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An overview of the intersection theory on moduli spaces of curves

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La musique est l'âme
de la géométrie

Paul Claudel

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Introduction

The topic of this thesis is the intersection theory on moduli spaces of curves. More precisely, the purpose of this work is to present the Chow ring of moduli spaces of stable genus g curves and to study the relations between some classes which will be called *tautological classes*. This is based on the concepts introduced by Mumford in his article *"Towards an enumerative geometry of the moduli space of curves"*[1]. All tautological classes belong to the *tautological ring*; the definition of tautological ring was given by Faber and Pandharipande[2]: it is a subring of the Chow ring generated by classes which are Chern classes of certain vector bundles and that turns out to be naturally related to geometrical instances. In the final part of the thesis we will focus on the special case of the moduli space of stable curves of genus 2 and on its Chow ring, computing explicitly the intersection product of the classes which generate the Chow ring of this moduli space.

In order to do that, we need first of all to introduce some preliminary notions; before discussing the intersection theory of the moduli spaces $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$, which will be treated in Chapter 4, there are three introductory chapters in which we will present some basics on moduli spaces and on intersection theory. Chapter 1 is dedicated to illustrate the notions of fine and coarse moduli space in full generality, Chapter 2 is dedicated to the special case of moduli spaces of curves and Chapter 3 is a basic introduction to intersection theory and Chow rings of algebraic varieties.

As announced, the last chapter of the thesis, Ch. 5, is the dedicated to the moduli space of genus 2 curves and on its Chow ring $A^*(\overline{\mathcal{M}}_2)$.

Overview on the contents

The problem of classifying objects up to isomorphism is a recurring notion in mathematics, let us think for instance of the classical problem of classifying

[1] Mum83.
[2] FP05.

finite-dimensional vector spaces up to isomorphism in linear algebra; analogously in algebraic geometry it is natural try to classify algebraic varieties up to isomorphism, such as for examples smooth curves of a fixed genus.

The idea behind the definition of moduli space is then to find a variety that classifies all objects under consideration up to isomorphism, and to reduce the study of the isomorphism classes to the study of this space. Specifically, we wish to construct a space parametrizing continuously varying families of objects keeping in mind, as a model, the features of the projective n-space $\mathbb{P}^n_{\mathbb{C}}$ that turns out to be the space parametrizing lines in $\mathbb{C}^{n+1}$ passing through the origin. There is in particular one property of this example that suggests how to define moduli spaces.

As we will see, for every subscheme $X \subseteq \mathbb{P}^n$, the families of lines parametrized by X correspond to the morphisms of schemes from X to $\mathbb{P}^n$. In other words, using categorical terminology, if we had defined a functor $\mathcal{F}$ which associates to every subscheme X the families of lines parametrized by X, the previous statement could be rephrased just by saying that the functor $\mathcal{F}$ is representable by $\mathbb{P}^n$. Hence, using category theory, it seems to be easy to generalize the particular example of the projective space to define the general notion of moduli space just by saying that, given a moduli functor $\mathcal{F}$, there exists a fine moduli space M if $\mathcal{F}$ if representable by M.

However, in many cases, fine moduli spaces do not exist, and this happens also for curves; hence, we have to introduce a weaker notion: coarse moduli spaces. Let us call h^M the functor that assigns to each object X the set of morphisms $\mathrm{Hom}(X, M)$; M is a moduli space if $\mathcal{F} \cong h^M$, isomorphism of functors, however, if the functor $\mathcal{F}$ is not representable but there is a natural transformation of functors $\mathcal{F} \to h^M$ and this is initial among such pairs with the same property, then M is said to be a coarse moduli space for the moduli functor $\mathcal{F}$.

In the thesis we are interested specially in moduli spaces of curves hence, after the general case, we will turn our attention to the space of isomorphism classes of curves. We are interested in particular in the moduli space parametrizing smooth curves of genus g, and being this space not compact, to its compactification, the moduli space parametrizing stable genus g curves. As already stated, the fine moduli spaces for genus g smooth or stable curves do not exist (unless in trivial cases) but the Theorem of Deligne-Mumford (Theorem 2.4.1, pag. 34) guarantees the existence of coarse moduli spaces; more precisely we will define the functors $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ respectively as the functors assigning to each scheme X the families of smooth and of stable curves with n marked points parametrized by X. By the Deligne-Mumford's theorem there exist coarse moduli spaces $M_{g,n}$ and $\overline{M}_{g,n}$ for the functors $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ which are normal algebraic varieties of dimension $3g - 3 + n$. We can hence develop an intersection theory on these varieties, studying their Chow rings.

The Chow ring is defined as the quotient of the cycle group $Z(X)$ by rational equivalence, this was introduced by Wei-Liang Chow (Fig. 1a) in his article of 1956 "*On the equivalence classes of cycles in an algebraic variety*"[3]. By definition the quotient has clearly a group structure, but moreover, if X is a smooth quasi-projective variety, then there is a product structure on $A(X)$ that makes it into an associative, commutative ring, graded by codimension (by Theorem 3.1.2, pag. 3.1.2). Hence it is well defined an intersection product between classes of $A(X)$. Moreover, the following generalization of the classical Bézout's theorem holds: if A, B are subvarieties of a smooth variety X such that every irreducible connected component C of the intersection has codimension $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$, then for each component C there is a positive integer $m_C(A, B)$, which will be called the intersection multiplicity of A and B along C, such that the product of the classes can be written as:

$$[A]\,[B] = \sum m_C(A, B)[C].$$

After the necessary recall of the definitions of moduli spaces and of the basics on intersection theory, we can finally enter into the main part of the thesis, which is discussed in Chapter 4 and Chapter 5.

We will follow the ideas presented by David Mumford (Fig. 1b), in his article of 1983 "*Towards an Enumerative Geometry of the Moduli Space of Curves*:"[4]

> The goal of this paper is to formulate and to begin an exploration of the enumerative geometry of the set of all curves of arbitrary genus g. By this we mean setting up a Chow ring for the moduli space $\mathcal{M}_g$ and its compactification $\overline{\mathcal{M}}_g$, defining what seem to be the most important classes in this ring and calculating the class of some geometrically important loci in $\overline{\mathcal{M}}_g$ in terms of these classes. We take as a model for this the enumerative geometry of the Grassmannians.

Mumford defined on the Chow ring of $\overline{\mathcal{M}}_g$ some particular classes which are Chern classes of some natural vector bundles that we have on $\overline{\mathcal{M}}_g$, and he called these classes the *tautological classes* of $\overline{\mathcal{M}}_g$. Along the pages of his article, he studied the relations among them in order to detect the dependency of these classes. Indeed, all the tautological classes turn out to belong to a subring of the Chow ring $A(\overline{\mathcal{M}}_g)$ that Carel Faber and Rahul Pandharipande (Fig. 1c) called the

[3] Cho56.
[4] Mum83.

(a) *Wei-Liang Chow (1911 - 1955)*

(b) *David Mumford (1937 -)*

(c) *Rahul Pandharipande (1969 -)*

Figure 1: Wei-Liang Chow, David Mumford and Rahul Pandharipande

tautological ring. This ring consists, according to the words of Ravi Vakil[5], of "*all the classes naturally coming from geometry*."

It is expected that no natural questions coming from geometrical instances reduces to questions in the nontautological part of the Chow ring, hence, it is sufficient, for studying many interesting questions in geometry, to study the tautological ring and the tautological relations among its classes.

We will conclude in fact Chapter 4 with a proposition (Prop. 4.8.5, pag. 4.8.5), which summarizes the tautological relations among classes deduced in the chapter. Finally, following the last part of Mumford's article[6], we will concentrate on the

[5] Vak03.

[6] Mum83.

moduli space of genus 2 stable curves, and we will describe the structure of its Chow ring, computing explicitly the intersection product of its classes.

Contents of the chapters

In this section, we will illustrate the contents of the chapters in more detail. The first chapter (Ch. 1, pag. 1) is devoted to an introduction to the notion of moduli space, more precisely the notion of fine and coarse moduli space.

This will be done in a purely categorical setting hence, for a better comprehension of the notion, in the first couple of sections we will try to give an intuitive idea of what a moduli space should be and which properties it should have. In particular, in Section 1.2, we will concentrate on the well-known example of projective n-space; afterwards in sections 1.3 and 1.4 we will define respectively fine and coarse moduli spaces; in the last section of the chapter (Sec. 1.5), we will illustrate the example of the Picard group: we will show that it can be viewed as a moduli space and this will naturally endow it with a scheme structure. The purely categorical results will be presented without proof, more details can be found in Appendix A.

Chapter 2 deals with moduli spaces of curves. First of all we will fix notation and we will recall the basic definitions; in particular we will try to justify why we expect that the moduli space of stable curves provides a compactification of the moduli space of smooth curves. In Section 2.4 we will then give the formal definition of families of smooth and stable genus g curves with n marked points, and we will state the Theorem of Deligne-Mumford (Theorem 2.4.1, pag. 34) on the existence of coarse moduli spaces. At the end of the chapter, in Section 2.5, we will describe some examples of moduli spaces of curves in low genus. The moduli spaces of smooth curves of genus 0 with n marked points are relatively easy to describe. However, for moduli spaces of stable curves with 3 or 4 marked points, the situation is more complicated; hence, we will also introduce in this section the combinatorial object that we will need to use for this purpose: the stable graph associated to a stable curve.

Chapter 3 consists in a basic introduction to intersection theory on algebraic varieties; we will first of all define the Chow ring of a variety in Section 3.1, together with a structural theorem for the Chow ring (Theorem 3.1.2, pag. 51). It is very difficult to be able to compute the Chow ring of a variety using directly the definition. This can be done in some simple case, as illustrated in Section 3.2, but in general we will need other tools. In Section 3.3 we will treat the Mayer-Vietoris theorem and excision, and in the following Section 3.4 we will discuss the behaviour of classes under pullback and pushforward maps. In Section 3.5 we will then be able to state the theorem which generalize the classical Bézout's theorem to the intersection of any two subvarieties A, B of a variety X, with the property that every irreducible component C of the intersection satisfies: $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$. Finally, in Section 3.6 we will prove the result that the Chow ring of the projective space is equal to $\mathbb{Z}[\zeta] / \left(\zeta^{n+1}\right)$, where ζ is the rational equivalence class of a hyperplane.

After the first three introductory chapters, in Chapter 4 we will start to develop the intersection theory of moduli spaces of curves. We will start by defining, according to Mumford's article, the *tautological classes*. The first couple of sections is introductory. In Section 4.1 we will give an intuitive idea of why we may wish to restrict our attention to tautological classes, and in Section 4.2 we will collect the definitions and the results on vector bundles that will be used in the next sections. The tautological classes will be then defined in Sections 4.4 and the tautological ring in 4.5. The remaining part of the chapter is devoted to finding relations among these classes, and in particular to introducing the main tool used by Mumford for this purpose: Grothendieck-Riemann-Roch formula (Theorem 4.7.1, pag. 4.7.1), stated in Section 4.7. In the last Section 4.8 we will finally deduce the tautological relations among classes.

In the last chapter of the thesis (Ch. 5) we will concentrate on the moduli space $\overline{\mathcal{M}}_2$ of stable curves of genus 2. In Section 5.1 we will write the tautological relations in the special case of $g = 2$ and $n = 0$. In Section 5.2 we will see the generators for the Chow ring of $\overline{\mathcal{M}}_2$ and in the last Section 5.3 we will write the multiplication table for the generators of the Chow ring.

1 Moduli space

The purpose of this chapter is to introduce the notion of moduli space, more precisely of fine and coarse moduli space, and to present the main results concerning them that will be used in the thesis. We'll start with a couple of preliminary sections whose aim is to give an idea what a moduli space should be and, after treating the example of the projective space in some detail, we will formalize the notion of fine and coarse moduli space in the next sections 1.3 and 1.4. In the last section 1.5, we will illustrate the example of the Picard group, showing that it can be seen as a moduli space and it has a scheme structure.

1.1 An intuitive motivation

The basic idea underlying the definition of moduli space is quite simple: in order to classify certain objects modulo an equivalence relation, it is natural to construct the quotient space of equivalence classes and to study its structure. This is, as said, the basic idea; this space, in fact, in order to be the moduli space of this family of object should satisfy also other properties, we look in particular for a space parametrizing *continuously varying families of objects*.

To classify algebraic curves up to isomorphism, it is natural to study the structure of the set of isomorphism classes:

$$\mathcal{M} = \{C \text{ smooth projective curve}\}/\cong.$$

Classifying mathematical objects up to isomorphism is indeed a recurring notion. A basic example is the classification in linear algebra of finite-dimensional vector spaces up to isomorphism by their dimension. This yields a bijection between the set of isomorphism classes and the positive integers:

$$\begin{array}{rcl} \left\{ \begin{array}{c} \text{finite dimensional} \\ \text{vector spaces} \end{array} \right\} / \cong & \longrightarrow & \mathbb{Z}_{\geq 0} \\ [\mathrm{V}] & \mapsto & \dim V \end{array}$$

Finite-dimensional vector spaces are uniquely classified by their dimension hence $V \cong V'$ if and only if $\dim V = \dim V'$. The classification obtained in this way is certainly a good result since the space parametrizing finite-dimensional vector spaces is the set of natural numbers, but we notice that the space of parameters is just a discrete set, the notion of continuously varying family is not attained.

1.2 The projective space $\mathbb{P}^n$

Let us see another example coming from algebraic geometry: the projective space $\mathbb{P}^n$.

The projective n-space is the space of equivalence classes of vectors in $\mathbb{C}^n$ modulo proportionality and it parametrizes lines through the origin in the space $\mathbb{C}^{n+1}$:

$$\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^n \setminus \{0\})/\sim = \{\ l\ :\ l \text{ is a line through the origin in } \mathbb{C}^{n+1}\}.$$

As in the previous case, this quotient space provides a good model for the space of lines, but in addition, this example in three aspects seems to be better then the previous one of vector spaces modulo isomorphism.

First of all, the categorical object we obtain belongs to the same category of objects we are working with: in algebraic geometry we usually work in the category of algebraic varieties or in the category of schemes and the projective space has naturally a scheme structure, it is not just a set.

The second reason is a topological one: projective space parametrizes continuously varying families of lines in $\mathbb{C}^{n+1}$; we can consider the sequence of lines:

$$L_n = \left\langle \begin{pmatrix} 1 \\ 1/n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle, L_0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle,$$

the sequence $(L_n)_{n\in\mathbb{N}}$, as shown in the figure 1.1, intuitively converges to the line L_0. In the complex topology of $\mathbb{P}^n$ the sequence of points L_n converges precisely to L_0.

Finally, the third reason, the most important, why this example better interprets the intuitive idea of a moduli space than the previous one, and that will also suggest us how we could define this space categorically in terms of the representability of

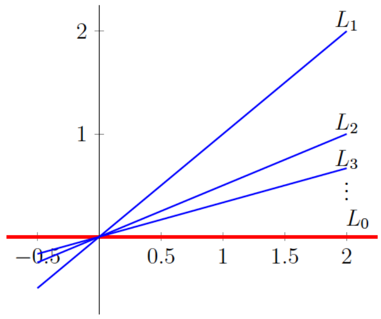


Figure 1.1: The convergence of lines L_n in $\mathbb{A}^2$

a suitable functor, is this: we defined a family of lines L_n that can be extended to a family

$$\left\{ L_t = \left\langle \begin{pmatrix} 1 \\ t \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle \mid t \in \mathbb{A}^1 \right\}$$

parametrized by the variety $\mathbb{A}^1$; this family has defining equations that are algebraic in the affine coordinate t on $\mathbb{A}^1$, so this family corresponds exactly to a morphism

$$\mathbb{A}^1 \to \mathbb{P}^n,\ t \mapsto L_t,$$

which is a morphism of schemes according to the scheme structure on $\mathbb{P}^n$. Generalizing this idea from $X = \mathbb{A}^1$ to any arbitrary scheme, we have that the scheme structure on $\mathbb{P}^n$ is such that this is true not only for $\mathbb{A}^1$: for every subscheme X, the families of lines parametrized by X correspond to morphisms of schemes from X to $\mathbb{P}^n$

$$\begin{aligned} \text{Families of lines parametrized by } X \quad &\leftrightarrow \quad \text{Morphisms } X \to \mathbb{P}^n \\ (L_t \mid t \in X) \quad &\leftrightarrow \quad (X \to \mathbb{P}^n,\ t \mapsto L_t). \end{aligned} \tag{1.1}$$

We will see in the next section 1.3 that the knowledge of the set of morphisms from X to $\mathbb{P}^n$ for every scheme X uniquely determines the scheme structure on $\mathbb{P}^n$, this is in fact a classical result in category theory; hence, in order to make precise the notion of moduli space, we have to make precise what mean by "family of lines parametrized by X"˙ Once defined this, the moduli space will be the unique object (that will be in fact unique up to a unique isomorphism) that realizes the bijection stated above.

1.3 Fine moduli space

In order to make the idea sketched in the previous sections more precise, it will be useful to interpret it in categorical language. All purely categorical results stated here will be presented without proof; a more extended treatment of these notions is included in Appendix A.

Let us denote the category of schemes over the complex field $\mathbb{C}$ by $\mathbf{Sch}_{\mathbb{C}}$, the class of objects by $\mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$, and the set of morphisms from X to Y for given $X, Y \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$ any two schemes, by $\mathrm{Hom}(X, Y)$. For any scheme $M \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$, we can define the functor

$$\begin{array}{rcl} h^M : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} & \to & \mathbf{Sets} \\ X & \mapsto & \mathrm{Hom}(X, M). \end{array}$$

This is a contravariant functor. In fact, for any $g : Y \to X$ morphism of schemes, we get a natural map in the opposite direction (the pullback map): $h^M(g) : h^M(X) \to h^M(Y)$, defined as

$$\begin{array}{rcl} h^M(g) : \mathrm{Hom}(X, M) & \to & \mathrm{Hom}(Y, M) \\ (X \xrightarrow{f} M) & \mapsto & (Y \xrightarrow{g} X \xrightarrow{f} M) \\ f & \mapsto & h^M(g)(f) = f \circ g. \end{array}$$

A classical result in category theory, explained with more details in the Appendix A, is the following:

Lemma 1.3.1 (Yoneda's Lemma)**.** *The functor*

$$\begin{array}{rcl} h^- : \boldsymbol{Sch}_{\mathbb{C}} & \to & Fct(\boldsymbol{Sch}_{\mathbb{C}}^{op} \to \boldsymbol{Sets}) \\ M & \mapsto & h^M \end{array}$$

is a fully faithful embedding.

For a proof see Theorem A.3.2, and Section A.3, page 106.

Remark 1.1. By the lemma above the category of schemes can be viewed as sitting inside the category of contravariant functors from schemes to sets.

The lemma also implies the following properties :

- given X, Y schemes, there is a one-to-one correspondence between morphisms of schemes from X and Y and natural transformations of functors between h^X and h^Y;
- $X \cong Y$ (isomorphism of objects) if and only if $h^X \cong h^Y$ (isomorphism of functors).

The first is just the definition of fully faithful functor, the second is Corollary A.3.3.

There is another notion that we need to recall :

Definition 1.1. A functor $F \in \mathrm{Fct}(\mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets})$ is said to be *representable by* M if $F \cong h^M$ for some $M \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$.

By the Yoneda's Lemma we obtain immediately the

Corollary 1.3.2. *Let F be a representable functor, then the object M representing it is unique up to a unique isomorphism.*

Remark 1.2. If we set $M = \mathbb{P}^n$, then $h^M(X)$ is precisely the set on the right hand-side of correspondence (1.1). Moreover, if we define a functor F that assigns to each scheme X the families of lines in $\mathbb{C}^{n+1}$ parametrized by X, using categorical terminology that correspondence can be rephrased exactly by saying that F is representable by M i.e. it is isomorphic to the functor h^M. This is the way in which we define moduli spaces.

Definition 1.2. A *moduli functor* F is a contravariant functor $F : \mathbf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Sets}$. More precisely, it is given by the following data:

- for every scheme X over $\mathbb{C}$, a set $F(X)$ (*the set of families parametrized by* X);
- for every morphism $f : Y \to X$ a map $F(f) : F(X) \to F(Y)$ (the *pullback of families under* f), satisfying
  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$;
  - $F(g) \circ F(f) = F(f \circ g)$ for every $g : Z \to Y$, $f : Y \to X$ morphisms of schemes.

We can finally give the definition of *fine moduli space*:

Definition 1.3 (Fine moduli space)**.** Given a moduli functor F, we say that the scheme M is a *fine moduli space* for F if the functor is representable by M i.e. $F \cong h^M$.

According to this definition, families of objects parametrized by X are just sets associated to each scheme. Intuitively, if we want to classify for example algebraic curves modulo isomorphism, the functor should assign to each scheme a subset of the quotient space of curves modulo the equivalence relation, and this is what we call the set of families parametrized by X. Hence, roughly speaking a family of objects in a category C modulo the equivalence relation $\sim$ parametrized by some scheme X should mean a collection of objects X_p, one for each $p \in X$

which are reasonably patched together according to the structure of parameter space.

Using category theory, we can make more explicit the previous definition by rephrasing and relating it to this more intuitive notion of family of objects.

Definition 1.4 (Fine moduli space – second form)**.** Let C be a subcategory of $\mathbf{Sch}_{\mathbb{C}}$ and $\sim$ be an equivalence relation on the objects of C that can be extended to families, a *fine moduli space* for the moduli problem $(C, \sim)$ is an object $M \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$ together with a family $\mathcal{U} \longrightarrow M$ which is universal in the following sense: for each family $\pi : \Xi \to X$ there is a unique morphism $f \in \mathrm{Hom}(X, M)$ such that $\Xi = X \times_M \mathcal{U} := f^*\mathcal{U}$.

We gave two different definitions hence we would like to prove their equivalence.

Proposition 1.3.3. *Definition 1.3 and Definition 1.4 are equivalent.*

Proof. ($\Rightarrow$) Let us suppose M is a fine moduli space according to Definition 1.3. Since the functor F is isomorphic to h^M there exists a unique family $\mathcal{U} \longrightarrow M$ associated with the identity morphism id_M (this is the element in $F(M)$ corresponding to the identity). If now $\Xi \longrightarrow X$ is any family, there exists, by definition of representable functor, a unique morphism $f : X \to M$ corresponding to the family in the one-to-one correspondence. Consider now the cartesian diagram:

$$\begin{tikzcd} X \times_M \mathcal{U} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{U} \arrow[d] \\ X \arrow[r, "f"] & M \end{tikzcd}$$

we get that $X \times_M \mathcal{U} \longrightarrow X$ is a family parametrized by X which induces the morphism f. But, by the uniqueness of f, it follows that $X \times_M \mathcal{U} \longrightarrow X$ must coincide with the family Ξ. Hence, M is a fine moduli space according with Definition 1.4.

($\Leftarrow$) The converse implication is then immediate. Since for every family $\Xi \to X$ there is a unique morphism $f : X \to M$, we have a bijection between $F(X)$ and $\mathrm{Hom}(X, M)$. Functoriality follows from the fact that $\Xi = X \times_M \mathcal{U}$ is the fibered product. $\square$

Remark 1.3. We defined moduli spaces in the category of schemes but all these definitions could be generalized to categories in which products and fibered products are defined (see Section A.2).

We started this chapter by looking at projective space, we can now see that it is really a fine moduli space for “lines through the origin in $\mathbb{C}^{n+1}$”.

Example 1.1 (The projective space $\mathbb{P}^n$). *First of all, we have to define the moduli functor that assigns to each scheme the lines parametrized by X and then to prove it is representable by $\mathbb{P}^n$.*

Let us start by defining the following functor:

$$F : \boldsymbol{Sch}_{\mathbb{C}}^{op} \to \boldsymbol{Sets}$$

- *on the objects: for every scheme X over $\mathbb{C}$, we define*

$$F(X) = \left\{ \begin{tikzcd} L \arrow[r, "i"] \arrow[d] & X \times \mathbb{C}^{n+1} \arrow[dl] \\ X & \end{tikzcd} \right\} / \sim$$

  where $L \to X$ is a line bundle on X which is a subbundle of the trivial bundle $X \times \mathbb{C}^{n+1} \to X$ (i.e. there is an injective map $i : L \to X \times \mathbb{C}^{n+1}$ of vector bundles such that the quotient is also a vector bundle). The equivalence relation defined on this set is the following: we set $(L \xrightarrow{i} X \times \mathbb{C}^{n+1}) \sim (L' \xrightarrow{i'} X \times \mathbb{C}^{n+1})$ if and only if there is an isomorphism of line bundles $L \xrightarrow{\cong} L'$ making the natural diagram commute;

$$\begin{tikzcd} X \times \mathbb{C}^{n+1} \arrow[drr] & L \arrow[l] \arrow[r, "\cong"] & L' \arrow[r] & X \times \mathbb{C}^{n+1} \arrow[dl] \\ & & X & \end{tikzcd}$$

- *on the morphisms; given a morphism of schemes $f : X' \to X$ we define the pullback as:*

$$\begin{array}{rcl} F(f) : F(X) & \to & F(X') \\ (L \xrightarrow{i} X \times \mathbb{C}^{n+1}) & \mapsto & \left(f^*L \xrightarrow{f^*i} f^*(X \times \mathbb{C}^{n+1}) = X' \times \mathbb{C}^{n+1} \right). \end{array}$$

Since we have naturally that $F(id_X) = id_{F(X)}$ and for every morphisms $g : X'' \to X'$, $f : X' \to X$, there is a canonical isomorphism $g^* f^* L \cong (g \circ f)^* L$, the compatibility conditions of the pullback are satisfied.

It remains to check that F is representable and more precisely that $F \cong h^{\mathbb{P}^n}$, natural isomorphism of functors.

In order to prove we have a natural isomorphism of functors, we have to give, for every scheme X, a bijection between the set $F(X)$ and the set of morphisms $Hom(X, \mathbb{P}^n)$ such that, for every morphism of schemes $f : X' \to X$ satisfies the following commutative diagram:

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d, "F(f)"'] & Hom(X, \mathbb{P}^n) \arrow[d, "h^{\mathbb{P}^n}(f)"] \\ F(X') \arrow[r] & Hom(X', \mathbb{P}^n) \end{tikzcd}$$

Given $i : L \to X \times \mathbb{C}^{n+1}$ family in $F(X)$, we have that for any point $x \in X$ by definition $i(L_x) \subseteq \{x\} \times \mathbb{C}^{n+1}$ is a line through the origin in $\mathbb{C}^{n+1}$ so it should be sent to the morphism $X \to \mathbb{P}^n$ that maps x to the projective point $[i(L_x)] \in \mathbb{P}^n$.

It is clear what happens for closed points $x \in X$; let us see the general case. Fix X any scheme and $(L \xrightarrow{i} X \times \mathbb{C}^{n+1})$ a line bundle in $F(X)$. The injective map i of total spaces of vector bundles corresponds to the following short exact sequence of locally free sheaves:

$$0 \longrightarrow \mathcal{L} \xrightarrow{\iota} \mathcal{O}_X^{n+1} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{Q}$ is the cokernel vector bundle. Taking the dual, we get:

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow \mathcal{O}_X^{n+1} \xrightarrow{\iota^\vee} \mathcal{L}^\vee \longrightarrow 0$$

Since the kernel of a map of locally free sheaves is locally free, this is equivalent to just specify the surjection $\iota^\vee : \mathcal{O}_X^{n+1} \to \mathcal{L}^\vee$. Such a surjection is determined by specifying sections $s_0, s_1, \dots, s_n \in H^0(X, \mathcal{L}^\vee)$ without common zeroes. Then, setting $\mathcal{M} = \mathcal{L}^\vee$ we can define another functor F' as follows:

$$F'(X) = \left\{ (\mathcal{M}, s_0, s_1, \dots, s_n) \mid \begin{array}{c} \mathcal{M} \text{ is a locally free sheaf on } X \\ s_0, s_1, \dots, s_n \in H^0(X, \mathcal{M}) \text{ sections} \\ \text{without common zeroes} \end{array} \right\} / \sim$$

In this way we obtained a map $F(X) \to F'(X)$. One checks that this map is a bijection, that F' is in fact a moduli functor and that the maps $F(X) \to F'(X)$ define a natural equivalence of functors, i.e. $F \cong F'$. Hence we reduced the problem of the representability to showing $F' \cong h^{\mathbb{P}^n}$.

But this last fact is a classical result of the Algebraic Geometry (see e.g. [Har97, II, Theorem 7.1]). Hence, by using the fact that $F' \cong h^{\mathbb{P}^n}$ by transitivity we have the equivalence we were looking for.

It turns out that the universal family is given by the tautological line bundle:

$$\begin{tikzcd}
\{(l,v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in l\} = L \arrow[r, "i"] \arrow[d] & \mathbb{P}^n \times \mathbb{C}^{n+1} \arrow[dl] \\
\mathbb{P}^n &
\end{tikzcd}$$

The line bundle L above is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

Example 1.2 (Grassmannians)**.** *Generalizing the previous example, for $n, k \in \mathbb{N}$ fixed natural numbers, the Grassmannian $Gr(k, \mathbb{C}^n)$ is a fine moduli space for k-dimensional vector subspaces of $\mathbb{C}^n$, i.e. it is the object representing the functor that assigns to each scheme X the locally trivial families of k-dimensional vector spaces over X.*

Once defined what fine moduli spaces are, we are interested in the existence of these spaces. In general, given a moduli problem, the existence of a fine moduli space is not guaranteed, this happens for instance when there is an object we want to classify which has nontrivial automorphisms, as we can see with the moduli space of curves in the following example.

Let us consider C_0 a curve which has nontrivial automorphisms; in this case we can define a family $C \longrightarrow X$ for which every fibre is isomorphic to C_0 but the family C is not isomorphic to $C_0 \times X$ (an example how to construct such family is illustrated in Example 1.3). Now, let us suppose by contradiction there exists a fine moduli space M and a universal family $\mathcal{U} \longrightarrow M$; then by definition there would exist a morphism $h\colon X \to M$ such that $C \cong h^*\mathcal{U}$. Since each fibre is isomorphic to C_0, the map h factors through the closed point corresponding to the isomorphism class of C_0. But then we obtain $C \cong h^*\mathcal{U} = C_0 \times X$, which is a contradiction.

We can see such a construction in the following example.

Example 1.3. *Let us give a concrete construction of such a family of curves over $\mathbb{C}$. Let C_0 be a curve and σ be a nontrivial automorphism on C_0. We can define first an action of $\mathbb{Z}$ on $\mathbb{C}$ by setting:*

$$k * z = z + 2\pi i k$$

for $k \in \mathbb{Z}$ and $z \in \mathbb{C}$. Then we can define an action of $\mathbb{Z}$ on $\mathbb{C} \times C_0$ by

$$k * (z, P) = (z + 2\pi i k, \sigma^k(P))$$

for $P \in C_0$. These actions commute with the projection $\mathbb{C} \times C_0 \to \mathbb{C}$ and moreover the quotient $\mathbb{C}/2\pi i\mathbb{Z}$ is isomorphic to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ via the exponential map.

Then, if we consider the quotient $(\mathbb{C} \times C_0)/\mathbb{Z}$*, it gives a family of curves over* $\mathbb{C}^*$ *in which every fibre is isomorphic to* C_0*. But the family cannot be isomorphic to* $\mathbb{C}^* \times C_0$ *because, otherwise, the action of* $\mathbb{Z}$ *on* C_0 *would be trivial and we would get a contradiction.*

Remark 1.4. We would like to remark that the existence of automorphisms for some objects of the category does not prevent, in general, the existence of a *fine* solution of the moduli problem. In fact let us consider this particular case; let C be the category of finite sets, and $\sim$ the equivalence relation defined by:

$$A \sim B \iff \#(A) = \#(B).$$

Let $\mathcal{D}$ the category of all sets. Then the set $\mathbb{N}$ of natural numbers is a fine moduli space for this moduli problem. In fact the tautological family $\mathcal{U} \longrightarrow \mathbb{N}$ given by attaching the sets $0, 1, 2, \dots, n-1$ over the integers $n \in \mathbb{N}$, is the universal family over $\mathbb{N}$ for the moduli problem. Surely, the objects of C, which are just finite sets, do admit automorphisms.

