UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Corso di Laurea in Fisica

Tesi di Laurea

Meccanica quantistica supersimmetrica e localizzazione equivariante

Supersymmetric quantum mechanics and equivariant localization

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Introduzione

Lo scopo di questa tesi è presentare gli argomenti della *meccanica quantistica supersimmetrica*, della *localizzazione equivariante* e mostrare come quest'ultima, insieme ad altri tipi di localizzazione e invarianza di deformazione, emerga quando studiamo sistemi quantistici supersimmetrici.

Ottenere risultati esatti in meccanica quantistica è un compito difficile, ma in certe situazioni le caratteristiche del sistema in esame ci aiutano a eseguire calcoli esatti: è il caso dei sistemi quantistici supersimmetrici, una classe di sistemi per i quali possiamo fare ciò applicando strumenti matematici avanzati come quelli discussi in questa tesi.

Per *localizzazione* intendiamo il fenomeno di riduzione di proprietà globali di una varietà a proprietà globali di una sottovarietà propria: ad esempio, la riduzione dell'integrale di una forma differenziale a una somma finita di contributi da punti isolati. In particolare, la localizzazione equivariante è un tipo di localizzazione indotta da un'azione di gruppo. Con *invarianza per deformazione* intendiamo il fenomeno di invarianza di proprietà di un sistema quantistico quando applichiamo una deformazione continua dell'Hamiltoniano. Le idee di localizzazione e invarianza di deformazione di alcune proprietà di un sistema supersimmetrico, come l'*indice di Witten*, sono apparse nell'articolo di Witten [4] del 1982.

Inoltre, lavorando con un sistema basato su una varietà compatta, la supersimmetria offre un contesto interessante per enunciare e abbozzare una dimostrazione di alcuni profondi teoremi della geometria differenziale: *teorema di decomposizione di Hodge*, *disequazioni di Morse*, *teorema di Chern-Gauss-Bonnet* e una versione più debole del *teorema di Poincaré-Hopf*.

Nel Capitolo 1 presentiamo un'introduzione generale alla meccanica quantistica supersimmetrica con $\mathcal{N} = 2$ e discutiamo alcuni esempi di base. Nel Capitolo 2 presentiamo un'introduzione generale alle forme differenziali e alla localizzazione equivariante, enunciamo la *formula di localizzazione di Berline-Vergne* e discutiamo alcuni esempi di base. Nel Capitolo 3 presentiamo un particolare sistema quantistico supersimmetrico basato su una varietà compatta, e alcune sue deformazioni. Dimostriamo alcuni risultati utilizzando gli strumenti dei capitoli precedenti insieme ad altri teoremi, la cui dimostrazione è riportata in bibliografia. In particolare, calcoliamo l'indice di Witten del sistema in diversi modi, derivando come sottoprodotto i teoremi di topologia differenziale sopracitati. Tutti questi metodi di calcolo si basano su un tipo di deformazione: l'aggiunta di un termine d'interazione generato da una funzione di Morse, un campo vettoriale di Killing o il ridimensionamento dell'Hamiltoniano non modificano l'indice, quindi possiamo eseguire il calcolo esatto in un limite appropriato del parametro di deformazione, dove avviene la localizzazione. Nell'Appendice A dimostriamo la formula di localizzazione di Berline-Vergne, mentre nell'Appendice B abbozziamo una dimostrazione del lemma 3.10.

Nella tesi presentiamo e sviluppiamo le idee delle lezioni di Tong [3] e dell'articolo di Witten [4], chiarendo rigorosamente una buona quantità di passaggi matematici, facendo riferimento anche ad altre fonti.

Introduction

The aim of this thesis is to present the topics of *supersymmetric quantum mechanics*, *equivariant localization* and to show how the latter, together with other types of localization and deformation invariance, emerges when we study supersymmetric quantum systems.

Obtaining exact results in quantum mechanics is a difficult task, but in certain situations the features of the system under examination help us to perform exact calculations: this is the case of supersymmetric quantum systems, a class of systems for which we can do so by applying advanced mathematical tools such as those discussed in this thesis.

By *localization* we mean the phenomenon of reduction of global properties of a manifold to global properties of a proper submanifold: for example, the reduction of the integral of a differential form to a finite sum of contributions from isolated points. In particular, equivariant localization is a type of localization induced by a group action. By *deformation invariance* we mean the phenomenon of invariance of properties of a quantum system when we apply a continuous deformation of the Hamiltonian. The ideas of localization and deformation invariance of some properties of a supersymmetric system, such as the *Witten index*, appeared in Witten's paper [4] of 1982.

Furthermore, working with a system based on a compact manifold, supersymmetry offers an interesting framework to state and sketch a proof of some deep theorems of differential geometry: *Hodge decomposition theorem*, *Morse inequalities*, *Chern-Gauss-Bonnet theorem* and a weaker version of *Poincaré-Hopf theorem*.

In Chapter 1 we present a general introduction to $\mathcal{N} = 2$ supersymmetric quantum mechanics and discuss some basic examples. In Chapter 2 we present a general introduction to differential forms and equivariant localization, state the *Berline-Vergne localization formula* and discuss some basic examples. In Chapter 3 we present a particular supersymmetric quantum system based on a compact manifold, and some deformations of it. We prove some results using the tools of the previous chapters together with other theorems, whose proof are reported in the bibliography. In particular, we compute the Witten index of the system in different ways, deriving as a byproduct the differential topology theorems mentioned above. All these methods of computation are based on a type of deformation: adding an interaction term generated by a Morse function, a Killing vector field or scaling the Hamiltonian does not change the index, so we can perform the exact computation in a suitable limit of the deformation parameter, where localization occurs. In Appendix A we prove the Berline-Vergne localization formula, while in Appendix B we sketch a proof of lemma 3.10.

Throughout the thesis we present and develop the ideas of Tong's lectures [3] and Witten's paper [4], rigorously clarifying a good amount of mathematical steps, also referring to other sources.

Contents

Chapter 1

$\mathcal{N} = 2$ supersymmetric quantum mechanics

1.1 General theory

Given a quantum system $(\mathcal{H}, H)$, we say it is supersymmetric with $\mathcal{N} = 2$ *conserved supercharges* if there exists a densely defined linear operator $Q \neq Q^*$, called the *complex supercharge operator*, and a self-adjoint operator F, called the *fermion number operator*, whose spectrum is a set of natural numbers $\{0, \ldots, n\}$, such that

$$\begin{aligned} Q &\neq Q^* \\ Q^2 &= 0 \\ \left\{Q, (-1)^F\right\} &= 0 \\ H &= \{Q, Q^*\}, \end{aligned}$$

where $\{*, *\}$ is the anticommutator. The operator $(-1)^F$ is called the *parity operator*. The conserved supercharges are $Q + Q^*$ and $i(Q - Q^*)$. If $\widetilde{Q}$ is a conserved supercharge, then $H = \widetilde{Q}^2$ and the name is trivially justified by $\left[H, \widetilde{Q}\right] = 0$. Note that $H^* = H \neq 0$. The Hilbert space $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$, where

$$\mathcal{H}_B = \left\{\psi \in \mathcal{H} : (-1)^F \psi = \psi\right\}$$

is called *bosonic Hilbert space* and

$$\mathcal{H}_F = \left\{\psi \in \mathcal{H} : (-1)^F \psi = -\psi\right\}$$

is called *fermionic Hilbert space.* We sometimes require the stronger property that

$$[F, Q] = Q,$$

in which case we say the system is *completely graded*, implying also that $\left\{Q, (-1)^F\right\} = 0$. Now we prove some general results.

Proposition 1.1. *H is positive semi-definite and a state ψ has zero energy if and only if $Q\psi = Q^*\psi = 0$. In symbols,*

$$H \geq 0$$

$$\ker H = \ker Q \cap \ker Q^*.$$

Proof. For a state ψ,

$$\begin{aligned}(\psi, H\psi) &= (\psi, \{Q, Q^*\}\psi) \\ &= (\psi, QQ^*\psi) + (\psi, Q^*Q\psi) \\ &= (Q^*\psi, Q^*\psi) + (Q\psi, Q\psi) \\ &= \|Q^*\psi\|^2 + \|Q\psi\|^2 \\ &\geq 0,\end{aligned}$$

and $(\psi, H\psi) = 0$ if and only if $Q\psi = Q^*\psi = 0$. □

Proposition 1.2. *If $Q^*Q\psi = 0$, then $Q\psi = 0$. If $QQ^*\psi = 0$ then $Q^*\psi = 0$.*

Proof.

$$0 = (Q^*Q\psi, \psi) = (Q\psi, Q\psi) = \|Q\psi\|^2,$$

so $Q\psi = 0$. The proof of the second part is analogous. □

Proposition 1.3. *We have $[H, Q] = 0$.*

Proof.

$$\begin{aligned}[H, Q] &= [\{Q, Q^*\}, Q] \\ &= [QQ^*, Q] + [Q^*Q, Q] \\ &= QQ^*Q - QQ^*Q \\ &= 0.\end{aligned}$$

□

Proposition 1.4. *If the system is completely graded, we have $[H, F] = 0$. In the general case we have $\left[H, (-1)^F\right] = 0$.*

Proof. Since

$$\begin{aligned}FQ &= Q(F+1) \\ QF &= (F-1)Q,\end{aligned}$$

then we have

$$\begin{aligned}Q^*F &= (F+1)Q^* \\ FQ^* &= Q^*(F-1)\end{aligned}$$

and

$$\begin{aligned}[H, F] &= [\{Q, Q^*\}, F] \\ &= QQ^*F + Q^*QF - FQQ^* - FQ^*Q \\ &= Q(F+1)Q^* + Q^*(F-1)Q - Q(F+1)Q^* - Q^*(F-1)Q \\ &= 0.\end{aligned}$$

The proof of the second part is similar. □

Proposition 1.5. *If H gives a finite-dimensional eigenspaces decomposition*

$$\mathcal{H} = \bigoplus_{N>0} \mathcal{H}_B^{(N)} \oplus \mathcal{H}_F^{(N)} \oplus \mathcal{H}^{(0)},$$

where $\mathcal{H}_B^{(N)}$ and $\mathcal{H}_F^{(N)}$ are the bosonic and fermionic part of each energy eigenspace $\mathcal{H}^{(N)}$, then $\dim \mathcal{H}_B^{(N)} = \dim \mathcal{H}_F^{(N)}$.

