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The AdS$_5$ x S^5 string and the one-loop correction to the circular Wilson loop

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Abstract

The thesis deals with the computation of the AdS_5 x S_5 string one-loop correction to the vacuum expectation value of the circular Wilson loop in the k^{th} rank totally symmetric representation.
We initially provide the theoretical foundations of the computation: we recall the statement of the AdS/CFT correspondence in the case of $\mathcal{N} = 4$ super Yang-Mills and the type IIB string in AdS_5 x S^5, briefly summarizing the main properties of these theories. We also describe how the correspondence simplifies in the limit of large number of colours and strong coupling. Once given the definition of Wilson loop operator and some of its relevant properties in the Yang-Mills and $\mathcal{N} = 4$ Super Yang-Mills Theory, we provide its dual representation in String Theory according to the original prescription.
Then we review the derivation of the string side formula for the one-loop correction to the circular Wilson loop in the k^{th} rank totally symmetric representation. We start with the Green-Schwarz action of the Type IIB String Theory in AdS_5 x S^5 and study the quadratic fluctuations around the classical solution with appropriated boundary conditions. It is shown that the one loop correction is equivalent to the partition function of a 2d field theory on the k-wrapped version of the euclidean AdS_2 with S^1 boundary.
Coherently with the physical content of the computation, we recall the definition of partition function in Quantum Field Theory on a curved space time for the scalar and the fermionic field. In this formalism the partition function for a particular field theory becomes essentially the logarithm of the determinant of a second order differential operator. Given this relation, we illustrate a standard method to regularize a determinant and compute its finite part based on the definition of heat kernel of the considered operator.
Afterards we review the computation for the $k = 1$ case and then we generalize it to a non trivial k using the methods previously introduced. Finally we discuss the obtained result by comparing it with the known gauge side expression and another string side prediction computed with a different method.

parametrize

Contents

1 Introduction

One of the most remarkable aspects of the perturbative String Theory is the existence of dualities between gauge theories and string solutions on a certain geometrical background. These dualities have to be considered as exact equivalences between the two involved theories and in principle allow to compute relevant quantities as scattering amplitudes, correlation functions and expectation values of operators on a side using its equivalent formulation and vice versa. These surprising relations turn out to be very useful in the study of a theory. Infact, a problem which is very hard to solve in a certain formulation could become considerably easier in the dual version, allowing to choose the most suited description according to the situation.
One of the most interesting dualities consists in the conjectured relation between conformal field theories and string solutions in Anti de Sitter background, known as AdS/CFT correspondence [1]. The most studied example in the last years of this duality is the relation between the $\mathcal{N} = 4$ Super Yang-Mills Theory in four dimensions and the Type IIB Super String Theory in AdS_5 x S^5. In general the computations on the string side are not easier than on the gauge side, so we cannot hope to compute relevant quantities in the gauge theory using the string description at an exact level.
Instead, a regime in which the correspondence becomes extremely useful is the so called 't Hooft limit [2]. Once redefined the coupling as $\lambda = g_{YM}^2 N$, where N is the number of colours, the duality provides a relation with the string low energy parameter

$$\frac{\alpha'}{R^2} = \lambda^{-1/2}. \tag{1.1}$$

Here α' is the string scale and R the radious of curvature of the target space.
In addition, we have a relation between the string coupling g_s and the gauge coupling:

$$g_s = 4\pi g_{YM}^2 = \frac{4\pi\lambda}{N}. \tag{1.2}$$

The 't Hooft limit consists in taking $N \to \infty$ and $g_{YM} \to 0$ so that λ remains fixed. According to 1.2 this is equivalent to have the classical limit $g_s \to 0$ on the string side. The second step prescribes to consider λ to be very large. The 1.1 converts the strong coupling regime on the gauge side in the low energy limit of the string solution. What we learn in the case of the $\mathcal{N} = 4$ Super Yang-Mills is that this theory can be studied for large N and at strong coupling using the perturbative Type IIB Superstring Theory in a weakly curved AdS background. The perturbative expansion can be performed simultaneously in g_s and $\frac{\alpha'}{R^2}$, which

regulate the quantum and the low energy expansion of the string respectively, with leading order the classical Type IIB Supergravity in AdS_5 x S^5.
One of the most interesting quantities that one can compute in the SU(N) Yang-Mills Theory is the vacuum expectation value of the Wilson loop operator. The definition of this operator is [3]:

$$W(C) = \mathrm{Tr}_R \; P \; e^{i \oint_C dx^\mu A_\mu}, \qquad \mu = 0, .., 3 \tag{1.3}$$

where $Pe^{\oint}$ is the path ordered exponential, C is a generic closed curve in four dimensions, A_μ is the Yang-Mills gauge field and the trace is performed on the SU(N) indices in a certain representation R. The vacuum expectation value of this operator can provide many informations about a gauge theory and in particular can reveal if a theory is confining or not. This becomes evident if we considered a rectangular loop of sides of leght T and L in the fundamental representation. In the limit of large time T this loop can be seen as a static system quark-anti quark separated from a distance L and existing for a time T which couples to the gauge field. In this case one can show that $\langle W(c) \rangle \sim e^{-E(L)T}$, where $E(L)$ is the ground state energy of the system, function of the distance between the two static particles. It is clear that from a rectangular loop one can learn how two particles of opposite charges interact and understand the nature of the interaction. For example an area law for the rectangular loop would imply $E(L) \sim L$ and consequently the confinement of the quark-anti quark couple.
The generalization to the case of the $\mathcal{N} = 4$ Super Yang-Mills can be obtained by dimensional reduction starting from the standard Wilson loop 1.3 in ten dimension. The operator that is naturally considered in this case is:

$$W(C) = \mathrm{Tr}_R \; P \; e^{i \oint_C dx^\mu (A_\mu + i|dx|\phi^I(x)\theta^I)}, \tag{1.4}$$

where A_μ and ϕ^I, $I = 1, ..., 6$ are respectively the gauge field and the six scalars of the $\mathcal{N} = 4$ Super Yang-Mills multiplet. The parameters θ^I, which parametrize the extra six coordinates of the loop in the original ten dimensional theory, here play the role of coupling constants for the scalar fields.
The vacuum expectation value of this object can be computed using the duality with the Type IIB String Theory. The correspondence prescribes to consider N coincident D3-branes at the boundary of AdS_5 with an open string stretched in the target space ending on it [4]. The fluctuations of the string generate the $\mathcal{N} = 4$ Super Yang-Mills multiplet living on the four dimensional AdS_5 boundary. The end-point of the string is seen from the perspective of the boundary as an external charged quark that couples to the Yang-Mills gauge field and the six scalars. In this context we obtain a definition of the Wilson loop by requiring that the string

end-point describes a closed curve on the AdS boundary. Explicitly, the duality maps the vacuum expectation value of the Wilson loop in the partition function of a string ending on the boundary of AdS_5 with a 2d world sheet enclosed by the considered loop:

$$W(C) = \int_{\partial X = C} [DX][D\theta][Dg] e^{-S_{\text{type IIB string}}}, \tag{1.5}$$

where we have used the Euclidean signature.
At the leading order the computation reduces to the research of the surface of minimal area with the loop as boundary:

$$\langle W(c) \rangle \sim e^{-S_{\text{type IIB supergravity}}} \sim e^{-A_{min}(C)}. \tag{1.6}$$

Our aim to go beyond the classical Supergravity limit and compute the one-loop correction to the Wilson loop in the $\mathcal{N} = 4$ Super Yang-Mills Theory using the string description. In particular we take in consideration the circular case in the k^{th} rank symmetric representation.
We initially review in detail the derivation of the AdS_5 x S^5 string prescription for the one-loop correction to the circular Wilson loop, which has been obtained originally in [5]. We start considering the partition function of an open type IIB string in AdS_5 x S^5 with 2d world sheet ending on the boundary of AdS_5 and enclosed by a circular loop. Then, using the Green-Schwarz formalism, we perform the perturbative expansion at the first order in the string coupling $g_s = \frac{4\pi\lambda}{N}$ and at the second order in the low energy parameter $\frac{\alpha'}{R^2}$, considering small quantum fluctuations around the classical solution. On the gauge side this is equivalent to compute the vacuum expectation value of the Wilson loop for large N and at the second order in the strong coupling limit. What we find is that the one-loop correction is equivalent to the partition function of a 2d field theory which consists in 8 real scalars, 5 massless and 3 with $m^2 = 2$, and 8 Majorana fermions with $m^2 = 1$. The computation has to be performed on a classical 2d world sheet which is the k-wrapped version of the euclidean AdS_2 with S^1 boundary, or equivalently AdS_2 with a conical singularity of negative angular deficit $\delta = 2\pi(1-k)$.
The reason why we are interested in the circular Wilson loop is that it represents one of the rare cases in which the expression of the vacuum expectation value is known exactly on the gauge side. In fact, with the exception of the straight line operator which preserves a half of supersymmetries and is protected by quantum corrections, the other Wilson loops receive non trivial contributions beyond the tree level. In [11] N.Drukker and D.J. Gross show that the expectation values of the circular and linear operators are related by a multiplicative factor, exploiting the fact that a circle can be obtained from a straight line by a conformal transformation.

This allows to know the gauge side expression of the circular Wilson loop at every order. In particular the one-loop correction is found to be [7]:

$$\Gamma_1(k) = \Gamma_1(1) + \frac{3}{2}\log k + \frac{3}{2}\log\sqrt{\lambda}, \qquad\qquad \Gamma_1(1) = \frac{1}{2}\log\frac{\pi}{2}. \tag{1.7}$$

This important result provides a consistency check for the computation we aim to do and gives the possibility to test the correspondence between the gauge and string theories.
The computation of partition functions is a well known problem in Quantum Field Theory. Up to a constant which depends on the type of field we are considering, this is equal to the logarithm of the determinant of a differential operator. In general the determinant of this type of operators is divergent and we need a method to regularize it and extract its finite part.
A standard method to perform this operation is based on the definition of heat kernel of a differential operator $\hat{O}$

$$K(t) = e^{-t\hat{O}}, \qquad t > 0. \tag{1.8}$$

This relevant object can be used to define an operatorial version of the Riemannian zeta-function in terms of the operator eigenvalues:

$$\zeta(s;\hat{O}) = \sum_{\lambda}\lambda^{-s}. \tag{1.9}$$

Through an analytic continuation, one can exploit such a function to remove the divergences of the determinant and keep only its finite part. This one can be obtained by derive the ζ-function in $s = 0$:

$$\log(\det[\hat{O}]) = \sum_{\lambda}\log(\lambda) = -\frac{d}{ds}\zeta(s;\hat{O})\mid_{s=0}. \tag{1.10}$$

The heat kernel method is one the most used in the determinant regularization and it turns out to be particularly suited to treat the conical singularities [6].
We want to underline that the matching between the gauge expression given in 1.7 and the string side prediction seems to be a quite hard problem. In fact, a first attempt has already been made by M.Kruczensky and A.Tirziu in [19]. In simple terms they first decompose the relevant two-dimensional spectral problems in one dimensional ones and, using a method developed in a series of paper [20, 21, 22, 23], they compute the involved one dimensional determinants without solving directly the eigenvalues equation. Their result following this procedure is:

$$\bar{\Gamma}_1(k) = \frac{1}{2}\log 2\pi + \left(2k + \frac{1}{2}\right)\log k - \log k!. \tag{1.11}$$

As one can see, this expression differs from the Drukker and Gross finding and the string prediction seems to fail in matching the gauge side one. However, we are not completely convinced of how Kruczensky and Tirziu take into account the conical singularity on the world sheet and, although the subtleties and complications of the computation, we try to give a different answer using the heat kernel method briefly summarized above.

The thesis is organized in three main parts:
In the first part we provide the theoretical background of the computation. We first give the general statement of the AdS/CFT correspondence in the case of the AdS$_5$ x S^5 Type IIB String and $\mathcal{N} = 4$ Super Yang-Mills Theory. We recall the main properties of the two theories and explore the main consequences of taking the large N and strong coupling limit on both sides. We go on with the Wilson loop operator: the definition and some of its relevant properties in the Yang-Mills Theory are exposed, as well as its generalization to the $\mathcal{N} = 4$ Super Yang-Mills case and its dual description in String Theory according to the AdS/CFT correspondence. In the last paragraph of this section we review the derivation of the string side formula for the one-loop correction to the vacuum expectation value of the circular Wilson loop in the k^{th} rank symmetric representation.
In the second part we describe in details the method that we use to perform the computation. We first define the partition function for a generic quantum field theory and in a second moment we specialize to the cases of our interest: the real scalar and fermionic fields on a curved geometry. For both the theories we give the demonstration for a compact Riemannian manifold that the partition function can be expressed as the logarithm of the determinant of a relevant differential operator. Then we illustrate the main aspects of the heat kernel method to regularize the partition function and extract its finite part. Particular emphasis in given to the techniques that can be used to compute the heat kernel in spaces with conical singularities in the bosonic and fermionic cases.
In the third part we proceed with the computation of the one-loop correction. Once reviewed the $k = 1$ loop which has been already treated in [7], we consider the case $k > 1$ using the method described in the second section.
Then, since we know exactly the gauge side expression and have at disposition another string computation obtained with a different method, we catch the opportunity to organize the final discussion as a double comparison between these findings and our result.
In Appendix A we recall important aspects about the spinor bundle of a fermionic field on a curved manifold which are relevant for the computations.

2 The Circular Wilson Loop in The AdS/CFT Correspondence

2.1 The $\mathcal{N} = 4$ Super Yang-Mills and The AdS$_5$ x S^5 Type IIB String

2.1.1 The AdS/CFT Correspondence

The AdS/CFT correspondence in its strongest form conjectures the exact equivalence between a conformal field theory and a string theory with an anti-de Sitter background as target space [1]. It is not known at the moment how to prove such dualities. In literature is usually given an intuitive argument in the case of the $\mathcal{N} = 4$ Super Yang-Mills Theory in four dimensions and the Type IIB String Theory in AdS$_5$ x S^5.
Let us consider N D3-branes sitting very close to each other and extended along a four dimensional plane in a ten dimensional space-time. In this background we can have both closed and open type IIB strings. The closed strings propagate in the bulk space and their fluctuations fill the empty space with the supergravity multiplet and the massive states of the type IIB theory. The open strings describe the excitations of the D3-branes, which contain in the massless sector the $\mathcal{N} = 4$ Super Yang-Mills multiplet. In this configuration the string theory defined on the bulk space and the gauge field theory living on the branes are strongly interacting and not much can be said about them. The coupling constant of the interaction lagrangian is proportional to the sigma-model parameter $\frac{\alpha'}{R^2}$, where α' is the string scale and R the radious of curvature of the target space. Taking the low energy limit $\frac{\alpha'}{R^2} << 1$ the open and closed strings become two distinct and completely decoupled faces of the physical system. From the perspective of the D3-branes, the interaction terms with the closed string can be neglected and the massive modes become infinitely heavy and non-dynamical. This limit leaves on the four dimensional world volume the pure $\mathcal{N} = 4$ Super Yang-Mills Theory generated by the open strings fluctuations. From the point of view of the bulk space, the system is described by the low energy limit of the closed string excitations, which define the Type IIB Supergravity. In this optics the branes can be seen as massive charged objects which act as sources for the supergravity fields. The D3-brane Type IIB Supergravity solution is found to be [24]:

$$ds^2 = f^{-1/2} dx^\mu dx_\mu + f^{1/2}(dr^2 + r^2 d\Omega_5^2),$$
$$f = 1 + \frac{R^4}{r^4}. \tag{2.1}$$

Here the index $\mu = 0, .., 3$ can be contracted with the Minkoskian or Euclidean metric and Ω_5 is the five dimensional solid angle.
In the limit of large radious of curvature $R >> 1$, we can approximate $f \sim \frac{R^4}{r^4}$ and the metric reduces to:

$$ds^2 = \frac{r^2}{R^2} dx^\mu dx_\mu + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2. \tag{2.2}$$

Setting $\frac{r}{R^2} = \frac{1}{y}$ we recognize the geometry of AdS$_5$ x S^5 in Poincarè coordinates:

$$ds^2_{\mathrm{AdS_5\ x\ S^5}} = R^2 \frac{dx_\mu dx^\mu + dy^2}{y^2} + R^2 d\Omega_5^2. \tag{2.3}$$

In conclusion the low energy limit leaves on the target space the Type IIB Supergravity defined on a weakly curved AdS$_5$ x S^5 background.
On the base of these considerations, one is naturally led to identify this two descriptions and conjecture the duality between the $\mathcal{N} = 4$ Super Yang-Mills Theory in four dimension and the Type IIB String in AdS$_5$ x S^5.
The correspondence prescribes how to compute relevant quantities like correlators and expectation values of operators of the gauge theory using the dual string description. Assuming to work in the coordinates given above, the boundary of AdS$_5$ can be defined by setting $y = 0$ only after a procedure of conformal compactification which removes the singularity of the metric in this set of points. Once solved this ambiguity, it is induced on this space the metric of a four dimensional plane. Now it is consistent to ask that the N D3-branes containing the CFT are arranged along the AdS$_5$ boundary. The precise statement of the correspondence [1] is that for any operator O of the CFT living on the conformal boundary of AdS$_5$, there is a field, which we generically indicate with ϕ, belonging to the bulk theory and carrying the same conformal quantum numbers. A common example is the correspondence between the stress tensor of the CFT and the Graviton of the theory in the target space.
The boundary condition of the field, which we indicate with $\phi(x, y = 0) = \phi_0$, couples to the operator O and acts as an external source for the generating functional of the correlators $\langle O(x_1)...O(x_n)\rangle$. The correspondence claims that this functional is equal to the partition function of the dual field with boundary condition ϕ_0:

$$Z_{bulk}[\phi(x,y=0)=\phi_0(x)] = \left\langle e^{\int d^4x\ \phi_0(x)O(x)} \right\rangle_{CFT}. \tag{2.4}$$

The correlators of the operator can be obtained by deriving the field partition function respect to ϕ_0 and then setting $\phi_0 = 0$.
One of the main ways to test the correspondence is to compute operator expectation values and correlators using this prescription and compare the result with the gauge side expression whether it is known.
In the following paragraphs we recall the main properties of the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions and the type IIB string theory in AdS_5 x S^5. We consider in particular the Euclidean version of these theories. The reason is that the string partition functions which the correspondence propose to compute are well defined only on a space with Euclidean signature (see section 3).

2.1.2 The $\mathcal{N} = 4$ Super Yang-Mills Theory

The $\mathcal{N} = 4$ Super Yang-Mills in four dimension is a supersymmetric field theory characterized by a unique fields multiplet transforming in the adjoint representation of SU(N). The degrees of freedom of this theory are a gauge field A_μ, $\mu = 0,..,3$, six scalars field ϕ^I, $I = 1,...,6$, and four Weyl spinors $\chi_{\alpha i}, \bar{\chi}_{\dot{\alpha}\bar{i}}$, $i, \bar{i} = 1,..,4$. Here $\alpha, \dot{\alpha} = 1,2$, are respectively the left and right chiral indices. This field content can be obtained from the $\mathcal{N} = 1$ super Yang-Mills theory in ten dimensions by dimensional reduction. This one contains a gauge field A_M, $M = 0,...,9$ and a Majorana-Weyl Spinor Ψ of 16 real components. All these degrees of freedom transorm in the adjoint representation of SU(N). Compactifying 6 spatial dymensions on the six dimensional torus T^6, the gauge field A_M decomposes in the bosonic sector of the $\mathcal{N} = 4$ Super Yang-Mills Theory in four dimension:

$$A_M \longrightarrow A_\mu,\ \phi^I, \tag{2.5}$$

while the spinor field Ψ decomposes in the four Weyl spinors of the fermionic sector:

$$\Psi \longrightarrow \chi_{\alpha i}, \bar{\chi}_{\dot{\alpha}\bar{i}}. \tag{2.6}$$

The Euclidean low energy lagrangian describing the theory is:

$$\mathcal{L} = \frac{1}{g_{YM}^2} \mathrm{Tr} \left[\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^I D^\mu \phi^I - \sum_{I<J} [\phi^I, \phi^I]^2 + i\bar{\Psi}\Gamma^\mu D_\mu \Psi + i\bar{\Psi}\Gamma^I [\phi^I, \Psi] \right],$$

$$F_{\mu\nu} = [D_\mu, D_\nu],$$

$$D_\mu \phi^I = \partial_\mu \phi^I + i[A_\mu, \phi^I], \qquad D_\mu \Psi = \partial_\mu \Psi + i[A_\mu, \Psi]. \tag{2.7}$$

Here $F_{\mu\nu}$ is the Yang-Mills field strenght and D_μ is the SU(N) covariant derivative in the adjoint representation. To get a more compact notation in the lagrangian the Weyl spinors are left embedded in the unique fermionic field Ψ using the ten dimensional Dirac matrices (Γ^μ, Γ^I).
In the original ten dimensional theory the fermionic generators of the super Poincarè algebra can be groupped in a Majorana-Weyl spinor of 16 components. Since the compactification preserves the number of supersymmetries, the four dimensional theory have the same amount of charges that can be distributed in four Majorana spinors of four real indipendent components.
The $\mathcal{N} = 4$ super Yang-Mills is also a conformal field theory. The Euclidean conformal group in four dimension is $SO(1,5)$. We can obtain this group by adding tho the Poincarè group the scale transformation $x_\mu \longrightarrow \lambda x_\mu$ and the inversion $x_\mu \longrightarrow \frac{x_\mu}{x^2}$ which maps point at small distances to infinity and vice versa. The commutations between the usual fermionic generators of the Super Poincarè algebra $\{Q_A\}$, $A = 1, ...16$, and these additional transformations give rise to extra 16 supercharges $\{S_A\}$, which are customarily called special conformal generators. What we have complessively is a superconformal field theory in four dimensions with 32 supercharges.
The theory has also a global $SO(6)$ symmetry. A_μ does not carry any index respect to this group and it is also said a singlet. The six scalars carry an index I which can be naturally intepreted as vector representation label. Therefore they can be seen as the components of a vector of $SO(6)$. Thinking in the same way, the four fermions carry the index $i, \bar{i}$ of the fundamental of $SU(4)$, which is the spinorial representation of $SO(6)$. This is also called "R" symmetry and the generators commute with those of the conformal group but in general not with the supercharges.
In conclusion the bosonic sector of the symmetry group is $SO(1,5)$ x $SO(6)$, while the fermionic sector contains, as said before, 32 supercharges.

2.1.3 The type IIB super string in $\mathbf{AdS_5}$ x $\mathbf{S^5}$

Differently from the Bosonic Theory where we can have only one type of string, in the ten dimensional Super String Theory we dispose of a discrete number of choices which give rise to 5 types of theories. This variety is mainly due to supersymmetry and the consequent presence of fermions on the string world sheet.
A Dirac spinor in ten dimension has 32 complex components which correspond to 64 real ones. We can impose on this spinor either the Majorana condition or the Majorana-Weyl condition. The first one is equivalent to require that the spinor is real. This means to have 32 real components. In the second case we also project the spinor on a chiral subspace, remaining with 16 real components. In terms of supersymmetry generators these two possibilities correspond respectively to a theory with $\mathcal{N} = 2$ supersymmetries and a theory with $\mathcal{N} = 1$. The super strings with 32 supercharges are called of Type II, while those with a half of charges are the Type I and the Heterotic String. In the Type II family we have two further different theories according to an another possible choice that one can do. This consists in considering fermions of the opposite or the same chirality in the left and right mooving sector of the world sheet. In the first case we obtain the Type IIA Theory, while in the second one we have the theory in which we are interested: the Type IIB.
As mentioned above, this is a supersymmetric theory with $\mathcal{N} = 2$ supersymmetries, for a total of 32 supercharges. The bosonic massless fluctuations of the closed string are the Graviton G_{MN}, $M, N = 0, ...9$, two anti symmetric 2-forms, which are called the Neveu-Schwarz gauge field B_2 and the Ramond-Ramond gauge field F_2, two 0-forms, that are known as the Dilaton Φ and the Axion φ, and the Ramond-Ramond 4-form C_4 with self-dual field strenght F_5^*. Instead, the massless fermionic sector contains two spin 3/2 Gravitinos $\Psi^A_{M\,1,2}$ and two spin 1/2 Dilatinos $\psi^A_{1,2}$ of the same chirality. The type of forms present in the massless sector detrmine the type of D-branes that we can have in a string theory.
A D-brane can be thought as the generalization in higher dimension of electromagnetic charged particle to wich a form field can couple. In an arbitrary number of dimension d a n-form A_n with field strenght $F_{n+1} = dA_n$ can couple electrically to a $(n-1)$-brane with interaction term the pull-back of the form on the n-dimensional world volume V:

$$S_{\text{int}} = \int_V A_n. \tag{2.8}$$

Taking into account also magnetically charged objects, a n-form can couple magnetically to a $d-n-3$-brane introducing a $d-n-2$-form $\tilde{A}_{d-n-2}$ defined by the condition $d\tilde{A}_{d-n-2} = *F_{n+1}$, where $*F_{n+1}$ is a $d-n-1$ form and represents the hodge dual of the n-form field strenght. The interaction term can be written as the pull-back of $\tilde{A}_{d-n-2}$ on the $d-n-2$ dimensional world volume $\tilde{V}$:

$$S_{\text{int}} = \int_{\tilde{V}} \tilde{A}_n. \tag{2.9}$$

In the ten dimensional type IIB string theory we have complessively: two 0-forms which couple eletrically to a -1-brane and magnetically to a 7-brane, two 2-forms which couple electrically to a 1-brane and magnetically to a 5-brane and a 4-form which couple electrically to a 3-brane and magnetically to a 3-brane. In summary in this theory there are only the odd branes $-1, 3, 5, 7$.
The type IIB string that we want to consider propagates in AdS_5 x S^5.
The Euclidean AdS_5 is a five dimensional maximally symmetric space defined as an hypersurface embedded in $R^{1,5}$ by the condition :

$$-X_{-1}^2 + X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2. \tag{2.10}$$

Starting with the $R^{1,5}$ metric :

$$ds^2_{R^{1,5}} = \eta_{IJ} dX^I dX^J, \qquad\qquad \eta_{IJ} = diag(-1,1,1,1,1,1), \tag{2.11}$$

we can realize the embedding of AdS_5 in this space by choosing: $X_{-1} + X_4 = R/y$, $X_\mu = Rx_\mu/y$ for $\mu = 0, .., 3$. With this choice we recover the AdS_5 metric in Poincarè coordinates written in 2.3 (the AdS_5 part) and consequently the flat metric induced on the conformal boundary.
S^5 is the five dimensional sphere embedded in six dimension. It is a maximally symmetric space whose defining condition can be obtained from the 2.10 through the analityc continuation $X_{-1} \longrightarrow iX_{-1},\ R \longrightarrow iR$:

$$X_{-1}^2 + X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = R^2. \tag{2.12}$$

The metric of the product space can be written as sum of the two metrics:

$$ds^2_{\mathrm{AdS}_5\ \mathrm{x}\ \mathrm{S}^5} = R^2 \frac{dx_\mu dx^\mu + dy^2}{y^2} + R^2 d\Omega_5^2, \tag{2.13}$$

which we had already introduced in 2.3.
The Killing group of this background is $SO(1,5)$ x $SO(6)$, which is the product of the Killing groups of respectively AdS_5 and S^5. The Killing isometry is transported

as global symmetry on the world sheet of the type IIB string embedded in this space. In conclusion we have a string theory with a $SO(1,5)$ x $SO(6)$ symmetry group and 32 supercharges, which is the same symmetry content of the $\mathcal{N}=4$ super Yang-Mills previously discussed.

2.1.4 The correspondence in the 't Hooft limit

The AdS/CFT correspondence conjectures the exact equivalence between the two theories described above. At this level is hard to work out the consequences of this non trivial duality because both the sides are in general complicated system.
A more useful version of the correspondence which allows to make predictions regarding the gauge theory using the string description can be obtained by considering the so called 't Hooft limit at strong coupling[2]. In fact, if in this regime the gauge theory becomes strongly coupled, the string theory, on the contrary, becomes weakly interacting and can be treated perturbatively.
We start defining the 't Hooft coupling as $\lambda = g_{YM}^2 N$, where N is the number of colours of the SU(N) theory. The correspondence puts this coupling constant in relation with the parameter which governs the low energy expansion of the string action:

$$\frac{\alpha'}{R^2} = \lambda^{-1/2}. \tag{2.14}$$

The string scale α' is a parameter which measures the charactericstic lenght scale of the string fluctuations and has the dimension of a square lenght.
In general the string lagrangian contains an infinite number of interaction terms with coupling constants of increasing power of $\frac{\alpha'}{R^2}$.
The correspondence prescribes also a relation between the string coupling g_s and the Yang-Mills coupling:

$$g_s = 4\pi g_{YM}^2 = \frac{4\pi\lambda}{N}. \tag{2.15}$$

The string coupling g_s is the coupling constant which describes the interactions of the string. The expansion in g_s is equivalent to sum over world sheets with increasing genus in the string path integral and regulates the quantum corrections to the classical action.
The first step to realize the 't Hooft regime is to take the limit of a large number of colours $N \longrightarrow \infty$ and a small Yang-Mills coupling $g_{YM} \longrightarrow 0$ so that λ remains fixed. According to 2.15, this is equivalent to have $g_s << 1$ on the string side.
At this stage we have a duality between the $\mathcal{N}=4$ Super Yang-Mills Theory at

large N and the classical limit of the Type IIB String Theory in AdS_5 x S^5.
Since g_s is very small, the string theory is weakly interacting and becomes possible to work perturbatively, ignoring world sheets of higher genus where the computations are considerably hard. However, although the semplification of working with the world sheet of minimal genus, we still have to do with a complicated 2d-field theory with an infinite number of interaction terms. Therefore, if we want to get more profound consequences from the duality, we need another step which simplifies the computations on the string side.
The next passage consists in considering the 't Hooft coupling to be very large $\lambda >> 1$. The 2.14 tells us that this is equal to have $\frac{\alpha'}{R^2} << 1$ on the string side, or in other worlds to take the low energy limit of the string theory.
When α' is very small compared to the radious of curvature we can neglect the interaction terms in the string lagrangian with higher powers of the parameter, remaining with a treatable 2d-field theory with a finite number of couplings. From the point of view of the string spectrum, in this limit the high energy modes of the string become negligible and only the low energy dynamics is relevant. The massive states of the string, whose squared mass goes like $\sim \frac{1}{\alpha'}$, become infinitely massive and they cannot be excited. This means that their fluctuations can be ignored, leaving us with the low energy dynamics of the massless excitations described in the previous paragraph.
The correspondence at the leading order of this limit becomes an equivalence between the $\mathcal{N} = 4$ SU(N) Super Yang-Mills Theory for large N at strong coupling and the classical low energy limit of the Type IIB String, which is also called the classical Type IIB Supergravity.
At this level the duality becomes extremely fruitful: it can be used to compute corrections to correlation functions and vacuum expectation values of operators for the strongly coupled gauge theory on the string side, where is possible to perform a perturbative expansion in either the string coupling or the low energy parameter.
Our aim is to use the correspondence in this limit to compute the vacuum expectation value of the circular Wilson loop operator for the $\mathcal{N} = 4$ SU(N) Super Yang-Mills Theory for large N at strong coupling. In particular we want to go beyond the classical Supergravity level and truncate the perturbative expansion at the first order in g_s and the second order in $\frac{\alpha'}{R^2}$.