1.4 Coarse moduli space

In the previous section, we defined the notion of fine moduli space; we observed that in many cases fine moduli spaces do not exist, for instance when we want to classify curves with nontrivial automorphisms. Hence, we need a weaker notion for a scheme to "approximately represent" a moduli functor: this is what will be called a coarse moduli space. Since the request of the representability is too strong, we substitute the natural isomorphism of Definition 1.3 with a natural transformation.

Definition 1.5 (Coarse moduli space)**.** Given a moduli functor F, we say that the pair (M, Φ) is a *coarse moduli space* for F, if M is a scheme, and Φ is a natural transformation of functors $\Phi : F \to h^M$ such that:

1. (M, Φ) is initial among all such pairs with the same property, more precisely, for every scheme M' and every natural transformation $\Phi' : F \to h^{M'}$, there exists a unique natural transformation $\Psi : h^M \to h^{M'}$ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[rr, "\Phi"] \arrow[dr, "\Phi'"'] & & h^M \arrow[dl, "\Psi"] \\ & h^{M'} & \end{tikzcd}$$

2. the map Φ induces a bijection on $\mathbb{C}$-points:

$$\Phi_{\mathrm{Spec}(\mathbb{C})}:\ F(\mathrm{Spec}(\mathbb{C})) \xrightarrow{\cong} h^M(\mathrm{Spec}(\mathbb{C})) \cong \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}), M) \cong M(\mathbb{C})$$

Remark 1.5. Obviously, by definition, every fine moduli space is also a coarse moduli space.

Remark 1.6. Another immediate consequence of the definition is that, since the couple (M,Φ) is initial among those couples, if a coarse moduli space exists, then it is unique up to a unique isomorphism.

Remark 1.7. We already observed in Remark 1.1 that the functor h^- embeds the category of schemes in the category of contravariant functors from schemes to sets. In view of the definitions given, we can rephrase this by saying h^- embeds the category of schemes into the category of moduli functors. Functors which belong to the image of h^- are the representable functors and they have by definition a fine moduli space.

However, there are bigger classes of moduli functors having only a coarse moduli space or even just a pair (M,Φ) satisfying only part 1. of Definition 1.5. This is represented in the following picture Fig.1.2.

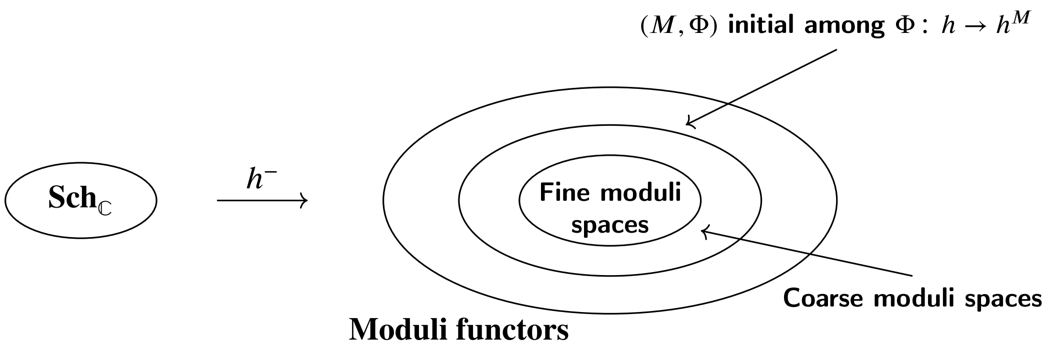


Figure 1.2: The Yoneda embedding of the category of schemes to the category of moduli functors

In the next chapter we will concentrate on the moduli spaces of curves and we will see that they have a coarse moduli space but not a fine one. Before studying them, we conclude this chapter with another example that illustrates how the moduli space structure could define a scheme structure on a group and how the presence of automorphisms obstructs the existence of fine moduli spaces.

1.5 The Picard scheme

Let us consider now the space that classifies line bundles up to isomorphism, the Picard group of a scheme:

$$\mathrm{Pic}(Y) = \{\mathcal{L} \mid \mathcal{L} \text{ is an invertible sheaf on } Y\}/\sim .$$

Invertible sheaves form a group under tensor product hence this set, which is a quotient of a group, has naturally a group structure; what we want to see is that, in many situations, it has also a scheme structure inherited from the fact it can be seen as a fine moduli space.

We start by defining the moduli functor associated to it; in order to do this, we have to define first what a family of line bundles on Y over the base X is. The most natural way to this is by taking the product $X \times Y$ and looking at the data of line bundles up to isomorphism on $X \times Y$.

Definition 1.6 (Absolute Picard funtor)**.** Let Y be a scheme, we define the *absolute Picard functor* by setting:

$$\begin{array}{rcl} \mathrm{Pic}_Y : \mathrm{Sch}^{\mathrm{op}} & \to & \mathrm{Set} \\ X & \mapsto & \mathrm{Pic}_Y(X) = \mathrm{Pic}(X \times Y) \\ (f : X' \to X) & \mapsto & \begin{array}{ccc} \mathrm{Pic}_Y(X) & \xrightarrow{(f\times \mathrm{id}_Y)^*} & \mathrm{Pic}_Y(X') \\ \mathcal{L} & \mapsto & f^*\mathcal{L} \end{array} \end{array}$$

This is the most natural choice for the moduli functor but, if Y is nonempty, it is possible to prove that it is never representable.

Proposition 1.5.1. *Let Y be a scheme, $Y \neq \emptyset$, then the functor Pic_Y defined above is not representable.*

Proof. Let X be any scheme with a nontrivial line bundle $\mathcal{M}$ (e.g. $X = \mathbb{P}^1$ with $\mathcal{M} = \mathcal{O}_{\mathbb{P}^1}(1)$). Let $\pi_X : X \times Y \to X$ be the projection on the first factor, we can define then:

$$\mathcal{M}_X = \pi_X^*\mathcal{M} \in \mathrm{Pic}(X \times Y) = \mathrm{Pic}_Y(X).$$

We claim that this object is nontrivial, i.e. not isomorphic to $\mathcal{O}_{X\times Y}$. In fact, since Y is supposed to be nonempty it has a $\mathbb{C}$-point, the projection π_X has a section; then the pullback of $\mathcal{M}_X$ under this section is $\mathcal{M}$, which is nontrivial.

Let us assume, by contradiction, that Pic_Y were representable by a scheme P together with a universal object $\mathcal{U} \in \mathrm{Pic}_Y(P)$. Then, there exists a unique morphism $g : X \to P$ such that $(g \times \mathrm{id}_Y)^*\mathcal{U} = \mathcal{M}_X$; and similarly, there is a unique morphism $p :$ $\mathrm{Spec}(\mathbb{C}) = \{\mathrm{pt}\} \to P$ associated to the trivial line bundle $\mathcal{O}_{\mathrm{pt}\times Y}$, i.e. $(p \times \mathrm{id}_Y)^*\mathcal{U} = \mathcal{O}_{\mathrm{pt}\times Y}$.

Let now $(U_i)_{i\in I}$ be an open covering of X that trivialize the line bundle $\mathcal{M}$. Then the pullbacks $\mathcal{M}_{U_i}$ of $\mathcal{M}_X$ to $U_i \times Y$ are trivial. In other words, they are pullbacks of $\mathcal{O}_{\mathrm{pt}\times Y}$ under the maps $U_i \to \{\mathrm{pt}\}$. Hence the unique map $U_i \to P$ associated to the bundle $\mathcal{M}_{U_i}$ must factor through the map $p\colon \{\mathrm{pt}\} \to P$. We have then a diagram of maps as follows:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "g'"] & P \\ \bigsqcup_{i\in I} U_i \arrow[u] \arrow[r] & \{pt\} \arrow[u, "p"'] \end{tikzcd}$$

Since the restrictions of the map $g\colon X \to P$ to the open coverings U_i all factor through p, the map g itself must factor through this morphism, but this would lead to a contradiction in fact it would imply:

$$\mathcal{M}_X = (g\times \mathrm{id}_Y)^*\mathcal{U} = (g'\times \mathrm{id}_Y)^*(p\times \mathrm{id}_Y)^*\mathcal{U} = (g'\times \mathrm{id}_Y)^*\mathcal{O}_{\mathrm{pt}\times Y} = \mathcal{O}_{X\times Y},$$

and this contradicts the statement that $\mathcal{M}_X$ is nontrivial. □

As remarked before, also for the Picard group, the presence of automorphisms prevents the existence of a moduli space. In particular, in the proof of Proposition 1.5.1 we reached a contradiction by using the nontrivial family $\mathcal{M}_X \in \mathrm{Pic}(X\times Y)$ obtained by pulling back a line bundle $\mathcal{M} \in \mathrm{Pic}(X)$ form the base.

In this case however, we can obtain a fine moduli space by modifying the definition of moduli functor given before;

Definition 1.7 (Relative Picard functor)**.** We define the *relative Picard functor* $\mathrm{Pic}_{Y/\mathbb{C}}\colon \mathrm{Sch}^{op} \to \mathrm{Set}$ of the scheme Y by setting:

$$\begin{array}{rcl} \mathrm{Pic}_{Y/\mathbb{C}}\colon \mathrm{Sch}^{op} & \to & \mathrm{Set} \\ X & \mapsto & \mathrm{Pic}_{Y/\mathbb{C}}(X) := \mathrm{Pic}(X\times Y)/\pi_X^*\mathrm{Pic}(X) \end{array}$$

where $\pi_X\colon X\times Y \to X$ is the projection onto the first factor, and $\pi_X^*\mathrm{Pic}(X) \subset \mathrm{Pic}(X\times Y)$ is the set of line bundles that are pullbacks from X. Given $f\colon X' \to X$, the corresponding morphism $\mathrm{Pic}_{Y/\mathbb{C}}(X) \to \mathrm{Pic}_{Y/\mathbb{C}}(X')$ is still given by the pullback $\mathcal{L} \mapsto f^*\mathcal{L}$ of line bundles under $f\times \mathrm{id}_Y$ (and this respects the equivalence relation we quotient by).

For the relative Picard functor we have the following result:

Theorem 1.5.2. *If Y is an integral, projective variety over $\mathbb{C}$, then the functor $Pic_{Y/\mathbb{C}}$ is representable by a separated scheme, locally of finite type, denoted as well as* $\boldsymbol{Pic}_{Y/\mathbb{C}}$.

Proof. For a proof see [Kle05, Theorem 4.8]. □

The theorem states that for any integral, projective variety Y over $\mathbb{C}$ the relative Picard functor has a fine moduli space $\mathbf{Pic}_{Y/\mathbb{C}}$ while the absolute Picard functor does not. Anyway, we can prove now the following result:

Proposition 1.5.3. *For any scheme Y such that the relative Picard functor $Pic_{Y/\mathbb{C}}$ has a fine moduli space $\mathbf{Pic}_{Y/\mathbb{C}}$, the natural map*

$$\Phi : \, Pic_Y \to Pic_{Y/\mathbb{C}} \cong h^{\mathbf{Pic}_{Y/\mathbb{C}}}$$

makes $(\mathbf{Pic}_{Y/\mathbb{C}}, \Phi)$ into a coarse moduli space for Pic_Y.

Proof. To show property 1. of Definition 1.5, let $\Phi' : \, \mathrm{Pic}_Y \to h^{M'}$ be a natural transformation, we have to prove that, for a given X, the map

$$\mathrm{Pic}_Y(X) \to \mathrm{Hom}(X, M')$$

induced by Φ' factors through $\mathrm{Pic}_Y(X)/\pi_X^*\mathrm{Pic}(X)$. To this end, let $\mathcal{L}_1, \mathcal{L}_2$ be line bundles on $X \times Y$ satisfying $\mathcal{L}_2 = \mathcal{L}_1 \otimes \pi_X^*\mathcal{M}$, for an assigned line bundle $\mathcal{M}$ on X. Let moreover $g_1, g_2 : \, X \to M'$ be the maps induced from $\mathcal{L}_1, \mathcal{L}_2$ via Φ'. We are done when we showed $g_1 = g_2$.

If $(U_i)_{i\in I}$ is an open covering of X that trivializes $\mathcal{M}$, then the pullbacks of $\mathcal{L}_1, \mathcal{L}_2$ to $U_i \times Y$ coincide for all i. By the functoriality of Φ', we get that

$$g_{1|U_i} = g_{2|U_i}.$$

But since morphisms are determined by their restriction to an open cover we can also deduce $g_1 = g_2$.

For the second part of the definition:

$$\begin{aligned} &\mathrm{Pic}_Y(\mathrm{Spec}(\mathbb{C})) = \mathrm{Pic}(\mathrm{Spec}(\mathbb{C}) \times Y) = \\ &\qquad = \mathrm{Pic}(\mathrm{Spec}(\mathbb{C}) \times Y)/\mathrm{Pic}(\mathrm{Spec}(\mathbb{C})) = \mathrm{Pic}_{Y/\mathbb{C}}(\mathrm{Spec}(\mathbb{C})) = \mathbf{Pic}(\mathbb{C}). \end{aligned}$$

□

Remark 1.8. The representability of the relative Picard functor has many consequences:

1. A line bundle $\mathcal{L}$ on $\mathbf{Pic}_{Y/\mathbb{C}} \times Y$ representing the universal family of the moduli space is called a *Poincaré line bundle.* Given a $\mathbb{C}$-point $[M] \in \mathbf{Pic}_{Y/\mathbb{C}}$, the restriction of $\mathcal{L}$ to $[M] \times Y \subset \mathbf{Pic}_{Y/\mathbb{C}} \times Y$ is isomorphic to $\mathcal{M}$. The line bundle $\mathcal{L}$ is unique up to tensoring with line bundles pulled back from $\mathbf{Pic}_{Y/\mathbb{C}}$.

2. The set $\mathrm{Pic}(Y)$ of line bundles on Y has a natural structure of abelian group, where the multiplication is given by the tensor product of line bundles. From this, we can deduce, $\mathbf{Pic}_{Y/\mathbb{C}}$ has the structure of an *algebraic group*, i.e. the natural maps:

$$\begin{array}{rcl} \mathbf{Pic}_{Y/\mathbb{C}} \times \mathbf{Pic}_{Y/\mathbb{C}} & \to & \mathbf{Pic}_{Y/\mathbb{C}} \\ ([\mathcal{L}_1],[\mathcal{L}_2]) & \mapsto & [\mathcal{L}_1 \otimes \mathcal{L}_2] \end{array} \qquad \begin{array}{rcl} \mathbf{Pic}_{Y/\mathbb{C}} & \to & \mathbf{Pic}_{Y/\mathbb{C}} \\ \mathcal{L} & \mapsto & \mathcal{L}^\vee \end{array}$$

   are algebraic morphisms which together with the inclusion $[\mathcal{O}_Y] \in \mathbf{Pic}_{Y/\mathbb{C}}$ define multiplication, inverse and neutral element of a group structure on $\mathbf{Pic}_{Y/\mathbb{C}}$.

3. We denote by $\mathbf{Pic}^0_{Y/\mathbb{C}} \subset \mathbf{Pic}_{Y/\mathbb{C}}$ the connected component of $\mathbf{Pic}_{Y/\mathbb{C}}$ containing the trivial bundle $[\mathcal{O}_Y]$. It gives a subgroup of $\mathbf{Pic}_{Y/\mathbb{C}}$. For $Y = C$ a smooth, projective, algebraic curve, the scheme $\mathbf{Pic}^0_{C/\mathbb{C}} = \mathrm{Jac}(C)$ is called the *Jacobian* of C.

Example 1.4. *For a point $Y = \{pt\} = Spec(\mathbb{C})$ we have that for any scheme S we get:*

$$Pic_{pt/\mathbb{C}}(S) = Pic(pt \times S)/Pic(S) = \{[\mathcal{O}_S]\}.$$

Hence

$$\boldsymbol{Pic}_{Y/\mathbb{C}} = Spec(\mathbb{C}).$$

Remark 1.9. For $Y = \mathbb{A}^n$ the same is not true even though $\mathrm{Pic}(\mathbb{A}^n) = \{\mathcal{O}_{\mathbb{A}^n}\}$.

Example 1.5. *For any $n \geq 1$ we have that*

$$\boldsymbol{Pic}_{\mathbb{P}^n/\mathbb{C}} = \bigsqcup_{m \in \mathbb{Z}} \{\mathcal{O}_{\mathbb{P}^n}(m)\}$$

is a countable union of isolated points.

Example 1.6. *Given an elliptic curve (E, p_0) we have:*

$$\boldsymbol{Pic}^0_{E/\mathbb{C}} = E$$

and for $\Delta \subset E \times E$ the diagonal, the line bundle

$$\mathcal{L} = \mathcal{O}_{E \times E}(\Delta \setminus E \times p_0) \in Pic(\boldsymbol{Pic}^0_{E/\mathbb{C}} \times E)$$

is a Poincaré line bundle over $\boldsymbol{Pic}^0_{E/\mathbb{C}}$.

2 Moduli space of curves

In this chapter we will concentrate on the special case of curves, defining moduli functors and moduli spaces for algebraic curves over the complex field $\mathbb{C}$. The assumption that the curves are complex is mostly for convenience; many result can be extended to any algebraically closed field, sometimes assuming it has characteristic 0.

The goal of the chapter is to give an idea how moduli theory applies to the particular case of curves in order to be able to give the statement of the Deligne-Mumford Theorem (Theorem 2.4.1, pag. 34) that proves that under certain hypotheses, moduli spaces for curves exist. We will start by making precise the terminology that will be used, and showing, in a particular example, which obstructions to the existence of a moduli space can occur. Then, in Section 2.3 we will introduce the notion of stable curves that is needed to compactify moduli spaces of smooth curves, and finally, in the Section 2.4 we'll give the precise definitions of family of smooth and stable curves, of their respective moduli functors, and the statement of the Theorem of Deligne-Mumford on the existence of coarse moduli spaces of smooth and stable genus g curves, respectively $\boldsymbol{M}_g$, $\overline{\boldsymbol{M}}_g$, and of smooth and stable n-pointed genus g curves $\boldsymbol{M}_{g,n}$, $\overline{\boldsymbol{M}}_{g,n}$. A final section will be dedicated to present some concrete examples of moduli spaces in low genus (Section 2.5); for this purpose we will introduce the notion of stable graph and of dual graph of a stable curve, and we will see how we can use graphs to describe, with combinatorial techniques the structure of low genus moduli spaces $\overline{\boldsymbol{M}}_{0,n}$.

This chapter is intended to be an explanatory overview of the topic. For this reason, many of the results stated are presented without proof.

2.1 Terminology and notation

In this first section, we introduce the terminology and the notation that will be used in the thesis.

Definition 2.1 (Variety)**.** A *variety* is a reduced, separated scheme of finite type over the base field (in our case $\mathbb{C}$).

Definition 2.2 (Curve)**.** A *(complex) curve* is a one-dimensional variety $C \to \operatorname{Spec}(\mathbb{C})$: in particular it is a scheme of finite type over $\mathbb{C}$ such that all irreducible components have dimension 1.

Definition 2.3 (Genus)**.** Let C be a projective curve, we define:

1. the *geometric genus* $p_g(C)$ as:

$$p_g(C) := h^{1,0}(C) = \dim H^0(C, \Omega^1_C)$$

   if C is smooth, where Ω^1_C is the cotangent line bundle of C; if C is singular, we define its geometric genus as the genus of its normalization;

2. the *arithmetic genus* as:

$$p_a(C) := 1 - \chi(C, \mathcal{O}_C) = 1 - \dim H^0(C, \mathcal{O}_C) + \dim H^1(C, \mathcal{O}_C).$$

Proposition 2.1.1. *If C is a smooth, irreducible, projective curve we have $p_g(C) = p_a(C)$.*

Proof. Since C is irreducible and projective, we have that it is connected, hence $H^0(C, \mathcal{O}_C)$; by Serre duality, moreover, $H^1(C, \mathcal{O}_C) \cong H^0(C, \Omega^1_C)^\vee$. Then:

$$p_a(C) = 1 - \dim H^0(C, \mathcal{O}_C) + \dim H^1(C, \mathcal{O}_C) = 1 - 1 + \dim H^0(C, \Omega^1_C)^\vee = p_g(C).$$

□

Remark 2.1. The smoothness condition is necessary. In fact, let us consider the nodal cubic curve $C \subseteq \mathbb{P}^2$ defined by

$$C = \{[X : Y : Z] \in \mathbb{P}^2 \mid ZY^2 + X^3 - ZX^2 = 0\};$$

in this case the geometric and the arithmetic genus are different.

Using the formulas for plane curves, we get:

$$p_a(C) = \frac{(d-1)(d-2)}{2} \qquad p_g(C) = \frac{(d-1)(d-2)}{2} - \sum_{\substack{P \text{ ordinary} \\ \text{singularity} \\ \text{of order } r}} \frac{r(r-1)}{2}$$

hence

$$p_a(C) = \frac{2 \cdot 1}{2} = 1 \qquad p_g(C) = 1 - \frac{2 \cdot 1}{2} = 0.$$

In conclusion, for C a nodal cubic we have $p_g(C) \lneqq p_a(C)$.

Remark 2.2. We recall there is a fundamental relation between algebraic curves and *Riemann surfaces*. Indeed, let us consider a smooth, irreducible, projective curve C, and let $S = C(\mathbb{C})$ be the set of its complex points endowed with the complex topology. Then S is a compact, connected, complex manifold of complex dimension 1; hence, viewed as a real manifold, it is a compact, connected, orientable real surface without boundary.

- S is compact since the curve C is projective;
- the curve C is 1-dimensional over $\mathbb{C}$, hence S is a 2-dimensional real manifold:
- S is a smooth manifold since C is smooth as an algebraic variety;
- S is orientable since it is a complex manifold.

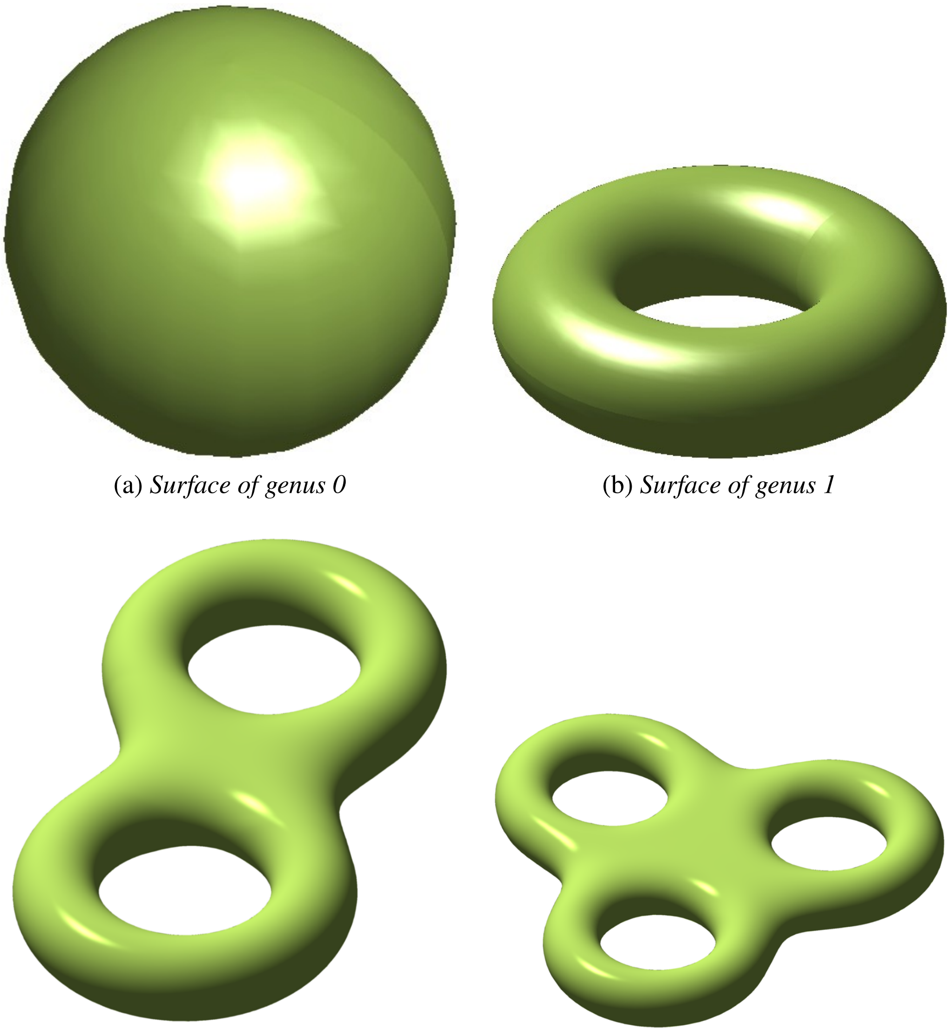
There is a complete classification of compact, connected, oriented real surfaces without boundary, and this classification is provided by the topological genus, in fact every Riemann surface is of the form of "a doughnut with g holes" (see Figure 2.1 for examples of low genus surfaces). For $S = C(\mathbb{C})$ it turns out that the topological genus of S is equal to the geometric (or arithmetic) genus of the algebraic curve C.

Another notion we need to recall is the notion of nodal curve:

Definition 2.4 (Nodal curve)**.** Let C be a complex curve.

1. A closed point $q \in C$ is said to be a *node* if it satisfies one of the following equivalent conditions:
   - there exists a neighbourhood of $q \in C(\mathbb{C})$ which is analytically isomorphic a neighbourhood of the origin in the locus: $\{(x, y) \in \mathbb{C}^2 \mid x \cdot y = 0\} \subseteq \mathbb{C}^2$;
   - the completion $\widehat{\mathcal{O}}_{C,q}$ of the local ring of C at the point q is isomorphic to $\mathbb{C}[[x, y]]/(xy)$.
2. The curve C is said to be *nodal* if every closed point $q \in C$ is either a smooth point or a node.

Example 2.1. *The nodal cubic of Remark 2.1 is a nodal curve.*



(a) *Surface of genus 0*

(b) *Surface of genus 1*

(c) *Surface of genus 2*

(d) *Surface of genus 3*

Figure 2.1: Riemann surfaces of genus g

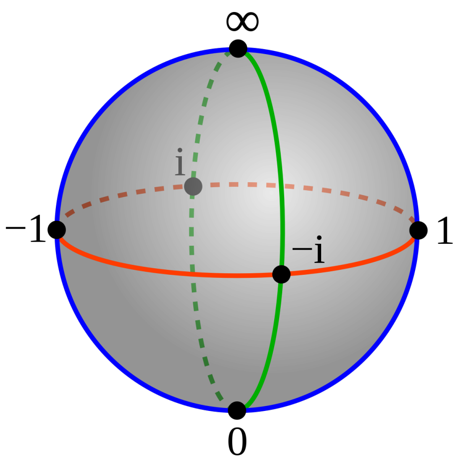


Figure 2.2: The Riemann Sphere

For future reference, we introduce next a couple of basic examples of algebraic curves: the projective line and plane curves, in particular plane cubics.

Example 2.2 (The projective line)**.** *The first example of algebraic curve we can think of is the projective line* $\mathbb{P}^1$*. It is covered by the two open affine subsets:*

$$U_0 = \mathbb{P}^1 \setminus \{[0 : 1]\} \cong \mathbb{A}^1 \quad \text{and} \quad U_1 = \mathbb{P}^1 \setminus \{[1 : 0]\} \cong \mathbb{A}^1,$$

overlapping in $U_0 \cap U_1 = \mathbb{A}^1 \setminus \{0\}$*. The complex points of the projective line are the points of the 2-sphere* $\mathbb{S}^2$ *by identifying the points of the form* $[1 : z]$ *with the points of the affine plane, under the inverse of the stereographic projection, and the point* $[0 : 1]$ *with the point at infinity, as shown in Figure 2.2; this is a Riemann surface of genus 0.*

Example 2.3 (Plane curves)**.** *A second source of examples are curves which are subvarieties of the projective plane* $\mathbb{P}^2$ *defined by a homogeneous equation of degree* d*.*

$d = 1$ *The varieties defined by degree 1 equations are lines, and they are all isomorphic to* $\mathbb{P}^1$*.*

$d = 2$ *The varieties defined by degree 2 equations are conics and, if they are irreducible, they are isomorphic to* $\mathbb{P}^1$*.*

$d = 3$ *Plane cubics.*

Let us consider the family of cubic curves $(E_t)_{t\in\mathbb{C}} \subseteq \mathbb{P}^2$ *defined by*

$$E_t = \{[X : Y : Z] \in \mathbb{P}^2 \mid Y^2Z + X(X - Z)(X - tZ) = 0\}, \quad t \in \mathbb{C}. \tag{2.1}$$

For $t \neq 0, 1$ these curves are smooth and projective, and they are all homeomorphic to a torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, *a surface of topological genus 1 (Fig. 2.1b).*

2.2 The classification of curves

After these preliminary definitions and examples, we can go back to the main problem we want to discuss in this chapter: the existence of moduli spaces for genus g smooth curves. The idea behind the study of these spaces is that this should allow to classify algebraic curves up to isomorphism, describing the structure of the set:

$$\mathcal{M} = \left\{ \begin{array}{c} C \text{ smooth, irreducible,} \\ \text{complex projective curve} \end{array} \right\} / \sim .$$

Remark 2.3.

- The requirement that the curves are smooth is just for simplicity, since singularities can add complications to our classification. Anyway, we will relax this condition later admitting the curves to have nodal singularities.
- Once we assume the curve C is smooth, irreducibility is equivalent to connectedness hence, since a disconnected curve is the disjoint union of connected ones, the problem of classification reduces to the classification of connected curves.
- The assumption that the curve is projective does not affect the generality of the classification, in fact, given any curve C' there exists an embedding $C' \hookrightarrow C$ into a smooth, irreducible and projective curve C (see [Vak17, Theorem 17.4.2]). By dimension reasons, the complement $C \setminus C'$ is a finite union of points and it will be easier to simply classify the data of C together with these points from which we can reconstruct the original curve C'. These will be curves with some marked points.

As remarked in the previous examples 2.2 and 2.3, the genus plays an important role in the problem of classification. Two curves of different genera cannot be isomorphic hence there is a well defined map:

$$\begin{array}{rccl} \mathcal{G} \colon & \mathcal{M} & \to & \mathbb{N} \\ & [C] & \mapsto & \mathcal{G}([C]) = p_a(C) \end{array}$$

where $p_a(C)$ is the arithmetic genus of C. The existence of this map suggests to reduce the study of $\mathcal{M}$ to the study of the spaces:

$$\mathcal{M}_g = \mathcal{G}^{-1}(g)$$

defined as the preimage of g under the map $\mathcal{G}$.

In other words, the initial problem can then be solved by studying the structure of sets $\mathcal{M}_g$ of isomorphism classes of genus g curves up to isomorphism for each $g \in \mathbb{N}$; we would like to discuss if these spaces $\mathcal{M}_g$ are, or not, fine or coarse moduli spaces for the corresponding moduli problem, i.e. if the functor that assigns to any scheme X the families of genus g curves parametrized by X admits a moduli space. For $g = 0$ the solution is trivial.

The space $\mathcal{M}_0$

Example 2.4. *Since every smooth, irreducible complex projective curve C of genus 0 is isomorphic to $\mathbb{P}^1$, (see [Vak17, Section 19.3]) the set $\mathcal{M}_0$ consists of the unique element $[\mathbb{P}^1]$, hence:*

$$\mathcal{M}_0 = \left\{ \left[\mathbb{P}^1 \right] \right\}.$$

The classification is then completed in genus 0.

