Università degli Studi di Padova

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Corso di Laurea Magistrale in Matematica

Derived moduli spaces of G-bundles

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21 luglio 2021

A thesis for the ALGANT Master Program
advised by Benjamin Hennion
at Université Paris-Saclay

Contents

Introduction

In this work I studied the basic definitions of derived algebraic geometry, starting from the generalization of classical stacks in groupoids (which will be called, in a more modern way, 1-stacks) to simplicial stacks to then introduce derived stacks and then study some concrete examples. The main reference has been undoubtedly [HAGII], which is where the approach to derived algebraic geometry with model categories comes from (studied in a much more general way). Not all the necessary background material has been covered here, for time (and space) issues: first of all the basic theory of model categories is assumed to be known to the reader, and only some more advanced definitions are recalled. The main references used for model categories are [Hov99] and [Hir03]. Also, the basic theory of simplicial sets is constantly used: a good reference is [GJ99]. Basic facts about stacks in groupoids, using pseudofunctors or fibered categories, are also assumed to be known: the reference here is [Vis08]. Finally, scheme theory contained in [Har77] will be used, if needed.

Here follows a more detailled abstract of this work. In Chapter 1 we recall some definitions coming from the theory of model categories, like what are simplicial and monoidal model categories, the main properties of the left Bousfield localization and of homotopy function complexes. These results are just stated: although the theory of model categories is interesting, we just need to know how to manipulate such objects (for example mapping sets, and their properties, will be heavily used in the rest of the work).

In Chapter 2, after a brief recall of some algebraic geometry definitions, we introduce simplicial presheaves and stacks (which will be the fibrant objects of the local model structure on $SPr(\mathsf{Aff})$, for the étale topology, or, more simply, simplicial presheaves satisfying an homotopical descent condition). We also mention the link between classical stacks in groupoids and simplicial stacks: composing the former with the nerve functor we obtain a particular class of simplicial stacks (we just call them stacks from now on), whose homotopy groups π_n are all trivial for $n \geq 2$. The descent condition with the homotopy limit, in this case, just gives back the more well-known 2-descent condition: a more extensive explanation can be found in [Hen11]. Finally we define, recursively, n-geometric stacks: the most intuitive way to think about them is using quotients; for example a 1-geometric stack is obtained as a quotient of a scheme by a smooth action of a group scheme. The whole theory of schemes is, of course, embedded into the theory of stacks using the Yoneda lemma (indeed representable stacks are just isomorphic to affine schemes).

In Chapter 3 we take a little break to carefully study a classical example: the moduli stack of G-bundles on a scheme X over S. The goal is to prove that, under suitable condition on X, it is an algebraic stack with a schematic diagonal map (using classical wording), covered by open substacks of finite presentation over S. This part is mainly a rewriting of [Wan11]. We use the classical language of pseudofunctors and 2-categories: some definitions, like the 2-fibered product, are recalled in the beginning of the chapter. The interest of this chapter, which can seem a bit unrelated from the rest of the work, is its final theorem, which will be used later to apply Lurie's representability criterion, Theorem 5.2.2. And also, it is a good example of how complex the problem of representability can be, and how non-trivial (and long and tedious) can be the process of finding the right open covering of a given stack.

We finally begin to dive into the *derived* world in Chapter 4, where we start recalling some results of simplicial algebra, like the definition of simplicial modules and of stable simplicial modules (needed for the cotangent complex). We generalize then some classical definitions related to modules and maps (like flat, étale or projective) to our simplicial (homotopical) case. We finally define derived stacks: the idea is quite similar to the definition of stacks in Chapter 2, what changes here is that simplicial presheaves are taken on the category of $\mathsf{dAff} = \mathsf{sComm}^{\mathrm{op}}$, i.e. we don't use just rings but simplicial rings. We need to consider two successive left Bousfield localizations of $SPr(\mathsf{dAff})$ (starting with the obvious projective model structure) to get to the analogue of the local model structure (w.r.t. the simplicial version of the étale topology), because we need to take into account the nontrivial model structure on sComm. Finally, derived stacks are just fibrant objects in this model category, i.e. simplicial presheaves which preserve equivalences between simplicial rings and satisfy an homotopical descent condition. Again, we define

n-geometric derived stacks in the same way as before. The basic objects here will be affine derived schemes, written as $\mathbb{R}\underline{\mathrm{Spec}}\,A$ using the model Yoneda lemma, Proposition 4.5.2. Obviously stacks embed into derived stacks, since any ring is a discrete simplicial ring, and we can also restrict a derived stack only on classical rings: truncation and extension are the functorial way to pass from stacks to derived stacks. Finally we introduce the definition of cotangent complex (and of higher tangent spaces) of a derived stack, which can be thought as an homotopical generalization of the cotangent sheaf for smooth schemes, or equivalently as an homotopical corepresentative of derived derivations.

We gathered enough theory and terminology to be able to see some concrete examples of derived stack: this is what is done in Chapter 5. We mainly consider the derived stack of local systems on a simplicial set (or, equivalently, on a topological space) and the derived stack of vector bundles on a scheme, which is a particular case of a mapping derived stack. They are both generalizations of well known classical stacks: the classical part of the derived stack of vector bundles is $\mathrm{Bun}_{\mathrm{GL}_n}$, studied in detail in Chapter 3. We compute, in both cases, the cotangent complex at a point (and then globally) and finally, for the derived stack of vector bundles, we apply Lurie's representability criterion and prove it is 1-geometric.

Notation and conventions

For us all rings are associative, unital and commutative. We will never mention universes, and totally ignore any related set-theoretic issue: the rigorous treating of the subject can be done, and can be found in [HAGII].

We will often abbreviate "with respects to" as w.r.t. as well as "left lifting property" as LLP. The category of simplicial sets sSet will be endowed with the Quillen model structure, and we will write Top to mean the category of compactly generated weak Hausdorff spaces with its Quillen model structure (so that sSet and Top are Quillen equivalent model categories). We will often write $X \in \mathcal{C}$ or $f\colon X \to Y \in \mathcal{C}$ to mean $X \in \mathrm{Ob}(\mathcal{C})$ or $f\colon X \to Y \in \mathrm{Mor}(\mathcal{C})$. Furthermore we will equally denote the hom sets in $\mathcal{C}$ either by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ or $\mathcal{C}(X,Y)$. Finally, we will both use the terms trivial and acyclic (which mean the same thing in the context of model category) as well as the words filtrant and directed (for colimits).

1 Recalls of model categories

An omnipresent topic in this work is model categories and simplicial homotopy theory. Two great references are [GJ99] and [Hov99]. The main reference for Bousfield localization machinery is [Hir03]. We will write in this section just some results and definitions. This means that this section is just an occasion to spell out and write properly the essential definitions mentioned around in the literature, and fix some background terminology.

1.1 Cellular model categories

Starting slowly, we recall the following classical definitions. Our goal is to be able to state a theorem of existence of left Bousfield localizations, which we will implicitely use/assume in the rest of our work. Of course we are not going to write everything from scratch, the choice of the definitions is simply given by the state of the knowledge of the author at the writing moment.

Definition 1.1.1. Let $\mathcal{M}$ be a category with filtered colimits, $J \subset \mathrm{Mor}(\mathcal{M})$, $X \in \mathrm{Ob}(\mathcal{M})$. We that X is $\mathbb{N}$-*small relative to* J if the functor $\mathrm{Hom}_{\mathcal{M}}(X, -)$ commutes with directed colimits of diagrams like

$$Y_0 \to Y_1 \to \cdots \to Y_n \to Y_{n+1} \to \dots$$

with every map $Y_i \to Y_{i+1}$ belonging to J.

This definition can be generalized for any regular cardinal κ, talking then about κ-small objects, where the condition simply changes in the fact that the colimit is indexed by κ and not by $\mathbb{N}$ (for the precise statement see [Hov99, Definition 2.1.3]). We then say that $X \in \mathrm{Ob}(\mathcal{M})$ is *small* if there exists a cardinal κ such that X is κ-small relative to $\mathrm{Mor}(\mathcal{M})$.

We will now talk about I-cell complexes, which can be thought as a categorical version of the classic topological cell complexes, to be able to then give the definition of a cellular model category. We will follow [Hir03].

Definition 1.1.2. Let $\mathcal{M}$ be a cocomplete category and $I \subset \mathrm{Mor}(\mathcal{M})$. A morphism $f\colon X \to Y \in \mathrm{Mor}(\mathcal{M})$ is a *relative I-cell complex* if it is a transfinite composition of pushouts of elements of I. A *presentation* of f is the datum of a λ-sequence (λ being a cardinal)

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \dots \qquad (\beta < \lambda)$$

such that any map $X_\beta \to X_{\beta+1}$ is a pushout of a diagram like

$$\begin{array}{ccc} \coprod_{s \in S_\beta} C_s & \longrightarrow & X_\beta \\ \downarrow & & \downarrow \\ \coprod_{s \in S_\beta} D_s & \longrightarrow & X_{\beta+1} \end{array}$$

where $S_\beta \in \mathsf{Set}$, $C_s \to D_s \in I$ for every $s \in S_\beta$. The datum of f together with a presentation is called a *presented relative I-cell complex.* If $X = \emptyset$ (initial object of $\mathcal{M}$) then we talk about I-cell complexes. The *size* of f is the cardinality of the set of cells of f, which is $\coprod_{\beta<\gamma} S_\beta$. For $\beta < \gamma$, the *β-skeleton* of f is X_β.

A *subcomplex* of f is a presented relative I-cell complex $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ with corresponding sequence being the upper row of the diagram

$$\begin{array}{ccccccc} X = \widetilde{X_0} & \longrightarrow & \widetilde{X_1} & \longrightarrow & \widetilde{X_2} & \longrightarrow & \dots \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow & & \downarrow & & \\ X = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots \end{array}$$

with the obvious compatibility conditions (i.e. everytime we attach a subset of the cells in I, in the same way). For details see [Hir03, Definition 10.6.2-10.6.7].

Observe that CW-complexes are exactly I-cell complexes in Top for $I = \{\mathbb{S}^n \hookrightarrow \mathbb{D}^{n+1}\}_n$ (generating cofibrations for the Quillen model structure).

Definition 1.1.3. Let $\mathcal{M}$ be a cocomplete category, $I \subset \mathrm{Mor}(\mathcal{M})$ and λ a cardinal. An object $X \in \mathrm{Ob}(\mathcal{M})$ is *γ-compact relative to I* if for every presented relative I-cell complex $f\colon X \to Y$, every map $g\colon W \to Y$ factors through a subcomplex of f of size at most γ. We say that X is *compact relative to I* if it is γ-compact relative to I for some cardinal γ.

Remark 1.1.4. There exist different notions of compact objects, the last one is from [Hir03]. Instead, following [Hov99], a compact object is $X \in \mathrm{Ob}(\mathcal{M})$ such that $\mathrm{Hom}_{\mathcal{M}}(X, -)$ commutes with κ-directed colimit, for κ a regular cardinal.

Definition 1.1.5. A morphism $f\colon X \to Y \in \mathrm{Mor}(\mathcal{M})$ is an *effective monomorphism* if

1. it has a cokernel pair, i.e. the pushout $Y \coprod_X Y$ exists;
2. it is the equalizer of the canonical maps $Y \rightrightarrows Y \coprod_X Y$.

Definition 1.1.6. Let $\mathcal{M}$ be a model category.

1. $\mathcal{M}$ is *left proper* if weak equivalences are stable by pushouts along cofibrations.
2. $\mathcal{M}$ is *right proper* if weak equivalences are stable by pullbacks along fibrations.
3. $\mathcal{M}$ is *proper* if it is both left and right proper.

Let's now introduce the loop and suspension functor of a pointed model category, to then define stable model categories.

Definition 1.1.7. Let $(\mathcal{M}, *)$ be a pointed model category. The *suspension functor* is given by

$$\Sigma\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M}), \qquad X \mapsto * \coprod_X^{\mathbb{L}} *$$

and the *loop functor* is given by

$$\Omega\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M}), \qquad X \mapsto * \times_X^h *.$$

They form an adjunction $\Sigma \dashv \Omega$ on the homotopy category.

Definition 1.1.8. Let $(\mathcal{M}, *)$ be a pointed model category. We say that $\mathcal{M}$ is a *stable model category* if the suspension functor $\Sigma\colon \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ is an equivalence of categories.

Time for a little tour into homotopy limits and colimits in model categories. We will often use homotopy pullbacks and homotopy pushouts, so let's recall a practical way to compute them.

Theorem 1.1.9. Let $\mathcal{M}$ be a model category. Consider the following square

$$\begin{array}{ccc} a & \xrightarrow{g} & c \\ \downarrow{\scriptstyle f} & & \\ b & & \end{array}$$

If $\mathcal{M}$ is left proper, or a, b, c are cofibrant, then the homotopy pushout of the above square can be computed by replacing g by a cofibration and then computing normal pushout. Consider now the square

$$\begin{array}{ccc} & & y \\ & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{g} & z \end{array}$$

If $\mathcal{M}$ is right proper, or x, y, z are fibrant, then the homotopy pullback of the above square can be computed by replacing f by a fibration and then computing normal pullback.

For example Top with the Quillen model structure is proper, so we can use the classic formulas with the cylinder and the path space.

Definition 1.1.10. Let $\mathcal{M}$ be a model category. We say that $\mathcal{M}$ is *cofibrantly generated* if there exist two sets I, J of maps in $\mathcal{M}$ such that

1. the domains of maps in I are small relative to I-cell complexes;
2. the domains of maps in J are small relative to J-cell complexes;
3. fibrations are exactly the maps with the LLP w.r.t. J;
4. trivial fibrations are exactly the maps with the LLP w.r.t. I.

In this case I is the set of generating cofibrations and J the set of generating trivial cofibrations.

Recall that cofibrations (resp. acyclic cofibrations) in a cofibrantly generated model category $\mathcal{M}$ are exactly retracts of relative I-cell complexes (resp. retracts of relative J-cell complexes).

Definition 1.1.11. Let $\mathcal{M}$ be a model category. We say that $\mathcal{M}$ is a *combinatorial model category* if it is cofibrantly generated and locally presentable, i.e. every object is a filtrant colimit of Hovey-compact objects.

Definition 1.1.12. Let $\mathcal{M}$ be a model category. We say that $\mathcal{M}$ is *cellular* if it is cofibrantly generated by I and J such that

1. the domains and the codomains of maps in I are compact relative to I;
2. the domains of maps in J are small relative to I;
3. the cofibrations are effective monomorphisms.

Basically cellular model categories are cofibrantly generated model categories in which the I-cell complexes are well behaved, in a certain sense. For a more thourough treatment of cellular model categories we refer to [Hir03, Chapter 12].

1.2 Simplicial model categories

Here follows a bunch of technical definitions, relating enriched and monoidal categories to their model structures (and hence bringing a ton of compatibility requirements between the different structures). We will follow [Hov99, Chapter 4]. Let's first recall the classic definitions of adjunction of two variables and closed monoidal categories.

Definition 1.2.1. Let $\mathcal{M}, \mathcal{D}$ and $\mathcal{E}$ be categories. An *adjunction of two variables* from $\mathcal{M} \times \mathcal{D}$ to $\mathcal{E}$ is a quintuple $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l)$ where

$$\otimes \colon \mathcal{M} \times \mathcal{D} \to \mathcal{E}, \quad \mathrm{Hom}_r \colon \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{M}, \quad \mathrm{Hom}_l \colon \mathcal{M}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D},$$

$$\mathcal{M}(C, \mathrm{Hom}_r(D, E)) \xrightarrow{\varphi_r^{-1}} \mathcal{E}(C \otimes D, E) \xrightarrow{\varphi_l} \mathcal{D}(D, \mathrm{Hom}_l(C, E))$$

where φ_l and φ_r are natural isomorphisms.

Definition 1.2.2. A *closed monoidal structure* on a category $\mathcal{M}$ is an octuple

$$(\otimes, a, l, r, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l)$$

where $(\otimes, a, l, r)$ is a monoidal structure on $\mathcal{M}$ (with associator, left and right unitor) and

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l) \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$

is an adjunction of two variables. We call $\mathcal{M}$ a *closed monoidal category*.

Let's now get back into the model world and try to generalize the previous definitions.

Definition 1.2.3. Let $\mathcal{M}, \mathcal{D}$ and $\mathcal{E}$ be model categories. An adjunction of two variables

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l, \varphi_r, \varphi_l) \colon \mathcal{M} \times \mathcal{D} \to \mathcal{E}$$

is called a *Quillen adjunction of two variables* if, given a cofibration $f \colon U \to V$ in $\mathcal{M}$ and a cofibration $g \colon W \to X$ in $\mathcal{D}$, the pushout-product

$$f \,\square\, g \colon P(f,g) \coloneqq (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X$$

is a cofibration in $\mathcal{E}$ which is trivial if either f or g is. We will say, with an abuse of notation, that $\otimes$ is a *Quillen bifunctor*, meaning that it is part of a Quillen adjunction in two variables $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$.

As expected, the total derived functors $(\otimes^{\mathbb{L}}, \mathbb{R}\,\mathrm{Hom}_r, \mathbb{R}\,\mathrm{Hom}_l, \mathbb{R}\varphi_r, \mathbb{R}\varphi_l)$ define an adjunction of two variables $\mathrm{Ho}(\mathcal{M}) \times \mathrm{Ho}(\mathcal{D}) \to \mathrm{Ho}(\mathcal{E})$. Here it is a classic lemma about Quillen adjunctions in two variables.

Lemma 1.2.4. Let $\mathcal{M}, \mathcal{D}$ and $\mathcal{E}$ be model categories and let $\otimes \colon \mathcal{M} \times \mathcal{D} \to \mathcal{E}$ be an adjunction of two variables. Then the following are equivalent:

1. $\otimes$ is a Quillen bifunctor.
2. Given a cofibration $g \colon W \to X$ in $\mathcal{D}$ and a fibration $p \colon Y \to Z$ in $\mathcal{E}$, the induced map
$$\mathrm{Hom}_{r,\square}(g,p) \colon \mathrm{Hom}_r(X,Y) \to \mathrm{Hom}_r(X,Z) \times_{\mathrm{Hom}_r(W,Z)} \mathrm{Hom}_r(W,Y)$$
is a fibration in $\mathcal{M}$, trivial if either g or p is so.
3. Given a cofibration $f \colon U \to V$ in $\mathcal{M}$ and a fibration $p \colon Y \to Z$ in $\mathcal{E}$, the induced map
$$\mathrm{Hom}_{l,\square}(f,g) \colon \mathrm{Hom}_l(V,Y) \to \mathrm{Hom}_l(V,Z) \times_{\mathrm{Hom}_l(U,Z)} \mathrm{Hom}_l(U,Y)$$
is a fibration in $\mathcal{D}$, trivial if f or p is so.

Proof. See [Hov99, Lemma 4.2.2]. ■

The following remark sheds some light on the terminology "Quillen bifunctor".

Remark 1.2.5. Let $\otimes \colon \mathcal{M} \times \mathcal{D} \to \mathcal{E}$ be a Quillen bifunctor. If $C \in \mathrm{Ob}(\mathcal{M})$ is cofibrant then $C \otimes - \colon \mathcal{D} \to \mathcal{E}$ is a left Quillen functor, with right adjoint $\mathrm{Hom}_l(C, -)$. If $E \in \mathcal{E}$ is fibrant, the functor $\mathrm{Hom}_r(-, E) \colon \mathcal{D} \to \mathcal{M}^{\mathrm{op}}$ is left Quillen, with right adjoint $\mathrm{Hom}_l(-, E) \colon \mathcal{M}^{\mathrm{op}} \to \mathcal{D}$.

We have enough tools to define the model version of monoidal category.

Definition 1.2.6. Let $\mathcal{M}$ be a model category. It is a *monoidal model category* if it is a closed category with a monoidal structure satisfying the following conditions.

1. The monoidal structure $\otimes \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a Quillen bifunctor.
2. Let $q \colon QS \to S$ be the cofibrant replacement for the unit S, obtained by using MC5 axiom on the map $\emptyset \to S$. Then the natural map $q \otimes \mathrm{id}_X \colon QS \otimes X \to S \otimes X$ is a weak equivalence if X is cofibrant. Similarly, the natural map $\mathrm{id}_X \otimes q \colon X \otimes QS \to X \otimes S$ is a weak equivalence if X is cofibrant.

An example of a (symmetric) monoidal model category is sSet with the cartesian product $\times$. The adjoint is the internal hom $\underline{\mathrm{Hom}}$. Another example is $\mathrm{Ch}(R)$ with the tensor product of complexes. Let's recall that if $\mathcal{M}$ is a monoidal category, an $\mathcal{M}$-module $\mathcal{D}$ is a category endowed with a functor $\mu \colon \mathcal{M} \times \mathcal{D} \to \mathcal{D}$ satisfying the classic module axioms, for example the following diagram must commute

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{D} & \xrightarrow{\quad\quad \mu \quad\quad} & \mathcal{D} \\ \Big\uparrow{\scriptstyle \otimes \times \mathrm{id}_{\mathcal{D}}} & & \Big\uparrow{\scriptstyle \mu} \\ (\mathcal{M} \times \mathcal{M}) \times \mathcal{D} \simeq \mathcal{M} \times (\mathcal{M} \times \mathcal{D}) & \xrightarrow{\mathrm{id}_{\mathcal{M}} \times \mu} & \mathcal{M} \times \mathcal{D} \end{array}$$

Definition 1.2.7. Let $\mathcal{M}$ be a monoidal model category. An $\mathcal{M}$*-model category* is an $\mathcal{M}$-module $\mathcal{D}$ with a model structure, making it a model category, such that

1. the action map $\mu\colon \mathcal{M} \times \mathcal{D} \to \mathcal{D}$ is a Quillen bifunctor;
2. if $q\colon QS \to S$ is the cofibrant replacement for S (unit) in $\mathcal{M}$, then the map $\mathrm{id}_X \otimes q\colon X \otimes QS \to X \otimes S$ is a weak equivalence for all cofibrant objects X.

The second condition is automatic if S is cofibrant in $\mathcal{M}$.

Definition 1.2.8. Let $\mathcal{M}$ be a model category. It is a *simplicial model category* if it is an sSet-model category.

In practice this means that we have a left adjoint $\otimes\colon \mathsf{sSet} \times \mathcal{M} \to \mathcal{M}$ with a right adjoint exponential map, satisfying the Quillen assumptions. For a more detailled treatment we send the interested reader to [Hov99, p. 4.2].

1.3 Homotopy function complexes

Here we briefly define the homotopy function complexes on a model category $\mathcal{M}$. Our main reference is [Hir03, Chapter 15-17].

1.3.1 Reedy diagrams

First of all let's recall the main definitions and results on Reedy diagram categories.

Definition 1.3.1. A category $\mathcal{R}$ is a *Reedy category* if it is small and it has two subcategories $\overrightarrow{\mathcal{R}}$ and $\overleftarrow{\mathcal{R}}$, both containing all $\mathrm{Ob}(\mathcal{R})$, and a degree function $\delta\colon \mathrm{Ob}(\mathcal{R}) \to \mathbb{N}$ satisfying

1. every non-identity map in $\overrightarrow{\mathcal{R}}$ raises degree (called direct maps);
2. every non-identity map in $\overleftarrow{\mathcal{R}}$ lowers degree (called inverse maps);
3. every morphism $g \in \mathrm{Mor}(\mathcal{R})$ admits a unique factorization $g = \overrightarrow{g} \circ \overleftarrow{g}$ where $\overrightarrow{g} \in \overrightarrow{\mathcal{R}}$ and $\overleftarrow{g} \in \overleftarrow{\mathcal{R}}$.

Observe that if $\mathcal{R}$ is Reedy than $\mathcal{R}^{\mathrm{op}}$ is also Reedy, with the same degree, putting $\overrightarrow{\mathcal{R}^{\mathrm{op}}} = (\overleftarrow{\mathcal{R}})^{\mathrm{op}}$ and similar.

Example 1. The category Δ is Reedy, where the degree of $[n]$ is n, direct maps are injections and inverse maps are surjections.

Definition 1.3.2. Let $\mathcal{R}$ be a Reedy category and $\alpha \in \mathrm{Ob}(\mathcal{R})$.

1. The *latching category* $\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ of $\mathcal{R}$ at α is the full subcategory of the comma category $(\overrightarrow{\mathcal{R}} \downarrow \alpha)$ containing all objects but id_α.
2. The *matching category* $\partial(\alpha \downarrow \overleftarrow{\mathcal{R}})$ of $\mathcal{R}$ at α is the full subcategory of the comma category $(\alpha \downarrow \overleftarrow{\mathcal{R}})$ containing all objects but id_α.

Definition 1.3.3. Let $\mathcal{M}$ be a model category, $\mathcal{R}$ be a Reedy category and consider the diagram category $\mathcal{M}^{\mathcal{R}}$. Let $\alpha \in \mathrm{Ob}(\mathcal{R})$ and $\mathbf{X} \in \mathcal{M}^{\mathcal{R}}$ (we use the notation $\mathbf{X}$ also to mean the induced $\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)$-diagram defined by $\mathbf{X}_{(\beta \to \alpha)} = \mathbf{X}_\beta$ and similar).

1. The *latching object* of $\mathbf{X}$ at α is $L_\alpha \mathbf{X} = \varinjlim_{\partial(\overrightarrow{\mathcal{R}} \downarrow \alpha)} \mathbf{X}$ and the *latching map* of $\mathbf{X}$ at α is the natural map $L_\alpha \mathbf{X} \to \mathbf{X}_\alpha$.
2. The *matching object* of $\mathbf{X}$ at α is $M_\alpha \mathbf{X} = \varprojlim_{\partial(\alpha \downarrow \overleftarrow{\mathcal{R}})} \mathbf{X}$ and the *matching map* of $\mathbf{X}$ at α is the natural map $\mathbf{X}_\alpha \to M_\alpha \mathbf{X}$.

The latching and matching object constructions are clearly functorial, i.e. if $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is a morphism in $\mathcal{M}^{\mathcal{R}}$, then, for every $\alpha \in \mathcal{R}$ we have a commutative diagram (natural in α)

$$\begin{array}{ccccc} L_\alpha\mathbf{X} & \longrightarrow & \mathbf{X}_\alpha & \longrightarrow & M_\alpha\mathbf{X} \\ \downarrow{\scriptstyle L_\alpha\varphi} & & \downarrow{\scriptstyle \varphi_\alpha} & & \downarrow{\scriptstyle M_\alpha\varphi} \\ L_\alpha\mathbf{Y} & \longrightarrow & \mathbf{Y}_\alpha & \longrightarrow & M_\alpha\mathbf{Y} \end{array}$$

Definition 1.3.4. Let $\mathcal{R}$ be a Reedy category, $\mathcal{M}$ a model category, $\mathbf{X}, \mathbf{Y} \in \mathcal{M}^{\mathcal{R}}$ and $\varphi\colon \mathbf{X} \to \mathbf{Y}$.

1. If $\alpha \in \mathrm{Ob}(\mathcal{R})$, the *relative latching map of* φ *at* α is the map $\mathbf{X}_\alpha \coprod_{L_\alpha\mathbf{X}} L_\alpha\mathbf{Y} \to \mathbf{Y}_\alpha$.
2. If $\alpha \in \mathrm{Ob}(\mathcal{R})$, the *relative matching map of* φ *at* α is the map $\mathbf{X}_\alpha \to \mathbf{Y}_\alpha \times_{M_\alpha\mathbf{Y}} M_\alpha\mathbf{X}$.

We have finally enough words to describe the Reedy model structure of the diagram category $\mathcal{M}^{\mathcal{R}}$.

Definition 1.3.5. Let $\mathcal{R}$ be a Reedy category, $\mathcal{M}$ a model category and $\mathbf{X}, \mathbf{Y} \in \mathcal{M}^{\mathcal{R}}$.

1. A map $f\colon \mathbf{X} \to \mathbf{Y}$ is a *Reedy weak equivalence* if it is pointwise a weak equivalence of $\mathcal{M}$.
2. A map $f\colon \mathbf{X} \to \mathbf{Y}$ is a *Reedy cofibration* if, for every $\alpha \in \mathrm{Ob}(\mathcal{R})$, the relative latching map
$$\mathbf{X}_\alpha \coprod_{L_\alpha\mathbf{X}} L_\alpha\mathbf{Y} \to \mathbf{Y}_\alpha$$
is a cofibration in $\mathcal{M}$.
3. A map $f\colon \mathbf{X} \to \mathbf{Y}$ is a *Reedy fibration* if, for every $\alpha \in \mathrm{Ob}(\mathcal{R})$, the relative matching map
$$\mathbf{X}_\alpha \to \mathbf{Y}_\alpha \times_{M_\alpha\mathbf{Y}} M_\alpha\mathbf{X}$$
is a fibration in $\mathcal{M}$.

Here's our final theorem about the existence of the Reedy model structure.

Theorem 1.3.6. Let $\mathcal{R}$ be a Reedy category and $\mathcal{M}$ be a model category.

1. The category $\mathcal{M}^{\mathcal{R}}$ with Reedy weak equivalences, Reedy cofibrations and Reed fibrations is a model category.
2. If $\mathcal{M}$ is a left/right proper model category, then also $\mathcal{M}^{\mathcal{R}}$ is so.

Proof. See [Hir03, Theorem 15.3.4]. ■

Under certain assumptions (cofibrant/fibrant constants) on $\mathcal{M}^{\mathcal{R}}$, we can compute homotopy limits and colimits (right and left Quillen adjoint of the constant diagram functor). For more details see [Hir03, p. 15.10]. Our main interest and application of Reedy model structures will be for simplicial and cosimplicial diagrams of a model category.

Let's now introduce simplicial and cosimplicial resolutions.

Definition 1.3.7. Let $X \in \mathrm{Ob}(\mathcal{M})$ and let $\mathrm{cc}_*X \in \mathcal{M}^{\Delta}$ be the corresponding constant cosimplicial object and $\mathrm{cs}_*X \in \mathcal{M}^{\Delta^{\mathrm{op}}}$ the corresponding constant simplicial object.

- A *cosimplicial resolution* of X is a cofibrant approximation $\widetilde{\mathbf{X}} \to \mathrm{cc}_*X$ in the Reedy model category $\mathcal{M}^{\Delta}$. A *fibrant cosimplicial resolution* is a cosimplicial resolution in which the weak equivalence $\widetilde{\mathbf{X}} \to \mathrm{cc}_*X$ is a Reedy trivial fibration.
- A *simplicial resolution* of X is a fibrant approximation $\mathrm{cs}_*X \to \widehat{\mathbf{X}}$ in the Reedy model category $\mathcal{M}^{\Delta^{\mathrm{op}}}$. A *cofibrant simplicial resolution* is a simplicial resolution in which the weak equivalence $\mathrm{cs}_*X \to \widehat{\mathbf{X}}$ is a Reedy trivial cofibration.

Example 2. In the category Top a cosimplicial resolution of X is given by $\{X \times |\Delta^n|\}_n$, with faces and degeneracies induced by the ones of the cosimplicial space $[n] \mapsto |\Delta^n|$.

We can give definitions of functorial simplicial/cosimplicial resolutions, maps between them and then prove that they are all Reedy weak equivalences. In particular, in the same spirit of the "Comparison theorem" of homological algebra, one can prove that fibrant cosimplicial resolutions are the final cosimplicial resolutions (there exists a unique, up to homotopy, weak equivalence). We have a similar (initial instead of final) result for cofibrant simplicial resolutions.

Proposition 1.3.8. Let $\mathcal{M}$ be a model category, $X \in \mathrm{Ob}(\mathcal{M})$ and $\widetilde{\mathbf{X}} \to \mathrm{cc}_* X$ a cosimplicial resolution. Then $\widetilde{\mathbf{X}}^0 \to X$ is a cofibrant approximation and

$$\widetilde{\mathbf{X}}^0 \coprod \widetilde{\mathbf{X}}^0 \xrightarrow{d^0 \coprod d^1} \widetilde{\mathbf{X}}^1 \xrightarrow{\quad s^0 \quad} \widetilde{\mathbf{X}}^0$$

is a cylinder object for $\widetilde{\mathbf{X}}^0$. Analogue for simplicial resolution and path objects.

Proof. See [Hir03, Prop 16.1.6]. ■

Finally we can talk about homotopy function complexes, which are a way to associate a simplicial space of morphisms between two objects of any model category such that the set of connected components is isomorphic to the set of maps, in the homotopy category, between these two objects. If we work in a simplicial model category then this new space, between a cofibrant and a fibrant object, will be the same as the simplicial space of morphisms coming from the sSet-enrichment.

Definition 1.3.9. Let $\mathcal{M}$ be a model category, $X, Y \in \mathrm{Ob}(\mathcal{M})$.

1. A *left homotopy function complex* from X to Y is a triple

$$\left(\widetilde{\mathbf{X}}, \widehat{Y}, \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{Y})\right)$$

   where

   - $\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X,
   - $\widehat{Y}$ is a fibrant approximation of Y,
   - $\mathcal{M}(\widetilde{\mathbf{X}}, \widehat{Y})$ is the simplicial set obtained by applying componentwise the contravariant functor $\mathcal{M}(-, \widehat{Y}) = \mathrm{Hom}_{\mathcal{M}}(-, \widehat{Y})$ to the cosimplicial diagram $\widetilde{\mathbf{X}}$.

2. A *right homotopy function complex* from X to Y is a triple

$$\left(\widetilde{X}, \widehat{\mathbf{Y}}, \mathcal{M}(\widetilde{X}, \widehat{\mathbf{Y}})\right)$$

   where

   - $\widetilde{X}$ is a cofibrant approximation of X;
   - $\widehat{\mathbf{Y}}$ is a simplicial resolution of Y;
   - $\mathcal{M}(\widetilde{X}, \widehat{\mathbf{Y}})$ is the simplicial set obtained by applying componentwise the covariant functor $\mathcal{M}(\widetilde{X}, -) = \mathrm{Hom}_{\mathcal{M}}(\widetilde{X}, -)$ to the simplicial diagram $\widehat{\mathbf{Y}}$.

3. A *two-sided homotopy function complex* from X to Y is a triple

$$\left(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}, \operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})\right)$$

   where

   - $\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X;
   - $\widehat{\mathbf{Y}}$ is a simplicial resolution of Y;
   - $\operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})$ is the diagonal of the bisimplicial set $([n], [k]) \mapsto \mathcal{M}(\widetilde{\mathbf{X}}^n, \widehat{\mathbf{Y}}_k)$.

Maps of left/right/two-sided homotopy function complexes are defined in the obvious way. We will talk about *homotopy function complex* between X and Y to mean one of the three complexes defined in Definition 1.3.9. Choosing a simplicial/cosimplicial resolution functor (e.g. $\Gamma\colon \mathcal{M} \to \mathcal{M}^{\Delta}$ with a natural transformation $\Gamma \to \mathrm{cc}_*$ etc) we can then talk about *functorial homotopy function complexes.*

Proposition 1.3.10. Let $\mathcal{M}$ be a model category and $X, Y \in \mathrm{Ob}(\mathcal{M})$. Then each left/right/two-sided homotopy function complex from X to Y is a fibrant simplicial set. Moreover, any change of left/right/two-sided homotopy function complex is a weak equivalence of fibrant simplicial sets.

Proof. See [Hir03, Proposition 17.1.3, 17.1.6, 17.2.3, 17.2.6, 17.3.2, 17.3.4]. ■

Finally we want to prove that all homotopy function complexes from X to Y are weakly equivalent. For example, starting from a two-sided homotopy function complex

$$\left(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}, \operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})\right)$$

we consider $\widetilde{\mathbf{X}}^0 \to X$, which is a cofibrant approximation by Proposition 1.3.8. Then

$$\left(\widetilde{\mathbf{X}}^0, \widehat{\mathbf{Y}}, \mathcal{M}(\widetilde{\mathbf{X}}^0, \widehat{\mathbf{Y}})\right)$$

is a right homotopy function complex, and the canonical map $\widetilde{\mathbf{X}} \to \mathrm{cc}_* \widetilde{\mathbf{X}}^0$ induces a morphism

$$\operatorname{diag} \mathcal{M}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}) \to \mathcal{M}(\widetilde{\mathbf{X}}^0, \widehat{\mathbf{Y}}).$$

Using this reasoning, and other analogue versions to build maps between the different homotopy function complexes, one can prove the following theorem.

Theorem 1.3.11. Let $\mathcal{M}$ be a model category and $X, Y \in \mathrm{Ob}(\mathcal{M})$. Then any two homotopy function complexes from X to Y are weakly equivalent Kan complexes.

Proof. See [Hir03, Theorem 17.1.11, 17.2.11, 17.3.9, 17.4.6]. ∎

By Theorem 1.3.11 we can give the following final definition.

Definition 1.3.12. Let $\mathcal{M}$ be a model category and $X, Y \in \mathrm{Ob}(\mathcal{M})$. We will denote by $\mathrm{Map}_{\mathcal{M}}(X, Y) \in$ sSet an homotopy function complex from X to Y. This makes sense since we are only interested in its homotopical properties, and all homotopy function complexes are weakly equivalent Kan complexes. We will call it, sometimes, the *mapping space* from X to Y.

Theorem 1.3.13. Let $\mathcal{M}$ be a model category, $X, Y \in \mathrm{Ob}(\mathcal{M})$ and consider $\mathrm{Map}_{\mathcal{M}}(X, Y)$ an homotopy function complex. Then $\pi_0(\mathrm{Map}_{\mathcal{M}}(X, Y))$ is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y)$.

Proof. See [Hir03, Theorem 17.7.2]. ∎

Let's also state a recognition result.

Theorem 1.3.14. Let $\mathcal{M}$ be a model category and $g \colon X \to Y \in \mathrm{Mor}(\mathcal{M})$. The following statements are equivalent:

1. g is a weak equivalence in $\mathcal{M}$;
2. for every $W \in \mathrm{Ob}(\mathcal{M})$ the map g induces a weak equivalence of simplicial sets $g_* \colon \mathrm{Map}_{\mathcal{M}}(W, X) \to \mathrm{Map}_{\mathcal{M}}(W, Y)$;
3. for every $Z \in \mathrm{Ob}(\mathcal{M})$ the map g induces a weak equivalence of simplicial sets $g^* \colon \mathrm{Map}_{\mathcal{M}}(Y, Z) \to \mathrm{Map}_{\mathcal{M}}(X, Z)$.

Proof. See [Vez13, Theorem 1.7.16]. ∎

Let's now state a more concrete computational version of this result, valid in the context of simplicial model categories.

Lemma 1.3.15. If $\mathcal{M}$ is a simplicial model category, then for each cofibrant object X, $\{X \otimes \Delta^n\}_n$ is a cosimplicial resolution of X.

Using this we can then say that:

1. in sSet the mapping space $\mathrm{Map}_{\mathsf{sSet}}(\mathcal{X}, \mathcal{Y})$ is just $\underline{\mathrm{Hom}}(\widetilde{\mathcal{X}}, \widehat{\mathcal{Y}})$, where $\widetilde{\mathcal{X}}$ is a cofibrant approximation of $\mathcal{X}$, $\widehat{\mathcal{Y}}$ is a fibrant approximation of $\mathcal{Y}$ and $\underline{\mathrm{Hom}}$ is the classic sSet-enrichment of sSet;
2. in $\mathrm{Ch}(R)$ we have a natural simplicial structure: the n-simplices of $\underline{\mathrm{Hom}}(E, F)$ are the chain maps of degree n, or equivalently the set $\mathrm{Hom}_{\mathrm{Ch}(R)}(E, F[-n])$. This implies that $\pi_i(\mathrm{Map}_{\mathrm{Ch}(R)}(E, F)) \simeq \pi_0(\mathrm{Map}_{\mathrm{Ch}(R)}(E, F[-i])) \simeq \mathrm{Hom}_{D(R)}(E, F[-i])$ using Theorem 1.3.13.

We also have a result about the relation between the mapping complex and Quillen adjunctions.

Theorem 1.3.16. Let $\mathcal{M}$ and $\mathcal{N}$ be small model categories and let $F\colon \mathcal{M} \rightleftarrows \mathcal{N} :U$ be a Quillen adjunction, let $X \in \mathcal{M}$ be cofibrant and $Y \in \mathcal{N}$ be fibrant.

1. If $\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X then $F\widetilde{\mathbf{X}}$ is a cosimplicial resolution of FX, and the adjunction induces a natural isomorphism $\mathcal{N}(F\widetilde{\mathbf{X}}, Y) \simeq \mathcal{M}(\widetilde{\mathbf{X}}, UY)$.
2. If $\widehat{\mathbf{Y}}$ is a simplicial resolution of Y, then $U\widehat{\mathbf{Y}}$ is a simplicial resolution of UY and the adjunction induces a natural isomorphism $\mathcal{N}(FX, \widehat{\mathbf{Y}}) \simeq \mathcal{M}(X, U\widehat{\mathbf{Y}})$.
3. If $\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X and $\widehat{\mathbf{Y}}$ is a simplicial resolution of Y, then the adjunction induces a natural isomorphism $\operatorname{diag}\mathcal{N}(F\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}}) \simeq \operatorname{diag}\mathcal{M}(\widetilde{\mathbf{X}}, U\widehat{\mathbf{Y}})$.