Proof. We claim that, given a conserved supercharge $\widetilde{Q}$, it is an isomorphism between $\mathcal{H}_B^{(N)}$ and $\mathcal{H}_F^{(N)}$ with $N \neq 0$. Firstly, $\widetilde{Q}\mathcal{H}_B^{(N)} \subseteq \mathcal{H}_F^{(N)}$, since $[H, Q] = 0$ and $\left\{Q, (-1)^F\right\} = 0$. Then, it is also invertible with inverse $\frac{1}{E_N}\widetilde{Q}$, where $E_N > 0$ is the energy of $\mathcal{H}^{(N)}$. In fact, if restricted on $\mathcal{H}_B^{(N)}$,

$$\frac{1}{E_N}\widetilde{Q}^2 = \frac{1}{E_N}H = 1.$$

□

We now restrict our study to systems which fullfill the conditions of the previous proposition. Moreover, we require that the energies of the eigenspace are such that $E_N \overset{N\to\infty}{\longrightarrow} \infty$ and we assume they are increasing. This will be the case of all the systems we care about in this thesis.

Definition 1.1. We define the Witten index $\mathcal{W}$ as

$$\mathcal{W} = \dim \mathcal{H}_B^{(0)} - \dim \mathcal{H}_F^{(0)}.$$

For $\beta > 0$, if the following is a trace-class operator, we have

$$\operatorname{tr}\left((-1)^F e^{-\beta H}\right) = \mathcal{W},$$

since

$$\begin{aligned}\operatorname{tr}\left((-1)^F e^{-\beta H}\right) &= \sum_{N>0} e^{-\beta E_N}\left(\dim \mathcal{H}_B^{(N)} - \dim \mathcal{H}_F^{(N)}\right) + \dim \mathcal{H}_B^{(0)} - \dim \mathcal{H}_F^{(0)} \\ &= \dim \mathcal{H}_B^{(0)} - \dim \mathcal{H}_F^{(0)}.\end{aligned}$$

Proposition 1.6. *If there are no convergence issue in the following proof, every state ψ in the domain of Q and Q^* can be decomposed as*

$$\psi = \psi_0 + Q\psi_1 + Q^*\psi_2,$$

where $H\psi_0 = 0$, and the three addends are unique. In symbols,

$$\mathcal{H} = \ker H \oplus \operatorname{im} Q \oplus \operatorname{im} Q^*.$$

Proof. Firstly, we decompose the state in the energy eigenspaces

$$\psi = \psi_0 + \sum_{N=1}^{\infty} \phi_N$$

$$H\phi_0 = 0$$

$$H\phi_N = E_N\phi_N,$$

with $E_N > 0$. We have

$$\psi = \psi_0 + Q\sum_{N=1}^{\infty}\frac{1}{E_N}Q^*\phi_N + Q^*\sum_{N=1}^{\infty}\frac{1}{E_N}Q\phi_N.$$

The only issue is about the convergence of the sum, but we assume it can be solved. For the uniqueness, we suppose

$$\psi_0 + Q\psi_1 + Q^*\psi_2 = \psi_0' + Q\psi_1' + Q^*\psi_2'$$

$$Q^*Q\left(\psi_1 - \psi_1'\right) = 0,$$

so $Q\psi_1 = Q\psi_1'$, the same for $Q\psi_2$ and finally for ψ_0. □

This proposition will be the key to prove the Hodge decomposition theorem.

Proposition 1.7. *For every energy eigenspace, there is a state ψ_1 such that $Q\psi_1 = 0$ and a state ψ_2 such that $Q^*\psi_2 = 0$.*

Proof. If $H\phi = E\phi$, $HQ\phi = QH\phi = EQ\phi$ and $QQ\phi = 0$, so the first part of the proposition is true with $\psi_1 = Q\phi$. The second part is proved in the same way. □

Proposition 1.8. *Assuming the same hypotheses of the proof of decomposition, there is an isomorphism between* $\ker H$ *and the quotient set*

$$\frac{\ker Q}{\operatorname{im} Q},$$

given by

$$\psi \mapsto [\psi] .$$

Proof. We have to find an inverse to this map. Our claim is the map

$$[\psi] = [\psi_0 + Q\psi_1 + Q^*\psi_2] \mapsto \psi_0.$$

Firstly, we verify it is well-defined, so

$$[\psi + Q\phi] = [\psi_0 + Q(\psi_1 + \phi) + Q^*\psi_2] \to \psi_0.$$

Then, we verify it is actually the inverse map. Since $Q\psi = 0$, $QQ^*\psi_2 = 0$ and $Q^*\psi_2 = 0$, so

$$[\psi_0] = [\psi_0 + Q\psi_1] = [\psi] .$$

□

Remark. If we call

$$\mathcal{H}_k = \{\psi \in \mathcal{H} : F\psi = k\psi\} ,$$

in a completely graded system we have a chain complex

$$0 \xleftarrow{Q^*} \mathcal{H}_0 \xleftarrow{Q^*} \mathcal{H}_1 \xleftarrow{Q^*} \dots \xleftarrow{Q^*} \mathcal{H}_{n-1} \xleftarrow{Q^*} \mathcal{H}_n \xleftarrow{Q^*} 0$$

and a cochain complex

$$0 \xrightarrow{Q} \mathcal{H}_0 \xrightarrow{Q} \mathcal{H}_1 \xrightarrow{Q} \dots \xrightarrow{Q} \mathcal{H}_{n-1} \xrightarrow{Q} \mathcal{H}_n \xrightarrow{Q} 0.$$

Regardless of whether the system is completely graded, we call a state of $\mathcal{H}_k$ a state of *degree* k, a state in $\ker Q$ a Q-*closed* state and a state of $\operatorname{im} Q$ a Q-*exact* state.

Corollary 1.9. *We have*

$$\mathcal{W} = \sum_{k=0}^{n} (-1)^k \dim \ker H|_{\mathcal{H}_k} = \sum_{k=0}^{n} (-1)^k \dim \left(\frac{\ker Q|_{\mathcal{H}_k}}{\operatorname{im} Q|_{\mathcal{H}_{k-1}}} \right).$$

Proof. The isomorphism of the previous proposition restricts to isomorphisms

$$\ker H|_{\mathcal{H}_k} \to \frac{\ker Q|_{\mathcal{H}_k}}{\operatorname{im} Q|_{\mathcal{H}_{k-1}}},$$

hence

$$\begin{aligned}
\mathcal{W} &= \dim \mathcal{H}_B^{(0)} - \dim \mathcal{H}_F^{(0)} \\
&= \dim \ker H|_{\mathcal{H}_B} - \dim \ker H|_{\mathcal{H}_F} \\
&= \sum_{(-1)^k=1} \dim \ker H_{\mathcal{H}_k} - \sum_{(-1)^k=-1} \dim \ker H_{\mathcal{H}_k} \\
&= \sum_{k=0}^{n} (-1)^k \dim \ker H|_{\mathcal{H}_\|} \\
&= \sum_{k=0}^{n} (-1)^k \dim \left(\frac{\ker Q|_{\mathcal{H}_k}}{\operatorname{im} Q|_{\mathcal{H}_{k-1}}} \right).
\end{aligned}$$

□

1.2 Basic examples

Example 1.1 (Particle in a line)**.** A simple example with $\mathcal{N} = 2$ conserved supercharges and dimension $n = 1$ is the system with Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ and

$$Q = (D + W'(X)) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$F \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix},$$

with D the derivative and W a smooth function. With this setup we have

$$Q^* = (-D + W'(X)) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$H = -D^2 + W'(X)^2 - W''(X) \otimes \sigma_3.$$

In order to calculate the Witten index, we have to solve the zero-energy states equation

$$Q \begin{pmatrix} \phi \\ \psi \end{pmatrix} = Q^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$$

$$\begin{cases} \phi' + W'\phi = 0 \\ -\psi' + W'\psi = 0 \end{cases}$$

$$\begin{cases} \phi = Ae^{W} \\ \psi = Be^{-W} \end{cases},$$

but we have to check if $\phi, \psi \in L^2(\mathbb{R})$.

1. If $W(x) \overset{|x|\to\infty}{\longrightarrow} \infty$, then $e^W \notin L^2(\mathbb{R})$ and $e^{-W} \in L^2(\mathbb{R})$, so there are no bosonic ground states and $\mathcal{W} = -1$.

2. If $W(x) \overset{|x|\to\infty}{\longrightarrow} -\infty$, then $e^W \in L^2(\mathbb{R})$ and $e^{-W} \notin L^2(\mathbb{R})$, so there are no fermionic ground states and $\mathcal{W} = 1$.

3. In all the other cases, $e^W, e^{-W} \notin L^2(\mathbb{R})$, so there are no zero-energy states and $\mathcal{W} = 0$.

Example 1.2 (Particle in a circle)**.** Another example with $\mathcal{N} = 2$ and $n = 1$ is the system with Hilbert space $L^2(S^1)$ and the same supercharge as the previous example, now with W a smooth function on S^1. The solutions to the zero-energy states equation are always in $L^2(S^1)$, so for every bosonic state

$$\begin{pmatrix} Ae^W \\ 0 \end{pmatrix}$$

there is a fermionic state

$$\begin{pmatrix} 0 \\ Ae^{-W} \end{pmatrix},$$

and $\mathcal{W} = 0$.