The space $\mathcal{M}_1$

The case of genus 1 curves is less trivial than the previous one. The family defined in (2.1) at page 22, provides an example of curves of genus 1. We have moreover the following fact:

Proposition 2.2.1. *Every smooth, irreducible, complex, projective curve C of genus 1 is isomorphic to one of the curves E_t defined in (2.1) for some $t \in \mathbb{C} \setminus \{0, 1\} := U$. Moreover, for $t_1, t_2 \in U$ we have $E_{t_1} \cong E_{t_2}$ if and only if*

$$t_2 \in \left\{ t_1, \frac{1}{t_1}, 1 - t_1, \frac{1}{1 - t_1}, \frac{t_1 - 1}{t_1}, \frac{t_1}{t_1 - 1} \right\}. \tag{2.2}$$

Proof. For a proof see, for example, [Har97, Section IV, 4] or [Vak17, Section 19.8]. □

Remark 2.4. While the topological spaces of the $\mathbb{C}$-points $E_t(\mathbb{C})$ are all homeomorphic to a torus for $t \neq 0, 1$, it turns out that the algebraic curves E_t are not all isomorphic. More precisely, it follows from the previous proposition that there are uncountably infinitely many isomorphism classes of curves in $\mathcal{M}_1$.

The family $(E_t)_{t \in U}$ provides a good parameter space of curves, in fact the equations defining E_t depend continuously on the parameter t, every genus 1 curve appears as a member of this family, and finally, given two such elements, we have a criterion to detect if they are isomorphic.

We can get rid of this redundancy by considering the quotient of U by the action of a suitable group; we define the action of the symmetric group Sym_3 on U in the following way:

$$(12) \cdot t = \frac{1}{t} \qquad (23) \cdot t = 1 - t$$

since (12) and (23) generate Sym_3 the action on the other elements is then:

$$\begin{aligned}
(1) \cdot t &= t \\
(132) \cdot t = (23) \cdot (12) \cdot t &= (23) \cdot \frac{1}{t} = \frac{t-1}{t} \\
(123) \cdot t = (12) \cdot (23) \cdot t &= (12) \cdot (1-t) = \frac{1}{1-t} \\
(13) \cdot t = (23) \cdot (123) \cdot t &= (23) \cdot \frac{1}{1-t} = \frac{t}{t-1}
\end{aligned}$$

Hence, the orbit of $t_1 \in U$ under Sym_3 is precisely the set in (2.2) and we have a natural identification:

$$\mathcal{M}_1 = U/\mathrm{Sym}_3$$

where U/Sym_3 is the set of orbits in U under the Sym_3-action.

The space of parameters U has naturally a scheme structure because $U = \mathbb{C} \setminus \{0,1\}$ whose points are the complex points of the scheme $\mathbb{A}^1 \setminus \{0,1\}$. Moreover, the action of Sym_3 is algebraic, hence it turns out that the quotient $(\mathbb{A}^1 \setminus \{0,1\})/\mathrm{Sym}_3$ also inherits a scheme structure.

This quotient scheme is given by the affine line together with the morphism:

$$\begin{array}{rccc}
j: & \mathbb{A}^1 \setminus \{0,1\} & \to & \mathbb{A}^1 \\
 & t & \mapsto & 2^8 \dfrac{(t^2-t+1)^3}{t^2(t-1)^2}
\end{array}$$

that is known as the *j-invariant.* This is an algebraic morphism and the fibres of closed points are exactly the orbits of the action of Sym_3 on U.

Resuming the facts collected until here, we found a precise description of the space $\mathcal{M}_1$: the scheme $\mathbb{A}^1$ together with the j-invariant morphism and the family $(E_t)_t$ provides a solution for the problem of classifying genus 1 curves up to isomorphism; in particular, we have the following statements for $g = 1$.

- There exists a variety $U_g = U$ and family C_t of genus g curves parametrized by $t \in U_g$ such that every smooth, irreducible, projective genus g curve is isomorphic to a fibre C_t.

- U_g is a smooth and connected variety, in particular two genus g curves can be deformed one into the other by using this family.
- There exists a variety M_g and a surjective morphism $U_g \to M_g$ which identifies two closed points t_1, t_2 if, and only if, $C_{t_1} \cong C_{t_2}$; in particular the closed points of M_g are in bijection with the smooth, irreducible, projective genus g curves up to isomorphism.

The space M_g appearing in the previous statements seems to be a reasonable answer to the problem of classification, and all these three statements are still true for $g \geq 1$. However, the presence of nontrivial automorphisms of curves obstruct the existence of a moduli space:

- for $g \geq 2$ the variety M_g is not smooth; for $g \geq 4$ its singular locus corresponds to the isomorphism classes of curves having a nontrivial automorphism;
- for $g \geq 1$, the family $(C_t)_{t \in U_g}$ above does not descend to a family over M_g. Hence there does not exist a family of curves parametrized by M_g such that the member of the family associated to the element $[C] \in M_g$ is isomorphic to C. However, there does exist such a family over the complement of the locus of curves having nontrivial automorphism.

From this observation we conclude the space $\mathcal{M}_1$ described above is not a moduli space for smooth genus 1 curves; the main result that we will present in this chapter is intended to explain under which conditions a moduli space for genus g curves does exist.

Before going on, we would like to make another observation: even in the case $g = 1$, the set of $\mathbb{C}$-points of $\mathcal{M}_1$, being identified with $\mathbb{A}^1$, is not compact, and this is true also for genera $g \geq 1$. Without the compactness, their cohomology groups lack some properties such as the Poincaré duality. In the example presented, the space $\mathcal{M}_1$ can be identified with $\mathbb{A}^1$ which is contractible and has the cohomology of a point. Hence, we look for a compactification of such spaces; given $\mathcal{M}_g$ defined as before, we look for a space $\overline{\mathcal{M}}_g$ containing $\mathcal{M}_g$ as a dense open subset that should be itself a moduli space of geometric objects generalizing the smooth genus g curves which $\mathcal{M}_g$ classifies.

Going back to the family E_t presented above, we can understand by this particular example how to admit in our classification also some non-smooth curves in order to make $\mathcal{M}_1$ into a compact space. In fact, let us recall the curves E_t are parametrized by $\mathbb{C} \setminus \{0, 1\}$. We excluded $t = 0, 1$ because the defining equations would give in these cases a singular curve in $\mathbb{P}^2$.

For $t = 0$, we would get:

$$E_0 = \left\{ [X : Y : Z] \in \mathbb{P}^2 \mid Y^2 Z + X^2(X - Z) = 0 \right\}.$$

The point $P = [0 : 0 : 1]$ is a singular point, hence the curve E_0 is not smooth; restricting to the affine chart of $\mathbb{P}^2$ defined by $\{Z \neq 0\}$, and dehomogeneiting with respect to the variable Z, the equation becomes:

$$y^2 + x^2(x - 1) = 0.$$

In a small neighbourhood of the origin $(0, 0)$, $x - 1 \approx -1$, hence the equation around the origin can be approximated by:

$$y^2 - x^2 = 0 \iff (y - x)(y + x) = 0,$$

that are just two lines meeting transversally in the origin.

This singularity is a node. The Figure 2.3a represents this nodal cubic, and in the Figure 2.3b we can see how the torus $E_t(\mathbb{C})$ degenerates when t tends to 0.

Analogously for $t = 1$, we have:

$$E_1 = \left\{ [X : Y : Z] \in \mathbb{P}^2 \mid Y^2 + X(X - Z)^2 = 0 \right\};$$

the only singular point is $Q = [1 : 0 : 1]$; if we restrict again to the affine chart defined by $\{Z \neq 0\}$ dehomogeneiting with respect to Z, we obtain the equation:

$$y^2 + x(x - 1)^2 = 0.$$

After the change of variable

$$\begin{array}{rcl} x & \mapsto & t + 1 \\ y & \mapsto & v \end{array}$$

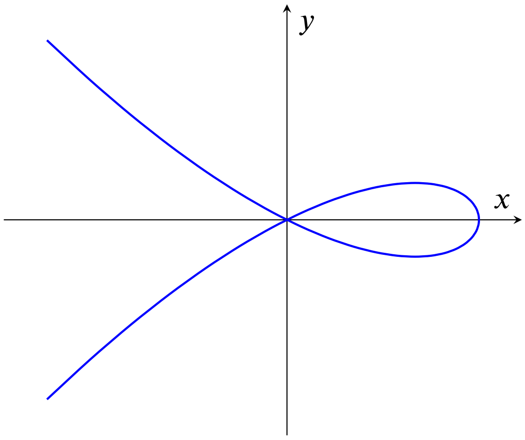
which moves the singular point to the origin of the plane (t, v), the equation becomes:

$$v^2 + t^2(t - 1) = 0$$

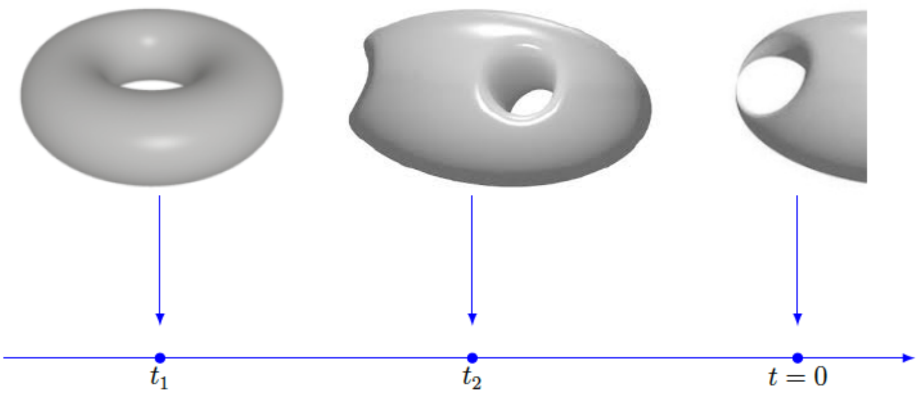
which is again the equation of a nodal cubic. In this example smooth curves E_t degenerate to nodal curves when t tends to 0 or 1.

Hence, by these observations, the natural way to try to compactify the spaces $\mathcal{M}_g$ of smooth curves C is by admitting curves have nodal singularities.

Then our new goal is studying the moduli functor of nodal curves up to isomorphism and the existence of its moduli space.



(a) *The real part of nodal cubic in the affine plane*



(b) *The degeneration of the family E_t for $t \to 0$*

Figure 2.3

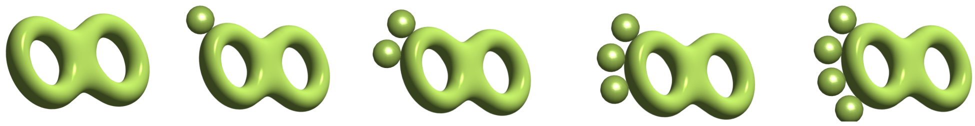


Figure 2.4: First elements of the sequence C_n

Remark 2.5. We remark that this idea is consistent with the classification by the arithmetic genus, in fact, in a family of nodal curves the arithmetic genus stays constant, while the geometric genus can change. For example, in the family of curves E_t of smooth plane cubics, the general curve E_t is smooth and for $t \neq 0, 1$:

$$p_a(C) = p_g(C) = 1;$$

the nodal cubic E_0 still has arithmetic genus 1, even if the geometric genus is 0.

This remark can be generalized by saying that smooth curves of genus g degenerate to nodal curves of arithmetic genus g; hence, the classification by the arithmetic genus is well-posed.

2.3 Stable curves

The idea of studying moduli functors of nodal curves instead of smooth curves is motivated by the fact that we would like to obtain compact spaces, as described in the previous section. But let us consider the following sequence of curves: let C_0 be a smooth curve of genus 2, and let C_n be the curve given by C_0 with n spheres attached (as represented in Figure 2.4 for $n = 0, 1, 2, 3, 4$). The sequence $(C_n)_{n \in \mathbb{N}}$ is a sequence of nodal curves of genus 2 corresponding to a sequence of closed points in the relative moduli space. If we had a compact moduli space, this sequence should have a convergent subsequence in the complex topology but intuitively this sequence of curves cannot converge to a curve of finite type.

Hence, this example suggests that taking all nodal curves seems to be "too much" in order to obtain a compact moduli space, and that restricting instead to stable curves should better work. Let us first recall the properties of normalization and then the definition of stable curve.

Let C be a complex, projective and nodal curve. Its normalization is then a complex, projective, smooth curve $\widetilde{C}$, together with a morphism $\nu : \widetilde{C} \to C$ which is an isomorphism over the smooth locus of C and such that every node $q \in C$ has exactly two preimages $q', q'' \in \widetilde{C}$. Moreover, we have the normalization exact sequence:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_* \mathcal{O}_{\widetilde{C}} \longrightarrow \bigoplus_{\substack{q \text{ node} \\ \text{of } C}} \mathbb{C}_q \longrightarrow 0 \tag{2.3}$$

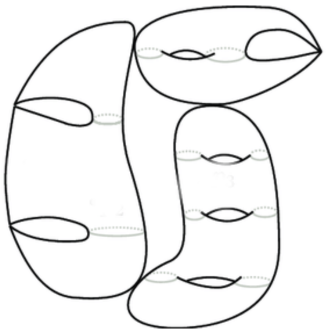


Figure 2.5: Example of curve with $p_g(C) = 4$ and $p_a(C) = 8$

where the first map sends a (local) function f on C to $f \circ \nu$ on $\widetilde{C}$ and the second map sends a (local) function g on $\widetilde{C}$ to $(g(q') - g(q''))_q$ where q runs through the nodes of C, q', q'' are the preimages of q under ν and where we choose an order on the preimages.

Remark 2.6. Using the exact sequence in (2.3), we have:

$$p_a(C) = p_a(\widetilde{C}) + \#\{\text{nodes of } C\}$$

$$p_a(C) = p_g(\widetilde{C}) + 1 - \#\{\text{components of } \widetilde{C}\} + \#\{\text{nodes of } C\}.$$

Example 2.5. *For example, if we consider the curve represented in Figure 2.5, we have:*

$$p_g(C) = p_g(\widetilde{C}) = 4$$

$$\begin{aligned} p_a(C) &= p_g(\widetilde{C}) + 1 - \#\{\textit{components of } \widetilde{C}\} + \#\{\textit{nodes of } C\} = \\ &= 4 + 1 - 3 + 6 = \\ &= 8 \end{aligned}$$

Remark 2.7. We observe that the data of a nodal curve C together with the set of nodes $\{q_1, q_2, \dots, q_N\}$ is equivalent to the data of its normalization $\widetilde{C}$ together with the set of preimages: $\big\{\{q_1', q_1''\}, \dots, \{q_N', q_N''\}\big\}$ because there is a unique way to glue the components of $\widetilde{C}$ by identifying the points in the pairs $\{q_i', q_i''\}$ to form nodes:

$$C, \{q_i\}_{i=1}^N \underset{glueing}{\overset{normalization}{\rightleftarrows}} \widetilde{C}, \big\{\{q_i', q_i''\}\big\}_{i=1}^N$$

Hence, the data of a morphism $\varphi : C \to X$ is equivalent to the data of a morphism $\widetilde{\varphi} : \widetilde{C} \to X$ such that $\widetilde{\varphi}(q_i') = \widetilde{\varphi}(q_i'')$ for all i.

$$\varphi : C \to X \underset{glueing}{\overset{normalization}{\rightleftarrows}} \widetilde{\varphi} : \widetilde{C} \to X \text{ such that } \widetilde{\varphi}(q_i') = \widetilde{\varphi}(q_i'') \tag{2.4}$$

We showed how the presence of automorphism can be a source of trouble. For this reason we restrict to curves with a finite group of automorphism; we introduce then the notion of *stable curves.*

Definition 2.5. A connected nodal, complex, projective curve C is called *stable* if the group of automorphism

$$\mathrm{Aut}(C) = \{\varphi : C \to C \mid \varphi \text{ is isomorphism}\}$$

is finite.

Definition 2.6. Given $p_1, p_2, \dots, p_n \in C$ distinct points, we say that $(C, p_1, \dots, p_n)$ is an *n-pointed curve*; the points $p_1, \dots p_n$ are called the *marked points.*

A connected, nodal and projective n-pointed curve $(C, p_1, \dots, p_n)$ is called *stable* if the group of automorphism of C fixing all p_i

$$\mathrm{Aut}(C, p_1, \dots, p_n) = \{\varphi : C \to C \mid \varphi \text{ is an isomorphism and } \varphi(p_i) = p_i \ \forall i = 1, \dots n\}$$

is finite.

We recall now some basic facts about automorphisms of curves. The reference for these statements is [Vak17].

Basic facts 2.3.1. *Let C be a smooth, irreducible, complex, projective curve of genus g.*

- *For $g = 0$ we have $C \cong \mathbb{P}^1$ and $Aut(\mathbb{P}^1) = PGL_2(\mathbb{C})$, where*

$$PGL_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(Mat_{2\times 2}(\mathbb{C})) \cong \mathbb{P}^3 \mid ad - bc \neq 0 \right\} = GL_2(\mathbb{C})/\mathbb{C}^*$$

is the projective linear group and the action of an element of $PGL_2(\mathbb{C})$ on the elements of $\mathbb{P}^1$ is defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} . [X : Y] := [aX + bY : cX + dY]$$

This action is transitive: for every $p_1, p_2, p_3 \in \mathbb{P}^1$ pairwise distinct closed points, there exists a unique element of the projective linear group sending *them respectively to* $0, 1, \infty \in \mathbb{P}^1$. *Equivalently, we can say the following morphism:*

$$\begin{array}{rcl} PGL_2(\mathbb{C}) & \to & (\mathbb{P}^1)^3 \setminus \Delta \\ A & \mapsto & [A.[0:1], A.[1:1], A.[1:0]], \end{array}$$

where $\Delta \subset \mathbb{P}$ *is the diagonal, is an isomorphism.*

- *For* $g = 1$ *the curve* $C = E$ *has a group of automorphisms that is isomorphic to the semidirect product of the form* $Aut(E) \cong E(\mathbb{C}) \rtimes G$, *for* G *one of the finite groups* $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}$. *The normal subgroup* $E(\mathbb{C})$ *acts transitively, hence for every* $p, p' \in E(\mathbb{C})$ *there exists a unique element of* $E(\mathbb{C})$ *sending* p *to* p'.

- *For* $g \geq 2$ *the automorphism group* $Aut(C)$ *is finite, of order at most* $84(g-1)$.

Every nontrivial automorphism of a curve C of genus g fixes at most $2g + 2$ points. A consequence of this fact is the following:

Proposition 2.3.2. *If* $(C, p_1, \ldots, p_n)$ *is a smooth curve and* $n \geq 2g + 3$, *then* $Aut(C, p_1, \ldots, p_n) = \{id_C\}$.

As a consequence of the basic facts recalled in 2.3.1, we have the following:

Proposition 2.3.3. *Let* C *be a smooth, complex, irreducible, projective curve of genus* g*; let* $p_1, p_2, \ldots p_n \in C$ *be fixed distinct points. Then* $Aut(C, p_1, \ldots, p_n)$ *is finite if and only if* $2g - 2 + n > 0$.

Proof. If $2g - 2 + n \leq 0 \Rightarrow 2g + n \leq 2 \Rightarrow g \in \{0, 1\}$ and $n \leq 2 - 2g$. Then the claim follows immediately from 2.3.1. □

From this proposition we can deduce the analogous for nodal n-pointed curves. For simplicity we will call *special points* the preimages in $\widetilde{C}$ of the nodes and the preimages of marked points.

Proposition 2.3.4. *Let* C *be a connected, nodal, complex projective curve, and* $p_1, p_2, \ldots, p_n \in C$ *be pairwise distinct smooth points. Then* $(C, p_1, \ldots, p_n)$ *is stable if, and only if, every irreducible component* $\widetilde{C}_v \subseteq \widetilde{C}$ *of the normalization satisfies one of the following properties:*

- $\widetilde{C}_v$ *has genus 0 and contains at least three special points, or*

- $\widetilde{C}_v$ *has genus 1 and contains at least 1 special point, or*

- $\widetilde{C}_v$ *has genus at least 2.*

Proof. Combining the correspondence (2.4) of Remark 2.7 with the universal property of normalization, we obtain that any automorphism $\varphi : C \to C$ corresponds to an automorphism $\widetilde{\varphi} : \widetilde{C} \to \widetilde{C}$ mapping each pair $(q_i', q_i'') \in \widetilde{C}$ of preimages of nodes to some other such pair. In particular, we get the following group homomorphism:

$$\mathrm{Aut}(C, p_1, \ldots, p_n) \quad \to \quad \mathrm{Sym}(\{\text{components } \widetilde{C}_v \text{ of } \widetilde{C}\}) \times \mathrm{Sym}\left(\left\{ \{q_j', q_j''\} \mid j = 1, \ldots, l \right\}\right)$$

sending an automorphism to the permutation on the set of components of the normalization $\widetilde{C}$ and the set of preimages of nodes. Since the permutation groups are finite, we can conclude $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is finite if and only if the kernel K of the morphism described above is finite.

An element of the kernel consists in a collection of automorphisms $\varphi_v : \widetilde{C}_v \to \widetilde{C}_v$ of the components of $\widetilde{C}$ which fix all marked points p_i and fix or permute the points q_i', q_i'' (in the case they both belong to the same component $\widetilde{C}_v$). In both cases if we set m the number of points q_i', q_i'' and p_i, we have the automorphisms of the kernel are those fixing m points.

By Proposition 2.3.3 we have that the group of automorphisms of $\widetilde{C}_v$ fixing a number m of distinct points of $\widetilde{C}_v$ is finite if and only if $\widetilde{C}_v$ is of genus 0 with $m \geq 3$, of genus 1 with $m \geq 1$ or of genus at least 2. This proves the claim. □

2.4 Moduli spaces of smooth and stable curves

In this section we will finally state the main result on the existence of moduli spaces of curves, the Deligne-Mumford theorem. Before stating the theorem we introduce the notions of family of smooth and of stable curves; this will allow us to define precisely the moduli functor both for n-pointed genus g smooth and stable curves and to present the statement of the theorem.

Definition 2.7. Let $g, n \in \mathbb{N}$, natural numbers; we define an *n-pointed family of smooth genus g curves* over a scheme S to be the data of

$$(\pi : C \to S; p_1, p_2, \ldots, p_n : S \to C)$$

where:

- π is a smooth, proper, surjective, finitely presented morphism of schemes such that the fibre C_x over any closed point $x \in S$ is a smooth, projective connected curve of arithmetic genus g;

- the morphisms $p_1, p_2, \dots, p_n$ are pairwise disjoint sections of π.

Definition 2.8. Let $g, n \in \mathbb{N}$, natural numbers; we define an *n-pointed family of stable genus g curves* over a scheme S the data of

$$(\pi : C \to S; p_1, p_2, \dots, p_n : X \to C)$$

where:

- π is a flat, proper, surjective, finitely presented morphism of schemes such that the fibre C_x over any closed point $x \in S$ is a stable, projective connected curve of arithmetic genus g;

- the morphisms $p_1, p_2, \dots, p_n$ are pairwise disjoint sections of π with image in the smooth locus of π.

Definition 2.9. Given $(C/S, p_1, p_2, \dots, p_n), (C'/S, p'_1, p'_2, \dots, p'_n)$ two families of smooth/stable curves, they are said to be isomorphic if there exists an isomorphism $\varphi : C \to C'$ over S such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[rr, "\varphi"] \arrow[dr, "\pi"] & & C' \arrow[dl, "\pi'"'] \\
& S \arrow[ul, bend left, "p_i"] \arrow[ur, bend right, "p'_i"'] &
\end{tikzcd}$$

i.e. if $\varphi \circ p_i = p'_i$ for all $i = 1, \dots, n$.

Definition 2.10 (Pullback)**.** Given $(C/S, p_1, p_2, \dots, p_n)$ a family of curves over S, and given $f : T \to S$ a morphism of schemes, we define the pullback of $(C/S, p_1, p_2, \dots, p_n)$ under f to be the family $(C_T/T, p_{1,T}, p_{2,T}, \dots, p_{n,T})$ where C_T is the fibered product: $C_T = C \times_S T$, and the sections $p_{i,T}$ are the induced sections $p_{i,T} = (p_i \circ f) \times \mathrm{id}_T$.

$$\begin{tikzcd}
C_T \arrow[r] \arrow[d] & C \arrow[d] \\
T \arrow[r, "f"] \arrow[u, bend left, "p_{i,T}"] & S \arrow[u, bend right, "p_i"']
\end{tikzcd}$$

We can now give the precise definition of the moduli functors:

Definition 2.11. Given $g, n \in \mathbb{N}$ natural numbers, we define the moduli functor $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ as follows: for every scheme $S \in \mathrm{Ob}(\mathbf{Sch}_{\mathbb{C}})$, they associate with it the sets:

$$\mathcal{M}_{g,n}(S) = \left\{ \begin{array}{l} (\pi : C \to S; p_1, p_2, \dots, p_n : S \to C) \\ \text{family of smooth curves over } S \end{array} \right\} / \sim$$

$$\overline{\mathcal{M}}_{g,n}(S) = \left\{ \begin{array}{l} (\pi : C \to S; p_1, p_2, \dots, p_n : X \to C) \\ \text{family of stable curves over } S \end{array} \right\} / \sim$$

Given a morphism $f : T \to S$, the maps

$$\overline{\mathcal{M}}_{g,n}(f) : \overline{\mathcal{M}}(S) \to \overline{\mathcal{M}}(T)$$

$$\mathcal{M}_{g,n}(f) : \mathcal{M}(S) \to \mathcal{M}(T)$$

are defined by the pullback of the families of curves over S to T.

We can finally state the main theorem on moduli space of curves,

Theorem 2.4.1 (Deligne-Mumford)**.** *Let $g, n \in \mathbb{N}$ natural numbers satisfying $2g - 2 + n > 0$. Then:*

1. *there exist coarse moduli spaces $M_{g,n}$ and $\overline{M}_{g,n}$ respectively of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$;*
2. *they are normal algebraic varieties of dimension $3g - 3 + n$ and there is a natural inclusion $M_{g,n} \subseteq \overline{M}_{g,n}$ as a nonempty, open and dense subvariety;*
3. *the variety $\overline{M}_{g,n}$ is irreducible and projective;*
4. *the boundary $\partial \overline{M}_{g,n} = \overline{M}_{g,n} \setminus M_{g,n}$, that is the complement of the locus of smooth curves, is a Weil divisor;*
5. *the locus $\overline{M}^{0}_{g,n} \subseteq \overline{M}_{g,n}$ of n-pointed curves with trivial automorphism group is an open and smooth subvariety. It is a fine moduli space for the moduli functor $\overline{\mathcal{M}}^{0}_{g,n}$ of stable curves with trivial automorphism group and thus has a universal family*
$$\overline{C}^{0}_{g,N} \xrightarrow{\pi} \overline{M}^{0}_{g,n}.$$

Remark 2.8. The hypothesis $2g - 2 + n > 0$ in the statement of the theorem ensures, by Proposition 2.3.3, that any smooth curve is stable. More precisely, with this condition we are excluding the following cases:

$$(g, N) = (0, 0), (0, 1), (0, 2), (1, 0).$$

Under this assumption, the set $\overline{M}^0_{g,n}$ of curves with trivial automorphism group introduced in the Theorem 2.4.1-cond. 5, is nonempty, if, and only if,

$$(g, n) \neq (1, 1), (2, 0).$$

2.5 Examples of moduli spaces of curves

We would like to conclude this chapter with some examples describing in some details moduli spaces of curves of low genus.

Smooth curves of genus 0: $M_{0,n}$

Let us start by discussing moduli spaces of smooth curve $M_{0,n}$ for given $n \in \mathbb{N}$ whose existence is guaranteed by Deligne-Mumford Theorem. The hypotheses of the theorem are satisfied if:

$$2g - 2 + n = -2 + n > 0 \iff n \geq 3$$

so let us start from the case $n = 3$. Since every smooth genus 0 curves C is isomorphic to $\mathbb{P}^1$, then every curve with three marked points $(C, p_1, p_2, p_3) \in \mathcal{M}_{0,3}(\mathrm{Spec}(\mathbb{C}))$ is isomorphic to $(\mathbb{P}^1, p_1, p_2, p_3)$. Moreover, since there exists an automorphism of the projective line sending any triple of points p_1, p_2, p_3 respectively to $0, 1, \infty$, we have:

$$(C, p_1, p_2, p_3) \cong (\mathbb{P}^1, p_1, p_2, p_3) \cong (\mathbb{P}^1, 0, 1, \infty).$$

Thus, up to isomorphism, there exists a unique smooth curve of genus 0 with three distinct marked points. Therefore, the moduli space $M_{0,3}$ should be given by just a point. We have in fact the following:

Proposition 2.5.1. *The variety $M_{0,3} = \{point\} = Spec(\mathbb{C})$ is a fine moduli space for the functor $\mathcal{M}_{0,3}$ and the universal family is given by*

$$\pi : \mathbb{P}^1 \to Spec(\mathbb{C})$$

$$p_1 = 0, p_2 = 1, p_3 = \infty : Spec(\mathbb{C}) \to \mathbb{P}^1.$$

The general case for $n \geq 4$ is not more difficult than this one. Indeed, given any curve $(C, p_1, p_2, \dots, p_n) \in \mathcal{M}_{0,n}$ we still have that $C \cong \mathbb{P}^1$ since C has genus 0. Moreover, there exists a unique element $B \in \mathrm{PGL}_2(\mathbb{C})$ of the automorphisms group of $\mathbb{P}^1$ sending p_1, p_2, p_3 respectively to $0, 1, \infty$. Hence, for $p'_j = Bp_j$, we have:

$$(C, p_1, p_2, \dots, p_n) \cong (\mathbb{P}^1, 0, 1, \infty, p'_4, \dots, p'_n).$$

The elements $p'_4, \dots, p'_n \in \mathbb{P}^1$ are pairwise distinct and also distinct from $0, 1, \infty$ and they uniquely determine the isomorphism class of $(C, p_1, \dots, p_n)$.

We have than the following:

Proposition 2.5.2. *For $n \geq 3$ the moduli functor $\mathcal{M}_{0,n}$ is representable by the variety*

$$M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta,$$

where $\Delta = \{(q_i)_i \mid \exists\, i \neq j \text{ with } q_i = q_j\}$ is the diagonal.

The universal family of $M_{0,n}$ is given by

$$\pi : \mathbb{P}^1 \times \left((\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta\right)$$

$$p_1 = 0, p_2 = 1, p_3 = \infty : \operatorname{Spec}(\mathbb{C}) \to \mathbb{P}^1$$
$$(p_4, \dots .p_n) : \operatorname{Spec}(\mathbb{C})^{n-3} \to \left((\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta\right)$$

Remark 2.9. We observe that in this case we have the following facts:

1. the space $M_{0,n}$ exists as a fine moduli space;
2. $M_{0,n}$ is a smooth and irreducible variety of dimension $3g - 3 + n = n - 3$;
3. every smooth genus 0 curve $(C, p_1, \dots, p_n)$ has trivial automorphism group, hence $M_{0,n} \subseteq \overline{M}^0_{0,n}$. This also explain why in this case the moduli space is a fine space and not just a coarse one, and why $M_{0,n}$ is also smooth instead of normal.

In order to study moduli spaces of stable curves we need to introduce the notion of dual graph of a curve.

Stable graphs associated to nodal curves

A very useful tool for looking at moduli spaces of stable curves of genus g is the dual graph of a stable curve. Given a stable curve $(C, p_1, \dots, p_n)$, the dual graph is a combinatorial object Γ_C that allows us to describe the topological type of the stable curve: how many components it has and of which genus, and how they intersect among themselves.

The dual graph associated to a stable curve $(C, p_1, \dots, p_n)$ is a graph with some decorations; we start by giving an informal idea, the precise definition of dual graph will be given in 2.14. In the graph associated to C we would like to have the following correspondences:

- the vertices v correspond to irreducible components C_v of C and are labelled with the geometric genus of the component;

- the edges correspond to the nodes of the curve, where a node connecting two components C_v, C_w gives an edge between v and w;
- the legs, labelled from 1 to n, attached to the vertices correspond to the marked points $p_1, \dots, p_n$; they are attached to the vertex v if the respective marked point belongs to the component C_v.