Proof. See [Hir03, Proposition 17.4.16]. ■

This implies that a Quillen adjunction induces the isomorphisms we expect also between mapping complexes, in Ho(sSet). From now on we will use the derived adjunction inside the mapping complexes to denote an appropriate representative.

Finally let's state a theorem relating homotopy function complexes with homotopy limits and colimits.

Theorem 1.3.17. Let $\mathcal{M}$ be a framed model category (i.e. fix a simplicial and cosimplicial resolution functor) and let I be a small indexing category.

1. If $\mathbf{X}$ is an objectwise cofibrant diagram in $\mathcal{M}^I$ and Y is fibrant in $\mathcal{M}$, then
$$\operatorname{Map}_{\mathcal{M}}(\operatorname{Hocolim}\mathbf{X}, Y) \simeq \operatorname{Holim}_i \operatorname{Map}_{\mathcal{M}}(\mathbf{X}_i, Y)$$
in Ho(sSet).
2. If X is cofibrant in $\mathcal{M}$ and $\mathbf{Y}$ is an objectwise fibrant diagram in $\mathcal{M}^I$, then
$$\operatorname{Map}_{\mathcal{M}}(X, \operatorname{Holim}\mathbf{Y}) \simeq \operatorname{Holim}_i \operatorname{Map}_{\mathcal{M}}(X, \mathbf{Y}_i)$$
in Ho(sSet).

Proof. See [Hir03, Theorem 19.4.4]. ■

1.4 Bousfield localization

Here we will briefly define the left Bousfield localization of a model category, and state an existence theorem. For time (and space) issues we again won't give a full treatment of the subject: already spelling out all precise definitions is too much, and we will just be happy with an intuition. For a more precise and general treatment of the subject see [Hir03, Chapter 3-4].

Definition 1.4.1. Let $\mathcal{M}$ be a model category and $\mathcal{C}$ a class of maps in $\mathcal{M}$.

1. An object $W \in \operatorname{Ob}(\mathcal{M})$ is *$\mathcal{C}$-local* if it is fibrant and for every $f\colon A \to B \in \mathcal{C}$, the induced map of homotopy function complexes
$$f^*\colon \operatorname{Map}(B, W) \to \operatorname{Map}(A, W)$$
is a weak equivalence of simplicial sets. If $\mathcal{C}$ is a single map f we say *f-local* and if $\mathcal{C}$ is just the map $A \to *$ then we say *A-local*.
2. A map $g\colon X \to Y$ in $\mathcal{M}$ is a *$\mathcal{C}$-local equivalence* if for every $\mathcal{C}$-local object W, the induced map
$$g^*\colon \operatorname{Map}(Y, W) \to \operatorname{Map}(X, W)$$
is a weak equivalence of simplicial sets.

Proposition 1.4.2. If $\mathcal{M}$ is a model category and $\mathcal{C}$ a class of maps in $\mathcal{M}$, then any weak equivalence of $\mathcal{M}$ is a $\mathcal{C}$-local equivalence.

Proof. See [Hir03, Proposition 3.1.5]. ■

Theorem 1.4.3. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a Quillen adjunction.Let $\mathcal{C}$ be a class of maps in $\mathcal{M}$. Then the following are equivalent.

(a) The total left derived functor $\mathbb{L}F\colon \operatorname{Ho}(\mathcal{M}) \to \operatorname{Ho}(\mathcal{N})$ satisfies $\mathbb{L}F(\mathcal{C}) \subset \operatorname{Iso}(\operatorname{Ho}(\mathcal{N}))$.

(b) The functor F takes cofibrant approximations of elements of $\mathcal{C}$ into weak equivalences in $\mathcal{N}$.

(c) The functor U takes fibrant objects of $\mathcal{N}$ into $\mathcal{C}$-local objects of $\mathcal{M}$.

(d) The functor F takes $\mathcal{C}$-local equivalences between cofibrant objects into weak equivalences in $\mathcal{N}$.

Proof. See [Hir03, Theorem 3.1.6]. ■

We are finally ready to define left Bousfield localizations.

Definition 1.4.4. Let $\mathcal{M}$ be a model category and $\mathcal{C}$ a class of maps in $\mathcal{M}$. The *left Bousfield localization* of $\mathcal{M}$ with respect to $\mathcal{C}$, if it exists, is a model category structure $L_{\mathcal{C}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that

(a) the class of weak equivalences of $L_{\mathcal{C}}\mathcal{M}$ is the class of $\mathcal{C}$-local equivalences of $\mathcal{M}$;

(b) the class of cofibrations of $L_{\mathcal{C}}\mathcal{M}$ is the class of cofibrations of $\mathcal{M}$;

(c) fibrations are defined by lifting properties.

Remark 1.4.5. As already written in Definition 1.4.4, the left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{C}$ might not exist. We will come back to the problem of existence later.

Let's immediately note some properties of the left Bousfield localization.

Proposition 1.4.6. Let $\mathcal{M}$ be a model category and $\mathcal{C}$ a class of maps in $\mathcal{M}$. Let $L_{\mathcal{C}}\mathcal{M}$ be the left Bousfield localization of $\mathcal{M}$ w.r.t. $\mathcal{C}$ (assume it exists). Then

(a) every weak equivalence of $\mathcal{M}$ is a weak equivalence of $L_{\mathcal{C}}\mathcal{M}$;

(b) the trivial fibrations of $L_{\mathcal{C}}\mathcal{M}$ are exactly the trivial fibrations of $\mathcal{M}$;

(c) every fibration of $L_{\mathcal{C}}\mathcal{M}$ is also a fibration of $\mathcal{M}$.

Moreover, the identity functors

$$\operatorname{id}_{\mathcal{M}}\colon \mathcal{M} \rightleftarrows L_{\mathcal{C}}\mathcal{M} : \operatorname{id}_{\mathcal{M}}$$

are a Quillen adjunction.

Proof. See [Hir03, Proposition 3.3.3]. ■

We also have a universal property, coming from the fact that the Bousfield localization is indeed a certain kind of *localization*.

Proposition 1.4.7. Let $\mathcal{M}$ be a model category and $\mathcal{C}$ a class of maps in $\mathcal{M}$. Let $L_{\mathcal{C}}\mathcal{M}$ be the left Bousfield localization of $\mathcal{M}$ w.r.t. $\mathcal{C}$ (assume it exists). Then the identity functor $j\colon \mathcal{M} \to L_{\mathcal{C}}\mathcal{M}$ is a left localization of $\mathcal{M}$ with respect to $\mathcal{C}$. This means the following things:

1. j is a left Quillen functor;
2. the total left derived functor $\mathbb{L}j\colon \operatorname{Ho}(\mathcal{M}) \to \operatorname{Ho}(L_{\mathcal{C}}\mathcal{M})$ takes the (images of) elements of $\mathcal{C}$ in $\operatorname{Ho}(\mathcal{M})$ into isomorphisms in $\operatorname{Ho}(L_{\mathcal{C}}\mathcal{M})$.
3. j is initial among such functors, i.e. if $\varphi\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor such that $\mathbb{L}\varphi(\mathcal{C}) \subset \operatorname{Iso}(\operatorname{Ho}(\mathcal{N}))$, then there exists a unique left Quillen functor $\delta\colon L_{\mathcal{C}}\mathcal{M} \to \mathcal{N}$ such that $\varphi = \delta \circ j$.

Proof. See [Hir03, Theorem 3.3.19]. ■

The key technical result is the following theorem. While all the previous results were completely symmetric, i.e. admitting complete dual formulations for right Bousfield localizations, the following theorem is not symmetric and it only holds for the left case. This is due to the requested properties, like being cofibrantly generated, not being self-dual.

Theorem 1.4.8. Let $\mathcal{M}$ be a left proper cellular model category and let S be a set of maps in $\mathcal{M}$.

1. The left Bousfield localization $L_S\mathcal{M}$ exists.
2. The fibrant objects of $L_S\mathcal{M}$ are the S-local objects of $\mathcal{M}$.
3. $L_S\mathcal{M}$ is a left proper cellular model category.
4. If $\mathcal{M}$ is a simplicial model category, then $L_S\mathcal{M}$ is also a simplicial model category.

Proof. See [Hir03, Theorem 4.1.1]. ∎

2 Stacks

2.1 Recalls of algebraic geometry

In this work we will use, when needed, classical algebraic geometry, where classical means [Har77]. Let's just recall some definitions to warm up.

Definition 2.1.1. Let $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ a morphism in Aff, the category of affine schemes, corresponding to a map of rings $B \to A$.

- The map f is *smooth* if it is flat, of finite presentation and B is a $B \otimes_A B$-module (by the multiplication map) of finite Tor dimension. This last condition is equivalent as asking $\Omega^1_{B/A}$ to be a projective (= locally free) B-module.
- The map f is *étale* if it is smooth and, in particular, B is a flat $B \otimes_A B$-module. This last condition is equivalent as asking $\Omega^1_{B/A} = 0$, the B-module of Kähler differentials.

Clearly étale implies smooth (precisely étale means smooth with relative dimension 0). One intuitive way to think is to see smooth maps as the analogue of submersions in differential geometry, and étale maps as the analogue of local diffeomorphisms (inducing isomorphisms on tangent spaces and hence local diffeomorphisms).

Of course these definitions are stable by composition and base change. For a more detailed exposition see [Har77, Chapther III, 10].

We will assume to be known the notion of Grothendieck topology and the general notion of sheaf on a site, for which our main reference will be [Vis08]. We will sometimes use pretopologies (i.e. set of arrows $\{U_i \to U\}$ covering U) and sometimes sieves (i.e. subfunctors of h_U), recalling that there is a canonical way to pass from one to the other (we have a one-to-one correspondence considering only saturated pretopologies). Let's now recall the definitions of fppf, fpqc and étale topology.

Definition 2.1.2. Let $f\colon X \to Y$ a faithfully flat morphism of schemes. If Y can be covered by open affine subschemes $\{V_i\}$ such that each V_i is the image of a quasi-compact open subset of X, then we say that the map f is *fpqc*.

See [Vis08, Proposition 2.33] for some equivalent characterizations.

Definition 2.1.3. Let's work in the category of affine schemes Aff (or also in the whole Sch).

- An *étale* covering $\{U_i \to U\}_i$ is a jointly surjective collection of étale maps locally of finite presentation.
- An *fppf* covering $\{U_i \to U\}_i$ is a jointly surjective collection of flat maps locally of finite presentation.
- An *fpqc* covering $\{U_i \to U\}_i$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fpqc (see Definition 2.1.2).

Observe that the fpqc topology is finer than the fppf topology, which is finer than the étale topology, which is in turn finer than the Zariski topology. A lot of properties of morphisms are local on the codomain in the fpqc topology, see [Vis08, Proposition 2.36].

2.2 Higher stacks

Let's consider the category of affine schemes $\mathsf{Aff} = \mathsf{Comm}^{\mathrm{op}}$ (in the following we will always use this equivalence implicitely) endowed with the étale Grothendieck topology, generated by coverings of $\operatorname{Spec} A$ of the type $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}_i$ such that any $\operatorname{Spec} A_i \to \operatorname{Spec} A$ is an étale map and the family of functors $\{- \otimes A_i : A-\mathsf{Mod} \to A_i - \mathsf{Mod}\}$ is conservative (this is an equivalent version of Definition 2.1.3). In an analogue way we consider the site Aff/X for a given X affine scheme. We will assume to be based on $k \in \mathsf{Comm}$ (i.e. we implicitely consider the comma site $\mathsf{Aff}/\operatorname{Spec} k$) but, for readability, we will drop the comma notation.
We can then talk about sheaves on Aff and we'll focus on simplicial presheaves $SPr(\mathsf{Aff}) \simeq \mathsf{sSet}^{\mathsf{Aff}^{\mathrm{op}}}$, which we endow with the projective model structure (called *global model structure*). This means that weak equivalences and fibrations are defined pointwise. One can prove this defines a cofibrantly generated model structure, proper and cellular.
We will now use the étale topology on $\mathsf{Aff} \simeq \mathrm{Ho}(\mathsf{Aff})$ (using trivial model structure) to define a local model structure, obtained as a left Bousfield localization of the global structure. Given a presheaf $F : \mathsf{Aff}^{\mathrm{op}} \to \mathsf{sSet}$ we can consider a presheaf of sets $\mathsf{Aff} \ni X \mapsto \pi_0(F(X))$, which we sheafify to get $\pi_0(F)$. Similarly, for $X \in \mathsf{Aff}$ and $s \in F(X)_0$, we define $\pi_j(F, s)$ to be the sheafification of the presheaf of groups

$$(\mathsf{Aff}/X)^{\mathrm{op}} \ni (f : Y \to X) \mapsto \pi_j(F(Y), f^*(s)).$$

Definition 2.2.1. Using the same notation as above, the sheaves $\pi_0(F)$ and $\pi_i(F, s)$ are called the *homotopy sheaves of F*.

Observe that they are clearly functorial in F. To compute homotopy groups we can either choose a fibrant replacement in sSet and apply the classical combinatorial definition, or just consider the topological homotopy group of the geometric realization.

Definition 2.2.2 (Local model structure)**.** Let $f : F \to F'$ be a map in $SPr(\mathsf{Aff})$.

1. The map f is a *local equivalence* if $\pi_0(f)$ is an isomorphism of sheaves, as well as any $\pi_j(F, s) \to \pi_j(F', f(s))$, for all $X \in \mathsf{Aff}$ and $s \in F(X)_0$.
2. The map f is a *local cofibration* if it is a global cofibration.

Local fibrations are defined by lifting properties. This structure is called the *local model structure* on $SPr(\mathsf{Aff})$ (it can be proved it actually defines a model structure, see [Bla01]), and we will use it from now on.

We can give a characterization of fibrant objects in $SPr(\mathsf{Aff})$, thanks to a general theorem of [DHI03]. We will need a general definition.

Definition 2.2.3 (Hypercovering)**.** Given $X \in \mathsf{Aff}$, the data of a morphism $H \to X$ in $SPr(\mathsf{Aff})$ (implicitely using Yoneda embedding on X) is called an (étale) *hypercovering* if it satisfies:

1. for any integer n, the presheaf of sets H_n is a disjoint union of representable presheaves
$$H_n \simeq \coprod_i H_{n,i}$$
for $H_{n,i} \in \mathsf{Aff}$ (Yoneda);
2. for any $n \geq 0$ the morphism of presheaves of sets
$$H_n \simeq H^{\Delta^n} \simeq \mathcal{H}\mathrm{om}(\Delta^n, H) \to \mathcal{H}\mathrm{om}(\partial\Delta^n, H) \times_{\mathcal{H}\mathrm{om}(\partial\Delta^n, X)} \mathcal{H}\mathrm{om}(\Delta^n, X)$$
induces an epimorphism on associated sheaves.

Here Δ^n and $\partial\Delta^n$ are considered as constant simplicial presheaves, while $\mathcal{H}\mathrm{om}$ is the presheaf of morphisms, valued in sets, between simplicial presheaves.

Example 3. Let $\{U_i \to X\}_{i \in I}$ be an étale covering in Aff. Then the Čech nerve $H \in SPr(\mathsf{Aff})$ of this covering (more classicaly denoted by the Čech nerve of $U = \coprod_i U_i$), given by

$$H_n = \coprod_{(i_0, i_1, \ldots, i_n) \in I^{n+1}} U_{i_0} \times_X U_{i_1} \times_X \cdots \times_X U_{i_n}$$

with face maps given by projections and degeneracies given by diagonal (implicitely using Yoneda as always), is a basic example of hypercovering.

We can restate the second condition of hypercoverings just by saying that given $Y \in \mathsf{Aff}$ and a commutative square

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & H(Y) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & X(Y) = \operatorname{const}\operatorname{Hom}(X,Y) \end{array}$$

in sSet, then there exists a covering sieve $u \subset h_Y$ such that for any $f\colon U \to Y$ in $u(U)$, there exists a dashed lift

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & H(U) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & X(U) = \operatorname{const}\operatorname{Hom}(X,U) \end{array}$$

making the diagram commutative. This is indeed a local lifting property, analogue to the lifting property characterizing acyclic fibrations in simplicial sets. Let's finally observe that, since X is a 0-truncated simplicial presheaf (i.e. all homotopy sheaves π_i are zero for $i \geq 1$), the restriction $X^{\Delta^n} \to X^{\partial\Delta^n}$ is actually an isomorphism for $n > 1$, so that the second condition becomes only dependent on H for $n > 1$ (requiring $H_n \to H^{\partial\Delta^n}$ to be an iso).

Given an hypercovering we define an augmented cosimplicial diagram in sSet

$$F(X) \to ([n] \mapsto F(H_n))$$

where $F(H_n) = \prod_i F(H_{n,i})$.

Theorem 2.2.4. A simplicial presheaf F over Aff is fibrant if and only if

1. for any $X \in \mathsf{Aff}$, $F(X)$ is fibrant;
2. for any $H \to X$ hypercovering, the natural map

$$F(X) \to \operatorname{Holim}_{[n]\in\Delta} F(H_n)$$

is an equivalence in sSet.

Proof. See [DHI03]. ■

If F is a discrete presheaf, i.e. a presheaf of sets, then the second condition boils down to the usual descent condition.

Definition 2.2.5. An $F \in SPr(\mathsf{Aff})$ is called a *stack* if it satisfies the descent condition of Theorem 2.2.4. The homotopy category $\operatorname{Ho}(SPr(\mathsf{Aff}))$ is called the *category of stacks* and its morphisms are denoted by $[F, F']$.

Remark 2.2.6. Although for F a presheaf of sets, the stack condition is equivalent to the set-theoretic sheaf condition, for a general simplicial presheaf F being a sheaf of simplicial sets and being a stacks are two different concepts: indeed the homotopy limit can be different from the standard limit, and we have a more "relaxed" condition. Moreover, one can prove that if F is groupoid-valued (as in the classic theory of 1-stacks), then the homotopy limit boils down to the classic 2-Ker condition, see [Hen11]. More precisely, to pass from groupoid-valued stacks to simplicial stacks one just composes with the nerve functor (and obtains a 1-truncated simplicial presheaf).

Let's observe that stacks are really a generalization of sheaves, by the full embedding

$$\operatorname{Sh}(\mathsf{Aff}) \to \operatorname{Ho}(SPr(\mathsf{Aff}))$$

considering any sheaf as a constant simplicial presheaf. This functor has a left adjoint $F \mapsto \pi_0(F)$.

Remark 2.2.7. To define stacks we work on the category of affine schemes Aff, and not on the category Sch of general schemes, which is more usual in the classic context of stacks in groupoids. This doesn't affect at all the degree of generality of the theory: covering any scheme by affine opens we can reduce ourselves to only look at affine schemes, and we can obtain a "bigger" stack on Sch using the descent condition. This will remain valid also in the next chapters, where we will generalize this construction to the simplicial world.

2.2.1 Structure of $SPr(\mathsf{Aff})$

It will be useful to know a little better the category $SPr(\mathsf{Aff})$, in particular its canonical sSet-enrichment. The following reasoning will hold in great generality for the category of simplicial objects of any cocomplete category $\mathcal{C}$.

Definition 2.2.8. Let $K \in \mathsf{sSet}$ and $F \in s\mathcal{C}$, and define $F \otimes K$ by

$$(F \otimes K)_n := \coprod_{k \in K_n} F_n \in \mathcal{C}.$$

Given $\phi\colon [n] \to [m] \in \Delta$ we define a map $\phi^*\colon (F \otimes K)_m \to (F \otimes K)_n$ by

$$\textstyle\coprod_{k\in K_m} F_m \xrightarrow{\coprod \phi^*} \coprod_{k\in K_m} F_n \longrightarrow \coprod_{k\in K_n} F_n$$

where the first map is induced by $\phi^*\colon F_m \to F_n$ and the second by $\phi^*\colon K_m \to K_n$.

Theorem 2.2.9. Suppose that $\mathcal{C}$ is bicomplete (for example any category of presheaves of sets). Then with this bifunctor $- \otimes -\colon s\mathcal{C} \times \mathsf{sSet} \to s\mathcal{C}$, the category $s\mathcal{C}$ becomes a simplicial category with

$$\underline{\mathrm{Hom}}_{s\mathcal{C}}(A, B)_n = \mathrm{Hom}_{s\mathcal{C}}(A \otimes \Delta^n, B).$$

Proof. See [GJ99, Chapter II, Theorem 2.5]. ∎

Let's now state a version of Yoneda lemma valid for general simplicial presheaves $SPr(\mathcal{C})$. Let's observe beforehand that the action of sSet on the simplicial category $SPr(\mathcal{C})$ (using Theorem 2.2.9) is given by

$$F \otimes K\colon [n] \mapsto \coprod_{K_n} F_n \rightsquigarrow F \otimes K \simeq F \times K$$

where $F \in SPr(\mathcal{C})$, $K \in \mathsf{sSet}$ and in the last passage K is considered as a constant presheaf.

Lemma 2.2.10. Let $\mathcal{C}$ be a category and $F \in SPr(\mathcal{C})$ be a simplicial presheaf. For any $X \in \mathrm{Ob}(\mathcal{C})$ there is a natural isomorphism

$$\underline{\mathrm{Hom}}_{SPr(\mathcal{C})}(h_X, F) \simeq F(X)$$

of simplicial sets, where $h_X = \mathrm{const}\,\mathrm{Hom}_{\mathcal{C}}(-, X)$ is the Yoneda discrete simplicial presheaf.

Proof. See [Vez13, Theorem 5.3.6]. ∎

Let's now deduce a useful corollary for the case of $SPr(\mathsf{Aff})$, endowed with the local model structure. It obviously can be generalized, being careful with fibrant and cofibrant objects.

Corollary 2.2.10.1. Let $F \in SPr(\mathsf{Aff})$ be a fibrant simplicial presheaf (hence a stack, according to our previous definition). For any $X \in \mathrm{Ob}(\mathsf{Aff})$ there is a natural isomorphism

$$\pi_0(\underline{\mathrm{Hom}}_{SPr(\mathsf{Aff})}(h_X, F)) \simeq \mathrm{Hom}_{\mathrm{Ho}(SPr(\mathsf{Aff}))}(h_X, F) \simeq \pi_0(F(X))$$

of sets.

Definition 2.2.11. Let $F \longrightarrow H \longleftarrow G$ be a diagram of stacks in $\mathrm{Ho}(SPr(\mathsf{Aff}))$. We will denote by $F \times^h_H G$ the homotopy fiber product of some lift of such diagram in $SPr(\mathsf{Aff})$ (this construction is thus not functorial in $\mathrm{Ho}(SPr(\mathsf{Aff}))$).

2.3 Geometric stacks

Here we will give a new definition of schemes, and then of geometric n-stacks, that need to be thought as quotients of schemes. More precisely the basic idea is the following: we consider a representable stack X and a groupoid object X_1 (which is itself a representable stack) acting smoothly on X, and then we consider the quotient. Sometimes X_1 needs not to be representable, but it can be itself a quotient of a representable stack, and this is the main motivation behind the recursive definition of geometric n-stacks (where we assume X_1 to be $(n-1)$-geometric). A more extensive explanation, although in a more general context, can be found at [HAGII, p. 1.3.3].

Recall that any affine scheme $\operatorname{Spec} A$ can be identified, through the Yoneda map, to the presheaf

$$\operatorname{Spec} B \mapsto \operatorname{Hom}(\operatorname{Spec} B, \operatorname{Spec} A) = \operatorname{Hom}(A, B)$$

which is actually a sheaf for the étale topology (faithfully flat descent, see [Vis08, Theorem 2.55]), and hence it can be considered as a constant simplicial stack in $\operatorname{Ho}(SPr(\mathsf{Aff}))$. The Yoneda embedding

$$h\colon \mathsf{Aff} \to \operatorname{Ho}(SPr(\mathsf{Aff}))$$

is fully faithful.

Definition 2.3.1. Any stack isomorphic, in the homotopy category $\operatorname{Ho}(SPr(\mathsf{Aff}))$, to one $\operatorname{Spec} A$ (using Yoneda embedding) is called an *affine scheme*, or a *representable stack*.

We can now define schemes.

Definition 2.3.2.

1. A morphism $F \to \operatorname{Spec} A$ is a *Zariski open immersion* if F is a sheaf (i.e. 0-truncated), i is a monomorphism of sheaves and there exists a family of classical Zariski open immersions $\{\operatorname{Spec} A_i \to \operatorname{Spec} A\}_i$ such that the map
$$\coprod_i \operatorname{Spec} A_i \to \operatorname{Spec} A$$
factors through an epimorphism of sheaves to F.

2. A morphism $F \to F'$ is a Zariski open immersion if it is locally so, i.e. for any affine scheme $\operatorname{Spec} A$ and any map $\operatorname{Spec} A \to F'$, the induced map
$$F \times^h_{F'} \operatorname{Spec} A \to \operatorname{Spec} A$$
is a Zariski open immersion as in the previous point.

3. A stack F is a *scheme* if there exists a family of affine schemes $\{\operatorname{Spec} A_i\}_i$ with open immersions $\operatorname{Spec} A_i \to F$ such that the induced morphism of sheaves
$$\coprod_i \operatorname{Spec} A_i \to F$$
is an epimorphism. Such a family is called a *Zariski atlas* for F.

4. A morphism of schemes $F \to F'$ is called *smooth* if it is "locally smooth", i.e. if there exist Zariski atlases $\{\operatorname{Spec} A_i \to F\}$ and $\{\operatorname{Spec} B_j \to F'\}$ such that we have commutative squares
$$\begin{array}{ccc} F & \longrightarrow & F' \\ \uparrow & & \uparrow \\ \operatorname{Spec} A_i & \longrightarrow & \operatorname{Spec} B_j \end{array}$$
with the downward morphism being (classically) smooth (here for any i we find $j = j(i)$).

Finally we are ready to define geometric stacks.

Definition 2.3.3.

1. A stack F is (-1)-*geometric* if it is representable (i.e. an affine scheme).

2. A morphism of stacks $F \to F'$ is (-1)-*representable* if for any representable stack X and any map $X \to F'$, the homotopy pullback $F \times^h_{F'} X$ is (-1)-geometric.

3. A (-1)-geometric morphism $F \to F'$ is (-1)-*smooth* if for any representable stack X and any map $X \to F'$, the induced morphism $F \times^h_{F'} X \to X$ is a smooth morphism between representable stacks.

Let $n > 0$ and assume the notions of $(n-1)$-geometric stack, morphism and smooth morphism to be defined. Then, by recursion on n, we can define the following.

1. An stack F is n-geometric if there exists a family of maps $\{U_i \to F\}_{i\in I}$ such that
   (a) each U_i is representable,
   (b) each map $U_i \to F$ is $(n-1)$-smooth,
   (c) the total morphism $\coprod_{i\in I} U_i \to F$ is an epimorphism.

   Such family is a *smooth n-atlas.*

2. A morphism $F \to F'$ is n-representable if for any representable stack X and any map $X \to F'$, the stack $F \times^h_{F'} X$ is n-geometric.

3. An n-geometric morphism $F \to F'$ is n-smooth if for any representable stack X and any map $X \to F'$, there exists a smooth n-atlas $\{U_i\}$ of $F \times^h_{F'} X$ such that each composite map $U_i \to X$ is smooth.

Observe that, since Zariski open immersions are smooth, schemes are 0-geometric stacks. One can prove, although non-trivially, that if F is an n-geometric stack, then its diagonal is $(n-1)$-representable. Our definition makes sense, as justified by the following statement.

Proposition 2.3.4.

1. Any $(n-1)$-representable (resp. $(n-1)$-smooth) morphism is n-representable (resp. n-smooth).

2. All n-representable (resp. n-smooth) morphisms are stable by isomorphisms, homotopy pullbacks and compositions.

Proof. See [HAGII, Proposition 1.3.3.3]. ■

To conclude, let's state some other important properties of n-geometric stacks and n-smooth maps.

Proposition 2.3.5. Let $f\colon F \to G$ be a morphism of stacks, where G is n-geometric. Suppose there exists a smooth n-atlas $\{U_i\}$ of G such that each stack $F \times^h_G U_i$ is n-geometric. Then F is also n-geometric. Furthermore, if each projection $F \times^h_G U_i \to U_i$ is n-smooth, then f is also n-smooth.

Proof. The slogan of this statement could be that n-geometricity and n-smoothness are local on n-geometric targets. See [HAGII, Proposition 1.3.3.4] for the proof. ■

Proposition 2.3.6. Let f be an n-representable morphism. If f is m-smooth, for $m \geq n$, then it is n-smooth.

Proof. See [HAGII, Proposition 1.3.3.6]. ■

A vey important corollary is the following.

Corollary 2.3.6.1. Let $n \geq 0$; then the full subcategory of n-geometric stacks in $\mathrm{St}(k) = \mathrm{Ho}(SPr(\mathsf{Aff}_{/k}))$ is stable by homotopy pullbacks and by disjoint unions.

Proof. See [HAGII, Corollary 1.3.3.5]. ■

We conclude with one last definition.

Definition 2.3.7. A stack is a *geometric stack* if it is n-geometric for some n. A morphism of stacks is *smooth* (resp. *representable*) if it is n-smooth (resp. n-representable) for some n.

Geometric stacks can be, maybe more intuitively, described, as announced in the beginning, in terms of quotients by groupoid actions. For time issues we won't report here this point of view, which can be found at [HAGII, pp. 1.3.4, 1.3.5].

3 Moduli stack of G-bundles

3.1 Recalls on classical 1-stacks

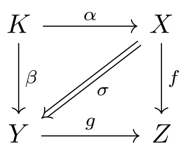
We will now focus on a concrete classical example of stack: the moduli stack of G-bundles for an S-scheme $X \to S$ (we can suppose to work, in general, in the category Sch_k for k a base field). We will resume the results proved in [Wan11]. This chapter will be almost self-contained and we won't use any theory introduced up to now (no model category theory at all). We will use the more classical language of stacks in groupoids and pseudofunctors, equivalent to the formulation in terms of fibered categories in groupoids: for a detailed explanation see [Vis08]. We will just recall a few definitions and we will completely skip any fibered category notion, for time and readability issues.

Definition 3.1.1. Let F be the datum of a map of sets/classes $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathsf{Cat})$ and, for every $X, Y \in \mathcal{C}$ a map of sets $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathsf{Cat}}(FY, FX)$. We say that F is a (contravariant) *pseudofunctor* or *quasifunctor* or *lax* 2*-functor* if the "functor conditions" hold up to isomorphism (i.e. it is a more relaxed definition of a functor). In particular for every $U \in \mathcal{C}$ we have an isomorphism of functors $\epsilon_U \colon F(\mathrm{id}_U) \simeq \mathrm{id}_{FU}$ and for each pair of maps $U \xrightarrow{f} V \xrightarrow{g} W$ an isomorphism $\alpha_{f,g} \colon F(f) \circ F(g) \simeq F(g \circ f)$. There are some compatibility assumptions, like associativity of composition $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$, expressed by

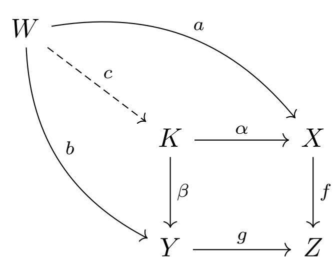
$$\begin{array}{ccc} F(f) \circ F(g) \circ F(h) & \xrightarrow{\alpha_{f,g}(F(h))} & F(g \circ f) \circ F(h) \\ \downarrow{\scriptstyle F(f)\alpha_{g,h}} & & \downarrow{\scriptstyle \alpha_{g \circ f, h}} \\ F(f) \circ F(h \circ g) & \xrightarrow{\alpha_{f, h \circ g}} & F(h \circ g \circ f) \end{array}$$

We will use a bit of 2-category theory, in particular we will be interested in computing 2-limits (mainly 2-fibered products), which ideally satisfy the same universal property as their 1-counterpart but diagrams commute *up to isomorphism*. Let's just give a definition.

Definition 3.1.2. Consider the square



and suppose it commutes up to isomorphism, i.e. there exists a natural transformation $\sigma \colon f \circ \alpha \to g \circ \beta$ which is an isomorphism in $\mathrm{Hom}(K, Z)$. We say that K is a *2-fibered product* of the square if for every $W \xrightarrow{(a,b)} X \times Y$ and $\tau \colon f \circ a \xrightarrow{\sim} g \circ b$ there exists a unique map (up to isomorphism) $c \colon W \to K$ such that the square



commutes up to isomorphism (or, in more modern terminology, 2-commutes). Explicitely this means we have isomorphism $\pi_a\colon a \xrightarrow{\sim} \alpha \circ c$ and $\pi_b\colon \beta \circ c \xrightarrow{\sim} b$ making the square

$$\begin{array}{ccc} f\circ a & \xrightarrow{\tau} & g\circ b \\ \downarrow{\scriptstyle f_*\pi_a} & & \downarrow{\scriptstyle g_*\pi_b} \\ f\circ\alpha\circ c & \xrightarrow{c^*\sigma} & g\circ\beta\circ c \end{array}$$

(1-)commute in the category $\mathrm{Hom}(W, Z)$.

Finally let's recall the classical definition of stacks in terms of 2-kernels.

Definition 3.1.3. Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Grpd}$ be a pseudofunctor, where $\mathcal{C}$ is a site. Then we say that F is a (1-)*stack* if for any $X \in \mathrm{Ob}(\mathcal{C})$ and for any covering $\{U_i \to X\}$ we have the following equivalence of categories

$$F(X) \simeq 2-\ker\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{i,j}) \rightrightarrows\!\!\!\!\!\rightarrow \prod_{i,j,k} F(U_{i,j,k}) \right).$$

The condition is very intuitive: it is exactly the sheaf condition "up to isomorphism". It means exactly that we can glue (uniquely) local data, whose restriction on intersections are isomorphic, and such that these isomorphisms (which before were supposed to be equalities) respect a natural cocycle condition (this explains the triple intersection term above). This is just a particular case of our definition of stacks (sometimes called "higher stacks"): applying the nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$ we get back to simplicial stacks, in particular 1-stacks. It can be proved that this is indeed an equivalence, see [Hen11, Thm 3.5.2] or [HAGII, p. 2.1.2]. Let's finally recall the 2-Yoneda lemma, fundamental generalization of classical Yoneda to the context of pseudofunctors (or fibered categories, depending on the tastes).

Lemma 3.1.4 (2-Yoneda lemma)**.** Given a pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Grpd}$ and an element $X \in \mathcal{C}$, then we have an equivalence of categories

$$\mathrm{Hom}(h_X, F) \simeq F(X)$$

where $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$ and $\mathrm{Hom}_{\mathcal{C}}(h_X, F)$ is the category of morphism between pseudofunctors (or fibered categories over $\mathcal{C}$). This equivalence is natural in X.

Proof. See [Vis08, p. 3.6.2]. ∎

3.2 Introduction and basic concepts

We will work in $\mathsf{Sch}_{/k}$ endowed with the fpqc topology, see Definition 2.1.3, and we will consider a scheme X both as a scheme and as a fpqc-sheaf (motivated by the fact that fpqc is subcanonical, again see [Vis08, Thm 2.55]). Let's recall the notions of (quasi)-projective morphism given by Grothendieck in [EGAII, pp. 5.3, 5.5], which are slightly different from the ones in [Har77].

Definition 3.2.1. Let $f\colon X \to S$ be a morphism of schemes. We say that

- f is *quasi-projective* if it is of finite type and there exists a relatively ample invertible sheaf $\mathcal{L}$ on X (relative means that for every affine open $V \subset S$, the sheaf $\mathcal{L}\big|_{f^{-1}(V)}$ is ample);
- f is *projective* if there exists a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type, such that X is S-isomorphic to a closed subscheme of $\mathbb{P}(\mathcal{E})$;
- f is *strongly projective* (resp. strongly quasi-projective) if it is finitely presented and there exists a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of constant finite rank such that X is S-isomorphic to a closed (resp. retrocompact, i.e. having inclusion map quasi-compact) subscheme of $\mathbb{P}(\mathcal{E})$.

Given $X, Y \in \mathsf{Sch}_{/k}$ we will write $X \times Y = X \times_{\mathrm{Spec}\,k} Y$ and $X(Y) = \mathrm{Hom}_k(Y, X)$. Given a fibered product over S, we write $\mathrm{pr}_i\colon X_1 \times_S X_2 \to X_i$ for the canonical map, for $i = 1, 2$. Given an $\mathcal{O}_X$-module $\mathcal{F}$ and an S-scheme T, we write $X_T = X \times_S T$ and $\mathcal{F}_T = \mathrm{pr}_X^* \mathcal{F}$.
In particular, for $S = \mathrm{Spec}\,\kappa(t)$ with $t \in T$, we will write $X_t \coloneqq X_S$ and $\mathcal{F}_t = \mathcal{F}_S$: this notation can be confused with taking stalks but usually context is enough to understand the correct meaning. Given a locally free $\mathcal{O}_X$-module of finite rank $\mathcal{E}$ we write $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ for its dual, which is again locally free of finite rank.

We will work with an algebraic group G over k, which will mean an affine group scheme of finite type over k.

Definition 3.2.2. Let $S \in \mathsf{Sch}_{/k}$ and $\mathcal{P}$ be a sheaf on $(\mathsf{Sch}_{/S})_{\text{fpqc}}$ (which corresponds to a sheaf on $(\mathsf{Sch}_{/k})_{\text{fpqc}}$ equipped with a morphism to S). We say that $\mathcal{P}$ is a right (left) G-bundle over S if it is a right (left) $G\big|_S$-torsor (recall that if $H \in \mathsf{Grp}$, an H-torsor is a set on which H acts simply transitively).

Of course the definition can be given in the more general case for $\mathcal{G}$ a sheaf of groups.

Proposition 3.2.3. Let $\mathcal{P}$ be a right G-bundle over S. It is then representable by an affine scheme over S and it is fppf-locally trivial. If G is smooth, then it is also étale-locally trivial.

The idea is that since affine morphisms are effective for fpqc descent (i.e. the fibered category $\mathsf{Aff}/S \to \mathsf{Sch}/S$ is a stack for fpqc topology, see [Vis08, Thm 4.33]), using the fpqc local trivializations of $\mathcal{P} \to S$ (by definition there exists an fpqc covering $\{U_i \to S\}_i$ such that $\mathcal{P}_{U_i} \cong U_i \times G_{U_i}$, and the isomorphism over $U_{i,j}$ will determine a cocycle $g_{i,j}$) we deduce that $\mathcal{P}$ is representable by a scheme affine over S. Since $G \to \operatorname{Spec} k$ is fppf we can also say $\mathcal{P}$ is locally trivial in the fppf topology.

Let's observe that, in general, for a group sheaf $\mathcal{G}$ on a site $\mathcal{C}$ and $S \in \mathcal{C}$, we have an isomorphism of sheaves $\mathcal{G}\big|_S \simeq \operatorname{Isom}(\mathcal{G},\mathcal{G})$ where the right hand side corresponds to right $\mathcal{G}$-equivariant morphisms ($\mathcal{G}$ acting by right multiplication on itself). This isomorhism is given by

$$\mathcal{G}(S) \ni g \mapsto \ell_g$$

where ℓ_g is the left multiplication by g.

Let's recall the classical definition of algebraic spaces, given, for example, at [Stacks, Definition 025Y].

Definition 3.2.4. A pseudofunctor $F\colon (\mathsf{Sch}_S)^{\text{op}} \to \mathsf{Grpd}$ is an *algebraic space* if it is an fppf sheaf of sets, the diagonal $F \to F \times_S F$ is representable (by a scheme) and it admits an étale surjective atlas $U \to F$, for U a scheme.

Definition 3.2.5. A morphism of pseudofunctors $\mathcal{X} \to \mathcal{Y}$ is *representable* (resp. *schematic*) if for any scheme S mapping to $\mathcal{Y}$, the 2-fibered product $\mathcal{X} \times_{\mathcal{Y}} S$ is isomorphic to an algebraic space (resp. a scheme).

We will talk about a specific class of stacks, the algebraic stacks. The definition we use is the following (it requires some more condition than the one stated before).

Definition 3.2.6. An *algebraic stack* $\mathcal{X}$ over a scheme S is a stack in groupoids on $(\mathsf{Sch}_{/S})_{\text{fppf}}$ such that the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and there exists a scheme U with a smooth surjective morphism $U \to \mathcal{X}$ (a smooth atlas).

We will study some properties of quotient stacks and Hom-stacks, to then finally use them together to prove some properties of the stack of G-bundles on $X \to S$ (which we will assume to have some nice properties), which will be defined later on.

3.3 Quotient stacks

Fix a k-scheme Z with a right G-action $\alpha\colon Z \times G \to Z$ (satisfying all the properties we expect a right action to satisfy).

