satisfies the following properties.*

1. $\sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \alpha' n(a/\pi)\varphi_0 = (\Lambda - \Lambda^{-1})\mathbf{1}_{\mathcal{O}_F}$.
2. $\varphi_0|T = \Lambda\varphi_0$.

PROOF. For the first statement one shall apply Lemma 3.1 to obtain

$$\sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \alpha' n(a/\pi)\varphi_0(x) = \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \alpha'\varphi_0(x+a/\pi) = \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \Lambda\varphi_0((x+a)/\pi).$$

Now, one shall see this is equal to $(\Lambda - \Lambda^{-1})\mathbf{1}_{\mathcal{O}_F}$, i.e. it is zero for all x with valuation strictly less than 0, and exactly $\Lambda - \Lambda^{-1}$ for any other x. Let us assume first $\mathrm{val}(x) < -1$. Then, $\mathrm{val}((x+a)/\pi) = -1 + \mathrm{val}(x) < -2$ for all the a in the sum and

$$\Lambda \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \varphi_0((x+a)/\pi) = \Lambda \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \Lambda^{2(-1+\mathrm{val}(x))} = \Lambda q\Lambda^{2(-1+\mathrm{val}(x))} = 0.$$

Next, one may assume $\mathrm{val}(x) > -1$. Then, for all $a \in (\mathcal{O}_F/\pi\mathcal{O}_{\mathcal{F}})^\times$, $\mathrm{val}((x+a/)\pi) = -1 + \mathrm{val}(x+a) = -1$ for all a but for the one representing $-\pi x$ in the residue field. Let us denote this special a by a_0 and just note that for this a_0 we get $\mathrm{val}((x+a_0)/\pi) = -1 + \mathrm{val}(a+x) \geq 0$, so one gets

$$\Lambda \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \varphi_0((x+a)/\pi) = \Lambda + \Lambda \sum_{a\neq a_0\in\mathcal{O}_F/\pi\mathcal{O}_F} \Lambda^{-2} = \Lambda + (q-1)\Lambda^{-1} = \Lambda - \Lambda^{-1}.$$

Lastly, one may assume $\mathrm{val}(x) = -1$. Then, $\mathrm{val}((x+a)/\pi) = -2$ for all $a \in \mathcal{O}_F/\pi\mathcal{O}_F$. Hence,

$$\Lambda \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \varphi_0((x+a)/\pi) = \Lambda \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \Lambda^{-4} = q\Lambda^{-3} = 0,$$

concluding the proof of the first statement.

For the second statement, recall how T acts on the K-invariants

$$v|T = T\phi_v^{KZ}(\mathbf{1}_{KZ}) = \phi_v^{KZ}(\mathbf{1}_{KZ\alpha K}).$$

In particular,

$$\varphi_0|T = \phi_{\varphi_0}^{KZ}(\mathbf{1}_{KZ\alpha K}).$$

By the explicit map of the Frobenius reciprocity one gets

$$\varphi_0|T = \phi_{\varphi_0}^{KZ}\mathbf{1}_{KZ\alpha K} = \sum_{g_i\in KZ\backslash KZ\alpha K} g_i^{-1}\varphi_0.$$

Now, one wants to decompose $KZ\alpha K$ in left cosets of KZ, so one notes $KZ\alpha K = K\alpha KZ$ and

$$k\alpha = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} = \begin{pmatrix} \pi & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a'-ac' & (b'-ad')\pi \\ c'\pi & d'\pi^2 \end{pmatrix}\begin{pmatrix} 1/\pi & 0 \\ 0 & 1/\pi \end{pmatrix},$$

for any non trivial $k \in K$ and $a \in \mathcal{O}_F$. Also, note

$$\begin{pmatrix} \pi & a \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \pi & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\pi & -a/\pi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/\pi(b-a) \\ 0 & 1 \end{pmatrix},$$

for $a, b \in \mathcal{O}_F$. This product is in KZ if and only if $a - b \in \pi\mathcal{O}_F$. Putting everything together one gets the decomposition

$$KZ\alpha K = \bigsqcup_{a\in\mathcal{O}_F/\pi\mathcal{O}_F}\begin{pmatrix} \pi & a \\ 0 & 1 \end{pmatrix} KZ \sqcup \alpha KZ.$$

So one shall note $(KZ\alpha K)^{-1} = KZ\alpha^{-1}K = KZ\alpha K$, so taking inverses of the left cosets above gives the right cosets

$$KZ\alpha K = \bigsqcup_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} KZ\begin{pmatrix} \pi & a \\ 0 & 1 \end{pmatrix}^{-1} \sqcup KZ\alpha^{-1}.$$

Then,

$$\varphi_0|T = \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F}\begin{pmatrix} \pi & a \\ 0 & 1 \end{pmatrix}\varphi_0 + \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}\varphi_0 = \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F}\alpha' n(a/\pi)\varphi_0 + \alpha\varphi_0.$$

Now, one shall apply Lemma 3.1 and first part of this Proposition to get

$$\begin{aligned}(\varphi_0|T)(x) &= \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F}\alpha' n(a/\pi)\varphi_0(x) + \alpha\varphi_0(x) \\ &= (\Lambda - \Lambda^{-1})\mathbf{1}_{\mathcal{O}_F}(x) + \Lambda^{-1}\varphi_0(\pi x) \\ &= \Lambda\varphi_0(x),\end{aligned}$$

where the last equality comes from considering x with positive valuation, and with negative valuation and comparing it to $\Lambda\varphi_0(x)$. □

It is clear now that

$$\phi_{\varphi_0}^{KZ}(T-\Lambda)(\mathbf{1}_{KZ}) = \varphi_0|T - \Lambda\phi_{\varphi_0}^{KZ}(\mathbf{1}_{KZ}) = \Lambda\varphi_0 - \Lambda\varphi_0 = 0.$$

Then, by G-equivariance one gets the following.

COROLLARY 3.3. $\phi_{\varphi_0}^{KZ}$ *induces a map*

$$\mathrm{ind}_{KZ}^G 1/(T-\Lambda)\mathrm{ind}_{KZ}^G 1 \to \mathrm{ind}_B^G(\chi \otimes \chi^{-1}).$$

REMARK. One shall notice φ_0 is not an element in the representation space of $\mathrm{ind}_B^G \chi \otimes \chi^{-1}$, but it corresponds uniquely to an element there, namely f_0 via the correspondance $\varphi \leftrightarrow f$. This correspondance is a G-map if one choose the appropiate G-action on the φ's, so the T-action on φ_0 determines the T-action on f_0. In [BL95], they use this correspondance to give an equivalent representation space with the φ's, but it is no more than the aforementioned fact that one can recover $f(1_G)$ from φ. Hence, here, one shall work with the φ's but the results will be given in terms of the f's.