The rigorous definition of stable graph is the following:

Definition 2.12 (Stable graph)**.** A *stable graph* Γ is given by the following data:

$$\Gamma = \big(V, H, L, g : V \to \mathbb{Z}_{\geq 0}, v : H \to V, \iota : H \to H, l : L \to \{1, \dots, n\}\big)$$

where

1. $V = V(\Gamma)$ is a finite set, the set of vertices, and $g : V \to \mathbb{Z}_{\geq 0}$ is a map associating a genus to each vertex v.
2. $H = H(\Gamma)$ is a finite set, the set of half-edges. The map $v : H \to V$ associates to each half-edge h a vertex $v(h)$, the vertex incident to h. We denote by
$$H(v) = \{h \in H \mid v(h) = v\}$$
the half-edges incident at v and by $n(v) = \#H(v)$ the number of these half-edges. The map $\iota : H \to H$ is an involution, thus it decomposes H into pairs of half-edges switched by ι and fixed points of ι.
3. The pairs $e = \{h, h'\}$ of distinct half-edges exchanged by ι are called the edges $E = E(\Gamma)$ of Γ.
4. The set $L = L(\Gamma) \subset H$ is the set of half-edges fixed by ι, called the legs of Γ. The map $l : L \to \{1, \dots, n\}$ is a bijection.
5. The graph is connected.
6. The graph satisfies the following stability condition: for each vertex $v \in V$ we have:
$$2g(v) - 2 + n(v) > 0.$$

Definition 2.13. An *isomorphism* of stable graphs $\varphi : \Gamma \to \Gamma'$ is a collection of bijective maps
$$\varphi_V : V \to V', \varphi_H : H \to H'$$
of their sets of vertices and half-edges which are compatible with the functions g, v, ι, i.e.

$$g'(\varphi_V)(v)) = g(v), v'(\varphi_H(h)) = \varphi_V(v(h)), \iota'(\varphi_H(h)) = \varphi_H(\iota(h)), l'(\varphi_H(h)) = l(h).$$

We denote by $\mathrm{Aut}(\Gamma)$ the set of automorphisms of a graph, that is a group with the respect to the composition. We finally define the genus of a stable graph as:

$$g(\Gamma) = \left(\sum_{v \in V(\Gamma)} g(v) \right) + 1 + \#E(\Gamma) - \#V(\Gamma). \tag{2.5}$$

We can now define rigorously the dual graph of a stable curve.

Definition 2.14 (Dual graph)**.** Given a stable curve $(C, p_1, \ldots, p_n)$ the *dual graph* associated to it is the stable graph Γ_C defined as follows.

- The vertices $v \in V$ of Γ are in one-to-one correspondence with the irreducible components C_v of C (which canonically correspond to the components $\widetilde{C}_v$ of the normalization $\widetilde{C}$).

  $$V \cong \left\{ C_v \mid \text{component of } C \right\} = \left\{ \widetilde{C}_v \mid \text{component of } \widetilde{C} \right\}$$

  The map $g : V \to \mathbb{Z}_{\geq 0}$ sends a vertex v to the genus $g\left(\widetilde{C}_v\right)$ of the component in the normalization.

- The half-edges $h \in H$ of Γ are in one-to-one correspondence with the union of the preimages $q', q'' \in \widetilde{C}$ of nodes $q \in C$ under the normalization map $\nu : \widetilde{C} \to C$ and the marked points $p_1, \ldots, p_n \in C$.

  $$H \cong \left(\coprod_{\substack{q \text{ node in } C \\ \nu^{-1}(q) = \{q', q''\}}} \{q', q''\} \right) \sqcup \{p_1, \ldots, p_n\}$$

  The map $v : H \to V$ sends half-edges of the form q', q'' to the vertex v for the component $\widetilde{C}$ of the normalization containing them, and the half-edges of the form p_i to the vertex v for the component C_v of C containing them. The involution ι exchanges the preimages of nodes ($\iota(q') = q'', \iota(q'') = q'$) and fixes the marked points ($\iota(p_i) = p_i$).

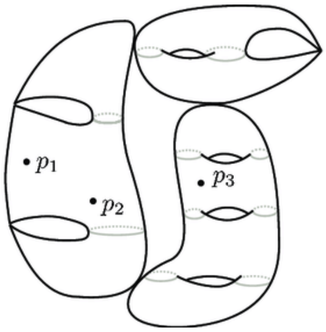
- The legs $L \subset H$ are precisely the marked points

  $$L = \{p_1, \ldots, p_n\}$$

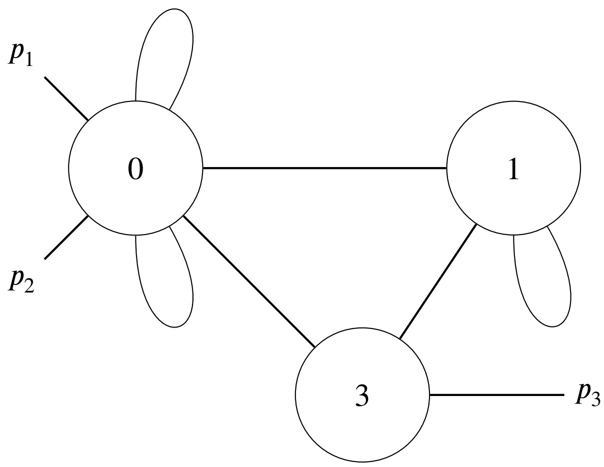
  and the map $l : L \to \{1, \ldots, n\}$ sends each p_i to i.

Remark 2.10. By definition the genus $g(\Gamma_C)$ of the dual graph of a curve $(C, p_1, \ldots, p_n)$ equals the arithmetic genus of C.

Example 2.6. *Let consider the curve C of the Example 2.5 with three marked points:*



it is a stable curve and the stable graph associated to C is given by:



We can compute, the genus of the curve using the dual graph and the formula 2.5; we have:

$$\begin{aligned} g(\Gamma) &= \sum_{v \in V(\Gamma)} g(v) + 1 - \#E(\Gamma) - \#V(\Gamma) \\ &= (0+1+3)+1+6-3= \\ &= 8 \end{aligned}$$

which coincide with the genus computed in the Example 2.5.

Remark 2.11. As remarked at the beginning of the section, stable graphs are very useful to study moduli spaces of curves with a combinatorial approach, as the following theorem and examples illustrate. They can in fact describe important information about the curves in a purely combinatorial way.

However, as it usually happens dealing with combinatorial techniques, the number of isomorphism classes of stable graphs grows drastically with g and n.

For instance, for $g = 1$, $n = 5$, the number of isomorphism classes that we have is 1576. [Sch20] Anyway, for low g and n everything is treatable.

Theorem 2.5.3. *Let $g, n \geq 0$ with $2g - 2 + n > 0$, then for any stable graph Γ of genus g with n legs, the set*

$$M^{\Gamma} = \{(C, p_1, \dots, p_n) \mid \Gamma_C \cong \Gamma\} \subseteq \overline{M}_{g,n}$$

of curves with stable graph isomorphic to Γ is a nonempty, irreducible, locally closed subset of $\overline{M}_{g,n}$. In particular, the space $\overline{M}_{g,n}$ is the disjoint union

$$\overline{M}_{g,n} = \coprod_{\Gamma} M^{\Gamma},$$

where Γ runs through isomorphism classes of stable graphs. We have

$$\dim M^{\Gamma} = \sum_{v \in V(\Gamma)} [3g(v) - 3 + n(v)] = \dim \overline{M}_{g,n} - \#E(\Gamma.)$$

The proof of the theorem is based on the existence of this *glueing morphism*:

Lemma 2.5.4. *Let Γ be a stable graph of genus g with n legs, then there exists a morphism*

$$\xi_{\Gamma} : \overline{M}_{\Gamma} := \prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)} \to \overline{M}_{g,n}$$

which send an element $\left(C_v, (q_h)_{h \in H(v)}\right)_{v \in V(\Gamma)}$ to the curve $(C, p_1, \dots, p_n)$ obtained by glueing all pairs $q_h, q_{h'}$ of points corresponding to pairs $\{h, h'\}$ forming edges of Γ, and setting $p_i \in C$ to be the image of the marked point $q_{l^{-1}(i)}$ belonging to the half-edge $l^{-1}(i) \in H(\Gamma)$.

The morphism is finite and its image is the closure $\overline{M^{\Gamma}}$ of M^{Γ}.

The following is just an idea of the proof:

Proof. It is possible to check the domain $\overline{M}_{\Gamma}$ of the map ξ_{Γ} is a coarse moduli space for the moduli functor

$$\begin{array}{rcl} \overline{\mathcal{M}}_{\Gamma} : \mathbf{Sch}^{\mathrm{op}} & \to & \mathbf{Sets} \\ S & \mapsto & \overline{\mathcal{M}}_{\Gamma}(S) = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}(S). \end{array}$$

Then, we could obtain the map ξ_{Γ} defined above by constructing a natural transformation

$$\widehat{\xi_{\Gamma}} : \overline{\mathcal{M}}_{\Gamma} \to \overline{\mathcal{M}}_{g,n}$$

between moduli functors and using the properties of coarse moduli space.

Let S be a scheme, the natural transformation $\widehat{\xi_\Gamma}$ takes an element of $\overline{\mathcal{M}}_\Gamma(S)$, that consist of a collection of stable curves over S

$$\left(\pi_v : C_v \to S, \left(q_h : S \to C_v\right)_{h\in H(\Gamma)}\right)_{v\in V(\Gamma)},$$

and glues the curves of this element to a stable curve in $\overline{\mathcal{M}}_{g,n}(S)$ by identifying the sections $q_h, q_{h'}$ corresponding to pairs $\{h, h'\}$ forming edges of Γ.

For instance, in the case of two curves $\pi_1 : C_1 \to S, \pi_2 : C_2 \to S$ glued along sections $q_1 : S \to C_1, q_2 : S \to C_2$, the glued family can be obtained as the union of the images of C_1 and C_2 inside the fiber product $C_1 \times_S C_2$, under the maps:

$$\begin{tikzcd}
C_1 \arrow[dr, "\mathrm{id}_{C_1}\times(q_2\circ\pi_1)"'] & & C_2 \arrow[dl, "(q_1\circ\pi_2)\times\mathrm{id}_{C_2}"] \\
& C_1 \times_S C_2 &
\end{tikzcd}.$$

Once constructed the natural transformation $\widehat{\xi_\Gamma}$, the map of coarse moduli spaces can be obtained by considering the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}_\Gamma \arrow[r] \arrow[d, "\widehat{\xi_\Gamma}"'] & h^{\overline{M}_\Gamma} \arrow[d, dashed, "\xi_\Gamma"] \\
\overline{\mathcal{M}}_{g,n} \arrow[r] & h^{\overline{M}_{g,n}}
\end{tikzcd}$$

The horizontal arrows come from the fact that $\overline{M}_\Gamma$ and $\overline{M}_{g,n}$ are coarse moduli spaces of the functors on the right of the diagram; moreover, by definition of coarse moduli space, the map $\overline{\mathcal{M}}_\Gamma \to h^{\overline{M}_{g,n}}$ (given by the composition of the bottom row arrow with the map $\widehat{\xi_\Gamma}$), factors through $h^{\overline{M}_\Gamma}$, giving the desired dashed arrow that makes commute the diagram.

From the definition of $\widehat{\xi_\Gamma}$, we have that ξ_Γ acts on the $\mathbb{C}$-points of $\overline{M}_\Gamma$ as in the statement of the Lemma 2.5.4 and it is possible to prove that it is finite.

It remains to check its image is the closure $\overline{M^\Gamma}$ of M^Γ; let $M_\Gamma = \prod_{v\in V(\Gamma)} M_{g(v),n(v)} \subseteq \overline{M}_\Gamma$, then we have:

$$\xi_\Gamma(M_\Gamma) = M^\Gamma.$$

Indeed, given a curve in M^Γ we can certainly obtain it by glueing a collection of smooth curves C_v under the map ξ_Γ (for instance if C_v are the components of the normalization of C). Conversely, any curve obtained by glueing of smooth curves in M_Γ has stable graph Γ.

In conclusion, since ξ_Γ is proper, its image is closed, hence

$$M^\Gamma \subseteq \xi_\Gamma(M_\Gamma) \Rightarrow \overline{M^\Gamma} \subseteq \xi_\Gamma(\overline{M}_\Gamma);$$

on the other hand, since the moduli spaces of curves are irreducible, so is the product $\overline{M}_\Gamma$ and thus the nonempty open subset M_Γ is dense in $\overline{M}_\Gamma$. This allow us to deduce the converse inclusion:

$$\xi_\Gamma(\overline{M}_\Gamma) \subseteq \overline{\xi_\Gamma(M_\Gamma)} = \overline{M^\Gamma}$$

and this proves the lemma. □

We can now go back to the proof of Theorem 2.5.3.

Proof. We saw in the proof of Lemma 2.5.4 that $M^\Gamma = \xi_\Gamma(M_\Gamma)$ is the image under ξ_Γ of $M_\Gamma = \prod_{v\in V(\Gamma)} M_{g(v),n(v)}$.

We first observe that, since M_Γ is nonempty and irreducible, so M^Γ is nonempty and irreducible; moreover, also the closure $\overline{M^\Gamma}$ is the image of $\overline{M}_\Gamma$. Repeating, with some adaptation the previous argument we have:

$$\xi_\Gamma(\overline{M}_\Gamma \setminus M_\Gamma) = \overline{M^\Gamma} \setminus M^\Gamma.$$

Since $\overline{M}_\Gamma \setminus M_\Gamma$ is closed and ξ_Γ is a proper map, then $\overline{M^\Gamma} \setminus M^\Gamma$ is closed in $\overline{M^\Gamma}$ and thus M^Γ is locally closed inside $\overline{M}_{g,n}$ since it is open in the closed subset $\overline{M^\Gamma}$.

To conclude the proof it remains to check thef formula for the dimension but one can see the formula hold for M_Γ and since ξ_Γ is finite then:

$$\dim \overline{M^\Gamma} = \dim \overline{M}_\Gamma = \dim \overline{M}_{g,n} - \#E(\Gamma).$$

□

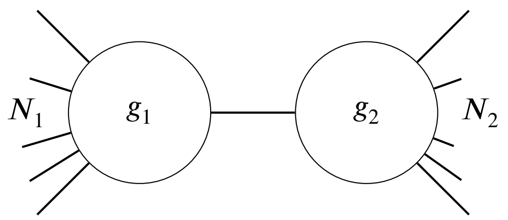
Definition 2.15. The sets M^Γ are called the *strata* of $\overline{M}_{g,n}$ and the decomposition is called *stratification according to the dual graph.*

From the previous theorem we can deduce the following corollary:

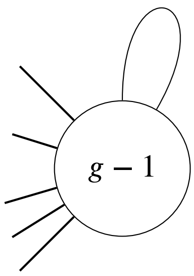
Corollary 2.5.5. *A stable graph of genus g with n legs has at most* $3g-3+n$ *edges.*

The irreducible components of the boundary of $\overline{M}_{g,n}$ are called the *boundary divisors*. In fact they are the $\overline{M^\Gamma}$ with $\#E(\Gamma) = 1$, and by Theorem 2.5.3, they have codimension 1 in $\overline{M}_{g,n}$.

The stable graphs Γ with $\#E(\Gamma) = 1$ are of one of the two forms:



or



with $g \geq 1$, $g_1 + g_2 = g$, $N_1 \cup N_2 = \{1, \dots n\}$ satisfying the stability condition, i.e. if $g_1 = 0$, $n_1 = \#N_1 \geq 2$ and similarly if $g_2 = 0$ we ask $n_2 = \#N_2 \geq 2$.

Stable curves of genus 0: $\overline{M}_{0,n}$

Using the dual graph associated to a stable curve it is possible to prove that the automorphism group of a genus 0 stable curves is trivial and to generalize the result obtained for the moduli space of smooth curves.

Proposition 2.5.6. *Any stable curve $(C, p_1, p_2, \dots, p_n)$ of genus 0 has trivial automorphism group:*

$$Aut(C, p_1, p_2, \dots, p_n) = \{id_C\}.$$

This yields the following:

Corollary 2.5.7. *For $n \geq 3$, the space $\overline{M}_{0,n}$ is a fine moduli space for the functor $\overline{\mathcal{M}}_{0,n}$, and it is a smooth irreducible projective variety of dimension $n - 3$.*

Proof. By the previous Corollary 2.5.6 we have $\overline{M}^0_{0,n} = \overline{M}_{0,n}$, so the statement follows from Theorem of Deligne-Mumford (page 34). □

$\overline{M}_{0,3}$

We can finally discuss some concrete examples. For the space $\overline{M}_{0,3}$, by the Corollary 2.5.5, a graph of genus 0 with n legs has at most $3g - 3 + n = n - 3$ edges. Hence, for $n = 3$ the only stable graph is given by the trivial one.

This implies $\overline{M}_{0,3} = M_{0,3} = \{pt\}$ is given by just a point.

$\overline{M}_{0,4}$

For $n = 4$, by Corollary 2.5.5, every graph has at most $3g - 3 + n = -3 + 4 = 1$ edge, hence Γ can have 0 or 1 edges.

Moreover, for the stability condition, each vertex satisfies

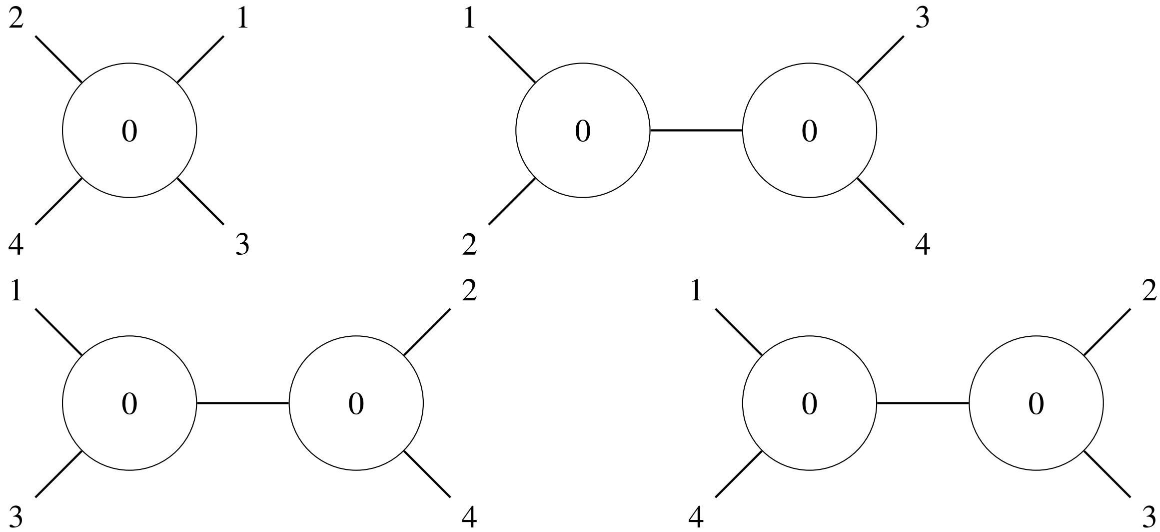
$$2g(v) - 2 + n(v) > 0 \iff n(v) > 2;$$

then, each vertex must be incident to at least three half-edges.

There are two possibilities:

1. A graph with only one vertex of genus 0 and four legs.
2. A graph with two vertices of genus 0, two legs for each vertex and an edge joining them.

These are the possible stable graphs Γ with $(g, n) = (0, 4)$:



By Theorem 2.5.3, we have that:

$$\overline{M}_{0,4} = M^{\Gamma_0} \sqcup M^{\Gamma_1} \sqcup M^{\Gamma_2} \sqcup M^{\Gamma_3}$$

where $M^{\Gamma_0} = M_{0,4} = \mathbb{A}^1 \setminus \{0, 1\}$ and the $M^{\Gamma_i} (i = 1, 2, 3)$ are irreducible, locally closed subset of $\overline{M}_{0,4}$ of dimension 0, then points (in fact by Corollary 2.5.7 $\dim \overline{M}_{0,4} = 4 - 3 = 1$ and by Theorem 2.5.3: $\dim M^{\Gamma_i} = \dim \overline{M}_{0,4} - \#E(\Gamma) = 0$).

Since $\mathbb{P}^1$ is the only smooth, irreducible, projective variety of dimension 1 which contains $M_{0,4} = \mathbb{A}^1 \setminus \{0, 1\}$ as an open subvariety we can finally conclude:

$$\overline{M}_{0,4} \cong \mathbb{P}^1.$$

Since $\overline{M}_{0,4}$ is a fine moduli space, the curves which have as dual graph M^{Γ_i} for $i = 0, 1, 2, 3$ fit together into a universal curve $\pi : \overline{C}_{0,4} \to \overline{M}_{0,4}$, the universal family for the moduli functor $\overline{\mathcal{M}}_{0,4}$.

By proposition 2.5.2 we have that the universal family over $M_{0,4}$ is the trivial family $M_{0,4} \times \mathbb{P}^1 \to M_{0,4}$. How to fill in the missing fibres over $0, 1, \infty$ in $\overline{M}_{0,4}$ is represented in Figure 2.6[1].

Thus, the universal curve can be obtained in the following way: we consider the blow-up map of $\mathbb{P}^1 \times \mathbb{P}^1$ at the three points: $(0,0), (1,1), (\infty,\infty)$ and the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor. Hence the universal curve is given by the following composition:

$$\pi : \overline{C}_{0,4} = \mathcal{B}l_{(0,0),(1,1),(\infty,\infty)}\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 = \overline{M}_{0,4}.$$

The sections $p_1, \ldots, p_4 : \overline{M}_{0,4} \to \overline{C}_{0,4}$ are the strict transforms of the four maps:

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^1 \times \mathbb{P}^1 \\ q & \mapsto & (q,0), (q,1), (q,\infty), (q,q). \end{array}$$

The singular fibres of π over $0, 1, \infty \in \mathbb{P}^1$ are the unions of the strict transforms of the fibre of the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and the exceptional divisor of the blow-up, which meet transversally and thus form a nodal curve.

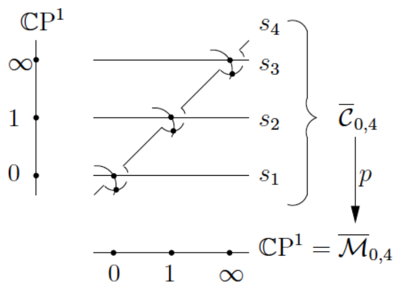


Figure 2.6: The universal family of curves over $\overline{M}_{0,4}$

[1]The picture is taken from [Zvo14]

Curves of genus 1: the case of $\overline{M}_{1,1}$

The case of curves of genus 1 is much harder than the genus 1 case and there is no recursive construction as for $\overline{M}_{0,n}$.

However, it is possible to make precise the example discussed in Subsection 2.2 obtaining that $M_{1,1} \cong \mathbb{A}^1$ and, as it should be being its compactification, deduce that $\overline{M}_{1,1} \cong \mathbb{P}^1$.

The moduli space $\overline{M}_{1,1}$ is obtained from $M_{1,1}$ by adding one point corresponding to the singular stable curve represented in Fig. 2.7[2]

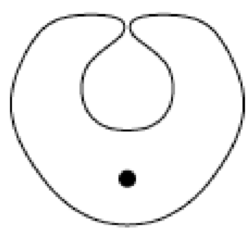


Figure 2.7: The singular stable curve in $\partial\overline{M}_{1,1}$

[2]Figure taken from [Zvo14]

3 Basics on intersection theory

This chapter is intended as a brief introduction to intersection theory. In the the next chapters we will develop, following Mumford's ideas ([Mum83]), a theory of intersection on moduli spaces of curves. Hence, in this chapter, we would like to introduce, in many cases without proofs, the results that will be used later.

The model for the development of intersection theory is the classical theorem of Bézout for plane curves; from this theorem we know that if two plane curves $\mathcal{A}, \mathcal{B} \subseteq \mathbb{P}^2$ intersect transversely, then they intersect in $(\deg \mathcal{A})(\deg \mathcal{B})$ points. Moreover, once the notion of *intersection multiplicity* is introduced, the theorem can be generalized: if $\mathcal{A}, \mathcal{B} \subseteq \mathbb{P}^2$ are two plane curves without common components, their intersection consists of a collection of $(\deg \mathcal{A})(\deg \mathcal{B})$ points counted with multiplicities.

In modern geometry we need to understand intersections of subvarieties in much greater generality, and this can be done by studying the Chow ring $A(X)$ of the ambient space X, an assigned scheme containing the varieties. The basic idea is to associate to every subscheme $A \subseteq X$ a class $[A] \in A(X)$ generalizing the notion of degree of a curve in $\mathbb{P}^2$.

There are in fact two aspects of the Bézout's theorem that seem to have far reaching consequences if correctly generalized.

1. The cardinality of the intersection does not depend on the choice of curves, it is sufficient to know the degrees of the curves to deduce in how many points (if necessary counted with multiplicities) the two curves intersect.

   Let us observe that, once assumed this invariance, let d and e respectively the degrees of $\mathcal{A}$ and $\mathcal{B}$, the claim of the theorem follows from the fact that a union of d general lines meets a union of e general lines in $d \cdot e$ points.

2. The answer to the problem of the cardinality of intersection is a product: $d \cdot e$; this suggests that the space of classes should have some sort of product structure.

This leads to the definition of Chow group of a scheme; we will see that, under certain assumptions, the rational equivalence class of the intersection $A, B \subseteq X$ depends only on the rational equivalence classes of the two subvarieties A and B and this gives, in the particular case of the Chow group of a smooth variety, a graded ring structure.

In this way, we will be able to obtain a generalized version of Bézout's Theorem:

Theorem 3.0.1. *If $A, B \subseteq X$ are subvarieties of a smooth variety X and* $\operatorname{codim}(A \cap B) = \operatorname{codim} A + \operatorname{codim} B$, *then we can associate to each irreducible component C_i of $A \cap B$ a positive integer $m_{C_i}(A, B)$ in such a way that we have*

$$[A][B] = \sum m_{C_i}(A, B) \cdot [C_i].$$

The integer m_{C_i} is called the *intersection multiplicity of A and B along C_i.*

Remark 3.1. Theorem 3.0.1 deals with the case where A, B are subvarieties which meet only in codimension $\operatorname{codim} A + \operatorname{codim} B$ (the case of dimensionally proper intersection); it is possible, however, to generalize this result to the case where the components of the intersection have arbitrary codimension.

3.1 The Chow ring

We can now turn to the definition and basic properties of the Chow ring of a scheme. The goal of this first section is to give the prerequisites to understand the definition of Chow ring and the theorem characterizing its structure.

We start by defining the group of cycles:

Definition 3.1. Let X be a scheme. We define the *group of cycles* on X, denoted as $Z(X)$, to be the free abelian group generated by the set of subvarieties of X.

Remark 3.2. The group $Z(X)$ is graded by dimension; let $Z_k(X)$ be the group of cycles that are formal linear combination of subvarieties (reduced irreducible subschemes) of dimension k, which will be called *k-cycles*, then:

$$Z(X) = \bigoplus_{k=0}^{\dim X} Z_k(X).$$

Definition 3.2. A cycle $Z = \sum n_i Y_i$, for Y_i given subvarieties, is said to be *effective* if all the coefficients n_i are nonnegative. A *divisor* is an $(n-1)$-cycle on a pure n-dimensional scheme.

Remark 3.3. We notice that it follows immediately from the definition that:

$$Z(X) = Z(X_{red});$$

i.e. $Z(X)$ does not depend on the nonreduced structure X may have.

It is possible now to associate to any subscheme $Y \subseteq X$ an effective cycle, this can be done in the following way.

Let $Y_1, Y_2, \dots, Y_s$ be the irreducible components of the reduced scheme Y_{red}; since the schemes are Noetherian, each local ring $\mathcal{O}_{Y,Y_i}$ has a finite composition series of length l_i. We then define:

$$\langle Y \rangle = \sum_{i=1}^{s} l_i Y_i.$$

The coefficient l_i is called the *multiplicity* of the scheme Y along the irreducible component Y_i and it will be written as: $\mathrm{mult}_{Y_i}(Y)$.

In a certain sense, cycles may be viewed as a coarse approximation to subschemes.

The Chow group is defined as the quotient of the group of cycles modulo rational equivalence; we have then to introduce first this equivalence relation:

Definition 3.3 (Rational equivalence)**.** Let us define $\mathrm{Rat}(X) \subseteq Z(X)$ to be the subgroup generated by all the differences of the form

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle,$$

where $t_0, t_1 \in \mathbb{P}^1$ and Φ is a subvariety of $\mathbb{P}^1 \times X$ not contained in any fibre $\{t\} \times X$.

Two cycles are said to be *rationally equivalent* if their difference is in $\mathrm{Rat}(X)$; two subschemes are said to be rationally equivalent if their associated cycles are rationally equivalent.

We can finally give the following:

Definition 3.4 (Chow group)**.** We define the *Chow group* of a scheme X to be the quotient:

$$A(X) = Z(X)/\mathrm{Rat}(X)$$

the group of rational equivalence classes of cycles on X. Given $Y \in Z(X)$ the class of Y in the Chow group $A(X)$ will be denoted by $[Y] \in A(X)$; if $Y \subseteq X$ is a subscheme of X, we usually write $[Y]$ to denote the class of the cycle $\langle Y \rangle$ associated to Y.

Remark 3.4. For the moment, being a quotient of a group it is clear only the group structure of $A(X)$, and this is the reason for the name used.

From the principal ideal theorem it follows that also the Chow group is graded by dimension:

Proposition 3.1.1. *If X is a scheme then the Chow group of X is graded by dimension; that is,*

$$A(X) = \bigoplus A_k(X)$$

where $A_k(X)$ is the group of rational equivalence classes of k-cycles.

Proof. Let $\Phi \subseteq \mathbb{P}^1 \times X$ be an irreducible variety not contained in any fibre over X, then, if we consider the affine open set $U := \Phi \cap \left(\mathbb{A}^1 \times X\right) \subseteq \Phi$, the scheme $\Phi \cap \left(\{t_0\} \times X\right)$ is defined by the zero locus of the polynomial $t - t_0$, which is not a zero divisor.

Hence, by the principal ideal theorem, it follows that the components of the intersection are all of codimension exactly 1 in Φ, and similarly for $\Phi \cap \left(\{t_1\} \times X\right)$. Hence all the varieties involved in the rational equivalence defined by Φ have the same dimension. □

In the previous proposition we proved the graduation of $A(X)$ by dimension; if X is equidimensional we may define the codimension of a subvariety $Y \subseteq X$ as $\dim X - \dim Y$. This shows that $A(X)$ could also be graded by codimension.

Let suppose X is a smooth equidimensional variety, the group of codimension-c cycles modulo rational equivalence will be denoted as

$$A^c(X) = A_{\dim X - c}.$$

After the notion of rational equivalence, there is another notion that has to be introduced:

Definition 3.5. Let X be a scheme, let A, B subvarieties of X, we say that A and B intersect *transversely* at a point p if A, B and X are all smooth at p and the tangent spaces to A and B at p span together the tangent space to X at p; i.e.

$$T_pA + T_pB = T_pX,$$

or equivalently

$$\operatorname{codim}(T_pA \cap T_pB) = \operatorname{codim} T_pA + \operatorname{codim} T_pB.$$

Definition 3.6. We say that $A, B \subseteq X$ subvarieties are *generically transverse*, or they intersect *generically transversely*, if A and B meet transversely at a general point of each component C of $A \cap B$.

Two cycles $A = \sum n_i A_i$ and $B = \sum m_j B_j$ are generically transverse if each A_i is generically transverse to each B_j.

Definition 3.7. Given finitely many subvarieties $A_i \subseteq X$, we say they intersect transversely at a smooth point $p \in X$ if p is a smooth point on each variety A_i and $\operatorname{codim}(\bigcap T_p A_i) = \sum \operatorname{codim} T_p A_i$.

We say they intersect generically transversely if there is a dense set of points in the intersection at which they are transverse.

Example 3.1. *If* $\dim A + \dim B = \dim X$*, then A and B are generically transverse if and only if they are transverse everywhere; their intersection in fact consists of finitely many points and they should intersect transversely at each of them.*

If $\operatorname{codim} A + \operatorname{codim} B > \dim X$*, then A and B are generically transverse if and only if they are disjoint.*

We are now interested in the ring structure of Chow groups; according to the intuition from Bézout's theorem, we would like to have an intersection of classes which define a product structure on the Chow group and that make it into a ring. This is precisely the case, proven in [Ful98], if we assume that X is a smooth variety.