3.3.1 Definition and first properties

Definition 3.3.1. The pseudofunctor $[Z/G]\colon (\mathsf{Sch}_{/k})^{\text{op}} \to \mathsf{Grpd}$ is defined by

$$[Z/G](S) = \{\text{Right } G\text{-bundles } \mathcal{P} \to S \text{ endowed with a } G\text{-equivariant morphism } \mathcal{P} \to Z\}\,.$$

A morphism from $\mathcal{P} \to Z$ to $\mathcal{P}' \to Z$ is simply a G-equivariant morphism $\mathcal{P} \to \mathcal{P}'$ over $S \times Z$. For $* = \operatorname{Spec} k$ endowed with the trivial action, we call $BG = [*/G]$.

Seeing schemes as fpqc sheaves, we notice immediately that $[Z/G]$ is an fpqc stack. The main result of this section will be that it is an algebraic stack.

Definition 3.3.2. Let $\mathcal{Y}$ be a k-stack and $\sigma_0\colon Z \to \mathcal{Y}$ a morphism of stacks. Then σ_0 is G-invariant if it satisfies the following conditions.

(1) The diagram

$$\begin{array}{ccc} Z \times G & \xrightarrow{\alpha} & Z \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \sigma_0} \\ Z & \xrightarrow{\sigma_0} & \mathcal{Y} \end{array}$$

is 2-commutative, i.e. there exists a 2-isomorphism $\rho\colon \mathrm{pr}_1^* \sigma_0 \to \alpha^* \sigma_0$ (here we use a notation inspired to fibered categories, simply $\mathrm{pr}_1^* \sigma_0$ means $\sigma_0 \circ \mathrm{pr}_1$).

(2) For a scheme S, given $z \in Z(S)$ and $g \in G(S)$, let $\rho_{z,g}$ denote the corresponding 2-morphism

$$z^*\sigma_0 \simeq (z,g)^* \mathrm{pr}_1^* \sigma_0 \xrightarrow{(z,g)^*\rho} (z,g)^*\alpha^*\sigma_0 \simeq (z.g)^*\sigma_0.$$

Then these 2-morphisms must satisfy the natural associativity condition, namely for $g_1, g_2 \in G(S)$, we require

$$\begin{array}{ccc} z^*\sigma_0 & \xrightarrow{\rho_{z,g_1}} & (z.g_1)^*\sigma_0 \\ \downarrow{\scriptstyle \rho_{z,g_1g_2}} & & \downarrow{\scriptstyle \rho_{z.g_1,g_2}} \\ (z.(g_1g_2))^*\sigma_0 & = & ((z.g_1).g_2)^*\sigma_0 \end{array}$$

to (1)-commute.

Let's consider a 2-cartesian diagram

$$\begin{array}{ccc} Z \times_{\mathcal{Y}} S & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle \sigma_0} \\ S & \longrightarrow & \mathcal{Y} \end{array}$$

with σ_0 G-invariant and S a scheme. Then $Z \times_{\mathcal{Y}} S$ is a sheaf of sets and it comes equipped with a (unique) right G-action making $Z \times_{\mathcal{Y}} S \to S$ G-invariant and $Z \times_{\mathcal{Y}} S \to Z$ G-equivariant. Let's recall that the 2-fibered product is built, for a scheme T, setting

$$(Z \times_{\mathcal{Y}} S)(T) = \left\{ (a, b, \phi) \mid a \in Z(T), b \in S(T), \phi\colon a^*\sigma_0 \xrightarrow{\sim} b^*\sigma_0 \right\}$$

so that the action is given by

$$(a, b, \phi).g := (a.g, b, \phi \circ \rho_{a,g}^{-1})$$

for any $g \in G(T)$. Since $\rho_{a.g_1,g_2} \circ \rho_{a,g_1} = \rho_{a,g_1g_2}$ this defines a natural G-action.

Definition 3.3.3. Let $\mathcal{Y}$ be a k-stack and $\sigma_0\colon Z \to \mathcal{Y}$ a G-invariant morphism of stacks. We say σ_0 is a *G-bundle* if for any $\mathsf{Sch} \ni S \to \mathcal{Y}$ the induced G-action on $Z \times_{\mathcal{Y}} S \to S$ gives a (classical) G-bundle. This implies that σ_0 is schematic.

We will need lot of lemmas, hold tight. Observe that we have a trivial element $\tau_0 \in [Z/G](Z)$, corresponding to the trivial bundle $\mathrm{pr}_1\colon Z \times G \to Z$ equipped with the G-equivariant morphism $\alpha\colon Z \times G \to G$.

Lemma 3.3.4. The diagram

$$\begin{array}{ccc} Z \times G & \xrightarrow{\alpha} & Z \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \tau_0} \\ Z & \xrightarrow{\tau_0} & [Z/G] \end{array}$$

is 2-cartesian, and τ_0 is a G-equivariant morphism.

Proof. Observe that $\tau_0(S)\colon Z(S) \to [Z/G](S)$ sends $f\colon S \to Z$ to the trivial bundle $f^*\tau_0 = \alpha \circ (f \times \mathrm{id})\colon S \times G \to Z$. Suppose that for a scheme S we have $z, z' \in Z(S)$ and an isomorphism $\phi\colon z'^*\tau_0 \xrightarrow{\sim} z^*\tau_0$. Explicitely, ϕ corresponds to an element $g \in G(S)$ such that

$$\begin{array}{ccc} S \times G & \xrightarrow{\quad r_g \quad} & S \times G \\ & {\scriptstyle \alpha(z' \times \mathrm{id})} \searrow \quad \swarrow {\scriptstyle \alpha(z \times \mathrm{id})} & \\ & Z & \end{array}$$

commutes (over S). This basically means $z.g = z'$. So we associate $(z, g) \in (Z \times G)(S)$ to the point (z, z', ϕ) of the 2-fibered product. Conversely, given $z \in Z(S)$ and $g \in G(S)$ they uniquely determine $z' = z.g$ and a G-equivariant isomorphism $z'^* \tau_0 \to z^* \tau_0$. This proves that the map

$$(\alpha, \mathrm{pr}_1) \colon Z \times G \to Z \times_{[Z/G]} Z$$

is indeed an isomorphism.

Let now $S = Z \times G$ and consider $\mathrm{id}_{Z \times G}$. We obtain, by definition of 2-fibered product, an isomorphism

$$\rho^{-1} \colon \alpha^* \tau_0 \to \mathrm{pr}_1^* \tau_0$$

which is explicitely defined by $(z, g_1, g_2) \mapsto (z, g_1, g_1 g_2)$. Therefore for a scheme S, $z \in Z(S)$ and $g \in G(S)$, the morphism $\rho_{z,g}$ corresponds (using again Yoneda and the definition of τ_0) to the morphism of schemes $S \times G \to S \times G$ given by $(s, g_0) \mapsto (s, g(s)^{-1} g_0)$. Using $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ we deduce that τ_0 is G-invariant. ■

Lemma 3.3.5. Let $\tau = (f \colon \mathcal{P} \to Z) \in [Z/G](S)$ for a scheme S and suppose that $\mathcal{P}$ admits a section $s \colon S \to \mathcal{P}$. Then the G-equivariant morphism $\widetilde{s} \colon S \times G \to \mathcal{P}$ induced by s gives an isomorphism $(f \circ s)^* \tau_0 \to \tau \in [Z/G](S)$.

Proof. Call $a = f \circ s \colon S \to \mathcal{P} \to Z$. We have a cartesian diagram

$$\begin{array}{ccccc} S \times G & \xrightarrow{a \times \mathrm{id}} & Z \times G & \xrightarrow{\ \alpha\ } & Z \\ \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1} & & \\ S & \xrightarrow{\quad a \quad} & Z & & \end{array}$$

so that, as observed in the preceding proof, $a^* \tau_0 = (\alpha(a \times \mathrm{id}) \colon S \times G \to Z)$. The diagram

$$\begin{array}{ccc} S \times G & \xrightarrow{\quad \widetilde{s} \quad} & \mathcal{P} \\ & {\scriptstyle \alpha(a \times \mathrm{id})} \searrow \quad \swarrow {\scriptstyle f} & \\ & Z & \end{array}$$

is commutative by G-equivariance. This proves $\widetilde{s}$ is a morphism in $[Z/G](S)$, inducing the claimed isomorphism. ■

Lemma 3.3.6. Let $\mathcal{Y}$ be a k-stack and $\sigma_0 \colon Z \to \mathcal{Y}$ a G-bundle. Then there exists an isomorphism $\mathcal{Y} \to [Z/G]$ of stacks making the following triangle

$$\begin{array}{ccc} & Z & \\ & {\scriptstyle \sigma_0} \swarrow \quad \searrow {\scriptstyle \alpha \colon Z \times G \to Z} & \\ \mathcal{Y} & \xrightarrow{\quad \sim \quad} & [Z/G] \end{array}$$

2-commutative.

Remark 3.3.7. Observe how we are already using the 2-Yoneda lemma. The statement above implies that, in the particular situation where $\mathcal{Y} = Y$ is a scheme, then $Y \simeq [Z/G]$, i.e. the two notions of quotients, as scheme and as stack, coincide.

Proof. First let's define the morphism $F \colon \mathcal{Y} \to [Z/G]$ by sending $\sigma \in \mathcal{Y}(S)$ to

$$\begin{array}{ccc} F(\sigma) \coloneqq Z \times_{\mathcal{Y}} S & \longrightarrow & Z \\ \downarrow & & \downarrow {\scriptstyle \sigma_0} \\ S & \xrightarrow{\quad \sigma \quad} & \mathcal{Y} \end{array}$$

for any scheme S. As said before, since σ_0 is a G-bundle, $Z \times_{\mathcal{Y}} S \to Z$ is an object of $[Z/G](S)$. Let now $\sigma, \sigma' \in \mathcal{Y}(S)$ and write $\mathcal{P} = Z \times_{\mathcal{Y},\sigma} S$ and $\mathcal{P}' = Z \times_{\mathcal{Y},\sigma'} S$. We write, as before, $\mathcal{P}(T) = \{(a, b, \phi)\}$ with $a \in Z(T), b \in S(T)$ and $\phi\colon a^*\sigma_0 \xrightarrow{\sim} b^*\sigma$. A morphism $\psi\colon \sigma \to \sigma'$ in $\mathcal{Y}(S)$ induces a G-equivariant morphism $\mathcal{P} \to \mathcal{P}'$ over $S \times Z$ by

$$(a, b, \phi) \mapsto (a, b, b^*\psi \circ \phi). \tag{3.1}$$

This proves that F is indeed a morphism of stacks. Our plan now is the following: we first prove the triangle is 2-commutative and then that F is fully faithful and essentially surjective (pointwise, see [Vis08, p. 3.5.2]) to deduce it is an isomorphism.

2-commutativity:

We must give an isomorhism $\tau_0 \to F(\sigma_0)$ in $[Z/G](Z)$. We have already seen that $(\alpha, \mathrm{pr}_1)\colon Z \times G \to Z \times_{\mathcal{Y}} Z$ is a G-equivariant isomorphism of sheaves over $Z \times Z$, and this is the searched map.

Fully faithfulness:

To prove F is fully faithful we can prove, since $\mathcal{Y}$ and $[Z/G]$ are stacks, that for any scheme S and $\sigma, \sigma' \in \mathcal{Y}(S)$ the induced morphism of sheaves of sets

$$F\colon \mathrm{Isom}_{\mathcal{Y}(S)}(\sigma, \sigma') \to \mathrm{Isom}_{[Z/G](S)}(F(\sigma), F(\sigma'))$$

is an isomorphism. Let $\mathcal{P}, \mathcal{P}'$ be as before. We can choose an fppf covering $\{S_i \to S\}_i$ trivializing both G-bundles, and since the Hom above are sheaves, it suffices to study this "trivial" case. Let $\psi\colon \mathcal{P} \to \mathcal{P}'$ be a G-equivariant morphism over $S \times Z$. Since $\mathcal{P}$ is trivial, there exists a section $s \in \mathcal{P}(S)$, corresponding to $(a, \mathrm{id}_S, \phi\colon a^*\sigma_0 \xrightarrow{\sim} \sigma)$ for $a \in Z(S)$. The map ψ sends s to some element $(a, \mathrm{id}_S, \phi'\colon a^*\sigma_0 \xrightarrow{\sim} \sigma) \in \mathcal{P}'(S)$, which is a section of $\mathcal{P}'$. Define the following morphism of sets

$$L\colon \mathrm{Hom}_{[Z/G](S)}(F(\sigma), F(\sigma')) \to \mathrm{Hom}_{\mathcal{Y}(S)}(\sigma, \sigma')$$

by $\psi \mapsto \phi' \circ \phi^{-1}\colon \sigma \to \sigma'$. Now we just need to check F (on the hom sets) and L are inverse to each other. Starting with $\psi\colon \sigma \simeq \sigma'$, we know $F(\psi)(s) = (a, \mathrm{id}_S, \psi \circ \phi) \in \mathcal{P}(S)$ by Eq. (3.1). Thus $LF(\psi) = (\psi \circ \phi) \circ \phi^{-1} = \psi$ so $LF = \mathrm{id}$. Starting instead with $\Psi\colon \mathcal{P} \to \mathcal{P}'$ then $L(\Psi) = \phi' \circ \phi^{-1}$ and hence

$$FL(\Psi)\colon s \mapsto (a, \mathrm{id}_S, (\phi' \circ \phi^{-1}) \circ \phi) \in \mathcal{P}'(S).$$

Since a G-equivariant morphism of trivial bundles $\mathcal{P} \to \mathcal{P}'$ is determined by the image of a single section $s \in \mathcal{P}(S)$ we deduce $\Psi = LF(\Psi)$. We conclude that F is fully faithful.

Essential surjectivity:

Finally we'll prove that F is essentially surjective and hence an isomorphism of stacks. Let $\tau = (f\colon \mathcal{P} \to Z) \in [Z/G](S)$ and let $\{j_i\colon S_i \to S\}_i$ be an fppf covering trivializing $\mathcal{P}$, so that we have sections $s_i \in \mathcal{P}(S_i)$. Write f_i the restriction of f to $\mathcal{P}\big|_{S_i}$. By Lemma 3.3.5 we have isomorphisms $(f_i \circ s_i)^*\tau_0 \simeq j_i^*\tau$. From the 2-commutativity of the triangle proved above, we already have $\tau_0 \simeq F(\sigma_0)$. Therefore

$$F((f_i \circ s_i)^*\sigma_0) \simeq (f_i \circ s_i)^* F(\sigma_0) \simeq (f_i \circ s_i)^*\tau_0 \simeq j_i^*\tau.$$

and we conclude that F is an isomorphism using [Stacks, Lemma 046N]. ■

3.3.2 Twisting by a torsor

Let's now introduce a useful construction, the twist by a torsor. Let's place ourselves in the most general setting, where $\mathcal{C}$ is a subcanonical site with a terminal object and a sheaf of groups $\mathcal{G}$. Let $S \in \mathcal{C}$ and let $\mathcal{P}$ be a right $\mathcal{G}\big|_S$-torsor over S. Let now $\mathcal{F}$ be a sheaf of sets on $\mathcal{C}$ endowed with a left $\mathcal{G}$-action. Then $\mathcal{G}\big|_S$ acts on the right on $\mathcal{P} \times \mathcal{F}$ by $(p, z).g := (p.g, g^{-1}.z)$. We build the presheaf $\mathcal{Q}$ on $\mathcal{C}/S$ by $\mathcal{Q}(U) = (\mathcal{P}(U) \times \mathcal{F}(U))/\mathcal{G}(U)$. Then we define

$$_{\mathcal{P}}\mathcal{F} = (\mathcal{P} \times \mathcal{F})/\mathcal{G} = \mathcal{P} \overset{\mathcal{G}}{\times} \mathcal{F}$$

to be the sheafification of $\mathcal{Q}$ and we call it the *twist of* $\mathcal{F}$ *by* $\mathcal{P}$. Since sheaves on $\mathcal{C}$ form a stack ([Vis08, p. 3.2]) we can describe $_{\mathcal{P}}\mathcal{F}$ by giving a descent datum. Let $\{S_i \to S\}$ be a trivializing fppf cover of $\mathcal{P}$. We then have a descent datum of $\mathcal{P}$ giving by $(\mathcal{G}\big|_{S_i}, g_{i,j})$ for some cocycle $g_{i,j} \in \mathcal{G}(S_{i,j})$. Observe that

$$(\mathcal{P}\big|_{S_i} \times \mathcal{F})/\mathcal{G} \simeq (\mathcal{G}\big|_{S_i} \times \mathcal{F})/\mathcal{G} \simeq \mathcal{F}\big|_{S_i},$$

and $\mathcal{Q}\big|_{S_i} \simeq \mathcal{F}\big|_{S_i}$ is already a sheaf on $\mathcal{C}/S_i$. By the definition of the group action of $\mathcal{P} \times \mathcal{F}$ we see that the transition maps

$$\varphi_{i,j} \colon \left(\mathcal{F}\big|_{S_j}\right)\Big|_{S_{i,j}} \to \left(\mathcal{F}\big|_{S_i}\right)\Big|_{S_{i,j}}$$

are given by left multiplication of $g_{i,j}$. Since sheafification commutes with the restrictions $\mathcal{C}/S_i \to \mathcal{C}/S$ we obtain that

$$\left(\mathcal{F}\big|_{S_i}, \varphi_{i,j}\right)$$

is a descent datum for ${}_{\mathcal{P}}\mathcal{F}$.

3.3.3 Change of space

Let $\beta \colon Z' \to Z$ be a G-equivariant morphism of schemes endowed with a right G-action. Then there is a natural morphism of stacks $[Z'/G] \to [Z/G]$ defined by

$$(\mathcal{P} \to Z') \mapsto (\mathcal{P} \to Z' \xrightarrow{\beta} Z).$$

We will prove that, under certain assumptions, this morphism is schematic. First, a technical lemma.

Lemma 3.3.8. Let $\beta_i \colon Z_i \to Z$ be G-equivariant morphisms of schemes for $i = 1, 2$. Then the square

$$\begin{array}{ccc} [(Z_1 \times_Z Z_2)/G] & \longrightarrow & [Z_1/G] \\ \downarrow & & \downarrow \\ [Z_2/G] & \longrightarrow & [Z/G] \end{array}$$

is 2-cartesian (the maps are induced by the natural projections $\mathrm{pr}_i \colon Z_1 \times_Z Z_2 \to Z_i$).

Proof. See [Wan11, Lemma 2.3.2]. ■

The key for the next proof will be the following technical lemma, which roughly says that algebraic spaces are stable under fppf descent.

Lemma 3.3.9. Let S be a scheme and $F \colon (\mathsf{Sch}_{/S})^{\mathrm{op}}_{\mathrm{fppf}} \to \mathsf{Set}$ be a functor. Let $\{S_i \to S\}_i$ be a covering of $(\mathsf{Sch}_{/S})_{\mathrm{fppf}}$. Assume that

(1) F is a sheaf,

(2) each $F_i = h_{S_i \to S} \times F$ is an algebraic space,

(3) $\coprod_{i \in I} F_i$ is an algebraic space.

Then F is an algebraic space.

Proof. See [Stacks, Lemma 04Sk]. ■

Lemma 3.3.10. The morphism $[Z'/G] \to [Z/G]$ is representable. If the morphism of schemes $Z' \to Z$ is affine (resp. quasi-projective with a G-equivariant relatively ample invertible sheaf), then the morphism of quotient stacks is schematic and affine (resp. quasi-projective). Any fppf target-local property of $Z' \to Z$ is inherited by the map of quotient stacks.

Proof. Let S be a scheme and $(f \colon \mathcal{P} \to Z) \in [Z/G](S)$. We know $S \simeq [\mathcal{P}/G]$ thanks to Remark 3.3.7. By Lemma 3.3.8 we have a cartesian square

$$\begin{array}{ccc} [(Z' \times_Z \mathcal{P})/G] & \longrightarrow & [Z'/G] \\ \downarrow & & \downarrow \\ S & \xrightarrow{f \colon \mathcal{P} \to Z} & [Z/G] \end{array}$$

and we must prove that $[(Z' \times_Z \mathcal{P})/G]$ is representable by an algebraic space. We will verify the assumptions of Lemma 3.3.9 to conclude. Since any category equivalent to a set is an equivalence relation, i.e. a groupoid with no automorphisms (see [Vis08, p. 3.5.3]), the stack isomorphism $S \simeq [\mathcal{P}/G]$ implies that $[\mathcal{P}/G](T)$ is an equivalence relation for any scheme T. Therefore $[(Z' \times_Z \mathcal{P})/G](T)$ is also an equivalence relation, so it is isomorphic to a sheaf of sets, verifying the assumption (1).

Choose an fppf covering $\{S_i \to S\}_i$ trivializing $\mathcal{P}$ and let $s_i \in \mathcal{P}(S_i)$ be sections. Using Lemma 3.3.8 we have the 2-cartesian diagram

$$\begin{array}{ccc} [Z' \times_Z \mathcal{P}_{S_i}/G] & \longrightarrow & [Z' \times_Z \mathcal{P}/G] \\ \downarrow & & \downarrow \\ S_i & \longrightarrow & S \end{array}$$

and we claim that $[Z' \times_Z \mathcal{P}_{S_i}/G]$ is represented by the scheme $Z' \times_{Z, f\circ s_i} S_i$ (varying i they form a covering of Z', by the axioms of Grothendieck topologies). We have a morphism $S_i \times G \xrightarrow{\sim} \mathcal{P}_{S_i} \to Z$ given by $f \circ \widetilde{s_i} = f \circ \alpha \circ (s_i \times \mathrm{id}_G)$. There is a G-equivariant isomorphism

$$\gamma_i \colon (Z' \times_{Z, f\circ s_i} S_i) \times G \to Z' \times_Z (S_i \times G)$$

over S_i, where on the left G acts as a trivial bundle, while on the right G acts both on Z' and $S_i \times G$. More precisely, given $a \in Z'(T)$, $g \in G(T)$ for T an S_i-scheme, we have $\gamma_i(a, g) = (a.g, g)$ (we can verify that γ_i, naturally defined as $(\alpha'(\mathrm{pr}_{Z'} \times \mathrm{id}_G), \mathrm{pr}_{S_i} \times \mathrm{id}_G)$ has this form and is indeed a G-equivariant isomorphism). Applying Remark 3.3.7 we have an isomorhism

$$\Phi_i \colon Z' \times_{Z, f\circ s_i} S_i \simeq [(Z' \times_{Z, f\circ s_i} S_i) \times G/G] \xrightarrow{\gamma_i} [(Z' \times_Z (S_i \times G))/G] \xrightarrow{\mathrm{id} \times \widetilde{s_i}} [(Z' \times_Z \mathcal{P}_{S_i})/G]$$

over S_i, defined explicitely on T-points by sending $a \in (Z' \times_{Z, f\circ s_i} S_i)(T)$ to

$$\begin{array}{ccccccc} T \times G & \xrightarrow{a \times \mathrm{id}} & (Z' \times_{Z, f\circ s_i} S_i) \times G & \xrightarrow{\gamma_i} & Z' \times_Z (S_i \times G) & \xrightarrow{\mathrm{id} \times \widetilde{s_i}} & Z' \times_Z \mathcal{P} \\ \downarrow & & & & & & \\ T & & & & & & \end{array}$$

After restricting to the chosen covering, $[(Z' \times_Z \mathcal{P})/G] \times_S S_i \simeq [(Z' \times_Z \mathcal{P}_{S_i})/G]$ becomes representable by a scheme (using the isomorphism Φ_i above). Then using Lemma 3.3.9 we deduce that $[(Z' \times_Z \mathcal{P})/G]$ is representable by an algebraic space. Let's take a break and try to describe its descent datum, which will be useful in the future.

Descent datum:

Denoting $S_{i,j} = S_i \times_S S_j$ we have a cocycle $g_{i,j} \in G(S_{i,j})$ such that $s_j = s_i.g_{i,j} \in \mathcal{P}(S_{i,j})$. The action of $g_{i,j}^{-1}$ on Z' induces an isomorphism

$$\varphi_{i,j} \colon Z' \times_{Z, f\circ s_j} S_{i,j} \to Z' \times_{Z, (f\circ s_j).g_{i,j}^{-1}} S_{i,j} = Z' \times_{Z, f\circ s_i} S_{i,j}.$$

Given $a \in (Z' \times_{Z, f\circ s_j} S_{i,j})(T)$, the square

$$\begin{array}{ccc} T \times G & \xrightarrow{\Phi_j(a) = (\mathrm{id} \times \widetilde{s_j}) \circ \gamma_j \circ (a \times \mathrm{id})} & Z' \times_Z \mathcal{P}_{S_{i,j}} \\ \downarrow \ell_{g_{i,j}} & & \| \\ T \times G & \xrightarrow{\Phi_i(\varphi_{i,j}(a)) = (\mathrm{id} \times \widetilde{s_i}) \circ \gamma_i \circ (\varphi_{i,j}(a) \times \mathrm{id})} & Z' \times_Z \mathcal{P}_{S_{i,j}} \end{array}$$

is commutative (easy computation). Therefore $\Phi_j(a)$ and $(\Phi_i \circ \varphi_{i,j})(a)$ are isomorphic in $[(Z' \times_Z \mathcal{P}_{S_{i,j}})/G]$. We can thus conclude that $(Z' \times_{Z, f\circ s_i} S_i, \varphi_{i,j})$ is a descent datum of $[(Z' \times_Z \mathcal{P})/G]$ with respect to the chosen covering $\{S_i \to S\}_i$.

If $Z' \to Z$ is affine, then by base change so is $Z' \times_{Z, f\circ s_i} S_i \to S_i$. Affine morphisms are effective under descent (see [Vis08, p. 4.33]) so we deduce that $[(Z' \times_Z \mathcal{P})/G]$ is representable by a scheme affine over S.

Suppose now $Z' \to Z$ is quasi-projective with a G-equivariant relatively ample invertible $\mathcal{O}_{Z'}$-module $\mathcal{L}$. From the description of the descent datum we have a commutative square

$$\begin{array}{ccc} Z' \times_{Z, f\circ s_j} S_{i,j} & \xrightarrow{\mathrm{pr}_{1,j}} & Z' \\ \downarrow{\scriptstyle \varphi_{i,j}} & & \downarrow{\scriptstyle g_{i,j}^{-1}} \\ Z' \times_{Z, f\circ s_i} S_{i,j} & \xrightarrow{\mathrm{pr}_{1,i}} & Z' \end{array}$$

and the G-equivariant structure of $\mathcal{L}$ gives an isomorphism

$$\varphi_{i,j}^* \operatorname{pr}_{1,i}^* \mathcal{L} \simeq \operatorname{pr}_{1,j}^* \mathcal{L}$$

of invertible sheaves relatively ample over $S_{i,j}$. Since the $(g_{i,j})$ are a cocycle, the associativity property of G-equivariance implies that the above isomorphisms also satisfy the corresponding cocycle condition. By descent ([SGAI, VIII, Proposition 7.8]) we can then conclude that $[(Z' \times_Z \mathcal{P})/G]$ is representable by a quasi-projective scheme over S. ■

Corollary 3.3.10.1. For a k-scheme Z with a right G-action and a right G-bundle $\mathcal{P}$ over a k-scheme S, there is an isomorphism ${}_{\mathcal{P}}Z \simeq [(Z \times \mathcal{P})/G]$ over S.

Proof. From the proof of Lemma 3.3.10 we know that $[(Z \times \mathcal{P})/G]$ has a descent datum $(Z \times S_{i,j}, \varphi_{i,j})$. This is the same descent datum as ${}_{\mathcal{P}}Z$, as explained in Section 3.3.2. This also implies that

$$\begin{array}{ccc} {}_{\mathcal{P}}Z & \longrightarrow & [Z/G] \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathcal{P}} & BG \end{array}$$

is cartesian. ■

Corollary 3.3.10.2. Given $\tau = (p\colon \mathcal{P} \to S, f\colon \mathcal{P} \to Z) \in [Z/G](S)$ for a k-scheme S,

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tau_0} \\ S & \xrightarrow{\tau} & [Z/G] \end{array}$$

is cartesian. In particular, $\tau_0\colon Z \to [Z/G]$ is schematic, affine and fppf.

Proof. Using Yoneda and Lemma 3.3.8 we have a cartesian square

$$\begin{array}{ccc} [((Z \times G) \times_{\alpha, Z, f} \mathcal{P})/G] & \longrightarrow & [Z \times G/G] \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ [\mathcal{P}/G] & \longrightarrow & [Z/G] \end{array}$$

where $Z \times G$ is the trivial bundle over Z. There is a G-equivariant morphism

$$(f \times \mathrm{id}, \alpha_{\mathcal{P}})\colon \mathcal{P} \times G \to (Z \times G) \times_{\alpha, Z, f} \mathcal{P}$$

where, again, $\mathcal{P} \times G$ is the trivial bundle over $\mathcal{P}$. Action of G and projection induce a G-equivariant map $\mathcal{P} \times G \to \mathcal{P} \times_S \mathcal{P}$ over $\mathcal{P} \times \mathcal{P}$. Therefore the following diagram

$$\begin{array}{ccccc} S & \xleftarrow{p} & \mathcal{P} & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{P}}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{P}\times G}} & & \downarrow{\scriptstyle \mathrm{id}_{Z\times G}} \\ & & [\mathcal{P} \times G/G] & & \\ & & \downarrow{\scriptstyle (f\times \mathrm{id}, \alpha_{\mathcal{P}})} & & \\ [\mathcal{P}/G] & \longleftarrow & [((Z \times G) \times_{\alpha, Z, f} \mathcal{P})/G] & \longrightarrow & [Z \times G/G] \end{array}$$

is 2-commutative. Hence, applying Remark 3.3.7, we deduce that the initial cartesian square is isomorphic to the desired one. Moreover, we see that the morphism $f^*\tau_0 \to p^*\tau$ is defined by the action and first projection morphism. ■

3.3.4 Change of group

Let $H \hookrightarrow G$ be a closed subgroup of G and let's investigate the relation between BH and BG.

Let's first consider the action by left multiplication of H on G, giving rise to the fppf sheaf of right cosets $H\backslash G$. By [DG70, Thm 5.4] the sheaf $H\backslash G$ is representable by a quasi-projective k-scheme with a G-equivariant ample invertible sheaf. The map $H \times G \to G \times_{H\backslash G} G$, given by the multiplication and the projection, is an isomorphism by [DG70, p. 2.4], i.e. the following square

$$\begin{array}{ccc} H \times G & \xrightarrow{\mu} & G \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \pi} \\ G & \longrightarrow & H\backslash G \end{array}$$

is cartesian. Therefore the projection $\pi\colon G \to H\backslash G$ is a left H-bundle. All the previous discussion, where we used right actions, can be rewritten using left actions; we will use the notation $[H\backslash G]$ to denote the quotient stack of G by the left action of H, sending S to the groupoid of left H-bundles $\mathcal{P} \to S$ equipped with an H-invariant morphism to G. Observe that Remark 3.3.7 implies that id_G induces an isomorphism $H\backslash G \xrightarrow{\sim} [H\backslash G]$ defined on a scheme S by

$$\begin{array}{ccccc} G \times_{H\backslash G} G & \dashrightarrow & G & \xrightarrow{\mathrm{id}_G} & G \\ \vdots & & \downarrow{\scriptstyle \pi} & & \\ S & \longrightarrow & H\backslash G & & \end{array}$$

Lemma 3.3.11. The left H-bundle $G \to H\backslash G$ defines a right G-bundle $H\backslash G \to [H\backslash *]$.

Proof. Let $\mu\colon G \times G \to G$ be the multiplication map and consider $(\mathrm{pr}_1, \mu)\colon G \times G \to G \times G$ where H acts on the first coordinate on the lhs and diagonally on the rhs, so that this morphism is H-equivariant. We have 2-commutative squares

$$\begin{array}{ccccc} H\backslash G & \xleftarrow{\mathrm{pr}_1} & H\backslash G \times G & \xrightarrow{\overline{\mu}} & H\backslash G \\ \downarrow{\scriptstyle \mathrm{id}_G} & & \downarrow{\scriptstyle (\mathrm{pr}_1,\mu)} & & \downarrow{\scriptstyle \mathrm{id}_G} \\ [H\backslash G] & \xleftarrow{\mathrm{pr}_1} & [H\backslash (G \times G)] & \xrightarrow{\mathrm{pr}_2} & [H\backslash G] \end{array}$$

where the 2-isomorphisms are identities (and the action of H on $G \times G$ in the middle top is on the left component). Thus we have a 2-commutative diagram

$$\begin{array}{ccc} H\backslash G \times G & \xrightarrow{\overline{\mu}} & H\backslash G \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ H\backslash G & \longrightarrow & [H\backslash *] \end{array}$$

with 2-morphism $\mathrm{id}_{G\times G}$, which gives $H\backslash G \to [H\backslash *]$ the structure of a right G-invariant morphism.

Let $\mathcal{P} \in [H\backslash *](S)$ be a left H-bundle over S and let $\{S_i \to S\}_i$ be an fppf covering trivializing $\mathcal{P}$. We will conclude by considering the cartesian square

$$\begin{array}{ccc} H\backslash G \times_{[H\backslash *]} S & \longrightarrow & H\backslash G \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathcal{P}} & [H\backslash *] \end{array}$$

and proving that the leftmost arrow is a G-bundle (S is a scheme obviously). Applying the same exact argument as in the proof of Lemma 3.3.10 we obtain the cartesian square

$$\begin{array}{ccc} S_i \times G & \longrightarrow & [H\backslash G] \\ \big\downarrow & & \big\downarrow \\ S_i & \xrightarrow{H \times S_i} & [H\backslash *] \end{array}$$

where the top arrow corresponds to $H \times S_i \times G \to G\colon (h, a, g) \mapsto hg$. The morphism $S_i \times G \to [H\backslash G] \simeq H\backslash G$ equals $\pi \circ \mathrm{pr}_2$ (easy computation) , which is G-equivariant. This means that

$$S_i \times G \to H\backslash G \times_{[H\backslash *], H\times S_i} S_i \simeq (H\backslash G \times_{[H\backslash *],\mathcal{P}} S) \times_S S_i$$

is a G-equivariant isomorphism. Thus $H\backslash G \times_{[H\backslash *],\mathcal{P}} S$ is a right G-bundle, and this suffices to prove that $H\backslash G \to [H\backslash *]$ is also a right G-bundle. ■

Let's now observe that combining Lemma 3.3.6 and Lemma 3.3.11 we have an isomorhism $[H\backslash *] \simeq [(H\backslash G)/G]$ sending a left H-bundle $\mathcal{P}$ over S to $H\backslash G \times_{[H\backslash *],\mathcal{P}} S \to H\backslash G$. Using inverse action we pass to right H-bundles, obtaining $BH \simeq [H\backslash *]$. Applying Corollary 3.3.10.1 we observe that the map $BH \simeq [H\backslash *] \simeq [(H\backslash G)/G] \to BG$ sends a right H-bundle $\mathcal{P}$ to the fiber bundle ${}_{\mathcal{P}}G$, where G is twisted using the left action. We have also a right G-action on such twist, induced bu the right multiplication by G on itself.

Proposition 3.3.12. The morphism $BH \to BG$ sending $\mathcal{P} \mapsto {}_{\mathcal{P}}G$ is schematic, finitely presented and quasi-projective.

Proof. Consider the map $[(H\backslash G)/G] \to BG$: by Corollary 3.3.10.1 we have a 2-cartesian square

$$\begin{array}{ccc} {}_{\mathcal{E}}(H\backslash G) & \longrightarrow & BH \simeq [(H\backslash G)/G] \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow{\mathcal{E}} & BG \end{array}$$

Since $H\backslash G$ is quasi-projective with a G-equivariant ample invertible sheaf over $* = \operatorname{Spec} k$, then, using Lemma 3.3.10 we deduce that ${}_{\mathcal{E}}(H\backslash G)$ is representable by a scheme quasi-projective and of finite presentation over S. This proves the proposition. ■

3.3.5 $[Z/G]$ is algebraic

We'll state the last technical lemmas and then, finally, we will be able to prove that the quotient stack is an algebraic stack, under certain assumptions.

Lemma 3.3.13. For algebraic groups G, G', the morphism $BG \times BG' \to B(G \times G')$ sending a G-bundle $\mathcal{P} \to S$ and a G'-bundle $\mathcal{P}' \to S$ to $\mathcal{P} \times_S \mathcal{P}'$ is an isomorphism.

Proof. Let, as in the statement, $(\mathcal{P}, \mathcal{P}') \in (BG \times BG')(S)$ and choose an fppf covering $\{S_i \to S\}$ trivializing both. This means that we have descent data

$$(S_i \times G, g_{i,j}), \quad (S_i \times G', g'_{i,j})$$

for $g_{i,j} \in G(S_{i,j}), g'_{i,j} \in G'(S_{i,j})$. Then

$$(S_i \times G \times G', (g_{i,j}, g'_{i,j}))$$

is a descent datum for $\mathcal{P} \times_S \mathcal{P}'$. Using the fact that both $BG \times BG'$ and $B(G \times G')$ are stacks, we see that this morphism is an isomorphism, whose inverse is $\mathcal{E} \mapsto ({}_{\mathcal{E}}G, {}_{\mathcal{E}}G')$ where $G \times G'$ acts by projections on G and G'. ■

Let's now study a particular case of our problem, with the simplest quotient stack BG. We will invoke the well known Artin's theorem, which we only state here.

Theorem 3.3.14. Let $\mathcal{X}$ be an S-stack satisfying the three following conditions:

(1) $\mathcal{X}$ is an fppf S-stack;

(2) the diagonal map $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, separated and quasi-compact;

(3) there exists an S-algebraic space Y and a map $Q\colon Y \to \mathcal{X}$ of S-stacks which is representable and fppf.

Then $\mathcal{X}$ is an algebraic stack.

Proof. See [LM00, Thm 10.1]. ■

Lemma 3.3.15. The k-stack BG is an algebraic stack with a schematic, affine diagonal. Specifically, for right G-bundles $\mathcal{P}, \mathcal{P}'$ over S, there is an isomorphism

$$\mathrm{Isom}_{BG(S)}(\mathcal{P}, \mathcal{P}') \simeq {}_{\mathcal{P}\times_S \mathcal{P}'}(G)$$

as sheaves of sets on $\mathsf{Sch}_{/S}$, where $G \times G$ acts on G from the right by $g.(g_1, g_2) = g_1^{-1} g g_2$.

Proof. Let $S \in \mathsf{Sch}_{/k}$ and $\mathcal{P}, \mathcal{P}' \in BG(S)$; we have a 2-cartesian square

$$\begin{array}{ccc}
\mathrm{Isom}_{BG(S)}(\mathcal{P}, \mathcal{P}') & \longrightarrow & BG \simeq [(G\backslash G \times G)/G \times G] \simeq [G/G \times G] \\
\downarrow & & \downarrow \Delta_{BG} \\
S & \xrightarrow{(\mathcal{P}, \mathcal{P}')} & BG \times BG \simeq B(G \times G)
\end{array}$$

where in the upper row we observed that there is an isomorphism $G\backslash(G \times G) \cong G$ given by $(g_1, g_2) \mapsto g_1^{-1} g_2$, where G acts diagonally on the left. Since the lhs has a G-equivariant right $G \times G$-action, the rhs G inherits a right $(G \times G)$-action given by $g.(g_1, g_2) = g_1^{-1} g g_2$, as claimed in the statement. By Proposition 3.3.12 and Lemma 3.3.13 we get an isomorhism $\mathrm{Isom}_{BG(S)}(\mathcal{P}, \mathcal{P}') \simeq {}_{\mathcal{P}\times_S \mathcal{P}'}(G)$ over S.

Since G is affine over k, by assumption, the associated fiber bundle ${}_{\mathcal{P}\times_S \mathcal{P}'}(G)$ is representable by an affine scheme over S. Hence the diagonal Δ_{BG} is schematic and affine. By Corollary 3.3.10.2 the morphism $* \to BG$ is schematic, affine and fppf. We can conclude that BG is an algebraic stack using Theorem 3.3.14. ■

Here is the main result of this whole section.

Theorem 3.3.16. The k-stack $[Z/G]$ is an algebraic stack with a schematic, separated diagonal. If Z is quasi-separated (i.e. the diagonal map is quasi-compact) then the diagonal $\Delta_{[Z/G]}$ is quasi-compact. If Z is separated then $\Delta_{[Z/G]}$ is affine.