3.2 The Unramified Principal Series

Now, one knows $\mathcal{F}(K)$ is a free $\mathcal{H}_K$-module and one may also consider R as a $\mathcal{H}_K$-module by considering the action $T \mapsto \Lambda$, i.e. $Tr = \Lambda r$ for any $r \in R$. Then, one shall consider the representation $\mathrm{ind}_{KZ}^G 1 \otimes_{\mathcal{H}_K} R$, which is no more than the tensor product of both representations spaces as $\mathcal{H}_K$-modules. One denotes this representation space as $\mathcal{V}$. Now, one is interested in the specialization of $\mathcal{V}$ to $\Lambda = \lambda$ for each $\lambda \in E^\times$. One shall denote this specialization as $\mathcal{V}_\lambda$. One has the obvious homomorphism

$$\mathcal{F}(K) \to \mathcal{V}_\lambda, \quad f \mapsto f \otimes 1,$$

which is surjective as $\mathcal{F}(K)$ and with kernel generated by $(T-\lambda)$, as the action of T on $\mathcal{V}_\lambda$ is the one of λ. All in all, one gets the isomorphism

$$\mathcal{V}_\lambda \cong \mathcal{F}(K)/(T-\lambda)\mathcal{F}(K).$$

Now, one can also specialize the unramified character $\chi(\pi) = \Lambda$ to $\chi_\lambda(\pi) = \lambda$. Then, one can consider the induced representation $\mathrm{ind}_B^G(\chi_\lambda \otimes \chi_\lambda^{-1})$ and denote its space of representation as $\mathcal{F}(\chi_\lambda, \chi_\lambda^{-1})$, which can be considered as a specialization of $\mathcal{F}(\chi, \chi^{-1})$. By tensoring with R as before, $\phi_{\varphi_0}^{KZ}$ extends to an R-morphism $\mathcal{V} \to \mathcal{F}(\chi, \chi^{-1})$. Then, it restricts by specialization to an E-morphism $\mathcal{V}_\lambda \to \mathcal{F}(\chi_\lambda, \chi_\lambda^{-1})$.

Recall the definition of $f_0 \in \mathcal{F}(\chi, \chi^{-1})$. One can consider its specialization denoted also $f_0 \in \mathcal{F}(\chi_1, \chi_1^{-1})$. Recall the Iwasawa decomposition $G = KB$. Then, any $g \in G$ can be written as $g = kb$ with $k \in K$ and $b \in B$ so

$$f_0(g) = f_0(kb) = 1(b) f_0(k) = 1,$$

by definition of f_0. Thus, it generates a one dimensional trivial G-subrepresentation of $\mathrm{ind}_B^G(\chi_1 \otimes \chi_1^{-1})$. One defines the *special representation* of G, Sp, as the quotient $\mathrm{ind}_B^G(\chi_1 \otimes \chi_1^{-1})/\mathrm{triv}$, and it can be realized over $\mathcal{F}(\mathrm{Sp}) := \mathcal{F}(\chi_1, \chi_1^{-1})/E f_0$. Note it can also be realized as a quotient of $\mathrm{ind}_B^G(\chi_{-1} \otimes \chi_{-1}^{-1})$ too. Just note the *twisting* map $\mathcal{F}(\chi_\lambda, \chi_\lambda^{-1}) \to \mathcal{F}(\chi_{-\lambda}, \chi_{-\lambda}^{-1})$ given by

$$f(g) \mapsto f^-(g) = \begin{cases} f(g), & \text{if } g = 1_G, \\ (-1)^{\mathrm{val}(x)} f(g), & \text{if } g = wn(x),\ x \in F, \end{cases}$$

extends to a G-isomorphism by definition. Then, the quotient map

$$\mathcal{F}(\chi, \chi^{-1}) \to \mathcal{F}(\chi, \chi^{-1})/((\Lambda^{-2} - 1)\mathcal{F}(\chi, \chi^{-1}) + R f_0),$$

as an R linear G morphism

$$\psi : \mathcal{F}(\chi, \chi^{-1}) \to \mathcal{F}(\mathrm{Sp}) \otimes_E R/(\Lambda^{-2} - 1)R.$$

Since it is a quotient map, it is clearly surjective. One shall prove the following theorem.

THEOREM 3.4. *The short sequence*

$$0 \longrightarrow \mathrm{ind}_{KZ}^G 1 \otimes_{\mathcal{H}_K} R \xrightarrow{\phi_{\varphi_0}^{KZ}} \mathrm{ind}_B^G(\chi \otimes \chi^{-1}) \xrightarrow{\psi} \mathrm{Sp} \otimes R/(\Lambda^{-2} - 1)R \longrightarrow 0,$$

is exact.

PROOF. Note one already knows the map ψ is surjective, so one is left to prove the map $\phi_{\varphi_0}^{KZ}$ is injective and its image is the kernel of the last map in the exact sequence. By passing to the representation spaces one has to prove the short sequence of R-modules

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{F}(\chi, \chi^{-1}) \longrightarrow \mathcal{F}(\mathrm{Sp}) \otimes_E R/(\Lambda^{-2} - 1)R \longrightarrow 0,$$

is exact. One shall first note that $\mathcal{V}$ is a free R-module as the representation space of $\mathrm{ind}_{KZ}^G 1$ is a free $\mathcal{H}_K$-module and, in particular, a free E-module. Also, $\mathcal{F}(\chi, \chi^{-1})$ is a free R-module since both $R f_0$ and $\mathcal{F}(\chi, \chi^{-1})/R f_0$ are free R-modules.

Now, one proves injectivity of the first map. Let $\phi_{\varphi_0}^{KZ}(c) = 0$ for $c \in \mathcal{V}$. One shall see $c = 0$. Recall that $\mathcal{F}(K) \cong C_0(\Delta)$, so $V \cong \mathcal{F}(K) \otimes R \cong C_0(\Delta)$

where coefficients are now in R. Thus, one may write $c = \sum_{v\in S} \alpha_v v$ with $0 \neq \alpha_v \in R$ and S a finite subset of vertices of Δ. Then, one may define $s := \min\{h(v) : v \in S\}$. Also, note all vertices in S are under some v_r for some big enough r. In other words, for any $v \in S$, $h(v) + d(v, v_r) = r$. Now, one can assume S is the set $\{v : h(v) \geq s, \text{and } v \text{ under } v_r\}$ by letting $a_v = 0$ in c if necessary. Now, note the trivial equality