Let us observe that Chow groups for quasi-projective varieties behave similarly to homology theory: they are abelian groups associated to a geometric object that are described by the quotient modulo the rational equivalence relation. In the case of a smooth variety, the intersection product makes the Chow groups into a graded ring, and in this case the Chow group will be called the Chow ring. This is analogous to the ring structure on the homology of smooth compact manifolds that can be imported, using Poincaré duality, from the natural ring structure on cohomology.

Theorem 3.1.2. *Let X be a smooth quasi-projective variety, then there exists a unique product structure on $A(X)$ satisfying the following condition:*
if two subvarieties A, B of X are generically transverse, then

$$[A][B] = [A \cap B].$$

This structure turns

$$A(X) = \bigoplus_{c=0}^{\dim X} A^c(X)$$

into an associative, commutative ring, graded by codimension, called the Chow ring *of X.*

The construction of the intersection ring was based by Chow in its article "*On the equivalence classes of cycles in an algebraic variety*[1]", on the moving lemma, of which we recall the statement. For proof of Theorem 3.1.2 see [Ful98, Section 8.3].

[1] Cho56.

Theorem 3.1.3 (Moving lemma)**.** *Let X be a smooth quasi-projective variety.*

1. *For every $\alpha, \beta \in A(X)$ there are generically transverse cycles $A, B \in Z(X)$ with $[A] = \alpha$ and $[B] = \beta$.*

2. *The class $[A \cap B]$ is independent of the choice of such cycles A and B.*

Proof. For a proof see [Ful98, Section 11.4]. □

Remark 3.5. The statement of the theorem is not necessarily true without the smoothness assumption. Let consider, for example, $X \in \mathbb{P}^3$ a quadric cone. Then, any two cycles representing the class of a line of X, meet at the origin, which is a singular point of the cone, hence they cannot be generically transverse.

Kleiman's transversality theorem

There is a particular situation in which the proof of the first part of the moving lemma becomes easier. If there is a group of automorphisms acting on X transitively, we can use it to move any two cycles to make them transverse. In particular we have the following result:

Theorem 3.1.4 (Kleiman's theorem in characteristic 0)**.** *Let G be an algebraic group acting transitively on a variety X over an algebraically closed field of characteristic 0, and let $A \subseteq X$ be a subvariety.*

1. *If $B \subseteq X$ is a subvariety, then there is an open dense subset $U \subseteq G$ such that gA is generically transverse to B for any element $g \in U$.*

2. *If $\varphi : Y \to X$ is a morphism of varieties, then for general $g \in G$ the preimage $\varphi^{-1}(gA)$ is generically reduced and of the same codimension as A.*

3. *If G is affine, then $[gA] = [A] \in A(X)$ for any $g \in G$.*

Proof. (1.) The statement in (1.) is a particular case of (2.) with $Y = B$.
(2.) Let n, a, b, m be the dimension respectively of X, A, Y and G, and let $x \in X$. The map

$$\begin{array}{ccc} G & \to & X \\ g & \mapsto & gx \end{array}$$

is surjective by hypothesis and its fibers are the cosets of the stabilizer of x in G. Since all fibers have the same dimension, this must be equal to $m - n$. Let now

$$\Gamma = \{(x, y, g) \in A \times Y \times G \mid gx = \varphi(y)\},$$

Since G acts transitively on X, the projection $\pi : \Gamma \to A \times Y$ is surjective, its fibers are the cosets of stabilizers of points in X, which have dimension $m - n$ by the previous observation. It follows that the dimension of Γ is:

$$\dim \Gamma = a + b + m - n.$$

Moreover, the fibre over g of the projection $\widetilde{\pi} : \Gamma \to G$ is isomorphic to $\varphi^{-1}(gA)$. Thus, either this intersection is empty for general g or it has dimension $a + b - n$, as required in the claim.

Now, since X is a variety, it is smooth at a general point and since G acts transitively, this implies all points of X are smooth, hence X is a smooth variety. Since any algebraic group in characteristic 0 is smooth ([Mum66]), the fibres of the projection to $A \times Y$ are also smooth, so Γ itself is smooth over $A_{sm} \times Y_{sm}$. Since field extensions in characteristic 0 are separable, the projection $\Gamma \setminus \Gamma_{sing} \to G$ is smooth over a nonempty open set of G, where we denoted by Γ_{sing} the singular locus of Γ. That is, the general fibre of the projection of Γ to G is smooth outside Γ_{sing}. If the projection of Γ_{sing} to G is not dominant,then $\varphi^{-1}(gA)$ is smooth for general g.

To complete the proof of generic transversality, we may assume that the projection $\Gamma_{sing} \to G$ is dominant. Since G is smooth, the principal ideal theorem shows that every component of every fibre of $\Gamma \to G$ has codimension less than or equal to the dimension of G, and thus every component of the general fibre has codimension exactly $\dim G$ in Γ. Since $\Gamma_{sing} \to G$ is dominant, its general fibre has dimension $\dim \Gamma_{sing} - \dim G < \dim \Gamma - \dim G$, so no component of a general fibre can be contained in Γ_{sing}. Thus $\varphi^{-1}(gA)$ is generically reduced for general $g \in G$.

(3.) We will prove this part only for the case where G is a product of copies of GL_n, as this is the only case it will be used. For the general result, see [Bor91, Theorem 18.2].

In this case G is an open set in a product M of vector spaces of matrices. Let L be the line joining 1 to g in M. The subvariety

$$Z = \left\{(g, x) \in (G \cap L) \times X \mid g^{-1}x \in A\right\}$$

gives a rational equivalence between A and gA and this proves the third part of the theorem. □

3.2 Techniques and computations

Definition 3.8. Let X be a scheme, we define the *fundamental class* of X, the class:

$$[X] \in A(X).$$

This is always nonzero, and we have the following:

Proposition 3.2.1. *Let X be a scheme. Then*

1. $A(X) = A(X_{red})$;
2. *if X is irreducible of dimension k, then $A_k(X) \cong \mathbb{Z}$ and it is generated by the fundamental class of X. More generally, if X is not irreducible and $X_1, X_2, \dots, X_m$ are its irreducible components, then the classes $[X_i]$ generate a free abelian subgroup of rank m in $A(X)$.*

Proof. (1.) Since both groups of cycles and of rational equivalences are generated by varieties, we have: $Z(X) = Z(X_{red})$ and $\mathrm{Rat}(X) = \mathrm{Rat}(X_{red})$, hence the claim.
(2.) By definition, the classes $[X_i]$ are among the generators of $A(X)$. Moreover, the group $\mathrm{Rat}(X)$ is generated by varieties in $\mathbb{P}^1 \times X$ each of which is contained in some $\mathbb{P}^1 \times X_i$. □

Example 3.2. *This result allows to compute the Chow ring in the particular case of zero-dimensional schemes. From the previous proposition it follows that the only nonzero Chow group of a zero-dimensional scheme is the free abelian group generated by its irreducible components, hence by the points in the support of the zero-dimensional scheme.*

Example 3.3 (Divisors)**.** *For 1-dimensional varieties, it is possible to see that the Chow group of 0-cycles on a curve is the divisor class group.*

In fact the group of cycles rationally equivalent to 0 in $Rat(X)$ can be expressed in terms of divisor classes.

Let us first suppose X is an affine variety, and let $f \in \mathcal{O}_X$ be a nonzero function on X. Then by Krull's principal ideal theorem the irreducible components of the subscheme defined by f are all of codimension 1, hence the cycle defined by this subscheme is a divisor, called the divisor of f, denoted as $Div(f)$.

Hence, if Y is any irreducible codimension-1 subscheme of X, and we denote as $ord_Y(f)$ the order of vanishing of f along Y, we have:

$$Div(f) = \sum_{\substack{Y \subseteq X\, irreducible \\ \operatorname{codim}_X Y = 1}} ord_Y(f)\langle Y \rangle.$$

Given $f, g \in \mathcal{O}_X$ and $\alpha = f/g$, we can define

$$Div(\alpha) = Div(f/g) := Div(f) - Div(g),$$

which is well defined.

It remains only to extend the definition of divisor associated to a rational function to varieties that are not affine; the field of rational functions on a variety X is isomorphic to the field of rational functions on any open affine subset $U \subseteq X$, hence, if α is a rational function on X, then we get a divisor $Div(\alpha_{|U})$ for every open affine U by restricting α. These divisors agree on the overlaps then it is well defined a divisor $Div(\alpha)$ on X.

In conclusion, to each rational function α on X it is possible to associate a divisor on X: $Div(\alpha)$.

The next proposition expresses how the rational group can be described in terms of the divisors of rational functions:

Proposition 3.2.2. *Let X be a scheme, then the group $Rat(X) \subseteq Z(X)$ is generated by all divisors of rational functions on all subvarieties of X. In particular, if X is irreducible of dimension n, then $A_{n-1}(X)$ is isomorphic to the divisor class group of X.*

Proof. For a proof see [Ful98, Proposition 1.6]. □

Example 3.4. *By the previous Proposition 3.2.2, it follows that two 0-cycles on a given curve C are rationally equivalent if and only if they differ by the divisor of a rational function. In particular, the cycles associated to two points are rationally equivalent if and only if there is a rational function defining such a rational equivalence, and this is equivalent to say that C is birational to $\mathbb{P}^1$.*

We conclude this section with the following proposition that characterizes the Chow ring of affine spaces:

Proposition 3.2.3. *Let $\mathbb{A}^n$ the affine n-space, we have:*

$$A(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n].$$

Proof. Let $Y \subseteq \mathbb{A}^n$ be a proper subvariety not containing the origin of the affine space, and let $z_1, \dots, z_n$ affine coordinates. Let define:

$$W = \{(t, tz) \in (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n \mid z \in Y\} = V(\{f(z/t) \mid f(z) \text{ vanishes on } Y\}).$$

The fibre of W over a point $t \in \mathbb{A}^1 \setminus \{0\}$ is tY, that is the image of Y under the automorphism of $\mathbb{A}^n$ defined by the multiplication by t.

Let now consider $\overline{W}$ be the closure of W in $\mathbb{P}^1 \times \mathbb{A}^n$; since W is irreducible in an open and dense subset, then also $\overline{W}$ it is.

Since the origin does not lie in Y, there is some polynomial $g(z)$ that vanishes on Y and has nonzero constant term c; then, the function $G(t, z) = g(z/t)$ on $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n$ extends to a regular function on $(\mathbb{P}^1 \setminus \{0\}) \times \mathbb{A}^n$ with constant value c on the fibre $\{\infty\} \times \mathbb{A}^n$. Thus, the fibre of $\overline{W}$ over the point $t = \infty$ is empty and this proves the equivalence:

$$Y \sim 0.$$

□

3.3 Mayer-Vietoris and excision

The following section is dedicated to excision and Mayer-Vietoris theorems: these theorems provide a couple of basic tools which allow to calculate the Chow rings of many varieties. Using these results it is easy to find in fact generators for the Chow groups of projective spaces and Grassmannians.

Proposition 3.3.1. *Let X be a scheme. Then,*

1. *(Mayer-Vietoris) If X_1, X_2 are closed subschemes of X, then there exists a right exact sequence:*

$$A(X_1 \cap X_2) \longrightarrow A(X_1) \oplus A(X_2) \longrightarrow A(X_1 \cup X_2) \longrightarrow 0;$$

2. *(Excision) If $Y \subseteq X$ is a closed subscheme and $U = X \setminus Y$ is its complement, then the inclusion and the restriction map of cycles give the following right exact sequence:*

$$A(Y) \longrightarrow A(X) \longrightarrow A(U) \longrightarrow 0.$$

Moreover, if X is smooth, then the map $A(X) \to A(U)$ is a ring homomorphism.

Remark 3.6. As a preliminary, let us observe that the definition of the Chow group can be reformulated by saying that there exists a right exact sequence:

$$Z(\mathbb{P}^1 \times X) \longrightarrow Z(X) \longrightarrow A(X) \longrightarrow 0,$$

where the left-hand map sends a subvariety $\Phi \subseteq \mathbb{P}^1 \times X$ to 0 if Φ is contained in a fibre, and to

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle$$

otherwise.

Proof. (2.) There is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z(Y\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial_Y"] & Z(X\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial_X"] & Z(U\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial_U"] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & Z(U) \arrow[r] \arrow[d, "\partial_U"] & 0 \\
& A(Y) \arrow[r] \arrow[d] & A(X) \arrow[r] \arrow[d] & A(U) \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

in fact, the columns are exact by the previous remark, the map $Z(Y) \to Z(X)$ takes the class $[A] \in Z(Y)$, for a given $A \subseteq Y$ subvariety, to $[A]$ itself considered as a class in X. Similarly for the map $Z(Y\times\mathbb{P}^1) \to Z(X\times\mathbb{P}^1)$.

The map $Z(X) \to Z(U)$ is defined as the map that takes each free generator $[A]$ to the generator $[A\cap U]$; similarly for $Z(X\times\mathbb{P}^1) \to Z(U\times\mathbb{P}^1)$. Moreover, the second and the third rows are exact trivially.

The exactness of the first row and the surjectivity of the map $A(X) \to A(U)$ the follows by diagram chasing, and this yields the thesis of part 2.
(1.) Let now $Y = X_1 \cap Y_2$, and $X = X_1 \cup X_2$. We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z(Y\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial"] & Z(X_1\times\mathbb{P}^1)\oplus Z(X_2\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial\oplus\partial"] & Z(X\times\mathbb{P}^1) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X_1)\oplus Z(X_2) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & 0 \\
& A(Y) \arrow[r] & A(X_1)\oplus A(X_2) \arrow[r] & A(X) \arrow[r] & 0
\end{tikzcd}$$

Arguing as in the previous part we obtain the thesis. □

Corollary 3.3.2. *Let $U \subseteq \mathbb{A}^n$ a nonempty open subset, then*

$$A(U) = A_n(U) = \mathbb{Z}[U].$$

In general it is quite difficult to compute Chow groups of a given variety and the knowledge of them is often very partial; however, when the variety admits an affine stratification we can know them completely. This is precisely the case of projective spaces and Grassmannians.

Let us recall the definition of affine stratification; this is a generalization of the idea introduced it in the particular case of spaces $M_{g,n}$ in Theorem 2.5.3 and Definition 2.15.

Definition 3.9. • A scheme X is said to admit a *stratification* if X is the disjoint union of a finite collection of irreducible, locally closed subschemes U_i:

$$X = \bigcup_{i \in \mathcal{I}} U_i$$

and, in addition, the closure of each U_i is a union of U_j, or equivalently, if $\overline{U_i}$ meets U_j, then $U_j \subseteq \overline{U_i}$.

- The sets U_i are called the *strata* of the stratification.
- The closures $Y_i = \overline{U_i}$ are called the *closed strata*.

Remark 3.7. Let us observe that the stratification can be recovered from the closed strata Y_i, in fact:

$$U_i = Y_i \setminus \bigcup_{Y_j \not\subseteq Y_i} Y_j.$$

Definition 3.10. A stratification of X, any scheme, with strata U_i is said to be:

- *affine* if each open stratum is isomorphic to an affine space $\mathbb{A}^k$;
- *quasi-affine* if each stratum U_i is isomorphic to an open subset of some affine space.

Example 3.5. *A flag of projective spaces:*

$$\mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \cdots \subseteq \mathbb{P}^n$$

gives an affine stratification of $\mathbb{P}^n$; the closed strata are the spaces $\mathbb{P}^i$ and the open strata are the affine spaces:

$$U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1} \cong \mathbb{A}^i.$$

Proposition 3.3.3. *Let X be a scheme which admits a quasi-affine stratification, then $A(X)$ is generated by the classes of closed strata.*

Proof. Let us prove the proposition by induction on the number of the strata U_i. The inductive basis for a single stratum is precisely Corollary 3.3.2. Let us prove now the inductive step.

Let U_0 be a minimal stratum. Since the closure of U_0 is a union of strata, then U_0 must be closed. Hence, the open subset $U = X \setminus U_0$ is stratified by the strata other than U_0.

By induction, $A(U)$ is generated by the classes of the closures of these strata, and by Corollary 3.3.2, $A(U_0)$ is generated by the class $[U_0]$. Now, by excision, the sequence:

$$\mathbb{Z} \cdot [U_0] = A(U_0) \longrightarrow A(X) \longrightarrow A(X \setminus U_0) \longrightarrow 0$$

is right exact. Since the classes in $A(U)$ of the closed strata in U comes from the classes of the same closed strata in X, it follows that $A(X)$ is generated by the classes of the closed strata as we wanted to prove. $\square$

In the case the stratification is an affine stratification we could have also more:

Theorem 3.3.4. *The classes of the strata in an affine stratification of a scheme form a basis for the Chow ring $A(X)$.*

3.4 Proper pushforward and flat pullback

From Proposition 3.2.2, it follows that if $Y \subseteq X$ is a closed subscheme, then the identification of cycles on $\mathbb{P}^1 \times Y$ with cycles on $\mathbb{P}^1 \times X$ induces a well defined map between $\mathrm{Rat}(Y) \to \mathrm{Rat}(X)$, and thus a map $A(Y) \to A(X)$.

Hence the inclusion

$$i : Y \to X$$

induces a map

$$A(Y) \to A(X).$$

If we consider now the intersection of a subvariety of X with the open set $U = X \setminus Y$, we obtain a subvariety of U, so there is a restriction homomorphism $Z(X) \to Z(U)$. The rational equivalences restrict too, so we get a homomorphism of Chow groups

$$A(X) \to A(U).$$

This suggests that Chow groups could have a double behaviour with respect to morphisms of varieties: a covariant and a contravariant behaviour depending if we think them respectively as the analogous of homology or cohomology groups.

Let us recall that a smooth complex projective variety of dimension n is a compact oriented $2n$-manifold, in particular $H_{2m}(X, \mathbb{Z})$ can be identified canonically with $H^{2n-2m}(X, \mathbb{Z})$.

If we think $A(X)$ as the analogous of the ring $H_*(X, \mathbb{Z})$, then we should expect the functor that assigns to each variety the m-th Chow group is a covariant functor from the category of smooth projective varieties to the category of groups, assigning to each morphism of varieties a morphism between the respective Chow groups. This allow to define a pushforward map preserving dimension.

If we think instead $A(X)$ as the analogous of $H^*(X, \mathbb{Z})$, in this case we should expect to obtain a contravariant functor, and a pullback map preserving the codimension.

Let us observe that the two maps of the previous examples constitute a a special case respectively of *pushforward* and of *pullback* of morphisms. In the next two sections these notions will be explained in some detail.

Proper pushforward

Let $f : Y \to X$ be a proper map of schemes, then the image of a subvariety $A \subseteq Y$ is a subvariety $f(A) \subseteq X$. We are expected to use the morphism f to send any cycle in $A(Y)$ to a cycle in $A(X)$.

If $A \subseteq Y$ is a subvariety and $\dim A = \dim f(A)$ then $f_{|A} : A \to f(A)$ is generically finite, i.e. the fraction field $k(A)$ is a finite field extension of $k(f(A))$. Geometrically, this condition can be rephrased by saying that, given $x \in f(A)$, the preimage $Y = f_{|A}^{-1}(x)$ in A is a finite scheme.

In this case we have that the degree of the field extension:

$$n = [k(A) : k(f(A))]$$

is equal to the degree of Y over x for a dense open subset of $x \in f(A)$. This common value is called the degree of the covering of $f(A)$ by A. We can then define the pushforward as follows counting $f(A)$ with multiplicity n in the pushforward cycle:

Definition 3.11 (Pushforward of cycles)**.** Let $f : Y \to X$ be a proper morphism of schemes, let $A \subseteq Y$ be a subvariety.

1. If $f(A)$ has strictly lower dimension than A, we define: $f_*\langle A\rangle = 0$.
2. If $\dim f(A) = \dim A$ and $f_{|A}$ has degree n, then we set $f_*\langle A\rangle = n \cdot \langle f(A)\rangle$.
3. For a generical cycle, we extend f_* by linearity setting:

$$f_*\left(\sum m_i\langle A_i\rangle\right) = \sum m_i f_*\langle A_i\rangle.$$

This definition is compatible with rational equivalence and we have the following:

Theorem 3.4.1. *Let $f : Y \to X$ be a proper morphism of schemes, then the map $f_* : Z(Y) \to Z(X)$ defined above induces a map of groups:*

$$f_* : A_k(Y) \to A_k(X) \qquad \textit{for each } k = 0, \dots, n$$

called pushforward *maps.*

Proof. For a proof see [Ful98, Section 1.4]. □

This theorem guarantees the existence of a pushforward map

$$f_* : A(X) \to A(X)$$

preserving dimension.

We conclude this section with the following proposition that gives a sufficient condition for a class in $A(X)$ being nonzero.

Proposition 3.4.2. *Let X be a proper scheme over Spec*$(\mathbb{C})$; *there is a unique map:*

$$\begin{array}{rcl} \deg : A(X) & \to & \mathbb{Z} \\ [p] & \mapsto & 1 \\ [Y] & \mapsto & 0 \end{array}$$

taking the class of each closed point $p \in X$ to 1 and vanishing on the class of any cycle of pure dimension greater than 0.

Remark 3.8. If A is a k-dimensional subvariety of a smooth projective variety X and B is a k-codimensional subvariety of X such that $A \cap B$ is finite and nonempty, then the map:

$$\begin{array}{rcl} A_k(X) & \to & \mathbb{Z} \\ [Z] & \mapsto & \deg([Z][B]) \end{array}$$

sends $[A]$ to a nonzero integer, and, in particular, no integer multiple of $[A]$ can be 0.

Flat pullback

Before defining the notion of pullback we have to give the following:

Definition 3.12. Let $f : Y \to X$ be a morphism of smooth varieties. A subvariety $A \subseteq X$ is said to be *generically transverse* to f if the preimage $f^{-1}(A)$ is generically reduced and $\operatorname{codim}_Y(f^{-1}(A)) = \operatorname{codim}_X(A)$.

Then we can state the following fundamental theorem:

Theorem 3.4.3. *Let $f : Y \to X$ be a map of smooth quasi-projective varieties.*

1. *There is a unique map of groups:*

$$\begin{array}{rcl} f^*: A^c(X) & \to & A^c(Y) \\ [A] & \mapsto & f^*([A]) = \left[f^{-1}(A)\right] \end{array}$$

for every subvariety A generically transverse to f. The map f^* is a ring homomorphism and makes A into a contravariant functor from the category of smooth projective varieties to the category of graded rings.

2. *(Push-pull formula) The map $f_*: A(Y) \to A(X)$ is a map of graded modules over the graded ring $A(X)$. Hence, if $\alpha \in A^k(X)$ and $\beta \in A_l(Y)$, then:*

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta \in A_{l-k}(X).$$

Proof. See [Ful98, Section 1.7]. □

Example 3.6. *Let us consider the case of the inclusion map of a closed subvariety $Y \subseteq X$, and let us suppose that X and Y are smooth. In this case we have that, if A is any subvariety of X, the element $[A][Y]$ is represented by the same cycle as $i^*([A])$. More precisely, we have the following equality:*

$$[A][Y] = i_*(i^*[A]).$$

The flat case

Let us finally consider the flat case, i.e. the case of a flat morphism of scheme $f: Y \to X$.

The flat case is simpler than the projective one almost for two reasons: the first is that the preimage of a subvariety of codimension c is always of codimension c, hence the first hypothesis to have that a variety is generically transverse to f is always satisfied, and the second is that rational functions on the target pull back to rational functions on the source.

We have for a flat morphism of schemes the following:

Theorem 3.4.4. *Let $f: Y \to X$ be a flat map of schemes. The map $f^*: A(X) \to A(Y)$ defined on cycles by setting:*

$$f^*(\langle A \rangle) = \langle f^{-1}(A) \rangle \qquad \textit{for every subvariety } A \subseteq X$$

preserves rational equivalence and thus induces a map of Chow groups preserving grading by codimension.

Proof. See [Ful98, Lemma 1.7.1] □

Remark 3.9. Obviously if X and Y are smooth and f is flat, the two pullback map agree.

3.5 Intersection multiplicity

In conclusion we have the following generalized version of Bézout's theorem:

Theorem 3.5.1. *Let $A, B \subseteq X$ be subvarieties of a smooth variety X such that every irreducible component C of the intersection $A \cap B$ has codimension* $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$. *For each such component C there is a positive integer $m_C(A, B)$, called the* intersection multiplicity *of A and B along C, such that:*

1. $[A]\,[B] = \sum m_C(A, B)\,[C] \in A(X)$;
2. *$m_C(A, B) = 1$ if and only if A and B intersect transversely at a general point of C;*
3. *$m_C(A, B)$ depends only on the local structure of A and B at a general point of C.*

Definition 3.13. Let A and B be two subschemes of a given scheme X; they are said to be *dimensionally transverse* if for every irreducible component C of $A \cap B$ we have: $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$.

Two cycles $A = \sum m_i A_i$ and $B = \sum n_j B_j$ are said to be dimensionally transverse if A_i and B_j are dimensionally transverse for every i, j.

The following proposition explains the relation between generic transversality and dimensional transversality.

Proposition 3.5.2. *Let A and B subschemes of a variety X; they are generically transverse if and only if they are dimensionally transverse and each irreducible component of $A \cap B$ contains a point where X is smooth and $A \cap B$ is reduced.*

Proof. Let A and B generically transverse. Then, by definition each irreducible component C of $A \cap B$ contains a smooth point p. Hence, C is smooth at p and thus, in particular C is reduced at p.

Viceversa, let C be an irreducible component of $A \cap B$. Since the set of smooth points of X is open, and since by hypothesis C contains one, the smooth points of X contained in C form an open and dense subset. But $A \cap B$ is generically reduced hence the open set where C is reduced is also dense, and the same is true for the sooth locus of C. This there is a point $p \in C$ that is smooth both on C and on C. It remains to check that A and B are smooth at p.

Let us consider the tangent space to C at the point p; it is the intersection of the tangent spaces T_pA and T_pB in T_pX. Since C and X are smooth at p,

$$\begin{aligned}\dim C = \dim T_pC &= \dim T_pA + \dim T_pB - \dim T_pX \\ &= \dim T_pA + \dim T_pB - \dim X.\end{aligned}$$

By hypothesis:

$$\dim C = \dim A + \dim B - \dim X.$$

Since $\dim T_p A \geq \dim A$ and $\dim T_p B \geq \dim B$, we must have necessarily that

$$\dim A = \dim T_p A \qquad \text{and} \qquad \dim B = \dim T_p B,$$

and this proves A and B are smooth at p as well. □

3.6 The Chow ring of $\mathbb{P}^n$

We conclude this chapter with a concrete example of Chow rings of a smooth algebraic variety. We computed above the Chow ring of affine space and of affine subvarieties; the case of projective space is much more interesting.

For the projective space we have the following theorem:

Theorem 3.6.1. *The Chow ring of $\mathbb{P}^n$ is*

$$A(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1})$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the rational equivalence class of a hyperplane; more generally the class of a variety of codimension k and degree d is $d\zeta^k$.

Remark 3.10. Notice that, in particular, the theorem implies that $A^m(\mathbb{P}^n) \cong \mathbb{Z}$ for $0 \leq m \leq n$, generated by the class of an $(n-m)$-plane.

From the previous theorem it is also possible to deduce the following two results:

Corollary 3.6.2. *A morphism from $\mathbb{P}^n$ to a quasi-projective variety of dimension strictly less than n is constant.*

Proof. Let $\varphi : \mathbb{P}^n \to X \subseteq \mathbb{P}^m$ be the map of the statement, which we may assume to be surjective onto X. The preimage of a general hyperplane section of X is disjoint from the preimage of a general point of X. But if $0 < \dim X < n$ then the preimage of a hyperplane section of X has dimension $n-1$ and the preimage of a point has dimension greater then 0. Since any two such subvarieties of $\mathbb{P}^n$ must intersect, this is a contradiction. □

Corollary 3.6.3. *If $X \subseteq \mathbb{P}^n$ is a variety of dimension m and degree d, then*

$$A_m(\mathbb{P}^n \setminus X) \cong \mathbb{Z}/(d),$$

while if $m < m' \leq n$ then

$$A_{m'}(\mathbb{P}^n \setminus X) \cong \mathbb{Z}.$$

In particular, m and d are determined by the isomorphism class of $\mathbb{P}^n \setminus X$.

Proof. By the excision we have the following exact sequences:

$$A_i(X) \longrightarrow A_i(\mathbb{P}^n) \longrightarrow A_i(\mathbb{P}^n \setminus X) \longrightarrow 0.$$

Furthermore $A_m(X) \cong \mathbb{Z}$ by Prop. 3.2.1, while $A_{m'}(X) = 0$ for $m < m' \leq n$. By Theorem 3.6.1, we have that $A_i(\mathbb{P}^n) \cong \mathbb{Z}$ for $0 \leq i \leq n$, and the image of the generator of $A_m(X)$ in $A_m(\mathbb{P}^n)$ is d times the generator of $A_i(\mathbb{P}^n)$. The result of the corollary then follows. $\square$

4 The tautological ring

In the previous chapters we introduced the notion of moduli space, focusing specifically on moduli spaces of curves, and the notion of Chow ring of an algebraic variety together with the main tools of intersection theory.

We can finally start to develop the intersection theory of moduli spaces in a more systematic way. The next steps will be the definition of the tautological classes, and the study of the relations among them; this is needed in order to be able to compute the intersection numbers between the classes.

These are the ideas presented in the first part of Mumford's article[1], the second part of it is dedicated to the special case of Chow ring of the moduli space $\overline{\mathcal{M}}_2$. The application of the general results in the case of the moduli space of curves of genus 2 will be discussed in Chapter 5 of the thesis.

Before starting the detailed discussion on the tautological ring, let us observe that moduli space can be considered from many and different points of view; they can in fact be defined also as smooth Deligne-Mumford stacks or as smooth complex orbifolds. From now on, we will refer to moduli spaces considering them as smooth orbifolds, and all the result stated will be thought in this context. The notion of orbifold is recalled in Appendix B, pag. 111.

From now one, when we refer to the Chow ring A^* or to the cohomology ring H^*, they are always taken with coefficients in $\mathbb{Q}$.

4.1 Tautological classes: an intuitive motivation

In order to study the topological structure of moduli spaces of curves, it is natural to look at their cohomology rings and Chow rings. Notice that, since both the generators and the relations of the Chow ring are subsets of the respective sets in cohomology, there is a natural map:

$$A^k \longrightarrow H^{2k}$$

[1] Mum83.

where the index for Chow groups corresponds to the complex codimension, hence its appears doubled in cohomology groups.

However, since in general homology and cohomology are difficult to treat, following the idea of Mumford, it appears that a fruitful approach to the study of full cohomology ring as well as the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ is to restrict our attention to a special subring, i.e. to the ring generated by some special classes. Mumford in fact, in the beginning of his article "*Towards an enumerative geometry of the moduli space of curves*[2]," writes:

> The goal of this paper is to formulate and to begin an exploration of the enumerative geometry of the set of all curves of arbitrary genus g. By this we mean setting up a Chow ring for the moduli space $\mathcal{M}_g$ and its compactification $\overline{\mathcal{M}}_g$, defining what seem to be the most important classes in this ring and calculating the class of some geometrically important loci in $\overline{\mathcal{M}}_g$ in terms of these classes. We take as a model for this the enumerative geometry of the Grassmannians.

Following this idea, Mumford defined the *tautological classes*, which consists, according to the description given by Ravi Vakil, in [Vak03], of *all the classes naturally coming from geometry* and this gives rise to a subring of the Chow ring, called the tautological subring, denoted by $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq A^*(\overline{\mathcal{M}}_{g,n})$.[3] All this, according again to Vakil's article should work for three reasons at least:

- many interesting questions in geometry boil down to the tautological subring;
- no natural questions seem to reduce to questions in the nontautological part of the Chow ring;
- the tautological subring has, at least conjecturally, a great deal of structure.