Example 1.3 (Pauli equation)**.** The Pauli Hamiltonian for a $\frac{1}{2}$-spin particle constrained on the xy plane (the Hilbert space is $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$) and immersed in an orthogonal uniform magnetic field

$$\mathbf{B}(x, y) = (0, 0, B_z(x, y))$$

has a supersymmetric form with supercharge

$$Q = ((P_x - A_x) + i\,(P_y - A_y)) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{B} = \nabla \times \mathbf{A}$, choosing a particular gauge for the vector potential $\mathbf{A}$. We have

$$\begin{aligned} QQ^* &= \left((P_x - A_x)^2 + (P_y - A_y)^2 - i\,[P_x - A_x, P_y - A_y]\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \left((\mathbf{P} - \mathbf{A})_T^2 - i\,[A_y, P_x] + i\,[A_x, P_y]\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \left((\mathbf{P} - \mathbf{A})_T^2 + (\nabla \times \mathbf{A})_z\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$Q^*Q = \left((\mathbf{P} - \mathbf{A})_T^2 - (\nabla \times \mathbf{A})_z\right) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$H = (\mathbf{P} - \mathbf{A})_T^2 + (\nabla \times \mathbf{A})_z \otimes \sigma_z.$$

With some arguments from the first example, supersymmetric quantum mechanics can also be useful to exactly solve a class of quantum system which share a property named *shape invariance*, but we do not talk about it in this thesis.

Chapter 2

Differential forms and equivariant localization

In this chapter, we work with a n-dimensional orientable compact Riemannian manifold (M, g). When a compact group G acts on the manifold, we can derive interesting facts about differential forms which are invariant for the group action. We restrict our study to isometric actions of S^1, which are generated by Killing vector fields V.

2.1 Differential forms

We work with the vector space of (complex inhomogeneous) differential forms

$$\Omega(M) = \Omega = \mathbb{C} \otimes \bigoplus_{k=0}^{n} \Omega_k(M),$$

and the k-degree component of a differential form ω is denoted by ω_k. Unless otherwise specified, all the operators $\Omega_k \to \Omega_{k'}$ are extended by linearity. The integral of a differential form on a k-dimensional submanifold $N \subseteq M$ is defined as the integral of its k-degree component

$$\int_N \omega = \int_N \omega_k.$$

Proposition 2.1. *For a differential form ω,*

$$\int_N d\omega = \int_{\partial N} \omega.$$

.

Proof. Let $k = \dim N$. Since $\dim \partial N = k - 1$, by Stokes theorem

$$\int_N d\omega = \int_N (d\omega)_k = \int_N d(\omega_{k-1}) = \int_{\partial N} \omega_{k-1} = \int_{\partial N} \omega.$$

□

Notation 2.1. For a differential form ω, we use the notation

$$(-1)^F \omega = \sum_{k=0}^{n} (-1)^k \omega_k.$$

ω is called even if $(-1)^F \omega = \omega$, odd if $(-1)^F \omega = -\omega$.

Remark. Obviously, $(-1)^F (-1)^F \omega = \omega$, $(-1)^F d\omega = -d(-1)^F \omega$ and analogous relations. It is no chance that the symbol used is the same as in the previous chapter: we will see why in the next chapter.

Proposition 2.2. *For differential forms ω, ξ and a vector field V,*

$$\begin{aligned} d(\omega \wedge \xi) &= d\omega \wedge \xi + (-1)^F \omega \wedge d\xi \\ i_V(\omega \wedge \xi) &= i_V\omega \wedge \xi + (-1)^F \omega \wedge i_V\xi. \end{aligned}$$

Proof.

$$\begin{aligned} d(\omega \wedge \xi) &= \sum_{i,j=0}^{n} d(\omega_i \wedge \xi_j) \\ &= \sum_{i,j=0}^{n} d\omega_i \wedge \xi_j + \sum_{i,j=0}^{n} (-1)^i \omega_i \wedge \xi_j \\ &= d\omega \wedge \xi + (-1)^F \omega \wedge d\xi. \end{aligned}$$

The other one is analogous. □

Definition 2.1. Given an analytic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with Taylor series

$$f(z_0, \ldots, z_n) = \sum_{i_0,\ldots,i_n=0}^{\infty} a_{i_0\ldots i_n} z_0^{i_0} \ldots z_n^{i_n},$$

we define for a differential form ω

$$f(\omega) = \sum_{i_0,\ldots,i_n=0}^{\infty} a_{i_0\ldots i_n} \omega_0^{i_0} \wedge \ldots \wedge \omega_n^{i_n},$$

which is well-defined because $\omega_k^i = 0$ for $k \neq 0$ and $i > \frac{n}{k}$, so

$$\begin{aligned} f(\omega) &= \sum_{i_1 \leq n} \cdots \sum_{i_k \leq \frac{n}{k}} \cdots \sum_{i_n \leq 1} \left(\sum_{i_0=0}^{\infty} a_{i_0\ldots i_n} \omega_0^{i_0} \right) \omega_1^{i_1} \wedge \ldots \wedge \omega_n^{i_n} \\ &= \sum_{i_1 \leq n} \cdots \sum_{i_k \leq \frac{n}{k}} \cdots \sum_{i_n \leq 1} b_{i_1\ldots i_n} \omega_1^{i_1} \wedge \ldots \wedge \omega_n^{i_n}. \end{aligned}$$

Remark. If ω is of 0-degree, $f(\omega) = f(\omega, 0, \ldots, 0)$.

Remark. We denote the function $e^{(z_0,\ldots,z_n)} = e^{z_0+\ldots+z_n}$. It is easy to prove that $e^\omega = e^{\omega_0} \wedge \ldots \wedge e^{\omega_n}$ and $\frac{d}{d\lambda} e^{\lambda\omega} = \omega \wedge e^{\lambda\omega}$.

Proposition 2.3. *Given an analytic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$, a differential form ω and a vector field V,*

$$\begin{aligned} df(\omega) &= \sum_{k=0}^{n} d\omega_k \partial_k f(\omega) \\ i_V f(\omega) &= \sum_{k=0}^{n} i_V \omega_k \partial_k f(\omega). \end{aligned}$$

Remark. $de^\omega = d\omega \wedge e^\omega$ and $i_V e^\omega = i_V\omega \wedge e^\omega$.

Definition 2.2. Given a vector field V, we define the V-equivariant exterior derivative

$$d_V = d + i_V$$

and the vector space of V-equivariant differential forms

$$\Omega^V_{\mathbb{C}}(M) = \ker \mathcal{L}_V.$$

Remark. $d_V(\omega \wedge \xi) = d_V\omega \wedge \xi + (-1)^F \omega \wedge d_V\xi$ and $d_V e^\omega = d_V\omega \wedge e^\omega$.

Proposition 2.4. *If ω is a V-equivariant differential form, $d_V^2\omega = 0$.*

Proof. Since $\omega \in \ker \mathcal{L}_V$, $\mathcal{L}_V\omega = 0$.

$$d_V^2 = (d + i_V)^2 = d^2 + \{d, i_V\} + i_V^2 = \mathcal{L}_V,$$

therefore $d_V^2\omega = \mathcal{L}_V\omega = 0$. □

Definition 2.3. An (V-)equivariant form ω is called (V-)equivariantly closed if $d_V\omega = 0$, while it is called equivariantly exact if $\omega = d_V\xi$ for some equivariant form ξ.

2.2 Equivariant localization

Lemma 2.5 (Equivariant localization lemma)**.** *Let α be a V-equivariantly closed differential form on a manifold M. Let β be a V-equivariant differential form. Then, for every $\lambda \in \mathbb{R}$,*

$$\int_M \alpha = \int_M \alpha \wedge e^{-\lambda d_V \beta}.$$

Proof. We prove that

$$\frac{d}{d\lambda}\int_M \alpha \wedge e^{-\lambda d_V\beta} = 0.$$

Firstly,

$$\frac{d}{d\lambda}\int_M \alpha \wedge e^{-\lambda d_V\beta} = \int_M \alpha \wedge \frac{d}{d\lambda} e^{-\lambda d_V\beta} = -\int_M \alpha \wedge d_V\beta \wedge e^{-\lambda d_V\beta}.$$

On the other hand, since $d_V\alpha = 0$ and $d_V^2\beta = 0$,

$$\begin{aligned} d_V((-1)^F \alpha \wedge \beta \wedge e^{-\lambda d_V\beta}) &= \\ &= d_V (-1)^F \alpha \wedge \beta \wedge e^{-\lambda d_V\beta} + \alpha \wedge d_V\beta \wedge e^{-\lambda d_V\beta} \\ &+ \alpha \wedge (-1)^F \beta \wedge d_V e^{-\lambda d_V\beta} \\ &= \alpha \wedge d_V\beta \wedge e^{-\lambda d_V\beta} - \lambda\alpha \wedge (-1)^F \beta \wedge d_V^2\beta e^{-\lambda d_V\beta} \\ &= \alpha \wedge d_V\beta \wedge e^{-\lambda d_V\beta}. \end{aligned}$$

Therefore, since $\int_M i_V\omega = 0$,

$$\begin{aligned} \frac{d}{d\lambda}\int_M \alpha \wedge e^{-\lambda d_V\beta} &= -\int_M \alpha \wedge d_V\beta \wedge e^{-\lambda d_V\beta} \\ &= -\int_M d_V(\alpha^* \wedge \beta \wedge e^{-\lambda d_V\beta}) \\ &= -\int_M d(\alpha^* \wedge \beta \wedge e^{-\lambda d_V\beta}) \\ &= 0. \end{aligned}$$

□

Theorem 2.6. *Let α be a V-equivariantly closed differential form. If V is Killing with respect to the metric g, then*

$$\int_M \alpha = \lim_{\lambda\to\infty} \int_M \alpha \wedge e^{-\lambda d_V i_V g}.$$

Proof. We could apply Lemma 2.5 with $\beta = i_V g$ if $\mathcal{L}_V i_V g = 0$. This is the case, since

$$\mathcal{L}_V i_V g = i_V \mathcal{L}_V g + i_{[V,V]} g = 0.$$

The hypotheses of the lemma are fulfilled, so

$$\int_M \alpha = \int_M \alpha \wedge e^{-\lambda d_V i_V g} = \lim_{\lambda\to\infty} \int_M \alpha \wedge e^{-\lambda d_V i_V g}.$$