$$v = \Lambda^{-1}a(v) - \Lambda^{-1}(a(v) - \Lambda v), \tag{3.1}$$

where $a(v)$ denotes the antecedent of v. Hence, one may write

$$\begin{aligned} c = \sum_{v\in S} \alpha_v v &= \sum_{v\in S} \alpha_v(\Lambda^{-1}a(v) - \Lambda^{-1}(a(v) - \Lambda v)) \\ &= Pv_r + \sum_{v\neq v_s} P_v(a(v) - v), \end{aligned}$$

where $P, P_v \in R$. Now, one recalls the action of T on v_0 is

$$v_0|T = \sum_{v\in\mathcal{N}_0} v,$$

where $\mathcal{N}_0$ is the set of neighbors of v_0. In other words, they are v_1, the antecedent of v_0, and all the vertices of height -1, i.e. those whose antecedent is exactly v_0. Then, by G-equivariance

$$v|T = a(v) + \sum_{a(v')=v} v'.$$

Now, one considers the action of $T - \Lambda$ and by (3.1)

$$\begin{aligned} v|(T-\Lambda) = a(v) + \sum_{a(v')=v} v' - \Lambda v &= a(v) - \Lambda v + \Lambda^{-1} \sum_{a(v')=v} (a(v') - a(v') + \Lambda v') \\ &= a(v) - \Lambda v + \Lambda^{-1}qv - \Lambda^{-1} \sum_{a(v')=v} (a(v') - \Lambda v') \\ &= a(v) - \Lambda v - \Lambda^{-1} \sum_{a(v')=v} (a(v') - \Lambda v'), \end{aligned}$$

and one gets

$$a(v) - \Lambda v \equiv \Lambda^{-1} \sum_{a(v')=v} (a(v') - \Lambda v') \pmod{(T-\Lambda)\mathrm{ind}_{KZ}^G 1}. \tag{3.2}$$

Then, one may rewrite the summand $\sum_{v\neq v_r} P_v(a(v) - v)$ with vertices of minimal height in S, i.e. of height s, by applying (3.2) as many times as needed to get

$$c \equiv Pv_r + \sum_{\{v\in S : h(v)=s\}} P'_v(a(v) - \Lambda v) \pmod{(T-\Lambda)\mathrm{ind}_{KZ}^G 1},$$

where the P'_v's are again in R. Now, one shall apply $\phi^{KZ}_{\varphi_0}$ to this expression noting it vanishes on $(T-\Lambda)\mathrm{ind}^G_{KZ}1$ and also in c to get

$$0 = \phi^{KZ}_{\varphi_0}(c) = P\phi^{KZ}_{\varphi_0}(v_r) + \sum_{\{v\in S: h(v)=s\}} P'_v \phi^{KZ}_{\varphi_0}(a(v) - \Lambda v).$$

Now, one shall see that $\phi^{KZ}_{\varphi_0}(v_r)$ has a non compact support and the $\phi^{KZ}_{\varphi_0}(a(v) - \Lambda v)$ all have disjoints compact supports, forcing all the P, P'_v's to be 0 and proving injectivity. First, note

$$\phi^{KZ}_{\varphi_0}(v_r) = \phi^{KZ}_{\varphi_0}(\alpha^r v_0) = \phi^{KZ}_{\varphi_0}(v_0) = \phi^{KZ}_{\varphi_0}(\mathbf{1}_{KZ}) = \varphi_0,$$

where one uses the usual identification $v_0 \leftrightarrow \mathbf{1}_{KZ}$. Then its support is G which is non compact. Now, one notes that by Lemma 1.7, one may replace the vertices v in $\phi^{KZ}_{\varphi_0}(a(v) - v)$ by nv_s where n ranges over all $n \in N_{-r}/N_{-s}$. Then, each $n \in N_{-r}/N_{-s}$ may be written as $n = n(x)$ with $x \in \pi^{-r}\mathcal{O}_F$ and one gets that each term

$$\begin{aligned}
\phi^{KZ}_{\varphi_0}(a(n(x)v_s) - \Lambda n(x)v_s) &= \phi^{KZ}_{\varphi_0}(n(x)v_{s+1} - \Lambda n(x)v_s) \\
&= \phi^{KZ}_{\varphi_0}((n(x)\alpha^{s+1} - \Lambda n(x)\alpha^s)(\mathbf{1}_{KZ})) \\
&= (n(x)\alpha^{s+1} - \Lambda n(x)\alpha^s)\varphi_0 \\
&= n(x)\alpha^s(\alpha - \Lambda)\varphi_0 \\
&= (n(x)\alpha^s)(\Lambda^{-1} - \Lambda)(\mathbf{1}_{\mathcal{O}_F}),
\end{aligned}$$

and one is left to see what the function $n(x)\alpha^s\mathbf{1}_{\mathcal{O}_F}$ is. Just note $\mathbf{1}_{\mathcal{O}_F}$ is given by $\varphi(y)$, 1-valued in $\mathcal{O}_F$ and 0 everywhere else. Thus, $n(x)\alpha^s\varphi(y) = \Lambda^{-s}\varphi(x+\pi^s y)$, so it is the function Λ^{-s}-valued if $x + \pi^s y \in \mathcal{O}_F$, i.e. if $y \in -x + \pi^{-s}\mathcal{O}_F$ and 0 everywhere else. In other words,

$$\phi^{KZ}_{\varphi_0}(a(n(x)v_s) - \Lambda n(x)v_s) = \Lambda^{-s}(\Lambda^{-1} - \Lambda)\mathbf{1}_{-x+\pi^{-s}\mathcal{O}_F},$$

which has compact support. Also, note for each $x \in \pi^{-r}\mathcal{O}_F/\pi^{-s}\mathcal{O}_F$, these supports are all disjoint, finally proving $c = 0$ as wanted.