We expect, then, that any calculation on the Chow ring that comes from geometric instances will require only the knowledge of the tautological ring. Although it is known that for large g and n the rank of the tautological ring is much smaller than that of the full cohomology ring of $\overline{\mathcal{M}}_{g,n}$, most naturally geometrically defined classes happen to be tautological and it is actually not so simple to construct examples of nontautological classes in the Chow ring (for a construction of a nontautological class one can see the article of Graber and Pandharipande [GP01]).

[2] Mum83.

[3] Mumford in his article only defined tautological classes, as said in the introduction the tautological ring was introduced later by Faber and Pandharipande in [FP05].

Following Mumford's idea, we can generalize the description of the Chow ring for Grassmannians in order to identify a natural set of tautological classes inside the Chow ring of moduli spaces, and to find some *tautological relations*, that are relations between tautological classes. We are expected to obtain in this way a sort of presentation of the tautological ring.

There are different examples of classes that are expected to be tautological.

ψ-classes These classes are defined as Chern classes of certain natural vector bundles on the moduli space.

λ-classes Another natural vector bundle on moduli spaces is obtained by extending to the whole $\overline{\mathcal{M}}_{g,n}$ the *Hodge bundle* which associates to any smooth curve the g-dimensional vector space of differentials: we obtain then a vector bundle Λ on $\overline{\mathcal{M}}_{g,n}$; the Chern classes of this vector bundle define the λ-classes.

κ-classes These are the classes which appeared in the original definition of tautological ring given by Mumford, that can be obtained by push forward of ψ-classes by the projection from the universal curve; for these classes we will give an equivalent but different definition.

4.2 Preliminary notions

In the next sections we will introduce tautological classes; to do that we need several notions that will be collected in this section. More precisely we will define here: forgetful and attaching maps, the relative cotangent line bundle (pag. 71), the Hodge bundle (pag. 71) and Chern classes (pag 72).

Forgetful and attaching maps

Forgetful maps

The idea of a forgetful map is quite simple; we would like to assign to a genus g stable curve $(C, p_1, \dots, p_{n+m})$ the curve $(C, p_1, \dots, p_n)$, simply forgetting last m marked points. The problem is that the curve obtained may not be stable.

Let us assume that $2g - 2 + n > 0$, thus, either the curve $(C, p_1, \dots, p_n)$ is stable, or it has at least one genus 0 component with one or two special points. If this is the case, this component can be contracted to a point and we can now, for the curve obtained, ask again if it is stable or not. If not, we can find another component to contract. Since the number of irreducible components is finite and it decreases with each contraction, we get that we can obtain a stable curve after finitely many steps.

At the end we will obtain a stable curve $(\widehat{C}, \widehat{p}_1, \dots, \widehat{p}_n)$ together with a *stabilization map*:

$$(C, p_1, \dots, p_n) \to (\widehat{C}, \widehat{p}_1, \dots, \widehat{p}_n).$$

Definition 4.1. We define the *forgetful map*

$$p : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$$

as the map that assigns to a curve $(C, p_1, \dots, p_{n+m}) \in \overline{\mathcal{M}}_{g,n+m}$ the stabilization of the curve $(C, p_1, \dots, p_n)$.

We have the following:

Proposition 4.2.1. *The universal curve $\overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ and the forgetful map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ are isomorphic as families over $\overline{\mathcal{M}}_{g,n}$.*

Proof. See [Knu83]. □

Attaching maps

Let $I \sqcup J$ be a partition of the set $\{1, 2, \dots, n+2\}$ into two disjoint subsets such that $n+1 \in I$, $n+2 \in J$, and let $g_1, g_2 \in \mathbb{N}$ such that $g_1 + g_2 = g$.

Let us denote by $\overline{\mathcal{M}}_{g_1,I}$ the moduli space of stable curves whose marked points are labelled by the elements of I, and analogously with $\overline{\mathcal{M}}_{g_2,J}$ the moduli space of stable genus g_2 curves with marked points labelled by the elements of J.

Definition 4.2. We define the *attaching map of separating type*

$$q : \overline{\mathcal{M}}_{g_1,I} \times \overline{\mathcal{M}}_{g_2,J} \to \overline{\mathcal{M}}_{g,n}$$

to be the map which assigns to two stable curves the stable curve obtained by identifying the marked points labelled with numbers $n+1$ and $n+2$.

We define the *attaching map of nonseparating type*

$$q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$$

to be the map which assigns to a stable curve the stable curve obtained by identifying the marked points labelled with numbers $n+1$ and $n+2$.

The relative cotangent line bundle

Let us consider
$$p\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$
the universal curve over $\overline{\mathcal{M}}_{g,n}$; let $\Delta \subseteq \overline{\mathcal{C}}_{g,n}$ be the set of nodes in the singular fibres.

Over $\overline{\mathcal{C}}_{g,n} \setminus \Delta$ there is a holomorphic line bundle $\mathcal{L}$ cotangent to the fibres of the universal curve. We claim that this line bundle can be extended to the whole universal curve.

Locally around the node we can choose coordinates such that the node is of the form $p\colon (x, y) \to xy$. The line bundle $\mathcal{L}$ is then generated by the sections $\frac{dx}{x}$ and $\frac{dy}{y}$ modulo the relation:
$$\frac{d(xy)}{xy} = \frac{dx \cdot y}{xy} + \frac{x \cdot dy}{xy} = \frac{dx}{x} + \frac{dy}{y} = 0.$$
Since the restriction of the 1-form $d(xy)$ on every fibre of p vanishes, the line bundle thus obtained is indeed identified with the cotangent line bundle to the fibres of $\overline{\mathcal{C}}_{g,n}$.

We can then give the

Definition 4.3 (Relative cotangent line bundle)**.** The extension of the line bundle $\mathcal{L}$ to the whole universal curve is called the *relative cotangent line bundle.*

Let us observe that the restriction of the line bundle $\mathcal{L}$ to a fibre C of the universal curve is a line bundle over C. In particular, if the curve C is smooth, then $\mathcal{L}_{|C}$ is the cotangent line bundle and its holomorphic sections are the abelian differentials, i.e. holomorphic differential 1-forms. Generalizing the definition of abelian differential to stable curves we are lead to the definition of the *Hodge bundle*.

The Hodge bundle

Definition 4.4 (Abelian differential)**.** We define an *abelian differential* on a stable curve C as a meromorphic 1-form α on each component of C satisfying the following properties:

- the only poles of α are the nodes of C;
- α has at most simple poles;
- the residues of the poles on two branches meeting at a node are opposite to each other.

Example 4.1. *Let us consider the Riemann sphere with two points identified and a marked point. On the Riemann sphere we have the coordinate z such that the marked point is at $z = 1$, and the identified points are situated at $z = 0$ and $z = \infty$. In this coordinate, the abelian differentials on the curve have the form: $\lambda\frac{dz}{z}$ and the residues at $z = 0$ and $z = \infty$ are equal to λ and $-\lambda$ respectively.*

Since the space of abelian differentials has the same dimension for each fibre of the universal curve, we can give the following definition:

Definition 4.5 (Hodge bundle)**.** We define the *Hodge bundle* Λ over $\overline{\mathcal{M}}_{g,n}$ to be the rank g vector bundle whose fibre over $t \in \overline{\mathcal{M}}_{g,n}$ is given by the space of abelian differentials on the curve associated to t.

Remark 4.1. More generally it is possible to consider meromorphic forms on a stable curve with poles of orders $k_1, k_2, \dots, k_n$ respectively at marked points $x_1, x_2, \dots, x_n$; in this case we are considering meromorphic sections of the cotangent line bundle $\mathcal{L}$ with poles as in the previous definition on the nodes and we will say shortly they are the sections of $\mathcal{L}(\sum k_i x_i)$.

Chern classes

Chern classes are particular characteristic classes which we associate to complex vector bundles.

Let $p : E \to M$, be a complex vector bundle of rank r, with base space M a compact, orientable, smooth manifold of real dimension n.

Definition 4.6.

- A section of the vector bundle p is called a *generic section* if it intersects the zero section $\mathcal{O}_M$ transversally.
- Two sections are said to have *transversal intersection* if at each point of the intersection, the tangent spaces of the sections generate the tangent space of the ambient space.

We can define Chern classes in terms of the zero sets of generic sections; for a generic section s, we denote with $Z(s)$ the zero set of s. Since s is transverse to the zero section, $Z(s)$ is a submanifold of M of complex codimension r, and of real codimension $2r$.

By Poincaré duality, the class $[Z(s)] \in H_{n-2r}(M)$ corresponds to a cohomology class in $H^{2r}(M)$.

Definition 4.7 (Chern class)**.** Let $p : E \to M$ be a vector bundle of rank r with base space M, let s be a generic section, we define:

the Euler class or the r^{th} Chern class of the vector bundle:

$$e(E) = \mathrm{c}_r(E) := [Z(s)] \in A^r(M) \subseteq H^{2r}(M);$$

the i^{th} Chern class

$$\mathrm{c}_i(E) = \left[Z(s_1 \wedge s_2 \wedge \cdots \wedge s_{r-i+1})\right] \in A^i(M) \subseteq H^{2i}(M)$$

for $s_1, s_2, \dots, s_{r-i+1}$ generic sections.

Remark 4.2. Notice that $Z(s_1 \wedge \cdots \wedge s_{r-i+1})$ can be viewed as the set of points $p \in M$ where the sections $s_1(p), s_2(p), \dots, s_{r-i+1}(p)$ become linearly dependent.

We also have that:

$$\mathrm{c}_1(E) = \left[Z(s_1 \wedge \cdots \wedge s_r)\right].$$

We set then

$$\mathrm{c}_0(E) = 1$$

and we define:

Definition 4.8. The sum

$$\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \mathrm{c}_2(E) + \cdots \in H^*(M, \mathbb{Z})$$

is called the *total Chern class* of E.

The following theorem collects the main properties of Chern classes:

Theorem 4.2.2. *There is a unique sequence of functions* $\mathrm{c}_1, \mathrm{c}_2, \dots$ *assigning to each complex vector bundle* $p: E \to M$ *a class* $\mathrm{c}_i \in H^{2i}(M, \mathbb{Z})$ *depending only on the isomorphism type of E and satisfying:*

1. $\mathrm{c}_i(f^*E) = f^*\left(\mathrm{c}_i(E)\right)$ *for a pullback* f^*E;
2. $\mathrm{c}(E_1 \oplus \mathrm{c}(E_2)) = \mathrm{c}(E_1) \cup \mathrm{c}(E_2)$;
3. $\mathrm{c}_i(E) = 0$ *if* $i > \mathrm{rk}\, E$.

4.3 The ψ-classes

We can finally go back to the initial program and introduce some tautological classes; let us start with the ψ-classes.

Definition 4.9 (Cotangent line at marked points)**.** Let $\mathcal{L}$ be the cotangent line bundle introduced in Section 4.2, and let

$$p : \overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$

be the universal curve over $\overline{\mathcal{M}}_{g,n}$; for every $i = 1, \dots, n$ we can consider

$$s_i : \overline{\mathcal{M}}_{g,n} \to \overline{C}_{g,n}$$

to be the section of $\mathcal{L}$ corresponding to the i^{th} marked point, so that $p \circ s_i = \mathrm{id}$. Then we define the line bundle:

$$\mathcal{L}_i = s_i^*(\mathcal{L}).$$

Remark 4.3. We obtained in this way , $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ holomorphic line bundles over $\mathcal{M}_{g,n}$. The fibre of $\mathcal{L}_i$ over a point $x \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line to the curve C_x at the i^{th} marked point.

Definition 4.10 (ψ-classes)**.** We define the ψ-classes as the first Chern classes of the line bundles $\mathcal{L}_i$,

$$\psi_i = c_1(\mathcal{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

ψ-classes in genus 0

For moduli spaces $\overline{\mathcal{M}}_{0,n}$ of curves of genus 0, it is possible to construct an explicit section of the line bundle $\mathcal{L}_i$ and consequently to express its first Chern class ψ_i as a linear combination of divisors.

Given pairwise distinct $i, j, k \in \{1, 2, \dots, n\}$, let us denote by $\delta_{i|jk}$ the set of stable genus 0 curves with a node separating the i^{th} marked point from the j^{th} and the k^{th} (see Fig. 4.1).

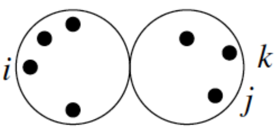


Figure 4.1: An element of the divisor $\delta_{i|jk}$

The set $\delta_{i|jk}$ is a divisor on $\overline{\mathcal{M}}_{0,n}$ and we denote by $\left[\delta_{i|jk}\right] \in A^1(\overline{\mathcal{M}}_{0,n})$ its class in the Chow ring. Then we have the following:

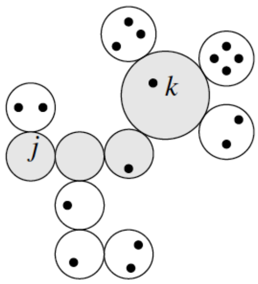
Proposition 4.3.1. *On $\overline{\mathcal{M}}_{0,n}$ we have, for any $j,k \in \{1,2,\dots,n\} \setminus \{i\}$, the following equality:*

$$\psi_i = \left[\delta_{i|jk}\right]$$

Proof. In order to construct a holomorphic section of $\mathcal{L}_i$, we construct first a meromorphic section α of the cotangent line bundle $\mathcal{L}$ over the universal curve and then we take its restriction to the i^{th} section s_i; the class ψ_i will be then represented by the divisors of its zeroes according to the definition of Chern class.

Let us observe that each fibre of the universal curve is a stable curve, and on each stable curve there is a unique meromorphic 1-form with simple poles at the j^{th} and k^{th} marked points with residues 1 and -1 respectively, the existence being guaranteed by *Weierstrass' product theorem* for compact Riemann surfaces. This form gives a section of $\mathcal{L}$ on each stable curve and the collection of all these defines the desired section α on $\mathcal{L}$ over the whole universal curve.

Let us now consider the following picture taken from [Zvo14]:



a stable genus 0 curve C is a tree of spheres; one of these spheres contains the j^{th} marked point and another one contains the k^{th}. Since the curve is connected, there exists a chain of spheres connecting these two.

On every sphere of the chain, the 1-form $\alpha_{|C}$ has two simple poles, one with residue 1 (at the j^{th} marked point or at the node leading to the sphere with it) and one with residue -1 (at the k^{th} marked point or at the node leading to the sphere with it). The 1-form then vanishes on the spheres that do not belong to the chain.

In conclusion α determines a nonvanishing cotangent vector at the i^{th} marked point if and only if the i^{th} marked point belongs to this chain; equivalently, we have that $\alpha_{|s_i}$ vanishes if and only if the curve C contains a node which separates the i^{th} marked point from j^{th} and k^{th} ones. This is equivalent to say that the curve C belongs to $\delta_{i|jk}$. Moreover, by a local coordinates computation it is possible to check that $\alpha_{|s_i}$ has a simple zero along $\delta_{i|jk}$ and this concludes the proof. □

Lemma 4.3.2. *Let D_i be the divisor of the i^{th} special section in the universal curve $p\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Then we have: $p_*(D_i^{k+1}) = (-\psi_i)^k$.*

4.4 Tautological classes on moduli space

In the previous section we defined ψ-classes on the moduli space $\overline{\mathcal{M}}_{g,n}$; we are going to define in this chapter the other classes introduced in the presentation of the chapter. Before to do this, we need to define some tautological classes on the universal curve $\overline{C}_{g,n}$, the tautological classes on $\overline{\mathcal{M}}_{g,n}$ will be constructed by pushforward of these classes.

Definition 4.11 (Tautological classes on $\overline{C}_{g,n}$).

- D_i is defined as the divisor given by the ith section of the universal curve. More precisely, D_i is the divisor such that the intersection of it with a fibre C of $\overline{C}_{g,n}$ is the i^{th} marked point on C. By abuse of notation we denote by $D_i \in A^1(\overline{C}_{g,n}, \mathbb{Q})$ also the class of the divisor.
- D is the divisor defined as $D = \sum_{i=1}^n D_i$.
- $K = \mathrm{c}_1(\mathcal{L}^{\log}) \in A^1(\overline{C}_{g,n}, \mathbb{Q})$, where $\mathcal{L}^{\log} = \mathcal{L}(D) = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is the line bundle $\mathcal{L}$ twisted by the divisor D.
- Δ is defined as the codimension 2 subvariety of $\overline{C}_{g,n}$ consisting of the nodes of the singular fibres. By abuse of notation we denote by $\Delta \in A^2(\overline{C}_{g,n}, \mathbb{Q})$ also its class.
- Let N be the normal vector bundle to Δ in $\overline{C}_{g,n}$. Then we define:

$$\Delta_{k,l} = (-\,\mathrm{c}_1(N))^k \Delta^{l+1}.$$

Before introducing the tautological classes on $\overline{\mathcal{M}}_{g,n}$ we just recall the following proposition on the intersection that will be used later:

Proposition 4.4.1. *For al $1 \leq i, j \leq n$, with $i \neq j$ we have:*

$$KD_i = D_iD_j = K\Delta = D_i\Delta = 0 \in A^*(\overline{C}_{g,n}).$$

Proof. The divisors D_i and D_j clearly do not intersect so the intersection of the corresponding classes vanishes and this proves $D_iD_j = 0$. Similarly, since the marked points are distinct from the nodes, the divisor D_i does not intersect Δ, hence also $D_i\Delta = 0$.

For the remaining equalities, notice that the restriction of the line bundle $\mathcal{L}^{\log}$ to D_i is trivial; indeed, the sections of $\mathcal{L}^{\log}$ are 1-forms with simple poles at the marked points, and the fibre at the marked points is the line of residues so it is canonically identified with $\mathbb{C}$.

So now, the intersection KD_i is the first Chern class of the restriction of $\mathcal{L}^{\log}$ to D_i, hence it vanishes. Finally, $\mathcal{L}^{\log}$ restricted to Δ is not necessarily trivial, however its pullback to the double-sheeted covering $\widetilde{\Delta}$ is trivial. Therefore $K\Delta = 0$. □

The tautological classes on $\overline{\mathcal{M}}_{g,n}$

We can now concentrate our attention on the tautological classes on the moduli space $\overline{\mathcal{M}}_{g,n}$. Let $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve. On the moduli space $\overline{\mathcal{M}}_{g,n}$ we can define the following classes:

κ-classes $\kappa_m = p_*(K^{m+1}) \in A^m(\overline{\mathcal{M}}_{g,n})$.

ψ-classes $\psi_i = -p_*(D_i^2) \in A^1(\overline{\mathcal{M}}_{g,n})$.

δ-classes $\delta_{k,l} = p_*(\Delta_{k,l}) \in A^{2l+k+1}(\overline{\mathcal{M}}_{g,n})$.

λ-classes $\lambda_i = \mathrm{c}_i(\Lambda) \in A^i(\overline{\mathcal{M}}_{g,n})$ where Λ is the Hodge bundle and c_i is the i^{th} Chern class.

Remark 4.4. Let us observe that, with the exception of the λ-classes, the tautological classes on $\overline{\mathcal{M}}_{g,n}$ are pushforwards of tautological classes on $\overline{\mathcal{C}}_{g,n}$ and their products.

Remark 4.5. Let us also notice that by Lemma 4.3.2, the definition of ψ-classes given here coincide with Definition 4.10 given above.

It is also possible to check that this definition of κ-classes coincide with the definition given by Mumford as discussed in the first introductory section.

Example 4.2. *By definition* $\delta_{0,0} = p_*(\Delta_{0,0})$, *where*

$$\Delta_{0,0} = \Delta$$

is the class corresponding to the divisor of nodes. Hence $\delta_{0,0}$ *is the pushforward of* Δ *via* p.

4.5 The tautological ring

In the first introductory Section 4.1 we described the idea of Mumford to reduce the study of intersection theory on moduli spaces to the study of intersection of tautological classes.

After the definition of tautological classes, we can then define the tautological ring, according to the definition given by Faber and Pandharipande in [FP11]:

Definition 4.12 (Tautological ring)**.** We define the system of tautological rings:

$$\left(R^*(\overline{\mathcal{M}}_{g,n}) \subseteq A^*(\overline{\mathcal{M}}_{g,n}) \right)_{g,n}$$

as the smallest system of $\mathbb{Q}$-subalgebras satisfying the following: the system is closed under pushforward by forgetful and attaching maps.

This definition can be extended:

Definition 4.13. We define the tautological ring of any open subset U of $\overline{\mathcal{M}}_{g,n}$ to be the image of $R * (\overline{\mathcal{M}}_{g,n})$ under the restriction map $A^*(\overline{\mathcal{M}}_{g,n}) \to A^*(U)$.

Remark 4.6. The first definition of tautological ring was given in [FP05] and it was the following.

Definition 4.14. We define the system of tautological rings:

$$\left(R^*(\overline{\mathcal{M}}_{g,n}) \subseteq A^*(\overline{\mathcal{M}}_{g,n}) \right)_{g,n}$$

as the smallest system of $\mathbb{Q}$-subalgebras satisfying the following:

1. $\psi_1, \psi_2, \dots, \psi_n \in R^*(\overline{\mathcal{M}}_{g,n})$;

2. the system is closed under pushforward by forgetful and attaching maps.

However, Faber and Pandharipande in [FP11] observed that condition 2. in Def. 4.14 implies 1. hence the definition can be reformulated as in Def. 4.12.

There is a canonical quotient:

$$\mathbb{Q}\left[\psi_1, \psi_2, \dots, \psi_n\right] \xrightarrow{q} R^*(\overline{\mathcal{M}}_{g,n}) \to 0;$$

the kernel of q is the ideal of relations among the ψ classes.

Let us notice that, for example, the classes represented by boundary strata all lie in the tautological ring since they are images of the fundamental class under attaching maps. Also the self-intersection of a boundary stratum lies in the tautological rings for the same reason.

In the previous section we introduced some classes that have been called tautological, the following theorem justifies the name:

Theorem 4.5.1. *The classes κ_m, $\delta_{k,l}$ and λ_i all lie in the tautological ring.*

Proof. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. In this case we have that

$$\kappa_m = \pi_*(\psi_{n+1}^{m+1}).$$

Moreover, the class $\delta_{k,l}$ is the sum of pushforwards under the attaching maps of the class $(\psi_{n+1} + \psi_{n+2})^k (\psi_{n+1}\psi_{n+2})^l$, hence all these classes belong to the tautological ring.

The classes λ_i, as we will see as a consequence of Theorem 4.8.4, can be expressed via the ψ, κ and δ classes, hence it will follow that they lie in the tautological ring too. □

4.6 Characteristic classes and Grothendieck group

Before going on, we need a short overview on characteristic classes of vector bundles.

Let us start by defining them. Let X be a complex manifold and

$$L \to X$$

be a vector bundle of rank k.

Definition 4.15. We say that the L can be *exhausted* by line bundles if there exist a line subbundle L_1 of L, a line subbundle L_2 of the quotient L/L_1, and so on, until the last quotient is itself a line bundle L_k.

This is equivalent to ask L to have a complete flag of subbundles, with graded pieces $L_1, L_2, \dots, L_k$. The simplest case is $L = \bigoplus L_i$.

Definition 4.16 (Chern roots)**.** Let L be a vector bundle on X; if L can be exhausted by line bundles, the first Chern classes

$$r_i = \mathrm{c}_1(L_i)$$

are called the *Chern roots* of L.

Definition 4.17 (Total Chern class, Todd class, Chern character)**.** Let L be a vector bundle on X with Chern roots $r_1, r_2, \dots, r_k$. We define:

the *total Chern class* as:

$$\mathrm{c}(L) = \prod_{i=1}^{k} (1 + r_i);$$

the *Todd class* as:

$$\mathrm{td}(L) = \prod_{i=1}^{k} \frac{r_i}{e^{r_i} - 1};$$

the Chern character as:

$$\mathrm{ch}(L) = \sum_{i=1}^{k} e^{r_i}.$$

The homogeneous degree i parts of these classes will be denoted respectively by c_i, td_i and ch_i. In particular $\mathrm{c}_i(L)$ is called the i^{th} Chern class of L.

Remark 4.7. Definition 4.7 and Definition 4.17 of the i^{th} Chern class are equivalent; Definition 4.8 and Definition 4.17 of total Chern class are equivalent.

The knowledge of the total Chern class together with the first i-ths classes of a vector bundle allows to compute the Todd class and the Chern character, with the exception of ch_0, which is equal to the rank of the bundle.

Let us start with the study of the Chern character; we have the following equality:

$$\begin{aligned}
\mathrm{ch}(L) &= \sum_{i=1}^{k} e^{r_i} = \\
&= \left[1 + r_1 + \frac{r_1^2}{2!} + \frac{r_1^3}{3!} + \frac{r_1^4}{4!} + \cdots\right] + \cdots + \left[1 + r_k + \frac{r_k^2}{2!} + \frac{r_k^3}{3!} + \frac{r_k^4}{4!} + \cdots\right] \\
&= k + \sum_{i=1}^{k} r_i + \frac{1}{2!}\sum_{i=1}^{k} r_i^2 + \frac{1}{3!}\sum_{i=1}^{k} r_i^3 + \frac{1}{4!}\sum_{i=1}^{k} r_i^4 + \cdots .
\end{aligned}$$

From the previous equation it follows:

$$\mathrm{ch}_0 = k \qquad \mathrm{ch}_1 = \sum r_i \qquad \mathrm{ch}_2 = \frac{1}{2}\sum r_i^2 \qquad \cdots$$

Moreover we also observe that:

$$\sum r_i = \mathrm{c}_1 \qquad \sum_{i<j} r_i r_j = \mathrm{c}_2 \qquad \sum_{i<j<k} r_i r_j r_k = \mathrm{c}_3 .$$

Hence the collection of these equalities allows to compute explicitly the Chern characters as in the following:

Example 4.3. *Let us compute the first three* $\mathrm{ch}_1, \mathrm{ch}_2\, \mathrm{ch}_3$ *in terms of the first three Chern classes:*

$$\begin{aligned}
\mathrm{ch}_1 &= \sum r_i = \mathrm{c}_1 \\
\mathrm{ch}_2 &= \frac{1}{2}\sum r_i^2 = \frac{1}{2}\left(\sum r_i\right)^2 - \sum_{i<j} r_i r_j = \frac{1}{2}\mathrm{c}_1^2 - \mathrm{c}_2 .
\end{aligned}$$

Similarly we can compute ch_3*:*

$$\begin{aligned}\mathrm{ch}_3 &= \frac{1}{6}\sum r_i^3 = \\ &= \frac{1}{6}\left(\sum r_i\right)^3 - \frac{1}{2}\left(\sum r_i\right)\left(\sum_{i<j} r_i r_j\right) + \frac{1}{2}\sum_{i<j<k} r_i r_j r_k = \\ &= \frac{1}{6}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3.\end{aligned}$$

Proposition 4.6.1. *The Chern character of a vector bundle L of rank k can be written as:*

$$\begin{aligned}\mathrm{ch}(L) = k + \mathrm{c}_1 + &\frac{1}{2}\left(\mathrm{c}_1^2 - \mathrm{c}_2\right) + \frac{1}{6}\left(\mathrm{c}_1^3 - 3\,\mathrm{c}_1\,\mathrm{c}_2 + 3\,\mathrm{c}_3\right) + \\ &+ \frac{1}{24}\left(\mathrm{c}_1^4 - 4\,\mathrm{c}_1^2\,\mathrm{c}_2 + 4\,\mathrm{c}_1\,\mathrm{c}_3 + 2\,\mathrm{c}_2^2 - 4\,\mathrm{c}_4\right) + \textit{ high order terms}\ldots \qquad (4.1)\end{aligned}$$

Remark 4.8. Notice that in particular for a line bundle $\mathcal{L}$ we have that $k = 1$, hence there is only one Chern root:

$$r_1 = \mathrm{c}_1(\mathcal{L})$$

and the Chern character is given by:

$$\mathrm{ch}(L) = e^{\mathrm{c}_1} = 1 + \mathrm{c}_1(\mathcal{L}) + \frac{\mathrm{c}_1^2(\mathcal{L})}{2!} + \frac{\mathrm{c}_1^3(\mathcal{L})}{3!} + \frac{\mathrm{c}_1^4(\mathcal{L})}{4!} + \cdots.$$

In this case we have:

$$\mathrm{ch}_i = \frac{\mathrm{c}_1^i}{i!}$$

For Todd classes, let us recall that the function $f(x) = \frac{x}{e^x-1}$ can be written as the following series:

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n\geq 1}\frac{B_{2n}}{(2n)!}x^{2n},$$

where B_n are the *Bernoulli numbers:*

$$B_2 = \frac{1}{6},\quad B_4 = -\frac{1}{30},\quad B_6 = \frac{1}{42},\quad B_8 = -\frac{1}{30},\quad B_{10} = \frac{5}{66},\quad B_{12} = -\frac{691}{2730},\cdots$$

hence we get:

$$\mathrm{td}(L) = \prod\left(\frac{r_i}{e^{r_i} - 1}\right) = \prod\left(1 - \frac{r_i}{2} + \sum_{n\geq 1}\frac{B_{2n}}{(2n)!}(r_i)^{2n}\right).$$

In particular:

$$\mathrm{td}_1 = -\frac{1}{2}\sum r_i = -\frac{1}{2}\,\mathrm{c}_1;$$

$$\mathrm{td}_2 = \frac{1}{4}\sum_{i<j} r_i r_j + \frac{1}{12}\sum (r_i)^2 = \frac{1}{4}\sum_{i<j} r_i r_j + \frac{1}{12}\left(\sum r_i\right)^2 - \frac{1}{6}\sum_{i<j} r_i r_j = \frac{1}{12}\,\mathrm{c}_1 + \frac{1}{12}\,\mathrm{c}_2\,.$$

More generally:

$$\mathrm{td}(L) = 1 + \frac{1}{2}\,\mathrm{c}_1 + \frac{1}{12}\left(\mathrm{c}_1^2 + \mathrm{c}_2\right) + \frac{1}{24}\,\mathrm{c}_1\,\mathrm{c}_2 + \frac{1}{720}\left(-\,\mathrm{c}_1^4 + 4\,\mathrm{c}_1^2\,\mathrm{c}_2 + 3\,\mathrm{c}_2^2 + \mathrm{c}_1\,\mathrm{c}_3 - \mathrm{c}_4\right) + \ldots$$

Remark 4.9. It is possible to prove that the Chern character is additive i.e. given E_1, E_2 vector bundles:

$$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2)$$

Remark 4.10. Let L be a vector bundle, since $\mathrm{ch}(L)$ is a symmetric polynomial in the Chern roots, it is possible to prove that $\mathrm{ch}(L)$ is a well-defined element of $A^*(X)_{\mathbb{Q}} := A^*(X) \otimes \mathbb{Q}$.

Analogously the Todd class too belongs to the Chow ring.

The Grothendieck group

For an easier comprehension of the Grothendieck-Riemann-Roch formula, in this section 4.6 on the Grothendieck group, and in the next Section 4.7 with the statement of GRR formula, the Chern classes, Chern characters and Todd classes will be considered as cohomology classes. Hence the full section will be treated in cohomology.

Anyway, as we will observe in Remark 4.11, the same result could be obtained also in the Chow ring.

Definition 4.18 (Grothendieck group)**.** Let X be a complex manifold, and let us consider the free abelian group generated by all vector bundles defined over X. In this group, to any three vector bundles sitting in a short exact sequence:

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

we assign the relation:

$$V_1 - V_2 + V_3 = 0.$$

We define the *Grothendieck group* $K^0(X)$ to be the group of vector bundles modulo this relation.

