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Axion Wormholes in Gauged Supergravity and their String Theory Uplift

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Abstract

Euclidean wormholes have attracted a renewed interest in Quantum Gravity as relevant saddle points in the path integral. To understand them, we still need to fully describe them from a String Theory point of view. The best way we can do this is by dimensional uplift from a lower dimensional Supergravity theory, where we can describe them properly. One way to have such a solution for space-time geometry is by using axions. This is why we try to embed the four-dimensional Giddings-Strominger wormhole solution in String Theory. The Supergravity model we chose is a $\mathcal{N} = 8$ dyonically gauged $ISO(7)$ maximal Supergravity, whose embedding into massive Type IIA String Theory is well known and reviewed in this work. We tried to recover for the first time wormhole solutions in such a workframe and managed to find significant bounds for this solution. In order to find these results, we analyzed the particle content of dyonic $ISO(7)$ theory in multiple subsectors of interest and identified good axio-dilaton pair candidates. Then we proceeded to approximate our solutions to expand our theory around a vacuum Giddings-Strominger solution with AdS_4 asymptotic behavior with a method implemented in previous works. In conclusion, we attempted to embed in this theory a gauged Supergravity theory with no scalar and found possible embeddings within the vacua of our model. This made it possible to assert that traversable wormhole solutions can be successfully embedded in String Theory starting from this never before analyzed model.

Contents

Chapter 0

Overview

This thesis work is partitioned in the following manner:

- Chapter 1 is dedicated to an introduction to the main theoretical frameworks we work in, namely String Theory and Supergravity. Furthermore, a short overview is given on Anti-deSitter geometries, which we will often encounter in the work, instanton solutions and the traversable wormhole features.

- In Chapter 2 we discuss the process of dimensional reduction and compactification, outlining the main results one wishes to recover using such a method. We also retrace Kaluza-Klein dimensional reduction, the first attempt at recovering new field content by means of dymensional reduction, and Freund-Rubin dimensional compactification, which set up the research field of flux compactification.

- The content of Chapter 3 is centered on introducing Giddings-Strominger wormholes. We first repropose and analyze the Lagrangian treated in their first paper introducing the matter and then give some stability bound for the solution.

- Chapter 4 is concerned with the dimensional reduction of a massless Type IIA String theory to a four-dimensional Supergravity theory. First we introduce the lower dimensional wormhole solution and then find a match between such a configuration and a the aforementioned ten-dimensional String theory. We survey the obtained solution before passing on to an introduction to a Type IIB String theory reduction to the same four-dimensional wormhole solution.

- In Chapter 5, we delve into the proper embedding of the $ISO(7)$ dyonically gauged Supergravity into massive Type IIA.

- With Chapter 6, we shift our focus to the $SU(3)$-invariant subsector of the $ISO(7)$ model, showing its action, its equations of motion and analyzing the vacuum configurations of the theory, trying to find new ones. We also shortly review the attractor solution of this subsector.

- In Chapter 7, we examine the possibility of having a Giddings-Strominger wormhole solution in the dyonically gauged $ISO(7)$ model, with particular stress on its $SU(3)$-invariant subsector. We apply field truncations in order to recover a desired axio-dilaton pair, we give a general solution for a generalized GS wormhole with an AdS background and find asymptotic solutions by means of justified approximations. Finally, we embed in our model a previously studied gauged $\mathcal{N} = 2$ Supergravity theory with no scalar multiplets. We find positive response when asked if wormhole solutions can eventually be embedded in String Theory in our model.

- Chapter 8 contains the comprehensive conclusions we drew from this thesis work.

- In Chapter 9 we report some outlooks on the traversable wormhole solutions we found of particular interest.

- In Appendix A we derive the switch in sign of the kinetical term of the action for the axion from a path integral perspective.
- Appendix B is focused on giving the explicit form of the scalar matrix of the $SU(3)$ and G_2-invariant subsectors of the $ISO(7)$ theory.
- In Appendix C we check the consistency of the truncation of massless Type IIA String theory of Chapter 4.

Chapter 1

Introduction

The tools and methods we used in this thesis project are the ones provided by two main frameworks we used to derive our results: String Theory and Supergravity. We will shortly introduce to the most prominent features of the two that were used in this work in Chapter 1.1 and 1.2.
The model we work with in this thesis displays a spacially asymptotic behaviour typical of many theories derived from the String Theory frame. This is the geometry of a so called Anti-deSitter space, which will be concisely treated in Chapter 1.3.
String theory and Supergravity have each their own proper formalism, but they can be related looking at low-energy modes of the former, a concept which will profusely be used in our work to derive a lower dimensional theory.
We will work with Euclidean actions, objects arising in the path integral formulation, yielding the so called instanton solutions. We will briefly introduce to the latter and their use in Quantum Gravity in Chapter 1.4.
We will focus on a peculiar geometry solution, namely the traversable wormhole of Giddings and Strominger, which will be described in Chapter 3 in its main aspects. These solutions are an example of traversable wormholes, whose features and criticalities will be discussed in Chapter 1.5.

1.1 String Theory

Many references have been used during this Master's thesis work to grasp the fundamentals of String theory. To mention only some of them, we would like to refer to [1], [2], [3], [4] and [5].

1.1.1 Birth and growth of String Theory

String theory was born in the 1960s as a tool to explain some features of QCD concerning hadrons. When QCD was introduced, String Theory was not immediately discarded as it had been proven that it could describe a spin-2 particle which is identified as the graviton $g_{\mu\nu}$, the boson associated to gravity. The theory could not only incorporate gravity, but also a much broader spectrum of particle, including matter, configurating as a Unification Theory and as such garnering increasing attention. Therefore, String theory represents a theory of gravity with a consistent quantum formalism.

The first kind of string that was introduced was the so called "bosonic string", carrying as its name suggests bosonic degrees of freedom only. Such a spectrum presented a pivotal flaw, an imaginary mass state known as tachyon, bringing in an undesired instability of the vacuum. The effort to correct such an unacceptable prediction of the bosonic string let to requiring supersymmetry invariance on the string and inherently to introducing fermionic degrees of freedom. Moreover, theoretical physics needed to erase undesired anomalies arising in the theory. The Green-Schwarz mechanism introduces additional fields exploiting SuperSymmetry, called "anomalous gauge fields" or "anomalous currents" that couple to the gauge fields of the theory in a way that cancels out anomalies. In such a way the Superstring was born and a period known as the **First Superstring Revolution**, taking place between the mid-eighties and the mid-ninties, inaugurated a thriving age for String Theory.

Superstring can be formulated in five different frameworks: Type IIA theory, Type IIB, Type I, heterotic SO(32) and heterotic $E_8 \times E_8$ theories. All of these theories live in the critical 10 dimensions of Superstring Theory. It later turned out that all of these theories are related by duality relations and that they all can be seen as dual theories of an eleven-dimensional theory: the M theory. The period during which these relations were found is known as the **Second Superstring Revolution**, and the plethora of newfound concepts in String Theory included the understanding of p-branes and the introduction of AdS/CFT correspondence.

1.1.2 The bosonic string

To understand the very basics of string theory, we introduce the first action one deals with when studying strings, the bosonic string action

$$S = -\frac{T_s}{2} \int d\tau d\sigma \sqrt{-\gamma} \gamma_{ab} G_{MN} \partial^a X^M \partial^b X^N \tag{1.1}$$

where γ_{ab} is a metric which defines the so called "world-sheet", the two-dimensional surface on along which the one-dimensional string moves. T_s is the so called string tension, and it can be expressed as

$$T_s = \frac{1}{2\pi\alpha'} \tag{1.2}$$

where α' is called Regge slope and is a parameter of high interest since it sets the string scale $l_s \equiv \sqrt{\alpha'}$.
From our definitions, we notice that, depending on the presence of bounds on our world-sheet, one can have both open and closed strings. Moreover, one can orientate the world-sheet manifoldt in order to have oriented strings.
One finds a critical dimensionality for bosonic string theory, which is D=26. To obtain the physical content of the theory, i.e. the "spectrum", one needs to require on-shellness. By doing so with this amount of dimensions, one find that the spectrum of the string, arising from "vibration" of the latter, contains a spin = 2 bosonic state which incorporates gravity naturally in the theory.
Naturally, if one is interested in studying the phenomenological implications of String Theory, what is relevant will be studying the low energy limits of the framework. With such a purpose, we are brought to derive the lowest-lying states in the tower of increasingly massive particles the theory predicts. We find that there exist some massless states, which are interpreted as different particles. These present different values of the spin and are
- a symmetric rank-2 tensor g_{MN}, the **dilaton** or metric;
- an **anti-symmetric 2-form** B_{MN};
- a scalar field ϕ, called the **dilaton**.

As anticipated, the theory predicts the existence of a $m^2 < 0$ particle, the tachyon, later eliminated by supersymmetry.

1.1.3 The superstring

The request of supersymmetric invariance in a theory expands the usual Poincaré symmetry group introducing some new conserved charges named "supercharges". The new transformations under which our theory needs to be invariant mix bosonic and fermionic degrees of freedom. The number of conserved supercharges in the theory can vary, thus giving birth to so called extended supersymmetry.
As anticipated, the theories arising from imposing supersymmetry to the bosonic string are 5, and they are the ones written at the beginning of this paragraph. All of these theories have a critical number of dimensions equal to 10. In 10 dimensions, killing spinors which are Weyl-Majorana have each 16 independent components. An $\mathcal{N} = 2$ supersymmetry, as we have in the case of Type IIB and Type IIA theories, thus requires the conservation of $16 \times 2 = 32$ independent supercharges.

Super-transformations invariance imposes some boundary conditions on the fermionic degrees of freedom. Such boundary conditions can be satisfied with two different periodic impositions on the functions, thus yielding different sectors on open and closed strings, differing in the boundary condition choice. These are called the Neveu-Schwarz and Ramond sectors.
Distinguishing between left-moving and right-moving modes on the string, one can consider these modes having different or the same chiralities. Upon making this choice and quantizing the theory, one finds that the two resulting theories are different, and they provide a different spectrum, sectorized accordingly to the previously defined boundary conditions. Thus, one finds the two theories, which are called **Type IIA** in the case of opposed chiralities for left and right-moving modes and **Type IIB** in the case of identical chirality, display the following ground-state content [2]:

	NS-NS sector	NS-R = SN-R sector	R-R sector
Type IIA	dilaton ϕ two-form B_{MN} graviton g_{MN}	gravtino ψ^{α}_{μ} dilatino λ^{α} (opposite chiralities)	1-form A_M 3-form A_{MNL}
Type IIB	dilaton ϕ two-form B_{MN} graviton g_{MN}	gravtino ψ^{α}_{μ} dilatino λ^{α} (same chiralities)	0-form χ 2-form A_{MN} 4-form A_{MNLO}

Table 1.1: Massless particle content for Type IIA and Type IIB String Theory.

The main focus in this thesis, at least for what concerns the particle content of the action, will be on bosonic degrees of freedom, thus we will neglect for now the mixed sectors. As we notice, the NS-NS sector is the same for both the theories, and it yields the action, in the Jordan frame (i.e. the workframe with a coupled Ricci scalar in the Lagrangian) [6]:

$$S_{NS} = \frac{1}{2k^2}\int d^{10}x\sqrt{-G}e^{-2\phi}\Big(\mathcal{R} + 4\partial_M\phi\partial^M\phi - \frac{1}{2}H_3^2\Big) \tag{1.3}$$

where H_3 is the field-strength tensor associated to B_2.
Upon defining the n-forms of the R-R sector, one can derive the associated field-strengths which will enter the Lagrangian.
Since the field-strengths we will deal with in dimensionally reducing String Theory to 4-dimensional Supergravity will be the ones present in Type IIA, we only write them down as:

$$F_2 = dA_1 \qquad\qquad F_4 = dA_3 - dB \wedge C_1 \tag{1.4}$$

They enter the String R-R action as (see also (6.7))

$$S_{R,A} = \frac{1}{2k^2}\int - \sum_{n=2,4} *F_n \wedge F_n - \frac{1}{2}B \wedge dA_3 \wedge dA_3 \tag{1.5}$$

This action is built in such a way to yield the correct equations of motion.

We also notice that the gravitational term in (1.3) is not the usual one, but is multiplied by a factor $e^{\cdot 2\phi}$, which puts the action in the so called "string frame" and emerges in taking the low-energy approximation of the string action. One can reabsorb it in the metric obtaining an action in the Einstein frame, leaving just a constant equal to an exponentiation of the vacuum expectation value for the dilaton. This term $e^{-2\phi_0}$ contributes to the gravitational coupling. Indeed, by referring to the 4-dimensional gravitational coupling, i.e. Planck mass, one can write

$$M^2_{p,10} = \frac{1}{k^2_{10}g^2_s} \tag{1.6}$$

where we have defined the **string coupling** $g_s = e^{\phi_0}$.

1.1.4 Gauge fields and Dp-branes

In string theory, branes and gauge fields are interconnected in a profound way, providing a crucial aspect of the theory's framework [7]. The gauge fields arise, in Type IIA and IIB, from the R-R sector of the theory, as it has been made clear in the previous section. Let us summarize their relation:

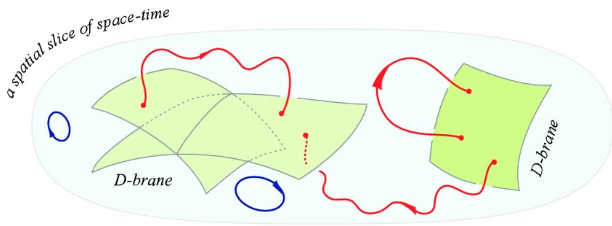

Figure 1.1: [8] Example of open strings interacting with D2-branes.

∘ **Branes**: Branes are extended objects (membranes) of various dimensions (0-dimensional point particles, 1-dimensional strings, 2-dimensional membranes, etc.) that are allowed to exist within the spacetime of string theory. The most well-known examples are D-branes (Dirichlet-branes), which are surfaces where open strings can end with specific boundary conditions for the gauge fields on their surface.
∘ **Gauge Fields**: Gauge fields are fundamental fields in quantum field theory that are associated with forces, such as electromagnetism (described by the electromagnetic field) or the strong and weak nuclear forces (described by gluon and W/Z boson fields, respectively). They mediate the interactions between particles with specific charges.
∘ **Open Strings and D-branes**: Open strings have endpoints that can attach to D-branes, leading to the emergence of gauge fields on the brane's surface. The endpoints of open strings can move freely on the brane, allowing the interaction of particles with the gauge fields living on the brane. Realizing non-Abelian gauge theories on branes in string theory involves the use of multiple coincident D-branes. Non-Abelian gauge theories arise from non-commuting gauge fields, which are associated with non-Abelian gauge groups like $SU(N)$ or $SO(N)$. In string theory, each open string has Chan-Paton factors associated with its endpoints. These factors represent the charges of the open string endpoints under the gauge group. In the case of multiple D-branes, the Chan-Paton factors allow for interactions between different D-branes, leading to non-Abelian gauge theories.

In summary, branes and gauge fields in string theory are intertwined due to the interactions of open strings with D-branes, which give rise to gauge fields on the brane's surface. The gauge fields introduced in the theory can either be linked to an Abelian or non-Abelian gauge theory, depending on the presence of coincident Dp-branes.

1.2 Supergravity

Supergravity contemplates a very broad landscape of concepts and subtleties that we will not be able to cover in the least. We refer to [9], [10] for a full coverage of the subject.

Supersymmetry as generally intended discusses globally symmetric theories under transformations mixing bosonic and fermionic d.o.f.'s of what is known as the "**chiral multiplet**", containing, for $\mathcal{N}=1$, one scalar ϕ and one chiral fermion field χ.
When one tries to make this invariance local (depending on specific coordinates in space-time), new degrees of freedom arise as a consequence. This leads to the formulation of the pure $\mathcal{N}=1$ Supergravity Lagrangian as

$$\mathcal{L}_{\text{sugra}} = \frac{M_p^2}{2}\sqrt{-g}\mathcal{R} - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \mathcal{L}_{\text{int}}(g_{\mu\nu},\psi_\mu) \tag{1.7}$$

where $\gamma^{\mu\nu\rho}$ is an anti-symmetrization of γ matrices, while:
• $g_{\mu\nu}$ is a spin-2 tensor field know as the **graviton** (or metric);
• ψ_μ is a spin-3/2 field knows as the **gravitino**, holding both space-time and spinorial indices.

This lagrangian is invariant under supersymmetry transformations [9]:

$$\begin{aligned} \delta g_{\mu\nu} &= \frac{1}{M_p}\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} \\ \delta\psi_\mu &= M_p\partial_\mu\epsilon \end{aligned} \tag{1.8}$$

where we have introduced a spinorial quantity, ϵ. Due to the fact that it has the same role that the killing vector holds in regular symmetric shifts, it is called **killing spinor**.

Important considerations can then be made on the geometry of manifolds in Supersymmetry: first of all, it is useful in general to formulate a theory with a number n_C of chiral multiplets. In this case we write the generalized scalar fields kinetic term as

$$g_{m\bar{n}}(\phi,\phi^*)(\partial_\mu\phi^m)(\partial^\mu\phi^{*\bar{n}}) \tag{1.9}$$

where $g_{m\bar{n}}$ is the metric parametrizing a scalar manifold $\mathcal{M}_{\text{scalar}}$ which is called "**target space**".
Also, a function called **superpotential** $W(\phi,\phi^*)$ arises, which can be recast in such a way to yield a scalar potential for our n_C scalar fields.
Moreover, for geometric and symmetry considerations, the $\mathcal{N}=1$ target space for scalars is a complex manifold with holonomy group $\subseteq U(n_C)$, called **Kahler manifold**.
On top of that, this structure introduces another potential, called accordingly **Kahler potential** $K(\phi,\phi^*)$, so that the target space metric $g_{m\bar{n}}$ can be written as $\partial_m\partial_{\bar{n}}K$.
When coupled to Supergravity, these Kahler manifolds gets restricted to a subclass known as **Kahler-Hodge** manifolds.

1.2.1 Extended Supergravity

One may want to extend the $\mathcal{N}=1$ case to a more generic theory with a higher number of supercharges. In the case of interest of $\mathcal{N}=2$ Supergravity, we introduce new multiplets bringing new fields into the theory. These are called $\mathcal{N}=2$ **vector multiplet** and **hypermultiplet** and are made of a $\mathcal{N}=1$ vector multiplet $(A_\mu,\,\lambda)$ plus a chiral multiplet $(\chi,\,\phi)$, where ϕ is complex and thus hold two d.o.f.'s, and two chiral multiplets with 4 scalar degrees of freedom in total, respectively.

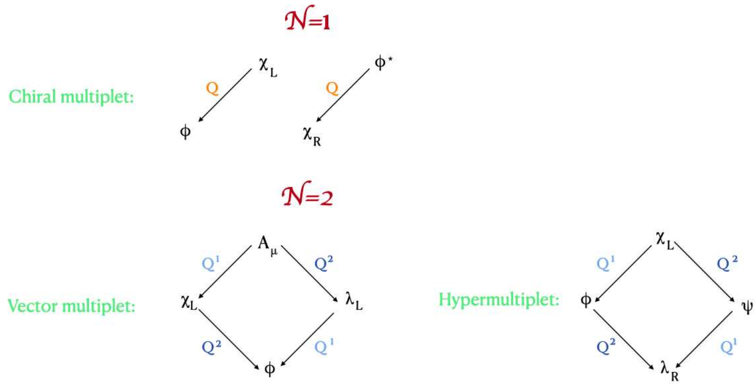

Figure 1.2: The structure of a chiral multiplet in $\mathcal{N}=1$ Supersymmetry and the structures of the vector and hypermultiplet in $\mathcal{N}=2$ Supersymmetry. We called the two different Supercharges Q^1 and Q^2. All of the fermions are Majorana, all of the scalars are complex. In $\mathcal{N}=1$, we showed the full structure of the multiplet, comprehending the "anti-multiplet", the complex conjugate of the first one. For $\mathcal{N}=2$, both for vector and hypermultiplet, a mirror structure for anti-particles is intended in order to conserve CPT-invariance. The vector multiplet can be regarder at as being built up of one vector and one chiral multiplet in $\mathcal{N}=1$. The hypermuliplet can be viewed as being composed of two $\mathcal{N}=1$ chiral multiplets. For symmetry reasons ϕ and ψ need to be different, complex scalars, as much as χ and λ need to represent different Majorana fermions. Therefore we will have 4 different bosonic degrees of freedom.