2.2 The Wilson Loop Operator

2.2.1 The Yang-Mills Wilson Loop

The Wilson loop is a gauge invariant operator whose vacuum expectation value can diagnose if a theory is confining or not. A SU(N) gauge theory is said confining if all

the finite energy states are singlets respect to the symmetry group, or equivalently there cannot exist free charged particles. The confinement is a non-perturbative phenomenum and cannot be studied with the usual perturbative techniques. The Wilson loop has been introduced by Wilson in 1974 in the attempt to quantize a Yang-Mills gauge theory on a lattice and explain the quark confinement in QCD. The lattice gauge theory is the most natural enviroment where defining this operator, but the definition can be generalized to the continuum case [3].
Let us start defining the Wilson link between two points x^μ and $x^\mu + \varepsilon^\mu$ in a four dimensional Euclidean spacetime, where ε^μ is infinitesimal:

$$W(x+\varepsilon, x) = e^{ig\varepsilon^\mu A_\mu(x)}, \tag{2.16}$$

where g is the coupling constant and A_μ is a N x N hermitian traceless matrix gauge field. In order to determine the transformation property of the Wilson link under a gauge transormation, we expand it at the first order in ε:

$$W(x+\varepsilon, x) = I + ig\varepsilon^\mu A_\mu(x) + O(\varepsilon^2). \tag{2.17}$$

A_μ transforms under a local SU(N) transformation as:

$$A_\mu \to U(x) A_\mu U(x)^+ + \partial_\mu U(x) U(x)^+, \tag{2.18}$$

where $U(x)$ is a local SU(N) matrix. Consequently the Wilson link transformation pattern is:

$$\begin{aligned} W(x+\varepsilon, x) \to I + ig\varepsilon^\mu U(x) A_\mu U(x)^+ + ig\varepsilon^\mu \partial_\mu U(x) U(x)^+ + O(\varepsilon^2) = \\ \left[\left(I + ig\varepsilon^\mu \partial_\mu\right) U(x)\right] U(x)^+ + ig\varepsilon^\mu U(x) A_\mu U(x)^+ + O(\varepsilon^2). \end{aligned} \tag{2.19}$$

Writing $(I+\varepsilon^\mu \partial_\mu) U(x) = U(x+\varepsilon) + O(\varepsilon^2)$ in the first term and replacing U(x) with $U(x+\varepsilon) + O(\varepsilon)$ in the second one we get :

$$W(x+\varepsilon, x) \to U(x+\varepsilon)(I + ig\varepsilon^\mu A_\mu) U(x)^+ + O(\varepsilon^2), \tag{2.20}$$

or equivalently

$$W(x+\varepsilon, x) \to U(x+\varepsilon) W(x+\varepsilon, x) U(x)^+, \tag{2.21}$$

where we are committing an error of order $O(\varepsilon^2)$.
Furthermore, the definition of Wilson link implies $W^+(x+\varepsilon,x) = e^{-ig\varepsilon^\mu A_\mu(x)} = W(x-\varepsilon,x)$. Shifting the variable $x \to x+\varepsilon$ and ignoring terms of order $O(\varepsilon^2)$ we find:

$$W^+(x+\varepsilon,x) = e^{-ig\varepsilon^\mu A_\mu(x+\varepsilon)} = W(x,x+\varepsilon). \tag{2.22}$$

Now we consider a path P in the space time between two points x and y made of a sequence of infinitesimal vector ε_j, $j = 1,...,n$, with $y = x + \varepsilon_1 + ... + \varepsilon_n$. The Wilson line between x and y is defined as the ordered multiplication of the Wilson links corresponding to the infinitesimal displacements ε_j:

$$W_P(y,x) = W(y,y-\varepsilon_n)W(y-\varepsilon_n,y-\varepsilon_n-\varepsilon_{n-1})....W(x+\varepsilon_2+\varepsilon_1,x+\varepsilon_1)W(x+\varepsilon_1,x). \tag{2.23}$$

Using 2.20 and 2.21 the Wilson line transforms under a local gauge transformation as:

$$W_P(y,x) \to U(y)W_P(y,x)U^+(x), \tag{2.24}$$

and the hermitian coniugation has the effect to reverse the order of the product:

$$W_P(y,x)^+ = W_{-P}(x,y), \tag{2.25}$$

where $-$P is the reverse of the path P.
Now we can consider a closed path C wich starts and end at the same point x and take the trace of the correspondent Wilson line:

$$W_C = \mathrm{Tr}\ W_C(x,x). \tag{2.26}$$

This is the so called Wilson loop operator and can be written simbolically as path-ordered exponential:

$$W_C = \mathrm{Tr}\left[P\ e^{ig\oint_C dx^\mu A_\mu(x)}\right]. \tag{2.27}$$

Since the closed Wilson line transforms under SU(N) as $W_C(x,x) \to U(x)W_C(x,x)U(x)^+$, the Wilson loop is manifestly locally gauge invariant. The trace of the Wilson line

can be taken in a generic SU(N) representation. As for the Wilson line, the Hermitian conjugation inverts the path verse: $W_C^+ = W_{-C}$.
The quantity in which we are interested is the vacuum expectation value (vev) of the Wilson loop operator:

$$\langle 0|W_C|0\rangle = \int DA_\mu \mathrm{Tr}\left[P\ e^{ig\oint_C dx^\mu A_\mu(x)}\right]\ e^{-S_{YM}[A_\mu(x)]} = \\ = \left\langle \mathrm{Tr}\left[P\ e^{i\oint_C dx^\mu A_\mu(x)}\right]\right\rangle, \tag{2.28}$$

where $S_{YM}[A_\mu(x)] = \int d^4x\ \frac{1}{2g^2}\mathrm{Tr}\ [F_{\mu\nu}F^{\mu\nu}]$.
To get an intuition of the physical meaning of this object we consider the case of a rectangular loop of size R x T in the fundamental representation.

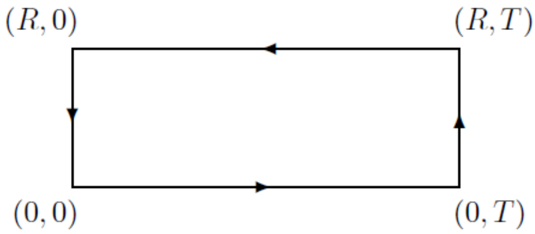

Figure 1: Rectangular loop

The sides of the rectangle are taken to be along the time direction x_0 and the spatial direction x_1 as shown in figure 1.
The phase phactor $ig\oint_C dx^\mu A_\mu(x)$ can be formally seen as an interaction term where the loop $i\oint_C dx^\mu$ represents a charged particle in the fundamental representation mooving along the contour C and acting as an external non dynamical source for the gauge field. In the limit $T >> R$ the rectangular loop tends to two parallel lines with opposite orientation along the time direction separated by a distance R. A charged particle propagating backwards in time is equivalent to a particle with opposite charge mooving in the positive time direction. Therefore the loop in figure can be interpreted as a static couple particle-anti particle separated by a distance R and existing for a long time T. In this limit the vev of the Wilson loop is related to the particle-anti-particle groundstate interaction energy $E_0(R)$:

$$W(\mathrm{R\ x\ T}) \propto e^{-E_0(R)T}, \quad T >> R \tag{2.29}$$

This relation can be easily proved [8] choosing the axial gauge $A_0 = 0$. With this choice we have to take into account only the circuitation along the vertical segments in figure.
The Wilson line relative to a segment $[0, R]$ at a fixed time t is :

$$W_{ij}(t) = \left[P \ e^{\ ig \int_0^R dx^1 A_1(x^1,...,t)} \right]_{ij}, \tag{2.30}$$

where $i, j = 1, ..., N$ are the gauge indices of the fundamental representation.
The vev of the Wilson loop can be rewritten as:

$$\langle 0|W(\text{R x T})|0\rangle = \left\langle 0|W_{ij}(0)W^+_{ji}(T)|0\right\rangle. \tag{2.31}$$

Using the relations:

$$W_{ij}(t) = U^+(t)W_{ij}(0)U(t), \qquad\qquad |0\rangle_H = U^+(t)\,|0\rangle_S, \tag{2.32}$$

where H, S denote respectively the Heisenberg and Schroedinger picture and $U(t)$ is the time evolution operator, we obtain the action of the Wilson line on the vacuum:

$$W_{ij}(t)\,|0\rangle_H = U^+(t)W_{ij}(0)U(t)\,|0\rangle_H = U^+(t)W_{ij}(0)\,|0\rangle_S. \tag{2.33}$$

Inserting in the expression 2.31 the completeness relation:

$$I = \sum_n |n\rangle\langle n| \tag{2.34}$$

where $|n\rangle$ are the energy eigenstates of the system gauge field-external charges, we find:

$$\begin{aligned} \langle 0|W(\text{R x T})|0\rangle &= \sum_n \langle 0|\,W_{ij}(0)\,|n\rangle\langle n|\,W^+_{ji}(T)\,|0\rangle \\ &= \sum_n \langle 0|\,W_{ij}(0)\,|n\rangle\langle n|\,U^+(T)W^+_{ji}(0)\,|0\rangle \\ &= \sum_n \langle 0|\,W_{ij}(0)\,|n\rangle|^2 e^{-E_n(R)T}, \end{aligned} \tag{2.35}$$

where in the second line we have used the 2.33. Here $E_n(R)$ represent the energy of the state $|n\rangle$ and is in general a function of the distance R . It is clear that when $T \to \infty$ only the low energy state which survives in the sum 2.35 and we find:

$$W(\mathrm{R\ x\ T}) \longrightarrow e^{-E_0(R)T}, \quad T >> R. \tag{2.36}$$

So from the rectangular Wilson loop we can learn how two static particles of opposite charge interact, obtaining useful information about the nature of the interaction.
To give a concrete example, we consider an abelian $U(1)$ gauge theory, where the vev of the rectangular loop can be computed exactly [3].
In the abelian case the phase factor becomes a true integral, since fields of different Wilson links commute. The expression for a generic loop is:

$$\langle 0|W_C|0\rangle = \int DA_\mu \; e^{ig\oint_C dx^\mu A_\mu(x)} \; e^{-S[A_\mu(x)]}, \tag{2.37}$$

where $S[A_\mu(x)] = \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}$.
If we formally identify the loop $ig\oint_C dx^\mu$ as an external source $J^\mu(x)$, the expression above is equivalent to the generating functional of the green functions for the free electromangnetic theory:

$$\langle 0|W_C|0\rangle = Z[\oint_C dx^\mu] = \exp\left(-\frac{1}{2}g^2\oint_C dx^\mu \oint_C dy^\mu \Delta_{\mu\nu}(x-y)\right), \tag{2.38}$$

where $\Delta_{\mu\nu}(x-y)$ is the free photon propagator.
In Feynman gauge we find:

$$
\begin{aligned}
\Delta_{\mu\nu}(x-y) =&\delta_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-y)}}{k^2} = \\
=&\delta_{\mu\nu}\frac{4\pi}{(2\pi)^4}\int_0^\infty \frac{k^3dk}{k^2}\int_0^\pi d\theta \sin^2\theta e^{ik|x-y|\cos\theta} = \\
=&\delta_{\mu\nu}\frac{4\pi}{(2\pi)^4}\int_0^\infty kdk\ \frac{\pi J_1(k|x-y|)}{k|x-y|} = \\
=&\frac{\delta_{\mu\nu}}{4\pi^2(x-y)^2}\int_0^\infty du J_1(u) = \\
=&\frac{\delta_{\mu\nu}}{4\pi^2(x-y)^2}
\end{aligned}
\tag{2.39}
$$

where $J_1(u)$ is a Bessel function.
Now we compute the 2.38 for the rectangular loop. The double line integral that we have to evaluate is :

$$
\oint_{\mathrm{R\ x\ T}}\oint_{\mathrm{R\ x\ T}}\frac{dx\cdot dy}{(x-y)^2} \tag{2.40}
$$

When x and y are on the same side of the rectangle we get:

$$
\int_0^L\int_0^L\frac{dxdy}{(x-y)^2} = 2L/a - 2\log(L/a) + O(1) \tag{2.41}
$$

where L is the lenght of the considered side (R or T), a is a short distance cut-off introduced to regularize the divergent integral and $O(1)$ is a numerical constant depending on the regularization procedure. If x and y are on perpendicular sides we have $dx\cdot dy = 0$ and the double line integral vanishes. If x is on the side of lengt R and y is on the other one, the contribution goes like R^2/T^2, which can be neglected in the limit $T >> R$. Finally, if x is on the side of lenght T and y on the other one we get:

$$
\int_0^T\int_0^T\frac{dxdy}{(x-y)^2+R^2} = \pi T/R - 2\log(T/R) - 2 + O(R^2/T^2). \tag{2.42}
$$

Adding all the contributions, we obtain in the limit of large T :

$$\oint_{\mathrm{R\ x\ T}} \oint_{\mathrm{R\ x\ T}} \frac{dx \cdot dy}{(x-y)^2} = (4/a - 2\pi/R)\,T + O(\log T). \tag{2.43}$$

Expressing the coupling in terms of the structure constant $\alpha^2 = g^2/4\pi$ we find:

$$\langle 0|W_{\mathrm{R\ x\ T}}|0\rangle = \exp\left[-\left(\frac{2\alpha}{\pi a} - \frac{\alpha}{R}\right)T\right]. \tag{2.44}$$

Comparing this result with the general expression given in 2.36 we can extract the interaction energy of the particle-anti-particle pair:

$$E_{\mathrm{pair}}(R) = \frac{2\alpha}{\pi a} - \frac{\alpha}{R}. \tag{2.45}$$

The first term in the expression represent the divergent self-interaction energy of the two point particles, while the second one is the well known Coulomb potential $V(R) = -\alpha/R$. What we lear from the Wilson loop is that the electromagnetism is not a confining theory, since the two particles are free at large distances.
In the case of a non-abelian SU(N) Yang-Mills theory on a lattice is possible to prove the area law for a generic Wilson loop in the fundamental representation [3], [8]:

$$\langle 0|W_C|0\rangle \sim e^{-\tau(g)A_{\min}(C)}, \tag{2.46}$$

where $A_{\min}(C)$ is the minimal area enclosed by the contour C and $\tau(g)$ is a function of the Yang-Mills coupling. Is easy to see that the area law for the Wilson loop implies confinement. This is clear when we consider again the rectangular loop R x T with $T >> R$. In this case:

$$\langle 0|W_C|0\rangle \sim e^{-\tau(g)RT} \sim e^{-E(R)T}. \tag{2.47}$$

From this expression we see that the interaction potential of a quark-anti quark pair in the fundamental representation is $V(R) = \tau(g)R$. The two particles interact with a linear potential wich increases with the mutual distance. Because it requires an infinite amount of energy to bring the charges at an infinite distance, these are confined. The constant $\tau(g)$ is also called the string tension, since from the linearity of the potential one can think that that the two quarks are connected by a string with energy per unit lenght (i.e. the string tension) $\tau(g)$.
It is generally assumed that if the area law holds for generic loops of large area, the SU(N) theory is confining. It is not clear whether the Yang-Mills theories, and in

particular the QCD, have the same behavour in the continuum limit, even if the experiments strongly suggest this.

2.2.2 The $\mathcal{N} = 4$ Super Yang-Mills Generalization

We want to generalize the previous definition of Wilson loop to the case of the $\mathcal{N} = 4$ Super Yang-Mills Theory. We start from the definition of Wilson loop for the $\mathcal{N} = 1$ Super Yang-Mills Theory in ten dimension:

$$W_C = \mathrm{Tr}\left[P\ e^{i\sqrt{\frac{\lambda}{N}}\oint_C dX^M A_M(x)}\right], \tag{2.48}$$

where we have expressed the Yang-Mills coupling in terms of the 't Hooft coupling. Here C is a closed contour in ten dimension parametrized by $X^M(s) = \big(x^\mu(s), y^I(s)\big)$, where $x^\mu(s)$ and $y^I(s)$ describe respectively the projection of the loop on R^4 and the orthogonal space R^6. As explained before, the $\mathcal{N} = 4$ Super Yang-Mills can be obtained from this theory by compactification of six spatial coordinates. In particular $A_M(x)$ decomposes in the four dimensional gauge field $A_\mu(x)$ and the six scalars $\phi^I(x)$ transforming in the adjoint representation.
Rewriting the integral measure as $dX^M = ds\ \dot{X}^M(s)$, where $\dot{X}^M(s) = (\dot{x}^\mu(s), \dot{y}^I(s))$ is the loop tangent vector, and performing the dimensional reduction, we obtain the following definition of Wilson loop:

$$W_C \longrightarrow \mathrm{Tr}\left[P\ e^{i\sqrt{\frac{\lambda}{N}}\oint_C ds\big(\dot{x}^\mu(s)A_\mu(x)+\dot{y}^I(s)\phi^I(x)\big)}\right]. \tag{2.49}$$

In this case the contour integration C has to be intended as a curve in four dimension obtained by projecting the original ten dimensional loop on R^4.
Since we are dealing with a four dimensional theory, the extra six components $y^I(s)$ of the tangent vector cannot be indipendent degrees of freedom and have to be expressed in terms of $\dot{x^\mu}(s)$. To achieve this, we first reparametrize the six coordinates as $y^I(s) = |\dot{y}(s)|\theta^I(s)$, with $\theta^I(s) = \dot{y^I}(s)/|\dot{y}(s)|$ and $\theta^2 = 1$.
In the most applications the unit lengh tangent vector $\theta^I(s)$ is taken to be indipendent of s, but in general it can be not costant along the loop.
The module $|\dot{y^I}(s)|$ should be real by definition, but, if we forget that the expression $\dot{y^I}(s)\phi^I(x)$ descends from a ten dimensional loop and we reinterpret it from the perspective of the $\mathcal{N} = 4$ Super Yang-Mills as a coupling term for the six scalars, we can relax this constraint and choose it to be complex. There are many choices of possible reparametrization and each of them exhibits very different properties. We are not interested in a complete classification of all the possible Wilson loops, but rather in the most common choice that is :

$$|\dot{y}| = i|\dot{x}|, \qquad \theta^I = \text{constant}. \tag{2.50}$$

There are many reasons why this type of loop is the most sudied. First of all it admits a dual description in the AdS_5 x S^5 string theory, as we will explain later, and therefore can be studied in the 't Hooft limit using the AdS/CFT correspondence. Moreover, the constraint above gives to the Wilson loop very special properties: UV finiteness and local supersymmetry [2], [9].
Although the $\mathcal{N} = 4$ super Yang-Mills is a finite UV theory, expectation values of composite operators may require a regularation. It turns out that with the choice $|\dot{y}| = i|\dot{x}|$ the divergences coming from the gauge field exactly cancel with the scalars ones. We can see it expanding the Wilson loop for small values of the 't Hooft coupling:

$$\begin{aligned}
&\left\langle \text{Tr}\left[P\ e^{i\sqrt{\frac{\lambda}{N}}\oint_C ds\ [\dot{x^\mu}(s)A_\mu(x)+\dot{y^I}(s)\phi^I(x)]}\right]\right\rangle = \dim(\text{Rep})+\\
&+i\sqrt{\frac{\lambda}{N}}\text{Tr}\ P\ \oint_C ds\left(\dot{x^\mu}(s)\left\langle A_\mu(x)\right\rangle + \dot{y^I}(s)\left\langle \phi^I(x)\right\rangle\right)\\
&-\frac{\lambda}{N}\text{Tr}\ P\ \oint_C ds\oint_C dt\left[\left\langle A_\mu(x)A_\nu(x)\right\rangle \dot{x^\mu}(s)\dot{x^\nu}(t) + + \left\langle \phi^I(x)\phi^J(x)\right\rangle \dot{y^I}(s)\dot{y^J}(t)\right]\\
&-\frac{\lambda}{N}\text{Tr}\ P\ \oint_C ds\oint_C dt\left[\left\langle A_\mu(x)\phi^J(x)\right\rangle \dot{x^\mu}(s)\dot{y^J}(t) + \left\langle \phi^I(x)A_\nu(x))\right\rangle \dot{x^\nu}(t)\dot{y^I}(s)\right] + O(\lambda^2)\\
&= \dim(\text{Rep})\left(1-\frac{\lambda}{N}\ P\ \oint_C ds\oint_C dt\left[\frac{\delta_{\mu\nu}}{(x(s)-x(t))^2}\dot{x^\mu}(s)\dot{x^\nu}(t) + \frac{\delta_{IJ}}{(x(s)-x(t))^2}\dot{y^I}(s)\dot{y^J}(t)\right]\right)\\
&+O(\lambda^2) = \dim(\text{Rep})\left(1-\frac{\lambda}{N}\ P\ \oint_C ds\oint_C dt\left[\frac{|\dot{x}|^2+|\dot{y}|^2}{(x(s)-x(t))^2}\right]\right) + O(\lambda^2)
\end{aligned} \tag{2.51}$$

In the fifth line we have used the expression for the free propagator of the gauge field and the six scalars, while all the correlators in the fourth line vanish because for reason of gauge and space-time symmetry.
We see that the double line integral in the last line is divergent. Infact when $x(s) \to x(t)$ the integrand goes like $\frac{1}{\epsilon^2}$ and the integral like $\frac{1}{\epsilon}$.
Thus the divergent part of the first perturbative correction is:

$$\langle W_C \rangle^{\text{div}}_{O(\lambda)} \sim \frac{\lambda}{\epsilon} P \oint_C ds(|\dot{x}|^2 + |\dot{y}|^2) \tag{2.52}$$

If $|\dot{y}| = i|\dot{x}|$ the gauge field contribution cancels the scalars one and the Wilson loop is finite at this order. If we consider only smooth curves, this seems to hold at every order and can be verified also for large values of the coupling using the AdS/CFT correspondence. On the contrary, if singularities like cusps are present along the contour, there could be divergences also for this loop.
As mentioned before, another important property of this reparametrization is the large amount of preserved supersymmetry.
The global transformations which leaves the action 2.7 invariant for the bosonic fields are:

$$\delta_\varepsilon A_\mu(x) = \bar{\Psi}\Gamma_\mu \varepsilon, \qquad \delta_\varepsilon \Phi^I(x) = \bar{\Psi}\Gamma^I \varepsilon, \tag{2.53}$$

where the parameter ε is a 16 components Majorana-Weyl spinor.
The Wilson loop transorms under these as:

$$\delta_\varepsilon W_C = \text{Tr}\left[P \left(i\sqrt{\frac{\lambda}{N}} \oint_C ds\ \bar{\Psi}\left(\dot{x^\mu}(s)\Gamma_\mu + i|\dot{x}(s)|\theta^I \Gamma^I \right)\varepsilon \right) e^{(\dots)} \right]. \tag{2.54}$$

We see that $\delta_\varepsilon W_C = 0$ if ε satisfies the equation:

$$\begin{gathered} \left(\dot{x^\mu}(s)\Gamma_\mu + i|\dot{x}(s)|\theta^I \Gamma^I \right)\varepsilon = \Gamma'(s)\varepsilon = 0, \\ \Gamma'(s) = \dot{x^\mu}(s)\Gamma_\mu + i|\dot{x}(s)|\theta^I \Gamma^I. \end{gathered} \tag{2.55}$$

The number of indipendent solutions of this equation determines the number of conserved supersymmetries. The matrix $\Gamma'(s)$ is a combination of 10-d Dirac matrices of 32 x 32 components, but, taking into account that it acts on a Weyl spinor, one can consider the chiral projection which is a 16 x 16 matrix. It is easy to see that $\Gamma'(s)$ squares to 0:

$$
\begin{aligned}
&\left(\dot{x^\mu}(s)\Gamma_\mu + i|\dot{x}(s)|\theta^I\Gamma^I\right)^2 = \\
&= \dot{x^\mu}(s)\dot{x^\nu}(s)\Gamma_\mu\Gamma_\nu - \dot{x}(s)^2\theta^I\theta^J\Gamma^I\Gamma^J + i\dot{x^\mu}(s)|\dot{x}(s)|\theta^I\ \{\Gamma_\mu,\Gamma_I\} = \\
&= \frac{1}{2}\dot{x^\mu}(s)\dot{x^\nu}(s)\{\Gamma_\mu,\Gamma_\nu\} - \frac{1}{2}\dot{x}(s)^2\theta^I\theta^J\{\Gamma^I,\Gamma^J\} + i\dot{x^\mu}(s)|\dot{x}(s)|\theta^I\ \{\Gamma_\mu,\Gamma_I\} = \\
&= \dot{x}(s)^2 - \dot{x}(s)^2 = 0,
\end{aligned}
\tag{2.56}
$$

where we have used the Clifford algebra: $\{\Gamma_\mu,\Gamma_\nu\} = 2\delta_{\mu\nu}$, $\{\Gamma^I,\Gamma^J\} = 2\delta^{IJ}$, $\{\Gamma_\mu,\Gamma_I\} = 0$.

As a consequence of this property the kernel and the image of $\Gamma'(s)$ coincide and both have dimension 8. This means that for each s the 2.55 has 8 indipendent solutions and the loop retains locally a half of supersymmetry. However, supersymmetry is not a local symmetry of the action. If we want to have global supersymmetry we need to require that the basis of solutions is the same for each point of the loop, or equivalenty that Γ' does not depend of s. This happens when $\dot{x^\mu}(s)$ is constant, or equivalently when C is a straight line. In this case W_C is a 1/2 operator which commutes with half of supercharges. Consistently with this property, the straight line loop seems to be protected from radiative corrections and can be exactly computed at tree level. What one finds in the straight line case is that the Wilson loop phase factor vanishes, since the integration contour is covered in both the directions and the two contributes cancel each other. Consequently the vev of the corresponding Wilson loop is :

$$
\langle W(\text{Straight Line})\rangle = 1. \tag{2.57}
$$

The straight line loop is the operator that preserves the higher number of supersymmetries. Other types of loop commute with less supercharges and in general they can receive non trivial quantum corrections [10].

The case in which we are interested is the circular loop. The circular Wilson loop does not preserve any supersymmetry, even if it commutes with 8 linear combination of the Poincarè supercharges and the special conformal generators [10].

It is important to point out that a straight line and a circle are related by conformal transformation. For example a circle on the plane passing through the origin and parametrized by $x(s) = (1+\cos s, \sin s)$ transforms under the inversion $x_\mu \to \frac{x_\mu}{x^2}$ as:

$$
x(s) = (1+\cos s, \sin s) \longrightarrow x(s) = \frac{1}{2}(1, \tan s/2). \tag{2.58}
$$

Since the $\mathcal{N} = 4$ super Yang-Mills is a conformal theory one could think that also the vev of the circular Wilson loop is '1'. However, this is not the case. Infact, when one performs the necessary transformation to turn a straight line into a circle, non trivial quantum anomalies arise and the vev of the circular Wilson loop differs from the straight one by a multiplicative conformal anomaly which can be computed exactly. The transformation properties of a generic Wilson loop under the inversion $x_\mu \to \frac{x_\mu}{x^2}$ has been studied in [11]. In this paper is shown that the circular and the straight line cases in the fundamental representation are related in the 't Hooft limit by:

$$\langle W(\text{Circle})\rangle_{N\to\infty,\lambda=\text{fixed}} = \frac{2}{\sqrt{\lambda}} J_1(\sqrt{\lambda})\, \langle W(\text{StraighLine})\rangle = \frac{2}{\sqrt{\lambda}} J_1(\sqrt{\lambda}), \tag{2.59}$$

where J_1 is a Bessel function.
This result can be generalized to the case of a circular Wilson loop in the k^{th} rank symmetric representation:

$$\langle W(\text{Circle})\rangle^{k^{\text{th}}\text{rank}}_{N\to\infty,\lambda=\text{fixed}} = \frac{2}{k\sqrt{\lambda}} J_1(k\sqrt{\lambda}) = e^{k\sqrt{\lambda}-\Gamma_1(k)+\dots}. \tag{2.60}$$

For large λ and fixed k the leading term is $\langle W_C\rangle = e^{k\sqrt{\lambda}}$. The first subleading correction goes like $O(\lambda^0)$ and reads:

$$\Gamma_1(k) = \Gamma_1(1) + \frac{3}{2}\log k + \frac{3}{2}\log\sqrt{\lambda}, \qquad \Gamma_1(1) = \frac{1}{2}\log\frac{\pi}{2}. \tag{2.61}$$

As explained previously, the strong coupling limit of the $\mathcal{N} = 4$ super Yang-Mills theory is conjectured to be equivalent to the classical low energy limit of the type IIB string in AdS_5 x S^5. Our purpose is to test this duality in the 't Hooft limit. Explicitly we aim to compute the equivalent on the string side of the first subleading correction at strong coupling to the vev of the circular Wilson loop in the k^{th} rank symmetric representation and compare it with the gauge side result given above. In the next section we give the string intepretation of the Wilson loop and the prescription to compute its perturbative corrections on the string side.

2.2.3 The Wilson Loop in String Theory

The Wilson loop of the $\mathcal{N} = 4$ Super Yang-Mills Theory can be represented through the AdS/CFT correspondence as an open string in AdS_5 x S^5 which ends on the boundary of AdS_5 [4].

A convenient choice of coordinates for AdS_5 x S^5 to provide this interpretation is the one introduced in section 2.1.3. In this frame AdS_5 is described by the Poincare coordinates (X^μ, Y) and the boundary is identified by the condition $Y = 0$.
Now let us consider $N + 1$ coincident, flat, four dimensional D3-branes extended in AdS_5 x S^5 and connected by an open type IIB string. The fluctuations of the string generate a $\mathcal{N} = 4$ Super Yang-Mills Theory with SU(N+1) gauge group which lives on the world volume of the branes and is described at low energy by the action given in 2.7. The potential which determines the interaction between the six scalars is:

$$V(\phi^I) = \sum_{I<J} [\phi^I, \phi^I]^2. \tag{2.62}$$

It is easy to see that this potential leads to a spontaneous symmetry breaking of the gauge group SU(N+1). In fact it is semipositive definite and therefore it vanishes only when all the scalar fluctuations commute, or equivalently when they are diagonal:

$$\langle \phi^I \rangle = \mathrm{diag}(\phi^I_1, ..., \phi^I_{N+1}), \tag{2.63}$$

where the diagonal element ϕ^I_j, $I = 1, ..6,\ j = 1, ...N + 1$, represents the coordinate of the n^{th} brane respect to the I^{th} transverse direction to the brane.
We break the gauge group SU(N+1) to SU(N) x U(1) choosing the vacuum expectation value:

$$\langle \phi^I \rangle = \mathrm{diag}(0, ...0, U\theta^I), \qquad \theta^2 = 1, \tag{2.64}$$

In this configuration we have an open string ending on N coincident D3-branes extended along the flat conformal boundary of AdS_5 and a D3-brane which has been separated from the other ones at a certain distance U from the boundary. Furthermore, the system of branes sits at a fixed position θ^I in S^5 . After the symmetry breaking we remain with a $\mathcal{N} = 4$ super Yang-Mills theory with SU(N) gauge group defined on the boundary of AdS_5 and a U(1) theory living on the isolated brane. We have also N massive W-bosons living in the bulk space which arise from the symmetry breakdown of the original gauge group. The mass of these fields is equal to the mass of the open string stretched from the boundary of AdS at the distance U :

$$M_W = TU, \tag{2.65}$$

where $T = \frac{1}{2\pi\alpha'}$ is the string tension.
These bosons couple to the fields living on the branes: they are charged under the U(1) gauge field, transform in the fundamental representation of SU(N) and interact with the six scalars. In this configuration the SU(N) theory and the U(1) theory are not separated and there are interactions between them.
In order to completely separate the two theories we have to send the single brane to an infinite distance $U \to \infty$ from the other N in AdS_5 . In this limit the U(1) theory can be ignored and we remain with the $\mathcal{N} = 4$ super Yang-Mills coupled to the W-bosons. These fields become in this limit infinitely massive and play the role of an external non-abelian source for the fields on the boundary of AdS_5. According to the 2.65, we can naturally think to have an open string stretched in AdS_5 and composed of heavy non dynamical quarks (intended as the W-bosons) in the fundamental representation of SU(N) which ends on the D3-branes at some fixed position θ^I in S^5. The end-point of the string is seen from the perspective of the brane as a charged particle which mooves along a curve on the world volume and couples to the fields of the super Yang-Mills theory.
The representation of the Wilson loop in this context can be obtained by requiring that the trajectory of the string end-point forms a closed contour on the AdS boundary. To clarify this passage we need to recall the general prescription of the AdS/CFT correspondence. As explaned in section 2.1.1, for each operator of the CFT on the brane there exhists a field of the bulk theory whose boundary condition plays the role of source in the generating functional of the operator correlators. Furthermore, this is identified by the conjecture with the field partition function under appropriate boundary conditions.
If we formally think a loop as a source for the gauge field and the scalars, we can see the vev of the corresponding Wilson loop as a generating functionional of the fields correlators:

$$Z\left[\oint_C ds\dot{x^\mu}(s), \oint_C ds|\dot{y(s)}|\theta^I\right] = \left\langle \mathrm{Tr}\left[P\ e^{i\sqrt{\frac{\lambda}{N}}\oint_C ds\ [\dot{x^\mu}(s)A_\mu(x)+|\dot{y(s)}|\theta^I\phi^I(x)]}\right]\right\rangle. \quad (2.66)$$

Here $J^\mu = \oint_C ds\dot{x^\mu}(s)$ and $J^I = \oint_C ds|\dot{y(s)}|\theta^I$ are the sources which couple respectively to the gauge field $A_\mu(x)$ and the scalars $\phi^I(x)$. These currents, from the perspective of the AdS_5 x S^5 string, are the boundary conditions of the open string stretched in AdS_5 and ending on its boundary at some point of S^5. Therefore, according to the AdS/CFT prescription, the equivalent of the Wilson loop on the string side is the partition function of a type IIB open string ending on the boundary of AdS_5 [4]

$$\langle W(C)\rangle = \sum_{\text{topologies}} (g_s)^{-\chi} \int_{\partial X = C} [DX][D\Theta][Dg] e^{-S_{\text{type IIB string}}}, \tag{2.67}$$

where X^M, $M = 0, ...9$, and Θ^J, $J = 1, 2$ are bosonic and fermionic (Green-Schwarz) variables wich describe the string embedding in the target space and g is the world sheet metric. The sum in the path integral is performed over worldsheet of Euler character χ. The string partition function has to be computed with boundary condition the considered loop:

$$\begin{gathered} X^\mu(\sigma_0, \sigma_1 = 0) = x^\mu(\sigma_0), \qquad Y(\sigma_0, \sigma_1 = 0) = 0, \qquad X^I(\sigma_0, \sigma_1 = 0) = \theta^I, \\ \partial_0 X^\mu(\sigma_0, \sigma_1 = 0) = \dot{x}^\mu(\sigma_0), \qquad \partial_0 X^I(\sigma_0, \sigma_1 = 0) = \dot{y}^I(\sigma_0) = |\dot{y}(\sigma_0)|\theta^I. \end{gathered} \tag{2.68}$$

Here $\sigma_{0,1}$ are the worldsheet coordinates and $\sigma_1 = 0$ parametrizes the string endpoint. Since the correspondence prescribes to have a string lying in the anti de Sitter space, the world sheet has to be thought as a two-dimensional surface embedded in AdS_5 and enclosed by the contour C on the boundary. It is important to note that this requirement is consistent if the S^5 point at which the string ends is the same for each point of the loop, or equivalently if θ^I is fixed. On the contrary, if $\theta^I(\sigma_0)$ depended on σ_0 the string would end for each point of the loop at a different point in S^5 and one should consider also a wrapping around the five sphere.