Hence, the Chern character, being additive, determines a group morphism from the Grothendieck group to the cohomology group:

$$\mathrm{ch}\colon K^0(X) \to H^*(X, \mathbb{Q}).$$

Let $p\colon X \to Y$ be a proper morphism of complex manifolds, then it induces a map between cohomology rings:

$$p_*\colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$

for example, it is defined for any cycle $C \subseteq Y$ in general position, by

$$\langle p_*\alpha, C\rangle = \langle \alpha, p^{-1}(C)\rangle.$$

The morphism p induces also another morphism between the Grothendieck groups: let X_y be the fibre over a point $y \in Y$ and V a vector bundle, we can consider the cohomology spaces $H^k(X_y, V)$. For each k, these vector spaces can be glued together into a sheaf over Y, that will be denoted as $R^k p_*(V)$. We the define:

$$p_!(V) = R^0 p_*(V) - R^1 p_*(V) + R^2 p_*(V) - \cdots.$$

Now we have the following diagram:

$$\begin{tikzcd} K^0(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "p_!"'] & H^*(X, \mathbb{Q}) \arrow[d, "p_*"] \\ K^0(Y) \arrow[r, "\mathrm{ch}"'] & H^*(Y, \mathbb{Q}) \end{tikzcd}$$

It is natural to ask about the commutativity of this; the answer is given precisely by the Grothendieck-Riemann-Roch (GRR) formula: it is not commutative but it can be made to commute by adding a multiplicative factor $\mathrm{td}(p)$.

4.7 Grothendieck-Riemann-Roch formula and technical lemmas

We collected now all the elements to state Grothendieck-Riemann-Roch formula (GRR).

Let $p\colon X \to Y$ be a morphism of complex manifolds with compact fibres and let V a vector bundle over X.

We denote by $\mathrm{td}(p)$ the following:

$$\mathrm{td}(p) = \frac{\mathrm{td}(T^\vee X)}{\mathrm{td}(p^*(T^\vee Y))}.$$

Theorem 4.7.1 (GRR formula)**.** *Let $\mathcal{F} \in K^0(X)$, then the following equality holds:*

$$\mathrm{ch}(p_!\mathcal{F}) = p_* \left[\mathrm{ch}(\mathcal{F})\, \mathrm{td}(p)\right].$$

Proof. [Ful98] □

Remark 4.11. In the statement of Theorem 4.7.1 and in the previous section 4.6 we considered the Chern character of a complex manifold X as a class in the cohomology ring $H^*(X, \mathbb{Q})$, therefore all the treatment has been done in cohomology. Anyway, the same can be done in the Chow ring; in this case the GRR-formula states the commutativity of the following diagram:

$$\begin{tikzcd}
K^0(X) \arrow[r, "\mathrm{ch}\cdot\mathrm{td}(X)"] \arrow[d, "p_!"'] & A^*(X, \mathbb{Q}) \arrow[d, "p_*"] \\
K^0(Y) \arrow[r, "\mathrm{ch}\cdot\mathrm{td}(Y)"'] & A^*(Y, \mathbb{Q})
\end{tikzcd}$$

For the proof of the GRR-formula in the Chow ring one can see the article of Borel and Serre [BS58].

We collect in this section also some lemmas that will be used to establish relations among classes:

Lemma 4.7.2. *Let $D \subseteq X$ be a smooth subvariety of codimension 1, then we have:*

$$\frac{1}{\mathrm{td}(\mathcal{O}_D)} = \frac{D}{1 - e^{-D}}.$$

Proof. We can in fact consider the following short exact sequence:

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0;$$

recalling that $\mathrm{td}(\mathcal{O}_X) = 1$ and that Todd classes have the multiplicative property in short exact sequences we get the claim of the lemma. □

Lemma 4.7.3. *Let $\Delta \subseteq X$ be a smooth codimension 2 subvariety and let v_1, v_2 be the Chern roots of its normal vector bundle. Then, if B_n are the Bernoulli numbers we have:*

$$\frac{1}{\mathrm{td}(\mathcal{O}_\Delta)} = 1 + \Delta \sum_{k\geq 1} \frac{B_{2k}}{(2k)!} \frac{v_1^{2k-1} + v_2^{2k-1}}{v_1 + v_2}.$$

Proof. Let us consider a tubular neighbourhood of Δ. Since the characteristic classes are topological invariants of vector bundles, the characteristic classes of a sheaf supported on Δ depend only on its resolution in the neighbourhood of Δ in $\overline{\mathcal{C}}_{g,n}$, which is topologically as the neighbourhood of Δ in the total space of the normal bundle N. For the normal bundle we have the following exact sequence of sheaves (given by the Koszul resolution):

$$0 \to \bigwedge^2 N^\vee \to N^\vee \to \mathcal{O}_N \to \mathcal{O}_\Delta \to 0$$

where $\mathcal{O}_N$ is the sheaf of holomorphic functions on the total space of N. Hence we have:

$$
\begin{aligned}
\frac{1}{\mathrm{td}(\mathcal{O}_\Delta)} = \frac{\mathrm{td}(N^\vee)}{\mathrm{td}(\bigwedge^2 N^\vee)} &= \frac{v_1}{e^{v_1}-1}\cdot\frac{v_2}{e^{v_2}-1}\cdot\frac{e^{v_1+v_2}-1}{v_1+v_2} = \\
&= \frac{v_1v_2}{v_1+v_2}\cdot\frac{e^{v_1+v_2}-1}{(e^{v_1}-1)(e^{v_2}-1)} = \\
&= \frac{v_1v_2}{v_1+v_2}\cdot\frac{e^{v_1+v_2}-e^{v_1}+e^{v_1}-1}{(e^{v_1}-1)(e^{v_2}-1)} = \\
&= \frac{v_1v_2}{v_1+v_2}\cdot\left[\frac{e^{v_1}(e^{v_2}-1)}{(e^{v_1}-1)(e^{v_2}-1)}+\frac{(e^{v_1}-1)}{(e^{v_1}-1)(e^{v_2}-1)}\right] = \\
&= \frac{v_1v_2}{v_1+v_2}\cdot\left[\frac{e^{v_1}}{(e^{v_1}-1)}+\frac{1}{(e^{v_2}-1)}\right] = \\
&= \frac{v_1v_2}{v_1+v_2}\cdot\left[1+\frac{1}{(e^{v_1}-1)}+\frac{1}{(e^{v_2}-1)}\right] = \\
&= \frac{v_1v_2}{v_1+v_2}+\frac{v_2}{v_1+v_2}\cdot\frac{v_1}{e^{v_1}-1}+\frac{v_1}{v_1+v_2}\cdot\frac{v_2}{e^{v_2}-1} = \\
&= \frac{v_1v_2}{v_1+v_2}+\frac{v_2}{v_1+v_2}\cdot\left[1-\frac{v_1}{2}+\sum_{k\geq 1}\frac{B_{2k}}{(2k)!}v_1^{2k}\right]+\frac{v_1}{v_1+v_2}\cdot \\
&\cdot\left[1-\frac{v_2}{2}+\sum_{k\geq 1}\frac{B_{2k}}{(2k)!}v_2^{2k}\right] \\
&= \frac{v_1v_2}{v_1+v_2}+\frac{v_2}{v_1+v_2}-\frac{v_1v_2}{2(v_1+v_2)}+\frac{v_1}{v_1+v_2}-\frac{v_1v_2}{2(v_1+v_2)}+ \\
&+\sum_{k\geq 1}\frac{B_{2k}}{(2k)!}\frac{v_1^{2k}v_2+v_1v_2^{2k}}{v_1+v_2} = \\
&= \left[\frac{v_2}{v_1+v_2}+\frac{v_1}{v_1+v_2}\right]+\left[\frac{v_1v_2}{v_1+v_2}-\frac{v_1v_2}{2(v_1+v_2)}-\frac{v_1v_2}{2(v_1+v_2)}\right]+ \\
&+\sum_{k\geq 1}\frac{B_{2k}}{(2k)!}\frac{v_1^{2k}v_2+v_1v_2^{2k}}{v_1+v_2} = \\
&= 1+v_1v_2\sum_{k\geq 1}\frac{B_{2k}}{(2k)!}\frac{v_1^{2k-1}+v_2^{2k-1}}{v_1+v_2} = \\
&= 1+\Delta\cdot\sum_{k\geq 1}\frac{B_{2k}}{(2k)!}\frac{v_1^{2k-1}+v_2^{2k-1}}{v_1+v_2}
\end{aligned}
$$

□

Let now, Δ, D_i and $\mathcal{L}^{\log}$ as defined in Definition 4.11;

Lemma 4.7.4. *We have:*

$$\mathrm{td}(p) = \frac{\mathrm{td}(\mathcal{L}^{\log})}{\mathrm{td}(\mathcal{O}_\Delta)\prod_{i=1}^n \mathrm{td}(\mathcal{O}_{D_i})}.$$

Proof. For a proof see Cor. 3.26 in [Zvo14]. □

4.8 Tautological relations

In this section, following Mumford's article we would like to apply the GRR formula to the case where the morphism is the projection from the universal curve $p\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, and the vector bundle over $\overline{\mathcal{C}}_{g,n}$ is the trivial line bundle $\mathcal{O} := \mathcal{O}_{\overline{\mathcal{C}}_{g,n}}$, in order to discover some relations among classes, getting the so called *tautological relations*.

Taking a look at the GRR formula,

$$\mathrm{ch}(p_! \mathcal{F}) = p_* \left[\mathrm{ch}(\mathcal{F})\,\mathrm{td}(p)\right]$$

in order to discover relations among classes we need to express the left-hand side and the right-hand side of this formula via the ψ, κ, δ and λ classes on moduli spaces.

The left-hand side: $p_!\mathcal{O}$

Proposition 4.8.1. *Let C be a stable curve, and $\mathcal{O}$ be the trivial line bundle over $\overline{\mathcal{C}}_{g,n}$; let Λ be the Hodge bundle, and $\mathbb{C}$ be the trivial line bundle over $\overline{\mathcal{M}}_{g,n}$, then:*

$$p_!\mathcal{O} = \mathbb{C} - \Lambda^\vee.$$

Proof. The proof is a consequence of the fact that $R^1(p, \mathcal{O})$ is the dual of the Hodge bundle, which implies the claim. □

As a corollary we are able to rewrite the left hand-side of the GRR formula:

Corollary 4.8.2. *Let $\mathcal{O}$ and Λ as above, then:*

$$\mathrm{ch}(p_!\mathcal{O}) = 1 - \mathrm{ch}(\Lambda^\vee).$$

Remark 4.12. We observed before that the classes ch_i can be expressed using the first i Chern classes: c_i; since these are by definition the λ classes, this corollary allows to express the left hand-side of the GRR formula for $\mathcal{O}_{\overline{\mathcal{C}}_{g,n}}$ in terms of λ-classes; to get our goal we have now to be able to write the right-hand side in terms of the other classes.

The right-hand side: td(p)

The right hand-side of the GRR formula involves td(p); let us start by writing td(p) using K, Δ and D_i's with the following proposition:

Theorem 4.8.3. *Let K, Δ, D_i defined as in Definition 4.11; then we have:*

$$\operatorname{td}(p) = 1 - \frac{1}{2}\left(K - \sum D_i\right) + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!}\left[K^{2k} - \sum_{i=1}^{n} D_i^{2k} + \Delta \cdot \frac{v_1^{2k-1} + v_2^{2k-1}}{v_1 + v_2}\right].$$

Proof. First of all, by Lemma 4.7.4 we have:

$$\operatorname{td}(p) = \frac{\operatorname{td}(\mathcal{L}^{\log})}{\operatorname{td}(\mathcal{O}_\Delta)\prod_{i=1}^{n}\operatorname{td}(\mathcal{O}_{D_i})};$$

the Todd class of $\mathcal{L}^{\log}$ can be computed to get:

$$\operatorname{td}(\mathcal{L}^{\log}) = \frac{K}{e^K - 1}.$$

Moreover, the expressions of $\frac{1}{\operatorname{td}(\mathcal{O}_\Delta)}$ and of $\frac{1}{\operatorname{td}(\mathcal{O}_{D_i})}$ are given by Lemmas 4.7.2 and 4.7.3 respectively. Hence, putting there results together we then obtain:

$$\begin{aligned}\operatorname{td}(p) &= \frac{\operatorname{td}(\mathcal{L}^{\log})}{\operatorname{td}(\mathcal{O}_\Delta)\prod_{i=1}^{n}\operatorname{td}(\mathcal{O}_{D_i})} = \\ &= \frac{K}{e^K - 1}\prod_{i=1}^{n}\frac{D_i}{1 - e^{-D_i}}\left(1 + \Delta\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\frac{v_1^{2k-1} + v_2^{2k-1}}{v_1 + v_2}\right),\end{aligned}$$

taking into account Proposition 4.4.1 we get the following equality:

$$\operatorname{td}(p) = 1 + \left(\frac{K}{e^K - 1} - 1\right) + \sum_{i=1}^{n}\left(\frac{D_i}{1 - e^{-D_i}} - 1\right) + \left(\Delta\sum_{k=1}\frac{B_{2k}}{(2k)!}\frac{v_1^{2k-1} + v_2^{2k-1}}{v_1 + v_2}\right).$$

Expanding the power series we have the claim:

$$\operatorname{td}(p) = 1 - \frac{1}{2}\left(K - \sum D_i\right) + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!}\left[K^{2k} - \sum_{i=1}^{n} D_i^{2k} + \Delta \cdot \frac{v_1^{2k-1} + v_2^{2k-1}}{v_1 + v_2}\right].$$

□

In conclusion, according to the GRR formula, the push forward p_* of $\mathrm{td}(p)$ is equal to the class $1 - \mathrm{ch}(\Lambda^\vee)$, since $\mathrm{ch}(\mathcal{O}) = 1$:

$$1 - \mathrm{ch}(\Lambda^\vee) = p_*\left(1 - \frac{1}{2}\left(K - \sum D_i\right) + \sum_{k\geq 1}\frac{B_{2k}}{(2k)!}\left[K^{2k} - \sum_{i=1}^{n} D_i^{2k} + \Delta\cdot\frac{v_1^{2k-1} + v_2^{2k-1}}{v_1 + v_2}\right]\right). \tag{4.2}$$

More explicitly, using the fact that:

$$\mathrm{ch}_k(\Lambda^\vee) = \begin{cases} \mathrm{ch}_k(\Lambda) & \text{if } k \text{ is even} \\ -\mathrm{ch}_k(\Lambda) & \text{if } k \text{ is odd} \end{cases}$$

$$\begin{aligned} p_*(K^{2k}) &= \kappa_{2k-1}, \\ \psi_i^{2k-1} &= p_*(D_i^{2k}), \end{aligned}$$

and defining:

$$\delta_{2k-1}^{\Lambda} := p_*\left(\Delta\frac{v_1^{2k-1} + v_2^{2k-1}}{v_1 + v_2}\right),$$

we obtain the following crucial theorem:

Theorem 4.8.4. *We have the following:*

$$\mathrm{ch}_0(\Lambda) = g; \tag{4.3a}$$

$$\mathrm{ch}_{2k}(\Lambda) = 0; \tag{4.3b}$$

$$\mathrm{ch}_{2k-1}(\Lambda) = \frac{B_{2k}}{(2k)!}\left[\kappa_{2k-1} - \sum_{i=1}^{n}\psi_i^{2k-1} + \delta_{2k-1}^{\Lambda}\right]. \tag{4.3c}$$

Proof. For the proof of the equality (4.3a) in degree zero, by (4.2) we get:

$$1 - \mathrm{ch}_0(\Lambda^\vee) = 1 - \mathrm{ch}_0(\Lambda) = -\frac{1}{2}p_*\left(K - \sum D_i\right) = -\frac{1}{2}(2g - 2 + n - n) = 1 - g$$

hence:

$$\mathrm{ch}_0(\Lambda) = g.$$

□

Using the relation:

$$v_1^p + v_2^p = \sum_{l=0}^{[p/2]}(-1)^l\frac{p}{p-l}\binom{p-l}{l}(v_1 v_2)^l(v_1 + v_2)^{p-2l}$$

we can finally express the class δ^{Λ}_{2k-1} via the standard classes $\delta_{k,l}$ with the following equality:

$$\delta^{\Lambda}_{2k-1} = \sum_{l=0}^{k-1}(-1)^l \frac{2k-1}{2k-1-l}\binom{2k-1-l}{l}\delta_{2k-2-2l,l}.$$

This theorem together with the observations of Remark 4.12 allows to get the desired relations among the tautological classes we were looking for. In particular, we can write them explicitly for low classes.

Proposition 4.8.5. *We have the following relations:*

$$\lambda_1 = \frac{1}{12}\left(\kappa_1 - \sum_{i=1}^{n}\psi_i + \delta_{0,0}\right); \tag{4.4a}$$

$$\lambda_2 = \frac{1}{2}\lambda_1^2; \tag{4.4b}$$

$$\lambda_3 = \frac{1}{6}\lambda_1^3 - \frac{1}{360}\left(\kappa_3 - \sum_{i=1}^{n}\psi_i^3 + \delta_{2,0} - 3\delta_{0,1}\right). \tag{4.4c}$$

Proof. For the equality (4.4a); by Example 4.3 we have:

$$\lambda_1 = \mathrm{c}_1(\Lambda) = \mathrm{ch}_1(\Lambda).$$

Moreover by Theorem 4.8.4, with $k = 1$:

$$\mathrm{ch}_1 = \frac{B_2}{2!}\left[\kappa_1 - \sum_{i=1}^{n}\psi_i^1 + \delta_1^{\Lambda}\right] =$$
$$= \frac{1}{6}\cdot\frac{1}{2}\left[\kappa_1 - \sum_{i=1}^{n}\psi_i + \left(\frac{1}{1}\binom{1}{1}\delta_{0,0}\right)\right] = \frac{1}{12}\left(\kappa_1 - \sum_{i=1}^{n}\psi_i + \delta_{0,0}\right)$$

and this proves the first equality.

For the second equality (4.4b), again by the computation of Example 4.3, we have:

$$\lambda_2 = \mathrm{c}_2(\Lambda) = \frac{1}{2}\mathrm{c}_1^2 - \mathrm{ch}_2 = \frac{1}{2}\lambda_1^2 - \mathrm{ch}_2;$$

moreover by Theorem 4.8.4: $\mathrm{ch}_2 = 0$ hence:

$$\lambda_2 = \frac{1}{2}\lambda_1^2.$$

For the third equality (4.4c), Example 4.3 gives:

$$\lambda_3 = \mathrm{c}_3(\Lambda) = 2\,\mathrm{ch}_3 - \frac{1}{3}\lambda_1^2 + \lambda_1\lambda_2 = \frac{1}{6}\lambda_1^3 + 2\,\mathrm{ch}_3;$$

on the other hand, using Theorem 4.8.4 with $k = 2$ we get:

$$\begin{aligned}
\mathrm{ch}_3 &= \frac{B_4}{4!}\left[\kappa_3 - \sum_{i=1}^{n}\psi_i^3 + \delta_3^{\Lambda}\right] = \\
&= -\frac{1}{30}\cdot\frac{1}{24}\left[\kappa_3 - \sum_{i=1}^{n}\psi_i^3 + \left(\sum_{l=0}^{1}(-1)^l\frac{3}{3-l}\binom{3-l}{l}\delta_{2-2l,l}\right)\right] = \\
&= -\frac{1}{720}\left[\kappa_3 - \sum_{i=1}^{n}\psi_i^3 + \left(\delta_{2,0} - \frac{3}{2}\cdot 2\delta_{0,1}\right)\right] = \\
&= -\frac{1}{720}\left(\kappa_3 - \sum_{i=1}^{n}\psi_i^3 + \delta_{2,0} - 3\delta_{0,1}\right).
\end{aligned}$$

In conclusion:

$$\lambda_3 = \frac{1}{6}\lambda_1^3 - \frac{1}{360}\left(\kappa_3 - \sum_{i=1}^{n}\psi_i^3 + \delta_{2,0} - 3\delta_{0,1}\right).$$

□

5 The moduli space $\overline{\mathcal{M}}_2$

5.1 The tautological relations in genus 2

First of all, we can specialize the calculations of the previous chapter to the case of genus 2.

From the Hodge bundle Λ we get the two λ-classes:

$$\lambda_1 \in A^1(\overline{\mathcal{M}}_2)$$

$$\lambda_2 \in A^2(\overline{\mathcal{M}}_2)$$

and by Proposition 4.8.5 they satisfy:

$$\lambda_2 = \frac{1}{2}\lambda_1^2.$$

Moreover, from the class K on $\overline{C}_2$, we get:

$$\kappa_i \in A^i(\overline{\mathcal{M}}_2), \qquad \text{for } i = 1, 2, 3.$$

Considering the space $\overline{\mathcal{M}}_2$, since it does not have marked point ψ-classes are zero.

Proposition 4.8.5 also gives the following equality:

$$\lambda_1 = \frac{1}{12}(\kappa_1 + \delta) \tag{5.1}$$

where $\delta = \delta_{0,0} - \sum_{i=1}^{n} \psi_i = \delta_{0,0}$ is the fundamental class of $\overline{\mathcal{M}}_2 \setminus \mathcal{M}_2$. Let us observe that $\overline{\mathcal{M}}_2 \setminus \mathcal{M}_2$ has two components: Δ_0 and Δ_1 :

- Δ_0 the closure of the locus of irreducible singular curves;

- Δ_1 the locus of singular curves $C_1 \cup C_2$, with $C_1 \cap C_2 = \{\mathrm{pt}\}$ and $p_a(C_1) = p_a(C_2) = 1$.

Hence, by definition:

$$\delta = [\Delta_0] + [\Delta_1] .$$

In codimension 3 we obtain moreover the following equality, by Prop 4.8.5:

$$\begin{aligned}\lambda_3 &= \frac{1}{6}\lambda_1^3 - \frac{1}{360}\left(\kappa_3 - \sum_{i=1}^{n}\psi_i^3 + \delta_{2,0} - 3\delta_{0,1}\right) = \\ &= \frac{1}{6}\lambda_1^3 - \frac{1}{360}\left(\kappa_3 + \delta_{2,0} - 3\delta_{0,1}\right).\end{aligned}$$

Mumford, using a formula for the Chern classes of the conormal bundle gets the following equality:

$$p^*\lambda_2 - K \cdot p^*\lambda_1 + K^2 - [\Delta_1^*] = 0;$$

(for a proof see [Mum83, §5]). Multiplying this by K and K^2, we get the following formulas:

$$K \cdot p^*\lambda_2 - K^2 \cdot p^*\lambda_1 + K^3 = 0$$

$$K^2 \cdot p^*\lambda_2 - K^3 \cdot p^*\lambda_1 + K^4 = 0$$

and taking π_* we finally get:

$$\kappa_1 = 2\lambda_1 + \delta_1 \tag{5.2a}$$

$$\kappa_2 = \kappa_1 \cdot \lambda_1 - 2\lambda_2 = \lambda_1 \cdot (\lambda_1 + \delta_1) \tag{5.2b}$$

$$\kappa_3 = \kappa_2\lambda_1 - \kappa_1\lambda_2 = \frac{1}{2}\lambda_1^2\delta_1. \tag{5.2c}$$

Combining now (5.1) and (5.2b) we can express both κ_1 and λ_1 in terms of δ_0, δ_1 :

Proposition 5.1.1. *We have the following tautological relations:*

$$10\lambda_1 = \delta_0 + 2\delta_1 \tag{5.3a}$$

$$5\kappa_1 = \delta_0 + 7\delta_1. \tag{5.3b}$$

Proof. For the proof of (5.3a), we have by (5.1):

$$\begin{aligned}\lambda_1 = \frac{1}{12}\kappa_1 + \frac{1}{12}\delta_1 + \frac{1}{12}\left(\delta_0 + \delta_1\right) = \\ = 2\frac{1}{12}\lambda_1 + \frac{1}{12}\delta_0 + \frac{1}{6}\delta_1 = \\ = \frac{1}{6}\lambda_1 + \frac{1}{12}\delta_0 + \frac{1}{6}\delta_1\end{aligned}$$

hence:

$$\left(1 - \frac{1}{6}\right)\lambda_1 = \frac{1}{12}\delta_0 + \frac{1}{6}\delta_1 \Rightarrow \frac{5}{6}\lambda_1 = \frac{1}{12}\delta_0 + \frac{1}{6}\delta_1$$

and finally, multiplying by 12:

$$10\lambda_1 = \delta_0 + 2\delta_1.$$

Similarly, to prove (5.3b), using by the first part of the proposition that $10\lambda_1 = \delta_0 + 2\delta_1$ we get:

$$5\kappa_1 = 10\lambda_1 + \delta_1 = \left(\delta_0 + 2\delta_1\right) + \delta_1 = \delta_0 + 7\delta_1$$

and this proves the claim. □

Remark 5.1. As κ_1 is ample, this implies moreover that $\mathcal{M}_2$ is affine.

Let us finally resume the tautological relations obtained for the classes of $\overline{\mathcal{M}}_2$:

Theorem 5.1.2 (Tautological relations for $\overline{\mathcal{M}}_2$)**.** *The tautological classes of $\overline{\mathcal{M}}_2$ satisfy the following tautological relations:*

$$\begin{aligned}10\lambda_1 &= \delta_0 + 2\delta_1;\\ 5\kappa_1 &= \delta_0 + 7\delta_1;\\ \lambda_2 &= \frac{1}{2}\lambda_1^2;\\ \kappa_2 &= \lambda_1^2 + \lambda_1\delta_1;\\ \kappa_3 &= \frac{1}{2}\lambda_1^2\delta_1.\end{aligned}$$

Remark 5.2. Let us observe that all the tautological classes of $R^*(\overline{\mathcal{M}}_2)$ can be expressed via δ_0 and δ_1.

5.2 Generators of $A^*(\overline{\mathcal{M}}_2)$

In the case of moduli space of curves of genus 2 we can give also a description of the Chow ring. The following two sections are dedicated to the study of the intersection product in $A^*(\overline{\mathcal{M}}_2)$; we start by describing the generators of $A^*(\overline{\mathcal{M}}_2)$.

We have in particular the following theorem:

Theorem 5.2.1. *The space $\overline{\mathcal{M}}_2$ is the disjoint union of 7 cells:*

$$\overline{\mathcal{M}}_2 = \mathcal{M}_2 \coprod Int(\Delta_0) \coprod Int(\Delta_1) \coprod Int(\Delta_{00}) \coprod Int(\Delta_{01}) \coprod \{C_{000}\} \coprod \{C_{001}\}.$$

In particular, $A^*(\overline{\mathcal{M}}_2)$ is generated by:

1. *1 in codimension 0;*
2. *δ_0, δ_1 in codimension 1,*
3. *δ_{00}, δ_{01} in codimension 2,*
4. *the class $[x]$ of a point in codimension 3: let us call this class $p = [x]$.*

Proof. We sketch the proof following Mumford's article; to get generators of $A_*(\overline{\mathcal{M}}_2)$ we would like to use the fact that, if $Y \subseteq X$ is a closed subvariety, we have the following exact sequence:

$$A_*(Y) \longrightarrow A_*(X) \longrightarrow A_*(X \setminus Y) \longrightarrow 0.$$

It is known, by [Igu60], that $\mathcal{M}_2$ is isomorphic to $\mathbb{C}^3$ modulo the action of $\mathbb{Z}/5\mathbb{Z}$ defined by:

$$(x, y, z) \mapsto (\zeta x, \zeta^2 x, \zeta^3 y).$$

The map:

$$A_*(\mathbb{C}^3) \to A_*(\mathcal{M}_2)$$

is surjective hence:

$$A_k(\mathcal{M}_2) = 0 \qquad \text{if } k < 3.$$

Thus:

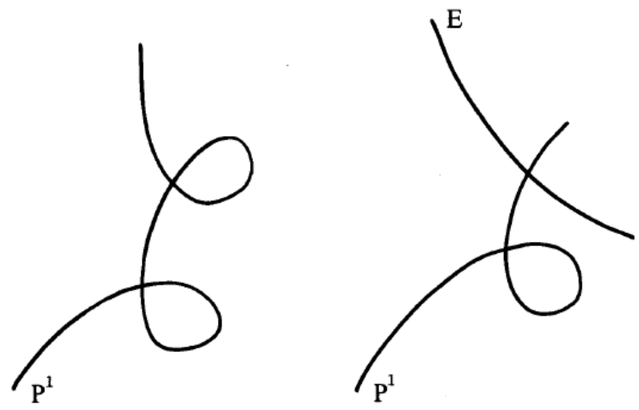
$$A_k(\Delta_0) \oplus A_k(\Delta_1) \longrightarrow A_k(\overline{\mathcal{M}}_2)$$

is surjective too if $k < 3$. In particular, if $k = 2$ we obtain that $A_2(\overline{\mathcal{M}}_2)$ is generated by δ_0 and δ_1, which are the classes respectively of Δ_0 and Δ_1.

Let us define now the following codimension 1 subsets. For clarity we include after the definitions, the pictures of Mumford's article:

$$\Delta_{00} = \left\{ \begin{array}{l} \text{Closure of curve in } \overline{\mathcal{M}}_2 \text{ parametrizing} \\ \text{irreducible rational curves with 2 nodes} \end{array} \right\}$$

$$\Delta_{01} = \Delta_0 \cap \Delta_1 = \\ = \left\{ \begin{array}{l} \text{Curve in } \overline{\mathcal{M}}_2 \text{ parametrizing curves} \\ C_1 \cup C_2, \text{where } C_1 \cap C_2 = \{x\}, \\ C_1 \text{ is elliptic or rational with one} \\ \text{node and } C_2 \text{ is rational with one node} \end{array} \right\}$$



curves in Δ_{00} curves in Δ_{01}

Δ_{00} contains, besides the irreducible curves illustrated, also the two reducible curves of the following picture:

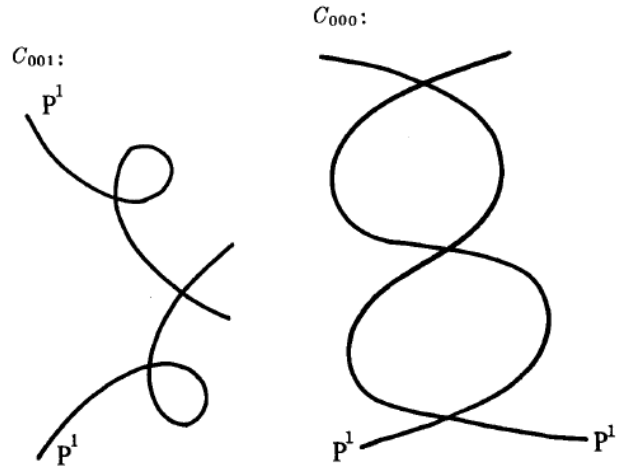


Figure 5.1: The pictures of C_{001} and C_{000}

Let us observe now that $\mathrm{Int}(\Delta_1) = \left(\Delta_1 \setminus \Delta_{01}\right)$ is the locus of curves of the form $C_1 \cap C_2$ where $C_1 \cap C_2 = \{x\}, C_1, C_2$ smooth elliptic. It is isomorphic to the symmetric product of two copies of the moduli space of 1-pointed genus one smooth curves: $\mathrm{Sym}M_{1,1}$, and this is known to be isomorphic to $\mathbb{C}^2$:

$$\mathrm{Int}(\Delta_1) \cong \mathbb{C}^2.$$

Moreover, $\mathrm{Int}(\Delta_0) = \Delta_0 \setminus \left(\Delta_{00} \cup \Delta_{01}\right)$ is the locus of irreducible elliptic curves with one node, which can be viewed as the space $\mathcal{M}_{1,2}$ of triples (E, p_1, p_2), where E is a smooth elliptic curve of genus 2 and $p_1, p_2 \in E, p_1 \neq p_2$, modulo the involution interchanging the two points.

We already observed that each genus 1 curve is isomorphic to an elliptic curve of the form:

$$y^2 = x(x-1)(x-\lambda), \qquad \text{with } \lambda \neq 0, 1.$$

If we take p_1 to be the point at infinity, and $p_2 = (x, y)$, interchanging p_1 with p_2 is equivalent to mapping the point (x, y) to the point $(x, -y)$. So we get a surjective map:

$$\{(x, \lambda) \mid \lambda \neq 0, 1\} \to \mathrm{Int}(\Delta_0).$$

Combining all this, we obtain the following results:

$$\begin{array}{ll} A_k(\Delta_1 \setminus \Delta_{0,1}) = 0 & \text{for } k < 2, \\ A_k(\Delta_0 \setminus (\Delta_{00} \cup \Delta_{01})) = 0 & \text{for } k < 2. \end{array}$$

Thus $A_1(\overline{\mathcal{M}}_2)$ is generated by:

$$\delta_{00} = \left[\Delta_{00}\right] \quad \text{and} \quad \delta_{01} = \left[\Delta_{01}\right].$$

Finally, since $\overline{\mathcal{M}}_2$ is unirational, all points are rationally equivalent. □

5.3 Multiplication in $A^*(\overline{\mathcal{M}}_2)$

We can now conclude the chapter by giving the multiplication rules of $A^*(\overline{\mathcal{M}}_2)$.