Now we prove the image of $\phi^{KZ}_{\varphi_0}$ is exactly $(\Lambda^{-2} - 1)R$. Clearly, $\varphi_0 = \phi^{KZ}_{\varphi_0}(\mathbf{1}_{KZ})$ is in the image, and all its G-translates by G-equivariance. But, recall the first property in Proposition 3.2. This implies the map corresponding to $(\Lambda - \Lambda^{-1})\mathbf{1}_{\mathcal{O}_F}$ is in the image. But, applying $n(-x)\alpha^s$ as before, the map corresponding to $\Lambda^{-s}(\Lambda - \Lambda^{-1})\mathbf{1}_{x+\pi^s\mathcal{O}_F}$ is also contained in the image. Since the image is an R-module, one also has $-\Lambda^{s-1}\Lambda^{-s}(\Lambda - \Lambda^{-1})\mathbf{1}_{x+\pi^s\mathcal{O}_F} = (\Lambda^{-2} - 1)\mathbf{1}_{x+\pi^s\mathcal{O}_F}$ for all $x \in F$ and $s \in \mathbb{Z}$. Thus, since these form an R-basis for $(\Lambda^{-2} - 1)\mathcal{F}(\chi, \chi^{-1})/Rf_0$, where one thinks on the correspondent functions in this space, and one also has Rf_0, the image contains $(\Lambda^{-2} - 1)\mathcal{F}(\chi, \chi^{-1}) + Rf_0$. For the converse, let $c = \sum r_i v_i$ be a 0-chain over R. One shall see its image is contained in $(\Lambda^{-2}-1)\mathcal{F}(\chi,\chi^{-1})+Rf_0$. Note, for any vertex $v\in V(\Delta)$, there is a unique path from v to v_0, and denote the vertices in this path as $v=v^1,v^2,\ldots,v^n=v_0$. Then, $v=v_0+\sum_{k=1}^{n-1}v^k-v^{k+1}$ and by multiplying by Λ^{-1}, summing both equalities and rearranging terms one gets

$$v=v_0+(1+\Lambda^{-1})\left(\sum_{k=1}^{n-1}v^k-\Lambda^{-1}v^{k+1}-\sum_{k=1}^{n-1}v^{k+1}-\Lambda^{-1}v^k\right).$$

Since G preserves adjacency and it is transitive, one can rewrite this as $v=v_0+(1+\Lambda^{-1})\sum g_k(v_0-\Lambda^{-1}v_1)=v_0+(1+\Lambda^{-1})\sum g_k(v_0-\Lambda^{-1}\alpha v_0)$. Now, $c=\sum r_iv_i=r_0'v_0+\sum r_j'g_j(v_0-\Lambda^{-1}\alpha v_0)$ with $r_0,r_j\in R$, and applying $\phi_{\varphi_0}^{KZ}$, under the identification $C_0(\Delta)\cong\mathcal{V}$ of course,

$$\begin{aligned}\phi_{\varphi_0}^{KZ}(c)=\phi_{\varphi_0}^{KZ}(\textstyle\sum r_iv_i)&=\phi_{\varphi_0}^{KZ}(r_0'v_0+\textstyle\sum r_j'g_j(v_0-\Lambda^{-1}\alpha v_0))\\&=r_0'\varphi_0+\textstyle\sum r_jg_j(\varphi_0-\Lambda^{-1}\alpha\varphi_0).\end{aligned}$$

Now, one just notes that $\varphi_0-\Lambda^{-1}\alpha\varphi_0=(1-\Lambda^{-2})\mathbf{1}_{\mathcal{O}_F}$. Then, $\phi_{\varphi_0}^{KZ}(c)\in(1-\Lambda^{-2})\mathcal{F}(\chi,\chi^{-1})+Rf_0$ concluding the proof. □

Remark. One may use the argument in the last proof to prove that the G-translates of f_0 generate $\mathcal{V}_\lambda$ over E, for $\lambda\neq\pm1$. One just follows the previous argument to prove the G-translates of f_0 generate $(1-\Lambda^{-2})\mathcal{F}(\chi,\chi^{-1})/Rf_0$, so when one considers the specialization to $\Lambda=\lambda\neq\pm1$, this is the whole $\mathcal{F}(\chi_\lambda,\chi_\lambda^{-1})/Ef_0$, and together with f_0 it generates the whole space $\mathcal{F}(\chi_\lambda,\chi_\lambda^{-1})$. Therefore, considering $\mathcal{V}_\lambda$ as a subspace of $\mathcal{F}(\chi_\lambda,\chi_\lambda^{-1})$ by the previous theorem, it must be generated by the G-translates of f_0.

3.3 Irreducibility of $\mathrm{ind}_B^G(\chi_\lambda\otimes\chi_\lambda^{-1})$

Let us prove that $\mathrm{ind}_B^G(\chi_\lambda\otimes\chi_\lambda^{-1})$ is irreducible for $\lambda\in E^\times\setminus\{\pm1\}$. For that, one shall study the action of $\mathcal{H}_I$ on $\mathcal{F}(\chi_\lambda,\chi_\lambda^{-1})^I$. One starts by noting this space is two dimensional over E. For that, it is enough to prove the following group decomposition: $G=BI\sqcup B\beta I$. Then, any element of $\mathcal{F}(\chi_\lambda,\chi_\lambda^{-1})^I$ is completely determined by its image on 1_G and β. Then, assume $M\notin BI$, i.e. the back-left entry of the matrix has valuation 0. Then,

$$M:=\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1&c^{-1}(\pi+a)\\0&1\end{pmatrix}\begin{pmatrix}0&1\\\pi&0\end{pmatrix}\begin{pmatrix}c/\pi&d/\pi\\-\pi&-dc^{-1}(\pi+a)+b\end{pmatrix}\in B\beta I.$$

Now, one considers the E-basis f_1,f_2 where $f_1(1_G)=1$ and $f_1(\beta)=0$ and $f_2(1_G)=0$ and $f_2(\beta)=1$. Now, recall the Iwasawa decomposition $G=BK$. Then, $\mathcal{F}(\chi_\lambda,\chi_\lambda^{-1})^K$ is one dimensional, generated by f_0 which is defined by $f_0(1_G)=1$, i.e. $f_0(bk)=f_0(b)=(\chi_\lambda\otimes\chi_\lambda^{-1})(b)$. One notes that

$f_0(\beta) = f_0(\alpha' k) = (\chi_\lambda \otimes \chi_\lambda^{-1})(\alpha') = \lambda^{-1}$ where k is the matrix permuting rows. Thus, $f_0 = f_1 + \lambda^{-1} f_2$. One only needs a little lemma to prove the irreducibility.

Lemma 3.5. *Let $\lambda \in E^\times \setminus \{\pm 1\}$. Then,*

1. $f_1 | T_{-1,0} = -f_2$,
2. $f_2 | T_{-1,0} = \lambda f_2$.