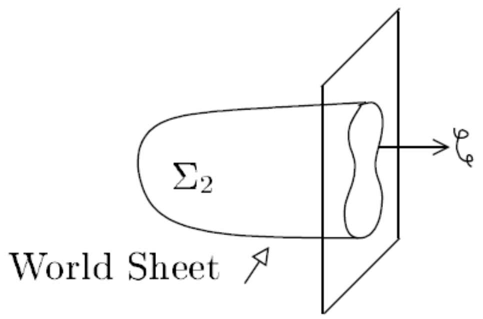

Figure 2: World sheet embedded in AdS_5 ending on the loop

This alternative formulation of Wilson loop turns out be extrmely useful when one is interested to compute the vev of the operator for large N at strong coupling. Infact, as explained in the section 2.1.4, in this limit the dual string theory can be treated perturbatively and become relatively simple to approach.
The limit $N \to \infty$ on the gauge side is translated in $g_s \to 0$ on the string side and

has the effect to project the string path integral at the leading order on the world sheet with minimum χ:

$$\langle W(C)\rangle \to \frac{1}{g_s}\int_{\partial X=C}[DX][D\Theta][Dg]e^{-S_{\text{type IIB string}}}. \tag{2.69}$$

Even if one can neglect world sheets of higher genus, the vev of the Wilson loop is still hard to compute because the action of the string contains an infinite number of coupling terms. To give a concrete idea, let us write the bosonic part of the type IIB string embedded in AdS_5 x S^5 is:

$$S_B = \frac{R^2}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ij}\partial_i X^M\partial_j X^N G_{MN}(X) \tag{2.70}$$

where $G_{MN}(X)$ is the AdS_5 x S^5 metric and $i,j=0,1$ are the worldsheet indices. According to the non-linear sigma model, one can shift the path integral variable by a quantum fluctuation $\frac{\sqrt{2\pi\alpha'}}{R}\xi$ and expand the metric in normal coordinates around the classical solution $\bar{X}$:

$$G_{MN}\left(\bar{X}+\frac{\sqrt{2\pi\alpha'}}{R}\xi\right) = \delta_{MN} - \frac{2\pi\alpha'}{R^2}\mathcal{R}_{MPNQ}(\bar{X})\xi^P\xi^Q + O\left(\frac{\alpha'^4}{R^2}\right), \tag{2.71}$$

where $\mathcal{R}_{MPNQ}$ is the Riemann tensor.
In this non-perturbative model the target space metric is thought as an infinite collection of coupling terms involving the bosonic fluctuations and going like increasing powers of $\frac{\alpha'}{R^2}$. In the limit in which the 't Hooft coupling is very large, the low energy parameter $\frac{\alpha'}{R^2}$ is very small and one can truncate the expansion of the metric at some order, obtaining a 2d-field theory on the string world sheet with a finite number of interaction terms.
At the leading order the fermionic sector does not give any contribution, while the bosonic action is proportional to the area of the classical world sheet embedded in the target space with certain boundary conditions.
More precisely, the computation of the vev of the Wilson loop at the classical supergravity level reduces to the problem of finding the surface of minimal area lying in AdS_5 and ending on the considered loop at boundary of this space:

$$\langle W(C)\rangle \sim e^{-\frac{\sqrt{\lambda}}{2\pi}A_{\min}(C)}, \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}. \tag{2.72}$$

The problem of finding a minimal surface at fixed boundary conditions in the context of the AdS_5 x S^5 string has been carefully studied in [9]. What one finds is that this

is a well defined problem if the string boundary conditions satisfy the additional constraint $|\dot{y(\sigma_0)}| = i|\dot{x(\sigma_0)}|$.
Therefore the string definition of Wilson loop is compatible with the parametrization of the scalars coupling that we have studied on the gauge side in the previous section. A simple case that we can immediately consider to test the correspondence is the straight line loop. Once subtracted the divergence with a renormalization procedure [2], the area of the surface enclosed by a straight line is null and we simply recover the gauge side result:

$$\langle W(\text{Straight Line})\rangle = 1. \tag{2.73}$$

In the next section we study the quantum quadratic fluctuations around the classical supergravity action of the type IIB string and derive the correction at the first order in g_s and second order in $\frac{\alpha'}{R^2}$ to the string partition function describing the circular loop.

2.3 The Semiclassical Partition Function of The AdS$_5$ x S^5 String

2.3.1 The Green-Schwarz Action

We review the derivation of the string partition function which has the intepretation on the gauge side of one-loop correction to the vev of the circular Wilson loop operator. The original derivation can be found in [5].
In this section we will use for convenience the following notation: $i, j = 0, 1$ and $\alpha, \beta = 0, 1$ denote respectively the 2d-world and tangent space indices; $a, b = 0, ...4$ and $p, q = 1, ..5$ are the tangent space indices of AdS$_5$ and S$_5$; $\mu, \nu = 0, ..., 9$ and $\hat{a}, \hat{b} = 0, .., 9$ are the curved and tangent space indices of the ten dimensional space-time. More informations regarding the main aspects of the tangent and spinor bundle that we will use in this section can be found in Appendix A.
In order to quantize the Type IIB String in AdS$_5$ x S^5 it is preferible to use the GS action, since it is difficult to use the RNS formalism in precence of the Ramond-Ramond 5-form included in this background. The firs step is to construct the GS action of a type IIB string in AdS$_5$ x S^5. In the RNS formalism one starts with the well known bosonic string action and introduces on the world sheet the necessary fermionic degrees of freedom to obtain complessively a 2-d supersymmetric theory. However in this formalism the supersymmetry of the space-time in which the string propagates is not manifest. The GS formalism has the prerogative to achieve the space-time supersymmetry by mapping the string world sheet directly into the su-

perspace. The fields describing the world sheet degrees of freedom in this formalism are:

$$X^\mu(\sigma_0, \sigma_1), \qquad \Theta^I(\sigma_0, \sigma_1), \tag{2.74}$$

where X^μ are bosonic fields describing the embedding of the string in AdS_5 x S^5 and Θ^I, $I = 1, 2$ stand for two positive chirality spinors satisfying the Majorana condition:

$$\Psi^I = \begin{pmatrix} \Theta^I \\ 0 \end{pmatrix}, \qquad \Psi^I = C\bar{\Psi}^{I^T}. \tag{2.75}$$

.

The Weyl condition projects the Dirac spinor Ψ^I into the positive chirality subspace, while the Majorana constraint is equivalent to a reality request. Therefore the fermionic degrees of freedom are described by two ten dimensional Majorana-Weyl spinors of 16 real components, reflecting the fact that we are dealing with a $\mathcal{N} = 2$ supersymmetric theory of 32 supercharges. The two spinors are choosen of the same chirality compatibly with the Type IIB String theory.
The supersymmetry transformations

$$\delta_\varepsilon X^\mu = \bar{\varepsilon^I}\Gamma^\mu\Theta^I \qquad \delta_\varepsilon\Theta^I = \varepsilon^I \tag{2.76}$$

are seen from the world sheet point of view as a global symmetry involving both the bosonic and the fermionic fields. Here Γ^μ are real 10-d Dirac gamma matrices.
The bosonic part of the action for a string in AdS_5 x S^5 is [5]

$$S_{\mathrm{B}} = \frac{R^2}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ij}\partial_i X^\mu\partial_j X^\nu G_{\mu\nu}(X), \tag{2.77}$$

where $G_{\mu\nu}(X)$ is the AdS_5 x S^5 metric and g_{ij} is the worldsheet metric.
S_{B} is equal to the Polyakov action for a bosonic string on a curved background and from the perspective of the world sheet it describes a 2-d field theory of ten scalars with the metric playing the role of a non linear self-interaction.
The structure of the full GS fermionic action in AdS_5 x S^5 is rather complicated, but, since we are interested in studyng the quadratic fluctuation around the classical solution, we only need its quadratic part which is relatively simple [12]:

$$S_{2\mathrm{F}} = \frac{1}{2\pi\alpha'}\int d^2\sigma\left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}S^{IJ}\right)\bar{\Theta}^I\rho_i D_j\Theta^J. \tag{2.78}$$

Here ρ_i are the projections of the 10-d Dirac gamma matrices on the world sheet

$$\rho_i = \Gamma_{\hat{a}} E^{\hat{a}}_{\mu} \partial_i X^{\mu} = \left(\Gamma_a E^a_{\mu} + \Gamma_p E^p_{\mu}\right) \partial_i X^{\mu} \tag{2.79}$$

and $E^{\hat{a}}_{\mu} = (E^a_{\mu}, E^p_{\mu})$ is the vielbein of the AdS$_5$ x S^5 metric $G_{\mu\nu} = E^{\hat{a}}_{\mu} E^{\hat{b}}_{\nu} \delta_{\hat{a}\hat{b}}$.
The 10-d Dirac matrices $\Gamma^{\hat{a}}$ are split in 5+5 as $\Gamma^a = \gamma^a \otimes I_4 \otimes \sigma_1$ and $\Gamma^p = I_4 \otimes \gamma^p \otimes \sigma_2$, where $\sigma_{1,2}$ are the Pauli matrices, (γ^a, γ^p) are the Dirac 4 x 4 matrices of the AdS$_5$ and S^5 spinor bundle and I_4 is a 4 x 4 unit matrix.
The explicit form of the covariant derivative $D_i \Theta^I$ is:

$$D_i \Theta^I = \left(\delta^{IJ} \mathcal{D}_i - \frac{i}{2} \epsilon^{IJ} \tilde{\rho}_i\right) \Theta^J, \qquad \mathcal{D}_i = \partial_i + \frac{1}{4} \partial_i X^{\mu} \Omega^{\hat{a}\hat{b}}_{\mu} \Gamma_{\hat{a}\hat{b}}. \tag{2.80}$$

The first piece consists in the projection on the world sheet of the 10-d spinor bundle derivative. This one contains the spin connection $\Omega^{\hat{a}\hat{b}}_{\mu}$ and the generators in the spinorial representation of the $SO(10)$ rotations $\frac{1}{2}\Gamma_{\hat{a}\hat{b}} = \frac{1}{4}\left[\Gamma_{\hat{a}}, \Gamma_{\hat{b}}\right]$. The presence of the spinor derivative makes the fermionic action manifestly invariant under local $SO(10)$ transformations on the spinor bundle. This represents a local symmetry for the GS action, since the parameters of the rotations depends on X^{μ} which are themselves functions of the world sheet coordinates.
The second term in the derivative originates from the coupling of the space-time fermions to the RR field strenght and contains the matrices $\tilde{\rho}_i$ defined as:

$$\tilde{\rho}_i = \left(\Gamma_a E^a_{\mu} + i \Gamma_p E^p_{\mu}\right) \partial_i X^{\mu}. \tag{2.81}$$

These matrices differ in general from the ρ_i unless the worldsheet solution is constant on S^5.
The tensor S^{IJ} is defined by $S^{11} = 1$, $S^{22} = -1$, $S^{12} = S^{21} = 0$ and the convention for the epsilon symbol is $\epsilon^{12} = 1 = -\epsilon^{21}$, $\epsilon^{11} = \epsilon^{22} = 0$.
From the perspective of the worldsheet the action in 2.78 describes a 2-d field theory of 16 Majorana fermions which are embedded in the two Majorana-Weyl space-time spinors.
The GS action $S_{GS} = S_B + S_{2F}$ inherits as additional global symmetry the Killing isometry of the AdS$_5$ x S^5 background with group $SO(1,5)$ x $SO(6)$. Moreover it is manifestly invariant under local change of coordinates of the target space.
Although the Type IIB String Theory is by construction supersymmetric, the GS action does not seem to have world sheet supersymmetry. Infact a superficial count of the degrees of freedom gives 10 scalars and 16 Majorana fermions. However we still have to take into account the gauge symmetries of the world sheet, which reveal

the presence of unphysical degrees of freedom.
The GS action is left invariant by the transformation [12]:

$$\delta_k \Theta^I = \tilde{\rho}_i^{\dagger} k^{iI} + ..., \qquad \delta_k X^\mu = \bar{\Theta}^I \Gamma^\mu \delta_k \Theta^I, \tag{2.82}$$

where k^{iI} are Majorana-Weyl spinor of 16 components.
The variations above represent the leading order term of the k-symmetry tranformation, a local symmetry with fermionic parameters satisfyng the anti self-dual and self-dual constraints:

$$\frac{1}{\sqrt{g}}\epsilon^{ij} k_j^1 = -k^{i1}, \qquad \frac{1}{\sqrt{g}}\epsilon^{ij} k_j^2 = k^{i2}. \tag{2.83}$$

These relations imply that k^{1I} and k^{2I} are not indipendent and allow to reorganize the k-symmetry transformation in terms of only two Majorana-Weyl spinor k^I as:

$$\delta_k \Theta^1 = \rho_+ k^1, \qquad \delta_k \Theta^2 = \rho_- k^2, \qquad \delta_k X^\mu = \bar{\Theta}^I \Gamma^\mu \delta_k \Theta^I. \tag{2.84}$$

The matrices $\rho_\pm$ behave as projectors of two orthogonal 8-d subspaces belonging to the positive chirality space [12]. Hence the transformation depends in total on 16 fermionic parameters and allows to gauge away the same number of fermionic degrees of freedom. A natural gauge fixing in this context is:

$$\Theta^1 = \Theta^2. \tag{2.85}$$

This choice is possible only in the Type IIB Theory because the two spinors are of the same chirality. After the fermionic gauge fixing we remain with a 2-d field theory with 10 scalars and 8 two dimensional Majorana spinors.
Another symmetry we have to consider is the invariance of the action under 2-d diffeomorphism involving the world sheet coordinates and Weyl transformation of the metric $g_{ij} \to \Omega^2 g_{ij}$. The combination of these two determines the conformal invariance at classical level of the GS string. The preservation of this property at quantum level is crucial for the UV finiteness of the string partition function. In fact, since conformal anomalies are associated with UV divergences, the cancellation of the first ones should guarantee a good UV behavour of the partition function [13]. The Weyl anomaly wich arises at the quantum level is proportional to the central charge of the conformal field theory [14]. In the bosonic sector we have ten 2-d scalars each one carrying central charge 1. The diffeomorphism and Weyl invariance of the GS action require to gauge fix the metric in the path integral with the usual

Faddev-Popov method. The metric gauge fixing leads to the origin of two bosonic conformal ghosts embedded in a 2-d vector. The total contribution to the central charge of the CFT ghost is -26. Looking at the fermionic sector, if we treat the 8 Majorana fermions as normal 2-d fermions and assign central charge 1/2, we have in total $c = 10 \text{ x } 1 + 8 \text{ x } 1/2 - 26 = -12$. Therefore the conformal anomaly apparently does not cancel. However, the fermionic GS action is built with the projection of the 10-d Dirac matrices $\rho_i = \partial_i X^{\hat{a}} \Gamma_{\hat{a}}$, instead of the 2-d gamma matrices as in the standard action for a two dimensional spinor. The dependence on the projected matrices on the target space metric leads to a different behaviour of the kinetic term under conformal transformation. This is reflected in the central charge carried by the GS fermions, which is found to be four times bigger than the usual one [15].
The count of the total central charge now gives :

$$c = 10 + 8 \text{ x } 4 \text{ x } \frac{1}{2} - 26 = 0 \tag{2.86}$$

As a consequence, since in a covariant regularization the cutoff is coupled to the conformal factor, we expect that the cancellation of the Weyl anomaly leads to the cancellation of the UV divergences at any loop order.
The physical effect of the conformal ghosts which arise from the gauge fixing is to cancel two unphysical degrees of freedom of the bosonic sector, remaining with a theory where on-shell the bosonic degrees of freedom exactly match the number of the fermionic ones. In particular, after the gauge fixing we have a theory with world sheet supersymmetry containing 8 scalars and 8 Majorana fermions.

2.3.2 Quadratic fluctuation of the bosonic action

We start with the bosonic sector and expand the Polyakov action around the classical solution :

$$X^\mu \to \bar{X}^\mu + \xi^\mu. \tag{2.87}$$

In this context $\bar{X}^\mu$ describes the classical world sheet of minimal area embedded in AdS_5 with boundary the considered loop.
The target space metric can be Taylor expanded for small fluctuations around the point $\bar{X}$:

$$G_{\mu\nu}(\bar{X} + \xi) = G_{\mu\nu}(\bar{X}) + \partial_\rho G_{\mu\nu}(\bar{X})\xi^\rho + \frac{1}{2}\partial_\rho\partial_\sigma G_{\mu\nu}(\bar{X})\xi^\rho\xi^\sigma + O(\xi^3). \tag{2.88}$$

The analysis of the quadratic fluctuations is considerably simplified if we pick Riemann normal coordinates in the point $\bar{X}$ such that:

$$G_{\mu\nu}(X) = \delta_{\mu\nu} - \mathcal{R}_{\mu\rho\nu\sigma}(\bar{X})\xi^\rho\xi^\sigma + O(\xi^3). \tag{2.89}$$

Expanding the lagrangian at the second order in ξ with this choice of coordinates we find:

$$\begin{aligned} G_{\mu\nu}(X)\partial_i X^\mu \partial_j X^\nu &\rightarrow \left(\delta_{\mu\nu} - \mathcal{R}_{\mu\rho\nu\sigma}(\bar{X})\xi^\rho\xi^\sigma\right)\left(\partial_i \bar{X}^\mu + \partial_i \xi^\mu\right)\left(\partial_j \bar{X}^\nu + \partial_j \xi^\nu\right) + O(\xi^3) = \\ &= \partial_i \bar{X}^\mu \partial_j \bar{X}_\mu + \partial_i \xi^\mu \partial_j \xi_\mu - \mathcal{R}_{\mu\rho\nu\sigma}(\bar{X})\xi^\rho\xi^\sigma \partial_i \bar{X}^\mu \partial_j \bar{X}^\nu + \text{linear terms} + O(\xi^3) \end{aligned} \tag{2.90}$$

The first piece in the expansion is the leading order term and determine the tree level contribution to the partition function. We ignore in the following steps the linear term in ξ, since they can always be absorbed by a shift of the fields.
Now we project the fields on the tangent space:

$$\xi^\mu = E^\mu_{\hat{a}}\zeta^{\hat{a}}, \qquad \zeta^{\hat{a}} = E^{\hat{a}}_\mu \xi^\mu. \tag{2.91}$$

The second term in the expansion becomes:

$$\begin{aligned} \partial_i \xi^\mu \partial_j \xi_\mu &= \partial_i (E^{\mu\hat{a}}\zeta^{\hat{a}})\partial_j (E^{\hat{b}}_\mu \zeta^{\hat{b}}) = \partial_i \zeta^{\hat{a}} \partial_j \zeta^{\hat{a}} + \zeta^{\hat{a}} \partial_j \zeta^{\hat{b}} E^{\hat{b}}_\mu \partial_i \bar{X}^\rho \partial_\rho E^{\mu\hat{a}} \\ &+ \zeta^{\hat{b}} \partial_i \zeta^{\hat{a}} E^{\hat{a}\mu} \partial_j \bar{X}^\sigma \partial_\sigma E^{\hat{b}}_\mu + \zeta^{\hat{a}}\zeta^{\hat{b}} \partial_i \bar{X}^\rho \partial_\rho E^{\hat{a}\mu} \partial_j \bar{X}^\sigma \partial_\sigma E^{\hat{b}}_\mu \\ &= D_i \zeta^{\hat{a}} D_j \zeta^{\hat{a}}, \end{aligned} \tag{2.92}$$

where the covariant derivative D_i contains the projection of the target space spin connection on the world sheet:

$$\begin{aligned} D_i \zeta^{\hat{a}} = \partial_i \zeta^{\hat{a}} + \omega_i^{\hat{a}\hat{b}} \zeta^{\hat{b}}, &\qquad \omega_i^{\hat{a}\hat{b}} = \partial_i \bar{X}^\mu \Omega_\mu^{\hat{a}\hat{b}}, \\ \Omega_\mu^{\hat{a}\hat{b}} = E^{\nu\hat{b}}\Gamma^\rho_{\mu\nu}E^{\hat{a}}_\rho - E^{\nu\hat{b}}\partial_\mu E^{\hat{a}}_\nu &= -E^{\nu\hat{b}}\partial_\mu E^{\hat{a}}_\nu. \end{aligned} \tag{2.93}$$

Here we have used the fact that the Cristoffel symbols $\Gamma^{\rho}_{\mu\nu}$ vanishes in normal coordinates.
In 2.92 and 2.93 we have raised all the tangent space indices for convenience of notation. This does not represent a problem if the target space metric has Euclidean signature because in this case the tangent space indices are raised and lowered with the Euclidean flat metric $\delta_{\hat{a}\hat{b}}$.
The term involving the Riemann tensor becomes:

$$
\begin{aligned}
&-\mathcal{R}_{\mu\rho\nu\sigma}\xi^{\rho}\xi^{\sigma}\partial_i\bar{X}^{\mu}\partial_j\bar{X}^{\nu} = -\mathcal{R}_{\hat{a}\hat{b}\hat{c}\hat{d}}E^{\hat{a}}_{\mu}E^{\hat{b}}_{\rho}E^{\hat{c}}_{\nu}E^{\hat{d}}_{\sigma}\xi^{\rho}\xi^{\sigma}\partial_i\bar{X}^{\mu}\partial_j\bar{X}^{\nu} \\
&= -\eta^{\hat{a}}_i\eta^{\hat{c}}_j\mathcal{R}_{\hat{a}\hat{b}\hat{c}\hat{d}}\zeta^{\hat{b}}\zeta^{\hat{d}},
\end{aligned}
\tag{2.94}
$$

where

$$
\eta^{\hat{a}}_i = \partial_i\bar{X}^{\mu}E^{\hat{a}}_{\mu} \tag{2.95}
$$

is the projection on the world sheet of the target space vielbein.
Since we are dealing with a background space which is the Cartesian product of two manifold, the whole tangent space is the direct sum of the two relative tangent spaces and the contributions to the action given in 2.92 and 2.94 can be split in two parts describing the fluctuations of AdS_5 and S^5.
Putting all the pieces together we get:

$$
S_{2B} = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}D_i\zeta^a D_j\zeta^a + X_{ab}\zeta^a\zeta^b + g^{ij}D_i\zeta^p D_j\zeta^p + X_{pq}\zeta^p\zeta^q\right), \tag{2.96}
$$

where

$$
X_{ab} = -g^{ij}\eta^c_i\eta^d_j\mathcal{R}_{acbd}, \qquad X_{pq} = -g^{ij}\eta^r_i\eta^s_j\mathcal{R}_{prqs}. \tag{2.97}
$$

In the action above we have rescaled the fluctuations by $\frac{\sqrt{2\pi\alpha'}}{R}$ to cancel the factor in front of the action.
The appereance of the covariant derivatives in the bosonic action is a consequence of the projection of the fluctuations on the tangent space. After this operation the invariance under local change of coordinates has been substituted by the invariance under $SO(10)$ rotation on the AdS_5 x S^5 tangent space.
Now we have to impose the necessary boundary conditions to have a string world

sheet of minimal area embedded in AdS_5 and describing a circular loop at the AdS_5 boundary. The problem of finding the minimal surface with boundary a circle in AdS_5 x S^5 has been solved in [9, 16]. The classical world sheet embedded in AdS_5 wich minimizes the area enclosed by a circular loop is found to be the Euclidean AdS_2 with boundary S^1.
We rewrite the target space metric given in 2.3 in polar coordinates:

$$ds^2_{\mathrm{AdS}_5 \text{ x } \mathrm{S}^5} = \frac{1}{y^2}\left(dr^2 + r^2 d\phi^2 + dx^s dx^s + dy^2\right) + d\Omega_5^2, \qquad s = 2, 3. \tag{2.98}$$

Here we have set the radious of curvature equal to 1, but it can be easily restored when is necessary.
To fix the conformal invariance we choose for the world sheet the induced metric on AdS_2 from AdS_5 by setting $x^s =$ constant, $\Omega_5 =$ constant and $y = \sqrt{1-r^2}$:

$$ds^2_{\mathrm{AdS}_2} = \frac{1}{1-r^2}\left(r^2 d\phi^2 + \frac{dr^2}{1-r^2}\right)$$

$$y = \sqrt{1-r^2}, \qquad g_{ij} = \begin{pmatrix} \frac{r^2}{y^2} & 0 \\ 0 & \frac{1}{y^4} \end{pmatrix}, \qquad \sqrt{g} g^{ij} = \begin{pmatrix} \frac{1}{ry} & 0 \\ 0 & ry \end{pmatrix}, \tag{2.99}$$

$$\sqrt{g} = \frac{r}{y^3},$$

where the worldsheet coordinates are $\sigma_0 = \phi \in [0, 2\pi]$ and $\sigma_1 = r \in [0, 1]$.
In this frame the conformal boundary is a circle identified by the constraint $r = 1$ and described by the polar angle $\phi \in [0, 2\pi]$.
To check that this metric describes the Euclidean AdS_2 we can put it in a more standard form. Setting $r = \tanh\xi$ we get:

$$ds^2_{\mathrm{AdS}_2} = d\xi^2 + \sinh^2\xi d\phi^2, \tag{2.100}$$

with $\phi \in [0, 2\pi]$, $\xi \in [0, \infty]$.
The choice of coordinates for the target space and the world sheet simplifies considerably the action of the bosonic quadratic fluctuations. The mass matrices in the present case become:

$$X^{ab} = 2\delta^{ab} - g^{ij}\eta_i^a\eta_j^b, \qquad \eta_0^a = \left(\frac{r}{y}, 0, 0, 0, 0\right), \qquad \eta_1^a = \left(0, \frac{1}{y}, 0, 0, -\frac{r}{y^2}\right),$$

$$X^{pq} = 0, \qquad \eta_i^r = 0. \tag{2.101}$$

The mass matrix and the projection of the vielbeins relative to the sphere tangent space vanish because the classical world sheet is completely embedded in AdS_5.
In these coordinates the non-zero components of the target space spin connection are:

$$\Omega_0^{01} = 1, \qquad \Omega_0^{04} = -\frac{r}{y}, \qquad \Omega_1^{14} = \Omega_2^{24} = \Omega_3^{34} = -\frac{1}{y}, \tag{2.102}$$

and the non trivial worldsheet covariant derivatives are:

$$D_0\zeta^0 = \partial_0\zeta^0 + \zeta^1 - \frac{r}{y}\zeta^4, \qquad D_0\zeta^1 = \partial_0\zeta^1 - \zeta^0, \qquad D_0\zeta^4 = \partial_0\zeta^4 + \frac{r}{y}\zeta^0,$$

$$D_1\zeta^1 = \partial_1\zeta^1 - \frac{1}{y}\zeta^4, \qquad D_1\zeta^4 = \partial_1\zeta^4 + \frac{1}{y}\zeta^1, \tag{2.103}$$

while all the other ones are simply partial derivative $D_i = \partial_i$.
In this tangent space basis the mass matrix relative to the AdS_5 fluctuations is not diagonal. In order to find the structure of bosonic multiplet we need to perform a local $SO(10)$ rotation on the AdS_5 tangent space:

$$\begin{pmatrix}\tilde{\zeta}^1\\ \tilde{\zeta}^4\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\zeta^1\\ \zeta^4\end{pmatrix}, \qquad \cos\alpha = y, \quad \sin\alpha = r. \tag{2.104}$$

In this basis the mass matrix becomes diagonal:

$$\tilde{X}_{ab} = \mathrm{diag}(1,1,2,2,2) \tag{2.105}$$

and the only non trivial covariant derivatives remains:

$$D_0\zeta^0 = \partial_0\zeta^0 + \frac{1}{y}\tilde{\zeta}^1, \qquad D_0\tilde{\zeta}^1 = \partial_0\tilde{\zeta}^1 - \frac{1}{y}\zeta^0. \tag{2.106}$$

The dynamics of the transverse fluctuations ζ^s, $\tilde{\zeta}^4$, ζ^q relative to the x^s, y and S^5 coordinates is described by the action:

$$S^T_{2B}(\zeta^T) = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}\partial_i\zeta^s\partial_j\zeta^s + 2\zeta^s\zeta^s + g^{ij}\partial_i\tilde{\zeta}^4\partial_j\tilde{\zeta}^4 + 2\tilde{\zeta}^4\tilde{\zeta}^4 + g^{ij}\partial_i\zeta^q\partial_j\zeta^q\right), \tag{2.107}$$

while the action describing the longitudinal modes ζ^0, $\tilde{\zeta}^1$ corresponding to the world sheet coordinates ϕ, r is:

$$S^L_{2B}(\zeta^L) = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}D_i\zeta^0 D_j\zeta^0 + \zeta^0\zeta^0 + g^{ij}D_i\tilde{\zeta}^1 D_j\tilde{\zeta}^1 + \tilde{\zeta}^1\tilde{\zeta}^1\right). \tag{2.108}$$

The bosonic action is complete only once we have added also the contribution of the conformal ghosts. The action which arises from the Faddev-Popov gauge fixing in the string path integral is [17]:

$$S_{\text{ghost}}(\epsilon) = \frac{1}{2}\int d^2\sigma\sqrt{g}\left(g^{ij}\nabla_i\epsilon^\alpha\nabla_j\epsilon^\alpha - \frac{1}{2}\mathcal{R}\epsilon^\alpha\epsilon^\alpha\right), \qquad \mathcal{R} = -2, \tag{2.109}$$

where ϵ^α are the Faddev-Popov ghosts. These behave as 2 real anti commuting scalars with $m^2 = 1$. The covariant derivative ∇_i contains the world sheet spin connection:

$$\nabla_i\epsilon^\alpha = \partial_i\epsilon^\alpha + \omega_i^{\alpha\beta}\epsilon^\beta. \tag{2.110}$$

The only non-zero component of the connection is $\omega_0^{01} = \frac{1}{y}$ and the non-trivial covariant derivatives are

$$\nabla_0\epsilon^0 = \partial_0\epsilon^0 + \frac{1}{y}\epsilon^1, \qquad \nabla_0\epsilon^1 = \partial_0\epsilon^1 - \frac{1}{y}\epsilon^0. \tag{2.111}$$

We note that the actions describing the ghosts and the longitudinal fluctuations are equal. In fact the covariant derivatives above coincide with those written in 2.106 and mass term in the ghosts action is the same as in 2.108. It is shown later that the different statistics leads to the mutual cancellation of the contributions to the partition function between ghosts and longitudinal modes, leaving us with only the transverse fluctuations of the AdS_5 x S^5 tangent space. In conclusion the bosonic sector contains three scalars with squared mass 2 and five massless scalars which propagate on the Euclidean AdS_2.

2.3.3 Quadratic fluctuation of the fermionic action

The quadratic part of the fermionic action is already given in 2.78 and we only need to rescale the fermionic fields by $\sqrt{2\pi\alpha'}$ to cancel the factor in front of the action. The lagrangian before the gauge fixing of the k-symmetry is:

$$L_{2F} = \left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}S^{IJ}\right)\bar{\Theta}^I \rho_i D_j \Theta^J,$$

$$D_i\Theta^I = \mathcal{D}_i\Theta^I - \frac{i}{2}\epsilon^{IK}\tilde{\rho}_i\Theta^K, \qquad \mathcal{D}_i\Theta^I = \partial_i\Theta^I + \frac{1}{4}\partial_i\bar{X}^\mu\Omega_\mu^{ab}\Gamma_{ab}\Theta^I, \quad (2.112)$$

$$\rho_i = \eta_i^a\Gamma_a = \tilde{\rho}_i, \qquad \eta_i^a = \partial_i\bar{X}^\mu E_\mu^a, \qquad \eta_i^p = \partial_i\bar{X}^\mu E_\mu^p = 0.$$

Here we have imposed the gauge fixing of the world sheet metric discussed in the previous section. In particular, since the classical world sheet solution is constant in S^5, the projection of the relative vielbeins $\eta_i^p = \partial_i\bar{X}^\mu E_\mu^p$ vanishes and one can take into account only the AdS_5 tangent space. The spin connection Ω_μ^{ab} is the same given in 2.102.