□

Corollary 2.7 (Berline-Vergne localization formula)**.** *With the previous hypotheses, if M is $2n$-dimensional and the zeros of V are isolated,*

$$\int_M \alpha = \sum_{V(x)=0} \frac{(2\pi)^n}{\mathrm{pf}(dV^\flat(x))} \alpha_0(x).$$

Proof. See Appendix A. □

Example 2.1. A basic example of application of the Berline-Vergne localization formula is the integration on a sphere

$$S^2 = \left\{x \in \mathbb{R}^3 : \|x\| = 1\right\}$$

of a 2-form

$$e^{tH}\omega,$$

where ω is the volume 2-form and H is the height function

$$H(x, y, z) = z.$$

Our Killing vector field V is the generator of rotation around an axis of the sphere. The integrand is not equivariantly closed but it is equal to the top-degree component of

$$\frac{1}{t} e^{t(\omega+H)}.$$

After choosing as Killing field

$$V(x, y, z) = (y, -x, 0),$$

our claim is that $\omega + H$ is equivariantly closed (and so the new integrand). In angular coordinates, excluding the poles, we have

$$\begin{array}{ccl} \omega & \mapsto & \widetilde{\omega}(\phi,\theta) = \sin\theta d\phi \wedge d\theta \\ V & \mapsto & \widetilde{V}(\phi,\theta) = (1,0) \\ H & \mapsto & \widetilde{H}(\phi,\theta) = \cos\theta, \end{array}$$

so we verify

$$i_{\widetilde{V}}\widetilde{\omega}(\theta,\phi) + d\widetilde{H}(\theta,\phi) = \sin\theta d\theta - \sin\theta d\theta = 0.$$

At the poles the relation holds trivially, since they are critical points of H and zeros of V. Our integral can be written as the integral of an equivariantly closed form, namely

$$\int_{S^2} e^{tH}\omega = \frac{1}{t}\int_{S^2} e^{t(\omega+H)}.$$

The last steps to apply the localization theorem are computing $\alpha_0 = \frac{1}{t}e^{tH}$ and $\mathrm{pf}(dV^\flat)$ at the poles, the only zeros of V. For the north pole, we choose projection coordinates

$$(x, y, z) \mapsto (x, y),$$

with which

$$V \mapsto \overline{V}(x, y) = (y, -x)$$
$$g \mapsto g_{\mu\nu}(0, 0) = \delta_{\mu\nu},$$

so

$$\partial_\mu V_\nu(0, 0) = \partial_\mu(g_{\nu\lambda}V^\lambda)(0, 0) = \partial_\mu V^\nu(0, 0),$$

corrisponding to a matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

resulting in $\mathrm{pf}(dV^\flat) = 1$ at the north pole, while we can check that at the south pole it changes sign. The final result is

$$\int_{S^2} e^{tH}\omega = 2\pi\left(\frac{e^t}{t} - \frac{e^{-t}}{t}\right) = 4\pi\frac{\sinh t}{t}.$$

Example 2.2 (Duistermaat–Heckman formula)**.** A generalization of the previous example involves $2n$-dimensional symplectic manifold (M, ω) with Hamiltonian function H. We know that, for a Hamiltonian vector field V, since $d\omega = 0$,

$$d_V(\omega + H) = i_V\omega + dH = 0,$$

If M is also a Riemannian manifold such that V is Killing, then

$$\int_M e^{tH}\omega^n = \frac{n!}{t^n}\int_M e^{t(\omega+H)} = \frac{n!}{t^n}\sum_{V(x)=0}\frac{(2\pi)^n}{\mathrm{pf}(dV^\flat(x))}e^{tH(x)}.$$

In view of this last example, equivariant localization can be useful to compute the partition function of classical systems when certain symmetries are available.

Chapter 3

$\mathcal{N} = 2$ supersymmetric quantum mechanics in differential forms Hilbert space

In this chapter we present a $\mathcal{N} = 2$ supersymmetric quantum system and discuss about its Witten index. We work with a n-dimensional orientable compact Riemannian manifold (M, g), unless otherwise specified. The first thing to do is define the right Hilbert space, so we start by talking about Hilbert spaces of sections of a fiber bundle, a generalization of classic L^2 spaces. The issue of domains of operators is neglected, but it can be solved defining Sobolev spaces.

3.1 Hilbert space: $L^2(E)$ for a vector bundle E

Definition 3.1. A *Hermitian vector bundle* (E, h) is a complex vector bundle $E \to M$ with a Hermitian product $h_x : E_p \times E_p \to \mathbb{C}$ on each fiber, such that, for every two smooth sections $X, Y \in \Gamma(E)$,

$$x \mapsto h_x(X(x), Y(x))$$

is a smooth function.

Notation 3.1. Let $\mathrm{vol} \in \Omega_n$ be the volume form induced by the metric of M. We call (M, μ) the measure space with the Borel measure

$$\mu(U) = \int_U \mathrm{vol}\,.$$

We denote by $L^1(M)$ the space of integrable functions $f : M \to \mathbb{C}$, and their Lebesgue integral by

$$\int_M f \,\mathrm{vol}\,.$$

We use the notation

$$\int_M f \,\mathrm{vol} = \int_M f(x) \,\mathrm{vol}_x\,.$$

Proposition 3.1. *Given a Hermitian vector bundle E, we define $L^2(E)$ as a space of equivalence classes of sections X such that*

$$x \mapsto h_x(X(x), X(x))$$

is integrable, where $X \sim Y$ if $X = Y$ almost everywhere. The Hermitian vector space $(L^2(E), (*, *))$, with

$$(X, Y) = \int_M h_x(X(x), Y(x)) \,\mathrm{vol}_x,$$

is a separable Hilbert space, and $\Gamma(E)$ is dense in it.

We use as vector bundle the vector space of (complex inhomogeneous) completely antisymmetric tensors

$$E = \mathbb{C} \otimes \bigoplus_{k=0}^{n} \Lambda^k T^* M,$$

whose smooth sections are differential forms in

$$\Gamma(E) = \Omega(M).$$

The Hermitian product on each fiber $E_x = \mathbb{C} \otimes \bigoplus_{k=0}^{n} \Lambda^k (T_x^* M)$ is given by

$$h_x(\omega, \xi) \,\mathrm{vol}_x = (\omega \wedge \star\xi)_n \,,$$

where $\star$ is the Hodge star extended by antilinearity. From now on we will use $L^2(E)$ as Hilbert space.

Definition 3.2. Given a 1-form ψ, we define two linear operators, called respectively *fermion creation* and *annihilation* operators:

$$\begin{aligned} a^*(\psi) &= \psi \wedge \\ a(\psi) &= i_{\psi^\#}, \end{aligned}$$

where $\#$ is the metric isomorphism $T^*M \to TM$ extended by linearity. The operator F such that

$$F\omega_k = k\omega_k$$

is our fermion number operator.

Remark. The symbol $(-1)^F$ introduced in the previous chapter acts as the operator denoted in the same way.

Proposition 3.2. *Fermion creation and annihilation operators are one the adjoint of the other, and*

$$\begin{aligned} \{a^*(\psi), a(\eta)\} &= g(\psi^\#, \eta^\#) \\ \{a^*(\psi), a^*(\eta)\} &= 0. \end{aligned}$$

We call *Hodge Laplacian* the self-adjoint operator $\Delta = \{d, d^*\} = (d + d^*)^2$, where d^* is the adjoint of d. Given a local orthonormal basis $(e_1, \ldots, e_n)$ of TM, we can show that

$$d\omega = \sum_{i=1}^{n} a^*(e^i) \nabla_{e_i}$$

$$d^*\omega = -\sum_{i=1}^{n} a(e^i) \nabla_{e_i}.$$

3.2 Hamiltonian: Hodge theory and Witten deformation

We have to specify the supersymmetric Hamiltonian, so we choose as in [4] (page 665) the supercharge

$$Q_\lambda = e^{-\lambda W} d e^{\lambda W},$$

with $\lambda \geq 0$ and W a smooth function on M. This deformation of d is called *Witten deformation*. We easily check that

$$Q_\lambda^2 = e^{-\lambda W} d^2 e^{\lambda W} = 0$$
$$F Q_\lambda = e^{-\lambda W} F d e^{\lambda W} = e^{-\lambda W} d(F+1) e^{\lambda W} = Q_\lambda (F+1).$$

The system is thus completely graded. It follows directly that

$$Q_\lambda = \lambda dW \wedge + d = \lambda a^*(dW) + d$$

and

$$\begin{aligned} H_\lambda &= \{\lambda a^*(dW) + d, \lambda a(dW) + d^*\} \\ &= \{d, d^*\} + \lambda \{d, i_{dW^\#}\} + \lambda \{d, i_{dW^\#}\}^* + \lambda^2 \{a^*(dW), a(dW)\} \\ &= \Delta + \lambda (\mathcal{L}_{dW} + \mathcal{L}_{dW}^*) + \lambda^2 dW^2, \end{aligned}$$

where $dW^2 = g(dW^\#, dW^\#)$. We call $H_0 = H$. Note that example 1.2 was just this model with $\lambda = 1$ and $M = S^1$. We assume the true fact that the eigenspaces of H_λ are all finite-dimensional, together with the other hypotheses used in the proof of 1.6: a proof of this properties can be found in literature under the name of *Hodge theorem*. We can compute the Witten index with its definition

$$\mathcal{W}_\lambda = \dim \ker H|_{\Omega_B} - \dim \ker H|_{\Omega_F}.$$

Remark. As a byproduct, the application of 1.6 tells us that every differential form ω can be decomposed uniquely as

$$\omega = d\alpha + d^*\beta + \gamma,$$

where $\Delta\gamma = 0$. This is called *Hodge decomposition theorem*.