The structures of these multiplets set different conditions on the geometry of their target spaces. The total scalar space indeed factorizes into

$$\mathcal{M}^{\mathcal{N}=2}_{\text{scalar}} = \mathcal{M}_{\text{vector}} \times \mathcal{M}_{\text{hyper}} \tag{1.10}$$

where $\mathcal{M}_{\text{vector}}$ turns out to be a so-called **Special Kahler manifold**, while $\mathcal{M}_{\text{hyper}}$ in Supergravity is a **Quaternionic-Kahler manifold**.
Geometry conditions on Special Kahler manifold set a particular form for the Kahler potential $K = -\log i(X^I \bar{F}_I - \bar{X}^I F_I)$ where X^I are the scalar fields and F_I are holomorphic functions of them and are usually collected in a symplectic vector $\mathcal{V} := (X^I \;\; F_J)^T$. This structure introduces a function $F(X)$ known as the **prepotential** in terms of which both the Kahler potential and the gauge kinetic matrix $\mathcal{N}_{IJ}$ can be written.
The latter arises in extended theories (with vector fields introduced by means of vector multiplets or the standard Supergravity multiplet in $\mathcal{N}=2$), where the symmetry requirements induce a structure of the kinetic gauge term that depends on scalar fields. We can generally write

$$e^{-1}\mathcal{L}_{\text{kin}} = \frac{1}{4}(Im\mathcal{N}_{IJ})F^I_{\mu\nu}F^{\mu\nu J} - \frac{1}{8}(Re\mathcal{N}_{IJ})\epsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}, \qquad Im\mathcal{N}_{IJ} = \mathcal{I}_{IJ}, \;\; Re\mathcal{N}_{IJ} = \mathcal{R}_{IJ} \tag{1.11}$$

It is important to mention that ungauged theories display a duality under the exchange of electric and magnetic fields, that requires a particular formalism in building our action. This duality shows effectively as a **symplectic**-transformations invariance, rotating electric and magnetic vector fields.

1.2.2 Gauged extended Supergravity

We will be interested in our work to turn on gauge fields and thus implement minimal coupling between vector bosons and matter in extended Supergravity with n_V vectorial multiplet (inherently, n_V vector bosons).
One may wonder why we want a gauged theory. The answer is that if we want a scalar potential in extended supergravities, which we do want since a mass for the scalars is experimentally required, the only way to achieve this is by gauging isometries of the scalar manifold in the Supergravity theory.
Due to the request of invariance under multiple supersymmetries, the freedom of choosing couplings will be further restricted and this will reflect on the potential as well.

To maintain the symplectic-convariant structure of the action we talked about above, which is generally undermined by gauging, a particular formalism, know as **embedding tensor formalism** is introduced. First, we need to gauge a subgroup of $G_U \subseteq Sp(2n_V, R)$, which is the subgroup of the symplectic group that leaves the equations of motion of the Lagrangian invariant, implementing a symmetry called U-symmetry. We associate to each of the gauged generators t_α a combination of vector fields that are going to become the gauge fields of our theory. In turn, we define the embedding tensor

$$\Theta^\alpha_M, \qquad \alpha = 1, ..., \text{dim}_{G_U}, \;\; M = 1, ..., 2n_V \tag{1.12}$$

so that the M-dimensional gauge algebra generators X_M and inherently covariant derivatives D_μ can be written as

$$X_M = \Theta^\alpha_M t_\alpha \qquad\qquad D_\mu = \partial_\mu - A^M_\mu \Theta^\alpha_M t_\alpha \quad \text{where } A^M_\mu = (A^I_\mu, A_{\mu I}) \tag{1.13}$$

The embedding tensor thus tells us which ones among the electric and magnetic vector fields enter the gauging. This clearly breaks the previous invariance under G_U.
Furthermore, the gauging introduces tensor fields in the theory, needed to effectively have a gauge invariance of the theory.

We shall mention also that in gauged extended SuGra, the scalar manifold is usually represented, to simplify calculations, by a **coset** $\frac{G}{H}$. Given a basis of the algebra of the coset manifold $t_{G/H}$, one can parametrize the scalars by giving a so-called **coset representative** $\mathcal{V}(\phi) = e^{t_1\phi_1}...e^{t_{(\text{d}[G]-\text{d}[H])}\phi_{(\text{d}[G]-\text{d}[H])}}$ or in group indices as a vector $\mathcal{V}^{ij}_M$, where $M = 1, ..., \text{dim}_{G/H}$ are indices parametrizing the generators of the coset, while i, j put the representative in a representation of G.

1.3 Anti-deSitter

Anti-deSitter spaces play a fundamental role in gauged Supergravity theories and often appear as vacuum solutions of Supergravity theories. They also hold a prominent spot in String Theory as part of the AdS/CFT correspondence [11]. The discussion follows [12] and [13].

The AdS metric is a solution of the generically D-dimensional gravitational action

$$S = \frac{1}{2k_D^2}\int d^D x\sqrt{|g|}(\mathcal{R} - 2\Lambda) \tag{1.14}$$

where we take a negative valued Cosmological Constant $\Lambda = -\frac{(D-1)(D-2)}{2l^2}$, where l is called carachteristic AdS length and it indeed has the dimension in natural units of $[\mathrm{E}]^{-1}$.
The action (1.14) yields the vacuum equation of motion

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \frac{(D-1)(D-2)}{2l^2}g_{\mu\nu} \tag{1.15}$$

which as expected reduces to vacuum Minkowski for $l \to \infty$.
The equation of motion thus yields a Ricci tensor proportional to the metric

$$R_{\mu\nu} = k(D-1)g_{\mu\nu} \tag{1.16}$$

and thus the AdS_D metric is an Einstein metric.

We can embed our D-dimensional Anti-deSitter space in a pseudo-Euclidean (D+1)-dimensional space, parametrized by X^A, with $A = 0, .., D$. The signature of such a higher dimensional space is

$$\eta_{AB} = \begin{bmatrix} -1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{1.17}$$

such that the embedding of AdS in this (D+1)-dimensional space defines a (D+1)-hyperboloid of equation

$$-(X^0)^2 + \sum_{i=1}^{D-1}(X^i)^2 - (X^D)^2 = -l^2 \tag{1.18}$$

As it is clear from expression (1.18), the isometry group of AdS_D is SO(D-1,2).

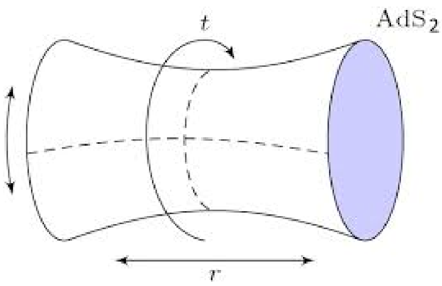

Figure 1.3: [14] Embedding of a 2-dimensional AdS space in 3 dimensions.

We now wish to define a coordinate parametrization of the AdS_D metric. We can induce such a metric from the hyperboloid embedding (1.18), finding a set of coordinates x^μ, with $\mu = 0, ..., D-1$ such that the previously referenced relation is satisfied.
• If one choses to define

$$\begin{aligned} X^i &= rx^i & \sum_{i=1}^{D-1}(x^i)^2 &= 1 \\ X^0 &= \sqrt{l^2+r^2}\sin(t/l) & X^D &= \sqrt{l^2+r^2}\cos(t/l) \end{aligned} \tag{1.19}$$

where the x^1 parametrize a (D-1)-dimensional sphere of unitary radius.
The induced metric turns out to be

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right)dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega^2_{D-2} \tag{1.20}$$

where we notice we have defined a spherically symmetric metric with periodic time t with period $\frac{2\pi}{l}$ and we considered the observer in the AdS_D space as sitting at r=0.
We notice that, differently from dS space, we have no cosmological horizon.
• We can also give a covering of AdS_D in a conformally flat parametrization, by scaling time $t = l\tau$ and defining a new "radial" coordinate $\rho = \sqrt{1 + r^2/l^2}$, obtaining

$$ds^2 = \frac{l^2}{\cos^2\rho}[-d\tau^2 + (d\rho^2 + \sin^2\rho d\Omega^2_{D-2})] \tag{1.21}$$

where the conformal factor $\frac{l^2}{\cos^2\rho}$ multiplies a static flat space.

1.3.1 AdS-Schwarschild space-time

With the purpose of finding AdS-Wormhole metric solutions in mind, we write down the main features of a metric describing a black hole in an AdS background.
Since both Schwarzschild and AdS are Einstein (they are both vacuum solutions) and spherically symmetric, we use both conditions to find, in four dimensions,

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega^2_2 \tag{1.22}$$

with

$$h(r) = \left(1 + \frac{r^2}{l^2} - \frac{r_0}{r}\right) \tag{1.23}$$

where the term $\propto \frac{1}{r}$ is the usual term appearing in the gravitational potential in flat space-time, with r_0 proportional to the mass of the body and is identified as

$$r_0 = \frac{2GM}{c^2} \tag{1.24}$$

The metric displays a curvature singularity at r = 0 and a coordinate singularity at a single finite value of r that acts as a event horizon, beyond which light cannot escape.
The metric has two notable continuous limits:

- for $r^0 \to 0$ we recover AdS metric;
- for $l \to \infty$ we recover the Schwarzschild metric.

1.4 Instantons

The discussion we do in the following is introductory to the topic. A broad coverage of the definition and general use of instantons can be found in [15] and [16], while for a discourse on the features of instantons in Supergravity one should check [17].

Instantons are a type of classical solution to the equations of motion in quantum field theories. These solutions are typically characterized by having finite action, and they play a significant role in understanding various nonperturbative phenomena in particle physics and condensed matter physics.
Instantons are essential in the study of quantum tunneling and the breaking of symmetries. For instance, in QCD, instantons are associated with non-perturbative effects inducing chiral symmetry

breaking, a spontaneous symmetry breaking which leads to the quarks having mass. Instantons appear in theories where the classical equations of motion allow for transitions between different classical vacua (lowest energy states) that would otherwise be forbidden by classical considerations. In quantum mechanics, particles can tunnel through energy barriers that they wouldn't classically have enough energy to overcome. Instantons provide a mathematical framework to describe such tunneling processes in quantum field theories.

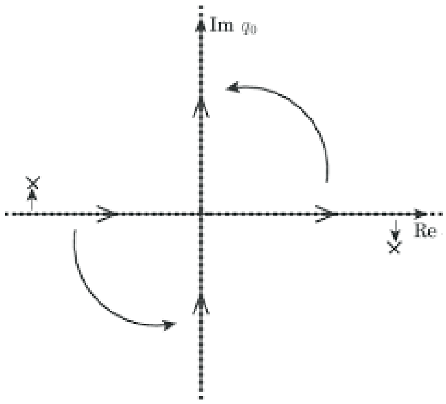

Figure 1.4: [18] Depiction of a Wick rotation.

In quantum mechanics, to transition from classical to quantum descriptions, one often employs the concept of a path integral, where all possible paths of a particle or field contribute to the final quantum state. To perform these path integrals and evaluate probabilities, it's often advantageous to work with the Euclidean action, which is obtained by Wick rotating the time coordinate.
The Wick rotation is a mathematical procedure in which one analytically continues the real time axis (which can be both positive and negative) to a purely imaginary axis. This involves a rotation of the time coordinate by 90 degrees in the complex plane, which transforms the Minkowski spacetime (characterized by a metric with both positive and negative signs) into a Euclidean spacetime (with a purely positive metric). Mathematically, this can be expressed as t $\rightarrow -i\tau$, where t is real time and τ is imaginary time.

Quantum gravity deals with the interplay of quantum mechanics and gravity, and path integrals are a crucial tool in such studies. Wick rotation simplifies these path integrals by transforming the oscillatory nature of quantum fluctuations in Minkowski spacetime into an exponentially damped behavior in Euclidean spacetime. This makes the path integral more manageable mathematically.
In many cases in fact, calculations involving quantum gravity yield divergent results due to the highly curved spacetime near black holes or singularities. Wick rotation can help in regularizing these divergences, making it easier to apply regularization and renormalization techniques to obtain finite, meaningful results.
In the context of quantum gravity, the role of instantons is not as well-established or understood as in other areas of theoretical physics like quantum field theory [19].
In theories of quantum gravity, such as String theory, instantons can arise as solutions to the equations of motion. These instantons are often associated with nonperturbative effects, which means they provide insights into the behavior of the theory beyond what can be described by the standard perturbative methods.
One of the intriguing aspects of instantons in the context of quantum gravity is their potential role in the understanding of black hole physics. Instantons have been used to study tunneling processes that could lead to evaporation of black holes [20]. These processes are related to the Hawking radiation phenomenon, where black holes are predicted to emit radiation due to quantum effects near their event horizons.

1.5 Traversable wormholes

The term wormhole was introduced for the first time in a paper by Fuller and Wheeler [21]. The very first wormhole wormhole solution to ever be found can be considered the Einstein-Rosen bridge [22], which connected anyway two causally disconnected universes. This idea was later expanded to that of a wormhole linking two patches of the same universe with the same boundary behaviors. We will refer to this, upon making some assumption, as to the traversable wormhole solution.

Many choices have been made in history to indicate a metric which is the one of a traversable wormhole space-time solution. To make general considerations and outline the main criticities of such a solution, we'll first choose, following [23], the metric ansatz for a Lorentzian traversable Wormhole as (what will turn out to be important here, is that g_{tt} should be defined negative, troath incuded, and $\Phi(r)$ should be everywhere differentiable):

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.25}$$

where $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate.
The metric has a singularity at $b(r_0) = r_0$, representing the radius at which we find the wormhole throat. This is a cordinate singularity that does not represent a physical horizon, thus keeping the wormhole traversable, as long as $g_{tt} = -e^{2\Phi(r)} \neq 0$.

To picture the structure of such a geometry, one can take a slice of space-time fixing the time t and considering $\theta = \pi/2$. This choice yields a 2D curved space

$$ds^2 = \frac{dr^2}{1 - b(r)/r} + r^2 d\phi^2 \tag{1.26}$$

which, embedded in a Euclidean 3D space parametrized by cylindrical coordinates gives

$$ds^2 = dz(r)^2 + dr^2 + r^2d\phi^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right]dr^2 + r^2d\phi^2 \Rightarrow \left(\frac{dz}{dr}\right) = \pm\left(\frac{r}{b(r)} - 1\right)^{-1/2} \tag{1.27}$$

This solution gives a vertical embedded surface at r_0, while for $r \to \infty$ the surface is asymptotically flat. Another requirement, known as the **"flaring-out condition"** [23] is imposed. This requires

$$\frac{d^2r}{d^2z} = \frac{b - b'r}{2b^2} > 0 \tag{1.28}$$

in the proximity of the throat r_0, establishing the openness of the geometry.
Calculating the Einstein tensor and exploiting the Einstein equations one can therefore derive the stress-energy tensor $T_{\mu\nu}$ needed for the wormhole geometry to arise. The components of the tensor which do not vanish can be proved to be the diagonal one. The remaining tensor components can thus be given a simple physical interpretation, and we take in consideration T_{tt}, which is interpreted as an energy density ρ and one can esily (see again [23]) derive

$$\rho(r) = \frac{1}{8\pi}\frac{b'(r)}{r^2} \tag{1.29}$$

and the (rr) component τ, the radial tension, as:

$$\tau(r) = \frac{1}{8\pi}\left[\frac{b}{r^3} - 2\left(\frac{b}{r}\right)\frac{\Phi'}{r}\right] \tag{1.30}$$

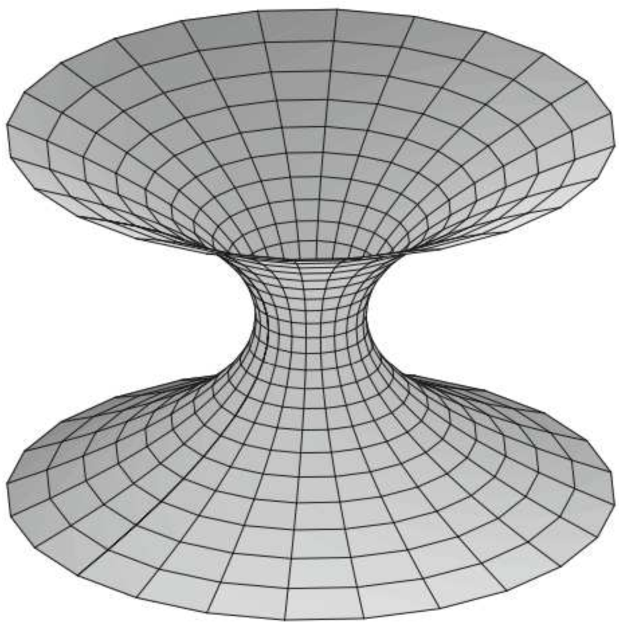

Figure 1.5: [23] Projection of a 4-dimensional wormhole in the 3 dimensions.

In the paper that pioneered the geometry of this open wormhole, [24], a function ξ, called **exoticity function**, is defined as

$$\xi = \frac{\tau - \rho}{|\rho|} = \frac{2b^2}{r|b'|}\frac{d^2r}{d^2z} - 2r\left(1 - \frac{b}{r}\right)\frac{\Phi'}{|b'|} \tag{1.31}$$

This expression implies that, given that Φ and b are regular functions for $r = r_0$, the second term in (1.31) vanishes and we are left, imposing (1.28) with a positive defined quantity in the proximity of the throat. The matter satisfying condition $\xi > 0$ and which must be present at the throat of the traversable wormhole is called **exotic matter** [25].
This matter is called exotic because it violates the **null energy condition**, which is satisfied for all known matter as of today. This implies that, taken a null vector k^μ and a stress-energy tensor $T_{\mu\nu}$, the condition $T_{\mu\nu}k^\mu k^\nu > 0$ holds. For a diagonal stress-energy tensor as the one we have in our case, this implies $\rho - \tau \geq 0$. This request is clearly violated by (1.31) at the troath.
Thus such a traversable wormhole requires the presence of matter going beyond the standard model, which is later identified in our calculations with the axion [26], providing the required negative energy density, or quantum effects such as the Casimir effect.

1.5.1 Euclidean wormholes and their uses in quantum gravity

Interest in Euclidean wormholes within the realm of quantum gravity has seen a recent and consistent upswing. The driving forces behind this surge can be traced back to two primary factors: a deeper exploration of the path integral for low-dimensional gravity, as discussed in [27], and a substantial body of research dedicated to the Swampland program [4], [28]. Recent investigations have shed light on wormholes as significant saddle points, yet a comprehensive understanding of these solutions within a higher-dimensional context remains elusive. As a result, our aim is to incorporate the wormhole solution into the framework of String theory, specifically within a ten-dimensional theory.
The most direct way for embedding such wormholes relies on axion fields, which naturally provide the requisite negative Euclidean energy-momentum tensor necessary to generate wormhole geometries (see also Chapter 3). One plausible interpretation of these Giddings-Strominger (GS) wormholes [29] is as instantons describing the birth or absorption of baby universes [17]. Since these baby universes carry a so called "axion charge", this could seemingly violate axion charge conservation, a phenomenon expected given that axion shifts are global symmetries. In this context, wormholes could serve as instantons generating Planck-suppressed potential terms for the axion [30]. Giddings-Strominger wormholes have been moreover interesting object when talking about the "axion-version" of the weak gravity conjecture. Therefore, we notice that the GS Euclidean wormhole solution pops up in several much recent topics of research and has been the main interest of many present-day works (just to mention some, [27], [31], [32], [33]). However, the ultimate fate of these wormholes hinges on various factors, including their stability, which, despite numerous papers on the topic, remained unresolved until recently. It has now been demonstrated that GS wormholes, in the absence of a dilaton, are perturbatively stable [31]. The question of wormhole stability in the presence of both axions and dilatons remains an ongoing area of investigation.
Remarkably, it has taken a considerable amount of time to integrate GS-like wormholes into string theory. The first proofs of regular GS wormholes in string theory surfaced in two of the papers this thesis work has been most inspired by, [34] and [35]. Their construction yields regular solutions in four flat Euclidean dimensions and is applicable to numerous compactifications without fluxes, particularly Calabi-Yau compactifications of type II Supergravity. In this work, we try to apply the same reasoning for a compactification on S^6 of massive Type IIA String theory.
In the context of AdS compactifications, the situation becomes considerably more intricate yet equally intriguing, given the availability of the AdS/CFT correspondence. Initial proposals for AdS embeddings of Euclidean wormholes in AdS were made, although these were not of the GS type, meaning they were not sourced by axion charges [36],[37]. Despite the absence of classical wormhole saddles in this scenario, the path integral over wormhole geometries can be performed though presenting many challenges.

Chapter 2

Dimensional Compactification

We give an overview on the process of dimensional reduction and compactification ([3], [38]), methods that will be vastly applied in the next chapters to restrict from ten-dimensional theories, namely massive and massless Type IIA and Type IIB String theories, to four dimensional theories which we will call Supergravities.

In Chapter 2.1 we shortly summarize the uses of dimensional reduction and the basics of dimensional compactification.
With Chapter 2.2 we reproduce the first attempt at dimensional reduction performed by Kaluza and Klein in the Twenties, from five to four dimensions. We refer to [39], [40] and [41] for our calculations.
Finally, in Chapter 2.3 we do the same for Freund-Rubin dimensional compactification, introducing to the appearance of scalar potential in the reduced theory and therefore to flux compactification and moduli stabilization.

2.1 Basics of dimensional compactification

In the pursuit of a unified theory that can reconcile the fundamental forces of nature, physicists have long been drawn to the concept of extra dimensions beyond the familiar three spatial dimensions and one time dimension. Among the various approaches to achieving this unification, Kaluza-Klein and Freund-Rubin compactifications stand as remarkable frameworks. These theories propose the existence of additional compact dimensions, curled up to subatomic scales, which are imperceptible at everyday scales but play a crucial role in shaping the dynamics of the universe.