We want to put the fermionic lagrangian in a more standard form in terms of the world sheet Dirac gamma matrices. The first step is to introduce the vielbeins e_i^α relative to the world sheet metric $g_{ij} = e_i^\alpha e_j^\beta\delta_{\alpha\beta}$ such that:

$$\rho_i = \eta_i^a\Gamma_a = e_i^\alpha\tilde{\Gamma}_\alpha,$$

$$e_0^\alpha = \left(\frac{r}{y}, 0\right), \qquad e_1^\alpha = \left(0, \frac{1}{y^2}\right), \quad (2.113)$$

where $\tilde{\Gamma}_\alpha$ are the world sheet Dirac matrices.

Then we perform on the AdS_5 spinor bundle the same rotation that we have used in the previous section:

$$\rho_0 = \eta_0^a \Gamma_a = \frac{r}{y}\Gamma_0 = e_0^\alpha \tilde{\Gamma}_\alpha = e_0^\alpha S \Gamma_\alpha S^{-1},$$

$$\rho_1 = \eta_1^a \Gamma_a = \frac{1}{y}\Gamma^1 - \frac{r}{y^2}\Gamma_4 = e_1^\alpha \tilde{\Gamma}_\alpha = e_1^\alpha S \Gamma_\alpha S^{-1},$$

$$\mathcal{D}_0 = \left(\partial_0 + \frac{1}{2}\Gamma_{01} - \frac{r}{2y}\Gamma_{04}\right) = S\hat{\nabla}_0 S^{-1}, \tag{2.114}$$

$$\mathcal{D}_1 = \left(\partial_1 - \frac{1}{2y}\Gamma_{14}\right) = S\hat{\nabla}_1 S^{-1},$$

$$\Theta^I = S\Psi^I, \qquad \bar{\Theta}^I = \bar{\Psi}^I S^{-1},$$

where the rotation matrix is

$$S = \exp\left(\frac{\alpha}{2}\Gamma_{14}\right). \tag{2.115}$$

The operator $\hat{\nabla}_i$ is the covariant derivative containing the spinor world sheet connection:

$$\hat{\nabla}_0 = \partial_0 + \frac{1}{2y}\Gamma_{01}, \qquad \hat{\nabla}_1 = \partial_1. \tag{2.116}$$

Under the rotation given above the lagrangian transforms as

$$\begin{aligned}
L_{2F} &= \left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}S^{IJ}\right)\bar{\Theta}^I \rho_i D_j \Theta^J \to \left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}S^{IJ}\right)\bar{\Psi}^I S^{-1} e_i^\alpha S \Gamma_\alpha S^{-1} S \hat{D}_j S^{-1} S \Psi^J = \\
&= \left(\sqrt{g}g^{ij}\delta^{IJ} - \epsilon^{ij}S^{IJ}\right)\bar{\Psi}^I e_i^\alpha \Gamma_\alpha \left(\hat{\nabla}_j \Psi^J - \frac{i}{2}\epsilon^{JK} e_j^\beta \Gamma_\beta \Psi^K\right) = \\
&= \sqrt{g}g^{ij}\bar{\Psi}^I e_i^\alpha \Gamma_\alpha \hat{\nabla}_j \Psi^I - \frac{i}{2}\sqrt{g}g^{ij}\epsilon^{IK} e_i^\alpha e_j^\beta \bar{\Psi}^I \Gamma_\alpha \Gamma_\beta \Psi^K - \epsilon^{ij}S^{IJ}\bar{\Psi}^I e_i^\alpha \Gamma_\alpha \hat{\nabla}_j \Psi^J \\
&+ \frac{i}{2}\epsilon^{ij}S^{IJ}\epsilon^{JK} e_i^\alpha e_j^\beta \bar{\Psi}^I \Gamma_\alpha \Gamma_\beta \Psi^K.
\end{aligned} \tag{2.117}$$

Choosing the gauge $\Psi^1 = \Psi^2 = \Psi$ we get:

$$
\begin{aligned}
&\sqrt{g} g^{ij} \bar{\Psi}^I e_i^\alpha \Gamma_\alpha \hat{\nabla}_j \Psi^I = 2\sqrt{g} \left(g^{ij} \bar{\Psi} e_i^\alpha \Gamma_\alpha \hat{\nabla}_j \Psi \right), \\
&- \frac{i}{2} \sqrt{g} g^{ij} \epsilon^{IK} e_i^\alpha e_j^\beta \bar{\Psi}^I \Gamma_\alpha \Gamma_\beta \Psi^K = 0, \\
&- \epsilon^{ij} S^{IJ} \bar{\Psi}^I e_i^\alpha \Gamma_\alpha \hat{\nabla}_j \Psi^J = 0, \qquad\qquad (2.118) \\
&\frac{i}{2} \epsilon^{ij} S^{IJ} \epsilon^{JK} e_i^\alpha e_j^\beta \bar{\Psi}^I \Gamma_\alpha \Gamma_\beta \Psi^K = i \epsilon^{ij} e_i^\alpha e_j^\beta \bar{\Psi} \Gamma_\alpha \Gamma_\beta \Psi = i e_0^\alpha e_1^\beta \bar{\Psi} \left[\Gamma_\alpha, \Gamma_\beta \right] \Psi = \\
&= 2 i e_0^\alpha e_1^\beta \bar{\Psi} \Gamma_{\alpha\beta} \Psi = 2i \frac{r}{y^3} \bar{\Psi} \Gamma_{01} \Psi = 2\sqrt{g} \left(i \bar{\Psi} \Gamma_{01} \Psi \right),
\end{aligned}
$$

and the fermionic lagrangian becomes:

$$
L_{2F} = 2\sqrt{g} \left(g^{ij} \bar{\Psi} e_i^\alpha \Gamma_\alpha \hat{\nabla}_j \Psi + i \bar{\Psi} \Gamma_{01} \Psi \right). \qquad (2.119)
$$

The first piece in the rotated lagrangian can be formally identified with the usual fermionic kinetic term containing the Dirac operator on a curved background, while the second piece descending from the RR coupling plays the role of a mass term. Since the spinor field Ψ contains eight non-interacting 2-d Majorana fermions, the world sheet gamma matrices can be factorized as the 2-d Dirac matrices γ_α times the 8 x 8 unit matrix I_8. Choosing for γ_α a representation in terms of the Pauli matrices $\gamma_{0,1} = \sigma_1, \sigma_2$, we get:

$$
\begin{aligned}
\Gamma_0 = \sigma_1 \otimes I_8, \qquad\qquad \Gamma_1 = \sigma_2 \otimes I_8, \\
\Gamma_{01} = \frac{1}{2} [\sigma_1, \sigma_2] \otimes I_8 = i \sigma_3 \otimes I_8.
\end{aligned} \qquad (2.120)
$$

We conclude that the lagrangian found above describes eight 2-d Majorana fermions of mass ± 1 propagating on the Euclidean version of AdS_2.

2.3.4 The Partition function of the Quadratic Fluctuations

Before giving the final expression of the one-loop correction to the circular Wilson loop we briefly review the tree level computation. According to 2.72 we need to find the area of the classical string worldsheet, which is in the circular case the Euclidean AdS_2 with S^1 boundary. Because AdS_2 is a non compact-space its area is infinite and requires a regularized definition. Explicitly one nedds to introduce a cutoff to regularize the volume integral and then add a counterterm to remove the divergent part of the expression. Choosing the coordinates given in 2.100 we obtain:

$$
\begin{aligned}
A_{\mathrm{AdS}_2} =& \int_0^{2\pi} d\phi \int_0^{\Lambda} d\chi \sinh\chi = \\
=& 2\pi(\cosh\Lambda - 1) \rightarrow 2\pi(e^{\Lambda} - 1), \qquad \Lambda \rightarrow \infty.
\end{aligned}
\tag{2.121}
$$

To renormalize the AdS_2 area we subtract to the above expression the measure of the boundary:

$$
L_{\mathrm{S}^1} = \sinh\Lambda \int_0^{2\pi} d\phi = 2\pi\sinh\Lambda \longrightarrow 2\pi e^{\Lambda}, \qquad \Lambda \rightarrow \infty. \tag{2.122}
$$

The renormalized definition of area we are looking for is:

$$
A_{\mathrm{AdS}_2} = -2\pi, \tag{2.123}
$$

and according to the 2.72 the leading order of the circular Wilson loop a strong coupling is

$$
\langle W(C) \rangle \sim e^{\sqrt{\lambda}}. \tag{2.124}
$$

This result is referred to a circular Wilson loop in the fundamental representation of the gauge group $SU(N)$. The k^{th} rank symmetric representation is equivalent to have k identical W-bosons which moves along the circular loop, or equivalenty k coincident strings which end on the boundary of AdS_5. Therefore the classical worldsheet which we have to consider in this case is the Euclidean AdS_2 with a S_1 boundary wrapped k times. This manifold can be described with the same coordinates given in 2.99 and 2.100 for the normal AdS_2, but now the periodicity of the angle changes from 2π to $2\pi k$. Consequently the renormalized area becomes:

$$A_{\mathrm{AdS}_2^k} = -2\pi k \tag{2.125}$$

and the circular Wilson loop in the k^{th} rank symmetric representation at tree level is:

$$\langle W(C)\rangle_{k^{th}\ \mathrm{rank}} \sim e^{k\sqrt{\lambda}}, \tag{2.126}$$

which matches the gauge side result given in section 2.2.2

According to what we found in the previous paragraphs, the string partition function which has the interpretation of $O(\lambda^0)$ correction to the circular Wilson loop is:

$$\begin{aligned}
&\langle W(C)\rangle_{\mathrm{one-loop}} = Z_{S_2} = \int [D\zeta][D\Psi][D\epsilon]e^{-S_2(\zeta,\Psi,\epsilon)}, \\
&S_2(\zeta,\Psi,\epsilon) = S_{2B}(\zeta) + S_{2F}(\Psi) + S_{\mathrm{ghost}}(\epsilon), \\
&S_{2B}(\zeta) = S_{2B}^T(\zeta^T) + S_{2B}^L(\zeta^L),
\end{aligned} \tag{2.127}$$

where $S_2(\zeta,\Psi,\epsilon)$ is the action describing the quantum, quadratic fluctuations of the Type IIB String in AdS_5 x S^5 which we have derived in 2.107, 2.108, 2.109 and 2.119.

For a circular Wilson loop in the k^{th} rank symmetric representation, the partition function Z_{S_2} has to be computed on a classical world sheet which is the k-wrapped version of the Euclidean AdS_2.

Strictly speaking, the quantity which can be directly compared to the gauge side result is the effective action of the string partition function:

$$\begin{aligned}
&\langle W(C)\rangle = Z_{S_2} = e^{-\Gamma}, \qquad \Gamma = \Gamma_0 + \Gamma_1 + ..., \\
&\Gamma_1 = -\log\left[\int [D\zeta][D\Psi][D\epsilon]e^{-S_2(\zeta,\Psi,\epsilon)}\right],
\end{aligned} \tag{2.128}$$

where Γ_1 is the one-loop correction which should match the expression given in 2.61.

In the next section we review the definition of partition function and effective action in Quantum Field Theory and provide a powerful method to compute them.

3 The Heat Kernel Method

3.1 Effective Action in Quantum Field Theory

3.1.1 The General Definition

Let us consider a generic field theory on a Riemannian manifold (M, g) of dimension d. We choose the Euclidean signature because the string partition function we want to compute is defined on a two-dimensional Euclidean world sheet. We indicate the fields of our theory with φ_r, where r can represent any type of index (gauge, spinorial, ecc..). The quantization of a field theory with the path integral approach usually starts with the definition of the generating functional of the Green functions:

$$Z[J] = \int [D\varphi] e^{-(S[\varphi] + \int d^d x \sqrt{g} J_r \varphi_r)}, \tag{3.1}$$

where $S[\varphi]$ is the euclidean action of the considered theory and J_r is an external non dynamical source for the fields. By taking derivatives of this functional respect to J we obtain the Green functions of the theory, wich are related to the Feynmann amplitudes through the LSZ formula. $Z[J]$ can be expressed in terms of another useful functional $W[J]$ which generates only the connected Green functions:

$$Z[J] = e^{-W[J]}, \qquad\qquad W[J] = -\log\left[Z[J]\right]. \tag{3.2}$$

$W[J]$ is mathematically the sum of all the connected Green functions with external legs corresponding to the external source. The functional derivation respect to J has the effect to truncate the external legs and return the connected diagrams.
Furthermore, this object is related to a third important functional through the so called Legendre transformation. Defining the "classical" field as:

$$\phi^c_r(J) = \frac{\delta W[J]}{\delta J_r}, \tag{3.3}$$

we can invert such relation to express J_r in terms of ϕ^c_r and define the generating functional of the 1PI functions:

$$\Gamma[\phi^c] = W[J(\phi^c)] - \int d^d x \sqrt{g} J_r(\phi^c)\phi^c_r. \tag{3.4}$$

This functional is also called effective action and can be thought as the quantum generalization of the classical action. In the classical limit it coincides with $S[\varphi]$ and ϕ^c coincides with the field φ satisfying the classical equations of motion (which justifies its name). Analitically the effective action is the sum of the 1PI functions of the theory with external legs corresponding to ϕ^c. This one plays the role of an external non dynamical field respect to which we can derive the functional to obtain the 1PI functions.
Inverting the effective action respect to $W[J]$ we can rewrite the 3.1 in the following way:

$$e^{-\left(\Gamma[\phi^c(j)]+\int d^dx\sqrt{g}J_r\phi^c_r(J)\right)} = \int [D\varphi]e^{-\left(S[\varphi]+\int d^dx\sqrt{g}J_r\varphi_r\right)}. \tag{3.5}$$

Then if we set all the external sources and classical fields to '0' we get what we call the partition function of a quantum field theory:

$$Z[J=0] = e^{-\Gamma[\phi^c=0]} = \int [D\varphi]e^{-S[\varphi]} \tag{3.6}$$

and, once inverted the expression, the vacuum effective action:

$$\Gamma = \Gamma[\phi^c = 0] = -\log\int [D\varphi]e^{-S[\varphi]}. \tag{3.7}$$

The vacuum effective action is the sum of only the 1PI vacuum diagrams of the theory (without external legs).
This object has important applications in Quantum Field theory and String theory and is the quantity in which we are mostly interested. The name partition function of the 3.6 is given in analogy with the thermal partition function which can be obtained from $Z[J]$ by setting the external source to '0' and compactifying the euclidean time on a finite interval $[0,\beta]$, where the period β is interpreted as the inverse of the temperature:

$$Z[\beta] = \int D\varphi e^{-\int_0^\beta dt\int d^{d-1}x\sqrt{g}L[\varphi]}. \tag{3.8}$$

Here $L[\varphi]$ is the lagrangian of the theory. If we impose for bosons and fermions periodic and anti periodic condition in the Euclidean time to have the right statistics, the expression above coincides with the gran canonical partition function up to a contribution due to the vacuum energy [6].
From now we concentrate on the cases of a real scalar field and a spinor field which are relevant to our purpose.

3.1.2 The bosonic effective action

The action describing a free scalar field $\phi(x)$ with mass m on an Euclidean curved manifold of dimension d and fixed metric $g_{\mu\nu}$, $\mu, \nu = 0, .., d-1$ is:

$$S[\phi, g_{\mu\nu}] = \frac{1}{2} \int d^d x \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi^2(x) \right). \tag{3.9}$$

As argued in [18], assuming appropriate boundary conditions on $\phi(x)$ such that one can integrate by parts omitting the boundary terms, we can rewrite the action as

$$S[\phi, g_{\mu\nu}] = \frac{1}{2} \int d^d x \sqrt{g} \phi(x) \hat{O} \phi(x), \tag{3.10}$$

where $\hat{O}$ is a second order differential operator containing the covariant Laplacian (D'Alembertian in Minkoskian signature) for a scalar field with metric $g_{\mu\nu}$:

$$\hat{O} = -\Box_g + m^2, \qquad \Box_g \phi(x) = \frac{1}{\sqrt{g}} \partial_\mu \left[\sqrt{g} g^{\mu\nu} \partial_\nu \phi(x) \right]. \tag{3.11}$$

The classical equation of motions for the scalar field can be written in terms of $\hat{O}$:

$$\hat{O} \phi(x) = 0. \tag{3.12}$$

According to the definition given in the previous paragraph, the effective action for the scalar fields can be obtained from

$$e^{-\Gamma[g_{\mu\nu}]} = \int [D\phi] e^{-S[\phi, g_{\mu\nu}]}. \tag{3.13}$$

Here the effective action depends on the manifold metric which is not quantized and has to be intended as a fixed background field.
Let us consider the following eigenvalue problem:

$$\left(-\Box_g + m^2 \right) \phi_n(x) = \lambda_n \phi_n(x) \tag{3.14}$$

where $\phi_n(x)$ are eigenfunctions of the operator with eigenvalue λ_n. For mathematical convenience we consider a compact manifold so that the spectrum of eigenvalues is discrete ($n = 0, 1, 2, ...$), but the following considerations can be extended to the non compact case. Choosing appropriate boundary conditions for $\phi(x)$ depending on the manifold, the operator $-\Box_g + m^2$ can be made self-adjoint with respect to the scalar product

$$(f,g) = \int d^d x \sqrt{g} f(x) g(x). \tag{3.15}$$

In general, a sufficient boundary condition to achieve this property should guarantee the possibility to integrate by parts ignoring boundary terms. For example in the non-compact case we need to assume that the scalar field tends to '0' sufficiently rapidly as $|x| \to \infty$.
The self-adjointness of the operator implies that the eigenvalues are real and the set of eigenfunctions ϕ_n forms a complete orthonormal basis in the space of functions:

$$(\phi_n, \phi_m) = \int d^d x \sqrt{g} \phi_n(x) \phi_m(x) = \delta_{nm}. \tag{3.16}$$

Moreover, because of the Euclidean signature of the metric, $\hat{O}$ is semi-positive definite and the spectrum is bounded from below ($\lambda_n \geq 0$).
The completeness and orthonormality of the eigenvectors allow to expand a generic function as

$$f(x) = \sum_n c_n \phi_n(x), \qquad c_n = (\phi_n, f) = \int d^d x \sqrt{g} \phi_n(x) f(x). \tag{3.17}$$

The coefficients c_n are real and represent the infinitely many numerable components of the function $f(x)$ respect to the orthonormal basis of the differential operator.
Now we can rewrite the action of the scalar field in an easier way:

$$\begin{aligned} S[\phi, g_{\mu\nu}] &= \frac{1}{2} \int d^d x \sqrt{g} \sum_{n,m} c_n c_m \phi_n(x) \hat{O} \phi_m(x) = \\ &= \frac{1}{2} \int d^d x \sqrt{g} \sum_{n,m} c_n c_m \lambda_m \phi_n(x) \phi_m(x) = \frac{1}{2} \sum_n \lambda_n c_n^2. \end{aligned} \tag{3.18}$$

Essentially we have diagonalized the action on the basis ϕ_n and express it in terms of the coefficients c_n. These ones are expressed in 3.17 as invariant integrals and by definition are indipendent from the choice of coordinates. Hence we are motivated to define an invariant path integral measure through the variables c_n as

$$D\phi = \prod_n \frac{dc_n}{\sqrt{2\pi}}. \tag{3.19}$$

Using these relations the path integral in 3.13 becomes

$$e^{-\Gamma[g_{\mu\nu}]} = \int \prod_n \frac{dc_n}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_n \lambda_n c_n^2} = \left[\prod_n \lambda_n\right]^{-1/2} = \left(\det[\hat{O}]\right)^{-1/2}. \tag{3.20}$$

The normalization constant $\frac{1}{\sqrt{2\pi}}$ in the path integral measure has been chosen to absorb the constant factor arising from the Gaussian integral.
If the mass of the scalar field is null, the Laplacian operator could admit some zero eigenvalue. In this eventuality the diagonalized action does not depend on the coefficients corresponding to the zero modes and the integral over these variables factorizes as an infinite constant. Therefore, to give a consistent definition to the effective action, one has to remove the divergent part due to the presence of zero modes through a renormalization procedure and express the path integral as determinant of the operator restricted to the non-zero subspace.
We can formally rewrite the effective action as

$$\Gamma[g_{\mu\nu}] = \frac{1}{2} \log\left(\det[-\Box_g + m^2]\right). \tag{3.21}$$

The computation of the effective action is reduced to the spectral problem of finding the determinant of a differential operator. In general this is not a well defined quantity, since the eigenvalues λ_n grows with n and the product $\prod_n \lambda_n$ diverges (in addition to the zero modes divergence). A finite result can be obtained only after a regularization and renormalization procedure wich leads to the definition of regularized functional determinant. These considerations hold also for a non compact manifold with the exception that the spectrum is continuous.

3.1.3 The fermionic effective action

The action describing a Dirac field Ψ of mass m on a curved Riemannian manifold (M, g) is

$$S[\bar{\Psi}, \Psi, g_{\mu\nu}] = \int d^dx \sqrt{g} \bar{\Psi} \hat{F} \Psi \tag{3.22}$$

where $\hat{F}$ contains the first order Dirac operator

$$\hat{F} = g^{\mu\nu}\gamma_\mu \nabla_\nu + m, \qquad\qquad \nabla_\nu = \partial_\nu + \frac{1}{8}\omega_\nu^{\alpha\beta}\left[\gamma_\alpha, \gamma_\beta\right]. \tag{3.23}$$

As said in section 2.3, ∇_ν indicates the spinor covariant derivative containing the spinor bundle connection $\omega_\nu^{\alpha\beta}$, $\alpha, \beta = 0, ..., d-1$, and the generators of the $SO(d)$ rotations in the spinorial representation. The Dirac gamma matrices satisfy the Clifford algebra with Euclidean signature:

$$\begin{aligned} \{\gamma_\mu, \gamma_\nu\} &= 2g_{\mu\nu}, \qquad\qquad \{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}, \\ \gamma_\mu &= e_\mu^\alpha \gamma_\alpha, \qquad\qquad \gamma_\alpha = e_\alpha^\mu \gamma_\mu, \end{aligned} \tag{3.24}$$

where e_μ^α are the vielbeins relative to the metric $g_{\mu\nu}$.
More details about the tangent and spinor bundle can be found in Appendix A.
Similarly to the scalar case, it is possible to prove a relation between the effective action for a Dirac or Majorana spinor and the determinant of the operator $\hat{F}$ which propagates the classical fields:

$$\hat{F}\Psi = 0. \tag{3.25}$$

As before, for mathematical convenience we consider the case of a compact manifold, but the following considerations can be extended to the non compact case.
Under suitable boundary conditions on the spinor field, the Dirac operator $\gamma^\mu \nabla_\mu$ is anti-hermitian respect to the natural scalar product [6]:

$$(\Phi, \Psi) = \int d^d x \sqrt{g}\bar{\Phi}\Psi. \tag{3.26}$$

The anti-hermiticity implies that the spectrum of the Dirac operator is pure immaginary and, assuming to have an even dimensional manifold, the anticommutation property with the chirality matrix $\gamma_* = i\gamma^0 \cdot \dots \cdot \gamma^{d-1}$ allows to show that the eigenvalues are distribuited symmetrically respect to the origin along the immaginary axis [6].
Therefore the eigenvalue problem corresponding to the operator $\hat{F}$ is:

$$\begin{aligned} &\left(\gamma^\mu \nabla_\mu + m\right)\psi_n = \lambda_n \psi_n, \\ &\lambda_n = \pm i\rho_n + m, \qquad \text{n} = 1, 2, ., \qquad \rho_n \in R > 0. \end{aligned} \tag{3.27}$$

The infinite numerable set of eigenfunctions ψ_n constitutes a complete othonormal basis respect to the scalar product 3.26:

$$(\psi_n, \psi_m) = \int d^d x \sqrt{g} \bar{\psi}_n(x) \psi_m(x) = \delta_{nm}. \tag{3.28}$$

A generic Dirac or Majorana spinor $\Psi(x)$ can be expanded on this basis as:

$$\Psi(x) = \sum_n c_n \psi_n(x), \qquad c_n = (\psi_n, \Psi) = \int d^d x \sqrt{g} \bar{\psi}_n(x) \Psi(x), \tag{3.29}$$

where, taking into account that spinors are anti-commuting objects at classical level, the set of coefficients $\{c_n\}$ forms an infinite numerable Grassman algebra:

$$c_n c_m = -c_m c_n, \qquad c_n^2 = 0. \tag{3.30}$$

As in the scalar case, we can diagonalize the action on the set of the operator eigenfunctions:

$$\begin{aligned} S[\bar{\Psi}, \Psi, g_{\mu\nu}] &= \int d^d x \sqrt{g} \sum_n \sum_m c_n^* c_m \bar{\psi}_n(x) \hat{F} \psi_m(x) = \\ &= \int d^d x \sqrt{g} \sum_n \sum_m \lambda_m c_n^* c_m \bar{\psi}_n(x) \psi_m(x) = \sum_n \lambda_n c_n^* c_n. \end{aligned} \tag{3.31}$$

The fermionic effective action can be obtained from the exponential relation:

$$e^{-\Gamma[g_{\mu\nu}]} = \int [D\bar{\Psi}][D\Psi] e^{-S[\bar{\Psi}, \Psi, g_{\mu\nu}]}. \tag{3.32}$$

The diagonalization of the action in terms of the Grassman variables c_n provides a natural measure for the Berezin path integral:

$$D\bar{\Psi} D\Psi = \prod_n dc_n^* dc_n. \tag{3.33}$$

The integration rules for a single degree of freedom are:

$$\int dc = 0, \qquad \int dc^* = 0, \qquad \int dcc = 1, \qquad \int dc^* c^* = 1. \tag{3.34}$$

Using these rules the fermionic path integral becomes:

$$\begin{aligned} e^{-\Gamma[g_{\mu\nu}]} =& \int \prod_n dc_n^* dc_n e^{-\sum_m \lambda_m c_m^* c_m} \prod_n \int dc_n^* dc_n e^{-\lambda_n c_n^* c_n} \\ =& \prod_n \int dc_n^* dc_n (1 - \lambda_n c_n^* c_n) = \prod_n \lambda_n \int dc_n^* c_n^* \int dc_n c_n = \\ =& \prod_n \lambda_n = \det[\hat{F}]. \end{aligned} \tag{3.35}$$

Therefore the effective action is:

$$\Gamma[g_{\mu\nu}] = -\log\left(\det\left[\gamma^\mu \nabla_\mu + m\right]\right). \tag{3.36}$$

Making a comparison with the scalar case, we see that the different statistics leads to a mignus sign in front of the effective action.
Even if the eigenvalues of the operator $\hat{F}$ are complex, since for each one there is the complex conjugate in the spectrum, the determinant is consequently real and positive.
Instead of the first order spectral problem given in 3.27, in literature is often studied the equivalent second order case relative to the "square" operator $\hat{F}^\dagger \hat{F}$. This one can be obtained using the Lichnerowicz formula:

$$\hat{F}^\dagger \hat{F} = (-\gamma^\mu \nabla_\mu + m)(\gamma^\mu \nabla_\mu + m) = -(\gamma^\mu \nabla_\mu)^2 + m^2 = -\nabla^2 + \frac{1}{4} R + m^2, \tag{3.37}$$

where ∇^2 is the spinor laplacian $\nabla^\mu \nabla_\mu$ and R is the Ricci scalar of the manifold. This operator is manifestly self-adjoint respect to the scalar product 3.26 and the spectrum contains real and positive eigenvalues.
The effective action 3.36 can be rewritten in terms of the square Dirac operator as:

$$\Gamma[g_{\mu\nu}] = -\log(\det[\hat{F}]) = -\frac{1}{2}\log(\det[\hat{F}^\dagger \hat{F}]) = -\frac{1}{2}\log(\det[-\nabla^2 + \frac{1}{4} R + m^2]). \tag{3.38}$$

As in the scalar case, the computation of the effective action consists evaluating the determinant of a second order differential operator. This quantity is divergent for the same reasons expalined in the previous paragraph and we need a regularization and renormalization procedure to get a finite and well defined result. In the next section we expose a standard method which is used to give a consistent definition of regularized functional determinant.

3.2 The Heat Kernel

3.2.1 The Definition of Heat Kernel

Let us consider a second order, self-adjoint and semi-positive differential operator $\hat{O}$ which acts on a field defined on a d-dimensional Riemannian manifold $\mathcal{M}$. We want to introduce an important object known as the heat kernel of the operator [6]. The first step is to consider the so-called heat equation:

$$(\partial_t + \hat{O})u(x;t) = 0, \qquad\qquad t > 0 \tag{3.39}$$

with initial condition

$$u(x;0) = f(x). \tag{3.40}$$

The solution to this equation can be written as

$$u(x;t) = e^{-t\hat{O}} f(x), \tag{3.41}$$

where $e^{-t\hat{O}}$ is the heat operator.
This construction determines also a (heat) kernel, function of two manifold points (x, y) and defined by:

$$u(x;t) = \int d^d y K(x, y; t) f(y). \tag{3.42}$$

By such relation, the heat kernel is itself solution to the heat equation:

$$(\partial_t + \hat{O})K(x, y; t) = 0, \tag{3.43}$$

with initial condition:

$$K(x,y;0) = \delta^d(x-y). \tag{3.44}$$

Denoting with $\{\,|x\rangle\}$ a set of eigenfunctions of the position opearator, one can easily see that the kernel satisfying the equation above can be written as the mixed matrix element of the heat operator:

$$K(x,y;t) = \langle y|\, e^{-t\hat{O}}\, |x\rangle. \tag{3.45}$$

The quantity which is relevant to the construction of a regularized definition of determinant is the trace of the heat operator:

$$K(t;\hat{O}) = \mathrm{Tr}\left[e^{-t\hat{O}}\right] = \int d^dx\sqrt{g}\langle x|\, e^{-t\hat{O}}\, |x\rangle = \int d^dx\sqrt{g}K(x,x;t) \tag{3.46}$$

In general an operator can carry extra indices of some gauge or rotation group. In that case we also have to take the trace over these indices.
If the spectrum of the operator $\hat{O}$ is discrete we can rewrite the traced heat kernel as

$$K(t;\hat{O}) = \sum_n \langle\psi_n|\, e^{-t\hat{O}}\, |\psi_n\rangle = \sum_n e^{-t\lambda_n}, \tag{3.47}$$

where $\{\psi_n\}$ is a set of normalized eigenfunctions of the operator and $\{\lambda_n\}$ the corresponding eigenvalues.
In the case of a continuous spectrum the eigenvalues are labelled by a continuous variable v and they are distributed along the real axis according to a density function $\mu(v)$, also called Plancherel measure, such that

$$K(t;\hat{O}) = \int_0^\infty dv\mu(v)e^{-t\lambda(v)}, \tag{3.48}$$

where $\lambda(v)$ is the eigenvalue function.
Let us give some examples. We consider a free Euclidean Laplacian $\Box = -\partial^\mu\partial_\mu$ on a torus T^n with periodic boundary conditions. Expanding a function $f(x)$ in Fourier series $f(x) = \sum_k c_k f_k(x)$, the heat operator can be diagonalized on the basis of plane waves $\{f_k\}$:

$$\Box f_k(x) = k^2 f_k(x), \qquad\qquad e^{-t\Box}f_k(x) = e^{-tk^2}f_k(x). \tag{3.49}$$

Therefore the traced kernel reads:

$$K(t;\Box) = \sum_k e^{-tk^2}. \tag{3.50}$$

We note that for $t > 0$ the sum exhibits a well behavour for large k, since the term of the series is exponentially suppressed in this limit. Hence $K(t;\Box)$ is a smooth function in t.
Another example is the free laplacian on R^n with Euclidean flat metric. In this case the solution to the heat kernel equation is:

$$\begin{aligned} K(x,y;t) =& \langle y| \, e^{-t\Box} \, |x\rangle = e^{-t\Box_x}\delta^d(x-y) = \frac{1}{(2\pi)^d}\int_{-\infty}^{\infty} d^d p \; e^{-t\Box_x} e^{ip\cdot(x-y)} = \\ =& \frac{1}{(2\pi)^d}\int_{-\infty}^{\infty} d^d p \; e^{-tp^2+ip\cdot(x-y)} = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x-y)^2}{4t}}. \end{aligned} \tag{3.51}$$

It is immediate to check that this kernel is smooth in the variable t. In fact $K(x,y;t) \to 0$ both for $t \to 0^+$ whether $x \neq y$ and for $t \to \infty$. However the traced kernel seems to be not well defined. It can be seen that, when one takes the limit $y \to x$, the dependence of the heat kernel on the manifold points disappears and the integration over the R^n volume is divergent, since we are dealing with a non compact space.
In general we can claim that for a compact manifold the traced kernel is a smooth function and can be expanded in series for small t:

$$K(t;\hat{O}) = \sum_{n=0}^{\infty} t^{\frac{n-d}{2}} a_n(\hat{O}), \tag{3.52}$$

where $a_n(\hat{O})$ are called Seeley-De Witt coefficients. It is possible to show that these coefficients can be expressed as integral over the manifold volume of local invariants. Moreover, their structure is universal and does not depend on the manifold or the Laplacian operator module normalization factors related to the spin structure [6]. This property is called universality of the heat kernel coefficients.
For example the first three coefficients for the scalar and spinor Laplacian are:

$$
-\Box + m^2 : \qquad \begin{aligned} a_0 &= \frac{\mathrm{Vol}(\mathcal{M})}{(4\pi)^{d/2}}, \qquad a_1 = 0, \\ a_2 &= \frac{1}{(4\pi)^{d/2}} \int_{\mathcal{M}} d^d x \sqrt{g} \left(\frac{\mathcal{R}}{6} - m^2 \right), \end{aligned}
$$

$$
-\nabla^2 + \frac{1}{4}\mathcal{R} + m^2 : \qquad \begin{aligned} a_0 &= \frac{\mathrm{Vol}(\mathcal{M})}{(4\pi)^{d/2}}, \qquad a_1 = 0, \\ a_2 &= \frac{1}{(4\pi)^{d/2}} \int_{\mathcal{M}} d^d x \sqrt{g} \left(\frac{\mathcal{R}}{12} + m^2 \right). \end{aligned} \tag{3.53}
$$

This construction can be generalized to the non compact case, with the difference that the volume integrals may be divergent and the coefficients could require a renormalized definition to make sense.
Observing that the variable t has the dimension of a square lenght and the heat kernel of the free Laplacian on R^n depends on the ratio $\frac{(x-y)^2}{t}$, we deduce that the asymptotic small t expansion in 3.52 is equivalent to consider the limit of small distances (or UV limit) of the kernel.