Proposition 3.3. *We have that, for each k,*

$$\dim \ker H_\lambda|_{\Omega_k} = \dim \ker H|_{\Omega_k}.$$

Proof. We use the degree-preserving isomorphism

$$\omega \mapsto e^{-\lambda W} \omega.$$

to show that

$$\ker Q_\lambda|_{\Omega_k} \simeq \ker Q|_{\Omega_k}$$
$$\operatorname{im} Q_\lambda|_{\Omega_k} \simeq \operatorname{im} Q|_{\Omega_k}$$

and then we conclude by the fact used in the proof of 1.9. If $d\omega = 0$, then

$$Q_\lambda e^{-\lambda W} \omega = e^{-\lambda W} d\omega = 0.$$

If $\omega = d\xi$, then

$$e^{-\lambda W} \omega = e^{-\lambda W} d\xi = Q_\lambda e^{-\lambda W} \xi.$$

The same facts are true for the inverse map

$$\omega \mapsto e^{\lambda W} \omega.$$

□

From the previous proposition we can conclude that

$$\mathcal{W}_\lambda = \mathcal{W}_0 = \mathcal{W},$$

so we understand that this type of deformation does not change the Witten index (and more generally the dimension of the kernel for each degree). We can exploit this invariance by studying the system when $\lambda \to \infty$, as happens in *Morse theory* (see next section for a brief introduction): we will find a formula for the Witten index localized on the critical point of W.

From 1.9 we deduce a first formula for the Witten index:

$$\begin{aligned} \dim \ker H|_{\Omega_k} &= \dim \left(\frac{\ker d|_{\Omega_k}}{\operatorname{im} d|_{\Omega_{k-1}}} \right) \\ &= \dim H^k(M) \\ &= b_k(M), \end{aligned}$$

the dimension of the k-th *de Rham cohomology group*, which is just the *Betti number* $b_k(M)$, a topological invariant. Hence, we have that

$$\mathcal{W} = \sum_{k=0}^{n} (-1)^k \, b_k(M) = \chi(M),$$

the *Euler characteristic*, another topological invariant. We have learnt that the Witten index of all these deformations depends only on the topology of the base manifold.

3.3 Localization via λ-invariance: Morse theory

The aim of this section is to compute the Witten index exploiting its λ-invariance, in particular for arbitrarily large λ. Morse theory is the study of the topology of a manifold by analyzing the behaviour of a particular class of functions, called *Morse functions.* A basic result are *Morse inequalities*, which we will prove as a byproduct of a localization argument.

Definition 3.3. A smooth function $f : M \to \mathbb{R}$ is called a *Morse function* if every critical point is regular, where regular means that the determinant of the Hessian of f in a chart is non-zero at the point. We define

$$\operatorname{Crit}(f) = \{x \in M : df(x) = 0\},$$

the set of critical points of f. Given a chart containing a critical point x, if the Hessian of f in the chart at the point has k positive eigenvalues, we say that x is a critical point of order k, or

$$x \in \operatorname{Crit}_k(f).$$

From now on we assume that W is a Morse function. Except for the critical points of W, the Hamiltonian H_λ becomes very large for large λ, so we expect that the eigenstates will localize at the critical points. This behaviour is encoded in the following lemma, which proof is technical, but relies on the approximation of the ground states with known local solutions of approximated Hamiltonian near the critical points (see B to understand how this idea is applied with a different deformation).

Lemma 3.4. *For every $c > 0$, there exists $\lambda_0 > 0$ such that, if $\lambda \geq \lambda_0$, the number of eigenvalues in $[0, c]$ of $H_\lambda|_{\Omega_k}$ equals the number of critical points of W of order k, named $m_k(W)$.*

Proof. See the section *Proof of Proposition 5.5* of [5] (page 83). □

Now, fixing $c, \lambda > 0$, let F_k be the subspace spanned by the eigenstates of $H_\lambda|_{\Omega_k}$ with eigenvalues in $[0, c]$. These sets form the cochain subcomplex

$$0 \xrightarrow{Q_\lambda} F_0 \xrightarrow{Q_\lambda} F_1 \xrightarrow{Q_\lambda} \dots \xrightarrow{Q_\lambda} F_{n-1} \xrightarrow{Q_\lambda} F_n \xrightarrow{Q_\lambda} 0.$$

We want to check if $Q_\lambda P_{F_k} = P_{F_{k+1}} Q_\lambda$, where P_V is the projection operator on a subspace V. If $\omega \in F_k$,

$$\omega = \sum_{r=0}^{l} \xi_r,$$

with $\xi_r \in \Omega_k$, $H_\lambda \xi_r = E_r \xi_r$ and $E_r \in [0, c]$. If $\omega \in F_{k+1}$,

$$\omega = \sum_{r=0}^{l} Q_\lambda \xi_r + \sum_{r=0}^{m} \zeta_r,$$

with $\zeta_r \in \Omega_{k+1}$, $H_\lambda \zeta_r = E'_r \zeta_r$ and $E'_r \in [0, c]$. Let $\omega \in \Omega_k$, with its eigenspace decomposition

$$\omega = \sum_{r=0}^{l} \xi_r + \sum_{N=1}^{\infty} \omega_N.$$

We have

$$Q_\lambda P_{F_k} \omega = \sum_{r=0}^{l} Q_\lambda \xi_r = P_{F_{k+1}} Q_\lambda \omega.$$

Now, if $V \subseteq \mathcal{H}$ is a vector subspace such that $P_{V_{k+1}} Q = Q P_{V_k}$,

$$\dim \left(\frac{\ker Q|_{V_k}}{\operatorname{im} Q|_{V_{k-1}}} \right) = \dim \left(\frac{\ker Q|_{\mathcal{H}_k}}{\operatorname{im} Q|_{\mathcal{H}_{k-1}}} \right) = b_k(M),$$

since we have the isomorphism

$$[\psi] \mapsto [\psi],$$

with inverse

$$[\psi] \mapsto [P_{V_k} \psi].$$

This is well-defined because, if $Q\psi = 0$, $Q P_{V_k} \psi = P_{V_{k+1}} Q \psi = 0$, and

$$[P_{V_k}(\psi + Q\phi)] = \left[P_{V_k}\psi + Q P_{V_{k+1}} \phi\right] = [P_{V_k}\psi].$$

We know from the previous lemma that, for $\lambda \geq \lambda_0$,

$$\begin{aligned} m_k(W) &= \dim F_k \\ &= \dim \ker Q_\lambda|_{F_k} + \dim \operatorname{im} Q_\lambda|_{F_k} \\ &= \dim \operatorname{im} Q_\lambda|_{F_k} + \dim \operatorname{im} Q_\lambda|_{F_k} + b_k(M), \end{aligned}$$

so we conclude that

$$\sum_{k=0}^{i} (-1)^k m_{i-k}(W) = \sum_{k=0}^{i} (-1)^k b_{i-k}(M) + \dim \operatorname{im} Q_\lambda|_{F_i}$$

$$\sum_{k=0}^{i} (-1)^k m_{i-k}(W) \leq \sum_{k=0}^{i} (-1)^k b_{i-k}(M),$$

the strong Morse inequality, and

$$\sum_{k=0}^{n} (-1)^k m_k(W) = \sum_{k=0}^{n} (-1)^k b_k(M) = \chi(M).$$

We understand that a combination of number and order of critical points of a Morse function is constrained to the topology of the base manifold. We found two ways of computing the Witten index of the theory, but there is a third one.

3.4 Localization via β-invariance: Chern-Gauss-Bonnet theorem

The aim of this section is to compute the Witten index exploiting the β-invariance, namely

$$\mathcal{W} = \operatorname{tr}\left((-1)^F e^{-\beta\Delta}\right),$$

in particular for arbitrarily small β. We use a heat kernel method, which is based on studying the behaviour of the heat equation

$$\frac{d}{dt}\psi_t = -\Delta\psi_t,$$

exactly solved by

$$e^{-t\Delta}\psi_0.$$

On a general manifold we can't find an explicit solution, so we try to guess an approximate solution for $t \sim 0$ and then verify it is actually a good approximation.

Definition 3.4. Given a densely defined self-adjoint operator A on $L^2(E)$ with eigenstates $(\psi_N)_{N\in\mathbb{N}}$ such that

$$A\psi_N = \lambda_N\psi_N,$$

we define its *kernel* as

$$K_A(x,y) = \sum_{N=0}^{\infty} \lambda_N\psi_N(y)h_x(*,\psi_N(x)),$$

for $x, y \in M$, when the sum converges in the topology of $\mathcal{L}(E_x, E_y)$.

Proposition 3.5. *The trace of an operator A is given by*

$$\operatorname{tr} A = \int_M \operatorname{tr} K_A(x,x)\operatorname{vol}_x.$$

Proof. From the definition of the trace,

$$\begin{aligned}
\int_M \operatorname{tr} K_A(x,x)\operatorname{vol}_x &= \sum_{N=0}^{\infty}\lambda_N \int_M h_x(\psi_N(x),\psi_N(x))\operatorname{vol}_x \\
&= \sum_{N=0}^{\infty}\int_M h_x(A\psi_N(x),\psi_N(x))\operatorname{vol}_x \\
&= \sum_{N=0}^{\infty}(A\psi_N,\psi_N) \\
&= \operatorname{tr} A.
\end{aligned}$$

□

We define the *heat kernel* of A as

$$\Phi_A(x,y,\beta) = K_{e^{-\beta A}}(x,y).$$

Note that as a function of y it solves the evolutionary problem

$$\frac{d}{d\beta}\psi_\beta = -A\psi_\beta.$$

Working again with differential forms, the Witten index is, for $\beta > 0$,

$$\mathcal{W} = \int_M \operatorname{tr}\left((-1)^F \Phi_\Delta(x,x,\beta)\right)\operatorname{vol}_x.$$