Proof. Consider the diagonal map $Z \to Z \times Z$, G-equivariant for the diagonal action of G on the rhs. It induces a map

$$[Z/G] \to [Z \times Z/G] \simeq [Z/G] \times_{BG} [Z/G]$$

which is representable by Lemma 3.3.10, the last isomorphism being the one of Lemma 3.3.8. Now, the diagonal $\Delta_{[Z/G]}$ is obtained by composition

$$[Z/G] \to [Z/G] \times_{BG} [Z/G] \to [Z/G] \times [Z/G]$$

where the last map is coming from the universal property of products. This last map is obtained by base change

$$\begin{array}{ccc}
[Z/G] \times_{BG} [Z/G] & \longrightarrow & [Z/G] \times [Z/G] \\
\downarrow & & \downarrow \\
BG & \xrightarrow{\Delta_{BG}} & BG \times BG
\end{array}$$

and hence it is representable (in particular schematic and affine) thanks to Lemma 3.3.15. This implies that the diagonal map $\Delta_{[Z/G]}$ is representable (this is the first condition to check to prove $[Z/G]$ is an algebraic stack). By Lemma 3.3.15 there exists a scheme U and a smooth surjective morphism $U \to BG$. The change of space $f\colon [Z/G] \to BG$ is representable by Lemma 3.3.10, so $U \times_{BG} [Z/G]$ is representable by an algebraic space. Therefore there exists a scheme V with an étale surjection $V \to U \times_{BG} [Z/G]$; the composition $V \to [Z/G]$ is then the searched smooth atlas, so we can conclude $[Z/G]$ is an algebraic stack.

As observed before, we know that $[Z/G] \times_{BG} [Z/G] \to [Z/G] \times [Z/G]$ is schematic and affine. Since f is representable, then its diagonal $\Delta_f\colon [Z/G] \to [Z/G] \times_{BG} [Z/G]$ (which is exactly the same morphism we used above) is schematic and separated, see [Stacks, Lemma 04YQ]. If Z is quasi-separated, then by descent ${}_{\mathcal{P}}Z \to S$ is quasi-separated for any $\mathcal{P} \in BG(S)$ (easy for trivial bundles then use descent). Recalling the pullback square of Corollary 3.3.10.1 we have just proved that $f\colon [Z/G] \to BG$ is quasi-separated. Thus, by [Stacks, Lemma 04YT] we see that Δ_f is quasi-compact.

Starting instead with Z separated we can do an analogue reasoning to deduce f is separated. Using [Stacks, Lemma 04YS] the diagonal Δ_f is, in this case, a closed immersion.

Composing the two maps Δ_f and $[Z/G] \times_{BG} [Z/G] \to [Z/G]$ now gives all the desired properties of the diagonal of $[Z/G]$. ■

3.4 Hom stacks

3.4.1 Hilb and Quot

First of all let's recall some definitions and results on Hilb and Quot constructions, which will be useful later on. Our main reference here is [Fan+13, Chapter 5].

Definition 3.4.1. Let S be a Noetherian scheme and $X \to S$ of finite type. Let $\mathcal{E}$ be a coherent sheaf on X. Let $\mathrm{Quot}_{\mathcal{E}/X/S}\colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Grpd}$ be the pseudofunctor defined by

$$(T \to S) \mapsto \left\{ (\mathcal{F}, q) \,\middle|\, \begin{array}{l} \mathcal{F} \in \mathrm{Coh}(X_T), \text{ flat over } T \text{ and with proper schematic support,} \\ q\colon \mathcal{E}_T \twoheadrightarrow \mathcal{F} \text{ is surjective} \end{array} \right\}.$$

A map $(F, q) \to (\mathcal{F}', q')$ is an isomorphism $f\colon \mathcal{F} \to \mathcal{F}'$ such that $f \circ q = q'$. It is indeed a pseudo-functor since properness and flatness are conserved by base change and tensor product is right exact.

Definition 3.4.2. Let $X \to S$ be of finite presentation and let's define $\mathrm{Hilb}_{X/S}\colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Grpd}$ by posing $\mathrm{Hilb}_{X/S} = \mathrm{Quot}_{\mathcal{O}_X/X/S}$. Explicitely we have

$$(T \to S) \mapsto \left\{ i\colon Z \hookrightarrow X_T \,\middle|\, \begin{array}{l} i \text{ is a closed immersion and } Z \to T \text{ is flat,} \\ \text{proper and finitely presented} \end{array} \right\}.$$

Let $X \to S$ be a morphism of finite type between Noetherian schemes and let $\mathcal{L}$ be a very ample sheaf on X and $\mathcal{F}$ a coherent sheaf on X with proper schematic support on S. Then for each $s \in S$ we can consider the Hilber polynomial Φ_s of $\mathcal{F}_s$ using the line bundle $\mathcal{L}_s$. By [Har77, III, Theorem 9.9] we know that if $\mathcal{F}$ is flat over S then the function $s \mapsto \Phi_s \in \mathbb{Q}[\lambda]$ is locally constant on s. We have shown

Proposition 3.4.3. In the above case, we have the stratification

$$\mathrm{Quot}_{\mathcal{E}/X/S} = \coprod_{\Phi \in \mathbb{Q}[\lambda]} \mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}.$$

The main result is the following.

Theorem 3.4.4. Let S be a scheme, $\pi\colon X \to S$ strongly (quasi-)projective and $\mathcal{L}$ a relatively very ample line bundle on X. Then for a coherent $\mathcal{O}_X$-module $\mathcal{E}$, quotient sheaf of some $\pi^*(W)(\nu)$ for W vector bundle on S and $\nu \in \mathbb{Z}$, and any polynomial $\Phi \in \mathbb{Q}[\lambda]$ the functor $\mathrm{Quot}^{\Phi,\mathcal{L}}_{\mathcal{E}/X/S}$ is representable by a (quasi-)projective S-scheme (which we'll denote in the same way).

Proof. See [AK80, Theorem 2.6]. ■

3.4.2 First definitions

We will now turn our attention to the so called "mapping stacks", which are nothing else than a particular case of the section stacks (all will be defined precisely later). They will be essential tools to define and understand our goal, the stack of G-bundles on $X \in \mathsf{Sch}_{/S}$.

Definition 3.4.5. Let $S \in \mathsf{Sch}$, $X \in \mathsf{Sch}_{/S}$ and $\mathcal{Y}\colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Grpd}$ be a pseudofunctor. We define the Hom 2-functor $\mathcal{H}\mathrm{om}_S(X, \mathcal{Y})\colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Grpd}$ by

$$\mathcal{H}\mathrm{om}_S(X, \mathcal{Y})(T) = \mathrm{Hom}_T(X_T, \mathcal{Y}_T) = \mathrm{Hom}_S(X_T, \mathcal{Y}).$$

Using the 2-Yoneda lemma we have a natural equivalence of categories $\mathrm{Hom}_S(X_T,\mathcal{Y}) \simeq \mathcal{Y}(X_T)$ and hence we deduce that if $\mathcal{Y}$ is an fpqc S-stack then $\mathcal{H}\mathrm{om}_S(X,\mathcal{Y})$ is as well.

We can now finally define our object of interest.

Definition 3.4.6. Let S be a k-scheme. Given $X \in \mathsf{Sch}_{/S}$, the *moduli stack of G-bundles on $X \to S$* is $\mathrm{Bun}_G \coloneqq \mathcal{H}\mathrm{om}_S(X, BG \times S)$.

Since BG is an fpqc stack, also Bun_G is so. Explicitely, for an S-scheme T, $\mathrm{Bun}_G(T)$ is the groupoid of G-bundles on X_T.

Our war strategy will be to study and prove general results about Hom-stacks, using the section stacks, and then deduce properties of Bun_G by using those cannons.

As announced we will now study the sheaf of sections.

3.4.3 Scheme of sections

Let S be a base scheme, $X \in \mathsf{Sch}_{/S}$ and fix a morphism of S-schemes $Y \to X$.

Definition 3.4.7. Using the same notations as above, we define the presheaf $\mathrm{Sect}_S(X,Y)\colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Set}$ by

$$(T \to S) \mapsto \mathrm{Hom}_{X_T}(X_T, Y_T) = \mathrm{Hom}_X(X_T, Y).$$

Since schemes are fpqc sheaves then the presheaf $\mathrm{Sect}_S(X,Y)$ is actually an fpqc sheaf of sets.

We will now state (resp. and sketch a proof) of (some) technical lemmas, the final goal being to prove representability of $\mathrm{Sect}_S(X,Y)$ under some particular assumptions.

Example 4. As mentioned before, understanding the sheaf of sections can be equivalent to understanding the mapping stack. Indeed, for $X, Z \in \mathsf{Sch}_{/S}$ and $\mathrm{pr}_1\colon X \times_S Z \to X$ we have the equality

$$\mathrm{Sect}_S(X, X \times_S Z) = \mathcal{H}\mathrm{om}_S(X, Z).$$

Lemma 3.4.8. Let $p\colon X \to S$ be a flat, finitely presented, proper morphism and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank. Then $\mathrm{Sect}_S(X, \underline{\mathrm{Spec}}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E})$ is representable by a scheme affine and finitely presented over S.

Proof. By [EGAG0, Corollaire 4.5.5] we can take an open affine covering of S and reduce to the case S is affine (roughly the corollary says that a sheaf $F\colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Set}$ is representable iff its restriction to any covering of S are so). We can assume S is also of finite type using Noetherian approximation and change of base, see [EGAIV, Tome 3, 8]. Write $\mathcal{F} = \mathcal{E}^\vee$, which is again a locally free $\mathcal{O}_X$-module. From [EGAIII, Tome 2, Thm 7.7.6, Remarque 7.7.9] there exists a coherent $\mathcal{O}_Y$-module $\mathcal{Q}$ equipped with natural isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{Q}_T, \mathcal{O}_T) \simeq \Gamma(T, (p_T)_*\mathcal{F}_T) = \Gamma(X_T, \mathcal{F}_T)$$

for any S-scheme T. Giving a section $X_T \to (\underline{\mathrm{Spec}}_X \mathrm{Sym}\,\mathcal{E}) \times_S T \simeq \underline{\mathrm{Spec}}_{X_T}(\mathrm{Sym}_{\mathcal{O}_{X_T}} \mathcal{E}_T)$ over X_T is the same thing as giving a morphism of $\mathcal{O}_{X_T}$-modules $\mathcal{E}_T \to \mathcal{O}_{X_T}$. This is nothing else than an element in $\Gamma(X_T, \mathcal{E}_T^\vee)$. Since $\mathcal{E}$ is locally free of finite rank, there is a canonical isomorphism $\mathcal{F}_T \simeq \mathcal{E}_T^\vee$. The above formula shows that an element of $\mathrm{Sect}_S(X, \underline{\mathrm{Spec}}_X \mathrm{Sym}\,\mathcal{E})(T)$ is naturally isomorphic to a map in $\mathrm{Hom}_S(T, \underline{\mathrm{Spec}}_S \mathrm{Sym}_{\mathcal{O}_S} \mathcal{Q}) \simeq \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{Q}_T, \mathcal{O}_T)$. We thus conclude that

$$\mathrm{Sect}_S(X, \underline{\mathrm{Spec}}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}) \simeq \underline{\mathrm{Spec}}_S \mathrm{Sym}_{\mathcal{O}_S} \mathcal{Q}$$

as sheaves over S. ■

Let's now prove, as a lemma, a (stronger) particular case of our final theorem of this section, when the map $Y \to X$ is affine.

Lemma 3.4.9. Let $p\colon X \to S$ be a flat, finitely presented, projective morphism. If $Y \to X$ is affine and finitely presented, then $\mathrm{Sect}_S(X,Y)$ is representable by a scheme affine and finitely presented over S.

Proof. As before, we can reduce to the case where S is affine and Noetherian, and X is a closed subscheme of $\mathbb{P}^r_S$ for some integer r. By assumption, $Y = \underline{\mathrm{Spec}}_X \mathcal{A}$ for $\mathcal{A}$ a quasi-coherent $\mathcal{O}_X$-algebra. Since Y is finitely presented over X and X is quasi-compact, there are finitely many local sections of $\mathcal{A}$ generating it as an $\mathcal{O}_X$-algebra. By extending coherent sheaves (see [EGAI, Corollaire 9.4.3]), there exists a coherent $\mathcal{O}_X$-module $\mathcal{F} \subset \mathcal{A}$ containing all such sections. Using [Har77, Chap 2, Corollary 5.18] (an easy corollary of Serre theorem for X projective, stating that we can find n big enough so that $\mathcal{F}(n)$ is globally generated) we can find a locally free resolution of $\mathcal{O}_X$-modules $\mathcal{E}_1 \twoheadrightarrow \mathcal{F}$. This induces a surjection of $\mathcal{O}_X$-algebras $\operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}_1 \twoheadrightarrow \mathcal{A}$, and let $\mathcal{I}$ be its kernel. Redoing this same reasoning on $\mathcal{I}$, we find a locally free $\mathcal{O}_X$-module $\mathcal{E}_2$ and a map $\mathcal{E}_2 \to \mathcal{I}$ whose image generates $\mathcal{I}$ as an $\mathcal{O}_X$-algebra. The short exact sequence of $\mathcal{O}_X$-algebras $\operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}_2 \to \operatorname{Sym}_{\mathcal{O}_X} \mathcal{E}_1 \to \mathcal{A} \to 0$ induces the cartesian square

$$\begin{array}{ccc} Y = \underline{\operatorname{Spec}}_X \mathcal{A} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathcal{E}_1 & \longrightarrow & \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathcal{E}_2 \end{array}$$

where $X \to \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathcal{E}_2$ is the zero section. We have a canonical isomorhism

$$\operatorname{Sect}_S(X, Y) \simeq \operatorname{Sect}_S(X, X) \times_{\operatorname{Sect}_S(X, \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathcal{E}_2)} \operatorname{Sect}_S(X, \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathcal{E}_1)$$

and observe that $S \simeq \operatorname{Sect}_S(X, X)$ (one way to see it is to verify that we can apply the conclusion of the previous lemma with $\mathcal{Q} = \mathcal{O}_S$, which satisfies the assumptions there). By Lemma 3.4.8, all the three sheaves in the fibered product are represented by schemes affine and finitely presented over S. Therefore $\operatorname{Sect}_S(X, Y)$ is representable by a scheme.

The morphism $S \simeq \operatorname{Sect}_S(X, X) \to \operatorname{Sect}_S(X, \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathcal{E}_2)$ sends $T \to S$ to $0 \in \Gamma(X_T, (\mathcal{E}_2)_T^\vee)$, using the correspondence of Lemma 3.4.8. Therefore

$$S \to \operatorname{Sect}_S(X, \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathcal{E}_2) \simeq \underline{\operatorname{Spec}}_S \operatorname{Sym} \mathcal{Q}_2$$

is the zero section, and in particular is a closed immersion. By base change, also $\operatorname{Sect}_S(X, Y) \to \operatorname{Sect}_S(X, \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathcal{E}_1)$ is a closed immersion and hence we conclude that $\operatorname{Sect}_S(X, Y)$ is affine and finitely presented over S. ∎

Lemma 3.4.10. Let $p\colon X \to S$ be proper. For any morphism $Y \to X$ and $U \hookrightarrow Y$ open immersion, $\operatorname{Sect}_S(X, U) \to \operatorname{Sect}_S(X, Y)$ is schematic and open.

Proof. For an S-scheme T, suppose there exists a map $f\colon X_T \to Y$ over X, which by 2-Yoneda corresponds to a map of sheaves $T \to \operatorname{Sect}_S(X, Y)$. The fibered product

$$\begin{array}{ccc} T \times_{\operatorname{Sect}_S(X,Y)} \operatorname{Sect}_S(X, U) & \longrightarrow & \operatorname{Sect}_S(X, U) \\ \downarrow & & \downarrow \\ T & \xrightarrow{X_T \to Y} & \operatorname{Sect}_S(X, Y) \end{array}$$

is a sheaf on $\mathsf{Sch}_{/T}$ and it sends $T' \to T$ to a singleton if $X_{T'} \to X_T \to Y$ factors through U, and to the empty set otherwise (this is just how points of fibered products behave in general). We claim that this sheaf is representable by the open subscheme

$$W = T \setminus p_T(X_T \setminus f^{-1}(U))$$

of T, where $p_T(X_T \setminus f^{-1}(U))$ is closed in T by properness of p. Suppose that the image of $T' \to T$ contains a point of $p_T(X_T \setminus f^{-1}(U))$, i.e. there are $t' \in T'$ and $x \in X_T$ mapping to the same point of T, and $f(x) \notin U$. Let K be the compositum field of $\kappa(t')$ and $\kappa(x)$ over $\kappa(p_T(x))$: this gives a point of $X_{T'}$ corresponding to $(t', x) \in T' \times X_T$. Thus the map $X_{T'} \to X_T \to Y$ does not factor through U, since $f(x) \notin U$.

Conversely, suppose there exists $x' \in X_{T'}$ with image outside of U. Then its image $x \in X_T$ is not in $f^{-1}(U)$, so $p_{T'}(x')$ maps to $p_T(x) \in p_T(X_T \setminus f^{-1}(U))$.

We then conclude that $T' \to T$ factors through $T \setminus p_T(X_T \setminus f^{-1}(U))$ if and only if $X_{T'} \to Y$ factors through U. Therefore $T \times_{\operatorname{Sect}_S(X,Y)} \operatorname{Sect}_S(X, U)$ is representable by this open subscheme of T. ∎

Lemma 3.4.11. Let S be a Noetherian scheme, and let $p\colon X \to S$ and $Z \to S$ be flat proper morphisms. Suppose there is a morphism $\pi : Z \to X$ over S. Then there exists an open subscheme $S_1 \subset S$ with the following universal property: for any locally Noetherian S-scheme T, the base change $\pi_T\colon Z_T \to X_T$ is an isomorphism if and only if $T \to S$ factors through S_1.

Proof. See [Wan11, Lemma 3.1.7]. ∎

Suppose we have a proper morphism $X \to S$ and a separated morphism $g\colon Y \to X$. For a section $f \in \mathrm{Sect}_S(X,Y)(T)$, the graph of f over X_T is a closed immersion $X_T \to X_T \times_{X_T} Y_T$. Indeed, $g_T \circ f = \mathrm{id}_{X_T}$ implies that f is separated, and we can obtain the graph of f as the following pullback

$$\begin{array}{ccc} X_T & \xrightarrow{(1,f)} & X_T \times_{X_T} Y_T \simeq Y_T \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \times_{X_T} \mathrm{id}_{Y_T}} \\ Y_T & \xrightarrow{\Delta_{g_T}} & Y_T \times_{X_T} Y_T \end{array}$$

and hence by separatedness of g_T (so that its diagonal is a closed immersion) we deduce our claim. Using $X_T \times_{X_T} Y_T \simeq Y_T$ we deduce that f itself if a closed immersion.

Therefore $f\colon X_T \hookrightarrow Y_T$ represents an element of $\mathrm{Hilb}_{Y/S}(T)$. We have defined an injection of sheaves

$$\mathrm{Sect}_S(X,Y) \to \mathrm{Hilb}_{Y/S}\,. \tag{3.2}$$

Lemma 3.4.12. Let $p\colon X \to S$ and $Y \to S$ be finitely presented, proper morphisms, and suppose that p is flat. Then Eq. (3.2) is an open immersion.

Proof. Since the statement is Zariski local on the base, we can assume S is affine and Noetherian by [EGAIV, Tome 3, 8]. Let $T \to \mathrm{Hilb}_{Y/S}$ represent $Z \subset Y_T$ a closed subscheme flat over an S-scheme T. Again, by the same argument, we can assume T Noetherian. Applying Lemma 3.4.11 to the composition $Z \to X_T$ we deduce that there exists an open subscheme $U \subset T$ such that for any locally Noetherian $T' \to T$, the base change $Z_{T'} \to X_{T'}$ is an isomorhism if and only if $T' \to T$ factors through U. An isomorhism $Z_{T'} \to X_{T'}$ is the same thing as giving a section $X_{T'} \simeq Z_{T'} \hookrightarrow Y_{T'}$. We assumed T' to be locally noetherian, but by Noetherian approximation the same reasoning holds in the more general case.

We conclude that $T \times_{\mathrm{Hilb}_{Y/S}} \mathrm{Sect}_S(X,Y)$ is represented by U, and hence we have the claimed open immersion. ∎

We are ready to prove our main result about the sheaf of sections.

Theorem 3.4.13. Let $X \to S$ be a flat, finitely presented, projective morphism, and let $Y \to X$ be a finitely presented, quasi-projective morphism. Then $\mathrm{Sect}_S(X,Y)$ is representable by a disjoint union of schemes which are finitely presented and locally quasi-projective on S.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module ample relative to $\pi\colon Y \to X$ and let $\mathcal{K}$ be an invertible $\mathcal{O}_X$-module ample relative to $p\colon X \to S$. For every $(n,m) \in \mathbb{N}^2$ choose an integer $\chi_{n,m}$. Define the subfunctor

$$\mathrm{Hilb}_{Y/S}^{(\chi_{n,m})} \subset \mathrm{Hilb}_{Y/S}$$

to have T-points the flat closed subschemes $Z \subset Y_T$ such that for all $t \in T$, the Euler characteristic $\chi((\mathcal{O}_Z \otimes \mathcal{L}^{\otimes n} \otimes \pi^*(\mathcal{K}^{\otimes m}))_t) = \chi_{n,m}$. We choose to look at these particular sheaves because, as we'll see, (some) will be (very) ample w.r.t. $Y \to S$, see [Stacks, Lemma 0C4K]. We claim that they form a disjoint open cover of $\mathrm{Hilb}_{Y/S}$ (making all possible choices of course). First of all, this is Zariski local on S so we may assume, by Noetherian approximation, S to be Noetherian. By [AK80, Corollary 2.7] (using $\mathrm{Hilb}_{Y/S} = \mathrm{Quot}_{\mathcal{O}_Y/Y/S}$) the functor $\mathrm{Hilb}_{Y/S}$ is representable by a locally Noetherian scheme.

Let's first of all verify

$$\mathrm{Hilb}_{Y/S} = \coprod_{s\in\mathbb{Z}} \mathrm{Hilb}_{Y/S}^{(s)}$$

where the coproduct is taken as sheaves. This means that we can assume T connected (and Noetherian, by approximation) so that using [EGAIII, Tome 2, Thm 7.9.4] the Euler characteristic of $(\mathcal{O}_Z \otimes \mathcal{L}^{\otimes n} \otimes \pi^*(\mathcal{K}^{\otimes m}))_t$ (seen as a function on T) is locally constant (hence constant by connectedness). This implies that it is a disjoint cover. To see they are open consider the cartesian square

$$\begin{array}{ccc} W & \longrightarrow & \mathrm{Hilb}_{Y/S}^{(\chi)} \\ \downarrow & & \downarrow \\ T & \xrightarrow{Z} & \mathrm{Hilb}_{Y/S} \end{array}$$

where $W = \{t \in T \mid \chi((\mathcal{O}_Z \otimes \mathcal{L}^{\otimes n} \otimes \pi^*(\mathcal{K}^{\otimes m}))_t) = \chi\}$ is an open subscheme of T, by the properties of Euler characteristic mentioned above and by the fact that the connected components of a locally Noetherian scheme are open [EGAI, Corollaire 6.1.9].

Now it suffices to show that for any choice of $(\chi_{n,m})$, the functor

$$\mathrm{Sect}_S(X, Y) \cap \mathrm{Hilb}_{Y/S}^{(\chi_{n,m})}$$

is representable by a scheme finitely presented and locally quasi-projective over S. Again, since this is Zariski local on S, we will assume S to be affine and Noetherian. By [EGAII, Prop 4.4.6, 4.6.12, 4.6.13], there exists a scheme $\overline{Y}$ projective over X, an open immersion $Y \hookrightarrow \overline{Y}$, an invertible module $\overline{\mathcal{L}}$ very ample relative to $\overline{Y} \to S$, and positive integers a and b such that

$$\mathcal{L}^{\otimes a} \otimes \pi^*(\mathcal{K}^{\otimes b}) \simeq \overline{\mathcal{L}}\big|_Y.$$

Let $\Phi \in \mathbb{Q}[\lambda]$ be a polynomial satisfying $\Phi(n) = \chi_{na,nb}$ for a particular choice of such integers. If it does not exist then $\mathrm{Hilb}_{Y/S}^{\chi_{na,nb}}$ is empty and the claim is trivial (so in practice now we focus on the "right" choice, using the Hilbert polynomial of Z w.r.t. $\overline{\mathcal{L}}$). Using Lemma 3.4.10, Lemma 3.4.12 and [AK80, Theorem 2.6, Step IV] we deduce that $\mathrm{Sect}_S(X, Y) \cap \mathrm{Hilb}_{Y/S}^{(\chi_{n,m})}$ is an open subfunctor of $\mathrm{Hilb}_{\overline{Y}/S}^{\Phi, \overline{\mathcal{L}}}$ (representable by [AK80, Corollary 2.8]). Since an open immersion to a Noetherian scheme is finitely presented we can conclude. ■

3.4.4 Morphisms between Hom stacks

The goal will be to use our results on the scheme of sections to deduce that the diagonal of Bun_G is schematic under certain conditions on X.

Lemma 3.4.14. Suppose $X \to S$ is flat, finitely presented, and projective. Let $F\colon \mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic morphism between pseudo-functors. If F is quasi-projective (resp. affine) and of finite presentation, then the corresponding morphism

$$\mathcal{H}\mathrm{om}_S(X, \mathcal{Y}_1) \to \mathcal{H}\mathrm{om}_S(X, \mathcal{Y}_2)$$

is schematic and locally of finite presentation (resp. affine and of finite presentation).

Proof. Let $\tau_2\colon X_T \to \mathcal{Y}_2$ represent, by Yoneda, a morphism $T \to \mathcal{H}\mathrm{om}_S(X, \mathcal{Y}_2)$. Since F is schematic, the 2-fibered product $\mathcal{Y}_1 \times_{\mathcal{Y}_2, \tau_2} X_T$ is representable by a scheme Y_T. We have thus the following 2-cartesian square

$$\begin{array}{ccc} Y_T & \xrightarrow{\tau_1} & \mathcal{Y}_1 \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle F} \\ X_T & \xrightarrow{\tau_2} & \mathcal{Y}_2 \end{array}$$

with the commuting 2-isomorphism $\gamma\colon F(\tau_1) \xrightarrow{\sim} \pi^*\tau_2$. Given a T-scheme T', suppose that $\tau_1'\colon X_{T'} \to \mathcal{Y}_1$ is a 1-morphism such that the square

$$\begin{array}{ccc} X_{T'} & \xrightarrow{\tau_1'} & \mathcal{Y}_1 \\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle F} \\ X_T & \xrightarrow{\tau_2} & \mathcal{Y}_2 \end{array}$$

2-commutes via a 2-morphism $\gamma'\colon F(\tau_1') \xrightarrow{\sim} \mathrm{pr}_1^*\,\tau_2$. Thus, by the universal property of 2-fibered products, there exists a unique morphism of schemes $f\colon X_{T'} \to Y_T$ over X_T and a unique 2-morphism $\phi\colon f^*\tau_1 \xrightarrow{\sim} \tau_1'$ such that

$$\begin{array}{ccccc} F(f^*\tau_1) & \xrightarrow{\sim} & f^*F(\tau_1) & \xrightarrow{f^*\gamma} & f^*\pi^*\tau_2 \\ \big\downarrow{\scriptstyle F(\phi)} & & & & \big\downarrow{\scriptstyle \sim} \\ F(\tau_1') & & \xrightarrow{\qquad\gamma'\qquad} & & \mathrm{pr}_1^*\,\tau_2 \end{array}$$

commutes. On the other hand, we have that

$$\left(\mathcal{H}\mathrm{om}_S(X,\mathcal{Y}_1)\times_{\mathcal{H}\mathrm{om}_S(X,\mathcal{Y}_2)} T\right)(T') = \left\{(T'\to T, \tau_1'\colon X_{T'}\to\mathcal{Y}_1, \gamma'\colon F(\tau_1')\xrightarrow{\sim}\mathrm{pr}_1^*\tau_2)\right\}$$

by the 2-Yoneda lemma. Thus for a T-scheme T' and a pair (τ_1',γ') as above, there exists a unique $f\in\mathrm{Hom}_{X_T}(X_{T'},Y_T)=\mathrm{Sect}_T(X_T,Y_T)(T')$ such that

$$(f^*\tau_1, f^*\gamma\colon F(f^*\tau_1)\xrightarrow{\sim}\mathrm{pr}_1^*\tau_2)\in\left(\mathcal{H}\mathrm{om}_S(X,\mathcal{Y}_1)\times_{\mathcal{H}\mathrm{om}_S(X,\mathcal{Y}_2)} T\right)(T')$$

and there is a unique 2-morphism $(f^*\tau_1,f^*\gamma)\simeq(\tau_1',\gamma')$ induced by ϕ. Therefore we have a cartesian diagram

$$\begin{array}{ccc} \mathrm{Sect}_T(X_T,Y_T) & \longrightarrow & \mathcal{H}\mathrm{om}_S(X,\mathcal{Y}_1) \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{H}\mathrm{om}_S(X,\mathcal{Y}_2) \end{array}$$

By assumptions we have $Y_T\to X_T$ finitely presented and quasi-projective (resp. affine) and hence, by Lemma 3.4.9 and Theorem 3.4.13 we deduce that $\mathrm{Sect}_T(X_T,Y_T)\to T$ is schematic and locally of finite presentation (resp. affine and of finite presentation). ■

Corollary 3.4.14.1. Suppose $X\to S$ is flat, finitely presented and projective. Then the diagonal of Bun_G is schematic, affine and finitely presented.

Proof. By Lemma 3.3.13 we deduce that the canonical map $\mathrm{Bun}_{G\times G}\to\mathrm{Bun}_G\times\mathrm{Bun}_G$ is an isomorphism. Applying Lemma 3.3.15 and Lemma 3.4.14 to $BG\to B(G\times G)$ we deduce that $\mathrm{Bun}_G\to\mathrm{Bun}_{G\times G}$ is schematic, affine and finitely presented, and we conclude. ■

Corollary 3.4.14.2. Let $H\hookrightarrow G$ be a closed subgroup of G. If $X\to S$ is flat, finitely presented and projective, then the corresponding morphism $\mathrm{Bun}_H\to\mathrm{Bun}_G$ is schematic and locally of finite presentation.

Proof. Immediate. ■

3.5 Presentation of Bun_G

Recall that we defined Bun_G of $X\to S$ in Definition 3.4.6 as $\mathcal{H}\mathrm{om}_S(X,BG\times S)$. We will prove that it is an algebraic stack and give a presentation. We will have lot of lemmas and we will mainly focus on the case $G=\mathrm{GL}_r$, to then pass to the general case using the well known fact that any affine algebraic group can be embedded into a linear group.

Lemma 3.5.1. The morphism from the k-stack

$$B_r\colon T\mapsto\{\text{locally free }\mathcal{O}_T\text{-modules of rank }r,\text{ with isomorphisms of }\mathcal{O}_T\text{-modules}\}$$

to $B\,\mathrm{GL}_r$, sending $\mathcal{E}\mapsto\mathrm{Isom}_T(\mathcal{O}_T^r,\mathcal{E})$ is an isomorhism.

Proof. From [Vis08, Theorem 4.2.3] we know that QCoh is an fpqc stack and since local freeness of rank r persists under fpqc maps (by [EGAIV, Tome 2, Proposition 2.5.2]) we deduce that B_r is an fpqc stack. We have a canonical simply transitive right action of GL_r on $\mathrm{Isom}_T(\mathcal{O}_T^r,\mathcal{E})$ given by right composition. Given a Zariski covering $\{T_i\subset T\}_i$ trivializing $\mathcal{E}$ we obtain a descent datum $(\mathcal{O}_{T_i}^r,g_{i,j})$ for $\mathcal{E}$, where $g_{i,j}\in\mathrm{GL}_r(T_i\cap T_j)$. Since $\mathrm{Isom}_{T_i}(\mathcal{O}_{T_i}^r,\mathcal{O}_{T_i}^r)\simeq T_i\times\mathrm{GL}_r$, we have a descent datum $(T_i\times\mathrm{GL}_r,g_{i,j})$ (the same as before) corresponding to $\mathrm{Isom}_T(\mathcal{O}_T^r,\mathcal{E})$. Conversely, given $\mathcal{P}\in B\,\mathrm{GL}_r(T)$ there exists a descent datum $(T_i\times\mathrm{GL}_r,g_{i,j})$ for some fppf covering $\{T_i\to T\}$. If $V=\mathbb{A}^r$ is the standard r-dimensional representation of GL_r, then the twist ${}_{\mathcal{P}}V$ is a module in $B_r(T)$ and has a descent datum given by $(\mathcal{O}_{T_i}^r,g_{i,j})$. This implies that $\mathcal{P}\mapsto{}_{\mathcal{P}}V$ is the inverse morphism. ■

This implies that any GL_r-bundle is Zariski locally trivial. From now on we'll implicitely use the isomorphism $B_r\simeq B\,\mathrm{GL}_r$ to pass between locally free modules and GL_r-bundles.

Fix a base k-scheme S and let $p\colon X\to S$ be a flat, strongly projective morphism, with a fixed ample invertible sheaf $\mathcal{O}(1)$ on X. From now on we will write Bun_r as a shorthand for $\mathrm{Bun}_{\mathrm{GL}_r}$, the stack of GL_r-bundles on X. Let's recall that an $\mathcal{O}_{X_T}$-module $\mathcal{F}$ is *relatively generated by global sections* if the counit $p_T^*p_{T*}\mathcal{F}\to\mathcal{F}$ is surjective.

Definition 3.5.2. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{X_T}$-module, flat over T. We say $\mathcal{F}$ is *cohomologically flat in degree* i if for any cartesian square

$$\begin{CD} X_{T'} @>{v}>> X_T \\ @V{p_{T'}}VV @VV{p_T}V \\ T' @>{u}>> T \end{CD}$$

the canonical morphism $u^*R^ip_{T*}\mathcal{F} \to R^ip_{T'*}(v^*\mathcal{F})$ from [Fan+13, p. 8.2.19.3] is an isomorphism.

We will use the main results about the base change formula, written in [Fan+13, Theorem 8.3.2]. In particular we will mainly use the following.

Theorem 3.5.3. Let

$$\begin{CD} X' @>{h}>> X \\ @V{f'}VV @VV{f}V \\ Y' @>{g}>> Y \end{CD}$$

be a cartesian square of schemes, with X and Y quasi-compact and separated. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. If the map g is flat, or $\mathcal{F}$ is flat on Y, then there is a natural isomorphism

$$Lg^*Rf_*\mathcal{F} \to Rf'_*(h^*\mathcal{F})$$

in $D(Y')$ (derived category of $\mathcal{O}_{Y'}$-modules).

In particular we deduce that the induced maps $g^*R^qf_*\mathcal{F} \to R^qf'_*(h^*\mathcal{F})$ are isomorphisms. We will use the following criterion for cohomological flatness.

Lemma 3.5.4. For an S-scheme T, let $\mathcal{F}$ be a quasi-coherent sheaf on X_T, flat over T. If p_T is separated, $p_{T*}\mathcal{F}$ is flat and $R^ip_{T*}\mathcal{F} = 0$ for $i > 0$, then $\mathcal{F}$ is cohomologically flat over T in all degrees.

Proof. Since the statement is local, we can assume T and T' to be affine. By the base change formula Theorem 3.5.3, we have a quasi-isomorphism

$$Lu^*Rp_{T*}\mathcal{F} \xrightarrow{\sim} Rp_{T'*}(v^*\mathcal{F})$$

of chain complexes of $\mathcal{O}_{T'}$-modules. Since $R^ip_{T*}\mathcal{F} = 0$ for $i > 0$, we have a quasi-isomorphism $p_{T*}\mathcal{F} \simeq Rp_{T*}\mathcal{F}$. By flatness of $p_{T*}\mathcal{F}$ we have $Lu^*p_{T*}\mathcal{F} \simeq u^*p_{T*}\mathcal{F}$ and hence $u^*p_{T*}\mathcal{F} \simeq Rp_{T'*}(v^*\mathcal{F})$. We conclude that

$$u^*R^ip_{T*}\mathcal{F} \simeq R^ip_{T'*}(v^*\mathcal{F})$$

for all i (consider $i = 0$ and $i > 0$ separately). This isomorphism corresponds to the canonical one given before, so we can conclude. ∎

Proposition 3.5.5. For an S-scheme T, let

$$\mathcal{U}_n(T) \coloneqq \left\{ \mathcal{E} \in \mathrm{Bun}_r(T) \,\middle|\, \begin{array}{l} R^ip_{T*}(\mathcal{E}(n)) = 0 \text{ for all } i > 0 \\ \text{and } \mathcal{E}(n) \text{ is relatively generated by global sections} \end{array} \right\}$$

be the full subgroupoid of $\mathrm{Bun}_r(T) = B_r(T)$. For $\mathcal{E} \in \mathcal{U}_n(T)$, the direct image $p_{T*}(\mathcal{E}(n))$ is flat, and $\mathcal{E}(n)$ is cohomologically flat over T in all degrees. In particular, the inclusion $\mathcal{U}_n \hookrightarrow \mathrm{Bun}_r$ makes $\mathcal{U}_n$ a pseudo-functor.

Proof. Since flatness is local we can assume T is affine. Call $\mathcal{F} = \mathcal{E}(n) \in \mathcal{U}_n(T)$. Then, remembering the assumption on $p\colon X \to S$, X_T is quasi-compact and separated, so we can choose a finite affine open covering $\mathfrak{U} = (U_j)_{j=1,\dots,N}$ of X_T and use it to compute Čech cohomology. We have an isomorphism

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} H^i(X_T, \mathcal{F}).$$

By [EGAIII, Tome 1, Proposition 1.4.10, Corollaire 1.4.11], $R^ip_{T*}\mathcal{F}$ is the quasi-coherent sheaf associated to $H^i(X_T, \mathcal{F})$, we deduce $\check{H}^i(\mathfrak{U}, \mathcal{F}) = 0$ for $i > 0$. This means that the augmented Čech complex

$$0 \longrightarrow \Gamma(X_T, \mathcal{F}) \longrightarrow \check{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \dots \longrightarrow \check{C}^{N-1}(\mathcal{U}, \mathcal{F}) \longrightarrow 0$$

is exact. Since $\mathcal{F}$ is T-flat, also $\check{C}^i(\mathfrak{U}, \mathcal{F}) = \prod_{j_0 < \cdots < j_i} \Gamma(U_{j_0} \cap \cdots \cap U_{j_i}, \mathcal{F})$ is flat over $\Gamma(T, \mathcal{O}_T)$. By induction and [Har77, III, Proposition 9.1] we conclude that $\Gamma(X_T, \mathcal{F})$ is flat over $\Gamma(T, \mathcal{O}_T)$, i.e. $p_{T_*}\mathcal{F}$ is flat over T. Using Lemma 3.5.4 we conclude that $\mathcal{F}$ is cohomologically flat in all degrees.

Now we want to show that for $T' \to T$, the pullback $\mathcal{E}_{T'}$ still lies in $\mathcal{U}_n(T')$, i.e. it is relatively generated by global sections and it satisfies $R^j p_{T'*}\mathcal{F}_{T'} = 0$ for all $j > 0$. This last fact holds by cohomologically flatness (i.e. base change isomorphism holds) so we concentrate on the first one. Again, since this property is local, let T and T' be affine schemes. Thus (relative becomes unnecessary and) there exists a surjection $\bigoplus \mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F}$, which pulls back to a surjection $\bigoplus \mathcal{O}_{X_{T'}} \twoheadrightarrow \mathcal{F}_{T'}$, showing that $\mathcal{F}_{T'}$ is also generated by global sections. Since pullback and tensor product commute, we conclude $\mathcal{E}_{T'} \in \mathcal{U}_n(T')$. ■

Remark 3.5.6. If S is affine then the ring $\Gamma(S, \mathcal{O}_S)$ is an inductive limit of finitely presented k-algebras and by [EGAIV, Tome 3, 8] we know that we can find $S_1 \to \operatorname{Spec} k$ affine of finite presentation, a map $S \to S_1$ and a flat projective morphism $p_1 \colon X_1 \to S_1$ which is equal to p after base change. From now on we will thus work with such p_1, because then we just need to base change to get back results from the original p. This will allow us to focus only on Noetherian S.

Lemma 3.5.7. The pseudofunctors $(\mathcal{U}_n)_{n\in\mathbb{Z}}$ form an open cover of Bun_r.

Proof. Fix $T \to S$ and $\mathcal{E} \in \mathrm{Bun}_r(T)$ (i.e. a map $T \to \mathrm{Bun}_r$). The 2-fibered product $\mathcal{U}_n \times_{\mathrm{Bun}_r} T \colon (\mathsf{Sch}_{/T})^{\mathrm{op}} \to \mathsf{Grpd}$ sends $T' \to T$ to the equivalence relation

$$\{(\mathcal{E}' \in \mathcal{U}_n(T'), \mathcal{E}' \simeq \mathcal{E}_{T'})\}\,.$$

To prove $\mathcal{U}_n$ is open then we just need to find an open subscheme $U_n \subset T$ with the universal property such that $T' \to T$ factors through U_n iff $R^i p_{T'*}(\mathcal{E}_{T'}(n)) = 0$ for $i > 0$ and $\mathcal{E}_{T'}(n)$ is relatively generated by global sections over T'. With the usual locality and Noetherian approximation-fu we can assume S and T to be affine and Noetherian (for the details see [Wan11, Lemma 4.1.5]).