Theorem 5.3.1. *The ring* $A^*(\overline{\mathcal{M}_2})$ *has a* $\mathbb{Q}$*-basis consisting of* $1, \delta_0, \delta_1, \delta_{00}, \delta_{01}, p$ *and a multiplication table given by:*

$$\delta_0^2 = \frac{5}{3}\delta_{00} - 2\delta_{01} \tag{5.5a}$$

$$\delta_0\delta_1 = \delta_{01} \tag{5.5b}$$

$$\delta_1^2 = -\frac{1}{12}\delta_{01} \tag{5.5c}$$

$$\delta_0\delta_{00} = -\frac{1}{4}p \tag{5.5d}$$

$$\delta_0\delta_{01} = \frac{1}{4}p \tag{5.5e}$$

$$\delta_1\delta_{00} = \frac{1}{8}p \tag{5.5f}$$

$$\delta_1\delta_{01} = -\frac{1}{48}p \tag{5.5g}$$

Remark 5.3. The multiplication table of Theorem 5.3.1 for $A^*(\overline{\mathcal{M}_2})$ can be given also using λ_1. More precisely, using the identity: $10\lambda_1 = \delta_0 + 2\delta_1$, it is possible to get the following equivalent multiplication table:

$$\delta_0^2 = \frac{5}{3}\delta_{00} - 2\delta_{01} \tag{5.6a}$$

$$\delta_0\delta_1 = \delta_{01} \tag{5.6b}$$

$$\delta_{00}\delta_1 = \frac{1}{8}p \tag{5.6c}$$

$$\delta_{00}\lambda_1 = 0 \tag{5.6d}$$

$$\delta_1\lambda_1 = \frac{1}{12}\delta_{01} \tag{5.6e}$$

$$\delta_0\lambda_1 = \frac{1}{6}\delta_{00} \tag{5.6f}$$

(5.6a), (5.6b) and (5.6c) are the same as (5.5a) ,(5.5b) and (5.5f).

For (5.6d):

$$\delta_{00}\lambda_1 = \delta_{00} \cdot \frac{1}{10}\delta_0 + \delta_{00} \cdot \frac{1}{5}\delta_1 = -\frac{1}{10} \cdot \frac{1}{4}p + \frac{1}{5} \cdot \frac{1}{8}p = 0.$$

For (5.6e):

$$\delta_1\lambda_1 = \delta_1 \cdot \frac{1}{10}\delta_0 + \delta_1 \cdot \frac{1}{5}\delta_1 = \frac{1}{10}\delta_0\delta_1 + \frac{1}{5}\delta_1^2 = \frac{1}{10}\delta_{01} - \frac{1}{5} \cdot \frac{1}{12}\delta_{01} = \frac{1}{12}\delta_{01}.$$

And finally for (5.6f):

$$\delta_0\lambda_1 = \delta_0 \cdot \frac{1}{10}\delta_0 + \delta_0 \cdot \frac{1}{5}\delta_1 = \frac{1}{10}\left(\frac{5}{3}\delta_{00} - 2\delta_{01}\right) + \frac{1}{5}\delta_{01} = \frac{1}{6}\delta_{00} - \frac{1}{5}\delta_{01} + \frac{1}{5}\delta_{01} = \frac{1}{6}\delta_{00}.$$

Obtaining the multiplication table of the Theorem from the identities (5.6a)-(5.6f) is an analogous computation.

A Basics on category theory

A.1 Categories and functors

Definition A.1. A *category* $\mathcal{C}$ consists of the following data:

1. a class $\mathrm{Ob}(\mathcal{C})$, whose elements are called the *objects* of $\mathcal{C}$;
2. for each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$, whose elements are called *morphisms* of A into B, represented with an arrow $f: A \to B$;
3. for each triple $A, B, C \in \mathrm{Ob}(\mathcal{C})$, a mapping called *composition law*:

$$\begin{array}{rcl} \circ: \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) & \to & \mathrm{Hom}_{\mathcal{C}}(A, C) \\ (g, f) & \mapsto & g \circ g \end{array}$$

   satisfying the following axioms:

   a) ASSOCIATIVITY: if $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ and $f: A \to B$, $g: B \to C$, $h: C \to D$ are morphisms, then the composition law is associative, i.e. $(h \circ g) \circ f = h \circ (g \circ f)$;
   b) IDENTITY MORPHISM: for every $A \in \mathrm{Ob}(\mathcal{C})$, there exists an element $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$, called *identity morphism* such that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ g = g$ for every $B \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$.

Definition A.2. A morphism $f: A \to B$ is said to be an *isomorphism* if there exists $g: B \to A$ such that: $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. If there exists, g is said to be an *inverse* of f.

Remark A.1. The inverse, if it exists, it is unique.

Definition A.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* $F: \mathcal{C} \to \mathcal{D}$ is given by the following data: to every object $C \in \mathrm{Ob}(\mathcal{C})$, the assignment of an object $F(C) \in \mathrm{Ob}(\mathcal{D})$ and to every morphism $f : C \to C'$ in $\mathcal{C}$ a morphism $F(f) : F(C) \to F(C')$ in $\mathcal{D}$, satisfying the following axioms:

- if $f : C \to C'$ and $g : C' \to C''$ are morphisms in $\mathcal{C}$ then
$$F(g \circ f) = F(g) \circ F(f);$$
- $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$ for every object $C \in \mathrm{Ob}(\mathcal{C})$.

Definition A.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ assigns to every object $C \in \mathrm{Ob}(\mathcal{C})$ an object $F(C) \in \mathrm{Ob}(\mathcal{D})$ and to every morphism $f : C \to C'$ in $\mathcal{C}$ a morphism $F(f) : F(C') \to F(C)$ in $\mathcal{D}$, satisfying the following axioms:

- if $f : C \to C'$ and $g : C' \to C''$ are morphisms in $\mathcal{C}$ then
$$F(g \circ f) = F(f) \circ F(g);$$
- $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$ for every object $C \in \mathrm{Ob}(\mathcal{C})$.

Example A.1 (The dual category). *Let $\mathcal{C}$ be a category, we define the* dual category *or the* opposite category *of $\mathcal{C}$ the category $\mathcal{C}^{op}$ defined as follows:*

- *$Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$;*
- *$Hom_{\mathcal{C}^{op}}(A, B) = Hom_{\mathcal{C}}(B, A)$*
- *if "$\circ$" is the composition law in $\mathcal{C}$ we define the composition law "$*$" in $\mathcal{C}^{op}$ as:*
$$[f * g]_{\mathcal{C}^{op}} = [g \circ f]_{\mathcal{C}} \tag{A.1}$$

The verifications to prove that $\mathcal{C}^{op}$ is a category are all straightforward, they follow for completeness.

Let $f^{op} \in Hom_{\mathcal{C}^{op}}(A, B)$, $g^{op} \in Hom_{\mathcal{C}^{op}}(B, C)$, $h^{op} \in Hom_{\mathcal{C}^{op}}(C, D)$; we have to prove that the composition law defined in (A.1) is associative and the existence of an identity morphism for each object. For the associativity, recalling that $f^{op} \in Hom_{\mathcal{C}^{op}}(A, B)$ if and only if $f \in Hom_{\mathcal{C}}(B, A)$, we have:

$$h^{op} * (g^{op} * f^{op}) = (f \circ g) \circ h = f \circ (g \circ h) = (h^{op} * g^{op}) * f^{op}$$

where the second equality holds in the category $\mathcal{C}$ for the associativity of the composition law "$\circ$" and the associativity is proven.

For the identity, since $\mathcal{C}$ is a category, for each object $A \in Ob(\mathcal{C})$ there exists an identity $id_A \in Hom_{\mathcal{C}}(A, A)$; we claim that, $id_A^{op} \in Hom_{\mathcal{C}^{op}}(A, A)$ is the identity *of A in the category C^{op}. In fact, let $f^{op} \in Hom_{C^{op}}(A, B)$, and $g^{op} \in Hom_{C^{op}}(B, A)$, we have:*

$$f^{op} * id_A^{op} = id_A \circ f = f = f^{op}$$

$$id_A^{op} * g^{op} = g \circ id_A = g = g^{op}$$

and this proves C^{op} is a category as well.

Remark A.2. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. In the follow we'll denote the contravariants functors in this way.

Definition A.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant (contravariant) functor. For all $A, B \in \mathrm{Ob}(\mathcal{C})$ the functor F induces a mapping $F_{AB} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ ($F_{AB} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$), defined by $F_{AB}(f) = F(f)$ for every $f : A \to B$.

The functor F is said to be:

1. *faithful* if all the maps F_{AB} are injective for every $A, B \in \mathrm{Ob}(\mathcal{C})$;

2. *full* if all these maps are surjective;

3. *fully faithful* if all these maps are bijective;

4. *essentially surjective* if for every $D \in \mathrm{Ob}(\mathcal{D})$ there exists $C \in \mathrm{Ob}(\mathcal{C})$ such that $F(C) \cong D$.

Definition A.6 (Natural transformation of functors)**.** Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be covariant functors. A *natural transformation* $\eta : F \to G$ is the assignment to each object $A \in \mathrm{Ob}(\mathcal{C})$ of a morphism $\eta_A : F(A) \to G(A)$ such that, for every morphism $f : A \to B$ in the category $\mathcal{C}$ the following diagram is commutative:

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

Definition A.7 (Natural isomorphism)**.** A natural transformation of functors $\eta : F \to G$ is called a *natural isomorphism* if $\eta_A : F(A) \to G(A)$ is an isomorphism for every $A \in \mathrm{Ob}(\mathcal{C})$.

A.2 Product and fibered product

Definition A.8 (Product)**.** Let C be a category. We say that the category C admits *products* if for each pair of objects $A, B \in \mathrm{Ob}(C)$ there exists an object $P \in \mathrm{Ob}(C)$ together with two morphisms $\pi_A : P \to A$ and $\pi_B : P \to B$

$$\begin{tikzcd} & P \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

satisfying the following universal property: for each $Q \in \mathrm{Ob}(C)$, with morphism $q_A : Q \to A$ and $q_B : Q \to B$, there exists a unique morphism $f : Q \to P$ making the following diagram commutes

$$\begin{tikzcd} & Q \arrow[d, "f"] \arrow[ddl, "q_A"'] \arrow[ddr, "q_B"] & \\ & P \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A & & B \end{tikzcd}$$

Corollary A.2.1. *If a product (P, π_A, π_B) exists, then it is unique up to a unique isomorphism.*

Remark A.3. This is a direct consequence of the universal property; in fact every object defined through a universal property is unique up to a unique isomorphism. For this reason, we'll prove the uniqueness only in this case, the case of the fibered product is analogous.

Proof. Let $A, B \in \mathrm{Ob}(C)$, and let (Π, π_A, π_B) and (P, p_A, p_B) two products. For the universal property of the product applied to the two product spaces, we obtain that there exist a unique $f : \Pi \to P$ and a unique $g : P \to \Pi$ making the following diagrams commute:

$$
\begin{tikzcd}
 & \Pi \arrow[dl, "\pi_A"'] \arrow[d, "f"] \arrow[dr, "\pi_B"] & \\
 & P \arrow[dl, "p_A"] \arrow[dr, "p_B"'] & \\
A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P \arrow[dl, "p_A"'] \arrow[d, "g"] \arrow[dr, "p_B"] & \\
 & \Pi \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}
$$

It follows that

$$\pi_A \circ (g \circ f) = (\pi_A \circ g) \circ f = p_A \circ f = \pi_A$$

$$\pi_B \circ (g \circ f) = (\pi_B \circ g) \circ f = p_B \circ f = \pi_B$$

and analogously

$$p_A \circ (f \circ g) = p_A, \quad p_B \circ (f \circ g) = p_B$$

hence $(g \circ f)$ and $(f \circ g)$ make commute the following diagrams:

$$
\begin{tikzcd}
 & \Pi \arrow[ddl, "\pi_A"'] \arrow[d, "f"] \arrow[ddr, "\pi_B"] & \\
 & P \arrow[d, "g"] & \\
A & \Pi \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P \arrow[ddl, "p_A"'] \arrow[d, "g"] \arrow[ddr, "p_B"] & \\
 & \Pi \arrow[d, "f"] & \\
A & P \arrow[l, "p_A"] \arrow[r, "p_B"'] & B
\end{tikzcd}
$$

But the identity morphisms id_P and id_Π satisfy the same universal property then, by the uniqueness of such maps:

$$g \circ f = \mathrm{id}_\Pi \quad f \circ g = \mathrm{id}_P$$

and this proves that $P \cong \Pi$ up to a unique isomorphism. □

Definition A.9 (Fibered product)**.** Let C be a category. We say that the category C admits *fibered products* if for each triple of objects $A, B, C \in \mathrm{Ob}(C)$ and each pair of morphisms $\varphi_A : A \to C$, $\varphi_B : B \to C$, there exists an object $P \in \mathrm{Ob}(C)$ together with two morphisms $\pi_A : P \to A$ and $\pi_B : P \to B$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A \arrow[dr, "\varphi_A"'] & & B \arrow[dl, "\varphi_B"] \\ & C & \end{tikzcd}$$

satisfying the following universal property: for each $Q \in \mathrm{Ob}(C)$, with morphism $q_A : Q \to A$ and $q_B : Q \to B$, making the following diagram commutes,

$$\begin{tikzcd} & Q \arrow[dl, "q_A"'] \arrow[dr, "q_B"] & \\ A \arrow[dr, "\varphi_A"'] & & B \arrow[dl, "\varphi_B"] \\ & C & \end{tikzcd}$$

there exists a unique morphism $f : Q \to P$ making the following diagram commutes

$$\begin{tikzcd} & Q \arrow[d, "f"] \arrow[ddl, "q_A"'] \arrow[ddr, "q_B"] & \\ & P \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A \arrow[dr, "\varphi_A"'] & & B \arrow[dl, "\varphi_B"] \\ & C & \end{tikzcd}$$

Remark A.4. Since it is defined by a universal property, the fibered product, if it exists, it us unique up to a unique isomorphism.

A.3 Yoneda Lemma

Let C be a category and let us denote with $C^\wedge := Fct(C^{\mathrm{op}}, \mathrm{Set})$ the category of contravariant functors from the category C to the category of sets.

For every $A \in \mathrm{Ob}(C)$ we define the following contravariant functor:

$$\begin{array}{rcl} h^A : C^{op} & \to & \mathrm{Set} \\ X & \mapsto & \mathrm{Hom}(X, A) \\ (f : X \to Y) & \mapsto & \begin{array}{ccc} \mathrm{Hom}(Y, A) & \xrightarrow{f^*} & \mathrm{Hom}(X, A) \\ \varphi & \mapsto & f^*(\varphi) = \varphi \circ f \end{array} \end{array}$$

since every morphism $f : A \to B$ induces a natural transformation $f_* : h^A \to h^B$ we can then define the following covariant functor:

$$\begin{array}{rcl} h_C : C & \to & C^\wedge \\ A & \mapsto & h^A = \mathrm{Hom}(-, A) \\ (f : A \to B) & \mapsto & \begin{array}{ccc} h^A & \xrightarrow{f_*} & h^B \\ \varphi & \mapsto & f_*(\varphi) = f \circ \varphi \end{array} \end{array}$$

The proof of the Yoneda Lemma is essentially based on the following result:

Lemma A.3.1. *For every $F \in C^\wedge$, we have:*

$$Hom_{C^\wedge}(h_C(A), F) \cong F(A) \tag{A.2}$$

for every element $A \in Ob(C)$, functorially in A.

Proof. Let us start by defining the map that induces the bijection (A.2); let $\eta \in \mathrm{Hom}_{C^\wedge}(h_C(A), F)$ be a natural transformation between $h_C(A) = h^A$ and F. For every $X \in \mathrm{Ob}(C)$, by definition of natural transformation, we have a morphism $\eta_X : \mathrm{Hom}(X, A) \to F(X)$, so let consider the morphism

$$\eta_A : \mathrm{Hom}(A, A) \to F(A)$$

Then, we define the following map:

$$\begin{array}{rcl} \Phi_A : \mathrm{Hom}_{C^\wedge}(h_C(A), F) & \to & F(A) \\ \eta & \mapsto & \eta_A(\mathrm{id}_A) \end{array}$$

We have to prove now this map is a bijection. Let us start from surjectivity; let $a \in F(A)$, we look for a natural transformation η such that $\Phi_A(\eta) = a$. Given $X \in \mathrm{Ob}(C)$ we define:

$$\begin{array}{rcl} \eta_X : h_C(A)(X) = \mathrm{Hom}(X, A) & \to & F(X) \\ f & \mapsto & F(f)(a). \end{array}$$

The collection of all such $(\eta_X)_{X\in \mathrm{Ob}(C)}$ define a natural transformation. In fact, given $X, Y \in \mathrm{Ob}(C)$ and $f : X \to Y$, if we consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}(Y, A) \arrow[r, "\eta_Y"] \arrow[d, "f^*"'] & F(Y) \arrow[d, "F(f)"] \\
\mathrm{Hom}(X, A) \arrow[r, "\eta_X"'] & F(X)
\end{tikzcd}$$

given $\varphi \in \mathrm{Hom}(Y, A)$, we have:

$$\begin{aligned}
&(F(f)\circ\eta_Y)(\varphi) = F(f)(\eta_Y(\varphi)) = F(f)(F(\varphi)(a)) = \\
&\quad = (F(f)\circ F(\varphi))(a) = F(\varphi\circ f)(a) = \eta_X(\varphi\circ f) = \eta_X(f^*(\varphi)) = (\eta_X\circ f^*)(\varphi)
\end{aligned}$$

hence the diagram commutes and the η defined above is a natural transformation.

Moreover, clearly this η satisfies $\Phi(\eta) = a$. In fact:

$$\Phi_A(\eta) = \eta_A(\mathrm{id}_A) = F(\mathrm{id}_A)(a) = \mathrm{id}_{F(A)}(a) = a$$

hence this is the natural transformation we were looking for. For the injectivity, let us suppose ξ is a natural transformation such that $\Phi_A(\xi) = a$. We claim that $\xi = \eta$. For any $X \in \mathrm{Ob}(C)$, and for any $f \in \mathrm{Hom}(X, A)$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(A, A) \arrow[r, "\xi_A"] \arrow[d, "f^*"'] & F(A) \arrow[d, "F(f)"] \\
\mathrm{Hom}(X, A) \arrow[r, "\xi_X"'] & F(X)
\end{tikzcd}$$

Since the diagram commutes:

$$\begin{aligned}
&\xi_X(f) = \xi_X(f^*(\mathrm{id}_A)) = (\xi_X\circ f^*)(\mathrm{id}_A) = \\
&\qquad = (F(f)\circ\xi_A)(\mathrm{id}_A) = F(f)(\xi_A(\mathrm{id}_A)) = F(f)(a) = \eta_X(f)
\end{aligned}$$

and this proves that Φ_A is also injective hence it is a bijection. It remains to check this is functorial in A, i.e. given $B \in \mathrm{Ob}(C)$, and a morphism $f : A \to B$, the bijections defined above make the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{C^\wedge}(h_C(B), F) \arrow[r, "\Phi_B"] \arrow[d, "h_f^*"'] & F(B) \arrow[d, "F(f)"] \\
\mathrm{Hom}_{C^\wedge}(h_C(A), F) \arrow[r, "\Phi_A"'] & F(B)
\end{tikzcd}$$

Let us recall that every morphism $f : A \to B$ induces a covariant functor $h_f : h_C(A) \to h_C(B)$; moreover h_f induces a pullback

$$h_f^* : \mathrm{Hom}_{C^\wedge}(h_C(B), F) \to \mathrm{Hom}_{C^\wedge}(h_C(A), F).$$

Given $\eta \in \mathrm{Hom}_{C^\wedge}(h_C(B), F)$ we have:

$$(\Phi_A \circ h_f^*)(\eta) = \Phi_A(h_f^*(\eta)) = \Phi_A(\eta \circ h_f) = (\eta_A \circ f_*)(\mathrm{id}_A) = \eta_A(f)$$

$$(F(f) \circ \Phi_B)(\eta) = F(f)(\Phi_B(\eta)) = F(f)(\eta_B(\mathrm{id}_B))$$

to conclude we only have to prove the equality:

$$\eta_A(f) = F(f)(\eta_B(\mathrm{id}_B)).$$

To do this, let consider the following commutative diagram, this commutes since η is a natural transformation:

$$\begin{tikzcd}
\mathrm{Hom}(B, B) \arrow[r, "\eta_B"] \arrow[d, "f^*"'] & F(B) \arrow[d, "F(f)"] \\
\mathrm{Hom}(A, B) \arrow[r, "\eta_A"'] & F(A)
\end{tikzcd}$$

hence, by the commutativity, we have:

$$\eta_A(f) = \eta_A(f^*(\mathrm{id}_B)) = (\eta_A \circ f^*)(\mathrm{id}_B) = (F(f) \circ \eta_B)(\mathrm{id}_B) = F(f)(\eta_B(\mathrm{id}_B))$$

and this concludes the proof. □

We can finally close the section with the

Theorem A.3.2 (Yoneda Lemma)**.** *The functor h_C is fully faithful.*

Proof. Let set $A := h_C(Y)$, then by Lemma A.2 we have immediately:

$$\mathrm{Hom}_{C^\wedge}(h_C(X), h_C(Y)) \cong \mathrm{Hom}(X, Y).$$

□

From Yoneda Lemma we can deduce the following:

Corollary A.3.3. *Let h^- the Yoneda functor as above, then:*

$$X \cong Y \text{ in } C \iff h^X \cong h^Y \text{ in } C^\wedge$$

B Orbifolds

Definition B.1 (Orbifold chart)**.** Let X be a topological space. A complex orbifold chart on X is the assignment of a map:

$$U/G \xrightarrow{\varphi} V \subseteq X,$$

where $U \subseteq \mathbb{C}^n$ is a contractible open set endowed with a biholomorphic action of a finite group G, $V \subseteq X$ is an open set, and φ is a homeomorphism between U/G and V considered as topological spaces.

Definition B.2 (Subchart)**.** An orbifold chart

$$U'/G' \xrightarrow{\varphi'} V' \subseteq X$$

is said to be a *subchart* of

$$U/G \xrightarrow{\varphi} V \subseteq X$$

if

- $V' \subseteq V$;
- there is a group homomorphism $G' \to G$ and a holomorphic embedding $U' \to U$ such that:
  1. the embedding and the group homomorphism commute with the group action;
  2. the G'-stabilizer of every point in U' is isomorphic to the G-stabilizer of its image in U;
  3. the embedding commutes with the isomorphisms φ and φ'.

Definition B.3. Two orbifold charts V_1 and V_2 are called *compatible* if every point of $V_1 \cap V_2$ is contained in some chart V_3 that is a subchart of both V_1 and V_2.

We can finally give the definition of *orbifold*.

Definition B.4 (Orbifold)**.** A *smooth complex orbifold* is a topological space X endowed with a maximal atlas of orbifold charts.

Definition B.5. Let X be an orbifold and $x \in X$ a point. The stabilizer of x is the stabilizer in G of a preimage of x in U under φin some chart.

Remark B.1. The previous definition is well-posed; if we choose another chart or another preimage we will get an isomorphic stabilizer, though the isomorphism is not canonical.

The definition of morphism of orbifolds in full generality would lead us to many technical difficulties, however it is easy to define a morphism of orbifolds in the case where the fibres of the morphism are manifolds. Even of this is only a particular situation, it will be enough for our purposes.

Definition B.6. Let X, Y be orbifolds. A *map of orbifolds* $f : X \to Y$ *with manifold fibres* is the data of a continuous map between the underlying topological spaces $\widehat{f} : \widehat{X} \to \widehat{Y}$ together with the choice for every $y \in Y$ of a chart $\varphi_y : U_y/G \to V_y$ containing y, of a holomorphic map $F : U_x \to U_y$, of a lifting of the G-action on U_y to U_x commuting with F and of an isomorphism φ_x of U_x/G with an open suborbifold of X, such that $\varphi_y \circ F = \widehat{f} \circ \varphi_x$.

B.1 Orbifolds and moduli spaces

The link between moduli spaces and orbifolds is given by the following theorem:

Theorem B.1.1. *There exists a smooth compact complex* $(3g-3+n)$*-dimensional orbifold* $\overline{\mathcal{M}}_{g,n}$*, a smooth compact complex* $(3g-2+n)$*-dimensional orbifold* $\overline{\mathcal{C}}_{g,n}$*, and a map* $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$*, such that*

1. $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ *is an open dense suborbifold and* $\mathcal{C}_{g,n} \subseteq \overline{\mathcal{C}}_{g,n}$ *its preimage under* p*;*

2. *the fibres of* p *are stable curves of genus* g *with* n *marked points;*

3. *each stable curve is isomorphic to exactly one fibre;*

4. *the stabilizer of a point* $t \in \overline{\mathcal{M}}_{g,n}$ *is isomorphic to the automorphism group of the corresponding stable curve* C_t*.*

For more details one can see [HM98, Chapter 4].

Definition B.7. The space $\overline{\mathcal{M}}_{g,n}$ is called the *Deligne-Mumford compactification* of the moduli space $\mathcal{M}_{g,n}$. The family $p : \overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is called the *universal curve*.

Definition B.8. The set $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ parametrizing singular stable curves is called the *boundary* of $\overline{\mathcal{M}}_{g,n}$.

Remark B.2. Le us notice that the boundary is a suborbifold of $\overline{\mathcal{M}}_{g,n}$ of codimension 1, hence a divisor. Anyway, the term boundary should not lead to any confusion: as already stated $\overline{\mathcal{M}}_{g,n}$ is a smooth orbifold and the boundary points are as smooth as any other point of it.

A generic point of the boundary corresponds to a stable curve with exactly one node. If a point t corresponds to a stable curve C_t with k nodes, there are k local components of the boundary that intersect transversally at t. Each of these components is obtained by smoothening $k-1$ out of k nodes of C_t. Hence, the boundary is a divisor with normal crossings in $\overline{\mathcal{M}}_{g,n}$. The figure below (Fig. B.1), which we took from [Zvo14], shows two components of the boundary divisor in $\overline{\mathcal{M}}_{g,n}$ and the corresponding stable curves.

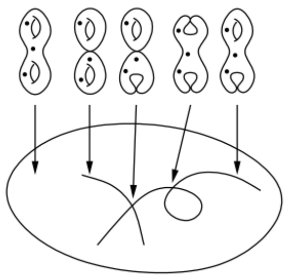


Figure B.1: Two components of the boundary divisor in $\overline{\mathcal{M}}_{g,n}$

Bibliography

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of Algebraic Curves: Volume II with a contribution by Joseph Daniel Harris*. Grundlehren der mathematischen Wissenschaften N. 268. Springer, 2011.

[Bor91] Armand Borel. *Linear algebraic groups*. New York: Springer, 1991 (cit. on p. 53).

[BS58] Armand Borel and Jean Pierre Serre. "Le théorème de Riemann-Roch". In: *Bulletin de la Société mathématique de France* 86 (1958), pp. 97–136. URL: http://www.numdam.org/article/BSMF_1958__86__97_0.pdf (cit. on p. 84).

[Cho56] Wei Liang Chow. "On the equivalence classes of cycles in an algebraic variety". In: *Annals of Mathematics* 64 (1956), pp. 450–479 (cit. on pp. ix, 51).

[DM69] Pierre Deligne and David Mumford. "The irreducibility of the space of curves of given genus". In: *Publications Mathématique de l'IHÉS* 36 (1969), pp. 75–109. URL: http://www.numdam.org/item/PMIHES_1969__36__75_0/.

[FP05] Carel Faber and Rahul Pandharipande. "Relative maps and tautological classes". In: *Journal of the European Mathematical Society* 7 (2005), pp. 13–49 (cit. on pp. vii, 68, 78).

[FP11] Carel Faber and Rahul Pandharipande. "Tautological and non-tautological cohomology of the moduli space of curves". In: (2011) (cit. on pp. 77, 78).

[Ful98] William Fulton. *Intersection theory*. 2nd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, 2. Springer, 1998 (cit. on pp. 51, 52, 55, 61, 62, 84).

[GP01] T. Graber and R. Pandharipande. "Constructions of nontautological classes on moduli spaces of curves". In: (2001). URL: https://arxiv.org/abs/math/0104057 (cit. on p. 68).

[Har97] Robin Hartshorne. *Algebraic Geometry*. 8th ed. Graduate Texts in Mathematics. Springer, 1997 (cit. on pp. 8, 23).

[HM98] Joe Harris and Ian Morrison. *Moduli of Curves*. Graduate tests in mathematics 187. Springer, 1998 (cit. on p. 112).

[Igu60] Jun-Ichi Igusa. "Arithmetic theory of moduli of genus two". In: 72 (1960), pp. 612–649 (cit. on p. 96).

[Kle05] Steven L. Kleiman. "The Picard scheme". In: *Fundamental algebraic geometry* Math. Surveys Monogr.123 (2005). Ed. by Amer. Math. Soc., pp. 235–321. URL: https://arxiv.org/abs/math/0504020 (cit. on p. 14).

[Knu83] Finn Knudsen. "The projectivity of the moduli space of stable curves, II". In: *Mathematica Scandinavica* 52 (1983), pp. 161–199. URL: http://eudml.org/doc/166839 (cit. on p. 70).

[Mum66] David Mumford. *Lectures on curves on an algebraic surface*. Ed. by Princeton University Press. Vol. Annals of Mathematics Studies. 59. 1966 (cit. on p. 53).

[Mum83] David Mumford. "Towards an Enumerative Geometry of the Moduli Space of Curves". In: *Arithmetic and Geometry: Papers Dedicated to I.R. Shafarevich on the Occasion of His Sixtieth Birthday. Volume II: Geometry*. Ed. by Michael Artin and John Tate. Boston, MA: Birkhäuser Boston, 1983, pp. 271–328. URL: https://doi.org/10.1007/978-1-4757-9286-7_12 (cit. on pp. vii, ix, x, 47, 67, 68, 94).

[Nam08] Norman Nam Van Do. *Intersection theory on moduli spaces of curves via hyperbolic geometry*. Doctoral thesis. Melbourne: Department of Mathematics and Statistics of the University of Melbourne, 2008.

[Pan18] Rahul Pandharipande. "A calculus for the moduli space of curves". In: *Proceedings of Symposia in Pure Mathematics* 97.1 (2018). URL: http://www.ams.org/books/pspum/097.1/.

[Sch20] Johannes Schmitt. *The moduli space of curves*. Lecture notes, draft. 2020. URL: https://www.math.uni-bonn.de/people/schmitt/moduli_of_curves (cit. on p. 40).

[Tav11] Mehdi Tavakol. *Tautological Rings of Moduli Spaces of Curves*. Doctoral thesis. Stockholm, 2011.

[Vak03] Ravi Vakil. “The moduli space of curves and its tautological ring”. In: *Notices of theAMS* 50.647-658 (2003). Ed. by Amer. Math. Soc., pp. 235–321. URL: `https://www.ams.org/notices/200306/fea-vakil.pdf` (cit. on pp. x, 68).

[Vak17] Ravi Vakil. *The rising sea. Foundations of algebraic geometry*. Lecture notes, draft. 2017. URL: `http://math.stanford.edu/%7Evakil/216blog/FOAGnov1817public.pdf` (cit. on pp. 22, 23, 30).

[Zvo14] Dimitri Zvonkine. *An introduction to the moduli spaces of curves and their intersection theory*. Lecture notes, draft. 2014. URL: `https://www-fourier.ujf-grenoble.fr/sites/ifmaquette.ujf-grenoble.fr/files/ete2011-zvonkine.pdf` (cit. on pp. 45, 46, 75, 87, 113).