3.2.2 The Heat Kernel on Manifolds with Conical Singularities

A well known type of geometrical defect on Riemannian manifolds is the conical singularities. Let us consider a 2-dimensional Riemannian manifold described by a radial coordinate $\rho \in [0, \alpha]$ and a polar angle of period β, with α, β real positive numbers. A conical singularity arises when $\beta \neq 2\pi$. In this case the manifold is singular in $\rho = 0$: since the lenght of the circle of fixed ρ does not equalize $2\pi\rho$, around $\rho = 0$ the manifold looks like a cone with tip in the singular point and angular deficit $\delta = 2\pi - \beta$.
It is possible to develop a formula, called Sommerfeld formula, which allows to find an expression for the heat kernel on such a manifold starting with the kernel of the regular space. A suitable derivation of the Sommerfeld formula for integer and half-odd integer spins was given by J.S.Dowker in [30, 31], while a review for the case of spin 1/2 can be found in [32]. In what follows we concentrate on the cases of spin 0 and 1/2 in two dimensions.

The Scalar Laplacian Let us consider the case of the bosonic Laplacian on a 2-d Riemannian manifold. We begin with the heat kernel of the Laplace operator $\Box = -\partial^\mu \partial_\mu$ on the two dimensional cone C_β. This space can be described by the metric:

$$ds^2_{c_\beta} = d\rho^2 + \rho^2 d\phi^2, \qquad\qquad \phi + \beta \sim \phi. \tag{3.54}$$

The regular plane R^2 is recovered by setting $\beta = 2\pi$.
The first step in the construction of the Sommerfeld formula is to find an appropriate representation for the kernel of the regular manifold. Without loss of generality and on the base of the rotation symmetry, we choose two points $(x(\phi), x(0))$ separated by an angle ϕ at the same fixed ρ.
According to the 3.51, the R^2 heat kernel between these points in polar coordinates $(\rho\cos\phi, \rho\sin\phi)$ reads:

$$K(x(\phi), x(0); t) = \frac{1}{(4\pi t)} e^{-\rho^2(1-\cos\phi)/2t}, \tag{3.55}$$

where we the distance between the points is $(x(\phi) - x(0))^2 = 2\rho^2(1 - \cos\phi)$.
Since the kernel above is a smooth function in the angle ϕ, we can rewrite it using the Cauchy representation:

$$K(x(\phi), x(0); t) = \frac{1}{2\pi i} \oint_C \frac{dz}{z - \phi} K(x(z), x(0); t), \tag{3.56}$$

where C is a contour in the complex which encircles the point $z = \phi$.

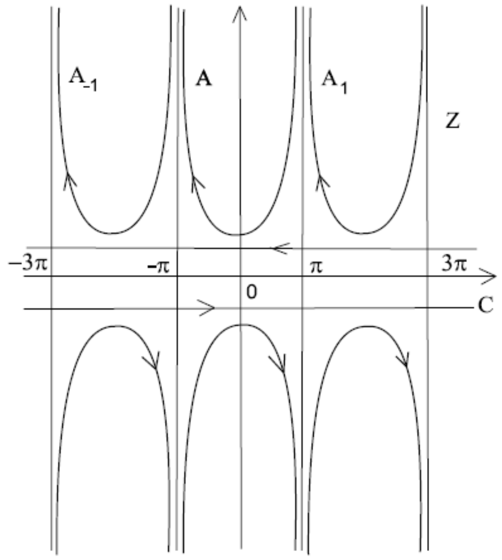

Figure 3: Family of contours encirclying $z = \phi$

Now we deform the circle to a family of contours $A_n = A_0 \bigcup A_n, n = \pm 1, \pm 2, ...$, in wich A_n goes from $(2n+1)\pi - \epsilon + i\infty$ to $(2n-1)\pi + \epsilon + i\infty$ in the higher half plane and from $(2n-1)\pi + \epsilon - i\infty$ to $(2n+1)\pi - \epsilon - i\infty$ in the lower one as shown in the figure 3.
Applying this contour prescription, the heat kernel can be expressed as the so-called 'sum over images':

$$K(x(\phi), x(0); t) = \frac{1}{2\pi i} \sum_{n=-\infty}^{n=\infty} \int_{A_n} \frac{dz}{z-\phi} K(x(z), x(0); t) =$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{n=+\infty} \int_{A_0} \frac{dz}{z-\phi-2\pi n} K(x(z), x(0); t) = \tag{3.57}$$

$$= \frac{1}{4\pi i} \int_{A_0} dz \cot\left(\frac{\pi(z-\phi)}{2\pi}\right) K(x(z), x(0); t).$$

In the second line we have shifted the variable by $2\pi n$ and used the periodicity property of the kernel, while in the third one we have used the identity

$$\sum_{n=-\infty}^{n=\infty} \frac{1}{z-na} = \frac{\pi}{a} \cot\left(\frac{\pi z}{a}\right), \qquad z \neq 0, \pm 1, \pm 2, ..., \quad a \in R. \tag{3.58}$$

We can repeat the same procedure changing the periodicity of the angle from 2π to $\beta \neq 2\pi$. Applying this variation we obtain the heat kernel on the singular cone:

$$K_\beta(x(\phi), x(0); t) = \frac{1}{2i\beta} \int_{A_0} dz \cot\left(\frac{\pi(z-\phi)}{\beta}\right). \tag{3.59}$$

Let us shift the integration variable $z \to z + \phi$:

$$K_\beta(x(\phi), x(0); t) = \frac{1}{2i\beta} \int_{A_0'} dz \cot\left(\frac{\pi z}{\beta}\right) K(x(z+\phi), x(0); t). \tag{3.60}$$

The contour A_0' is obtained by A_0 after as shift of ϕ and consists of two pieces: one in the upper half plane running from $\pi + \phi + i\infty$ to $-\pi + \phi + i\infty$ and one in the lower half plane going from $-\pi + \phi - i\infty$ to $\pi + \phi - i\infty$.
We continue by deforming A_0' in a small circle around $z = 0$ and another contour Γ' that is made of two vertical lines: one going from $-\pi + \phi - i\infty$ to $-\pi + \phi + i\infty$ and one from $\pi + \phi + i\infty$ to $\pi + \phi - i\infty$.
Since we are interested in the case of a manifold with boundary a circle wrapped an integer number of times, we set $\beta = 2\pi\alpha, \alpha = 1, 2, 3, \ldots$.
Near $z = 0$ the integrand has a pole of $\cot\left(\frac{\pi z}{\beta}\right) \sim \frac{\beta}{\pi z}$ with residual the regular kernel $K(x(z+\phi), x(0); t)$. This can be recognized as the contribution to the heat kernel of the regular part of the manifold. Then, denoting $\Gamma = -\Gamma'$ we get:

$$K_\alpha(x(\phi), x(0); t) = K(x(\phi), x(0); t) + \frac{i}{4\pi\alpha} \int_\Gamma dz \cot\left(\frac{z}{2\alpha}\right) K(x(z+\phi), x(0); t). \tag{3.61}$$

This is the Sommerfeld formula for the scalar Laplacian which provides a representation of the heat kernel on the cone C_β in terms of the regular kernel.
The same procedure can be applied to the case of an arbitrary manifold $\mathcal{M}$ where the heat kernel depends on the angular distance between two points. Denoting with (ρ, ϕ), (ρ', ϕ') the coordinates identifying two generic points on the singular manifold $\mathcal{M}_\beta$, the Sommerfeld formula for the heat kernel of the scalar Laplacian generalizes to:

$$K_\alpha(\rho, \rho', \phi, \phi'; t) = K(\rho, \rho', \phi, \phi'; t) + \frac{i}{4\pi\alpha} \int_\Gamma dz \cot\left(\frac{z}{2\alpha}\right) K(\rho, \rho', \phi+z, \phi'; t). \tag{3.62}$$

Here the angular distance becomes $\phi - \phi'$ and the contour Γ needs to be modified. Now the two vertical lines described before run from $-\pi + (\phi - \phi') + i\infty$ to $-\pi +$

$(\phi-\phi')-i\infty$ and from $\pi+(\phi-\phi')-i\infty$ to $\pi+(\phi-\phi')+i\infty$.
Taking coincident points we obtain:

$$K_\alpha(\rho=\rho',\phi=\phi';t)=K(\rho=\rho',\phi=\phi';t)+\frac{i}{4\pi\alpha}\int_\Gamma dz\cot\left(\frac{z}{2\alpha}\right)K(\rho=\rho',\phi'=\phi+z;t). \tag{3.63}$$

Finally, once defined the integrated heat kernel as:

$$K_z(t)=\int_{M_\beta}d\phi d\rho\sqrt{g}\;K(\rho=\rho',\phi'=\phi+z;t), \tag{3.64}$$

we get an expression for the traced heat kernel:

$$K_\alpha(t)=K_{z=0}(t)+\frac{i}{4\pi\alpha}\int_\Gamma dz\cot\left(\frac{z}{2\alpha}\right)K_z(t). \tag{3.65}$$

At concident points the angular distance $\phi-\phi'$ vanishes and the contour Γ reduces to two vertical lines going from $-\pi-i\infty$ to $-\pi+i\infty$ and from $\pi-i\infty$ to $\pi+i\infty$. We can see by the structure of the formula that the correction to the heat kernel due to the singularity is encoded in the second term as contour integral on the complex plane. We note also that if we set $\alpha=1$, $\beta=2\pi$, the singular term vanishes. In fact, in this case the integrand becomes periodic in z of 2π and the contributions coming from the integration along the two vertical lines in $-\pi,\pi$, covered in opposite directions, cancel each other. On the contrary, for $\alpha\neq 1$ the contour integral gives a non trivial contribution due to the different periodicity.

The Spinor Laplacian Similarly to the bosonic case it is possible to put in relation the spinor heat kernel on a manifold with conical singularity M_β with the spinor heat kernel on the correspondent regular space $M_{\beta=2\pi}$. As before we start by considering the cone C_β and then we generalize the formula.
In the case of the regular cone R^2 the spin connection is trivial in Cartesian coordinates and the fermionic Laplace operator $\Delta^{1/2}=-\nabla^2+\frac{1}{4}\mathcal{R}$ reduces to the scalar Laplacian $\Box=-\partial^\mu\partial_\mu$, with solution to the heat kernel equation:

$$K(x,y;t)=K^0(x,y;t)\,I, \tag{3.66}$$

where with $K^0(x,y;t)$ we indicate the scalar Laplacian heat kernel and I is the 2 x 2 identity matrix.

The Cartesian frame is not suited to treat the singularity and we first need to adopt the polar coordinates (ρ, ϕ). In these ones the spinor connection has a non trivial expression $\nabla_\mu = \partial_\mu + \omega_\mu^{[s]}$, with $\omega_\mu^{[s]} dx^\mu = -\frac{i}{2}\sigma_3 d\phi$. The change of coordinates can be performed with a rotation on the spinor bundle given by a unitary matrix $A(\phi)$, which turns the trivial derivative in the covariant derivative of the new frame:

$$A(\phi)^{-1}\partial_\mu A(\phi) = \nabla_\mu, \qquad A(\phi) = e^{\frac{i}{2}\sigma_3\phi}. \tag{3.67}$$

We see that the matrix $A(\phi)$ coincides exactly with a rotation matrix of angle ϕ in the spinor representation.
Since the spinor Laplacian operator tranforms with the same pattern:

$$\Delta^{1/2} \to A(\phi)^{-1}\Delta^{1/2}A(\phi), \tag{3.68}$$

the solution to the heat kernel equation in polar coordinates is:

$$K'(x, y; t) = A(\phi)K(x, y; t). \tag{3.69}$$

We see that, as a consequence of the different spin structures, the bosonic and fermionic kernels have different periodicity properties in the angle ϕ. While the first one is periodic in 2π, the second one inherits the antiperiodicity of the matrix $A(\phi)$, which changes sign after a complete rotation of 2π.
Now as before we choose two points $(x(\phi), x(0))$ and we express $K'(x, y; t)$ with the method of the images that we have used in the bosonic case.
We begin again with the Cauchy representation:

$$K'(x(\phi), x(0); t) = \frac{1}{2\pi i}\oint \frac{dz}{z - \phi} A(z)K(x(z), x(0); t), \tag{3.70}$$

where the contour of integration encircles the point $z = \phi$.
Deforming the contour integration as explained in the previous paragraph we get:

$$K'(x(\phi),x(0);t) = \frac{1}{2\pi i}\sum_{n=-\infty}^{n=\infty}\int_{A_n}\frac{dz}{z-\phi}A(z)K(x(z),x(0);t) =$$

$$= \frac{1}{2\pi i}\sum_{n=-\infty}^{n=+\infty}\int_{A_0}\frac{dz(-1)^n}{z-\phi-2\pi n}A(z)K(x(z),x(0);t) = \tag{3.71}$$

$$= \frac{1}{4\pi i}\int_{A_0}\frac{dz}{\sin(\frac{\pi(z-\phi)}{2\pi})}A(z)K(x(z),x(0);t),$$

In the second line we have shifted the variable by $2\pi n$ and used the antiperiodicity property of the rotation matrix, while in the third line we have used the relation

$$\sum_{n=-\infty}^{n=\infty}\frac{(-1)^n}{z-na} = \frac{\pi}{a}\frac{1}{\sin(\frac{\pi z}{a})} \qquad z =\neq 0, \pm 1, \pm 2, ..., \quad a \in R. \tag{3.72}$$

At this point we change the periodicity of the angle from 2π to $\beta \neq 2\pi$ and get the heat kernel on the cone:

$$K_\beta(x(\phi),x(0);t) = \frac{1}{2i\beta}\int_{A_0}\frac{dz}{\sin(\frac{\pi(z-\phi)}{\beta})}K(x(z),x(0);t). \tag{3.73}$$

Here we have suspended the notation K', K and indicated with K_β,$K = K_{\beta=1}$ respectively the singular kernel and regular Kernel in polar coordinates.
Proceeding identically to the scalar case, we shift the variable $z \to z + \phi$:

$$K_\beta(x(\phi),x(0);t) = \frac{1}{2i\beta}\int_{A'_0}\frac{dz}{\sin(\frac{\pi z}{\beta})}K(x(z+\phi),x(0);t), \tag{3.74}$$

where A'_0 is again given by two curves in the upper and lower half plane, the first one running from $\pi+\phi+i\infty$ to $-\pi+\phi+i\infty$ and the second one from $-\pi+\phi-i\infty$ to $\pi+\phi-i\infty$. Then we deform A'_0 in a small circle around $z=0$, which allows to isolate the contribution given by the regular kernel, and a contour Γ', from which arises the correction due to the singularity, made of two vertical lines that run from $-\pi+\phi-i\infty$ to $-\pi+\phi+i\infty$ and from $\pi+\phi+i\infty$ to $\pi+\phi-i\infty$.
Setting $\beta = 2\pi\alpha, \alpha = 1,2,3,...$ and calling $\Gamma = -\Gamma'$ we obtain:

$$K_\alpha(x(\phi),x(0);t) = K(x(\phi),x(0);t) + \frac{i}{4\pi\alpha}\int_\Gamma\frac{dz}{\sin(\frac{z}{2\alpha})}K(x(z+\phi),x(0);t). \tag{3.75}$$

This expression can be generalized to the case of an arbitrary 2-dimensional Riemannian manifold where the heat kernel depends on the angular distance between two points. Denoting with (ρ, ϕ) and (ρ', ϕ') the polar coordinates of two generic points on a singular manifold $\mathcal{M}_\beta$, the Sommerfeld formula for the spinor Laplacian is found to be:

$$K_\alpha(\rho, \rho', \phi, \phi'; t) = K(\rho, \rho', \phi, \phi'; t) + \frac{i}{4\pi\alpha} \int_\Gamma \frac{dz}{\sin(\frac{z}{2\alpha})} K(\rho, \rho', \phi + z, \phi'; t), \quad (3.76)$$

where, as in the bosonic case, we have to generalize the structure of Γ with the modification $\phi \to \phi - \phi'$.
The untraced spinor heat kernel at coincident points reads:

$$K_\alpha(\rho = \rho', \phi = \phi'; t) = K(\rho = \rho', \phi = \phi'; t) + \frac{i}{4\pi\alpha} \int_\Gamma \frac{dz}{\sin(\frac{z}{2\alpha})} K(\rho = \rho', \phi, \phi + z; t) \quad (3.77)$$

and, with the definition of integrated kernel:

$$K_z(t) = \int_{\mathcal{M}_\beta} \sqrt{g}\, K(\rho = \rho', \phi' = \phi + z; t), \quad (3.78)$$

we can get the final expression for the traced spinor heat kernel:

$$K_\alpha(t) = \mathrm{Tr} K_{z=0}(t) + \frac{i}{4\pi\alpha} \int_\Gamma \frac{dz}{\sin(\frac{z}{2\alpha})} \mathrm{Tr} K_z(t). \quad (3.79)$$

Here the trace is performed over the spinor indices and the integration contour reduces for $\phi = \phi'$ to two vertical lines going from $-\pi - i\infty$ to $-\pi + i\infty$ and from $\pi - i\infty$ to $\pi + i\infty$
Also in this case the singular kernel is given by the sum of the regular kernel and a contour integral describing the singular part, which vanishes when we come back to the regular case. In fact, if we set $\alpha = 1$ the integrand becomes the product of two antiperiodic function in z of 2π and therefore complessively periodic. As explained previously, this leads to the cancellation of the integrals over the two vertical lines and the consequent vanishing of the singular contribution.
The presence of conical singularities produces corrections to the Seeley-De Witt coefficients of the UV heat kernel expansions. It is possible to show that these contributions can be expressed as integral of local invariants over the set of singular

points Σ and, as the regular coefficients, have universal structure [6].
For example the first non trivial correction regards the coefficient a_2 and reads:

$$\begin{aligned}
&-\Box + m^2 : &&a_{2,\alpha} = \frac{\alpha}{12}\left(\frac{1}{\alpha^2} - 1\right)\int_\Sigma , \\
&-\nabla^2 + \frac{1}{4}\mathcal{R} + m^2 : &&a_{2,\alpha} = -\frac{\alpha}{12}\left(\frac{1}{\alpha^2} - 1\right)\int_\Sigma .
\end{aligned} \tag{3.80}$$

In the case of a discrete set of singularities the integral over Σ reduces to a sum over the singular points.

3.2.3 The Regularized Determinant

Now we can give the definition of regularized determinant of a second order differntial operator $\hat{O}$, using the concept of heat kernel defined in the previous section. We assume also that the operator is self-adjoint and positive-definite.
We start defining an operatorial version of the Riemannian ζ-function in terms of the operator eigenvalues:

$$\zeta(s;\hat{O}) = \sum_\lambda \lambda^{-s} \tag{3.81}$$

The analysis of the spectrum of Laplace type operators on Riemannian manifolds reveals that the series converges for $\Re s > \frac{d}{2}$, where d is the dimension of the considered manifold $\mathcal{M}$ [6].
However, as in the case of the usual ζ-function, this can be analitically continued on the entire complex plane to a meromorphic function of s with a finite number of poles on the real axis. These ones can be studied using the integral representation of the ζ-function. Using the relation:

$$\lambda^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}e^{-\lambda t}, \tag{3.82}$$

where $\Gamma(s)$ is the Euler Gamma function, we can express $\zeta(s;\hat{O})$ in terms of the heat kernel of the operator:

$$\zeta(s;\hat{O}) = \frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}\sum_\lambda e^{-\lambda t} = \frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}\ K(t;\hat{O}). \tag{3.83}$$

Having $K(t;\hat{O}) \sim e^{-\lambda_0 t}$ for $t \to \infty$, where $\lambda_0 > 0$ is the lowest eigenvalue of $\hat{O}$, the integrand has a good behavour for large t and the integral is convergent in the upper integration limit. It is instead clear that the poles of the ζ-function come from the lower one. We rewrite the above relation as:

$$\zeta(s;\hat{O}) = f_1(s) + f_2(s),$$
$$f_1(s) = \frac{1}{\Gamma(s)}\int_0^1 dt\ t^{s-1}\ K(t;\hat{O}), \qquad f_2(s) = \frac{1}{\Gamma(s)}\int_1^\infty dt\ t^{s-1}\ K(t;\hat{O}). \tag{3.84}$$

It is easy to see that $f_2(s)$ has no divergences and it is not relevant to the study of the poles. Expanding the heat kernel in powers of t in $f_1(s)$, one finds that this function is well defined only for $s > d/2$ (as anticipated previously). The analytic continuation can be performed by solving the integral for $s > d/2$ and then extending the domain of definition to the whole complex plane (except the possible poles). Explicitly one obtains:

$$f_1(s) = \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{a_n(\hat{O})}{s+\frac{n-d}{2}}. \tag{3.85}$$

This expression shows that the function $\Gamma(s)\zeta(s;\hat{O})$ has simple poles in $s = \frac{d-n}{2}$ with residues:

$$\text{Res}\ \Gamma(s)\zeta(s;\hat{O})_{s=\frac{d-n}{2}} = a_n(\hat{O}). \tag{3.86}$$

In particular, since $\Gamma(s)\zeta(s;\hat{O})$ has a simple pole in $s = 0$ and $\Gamma(s) \sim \frac{1}{s}$, the ζ-function is regular in $s = 0$ and therefore also the derivative $\zeta'(0;\hat{O})$ is well defined. Moreover,using the 3.85 and the properties of $\Gamma(s)$, one can compute the values of the ζ-function for non positive integers in terms of the Seeley coefficients:

$$\zeta(-k;\hat{O}) = (-1)^k k! a_{d+2k}(\hat{O}). \tag{3.87}$$

This special function can be used to define consistently the determinant of the operator $\hat{O}$. From the identity

$$-\frac{d}{ds}\lambda^{-s}\mid_{s=0} = \log(\lambda) \tag{3.88}$$

we are entitled to write:

$$\log(\det[\hat{O}]) = \sum_{\lambda} \log(\lambda) = -\frac{d}{ds}\zeta(s;\hat{O}) \mid_{s=0} . \tag{3.89}$$

This is known as the Ray-Singer definition of determinant. As said before, the derivative $\zeta'(0;\hat{O})$ is well defined because the ζ-function is regular in $s = 0$. This means that the Ray-Singer formula omits the divergent part of the determinant and keeps only the finite contribution. However, in some occasion could be of physical interest also knowing the structure of the divergences. To achieve this aim we need alternative definitions of determinant.
We take as starting point the asymptotic formula:

$$-\int_{\delta}^{\infty} \frac{dt}{t} e^{-\lambda t} = \log \lambda\delta + \gamma_E + O(\lambda\delta), \tag{3.90}$$

where $\gamma_E = 0,57721...$ is the Euler-Mascheroni constant.
This expression provides a motivation for the following definition of regularized determinant:

$$\log(\det[\hat{O}])_{\delta} = \sum_{\lambda} \log(\lambda) = -\int_{\delta}^{\infty} \frac{dt}{t}\ K(t;\hat{O}) \tag{3.91}$$

This definition is called “proper time cutoff”(PTC) regularization.
Similarly to the integral studied in 3.83 and 3.84, the divergent part of this expression comes from the lower limit of integration. To extract the divergence, we need to divide the domain of t in two intervals $[\delta, 1]$,$[1, \infty]$ and, focusing on the first one which contains the cutoff, expand the heat kernel in powers of t. Following this procedure and solving the integral we find:

$$\log(\det[\hat{O}])^{\text{div}}_{\delta} = \sum_{n=0}^{d-1} \frac{2a_n(\hat{O})}{n-d}\delta^{\frac{n-d}{2}} + a_D \log(\delta). \tag{3.92}$$

What we learn is that in PTC regularization the first $d-1$ heat kernel coefficients are related to the power divergences of the determinant , while the d^{th} coefficient is responsible for the logarithmic divergence.
There are many other definitions of regularized determinant that we can adopt and the structure of the divergences varies from one to another.
A different regularization can be obtained by setting $\delta = 0$ and shifting the power t^{-1} to t^{s-1}:

$$\log(\det[\hat{O}])_s = -\mu^{2s} \int_0^\infty dt\ t^{s-1}\ K(t;\hat{O}), \tag{3.93}$$

where we have inserted a constant μ of the dimension of a mass to have the right dimensionality of the expression. According to the 3.83 the definition above can be rewritten as:

$$\log(\det[\hat{O}])_s = -\mu^{2s}\Gamma(s)\zeta(s;\hat{O}). \tag{3.94}$$

This is the so-called "zeta-function regularized" determinant. Expanding around $s = 0$ we get:

$$\log(\det[\hat{O}])_s = -\left(\frac{1}{s} - \gamma_E + \log\mu^2\right)\zeta(0;\hat{O}) - \zeta'(0;\hat{O}) + O(s). \tag{3.95}$$

We see that this definition reproduces only the logarithmic divergence of the PTC regularized determinant. The divergent part has the structure of a simple pole and is proportional to the d^{th} Seeley-De Witt coefficient:

$$\log(\det[\hat{O}])_s^{\mathrm{div}} = -\frac{1}{s}\zeta(0;\hat{O}), \qquad\qquad \zeta(0;\hat{O}) = a_d(\hat{O}), \tag{3.96}$$

where we have used the 3.87.
Setting $\mu = 1$ and ignoring the term proportional to the Euler-Mascheroni constant, we recover the Ray-Singer definition for the finite part of the determinant.
In general the finite part is the same for all the definitions of determinant, while the structure of the divergences depend on the chosen regularization. More details can be found in the D.Fursaev and D.Vassilevich book [6].

4 The One-Loop Correction to the Circular Wilson Loop

4.1 The String Side Prescription

The string prescription for the one-loop correction to the circular Wilson loop in the k^{th} rank symmetric representation has been derived in section 2 and reads as the effective action for a multiplet composed of 3 scalars with $m^2 = 2$, 5 massless scalars, 2 world sheet longitudinal modes, 2 conformal ghosts and 8 Majorana fermions with

$m^2 = 1$ ($m = \pm 1$). According to the relations shown in the previous section, the explicit expression is:

$$\Gamma_1 = \frac{1}{2} \log \frac{[\det(-\Box + 2)]^3 [\det(-\Box)]^5 [\det(-\Delta^{\text{long}} + 1)]}{[\det(-\nabla^2 + \frac{1}{4}\mathcal{R} + 1)]^8 [\det(-\Delta^{\text{ghosts}} + 1)]} \tag{4.1}$$

Here the ghosts and fermions contributions are at the dominator because the corresponding fields are Grassmanian variables. The world sheet transverse modes are described by the scalar Laplacian $-\Box + m^2$, $m^2 = 0, 2$ given in 3.11 and the Majorana fermions are propagated by the spinor Laplacian $-\nabla^2 + \frac{1}{4}\mathcal{R} + 1$ introduced in 3.37. As anticipated in section 2, since the dynamics of the ghost and longitudinal modes multiplet is governed by the same operator $-\Delta^{\text{long}} + 1 = -\Delta^{\text{ghosts}} + 1$ discussed in 2.103 and 2.111, the relative contributions cancel each other, leaving us with the 8 transverse scalars and the 8 Majorana fermions:

$$\Gamma_1 = \frac{1}{2} \log \frac{[\det(-\Box + 2)]^3 [\det(-\Box)]^5}{[\det(-\nabla^2 + \frac{1}{4}\mathcal{R} + 1)]^8}. \tag{4.2}$$

The classical world sheet on which we have to perform the computation is the Euclidean AdS_2 with S^1 boundary wrapped k times, or equivalently AdS_2 with a conical singularity. In the following passages we prefer to work in polar coordinates:

$$ds^2_{\text{AdS}_2^\beta} = d\xi^2 + \sinh^2 \xi d\phi^2, \tag{4.3}$$

where $\phi \in [0, \beta]$, $\beta = 2\pi k$ and $\xi \in [0, \infty]$.
The singularity is situated in $\xi = 0$ and has negative angular deficit $\delta = 2\pi(1 - k)$. The effective action 4.2 has already been computed by M.Kruczensky and A.Tirziu in [19] with a method developed in a series of papers [20, 21, 22, 23]. In short this technique says that the ratio of two one-dimensional determinants is equal to the ratio of the respective wave-functions corresponding to the zero eigenvalue of the operators and evaluated at the boundary. In order to use this method the involved two-dimensional spectral problems requires first to be separated into one-dimensional ones. Then, one can obtain the relevant one-dimensional determinants by computing their ratio with a familiar one. Following this procedure they obtain:

$$\bar{\Gamma}_1(k) = \frac{1}{2} \log 2\pi + \left(2k + \frac{1}{2}\right) \log k - \log k!. \tag{4.4}$$

This result differs considerably from the gauge side expression given in 2.61 and there seems to be a discrepancy between the string and gauge theories predictions.

However, the way in which the presence of conical singularity is taken into account in [19] is not really convincing and the failure to match the Drukker and Gross result provides a motivation to perform the computation with a different method. Our aim is to obtain the Wilson loop effective action 4.2 with a direct computation of the determinants through the heat kernel technique. We first review the computation for $k = 1$ which has been already done in [7] and then we treat the case of a generic k. Afterwards we give a general discussion by comparing the result with the gauge side expression and the Kruczensky and Tirziu string side finding.

4.2 The Case $k = 1$

4.2.1 The Scalar Heat Kernel

In the case $k = 1$ the world sheet manifold reduces to the regular AdS_2. To perform the computation we need to find the heat kernel of the scalar and spinor Laplacian on this space. Let us start with the scalar case.
The spectrum of this operator on Euclidean maximally symmetric spaces has been carefully studied by R.Camporesi and A.Higuchi in [25]. Using an inductive procedure it is possible to solve the eigenvalue problem on the N-dimesnional sphere S^N for a generic N and then by analytic continuation find the spectrum of the Laplacian on the non compact partner H^N, which represent the N-dimensional hyperbolic space.
In the two dimensional case the compact version of AdS_2 is the two-sphere described in polar coordinates by the metric:

$$ds^2_{\mathrm{S}^2} = d\theta^2 + \sin^2\theta d\phi^2, \tag{4.5}$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The eigenfunctions of the Laplacian on the two-sphere are the well-known spherical harmonics with eigenvalues the "angular momenta":

$$-\Box\Phi_l = -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\Phi_l = l(l+1)\Phi_l \qquad l = 0, 1, 2, ... \tag{4.6}$$

where each eigenvalue has degeneracy $d_l = 2l + 1$. The eigenfunctions are assumed to satisfy periodic conditions in ϕ for a fixed θ.
The traced heat kernel corresponding to this eigenvalue problem is:

$$K_{\mathrm{S}^2}(t) = \sum_{l=0}^{\infty}(2l+1)e^{-tl(l+1)}. \tag{4.7}$$

Using the residue theorem, we can find an integral representation of the kernel expressing the sum over the positive integers as contour integral on the complex plane:

$$\sum_{l=0}^{\infty} f_l = \frac{1}{2i}\int_0^{\infty} dz \cot(\pi z) f(z). \tag{4.8}$$

According to this formula the heat kernel becomes:

$$K_{\mathrm{S}^2}(t) = \frac{1}{2i}\int_0^{\infty} dz (2z+1)\cot(\pi z) e^{-tz(z+1)}. \tag{4.9}$$

In order to perform the analytic continuation of the heat kernel we need to know the Laplacian spectrum on AdS_2. The structure of the eigenvalues has been found to be [25]:

$$-\Box\Phi_\lambda = -\left[\frac{\partial^2}{\partial\xi^2} + \coth\xi\frac{\partial}{\partial\xi} + \frac{1}{\sinh^2\xi}\frac{\partial^2}{\partial\phi^2}\right]\Phi_\lambda = (\lambda^2+\frac{1}{4})\Phi_\lambda \qquad \lambda\in[0,\infty]. \tag{4.10}$$

We note that the spectrum in the non-compact case is continuos and therefore we need the relative Plancherel measure to construct the kernel.
Once restored the radious in the previous relations $ds^2 \to R^2 ds^2$, $\lambda_l \to \lambda_l/R^2$, the analytic continuation to the AdS_2 geometry is realized by setting:

$$\theta \to i\xi, \qquad \phi\to\phi, \qquad R\to iR. \tag{4.11}$$

From the 4.10 we learn the correct transformation for the Laplacian eigenvalues is:

$$l \to i\lambda - \frac{1}{2}. \tag{4.12}$$

Starting from 4.9 and adding the mass contribution, the S^2 heat kernel with the above prescriptions becomes:

$$K_{\mathrm{S}^2}(t) \to K_{\mathrm{AdS}_2}(t) = -\int_0^{\infty} d\lambda\lambda\tanh(\pi\lambda) e^{-t\left(\lambda^2+\frac{1}{4}+m^2\right)}. \tag{4.13}$$

Here after the analytic continuation we have restored $R=1$.
In this expression we recognize the eigenvalues $\lambda^2+\frac{1}{4}+m^2$ in the exponential and read the scalar Plancherel measure:

$$\mu(\lambda) = -\lambda \tanh(\pi\lambda). \tag{4.14}$$

One can see that the heat kernel integral is convergent in both the limits of integration and therefore is a well defined object.