We will have in mind the following cartesian square

$$\begin{array}{ccc} X_t & \longrightarrow & X_T \\ \Big\downarrow{\scriptstyle p_t} & & \Big\downarrow{\scriptstyle p_T} \\ \operatorname{Spec}\kappa(t) & \longrightarrow & T \end{array}$$

where we observe that the upward arrow is a closed immersion by base change. Call $\mathcal{F} = \mathcal{E}(n)$ and define

$$U_n \coloneqq \left\{ t \in T \;\middle|\; \begin{array}{l} H^i(X_t, \mathcal{F}_t) = 0 \text{ for } i > 0 \\ \text{and } \varphi_t \colon p_t^* p_{t*}\mathcal{F}_t = \Gamma(X_t, \mathcal{F}_t) \otimes \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{F}_t \text{ surjective} \end{array} \right\}.$$

We claim that the set of points $t \in T$ where $\Gamma(X_t, \mathcal{F}_t) \otimes \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{F}_t$ is surjective is open. Indeed consider the set

$$F \coloneqq \{x \in X_T \mid (\varphi_t)_x \colon (p_t^* p_{t*}\mathcal{F}_t)_x \to (\mathcal{F}_t)_x \simeq \mathcal{F}_x \text{ is not surjective}\}$$

where the subscript x denotes taking the stalk at the point $x \in X_T$ (φ_t is indeed a map of $\mathcal{O}_{X_t}$-modules and hence of $\mathcal{O}_X$-modules by pushforward of a closed immersion). By Nakayama's lemma (using coherence of $\mathcal{F}$ and finiteness of $\operatorname{Spec}\kappa(t) \to T$, so that $\mathcal{F}_t$ is still coherent) we deduce that if $(\varphi_t)_x$ is surjective at $x \in X_T$, then it is surjective at any y in an open neighborhood of x in X_T (we find a lift of generators). This means exactly that F is closed in X_T and hence $p_T(F) \subset T$ is also closed since p_T is projective. Then its complement in T is open, and clearly for any t in it, the map φ_t is surjective: there cannot exist $x \in X_T$ mapping to t such that $(\varphi_t)_x$ is not surjective, since then $x \in F$ and hence $t = p_T(x) \in p_T(F)$.

Since X is quasi-compact, it can be covered by N affines, and so does X_t by base change. We can compute the cohomology of $\mathcal{F}_t$ using Čech cohomology, from which we immediately see $H^i(X_t, \mathcal{F}_t) = 0$ for $i > N$ and all t. Fixed i, the set of $t \in T$ for which $H^i(X_t, \mathcal{F}_t) = 0$ is open by upper semicontinuity of the fibers [Har77, III, Theorem 12.8] ($\mathcal{F}$ is flat over T since it is locally free and p_T is flat by base change). By intersecting a finite number of such sets, we deduce that U_n is open.

By [Har77, III, Theorem 12.11], for $t \in U_n$ we have $R^i p_{T*}\mathcal{F} \otimes \kappa(t) = 0$ for $i > 0$. Since p is proper by assumption, $R^i p_* \mathcal{F}$ is coherent by [EGAIII, Tome 1, Theorem 3.2.1]. Thus by base change ($\mathcal{F}$ is cohomologically flat by Proposition 3.5.5) and by Nakayama's lemma we obtain

$$R^i p_{U_n*}(\mathcal{F}_{U_n}) \simeq (R^i p_{T*}\mathcal{F})\big|_{U_n} = 0.$$

Again by Nakayama's lemma we have that $\mathcal{F}_{U_n}$ is relatively generated by global sections (by the choice of U_n). This proves that $\mathcal{E}_{U_n} \in \mathcal{U}_n(U_n)$, i.e. we have a 2-commutative square

$$\begin{array}{ccc} U_n & \xrightarrow{\mathcal{E}_{U_n}} & \mathcal{U}_n \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow{\mathcal{E}} & \mathrm{Bun}_r \end{array}$$

Suppose now there is a morphism $u\colon T' \to T$ such that $R^i p_{T'*}\mathcal{F}_{T'} = 0$ for $i > 0$ and $\mathcal{F}_{T'}$ is relatively generated by global sections. We need to prove it factors through U_n. Take $t' \in T'$ and let $t = u(t')$. By cohomological flatness (and structure of higher direct images) we deduce that $H^i(X_{t'}, \mathcal{F}_{t'}) = 0$ for $i > 0$. Since $\operatorname{Spec}\kappa(t') \to \operatorname{Spec}\kappa(t)$ is faithfully flat, by Theorem 3.5.3, we have

$$H^i(X_t, \mathcal{F}_t) \otimes_{\kappa(t)} \kappa(t') \simeq H^i(X_{t'}, \mathcal{F}_{t'}) = 0$$

and hence $H^i(X_t, \mathcal{F}_t) = 0$ for $i > 0$. Since $\mathcal{F}_{T'}$ is relatively generated by global sections, by assumptions, we have $\Gamma(X_{t'}, \mathcal{F}_{t'}) \otimes \mathcal{O}_{X_{t'}} \twoheadrightarrow \mathcal{F}_{t'}$; by faithfully flatness this implies $\Gamma(X_t, \mathcal{F}_t) \otimes \mathcal{O}_{X_t} \twoheadrightarrow \mathcal{F}_t$. Thus $t \in U_n$, i.e. the morphism u factors through U_n.

This proves that $\mathcal{U}_n$ is open in Bun_r. Finally, by Serre's cohomology theorem [Har77, III, Theorem 5.2], given $\mathcal{E} \in \mathrm{Bun}_r(T)$ there exists $n \in \mathbb{Z}$ such that $R^i p_{T*}(\mathcal{E}(n)) = 0$ for $i > 0$ and $\mathcal{E}(n)$ is generated by global sections. This implies that $\mathcal{E} \in \mathcal{U}_n(T)$ and therefore the $(\mathcal{U}_n)_{n\in\mathbb{Z}}$ form an open cover of Bun_r. ■

Remark 3.5.8. For an S-scheme T and $\mathcal{E} \in \mathcal{U}_n(T)$ we claim that $p_{T*}(\mathcal{E}(n))$ is a locally free $\mathcal{O}_T$-module of finite rank. The key idea is to use Noetherian approximation, and then the well known algebraic fact that a flat module of finite type over a noetherian ring is locally free of finite rank. See [Wan11, Remark 4.1.7].

For a polynomial $\Phi \in \mathbb{Q}[\lambda]$, define a pseudo-subfunctor $\mathrm{Bun}_r^\Phi \subset \mathrm{Bun}_r$ by

$$\mathrm{Bun}_r^\Phi(T) = \{\mathcal{E} \in \mathrm{Bun}_r(T) \mid \Phi(m) = \chi(\mathcal{E}_t(m))\, \forall\, t \in T,\, m \in \mathbb{Z}\}\,.$$

For a locally Noetherian S-scheme T and $\mathcal{E} \in \mathrm{Bun}_r(T)$, the Hilbert polynomial of $\mathcal{E}_t$ (the unique polynomial in $\mathbb{Q}[\lambda]$ sending $n \mapsto \chi(\mathcal{E}_t(n))$) is a locally constant function on T by [EGAIII, Tome 2, Theorem 7.9.4]. We deduce (hiding under our carpet all noetherian approximation arguments) that the $(\mathrm{Bun}_r^\Phi)_{\Phi\in\mathbb{Q}[\lambda]}$ form a disjoint open cover of Bun_r. Let

$$\mathcal{U}_n^\Phi \coloneqq \mathcal{U}_n \cap \mathrm{Bun}_r^\Phi$$

so that $\mathcal{U}_n$ has a disjoint open cover given by such opens. Let's try to understand them better: given $\mathcal{E} \in \mathcal{U}_n^\Phi(T)$ we know by Remark 3.5.8 that $p_{T*}(\mathcal{E}(n))$ is locally free of finite rank. By cohomological flatness of $\mathcal{E}(n)$ (see Proposition 3.5.5) and the assumption $R^i p_{T*}(\mathcal{E}(n)) = 0$ for $i > 0$, we have $H^0(X_t, \mathcal{E}_t(n)) \simeq p_{T*}(\mathcal{E}(n)) \otimes \kappa(t)$ is a vector space of dimension $\Phi(n)$ (we consider Euler characteristic but by assumption we have only 0-th homology). This implies that $p_{T*}(\mathcal{E}(n))$ is locally free of rank $\Phi(n)$.

Using [Stacks, Lemma 05UN] (which says that if we have a morphism from a pseudofunctor to a stack that is representable by an algebraic space, then the source is also a stack) and the fact that Bun_r is a stack, we deduce that all these pseudo-functors are also fpqc stacks.

Now we want to find smooth surjective morphisms to the $\mathcal{U}_n^\Phi$. We will introduce a bunch of new pseudofunctors.

Let's define the pseudofunctors $\mathcal{Y}_n^\Phi\colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Grpd}$ by

$$(T \to S) \mapsto \left\{ (\mathcal{E}, \phi, \psi) \;\middle|\; \begin{array}{l} \mathcal{E} \in \mathcal{U}_n^\Phi(T),\ \phi\colon \mathcal{O}_{X_T}^{\Phi(n)} \twoheadrightarrow \mathcal{E}(n) \text{ is surjective,} \\ \text{the adjoint morphism } \psi\colon \mathcal{O}_T^{\Phi(n)} \to p_{T*}(\mathcal{E}(n)) \text{ is an isomorphism} \end{array} \right\}.$$

A morphism $(\mathcal{E}, \phi, \psi) \to (\mathcal{E}', \phi', \psi')$ is an isomorphism $f\colon \mathcal{E} \to \mathcal{E}'$ satisfying $f(n) \circ \phi = \phi'$. This last condition is equivalent to $p_{T*}(f(n)) \circ \psi = \psi'$ by adjunction. To give an isomorphism ψ as above just means specifying $\Phi(n)$ elements of $\Gamma(X_T, \mathcal{E}(n))$ that form a basis of $p_{T*}(\mathcal{E}(n))$ as an $\mathcal{O}_T$-module. By how we defined morphism, $\mathcal{Y}_n^\Phi(T)$ is an equivalence relation.

Technical lemma.

Lemma 3.5.9. Suppose we have a cartesian square

$$\begin{CD} X_{T'} @>{v}>> X_T \\ @V{p_{T'}}VV @VV{p_T}V \\ T' @>{u}>> T \end{CD}$$

$\mathcal{M} \in \mathcal{O}_T - \mathsf{Mod}$, $\mathcal{N} \in \mathcal{O}_{X_T} - \mathsf{Mod}$ and $\phi\colon p_T^*\mathcal{M} \to \mathcal{N}$ a map of $\mathcal{O}_{X_T}$-modules. If $\psi\colon \mathcal{M} \to p_{T*}\mathcal{N}$ is its adjoint morphism, then the composition of $u^*(\psi)$ with the base change map $u^*p_{T*}\mathcal{N} \to p_{T'*}v^*\mathcal{N}$ corresponds, via the adjunction $(p_{T'*}, p_{T'}^*)$ to $v^*(\phi)\colon p_{T'}^*u^*\mathcal{M} \simeq v^*p_T^*\mathcal{M} \to v^*\mathcal{N}$.

Proof. See [Wan11, Lemma 4.1.8]. ∎

For $\mathcal{M} = \mathcal{O}_T^{\Phi(n)}$ and $\mathcal{N} = \mathcal{E}(n)$ in Lemma 3.5.9 we see that the pullbacks of ϕ and ψ are compatible, so that $\mathcal{Y}_n^\Phi$ is a pseudo-functor in a non-ambiguous way.

Lemma 3.5.10. For $\Psi \in \mathbb{Q}[\lambda]$, define the pseudofunctor $\mathcal{W}^\Psi\colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Grpd}$ by

$$\mathcal{W}^\Psi(T) := \left\{(\mathcal{F}, \phi) \mid \mathcal{F} \in \mathrm{Bun}_r^\Phi(T), \phi\colon \mathcal{O}_{X_T}^{\Psi(0)} \twoheadrightarrow \mathcal{F}\right\}$$

where a map $(\mathcal{F}, \phi) \to (\mathcal{F}', \phi')$ is an isomorphism $f\colon \mathcal{F} \to \mathcal{F}'$ satisfying $f \circ \phi = \phi'$. Then $\mathcal{W}^\Psi$ is representable by a strongly quasi-projective S-scheme.

Proof. As before the $\mathcal{W}^\Psi(T)$ are equivalence relations so, considering the sets of equivalence classes, $\mathcal{W}^\Psi$ is isomorphic to a functor

$$W^\Psi \subset \mathrm{Quot}^{\Psi, \mathcal{O}(1)}_{\mathcal{O}_X^{\Psi(0)}/X/S} =: Q.$$

By Theorem 3.4.4 we know that Q is representable by a scheme strongly projective over S. We will prove that $W^\Psi \hookrightarrow Q$ is schematic, open and finitely presented and this will imply that $\mathcal{W}^\Psi \simeq W^\Psi$ is representable by a strongly quasi-projective S-scheme. As usual, by the locality of the claim, we assume S affine and Noetherian. Let $(\mathcal{F}, \phi)\colon T \to Q$ and consider $W^\Psi \times_Q T$, the 2-fibered product. Consider the following set

$$U := \{x \in X_T \mid \text{the stalk } \mathcal{F} \otimes \mathcal{O}_{X_T,x} \text{ is free}\}.$$

Since $\mathcal{F}$ is coherent, by Nakayama's lemma, U is an open subset of X_T. We claim that the open subset

$$V := T \setminus p_T(X_T \setminus U) \subset T$$

represents $W^\Psi \times_Q T$. We need to prove that a morphism $T' \to T$ lands in V iff $\mathcal{F}_{T'}$ is locally free. If $T' \to T$ factors through V, then $X_{T'} \to X_T$ lands in U, which implies that $\mathcal{F}_{T'}$ is locally free (since, by construction, $\mathcal{F}\big|_U$ is locally free). Conversely, suppose $T' \to T$ is such that $\mathcal{F}_{T'}$ is locally free and assume by contradiction that there exists $t' \in T$ and $x \in X_T$ mapping to the same $t \in T$ such that $\mathcal{F} \otimes \mathcal{O}_{X_T,x}$ is not free. We have cartesian squares

$$\begin{CD} (X_{T'})_{t'} @>>> (X_T)_t @>>> X_T \\ @VVV @VVV @VVV \\ \operatorname{Spec}\kappa(t') @>>> \operatorname{Spec}\kappa(t) @>>> T \end{CD}$$

where the down-left arrow is faithfully flat. Then $\mathcal{F}_{t'} \simeq \mathcal{F}_t \otimes_{\kappa(t)} \kappa(t')$ is a flat $\mathcal{O}_{X_{t'}}$-module (change base of $\mathcal{F}_{T'}$ which is locally free). Since flatness is a local property for the quasi-compact faithfully flat maps (see [EGAIV, Tome 2, Proposition 2.7.1]), this implies that $\mathcal{F}_t$ is flat over $\mathcal{O}_{X_t}$. By the definition of Q (see Definition 3.4.1) $\mathcal{F}$ is T-flat. Therefore $\mathcal{F} \otimes \mathcal{O}_{X_T,x}$ is $\mathcal{O}_{T,t}$-flat and $\mathcal{F}_t \otimes \mathcal{O}_{X_t,x}$ is $\mathcal{O}_{X_t,x}$-flat and by [Mat70, 20.G] we conclude that $\mathcal{F} \otimes \mathcal{O}_{X_T,x}$ is $\mathcal{O}_{X_T,x}$-flat and hence a free module, contradiction! Hence $T' \to T$ must factor through V. We showed that W^Ψ is representable by an open subscheme of Q. Since S is Noetherian, Q is also Noetherian and hence $W^\Psi \hookrightarrow Q$ is finitely presented (being an open immersion between Noetherian schemes). ∎

Lemma 3.5.11. The pseudo-functor $\mathcal{Y}_n^\Phi$ is representable by a scheme Y_n^Φ which is strongly quasi-projective over S.

Proof. Let $\Psi \in \mathbb{Q}[\lambda]$ be defined by $\Psi(\lambda) := \Phi(\lambda + n)$, and consider $\mathcal{W}^\Psi$ as before. There is a morphism $\mathcal{W}^\Psi \to \mathrm{Bun}_r^\Phi$ (observe the change in the polynomial) given by $(\mathcal{F}, \phi) \mapsto \mathcal{F}(-n)$ (indeed $\chi(\mathcal{F}(-n)(m)) = \chi(\mathcal{F}(m-n)) = \Psi(m-n) = \Phi(m)$). The corresponding 2-fibered product $\mathcal{U}_n^\Phi \times_{\mathrm{Bun}_r^\Phi} \mathcal{W}^\Psi$ is isomorphic to the pseudo-functor $\mathcal{Z}_n^\Phi \colon (\mathsf{Sch}_{/S})^{\mathrm{op}} \to \mathsf{Grpd}$ defined by

$$\mathcal{Z}_n^\Phi(T) := \left\{ (\mathcal{E}, \phi) \mid \mathcal{E} \in \mathcal{U}_n^\Phi(T), \phi \colon \mathcal{O}_{X_T}^{\Phi(n)} \twoheadrightarrow \mathcal{E}(n) \right\}$$

where a morphism $(\mathcal{E}, \phi) \to (\mathcal{E}', \phi')$ is an isomorphism $f \colon \mathcal{E} \to \mathcal{E}'$ satisfying $f(n) \circ \phi = \phi'$. Since $\mathcal{U}_n^\Phi$ is open and schematic in Bun_r^Φ then we deduce that the morphism $\mathcal{Z}_n^\Phi \to \mathcal{W}^\Psi$ is schematic and open. Thus Lemma 3.5.10 implies that $\mathcal{Z}_n^\Phi$ is representable by an open subscheme $Z_n^\Phi \subset W^\Psi$. We claim that $Z_n^\Phi \hookrightarrow W^\Psi$ is finitely presented: using Noetherian approximation as in Remark 3.5.6 we reduce to S Noetherian, where this is trivial (any open subscheme of a Noetherian scheme is finitely presented). Therefore we proved that Z_n^Φ is quasi-projective over S.

We will prove that the morphism $\mathcal{Y}_n^\Phi \to \mathcal{Z}_n^\Phi$ sending $(\mathcal{E}, \phi, \psi) \mapsto (\mathcal{E}, \phi)$ is schematic, open and finitely presented, and this suffices to conclude the proof. We may assume S Noetherian as usual, using Remark 3.5.6 and [EGAG0, Corollaire 4.5.5]. Let $(\mathcal{E}, \phi) \in \mathcal{Z}_n^\Phi(T)$ and let ψ be the adjoint map $\mathcal{O}_T^{\Phi(n)} \to p_{T*}(\mathcal{E}(n))$. Let's define

$$U := T \setminus \mathrm{Supp}(\mathrm{coker}\, \psi)$$

and observe it is open in T. Indeed ψ means choosing $\Phi(n)$ sections in $\Gamma(X_T, \mathcal{E}(n))$ and U is the maximal subscheme of T where they generate $p_{T*}(\mathcal{E}(n))$, so it is open by Nakayama. Then we easily see that a map $u \colon T' \to T$ lands in U if and only if $u^*(\psi)$ is a surjection, which happens iff $u^*(\psi)$ is an isomorphism of sheaves, since both $\mathcal{O}_T^{\Phi(n)}$ and $p_{T*}(\mathcal{E}(n))$ are locally free $\mathcal{O}_T$-modules of rank $\Phi(n)$. From Proposition 3.5.5 we know that the base change morphism $u^* p_{T*}(\mathcal{E}(n)) \to p_{T'*}(\mathcal{E}_{T'}(n))$ is an isomorphism. Thus the compatibility statement of Lemma 3.5.9 says that $u^*(\psi)$ is an isomorphism if and only if the adjoint of $v^*(\phi)$ is an isomorphism, where $v \colon X_{T'} \to X_T$. This translates exactly to U being the representative of the 2-fibered product $\mathcal{Y}_n^\Phi \times_{\mathcal{Z}_n^\Phi} T$.

Taking $T = Z_n^\Phi$, Noetherian, we deduce that $\mathcal{Y}_n^\Phi \to \mathcal{Z}_n^\Phi$ is schematic, open and finitely presented and therefore we conclude that $\mathcal{Y}_n^\Phi$ is represented by a strongly quasi-projective S-scheme. ■

We are ready for the last lemma before our main result. We will connect with the theory of G-bundles we started studying.

Lemma 3.5.12. There is a canonical right $\mathrm{GL}_{\Phi(n)}$-action on Y_n^Φ such that the morphism $Y_n^\Phi \to \mathcal{U}_n^\Phi$ sending $(\mathcal{E}, \phi, \psi) \mapsto \mathcal{E}$ is a $\mathrm{GL}_{\Phi(n)}$-bundle. Therefore by Lemma 3.3.6 we have an isomorphism

$$\mathcal{U}_n^\Phi \simeq [Y_n^\Phi / \mathrm{GL}_{\Phi(n)}]$$

sending $\mathcal{E} \in \mathcal{U}_n^\Phi(T)$ to a $\mathrm{GL}_{\Phi(n)}$-equivariant map $\mathrm{Isom}_T(\mathcal{O}_T^{\Phi(n)}, p_{T*}(\mathcal{E}(n))) \to Y_n^\Phi$.

Proof. Define a map $\alpha \colon \mathcal{Y}_n^\Phi \times \mathrm{GL}_{\Phi(n)} \to \mathcal{Y}_n^\Phi$ by

$$((\mathcal{E}, \phi, \psi), g) \mapsto (\mathcal{E}, \phi \circ p_T^*(g), \psi \circ g)$$

over an S-scheme T. The diagram

$$\begin{array}{ccc} \mathcal{Y}_n^\Phi \times \mathrm{GL}_{\Phi(n)} & \xrightarrow{\alpha} & \mathcal{Y}_n^\Phi \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ \mathcal{Y}_n^\Phi & \longrightarrow & \mathcal{U}_n^\Phi \end{array}$$

is 2-commutative, evidently. Then α induces a $\mathrm{GL}_{\Phi(n)}$-action on Y_n^Φ taking equivalence classes, i.e. there exists a 2-commutative square

$$\begin{array}{ccc} Y_n^\Phi \times \mathrm{GL}_{\Phi(n)} & \longrightarrow & Y_n^\Phi \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \mathcal{Y}_n^\Phi & \xrightarrow{\alpha} & \mathcal{Y}_n^\Phi \end{array}$$

and the associated 2-morphisms satisfy the classic associativity conditions because $\mathcal{Y}_n^\Phi(T)$ is an equivalence relation. Thus we deduce that $Y_n^\Phi \to \mathcal{U}_n^\Phi$ is $\mathrm{GL}_{\Phi(n)}$-invariant. For $T \in \mathsf{Sch}_{/S}$ let $\mathcal{E}\colon T \to \mathcal{U}_n^\Phi$. Then the 2-fibered product is given by

$$\left(\mathcal{Y}_n^\Phi \times_{\mathcal{U}_n^\Phi} T\right)(T') = \left\{(\mathcal{N}, \phi, \psi, \gamma) \mid (\mathcal{N}, \phi, \psi) \in \mathcal{Y}_n^\Phi(T'), \gamma\colon \mathcal{N} \xrightarrow{\sim} \mathcal{E}_{T'}\right\}$$

where a map $(\mathcal{N}, \phi, \psi, \gamma) \to (\mathcal{N}', \phi', \psi', \gamma')$ is an isomorphism $f\colon \mathcal{N} \to \mathcal{N}'$ satisfying $f(n) \circ \phi = \phi'$ and $\gamma' \circ f = \gamma$. Then the above groupoid is an equivalence relation and $\mathcal{Y}_n^\Phi \times_{\mathcal{U}_n^\Phi} T$ is isomorphic to the functor $\mathcal{P}\colon (\mathsf{Sch}_{/T})^{\mathrm{op}} \to \mathsf{Set}$ given by

$$(T' \to T) \mapsto \left\{(\phi, \psi) \mid \phi\colon \mathcal{O}_{X_{T'}}^{\Phi(n)} \twoheadrightarrow \mathcal{E}_{T'}(n), \text{ the adjoint } \psi \text{ is an isomorphism}\right\}.$$

Let's observe that the surjectivity of ϕ is automatic: since $\mathcal{E}_{T'}(n)$ is relatively generated by global sections, if we have $\psi\colon \mathcal{O}_{T'}^{\Phi(n)} \xrightarrow{\sim} p_{T'*}(\mathcal{E}_{T'}(n))$, then the adjoint morphism

$$\phi\colon \mathcal{O}_{X_{T'}}^{\Phi(n)} \overset{\psi}{\simeq} p_{T'}^* p_{T'*}(\mathcal{E}_{T'}(n)) \twoheadrightarrow \mathcal{E}_{T'}(n)$$

is surjective (where the last morphism is the counit, by definition of adjunction). Thus, $\mathcal{P}(T')$ is just the choice of a basis for $p_{T'*}(\mathcal{E}_{T'}(n))$ and the induced right $\mathrm{GL}_{\Phi(n)}$-action on $\mathcal{P}$ is defined by $\psi.g = \psi \circ g$, for $g \in \mathrm{GL}_{\Phi(n)}(T')$. Recall that we know, by Remark 3.5.8, that the direct image $p_{T*}(\mathcal{E}(n))$ is locally free of rank $\Phi(n)$. By cohomological flatness of $\mathcal{E}(n)$ over T by Proposition 3.5.5 we have

$$\begin{aligned}\mathcal{P}(T' \xrightarrow{u} T) &\simeq \mathrm{Isom}_{T'}(\mathcal{O}_{T'}^{\Phi(n)}, p_{T'*}(\mathcal{E}_{T'}(n))) \simeq \mathrm{Isom}_{T'}(\mathcal{O}_{T'}^{\Phi(n)}, u^* p_{T*}(\mathcal{E}(n))) \simeq \\ &\simeq \mathrm{Isom}_{T'}(u^*\mathcal{O}_{T}^{\Phi(n)}, u^* p_{T*}(\mathcal{E}(n))) = \mathrm{Isom}_{T}(\mathcal{O}_{T}^{\Phi(n)}, p_{T*}(\mathcal{E}(n)))(T' \xrightarrow{u} T)\end{aligned}$$

so that $\mathcal{P} \simeq \mathrm{Isom}_T(\mathcal{O}_T^{\Phi(n)}, p_{T*}(\mathcal{E}(n)))$, where the latter is the obvious $\mathrm{GL}_{\Phi(n)}$-bundle. This proves that the map $\mathcal{Y}_n^\Phi \to \mathcal{U}_n^\Phi$ is a $\mathrm{GL}_{\Phi(n)}$-bundle. ■

Finalle we conclude this chapter stating our main result about Bun_G.

Theorem 3.5.13. Let $X \to S$ be a flat, finitely presented projective morphism. The S-stack Bun_G is an algebraic stack locally of finite presentation over S, with a schematic, affine diagonal of finite presentation. Additionally, Bun_G admits an open covering by algebraic substacks of finite presentation over S.

Proof. By Corollary 3.4.14.1 we already know that the diagonal of Bun_G is schematic, affine and finitely presented. By the classic [DG70, 2, Theorem 3.3] for any affine algebraic group G there exists $r \in \mathbb{N}$ such that $G \subset \mathrm{GL}_r$ is a closed subgroup. So we will first focus on this case and then generalize.

We saw in Lemma 3.5.1 that the morphisms $\mathcal{Y}_n^\Phi \to \mathcal{U}_n^\Phi$ are smooth and surjective, being $\mathrm{GL}_{\Phi(n)}$-bundles. After having assumed S Noetherian as usual, by Lemma 3.5.11 we know that each $\mathcal{Y}_n^\Phi$ is representable by a scheme Y_n^Φ finitely presented over S. The map

$$Y = \coprod_{n \in \mathbb{Z}, \Phi \in \mathbb{Q}[\lambda]} Y_n^\Phi \to \mathrm{Bun}_r = \mathrm{Bun}_{\mathrm{GL}_r}$$

is then smooth and surjective because the $(\mathcal{U}_n^\Phi)$ are an open cover of Bun_r by Lemma 3.5.7, and Y is locally of finite presentation over S. This is the searched smooth atlas for Bun_r, which is then an algebraic stack locally of finite presentation over S. From the already mentioned [Stacks, Lemma 05UN] we deduce that also all the $\mathcal{U}_n^\Phi \subset \mathrm{Bun}_r$ are algebraic stacks, and they are of finite presentation over S (since the Y_n^Φ are so).

Let's now pass to our initial G; by Corollary 3.4.14.2 we know that the corresponding map $\mathrm{Bun}_G \to \mathrm{Bun}_r$ is schematic and locally of finite presentation. Consider the 2-cartesian squares

$$\begin{array}{ccccc} \widetilde{Y_n^\Phi} & \longrightarrow & \widetilde{\mathcal{U}_n^\Phi} & \hookrightarrow & \mathrm{Bun}_G \\ \downarrow & & \downarrow & & \downarrow \\ Y_n^\Phi & \longrightarrow & \mathcal{U}_n^\Phi & \hookrightarrow & \mathrm{Bun}_r \end{array}$$

where by base change we deduce that the maps $\widetilde{Y_n^\Phi} \to \widetilde{\mathcal{U}_n^\Phi}$ are smooth and surjective, and the $(\widetilde{\mathcal{U}_n^\Phi})$ form an open covering of Bun_G. From Theorem 3.4.13 (using $\mathrm{Bun}_G = \mathcal{H}\mathrm{om}_S(X, BG \times S) \simeq \mathrm{Sect}_S(X, X \times_S BG)$) and Corollary 3.4.14.2, we deduce that $\widetilde{Y_n^\Phi}$ is representable by a disjoint union of schemes of finite presentation over S, let them by A_i (imagine fixed n and ϕ). Then, chosen i, we have an open immersion $A_i \hookrightarrow \widetilde{Y_n^\Phi}$, together with a smooth surjective morphism $\widetilde{Y_n^\Phi} \to \widetilde{\mathcal{U}_n^\Phi}$, where the last is an algebraic open substack of Bun_G. Applying [Stacks, Lemma 05UP] to such map, we deduce that there exists $\mathcal{X}_{n,\phi,i}$ algebraic open substack of $\widetilde{\mathcal{U}_n^\Phi}$ with a smooth surjective morphism $A_i \to \mathcal{X}_{n,\phi,i}$ and hence of finite presentation over S. This concludes our proof. ∎

4 Derived stacks

4.1 Recalls of simplicial algebra

From now on k will be a fixed ring. Let's recall a classical fact, which will be used in the following.

Proposition 4.1.1. Every simplicial set X is the homotopy colimit over its cells. More precisely, let

$$\widetilde{X}: \Delta^{\mathrm{op}} \to \mathsf{Set} \hookrightarrow \mathsf{sSet}$$

be the corresponding bisimplicial set, which in degree k is given by the constant simplicial set X_k. Then we have an isomorphism

$$\operatorname{Hocolim} \widetilde{X} \to X$$

in $\mathrm{Ho}(\mathsf{sSet})$.

Proof. Classic, see Proposition 6.5, nLab. ∎

With the same idea one can prove that for $X \in \mathsf{sSet}$, we have $X \simeq \operatorname{Hocolim}_{\Delta^n \to X} \Delta^n$. We will use this argument a lot in the next chapters. Recall that the category of simplicial abelian groups (abelian group objects in sSet) has a model structure, right-transferred from sSet (so that equivalences and fibrations are the same). In particular every simplicial abelian group is a Kan complex thanks to the following classic proposition.

Proposition 4.1.2. Let $\mathcal{G}$ a simplicial group. Then $\mathcal{G}$ is a Kan complex. Moreover, let $\mathcal{A} \to \mathcal{B}$ be a surjection of simplicial abelian groups; then it is a fibration.

Proof. See [GJ99, Lemma I.3.4, Proposition III.2.10]. ∎

It is also a monoidal model category with the componentwise tensor product, i.e. given $\mathcal{A}, \mathcal{B} \in \mathsf{sAb}$ we can define $\mathcal{A} \otimes \mathcal{B} \in \mathsf{sAb}$ by $(\mathcal{A} \otimes \mathcal{B})_n = \mathcal{A}_n \otimes \mathcal{B}_n$. The category sComm of simplicial rings also has a right-transferred model structure and so we can define simplicial modules.

Definition 4.1.3. Let $A \in \mathsf{sComm}$ and $M \in \mathsf{sAb}$. We say that M is a simplicial A-module if there exists a map

$$\mu: A \otimes_{\mathbb{Z}} M \to M$$

of simplicial abelian groups satisfying the classical module axioms. The category of simplicial A-modules, denoted by $sA - \mathsf{Mod}$, inherits a model structure as before.

Similarly we can define the category of simplicial A-algebras (with $A \in \mathsf{sComm}$) simply as the comma category A/sComm. Seeing every ring as a discrete simplicial ring we recover the particular cases of $sk - \mathsf{Mod}$ and $sk - \mathsf{Alg}$.

Theorem 4.1.4 (Dold-Kan)**.** Let $k \in \mathsf{Comm}$. The normalised cochain complex functor

$$N: sk - \mathsf{Mod} \to \mathrm{Ch}^{\leq 0}(k)$$

sending $A \in sk - \mathsf{Mod}$ to the cochain complex given where $N(A)^m$ is the free k-module on A_m modulo degenerate simplices, induces a Quillen equivalence between the category of simplicial k-modules and of (cohomologically graded) cochain complexes.

Proof. See [Wei94, p. 8.4.1]. ∎

We will use later on this equivalence between the homotopy categories $\mathrm{Ho}(sk-\mathsf{Mod})$ and $D^{\leq 0}(k)$.

Let's now look a little more in details the model structure of $sk-\mathsf{Alg}$; we will just state some properties, without proving them. For $n \geq 0$ let's define the n-sphere k-modules by $S^n_k := S^n \otimes k \in sk-\mathsf{Mod}$. For every $\mathcal{X} \in \mathsf{sSet}$ we can build the free k-algebra $k[\mathcal{X}] \in sk-\mathsf{Alg}$ (pointwise construction). In particular we have isomorphisms

$$\mathrm{Hom}_{\mathrm{Ho}(sk-\mathsf{Alg})}(k[S^n], A) = [k[S^n], A]_{sk-\mathsf{Alg}} \simeq \pi_n(A)$$

where the homotopy groups are of course based in 0. The inclusions $S^n \simeq \partial\Delta^{n+1} \hookrightarrow \Delta^{n+1}$ induce natural maps $k[S^n] \to k[\Delta^{n+1}]$ and one can prove that the set of morphisms

$$\{k[S^n] \to k[\Delta^{n+1}] \mid n \in \mathbb{N}\}$$

is a set of generating cofibrations of $sk-\mathsf{Alg}$, which is also compactly generated.

A finite cell object is an element $A \in sk-\mathsf{Alg}$ for which there exists a finite sequence in $sk-\mathsf{Alg}$

$$A_0 = k \to A_1 \to A_2 \to \cdots \to A_m = A$$

such that for any i there exists a push-out square in $sk-\mathsf{Alg}$ given by

$$\begin{array}{ccc} k[S^{n_i}] & \longrightarrow & A_i \\ \downarrow & & \downarrow \\ k[\Delta^{n_i+1}] & \longrightarrow & A_{i+1} \end{array}$$

Using [HAGII, Prop 1.2.3.5] we see that the finitely presented objects of the model category $sk-\mathsf{Alg}$ (see Definition 4.4.2) are exactly retracts of finite cell objects. This reasoning can be generalised to the category of $sA-\mathsf{Alg}$, with $A \in sk-\mathsf{Alg}$, by considering $A[S^n] = k[S^n] \otimes_k A$ and similar.

The functor $\pi_0\colon sk-\mathsf{Alg} \to k-\mathsf{Alg}$ is left Quillen (the right adjoint being the classic inclusion $i\colon k-\mathsf{Alg} \to sk-\mathsf{Alg}$) and preseves finitely presented morphisms, but i does not. This means that being of finite presentation in $sk-\mathsf{Alg}$ is a much stronger condition than to be just a finitely presented k-algebra. The model structures on simplicial modules and algebras are cofibrantly generated, proper and cellular.

Given A simplicial commutative k-algebra (recall that any simplicial group is a Kan complex), we can consider its homotopy groups (based at 0) together as $\pi_*(A) := \bigoplus_n \pi_n(A)$. This is a graded abelian group (use Dold-Kan correspondence to identify homotopy of A with the homology of the normalized chain complex), and moreover it is also a graded commutative algebra. In-fact, given $\alpha\colon \Delta^n/\partial\Delta^n \simeq S^n \to A \in \pi_n(A)$ and $\beta\colon S^m \to A \in \pi_m(A)$ maps of pointed simplicial sets, we can consider

$$\alpha.\beta\colon S^n \times S^m \longrightarrow A \times A \longrightarrow A \otimes_k A \xrightarrow{\mu} A$$

where the last morphism is the multiplication on A. Observing that $(S^n, *) \cup (*, S^m)$ is sent to zero by $\alpha.\beta$ we can factorize through the smash product (defined, as in topological spaces, by the quotient of the product by the wedge sum) to obtain

$$\alpha.\beta\colon S^n \wedge S^m \cong S^{n+m} \to A$$

which will be the product of α and β. Similarly, for $M \in sA-\mathsf{Mod}$, $\pi_*(M)$ is a graded $\pi_*(A)$-module.

Let $f\colon A \to B \in \mathsf{sComm}$ and consider the classical adjunction

$$- \otimes_A B\colon sA-\mathsf{Mod} \rightleftarrows sB-\mathsf{Mod} : f^*$$

where f^* is just restriction of scalars. It is a Quillen adjunction, which becomes a Quillen equivalence when f is a weak equivalence (see [Fre09, p. 11.2.10]). The left derived functor is denoted by

$$- \otimes^{\mathbb{L}}_A B\colon \mathrm{Ho}(sA-\mathsf{Mod}) \to \mathrm{Ho}(sB-\mathsf{Mod}).$$

As done before, any ring can be considered as a constant simplicial ring and this induces a fully faithful embedding

$$j\colon \mathsf{Comm} \to \mathrm{Ho}(\mathsf{sComm})$$

which possesses a left adjoint

$$\pi_0\colon \mathrm{Ho}(\mathsf{sComm}) \to \mathsf{Comm}.$$

In the same way, considering a $\pi_0(A)$-module as a constant simplicial A-module (using the obvious map $A \to \pi_0(A)$) we obtain the adjunction

$$\pi_0\colon \mathrm{Ho}(sA - \mathsf{Mod}) \rightleftarrows \pi_0(A) - \mathsf{Mod} : j$$

where j is again a full embedding.

Finally we can consider the loop and suspension functors. They are defined in any nice enough model category by $\Omega(X) = * \times^h_X *$ and $S(X) = * \coprod^h_X *$, where $*$ is the terminal object (see Definition 1.1.7).

We observe that in our case the suspension $S\colon \mathrm{Ho}(sk - \mathsf{Mod}) \to \mathrm{Ho}(sk - \mathsf{Mod})$ corresponds to the shift functor $E \mapsto E[1]$, using Dold-Kan correspondence. Symmetrically, the loop is the opposite shift $E \mapsto E[-1]$.

4.2 Stable modules

Let's now introduce the category of stable A-modules, denoted by $Sp(A - \mathsf{Mod})$. We will just use it when A is a simplicial ring, and there is a more trivial description which we will give at the end of this section; for now let's assume to be in a good enough model category $\mathcal{C}$ (the precise assumptions are exactly the Homotopical Algebraic Context one, found in [HAGII]). This section is just a summary of the highlights of [HAGII, p. 1.2.11].

Assume $\mathcal{C}$ is a pointed model category, $\mathbf{1} \in \mathrm{Ob}(\mathcal{C})$ its monoidal unit, and let S be the suspension functor. Let $S^1_{\mathcal{C}} \in \mathcal{C}$ be a cofibrant model for $S(\mathbf{1}) \in \mathrm{Ho}(\mathcal{C})$ and, given $A \in \mathsf{Comm}(\mathcal{C})$ consider

$$S^1_A \coloneqq S^1_{\mathcal{C}} \otimes A \in A - \mathsf{Mod}$$

the free A-module on $S^1_{\mathcal{C}}$. The functor

$$S^1_A \otimes_A -\colon A - \mathsf{Mod} \to A - \mathsf{Mod}$$

is left Quillen and has a right adjoint

$$\underline{\mathrm{Hom}}_A(S^1_A, -)\colon A - \mathsf{Mod} \to A - \mathsf{Mod}.$$

We define the category $Sp^{S^1_A}(A - \mathsf{Mod})$ of spectra of A-modules following [Hov01].