The Kaluza-Klein compactification [41], formulated by Theodor Kaluza and Oskar Klein in the early 20th century, integrates the gravitational field of general relativity with electromagnetism by introducing an extra spatial dimension. This additional dimension is assumed to be compactified, meaning it forms a closed, compact loop, and its effects manifest themselves as a new gauge field, a massless vector field corresponding to the electromagnetic force.
This is achieved by dividing the higher dimensional space-time into an internal and an external space, assuming we can describe it as the product $M_4 \times Y_n$, where in the specific case of the simplest Kaluza-Klein reduction Y_n is the 1-D circumference S^1. Therefore one expands the EH action background metric as:

$$g_{MN}(x^\mu, y) = \begin{bmatrix} g^{(0)}_{\mu\nu}(x^\mu, y) & 0 \\ 0 & g^{(0)}(x^\mu, y) \end{bmatrix} \tag{2.1}$$

known as **KK background metric**.
Therefore, having an S^1 background metric, we Fourier expand all our fields as

$$\phi(x^\mu, y) = \sum_n \phi_n(x^\mu) e^{\frac{iny}{R}} \tag{2.2}$$

where R is the radius of S^1 and we managed to separate the internal geometry depending part of the fields from the external coordinates depending one. This process is known as **KK expansion**, and in can obviously generalized to S^n using appropriate angular functions.
The last step is assuming our fields do not depend on the y coordinates, therefore eliminating all the "excitations" along S^1 in (2.2). This is why we say we only retain the "zero-modes" of our fields. This is called the **KK ansatz**, and it is applied to (2.1) as well as reported step by step in Chapter 2.2.

However, while the Kaluza-Klein theory (apparently) successfully unified gravity and electromagnetism, it raised an intriguing challenge related to the scalar potential.
The scalar potential problem in Kaluza-Klein compactification therefore arises due to the presence of an additional scalar field originating from the higher-dimensional metric components. This scalar field, often referred to as the "dilaton," interacts with matter in ways that are not directly observed in our four-dimensional spacetime. This raises questions about the stability of the theory and the nature of these scalar interactions. Inconsistencies in the scalar potential can lead to physical effects that contradict experimental observations, demanding a resolution to this problem for the theory to be physically viable.
The Freund-Rubin compactification, introduced later in the mid-20th century by Peter G. O. Freund and Marcel J. Rubin, extends the ideas of compactification to incorporate higher-dimensional theories with non-Abelian gauge fields. However, like the Kaluza-Klein theory, the Freund-Rubin approach also confronts the scalar potential problem as it deals with scalar fields associated with the compactification process.
Indeed, in this process, we take a manifold for d external and (D-d) internal space $M_D = X_d \times Y_{D-d}$. We assume in the theory we have a (D-d)-form. The curvature of the internal space will generate a potential, which will anyway be generically out of control and set the dimensions of the internal geometry to zero or infinity. We rely on the (D-d)-form to generate the desired scalar potential to stabilize the modulus.
In both Kaluza-Klein and Freund-Rubin compactifications, this scalar potential issue represents a significant theoretical challenge. Physicists continue to investigate ways to address this issue, seeking solutions that would harmonize the predictions of these higher-dimensional theories with the observed behavior of the physical universe, through the process of the so called **moduli stabilization**. Approaches such as stabilization mechanisms, modifications to the compactification scheme, and the incorporation of additional fields have been proposed to tackle the scalar potential problem and render these theories more consistent with empirical observations.

2.2 Kaluza-Klein Hilbert-Einstein action reduction

An extensive review of Kaluza and Klein approach to dimensional reduction is given in [42]. For explicit calculations similar to the ones we'll do, check [40] and [39].
We consider a 5-D space $\mathcal{M}$, which is a product of a 4-D space mapped by a Lorentz-signature metric $g_{\mu\nu}$ and S^1, a 1-D sphere with radius R.
A consistent truncation at 0^{th} order of Kaluza-Klein 5D theory needs a non-linearly modified metric of the form [39]:

$$G^{5D}_{MN} = \begin{bmatrix} g^{4D}_{\mu\nu} + e^{2\phi(x)} A_\mu A_\nu & e^{2\phi(x)} A_\mu \\ e^{2\phi(x)} A_\nu & e^{2\phi(x)} \end{bmatrix} \tag{2.3}$$

Hence we have

$$det[G^{5D}_{MN}] = \det[g_{\mu\nu} e^{2\phi(x)}] \tag{2.4}$$

in which we think of $A_\mu(x)$ as a $U(1)$ gauge field and $\phi(x)$ is a scalar field.
The EH action in 5D reads:

$$S^{5D} = \frac{1}{16\pi G'} \int d^5x \sqrt{-|G^{5D}|}\mathcal{R}_{5D} \tag{2.5}$$

where G' is the Newton constant in our 5-dim frame.
First we need to compute the 5-dim Ricci scalar:

$$\mathcal{R}_{5D} = \mathcal{R}_{4D} - 2e^{-\phi}\nabla^2 e^{\phi} - \frac{1}{4}e^{2\phi}F_{\mu\nu}F^{\mu\nu} \tag{2.6}$$

We then have:

$$\begin{aligned} S^{5D} =& \frac{1}{16\pi G'}\int d^5x\sqrt{-g}e^{\phi}\big(\mathcal{R}_{4D} - 2e^{-\phi}\nabla^2 e^{\phi} - \frac{1}{4}e^{2\phi}F_{\mu\nu}F^{\mu\nu}\big) \\ =& \frac{1}{16\pi G'}\int d^5x\sqrt{-g}\big(\mathcal{R}_{4D}^{Jordan} - 2\nabla^2 e^{\phi} - \frac{1}{4}e^{3\phi}F_{\mu\nu}F^{\mu\nu}\big) \end{aligned} \tag{2.7}$$

where $\nabla^2 e^{\phi} = \nabla_\mu \partial^\mu e^{\phi}$ is a total derivative in the complete space and we can cancel it in the 5D integral [40] if ϕ only depends on the four space-time coordinate, since:

$$\int d^4x\sqrt{-g}\nabla^2 e^{\phi} = \int d^4x\partial_\mu(\sqrt{-g}\nabla^\mu e^{\phi}) = 0 \tag{2.8}$$

Now, assuming, as we may when we compactify on S^1, to Fourier expand both scalar and vector fields in the 5^{th} dimension in such a way that we keep in KK ansatz only the 0^{th}, internal coordinate independent excitation, we are able integrate the S^1 volume out obtaining:

$$S^{4D} = \frac{2\pi R}{16\pi G'}\int \sqrt{-g}\big(\mathcal{R}_{4D}^{Jordan} - \frac{1}{4}e^{3\phi}F_{\mu\nu}F^{\mu\nu}\big) \tag{2.9}$$

whence we can extract:

$$G = \frac{G'}{2\pi R} \tag{2.10}$$

suggesting that if R $\ll$ 1 as expected, G is much larger than G'.
We rememeber:

$$\mathcal{R}_{4D}^{\text{Jordan}} = e^{\phi}\mathcal{R}_{4D} \tag{2.11}$$

and we can get back to the usual Einstein-frame action by setting:

$$g_{\mu\nu} = \Omega^2 g^E_{\mu\nu} \tag{2.12}$$

so that:

$$\sqrt{-g}e^{\phi}\mathcal{R}_{4D} = \sqrt{-g^E}\Omega^4 e^{\phi}(\mathcal{R}^E_{4D}\frac{1}{\Omega^2}) \tag{2.13}$$

where we are taking in consideration for the moment only the term in the Weyl transformation of the Ricci scalar we are interested in to find Ω, namely the $\mathcal{R}$ itself proportional one. Later, we'll discuss the full expression of the tranformed Ricci scalar.
This implies:

$$\Omega^2 = e^{-\phi} \tag{2.14}$$

Now, considering that we found that Ω is not a constant, we rewrite the full expression for the conformally rescaled R in D dimensions in the Einstein frame:

$$\mathcal{R}_d = e^{-\frac{2\phi}{2-d}}\big[\tilde{\mathcal{R}}_d - \frac{2(d-1)}{2-d}\tilde{\nabla}^2\phi - \frac{d-1}{d-2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}\phi\big] \tag{2.15}$$

and eliminating the total derivative part and adjusting for D=4, we finally get (rewriting R^E as R as well as the metric):

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big(\mathcal{R}_{4D} - \frac{3}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{4}e^{3\phi}F_{\mu\nu}F^{\mu\nu}\Big) \tag{2.16}$$

since the Weyl rescaling cancels in the EM term (easy looking at transformations for $g^{\mu\nu}$ and $\sqrt{-g}$).

2.3 Freund-Rubin reduction

Freund-Rubin approach to dimensional compactification ([13], [43], [44]) and first attempt to flux compactification is well reviewed in [45], we will only redo the calculations and outline the main steps. The action is a 6-dimensional one, where we take the ansatz:

$$\mathcal{M}_6 = \mathcal{X}_4 \times \mathcal{Y}_2 \tag{2.17}$$

and a metric G_{MN} defined on such a space, $M, N = 0, ..., 9$, and it shall contain the EH term plus a kinetic term for the 2-form F_{MN}:

$$S = \int d^6x\sqrt{-G}(M_{p6}^4\mathcal{R}_6 - M_{p6}^2|F|^2) \tag{2.18}$$

where M_{p6}^2 is the equivalent of the Planck mass squared in 6 dimension, intended as the coupling constant of gravity in such a space.
We call external coordinates x^μ and internal ones y^m and take the metric ansatz:

$$ds^2 = g_{\mu\nu}x^\mu x^\nu + R^2 h_{mn}y^m y^n \tag{2.19}$$

where R is the characteristic dimension associated to internal geometry, is a varying function whose value will be determined by the minimization of the potential and is called "modulus". h_{mn} is the unit volume metric of internal space and one can formally write $\tilde{g}_{mn} = R^2 h_{mn}$. As for the Kaluza-Klein case, all the scalar and vector fields stemming from the reduction are assumed to depend on x^μ solely. We wish to focus on the gravitational term first. In trying to recover a term ascribable to the 4-dimensional EH action as in (2.6), using (2.19) and the imposition of internal-coordinates dependence only, we find that the only non-zero Christoffel terms are (in the introduced index notation):

$$\begin{aligned}
\Gamma^m_{n\mu} &= \frac{1}{2R(x)^2}h^{ml}\partial_\mu(R(x)^2h_{nl}) \\
\Gamma^\mu_{mn} &= -\frac{1}{2}g^{\mu\lambda}\partial_\lambda(R(x)^2h_{mn}) \\
\Gamma^\mu_{\nu\rho} &= \frac{1}{2}g^{\mu\lambda}(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho})
\end{aligned} \tag{2.20}$$

which yield 8 non-vanishing Riemann tensor components with different internal-external coordinate combinations.
In the following we only report the 4 components entering the Ricci tensor and that we'll need for our reduction:

$$\begin{aligned}
R^\sigma_{\mu\nu\rho} &= \partial_\nu\Gamma^\sigma_{\mu\rho} - \partial_\rho\Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\rho} - \Gamma^\sigma_{\rho\lambda}\Gamma^\lambda_{\mu\nu} \\
R^\sigma_{m\nu r} &= -\frac{1}{2}\partial_\sigma(g^{\sigma\lambda}\partial_\lambda(R(x)^2h_{mr})) - \frac{1}{4}\Big[g^{\sigma\pi}(\partial_\nu g_{\lambda\pi} + \partial_\lambda g_{\nu\pi} - \partial_\pi g_{\nu\lambda})g^{\lambda\phi}\partial_\phi(R(x)^2h_{mr}) \\
&\quad - \frac{1}{R(x)^2}g^{\sigma\pi}\partial_\pi(R(x)^2h_{rl})h^{lp}\partial_\nu R(x)^2h_{mp})\Big] \\
R^s_{mnr} &= -\frac{1}{4R(x)^2}[h^{sp}\partial_\lambda(R(x)^2h_{np})g^{\lambda\pi}\partial_\pi(R(x)^2h_{mr}) - h^{sp}\partial_\lambda(R(x)^2h_{rp})g^{\lambda\pi}\partial_\pi(R(x)^2h_{mn})] \\
R^s_{\mu n\rho} &= -\frac{1}{2}\partial_\rho\left(\frac{1}{R(x)^2}h^{sp}\partial_\mu(R(x)^2h_{np})\right) - \frac{1}{4R(x)^2}[h^{sp}\partial_\rho(R(x)^2h_{lp})h^{lf}\partial_\mu(R(x)^2h_{nf}) \\
&\quad - R(x)^2h^{sp}\partial_\lambda(R(x)^2h_{pn})g^{\lambda\pi}(\partial_\mu g_{\rho\pi} + \partial_\rho g_{\mu\pi} - \partial_\pi g_{\mu\rho})]
\end{aligned} \tag{2.21}$$

and defining the Ricci tensor $\mathcal{R}_{MN} = \mathcal{R}^S_{MSN}$, we will find the surviving terms of such a tensor are:

$$\begin{aligned}
\mathcal{R}_{\mu\nu} &= \mathcal{R}^\sigma_{\mu\sigma\nu} + \mathcal{R}^s_{\mu s\nu} = \mathcal{R}^4_{\mu\nu} + ... \\
\mathcal{R}_{mn} &= \mathcal{R}^\sigma_{m\sigma n} + \mathcal{R}^s_{msn}
\end{aligned} \tag{2.22}$$

such that the effective Ricci scalar is:

$$\begin{aligned}\mathcal{R} = g^{MN}\mathcal{R}_{MN} &= g^{\mu\nu}\mathcal{R}_{\mu\nu} + \frac{h^{mn}}{R^2}\mathcal{R}_{mn} \\ &= \mathcal{R}_4 + ...\end{aligned} \tag{2.23}$$

so that one can see that we will recover a term:

$$M_{p6}^2 R^2 \int d^6x \sqrt{-g}\sqrt{h} R^2 \mathcal{R}_4 \tag{2.24}$$

and assuming again no dependence of the fields on y^m and remembering we normalized the volume of the internal geometry, we can write $M_{p6}^4 R^2$ as the Planck mass squared in 4 dimensions, M_p^2, thus yielding a total reduced action:

$$S = M_p^2 \int d^4x \sqrt{-g}\Big(R^2\mathcal{R}_4 + \Big[\int d^2\sqrt{h}\mathcal{R}_2\Big]\Big) + ... \tag{2.25}$$

where one can show that the formal second term is a topological invariant and is equal to (2-2g), a function of the genus g ("number of holes" of the internal manifold, so g=1 for a torus, 0 for a sphere, ...) of the internal manifold.
The dots ... stand for higher orders in derivative terms of $R(x)$ and we neglect them.
We see that in order to go back to the Einstein frame we need to introduce a term Ω^2, just as in (2.12), which implies:

$$\sqrt{-g}\mathcal{R}_4 = \sqrt{-g^E}\Omega^4 R^2(\mathcal{R}_4^E \frac{1}{\Omega^2}) \quad \Rightarrow \quad \Omega^2 = R^{-2} \tag{2.26}$$

so that passing from $g \to g^E = R^2 g$, we will eventually get the 10-dimensional gravitational action gives in the 4 dimensions:

$$S_R = M_{p4}^2 \int d^4x \sqrt{-\tilde{g}}[\mathcal{R}_4 - V(R)] \tag{2.27}$$

where $V(R)$ is a potential setting the value of the modulus $R(x)$.

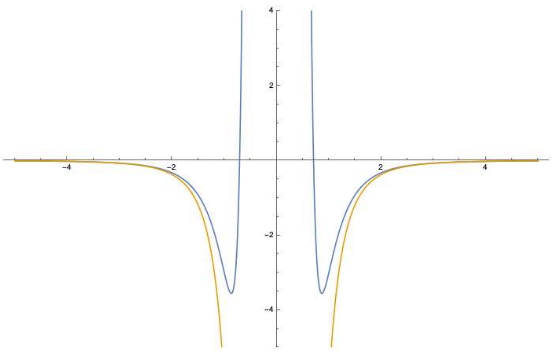

Figure 2.1: Graphical representation of the way flux compactification modifies the moduli potential and stabilizes the dimension of the compactified geometry to a finite value. In blue we draw the potential after compactifying the fluxes, in yellow we draw the potential with no fluxes.

After passing to the Einstein frame, the potential takes the form:

$$V(R) = \frac{(2g-2)}{R^4} \tag{2.28}$$

where it is clear the factor $1/R^4$ comes from the factor Ω^4 in the determinant.
Therefore, because of minimization of the potential, for a torus or a sphere, the value of R will tend to 0 and the coupling in 6 dimensions would become infinitely strong, thus busting our low-energy approximation. On the other hand, for an internal geometry with genus > 2, R would tend to ∞, thus giving too large internal dimensions.
We use then the 2-form, switching it on only in the internal space, not to spoil the external space isometries. We thus recover the magnetic charge of the flux F as:

$$\int_{\mathcal{Y}_2} F = n \tag{2.29}$$

where n is an integer after charge quantization. Thus, introducing (2.29) in (2.18), we recover a term which will have many $\propto$R terms, namely:

- a factor R^2 from the determinant of the metric in the Jordan frame;
- a factor R^{-4} from the internal terms of the metric contracting F_{mn};
- a factor R^{-4} coming from going to the Einstein frame.

Altogether, this gives a factor proportional to n^2/R^6, which we insert in the previously derived potential (2.28), so to get

$$V(R) \propto \frac{(2g-2)}{R^4} + k\frac{n^2}{R^6} \tag{2.30}$$

where k is an appropriate constant. As we can see, once one adjusts the coupling constants, it can yield a prediction for the v.e.v. of our theory that could in principle be correct. This of course requires our internal geometry to have $g < 2$.
In this sense, we can predict a way to justify compactification of internal dimensions, interpreting such a "strange" way physics chooses to behave as a result of energy minimization with a scalar field potential in the presence of magnetic fields named fluxes. This **"flux compactification"** thus is identified as a way to **stabilize the moduli** appearing when compactifying.

The issue with this solution is that the minimized potential introduces a cosmological constant inside the action, which turns out to be

$$V_0 = \Lambda = -O(1) \times \frac{1}{R} \tag{2.31}$$

thus yielding unacceptable values once we compare to data. Such a small measured value of Λ would in fact set $R \to \infty$, spoiling our assumption of dimensional compactification and yielding an almost flat 6-dimensional space-time.

Chapter 3

Giddings-Strominger Wormholes

Giddings-Strominger wormholes are wormhole solutions named after physicists Steven Giddings and Andrew Strominger, who proposed the idea in a paper titled "Axion Induced Topology Change in Quantum Gravity and String Theory" [29] in 1988.

The Giddings-Strominger wormholes [46], [47] are distinct, for instance, from Einstein-Rosen closed wormhole solutions in that they involve certain exotic properties and arise in the context of string theory, thus linking directly to quantum gravity. These wormholes are often associated with topological changes induced by axions, which are hypothetical particles that arise in some extensions of the Standard Model of particle physics. In a particle physics frame, they were introduced by means of an anomalous symmetry of the Lagrangian called Peccei-Quin symmetry, while in a cosmological context they are one of the candidates for Dark Matter.
The Giddings-Strominger wormholes are intriguing because they could potentially allow for a kind of "topology change" in spacetime, meaning that the structure of spacetime itself could undergo a transformation [48].

3.1 Action and equations of motion

If one takes an action in $d \geq 3$ with a non-null cosmological constant [35]:

$$S = \frac{1}{2k_d^2} \int \left(*(\mathcal{R}_d - 2\Lambda) - \frac{1}{2} G_{ij}(\varphi) d\varphi^i \wedge *d\varphi^j \right) \tag{3.1}$$

where ' stand for radial derivative and

- $\mathcal{R}_d$ is the d-dimensional Ricci scalar;
- k_d is the d-dimensional gravitational constant;
- Λ is a gravitational constant;
- G_{ij} is the target space metric defined on the space parametrized by scalars $\varphi_{i,j}$

one can find a spherically-symmetric solution being:

$$\begin{aligned}
&\mathrm{ds}^2 = f(r)^2 dr^2 + a(r)^2 d\Omega_{d-1}^2 \\
&\left(\frac{a'}{f}\right)^2 = 1 + \frac{a^2}{l^2} + \frac{\mathrm{c}}{2(d-1)(d-2)a^{2d-4}} \\
&\mathrm{c} = G_{ij}(\varphi) \frac{d\varphi^i}{dh} \frac{d\varphi^j}{dh} = G_{ij}(\varphi) \varphi^{i'} \varphi^{j'} \frac{a(r)^{2d-2}}{f(r)^2}
\end{aligned} \tag{3.2}$$

where $h(r)$ is a harmonic function such that $h' = \frac{f}{a^{d-1}}$, c is the geodesic velocity on the target space and h acts as an affine parameter.
The Cosmological Constant is defined as a function of the AdS length l as:

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2} \tag{3.3}$$

The fields φ^i are clearly only r-dependent, since we are considering a spherically-symmetric solution.