Proof. Since f_1 and f_2 are I-invariants and $\mathrm{ind}_B^G(\chi_\lambda \otimes \chi_\lambda^{-1})$, one may apply Lemma 2.3 so

$$f_1 | T_{10} = \beta f_1 = f_2,$$

by looking at the image on 1_G and on β, noting $\beta^2 = \pi 1_G$. Same reasoning gives $f_2 | T_{10} = f_1$. Now, one need to show $f_0 | T_{-1,0} = f_0 | (T - T_{10})$. For that, recall $\phi_{\psi_0}^{IZ}$ acts on edges as computing the origin

$$\begin{aligned} f_0 | T &= (\mathbf{1}_{KZ}) T \phi_{f_0}^{KZ} = (\mathbf{1}_{KZ\alpha K}) \phi_{f_0}^{KZ} = (\mathbf{1}_{KZ\alpha^{-1} I} + \mathbf{1}_{KZ\alpha I}) \phi_{f_0}^{KZ} \\ &= (\mathbf{1}_{IZ\beta I} + \mathbf{1}_{IZ\alpha^{-1} I}) \phi_{\psi_0}^{IZ} \phi_{f_0}^{KZ} = (\mathbf{1}_{IZ})(T_{10} + T_{-1,0}) \phi_{f_0}^{IZ} = f_0 | (T_{10} + T_{-1,0}). \end{aligned}$$

Now, one may compute

$$f_0 | T_{-1,0} = f_0 | (T - T_{10}) = \lambda f_0 - (f_2 + \lambda f_1) = (\lambda^2 - 1) f_2.$$

Now, note

$$T_{-10}^2 = (T_{10} T_{12} T_{10})^2 = T_{10} T_{12}^2 T_{10} = T_{-2,-1} = T_2 - T_{2,1} = (T^2 - 1) - T_{21},$$

and also

$$T_{21} = T_{10} T_{12} = T_{10} T_{12} T_{10}^2 = T_{-10} T_{10},$$

so we obtain

$$T_{10}^2 = (T^2 - 1) - T_{-10} T_{10}.$$

Now, one may compute

$$\begin{aligned} f_0 | T_{-10}^2 &= f_0 | (T^2 - 1 - T_{-10} T_{10}) \\ &= (\lambda^2 - 1) f_0 - (\lambda^2 - 1) f_1 = (\lambda^2 - 1) \lambda f_2, \end{aligned}$$

and together with $f_0 | T_{-1,0} = (\lambda^2 - 1) f_2$, it follows that $f_2 | T_{-10} = \lambda f_2$, and similarly noting $f_1 = f_0 - \lambda^{-1} f_2$) we get $f_1 | T_{-10} = -f_2$. □

Now, one is in position to prove the irreducibility of the principal series representation.

THEOREM 3.6. *For* $\lambda \neq 0, \pm 1$*, the representation* $\mathrm{ind}_B^G(\chi_\lambda \otimes \chi_\lambda^{-1})$ *is irreducible.*

PROOF. Let Y be a non zero G-submodule of the representation space $\mathcal{F}(\chi_\lambda, \chi_\lambda^{-1})$. Let us see it is the whole space, proving irreducibility. Since I_1 is a pro-p-group, $Y^{I_1} \neq \{0\}$ by Lemma 1.3. Since I/I_1 acts trivially on the whole space, it acts trivially on the I_1-invariants and one gets $Y^I \neq \{0\}$. Let f be a non zero element in Y^I, then one may write $f = \alpha_1 f_1 + \alpha_2 f_2$ with $\alpha_1, \alpha_2 \in E$. Now, $\beta f = f|T_{10} \in Y^I$ is also non zero. One may apply T_{-10} to both f and βf and by the previous lemma one gets

$$f|T_{-10} = (-\alpha_1 + \alpha_2\lambda)f_2, \quad f|T_{10}T_{-10} = \alpha_1 f_2 + \alpha_2 f_1|T_{-10} = (-\alpha_2 + \alpha_1\lambda)f_2,$$

and since $\lambda \neq \pm 1$, $f_2 \in Y^I$; thus, also $f_1 \in Y^I$ and these two elements give the element f_0. Then, by the last remark in the previous section, the G-translates of f_0 give the whole representation space, so Y is the whole space as wanted. □

Chapter 4

The Special Series

For the case $\lambda = \pm 1$ it is no longer true that the principal series representation $\mathrm{ind}_B^G(\chi_\lambda \otimes \chi_\lambda^{-1})$ is irreducible. However, it only has one irreducible quotient, namely the special series Sp. In section 1 we prove the irreducibility of the special series and in section 2 we prove it is the unique irreducible quotient of the principal series representation for $\lambda = \pm 1$. In this last section we also prove that Sp is the only irreducible representation contained in $\mathcal{V}_1$. We will follow [BL95] in this chapter. On the following we assume $\lambda = 1$ since the case $\lambda = -1$ can be reduced to the former by twisting.

4.1 Irreducibility of Sp

Let us prove the special series is irreducible. One first changes the realization space of Sp. For that, recall the Bruhat decomposition $G = B \sqcup BwN$. Then, since $wN \cong F$, one may see $B\backslash G \cong \mathbb{P}^1(F)$, and $\mathrm{ind}_B^G(\chi_1 \otimes \chi_1^{-1})$ may be realized over $\mathcal{F}(\mathbb{P}^1(F))$. Note the image at infinity of any element $f \in \mathcal{F}(\mathbb{P}^1(F))$ is determined by its restriction to the affine part F, so one shall denote by φ this restriction. Note G acts by right translation on $\mathbb{P}^1(F)$. One can realize Sp on $\mathcal{F}(\mathbb{P}^1(F))/\{const.\} = \mathcal{F}(\mathbb{P}^1(F))/E\varphi_0$ because $\lambda = 1$ implies $\varphi_0 = 1$. One shall study the action of I_1 on this space. One starts with its action on the projective space $\mathbb{P}^1(F)$.

Lemma 4.1. *I_1 has exactly 2 orbits on $\mathbb{P}^1(F)$, one for $0 := (1 : 0)$ and one for $\infty := (0 : 1)$, explicitly*

$$\mathcal{O}_0 = \{(1 : x) : x \in \mathcal{O}_{\mathcal{F}})\},$$
$$\mathcal{O}_\infty = \{(\pi x : 1) : x \in \mathcal{O}_F\}.$$

What is more, I_1 acts transitively on $\mathcal{O}_0 \times \mathcal{O}_\infty$, i.e. the stabilizers of 0 and ∞ act transitively on $\mathcal{O}_\infty$ and $\mathcal{O}_0$ respectively. Also, any element in I_1 stabilizes either 0 or ∞.

Proof. To compute these orbits just note $\mathbb{P}^1(F) = \mathcal{O}_0 \sqcup \mathcal{O}_\infty$ and

$$(1:0)\begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} = (a:b) = (1:a^{-1}b), \quad \begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} \in I_1,$$

where $a^{-1}b$ ranges over all $\mathcal{O}_F$. Similarly,

$$(0:1)\begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} = (c\pi:d) = (d^{-1}c\pi:1), \quad \begin{pmatrix} a & b \\ c\pi & d \end{pmatrix} \in I_1,$$

where again $d^{-1}c$ ranges over all $\mathcal{O}_F$. The above computations also show that the stabilizer of 0 are the above matrices such that $b = 0$ and the stabilizer of ∞ the ones such that $c = 0$. Then, the last assertion of the lemma follows from this. □

Lemma 4.2. *The following sequence of $\mathcal{H}_I$-modules*

$$0 \longrightarrow E\varphi_0 \longrightarrow (\mathcal{F}(\mathbb{P}^1(F)))^{I_1} \longrightarrow (\mathcal{F}(\mathbb{P}^1(F))/E\varphi_0)^{I_1} \longrightarrow 0,$$

is exact.