Proposition 3.6 (Weitzenböck identity)**.** *Let $x \in M$ and let $(e_1, \dots, e_n)$ be a local orthonormal basis of TM such that $\nabla_{e_i} e_j(x) = 0$. Then, defining the Bochner Laplacian*

$$\left(\widetilde{\Delta}\omega\right)_x = \sum_{i=0}^{n} \nabla_{e_i}\nabla_{e_i}\omega$$

and the Ricci endomorphism

$$\operatorname{Ric}\omega(V_1, \dots, V_k) = \sum_{i,j=0}^{n} (R(e_j, V_i)\omega)(V_1, \dots, e_j, \dots, V_k),$$

where R is Riemann curvature tensor and e_j is in the i-th place, we have

$$\Delta = \widetilde{\Delta} + \operatorname{Ric}.$$

Proof. See the section *The Weitzenböck Curvature* of [1] (page 343). □

Proposition 3.7. *Given a local orthonormal basis $(e_1, \dots, e_n)$ of TM, Ricci endomorphism acts as*

$$\operatorname{Ric} = R_{ijkl}a^*(e^i)a(e^j)a^*(e^k)a(e^l),$$

where

$$R_{ijkl} = g(R(e_i, e_j)e_k, e_l).$$

As a heat kernel approximation for $\beta \sim 0$, we define

$$\overline{\Phi}_\Delta(x, x, \beta) = \frac{1}{(4\pi\beta)^{\frac{n}{2}}} e^{-\beta \operatorname{Ric}_x},$$

an endomorphism of $\Lambda^k T_x^* M$. This is similar to the classic heat kernel on $\mathbb{R}^n$, except for the curvature part (Ric is 0 on a flat manifold). Given an endomorphism T of $\Lambda^k T_x^* M$ and an orthonormal basis $(e_1, \dots, e_n)$ of $T_x M$, it can be shown that T can be decomposed uniquely as

$$T = \sum_{IJ} c_{IJ} A_{IJ},$$

where I, J are multi-indexes $I = (I_1, \dots, I_k)$, with $k \leq n$, $I_1 < \dots < I_k$, and

$$A_{IJ} = a(e^{I_1}) \cdots a(e^{I_k}) a^*(e^{J_1}) \cdots a^*(e^{J_l}).$$

We denote by T_{2n} the component of T with multi-indexes $(1, \dots, n)$, $(1, \dots, n)$. We can also prove that

$$\operatorname{tr}\left((-1)^F T\right) = \operatorname{tr}\left((-1)^F T_{2n}\right).$$

Lemma 3.8. *There exists $C > 0$ such that, for all $\beta \in (0, 1]$,*

$$\|(\Phi_\Delta - \overline{\Phi}_\Delta)(x, x, \beta)_{2n}\| \leq C\beta,$$

with operator norm.

Proof. See the section *Proof of the Chern-Gauss-Bonnet Theorem* of [2] (page 115). □

Since

$$\left|\operatorname{tr}\left((-1)^F (\Phi_\Delta - \overline{\Phi}_\Delta)(x,x,\beta)_{2n}\right)\right| \le n \left\|(-1)^F (\Phi_\Delta - \overline{\Phi}_\Delta)(x,x,\beta)_{2n}\right\|$$
$$\le n\|(\Phi_\Delta - \overline{\Phi}_\Delta)(x,x,\beta)_{2n}\|$$
$$\le nC\beta$$

$$\operatorname{tr}\left((-1)^F (\Phi_\Delta - \overline{\Phi}_\Delta)(x,x,\beta)_{2n}\right) = o(1),$$

we conclude that

$$\begin{aligned}
\mathcal{W} &= \int_M \operatorname{tr}\left((-1)^F \Phi_\Delta(x,x,\beta)\right) \operatorname{vol}_x \\
&= \int_M \operatorname{tr}\left((-1)^F \Phi_\Delta(x,x,\beta)_{2n}\right) \operatorname{vol}_x \\
&= \int_M \operatorname{tr}\left((-1)^F \overline{\Phi}_\Delta(x,x,\beta)_{2n}\right) \operatorname{vol}_x + o(1) \\
&= \frac{1}{(4\pi\beta)^{\frac{n}{2}}} \int_M \operatorname{tr}\left((-1)^F \left(e^{-\beta \operatorname{Ric}_x}\right)_{2n}\right) \operatorname{vol}_x + o(1).
\end{aligned}$$

Expanding $e^{-\beta \operatorname{Ric}_x}$ with its series, the first term of order $2n$ is

$$(-1)^k \frac{\beta^k}{k!} \operatorname{Ric}_x^k,$$

with $k = \lceil \frac{n}{2} \rceil$. All the other terms of order $2n$ are $o(\beta^k)$. Hence,

$$\mathcal{W} = \frac{(-1)^k}{(4\pi)^{\frac{n}{2}} k!} \beta^{k-\frac{n}{2}} \int_M \operatorname{tr}\left((-1)^F \operatorname{Ric}^k\right) \operatorname{vol} + o(\beta^{k-\frac{n}{2}}) + o(1).$$

Taking the limit $\beta \to 0$, it follows that

- If n is odd, $k - \frac{n}{2} > 0$ and $\mathcal{W} = 0$.
- If n is even, $k = \frac{n}{2}$ and

$$\mathcal{W} = \frac{(-1)^k}{(4\pi)^k k!} \int_M \operatorname{tr}\left((-1)^F \operatorname{Ric}^k\right) \operatorname{vol}.$$

Defining the Euler form

$$\omega = \frac{(-1)^k}{(4\pi)^k k!} \operatorname{tr}\left((-1)^F \operatorname{Ric}^k\right) \operatorname{vol},$$

which has an expression in local coordinates in terms of Riemann curvature tensor (see [2], page 112), if M is even-dimensional we conclude that

$$\chi(M) = \mathcal{W} = \int_M \omega,$$

which is the *Chern-Gauss-Bonnet theorem.* Note that ω depends only on the Riemannian metric of M, so this is a topological constraint to the metric structure. We also proved that Euler characteristic of an odd-dimensional manifold is 0.

3.5 Another deformation: Killing vector field

In this last model, which also appeared in [4] (page 676), we use the same ideas of equivariant localization, Chapter 2. Let V be a Killing vector field with isolated zeros and let M be $2n$-dimensional. Our supercharge will be

$$Q_\lambda = d_{\lambda V} = d + \lambda i_V = d + \lambda a(V^\flat)$$

with $\lambda \geq 0$, but

$$\begin{aligned} Q_\lambda^2 &= \lambda\{d, i_V\} = \lambda \mathcal{L}_V \\ (Q_\lambda^*)^2 &= \lambda \mathcal{L}_V^*, \end{aligned}$$

so we restrict our Hilbert space to $\mathcal{H} = \overline{\ker \mathcal{L}_V} \cap \overline{\ker \mathcal{L}_V^*}$. For $\lambda \neq 0$, the supercharge does not increase the degree of the forms, but only change their parity (the system is not completely graded). We define the *Clifford operators*

$$\begin{aligned} c(V) &= a^*(V^\flat) - a(V^\flat) \\ \widehat{c}(V) &= a^*(V^\flat) + a(V^\flat), \end{aligned}$$

which, for a orthonormal basis $(e_1, \ldots, e_n)$, have properties

$$\begin{aligned} \widehat{c}(V)^2 &= g(V, V) \\ \{c(V), \widehat{c}(W)\} &= 0 \\ [\nabla_{e_i}, c(V)] &= c(\nabla_{e_i} V) \\ d + d^* &= \sum_{i=0}^{2n} c(e_i)\nabla_{e_i}. \end{aligned}$$

The Hamiltonian is thus

$$\begin{aligned} H_\lambda &= (Q_\lambda + Q_\lambda^*)^2 \\ &= (d + d^* + \lambda \widehat{c}(V))^2 \\ &= (d + d^*)^2 + \lambda \sum_{i=0}^{2n} \{c(e_i)\nabla_{e_i}, \widehat{c}(V)\} + \lambda^2 g(V, V) \\ &= \Delta + \lambda \sum_{i=0}^{2n} \left(c(e_i)\widehat{c}(\nabla_{e_i} V) + \{c(V), \widehat{c}(e_i)\} \nabla_{e_i}\right) + \lambda^2 g(V, V) \\ &= \Delta + \lambda \sum_{i=0}^{2n} c(e_i)\widehat{c}(\nabla_{e_i} V) + \lambda^2 g(V, V). \end{aligned}$$

Remark. Since it can be proved that $\mathcal{L}_V \star = \star \mathcal{L}_V$,

$$\mathcal{L}_V(\alpha \wedge \star\beta) = \mathcal{L}_V \alpha \wedge \star\beta + \alpha \wedge \star \mathcal{L}_V \beta$$

and

$$\begin{aligned} \int_M \mathcal{L}_V(\alpha \wedge \star\beta) &= \int_M \mathcal{L}_V \alpha \wedge \star\beta + \int_M \alpha \wedge \star \mathcal{L}_V \beta \\ &= \int_M d i_V(\alpha \wedge \star\beta) + \int_M i_V d(\alpha \wedge \star\beta) \\ &= 0. \end{aligned}$$

This implies that $\mathcal{L}_V^* = -\mathcal{L}_V$ and $\mathcal{H} = \overline{\ker \mathcal{L}_V}$, the closure of the vector space of V-equivariant differential forms. Note also that the restriction of the Hilbert space did not change the kernel of H_λ, since, if $H_\lambda \omega = 0$ for ω in the original Hilbert space, then

$$\begin{aligned}
0 &= H_\lambda \omega \\
&= \{Q_\lambda, Q_\lambda^*\}\, \omega \\
&= (\{Q_\lambda, Q_\lambda^*\} + \lambda \mathcal{L}_V + \lambda \mathcal{L}_V^*)\, \omega \\
&= \left(\{Q_\lambda, Q_\lambda^*\} + Q_\lambda^2 + {Q_\lambda^*}^2\right) \omega \\
&= (Q_\lambda + Q_\lambda^*)^2\, \omega
\end{aligned}$$

and similarly

$$0 = H_\lambda \omega = (Q_\lambda - Q_\lambda^*)^2\, \omega,$$

resulting in $Q_\lambda \omega = 0$ and $\mathcal{L}_V \omega = 0$.