Definition 4.2.1. With the notation above, let $Sp^{S^1_A}(A - \mathsf{Mod})$ be the category of whose objects are sequences $(X_n)_{n\in\mathbb{N}}$ together with structure maps $\sigma\colon S^1_A \otimes_A X_n \to X_{n+1}$ for all n. A morphism of spectra from X to Y is a collection of maps $f_n\colon X_n \to Y_n$ commuting with structure maps, i.e. making the following squares

$$\begin{array}{ccc} S^1_A \otimes_A X_n & \xrightarrow{\sigma_X} & X_{n+1} \\ \downarrow{\scriptstyle S^1_A \otimes_A f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ S^1_A \otimes_A Y_n & \xrightarrow{\sigma_Y} & Y_{n+1} \end{array}$$

commutative. We will call $Sp^{S^1_A}(A - \mathsf{Mod})$ the category of *stable A-modules*, and we will often denote it by $Sp(A - \mathsf{Mod})$.

We initially endow $Sp^{S^1_A}(A - \mathsf{Mod})$ with the projective model structure, with componentwise weak equivalences and fibrations. Then we will consider the left Bousfield localization of this structure, whose local objects are the stable modules $M_* \in Sp(A - \mathsf{Mod})$ such that each induced map

$$M_n \to \mathbb{R}\,\underline{\mathrm{Hom}}_A(S^1_A, M_{n+1})$$

is an isomorphism in $\mathrm{Ho}(A - \mathsf{Mod})$. Details can be found in [Hov01].

There exists a Quillen adjunction, functorial in A, given by

$$S_A\colon A - \mathsf{Mod} \rightleftarrows Sp(A - \mathsf{Mod}) : (-)_0$$

where the right adjoint acts by $M_* \mapsto M_0$, while the left adjoint is given by $S_A(M)_n = (S^1_A)^{\otimes_A n} \otimes_A M$. Under reasonable assumption, S_A is a fully faithful embedding, i.e. every A-module is a particular stable A-module.

Lemma 4.2.2. If the suspension functor $S\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ is fully faithful then for any $A \in \mathsf{Comm}(\mathcal{C})$ the functor S_A is fully faithful. Moreover, if $\mathcal{C}$ is a stable model category as in Definition 1.1.8, S_A is a Quillen equivalence.

Proof. See [HAGII, Lemma 1.2.11.2]. ■

One can then prove that the $\mathrm{Ho}(Sp(A-\mathsf{Mod}))$ is a closed symmetric monoidal category, with the symmetric monoidal structure inherited from $\mathrm{Ho}(A-\mathsf{Mod})$ and still denoted by $-\otimes_A^{\mathbb{L}}-$. In particular this means that for $M_*, N_* \in Sp(A-\mathsf{Mod})$ we have a stable A-module of morphisms

$$\mathbb{R}\,\underline{\mathrm{Hom}}_A^{Sp}(M_*, N_*) \in \mathrm{Ho}(Sp(A-\mathsf{Mod})).$$

Let's now give some more words.

Definition 4.2.3. A stable A-module $M_* \in \mathrm{Ho}(Sp(A-\mathsf{Mod}))$ is called 0-*connective* if it is isomorphic to some $S_A(M)$ for an A-module $M \in \mathrm{Ho}(A-\mathsf{Mod})$. By induction, for $n>0$, M_* is $(-n)$-*connective* if it is isomorphic to $\Omega(M_*')$ for some $-(n-1)$-connective M_*' (recall that Ω is the loop functor on $\mathrm{Ho}(Sp(A-\mathsf{Mod}))$).

Recall that by $(A-\mathsf{Mod})^\wedge$ we mean the prestack category on $A-\mathsf{Mod}$, i.e. the left Bousfield localization of the projective strucure on $SPr(A-\mathsf{Mod})$ with respect to the Yoneda images of weak equivalences of $A-\mathsf{Mod}$. Consider now the following "restricted" Yoneda embedding:

$$\underline{h}_s^- : Sp(A-\mathsf{Mod})^{\mathrm{op}} \to (A-\mathsf{Mod}^{\mathrm{op}})^\wedge, \qquad M_* \mapsto \Big(\underline{h}_s^{M_*} : N \in A-\mathsf{Mod} \mapsto Hom(M_*, \Gamma_*(S_A(N)))\Big)$$

where we chose a simplicial resolution functor Γ_* for the model category $Sp(A-\mathsf{Mod})$. One can prove the following

Proposition 4.2.4. For any $A \in \mathsf{Comm}(\mathcal{C})$, the functor $\underline{h}_s^-$ has a total right derived functor

$$\mathbb{R}\underline{h}_s^- : \mathrm{Ho}(Sp(A-\mathsf{Mod}))^{\mathrm{op}} \to \mathrm{Ho}((A-\mathsf{Mod}^{\mathrm{op}})^\wedge)$$

which commutes with homotopy limits.

Proof. See [HAGII, Proposition 1.2.11.3]. ■

We will use this functor later, talking about the cotangent complex.

Finally, as said in the beginning, let's specialize this very general definition to our well known simplicial case.

Remark 4.2.5. Let $A \in sk-\mathsf{Alg}$ and consider the homotopy category $\mathrm{Ho}(Sp(sA-\mathsf{Mod}))$. It can be described as follows: let $N(A)$ be the normalized dg-algebra of A by Dold-Kan (see Theorem 4.1.4 recalling that N is a lax symmetric monoidal functor). Then we can consider the model category of unbounded $N(A)$-dg-modules (weak equivalences are quasi-isomorphisms of complexes and fibrations are the ones of $\mathrm{Ch}(k)$) and its homotopy category $\mathrm{Ho}(N(A)-dg-\mathsf{Mod})$. One can then prove that $\mathrm{Ho}(Sp(sA-\mathsf{Mod}))$ and $\mathrm{Ho}(N(A)-dg-\mathsf{Mod})$ are equivalent. In particular if A is a commutative k-algebra, then $N(A)=A$ and we find

$$\mathrm{Ho}(Sp(sA-\mathsf{Mod})) \simeq D(A) \simeq \mathrm{Ho}(\mathrm{Ch}(A)).$$

4.3 Affine cotangent complex

Let's now introduce the definition of cotangent complex. We will start from the affine case and then globalize. We will need some notions of simplicial commutative algebra before.

Definition 4.3.1. Let $A \in \mathsf{sComm}$ and $M \in sA-\mathsf{Mod}$. The *trivial square zero extension* of A by M is the simplicial commutative ring $A \oplus M$ defined for every n by the classical trivial square zero extension $A_n \oplus M_n$ (faces an degeneracies are defined in the obvious way). In particular, this means that for $a, a' \in A_n$ and $m, m' \in M_n$ we have

$$(a, m)\cdot(a', m') := (aa', am' + a'm)$$

and sum is obvious.

It is a bi-augmented A-algebra $A \to A \oplus M \to A$ by the inclusion/projection respectively. Recall that the classical (i.e. using normal rings and modules) set $\mathrm{Der}_k(A, M)$ is defined as the set of k-linear sections of the projection $A \oplus M \to A$. One can prove that $\mathrm{Der}_k(A, -)$ defines a functor which is corepresentable by $\Omega^1_A \in A - \mathsf{Mod}$, the module of Kähler differentials.

We can generalize this to the simplicial context: let $A \in sk - \mathsf{Alg}$ and $M \in sA - \mathsf{Mod}$. Recall that sComm is a simplicial model category (see Definition 1.2.8) and, denoting by $\underline{\mathrm{Hom}}$ its simplicial Hom set, we can consider its derived version $\mathbb{R}\,\underline{\mathrm{Hom}}(A, B) = \underline{\mathrm{Hom}}(QA, B)$ (where QA is a cofibrant replacement for A). Similarly, the category $sA - \mathsf{Mod}$ is enriched over $sk - \mathsf{Mod}$, i.e. we have simplicial Hom sets which are moreover simplicial k-modules (just as it happens in the classical case). To refresh our memory, we have

$$\underline{\mathrm{Hom}}_{\mathsf{sComm}}(A, B)\colon [n] \mapsto \mathrm{Hom}_{\mathsf{sComm}}(A \otimes \Delta^n, B),$$
$$\underline{\mathrm{Hom}}_{sA-\mathsf{Mod}}(M, N)\colon [n] \mapsto \mathrm{Hom}_{sA-\mathsf{Mod}}(M \otimes_A A[\Delta^n], N).$$

which $(A \otimes \Delta^n)_m = \bigotimes_{\sigma\colon \Delta^m \to \Delta^n} A_m$ is just a particular case of Theorem 2.2.9, recalling that coproduct in Comm is the tensor product. See [SCR] for the details.

Definition 4.3.2. The simplicial set of derived derivations $\mathbb{D}er_k(A, M)$ is defined as the homotopy pullback (computed, equally, either in sSet or $sk - \mathsf{Mod}$)

$$\begin{array}{ccc} \mathbb{D}er_k(A, M) & \longrightarrow & \mathbb{R}\,\underline{\mathrm{Hom}}_{sk-\mathsf{Alg}}(A, A \oplus M) \\ \downarrow & & \downarrow \pi_* \\ * = k & \xrightarrow{\mathrm{id}_A} & \mathbb{R}\,\underline{\mathrm{Hom}}_{sk-\mathsf{Alg}}(A, A) \end{array}$$

i.e. the homotopy fiber at id_A of the natural map $\mathbb{R}\,\underline{\mathrm{Hom}}_{sk-\mathsf{Alg}}(A, A \oplus M) \to \mathbb{R}\,\underline{\mathrm{Hom}}_{sk-\mathsf{Alg}}(A, A)$. To be precise, the homotopy fiber is taken exactly at the image of id_A through the canonical map $\underline{\mathrm{Hom}}_{sk-\mathsf{Alg}}(A, A) \to \mathbb{R}\,\underline{\mathrm{Hom}}_{sk-\mathsf{Alg}}(A, A)$.

A more compact notation is

$$\mathbb{D}er_k(A, M) = \mathbb{R}\,\underline{\mathrm{Hom}}_{-/A}(A, A \oplus M)$$

where $\underline{\mathrm{Hom}}_{-/A}$ is the simplicial Hom of the comma category $sk - \mathsf{Alg}/A$.

Proposition 4.3.3. The functor

$$\mathbb{D}er_k(A, -)\colon \mathrm{Ho}(sA - \mathsf{Mod}) \to \mathrm{Ho}(\mathsf{sSet})$$

is co-represented by a simplicial A-module $\mathbb{L}_A = \mathbb{L}_{A/k} \in \mathrm{Ho}(sA - \mathsf{Mod})$, called the *cotangent complex of* A. This means that for any $M \in sA - \mathsf{Mod}$ one has

$$\mathbb{D}er_k(A, M) \cong \mathrm{Map}_{sA-\mathsf{Mod}}(\mathbb{L}_A, M) \in \mathrm{Ho}(\mathsf{sSet}).$$

Proof. The proof can be found in [Qui70; GH00]. One possible construction for $\mathbb{L}_A$ is by setting

$$\mathbb{L}_A \coloneqq \Omega^1_{QA} \otimes^{\mathbb{L}}_{QA} A \in \mathrm{Ho}(sA - \mathsf{Mod})$$

where QA is a cofibrant replacement for A, and Ω^1_{QA} is obtained by applying levelwise the construction of Kähler differentials. ∎

Let's observe that $\pi_0(\mathbb{L}_A) \simeq \Omega^1_{\pi_0(A)} \in \pi_0(A) - \mathsf{Mod}$ by adjunction. The construction $\mathbb{L}_A$ is functorial in A and therefore for any morphism $f\colon A \to B$ we have an induced morphism $\mathbb{L}_A \to \mathbb{L}_B$ in $\mathrm{Ho}(sA - \mathsf{Mod})$, which by adjunction induces a map

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B$$

in $\mathrm{Ho}(sB - \mathsf{Mod})$.

Definition 4.3.4. The homotopy cofiber of the above morphism is denoted by $\mathbb{L}_{B/A}$ (although it depends on f) and it is called the *relative cotangent complex of* B *over* A. More specifically we have, by definition, the homotopy cocartesian diagram

$$\begin{array}{ccc} \mathbb{L}_A \otimes^{\mathbb{L}}_A B & \xrightarrow{\mathbb{L}f} & \mathbb{L}_B \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{L}_{B/A} \end{array}$$

where the diagram must be taken in the category $sB - \mathsf{Mod}$ (so choosing representatives) and $*$ is the terminal object in $sB - \mathsf{Mod}$.

4.4 Some properties of modules and morphisms

As described in the previous section, given $A \in sk - \mathsf{Alg}$ we can consider the graded commutative k-algebra $\pi_*(A)$, functorial in A. In particular $\pi_i(A)$ is a $\pi_0(A)$-module and hence for a map $A \to B$ in $sk - \mathsf{Alg}$ we obtain a natural morphism

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$$

of $\pi_0(B)$-modules. More generally, for $M \in sA - \mathsf{Mod}$ we have a morphism of $\pi_0(A)$-modules $\pi_0(M) \to \pi_*(M)$ giving rise to

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(M) \to \pi_*(M).$$

Definition 4.4.1. Let $A \in sk - \mathsf{Alg}$ and $M \in sA - \mathsf{Mod}$. The simplicial A-module M is *strong* if the natural morphism

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(M) \to \pi_*(M)$$

is an isomorphism of graded $\pi_*(A)$-modules. A morphism $A \to B$ in $sk - \mathsf{Alg}$ is *strong* if B is strong as a simplicial A-module.

Let's now give some definitions of classical module properties, but seen in a *homotopical* point of view. This does not happen in the classical case, i.e. with normal rings, because the model structures are trivial and so it does not change to work in the normal category or in the homotopy one. Here, instead, there is a difference. All of the following can be carried out in a greater generality, in a monoidal model category satisfying some niceness properties (an Homotopical Algebraic Context), see [HAGII, Part 1]. This section makes more sense in a general HA context, for our purposes the only results that matter (that can be as well taken as definitions) are Lemma 4.4.4 and Theorem 4.4.10. We have chosen to include the more general definitions because they really give an intuition of how classic ones are generalized in an homotopical context.

Definition 4.4.2. Let $A \in sk - \mathsf{Alg}$ and $M \in sA - \mathsf{Mod}$.

1. M is *flat* as a simplicial A-module if the functor

$$- \otimes^{\mathbb{L}}_A M \colon \operatorname{Ho}(sA - \mathsf{Mod}) \to \operatorname{Ho}(sA - \mathsf{Mod})$$

   preserves homotopy pullbacks.

2. M is *projective* if it is a retract of $\coprod^{\mathbb{L}}_I A$, for some set I, in the homotopy category $\operatorname{Ho}(sA - \mathsf{Mod})$.

3. M is *perfect* if the natural map (coming from derived tensor adjunction)

$$M \otimes^{\mathbb{L}}_A M^\vee \to \mathbb{R}\,\underline{\operatorname{Hom}}_{sA-\mathsf{Mod}}(M, M)$$

   is an isomorphism in $\operatorname{Ho}(sA - \mathsf{Mod})$. Here we are using the derived tensor product, the derived dual $M^\vee := \mathbb{R}\,\underline{\operatorname{Hom}}_{sA-\mathsf{Mod}}(M, A)$ and the structure of simplicial A-module of the internal hom.

4. $f \colon M \to N \in sA - \mathsf{Mod}$ is *finitely presented* if for any filtered diagram $\{M \to Z_i\}_i$ in $sA - \mathsf{Mod}$, the natural morphism

$$\operatorname{Hocolim}_i \operatorname{Map}_{M/sA-\mathsf{Mod}}(N, Z_i) \to \operatorname{Map}_{M/sA-\mathsf{Mod}}(N, \operatorname{Hocolim}_i Z_i)$$

   is an isomorphism in $\operatorname{Ho}(\mathsf{sSet})$.

Here's a bunch of properties.

Proposition 4.4.3. Let $A \in sk - \mathsf{Alg}$ and $M \in sA - \mathsf{Mod}$.

1. The free A-module A^n is flat
2. Flat modules in $\mathrm{Ho}(sA - \mathsf{Mod})$ are stable by derived tensor products, finite coproducts and retracts.
3. Projective modules in $\mathrm{Ho}(sA - \mathsf{Mod})$ are stable by derived tensor products, finite coproducts and retracts.
4. If M is a flat/projective $sA - \mathsf{Mod}$, then $M \otimes^{\mathbb{L}}_A B$ (fixing $A \to B$ a map in sComm) is a flat/projective $sB - \mathsf{Mod}$.
5. A perfect module is flat.

Proof. See [HAGII, Prop 1.2.4.2]. ∎

While the previous definitions and properties can be generalized, we can use our definition of strongness Definition 4.4.1 to have more useful characterizations of such properties in $sk - \mathsf{Alg}$ and $sA - \mathsf{Mod}$.

Lemma 4.4.4. Let $A \in sk - \mathsf{Alg}$ and $M \in sA - \mathsf{Mod}$.

1. M is projective (resp. flat) if and only if it is strong and $\pi_0(M)$ is a projective (resp. flat) $\pi_0(A)$-module.
2. M is perfect if and only if it is strong and $\pi_0(M)$ is a projective $\pi_0(A)$-module of finite type.
3. M is projective and finitely presented if and only if it is perfect.

Proof. See [HAGII, Lemma 2.2.2.2]. ∎

Let's now focus on properties of morphisms. The philosophy will always be to look at π_0 and ask some strongness condition. As usual, following [HAGII] we will give general definitions (written, for readability, already in our specific case) and then specialize to our case with simplicial modules.

Definition 4.4.5. Let $f : A \to B$ in $sk - \mathsf{Alg}$.

1. The map f is a (model) *epimorphism* if for any $C \in sA - \mathsf{Alg}$, the simplicial set $\mathrm{Map}_{sA-\mathsf{Alg}}(B, C)$ is either empty or contractible.
2. The map f is *flat* if the induced functor
$$- \otimes^{\mathbb{L}}_A B : \mathrm{Ho}(sA - \mathsf{Mod}) \to \mathrm{Ho}(sB - \mathsf{Mod})$$
commutes with finite homotopy limits.
3. The map f is a *formal Zariski open immersion* if it is flat and the forgetful functor
$$f_* : \mathrm{Ho}(sB - \mathsf{Mod}) \to \mathrm{Ho}(sA - \mathsf{Mod})$$
is fully faithful.
4. The map f is *formally unramified* if $\mathbb{L}_{B/A} \simeq 0$ in $\mathrm{Ho}(sB - \mathsf{Mod})$.
5. The map f is *formally étale* if the natural morphism
$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B$$
is an isomorphism in $\mathrm{Ho}(sB - \mathsf{Mod})$.

Definition 4.4.6. A morphism $f : A \to B \in sk - \mathsf{Alg}$ is a Zariski open immersion/unramified/étale if it is finitely presented and a formal Zariski open immersion/formally unramified/formally étale.

Remark 4.4.7. Recall that in a category $\mathcal{C}$ with fiber products, a map $i : x \to y$ is a monomorphism if and only if $x \to x \times_y x$ is an isomorphism. In the model context this gives the definition of i being a monomorphism if the diagonal $x \to x \times^h_y x$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. Equivalently, i is a monomorphism if and only if for any $z \in \mathcal{C}$ the induced map $\mathrm{Map}_{\mathcal{C}}(z, x) \to \mathrm{Map}_{\mathcal{C}}(z, y)$ is a monomorphism in sSet. Moreover a map $f : K \to L$ of simplicial sets is a monomorphism if and only if for any $s \in L$ the homotopy fiber of f at s is either empty or contractible.

Thus a map $A \to B \in sk - \mathsf{Alg}$ is an epimorphism in the sense of Definition 4.4.5 iff it is a monomorphism (in the sense explained above) in the opposite model category, or equivalently if the map $B \otimes^{\mathbb{L}}_A B \to B$ is an isomorphism in $\mathrm{Ho}(sk - \mathsf{Alg})$.

All these definitions are stable by compositions, equivalences and homotopy pushouts. For a more detailled discussion see [HAGII, p. 1.2.6]. Let's now see these general definitions in our simplicial case.

Definition 4.4.8. A map $f\colon A \to B$ is *strongly flat/strongly étale/strong Zariski open immersion* if it is strong and if the morphism of affine schemes

$$\operatorname{Spec} \pi_0(B) \to \operatorname{Spec} \pi_0(A)$$

is flat/étale/Zariski open immersion.

Proposition 4.4.9. Let $f\colon A \to B$ in $sk - \mathsf{Alg}$; it is finitely presented in the sense of Definition 4.4.2 if and only if

1. the morphism $\pi_0(A) \to \pi_0(B)$ is a finitely presented morphism of rings (classical sense),
2. the cotangent complex $\mathbb{L}_{B/A} \in \operatorname{Ho}(sB - \mathsf{Mod})$ is finitely presented.

Proof. See [HAGII, Proposition 2.2.2.4]. ∎

Finally, the key result, confirming our philosophy.

Theorem 4.4.10. A morphism in $sk-\mathsf{Alg}$ is flat/Zariski open immersion/étale if and only if it is strongly flat/strong Zariski open immersion/strongly étale.

Proof. See [HAGII, Theorem 2.2.2.6]. ∎

4.5 Derived Stacks

We work in the category $\mathsf{dAff} := \mathsf{sComm}^{\mathrm{op}}$, or, if needed, we can work over a different base commutative ring k, and we will consider simplicial commutative k-algebras. We will write $\operatorname{Spec} A$ for $A \in \mathsf{sComm}$ just as a formal writing. We endow dAff with the opposite model structure of the classical one on sComm, the right-transferred structure from the forgetful functor to simplicial sets (concretely speaking weak equivalences and fibrations are taken in sSet). Let's consider $SPr(\mathsf{dAff}) = \operatorname{Funct}(\mathsf{dAff}^{\mathrm{op}}, \mathsf{sSet})$, equivalently thought as simplicial object in $\mathsf{Psh}(\mathsf{dAff})$. We will introduce three different model structures on this category, by successive left Bousfield localizations, and to avoid confusion we will use different names.

Projective model The first model structure, denoted again by $SPr(\mathsf{dAff})$, is simply the projective model structure on $\mathsf{sSet}^{\mathsf{dAff}^{\mathrm{op}}}$ with weak equivalences and fibrations defined componentwise.

Prestack category Let's now consider the Yoneda embedding

$$h\colon \mathsf{dAff} \to SPr(\mathsf{dAff}), \qquad X \mapsto h_X = \operatorname{Hom}(-, X)$$

extending the classical Yoneda map with constant simplicial sets. Given a weak equivalence $X \xrightarrow{\sim} Y$ in dAff we obtain a map $h_X \to h_Y$ which has no reason to be a weak equivalence a priori, and hence we cannot factorize h through $\operatorname{Ho}(\mathsf{dAff})$. For this we introduce a new model structure $\mathsf{dAff}^{\wedge}$ via left Bousfield localization of $SPr(\mathsf{dAff})$ with respect to all maps $h_X \to h_Y$ coming from $X \xrightarrow{\sim} Y$ in dAff.

This intermediate model structure does not appear when defining stacks since they are simplicial presheaves over Aff, whose model structure is trivial, and thus there's no need to worry about equivalences at the source.

Proposition 4.5.1. The fibrant objects in $\mathsf{dAff}^{\wedge}$ are the $F\colon \mathsf{dAff}^{\mathrm{op}} \to \mathsf{sSet}$ satisfying:

1. For any $X \in \mathsf{dAff}$, $F(X)$ is fibrant (i.e. a Kan complex).
2. For any $X \xrightarrow{\sim} Y$ in dAff, the induced $F(Y) \to F(X)$ is an equivalence in sSet.

Proof. See [HAGI, p. 4.1]. ∎

Recall that $\mathsf{dAff}^\wedge$ is a simplicial model category, being a left Bousfield localization of the simplicial model category $SPr(\mathsf{dAff})$. Thus, for fibrant and cofibrant objects, the homotopy function complex is exactly the internal simplicial hom. We will implicitely use from now on the following version of Yoneda lemma. Define

$$\underline{h}\colon \mathsf{dAff} \to \mathsf{dAff}^\wedge \colon x \mapsto (\underline{h}_x \colon y \mapsto \underline{\mathrm{Hom}}(y, x)).$$

Recall that, in the homotopy category, for x fibrant, this is exactly the homotopy function complex between y and x. By [HAGI, Lemma 4.2.1], the functor $\underline{h}$ preserves fibrant objects and weak equivalences between them, so we can right derive it to obtain $\mathbb{R}\underline{h} = \underline{h} \circ R$, where R is a functorial fibrant replacement in dAff.

Proposition 4.5.2 (Model Yoneda Lemma)**.** Let's consider the map $\mathbb{R}\underline{h}$ as defined above. Then

1. it is fully faithful;
2. the canonical map $h_x \to \mathbb{R}\underline{h}_x$ is an isomorphism in $\mathrm{Ho}(\mathsf{dAff}^\wedge)$;
3. for any fibrant object $F \in \mathsf{dAff}^\wedge$, using internal simplicial hom, we have

$$\mathbb{R}\,\underline{\mathrm{Hom}}(\mathbb{R}\underline{h}_x, F) \simeq \mathbb{R}\,\underline{\mathrm{Hom}}(h_x, F) \simeq F(x)$$

in $\mathrm{Ho}(\mathsf{sSet})$.

Proof. See [HAGI, Theorem 4.2.3]. ■

This model-version of Yoneda gives us an equivalent representation of the discrete presheaf h_X, which is clearly not discrete anymore.

Stack category Finally we want to introduce also a notion of local equivalences for morphisms in $\mathsf{dAff}^\wedge$, and for this we define the *étale* Grothendieck topology on $\mathrm{Ho}(\mathsf{dAff})$: a family of morphisms $\{A \to A_i\}_i$ in sComm is an étale covering if every $A \to A_i$ is étale (in simplicial sense, see Definition 4.4.5 and Theorem 4.4.10) and if the family of functors

$$\{- \otimes^{\mathbb{L}}_A A_i \colon \mathrm{Ho}(sA - \mathsf{Mod}) \to \mathrm{Ho}(sA_i - \mathsf{Mod})\}$$

is conservative. A completely analogue definition gives rise to the étale topology on $\mathrm{Ho}(\mathsf{dAff}/X)$, for $X \in \mathsf{dAff}$. We use this topology to define homotopy sheaves for objects $F \in \mathrm{Ho}(\mathsf{dAff}^\wedge)$, which we can assume to be fibrant (and hence to preserve weak equivalences). Given such an $F\colon \mathsf{dAff}^{\mathrm{op}} \to \mathsf{sSet}$ we consider the presheaf $X \mapsto \pi_0(F(X))$: since it sends equivalences in dAff to isomorphism of sets, we can factorize it to obtain

$$\pi_0^{pr}(F)\colon \mathrm{Ho}(\mathsf{dAff})^{\mathrm{op}} \to \mathsf{Set},$$

which we can finally sheafify to obtain the sheaf $\pi_0(F)$. Similarly, for $X \in \mathsf{dAff}$ and $s \in F(X)_0$ we can define a presheaf of groups sending $f\colon Y \to X$ to $\pi_j(F(Y), f^*(s))$, which again we can factorize through the homotopy category and the sheafify to obtain $\pi_j(F, s)\colon \mathrm{Ho}(\mathsf{dAff}/X)^{\mathrm{op}} \to \mathsf{Grp}$.

Definition 4.5.3. Using the same notations as above, $\pi_0(F)$ and $\pi_i(F, s)$ are called the *homotopy sheaves of* F.

As the careful reader may have noticed, we have defined them for a generic object of $\mathrm{Ho}(\mathsf{dAff}^\wedge)$ by choosing a particular fibrant approximation, so we should verify that this does not really depend on this arbitrary choice. This comes from general Bousfield localization properties: local weak equivalences between local fibrant objects are indeed global equivalences. We are ready to define the local model structure.

Definition 4.5.4. Let $f\colon F \to F' \in SPr(\mathsf{dAff})$.

- The map f is a *local equivalence* if the induced morphism $\pi_0(F) \to \pi_0(F')$ is an isomorphism of sheaves on $\mathrm{Ho}(\mathsf{dAff})$, and if for any $X \in \mathsf{dAff}$, $s \in F(X)_0$ the induced morphism $\pi_j(F, s) \to \pi_j(F', f(s))$ is an isomorphism of sheaves on $\mathrm{Ho}(\mathsf{dAff}/X)$.
- The map f is a *local cofibration* if it is a cofibration in $\mathsf{dAff}^\wedge$, i.e. a cofibration in $SPr(\mathsf{dAff})$.

Local fibrations are defined by lifting properties. We call this model structure the *local model structure* and we write it like $\mathsf{dAff}^\sim$.

We are not proving that this is actually a model structure, see [HAGI] for the details. As before, we have a nice characterization of fibrant objects in $\mathsf{dAff}^{\sim}$.

Proposition 4.5.5. A presheaf $F\colon \mathsf{dAff}^{\mathrm{op}} \to \mathsf{sSet}$ is fibrant if and only if it satisfies the following properties.

1. For any $X \in \mathsf{dAff}$, $F(X)$ is fibrant (i.e. a Kan complex).
2. For any $X \xrightarrow{\sim} Y$ in dAff, the induced $F(Y) \to F(X)$ is an equivalence in sSet.
3. Given $X, Y \in \mathsf{dAff}$, the natural morphism
$$F(X \times^h Y) \to F(X) \times F(Y)$$
is an isomorphism in $\mathrm{Ho}(\mathsf{sSet})$.
4. For any $X \in \mathsf{dAff}$ and $H \to X$ étale hypercovering, the natural map
$$F(X) \to \mathrm{Holim}_{[n]\in\Delta} F(H_n)$$
is an equivalence in sSet.

Finally we can define derived stacks.

Definition 4.5.6. An object $F \in SPr(\mathsf{dAff})$ is called a *derived stack* if it respects conditions (2), (3) and (4) of Proposition 4.5.5. The homotopy category $\mathrm{Ho}(\mathsf{dAff}^{\sim})$ is called the homotopy category of derived stacks (often we will just refer to its objects as derived stacks) and morphisms in this category are denoted by $[F, F']$.

A particular case is the following.

Definition 4.5.7. Let $A \in \mathsf{sComm}$ and consider $\mathrm{Spec}\, A \in \mathsf{dAff}$. We have $R(\mathrm{Spec}\, A) = \mathrm{Spec}\, Q(A)$, for Q a cofibrant replacement in sComm. We define
$$\mathbb{R}\underline{\mathrm{Spec}}\, A := \mathbb{R}\underline{h}_{\mathrm{Spec}\, A}\colon B \in \mathsf{sComm} \mapsto \underline{\mathrm{Hom}}(QA, -).$$
Any element of $\mathsf{dAff}^{\sim}$ isomorphic to $\mathbb{R}\underline{\mathrm{Spec}}\, A$ for some $A \in \mathsf{sComm}$ is called a *derived affine scheme*.

Finally let's talk about internal homs in the homotopy category $\mathrm{Ho}(\mathsf{dAff}^{\sim})$. A much more general and detailled treatment can be found at [HAGI, p. 3.6].

Proposition 4.5.8. The homotopy category of derived stacks $\mathrm{Ho}(\mathsf{dAff}^{\sim})$ is cartesian closed. Its internal homs are denoted by $\mathbb{R}\,\mathcal{H}\mathrm{om}(-,-)$. Explicitely, given F and G derived stacks, we have
$$\mathbb{R}\,\mathcal{H}\mathrm{om}(F, G) \simeq \mathcal{H}\mathrm{om}(F, R_{\mathrm{inj}}G),$$
where R_{inj} is the fibrant replacement in the model category $SPr(\mathsf{dAff})$ with the local injective model structure and $\mathcal{H}\mathrm{om}$ is the internal hom functor of $SPr(\mathsf{dAff})$. Furthermore, if G is a derived stack then $\mathbb{R}\,\mathcal{H}\mathrm{om}(F, G)$ is a derived stack.

Proof. See [HAGI, Proposition 3.6.1, Corollary 3.6.2, Definition 3.6.3]. ∎

4.6 Derived geometric stacks

We can give the same definitions of derived geometric n-stacks just as it is done for normal stacks in Section 2.3.

Let's first recall that we can consider the model Yoneda embedding
$$\mathbb{R}\underline{h}\colon\ \mathrm{Ho}(\mathsf{dAff}) \to \mathrm{Ho}(\mathsf{dAff}^{\wedge})$$
and we have a corresponding derived analogue of the faithfully flat descent (i.e. the classical theorem stating that $\mathrm{Spec}(A)\colon B \in \mathsf{Comm} \mapsto \mathrm{Hom}(A, B)$ is a sheaf for the fppf topology on Aff). This practically means that $\mathbb{R}\underline{\mathrm{Spec}}\, A\colon B \in \mathsf{sComm} \mapsto \underline{\mathrm{Hom}}(QA, B)$ satisfies the descent condition for étale hypercoverings, i.e. it is a derived stack. Let's recall our terminology.

Definition 4.6.1. Objects in the essential image of $\mathbb{R}\underline{h}$ are called *derived affine schemes*, or *representable derived stacks.*

Observe that, differently from affine schemes of Definition 2.3.1, they are not 0-truncated. Let's now give a general definition of a derived scheme.

Definition 4.6.2.

1. A map of derived stacks $F \to F'$ is a *monomorphism* if the induced $F \to F \times^h_{F'} F$ is an equivalence (see Remark 4.4.7).

2. A map of derived stacks $F \to F'$ is an *epimorphism* if the induced $\pi_0(F) \to \pi_0(F')$ is an epimorphism of sheaves.

3. Let $i\colon F \to \mathbb{R}\underline{\mathrm{Spec}}\,A$ be a morphism. It is a *Zariski open immersion* if it satisfies the following conditions.

   (a) The map i is a monomorphism.

   (b) There exists a family of (simplicial) Zariski open immersions $\{A \to A_i\}_i$ such that

   $$\coprod_i \mathbb{R}\underline{\mathrm{Spec}}\,A_i \to \mathbb{R}\underline{\mathrm{Spec}}\,A$$

   factors through an epimorphism to F.

4. A map $F \to F'$ is a Zariski open immersion if for any derived affine scheme X and any map $X \to F'$ we have that

   $$\begin{array}{ccc} F \times^h_{F'} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ F & \longrightarrow & F' \end{array}$$

   the induced map $F \times^h_{F'} X \to X$ is a Zariski open immersion (in the sense of the previous point).

5. A derived stack F is a *derived scheme* if there exists a family of derived affine schemes $\{\mathbb{R}\underline{\mathrm{Spec}}\,A_i\}_i$ and Zariski open immersions $\mathbb{R}\underline{\mathrm{Spec}}\,A_i \to F$ such that

   $$\coprod_i \mathbb{R}\underline{\mathrm{Spec}}\,A_i \to F$$

   is an epimorphism of sheaves. Such a family $\{\mathbb{R}\underline{\mathrm{Spec}}\,A_i \to F\}$ is a *Zariski atlas* for F.

We say that a morphism of derived schemes $X \to Y$ is smooth/flat/étale/finitely presented (etc) if there exist Zariski atlases $\{\mathbb{R}\underline{\mathrm{Spec}}\,A_i \to X\}$ and $\{\mathbb{R}\underline{\mathrm{Spec}}\,B_j \to Y\}$ with commutative squares (in $\mathrm{Ho}(\mathsf{dAff}^{\sim})$)

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ \mathbb{R}\underline{\mathrm{Spec}}\,A_i & \longrightarrow & \mathbb{R}\underline{\mathrm{Spec}}\,B_j \end{array}$$

such that the downward arrow has the desired properties (for each i, j). We have, as usual, stability by composition and homotopy base changes.

Now we are ready to define, by recursion, derived geometric stacks. The definition is exactly the same as for geometric stacks (we simply work in a different category so we need to use the word "derived"), but we will write it again here for readability. Since here the derived context is clear, we will just say "representable" to mean "representable derived stack".

Definition 4.6.3.

1. A derived stack F is (-1)-*geometric* if it is representable (i.e. a derived affine scheme).

2. A morphism of derived stacks $F \to F'$ is (-1)-*geometric* if for any representable derived stack X and any map $X \to F'$, the homotopy pullback $F \times^h_{F'} X$ is (-1)-geometric.

3. A (-1)-geometric morphism $F \to F'$ is (-1)-*smooth* if for any representable derived stack X and any map $X \to F'$, the induced morphism $F \times^h_{F'} X \to X$ is a smooth morphism between representable derived stacks.

Let $n > 0$ and assume the notions of derived $(n-1)$-geometric stack, morphism and smooth morphism to be defined. Then, by recursion on n, we can define the following.

1. A derived stack F is n-geometric if there exists a family of maps $\{U_i \to F\}_{i \in I}$ such that

   (a) each U_i is representable,

   (b) each map $U_i \to F$ is $(n-1)$-smooth,

   (c) the total morphism $\coprod_{i \in I} U_i \to F$ is an epimorphism.

   Such family is a *smooth n-atlas*.

2. A morphism $F \to F'$ is n-representable if for any representable derived stack X and any map $X \to F'$, the derived stack $F \times^h_{F'} X$ is n-geometric.

3. An n-geometric morphism $F \to F'$ is n-smooth if for any representable derived stack X and any map $X \to F'$, there exists a smooth n-atlas $\{U_i\}$ of $F \times^h_{F'} X$ such that each composite map $U_i \to X$ is smooth.

Observe that, since Zariski open immersions are smooth, derived schemes are derived 0-geometric stacks. All properties stated in the final part of Section 2.3 continue to hold in the derived context, so we won't write them again.

4.7 Truncations

Let's now explore the relation between stacks, as defined in Definition 2.2.5, and derived stacks. The initial idea is that we enlarged the domain to consider simplicial rings, which have a nontrivial model structure. By considering every ring as a discrete simplicial one $\mathsf{Comm} \to \mathsf{sComm}$ we have a functor $i\colon \mathsf{Aff} \to \mathsf{dAff}$, whose pullback functor

$$i^*\colon \mathsf{dAff}^\sim \to SPr(\mathsf{Aff})$$

can be proved to be a right Quillen adjoint, where $SPr(\mathsf{Aff})$ has the local model structure (see [HAGII, p. 2.2.4]). Its left Quillen adjoint is denoted by $i_!\colon SPr(\mathsf{Aff}) \to \mathsf{dAff}^\sim$. Passing to homotopy derived functors (and working over k) we obtain the adjunction

$$\mathbb{L}i_!\colon \mathrm{St}(k) = \mathrm{Ho}(SPr(\mathsf{Aff})) \rightleftarrows \mathrm{dSt}(k) = \mathrm{Ho}(\mathsf{dAff}^\sim) :\mathbb{R}i^*$$

on the homotopy categories of stacks and derived stacks. Let's state two important properties.

Lemma 4.7.1. The functor $\mathbb{L}i_!$ is fully faithful. Moreover the functor i^* is both right and left Quillen, and, in particular, it preserves weak equivalences.

Proof. See [HAGII, Lemma 2.2.4.1, Lemma 2.2.4.2]. ■

In particular, we have $\mathbb{L}i_!(\operatorname{Spec} A) = \mathbb{R}\underline{\operatorname{Spec}} A$ and $\mathbb{L}i_!$ commutes with homotopy colimits, so that writing any $F \in \mathrm{St}(k)$ as homotopy colimit of representable stacks (affine scheme) we get $\mathbb{L}i_! F$. Time for some terminology.

Definition 4.7.2.

1. The *truncation functor* is

$$t_0 := \mathbb{R}i^*\colon \mathrm{dSt}(k) \to \mathrm{St}(k).$$

2. The *extension functor* is the left adjoint to t_0

$$i := \mathbb{L}i_! \colon \mathrm{St}(k) \to \mathrm{dSt}(k).$$

3. A derived stack F is *truncated* if the adjunction counit map

$$it_0(F) \to F$$

is an isomorphism in $\mathrm{dSt}(k)$.

Concretely, the trunctation functor sends a functor $F \colon \mathsf{dAff}^{\mathrm{op}} \to \mathsf{sSet}$ to $t_0(F) \colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{sSet}$, i.e. we only compute F on classical nonsimplicial rings. In particular we have

$$t_0(\mathbb{R}\underline{\mathrm{Spec}}\, A) \simeq \mathrm{Spec}\, \pi_0(A).$$

Proposition 4.7.3.

1. The truncation functor t_0 commutes with homotopy limits and homotopy colimits. The extension functor i commutes with homotopy colimits, but not with homotopy limits.
2. The truncation functor t_0 preserves epimorphism of stacks (they are checked at the level of π_0).
3. The functor t_0 sends n-geometric derived stacks to n-geometric stacks, and flat (resp. smooth, étale) morphisms between n-geometric derived stacks to flat (resp. smooth, étale) morphisms to n-geometric stacks.
4. The functor i preserves homotopy pullbacks of n-geometric stacks along a flat morphism, sends n-geometric stacks to n-geometric derived stacks and flat (resp. smooth, étale) morphisms between n-geometric stacks to flat (resp. smooth, étale) morphisms between n-geometric derived stacks.

Proof. See [HAGII, Proposition 2.2.4.4]. ■

Let's conclude with another definition.