4.2.2 The Spinor Heat Kernel

The same steps can be repeated for the fermionic field with the difference that we have also the spinor bundle structure to take in consideration. We choose for the 2-d Dirac gamma matrices the representation introduced in 2.120:

$$\gamma^0 = \sigma^1, \qquad \gamma^1 = \sigma^2 \tag{4.15}$$

In the coordinates 4.5 the vielbeins and the tangent bundle connection are:

$$\begin{gathered} e^0_\mu = (1,0), \qquad e^1_\mu = (0,\sin\theta), \qquad \mu = \theta,\phi \\ \omega^{\alpha\beta}_\theta = 0, \qquad \omega^{00}_\phi = \omega^{11}_\phi = 0, \qquad \omega^{01}_\phi = -\omega^{10}_\phi = -\cos\theta, \end{gathered} \tag{4.16}$$

while the spinor connection and the covariant derivative are:

$$\begin{gathered} \omega^{[s]} = (0, -\frac{i}{2}\cos\theta\sigma^3), \qquad \Sigma_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha,\gamma_\beta] = \frac{i}{2}\sigma^3\epsilon_{\alpha\beta}, \qquad \epsilon_{00} = \epsilon_{11} = 0, \epsilon_{01} = -\epsilon_{10} = 1, \\ \nabla_\theta = \partial_\theta, \qquad\qquad \nabla_\phi = \partial_\phi - \frac{i}{2}\cos\theta\sigma^3. \end{gathered} \tag{4.17}$$

The eigenvalues of the spinor Laplacian are found to be [26]:

$$\begin{gathered} \left(-\nabla^2 + \frac{1}{4}\mathcal{R}\right)\Psi_l = (l+1)^2\Psi_l, \qquad l = 0,1,2,... \\ \nabla^2 = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - i\frac{\cos\theta}{\sin^2\theta}\sigma^3\frac{\partial}{\partial\phi} - \frac{1}{4}\cot^2\theta, \qquad\qquad \mathcal{R} = 2, \end{gathered} \tag{4.18}$$

with degeneracy $d_l = 2(l+1)$.
To be coherent with the fermionic spin structure one has to assume anti-periodic boundary conditions in the angle ϕ for a fixed θ. This can be motivated considering the passage from Cartesian to polar coordinates on the plane R^2 discussed in 3.2.2. In Cartesian coordinates the spinor connection is trivial and the spinor Laplacian reduces to the scalar one acting on a spinor Ψ which behaves as a couple of scalars satisfying periodic conditions. The change of coordinates is performed through a local rotation on the spinor bundle of angle ϕ given by the matrix $A(\phi) = e^{\frac{i}{2}\sigma^3\phi}$. Under such a rotation the spinor transforms as $\Psi' \to e^{\frac{i}{2}\sigma^3\phi}\Psi$ and gains an anti-periodicity property. In fact Ψ' changes sign after a complete rotation of 2π because of the matrix $A(\phi)$. Since the spinor representation is the same, this considerations can be generalized also to the sphere and AdS_2 cases.
The spinor Laplacian heat kernel on S^2 is therefore:

$$K_{\mathrm{S}^2}(t) = -\sum_{l=0}^{\infty}(2l+2)e^{-t(l+1)^2}, \tag{4.19}$$

where he have inserted a mignus sign to take into account the fermionic statistics. As in the scalar case, using the formula 4.8 we can translate the summation in an integral as:

$$K_{\mathrm{S}^2}(t) = -\frac{1}{2i}\int_0^{\infty} dz(2z+2)\cot(\pi z)e^{-t(z+1)^2}. \tag{4.20}$$

The structure of the AdS_2 spinor bundle can be obtained from the sphere by the analytic continuation 4.11:

$$\begin{gathered}
e^0_\mu = (1,0), \qquad e^1_\mu = (0,\sinh\xi), \qquad \mu = \xi,\phi, \\
\omega^{\alpha\beta}_\xi = 0, \qquad \omega^{00}_\phi = \omega^{11}_\phi = 0, \qquad \omega^{01}_\phi = -\omega^{10}_\phi = -\cosh\xi, \\
\omega^{[s]} = (0, -\frac{i}{2}\cosh\xi\sigma^3), \qquad \Sigma_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha,\gamma_\beta] = \frac{i}{2}\sigma^3\epsilon_{\alpha\beta}, \qquad \gamma^{0,1} = \sigma^{1,2}, \\
\nabla_\xi = \partial_\theta, \qquad \nabla_\phi = \partial_\phi - \frac{i}{2}\cosh\xi\sigma^3.
\end{gathered} \tag{4.21}$$

The eigenvalue problem for the spinor laplacian on AdS_2 is [26]

$$\left(-\nabla^2 + \frac{1}{4}\mathcal{R}\right)\Psi_\lambda = \lambda^2\Psi_\lambda, \qquad \lambda \in [0,\infty]$$

$$\nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{1}{\sinh^2\xi}\frac{\partial^2}{\partial\phi^2} - i\frac{\cosh\xi}{\sinh^2\xi}\sigma^3\frac{\partial}{\partial\phi} - \frac{1}{4}\coth^2\xi, \qquad \mathcal{R} = -2. \tag{4.22}$$

As in the scalar case the spectrum is labelled by a continous non negative variable. The prescription for the continuation of the eigenvalues in this case becomes:

$$l \to i\lambda - 1. \tag{4.23}$$

Following the rules of the analytic continuation and adding the mass contribution, the heat kernel for the spinor Laplacian on AdS_2 is found to be:

$$K_{\mathrm{S}^2}(t) \to K_{\mathrm{AdS}_2}(t) = \int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-t\left(\lambda^2+m^2\right)} \tag{4.24}$$

with fermionic Plancherel measure:

$$\mu(\lambda) = \lambda\coth(\pi\lambda). \tag{4.25}$$

Also in this case the heat kernel is a well defined integral in both the integration limits.

4.2.3 The ζ-Function

We have to compute the effective action for a multiplet of 3 scalars with $m^2 = 2$, 5 massless scalars and 8 Majorana fermions with $m^2 = 1$ propagating in AdS_2. We proceed as in the original computation [7] We use the definition of determinant based on the ζ-function given in the previous section. The traced heat kernels are:

$$K(t) = \int_0^\infty d\lambda\mu(\lambda)e^{-t\left(\lambda^2+M\right)},$$

$$\text{scalars}: \quad \mu(\lambda) = -\lambda\tanh(\pi\lambda), \qquad M = \frac{1}{4} + m^2, \tag{4.26}$$

$$\text{fermions}: \quad \mu(\lambda) = \lambda\coth(\pi\lambda), \qquad M = m^2.$$

The corresponding ζ-function can be written as:

$$\zeta(s;M) = \frac{1}{\Gamma(s)}\int_0^\infty d\lambda \mu(\lambda)\int_0^\infty dt t^{s-1} e^{-t(\lambda^2+M)} = \int_0^\infty d\lambda \frac{\mu(\lambda)}{(\lambda^2+M)^s}. \tag{4.27}$$

Starting with the bosonic case, we split the spectral density as $-\lambda \tanh \pi\lambda = -\lambda\left(1-\frac{2}{e^{2\pi\lambda}+1}\right)$ obtaining:

$$\zeta(s;M) = -\int_0^\infty d\lambda \frac{\lambda}{(\lambda^2+M)^s} + 2\int_0^\infty d\lambda \frac{\lambda}{(e^{2\pi\lambda}+1)(\lambda^2+M)^s}. \tag{4.28}$$

According to the definition of determinant that we are using, the divergent part is proportional to $\zeta(0;M)$, while the finite part is related to the derivative $\zeta'(0;M)$. We note that the first integral is well defined in the upper integration limit only for $\Re s > 1$. This is coherent with the discussion provided in section 2.2.3. To define consitently the finite and divergent parts of the determinant we need to perform an analytic continuation of the integral to a function defined on all the complex plane except a finite number of poles. This can be achieved by first doing the integral for $\Re s > 1$ and then extending the domain to the rest of the complex plane. Explicitly we get:

$$\zeta(s;M) = \frac{M^{(1-s)}}{2(1-s)} + 2\int_0^\infty d\lambda \frac{\lambda}{(e^{2\pi\lambda}+1)(\lambda^2+M)^s}. \tag{4.29}$$

What we find after the analytic continuation is a function with a good behaviour on all the complex plane except the point $s=1$ where it has a pole.
We first focus on the divergences of the determinant. Setting $s=0$ in the expression above we obtain:

$$\begin{aligned}\zeta(0;M) =& \frac{M}{2} + 2\int_0^\infty d\lambda \frac{\lambda}{(e^{2\pi\lambda}+1)} = \frac{1}{2}\left(m^2+\frac{1}{4}\right) + \frac{1}{24} = \\ =& \frac{m^2}{2} + \frac{1}{6} = \frac{1}{4\pi}\int_{\mathrm{AdS}_2} d^2x\sqrt{g}\left(\frac{\mathcal{R}}{6} - m^2\right),\end{aligned} \tag{4.30}$$

where in the last equality we recognize the structure of the Seeley-De Witt coefficient given in 3.53 with the definition of renormalized volume $V_{\mathrm{AdS}_2} = -2\pi$. Summing the contributions of the 3 scalars with $m^2 = 2$, $M = \frac{9}{4}$ and the 5 massless scalars with $M = \frac{1}{4}$ one finds that the total bosonic divergence is proportional to:

$$\zeta^{\text{scalars}}(0) = 5\zeta(0;9/4) + 3\zeta(0;1/4) = \frac{1}{4\pi}\int_{\text{AdS}_2} d^2x\sqrt{g}\left(8 \text{ x } \frac{\mathcal{R}}{6} - 3 \text{ x } 2\right) = $$
$$= \frac{1}{4\pi}\int_{\text{AdS}_2} d^2x\sqrt{g}\left(8 \text{ x } \frac{\mathcal{R}}{6} - 6\right) \tag{4.31}$$

Now let us compute the finite part of the scalar Laplacian determinant. The derivative of the ζ-function in $s = 0$ is:

$$\zeta'(0;M) = -\frac{1}{2}M(\log M - 1) - I_2(M), \qquad I_2(M) = 2\int_0^\infty d\lambda \frac{\lambda \log(\lambda^2 + M)}{e^{2\pi\lambda} + 1}. \tag{4.32}$$

$I_2(M)$ can be rewritten as:

$$I_2(M) = \int_0^M dx \frac{\partial I_2(x)}{\partial x} + I_2(0),$$
$$\frac{\partial I_2(x)}{\partial x} = 2\int_0^\infty d\lambda \frac{\lambda}{(e^{2\pi\lambda}+1)(\lambda^2 + x)} = -\frac{1}{2}\log x + \psi(\sqrt{x} + \frac{1}{2}), \tag{4.33}$$
$$I_2(0) = 2\int_0^\infty d\lambda \frac{\lambda \log \lambda^2}{e^{2\pi\lambda} + 1} = \frac{1}{12}(1 + \log 2) - \log A,$$

where A is the Glaisher constant defined by $\log A = \frac{1}{12} - \zeta_R'(-1) = 1.282...$, with ζ_R the Riemann zeta-function, and $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ is the polygamma function. Thus we find:

$$\zeta'(0;M) = -\frac{1}{2}M(\log M - 1) - \frac{1}{12}(1 + \log 2) + \log A + \int_0^M dx\left(\frac{1}{2}\log x - \psi(\sqrt{x} + \frac{1}{2})\right) =$$
$$= -\frac{1}{12}(1 + \log 2) + \log A - \int_0^M dx\ \psi(\sqrt{x} + \frac{1}{2}). \tag{4.34}$$

Setting $M = \frac{9}{4}$ we get:

$$\zeta'(0;9/4) = -\frac{1}{12}(1+\log 2) + \log A - \left(2 - \frac{19}{12}\log 2 + 3\log A - \frac{3}{2}\log\pi\right) = \\ = -\frac{25}{12} + \frac{3}{2}\log 2\pi - 2\log A, \tag{4.35}$$

and

$$\zeta'(0;1/4) = -\frac{1}{12}(1+\log 2) + \log A - \left(-\frac{7}{12}\log 2 + 3\log A - \frac{1}{2}\log\pi\right) = \\ = -\frac{1}{12} + \frac{1}{2}\log 2\pi - 2\log A \tag{4.36}$$

for $M = \frac{1}{4}$.

Putting all the contribution together the finite part of the scalar Laplacian determinant is found to be:

$$\log\left([\det(-\Box+2)]^3[\det(-\Box)]^5\right) = -\zeta'^{\,\text{scalars}}(0) = \\ = -3\zeta'(0;9/4) - 5\zeta'(0;1/4) = \frac{20}{3} - 7\log 2\pi + 16\log A. \tag{4.37}$$

The same steps can be repeated for the spinor Laplacian using the relation $\coth\pi\lambda = 1 + \frac{2}{e^{2\pi\lambda}-1}$. The ζ-function in this case is:

$$\zeta(s;M) = \frac{M^{(1-s)}}{2(s-1)} + 2\int_0^\infty d\lambda \frac{\lambda}{(e^{2\pi\lambda}-1)(\lambda^2+M)^s}. \tag{4.38}$$

The divergent part of the spinor Laplacian determinant is proportional to $\zeta(0;M)$:

$$\zeta(0;M) = -\frac{M}{2} + 2\int_0^\infty d\lambda \frac{\lambda}{(e^{2\pi\lambda}-1)} = -\frac{m^2}{2} + \frac{1}{12} = \\ = \frac{1}{4\pi}\int_{\mathrm{AdS}_2} d^2x\sqrt{g}\left(\frac{\mathcal{R}}{12} + m^2\right). \tag{4.39}$$

This is the result for a standard 2-d fermion with usual kinetic term. As discussed in sections 2.3.1 and 2.3.3, the kinetic term in the GS fermionic lagrangian depends from the projection of the 10-d Dirac matrices on the world sheet and is formally identified with the canonical kinetic term after a rotation on the target space spinor bundle. We have already mentioned that the conformal anomaly produced by the GS kinetic term is found to be 4 times bigger than the usual one and the fermionic central charge has to be multiplied by 4 to have a complessive cancellation of the GS anomaly. In the present situation, to take into account the different behavour under conformal transformations of the kinetic term and get the right conformal anomaly for the GS fermion, one should multiply by 4 the term proportional to $\mathcal{R}$ in the Seeley coefficient above [5]. Therefore, setting $m^2 = 1, M = 1$ and summing the contribution of the 8 fermions we find that the total divergence is proportional to:

$$
\begin{aligned}
\zeta(0)^{\text{fermions}} =& 8\zeta(0;1) = \frac{1}{4\pi}\int_{\text{AdS}_2} d^2x\sqrt{g}\left(8 \text{ x } 4 \text{ x } \frac{\mathcal{R}}{12} + 8 \text{ x } 1\right) = \\
=& \frac{1}{4\pi}\int_{\text{AdS}_2} d^2x\sqrt{g}\left(8 \text{ x } \frac{\mathcal{R}}{3} + 8\right).
\end{aligned}
\tag{4.40}
$$

Summing the divergences of the bosonic and fermionic sector one gets:

$$
\begin{aligned}
\hat{\Gamma}^{\text{div}}_{1,s} =& -\frac{1}{s}\left[\frac{1}{4\pi}\int_{\text{AdS}_2} d^2x\sqrt{g}\left((8 + 8 \text{ x } 2) \text{ x } \frac{\mathcal{R}}{6} + (8-6)\right)\right] = \\
=& -\frac{1}{s}\left[\frac{1}{4\pi}\int_{\text{AdS}_2} d^2x\sqrt{g}\,(4\mathcal{R} + 2)\right] = -\frac{1}{s}\left[\frac{1}{4\pi}\int_{\text{AdS}_2} d^2x\sqrt{g}\; 3\mathcal{R}\right].
\end{aligned}
\tag{4.41}
$$

We see that the divergences of GS string partition function do not cancel automatically after the gauge fixing just by adding the Seeley coefficients of the involved determinants. This is a general fact regarding the open GS string on a generic background, where the usual conformal anomaly should be accompanied by an appropriate measure factor contribution. This additional term is proportional to the Euler number $\int d^2x\sqrt{g}\,\mathcal{R}$ and exactly cancels the residual divergence in 4.41.
Thus the string partition function, and consequently the circular Wilson loop, turns out to be finite and conformal at one loop as expected.
A more general discussion about the cancellation of the GS string anomalies on a generic background can be found in [5].

The finite part of the fermionic Laplacian determinant is instead related to the derivative $\zeta'(0;M)$:

$$\zeta'(0;M) = \frac{1}{2}M(\log M - 1) - I_2(M), \qquad I_2(M) = 2\int_0^\infty d\lambda \frac{\lambda\log(\lambda^2+M)}{e^{2\pi\lambda}-1}$$

$$\begin{aligned} I_2(M) &= \int_0^M dx \frac{\partial I_2(x)}{\partial x} + I_2(0), \\ \frac{\partial I_2(x)}{\partial x} &= 2\int_0^\infty d\lambda \frac{\lambda}{(e^{2\pi\lambda}-1)(\lambda^2+x)} = \frac{1}{2}\log x - \frac{1}{2}(\sqrt{x})^{-1} - \psi(\sqrt{x}), \\ I_2(0) &= 2\int_0^\infty d\lambda \frac{\lambda\log\lambda^2}{e^{2\pi\lambda}-1} = \frac{1}{6} - 2\log A. \end{aligned} \tag{4.42}$$

Therefore:

$$\begin{aligned} \zeta'(0;M) &= \frac{1}{2}M(\log M - 1) - \frac{1}{6} + 2\log A - \int_0^M dx\left(\frac{1}{2}\log x - \frac{1}{2}(\sqrt{x})^{-1} - \psi(\sqrt{x})\right) = \\ &= -\frac{1}{6} + 2\log A + \sqrt{M} + \int_0^M dx\ \psi(\sqrt{x}). \end{aligned} \tag{4.43}$$

Setting $M=1$ we obtain:

$$\zeta'(0;1) = \frac{5}{6} + 2\log A - \log 2\pi. \tag{4.44}$$

Adding the contributions of all the 8 fermions one finds that the finite part of the spinor Laplacian determinant is:

$$\begin{aligned} &\log([\det(-\nabla^2 + \frac{1}{4}\mathcal{R} + 1)]^8) = -\zeta'^{\,\text{fermions}}(0) = -8\zeta'(0;1) = \\ &= -\frac{20}{3} - 16\log A + 8\log 2\pi. \end{aligned} \tag{4.45}$$

Ignoring the divergent parts of the determinants that cancel out as explained, the 1-loop correction 4.2 for the case $k = 1$ is:

$$\hat{\Gamma}_1(1) = \frac{1}{2}\left(5\log(\det[-\Box]) + 3\log(\det[-\Box + 2]) + 8\log(\det[-\nabla^2 + \frac{1}{4}\mathcal{R} + 1])\right) =$$

$$= -\frac{1}{2}\zeta'^{\,\text{scalars}}(0) - \frac{1}{2}\zeta'^{\,\text{fermions}}(0) = \frac{1}{2}\log 2\pi. \tag{4.46}$$

We note that our result matches that of Kruczensky and Tirziu, but differs from the gauge side expression in 2.61 by a term $\frac{3}{4}\log\lambda - \log 2$. However one should not worry about this discrepancy because there is a missing contribution which we have not discussed in the 4.2. The Faddev-Popov method applied in the quantization of the GS open string does not fix completely the diffeomorphism symmetry of the metric. In fact AdS_2 is a 2-d maximally symmetric space and the metric is left invariant by three Killing isometries which generate the group SL(2;R). The generators of this residual symmetry are called ghosts zero modes and their contribution to the string partition function requires to be treated separately and added by hand to the final result. In [11] it is claimed that each zero mode contributes to the string partition function with a multiplicative factor proportional to $\lambda^{-1/4}$. Including this missing contribution one obtains:

$$Z_{S_2} \to c\lambda^{-3/4} Z_{S_2}, \tag{4.47}$$

where Z_{S_2} is the partition function of the GS string quadratic fluctuations given in 2.127 and c is a normalization constant to be determined.
As a consequence the 1-loop effective action is shifted by $\frac{3}{4}\log\lambda - \log c$. It is hard in general to derive the correct normalization of the ghosts zero modes on the string side and could be possible to determine the constant c only through a comparison with the gauge side. In this case the AdS/CFT correspondence demands $c = 2$ to have a matching between the circular Wilson loop predictions.
This provides another possible reason to perform the computation for a non trivial k on the string side. In fact through a comparison between the already known gauge side result and the string side one could be useful to determine the probable dipendence of the normalization constant c on the integer k.

4.3 The Scalar Heat Kernel on AdS_2 with Conical Singularity

4.3.1 Applying the Sommerfeld Formula

To compute the 1-loop correction to the circular loop in the k^{th} rank symmetric representation we need the scalar and spinor heat kernel on AdS_2 with conical singularity of negative deficit $\delta = 2\pi(1-k)$.
We start with the bosonic case. To apply the Sommerfeld formula and get correction due to the singularity we need the expression for the untraced heat kernel of the regular AdS_2. The relative heat equation has been solved by Camporesi in [27] and the solution admits the integral representation:

$$K(x, x', t) = \frac{\sqrt{2}e^{-t(m^2+1/4)}}{(4\pi t)^{\frac{3}{2}}} \int_{d(x,x')}^{\infty} \frac{y e^{-\frac{y^2}{4t}} dy}{\sqrt{\cosh y - \cosh d(x, x')}} \tag{4.48}$$

where the dependence of the kernel in the couple of points (x, x') is encoded in the AdS_2 geodesic distance $d(x, x')$.
According to 3.65, the heat kernel in presence of a conical singularity is given by:

$$\begin{aligned}
K_\alpha(t) &= K_{z=0}(t) + \frac{i}{4\pi\alpha}\int_\Gamma dz \cot\left(\frac{z}{2\alpha}\right) K_z(t), \\
K_z(t) &= \int_0^{2\pi\alpha} d\phi \int_0^\infty d\xi \sinh\xi \; K(\xi = \xi', \phi' = \phi + z; t), \\
K(\xi = \xi', \phi' = \phi + z; t) &= \frac{\sqrt{2}e^{-t(m^2+1/4)}}{(4\pi t)^{\frac{3}{2}}} \int_{d(\xi,z)}^{\infty} \frac{y e^{-\frac{y^2}{4t}} dy}{\sqrt{\cosh y - \cosh d(\xi, z)}}, \\
K_{z=0}(t) &= \int_0^{2\pi\alpha} d\phi \int_0^\infty d\xi \sinh\xi \; K(\xi = \xi', \phi' = \phi; t) = \alpha K(t).
\end{aligned} \tag{4.49}$$

Here we have chosen to work in polar coordinates 4.3 and to avoid conflict of notation we have substituted the integer k with α. The contour Γ on the complex plane has been described in section 3.2.2 and is made of two vertical lines going from $-\pi + i\infty$ to $-\pi - i\infty$ and from $\pi - i\infty$ to $\pi + i\infty$. The geodesic distance $d(\xi, z)$ between the points (ξ, ϕ) and $(\xi, \phi + z)$ is defined by the relation $\sinh^2 d(\xi, z) = \sinh^2(\xi)\sin^2\frac{z}{2}$.
In the last expression the kernel at coincidents points does not depend on the integration coordinates and the integral reduces to the traced kernel for the regular

AdS$_2$ multiplied by α, once performed the renormalization of the infinite volume as explained in 2.3.4.
From now we focus on the computation of $K_z(t)$ which describes the correction to the kernel due to the singularity. The integral we have to calculate is:

$$K_z(t) = \frac{\sqrt{2}e^{-t(m^2+1/4)}}{(4\pi t)^{\frac{3}{2}}} \int_0^{2\pi\alpha} d\phi \int_0^\infty d\xi \sinh\xi \int_{d(\xi,z)}^\infty \frac{ye^{-\frac{y^2}{4t}}dy}{\sqrt{\cosh y - \cosh d(\xi,z)}}. \tag{4.50}$$

Performing the integration over the angle and denoting:

$$\begin{aligned} &\sinh\xi = x, \qquad d(x,z) = 2\text{arcsinh}\left(x\sin\frac{z}{2}\right), \\ &\cosh d = 1 + 2x^2\sin^2\frac{z}{2}, \qquad \cosh y = 1 + 2\sinh^2\frac{y}{2}, \end{aligned} \tag{4.51}$$

we obtain

$$\begin{aligned} K_z(t) =& \frac{\alpha e^{-t(m^2+1/4)}}{2(4\pi)^{\frac{1}{2}}t^{\frac{3}{2}}} \int_0^\infty \frac{dx x}{\sqrt{1+x^2}} \int_{d(x,z)}^\infty \frac{ye^{-\frac{y^2}{4t}}dy}{\sqrt{\sinh^2(\frac{y}{2}) - x^2\sin^2(\frac{z}{2})}} = \\ =& \frac{\alpha e^{-t(m^2+1/4)}}{2(4\pi)^{\frac{1}{2}}t^{\frac{3}{2}}\sin z/2} \int_0^\infty \frac{dx x}{\sqrt{1+x^2}} \int_{d(x,z)}^\infty \frac{ye^{-\frac{y^2}{4t}}dy}{A(y,z)^2 - x^2}, \\ A(y,z) =& \frac{\sinh y/2}{\sin z/2}. \end{aligned} \tag{4.52}$$

Now we interchange the integration limits as follows:

$$\int_0^\infty dx \int_{y=d(x,z)}^\infty dy = \int_0^\infty dy \int_0^{x=d^{-1}(y,z)} dx, \qquad x = d^{-1}(y,z) = \frac{\sinh\frac{y}{2}}{\sin\frac{z}{2}} = A(y,z) \tag{4.53}$$

and get:

$$K_z(t) = \frac{\alpha e^{-t(m^2+1/4)}}{2(4\pi)^{\frac{1}{2}}t^{\frac{3}{2}}\sin z/2} \int_0^\infty dyye^{-y^2/4t} \int_0^{A(y,z)} \frac{dxx}{\sqrt{1+x^2}\sqrt{A(y,z)^2 - x^2}}. \tag{4.54}$$

The integration over x gives:

$$\int_0^{A(y,z)} \frac{dxx}{\sqrt{1+x^2}\sqrt{A(y,z)^2-x^2}} = \frac{\pi}{2} - \arctan\left[\frac{\sin z/2}{\sinh y/2}\right]. \tag{4.55}$$

The expression for the traced kernel becomes:

$$\begin{aligned}
K_z(t) =& \frac{\alpha e^{-t(m^2+1/4)}}{2(4\pi)^{\frac{1}{2}} t^{\frac{3}{2}} \sin z/2} \int_0^\infty dyye^{-y^2/4t}\left(\pi/2 - \arctan\left[\frac{\sin z/2}{\sinh y/2}\right]\right) = \\
=& \frac{\alpha}{2(4\pi)^{\frac{1}{2}} t^{\frac{3}{2}} \sin z/2}(-2t) \int_0^\infty dy\partial_y[e^{-y^2/4t}]\left(\pi/2 - \arctan\left[\frac{\sin z/2}{\sinh y/2}\right]\right) = \\
=& \frac{\alpha}{(4\pi t)^{\frac{1}{2}} \sin z/2} \int_0^\infty dye^{-y^2/4t}\partial_y\left(\pi/2 - \arctan\left[\frac{\sin z/2}{\sinh y/2}\right]\right) = \\
=& \frac{\alpha}{(4\pi t)^{\frac{1}{2}} \sin z/2} \int_0^\infty dye^{-y^2/4t}\frac{\cosh y/2 \sin z/2}{\cosh y - \cos z}.
\end{aligned} \tag{4.56}$$

Changing the variable $y \to 2y$ we get the final expression:

$$\begin{aligned}
K_z(t) =& \frac{2\alpha e^{-t(m^2+1/4)}}{(4\pi t)^{\frac{1}{2}}} \int_0^\infty dye^{-y^2/t}\frac{\cosh y}{\cosh 2y - \cos z} = \\
=& \frac{\alpha e^{-t(m^2+1/4)}}{(4\pi t)^{\frac{1}{2}}} \int_0^\infty dye^{-y^2/t}\frac{\cosh y}{\sinh^2 y + \sin^2 z/2}
\end{aligned} \tag{4.57}$$

According to 4.49 the scalar heat kernel on AdS_2 with a conical singularity is:

$$\begin{aligned}
K_\alpha(t) =& \alpha K(t) + \frac{\alpha e^{-t(m^2+1/4)}}{(4\pi t)^{\frac{1}{2}}} \int_0^\infty dy \cosh y e^{-y^2/t}\frac{i}{4\pi\alpha}\int_\Gamma dzF(z;y,\alpha), \\
F(z;y,\alpha) =& \frac{\cot\frac{z}{2\alpha}}{\sinh^2 y + \sin^2 z/2}.
\end{aligned} \tag{4.58}$$

We note that $F(z,y;\alpha)$ becomes periodic in z of 2π when is set $\alpha = 1$. As explained in section 3.2.2, because of the structure of the contour Γ this should guarantee the automatic cancellation of the singular contribution for $\alpha = 1$.
Now we evaluate the contour integral on the complex plane by using the residuals theorem. The two vertical lines $(-\pi + i\infty, -\pi - i\infty)$ and $(\pi - i\infty, \pi + i\infty)$ contains one pole of $\cot\frac{z}{2\alpha}$ at $z = 0$ with residual:

$$ResF(z=0; y, \alpha) = \frac{2\alpha}{\sinh^2(y)} \tag{4.59}$$

Then, denoting $y = -iu$, $\sinh(y) = -i\sin(u)$ and rewriting $F(z; y, \alpha)$ as

$$F(z; u, \alpha) = \frac{\cot\frac{z}{2\alpha}}{(\sin z/2 + \sin u)(\sin z/2 - \sin u)}, \tag{4.60}$$

we see that there are other two poles on the immaginary axis at $z = \pm 2u$. Using the limit:

$$\lim_{z \to \pm 2u} \frac{\sin z/2 \pm \sin u}{z \pm 2u} = \frac{1}{2}\cos(u) \tag{4.61}$$

we find

$$ResF(z = \pm 2u; u, \alpha) = \frac{\cot\frac{u}{\alpha}}{\sin u \cos u} = -\frac{\coth\frac{y}{\alpha}}{\sinh y \cosh y}. \tag{4.62}$$

Therefore the contour integral gives:

$$\begin{aligned} \frac{i}{4\pi\alpha}\int_\Gamma dz F(z; y, \alpha) =& \frac{i}{4\pi\alpha} 2\pi i \left[\frac{2\alpha}{\sinh^2(y)} - \frac{2\coth\frac{y}{\alpha}}{\sinh y \cosh y}\right] = \\ =& \frac{1}{\sinh^2 y}\left[\frac{\tanh y}{\alpha \tanh\frac{y}{\alpha}} - 1\right] \end{aligned} \tag{4.63}$$

Finally, the AdS_2 with conical singularity scalar heat kernel is:

$$K_\alpha(t) = \alpha K(t) + \frac{\alpha e^{-t(m^2+1/4)}}{(4\pi t)^{\frac{1}{2}}} \int_0^\infty dy e^{-\frac{y^2}{t}} \frac{1}{\sinh y}\left[\frac{\coth\frac{y}{\alpha}}{\alpha} - \coth y\right] \tag{4.64}$$

This result is the same given in [33] without the details of the computation.
The integral is obviously convergent in the upper limit because of the presence of the exponential. Then for $y \to 0$ we have:

$$\frac{1}{\sinh y}\left[\frac{\coth\frac{y}{\alpha}}{\alpha} - \coth y\right] \sim \frac{1}{y}\left[\left(\frac{1}{y} + \frac{y}{3\alpha^2}\right) - \left(\frac{1}{y} + \frac{y}{3}\right)\right] = \frac{1}{3}\left[\frac{1}{\alpha^2} - 1\right] \quad (4.65)$$

Therefore the integral is well defined also in the lower limit of integration.
Furthermore, if we set $\alpha = 1$ we see that the contribution from the singularity vanishes, leaving us with the regular kernel $K(t)$ as expected.