We checked the consistency of such a solution in D=4, since it is a useful generalization of the calculations done in Chapter 4 for the specific case of D=4 Supergravity with $\Lambda \equiv 0$ arising from dimensional reduction of Type IIA String Theory on T^6.
The Einstein equation reads

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = G_{ij}(\varphi)\varphi'^i\varphi'^j - \frac{1}{2}g_{\mu\nu}G_{ij}(\varphi)\partial_\rho\varphi^i\partial^\rho\varphi^j \tag{3.4}$$

while the equations of motion for the scalar φ^i is

$$(\partial_i G_{ij})\partial_\mu\varphi^i\partial^\mu\varphi^j - \frac{1}{\sqrt{g}}\Big(\partial_\mu(\sqrt{g}g^{\mu\nu}G_{ij}\partial_\nu\varphi^j)_{i\neq j} + 2\partial_\mu(\sqrt{g}g^{\mu\nu}G_{ii}\partial_\nu\varphi^i)\Big) = 0 \tag{3.5}$$

and we are going to leave their solution to the specific definitions of the target space metric $G_{ij}(\varphi)$ we have case by case.
We find the only non-null Ricci tensor components to be:

$$\begin{cases} R_{rr} = 3\frac{a'(r)f'(r)-f(r)a''(r)}{a(r)f(r)} \\ R_{\theta\theta} = 2 + \frac{a(r)a'(r)f'(r)-f(r)(2a'(r)^2+a(r)a''(r))}{f(r)^3} \\ R_{\phi\phi} = \sin^2(\theta)\Big(2 + \frac{a(r)a'(r)f'(r)-f(r)(2a'(r)^2+a(r)a''(r))}{f(r)^3}\Big) \\ R_{\eta\eta} = \sin^2(\theta)\sin^2(\phi)\Big(2 + \frac{a(r)a'(r)f'(r)-f(r)(2a'(r)^2+a(r)a''(r))}{f(r)^3}\Big) \end{cases} \tag{3.6}$$

while the Ricci scalar $\mathcal{R}$ turns out to be:

$$\mathcal{R} = \frac{6[f(r)^3 + a(r)a'(r)f'(r) - f(r)(a'(r)^2 + a(r)a''(r))]}{a(r)^2 f(r)^3} \tag{3.7}$$

so that the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}$ can be defined as:

$$G_{\mu\nu} = \begin{bmatrix} -3\frac{f^2-a'^2}{a^2} & 0 & 0 & 0 \\ 0 & -1 - \frac{2a(r)a'(r)f'(r)-f(r)(a'(r)^2+2a(r)a''(r))}{f(r)^3} & 0 & 0 \\ 0 & 0 & \sin^2(\theta)G_{\theta\theta} & 0 \\ 0 & 0 & 0 & \sin^2(\theta)\sin^2(\phi)G_{\theta\theta} \end{bmatrix} \tag{3.8}$$

First we consider a vacuum solution with no scalar field in the (rr) component of the stress-energy tensor. As we can see, introducing a Cosmological Constant Λ as defined in (3.3) yields a solution of the type

$$-3\frac{f(r)^2 - a'(r)^2}{a(r)^2} - \frac{3f(r)^2}{l^2} = 0 \quad \Longrightarrow \quad \left(\frac{a'(r)}{f(r)}\right)^2 = 1 + \frac{a(r)^2}{l^2} \tag{3.9}$$

as in (3.2), apart from the field-dependent term.
Condition (3.9) implies $a''(r) = \frac{1}{a'(r)}\Big(f'(r)f(r)(1+\frac{a(r)^2}{l^2}) + \frac{a(r)a'(r)f(r)^2}{l^2}\Big)$ and one sees the equation satisfies the request for the three remaining components as well.
The generalization to a theory with scalar fields as in (3.5) is then straightforward once we remember

they only depend on the radial coordinate.
Indeed, once again taking in consideration just the (rr) component for simplicity, we obtain a term:

$$G_{ij}(\varphi)\varphi'^i\varphi'^j - \frac{1}{2}f(r)^2 G_{ij}(\varphi)\partial_r\varphi^i\partial^r\varphi^j = \frac{1}{2}G_{ij}(\varphi)\varphi'^i\varphi'^j \tag{3.10}$$

If we put together the solution for this "reduced" Einstein equation with (3.9), we obtain the complte result:

$$\left(\frac{a'(r)}{f(r)}\right)^2 = \left(1 + \frac{a(r)^2}{l^2} + \frac{G_{ij}\varphi'^i\varphi'^j a(r)^2}{6f(r)^2}\right) \tag{3.11}$$

Upon iserting the third of (3.2) into the second one, we obtain back the result.

3.2 Regularity of the solutions

The following discussion is based on and expands the results of [47].

We notice that $a(r)$ can be identified as the radius of the hyper-spheric S^{d-1} section of space-time at a certain radius r from the origin, thus to find whether the solution is an open wormhole geometry one needs to check $a(r=0) = a_0$ is bigger than 0.

Wormholes are defined to be c ≥ 0 "time-like" geodesics, but in full generality the metric signature is not defined, and one needs to check that regularity of the solution implies the statement above.
Indeed, in order to find regular wormhole metric solutions one needs to fix c$\geq$ 0, since the geodesic equation imposes, when used in the $a(r)$ e.o.m. for the scalar terms in the metric in (3.2):

$$a'^2 - 1 - \frac{\mathrm{c}}{2(d-1)(d-2)a^{2d-4}} = 0 \tag{3.12}$$

and therefore, setting

$$\mathrm{c} = -2(d-1)(d-2)a_0^{2d-4} \leq 0 \tag{3.13}$$

one gets a regular solution with $a(r) \to r^2$ for $r \to \pm\infty$ and $a(r=0) = a_0$.
If we chose c $= 0$, we would recover $a(r) = r$, thus an extremal instanton metric with flat Euclidean space, while setting $c \geq 0$ we would end up with a singular solution, as a' would diverge when $a \to 0$.

Thus, scalars travel along "time-like" geodesics in moduli space, but in order to find regular solutions for the fields, one needs to impose boundary conditions on the extrema of the fields.

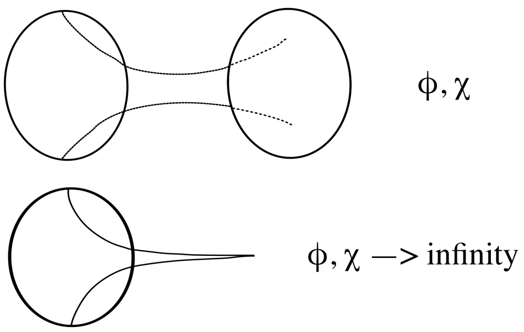

Figure 3.1: Mechanism keeping the wormhole stably open of closed, depending on the values the fields take in the space-time geometry.

We will first try to explain this simply and intuitively. We need to check whether the fields are regular along the trajectories they take within the target space due to the curvature of the physical space-time. In order to do this, we check the different values they take, being only interested in seeing whether they diverge for a space-time coordinate value. If this happens, as one can see from (3.2), the derivative of the fields diverges and so does the metric solution, yielding a metric that looks more like the one of a singularity than a smooth, traversable wormhole. Therefore, we need to check whether the geometry of the entire target space, encoded in the metric $G_{ij}(\varphi)$, allows us to have a "time-like" (it would be more sensible saying "axion-like") geodesic along which we can freely move, that is longer than the one treaded by our fields, i.e. the range of values they take. In this way, the fields will vary along what will be perceived by the observer "moving" in the target space as "time-like" trajectories and one will be able to probe their

variation step by step without "losing track" of a too fast soaring in their values.
This approach is particularly likeable because this request of "time-like"ness for geodesics in the target space to have a traversable wormhole somehow mirrors the request of being able to follow a (proper) time-like geodesic in space-time without falling in the case of a closed Schwarzschild-like wormhole geometry.

Passing to the matemathical side of the question, if one calls the space-time boundaries values of a field $\varphi_{\pm\infty}$ and its value at the neck φ_0, one can find that the distance between the extramal values in the target space along the geodesic is, using flat (3.2):

$$\begin{aligned} D_d = d[\varphi_{-\infty}, \varphi_{+\infty}] = 2d[\varphi_{-\infty}, \varphi_0] =& 2\int_{-\infty}^{0} dr\, \frac{\sqrt{|c|}}{a(r)} \\ =& \pi\sqrt{\frac{2(d-1)}{d-2}} \end{aligned} \tag{3.14}$$

and thus, to have bounded scalars, one needs to find a compact time-like geodesics in the moduli space geometry that is longer than the one in (3.14). All these considerations were made in a flat space ($l \to \infty$). Adding a cosmological constant Λ implies trading the integration constant r with h defined as above, and gives a dependence to the solution in (3.14) on the relation $\frac{a_0}{l}$. This relates the AdS scale and the width of the wormhole mouth, and one finds

$$\pi\sqrt{\frac{2(d-1)}{d-2}} = D_d(0) \geq D_d(\frac{a_0}{l}) \geq D_d(\infty) = \pi\sqrt{\frac{2(d-2)}{d-1}} \tag{3.15}$$

where we relax our condition in the case of a much curved background ($a_0 >> l$).

In an axion-only theory, one would find

$$ds^2 = -da^2 \tag{3.16}$$

and thus it would always be possible to find a fitting time-like geodesic due to the "wrong" sign for axions. Anyway, in a compactification of String Theory one always finds a dilaton joining the axion in such a way that

$$ds^2 = d\varphi^2 - e^{2\varphi} da^2 \tag{3.17}$$

and generalizing the factor multiplying the dilaton in the exponential back to β as in (3.1), one finds the maximal time-like geodesic length to be $\frac{2\pi}{\beta}$, so that one needs to require:

$$\frac{1}{\beta^2} > \frac{d-1}{2(d-2)} \tag{3.18}$$

In case of multiple axio-dilaton pairs, the constraint is relaxed and one can follow a "diagonal" path in the target space, finding the condition:

$$\sum_i \frac{1}{\beta_i^2} > \frac{d-1}{2(d-2)} \tag{3.19}$$

One can in principle have multiple axio-dilaton pairs, as emerging from several scalar components of compactified String Theory fields. This is what we are going to test in the next chapter.

Chapter 4

Reduction from a 10-dimensional IIA theory to a scalar 4d theory

What we wish to achieve in this chapter is truncating the reduced spectrum of massless Type IIA String theory to a four-dimensional Supergravity where we can recover a Giddings-Strominger wormhole solution for space-time geometry. We subsequently want to use the ansatz we imposed for dimensional reduction to build back our ten-dimensional string theory where we fixed the form of our fields from imposing the wormhole solution. This will yield the embedding of our Supergravity theory in a higher dimensional high-energy frame, thus justifying our geometry from a String point of view.

In Chapter 4.1 we introduce the action of the truncated massless Type IIA theory using an ansatz (see Appendix C to check the consistency of such a truncation and of the ansatz) and find the corresponding four-dimensional theory by dimensional compactification with Kaluza-Klein null modes. In doing this, we follow the steps of [35].
Chapter 4.2 deals with handling the scalar content of the theory in order to find a wormhole solution in Einstein equations. Inherently, we re-uplift our theory to the ten dimensions performing an embedding.
In Chapter 4.3 we check that this wormhole solution is indeed an open Giddings-Strominger solution by means of the criteria given in Chapter 2. We use equations of motion and regularity condition.
Finally, we conclude with Chapter 4.4, where we briefly introduce a Type IIB dimensional compactification to a Giddings-Strominger solution and introduce a method we will use later to check asymptotically the possibility of having axio-dilaton wormhole solutions in a certain frame.

4.1 Truncation of massless Type IIA String Theory

In the following, we will refer to the dimensional compactification studied in [35] and based on the previous work of [49].
The action we start with is a 10-dimensional one including EH term, a dilaton and the RR 3-form C_3 field strength.
The action is

$$\frac{1}{2k_{10}^2}\int d^{10}x\sqrt{-g}\Big(\mathcal{R}-\frac{1}{2}(\partial\phi)^2-\frac{1}{2*4!}e^{\phi/2}F_4^2\Big) \tag{4.1}$$

and we use it to describe D2-branes.
The aim is to reduce it dimensionally on a $\mathcal{T}^6$ to a 4-dimensional theory.
We use three ansatz to simplify our calculations in the reduction:
• The metric is written in a block form as

$$ds_{10}^2 = e^{2a\varphi}ds_4^2 + e^{2b\varphi}\mathcal{M}_{\alpha\beta}d\theta^\alpha d\theta^\beta \tag{4.2}$$

• We neglect all the vector and tensor fields coming from the 10-dimensional C_3 and only keep 4 of the 15 scalar fields and we identify the remaining fields as 4 axions. Namely we define

$$C_3 = \chi_1 d\theta^1 \wedge d\theta^2 \wedge d\theta^3 + \chi_2 d\theta^1 \wedge d\theta^4 \wedge d\theta^5 + \chi_3 d\theta^2 \wedge d\theta^5 \wedge d\theta^6 + \chi_4 d\theta^3 \wedge d\theta^4 \wedge d\theta^6 \tag{4.3}$$

We need the matrix $\mathcal{M}_{\alpha\beta}$ to describe the torus metric, as much as to have unit determinant, be positive and symmetric.
We consider a diagonal torus metric , writing $\mathcal{M} = diag(e^{\vec{\beta_1}\cdot\vec{\Phi}}, ...)$.
• We consider all fields to depend only on the main space coordinates, using Kaluza-Klein ansatz extended to the torus.
For a check on the consistency of this ansatz in the 10D frame and for massless Type IIA equations of motion refer to Appendix C.
First we need to rewrite the 10D scalar curvature $\hat{\mathcal{R}}$ as a sum of terms containing the 4-D $\mathcal{R}$ and the scalar fields contained within the metric.
For such a purpose we used the RGTC Mathematica package to calculate useful GR quantities and got from the metric of interest

$$\begin{aligned}\hat{\mathcal{R}} =& e^{-2a\varphi}\Bigg(\mathcal{R}_4 + \frac{1}{2}[-12(a^2+4ab+7b^2)(\partial\varphi)^2 - 2(2a+7b)((\sum_{i=1,6}\beta_i)\partial_\mu\varphi\partial^\mu\Phi - \\ & [\sum_{i,j=1,6,j\neq i}(\beta_i^2+\frac{1}{2}\beta_i\beta_j)]\sum_{a=1,5}(\partial\Phi_a)^2 - (12a+24b)(\partial^2\varphi) - 2\sum_{1=1,6}\beta_i(\partial^2\Phi)\Bigg)\end{aligned} \tag{4.4}$$

Neglecting the ∂^2 terms which are total derivatives, we observe that the mixed term vanishes for unitarity of $\mathcal{M}$ and we can get a standard kinetic term for the radion by setting

$$a = \frac{\sqrt{3}}{4} \tag{4.5}$$

$$a = -3b \tag{4.6}$$

and by normalizing the β_i internal products to

$$(\beta_i \cdot \beta_j) = 2\delta_{ij} - \frac{1}{3} \tag{4.7}$$

we obtain a canonical term for the Φ's as well.
Now, we move to the RR sector 3-form field strength.
Applying the exterior derivative to our previously defined 3-form, keeping in mind that the axion fields only depend on the 4d coordinates, we obtain the desired 4-form F_4, such that when contracting the indices with the already mentioned assumed metric, we obtain a term

$$\begin{aligned}&(4!)e^{-2a\varphi-6b\varphi+\phi/2}\Big[e^{-(\vec{\beta_1}+\vec{\beta_2}+\vec{\beta_3})\cdot\vec{\Phi}}(\partial\chi_1)^2 + e^{-(\vec{\beta_1}+\vec{\beta_4}+\vec{\beta_5})\cdot\vec{\Phi}}(\partial\chi_2)^2 + \\ & e^{-(\vec{\beta_2}+\vec{\beta_5}+\vec{\beta_6})\cdot\vec{\Phi}}(\partial\chi_3)^2 + e^{-(\vec{\beta_3}+\vec{\beta_4}+\vec{\beta_6})\cdot\vec{\Phi}}(\partial\chi_4)^2\Big]\end{aligned} \tag{4.8}$$

which is the kinetic term for axions.
Now we notice that the usual metric term in the action, switching from a 10d theory to a 4d one, changes only by a total exponential factor, namely

$$\sqrt{-g_{10}} = \sqrt{-g_4}[e^{4a\varphi+6b\varphi+\sum_{i=1,6}\vec{\beta_i}}] \tag{4.9}$$

and for unitarity of the $\mathcal{M}$ matrix and from the choices we made for a and b, this amounts to

$$\sqrt{g_4}e^{2a\varphi} \tag{4.10}$$

Thanks to our choices we can write the action directly in the Einstein frame and we get a final

$$S_{4d} = \frac{\text{Vol}(\mathcal{T}^6)}{2k_{10}^2} \int d^4x\sqrt{-g_4}\Big(\mathcal{R}_4 - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\sum_{a=1,5}(\partial\Phi_a)^2 + \mathcal{L}_{\text{axions}}\Big) \tag{4.11}$$

where

$$\begin{aligned}\mathcal{L}_{\text{axions}} = &-\frac{1}{2}(e^{\phi/2-6b\varphi})(e^{-(\vec\beta_1+\vec\beta_2+\vec\beta_3)\cdot\vec\Phi}(\partial\chi_1)^2 + e^{-(\vec\beta_1+\vec\beta_4+\vec\beta_5)\cdot\vec\Phi}(\partial\chi_2)^2\\ &+ e^{-(\vec\beta_2+\vec\beta_5+\vec\beta_6)\cdot\vec\Phi}(\partial\chi_3)^2 + e^{-(\vec\beta_3+\vec\beta_4+\vec\beta_6)\cdot\vec\Phi}(\partial\chi_4)^2)\end{aligned} \tag{4.12}$$

4.2 Wormhole solution in 4 dimensions

Now we need to solve the equations of motion for a wormhole geometry with the reduced 4 dimensional action.
We thus initially require an instanton metric of the most general form

$$ds^2 = e^{2B(r)}(dr^2 + r^2 d\Omega_3^2) \tag{4.13}$$

with fields as specified in the action only dependend on the radial coordinate.
We first simplify the metric requiring a unique axion coupling constant and therefore reduce to one effective axio-dilaton pair. In order to do this we simply consistently set $\vec\Phi = \vec 0$ and by setting $\chi_i = \frac{1}{2}\chi$ we derive an effective axion Lagrangian of the form

$$\mathcal{L}_{\text{axion}} = -\frac{1}{2}e^{\phi/2-6b\varphi}(\partial\chi)^2 \tag{4.14}$$

By orthogonality condition in the target space, we would recover an exponential term where the dilaton is multiplied by a constant $\beta = 1$, thus setting a definite axion charge. In full generality, here we apply the same reasoning but we leave a free parameter b standing for the effective axion coupling. With this choice we recover a simpler expression just in terms of the dilaton field:

$$\mathcal{L}_{\text{axion}} = -\frac{1}{2}e^{b\phi}(\partial\chi)^2 \tag{4.15}$$

With the assumption made above, we finally end up with an effective Lorentzian axio-dilaton-gravity coupled action

$$S = \frac{1}{2k_4^2}\int d^4x\sqrt{|g_4|}\big(\mathcal{R}_4 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{b\phi}(\partial\chi)^2\big) \tag{4.16}$$

Now that we have the desired action, we Wick-rotate the action to obtain the Euclidean-action frame, resulting in the rotation of the axion field $\chi \to -i\chi$ and we can write the 4-dimensional equations of motion as

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - e^{b\phi}\partial_\mu\chi\partial_\nu\chi\\ 0 &= \partial_\mu(\sqrt{g}g^{\mu\nu}e^{b\phi}\partial_\nu\chi)\\ 0 &= \frac{b}{2}e^{b\phi}(\partial\chi)^2 + \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi)\end{aligned} \tag{4.17}$$

We first try and solve the equations using coefficients derived from the conserved charges of the $SL(2,R)$ symmetry the Lagrangian enjoys (this is made clear writing the action in a Scalar-Matrix formalism, highlighting the geometry of the target space), following the approach of [49] and [34]. Taking the value $\tilde q$ as a proper combination of the three conserved charges we find the solutions of (4.17) as

$$\begin{aligned} ds^2 &= \left(1 + \frac{\tilde q^2}{r^4}\right)(dr^2 + r^2 d\Omega_3^2)\\ e^{b\phi(r)} &= \left(\frac{q_-}{\tilde q}\sin\left(bc\arctan\left(\frac{\tilde q}{r^2}\right) + C_1\right)\right)^2\\ \chi(r) &= \frac{2}{bq_-}\left(\tilde q\cot\left(bc\arctan\left(\frac{q}{r^2}\right) + C_1\right) - q_3\right)\end{aligned} \tag{4.18}$$

where c is given for a n-dimensional theory as $c = \sqrt{\frac{2(N-1)}{D-2}}$, which in our case is equal to $\sqrt{3}$, while C_1 is a constant of integration setting the value of the dilaton field at infinite radius.
To verify this choice of the metric solves the equations of motion we report the explicit form of the Riemann tensor:

$$R_{\mu\nu} = \begin{bmatrix} \frac{24\tilde{q}r^2}{(-q_3^2+q_-q_++r^4)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.19}$$

and as one can see, the tensor only has a non-null (rr) component, which is what we desired for a theory with only radius-dependent fields. Indeed, the tensor arising from the energy-momentum tensor in the first equation in (4.17) only involves radial derivatives of the fields and thus only has a non-null (rr) component as well, which turns out to satisfy Einsten.
For what concerns the equations of motion of the fields, the forms given in (4.18) satisfy our bound and are thus solutions of the system.
We need to impose some costraints to our solutions in order for them to be regular. We indeed notice that, for $bc > 2$, the cotangent defining the prospect of $\chi(r)$ field explodes, as the arcotangent reaches the value of $\pi/2$. Therefore, we need to set $bc < 2$, which we have in four dimensions for $b < \frac{2}{\sqrt{3}}$.
We notice that the metric we have as a solution of Einstein in (4.17) has the features we require for our wormhole solutions:

- it is symmetric under the reflection $r^2 \to \frac{q}{r^2}$,
- we have a fixed point in this reflection, namely

$$r_{SD}^2 = q \tag{4.20}$$

  defined as the self-dual radius,

- we can define a finite thickness of this throat as

$$\rho_{SD}^2 = 2q \tag{4.21}$$

  taken as the value of the minimum radius squared of the 3-dimensional "angular" section in Euclidean space-time.