Now, we want to apply equivariant localization to compute the inner product (α, β) of two states such that $\alpha \in \ker Q_\lambda$, $\beta \in \ker Q_\lambda^*$. We prove that $\alpha \wedge \star\beta$ is V-equivariantly closed. Since

$$\begin{aligned}
0 &= \int_M Q_\lambda(\alpha \wedge \star\beta) \\
&= \int_M Q_\lambda \alpha \wedge \star\beta + \int_M (-1)^F\, \alpha \wedge Q_\lambda \star \beta \\
&= (Q_\lambda \alpha, \beta) + \Big((-1)^F\, \alpha, \star^{-1} Q_\lambda \star \beta\Big) \\
&= (Q_\lambda \alpha, \beta) + \Big(\alpha, (-1)^F \star^{-1} Q_\lambda \star \beta\Big),
\end{aligned}$$

we have

$$\begin{aligned}
Q_\lambda^* &= (-1)^F \star^{-1} Q_\lambda \star \\
\star\, (-1)^F\, Q_\lambda^* &= Q_\lambda \star .
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q_\lambda(\alpha \wedge \star\beta) &= Q_\lambda \alpha \wedge \star\beta + (-1)^F\, \alpha \wedge Q_\lambda \star \beta \\
&= Q_\lambda \alpha \wedge \star\beta + (-1)^F\, \alpha \wedge \star\, (-1)^F\, Q_\lambda^* \beta \\
&= 0.
\end{aligned}$$

We have all the ingredients to apply the Berline-Vergne localization formula 2.7:

$$(\alpha, \beta) = \int_M \alpha \wedge \star\beta = \sum_{V(x)=0} \frac{(2\pi)^n}{\mathrm{pf}(dV^\flat(x))} (\alpha \wedge \star\beta)_0(x).$$

We now prove that for $\lambda \neq 0$ the Witten index is independent of λ.

Proposition 3.9. *If $\lambda, \lambda' \neq 0$, there is an isomorphism*

$$\ker H_\lambda|_{\Omega_B} \to \ker H_{\lambda'}|_{\Omega_B} \qquad\qquad \ker H_\lambda|_{\Omega_F} \to \ker H_{\lambda'}|_{\Omega_F},$$

given by

$$\omega \mapsto e^{-\log\left(\frac{\lambda'}{\lambda}\right) P} \omega,$$

where P is the projection operator on $\ker H$.

Proof. If $Q_\lambda \omega = 0$, $(d + \lambda i_V)\omega = 0$ and

$$\begin{aligned}
Q_{\lambda'} e^{-\log\left(\frac{\lambda'}{\lambda}\right)P} \omega &= (d + \lambda' i_V) e^{-\log\left(\frac{\lambda'}{\lambda}\right)P} (P\omega + (1-P)\omega) \\
&= (d + \lambda' i_V) \left(\frac{\lambda}{\lambda'} P\omega + (1-P)\omega \right) \\
&= d\omega + \lambda i_V P\omega + \lambda' i_V (1-P)\omega.
\end{aligned}$$

Now, since $d i_V P\omega = -i_V dP\omega = 0$,

$$i_V P\omega \in \ker H$$

and, since $d^* i_V^* P\omega = -i_V^* d^* P\omega = 0$,

$$i_V^* P\omega \in \ker H.$$

The latter means that $(1-P) i_V^* P = 0$, $P i_V (1-P) = 0$ and

$$i_V(1-P) \in \ker H^\perp.$$

We conclude that

$$\begin{aligned}
e^{\log\left(\frac{\lambda'}{\lambda}\right)P} Q_{\lambda'} e^{-\log\left(\frac{\lambda'}{\lambda}\right)P} \omega &= e^{\log\left(\frac{\lambda'}{\lambda}\right)P} (d\omega + \lambda i_V P\omega + \lambda' i_V (1-P)\omega) \\
&= \frac{\lambda'}{\lambda} d\omega + \lambda' i_V P\omega + \lambda' i_V (1-P)\omega \\
&= \frac{\lambda'}{\lambda} (d + \lambda i_V)\omega \\
&= \frac{\lambda'}{\lambda} Q_\lambda \omega
\end{aligned}$$

$$Q_{\lambda'} e^{-\log\left(\frac{\lambda'}{\lambda}\right)P} \omega = \frac{\lambda'}{\lambda} e^{-\log\left(\frac{\lambda'}{\lambda}\right)P} Q_\lambda \omega = 0.$$

By symmetry, we can prove the same fact about Q_λ^*. Finally, the map has inverse

$$\omega \mapsto e^{\log\left(\frac{\lambda'}{\lambda}\right)P} \omega$$

and, since $\left[H, (-1)^F\right] = 0$, we can check it preserves the parity of forms. $\square$

We deduce that, if $\lambda, \lambda' \neq 0$ and the eigenspaces are again all finite-dimensional,

$$\mathcal{W}_\lambda = \mathcal{W}_{\lambda'}.$$

As long as $\lambda \neq 0$, the Witten index is again independent of the deformation, but we can show that this is the case even if $\lambda = 0$. For now, we call

$$\mathcal{W}_0 = \mathcal{W}$$

and for $\lambda \neq 0$

$$\mathcal{W}_\lambda = \widetilde{\mathcal{W}}.$$

Again, in order to compute the index we could exploit its invariance for arbitrarily large λ, expecting a localization on the zeros of V, in a similar way to the Morse function localization. This is captured in the proof of the following lemma.

Lemma 3.10. *For every $c > 0$, there exists $\lambda_0 > 0$ such that, if $\lambda \geq \lambda_0$, the number of eigenvalues in $[0, c]$ of $H_\lambda|_{\Omega_B}$ equals the number of zeros of V.*

Proof. See Appendix B. □

The conclusions after this lemma are the same as after lemma 3.4. For $\lambda \neq 0$, the dimension of

$$\ker H_\lambda \cap \Omega_B,$$

the space of even ground states, is equal to the number of zeros of V (it is not true for $\lambda = 0$). It can be shown in the same way of B that there are no odd ground states. Since the restriction of the Hilbert space did not change the kernel of H_λ, this proposition is true even for the original Hilbert space. It also tells us that

$$\widetilde{\mathcal{W}} = \#\{V(x) = 0\}.$$

The *Poincaré-Hopf theorem* for Killing vector fields says that

$$\chi(M) = \#\{V(x) = 0\},$$

so we could prove it if only we knew that $\mathcal{W} = \widetilde{\mathcal{W}}$ and $\mathcal{W} = \chi(M)$. The second equality is true since the restriction of the Hilbert space did not change the dimension of cohomology groups: we can prove it in the same way as we did after 3.4. The first equality comes from general theory of elliptic operators.

Conclusions

We have seen a series of connections between supersymmetric quantum systems and geometrical properties of their base manifold. The themes of localization and deformation invariance were present throughout the presentation. It is also interesting and elegant how all the differential geometry theorems shown emerge in the framework of supersymmetric quantum mechanics. In two different models we noted how exploiting the invariance of the Witten index under deformations led us to localization arguments, the second one similar to the one that allowed us to prove the Berline-Vergne localization formula. All the theorems we have discussed are totally non-trivial and their proofs rely on difficult technical lemmas, but the intution behind them is somehow clear: we have managed to present almost all key steps and sources to fill the gaps. For further information about all the proofs, it is recommended to read [2] and [5].

Appendix A

Proof of Berline-Vergne localization formula 2.7

This proof follows the one in the section *Berline-Vergne localization formula* of [5] (page 29).

Firstly,

$$d_V i_V g = d i_V g + i_V i_V g = d i_V g + g(V,V),$$

so

$$\begin{aligned}(\alpha \wedge e^{-\lambda d_V i_V g})_n &= e^{-\lambda g(V,V)} (\alpha \wedge e^{-\lambda d i_V g})_n \\ &= \sum_{i+2j=2n} e^{-\lambda g(V,V)} (-1)^j \frac{\lambda^j}{j!} \alpha_i \wedge (d i_V g)^j \\ &= (*).\end{aligned}$$

Now we perform the calculation in convenient coordinates. Let x be a zero of V. In a sufficiently small neighborhood U of x (that does not contain other zeros of V), we use a chart $\phi : U \to U' \subseteq \mathbb{R}^{2n}$ such that

$$\phi(x) = 0$$

$$g_{\mu\nu}(0) = \delta_{\mu\nu}$$

$$g_{\mu\nu} V^\mu V^\nu(y) = y^T A y,$$

where A is a matrix. With $\partial_\mu V_\nu(0) = \widetilde{\Omega}_{\mu\nu}$, in these coordinates

$$\widetilde{\Omega}^T \widetilde{\Omega} = A$$

$$\mathcal{L}_V g \quad \mapsto \quad \left(V^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu V^\lambda + g_{\mu\lambda} \partial_\nu V^\lambda \right) dy^\mu \otimes dy^\nu .$$

So, since $\mathcal{L}_V g = 0$,

$$\widetilde{\Omega}_{\mu\nu} + \widetilde{\Omega}_{\nu\mu} = 0.$$

In matrix notation we have

$$-\widetilde{\Omega}^2 = \widetilde{\Omega}^T \widetilde{\Omega} = A.$$

The matrix Ω is even-dimensional and skew-symmetric, hence after a rotation of the chart it has the unique form

$$\Omega = \begin{pmatrix} \Omega_1 & & \\ & \ddots & \\ & & \Omega_m \end{pmatrix} \qquad \Omega_i = \begin{pmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{pmatrix}.$$

With these new rotated coordinates,

$$g(V,V) \quad \mapsto \quad \begin{cases} g_{\mu\nu}V^\mu V^\nu(y) & = \\ & = -y^T\Omega^2 y \\ & = \sum_{i=1}^n \omega_i^2\left(y_{2i-1}^2 + y_{2i}^2\right) \end{cases}$$

$$di_V g \quad \mapsto \quad \begin{cases} \partial_\mu(V^\lambda g_{\lambda\nu})(y) dy^\mu \wedge dy^\nu & = \\ & = (\Omega_{\mu\nu} + o(1))\, dy^\mu \wedge dy^\nu \\ & = -2\sum_{i=1}^n (\omega_i + o(1))\, dy^{2i-1}\wedge dy^{2i}. \end{cases}$$