4.3.2 Comparison with Orbifolds

A Z_N orbifold of AdS_2 is the quotient manifold AdS_2/Z_N which can be obtained from AdS_2 in polar coordinates by considering the identification:

$$\phi \sim \phi + \frac{2\pi}{N}. \quad (4.66)$$

In practice the periodicity of the polar angle is changed from 2π to $\frac{2\pi}{N}$, causing a conical singularity of positive deficit $\delta = 2\pi\left(1 - \frac{1}{N}\right)$.
A modified version of the Sommerfeld formula for orbifolds has been developed in [28], where is found the small t expansion of the heat kernel for the massless scalar Laplacian on AdS_2/Z_N:

$$K_{AdS_2/Z_N}(t) = \frac{1}{N}K_{AdS_2}(t) + \frac{1}{2}\left[\frac{N^2-1}{6N} - \frac{(N^2+11)(N^2-1)}{180N}t\right] + O(t^2). \quad (4.67)$$

Since the orbifold AdS_2/Z_N can be obtained from AdS_2 with conical singularity by an analytic continuation $\alpha \to \frac{1}{N}$, we dispose of a consistency check for the kernel found in 4.64. In detail one can choose $m = 0$, expand the scalar Laplacian heat kernel of AdS_2 with conical singularity for small t , set $\alpha = \frac{1}{N}$ and compare the finding with the expression above.
We first perform a change of variable $y \to y\sqrt{t}$:

$$
\begin{aligned}
K_\alpha(t) =&\alpha K(t) + \frac{\alpha e^{-\frac{t}{4}}}{(4\pi t)^{\frac{1}{2}}}\sqrt{t}\int_0^\infty dy e^{-y^2}\frac{1}{\sinh\sqrt{t}y}\left[\frac{\coth\frac{\sqrt{t}y}{\alpha}}{\alpha} - \coth\sqrt{t}y\right] = \\
=&\alpha K(t) + \frac{\alpha e^{-\frac{t}{4}}}{(4\pi)^{\frac{1}{2}}}\int_0^\infty dy e^{-y^2}\frac{1}{\sinh\sqrt{t}y}\left[\frac{\coth\frac{\sqrt{t}y}{\alpha}}{\alpha} - \coth\sqrt{t}y\right]
\end{aligned}
\tag{4.68}
$$

Then we can expand the expression for small t:

$$
\begin{aligned}
K_\alpha(t) =&\alpha K(t) + \frac{\alpha}{(4\pi)^{\frac{1}{2}}}\left(1-\frac{t}{4}\right)\int_0^\infty dy\left[\frac{1}{3}\left(\frac{1}{\alpha^2}-1\right) + \left(\frac{7\alpha^4-5\alpha^2-2}{90\alpha^4}\right)ty^2\right]e^{-y^2} + O(t^2) = \\
=&\alpha K(t) + \frac{\alpha}{2\sqrt{\pi}}\left(1-\frac{t}{4}\right)\sqrt{\pi}\left[\frac{1}{6}\left(\frac{1}{\alpha^2}-1\right) + \left(\frac{7\alpha^4-5\alpha^2-2}{360\alpha^4}\right)t\right] + O(t^2) \\
=&\alpha K(t) + \frac{1}{2}\left[\frac{\alpha}{6}\left(\frac{1}{\alpha^2}-1\right) + \alpha\left(\frac{7\alpha^4-5\alpha^2-2}{360\alpha^4} - \frac{1-\alpha^2}{24\alpha^2}\right)t\right] + O(t^2) = \\
=&\alpha K(t) + \frac{1}{2}\left[\frac{\alpha}{6}\left(\frac{1}{\alpha^2}-1\right) - \alpha\left(\frac{1+10\alpha^2-11\alpha^4}{180\alpha^4}\right)t\right] + O(t^2)
\end{aligned}
\tag{4.69}
$$

We see that the coefficient relative to the power t^0 reproduces the structure of the Seeley coefficient given in 3.80. Now setting $\alpha = \frac{1}{N}$ we get:

$$
\begin{aligned}
K_{\frac{1}{N}}(t) =&\frac{1}{N}K(t) + \frac{1}{2}\left[\frac{N^2-1}{6N} - \left(\frac{10N^2+N^4-11}{180N}\right)t\right] + O(t^2) \\
=&\frac{1}{N}K(t) + \frac{1}{2}\left[\frac{N^2-1}{6N} - \left(\frac{(N^2+11)(N^2-1)}{180N}\right)t\right] + O(t^2)
\end{aligned}
\tag{4.70}
$$

that matches 4.67.

4.4 The Spinor Heat Kernel on AdS_2 with Conical Singularity

4.4.1 The 2-d Spinor Propagator

We repeat the same steps for the fermionic case. The solution to the heat equation for the spinor Laplacian on AdS_2 has been found to be [27]:

$$K(x,x';t) = U(x,x')\frac{\sqrt{2}e^{-m^2 t}}{(4\pi t)^{\frac{3}{2}}}(\cosh d(x,x')/2)^{-1}\int_{d(x,x')}^{\infty} \frac{dyy\cosh\frac{y}{2}e^{-\frac{y^2}{4t}}}{\sqrt{\cosh y - \cosh d(x,x')}}, \tag{4.71}$$

where $U(x,x')$ is called the spinor propagator. This is a 2 x 2 matrix acting on the AdS_2 spinor bundle and satisfying the equation of the parallel transport along the geodesic connecting the two points (x,x'):

$$\begin{gathered} n^a(x,x')\nabla_a U(x,x') = 0 \\ U(x,x) = I \qquad\qquad n_a(x,x') = \nabla_a d(x,x') = \partial_a d(x,x') \end{gathered} \tag{4.72}$$

Here the point x' is kept fixed and the derivatives are taken respect to the variable x. The vector $n_a(x,x')$ defined above is the tangent vector to the geodesic at the point x and $a = 0,1$ is the tangent bundle index.
The general structure of the spinor propagator on hyperbolic spaces is not known in literature. In what follows we solve the parallel transport equation in the two dimensional case. It is preferible to work in polar coordinates with the spinor bundle structure:

$$\begin{gathered} g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\xi \end{pmatrix}, \qquad e^0_\mu = (1,0), \qquad e^1_\mu = (0,\sinh\xi), \qquad \mu = \xi,\phi \\ \omega^{\alpha\beta}_\xi = 0, \qquad \omega^{00}_\phi = \omega^{11}_\phi = 0, \qquad \omega^{01}_\phi = -\omega^{10}_\phi = -\cosh\xi, \\ \omega^{[s]} = (0, -\frac{i}{2}\cosh\xi\sigma^3), \qquad \Sigma_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha,\gamma_\beta] = \frac{i}{2}\sigma^3\epsilon_{\alpha\beta}, \qquad \gamma^{0,1} = \sigma^{1,2}. \end{gathered} \tag{4.73}$$

In [27] is shown that the equation above is equivalent to require that the spinor Laplacian acts on the propagator in the following way:

$$
\begin{aligned}
&\nabla_a U(\xi,\xi',\phi,\phi') = \partial_a U(\xi,\xi',\phi,\phi') + \omega_a^{[s]} U(\xi,\xi',\phi,\phi') = \\
&= A(\xi,\xi',\phi,\phi')\Sigma_{ab} n^b(\xi,\xi',\phi,\phi') U(\xi,\xi',\phi,\phi')
\end{aligned}
\tag{4.74}
$$

with

$$
A(\xi,\xi',\phi,\phi') = \tanh \frac{d(\xi,\xi',\phi,\phi')}{2}. \tag{4.75}
$$

The geodesic distance between the two points $(\xi,\phi), (\xi',\phi')$ is defined in polar coordinates by the relation $\cosh d(\xi,\xi',\phi,\phi') = \cosh\xi \cosh\xi' - \sinh\xi \sinh\xi' \cos(\phi-\phi')$. In what follows we omit in cartain passages the dependence from all the variables for simplicity of notation.
The partial derivatives in the chosen frame $\partial_a = e_a^\mu \partial_\mu$ are respectively:

$$
\partial_0 = \partial_\xi, \qquad \partial_1 = \frac{1}{\sinh\xi}\partial_\phi. \tag{4.76}
$$

The connection $\omega_a^{[s]}$ can be obtained by projecting the spinor connection $\omega_\mu^{[s]}$ in the vielbein basis:

$$
\omega_a^{[s]} = e_a^\mu \omega_\mu^{[s]}, \qquad \omega_0^{[s]} = 0, \qquad \omega_1^{[s]} = -\frac{i}{2}\coth\xi \sigma^3. \tag{4.77}
$$

The equation for the spinor propagator can be rewritten in the chosen frame as:

$$
\begin{aligned}
&\partial_\xi U = \frac{i}{2}\tanh d/2 \frac{\partial_\phi d}{\sinh\xi}\sigma^3 U, \\
&\frac{\partial_\phi U}{\sinh\xi} = \frac{i}{2}\left(\coth\xi - \tanh\frac{d}{2}\partial_\xi d\right)\sigma^3 U, \\
&U(\xi'=\xi,\phi'=\phi) = I.
\end{aligned}
\tag{4.78}
$$

Using the relations $\tanh\frac{d}{2} = \frac{\sinh d}{1+\cosh d}$ and $\partial_\phi d = \frac{\partial_\phi \cosh d}{\sinh d}$ the first equation becomes:

$$
\partial_\xi U = \frac{i}{2}\sigma^3 \frac{\partial_\phi \cosh d}{(1+\cosh d)\sinh\xi} U \tag{4.79}
$$

The solution to the this equation is:

$$U(\xi,\xi',\phi,\phi') = \exp\left[\frac{i}{2}\sigma^3 B(\xi,\xi',\phi,\phi')\right] C(\phi,\phi') \tag{4.80}$$

where:

$$B(\xi,\xi',\phi,\phi') = \int d\xi \frac{\partial_\phi \cosh d}{(1+\cosh d)\sinh\xi} \tag{4.81}$$

and $C(\phi,\phi')$ is a matrix depending only on the polar angles which can be determined with the second equation and the initial condition $U(\xi=\xi',\phi=\phi') = I$.
Using $\cosh d(\xi,\xi',\phi,\phi') = \cosh\xi\cosh\xi' - \sinh\xi\sinh\xi'\cos(\phi-\phi')$ we find:

$$\begin{aligned}
B(\xi,\xi',\phi,\phi') =& \int d\xi \frac{\sinh\xi\sinh\xi'\sin(\phi-\phi')}{(1+\cosh\xi\cosh\xi'-\sinh\xi\sinh\xi'\cos(\phi-\phi'))\sinh\xi} = \\
=& \int d\xi \frac{\sinh\xi'\sin(\phi-\phi')}{(1+\cosh\xi\cosh\xi'-\sinh\xi\sinh\xi'\cos(\phi-\phi'))} = \\
=& -2\arctan\left[\frac{\cosh(\frac{\xi-\xi'}{2})\cot(\frac{\phi-\phi'}{2})}{\cosh(\frac{\xi+\xi'}{2})}\right] = \\
=& 2\left(\arctan\left[\frac{\cosh(\frac{\xi+\xi'}{2})\tan(\frac{\phi-\phi'}{2})}{\cosh(\frac{\xi-\xi'}{2})}\right] - \frac{\pi}{2}\right).
\end{aligned} \tag{4.82}$$

Then, inserting the 4.80 in the second equation for the propagator we get a equation for $C(\phi,\phi')$:

$$\begin{aligned}
\partial_\phi C =& \frac{i}{2}\sigma^3\left(\cosh\xi - \sinh\xi\tanh\frac{d}{2}\partial_\xi d - \partial_\phi B\right) C = \\
=& \frac{i}{2}\sigma^3\left(\cosh\xi - \frac{\sinh\xi\partial_\xi\cosh d}{1+\cosh d} - \partial_\phi B\right) C = \\
=& \frac{i}{2}\sigma^3\left(\cosh\xi - \frac{\sinh\xi\,(\sinh\xi\cosh\xi' - \cosh\xi\sinh\xi'\cos(\phi-\phi'))}{1+\cosh\xi\cosh\xi'-\sinh\xi\sinh\xi'\cos(\phi-\phi')} - \partial_\phi B\right) C.
\end{aligned} \tag{4.83}$$

Computing $\partial_\phi B$ and inserting it in the equation above we find that the full r.h.s vanishes. Hence C is a constant matrix and according to the initial condition we choose:

$$C(\phi, \phi') = \exp\left(\frac{i}{2}\sigma^3\pi\right) \tag{4.84}$$

In conclusion the spinor propagator in the choosen frame is:

$$U(\xi, \xi', \phi, \phi') = \exp\left(i\sigma^3 \arctan\left[\frac{\cosh(\frac{\xi+\xi'}{2})\tan(\frac{\phi-\phi'}{2})}{\cosh(\frac{\xi-\xi'}{2})}\right]\right). \tag{4.85}$$

4.4.2 Applying the Sommerfeld Formula

Now we can apply the Sommerfeld formula to obtain the spinor heat kernel on AdS_2 with conical singularity.
According to 3.79 this is given by:

$$K_\alpha(t) = \alpha K(t) + \frac{i}{4\pi\alpha}\int_\Gamma dz \frac{1}{\sin\left(\frac{z}{2\alpha}\right)}\mathrm{Tr}K_z(t),$$

$$\mathrm{Tr}K_z(t) = \int_0^{2\pi\alpha} d\phi \int_0^\infty d\xi \sinh\xi \; \mathrm{Tr}K(\xi = \xi', \phi' = \phi + z; t),$$

$$\mathrm{Tr}K(\xi = \xi', \phi' = \phi + z; t) =$$

$$= \mathrm{Tr}U(\xi, z)\frac{\sqrt{2}e^{-m^2 t}}{(4\pi t)^{\frac{3}{2}}}(\cosh d(\xi, z)/2)^{-1}\int_{d(\xi,z)}^\infty \frac{y e^{-\frac{y^2}{4t}}\, dy}{\sqrt{\cosh y - \cosh d(\xi, z)}},$$

$$U(\xi, z) = \exp\left(i\sigma^3 \arctan\left[\cosh\xi \tan\frac{z}{2}\right]\right), \qquad TrU(\xi, z) = \frac{2}{\sqrt{1 + \cosh^2\xi \tan^2 z/2}}. \tag{4.86}$$

In the prescription above $K(t)$ indicates the regular fermionic kernel given in 4.24. The contour Γ is the same of the bosonic formula as explained in section 3.2.2.
From now we proceed identically to the scalar case. The integrated kernel we need to compute is:

$$\mathrm{Tr}K_z(t) = \frac{\sqrt{2}e^{-m^2t}}{(4\pi t)^{\frac{3}{2}}}\int_0^{2\pi\alpha} d\phi \int_0^\infty d\xi \sinh(\xi)\frac{\mathrm{Tr}U(\xi,z)}{\cosh d(\xi,z)/2}\int_{d(\xi,z)}^\infty \frac{y\cosh\frac{y}{2}e^{-\frac{y^2}{4t}}dy}{\sqrt{\cosh y - \cosh d(\xi,z)}}. \tag{4.87}$$

Performing the integration over the angle and a change of variable as below:

$$\sinh(\xi) = x, \qquad d = d(x) = 2\mathrm{arcsinh}\left(x\sin\frac{z}{2}\right), \qquad \cosh d = 1 + 2x^2\sin^2\frac{z}{2},$$

$$\cosh y = 1 + 2\sin^2 y/2, \qquad \cosh d(x,z)/2 = \sqrt{1+\sinh^2 d(x,z)/2} = \sqrt{1+x^2\sin^2 z/2},$$

$$TrU(\xi,z) = \frac{2}{\sqrt{1+\cosh^2\xi\tan^2 z/2}} = \frac{2}{\sqrt{1+(1+x^2)\tan^2 z/2}} = \frac{2\cos z/2}{\sqrt{1+x^2\sin^2 z/2}}, \tag{4.88}$$

we obtain:

$$\mathrm{Tr}K_z(t) = \frac{\alpha e^{-m^2t}}{(4\pi)^{\frac{1}{2}}t^{\frac{3}{2}}}\int_0^\infty \frac{dxx}{\sqrt{1+x^2}}\frac{\cos z/2}{1+x^2\sin^2 z/2}\int_{d(x,z)}^\infty \frac{y\cosh\frac{y}{2}e^{-\frac{y^2}{4t}}dy}{\sqrt{\sinh^2\frac{y}{2} - x^2\sin^2\frac{z}{2}}}. \tag{4.89}$$

Interchanging the integration limit as explained in 4.53 the expression above becomes:

$$\begin{aligned} \mathrm{Tr}K_z(t) &= \frac{\alpha e^{-m^2t}}{(4\pi)^{\frac{1}{2}}t^{\frac{3}{2}}\sin(z/2)}\int_0^\infty dyy\cosh\frac{y}{2}e^{-y^2/4t}G(y,z), \\ G(y,z) &= \int_0^{A(y,z)} dx\frac{x\cos z/2}{\sqrt{1+x^2}\sqrt{A(y,z)^2 - x^2}(1+x^2\sin^2 z/2)} \end{aligned} \tag{4.90}$$

where $A(y,z) = \sinh\frac{y}{2}/\sin\frac{z}{2}$.
Changing again variable $t = x^2$ and denoting $a(y,z) = A^2(y,z), b(z) = \sin^2 z/2$ we find:

$$
\begin{aligned}
G(y,z) =& \frac{\cos z/2}{2}\int_0^{a(y,z)} \frac{dt}{\sqrt{1+t}\sqrt{a(y,z)-t}(1+tb(z))} = \\
=& \frac{\arctan\left[\frac{\sqrt{1+ab}\sqrt{1+t}}{\sqrt{1-b}\sqrt{a-t}}\right]\cos z/2}{\sqrt{1-b}\sqrt{1+ab}}|_{t=0}^{t=a(y,z)} = \\
=& \frac{\cos\frac{z}{2}}{\sqrt{1-b}\sqrt{1+ab}}\left(\pi/2 - \arctan\left[\frac{\sqrt{1+ab}}{\sqrt{1-b}\sqrt{a}}\right]\right) = \\
=& \frac{\pi/2 - \arctan\left[\frac{\tan z/2}{\tanh y/2}\right]}{\cosh y/2},
\end{aligned}
\tag{4.91}
$$

where in the last line we used the expression for a and b.
The integration over y can be further simplified:

$$
\begin{aligned}
\mathrm{Tr}K_z(t) =& \frac{\alpha e^{-m^2 t}}{(4\pi)^{\frac{1}{2}} t^{\frac{3}{2}} \sin(z/2)}\int_0^\infty dy y e^{-y^2/4t}\left(\pi/2 - \arctan\left[\frac{\tan z/2}{\tanh y/2}\right]\right) = \\
=& \frac{\alpha e^{-m^2 t}}{(4\pi)^{\frac{1}{2}} t^{\frac{3}{2}} \sin(z/2)}(-2t)\int_0^\infty dy \partial_y[e^{-y^2/4t}]\left(\pi/2 - \arctan\left[\frac{\tan z/2}{\tanh y/2}\right]\right) = \\
=& \frac{2\alpha e^{-m^2 t}}{(4\pi t)^{\frac{1}{2}} \sin(z/2)}\int_0^\infty dy e^{-y^2/4t}\partial_y\left(\pi/2 - \arctan\left[\frac{\tan z/2}{\tanh y/2}\right]\right) = \\
=& \frac{2\alpha e^{-m^2 t}}{(4\pi t)^{\frac{1}{2}} \sin(z/2)}\int_0^\infty dy e^{-y^2/4t}\frac{\sin z}{2(\cosh y - \cos z)}.
\end{aligned}
\tag{4.92}
$$

Changin the variable $y \to 2y$ we get a final expression:

$$
\begin{aligned}
\mathrm{Tr}K_z(t) =&\frac{2\alpha e^{-m^2t}}{(4\pi t)^{\frac{1}{2}}\sin(z/2)}\int_0^\infty dy e^{-y^2/t}\frac{\sin z}{(\cosh 2y-\cos z)} = \\
=&\frac{2\alpha e^{-m^2t}}{(4\pi t)^{\frac{1}{2}}\sin z/2}\int_0^\infty dy e^{-y^2/t}\frac{2\sin z/2\cos z/2}{2(\sinh^2 y+\sin^2 z/2)} = \\
=&\frac{2\alpha e^{-m^2t}}{(4\pi t)^{\frac{1}{2}}}\int_0^\infty dy e^{-y^2/t}\frac{\cos z/2}{(\sinh^2 y+\sin^2 z/2)}.
\end{aligned}
\tag{4.93}
$$

According to the Sommerfeld formula for the fermionic case, the spinor heat kernel on AdS_2 with a conical singularity is:

$$
\begin{aligned}
&K_\alpha(t) = \alpha K(t) + \frac{2\alpha\, e^{-m^2t}}{(4\pi t)^{\frac{1}{2}}}\int_0^\infty dy e^{-y^2/t}\frac{i}{4\pi\alpha}\int_\Gamma dz F(z;y,\alpha), \\
&F(z;y,\alpha) = \frac{1}{\sin z/2\alpha}\ \frac{\cos\frac{z}{2}}{\sinh^2 y+\sin^2 z/2}
\end{aligned}
\tag{4.94}
$$

We observe that when is set $\alpha = 1$ the integrand $F(z,y;\alpha)$ reduces to the product of two anti periodic functions in z of 2π and therefore becomes complessively periodic. As in the scalar case, we expect that for $\alpha = 1$ the contribution due to the singularity vanishes because of the structure of the contour Γ.
Now we can evaluate the contour integral on the complex plane with the residuals theorem. As in the bosonic case the contour Γ contains one pole of $\frac{1}{\sin z/2\alpha}$ at $z=0$ and two immaginary poles $z = \pm 2iy$ of $\frac{1}{\sinh^2 y+\sin^2 z/2} = \frac{1}{(\sin z/2+i\sinh y)(\sin z/2-i\sinh y)}$.
The residuals are:

$$
ResF(z=0;y,\alpha) = \frac{2\alpha}{\sinh^2(y)}, \qquad ResF(z=\pm 2iy;y,\alpha) = -\frac{1}{\sinh y\sinh\frac{y}{\alpha}}.
\tag{4.95}
$$

Thus:

$$\begin{aligned}\frac{i}{4\pi\alpha}\int_\Gamma dzF(z;y,\alpha) =&\frac{i}{4\pi\alpha}2\pi i\left[\frac{2\alpha}{\sinh^2 y}-\frac{2}{\sinh y\sinh\frac{y}{\alpha}}\right]\\ =&\frac{1}{\sinh y}\left[\frac{1}{\alpha\sinh\frac{y}{\alpha}}-\frac{1}{\sinh y}\right].\end{aligned} \tag{4.96}$$

Finally the AdS_2 with conical singularity spinor heat kernel is:

$$K_\alpha(t)=\alpha K(t)+\frac{2\alpha e^{-m^2t}}{(4\pi t)^{\frac{1}{2}}}\int_0^\infty dy e^{-\frac{y^2}{t}}\frac{1}{\sinh(y)}\left[\frac{1}{\alpha\sinh(\frac{y}{a})}-\frac{1}{\sinh(y)}\right] \tag{4.97}$$

The integral is clearly well defined in the upper limit because of the integrand is suppressed by the exponential. Moreover for $y\to 0$ one finds:

$$\frac{1}{\sinh y}\left[\frac{1}{\alpha\sinh\frac{y}{\alpha}}-\frac{1}{\sinh y}\right]\sim\frac{1}{y}\left[\left(\frac{1}{y}-\frac{y}{6\alpha^2}\right)-\left(\frac{1}{y}-\frac{y}{6}\right)\right]=\frac{1}{6}\left[1-\frac{1}{\alpha^2}\right] \tag{4.98}$$

Therefore the integral is well defined also in the lower limit of integration.
Consistently with the Sommerfeld formula, if we set $\alpha=1$ the contribution from the singularity vanishes and the expression reduces to the regular kernel $K(t)$.

4.4.3 Comparison with Orbifolds

As in the scalar case we can check the consistency of the spinor heat kernel through a comparison with the orbifold expression after an analytic continuation.
The small t expansion for the massless spinor heat kernel on AdS_2/Z_N is found to be [28]:

$$K_{AdS_2/Z_N}(t)=\frac{1}{N}K_{AdS_2}(t)-\frac{1}{2}\left[\frac{N^2-1}{6N}-\frac{(7N^2+17)(N^2-1)}{720N}t\right]+O(t^2) \tag{4.99}$$

As in the bosonic case, once set $m=0$ we perform a change of variable $y\to y\sqrt{t}$:

$$K_\alpha(t)=\alpha K(t)+\frac{2\alpha}{(4\pi)^{\frac{1}{2}}}\int_0^\infty dy e^{-y^2}\frac{1}{\sinh(y\sqrt{t})}\left[\frac{1}{\alpha\sinh(\frac{y\sqrt{t}}{a})}-\frac{1}{\sinh(y\sqrt{t})}\right] \tag{4.100}$$

Then we can expand in t around $t = 0$:

$$
\begin{aligned}
K_\alpha(t) =& \alpha K(t) + \frac{2\alpha}{(4\pi)^{\frac{1}{2}}} \int_0^\infty dy \left(\frac{1}{6}\left(1 - \frac{1}{\alpha^2}\right) + \left(\frac{-17\alpha^4 + 10\alpha^2 + 7}{360\alpha^4} \right) t y^2 \right) e^{-y^2} + O(t^2) \\
=& \alpha K(t) + \frac{\alpha}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} \left(\frac{1}{6}\left(1 - \frac{1}{\alpha^2}\right) + \left(\frac{-17\alpha^4 + 10\alpha^2 + 7}{720\alpha^4} \right) t \right) \right] + O(t^2) \\
=& \alpha K(t) - \frac{1}{2} \left[\frac{\alpha}{6} \left(\frac{1}{\alpha^2} - 1 \right) + \alpha \left(\frac{17\alpha^4 - 10\alpha^2 - 7}{720\alpha^4} \right) t \right] + O(t^2)
\end{aligned}
\tag{4.101}
$$

Also in this case we can note in the expansion that the t^0 coefficient exhibits the structure of the Seeley coefficient given in 3.80.
Performing the analytic continuation $\alpha = \frac{1}{N}$ we get

$$
\begin{aligned}
K_{\frac{1}{N}}(t) =& \frac{1}{N} K(t) - \frac{1}{2} \left[\frac{N^2 - 1}{6N} - \left(\frac{-17 + 10N^2 + 7N^4}{720N} \right) t \right] + O(t^2) \\
=& \frac{1}{N} K(t) - \frac{1}{2} \left[\frac{N^2 - 1}{6N} - \left(\frac{(7N^2 + 17)(N^2 - 1)}{720N} \right) t \right] + O(t^2),
\end{aligned}
\tag{4.102}
$$

that matches 4.99.

4.5 The Partition Function

4.5.1 The Integral Representation

The heat kernel of the scalar and spinor Laplacian found in the previous sections can be used to define the correspondent regularized determinant according to definitions given in section 2.2.3.
In this case we prefer to work in PTC regularization:

$$
\begin{aligned}
&\log(\det[\hat{O}])_\delta = -\int_\delta^\infty \frac{dt}{t} K(t), \\
&K_\alpha(t) = \alpha K(t) + \frac{\alpha e^{-Mt}}{(4\pi t)^{\frac{1}{2}}} \int_0^\infty dy e^{-\frac{y^2}{t}} h(y,\alpha), \\
&\text{scalars :} \qquad M = m^2 + \frac{1}{4}, \qquad h(y,\alpha) = \frac{1}{\sinh y}\left(\frac{\coth\frac{y}{\alpha}}{\alpha} - \coth y\right), \\
&\text{fermions :} \qquad M = m^2, \qquad h(y,\alpha) = \frac{2}{\sinh y}\left(\frac{\mathrm{cosech}\frac{y}{\alpha}}{\alpha} - \mathrm{cosech} y\right).
\end{aligned} \tag{4.103}
$$

The expression for the regularized determinant is:

$$
\log(\det[\hat{O}])_{\alpha,\delta} = -\int_\delta^\infty \frac{dt}{t}\alpha K(t) - \int_\delta^\infty \frac{dt}{t}\frac{\alpha e^{-Mt}}{(4\pi t)^{\frac{1}{2}}} \int_0^\infty dy e^{-\frac{y^2}{t}} h(y,\alpha) \tag{4.104}
$$

Since we already know the partition function for the regular AdS_2, from this point we concentrate on the contribution of the conical singularities.
We first analyse the divergent part of the determinants. As explained in section 2.2.3 to find the divergences in PTC regularization we need to split the domain of integration in two intervals $[\delta, 1]$, $[1, \infty]$ and then expand the heat kernel in powers of t as follows:

$$
\log(\det[\hat{O}])_\delta = -\int_\delta^1 \frac{dt}{t}(a_0 + a_1 t + ...) - \int_1^\infty \frac{dt}{t}(a_0 + a_1 t + ...) \tag{4.105}
$$

The second integral is finite because the heat kernel is exponentially suppressed for large t and is clearly regular in $t = 1$. As it can be seen the divergence comes from the Seeley coefficient a_0 in the first integral and is logarithmic:

$$
\log(\det[\hat{O}])_\infty = a_0 \log(\delta) \tag{4.106}
$$

We have already found these coefficients in the previous sections:

$$\text{scalars :} \qquad K_\alpha(t) = \frac{\alpha}{12}\left(\frac{1}{\alpha^2}-1\right) + O(t), \qquad \log(\det[\hat{O}])_{\alpha,\infty} = \left[\frac{\alpha}{12}\left(\frac{1}{\alpha^2}-1\right)\right]\log(\delta),$$

$$\text{fermions :} \qquad K_\alpha(t) = -\frac{\alpha}{12}\left(\frac{1}{\alpha^2}-1\right) + O(t), \qquad \log(\det[\hat{O}])_{\alpha,\infty} = -\left[\frac{\alpha}{12}\left(\frac{1}{\alpha^2}-1\right)\right]\log(\delta). \tag{4.107}$$

Therefore the divergent part of the effective action 4.2 in the case $\alpha > 1$ is:

$$\begin{aligned}\hat{\Gamma}^{\text{div}}_{1,\delta}(\alpha) =& \hat{\Gamma}^{\text{div}}_{1,\delta}(1) + \left[8 \text{ x } \frac{\alpha}{12}\left(\frac{1}{\alpha^2}-1\right) - 8 \text{ x } \frac{\alpha}{12}\left(\frac{1}{\alpha^2}-1\right)\right]\log\delta = \\ =& \hat{\Gamma}^{\text{div}}_{1,\delta}(1) = 0.\end{aligned} \tag{4.108}$$

The vanishing of $\hat{\Gamma}^{\text{div}}_{1,\delta}(1)$ has already been discussed during the computation of the $\alpha = 1$ effective action and the same considerations hold also in PTC regularization. Since the contributions coming from the singularity are cancelled between scalars and fermions, the string partition function is finite also for a α-wrapped loop and the circular Wilson loop in the α^{th} rank symmetric representation is free of divergences as expected.
Now we can procede to compute the finite part of the effective action. Dropping the divergent parts which vanish in the sum we can remove the cutoff parameter and write:

$$\begin{aligned}\log(\det[\hat{O}])_\alpha =& -\int_0^\infty \frac{dt}{t}\frac{\alpha e^{-Mt}}{(4\pi t)^{\frac{1}{2}}}\int_0^\infty dy e^{-\frac{y^2}{t}} h(y,\alpha) = \\ =& -\int_0^\infty dy h(y,\alpha)\frac{\alpha}{(4\pi)^{\frac{1}{2}}}\int_0^\infty \frac{dt}{t^{\frac{3}{2}}}\, e^{-(Mt+\frac{y^2}{t})}\end{aligned} \tag{4.109}$$

The integral over t can be reconducted to the known integral:

$$\int_0^\infty \frac{dt}{t^{\frac{3}{2}}} e^{-(ct+\frac{a^2}{4t})} = 2\frac{\sqrt{\pi}}{a}e^{-\sqrt{c}a}, \qquad a, c \in R > 0. \tag{4.110}$$

With the identifications $c = M$, $y = \frac{a}{2}$, we find

$$\int_0^\infty \frac{dt}{t^{\frac{3}{2}}} e^{-(Mt+\frac{y^2}{t})} = \frac{\sqrt{\pi}}{y} e^{-2\sqrt{M}y}. \tag{4.111}$$

So the determinant simplifies as:

$$\log(\det[\hat{O}])_\alpha = -\frac{\alpha}{2}\int_0^\infty \frac{dy}{y} h(y,\alpha) e^{-2\sqrt{M}\ y}. \tag{4.112}$$

The multiplet describing the one-loop correction to the circular Wilson loop contains 5 massless scalars, 3 scalars with $m^2 = 2$ and 8 fermions with $m^2 = 1$:

$$\hat{\Gamma}_1(\alpha) = \frac{1}{2}\left(5\log(\det[-\Box])_\alpha + 3\log(\det[-\Box+2])_\alpha + 8\log(\det[-\nabla^2 + \frac{1}{4}\mathcal{R} + 1])_\alpha\right) \tag{4.113}$$

Since the heat kernel on AdS_2 with conical singularity is expressed as a sum of the regular AdS_2 kernel multiplied by α and a term wich has origin from the singularity, it is possible to write the one-loop partition function in the same way because of the linearity of the integration. The regular case has been found previoulsy and reads:

$$\hat{\Gamma}_1(\alpha = 1) = \frac{1}{2}\log 2\pi \tag{4.114}$$

The singular part can be added by taking the expression in 4.111 and summing all the contributions. According to 4.103 we have $M = \frac{1}{4}$ for the massless scalars, $M = \frac{9}{4}$ for the massive scalars and $M = 1$ for the fermions.
In conclusion the one-loop partition function for a generic α is:

$$\hat{\Gamma}_1(\alpha) = \frac{1}{2}\alpha\log 2\pi + \frac{1}{2}\left[-\frac{\alpha}{2}\int_0^\infty \frac{dy}{y} h^{\text{scalars}}(y,\alpha)\left(5e^{-y} + 3e^{-3y}\right) - \frac{\alpha}{2}\int_0^\infty \frac{dy}{y} 8h^{\text{fermions}}(y,\alpha)e^{-2y}\right] =$$

$$= \frac{1}{2}\alpha\log(2\pi) - I(\alpha),$$

$$I(\alpha) = \frac{1}{4}\int_0^\infty \frac{dy}{y\sinh y}\left[\left(\coth\frac{y}{\alpha} - \alpha\coth y\right)\left(5e^{-y} + 3e^{-3y}\right) + 16\left(\text{cosech}\frac{y}{\alpha} - \alpha\text{cosech}y\right)e^{-2y}\right]. \tag{4.115}$$

The integral $I(\alpha)$ is well defined for $y \to \infty$ because the integrand is exponentially suppressed. Furthermore, thanks to the cancellation of the Seeley-De Witt coefficients demonstrated in 4.108, it is perfectly regular also in the lower integration limit $y = 0$.