Definition 4.7.4. Given a stack $F \in \mathrm{Ho}(SPr(\mathsf{Aff}))$, a *derived extension* of F is the data of a derived stack $\widetilde{F} \in \mathrm{Ho}(\mathsf{dAff}^{\sim})$ and an isomorphism of stacks $F \simeq t_0\widetilde{F}$.

There is always a trivial derived extension, given by j, but lot of times most of stacks (coming from moduli problems) admit natural nontrivial derived extensions. We will see an example with the derived stack of local system.

4.8 Cotangent complex

Let's now pass to the global case, and let F be a derived stack and $X = \mathbb{R}\underline{\mathrm{Spec}}\, A$ a derived affine scheme. Let's choose an A-point of F (morphism of derived stacks)

$$x \colon X = \mathbb{R}\underline{\mathrm{Spec}}\, A \to F$$

and let's recall that by Yoneda (Proposition 4.5.2) we have $\mathbb{R}\,\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Spec}}\, A, F) \simeq F(A)$ in $\mathrm{Ho}(\mathsf{sSet})$.

Definition 4.8.1. Using the same notation as above, let $M \in sA - \mathsf{Mod}$, consider the level-wise trivial square zero extension $A \oplus M$ and let $X[M] := \mathbb{R}\underline{\mathrm{Spec}}(A \oplus M)$. We define

$$\mathbb{D}er_x(F, M) := Hofiber_x(F(X[M]) \to F(X)) \in \mathrm{Ho}(\mathsf{sSet})$$

where the map is induced by $X \to X[M]$ (which is induced by the canonical projection $A \oplus M \to A$). It is called the simplicial set of derived derivations of F at the point x with coefficients in M.

Using Yoneda one can rewrite

$$\mathbb{D}er_x(F, M) \simeq \mathrm{Map}_{X/\,\mathrm{dSt}}(X[M], F) \in \mathrm{Ho}(\mathsf{sSet}).$$

This definition is functorial in M and hence we get a functor

$$\mathbb{D}er_x(F, -) \colon sA - \mathsf{Mod} \to \mathsf{sSet}$$

(the definition with the mapping space is at the level of homotopy categories, while the one with the homotopy fibers is at the level of model categories).

Definition 4.8.2. Let F be a derived stack and let $A \in sk-\mathsf{Alg}$.

1. Let $x\colon X = \mathbb{R}\underline{\mathrm{Spec}}\,A \to F$ be an A-point. We say that F *has a cotangent complex at* x if there exists an integer $n \geq 0$, a $(-n)$-connective stable A-module $\mathbb{L}_{F,x} \in \mathrm{Ho}(Sp(sA-\mathsf{Mod}))$ and an isomorphism
$$\mathbb{D}er_x(F,-) \simeq \mathbb{R}\underline{h}_s^{\mathbb{L}_{F,x}}$$
in $\mathrm{Ho}((A-\mathsf{Mod}^{\mathrm{op}})^\wedge)$.

2. If F has a cotangent complex at x, the stable A-module $\mathbb{L}_{F,x}$ is called the *cotangent complex of* F *at* x.

3. If F has a cotangent complex at x, the *tangent complex of* F *at* x is then the stable A-module
$$\mathbb{T}_{F,x} \coloneqq \mathbb{R}\,\underline{\mathrm{Hom}}_A^{Sp}(\mathbb{L}_{F,x}, A) \in \mathrm{Ho}(Sp(sA-\mathsf{Mod})).$$

Suppose F has a cotangent complex at x. This means that for every $M \in sA-\mathsf{Mod}$ we have an isomorphism
$$\mathbb{D}er_x(F,M) \simeq \mathbb{R}\underline{h}_s^{\mathbb{L}_{F,x}}(M) \simeq \mathrm{Map}_{Sp(sA-\mathsf{Mod})}(\mathbb{L}_{F,x}, M)$$
in $\mathrm{Ho}(Sp(sA-\mathsf{Mod}))$, i.e. $\mathbb{L}_{F,x}$ corepresents $\mathbb{D}er_x(F,-)$ on the level of homotopy categories. In particular, if A is a discrete k-algebra, by Remark 4.2.5, we have $\mathrm{Ho}(Sp(sA-\mathsf{Mod})) \simeq D(A)$, so that we can consider $\mathbb{L}_{F,x}$ to be a chain complex in A (not necessarily bounded in non-negative degree, so not corresponding to a simplicial A-module by Dold-Kan). Clearly, for $F = \mathbb{R}\underline{\mathrm{Spec}}\,B$ we obtain again the previously built affine cotangent complex.

Let's now consider a morphism u in $\mathrm{Ho}(\mathsf{dAff}^\sim/F)$
$$\begin{array}{ccc} Y = \mathbb{R}\underline{\mathrm{Spec}}\,B & \xrightarrow{\quad u \quad} & X = \mathbb{R}\underline{\mathrm{Spec}}\,A \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle x} & \\ & F & \end{array}$$

Let $M \in sB-\mathsf{Mod}$, which is also a simplicial A-module using the restriction $A \to B$; we have a commutative diagram
$$\begin{array}{ccc} A \oplus M & \longrightarrow & B \oplus M \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$
inducing a commutative diagram of representable stacks
$$\begin{array}{ccc} X[M] & \longleftarrow & Y[M] \\ \uparrow & & \uparrow \\ X & \longleftarrow & Y \end{array}$$
inducing again a natural morphism
$$u^*\colon \mathbb{D}er_x(F,M) \to \mathbb{D}er_y(F,M).$$
By universal property, assuming the following objects exist, this induces a morphism
$$u^*\colon \mathbb{L}_{F,y} \to \mathbb{L}_{F,x} \otimes_A^{\mathbb{L}} B.$$

Definition 4.8.3. The derived stack F has a *global cotangent complex* if the following two are satisfied:

(1) For any simplicial ring A and any point $x\colon \mathbb{R}\underline{\mathrm{Spec}}\,A \to F$, there exists a cotangent complex $\mathbb{L}_{F,x} \in \mathrm{Ho}(Sp(A-\mathsf{Mod}))$.

(2) For any triangular diagram like the one above, the induced morphism $u^*\colon \mathbb{L}_{F,y} \to \mathbb{L}_{F,x} \otimes_A^{\mathbb{L}} B$ is an isomorphism in $\mathrm{Ho}(Sp(B-\mathsf{Mod}))$.

Proposition 4.8.4. Any representable derived stack $F = \mathbb{R}\underline{\mathrm{Spec}}\, A$ has a global cotangent complex.

Proof. See [HAGII, Proposition 1.4.1.8]. ■

Proposition 4.8.5. Let F be an n-geometric derived stack. Then F has a global cotangent complex, which is furthermore $(-n)$-connective.

Proof. See [HAGII, Proposition 1.4.1.11]. ■

There is also a notion of relative cotangent complex for a morphism $f\colon F \to F'$ of derived stacks. As before, let $X = \mathbb{R}\underline{\mathrm{Spec}}\, A$ and $x\colon X \to F$. The map f induces a morphism $\mathbb{D}er_x(F, M) \to \mathbb{D}er_{f(x)}(F', M)$ and hence a map

$$df_x\colon \mathbb{L}_{F',f(x)} \to \mathbb{L}_{F,x}$$

in $\mathrm{Ho}(Sp(sA-\mathsf{Mod}))$.

Definition 4.8.6. With the previous notations, we define

$$\mathbb{L}_{F/F',x} \coloneqq Hocofiber(df_x)$$

and we call it the *relative cotangent complex of f at x.*

We have used this definition for its shortness, one could also use [HAGII, Definition 1.4.1.14] and then prove the above one as a property. As before we have a notion of global relative cotangent complex.

Definition 4.8.7. Let $f\colon F \to G$ a morphism of derived stacks. We say that f has a *relative cotangent complex* if the following two conditions are satisfied.

1. For any $A \in sk-\mathsf{Alg}$ and any point $x\colon\ \mathbb{R}\underline{\mathrm{Spec}}\, A \to F$, the map f has a relative cotangent complex $\mathbb{L}_{F/G,x}$ at x.
2. For any commutative diagram

$$\begin{array}{ccc} Y = \mathbb{R}\underline{\mathrm{Spec}}\, B & \xrightarrow{\quad u \quad} & X = \mathbb{R}\underline{\mathrm{Spec}}\, A \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle x} & \\ & F & \end{array}$$

   in $\mathrm{Ho}(\mathsf{dAff}^{\sim}/F)$, the induced (same reasoning as before) morphism

$$u^*\colon \mathbb{L}_{F/G,y} \to \mathbb{L}_{F/G,x} \otimes^{\mathbb{L}}_A B$$

   is an isomorphism in $\mathrm{Ho}(Sp(sB-\mathsf{Mod}))$.

Finally let's state some of the main properties of cotangent complexes, which can be seen as homotopical analogues of the properties of the sheaf of differentials on a scheme (e.g. normal and conormal sequences).

Lemma 4.8.8. Let $f\colon F \to G$ be a morphism of derived stacks.

1. If F and G both have cotangent complexes, then f has a relative cotangent complex. For every point $x\colon\ \mathbb{R}\underline{\mathrm{Spec}}\, A \to F$ we have a natural homotopy cofiber sequence of stable A-modules

$$\mathbb{L}_{G,x} \longrightarrow \mathbb{L}_{F,x} \longrightarrow \mathbb{L}_{F/G,x}.$$

2. If f has a relative cotangent complex, for map of derived stacks $H \to G$, the morphism $F \times^h_G H \to H$ has a relative cotangent complex satisfying

$$\mathbb{L}_{F/G,x} \simeq \mathbb{L}_{F\times^h_G H/H,x}$$

   for any $x\colon\ \mathbb{R}\underline{\mathrm{Spec}}\, A \to F \times^h_G H$.

3. If for any point $x\colon X = \mathbb{R}\underline{\mathrm{Spec}}\, A \to F$, the map $F \times^h_G X \to X$ has a relative cotangent complex, then the morphism f has a relative cotangent complex. Furthermore, we have

$$\mathbb{L}_{F/G,x} \simeq \mathbb{L}_{F\times^h_G X/X,x}.$$

4. If for any point $x\colon X = \mathbb{R}\underline{\mathrm{Spec}}\, A \to F$, the derived stack $F \times^h_G X$ has a cotangent complex, then the morphism f has a relative cotangent complex. Furthermore, we have a natural homotopy cofiber sequence
$$\mathbb{L}_A \longrightarrow \mathbb{L}_{F \times^h_G X, x} \longrightarrow \mathbb{L}_{F/G, x}.$$

Proof. See [HAGII, Lemma 1.4.1.16]. ■

4.8.1 Postnikov towers

Let A be a simplicial commutative k-algebra.

Definition 4.8.9. A is said to be n-truncated if $\pi_i(A) = 0$ for all $i > n$.

We can consider the full subcategory of n-truncated simplicial k-algebras $sk-\mathsf{Alg}_{\leq n}$ (with the induced model structure) with the corresponding embedding between homotopy categories. This last functor has a left adjoint
$$\tau_{\leq n}\colon \mathrm{Ho}(sk-\mathsf{Alg}) \to \mathrm{Ho}(sk-\mathsf{Alg}_{\leq n})$$
called the *n-truncation functor.*

Definition 4.8.10. Let $A \in sk-\mathsf{Alg}$ and consider a diagram
$$A \to \cdots \to A_{\leq n} \to A_{\leq (n-1)} \to \cdots \to A_{\leq 1} \to A_{\leq 0} = \pi_0(A)$$
of simplicial commutative k-algebras such that

- each $A_{\leq n}$ is n-truncated;
- the map $d_n\colon A \to A_{\leq n}$ induces isomorphisms on π_i for $i \leq n$;
- the map $d_n\colon A \to A_{\leq n}$ is such that for every n-truncated simplicial k-algebra N, we have
$$d_n^*\colon \mathrm{Map}_{sk-\mathsf{Alg}}(A_{\leq n}, N) \xrightarrow{\sim} \mathrm{Map}_{sk-\mathsf{Alg}}(A, N) \in \mathrm{Ho}(\mathsf{sSet}).$$

Such a diagram is called a *Postnikov tower of A*. It is clearly uniquely defined in the homotopy category of $sk-\mathsf{Alg}$.

One has a natural isomorphism
$$A \simeq \mathrm{Holim}_n\, A_{\leq n}$$
in $\mathrm{Ho}(sk-\mathsf{Alg})$. Using the cotangent complex we can give an explicit formula for the Postnikov tower of A, by induction on n (starting with $A_{\leq 0} = \pi_0(A)$ and the trivial map $A \to \pi_0(A)$). It can be proven that for every $n > 0$ there is an homotopy cartesian diagram
$$\begin{array}{ccc} A_{\leq n} & \longrightarrow & A_{\leq (n-1)} \\ \downarrow & & \downarrow {\scriptstyle 0} \\ A_{\leq (n-1)} & \xrightarrow{k_n} & A_{\leq (n-1)} \oplus \pi_n(A)[n+1] \end{array}$$
where $\pi_n(A)[i] \coloneqq S^i \otimes \pi_n(A_i) \in \mathrm{Ho}(sk-\mathsf{Mod})$, 0 is the trivial derivation and k_n is a (uniquely determined) derivation, corresponding to an element of $[\mathbb{L}_{A_{\leq (n-1)}}, \pi_n(A)[n+1]]$. This is called the *$n$-Postnikov invariant of A*.

4.8.2 Obstruction theory

Definition 4.8.11. Given A simplicial ring, $M \in sA-\mathsf{Mod}$ and $d \in \pi_0(\mathbb{D}er(A, M))$ let's define $A \oplus_d \Omega M$ as the homotopy pullback of
$$\begin{array}{ccc} A \oplus_d \Omega M & \xrightarrow{p} & A \\ \downarrow & & \downarrow {\scriptstyle d} \\ A & \xrightarrow{s} & A \oplus M \end{array}$$

where $s\colon A \to A\oplus M$ is the trivial derivation. The map $p\colon A\oplus_d \Omega M \to A$ is called the natural projection.

Definition 4.8.12. A derived stack F is *inf-cartesian* if for any diagram like above, the square

$$\begin{array}{ccc} \mathbb{R}F(A \oplus_d \Omega M) & \longrightarrow & \mathbb{R}F(A) \\ \downarrow & & \downarrow \\ \mathbb{R}F(A) & \xrightarrow{\ r\ } & \mathbb{R}F(A \oplus M) \end{array}$$

is homotopy cartesian. A derived stack F has an *obstruction theory* if it has a global cotanget complex and it is inf-cartesian.

Recall that for $F \in \mathrm{dSt}(k)$ and $A \in sk - \mathsf{Alg}$ we write

$$\mathbb{R}F(A) := \mathbb{R}\,\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Spec}}\,A, F)$$

so that, by Yoneda lemma, we have $\mathbb{R}F(A) \simeq (RF)(A)$ where RF is a fibrant replacement of F.

5 Examples

We finally have enough theory to study some examples of derived stacks, generalizing well known classical stacks, like local systems or vector bundles. Unless otherwise specified, we will work over a base ring $k \in \mathsf{Comm}$ (although sometimes, for readability, we will just leave implicit the comma category notation). Recall that we use the following notations

$$\mathrm{St}(k) := \mathrm{Ho}(SPr(\mathsf{Aff}_{/\operatorname{Spec} k})), \qquad \mathrm{dSt}(k) := \mathrm{Ho}(\mathsf{dAff}^{\sim}_{/\operatorname{Spec} k}).$$

5.1 Local systems

Let's consider the 1-geometric stack

$$\mathbf{Vect}_n \colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{sSet}, \qquad \operatorname{Spec} A \mapsto \mathbf{Vect}_n(A)$$

where $\mathbf{Vect}_n(A)$ is the nerf of the groupoid of projective A-modules of rank n. It is exactly the nerf of the classical stack $B\,\mathrm{GL}_n$ of Chapter 3. We can see it as a derived stack thanks to the extension functor

$$i \colon \mathrm{St}(k) \to \mathrm{dSt}(k)$$

of Definition 4.7.2.

5.1.1 Quasi-coherent modules and vector bundles

Let's now take a little time to define a general version of the derived stacks of vector bundles, which a priori is unrelated to the extension $i(\mathbf{Vect}_n)$ (we will see they are equivalent after this section). This is just a summary of [HAGII, p. 1.3.7]. We will first define the derived stack of quasi-coherent modules, generalizing the classic case from algebraic geometry.

Definition 5.1.1. Let $A \in sk - \mathsf{Alg}$ and let's define the category $\mathsf{QCoh}(A)$ of *quasi-coherent sA-modules*. Its objects are families $(M_B)_B$ where $B \in sA - \mathsf{Alg}$ and $M_B \in sB - \mathsf{Mod}$, together with isomorphisms

$$\alpha_u \colon M_B \otimes_B B' \to M_{B'}$$

for every morphism of simplicial A-algebras $u \colon B \to B'$; we also require this family to satisfy a compatibility condition, namely that for any pair of maps

$$B \xrightarrow{u} B' \xrightarrow{v} B''$$

of $sA - \mathsf{Alg}$ we must have $\alpha_v \circ (\alpha_u \otimes_{B'} B'') = \alpha_{v \circ u}$. Such data is denoted simply by (M, α). A morphism $(M, \alpha) \to (M', \alpha')$ is given by a family of morphisms of sB-modules $f_B \colon M_B \to M'_B$ for every $B \in sA - \mathsf{Alg}$, such that for any $u \colon B \to B' \in sA - \mathsf{Alg}$ the diagram

$$\begin{array}{ccc} M_B \otimes_B B' & \xrightarrow{\alpha_u} & M_{B'} \\ \downarrow{\scriptstyle f_B \otimes_B B'} & & \downarrow{\scriptstyle f_{B'}} \\ (M'_B) \otimes_B B' & \xrightarrow{\alpha'_u} & M'_{B'} \end{array}$$

commutes.

Morally, this defines the category of quasi-coherent simplicial sheaves over the derived affine schemes $\mathbb{R}\underline{\mathrm{Spec}}\, A$. We have a natural projection functor $\mathsf{QCoh}(A) \to sA - \mathsf{Mod}$ (which will be denoted by $\Gamma(\mathbb{R}\underline{\mathrm{Spec}}\, A, -)$ in analogy with the global section functor) sending (M, α) to M_A. It is straightforward to check it is an equivalence of categories, just as in the classical case quasi-coherent modules over an affine scheme $\mathrm{Spec}\, B$ are equivalent to B-modules.

Moreover, the global section functor can be used to transport the model structure of $sA - \mathsf{Mod}$ on $\mathsf{QCoh}(A)$, using the well known Quillen's transfert theorem. In practice this means that weak equivalences and fibrations in $\mathsf{QCoh}(A)$ are simply checked on the global sections, while cofibrations are defined by lifting properties.

Given $f\colon A \to A' \in sk - \mathsf{Alg}$, corresponding to $u\colon \mathbb{R}\underline{\mathrm{Spec}}\, A' \to \mathbb{R}\underline{\mathrm{Spec}}\, A$, we have a pullback functor (induced by the tensor product $- \otimes_A A'$)

$$f^*\colon \mathsf{QCoh}(A) \to \mathsf{QCoh}(A')$$

and it is clearly a left Quillen functor.

Definition 5.1.2. The assignment $A \mapsto \mathsf{QCoh}(A)$ and $(f\colon A \to A') \mapsto f^*$ defines a (pseudo)-functor $\mathsf{QCoh}\colon \mathsf{dAff}^{\mathrm{op}}_{/\,\mathrm{Spec}\, k} \to \mathsf{Cat}$. It is a *cofibrantly generated left Quillen presheaf on* $\mathsf{dAff}_{/\,\mathrm{Spec}\, k}$, in the sense of [HAGII, Appendix B].

Since pullbacks are left Quillen, we have a (pseudo)-subfunctor QCoh^c_W, considering the subcategory of cofibrant objects and weak equivalences between them. Composing it with the nerve (and strictifying if needed) we obtain the simplicial presheaf

$$N(\mathsf{QCoh}^c_W)\colon \mathsf{dAff}^{\mathrm{op}}_{/\,\mathrm{Spec}\, k} \simeq sk - \mathsf{Alg} \to \mathsf{sSet}, \qquad A \mapsto N(\mathsf{QCoh}(A)^c_W).$$

Definition 5.1.3. The simplicial presheaf of *quasi-coherent modules* is $N(\mathsf{QCoh}^c_W)$ as defined above. It is denoted by $\mathbf{QCoh}$ and considered as an object in $\mathsf{dAff}^{\sim}_{/\,\mathrm{Spec}\, k}$.

Let's immediately observe that for any simplicial k-algebra A, $\mathbf{QCoh}(A)$ is weakly equivalent to the nerve of $sA - \mathsf{Mod}^c_W$, so in particular $\pi_0(\mathbf{QCoh}(A))$ is in bijection with isomorphism classes of $\mathrm{Ho}(sA - \mathsf{Mod})$. The main result on quasi-coherent modules is the following theorem.

Theorem 5.1.4. The simplicial preshaf $\mathbf{QCoh}$ is a derived stack.

Proof. See [HAGII, Theorem 1.3.7.2]. ∎

Finally we can talk about vector bundles of rank n. Recall from Lemma 4.4.4 that a simplicial A-module is projective of rank n if and only if it is strong and $\pi_0(M)$ is a (classical) projective $\pi_0(A)$-module of rank n. It is equivalent to ask for the existence of a covering family $A \to A'$ such that $M \otimes^{\mathbb{L}}_A A'$ is isomorphic to $(A')^n$ in $\mathrm{Ho}(sA' - \mathsf{Mod})$ (this is simply the homotopical version of projective of rank n iff locally free of rank n). We can then consider $\mathbf{Vect}_n(A) \subset \mathbf{QCoh}(A)$ to be the sub-simplicial set consisting of connected components corresponding to rank n vector bundles. Since projective modules are stable by base change, we see that $\mathbf{Vect}_n \subset \mathbf{QCoh}$ is indeed a sub-simplicial presheaf. One can prove the following.

Theorem 5.1.5. The simplicial presheaf $\mathbf{Vect}_n$ is a derived stack, and it is usually called the *derived stack of vector bundles of rank n*. Moreover, $\mathbf{Vect}_n$ is 1-geometric, finitely presented and its diagonal is a (-1)-representable morphism.

Proof. See [HAGII, Corollary 1.3.7.4, Corollary 1.3.7.12]. ∎

5.1.2 Derived stack of local systems

In the previous section, we re-defined $\mathbf{Vect}_n$ as a derived stack, so now we have two version of derived stacks of vector bundles: this one and the one obtained extending the classical one. We used the same notation $\mathbf{Vect}_n$ for both of them, and this abuse of notation is justified by the following lemma.

Lemma 5.1.6. There exists a natural isomorphism

$$i(\mathbf{Vect}_n) \simeq \mathbf{Vect}_n$$

in $\mathrm{dSt}(k)$.

Proof. See [HAGII, Lemma 2.2.6.1]. ■

Recalling that the category $\mathsf{dAff}^{\sim}$ is a simplicial model category (we start with a simplicial model category $SPr(\mathsf{dAff})$ and then we do two left Bousfield localizations, which keep the simplicial model structure), we can consider, fixed $K \in \mathsf{sSet}$, the exponentiation functor $F \mapsto F^K$ which is right Quillen and hence can be derived. Its right derived functor is denoted by $F \mapsto F^{\mathbb{R}K} \simeq (RF)^K$, where RF is a fibrant replacement of F in $\mathsf{dAff}^{\sim}$.

Definition 5.1.7. Let $K \in \mathsf{sSet}$, which we think embedded as constant simplicial presheaf in $\mathsf{dAff}^{\sim}$. The *derived moduli stack* of rank n local systems on K is

$$\mathbb{R}\mathbf{Loc}_n(K) \coloneqq \mathbf{Vect}_n^{\mathbb{R}K} = \mathbb{R}\,\mathcal{H}\mathrm{om}(K, \mathbf{Vect}_n) \in \mathrm{dSt}(k) = \mathrm{Ho}(\mathsf{dAff}_k^{\sim}),$$

see Proposition 4.5.8.

Let's recall that we also have a non derived version of this stack $\mathbf{Loc}_n(K)\colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Grpd}$ defined by $A \mapsto \mathrm{Funct}(\Pi^1(K), B\,\mathrm{GL}_n(A))$. One can prove the following.

Lemma 5.1.8. We have an isomorphism in $\mathrm{St}(k)$:

$$\mathbf{Loc}_n(K) \simeq t_0\mathbb{R}\mathbf{Loc}_n(K)$$

where t_0 is the truncation functor $t_0\colon \mathrm{dSt}(k) \to \mathrm{St}(k)$, right adjoint to i.

Proof. See [HAGII, Lemma 2.2.6.4]. ■

Also we have

Lemma 5.1.9. If K is a finite simplicial set, then the derived stack $\mathbb{R}\mathbf{Loc}_n(K)$ is a finitely presented 1-geometric derived stack.

Proof. See [HAGII, Lemma 2.2.6.3]. ■

We can give also a more intuitive topological notion of the derived stack of local system, to then finally prove it gives the same result as the simplicial one if we consider it on the geometric realization of $K \in \mathsf{sSet}$. Let $X \in \mathsf{Top}$ and $A \in sk-\mathsf{Alg}$; the category $sA-\mathsf{Mod}(X)$ is simply the category of presheaves on X valued in $sA-\mathsf{Mod}$. We can consider first the projective model structure and then its left Bousfield localization such that $\mathcal{E} \to \mathcal{F}$ is a weak equivalence if $\mathcal{E}_x \to \mathcal{F}_x$ is an equivalence in $sA-\mathsf{Mod}$ for all $x \in X$. We now focus on the subcategoy $sA-\mathsf{Mod}(X)^c_W$ of cofibrant objects and weak equivalences between them. If $A \to B$ is a map in $sk-\mathsf{Alg}$ we have a base change functor

$$sA-\mathsf{Mod}(X) \to sB-\mathsf{Mod}(X), \qquad \mathcal{E} \mapsto \mathcal{E} \otimes_A B$$

which is left Quillen and hence induces a functor

$$-\otimes_A B\colon sA-\mathsf{Mod}(X)^c_W \to sB-\mathsf{Mod}(X)^c_W.$$

This construction induces a pseudofunctor $A \mapsto sA-\mathsf{Mod}(X)^c_W$ from $sk-\mathsf{Alg}$ to Cat, which can be strictified as usual (see [Hen11] for the explicit procedure), from now on we will pretend it is a genuine functor. Let's define a subfunctor of it: for $A \in sk-\mathsf{Alg}$ we call $A-Loc_n(X)$ the full subcategory of $sA-\mathsf{Mod}(X)^c_W$ consisting of all $\mathcal{E}$ such that there exists an open covering $\{U_i\}$ of X such that each restriction $\mathcal{E}\big|_{U_i}$ is isomorphic, in $\mathrm{Ho}(sA-\mathsf{Mod}(U_i))$, to a constant presheaf with fibers a projective $sA-\mathsf{Mod}$ of rank n. This is a subfunctor of $A \mapsto sA-\mathsf{Mod}(X)^c_W$ and composing it with the nerve we obtain a presheaf $\mathbb{R}\mathbf{Loc}_n(X) \in \mathsf{dAff}^{\sim}$ given by

$$\mathbb{R}\mathbf{Loc}_n(X)(A) \coloneqq N(A-Loc_n(X)).$$

Proposition 5.1.10. Let $K \in \mathsf{sSet}$ and $|K| \in \mathsf{Top}$ be its geometric realization. Then the simplicial presheaf $\mathbb{R}\mathbf{Loc}_n(|K|)$ is a derived stack and there exists an isomorphism

$$\mathbb{R}\mathbf{Loc}_n(|K|) \simeq \mathbb{R}\mathbf{Loc}_n(K)$$

in $\mathrm{dSt}(k)$.

Proof. See [HAGII, Proposition 2.2.6.5]. ■

Let's now try to describe the cotangent complex of $\mathbb{R}\mathbf{Loc}_n(K)$ at a global point

$$E\colon * = \mathrm{Spec}(k) \to \mathbb{R}\mathbf{Loc}_n(K)$$

and recall that it belongs to $\mathrm{Ho}(\mathrm{Ch}(k)) = D(k)$ (derived category of k) and for any $M \in sk-\mathsf{Mod}$ (identified with a chain complex in negative degree by Dold-Kan when needed) it satisfies

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M) \simeq \mathrm{Map}_{\mathrm{Ch}(k)}(\mathbb{L}_{\mathbb{R}\mathbf{Loc}_n(K),E}, M) \in \mathrm{Ho}(\mathsf{sSet}).$$

Let's first observe that, using the adjunction with t_0 (whose left adjoint is the inclusion functor i) and Lemma 5.1.8, E corresponds to a map $\mathrm{Spec}(k) \to \mathbf{Loc}_n(K)$ in $\mathrm{St}(k)$, which then, by Corollary 2.2.10.1, corresponds to a functor $\Pi^1(K) \to \mathcal{G}(k)$, where $\mathcal{G}(k)$ is the groupoid of projective k-modules of rank n, i.e. E is a local system on K. We can then consider the homology complex of K with coefficients in the local system $E \otimes_k E^\vee$, where $E^\vee$ is the pointwise dual.

An intuitive way to think about local system on a space X (since Top and sSet are Quillen equivalent) is to think about locally constant sheaf of modules, so that the cohomology complex is just the sheaf cohomology one.

Let's quickly view two nice proofs of the universal coefficient theorems in a derived setting, which will help us understand the next part. We will just focus on abelian groups, to avoid the spectral sequence versions.

Proposition 5.1.11 (Universal Coefficients)**.** Let $X \in \mathsf{Top}$ be a nice space (e.g. a CW-complex) and $G \in \mathbb{Z}-\mathsf{Mod}$. We have the following two (non-naturally) split exact sequences

$$0 \longrightarrow H_n(X) \otimes G \longrightarrow H_n(X,G) \longrightarrow \mathrm{Tor}^1(H_{n-1}(X),G) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X),G) \longrightarrow H^n(X,G) \longrightarrow \mathrm{Hom}(H_n(X),G) \longrightarrow 0.$$

In the derived category $D(\mathbb{Z})$ this just amounts in saying

$$C_*(X,G) \simeq C_*(X) \otimes^{\mathbb{L}} G, \qquad C^*(X,G) \simeq \mathbb{R}\,\underline{\mathrm{Hom}}(C_*(X),G).$$

Proof. Since the homology complex $C_*(X)$ is bounded and free in each component and $\mathbb{Z}$ is a PID, we can write, in the derived category $D(\mathbb{Z})$

$$C_*(X,G) \overset{\mathrm{def}}{\simeq} C_*(X) \otimes G \simeq C_*(X) \otimes^{\mathbb{L}} G.$$

The proof of this fact goes as follows: we can find a quasi-isomorphism from a complex of free abelian groups F_* to $C_*(X)$, and the complex of free $\mathbb{Z}$-modules is itself quasi-isomorphic to the sum of its homology, using the fact that every subgroup of a free abelian group is free (we write every term $F_n \simeq \ker\partial_n \oplus \mathrm{im}\,\partial_n$). To get back the original statement we can just observe that in $D(\mathbb{Z})$ we have

$$C_*(X) \simeq \bigoplus_n H_n(X)[n] \implies C_*(X) \otimes^{\mathbb{L}} G \simeq \bigoplus_n (H_n(X) \otimes^{\mathbb{L}} G)[n].$$

Since $\mathbb{Z}$ has projective dimension 1 (i.e. any abelian group has a free resolution with only two nonzero terms) we see that in $D(\mathbb{Z})$, for any $A \in \mathsf{Ab}$, we have $A \otimes^{\mathbb{L}} G \simeq (A \otimes G) \oplus \mathrm{Tor}^1(A,G)[1]$. We then conclude

$$C_*(X,M) \simeq \bigoplus_n H_n(X,M)[n] \simeq \bigoplus_n \left((H_n(X) \otimes G) \oplus \mathrm{Tor}^1(H_{n-1}(X),G)\right)[n].$$

The cohomology version is analogue, observing

$$C^*(X,G) \overset{\mathrm{def}}{\simeq} \underline{\mathrm{Hom}}(C_*(X),G) \simeq \mathbb{R}\,\underline{\mathrm{Hom}}(C_*(X),G)$$

and $\mathbb{R}\,\underline{\mathrm{Hom}}(A,G) \simeq \mathrm{Hom}(A,G) \oplus \mathrm{Ext}^1(A,G)[1]$. ∎

Proposition 5.1.12. We have an isomorphism in $D(k)$:

$$\mathbb{L}_{\mathbb{R}\mathbf{Loc}_n(K),E} \simeq C_*(K, E \otimes_k E^\vee)[-1].$$

Proof. Let's fix $M \in sk-\mathsf{Mod}$ and try to compute $\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M)$, defined as the homotopy fiber of

$$\mathbb{R}\mathbf{Loc}_n(K)(\mathbb{R}\underline{\mathrm{Spec}}(k \oplus M)) \to \mathbb{R}\mathbf{Loc}_n(K)(\mathbb{R}\underline{\mathrm{Spec}}(k))$$

at E, where the morphism is induced by the projection $k \oplus M \to k$ (trivial square zero extension at every level). This corresponds to the simplicial mapping set of all maps $K \to \mathbf{Vect}_n(k \oplus M)$ lifting $E \colon K \to \mathbf{Vect}_n(k)$, i.e. we have

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M) \simeq \mathrm{Map}_{\mathsf{sSet}/\mathbf{Vect}_n(k)}(K, \mathbf{Vect}_n(k \oplus M)) \in \mathrm{Ho}(\mathsf{sSet}).$$

Let's try to describe in a different way the projection $\mathbf{Vect}_n(k \oplus M) \to \mathbf{Vect}_n(k)$; observe first of all that, thanks to the bi-augmentation $k \to k \oplus M \to k$, this map has a section. We claim that it is already a Kan fibration, so we can compute its homotopy fiber just with a normal pullback. Recalling the definitions, $\mathbf{Vect}_n(A)$, for $A \in sk-\mathsf{Alg}$, is just the nerf of the groupoid of projective sA-modules of rank n (see Definition 4.4.2). Then, to verify that it is a fibration, we just need to verify it lifts against generating acyclic cofibrations $\Lambda^n_k \to \Delta^n$. This holds by surjectivity and by the fact that we are taking nerves of groupoids (i.e. all maps are invertible). Thus, this is a fibration and we can just compute the normal fiber.

Observe now that any projective $s(k \oplus M)-\mathsf{Mod}$ is a cobase change of a projective $k-\mathsf{Mod}$. Indeed a projective $s(k \oplus M)-\mathsf{Mod}$ corresponds to an idempotent endomorphism of a free $s(k \oplus M)-\mathsf{Mod}$: fixing a base of such a free module (of finite type, since we are interested in rank n), the coefficients of the matrix of such endomorphism must lie in k (embdedded in $k \oplus M$), since multiplication on M is zero in $k \oplus M$. This means exactly that we can consider the corresponding endomorphism of the same free k-module, obtaining a projective k-module, and then base change to get back our original module.

Let's define a simplicial category $\mathcal{G}(k \oplus M)$ having as objects projective k-modules of rank n (same objects as $\mathcal{G}(k)$) and having as the simplicial set of morphisms

$$\mathcal{G}(k \oplus M)(P, P') \coloneqq \underline{\mathrm{Hom}}^W_{s(k\oplus M)-\mathsf{Mod}}(P \oplus (P \otimes_k M), P' \oplus (P' \otimes_k M))$$

where we consider the sub-simplicial set of (weak) equivalences in the simplcial hom-sets in the S-category $s(k \oplus M)-\mathsf{Mod}$. Observe that $P \oplus (P \otimes_k M) \simeq P \otimes_k (k \oplus M)$, which is a cobase change of $P \in sk-\mathsf{Mod}$ (seen as a constant simplicial k-module). It is then cofibrant, since $- \otimes_k (k \oplus M)$ preserves cofibrant objects (being left Quillen) and since P is cofibrant as $sk-\mathsf{Mod}$, being projective (any acyclic fibration is surjective levelwise and projective modules lift against surjections). Let's also observe that, since a surjection between simplicial abelian groups is a Kan fibration of simplicial sets by Proposition 4.1.2, every object of $s(k \oplus M)-\mathsf{Mod}$ is also fibrant. We have a natural map of S-categories $\mathcal{G}(k \oplus M) \to \mathcal{G}(k)$ being identity on objects and acting on simplicial sets of morphisms by

$$\underline{\mathrm{Hom}}^W_{s(k\oplus M)-\mathsf{Mod}}(P \oplus (P \otimes_k M), P' \oplus (P' \otimes_k M)) \to \underline{\mathrm{Hom}}^W_{sk-\mathsf{Mod}}(P, P') \to$$
$$\mathrm{const}\, \pi_0(\underline{\mathrm{Hom}}^W_{sk-\mathsf{Mod}}(P, P')) \simeq \mathcal{G}(k)(P, P')$$

where we consider $\mathcal{G}(k)$ as simplicial enriched category in the trivial way (constant morphism). Using [HAGII, Prop. A.0.6], we can say that the projection $\mathbf{Vect}_n(k \oplus M) \to \mathbf{Vect}_n(k)$ is isomorphic to the map

$$N(\mathcal{G}(k \oplus M)) \to N(\mathcal{G}(k)),$$

where we consider the nerve of an S-category, defined as the diagonal of the bisimplicial set sending $([n], [m])$ to chains of $(n+1)$ objects whose degree of maps is m. It is again a right Quillen adjoint, as in the classical case (explained, for example, in [DK79] and in [HAGII, Appendix A]). This means that N preserves (homotopy) pullbacks.

Let's now focus on the particular case where $K = \Delta^0$, where $E \colon \Delta^0 \to \mathbf{Vect}_n(k)$ is just a projective k-module of rank n. Using $\Delta^0 = N(1)$, where 1 is the singleton groupoid, we just need to find the fiber, in S-categories, of $\mathcal{G}(k \oplus M) \to \mathcal{G}(k)$ at $E \in \mathcal{G}(k)$. The object set is clearly a singleton, corresponding to the point E in $\mathcal{G}(k \oplus M)$, so the only problem is finding its simplicial set of endomorphisms, which is exactly the sub-simplicial set of endomorphism of E in $\mathcal{G}(k \oplus M)$ that gets sent to id_E. Observe that (by the tensor-forgetful adjunction) we have

$$\underline{\mathrm{Hom}}^W_{s(k\oplus M)-\mathsf{Mod}}(E \oplus (E \otimes_k M), E \oplus (E \otimes_k M)) \simeq \underline{\mathrm{Hom}}^W_{sk-\mathsf{Mod}}(E, E \oplus (E \otimes_k M)) \simeq$$
$$\underline{\mathrm{Hom}}^W_{sk-\mathsf{Mod}}(E, E) \times \underline{\mathrm{Hom}}_{sk-\mathsf{Mod}}(E, E \otimes_k M)$$

and hence the simplicial set of morphisms must be $\underline{\mathrm{Hom}}_{sk-\mathsf{Mod}}(E, E \otimes_k M) \simeq E \otimes_k E^\vee \otimes_k M$ (since E is of finite type). Therefore the searched fiber is the S-category $(*, E \otimes_k E^\vee \otimes_k M)$, whose simplicial nerve is the classifying space $K(E \otimes_k E^\vee \otimes_k M, 1)$, defined by $[n] \mapsto (E \otimes E^\vee \otimes M_n)^{\times n}$, with faces and cofaces being the obvious ones. Summarizing, we have this (homotopy) pullback diagram in sSet:

$$\begin{array}{ccc} K(E \otimes E^\vee \otimes M, 1) & \longrightarrow & \mathbf{Vect}_n(k \oplus M) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{\quad E \quad} & \mathbf{Vect}_n(k) \end{array}$$

Therefore, using the universal property of the pullback, we obtain

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(\Delta^0), M) \simeq \mathrm{Map}_{\mathsf{sSet}/\mathbf{Vect}_n(k)}(\Delta^0, \mathbf{Vect}_n(k \oplus M)) \simeq$$
$$\simeq \mathrm{Map}_{\mathsf{sSet}/\Delta^0}(\Delta^0, K(E \otimes E^\vee \otimes M, 1)) \simeq K(E \otimes E^\vee \otimes M, 1).$$

Expressing K as homotopy colimit of standard simplices Δ^n (the index category is given by maps $\Delta^m \to K$) we obtain

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M) \simeq \mathrm{Map}_{\mathsf{sSet}/\mathbf{Vect}_n(k)}(K, \mathbf{Vect}_n(k \oplus M)) \simeq$$
$$\mathrm{Holim}_{\Delta^n \to K} \mathrm{Map}_{\mathsf{sSet}/\mathbf{Vect}_n(k)}(\Delta^n, \mathbf{Vect}_n(k \oplus M))$$

and, by recalling that local systems only depend on the homotopy type of the base space (and Δ^n is contractible), using the above expression for Δ^0 (being careful that there E means the restriction of our local system to one simplex) we obtain

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M) \simeq \mathrm{Map}_{\mathsf{sSet}}(K, K(E \otimes_k E^\vee \otimes_k M, 1)).$$

Lemma 5.1.13. We have an isomorphism

$$\mathrm{Map}_{\mathsf{sSet}}(\mathcal{X}, K(\mathcal{G}, n)) \simeq \mathrm{Map}_{\mathrm{Ch}(k)}(k, C^*(\mathcal{X}, \mathcal{G})[n])$$

in $\mathrm{Ho}(\mathsf{sSet})$, where $\mathcal{X} \in \mathsf{sSet}$, $\mathcal{G} \in sk - \mathsf{Mod}$ and $K(\mathcal{G}, n)$ is the n-th classifying space, obtained applying n times, in sequence, the bar construction to the S-category $\mathcal{G}$, i.e. $K(\mathcal{G}, n) = B^n(\mathcal{G})$.