We can thus give a visual description of the system just described (in a 3-dimensional projection) as:

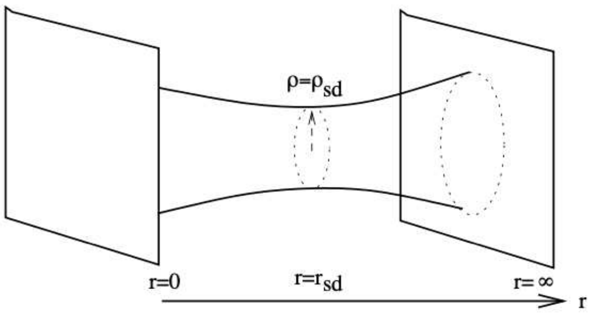

Figure 4.1: [49] Wormhole solution (4.18) as envisioned in a 3-dimensional projection of the solution.

Rewriting the constants in these expressions in a less explicit, at least in terms of the symmetries of the theory, but more direct formalism employed in [35], using the radius of the wormhole neck a_0, we can then consistently uplift this wormhole solution to:

$$\begin{aligned}
ds^2 &= A^2 \sin^2\left[\sqrt{3}\arctan\left(\frac{a_0^2}{2r^2}\right) + C_1\right]\left(1+\frac{a_0^4}{4r^4}\right)(dr^2 + r^2 d\Omega_3^2) + \sum_{i=1}^{6} d\theta_i^2 \\
e^{2\phi} &= A^2 \sin^2\left[\sqrt{3}\arctan\left(\frac{a_0^2}{2r^2}\right) + C_1\right] \\
C_3 &= \frac{i}{A}\cot\left[\sqrt{3}\arctan\left(\frac{a_0^2}{2r^2}\right) + C_1\right](d\theta_{123} + d\theta_{145} + d\theta_{256} + d\theta_{346})
\end{aligned} \tag{4.22}$$

where we can identify the constant A with $\frac{q_-}{\tilde{q}}$ by comparing these solutions to (4.18) and remembering the reduction ansatz (4.2) and the structure of the three-form C_3 as (4.3). For simplicity of notation we wrote:

$$d\theta_{ijk} = d\theta^i \wedge d\theta^j \wedge d\theta^k \tag{4.23}$$

Therefore, we succesfully managed to uplift the Supergravity theory showing a wormhole geometry solution to Strin theory, finding a proper analytic embedding of the solutions for our metric and scalar fields.

4.3 Regular Giddings-Strominger Wormhole Structure

4.3.1 Equations of motion

We now check that the four-dimensional theory results we just obtained from the compactification of type IIA on T^6 are compatible with those of a four-dimensional Giddings-Strominger Wormhole obtained in Chapter 3 with null Cosmological Constant $\Lambda = 0$.

First, we notice that in the properly truncated action (4.16), the target-space metric appearing in (3.1) is defined as

$$G_{ij}(\phi,\chi) = \begin{bmatrix} G_{\phi\phi} & G_{\phi\chi} \\ G_{\chi\phi} & G_{\chi\chi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -e^{b\phi} \end{bmatrix} \tag{4.24}$$

and, upon setting Λ to zero or equivalently the characteristic AdS length l to ∞, one can easily see that the remaining terms in the equations of motion (3.5) and Einstein equation (3.4) reduce indeed to the case of (4.17).
We can therefore identify the general functions multiplying the radial and angular parts of the Euclideanized metric $f(r)$ and $a(r)$, respectively, with

$$f^2(r) = \left(1 + \frac{\tilde{q}^2}{r^4}\right) \qquad\qquad a^2(r) = \left(1 + \frac{\tilde{q}^2}{r^4}\right) r^2 \tag{4.25}$$

The just identified functions need to satisfy condition (3.2). Let's check this out:
• on the left hand side we have:

$$\left(\frac{a'(r)}{f(r)}\right)^2 = \frac{(\tilde{q}^2 - r^4)^2}{(\tilde{q}^2 + r^4)^2} \tag{4.26}$$

• on the right hand side we find:

$$G_{ij}(\varphi)\varphi'^i \varphi'^j \frac{a(r)^2}{f(r)^2} = (\phi'^2 - e^{b\phi}\chi'^2) r^2 \tag{4.27}$$

and we thus find that all the constraints are indeed satisfied inserting (4.18):

$$\frac{r^2}{12}(\phi'(r)^2 - e^{b\phi}\chi'(r)^2) = -\frac{4c^2\tilde{q}^2 r^4}{3(\tilde{q}^2 + r^4)^2} \tag{4.28}$$

Setting $c = \sqrt{\frac{2(d-1)}{d-2}} = \sqrt{3}$, we see that the expressions (4.18) satisfy the requests (3.2) exactly, and therefore we found a GS Wormhole geometry.

4.3.2 Regularity

In order to check for regularity condition, we need to make sure that (3.19) is satisfied before truncating the different β_is to the single axion charge or equivalently that (3.18) is satisfied for the only effective coupling β.

As anticipated in Chapter 4.2, the effective coupling stemming from our reduction is

$$\beta = 1 \qquad \Rightarrow \qquad \frac{1}{\beta^2} = 1 > \frac{3}{4} \tag{4.29}$$

thus yielding a regular traversable wormhole.
If one wanted to recover the theory with four different axio-dilaton pair to check whether $\sum_i \frac{1}{\beta_i^2} > \frac{3}{4}$, one would find that, upon writing the axion Lagrangian as (4.12) and grouping the seven scalars of the theory $(\varphi, \phi, \vec{\Phi})$ into a 7-vector $\vec{t}$ [35], one would find a Lagrangian written as

$$\mathcal{L}_{\text{axions}} = \frac{1}{2}\sum_{i=1}^{4} e^{\vec{\alpha_i}\vec{t}}(\partial\chi_i)^2 \tag{4.30}$$

where $\vec{\alpha_i}$ are four orthonormalized 7-vectors.
Then what we are left to do is to match the effective dilaton number with the axion number by defining a new basis of scalars

$$s_i = \frac{1}{2}\vec{\alpha_i}\vec{t} \tag{4.31}$$

for $i = 1, ..., 4$ such that the remaining s_5, s_6, s_7 are orthogonal to them by construction and decouple. The final result would give four axio-dilaton pairs, each with $\beta_i = 2$, such that

$$\sum_{i=1}^{4} \frac{1}{\beta_i^2} = 1 > \frac{3}{4} \tag{4.32}$$

and our regularity condition is satisfied.

4.4 Reduction from a 10-dimensional IIB theory to a scalar 4d theory

We studied the reduction of Type IIB String Theory to Euclidean AdS_4 ([35], [50]) in order to recover some consideration that will be very useful when treating the necessary truncation in the case of massive Type IIA in Chapter 7.3.

We want to consider a consistent truncation of Type IIB on $T^{1,1}$. We support our Type IIB theory only with its $F_5 -$ flux. The geometric ansatz leads to

$$\begin{aligned}
&ds_{10}^2 = l^2\left(\frac{dr^2}{1+r^2} + r^2 d\Omega_4^2\right) + l^2(ds_{KE}^2 + \eta^2) \\
&e^{\Phi} = g_s \\
&H_3 = F_1 = F_3 = 0 \\
&F_5 = 4l^4(1 - i*)\text{vol}_{T^{1,1}}
\end{aligned} \tag{4.33}$$

where the i term appears after going to Euclidean signature, while the internal metric is expressed in such a way to highlight the $T^{1,1}$ nature of $U(1)$ fibration over an $S^2 \times S^2$ Kahler-Einstein [51] base.

We now wish to build a 4-dimensional wormhole solution. Therefore, looking at (3.2), we write

$$\begin{aligned}
&ds_{10}^2 = l^2 e^{-\frac{2}{3}(4u-v)}[f(r)^2 dr^2 + a(r)^2 d\Omega_4^2] + l^2(e^{2u} ds_{KE}^2 + e^{2v}\eta^2) \\
&e^{\Phi} = g_s e^{\phi} \\
&B_2 = l^2 g_s^{\frac{1}{2}} b\Phi_2 \\
&C_0 = i g_s^{-1}\chi \\
&C_2 = i l^2 g_s^{-\frac{1}{2}} c\Phi_2 \\
&F_5 = 4l^2(1 - i*)\mathrm{vol}_{T^{1,1}}
\end{aligned} \tag{4.34}$$

where we normalized $|\Phi|^2 = 1$ and set $\phi \to 0$ at the boundary.
From this reduction it is immediately clear which ones among the scalar fields will have a kinetic terms switching sign in the Euclidean.
One can show that the 5-dimensional effective action already in the Euclidean frame turns out to be

$$\begin{aligned}
S_5 =& \frac{1}{2k_5^2}\int\Big(*(\mathcal{R} - \mathcal{V}) - \frac{1}{2}d\phi \wedge *d\phi - \frac{1}{2}e^{-4u-\phi} db \wedge *db \\
&+ \frac{1}{2}e^{2\phi} d\chi \wedge *d\chi + \frac{1}{2}e^{-4u+\phi}(dc - \chi db) \wedge *(dc - \chi db) \\
&- \frac{28}{3} du \wedge *du - \frac{8}{3} du \wedge *dv - \frac{4}{3} dv \wedge *dv\Big)
\end{aligned} \tag{4.35}$$

where $\mathcal{V}$ is a scalar potential written as

$$\mathcal{V} = 2e^{-\frac{8}{3}(4u+v)}(2e^{4u+4v} - 12e^{6u+2v} + 4) \tag{4.36}$$

which we can see has a minimum $\mathcal{V}_0 = -12$ for both the scalar u and v sitting at 0, which yields AdS vacuum configurations.

As we can immediately see, the 5-dimensional action is much more complicated than (4.16), and finding an effective axio-dilaton pair to give regular GS wormhole solutions will indeed turn out to be analytically impossible, because of the dependence of the potential on the fields we would like to keep in order to have the wormhole geometry. If one therefore set these fields to a constant, obtaining a cosmological constant Λ, the possibility to derive the GS geometry would be lost, as one can see from (3.2). On the other hand, if we keep the fields the potential changes the equations of motion and the simple GS wormhole geometry cannot be recovered.

Nevertheless, we can examine some interesting cases.
We use the same strategy that we'll use for the dyonically gauged $ISO(7)$ Supergravity model in Chapter 7.3.
It consists in an asymptotic-case approximation, according to which we set the potential $\mathcal{V}$ to its minimum value, thus recovering a cosmological constant Λ, whose value is $\Lambda = -12$, as was proved earlier. Then, we just retain some of the fields in the action that we interpret as axio-dilaton pairs. This limit case is studied in detail in [35].
In the following, we analyze the possibilities case by case, as a useful exercise for our model:
• **Case 1**: we set b and c to a constant. u and v therefore decouple and we remain with an axio-dilaton pair given by (ϕ, χ). The coupling constant β is equal to 2, therefore the conditions of smoothness for the fields in (3.18) are not satisfied and the wormhole closes up.
• **Case 2**: setting u and v to 0, we keep the fields ϕ and c. Here we have a coupling constant 1, which satisfies the regularity condition, since

$$\frac{1}{\beta^2} = 1 \;\;>\;\; \frac{3}{4} = \left(\frac{D_4(0)}{2\pi}\right)^2 \tag{4.37}$$

Therefore, in this case, we have an open GS wormhole solution.
• **Case 3**: in the third case we switch off the χ field and keep c, while for the "dilaton-like" fields we keep ϕ, u and v.
What we then need to do is to find an effective group of dilaton fields, which will be given by linear combinations of the three previous fields, in order to be able to write $-4u + \phi$ as a single φ_1. We find

$$\varphi_1 = \frac{-4u + \phi}{\sqrt{2}} \qquad \varphi_2 = \frac{4u + \phi}{\sqrt{2}} \qquad \varphi_3 = \frac{4u + 4v}{\sqrt{2}} \tag{4.38}$$

so that the other two linear combinations decouple as we expected and the effective coupling constant turns out to be $\beta = \sqrt{2}$, thus yielding another case in which we can have a smooth GS wormhole limit geometry.

Chapter 5

Massive Type IIA String Theory reduction to maximal $\mathcal{N}$=8, D=4 $ISO(7)_c$ dyonically gauged Supergravity

Maximally gauged Supergravity can usually be extended with the usual symplectic formalism to dyonically gauged $\mathcal{N} = 8$ theories respecting the Supersymmetry and the gauge symmetry invariances. Introducing magnetic fields deforms the covariant derivative to a generic

$$D = d - g(\mathcal{A}^{\Lambda} - c\tilde{\mathcal{A}}_{\Lambda}) \tag{5.1}$$

where c is constant that distinguishes between electric and magnetic coupling constants.
In the ungauged case, one can set $c \to 0$ by a symplectic transformation. When adding gauging and inherently a non trivial covariant derivative (5.1), one breaks the symplectic transformations- invariance of the theory and the frame one works with becomes relevant. Depending on the chosen gauge group then, the theory acquires interesting properties. For instance, chosing the $ISO(7)$ group, one recovers, as we will show, several AdS vacuum solutions. This is not the case for the exclusively electrically gauged theory, which has been studied for instance in [52].

One of the most interesting features of these dyonic Suprgravity theories is they can descend from String theories.
This is the case for a dyonically gauged theories under an $ISO(7) = SO(7) \ltimes \mathbb{R}$ [53], whose symplectic deformation to a dyonic theory has been discussed in [54] and [55].
In the following we derive the work of [56] and [53] to discuss [57], summarizing the process of showing how $ISO(7)$ dyonically gauged $\mathcal{N} = 8$ D=4 Supergravity can be obtained as a consistent truncation of massive Type IIA String theory on the six-sphere S^6.

In Chapter 5.1 we write the massive Type IIA Lagrangian density and derive its equations of motion for scalars and vector fields and the Einstein equation.
In Chapter 5.2 we rewrite the fields of massive Type IIA in such a way to stress their covariance under $SO(1,3) \times SO(6)$ in the bosonic and fermionic case. Then we make the fields compatible with the tensor hierarchy of the 10-dimensional theory by performing a rewriting, indexing them in such a way to make them covariant under $SU(8)$.
In Chapter 5.3 we perform our dimensional reduction, displaying the chosen ansatz for the fields' structure. Then we embed our choice in the 10-dimensional form of the fields to rewrite the internal space components in terms of the quantities appearing in the 4-dimensional Supergravity action.
In Chapter 5.4 we perform a consistency check of the Supersymmetry transformations and Bianchi identities in the 10-dimensional theory. This is done using the ansatz of Chapter 5.3 in SUSY transformations and Bianchi identities of the string theory and matching them with the known $N = 8$, D=4 dyonically gauged $ISO(7)$ theory ones.
In Chapter 5.5 we write the dymensionally reduced Supergravity action and its equations of motion.

5.1 Massive type IIA theory

We can write the general Lagrangian for the bosonic content of massive type IIA theory [58],[59] as:

$$
\begin{aligned}
2\hat{k}^2\mathcal{L}_{10} =& \hat{\mathcal{R}}\hat{\text{vol}}_{10} + \frac{1}{2}d\hat{\phi}\wedge \hat{*}d\hat{\phi} - \frac{1}{2}e^{\frac{3}{2}\hat{\phi}}\hat{F}_{(2)}\wedge\hat{*}\hat{F}_{(2)} \\
&+ \frac{1}{2}e^{-\hat{\phi}}\hat{H}_{(3)}\wedge\hat{*}\hat{H}_{(3)} - \frac{1}{2}e^{\frac{1}{2}\hat{\phi}}\hat{F}_{(4)}\wedge\hat{*}\hat{F}_{(4)} \\
&- \frac{1}{2}(d\hat{A}_{(3)})^2\wedge\hat{B}_{(2)} - \frac{1}{6}md\hat{A}_{(3)}\wedge(\hat{B}_{(2)})^3 - \frac{1}{40}m^2(\hat{B}_{(2)})^5 - \frac{1}{2}m^2e^{\frac{5}{2}\hat{\phi}}\hat{\text{vol}}_{10}
\end{aligned}
\tag{5.2}
$$

where the ˆ symbol is used for 10 dimensional quantities.
As one can see, the content of the action is expressed in terms of:

- $\hat{H}_{(3)}=d\hat{B}_{(2)}$ the field strength tensor of the NS two-form;
- $\hat{F}_{(2)}$ as the field-strength of RR one-form;
- $\hat{F}_{(4)}$ as the field-strength of RR three-form.

The powers of one-forms indicate the degree of the form.
$\hat{k}$ is the gravitational coupling in the 10-dimensional theory and it is given as usual in terms of the string length.
The Romans mass [58] given by the 0-form $\hat{F}_{(0)}$ is identified with m, the magnetic coupling constant.
We have the following equations of motion deriving from (5.2):

$$
\begin{aligned}
&d(e^{\frac{1}{2}\hat{\phi}}\hat{*}\hat{F}_{(4)}) + \hat{H}_{(3)}\wedge\hat{F}_{(4)} = 0 \\
&d(e^{\frac{3}{2}\hat{\phi}}\hat{*}\hat{F}_{(2)}) + e^{\frac{1}{2}\hat{\phi}}\hat{H}_{(3)}\wedge\hat{*}\hat{F}_{(4)} = 0 \\
&d(e^{-\hat{\phi}}\hat{*}\hat{H}_{(3)}) + \frac{1}{2}\hat{F}_{(4)}\wedge\hat{F}_{(4)} + me^{\frac{3}{2}\hat{\phi}}\hat{*}\hat{F}_{(2)} + e^{\frac{1}{2}\hat{\phi}}\hat{F}_{(2)}\wedge\hat{*}\hat{F}_{(4)} = 0 \\
&d\hat{*}d\hat{\phi} + \frac{5}{4}m^2e^{\frac{5}{2}\hat{\phi}}\hat{\text{vol}}_{10} + \frac{3}{4}e^{\frac{3}{2}\hat{\phi}}\hat{F}_{(2)}\wedge\hat{*}\hat{F}_{(2)} + \frac{1}{2}e^{-\hat{\phi}}\hat{H}_{(3)}\wedge\hat{*}\hat{H}_{(3)} + \frac{1}{4}e^{\frac{1}{2}\hat{\phi}}\hat{F}_{(4)}\wedge\hat{*}\hat{F}_{(4)} = 0
\end{aligned}
\tag{5.3}
$$

These are, respectively, the equations of motion for $\hat{A}_{(3)}$, $\hat{A}_{(1)}$, $\hat{B}_{(2)}$ and $\hat{\phi}$.
To express the Einstein equation, we introduce the local indices in 10-dimension M,N = 0,...,9 and find the expression:

$$
\begin{aligned}
\hat{R}_{MN} =& \frac{1}{2}\partial_M\hat{\phi}\partial_N\hat{\phi} + \frac{1}{16}m^2e^{\frac{5}{2}\hat{\phi}}\hat{g}_{MN} + \frac{1}{12}e^{\frac{1}{2}\hat{\phi}}(\hat{F}_{MPQR}\hat{F}_N^{PQR} - \frac{3}{32}\hat{g}_{MN}\hat{F}_{PQRS}\hat{F}^{PQRS}) \\
&+ \frac{1}{4}e^{-\hat{\phi}}(\hat{H}_{MPQ}\hat{H}_N^{PQ} - \frac{1}{12}\hat{g}_{MN}\hat{H}_{PQR}\hat{H}^{PQR}) \\
&+ \frac{1}{2}e^{\frac{3}{2}\hat{\phi}}(\hat{F}_{MP}\hat{F}_N^P - \frac{1}{16}\hat{g}_{MN}\hat{F}_{PQ}\hat{F}^{PQ})
\end{aligned}
\tag{5.4}
$$

5.2 Massive type IIA fields rewriting

The fields in the ten-dimensional frame need first to be written in their explicitly SO(1,3)×SO(6) symmetrical form, in order to stress the degrees of freedom we will need in our 4-dimensional theory. In order to do this, we'll distinguish between the space-time coordinates x^μ and the internal space coordinates y^m.

5.2.1 Bosonic fields

The fields we take in consideration are

- the 10-dimensional metric $d\hat{s}^2_{10}$;
- the RR 1-form $\hat{A}_{(1)}$, depending on the 10 coordinates;
- the RR 3-form $\hat{A}_{(3)}$, depending on the 10 coordinates;
- the NS 2-form $\hat{B}_{(2)}$, depending on the 10 coordinates;
- the NS scalar field, the dilaton $\hat{\phi}$.