4.5.2 Numerical Analysis

It is not clear how to compute the integral $I(\alpha)$ analitically. A computation for some fixed values of α reveals the possibility to express it in terms of Riemann ζ-function and polygamma functions, but it seems hard to understand the general trend. Therefore we proceed with a numerical approach: we first compute the numerical value of $\hat{\Gamma}_1(\alpha)$ in two ranges of the integer α, one for small and another for large values, and make a direct comparison with the gauge side expression and the Kruczensky and Tirziu result. Afterwards we generate new points in a wide range of integers and fit them with a probable numerical law after a study of $\hat{\Gamma}_1(\alpha)$ for small and large α.
Let us start with a comparison with the gauge side expression. Ignoring the logarithmic term in the 't Hooft coupling, which can be attributed to the ghost zero modes, the law we take in consideration on the gauge side is:

$$\Gamma_1(\alpha) = \frac{1}{2}\log\frac{\pi}{2} + \frac{3}{2}\log\alpha. \tag{4.116}$$

We report below a comparative graphic between $\hat{\Gamma}_1(\alpha)$ and $\Gamma_1(\alpha)$ and two tables showing the numerical values of the two functions for small and large α:

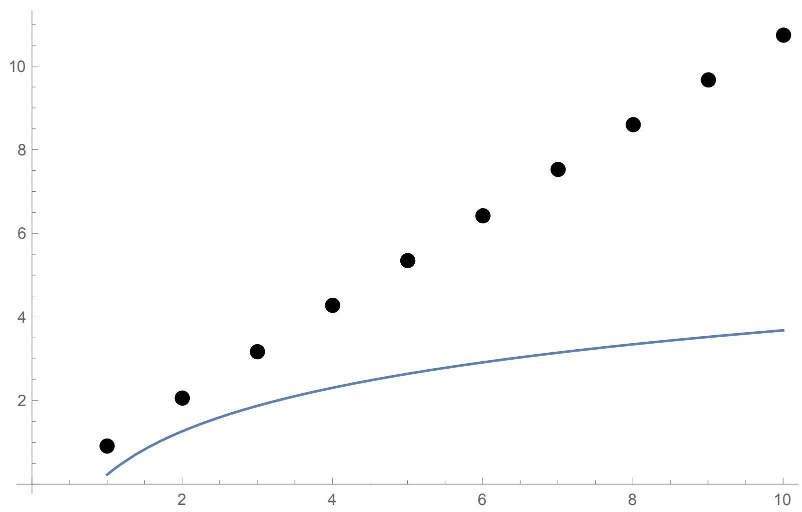

Figure 4: Comparison between $\hat{\Gamma}_1(\alpha)$ in black dots and $\Gamma_1(\alpha)$ in blue line

Range $[1, 10]$

α	$\hat{\Gamma}_1(\alpha)$	$\Gamma_1(\alpha)$
1	0.91894	0.22579
2	2.07494	1.26551
3	3.17845	1.87371
4	4.26882	2.30523
5	5.35390	2.63995
6	6.43634	2.91343
7	7.51727	3.14466
8	8.59725	3.34495
9	9.67660	3.52163
10	10.7555	3.67967

Range $[1000, 1009]$

α	$\hat{\Gamma}_1(\alpha)$	$\Gamma_1(\alpha)$
1000	1077.14	10.5874
1001	1078.22	10.5889
1002	1079.30	10.5904
1003	1080.37	10.5919
1004	1081.45	10.5934
1005	1082.53	10.5949
1006	1083.60	10.5964
1007	1084.68	10.5979
1008	1085.76	10.5994
1009	1086.84	10.6009

As one can deduce from the graphic and the tables, the discrepancy between our result and the gauge side expression, already manifested in the case $\alpha = 1$ without the zero modes correction, persists also for $\alpha > 1$. In particular our finding exhibits in prevalence a linear trend rather than logarithmic and grows definitely faster with α.

The same type of analysis can be performed for the Kruczensky and Tirziu result:

$$\bar{\Gamma}_1(\alpha) = \frac{1}{2}\log 2\pi + \left(2\alpha + \frac{1}{2}\right)\log\alpha - \log\alpha!. \tag{4.117}$$

The numerical computation gives:

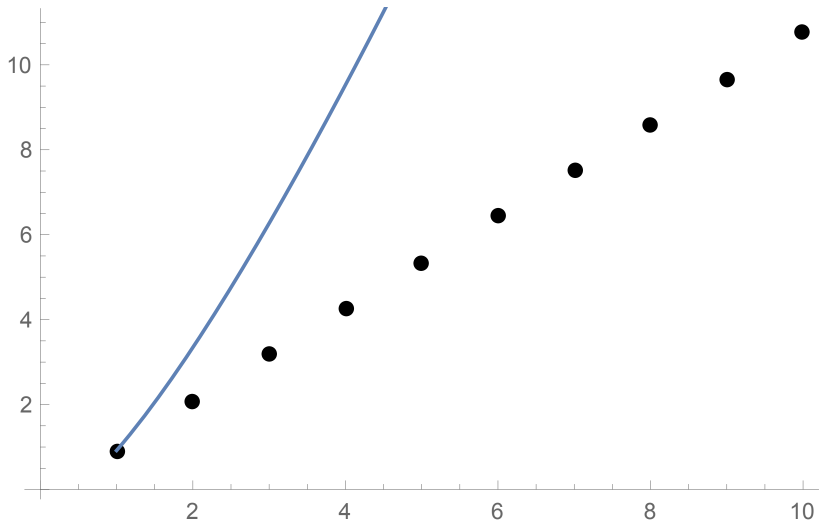

Figure 5: Comparison between $\hat{\Gamma}_1(\alpha)$ in black dots and $\bar{\Gamma}_1(\alpha)$ in blue line

Range $[1, 10]$

α	$\hat{\Gamma}_1(\alpha)$	$\bar{\Gamma}_1(\alpha)$
1	0.91894	0.91894
2	2.07494	3.34495
3	3.17845	6.26816
4	4.26882	9.52439
5	5.35390	13.0305
6	6.43634	16.7367
7	7.51727	20.6095
8	8.59725	24.6251
9	9.67660	28.7658
10	10.7555	33.0175

Range $[1000, 1009]$

α	$\hat{\Gamma}_1(\alpha)$	$\bar{\Gamma}_1(\alpha)$
1000	1077.14	7907.76
1001	1078.22	7916.66
1002	1079.30	7925.57
1003	1080.37	7934.48
1004	1081.45	7943.39
1005	1082.53	7952.31
1006	1083.60	7961.22
1007	1084.68	7970.13
1008	1085.76	7979.05
1009	1086.84	7987.97

Except for the case $\alpha = 1$ which we have already treated in section 4.2.1, even the alternative string finding is not compatible with our result. Infact the Kruczensky and Tirziu expression grows much faster with α and the gap between the numerical estimations becomes considerable for large integers.
To understand what type of law can fit the points generated numerically we begin with a study of $I(\alpha)$ for $\alpha \to \infty$ and $\alpha \to 0$. Strictly speaking, α should assume only integer values $1, 2, 3, ..$, but one can perform an analytic continuation and extend the domain of $I(\alpha)$ to the whole real axis. This provides useful indications about the dominant trend of the integral in the two limits of small and large integers and represents a starting point for developing a numerical law.
We first rewrite $I(\alpha)$ as sum of two pieces:

$$
\begin{aligned}
& I(\alpha) = I'(\alpha) - \alpha I'(1), \\
& I'(\alpha) = \frac{1}{4}\int_0^\infty \frac{dy}{y\sinh y}\left[\coth\frac{y}{\alpha}\left(5e^{-y}+3e^{-3y}\right)+16\mathrm{cosech}\frac{y}{\alpha}e^{-2y}\right].
\end{aligned}
\tag{4.118}
$$

The second piece is linear in α, so we concentrate on the study of $I'(\alpha)$. Let us start with the case of large integers. To facilitate the expansion in power of α we analytically continue the integral by setting $\alpha = \frac{1}{N}$ as in the study of the orbifolds. Taking the limit $N \to 0$ we get:

$$
\begin{aligned}
I'(N) \sim & \frac{1}{4}\int_0^\infty \frac{dy}{y\sinh y}\left[\left(\frac{1}{Ny}+\frac{Ny}{3}\right)\left(5e^{-y}+3e^{-3y}\right)+16\left(\frac{1}{Ny}-\frac{Ny}{6}\right)e^{-2y}\right]+O(N^3) = \\
= & \frac{1}{4}\int_0^\infty dy\left[\left(\frac{5e^{-y}+3e^{-3y}+16e^{-2y}}{y^2\sinh y}\right)\frac{1}{N}+\left(\frac{5e^{-y}+3e^{-3y}-8e^{-2y}}{3\sinh y}\right)N\right]+O(N^3).
\end{aligned}
\tag{4.119}
$$

What we learn is that in the limit of large integers the leading order of the effective action is linear:

$$
\hat{\Gamma}_1(\alpha) \sim c\alpha + O(1/\alpha), \qquad \alpha \to \infty, \tag{4.120}
$$

where c is a real constant. We had already observed this property from the graphics and comparative tables previously reported.
We can repeat the same study for $\alpha \to 0$, once analytically continued α to a real variable. First it is convenient to perform a change of variable $y \to \alpha y$:

$$
I'(\alpha) = \frac{1}{4}\int_0^\infty \frac{dy}{y\sinh \alpha y}\left[\coth y\left(5e^{-\alpha y}+3e^{-3\alpha y}\right)+16\mathrm{cosech} y e^{-2\alpha y}\right]. \tag{4.121}
$$

Then we can expand the integral for infinitesimal values of α:

$$
\begin{aligned}
I'(\alpha) \sim & \frac{1}{4}\int_0^\infty \frac{dy}{\alpha y}\left[\coth y\,(8-14\alpha y)+16\mathrm{cosech} y(1-2\alpha y)\right]+O(\alpha) = \\
= & \frac{1}{4}\int_0^\infty dy\left[\left(\frac{8\coth y+16\mathrm{cosech} y}{y}\right)\frac{1}{\alpha}-2\left(7\coth y+16\mathrm{cosech} y\right)\right]+O(\alpha).
\end{aligned}
\tag{4.122}
$$

Thus the leading order of the 1-loop correction in this limit is:

$$\hat{\Gamma}_1(\alpha) \sim \frac{b}{\alpha} + O(1), \qquad \alpha \to 0, \tag{4.123}$$

with b a real constant.
A first approach that one can adopt is to fit the numerical points only using the leading order terms of the effective action in the two limits of small and large integers.
A numerical estimation realized on 1000 points in the interval $[1, 1000]$ gives as result:

$F(\alpha) = \frac{b}{\alpha} + c\alpha$

coefficient	estimation
b	-0.158476
c	1.077140

A comparison between the numerical values of $\hat{\Gamma}_1(\alpha)$ and $F(\alpha)$ for small and large integers is reported below:

Range $[1, 10]$

α	$\hat{\Gamma}_1(\alpha)$	$F(\alpha)$
1	0.91894	0.91866
2	2.07494	2.07505
3	3.17845	3.17860
4	4.26882	4.26895
5	5.35390	5.35401
6	6.43634	6.43644
7	7.51727	7.51735
8	8.59725	8.59732
9	9.67660	9.67667
10	10.7555	10.7556

Range $[1000, 1009]$

α	$\hat{\Gamma}_1(\alpha)$	$F(\alpha)$
1000	1077.14	1077.14
1001	1078.22	1078.22
1002	1079.30	1079.30
1003	1080.37	1080.37
1004	1081.45	1081.45
1005	1082.53	1082.53
1006	1083.60	1083.60
1007	1084.68	1084.68
1008	1085.76	1085.76
1009	1086.84	1086.84

As it can be deduced, the linear term $c\alpha$ interprets accurately the behavour of the effective action for large integers. Also for small α there is a good agreement, even if one can observe in the numerical values of $F(\alpha)$ a slight overestimation, with the

exception of the case $\alpha = 1$ where $F(1)$ is minor than $\hat{\Gamma}_1(1)$.
Taking inspiration from the Kruczensky and Tirziu expression, we can modify the previous law by adding a constant term and logarithmic corrections like $\log\alpha$, $\alpha\log\alpha$, $\log\alpha!$, in order to improve the numerical agreement for small integers without changing the dominant trend in the limits previously studied.
A numerical fit based on the interval $[1, 1000]$ provides:

$$F(\alpha) = \frac{d}{\alpha} + e + f\alpha + g\log\alpha + h\alpha\log\alpha + l\log\alpha!$$

coefficient	estimation
d	-0.132233
e	0.293791
f	0.757384
g	0.159891
h	0.319758
l	-0.319758

We can see that the numerical fit adjusts the coefficients so that the leading orders for small and large integers are not changed respect to the previous estimation.
Infact the dominant term of the effective action for infinetisimal α is still $F(\alpha) \sim 1/\alpha$ and the combination of logarithmic corrections behaves linearly for $\alpha \to \infty$. This can be checked remembering the Stirling approximation:

$$\log\alpha! \sim \left(\alpha + \frac{1}{2}\right)\log\alpha - \alpha + \frac{1}{2}\log 2\pi, \qquad \alpha \to \infty. \tag{4.124}$$

In our case, noting that $l = -h$, $g \sim \frac{h}{2}$, we have for large α:

$$\begin{aligned} F(\alpha) \sim & e + f\alpha + h\left(\alpha\log\alpha + \frac{1}{2}\log\alpha - \log\alpha!\right) + O(1/\alpha) \sim \\ \sim & e + f\alpha + h\left(\alpha - \frac{1}{2}\log 2\pi\right) + O(1/\alpha) = \left(e - \frac{h}{2}\log 2\pi\right) + (f+h)\alpha + O(1/\alpha) \sim \\ \sim & c\alpha + O(1/\alpha), \qquad\qquad e \sim \frac{h}{2}\log 2\pi,\ f + h \sim c. \end{aligned} \tag{4.125}$$

Thus for $\alpha \to \infty$ we recover the limit found in 4.120 with the coefficient c estimated in the numerical fit.

A direct comparison between $F(\alpha)$ and $\hat{\Gamma}_1(\alpha)$ is given below:

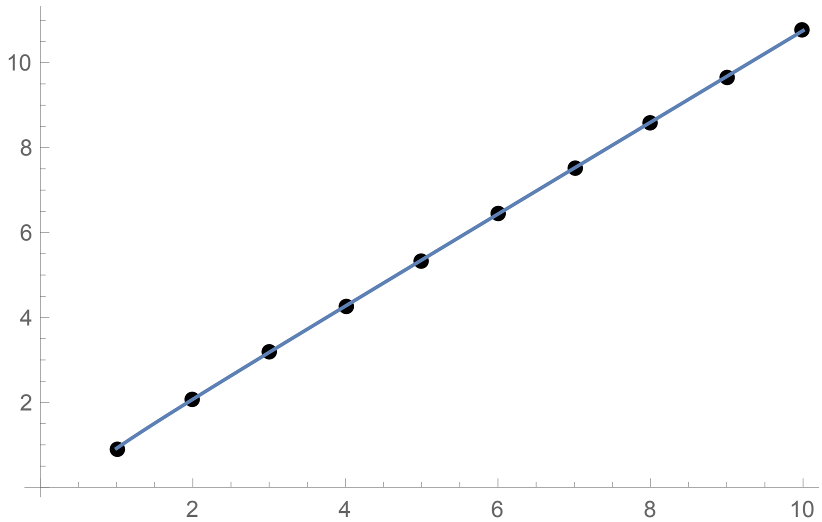

Figure 6: Comparison between $\hat{\Gamma}_1(\alpha)$ in black dots and $F(\alpha)$ in blue line

Range $[1, 10]$

α	$\hat{\Gamma}_1(\alpha)$	$F(\alpha)$
1	0.91894	0.91894
2	2.07494	2.07491
3	3.17845	3.17846
4	4.26882	4.26883
5	5.35390	5.35391
6	6.43634	6.43635
7	7.51727	7.51727
8	8.59725	8.59725
9	9.67660	9.67660
10	10.7555	10.7555

Range $[1000, 1009]$

α	$\hat{\Gamma}_1(\alpha)$	$F(\alpha)$
1000	1077.14	1077.14
1001	1078.22	1078.22
1002	1079.30	1079.30
1003	1080.37	1080.37
1004	1081.45	1081.45
1005	1082.53	1082.53
1006	1083.60	1083.60
1007	1084.68	1084.68
1008	1085.76	1085.76
1009	1086.84	1086.84

As it can be seen the applied modifications to the previous guess improve considerably the agreement between the numerical law with and the 1-loop correction for small integers, preserving at the same time the already good matching for large ones.

Finally we report a graphic which shows the three different predictions for the 1-loop correction to the circular Wilson loop:

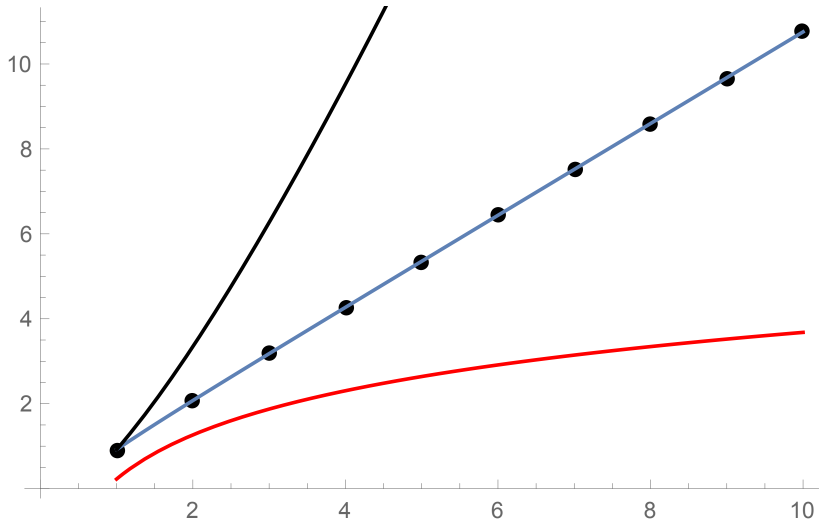

Figure 7: Comparison between $\hat{\Gamma}_1(\alpha)$ in black dots, $F(\alpha)$ in blue line, $\Gamma_1(\alpha)$ in red line and $\bar{\Gamma}_1(\alpha)$ in black line

4.5.3 Conclusion

In the thesis we have discussed the string description of the 1-loop correction to the circular Wilson loop in the k^{th} rank symmetric representation.
In the first section we have provided the recipe for the computation of the Wilson loop on the string side. We have first recalled the main concepts of the AdS/CFT correspondende in the case of the $\mathcal{N} = 4$ super Yang-Mills and the AdS_5 x S^5 type IIB string theories, whose relevant properties have been briefly summarized. It has been explained how the correspondende simplifies in the 't Hooft limit and can be obtained useful informations about a strongly coupled gauge theory through the dual perturbative string theory. Afterwards, we have focused the attention on the Wilson loop operator in the Yang-Mills and $\mathcal{N} = 4$ super Yang-Mills gauge theories, exposing its principal mathematical properties and its physical meaning. Once recalled the AdS_5 x S^5 string representation of the Wilson loop, we have reviewed the derivation of the 1-loop correction in the circular case by studying the quadratic fluctuation around the classical IIB string solution in AdS_5 x S^5. In summary the 1-loop correction in the k^{th} rank representation turns out to be equivalent to the partition function of a 2-d field theory living in the k-wrapped AdS_2 with S^1 boundary and containing 5 massless scalars, 3 scalars with $m^2 = 2$ and 8 Majorana fermions with $m^2 = 1$.

In section 2 we have exposed the necessary methods to perform the computation of the string partition function. First we have recalled the definition of partition function in Quantum Field Theory and showed how it can be expressed as determinant of a differential operator in the case of a scalar and fermionic field. Useful informations about the tangent and spinor bundle which are necessary to describe a spinor field on a curved background can be found in Appendix A. Then we have introduced two important objects as the heat kernel and the heat operator which are extremely useful to treat spectral problems related to differential operators on curved manifolds. Once we have provided the formula to obtain the scalar and fermionic Laplacian heat kernel on a 2-d manifold with conical singularity, we have given the definition of functional regularized determinant, showing how to compute its divergent and finite part.

Finally, in the third section we have performed the computation of the 1-loop correction using the methods previously discussed. First we have reviewed the case of a Wilson loop in the fundamental representation. The result obtained with the heat kernel method matches that of has been derived by Kruczensky and Tirziu with an alternative procedure, but differs from the gauge side expression which is known at any order for reason of conformal symmetry. As already discussed at the end of the section 2.2.1 this discrepancy may be cured including in the string partition function the contribution of the ghosts zero modes which arise from the residual SL(2,R) symmetry of the metric. The normalization factorof the zero modes is hard to determine on the string side and the AdS/CFT correspondence could be the only way to fix it. In the present case we should set the constant equal to 2 to have the matching between the gauge and string side predictions.

Then, after the computation of the scalar and spinor Laplacian heat kernel on AdS_2 with conical singularity, we have computed the partition function for a non trivial k. The numerical analysis reveals that our result is not compatible both with the gauge side and with the alternative string side finding. We remark that the matching between the gauge and string side expression for the circular Wilson loop is a delicate issue and, even the comfortable checks in our favour, also our attempt does not reach this aim. The discrepancy for $k > 1$ could be again explained with the necessity of including the zero modes contribution appropriately normalized. In this case the normalization factor could be a function of the integer k, but we cannot use the correspondence to determine it without an analytic expression of the string side result. Moreover, another possible explanation for the non matching is related to the presumed cancellation of the ghosts and longitudinal modes contributions from the string partition function. It has been pointed out in [29] that the ratio between the ghosts and longitudinal modes determinants could be different from 1, as instead we have assumed, once they are computed with appropriate boundary conditions. Thus it is an eventuality that, once added to the effective action the possible non trivial contributions of ghosts and longitudinal modes determinants and solved the problem of the ghosts zero modes normalization, the string side prediction could finally match the gauge side one also for arbitrary winding number.

A The Spinor Bundle

The fields are function on a d-dimensional manifold characterized by some internal structure. For example they may have a spin structure as the fermions which are particles of spin 1/2 or belong to some representation of a gauge group. In order to describe these additional structures we need the concept of fiber bundles [6]. Denoting with $\mathcal{M}$ the base manifold and with $\mathcal{F}$ the fiber manifold, a fiber bundle over $\mathcal{M}$ with fiber $\mathcal{F}$ is a manifold which locally looks like the product $\mathcal{M}$ x $\mathcal{F}$.
If the fiber bundle is a direct product also globally we call it trivial and write:

$$\xi = \mathcal{M} \text{ x } \mathcal{F}. \tag{A.1}$$

In what follows we will focus on this tyoe of fiber bundle.
A field on ξ is a rule which takes a value $\varphi(x)$ on the fiber for each point x of the base manifold. In the most of the physical applications the fields take value in a linear space $\mathcal{F} = R^k(\mathcal{C}^K)$ which is general the representation of a symmetry group. The most relevant examples of linear fiber bundles are the tangent and cotangent bundles.
The tangent space to a manifold $\mathcal{M}$ in a point x, indicated as $T_x(\mathcal{M})$, is a vector space of dimension d (the same of $\mathcal{M}$) spanned by the tangent vectors in the point x to all the curves which pass through x in $\mathcal{M}$. The tangent bundle $T(M)$ is defined naturally as a fiber bundle where the fiber manifold at the point x is given by the tangent space. The cotangent space $T_x(M)^*$ and cotangent bundle $T^*(M)$ are the dual spaces of the tangent space and tangent bundle respectively.
A natural local basis for the tangent space is given by the partial derivatives $\{\partial_\mu\}$, $\mu = 0, .., d-1$ and a generic vector v can be written as $v = v^\mu \partial_\mu$, where v^μ are the components of v respect to the basis.
The dual local frame for the cotangent space is given by the 1-differential forms $\{dx^\mu\}$ and a covector ω can be expanded on this basis as $\omega = \omega_\mu dx^\mu$ with components ω_μ.
Locally on a Riemannian manifold with metric $g_{\mu\nu}$ we can always introduce on the tangent space an orthonormal basis e_a^μ, $a = 0, ...d-1$, such that:

$$e_a^\mu e_b^\nu g_{\mu\nu} = \delta_{ab}. \tag{A.2}$$

The basis vector e_a^μ are called the vielbeins of the metric $g_{\mu\nu}$. Here we indicate with the Greek letters the components respect to the coordinate basis previously introduced and with the Latin letters the components respect to the tangent space basis given by the vielbeins. These objects can be used to tranform a coordinate index in a tangent index and vice versa:

$$v^a = e^a_\mu v^\mu, \qquad\qquad v^\mu = e^\mu_a v^a. \tag{A.3}$$

This construction allows also to define the dual basis $\delta^{ab} g_{\mu\nu} e^\nu_b = e^a_\mu$ such that are satisfied the following relations:

$$e^\mu_a e^b_\mu = \delta^b_a, \qquad e^a_\mu e^\nu_a = \delta^\nu_\mu, \qquad e^a_\mu e^b_\nu \delta_{ab} = g_{\mu\nu}. \tag{A.4}$$

We note that through the relation A.2 the Riemannian metric $g_{\mu\nu}$ induces locally a flat metric δ_{ab} on the tangent space which can be used to raise and lower the tangent indices by contraction. The symmetry group of the tangent bundle which leaves this metric locally invariant is the orthogonal group $O(d)$. Compatibly with the spin structure of a vector field, the components of a vector $v(x)$ respect to the vielbein basis transform locally in the fundamental representation of $O(d)$:

$$v'^a(x) = O^a_b(x) v^b(x), \tag{A.5}$$

where $O^a_b(x)$ are the components of a $O(d)$ matrix which depends on the point x.
In order to construct a locally $O(d)$ invariant action for a field theory defined on the tangent bundle $T(M)$, we need to define an appropriate covariant derivative:

$$\begin{aligned} \nabla_\mu v^a(x) &= \partial_\mu v^a(x) + \omega_{\mu\,b}^{\,a} v^b(x), \\ \nabla_\mu T^{ab}(x) &= \partial_\mu T^{ab}(x) + \omega_{\mu\,c}^{\,a} T^{cb}(x) + \omega_{\mu\,c}^{\,b} T^{ac}(x), \end{aligned} \tag{A.6}$$

where $\omega_{\mu\,b}^{\,a}$ is the tangent bundle connection and $T^{ab}(x)$ a second rank tensor field. The connection is not unique and there are many choices that we can do.
We require that the covariant derivative is consistent with all the structures that we have defined. In particular we ask that the derivative commutes with the contraction of the indices:

$$0 = \nabla_\mu \delta_{ab} = \omega_\mu^{ab} + \omega_\mu^{ba}. \tag{A.7}$$

This implies that the connection is a second rank antisymmetric tensor respect to the tangent indices and therefore it belongs to the Lie algebra of the tangent bundle

group.
Furthermore we require that the vielbeins are covariantly constant:

$$\nabla_\mu e^a_\nu = 0. \tag{A.8}$$

This condition provides an equation which is solved by the so-called Levi-Civita connection:

$$\omega^{ab}_\mu = e^{\nu b}\Gamma^\rho_{\mu\nu} e^a_\rho - e^{\nu b}\partial_\mu e^a_\nu. \tag{A.9}$$

The field strenght of this connection is given by the Riemann curvature tensor:

$$\partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{a}_{\mu\, c}\omega^{cb}_\nu - \omega^{a}_{\nu\, c}\omega^{cb}_\mu = \mathcal{R}^{ab}_{\mu\nu}. \tag{A.10}$$

The vector bundle language can be generalized to the case of a spinor field.
The fiber bundle which describes the spin structure of a fermionic field on a Riemannian manifold can be constructed by taking the spinorial representation of $O(d)$ as fiber over the base manifold. This type of fiber bundle is also called spinor bundle.
To construct the spinorial representation of the rotation group we need to introduce the Dirac gamma matrices γ^a, $a = 0, ..d-1$. These are Hermitian traceless $2^{[n/2]}$ x $2^{[n/2]}$ matrices which obey the Clifford anticommutation relation:

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} I, \tag{A.11}$$

where I is the unit $2^{[d/2]}$ x $2^{[d/2]}$ matrix. The identity is left implied in the following relations.
The generators of the $SO(d)$ rotations in the spinorial representation in terms of the Dirac matrices are:

$$\Sigma_{ab} = \frac{1}{4}\left[\gamma_a, \gamma_b\right]. \tag{A.12}$$

A spinor field $\psi(x)$ transform under a local rotation as:

$$\psi'(x) = S(x)\psi(x), \tag{A.13}$$

where $S(x)$ is a rotation matrix in the spinorial representation depending on x:

$$S(x) = \exp\left(\frac{1}{2}\Omega^{ab}(x)\Sigma_{ab}\right) \qquad\qquad \Omega^{ab}(x) = -\Omega^{ba}(x). \tag{A.14}$$

The Dirac gamma matrices carry the tangent space index and transform under a global rotation as a vector multiplet:

$$\gamma'^{\,a} = S\gamma^a S^\dagger = O^a_b\gamma^b. \tag{A.15}$$

If we want to construct an action for a spinor field theory which is simultaneously invariant under local change of coordinates and local spinor bundle rotation, we need to introduce a local set of gamma matrices which satisfy:

$$\gamma_\mu(x) = e^a_\mu(x)\gamma_a, \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{A.16}$$

The local matrices $\gamma_\mu(x)$ are related to the constant Dirac matrices Γ^a through a vielbein rotation and are function of the manifold point.
Then we need to generalize the definition of tangent bundle connection to the case of a spinor field. Since the components of the vector bundle connection are the generators of the rotation group in the vectorial representation, we expect that the spinor bundle connection $\omega^{[s]}_\mu$ is proportional to the same generators in the spinorial representation. The exact factor can be obtained by requiring that the gamma matrices commute with the covariant derivative:

$$[\nabla_\mu, \gamma^a] = \left[\omega^{[s]}_\mu, \gamma^a\right] + \omega^a_{\mu\,b}\gamma^b = 0. \tag{A.17}$$

This condition leads to the so-called spin connection:

$$\omega^{[s]}_\mu = \frac{1}{2}\omega^{ab}_\mu\Sigma_{ab} = \frac{1}{8}\omega^{ab}_\mu\left[\gamma_a, \gamma_b\right]. \tag{A.18}$$

The corresponding field strenght is again given in terms of the Riemann curvature tensor:

$$\partial_\mu\omega^{[s]}_\nu - \partial_\nu\omega^{[s]}_\mu + \omega^{[s]}_\mu\omega^{[s]}_\nu - \omega^{[s]}_\nu\omega^{[s]}_\mu = \frac{1}{4}\gamma^a\gamma^b\mathcal{R}_{ab\mu\nu} \tag{A.19}$$

and the action of the covariant derivative on the spinor field is:

$$\nabla_\mu\psi(x) = \partial_\mu\psi(x) + \omega^{[s]}_\mu\psi(x). \tag{A.20}$$

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