Proof. We will just sketch a proof to justify why this adjunction should hold, or better, how we must understand the object $C^*(\mathcal{X}, \mathcal{G})$ (which is weird, since $\mathcal{G}$ is a *simplicial* k-module). To motivate the generalization in the lemma we will assume $\mathcal{G}$ to be discrete and call it $G \in k - \mathsf{Mod}$ (so that if the property holds for discrete objects then definitions are taken so that it still holds in the general case).

By Proposition 4.1.1 we can write $\mathcal{X} \simeq \mathrm{Hocolim}_{\Delta^n \to \mathcal{X}} \Delta^n$ and using the properties of the mapping space (see Theorem 1.3.17) we can write

$$\mathrm{Map}_{\mathsf{sSet}}(\mathcal{X}, K(G, n)) \simeq \mathrm{Holim}_{\Delta^n \to \mathcal{X}} \mathrm{Map}_{\mathsf{sSet}}(\Delta^n, K(G, n))$$

and since $\Delta^n \simeq \Delta^0$ in $\mathrm{Ho}(\mathsf{sSet})$ (contractible) we have

$$\mathrm{Map}_{\mathsf{sSet}}(\mathcal{X}, K(G, n)) \simeq \mathrm{Holim}_{\Delta^n \to \mathcal{X}} \mathrm{Map}_{\mathsf{sSet}}(\Delta^0, K(G, n)) \simeq \mathrm{Holim}_{\Delta^n \to \mathcal{X}} K(G, n).$$

Similarly, writing the cohomology complex $C^*(\mathcal{X}, G)$ as the homotopy limit of the $C^*(\Delta^n, G) \simeq G$ we get, for the right hand side, the following expression

$$\mathrm{Map}_{\mathrm{Ch}(k)}(k, C^*(\mathcal{X}, G)[n]) \simeq \mathrm{Holim}_{\Delta^n \to \mathcal{X}} \mathrm{Map}_{\mathrm{Ch}(k)}(k, G[n]) \simeq \mathrm{Holim}_{\Delta^n \to \mathcal{X}} \mathrm{Map}_{\mathsf{sAb}}(k, DK(G[n]))$$

where we used the Dold-Kan equivalence, see Theorem 4.1.4. Let's observe now, using the free-forgetful adjunction, that

$$\mathrm{Map}_{\mathsf{sAb}}(k, DK(G[n])) \simeq \mathrm{Map}_{\mathsf{sSet}}(\Delta^0, DK(G[n])) \simeq DK(G[n]).$$

We are reduced to compare $K(G, n)$ and $DK(G[n])$ as simplicial set, which are clearly weak equivalent (the only nontrivial homotopy group of $K(G, n)$ is $\pi_n = G$, while the only nontrivial homology, and hence homotopy group using Dold-Kan, of $G[n]$ is $H^n = G$). ∎

Thus by the intuition given by the lemma we are confident enough to write

$$\mathrm{Map}_{\mathsf{sSet}}(K, K(E \otimes_k E^\vee \otimes_k M, 1)) \simeq \mathrm{Map}_{\mathrm{Ch}(k)}(k, C^*(K, E \otimes_k E^\vee \otimes_k M)[1]) \simeq \\ \simeq \mathrm{Map}_{\mathrm{Ch}(k)}(k[-1], C^*(K, E \otimes_k E^\vee \otimes_k M)).$$

By perfectness of $E \otimes E^\vee$ and (a general version of) Proposition 5.1.11 we have

$$C^*(K, E \otimes_k E^\vee \otimes_k M) \simeq \mathbb{R}\,\underline{\mathrm{Hom}}(C_*(K, E \otimes_k E^\vee), M)$$

and hence, using the adjunction with the (left derived) tensor product (although here $k[-1]$ is already projective, so it does not make any difference and it just produces a shift), we conclude

$$\mathbb{D}er_E(\mathbb{R}\mathbf{Loc}_n(K), M) \simeq \mathrm{Map}_{\mathrm{Ch}(k)}(k[-1], \mathbb{R}\,\underline{\mathrm{Hom}}(C_*(K, E \otimes_k E^\vee), M)) \simeq \\ \simeq \mathrm{Map}_{\mathrm{Ch}(k)}(C_*(K, E \otimes_k E^\vee)[-1], M).$$

Thus, we finally conclude that

$$\mathbb{L}_{\mathbb{R}\mathbf{Loc}_n(K), E} \simeq C_*(K, E \otimes_k E^\vee)[-1] \in \mathrm{Ho}(\mathrm{Ch}(k)).$$ ■

5.2 Derived mapping stack

Definition 5.2.1. Let X be a (classic) stack over $k \in \mathsf{Comm}$ and F a derived n-geometric stack. The *mapping derived stack* of morphisms between X and F is given by

$$\underline{\mathrm{Map}}(X, F) := \mathbb{R}\,\mathcal{H}\mathrm{om}(i(X), F) \in \mathrm{dSt}(k)$$

where $\mathbb{R}\,\mathcal{H}\mathrm{om}$ is the internal hom of the cartesian closed category $\mathrm{Ho}(\mathsf{dAff}^\sim)$, as defined in Proposition 4.5.8 (a priori this differs from the internal hom for the category of stacks $\mathrm{St}(k)$). It sends $A \in \mathsf{sComm}$ to $\mathbb{R}\,\underline{\mathrm{Hom}}(i(X) \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, F)$.

Let's recall the following criterion.

Theorem 5.2.2 (J. Lurie's representability criterion)**.** Let $F \in \mathrm{dSt}(k)$. The following are equivalent:

(1) F is an n-geometric derived stack.

(2) F satisfies the three following conditions.

  (a) The truncation $t_0(F)$ is an Artin $(n+1)$-stack, i.e. it is $(n+1)$-truncated and m-geometric for some m.

  (b) F has an obstruction theory.

  (c) For any $A \in sk - \mathsf{Alg}$, the natural map

$$\mathbb{R}F(A) \to \mathrm{Holim}_s\, \mathbb{R}F(A_{\leq s})$$

  is an isomorphism in $\mathrm{Ho}(\mathsf{sSet})$, where we consider the Postnikov tower of A.

Proof. See [HAGII, Appendix C]. ■

Let's now concentrate on a particular case, namely when X is a projective and flat k-scheme and $F = Y$ is also a projective smooth k-scheme. Let's recall the following.

Remark 5.2.3. Let $\tau\colon \mathrm{Spec}\, A \to \mathrm{Spec}\, k$ a map of schemes. Let's observe the classical adjunction

$$\tau^*\colon \mathcal{O}_{\mathrm{Spec}\, A} - \mathsf{Mod} \simeq A - \mathsf{Mod} \to \mathcal{O}_{\mathrm{Spec}\, k} - \mathsf{Mod} \simeq k - \mathsf{Mod} \colon (\tau)_*$$

passes to chain complexes $\mathrm{Ch}(A)$ and $\mathrm{Ch}(k)$ (endowed with the projective model structures) inducing a Quillen adjunction. The left derived functor $\mathbb{L}\tau^*$ corresponds to the left derived tensor product $A \otimes_k^{\mathbb{L}} -$, while $\mathbb{R}(\tau)_*$ is denoted also as $C^*(\mathrm{Spec}\, A, -)$ is the sheaf cohomology complex (indeed the non-derived version just amounts to taking global sections, i.e. $(\tau)_*\mathcal{F} = \Gamma(\mathcal{F}, \mathrm{Spec}\, A)$).

We are ready to prove the following.

Proposition 5.2.4. With the same notations as above, the derived stack $\underline{\mathrm{Map}}(X,Y)$ (we consider X and Y already embedded in $\mathrm{dSt}(k)$) is a 1-geometric derived stack. Moreover, for any $f\colon X\to Y$ the cotangent complex of $\underline{\mathrm{Map}}(X,Y)$ at the point f is

$$\mathbb{L}_{\underline{\mathrm{Map}}(X,Y),f}\simeq C^*(X,f^*T_Y)^\vee$$

where $T_Y=\mathcal{H}om_{\mathcal{O}_Y-\mathsf{Mod}}(\Omega^1_{Y/\operatorname{Spec}k},\mathcal{O}_Y)$ is the tangent sheaf of $Y\to\operatorname{Spec}k$ and $C^*(X,f^*T_Y)$ is the sheaf cohomology complex.

Proof. As usual observe that a point $\operatorname{Spec}k\to\underline{\mathrm{Map}}(X,Y)$ corresponds, using adjunctions and Yoneda, to an element of $\pi_0(\underline{\mathrm{Map}}(X,Y)(\operatorname{Spec}k))=\mathrm{Hom}_{\mathrm{dSt}(k)}(X,Y)$, i.e. to a map $f\colon X\to Y$. Let's first focus on finding the cotangent complex. To do so, we must understand derivations; denote by $S=\mathbb{R}\underline{\mathrm{Spec}}\,k$, pick $M\in sk-\mathsf{Mod}$ and let $S[M]=\mathbb{R}\underline{\mathrm{Spec}}(k\oplus M)$. Then

$$\mathbb{D}er_f(\underline{\mathrm{Map}}(X,Y),M)\overset{\text{def}}{=}\mathrm{Map}_{X/\,\mathrm{dSt}(k)}(X\times^h_S\mathbb{R}\underline{\mathrm{Spec}}(k\oplus M),Y)=\mathrm{Map}_{X\times_S S/\,\mathrm{dSt}(k)}(X\times^h_S S[M],Y).$$

Writing $X=\mathrm{Hocolim}_i\,U_i$ for $U_i=\mathbb{R}\underline{\mathrm{Spec}}\,A_i$ flat affine k-schemes (a suitable chart system of X) with A_i flat k-algebra for each i, we obtain

$$\mathrm{Map}_{X\times_S S/\,\mathrm{dSt}(k)}(X\times^h_S S[M],Y)\simeq\mathrm{Holim}_i\,\mathrm{Map}_{U_i/\,\mathrm{dSt}(k)}(U_i\times^h_S S[M],Y).$$

Observe that we implicitely used that $\mathrm{Hocolim}_i$ and $-\times^h_S S[M]$ commute (this derives from properties of t-model topoi, see [HAGI, Theorem 4.9.2]). Observe now that

$$U_i\times^h_S S[M]=\mathbb{R}\underline{\mathrm{Spec}}(A_i)\times^h_k\mathbb{R}\underline{\mathrm{Spec}}(k\oplus M)\simeq\mathbb{R}\underline{\mathrm{Spec}}(A_i\oplus(A_i\otimes^{\mathbb{L}}_k M))=U_i[A_i\otimes^{\mathbb{L}}_k M]$$

and, calling $\tau_i\colon U_i\to\operatorname{Spec}k$ the structure morphism, we get $A_i\otimes^{\mathbb{L}}_k M=\mathbb{L}\tau_i^*(M)$ (the pullback functor τ^* is left Quillen, and we are also identifying quasi-coherent modules on affine schemes with their global sections). Thus we can write

$$\mathrm{Map}_{X/\,\mathrm{dSt}(k)}(X\times^h_S S[M],Y)\simeq\mathrm{Holim}_i\,\mathrm{Map}_{U_i/\,\mathrm{dSt}(k)}(U_i[\mathbb{L}\tau_i^*(M)],Y)\simeq\mathrm{Holim}_i\,\mathrm{Map}_{\mathrm{Ch}(A_i)}(\mathbb{L}_{Y,g_i},\mathbb{L}\tau_i^*(M))$$

where the last function complexes is in $\mathrm{Ch}(A_i)$, g_i is the restriction of f to U_i and $\mathbb{L}_{Y,g_i}=g_i^*\mathbb{L}_Y$ is the global cotangent complex of Y (which is a smooth scheme, hence its cotangent complex is perfect, corresponding to the classical cotangent sheaf) at the point $g_i\colon U_i\to Y$. Recalling the the (global) tangent complex $\mathbb{T}_Y$ is defined as the dual complex of $\mathbb{L}_Y$, we obtain

$$\begin{aligned}\mathrm{Holim}_i\,\mathrm{Map}_{\mathrm{Ch}(A_i)}(g_i^*\mathbb{L}_Y,\mathbb{L}\tau_i^*(M))&\simeq\mathrm{Holim}_i\,\mathrm{Map}_{\mathrm{Ch}(A_i)}(\mathbb{L}\tau_i^*(M^\vee),g_i^*\mathbb{T}_Y)\simeq\\&\simeq\mathrm{Holim}_i\,\mathrm{Map}_{\mathrm{Ch}(k)}(M^\vee,C^*(U_i,g_i^*\mathbb{T}_Y))\end{aligned}$$

where the last passage is given by the universal property of the cohomology complex (right derived functor of global sections, which corresponds to the structure pushforward to $\operatorname{Spec}k$, see Remark 5.2.3).

Notice that we also used, en passant, the commutation between the (derived) pullback τ_i^* and the (derived) dual $(-)^\vee=\mathbb{R}\,\underline{\mathrm{Hom}}_k(-,k)$. This is possible because A_i is flat over k and hence

$$(\mathbb{L}\tau_i^*(M))^\vee=\mathbb{R}\,\underline{\mathrm{Hom}}_{\mathrm{Ch}(A_i)}(A_i\otimes^{\mathbb{L}}_k M,A_i)\simeq A_i\otimes^{\mathbb{L}}_k\mathbb{R}\,\underline{\mathrm{Hom}}_{\mathrm{Ch}(k)}(M,k)=\mathbb{L}\tau_i^*(M^\vee).$$

Finally, recalling that $X=\mathrm{Hocolim}_i\,U_i$, we obtain

$$\mathbb{D}er_f(\underline{\mathrm{Map}}(X,Y),M)\simeq\mathrm{Map}_{\mathrm{Ch}(k)}(M^\vee,C^*(X,f^*\mathbb{T}_Y))$$

so that, using again duality and $(M^\vee)^\vee\simeq M$, we can conclude

$$\mathbb{L}_{\underline{\mathrm{Map}}(X,Y),f}\simeq C^*(X,f^*\mathbb{T}_Y)^\vee.$$

This last passage is justified by the fact that $f^*\mathbb{T}_Y$ is a perfect complex in $\mathrm{Ch}(\mathcal{O}_X-\mathsf{Mod})$: this holds because $\mathbb{T}_Y$ is projective, since Y is smooth (so that also the pullback is projective, componentwise) and X is projective (so its higher cohomology groups will vanish, i.e. only a finite number of nonzero components will be present). ■

Let's state and prove a technical lemma about inf-cartesianity.

Lemma 5.2.5. Let F be a derived stack which is inf-cartesian. Then, for any $F' \in \mathrm{dSt}(k)$ the mapping derived stack $\underline{\mathrm{Map}}(F', F)$ is also inf-cartesian.

Proof. Choose F' derived stack and write it (using the "homotopical" version of the density theorem, the classical statement saying that any presheaf is a colimit of representable presheaves) as an homotopy colimit of representable derived stacks

$$F' \simeq \mathrm{Hocolim}_i\, U_i \in \mathrm{dSt}(k)$$

Then we obtain

$$\underline{\mathrm{Map}}(F', F) \simeq \mathrm{Holim}_i\, \underline{\mathrm{Map}}(U_i, F)$$

and, as inf-cartesian property is stable by homotopy limits, we can just focus on the case $F' = \mathbb{R}\underline{\mathrm{Spec}}\, B$ is representable. Let $A \in sk - \mathsf{Alg}$ and $M \in sA - \mathsf{Mod}$ with $\pi_0(M) = 0$ and choose a derivation $d \in \pi_0(\mathbb{D}er_k(A, M))$. Applying the derived stack $\underline{\mathrm{Map}}(F', F)$ to the homotopy pullback diagram defining $A \oplus_d \Omega M$ (see Definition 4.8.11) we obtain the commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Map}}(F', F)(A \oplus_d \Omega M) & \longrightarrow & \underline{\mathrm{Map}}(F', F)(A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Map}}(F', F)(A) & \longrightarrow & \underline{\mathrm{Map}}(F', F)(A \oplus M) \end{array}$$

and, observing $\underline{\mathrm{Map}}(F', F)(A) = \mathbb{R}\,\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Spec}}\, B \times^h_k \mathbb{R}\underline{\mathrm{Spec}}\, A, F) \simeq \mathbb{R}F(A \otimes^{\mathbb{L}}_k B)$ we get

$$\begin{array}{ccc} \mathbb{R}F((A \oplus_d \Omega M) \otimes^{\mathbb{L}}_k B) & \longrightarrow & \mathbb{R}F(A \otimes^{\mathbb{L}}_k B) \\ \downarrow & & \downarrow \\ \mathbb{R}F(A \otimes^{\mathbb{L}}_k B) & \longrightarrow & \mathbb{R}F((A \oplus M) \otimes^{\mathbb{L}}_k B). \end{array}$$

By assumption F is inf-cartesian, so we can consider this property with $A \otimes^{\mathbb{L}}_k B \in sk - \mathsf{Alg}$ and $d \otimes_k B \in \pi_0(\mathbb{D}er_k(A \otimes^{\mathbb{L}}_k B, M \otimes^{\mathbb{L}}_k B))$, and we obtain that the last diagram is homotopy cartesian (the functor $- \otimes^{\mathbb{L}}_k B$ commutes with homotopy pullbacks, since $- \otimes_k B$ is left Quillen). Thus we conclude that also $\underline{\mathrm{Map}}(F', F)$ is inf-cartesian. ∎

Let's now state a criterion to investigate the n-geometricity of the derived stack $\underline{\mathrm{Map}}(X, F)$ (with the same notations as above, i.e. X is a stack and F an n-geometric derived stack).

Theorem 5.2.6. With same notations as above, assume the following are satisfied:

(1) The stack
$$t_0(\underline{\mathrm{Map}}(X, F)) \simeq \mathbb{R}\,\mathcal{H}\mathrm{om}(X, t_0(F)) \in \mathrm{St}(k)$$
is n-geometric (where the last $\mathcal{H}$om is the internal hom for classical stacks).

(2) The derived stack $\underline{\mathrm{Map}}(X, F)$ has a global cotangent complex.

(3) The stack X can be written, in $\mathrm{St}(k)$, as an homotopy colimit $\mathrm{Hocolim}_i\, U_i$, where U_i is an affine scheme, flat over $\mathrm{Spec}\, k$.

Then, the derived stack $\underline{\mathrm{Map}}(X, F)$ is n-geometric.

Proof. The converse holds, clearly. Let us suppose $\underline{\mathrm{Map}}(X, F)$ respects the three hypotheses, and let's try to lift an n-atlas of $t_0(\underline{\mathrm{Map}}(X, F))$ to an n-atlas of $\underline{\mathrm{Map}}(X, F)$. We will use Theorem 5.2.2, i.e. we need to prove that $\underline{\mathrm{Map}}(X, F)$ satisfies conditions (a)-(c). The condition (a) is exactly our assumption (1). Having an obstruction theory means having a global cotangent complex and being inf-cartesian: the existence of cotangent complex is assumption (2), while for inf-cartesian we can use Lemma 5.2.5. It then remains just to show condition (c) of Lurie's criterion. Let's now write $X = \mathrm{Hocolim}_i\, U_i$, with U_i affine and flat over k, so that

$$\underline{\mathrm{Map}}(X, F) \simeq \mathrm{Holim}_i\, \underline{\mathrm{Map}}(U_i, F).$$

Since condition (c) is stable by homotopy limits, we can assume $X = \mathrm{Spec}\, B$ for B a commutative flat k-algebra. For any $A \in sk - \mathsf{Mod}$ the map

$$\underline{\mathrm{Map}}(X, Y)(A) \to \mathrm{Holim}_s\, \underline{\mathrm{Map}}(X, Y)(A_{\leq s})$$

is equivalent to (see the proof of preceeding lemma)

$$\mathbb{R}F(A \otimes_k B) \to \mathrm{Holim}_s\, \mathbb{R}F((A_{\leq s} \otimes_k B))$$

where we didn't write the left-derived tensor product since B is flat over k (and hence $A \otimes_k B$ corresponds to $A \otimes^{\mathbb{L}}_k B$). By flatness, we can write $(A_{\leq s}) \otimes_k B \simeq (A \otimes_k B)_{\leq s}$ (recall that $A_{\leq s}$ is obtained as an homotopy pullback) and therefore the above morphism is equivalent to

$$\mathbb{R}F(A \otimes_k B) \to \mathrm{Holim}_s\, \mathbb{R}F((A \otimes_k B)_{\leq s})$$

which is an equivalence since F is n-geometric (using the other direction of Theorem 5.2.2). ■

5.3 Derived moduli space of vector bundles

Finally we will focus our attention on the derived version of the stack of rank n vector bundles (i.e. GL_n-bundles) studied in Chapter 3. We start with similar assumptions, namely $p\colon X \to \mathrm{Spec}\, k$ a projective and flat map (so that X is a projective flat space over k). Here k will be a field.

Definition 5.3.1. The derived moduli space of GL_n-bundles of $p\colon X \to \mathrm{Spec}\, k$ is

$$\underline{\mathrm{Map}}(X, \mathbf{Vect}_n) := \mathbb{R}\,\mathcal{H}\mathrm{om}(X, \mathbf{Vect}_n) \simeq \mathbb{R}\,\mathcal{H}\mathrm{om}(X, B\,\mathrm{GL}_n)$$

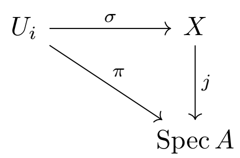
where we view $X, \mathbf{Vect}_n$ and $B\,\mathrm{GL}_n$ implicitely embedded in $\mathrm{dSt}(k)$.

We see that it is exactly the derived version of Definition 3.4.6. Recall that, as classic stacks, $\mathbf{Vect}_n \simeq B\,\mathrm{GL}_n$.

Remark 5.3.2. Let's take a little detour, which will be useful in the following proof; let $j\colon X \to \mathrm{Spec}\, A$ be a map of schemes and consider the direct image functor

$$j_* = \Gamma(X, -)\colon \mathrm{QCoh}(X) \to A - \mathsf{Mod}.$$

Write X as colimit of affine charts U_i and observe that, by the very definition of sheaf, we can write j_* as a limit of the functors of global sections on the open covering $\{U_i\}_i$ (considering also intersections). More specifically we have the following diagram



and we consider the (pseudo)-functors $\pi_* \circ \sigma^*$ (the index i is implicit), acting by $\mathcal{F} \mapsto \Gamma\left(U_i, \mathcal{F}|_{U_i}\right)$. We can thus write

$$j_* \simeq \varprojlim_i \pi_* \circ \sigma^*$$

and we will then use the derived version of this (which is nothing else then the derived base change formula), stating

$$C^*(X, -) = \mathbb{R}j^* \simeq \mathrm{Holim}_i\, C^*(U_i, \mathbb{L}\sigma^*(-)) \in D(A).$$

Proposition 5.3.3. The derived stack $\underline{\mathrm{Map}}(X, \mathbf{Vect}_n)$ has a global cotangent complex. In particular, given a point $x\colon \mathbb{R}\underline{\mathrm{Spec}}\, A \to \underline{\mathrm{Map}}(X, \mathbf{Vect}_n)$ with $A \in sk - \mathsf{Alg}$, corresponding to a vector bundle $\mathcal{E}$ on $X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A$, the cotangent complex is

$$\mathbb{L}_{\underline{\mathrm{Map}}(X, \mathbf{Vect}_n), \mathcal{E}} \simeq C_*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E}))[-1] \in \mathrm{Ho}(Sp(sA - \mathsf{Mod})).$$

Proof. By Yoneda lemma, x is an element of $\pi_0(\mathbb{R}\,\underline{\mathrm{Hom}}(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathbf{Vect}_n))$, corresponding to a rank n vector bundle $\mathcal{E}$ on $X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A$. Let's fix a system of affine (flat) k-charts so that

$$X \simeq \mathrm{Hocolim}_i\, U_i$$

where $U_i \simeq \operatorname{Spec} B_i$ for B_i a flat k-algebra. Then we have

$$\underline{\mathrm{Map}}(X, \mathbf{Vect}_n) \simeq \mathrm{Holim}_i\, \underline{\mathrm{Map}}(U_i, \mathbf{Vect}_n)$$

and we can consider x_i, obtained by composing x and the canonical projection $\underline{\mathrm{Map}}(X, \mathbf{Vect}_n) \to \underline{\mathrm{Map}}(U_i, \mathbf{Vect}_n)$. Since $U_i \simeq \operatorname{Spec} B_i \simeq \mathbb{R}\underline{\mathrm{Spec}}\, B_i$, by Yoneda and k-flatness of B_i we have

$$x_i \in \pi_0(\underline{\mathrm{Map}}(U_i, \mathbf{Vect}_n)(\mathbb{R}\underline{\mathrm{Spec}}\, A)) = \pi_0(\mathbb{R}\,\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Spec}}\, A \times^h \mathbb{R}\underline{\mathrm{Spec}}\, B_i, \mathbf{Vect}_n)) =$$
$$= \pi_0(\mathbb{R}\,\underline{\mathrm{Hom}}(\mathbb{R}\underline{\mathrm{Spec}}(B_i \otimes_k^{\mathbb{L}} A), \mathbf{Vect}_n)) = \pi_0(\mathbf{Vect}_n(B_i \otimes_k A))$$

so that x_i is just the datum of E_i, a projective $s(B_i \otimes_k A)$-module of rank n. It corresponds to the global sections of $\mathcal{E}_i$, the homotopy pullback of the vector bundle $\mathcal{E}$, in this diagram

$$\begin{array}{ccc} U_i \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A & \overset{\sigma}{\hookrightarrow} & X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A \\ & \searrow^{\pi} & \downarrow j \\ & & \mathbb{R}\underline{\mathrm{Spec}}\, A \end{array}$$

where we immediately notice that j is projective and flat by base change.

Observe now that given $M \in sA - \mathsf{Mod}$ we have

$$\underline{\mathrm{Map}}(U_i, \mathbf{Vect}_n)(\mathbb{R}\underline{\mathrm{Spec}}\, A) \simeq \mathbf{Vect}_n(B_i \otimes_k A),$$
$$\underline{\mathrm{Map}}(U_i, \mathbf{Vect}_n)(\mathbb{R}\underline{\mathrm{Spec}}\, A \oplus M) \simeq \mathbf{Vect}_n(B_i \otimes_k (A \oplus M)) \simeq \mathbf{Vect}_n((B_i \otimes_k A) \oplus (B_i \otimes_k M)).$$

To compute derivations for U_i and M, we need to find the homotopy fiber

$$\begin{array}{ccc} \bullet & \longrightarrow & \mathbf{Vect}_n((B_i \otimes_k A) \oplus (B_i \otimes_k M)) \\ \downarrow & & \downarrow \\ * = \Delta^0 & \xrightarrow{E_i} & \mathbf{Vect}_n(B_i \otimes_k A) \end{array}$$

and we see that, as in the proof of Proposition 5.1.12, we have

$$\bullet = K(E_i \otimes_{B_i \otimes A} E_i^{\vee_i} \otimes_{B_i \otimes A} (B_i \otimes M), 1)$$

where $E_i^{\vee_i} = \mathrm{Hom}_{s(B_i \otimes A)-\mathsf{Mod}}(E_i, B_i \otimes A)$ and $B_i \otimes_k M \in s(B_i \otimes_k A) - \mathsf{Mod}$. Let's observe that by thinking of derivations as maps towards our derived stack under $\mathbb{R}\underline{\mathrm{Spec}}\, A$, we see that in $\mathrm{Ho}(\mathsf{sSet})$ we have

$$\mathbb{D}er_x(\underline{\mathrm{Map}}(X, \mathbf{Vect}_n), M) \simeq \mathbb{D}er_x(\mathrm{Holim}_i\, \underline{\mathrm{Map}}(U_i, \mathbf{Vect}_n), M) \simeq$$
$$\simeq \mathrm{Holim}_i\, \mathbb{D}er_{x_i}(\underline{\mathrm{Map}}(U_i, \mathbf{Vect}_n), M).$$

By definition of derivations as homotopy fibers, and reasoning again as in Lemma 5.1.13 (even though here we need to use stable modules and not just chain complexes), we have

$$\mathbb{D}er_{x_i}(\underline{\mathrm{Map}}(U_i, \mathbf{Vect}_n), M) \simeq K(E_i \otimes_{B_i \otimes A} E_i^{\vee_i} \otimes_{B_i \otimes A} (B_i \otimes M), 1) \simeq$$
$$\simeq \mathrm{Map}_{\mathsf{sSet}}(\Delta^0, K(E_i \otimes_{B_i \otimes A} E_i^{\vee_i} \otimes_{B_i \otimes A} (B_i \otimes M), 1)) \simeq$$
$$\simeq \mathrm{Map}_{Sp(s(B_i \otimes A)-\mathsf{Mod})}(B_i \otimes A, (E_i \otimes_{B_i \otimes A} E_i^{\vee_i} \otimes_A M)[1])$$

where we used the fact that $C^*(\Delta_0, G) = G$ and $E_i \otimes_{B_i \otimes A} E_i^{\vee_i} \otimes_{B_i \otimes A} (B_i \otimes_k M) \simeq E_i \otimes_{B_i \otimes A} E_i^{\vee_i} \otimes_A M$.

Recall now that, B_i being k-flat, $B_i \otimes_k^{\mathbb{L}} A = B_i \otimes_k A$, and using the derived adjunction (where the right adjoint is the forgetful functor $s(B_i \otimes A) - \mathsf{Mod} \to sA - \mathsf{Mod}$) we obtain

$$\mathrm{Map}_{B_i \otimes A}(B_i \otimes_k^{\mathbb{L}} A, (E_i \otimes_{B_i \otimes A} E_i^{\vee_i} \otimes_A M)[1]) \simeq \mathrm{Map}_A(A[-1], {}_A(E_i \otimes_{B_i \otimes A} E_i^{\vee_i}) \otimes_A M)$$

where we do not need to (right) derive the forgetful functor $s(B_i \otimes A) - \mathsf{Mod} \to sA - \mathsf{Mod}$ since it is exact. For typography reasons we used $B_i \otimes A$ to mean the category of stable simplicial $(B_i \otimes A)$-modules, similar for A in the second mapping space.
Observe that we have the following homotopy cartesian diagram

$$\begin{array}{ccc} \mathcal{E}\mathrm{nd}(\mathcal{E}_i) & \longrightarrow & \mathcal{E}\mathrm{nd}(\mathcal{E}) \\ \downarrow & & \downarrow \\ U_i \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A & \longrightarrow & X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A \end{array}$$

and that

$$\Gamma(U_i \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E}_i)) = \underline{\mathrm{End}}_{\mathcal{O}_{U_i \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A}}(\mathcal{E}_i) \simeq \underline{\mathrm{End}}_{B_i \otimes A}(E_i) = E_i \otimes_{B_i \otimes A} E_i^{\vee_i}.$$

Since we don't need to derive the forgetful functor σ^* we then obtain

$$C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E})) \simeq \mathrm{Holim}_i\, C^*(U_i \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E}_i)) \simeq \mathrm{Holim}_i\, {}_A(E_i \otimes_{B_i \otimes A} E_i^{\vee_i})$$

where in the last passage we exploited the fact that $U_i \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A$, being (derived) affine, has no higher cohomology (this is just a fancy slogan: to be precise we used that on a derived affine schemes, giving a quasi-coherent (simplicial) module corresponds to give its global section, and $\mathcal{E}\mathrm{nd}(\mathcal{E}_i)$ is projective, hence fibrant, so no need to derive).

Let's now prove that the stable sA-module $C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E}))$ is perfect. First technical point: since in the category $Sp(sA - \mathsf{Mod})$ a square is homotopy cartesian if and only if it is homotopy cocartesian, any functor commutes with finite homotopy limits iff it commutes with finite homotopy colimits. We will use this property for derived tensor products (in our case the derived version is the same as non-derived since we work with flat schemes). Since $\mathcal{E}\mathrm{nd}(\mathcal{E})$ is projective of rank n, we have two maps

$$\mathcal{E}\mathrm{nd}(\mathcal{E}) \hookrightarrow (\mathcal{O}_X \otimes A)^n \longrightarrow \mathcal{E}\mathrm{nd}(\mathcal{E})$$

whose composite is the identity. They induce corresponding maps at the level of "complexes" so that we might just prove that $C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{O}_X \otimes A)$ is perfect (hence also a sum of n copies is so). Recalling the writing of X as homotopy colimit, we have

$$C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{O}_X \otimes A) \simeq \mathrm{Holim}_i\, C^*(U_i \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{O}_{U_i} \otimes A) \simeq \mathrm{Holim}_i(\mathcal{O}_{U_i} \otimes A)$$

where we used the fact that (derived) affine schemes have no higher cohomology. Since X is projective of finite type, it admits a finite number of affine charts, i.e. the homotopy limit above is finite and hence we can exchange it with the tensor product $- \otimes A$. We obtain

$$C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{O}_X \otimes A) \simeq (\mathrm{Holim}_i\, \mathcal{O}_{U_i}) \otimes A \simeq (\mathrm{Holim}_i\, C^*(U_i, \mathcal{O}_{U_i})) \otimes A \simeq C^*(X, \mathcal{O}_X) \otimes A.$$

Since X is projective over a field k, we know from classical algebraic geometry that its cohomology $H^*(X, \mathcal{O}_X)$ is a finite dimensional k-vector space, i.e. that $C^*(X, \mathcal{O}_X)$ is a perfect complex in $\mathrm{Ch}(k)$. By base change, also $C^*(X, \mathcal{O}_X) \otimes A$ is perfect as a stable sA-module, and hence we conclude that $C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{O}_X \otimes A)$ is perfect in $Sp(sA - \mathsf{Mod})$. We will then be able to do all the usual duality shenanigans. Since only the homotopy type of $M \in \mathrm{Ho}(sA - \mathsf{Mod})$ matters, we can replace M by a (fibrant) and cofibrant resolution in $Sp(sA - \mathsf{Mod})$, so that $- \otimes_A M \simeq - \otimes_A^{\mathbb{L}} M$. As written above, $- \otimes_A^{\mathbb{L}} M$ commutes with finite holimits (taken in $Sp(sA - \mathsf{Mod})$), and hence

$$\begin{aligned} &\mathrm{Holim}_i\, \mathrm{Map}_{Sp(sA-\mathsf{Mod})}(A[-1], {}_A(E_i \otimes_{B_i \otimes A} E_i^{\vee_i}) \otimes_A^{\mathbb{L}} M) \simeq \\ &\simeq \mathrm{Map}_{Sp(sA-\mathsf{Mod})}(A[-1], \mathrm{Holim}_i\, {}_A(E_i \otimes_{B_i \otimes A} E_i^{\vee_i}) \otimes_A^{\mathbb{L}} M) \simeq \\ &\simeq \mathrm{Map}_{Sp(sA-\mathsf{Mod})}(A[-1], C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes_A^{\mathbb{L}} M) \end{aligned}$$

and using Proposition 5.1.11 together with

$$\mathbb{R}\,\underline{\mathrm{Hom}}(C_*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E})), A) \otimes_A^{\mathbb{L}} M \simeq \mathbb{R}\,\underline{\mathrm{Hom}}(C_*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E})), M)$$

and the tensor-hom adjunction we conclude

$$\begin{aligned} \mathbb{D}er_x(\underline{\mathrm{Map}}(X, \mathbf{Vect}_n), M) &\simeq \mathrm{Map}_{Sp(sA-\mathsf{Mod})}(A[-1], C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E})) \otimes_A^{\mathbb{L}} M) \simeq \\ &\simeq \mathrm{Map}_{Sp(sA-\mathsf{Mod})}(C_*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E}))[-1], M) \end{aligned}$$

so that we proved $\mathbb{L}_{\underline{\mathrm{Map}}(X, \mathbf{Vect}_n), \mathcal{E}} \simeq C_*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\, A, \mathcal{E}\mathrm{nd}(\mathcal{E}))[-1]$.

Finally, let's prove the derived stack $\underline{\mathrm{Map}}(X, \mathbf{Vect}_n)$ admits a global cotangent complex using Definition 4.8.3. Let's consider the following commutative diagram in $\mathrm{Ho}(\mathsf{dAff}^{\sim})$

$$\begin{array}{ccc} \mathbb{R}\underline{\mathrm{Spec}}\,A & \xrightarrow{\quad f \quad} & \mathbb{R}\underline{\mathrm{Spec}}\,B \\ & {\scriptstyle \mathcal{E}} \searrow \quad \swarrow {\scriptstyle \mathcal{F}} & \\ & \underline{\mathrm{Map}}(X, \mathbf{Vect}_n) & \end{array}$$

where $\mathcal{E}$ (resp. $\mathcal{F}$) is a rank n vector bundle on $X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,A$ (resp. $X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,B$). More precisely this means that $\mathcal{E}$ is the homotopy pullback of $\mathcal{F}$ under the natural map $\mathrm{id}_X \times f \colon X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,A \to X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,B$ (and the same happens for the sheaf of endomorphisms). We have proved before that

$$\begin{aligned} \mathbb{L}_{\mathcal{E}} &\coloneqq \mathbb{L}_{\underline{\mathrm{Map}}(X, \mathbf{Vect}_n), \mathcal{E}} = C_*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,A, \mathcal{E}\mathrm{nd}(\mathcal{E}))[-1] \in Sp(sA - \mathsf{Mod}), \\ \mathbb{L}_{\mathcal{F}} &\coloneqq \mathbb{L}_{\underline{\mathrm{Map}}(X, \mathbf{Vect}_n), \mathcal{F}} = C_*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,B, \mathcal{E}\mathrm{nd}(\mathcal{F}))[-1] \in Sp(sB - \mathsf{Mod}) \end{aligned}$$

and now we want to prove that the natural map

$$u \colon \mathbb{L}_{\mathcal{E}} \to \mathbb{L}_{\mathcal{F}} \otimes_B^{\mathbb{L}} A$$

is a weak equivalence in $Sp(sA - \mathsf{Mod})$. Since they are perfect stable modules, it suffices to prove their dual (the cohomology complexes) are weakly equivalent. By Theorem 3.5.3 applied to the homotopy cartesian square

$$\begin{array}{ccc} X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,A & \xrightarrow{\mathrm{id}_X \times f} & X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,B \\ \downarrow {\scriptstyle p_A} & & \downarrow {\scriptstyle p_B} \\ \mathbb{R}\underline{\mathrm{Spec}}\,A & \xrightarrow{\quad f \quad} & \mathbb{R}\underline{\mathrm{Spec}}\,B \end{array}$$

and to the coherent locally free sheaf $\mathcal{E}\mathrm{nd}(\mathcal{F})$ on $X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,B$ we obtain

$$C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,B, \mathcal{E}\mathrm{nd}(\mathcal{F})) \otimes_B^{\mathbb{L}} A \xrightarrow{\sim} C^*(X \times^h \mathbb{R}\underline{\mathrm{Spec}}\,A, \mathcal{E}\mathrm{nd}(\mathcal{E}))$$

in $\mathrm{Ho}(Sp(sA - \mathsf{Mod}))$, which is exactly what we wanted (recalling that the derived pullback is the derived tensor product and the derived push-forward is the cohomology complex). ■

Proposition 5.3.4. The classical part $t_0\,\underline{\mathrm{Map}}(X, \mathbf{Vect}_n)$ is 1-truncated Artin geometric stack.

Proof. We have

$$t_0\,\underline{\mathrm{Map}}(X, \mathbf{Vect}_n) \simeq t_0 \mathbb{R}\,\mathcal{H}\mathrm{om}(X, \mathbf{Vect}_n) \simeq \mathcal{H}\mathrm{om}(t_0(X), t_0(\mathbf{Vect}_n)) \overset{\mathrm{def}}{\simeq} \mathrm{Bun}_{\mathrm{GL}_n}$$

where we see the stack in groupoids $\mathrm{Bun}_{\mathrm{GL}_n}$ as a stack in simplicial sets by using the nerve functor. Of course it is 1-truncated, and it is Artin (i.e. using smooth atlases) 1-geometric by Theorem 3.5.13. ■

Theorem 5.3.5. The derived stack $\underline{\mathrm{Map}}(X, \mathbf{Vect}_n)$ is a 1-geometric derived stack.

Proof. This derives from Theorem 5.2.6, by verifying the three asumptions. The first is Proposition 5.3.4, the second is Proposition 5.3.3 and the third one is obvious by our assumptions on X (just choose an affine covering). ■

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