We then write an expansion for the just mentioned fields, singling out the internal geometry components of the tensor fields as:

$$\begin{aligned}
\bullet\quad d\hat{s}^2_{10} =&\ \Delta^{-1} ds^2_4 + g_{mn}(dy^m + B^m)(dy^n + B^n)\\
\bullet\quad \hat{A}_{(3)} =&\ \frac{1}{6}A_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho + \frac{1}{2}A_{\mu\nu m}dx^\mu \wedge dx^\nu \wedge (dy^m + B^m)\\
&+ \frac{1}{2}A_{\mu mn}dx^\mu \wedge (dy^m + B^m) \wedge (dy^n + B^n)\\
&+ \frac{1}{6}A_{mnp}(dy^m + B^m) \wedge (dy^n + B^n) \wedge (dy^p + B^p)\\
\bullet\quad \hat{B}_{(2)} =&\ \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu + B_{\mu m}dx^\mu \wedge (dy^m + B^m) + \frac{1}{2}B_{mn}(dy^m + B^m) \wedge (dy^n + B^n)\\
\bullet\quad \hat{A}_{(1)} =&\ A_\mu dx^\mu + A_m(dy^m + B^m)
\end{aligned} \tag{5.5}$$

where

$$\Delta = \sqrt{\frac{det(g_{mn})}{det(\tilde{g}_{mn})}} \tag{5.6}$$

for a background internal metric $\tilde{g}_{mn}$ of S^6.
By dimensional compactification, using the KK ansatz, we then get the 4D Supergravity bosonic content

IIA field	SuGra field(s)	SO(6)
$\hat{ds}^2_{10}$	metric ds^2_4	**1**
	scalars g_{mn}	**21**
$\hat{A}_{(1)}$	scalars A_m	**6**
	vector A_μ	**1**
$\hat{B}_{(2)}$	scalars B_{mn}	**15**
	vectors $B_{\mu m}$	**6**
	two-form $B_{\mu\nu}$	**1**
$\hat{A}_{(3)}$	scalars A_{mnp}	**20**
	vectors $A_{\mu mn}$	**15**
	two-forms $A_{\mu\nu m}$	**6**
	three-form $A_{\mu\nu\rho}$	**1**
$\hat{\phi}$	scalar ϕ	**1**

Table 5.1: Bosonic fields derived in the 4-dimensional Supergravity theory by dimensional reduction of massive Type IIA String theory.

for a total of **1** metric, **21** + **6** + **15** + **20** + **1** = **73** scalar fields, **1** + **6** + **15** = **22** vector fields, **1** + **6** = **7** two-form and **1** three-form field. We will keep fewer fields by exploiting duality conditions binding some degrees of freedom.
It is important to remember that these fields depend both on x^μ and y^m coordinates.

5.2.2 Fermionic fields

In the massive IIA theory we then introduce fermionic fields by requiring supersymmetric invariance of our theory. This introduces the quantities:

- gravitino $\hat{\psi_M}$;
- dilatino $\hat{\lambda}$;
- supersymmetry parameter $\hat{\epsilon}$, appearing in the SuperSymmetry transformations.

These fields are all Majorana.
We wish to rewrite these fields in their components, stressing their $SO(1,3)\times SO(6) \to SO(1,3)\times SU(8)$ invariance, where we expanded the SO(6) invariance of the internal tangent space components of the fermionic fields to the R-symmetry group of maximal D=4, $\mathcal{N}$=8 Supergravity, stressing the covariance under such group of the internal degrees of freedom.
We introduce a rewriting of the fermion fields:

$$\begin{aligned}
\hat{\psi}'_\mu &= \Delta^{-\frac{1}{4}} e^\alpha_\mu (\hat{\psi}_\alpha + \frac{1}{2}\hat{\Gamma}_\alpha \hat{\Gamma}^a \hat{\psi}_a) \\
\hat{\psi}'_a &= \Delta^{-\frac{1}{4}} \hat{\psi}_a \\
\hat{\lambda}' &= \Delta^{-\frac{1}{4}} \hat{\lambda} \\
\hat{\epsilon}' &= \Delta^{\frac{1}{4}} \hat{\epsilon}
\end{aligned} \tag{5.7}$$

where $\alpha = 0, ..., 3$ and $a = 4, ..., 9$ are tangent space indices in the vector representation of SO(1,3) and SO(6), while $\Gamma_{\alpha,a}$ are the 4 and 6-dimensional gamma matrices.

5.2.3 Rewriting of the fields

In order to make the supersymmetry transformations compatible with the tensor hierarchy of D=4 $ISO(7)$ Supergravity (explicitly, the supersymmetric variations for each field in ten dimensions need to contain the same type of fields that the SUSY variations for their $ISO(7)$ theory counterpart do), we need to rearrange some degrees of freedom of the IIA theory. Writing the standard supersymmety variations of massive IIA with its fields in the form (5.5), we end up with equations that in some cases are compatible with the 4D theory, while in some other cases we need to erase some terms which do not appear in our Supergravity theory. In order to do so, we define new vector and tensor fields as combinations of the previously defined ones (in a $SO(6)$ covariant form):

$$\begin{aligned}
&\text{1-forms :} \quad && C^{m8}_\mu = B^m_\mu \qquad C^{78}_\mu = A_\mu \qquad \tilde{C}_{\mu mn} = A_\mu B_{mn} \qquad \tilde{C}_{\mu m7} = B_{\mu m} \\
&\text{2-forms :} \quad && C^8_{\mu\nu m} = -A_{\mu\nu m} + C^{n8}_{[\mu}\tilde{C}_{\nu]nm} + C^{78}_{[\mu}\tilde{C}_{\nu]m7} \qquad C^8_{\mu\nu 7} = -B_{\mu\nu} + C^{m8}_{[\mu}\tilde{C}_{\nu]m7} \\
&\text{3-forms :} \quad && C^{88}_{\mu\nu\rho} = A_{\mu\nu\rho} - C^{m8}_{[\mu}C^{n8}_{\nu}\tilde{C}_{\rho]mn} + C^{m8}_{[\mu}C^{78}_{\nu}\tilde{C}_{\rho]m7} + 3C^{78}_{[\mu}C^8_{\nu\rho]7}
\end{aligned} \tag{5.8}$$

We end up with $SL(6)$ covariant forms that can be regarded as components of $SL(7)$ covariant forms. We therefore derive electric vectors in the $\mathbf{7}'$, magnetic vectors in the **21**, two-forms in the **7** and three forms that are singlets under $SL(7)$. The newly defined tensors satisfy $SL(7)$ covariant supersymmetry variations, compatible with the D=4 ones.
In these transformations we also collect the scalar degrees of freedom in the relevant components of the newly defined generalized vielbein

$$V^{I8}_{AB} = (V^{m8}_{AB}, V^{78}_{AB}) \qquad\qquad \tilde{V}_{IJAB} = (\tilde{V}_{mnAB}, \tilde{V}_{m7AB}) \tag{5.9}$$

where m=1,...,6 are SO(6) indices, I,J=1,...,7 are $SL(7)$ indices and A,B=1,...,8 are $SU(8)$ indices and [AB] parametrizes the **28** of $SU(8)$.

Finally, we derive an expression for the fields that appear in the String theory and are compatible with the structure of the Supergravity:

Fields		$SL(7)$
metric	ds_4^2	$\mathbf{1}$
generalized vielbein	V_{AB}^{I8}, $\tilde{V}_{IJAB}$	$\mathbf{7}' + \mathbf{21}$
vectors	C_μ^{I8}, $\tilde{C}_{\mu IJ}$	$\mathbf{7}' + \mathbf{21}$
two-forms	$C_{\mu\nu I}^{8}$	$\mathbf{7}$
three-form	$C_{\mu\nu\rho}^{88}$	$\mathbf{1}$

Table 5.2: IIA fields in a $SL(7)$ representation

where all the fields depend on (x^μ, y^m) and AB indices run over the **28** representation of $SU(8)$. We also define the fermion fields in a $SU(8)$ covariant form

$$\psi_\mu^A(x, y) \tag{5.10}$$

$$\chi^{ABC}(x, y) \tag{5.11}$$

where clearly the gravitons are in the **8** of $SL(8)$, while the spin-$\frac{1}{2}$ fields $\chi(x, y)$, defined as an appropriate combination of internal-space components of the gravitino and the gaugino, are in the **56** irrepresentation of $SL(8)$.

5.3 The S^6 truncation

The previously defined $SO(1,3)\times SL(7)$-covariant formalism for bosons and $SO(1,3)\times SU(8)$-covariant formalism for fermions turns out to be particularly suitable to truncate type IIA Supergravity down to D=4, $\mathcal{N}$=8 maximal Supergravity in a restricted tensor hierarchy.
The bosonic content of a dyonically $ISO(7)$ gauged 4D, $\mathcal{N}$=8 Supergravity in a $SL(7)$ representation is [53]:

Fields		$SL(7)$
metric	$ds_4^2(x)$	$\mathbf{1}$
coset representatives	$\mathcal{V}^{IJij}(x)$, $\mathcal{V}^{I8ij}(x)$, $\tilde{\mathcal{V}}_{IJ}^{ij}(x)$, $\tilde{\mathcal{V}}_{I8}^{ij}(x)$	$\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7}$
vectors	$\mathcal{A}_\mu^{IJ}(x)$, $\mathcal{A}_\mu^{I}(x)$, $\tilde{\mathcal{A}}_{\mu IJ}(x)$, $\tilde{\mathcal{A}}_{\mu I}(x)$	$\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7}$
two-forms	$\mathcal{B}_{\mu\nu I}^{J}(x)$, $\mathcal{B}_{\mu\nu}^{I}(x)$	$\mathbf{48} + \mathbf{7}'$
three-forms	$\mathcal{C}_{\mu\nu\rho}^{IJ}(x)$	$\mathbf{28}'$

Table 5.3: Field content of a D=4, $\mathcal{N}$=8 dyonically gauged Supergravity in $SL(7)$-covariant representation.

where i,j parametrize the D=4 $SU(8)$ representation.
For the fermions, in an $SU(8)$ covariant formalism, we recover:

$$\psi_\mu^i(x) \qquad\qquad \chi^{ijk}(x) \tag{5.12}$$

where all fields depend only on the 4 space-time coordinates x^μ, since we are analyzing the zero modes of the fields following the KK ansatz (see Chapter 2) approach.

We notice that the $SL(7)$ representation in Supergravity differs from the type IIA case for many reasons, first of all being the fact that in the first case we have **21** and **7** both for the electric and magnetic case, while in String Theory we only have half the vector fields.
Moreover, in one case we have a **7** representation, while in the other we have the adjoint **7**$'$ one. These representations are inequivalent, thus yielding different supersymmetry variations.

5.3.1 The Kaluza-Klein ansatz

We now need to set our KK ansatz on the fields in massive IIA theory to reconduct them to our D=4 $ISO(7)$ gauged Supergravity particle content.
We start by simply assuming:

$$ds_4^2(x,y) = ds_4^2(x) \tag{5.13}$$

where the term on the right is the term in the 10-dimensional metric describing the geometry of the four space-time dimensions in the string frame, thus depending on all the 10 coordinates of type IIA theory, and we state this is dependent only on the x^μ coordinates previously defined.
We wish to set our KK ansatz on the vector and tensor fields using their $SL(6)$ representation, which comes particularly in handy for this purpose. We thus use our m,n coordinates parametrizing the representations of $SL(6)$ and we write

$$C_\mu^{m8}(x,y) = \frac{1}{2} g K^m_{IJ}(y)\mathcal{A}^{IJ}(x) \qquad C_\mu^{78}(x,y) = -\mu_I(y)\mathcal{A}^I_\mu(x) \tag{5.14}$$

$$\tilde{C}_{\mu mn}(x,y) = \tfrac{1}{4} K^{IJ}_{mn}(y)\tilde{\mathcal{A}}_{\mu IJ}(x) \qquad \tilde{C}_{\mu m7}(x,y) = -g^{-1}(\partial_m\mu^I)(y)\tilde{\mathcal{A}}_{\mu I}(x)$$

where the $\mu^I(y)$ (defined in the $\mathbf{7'}$) describe the geometry of S^6 in terms of R^7 coordinates by the defining S^6 as the geometrical space given by $\mu_I\mu^I$=1, while their derivatives $\partial_m\mu^I(y)$ are defined with respect to the 6 angles parametrizing S^6, y^m (defined in the $\mathbf{6'}$ of $SO(6)$). By defining the metric of the internal space as $\tilde{g}_{mn}$, we used its killing vectors written as K^m_{IJ} (in the $(\mathbf{6'},\mathbf{21})$ of $SO(6)\times SL(7)$) and their derivatives K^{IJ}_{mn}.
For the two-forms we choose

$$C^8_{\mu\nu m}(x,y) = -g^{-1}(\mu_I\partial_m\mu^J)(y)\mathcal{B}^I_{\mu\nu J}(x) \qquad C^8_{\mu\nu 7}(x,y) = \mu_I(y)\mathcal{B}^I_{\mu\nu}(x) \tag{5.15}$$

while the three forms are defined as

$$C^{88}_{\mu\nu\rho}(x,y) = (\mu_I\mu_J)(y)\mathcal{C}^{IJ}_{\mu\nu\rho}(x) \tag{5.16}$$

For the generalized vielbein we choose the ansatz

$$\begin{aligned}
V^{m8AB}(x,y) &= \frac{1}{2} g K^m_{IJ}(y)\eta^A_i(y)\eta^B_j(y)\mathcal{V}^{IJij}(x)\\
V^{78AB}(x,y) &= -\mu_I(y)\eta^A_i(y)\eta^B_j(y)\mathcal{V}^{I8ij}(x)\\
\tilde{V}^{AB}_{mn}(x,y) &= \frac{1}{4}K^{IJ}_{mn}(y)\eta^A_i(y)\eta^B_j(y)\tilde{\mathcal{V}}^{ij}_{IJ}(x)\\
\tilde{V}^{AB}_{m7}(x,y) &= -g^{-1}(\partial_m\mu^I)(y)\eta^A_i(y)\eta^B_j(y)\tilde{\mathcal{V}}^{ij}_{I8}(x)
\end{aligned} \tag{5.17}$$

Finally, the ansatz for the fermions turns out to be

$$\psi^A_\mu(x,y) = \eta^A_i(y)\psi^i_\mu(x) \qquad \chi^{ABC}(x,y) = \eta^A_i(y)\eta^B_j(y)\eta^C_k(y)\chi^{ijk}(x) \tag{5.18}$$

where we used the indices conventions:

- $SL(6)$ m,n = 1,...,6 indices;
- $SU(8)$ A,B = 1,...,8 indices;
- $SL(7)$ I,J = 1,...,7 indices;
- D=4 $SU(8)$ i,j = 1,...,8 indices.

We rotated from one index formalism to another in order to compare fields in the IIA and $ISO(7)$ gauged D=4 theories.
The $\eta_i^A(\mathrm{y})$ are the Killing spinors on S^6, rotating indices from the string frame $SU(8)$ representation to the D=4 one.

Finally, we can see the derivation of all the particle content of D=4 $ISO(7)$ dyonically gauged Supergravity as deriving from the massive type IIA 10-D string theory.
We can also notice that the Romans mass m never appears in these expressions, thus seemingly identifying the massive and non-massive cases reductions. We will see that this simmetry is broken at the level of the equations of motion.

5.3.2 The embedding

Now we need to insert the KK ansatz in the definitions of our 10-D fields and invert the rewriting of these ones performed previously in order to find the resulting expressions of the fields in (5.5).
After some calculations, inverting the definition of the tensorial fields C as defined in (5.8) to find the KK ansatz meaning for the original type IIA n-forms, we find the expressions:

$$\bullet \quad d\hat{s}_{10}^2 = \Delta^{-1} ds_4^2 + g_{mn} Dy^m Dy^n \tag{5.19}$$

$$\begin{aligned}\bullet \quad \hat{A}_{(3)} = \mu_I \mu_J (\mathcal{C}^{IJ} + \mathcal{A}^I \wedge \mathcal{B}^J + \frac{1}{6}\mathcal{A}^{IK} \wedge \mathcal{A}^{JL} \wedge \tilde{\mathcal{A}}_{KL} + \frac{1}{6}\mathcal{A}^I \wedge \mathcal{A}^{JK} \wedge \tilde{\mathcal{A}}_K) \\ +g^{-1}(\mathcal{B}^I_J + \frac{1}{2}\mathcal{A}^{IK} \wedge \tilde{\mathcal{A}}_{KJ} + \frac{1}{2}\mathcal{A}^I \wedge \tilde{\mathcal{A}}_J) \wedge \mu_I D\mu^J + \frac{1}{2} g^{-2} \tilde{\mathcal{A}}_{IJ} \wedge D\mu^I \wedge D\mu^J \\ -\frac{1}{2}\mu_I B_{mn} \mathcal{A}^I \wedge Dy^m \wedge Dy^n + \frac{1}{6} A_{mnp} Dy^m \wedge Dy^n \wedge Dy^p\end{aligned} \tag{5.20}$$

$$\bullet \quad \hat{B}_{(2)} = -\mu_I (\mathcal{B}^I + \frac{1}{2}\mathcal{A}^{IJ} \wedge \tilde{\mathcal{A}}_J - g^{-1} \tilde{\mathcal{A}}_I \wedge D\mu^I + \frac{1}{2} B_{mn} Dy^m \wedge Dy^n \tag{5.21}$$

$$\bullet \quad \hat{A}_{(1)} = -\mu_I \mathcal{A}^I + A_m Dy^m \tag{5.22}$$

where the covariant derivatives:

$$Dy^m = dy^m + \frac{1}{2} g K^m_{IJ} \mathcal{A}^{IJ} \qquad D\mu^I = d\mu^I - g\mathcal{A}^{IJ}\mu_J \tag{5.23}$$

were defined.
We notice how only the electric gauge fields $\mathcal{A}^{IJ}$ enter these expressions.
Now we need to find the embedding for the scalars. In order to do this we substitute our ansatz for the generalized vielbein (5.17) in the previously given expression as a combination of internal scalar d.o.f's for the latter. Then we take products of the vielbein in order to eliminate their dependence on the orthogonal S^6 killing spinors and we write such products naturally in terms of the $SL(7)$-covariant blocks of the scalar matrix $\mathcal{M}_{MN}$, quadratic in the $SL(7)/SU(8)$ coset representatives $\mathcal{V}_M^{ij}$, where $M, N = 1, ..., 7$ are $SL(7)$ indices.
We thus find the expressions:

$$\begin{aligned}&\mathcal{M}^{IJKL} K^m_{IJ} K^n_{KL} = 4g^{-2}\Delta^{-1} g^{mn} \\ &\mathcal{M}^{IJK8} K^m_{IJ} \mu_K = 2g^{-1}\Delta^{-1} g^{mn} A_n \\ &\mathcal{M}^{IJ}_{KL} K^m_{IJ} \partial_n \mu^K = -2\Delta^{-1} g^{mp} B_{pn} \\ &\mathcal{M}^{IJ}_{KL} K^m_{IJ} K^{KL}_{np} = 8g^{-1}\Delta^{-1} g^{mq}(A_{qnp} - A_q B_{np}) \\ &\mathcal{M}^{I8J8} \mu_I \mu_J = \Delta^{-1}(e^{-\frac{3}{2}\hat{\phi}} + g^{mn} A_m A_n)\end{aligned} \tag{5.24}$$

With these expression one can thus sequentially find the embedding of the D=4 scalars in IIA by first solving the first equation and relating the D=4 scalars to g^{mn} and then using the results to find with the second equation the embedding of the scalars in the internal-space components of $\hat{A}_{(1)}$ and so on. To explicitly see what the expression to insert inside the IIA Lagrangian is in order to recover D=4 $ISO(7)$ Supergravity, we invert (5.24) and write the internal-space components of the bosonic fields as:

$$\begin{aligned} g^{mn} &= \frac{1}{4} g^2 \Delta K^m_{IJ} K^n_{KL} \mathcal{M}^{IJKL} \\ A_m &= \frac{1}{2} g \Delta g_{mn} K^n_{IJ} \mu_K \mathcal{M}^{IJK8} \\ B_{mn} &= -\frac{1}{2} \Delta g_{mp} K^p_{IJ} \partial_n \mu^K \mathcal{M}^{IJ}_{K8} \\ A_{mnp} &= A_m B_{np} + \frac{1}{8} g \Delta g_{mq} K^q_{IJ} K^{KL}_{np} \mathcal{M}^{IJ}_{KL} \end{aligned} \tag{5.25}$$