Proof. First note I_1-invariants is obviously left exact. Then, note that since φ_0 is I_1-invariant, $E\varphi_0^{I_1} = E\varphi_0$. One is just left with surjectivity of the last map. Let $w \in (\mathcal{F}(\mathbb{P}^1(F))/E\varphi_0)^{I_1}$ and let φ be a pullback in $\mathcal{F}(\mathbb{P}^1(F))$. Let us see it is I_1-invariant. This is equivalent to being constant on the I_1-orbits, i.e. on $\mathcal{O}_0$ and $\mathcal{O}_\infty$ by the previous lemma. Since w is I_1-invariant, for any $i \in I_1$, $i\varphi - \varphi$ must be equal mod $E\varphi_0$, i.e. it must be a constant. Let $i \in I_1$. Then, if it stabilizes ∞

$$i\varphi(\infty) = \varphi(\infty i) = \varphi(\infty),$$

proving $i\varphi = \varphi$ in this case. Otherwise, if i stabilizes 0 then

$$i\varphi(0) = \varphi(0i) = \varphi(0),$$

and $i\varphi = \varphi$ also in this case. Thus, φ is I_1-invariant. □

Now, one is in position to prove the irreducibility of Sp.

Theorem 4.3. Sp *is irreducible.*

Proof. Let $W \subset \mathcal{F}(\mathbb{P}^1(F))/E\varphi_0$ be a non zero G-submodule. Since I_1 is a pro-p-group, by Lemma 1.3 $W^{I_1} \neq 0$. Now, let V the pullback of W to $\mathcal{F}(\mathbb{P}^1(F))$. By the previous lemma,

$$2 = \dim(\mathcal{F}(\mathbb{P}^1(F)))^{I_1} \geq \dim V^{I_1} = \dim E\varphi_0 + \dim W^{I_1} \geq 2,$$

which shows $V^{I_1} = (\mathcal{F}(\mathbb{P}^1(F)))^{I_1}$ as they are E-vector spaces of the same dimension contained one in the other. Hence, $\mathbf{1}_{\mathcal{O}_0}, \mathbf{1}_{\mathcal{O}_\infty} \in V^{I_1}$. Since $\mathcal{F}(\mathbb{A}^1(F))$

and $\mathbf{1}_{\mathcal{O}_\infty}$ generate the whole $\mathcal{F}(\mathbb{P}^1(F))$, it is enough to check $\mathcal{F}(\mathbb{A}^1(F)) \subset V$ to prove V is the whole space; thus, $W = \mathcal{F}(\mathbb{P}^1(F))/E\varphi_0$ proving irreducibility. Since $a + \pi^n\mathcal{O}_F$ for $n \in \mathbb{Z}$ and a ranging over all $\mathcal{O}_F/\pi^n\mathcal{O}_F$ constitute a basis for the topology on F, the functions $\mathbf{1}_{a+\pi^n\mathcal{O}_F}$ form an E-basis of $\mathcal{F}(\mathbb{A}^1(F))$ and note that by the Lemma 3.1

$$n(-a)(\alpha')^n\mathbf{1}_{\mathcal{O}_0} = \mathbf{1}_{a+\pi^n\mathcal{O}_F}, \quad n(-a), \alpha' \in B,$$

so the B-translates of $\mathbf{1}_{\mathcal{O}_0}$ generate $\mathcal{F}(\mathbb{A}^1(F))$ and since V is a G-module, $\mathcal{F}(\mathbb{A}^1(F)) \subset V$ as wanted. □

4.2 Properties of Sp

Now, one needs some extra result concerning the special representation.

Theorem 4.4. Sp *is the only quotient of* $\mathrm{ind}_B^G(\chi_1 \otimes \chi_1^{-1})$.

Proof. Since triv and Sp are both irreducible, $\mathrm{ind}_B^G(\chi_1 \otimes \chi_1^{-1})$ is of length two and it is enough to see Sp cannot be realized as a subrepresentation. Then, it is enough to see the exact sequence

$$0 \longrightarrow \mathrm{triv} \longrightarrow \mathrm{ind}_B^G(\chi_1 \otimes \chi_1^{-1}) \longrightarrow \mathrm{Sp} \longrightarrow 0,$$

is not left-split. For that, let $\psi : \mathcal{F}(\mathbb{P}^1(F)) \to E$, which is G-invariant. Then,

$$\psi(\mathbf{1}_{\mathcal{O}_0}) = \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \psi(\mathbf{1}_{a+\pi\mathcal{O}_F}) = \sum_{a\in\mathcal{O}_F/\pi\mathcal{O}_F} \psi(n(-a)\alpha'\mathbf{1}_{\mathcal{O}_0}) = q\psi(\mathbf{1}_{\mathcal{O}_0}) = 0.$$

Thus, as before since it is B-invariant, ψ is also 0-valued on all $\mathcal{F}(\mathbb{A}^1(F))$ and since

$$\psi(\mathbf{1}_{\mathcal{O}_\infty}) = \psi(\beta\mathbf{1}_{\mathcal{O}_0}) = \psi(\mathbf{1}_{\mathcal{O}_0}) = 0,$$

it is zero everywhere so $\psi = 0$ and the above short exact sequence does not left split. □

Let us consider the degree map deg: $\mathcal{V} \mapsto E$ defined before as $\deg(\sum \alpha_i v_i) = \sum \alpha_i)$.

Lemma 4.5. *The map* deg *satisfies the following properties:*

1. *It is a surjective G-morphism.*
2. *It is trivial on $(T-1)\mathcal{V}$.*
3. *The kernel of the induced map $\overline{\deg} : \mathcal{V}_1 = \mathcal{V}/(T-1)\mathcal{V} \to E$ is isomorphic to* Sp.

What is more, the exact sequence

$$0 \longrightarrow \mathrm{Sp} \longrightarrow \mathcal{V}_1 \longrightarrow \mathrm{triv} \longrightarrow 0,$$

is not split.