Now we compute $(*)$ in these coordinates.

$$\begin{aligned}(*) \mapsto & \sum_{i+2j=2n} e^{-\lambda\sum_{i=1}^n \omega_i^2\left(y_{2i-1}^2+y_{2i}^2\right)} 2^j \frac{\lambda^j}{j!}\widetilde{\alpha}_i(y) \wedge \left(\sum_{k=1}^n (\omega_i + o(1))\, dy^{2k-1}\wedge dy^{2k}\right)^j \\ = & \sum_{j=1}^n (\dots)\, \widetilde{\alpha}_{2(n-j)}(y) \wedge \sum_{k_1,\dots,k_j=1}^n \big(\omega_{k_1}\cdots\omega_{k_j} + o(1)\big)\, dy^{2k_1-1}\wedge dy^{2k_1}\wedge\cdots\wedge dy^{2k_j-1}\wedge dy^{2k_j}.\end{aligned}$$

With

$$\widetilde{\alpha}_{2(n-j)}(y)\wedge dy^{2k_1-1}\wedge dy^{2k_1}\wedge\cdots\wedge dy^{2k_j-1}\wedge dy^{2k_j} = f_j^{(k_1,\dots,k_j)}(y) dy^1\wedge\cdots\wedge dy^{2n},$$

applying 2.6 we conclude that

$$\begin{aligned}\int_U \alpha = & \\ & = \lim_{\lambda\to\infty}\sum_{j=1}^n 2^j\frac{\lambda^j}{j!}\sum_{k_1,\dots,k_j=1}^n \int_{U'} e^{-\lambda\sum_{i=1}^n \omega_i^2\left(y_{2i-1}^2+y_{2i}^2\right)}(\dots)\, f_j^{(k_1,\dots,k_j)}(y) dy_1\cdots dy_{2n} \\ & = \lim_{\lambda\to\infty}\sum_{j=1}^n 2^j\frac{\lambda^j}{j!}\sum_{k_1,\dots,k_j=1}^n \frac{1}{\omega_1^2\cdots\omega_n^2}\left(\frac{\pi}{\lambda}\right)^n \omega_{k_1}\cdots\omega_{k_n} f_j^{(k_1,\dots,k_j)}(0) \\ & = \frac{1}{n!}\frac{(2\pi)^n}{\omega_1^2\cdots\omega_n^2}\sum_{k_1,\dots,k_n=1}^n \omega_{k_1}\cdots\omega_{k_n} f_n^{(k_1,\dots,k_j)}(0) \\ & = \frac{(2\pi)^n}{\omega_1\cdots\omega_n}\alpha_0(x).\end{aligned}$$

The product $\omega_1\cdots\omega_n$ is just the Pfaffian of the antisymmetric matrix Ω, which as an operator $T_xM \to T_xM$ is identified through the metric with $dV^\flat$, hence

$$\int_U \alpha = \frac{(2\pi)^n}{\mathrm{pf}(dV^\beta(x))}\alpha_0(x).$$

If the neighborhood U does not contain any zeros of V it can be shown that

$$\int_U \alpha = 0,$$

so via partition of unity we get the thesis.

Appendix B

Proof of lemma 3.10

First part of this proof comes from section 4 of [5] (page 64), while second part is identical to the proof cited in 3.4.

We recognize as domain of Q_λ and Q_λ^* the Sobolev space $H^1(E)$ (see for example page 33 of [2] for a definition). First part of the proof is finding approximate local solutions of equation $H_\lambda \alpha = 0$ for $\lambda \to \infty$. Let U_x be disjoints neighborhoods of all the zeros x of V. We choose a chart $\phi : U_x \to B(0,4) \subseteq \mathbb{R}^{2n}$ around a zero x of V such that

$$\phi(x) = 0$$

$$V \quad \mapsto \quad \widetilde{V}(y) = \Omega y$$

for an antisymmetric (since V is Killing) matrix

$$\Omega = \begin{pmatrix} \Omega_1 & & \\ & \ddots & \\ & & \Omega_m \end{pmatrix} \qquad \Omega_i = \begin{pmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{pmatrix}.$$

If needed we choose a smaller a in order to be able to do so for all zeros. Now we write in these coordinates the operator H_λ, deformed by changing the metric to the flat one induced by this chart. With this prescription it becomes

$$-\left(\Delta + \lambda c^T \Omega \widehat{c} + \lambda^2 Y^T \Omega^2 Y\right),$$

where Δ is the ordinary Laplacian and $c = (c(e_1), \ldots, c(e_{2n}))$, $\widehat{c} = (\widehat{c}(e_1), \ldots, \widehat{c}(e_n))$ for the canonical basis of $\mathbb{R}^{2n}$. We want to split the new Hamiltonian in two components, the first being an harmonic oscillator with 0 ground energy. The first component is thus

$$A = -\sum_{i=0}^{2n} \partial_i^2 + \lambda^2 \sum_{i=0}^{n} \omega_i^2 \left(Y_{2i-1}^2 + Y_{2i}^2\right) - 2\lambda \sum_{i=0}^{n} \omega_i,$$

while the second is

$$B = -\lambda \sum_{i,j=0}^{2n} \Omega_{ij} c(e_i) \widehat{c}(e_j) + 2\lambda \sum_{i=0}^{n} \omega_i.$$

If we decompose the space of L^2 forms (our candidates as approximate solutions) of $\mathbb{R}^{2n}$ as $L^2(\mathbb{R}) \otimes \Lambda(\mathbb{R}^{2n})$, then A acts only on $L^2(\mathbb{R})$ and B acts only on $\Lambda(\mathbb{R}^{2n})$. Moreover, A is an harmonic oscillator with 0 ground energy and we already know its ground state from quantum mechanics: it is

$$\psi_\lambda(y) = \exp\left(-\frac{\lambda}{2} \sum_{i=0}^{n} \omega_i \left(y_{2i-1}^2 + y_{2i}^2\right)\right).$$

For the equation $B\alpha = 0$, we know from [5] (page 65) that it has only a solution and it is even, called β_λ. Given this approximate solution in a chart around x, we glue it to the manifold via a smoothing function γ such that

$$\begin{cases} \gamma(y) = 1 & |y| \leq 1 \\ \gamma(y) = 0 & |y| \geq 2, \end{cases}$$

getting

$$\alpha_\lambda = C_\lambda \phi^* \left(\gamma \psi_\lambda \beta_\lambda \right),$$

where C_λ is a normalization constant. We do the same thing for all the other zeros of V, getting an approximate kernel of H_λ restricted to even forms of dimension equal to m, the number of zeros. This approximate kernel is spanned by the approximate (even) solutions glued to the manifold for each zero:

$$E_\lambda = \operatorname{span}\left\{ \alpha_\lambda^{(1)}, \ldots, \alpha_\lambda^{(m)} \right\}.$$

Second part of the proof is showing that, for λ big enough,

$$\dim E_\lambda = \dim \ker H_\lambda \cap \Omega_B.$$

Let P_λ be the projection on E_λ and $P_\lambda^\perp$ the projection on E_λ^T. We start by decomposing $D_\lambda = (Q_\lambda + Q_\lambda^*)\,|_{\Omega_B}$ as

$$D_\lambda = D_\lambda^{(1)} + D_\lambda^{(2)} + D_\lambda^{(3)} + D_\lambda^{(4)}$$

$$\begin{aligned} D_\lambda^{(1)} &= P_\lambda D_\lambda P_\lambda & \qquad D_\lambda^{(3)} &= P_\lambda^\perp D_\lambda P_\lambda \\ D_\lambda^{(2)} &= P_\lambda D_\lambda P_\lambda^\perp & \qquad D_\lambda^{(4)} &= P_\lambda^\perp D_\lambda P_\lambda^\perp. \end{aligned}$$

1. We have $D_\lambda^{(1)} = 0$.

   Proof. From the definition of E_λ,

   $$P_\lambda = \sum_{i=0}^{m} \alpha_\lambda^{(i)} (*, \alpha_\lambda^{(i)}).$$

   If ω is odd, $P_\lambda \omega = 0$. If ω is even, $P_\lambda \omega$ is even, but $D_\lambda P_\lambda \omega$ is odd and $D_\lambda^{(1)} \omega = 0$. □

2. There exists $\lambda_1 > 0$ such that, for $\lambda \geq \lambda_1$, if $\omega \in E_\lambda^\perp \cap H_1 \cap \Omega_B$ and $\omega' \in E_\lambda$,

   $$\left\| D_\lambda^{(2)} \omega \right\| \leq \frac{1}{\lambda} \left\| \omega \right\| \qquad\qquad \left\| D_\lambda^{(3)} \omega' \right\| \leq \frac{1}{\lambda} \left\| \omega' \right\|.$$

   Proof. If $\omega \in E_\lambda^\perp \cap H_1 \cap \Omega_B$, then $P_\lambda^\perp \omega = \omega$ and

   $$D_\lambda^{(2)} \omega = P_\lambda D_\lambda \omega = \sum_{i=0}^{m} \alpha_\lambda^{(i)} (D_\lambda \omega, \alpha_\lambda^{(i)}) = \sum_{i=0}^{m} \alpha_\lambda^{(i)} (\omega, D_\lambda \alpha_\lambda^{(i)})$$

   $$\left\| D_\lambda^{(2)} \omega \right\| \leq \sum_{i=0}^{m} \left\| D_\lambda \alpha_\lambda^{(i)} \right\| \left\| \omega \right\|,$$

   but in coordinates it can be verified that

   $$\left\| D_\lambda \alpha_\lambda^{(i)} \right\| \leq C_1 e^{-C_2 \lambda},$$

   proving the first inequality. The second inequality follows from $D_\lambda^{(3)}$ being the adjoint of $D_\lambda^{(2)}$. □

The rest of the proof is as in [5], from page 84.

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