5.4 Consistency check

5.4.1 Supersymmetric transformations

One can then check the consistency of the truncation (5.19), (5.20), (5.21) by evaluating the supersymmetry transformations of the fields of type IIA String Theory. One would find that plugging in the reduction ansatz the S^6-dependent terms factorize both on the left and right hand sides of the equations and thus find back the supersymmetry transformations of D=4 $ISO(7)$ Supergravity.
Explicitly, the Supersymmetry transformations for the IIA vielbein is

$$\delta e^\alpha_\mu = \frac{1}{4} \bar{\epsilon}_A \gamma^\alpha \psi^A_\mu + \frac{1}{4} \bar{\epsilon}^A \gamma^\alpha \psi_{\mu A} \tag{5.26}$$

while for the tensor fields the transformations are

$$\delta C^{I8}_\mu = i V^{I8}_{AB} \left(\bar{\epsilon}^A \psi^B_\mu + \frac{1}{2\sqrt{2}} \bar{\epsilon}_C \gamma_\mu \chi^{ABC} \right) + \text{h.c.} \tag{5.27}$$

$$\delta \tilde{C}_{\mu IJ} = -i V_{IJAB} \left(\bar{\epsilon}^A \psi^B_\mu + \frac{1}{2\sqrt{2}} \bar{\epsilon}_C \gamma_\mu \chi^{ABC} \right) + \text{h.c.} \tag{5.28}$$

$$\begin{aligned} \delta C^8_{\mu\nu I} = &\left[\frac{2}{3} (V^{J8}_{BC} \tilde{V}^{AC}_{IJ} + \tilde{V}_{IJBC} V^{J8AC}) \bar{\epsilon}_A \gamma_{[\mu} \psi^B_{\nu]} \right] \\ &+ \frac{\sqrt{2}}{3} V^{J8}_{AB} \tilde{V}_{IJCD} \bar{\epsilon}^{[A} \gamma_{\mu\nu} \chi^{BCD]} + \text{h.c.} \Big] \\ &- C^{J8}_{[\mu} \delta \tilde{C}_{\nu] IJ} - \tilde{C}_{[\mu|IJ} \delta C^{J8}_{|\nu]} \end{aligned} \tag{5.29}$$

$$\begin{aligned} \delta C^{88}_{\mu\nu\rho} = &\left[\frac{4i}{7} V^{I8}_{BD} (V^{J8DC} \tilde{V}_{IJAC} + \tilde{V}^{DC}_{IJ} V^{J8}_{AC}) \bar{\epsilon}^A \gamma_{[\mu\nu} \psi^B_{\rho]} \right. \\ &- \frac{\sqrt{2}i}{3} V^{I8AE} V^{J8}_{[EB|} \tilde{V}_{IJ|CD]} \bar{\epsilon}_A \gamma_{\mu\nu\rho} \chi^{BCD} + h.c. \Big] \\ &+ 3 C^8_{[\mu\nu|I} \delta C^{I8}_{|\rho]} - C^{I8}_{[\mu} (C^{J8}_\nu \delta \tilde{C}_{\rho] IJ} + \tilde{C}_{\nu|IJ} \delta C^{J8}_{|\rho]}) \end{aligned} \tag{5.30}$$

Equation (5.26), upon using (5.13) and (5.18), reduces to the transformation of D=4 vielbein.
We notice that the internal space dependencies in the ansatz (5.14) and (5.17) are the same for each component. Thus, looking at (5.27) and (5.28), we see that the S^6 indeed factorize and we recover the D=4 vector Susy transformations for electric and magnetic gauge fields in the $SL(7)$ representation,

with D=4 $SU(8)$ indices i, j:

$$
\begin{aligned}
\delta\mathcal{A}_\mu^{IJ} &= i\mathcal{V}_{ij}^{IJ}\left(\bar{\epsilon}^i\psi_\mu^j + \frac{1}{2\sqrt{2}}\bar{\epsilon}_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.}\\
\delta\mathcal{A}_\mu^{I} &= i\mathcal{V}_{ij}^{I8}\left(\bar{\epsilon}^i\psi_\mu^j + \frac{1}{2\sqrt{2}}\bar{\epsilon}_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.}\\
\delta\tilde{\mathcal{A}}_{\mu IJ} &= -i\tilde{\mathcal{V}}_{IJij}\left(\bar{\epsilon}^i\psi_\mu^j + \frac{1}{2\sqrt{2}}\bar{\epsilon}_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.}\\
\delta\tilde{\mathcal{A}}_{\mu I} &= -i\tilde{\mathcal{V}}_{I8ij}\left(\bar{\epsilon}^i\psi_\mu^j + \frac{1}{2\sqrt{2}}\bar{\epsilon}_k\gamma_\mu\chi^{ijk}\right) + \text{h.c.}
\end{aligned}
\tag{5.31}
$$

Recovering Supersymmetry transformations for 2-forms has been more difficult, and one need to make considerations on the internal geometry to factorize the resulting terms. Anyway, by plugging the ansatz in (5.29), we find:

$$
\begin{aligned}
\delta\mathcal{B}^I_{\mu\nu J} =& \Bigg[\frac{2}{3}(\mathcal{V}^{IK}_{jk}\tilde{\mathcal{V}}^{ik}_{JK} + \mathcal{V}^{I8}_{jk}\tilde{\mathcal{V}}^{ik}_{J8} + \tilde{\mathcal{V}}_{JKjk}\mathcal{V}^{IKik} + \tilde{\mathcal{V}}_{J8jk}\mathcal{V}^{I8ik})\bar{\epsilon}_i\gamma_{[\mu}\psi^i_{\nu]}\\
&- \frac{\sqrt{2}}{3}(\mathcal{V}^{IK}_{ij}\tilde{\mathcal{V}}_{JKkl} + \mathcal{V}^{I8}_{ij}\tilde{\mathcal{V}}_{J8kl})\bar{\epsilon}^{[i}\gamma_{\mu\nu}\chi^{jkl]} + \text{h.c.}\Bigg]\\
&+ (\mathcal{A}^{IK}_{[\mu}\delta\tilde{\mathcal{A}}_{\nu]JK} + \mathcal{A}^{I}_{[\mu}\delta\tilde{\mathcal{A}}_{\nu]J} + \tilde{\mathcal{A}}_{[\mu|JK}\delta\mathcal{A}^{IK}_{|\nu]} + \tilde{\mathcal{A}}_{[\mu|J}\delta\mathcal{A}^{I}_{|\nu]}) - \frac{1}{7}\delta^I_J(\text{trace})
\end{aligned}
\tag{5.32}
$$

$$
\begin{aligned}
\delta\mathcal{B}^I_{\mu\nu} =& \Bigg[\frac{2}{3}(\mathcal{V}^{IJ}_{jk}\tilde{\mathcal{V}}^{ik}_{J8} + \tilde{\mathcal{V}}_{J8jk}\mathcal{V}^{IJik})\bar{\epsilon}_i\gamma_{[\mu}\psi^j_{\nu]} + \frac{\sqrt{2}}{3}\mathcal{V}^{IJ}_{ij}\tilde{\mathcal{V}}_{J8kl}\bar{\epsilon}^{[i}\gamma_{\mu\nu}\chi^{jkl]} + \text{h.c.}\Bigg]\\
&- (\mathcal{A}^{IJ}_{[\mu}\delta\tilde{\mathcal{A}}_{\nu]J} + \tilde{\mathcal{A}}_{[\mu|J}\delta\mathcal{A}^{IJ}_{|\nu]})
\end{aligned}
\tag{5.33}
$$

and doing the same for the three form in (5.30):

$$
\begin{aligned}
\delta\mathcal{C}^{IJ}_{\mu\nu\rho} =& \Bigg[-\frac{4i}{7}\Bigg(\mathcal{V}^{K(I}_{jl}(\mathcal{V}^{J)Llk}\tilde{\mathcal{V}}_{KLik} + \tilde{\mathcal{V}}^{lk}_{KL}\mathcal{V}^{J)L}_{ik}\\
&+ \mathcal{V}^{K(I}_{jl}(\mathcal{V}^{J)8lk}\tilde{\mathcal{V}}_{K8ik} + \tilde{\mathcal{V}}^{lk}_{K8}\mathcal{V}^{J)8}_{ik})\\
&+ \mathcal{V}^{(I|8}_{jl}(\mathcal{V}^{|J)Klk}\tilde{\mathcal{V}}_{K8ik} + \tilde{\mathcal{V}}_{K8ik} + \tilde{\mathcal{V}}^{lk}_{K8}\mathcal{V}^{|J)K}_{ik})\Bigg)\bar{\epsilon}^i\gamma_{[\mu\nu}\psi^j_{\rho]}\\
&+ i\frac{\sqrt{2}i}{3}\Bigg(\mathcal{V}^{K(I|hi}\mathcal{V}^{|J)L}_{[ij|}\tilde{\mathcal{V}}_{KL|kl]} + \mathcal{V}^{K(I|hi}\mathcal{V}^{|J)8}_{[ij|}\tilde{\mathcal{V}}_{K8|kl]}\\
&+ \mathcal{V}^{(I|8hi}\mathcal{V}^{|J)K}_{[ij|}\tilde{\mathcal{V}}_{K8|kl]}\Bigg)\bar{\epsilon}_h\gamma_{\mu\nu\rho}\chi^{jkl} + \text{h.c.}\Bigg]\\
&- 3\Bigg(\mathcal{B}^{(I}_{[\mu\nu|K}\delta\mathcal{A}^{J)K+\mathcal{B}_{[\mu\nu}(I}_{|\rho]}\delta\mathcal{A}^{J)}_{\rho]}\Bigg)\\
&+ \mathcal{A}^{K(I}_{[\mu}(\mathcal{A}^{J)L}_{\nu}\delta\tilde{\mathcal{A}}_{\rho]KL} + \tilde{\mathcal{A}}_{\nu KL}\delta\mathcal{A}^{J)L}_{\rho]}) + \mathcal{A}^{K(I}_{[\mu}(\mathcal{A}^{J)}_{\nu}\delta\tilde{\mathcal{A}}_{\rho]K} + \tilde{\mathcal{A}}_{\nu K}\delta\mathcal{A}^{J)}_{\rho]})\\
&+ \mathcal{A}^{(I}_{[\mu}(\mathcal{A}^{J)K}_{\nu}\delta\tilde{\mathcal{A}}_{\rho]K} + \tilde{\mathcal{A}}_{\nu K}\delta\mathcal{A}^{J)K}_{\rho]}).
\end{aligned}
\tag{5.34}
$$

These are exactly the D=4 $ISO(7)$ Supergravity transformations that one can find in [57].
The fact that BPS equations are satisfied in D=4 $\mathcal{N}$=8, should imply consistency also at the equations of motion level.
One would also find that these supersymmetry transformations are independent on the magnetic coupling constant m as well, thus giving, at this level, a seemingly equal embedding process both for dyonic and electric-only gauged Supergravity in massive and massless type IIA string theory.

5.4.2 Bianchi identities

The difference between the purely-electric and the dyonic theory is seen when dealing with gauging-dependent terms, namely covariant derivatives and field-strengths.
First Bianchi identities in ten dimensions need to be rewritten in terms of the D=4 field-strengths related to electrical and magnetic gauge fields and two/three-forms of the restricted tensor hierarchy. Therefore, one needs to check if the Bianchi identities of type IIA field-strengths are equivalent in shape to the Supergravity Bianchi identities. We find that such a request is satisfied.
Moreover, one can prove that hodge-duality relations bound magnetic field-strengths and scalar-dependent combinations of electric fields. Thus, by duality, the fact that Bianchi identities are satisfied with our chosen truncation bot in D=4 and D=10, partially implies the consistency of our truncation at the level of equations of motion as well.
Exploiting the so called "duality hierarchy", relating non-independent different-order tensorial degrees of freedom by hodge-duality, one can thus express the D=4 field-strengths by means of independent type IIA fields.
The field-strengths of type IIA String theory can be recovered by integrating the Bianchi identities

$$d\hat{F}_{(4)} - \hat{F}_{(2)} \wedge \hat{H}_{(3)} = 0 \qquad d\hat{H}_{(3)} = 0 \qquad d\hat{F}_{(2)} - m\hat{H}_{(3)} = 0 \tag{5.35}$$

finding

$$\begin{aligned} \hat{F}_{(4)} &= d\hat{A}_{(3)} + \hat{A}_{(1)} \wedge d\hat{B}_{(2)} + \frac{1}{2}m\hat{B}_{(2)} \wedge \hat{B}_{(2)} \\ \hat{H}_{(3)} &= d\hat{B}_{(2)} \\ \hat{F}_{(2)} &= d\hat{A}_{(1)} + m\hat{B}_{(2)} \end{aligned} \tag{5.36}$$

Inserting our reduction ansatz (5.19) in the previous relations, the resulting expressions turn out to be:

$$\begin{aligned} \hat{F}_{(4)} &= \mu_I\mu_J\mathcal{H}^{IJ}_{(4)} + g^{-1}\mathcal{H}^I_{(3)J} \wedge \mu_I D\mu^J + \frac{1}{2}g^{-2}\tilde{\mathcal{H}}_{(2)IJ} \wedge D\mu^I \wedge D\mu^J + ... \\ \hat{H}_{(3)} &= -\mu_I\mathcal{H}^I_{(3)} - g^{-1}\tilde{\mathcal{H}}_{(2)I} \wedge D\mu^I + ... \\ \hat{F}_{(2)} &= -\mu_I\mathcal{H}^I_{(2)} + g^{-1}(g\delta_{IJ}\mathcal{A}^J - m\tilde{\mathcal{A}}_I) \wedge D\mu^I + ... \end{aligned} \tag{5.37}$$

where the dots indicate terms containing Suprgravity scalars.
It is using the explicit expressions (5.37) in (5.35) one can indeed check that Bianchi identities for the fields in IIA imply that Bianchi identities are also satisfied in the D=4 theory.
Many terms in the expression above can actually be expressed as functions of other fields of the theory. Indeed, the hodge-duality conditions impose:

$$\tilde{\mathcal{H}}_{(2)IJ} = \frac{1}{2}\mathcal{I}_{[IJ][KL]}\mathcal{H}^{KL}_{(2)} + \mathcal{I}_{[IJ][K8]}\mathcal{H}^K_{(2)} + \frac{1}{2}\mathcal{R}_{[IJ][KL]}\mathcal{H}^{KL}_{(2)} + \mathcal{R}_{[IJ][K8]}\mathcal{H}^K_{(2)} \tag{5.38}$$

$$\tilde{\mathcal{H}}_{(2)I} = \frac{1}{2}\mathcal{I}_{[I8][KL]}\mathcal{H}^{KL}_{(2)} + \mathcal{I}_{[I8][K8]}\mathcal{H}^K_{(2)} + \frac{1}{2}\mathcal{R}_{[I8][KL]}\mathcal{H}^{KL}_{(2)} + \mathcal{R}_{[I8][K8]}\mathcal{H}^K_{(2)} \tag{5.39}$$

$$\mathcal{H}^J_{(3)I} = \frac{1}{12}(t_I^J)_M^P\mathcal{M}_{NP}D\mathcal{M}^{MN} - \frac{1}{7}\delta^J_I \tag{5.40}$$

$$\mathcal{H}^I_{(3)} = \frac{1}{12}(t_8^I)_M^P\mathcal{M}_{NP}D\mathcal{M}^{MN} \tag{5.41}$$

$$\mathcal{H}^{IJ}_{(4)} = \frac{1}{84}X^S_{NQ}((t_K^{(I|})_P^R\mathcal{M}^{|J)KN} + (t_8^{(I|})_P^R\mathcal{M}^{|J)8N})(\mathcal{M}^{PQ}\mathcal{M}_{RS} + 7\delta^P_S\delta^Q_R)vol_4 \tag{5.42}$$

where $\mathcal{I}_{[IJ][KL]}$ and $\mathcal{R}_{[IJ][KL]}$ are the imaginary and real part of the $SL(7)$-covariant scalar matrix of the gauged theory, while X^S_{NQ} is the constant tensor constructed by contracting the embedding tensor Θ^α_N of the dyonically $ISO(7)$ gauged theory with the generators of the symmetry group $E_{7(7)}$ $(t_\alpha)^S_Q$:

$$X^S_{NQ} = \Theta^\alpha_N(t_\alpha)^S_Q \tag{5.43}$$

One interesting term in our embedding formula (5.37) is the so called Freund-Rubin term $\mathcal{H}^{IJ}_{(4)}\mu_I\mu_J$, which can be related to the scalar potential of $ISO(7)$ Supergravity upon using Bianchi identities and an expression involving the four-form field strengths by

$$\mathcal{H}^{IJ}_{(4)}\mu_I\mu_J = -\frac{1}{3}g^{-1}V\text{vol}_4 + (\text{field strengths dependent terms}) \tag{5.44}$$

where V is indeed the scalar potential of the D=4 theory.
At a critical point of the potential, we can consistently set all gauge fields to zero, thus remaining with the relation

$$\mathcal{H}^{IJ}_{(4)0}\mu_I\mu_J = \frac{1}{3}g^{-1}V_0\text{vol}_4. \tag{5.45}$$

Thus, at the minimum of the potential, the Freund-Rubin term becomes constant, and can thus be seen as a Cosmological Constant in the lower dimensional theory.

5.5 The D=4 action

The dyonically $ISO(7)$ gauged Supergravity bosonic action [57] finally is:

$$\begin{aligned}\mathcal{L} =& R\text{vol}_4 - \frac{1}{48}D\mathcal{M}_{MN}\wedge *D\mathcal{M}^{MN} + \frac{1}{2}I_{\Lambda\Sigma}\mathcal{H}^{\Lambda}_{(2)}\wedge *\mathcal{H}^{\Sigma}_{(2)} + \frac{1}{2}\mathcal{R}_{\Lambda\Sigma}\mathcal{H}^{\Lambda}_{(2)}\wedge\mathcal{H}^{\Sigma}_{(2)} \\ & - V\text{vol}_4 - m\Big[\mathcal{B}^I\wedge(\tilde{\mathcal{H}}_{(2)I} - \frac{g}{2}\delta_{IJ}\mathcal{B}^J) - \frac{1}{4}\tilde{\mathcal{A}}_I\wedge\tilde{\mathcal{A}}_J\wedge(d\mathcal{A}^{IJ} + \frac{g}{2}\delta_{KL}\mathcal{A}^{IK}\wedge\mathcal{A}^{JL})\Big]\end{aligned} \tag{5.46}$$

where the scalar potential is defined as

$$V = \frac{g^2}{168}X_{MP}^{R}X_{NQ}^{S}\mathcal{M}^{MN}\left(\mathcal{M}^{PQ}\mathcal{M}_{RS} + 7\delta^P_S\delta^Q_R\right) \tag{5.47}$$

The theory is expressed in term of the $\mathbf{21}' + \mathbf{7}'$ electric gauge fields $\mathcal{A}^* = (\mathcal{A}^{IJ}, \mathcal{A}^I)$ and their corresponding field strengths $\mathcal{H}^{\Lambda}_{(2)} = (\mathcal{H}^{IJ}_{(2)}, \mathcal{H}^{I}_{(2)})$ as well as their magnetic duals $\tilde{\mathcal{A}}_\Lambda = \mathcal{A}_I$, because one can prove that the $\mathbf{21}$ $\mathcal{A}_{IJ}$ do not participate in the gauging, and $\tilde{\mathcal{H}}^{(2)}_{\Lambda}$.
Moreover we can write the scalar matrix $\mathcal{M}_{MN}$ in terms of the components of a complex scalar matrix $\mathcal{N} = \mathcal{R} + i\mathcal{I}$. Indeed, we define $\mathcal{M}$ contracting the $SU(8)$ indices of the coset representatives in such a way that

$$\mathcal{M}_{MN} = 2\mathcal{V}^{ij}_{(M}\mathcal{V}_{N)ij} = \begin{bmatrix}\mathcal{M}_{\Lambda\Sigma} & \mathcal{M}^{\Sigma}_{\Lambda} \\ \mathcal{M}^{\Lambda}_{\Sigma} & \mathcal{M}^{\Lambda\Sigma}\end{bmatrix} = \begin{bmatrix}-(\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & (\mathcal{R}\mathcal{I}^{-1})^{\Sigma}_{\Lambda} \\ (\mathcal{I}^{-1}\mathcal{R})^{\Lambda}_{\Sigma} & -(\mathcal{I}^{-1})^{\Lambda\Sigma}\end{bmatrix} \tag{5.48}$$

Now, having defined all the relations that connect this theory to the type IIA String Theory, we could insert relations (5.24) and invert relations (5.37) in order to recover the high-energy theory Lagrangian.