Proof. Surjective it is trivial as $ev \mapsto e$ for any $e \in E$. The second statement follows from the equality

$$\deg Tv = \deg \sum v' = q + 1 = 1 = \deg v,$$

where the sum above is taken over the neighbors of v, and there are exactly $q + 1$ of them. The fact that Sp is exactly the kernel of the induced map $\overline{\deg}$ follows from a general fact about Dedekind domains (see Lemma 31 in [BL95]). Then, one gets the short exact sequence

$$0 \longrightarrow \mathrm{Sp} \longrightarrow \mathcal{V}_1 \longrightarrow \mathrm{triv} \longrightarrow 0,$$

and one wants to prove it is not split, i.e. that triv cannot be realized as a subspace of $\mathcal{V}_1$. One shall prove it is not right split. By contradiction, assume it is right split, i.e. there exists a lifting of 1 to $\overline{c} \in \mathcal{V}_1$, represented by $c \in \mathcal{V}$. Since it is a finite sum of vertices, one may assume its support is in the ball $B_k(v_0)$ of radius k centered at v_0, i.e. all the vertices in c are at distance at most k from v_0. Since the lifting is G-equivariant, one has $g\overline{c} = \overline{c}$, in other words, $gc - c \in (T-1)\mathcal{V}$, i.e. $gc - c = (T-1)b$ for some $b \in \mathcal{V}$. Let us fix $g = \alpha^{2k+1}$ and therefore the previous b. Then, for $a \in \mathcal{V}_1$ one defines t_a as the minimal subtree of Δ containing all the vertices in the support of a, i.e. the vertices in the paths joining each pair of vertices in the support of a. Then, $t_c \subset B_k(v_0)$ and $t_{gc} \subset gB_k(v_0) = B_k(v_{2k+1})$. Thus,

$$t_{gc-c} \subset X = B_k(v_0) \sqcup B_k(v_{2k+1}),$$

where the union is disjoint because v_{2k+1} is at distance at least $k+1$ for any vertex at distance at most k from v_0. Now, recall T acts on b as a sum of the neighbors of b; thus, the support of $(T-1)b$ is exactly the support of b plus the neighbors of b. Thus, $t_b \subset t_{(T-1)b} = t_{gc-c} = X$. What is more, t_b can be obtained from t_{gc-c} by omitting the endpoints, since these correspond to the ones of Tb. Clearly, vertices in t_{gc-c} having neighbors outside X are endpoints, and this is the case for both v_k and v_{k+1}. Thus, $t_b \subset X \setminus \{v_k, v_{k+1}\}$. Thus, b can be uniquely decomposed as $b = b_1 + b_2$ with $t_{b_1} \subset B_1 = B_k(v_0) \setminus \{v_k\}$ and $t_{b_2} \subset B_2 = B_k(v_{2k+1}) \setminus \{v_{k+1}\}$. Note B_1 and B_2 are at distance at least 3 from each other, in the sense that any vertex of B_1 is at distance at least 3 from any vertex of B_2. Hence, B_1 and B_2 are disjoint. In particular, $(T-1)b_1 = c$, which implies $\overline{c} = 0$ so the sequence cannot right split. □

Chapter 5

Classification of the irreducible representations

Now, we are in possition to give a classification of the irreducible modular representations of $\mathrm{GL}_2(F)$. We will first look at the non supersingular case ($\lambda \neq 0$) and prove this classification under the assumption $V^K \neq 0$ following [BL95]. The non supersingular case in general follows the same guidelines (see [BL94]) and in section 2 we outline the strategy in [BL94] and present their main results of interest for us. In section 3 we deal with the supersingular case, for which we just refer the reader to [Bre03] and we present the main result, namely that these supersingular representations are irreducible. Combining all the previous results we present a complete classification of the irreducible modular representations.

5.1 The unramified case

THEOREM 5.1. *Let $\rho : G \to V$ be an irreducible representation with central character ω_p such that $V^K \neq 0$. Then,*

1. *ω_p is unramified.*
2. *There is an eigenvector $v \in V^K$ for $\mathcal{H}_K$ such that $v|T = \lambda v$. Furthermore, if $\lambda \neq 0$, ρ is admissible and one has the following classification.*
   a) *If $\lambda^2 = \omega_p(\pi)$, V is one dimensional and if one lets ϵ be an unramified character such that $\epsilon^2 = \omega_p$, ρ is equivalent to $g \mapsto \epsilon(\det g)$ if $\lambda = \epsilon(\pi)$ or to $g \mapsto (-1)^{\mathrm{val}(\det g)}\epsilon(\det(g))$ if $\lambda = -\epsilon(\pi)$.*
   b) *Otherwise, let $\mu_1, \mu_2 \in E$ such that $\mu_2/\mu_1 = \lambda$ and $\mu_1\mu_2 = \omega_p(\pi)$. Then, V^K is one dimensional and ρ is equivalent to $\mathrm{ind}_B^G(\chi_{\mu_1}, \chi_{\mu_2})$.*

PROOF. The first statement is clear. Let $\epsilon : F^\times \to E^\times$ be an unramified character such that $\epsilon^2 = \omega_p$. One may replace ρ with $\epsilon^{-1}\rho$, so that w_p becomes trivial. Then, there exists an eigenvector $v \neq 0$ fixed by K such that $v|T = \lambda v$ by Theorem 2.7. Assume $\lambda \neq 0$. First, let us prove (b), i.e. assume $\lambda \neq \pm 1$ too. In this case, one may consider the map $\phi_v^{KZ} : \mathcal{V} \to V$ which is non zero since $\phi_v^{KZ}(\mathbf{1}_{KZ}) = v \neq 0$. Also, note

$$\phi_v^{KZ}((T-\lambda)(\mathbf{1}_{KZ})) = v|T - \lambda v = 0,$$

by the choice of the v. It follows by G-equivariance, that $\phi_v^{KZ}((T-\lambda)\mathcal{V}) = 0$; thus, a nonzero quotient of $\mathcal{V}_\lambda$ is contained in V. But, $\mathcal{V}_\lambda$ is inside $\mathcal{F}(\chi_\lambda, \chi_\lambda^{-1})$ by Theorem 3.4 and the latter is irreducible, so $\mathcal{V}_\lambda$ is irreducible since $\lambda \neq 0, \pm 1$, so it is itself contained in V. V is also irreducible so $V \cong \mathcal{V}_\lambda$ and one gets $\rho \cong \mathrm{ind}_B^G(\chi_{\mu_1}, \chi_{\mu_2})$ as wanted. For admissibility note that for any open $U < G$ one has V^U is generated by the functions $\mathbf{1}_{BgU}$ with $g \in B\backslash G/U$, and this set is finite for B/G being compact and $B/G\backslash U$ discrete. It follows that for any compact $C < G$, C has a finite open convering $C = \bigcup U_i$ and V^C is generated over E by the functions $\mathbf{1}_{BgU_i}$ for $g \in B\backslash G/U_i$ for each U_i and there is a finite number of them. Thus, it is finite dimensional over E.

Now, one proves (a). The case $\lambda = -1$ reduces to the case $\lambda = 1$ via the twisting map which identifies $\mathcal{F}(\chi_{-1}, \chi_{-1}^{-1}) \cong \mathcal{F}(\chi_1, \chi_1^{-1})$. Now, one repeats the above argument to see V contains a nonzero quotient of $\mathcal{V}_1$. But by the last part of Lemma 4.5, the only nonzero quotient of $\mathcal{V}_1$ is triv. Thus, $\rho \cong$triv and $V \cong E$ is one dimensional. Clearly, ρ is admissible in this case as E^C must be of dimension 1 over E for any compact $C < G$. □

Remark. It is also possible to obtain 3(a) proving first $\mathrm{Sp}^K = 0$ so that necessarily $\rho \cong$ triv as it is done in [BL95], but they suggested this other approach too so we took it.

This theorem classifies roughly the modular representations of $\mathrm{GL}_2(F)$ when there is a K-invariant nonzero vector. One is just left to deal with the case $\lambda = 0$. However, it is not always the case that $V^K \neq 0$, so one should try to generalize this approach to this case.

5.2 The general case

Now, one considers a more general irreducible modular representation. In all the previous work, one assumed that the original representation was unramified, so the strategy was to start from the trivial character in K extending it to KZ letting the center act trivially via scalaras, and induce a representation in G. Now, this is no longer possible, one may consider also non trivial representations in K. However, one can show that all these irreducible representations can be obtained inflating an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$, namely a symmetric power Sym^r of the standard identity representation (see [BL94]). Thus, one should work with the more general Hecke algebras

$$\mathcal{H}(G, H, \sigma) := \mathrm{Hom}_G(\mathrm{ind}_H^G \sigma, \mathrm{ind}_H^G \sigma),$$

where H is a subgroup of G, usually KZ, IZ or B, and σ is an irreducible representation of H, non necessarily trivial. It is easy to classify these irreducible representations σ for $H = KZ$ (see Proposition 4 in [BL94]). Using this more general notion of a Hecke algebra, one can still prove that the Hecke algebra corresponding to the subgroup KZ coincides with $E[T]$ just as in the unramified case (see Proposition 8 in [BL94]). For the subgroup IZ, it coincides with the unramified case when the character factors through the determinant (see Lemma 10 and Proposition 11 in [BL94]). Otherwise, one gets the commutative algebra

$$\mathcal{H}(G, IZ, \epsilon) \cong E[T_{12}, T_{-10}]/(T_{12}T_{-10}, T_{-10}T_{12}),$$

see Proposition 13 in [BL94].

Using the language of local systems, one can prove the key result that the Hecke-submodules of the I-invariants are of finite codimension as an analogue to Proposition 2.6 (see Proposition 18 in [BL94]). Then, one constructs the (tamely ramified) principal series representations and prove when they are irreducible (see Proposition 29). Theorem 31 in [BL94] gives the following classification of the irreducible modular representations of $\mathrm{GL}_2(F)$.

THEOREM 5.2. *If $\rho : G \to V$ is an irreducible modular representation of G with central character, then it is in one of the following clases:*

1. *One dimensional of the form* $\det \cdot \chi$*,*
2. *Principal series representation* $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ *with $\chi_1 \neq \chi_2$,*
3. *Special series* $(\det \chi) \otimes Sp$*,*
4. *Supersingular.*

What is more, all these classes are non empty.

By Schur's Lemma, if one restricts to admissible representations, then ρ always has a central character and one needs no extra assumption on ρ. Then, one is just left to show which of the supersingular representations are actually irreducible. For the moment, for $\lambda = 0$, one only knows ρ is isomorphic to a quotient of the standard supersingular module. The goal of the next section is to establish which are these irreducible supersingular representations.

5.3 Supersingular representations

For the supersingular case, we refer the reader to [Bre03]. Theorem 1.1 in [Bre03] says that for all $r = 0, \dots, p-1$, the supersingular representations $\mathrm{ind}_{KZ}^{G}(\mathrm{Sym}^r \overline{\mathbb{F}_p})/(T)$ are irreducible. This gives finally a complete classification of the irreducible modular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

THEOREM 5.3 (MAIN THEOREM FOR $\mathbb{Q}_p$). *The irreducible modular representations of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *with central character are of the following form:*

1. *One dimensional of the form* $\det \cdot \chi$*,*
2. $(\lambda = 0)$ *Principal series representation* $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$ *with* $\chi_1 \neq \chi_2$*,*
3. $(\lambda = \pm 1)$ *Special series* $(\det \chi) \otimes Sp$*.*

However, the proof in [Bre03] works only for $\mathbb{Q}_p$, for finite extensions of $\mathbb{Q}_p$ one needs new ideas to determine when this supersingular representations are irreducible.

Lastly, I want to mention that [Bre03] also establishes a natural correspondence between supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and two dimensional irreducible Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ (see Corollary 1.4 in [Bre03]). This shows that supersingular representations are the natural choice for the Langlands correspondence in the modular case.

Modular representations are only a first step towards understanding the much more complex p-adic representations which appear in the p-adic Langlands program. The Langlands program is far from being finished, and it seems to be an active area of research nowadays. We hope this gentle introduction to the automorphic side of the Langlands program motivates the reader to learn more about it. We hope to see new ideas in the upcoming years which will push further the state of the art and shed new light on the mysterious Langlands correspondence.

Bibliography

[BL94] L. Barthel and R. Livné. Irreducible modular representations of GL_2 of a local field. *Duke Mathematical Journal*, 75(2):261–292, 1994.

[BL95] L. Barthel and R. Livné. Modular representations of GL_2 of a local field: The ordinary, unramified case. *Journal of Number Theory*, 5:1–27, 1995.

[Bre03] C. Breuil. Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbb{Q}_p)$: I. *Compositio Mathematica*, 138:165–188, 2003.

[Cas14] B. Casselman. *The Bruhat-Tits Tree of SL(2)*, 2014. Available online at `https://ncatlab.org/nlab/files/CasselmanOnBruhatTitsTree2014.pdf`.

[IB76] A.V. Zhelevinskii I.N. Bernshtein. Representations of the group $\mathrm{GL}(n, F)$. *Uspehi Mat. Nauk.*, 31(3):1–68, 1976.

[Vig89] M.F. Vignéras. Représentations modulaires de $\mathrm{GL}(2, F)$, en caractéristique l, f corps p-adique, $p \neq l$. *Compositio Mathematica*, 72:33–66, 1989.