Furthermore, from (5.36) one can recover an expression for the fluxes in the $ISO(7)$ four-dimensional theory as:

$$\begin{aligned}\mathcal{H}^{IJ}_{(2)} &= d\mathcal{A}^{IJ} - g\delta_{KL}\mathcal{A}^{IK}\wedge\mathcal{A}^{LJ} \\ \mathcal{H}^{I}_{(2)} &= d\mathcal{A}^{I} - g\delta_{JK}\mathcal{A}^{IJ}\wedge\mathcal{A}^{K} + \frac{1}{2}m\mathcal{A}^{IJ}\wedge\tilde{\mathcal{A}}_J + m\mathcal{B}^I \\ \tilde{\mathcal{H}}_{(2)IJ} &= d\tilde{\mathcal{A}}_{IJ} + g\delta_{K[I}\mathcal{A}^{KL}\wedge\tilde{\mathcal{A}}_{J]L} + g\delta_{K[I}\mathcal{A}^{K}\wedge\tilde{\mathcal{A}}_{J]} - m\tilde{\mathcal{A}}_I\wedge\tilde{\mathcal{A}}_J + 2g\delta_{K[I}\mathcal{B}^{K}_{J]} \\ \tilde{\mathcal{H}}_{(2)I} &= d\tilde{\mathcal{A}}_I - \frac{1}{2}g\delta_{IJ}\mathcal{A}^{JK}\wedge\tilde{\mathcal{A}}_K + g\delta_{IJ}\mathcal{B}^J\end{aligned} \tag{5.49}$$

Chapter 6

$\mathcal{N}=8, ISO(7)_c$ gauged model restricted to $\mathcal{N}=2$ SuperSymmetry with residual $SU(3)$ global and $SO(1,1)\times U(1)$ gauge symmetries

As we can notice from Chapter 5.5, the dyonically gauged $ISO(7)$ four-dimensional theory has a quite involved Lagrangian expression (5.46).
The formalism includes definition of a scalar matrix $\mathcal{M}_{MN}$ and subsequent definition of gauge-scalar matrices $\mathcal{R}_{\Lambda\Sigma}$ and $\mathcal{I}_{\Lambda\Sigma}$, which collect all the scalar degrees of freedom.
In some way, this formalism is useful for writing the action in a compact form, but keeps us from having a clear view on what are the standard kinetic and interaction terms for the scalar fields.
Because of this, in order to carry out our analysis of wormhole geometry, we exploit useful truncations to subsectors of the $ISO(7)$ theory which display consistency in terms of closure of their equations of motions and invariance under symmetry sub-groups of $ISO(7)$.
We first carry out a detailed analysis of the $SU(3)$ invariant subsector, with a residual $SO(1,1)\times U(1)$ gauge symmetry, while in the following we will also use particle content of the G_2 and $SO(4)$ subsectors.

In Chapter 6.1 we restrict the $ISO(7)$ dyonically gauged theory to its subsector with $SU(3)$ residual symmetry. We give its action with its scalar potential.
We derive the equations of motion for this subsector in Chapter 6.2. These will be used extensively in the following.
In Chapter 6.3 we analyze the vacua of the theory and try to find Minkowski, dS and AdS vacuum configurations.
Finally, in Chapter 6.4,for completeness, we rapidly analyze the supersymmetric black hole solution of the $SU(3)$ invariant subsector.

6.1 The $SU(3)$-invariant subsector

First, one need to give a branching of all the fields in the $ISO(7)$ dyonically gauged theory in terms of of their transformation rules under $SU(3)$. One should then only retain the fields that behave as singlets under the sub-group, and doing so we observe [57] that we are left with six scalar fields, one field stemming from $\mathcal{A}^{IJ}$, $\mathcal{A}^I$, $\tilde{\mathcal{A}}_{IJ}$ and $\tilde{\mathcal{A}}_I$ each, two invariant fields coming from $\mathcal{B}_I^{\,J}$, one from $\mathcal{B}^I$ and two other residual fields left from the three-form $\mathcal{C}^{IJ}$.

The $SU(3)$-invariant sector cerresponds to an extended $\mathcal{N}=2$ Supergravity theory, where we couple our supermultiplet to **one vector multiplet** and **one hypermultiplet** (see Chapter 1.2.1 for further reference).

The 2+4 real scalar fields paramatrize two coset spaces, which are one Special Kahler (SK) and one Quaternionic Kahler (QK) manifolds:

$$\frac{SU(1,1)}{U(1)} \times \frac{SU(2,1)}{SU(2) \times U(1)} \tag{6.1}$$

We derive from this scalar field truncation a gauge theory including an abelian dyonic gauge group $U(1) \times SO(1,1)$.

6.1.1 Particle content of the theory

We wish to recover a subsector of the main theory using only $SU(3)$ invariant quantities stemming from the Lagrangian (5.46).
In order to do this, we use a sub-group chain

$$SO(7) \supset SO(6) \supset SU(3) \tag{6.2}$$

to write our theory embedding the $SU(3)$ indices formalism in the SO(6) one of the main theory.
By doing so, we recover the following particle content:

- metric: $g_{\mu\nu}$
- scalar matrix: $\mathcal{M}_{MN} \to (\chi, \varphi) + (\phi, a, \zeta, \tilde{\zeta})$
- gauge vectors: $A^0, A^1, \tilde{A}_0, \tilde{A}_1$
- two-forms: B^0, B_1, B_2
- three-forms: C^0, C^1

We defined the restricted gauge vector boson particle content as

$$\begin{aligned}
&\mathcal{A}^I \to \mathcal{A}^1 \equiv A^0 \\
&\mathcal{A}^{IJ} \to \mathcal{A}^{ij} \equiv A^1 J^{ij} \\
&\tilde{\mathcal{A}}_I \to \tilde{\mathcal{A}}_1 \equiv \tilde{A}_0 \\
&\tilde{\mathcal{A}}_{IJ} \to \tilde{\mathcal{A}}_{ij} \equiv \frac{1}{3}\tilde{A}_1 J_{ij}
\end{aligned} \tag{6.3}$$

where we introduced the $SU(3)$-invariant two-form

$$J = e^2 \wedge e^3 + e^4 \wedge e^5 + e^6 \wedge e^7 \tag{6.4}$$

The index $\Lambda = 0, 1$ in A^Λ numbers the two gauge fields and we assign index ^{0} to the graviphoton (gauge field stemming from the gravity supermultiplet) and index ^{1} to the vector in the vector multiplet. The same is valid for the index $_\Lambda$ in $\tilde{A}_\Lambda$ for the magnetic fields.
A^0 and $\tilde{A}_0$ gauge dyonically the $SO(1,1)$ group (1 d.o.f.) while A^1 gauges electrically $U(1)$.
The two-form fields are restricted to

$$\begin{aligned}
&\mathcal{B}^I \to \mathcal{B}^1 \equiv B^0 \\
&\mathcal{B}_I^J \to \mathcal{B}_1^1 = \frac{6}{7}B_1, \qquad \mathcal{B}_i^j \equiv -\frac{1}{7}B_1\delta_i^j + \frac{1}{3}B_2 J_i^j
\end{aligned} \tag{6.5}$$

while for the three-forms we truncate

$$\mathcal{C}^{IJ} \to \mathcal{C}^{11} \equiv C^0, \qquad \mathcal{C}^{ij} \equiv C^1 \delta^{ij} \tag{6.6}$$

Of these, the three-forms, B_1 and B_2 and the magnetic potential $\tilde{A}_1$ do not enter the Lagrangian of the bosonic sector.

6.1.2 The Lagrangian and scalar potential

We restrict (5.46) to its $SU(3)$, $\mathcal{N}$=2 invariant subsector and, using the just introduced field truncations in ((6.3)-(6.6)), we find a final expression for the Lagrangian density $\mathcal{L}$ [57]:

$$
\begin{aligned}
\mathcal{L} =&\big(R - V\big)\mathrm{vol}_4 - \frac{3}{2}[d\varphi \wedge *d\varphi + e^{2\varphi} d\chi \wedge *d\chi] \\
&- 2d\phi \wedge *d\phi - \frac{1}{2}e^{2\phi}[D\zeta \wedge *D\zeta + D\tilde{\zeta} \wedge *D\tilde{\zeta}] \\
&- \frac{1}{2}e^{4\phi}[Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta} D\zeta)] \wedge *[Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta} D\zeta)] \\
&+ \frac{1}{2}\mathcal{I}_{\Lambda\Sigma} H^{\Lambda}_{(2)} \wedge *H^{\Sigma}_{(2)} + \frac{1}{2}\mathcal{R}_{\Lambda\Sigma} H^{\Lambda}_{(2)} \wedge H^{\Sigma}_{(2)} - mB^0 \wedge d\tilde{A}_0 - \frac{1}{2} gmB^0 \wedge B^0
\end{aligned}
\tag{6.7}
$$

To see the full derivation of scalar matrix $\mathcal{M}_{MN}$ that defines the kinetic and interaction terms for scalars, as well as for the derivation of the restricted gauge group invariance, refer to Appendix B.

The scalar field content is as usual divided between RR-sector and NS-NS derived fields. The scalars ζ and $\tilde{\zeta}$ come from the former, while a and ϕ come from the dimensional reduction process of the latter in the hypermultiplet scalar content. For what concerns the vector multiplet, we have φ stemming from the NS-NS string sector and χ coming from the R-R sector.

The restricted gauge symmetries are generated by the killing vectors:

$$k_R = \partial_a \tag{6.8}$$

$$k_{U(1)} = 3(\zeta\partial_{\tilde{\zeta}} - \tilde{\zeta}\partial_{\zeta}) \tag{6.9}$$

The 2-forms that define the electric gauge fields' fluxes are:

$$H^0 = dA^0 + \frac{1}{2}mB^0 \qquad\qquad H^1 = dA^1 \tag{6.10}$$

The electric and magnetic gauge fields define of course the covariant derivatives setting the interaction terms with the gauged scalars. These covariant derivatives are defined as:

$$Da = da + gA^0 - m\tilde{A}_0 \qquad\qquad D\zeta = d\zeta - 3gA^1\tilde{\zeta} \qquad\qquad D\tilde{\zeta} = d\tilde{\zeta} + 3gA^1\zeta \tag{6.11}$$

The scalar potential V_g appearing in (6.7) is explicitly [57]:

$$
\begin{aligned}
V_g =&\frac{1}{2}g^2[e^{4\phi-3\varphi}(1+e^{2\varphi}\chi^2)^3 - 12e^{2\phi-\varphi}(1+e^{2\varphi}\chi^2) - 24e^{\varphi} + \\
&+ \frac{3}{4}e^{4\phi+\varphi}(\zeta^2+\tilde{\zeta}^2)^2(1+3e^{2\varphi}\chi^2) + 3e^{4\phi+\varphi}(\zeta^2+\tilde{\zeta}^2)\chi^2(1+e^{2\varphi}\chi^2) + \\
&- 3e^{2\phi+\varphi}(\zeta^2+\tilde{\zeta}^2)(1-3e^{2\varphi}\chi^2)] - \frac{1}{2}gm\chi e^{4\phi+3\varphi}[3(\zeta^2+\tilde{\zeta}^2)+2\chi^2] + \frac{1}{2}m^2e^{4\phi+3\varphi}
\end{aligned}
\tag{6.12}
$$

which is obtained applying our scalar field truncation defined through the coset representatives in Appenidx B to (5.47).
We will often use a more direct rewriting for the ζ and $\tilde{\zeta}$ fields as the linear combination:

$$\tilde{\zeta} + i\zeta = 2\rho e^{i\beta} \tag{6.13}$$

6.2 Equations of motion

We derived from (6.7) the equations of motion for all the fields in our theory. We report in the following all the equations for:
- **Gauge fields**

$$d\mathcal{B}^0 = -e^{4\phi} * [Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta} D\zeta)] \tag{6.14}$$

$$d\tilde{\mathcal{A}}_0 + \frac{1}{2}g\mathcal{B}^0 = \mathcal{I}_{0\Lambda} * \mathcal{H}^\Lambda + \mathcal{R}_{0\lambda}H^\Lambda \tag{6.15}$$

$$d(\mathcal{I}_{0\Lambda} * H^\Lambda + \mathcal{R}_{0\Lambda}H^\Lambda) = \frac{1}{2}ge^{4\phi} * [Da + \frac{1}{2}(\zeta\ D\tilde{\zeta} - \tilde{\zeta}D\zeta)] \tag{6.16}$$

$$d(\mathcal{I}_{1\Lambda} * H^\Lambda + \mathcal{R}_{1\Lambda}H^\Lambda) = \frac{3}{2}ge^{4\phi}(\zeta^2 + \tilde{\zeta}^2) * [Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta}D\zeta)] - \frac{3}{2}ge^{2\phi}(\tilde{\zeta} * D\zeta - \zeta * D\tilde{\zeta}) \tag{6.17}$$

We notice that (6.14), (6.15) and (6.16) are redundant: we can obtain easily the first one from the second one and the third one.

• **Hypermultiplet scalars**

$$d[e^{4\phi} * (Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta}D\zeta))] = 0 \tag{6.18}$$

$$\frac{1}{2}d[e^{2\phi} * D\zeta] = \frac{3}{2}ge^{2\phi}\mathcal{A}^1 \wedge *D\tilde{\zeta} + \frac{1}{2}e^{4\phi}D\tilde{\zeta} \wedge *[Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta}D\zeta)] + \partial_\zeta V_g * 1 \tag{6.19}$$

$$\frac{1}{2}d[e^{2\phi} * D\tilde{\zeta}] = -\frac{3}{2}ge^{2\phi}\mathcal{A}^1 \wedge *D\zeta - \frac{1}{2}e^{4\phi}D\zeta \wedge *[Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta}D\zeta)] + \partial_{\tilde{\zeta}} V_g * 1 \tag{6.20}$$

$$\begin{aligned} 2d * d\phi =& e^{4\phi}[Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta}D\zeta)] \wedge *[Da + \frac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta}D\zeta)] \\ &+ \frac{1}{2}e^{2\phi}[D\zeta \wedge *D\zeta + D\tilde{\zeta} \wedge *D\tilde{\zeta}] + \partial_\phi V_g * 1 \end{aligned} \tag{6.21}$$

• **Vector multiplet scalars**

$$\frac{3}{2}d * d\varphi = \frac{3}{2}e^{2\varphi}d\chi \wedge *d\chi - \frac{1}{2}\partial_\varphi\mathcal{I}_{\Lambda\Sigma}\mathcal{H}^\Lambda \wedge *\mathcal{H}^\Sigma - \frac{1}{2}\partial_\varphi\mathcal{R}_{\Lambda\Sigma}\mathcal{H}^\Lambda \wedge \mathcal{H}^\Sigma + \partial_\varphi V_g * 1 \tag{6.22}$$

$$\frac{3}{2}d[e^{2\varphi} * d\chi] = -\frac{1}{2}\partial_\chi\mathcal{I}_{\Lambda\Sigma}\mathcal{H}^\Lambda \wedge *\mathcal{H}^\Sigma - \frac{1}{2}\partial_\chi\mathcal{R}_{\Lambda\Sigma}\mathcal{H}^\Lambda \wedge \mathcal{H}^\Sigma + \partial_\chi V_g * 1 \tag{6.23}$$

• **Einstein equations**

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{scalars}_{\mu\nu} + T^{vectors}_{\mu\nu} \tag{6.24}$$

$$T^{vectors}_{\mu\nu} = -\mathcal{I}_{\Lambda\Sigma}[\mathcal{H}^\Lambda_{\mu\rho}\mathcal{H}^{\Sigma\rho}_\nu - \frac{1}{4}g_{\mu\nu}\mathcal{H}^\Lambda_{\rho\sigma}\mathcal{H}^{\Sigma\rho\sigma}] \tag{6.25}$$

$$\begin{aligned} T^{scalars}_{\mu\nu} =& \frac{3}{2}(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi) + \frac{3}{2}e^{2\varphi}(\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}\partial_\rho\chi\partial^\rho\chi) \\ &+ 2(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi) + \frac{1}{2}e^{2\phi}(D_\mu\zeta D_\nu\zeta - \frac{1}{2}g_{\mu\nu}D_\rho\zeta D^\rho\zeta) \\ &+ \frac{1}{2}e^{2\phi}(D_\mu\tilde{\zeta}D_\nu\tilde{\zeta} - \frac{1}{2}g - \mu\nu D_\rho\tilde{\zeta}D^\rho\tilde{\zeta}) + \frac{1}{2}e^{4\phi}(\xi_\mu\xi_\nu - \frac{1}{2}g_{\mu\nu}\xi_\rho\xi^\rho) \\ &- g_{\mu\nu}V_g \end{aligned} \tag{6.26}$$

with $\xi_\mu = D_\mu a + \frac{1}{2}(\zeta D_\mu\tilde{\zeta} - \tilde{\zeta}D_\mu\zeta)$.

6.3 The vacua of the theory

We aim to find the possible vacuum sulutions of this theory by looking at the potential V_g in (6.12). We will examine the different cases and try to define whether our model yield a positive, negative or null **cosmological constant** Λ when minimized and set to a constant value. These cases correspond to a deSitter, an Anti-deSitter and a Minkowski space-time respectively.

All these vacuum solutions come with a certain extension of Supersymmetry, which we can easily check by imposing that the constant solutions for the fields satisfy BPS equations.

Furthermore, we notice that no minimum can be found in an exclusively electrically gauged theory. Indeed, setting m, i.e. the Romans mass, to 0 and reducing consistently to a solution with $\rho = 0 = \chi$ (this is done just in order to be able to give a graphical depiction of the potential) the potential takes the form:

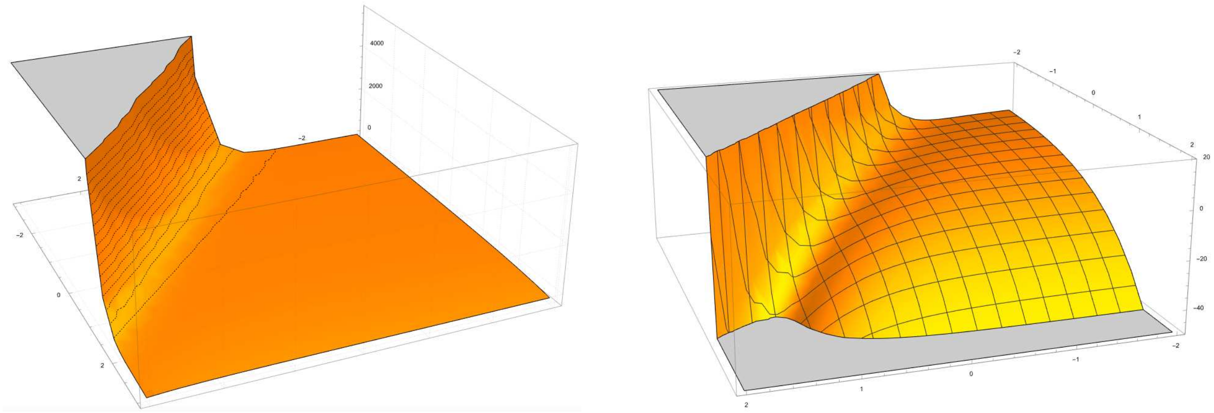

Figure 6.1: Plot of potential in an exclusively electrically gauged theory ($m = 0$) for a slice with $\chi(r)$ and $\rho(r)$ consistently set to 0. The second subfigure shows a closeup for small values of the ϕ and φ fields where we observe no critical points, to compare with (Figure 6.2).

which has no minima, and this can be shown to keep on being valid also for general values of χ and ρ.

6.3.1 Minkowski and deSitter vacua

The assumption of a Minkowski solution implies a minimization of the potential which leads us to an effective $\Lambda = 0$. This has not been found, thus we reject the possibility of having such a vacuum solution.

In our calculations we also found no positive value for a local minimum of the potential. This mirrors the fact that also no deSitter vacuum solutions have been found.

6.3.2 Anti-deSitter vacua

Many AdS vacua have been found in our $\mathcal{N} = 2$ sector which lead to an AdS vacuum. As anticipated, the potential (6.12) depends only on four degrees of freedom, namely φ, χ, ϕ and ρ. The latter is defined writing

$$\tilde{\zeta} + i\zeta = 2\rho e^{i\beta} \tag{6.27}$$

We shall derive the potential with respect to the fields it depends on, and find vacuum configurations giving a negative value of the potential in order to have an AdS solution.
Therefore we write:

- $$\begin{aligned}\partial_\chi V_g =&\frac{1}{8}g^2[6e^{4\phi-\varphi}\chi(1+e^{2\varphi}\chi^2)^2 - 24e^{2\phi+\varphi}\chi + \frac{9}{2}e^{4\phi+3\varphi}(4\rho^2)^2\chi + 6e^{4\phi+\varphi}(4\rho^2)\chi + \\ &+ 12e^{4\phi+3\varphi}(4\rho^2)\chi^3 + 18e^{2\phi+3\varphi}(4\rho^2)\chi - \frac{3}{8}gm\chi e^{4\phi+3\varphi}(4\rho^2)] - \frac{3}{4}gme^{4\phi+3\varphi}\chi^2 = 0\end{aligned} \tag{6.28}$$

- $$\begin{aligned}\partial_\phi V_g =&\frac{1}{8}g^2[4e^{4\phi-3\varphi}(1+e^{2\varphi}\chi^2)^3 - 24e^{2\phi-\varphi}(1+e^{2\varphi}\chi^2) + 48e^{4\phi+\varphi}\rho^4(1+3e^{2\varphi}\chi^2) \\ &+ 48e^{4\phi+\varphi}\rho^2\chi^2(1+e^{2\varphi}\chi^2) - 24e^{2\phi+\varphi}\rho^2(1-3e^{2\varphi}\chi^2)] \\ &- \frac{1}{2}gm\chi e^{4\phi+3\varphi}[12\rho^2 + 2\chi^2] + \frac{1}{2}m^2e^{4\phi+3\varphi} = 0\end{aligned} \tag{6.29}$$

-

$$
\begin{aligned}
\partial_\varphi V_g =& \frac{1}{8} g^2 [-3e^{4\phi-3\varphi}(1+e^{2\varphi}\chi^2)^3 + 3e^{4\phi-3\varphi}(1+e^{2\varphi}\chi^2)^2(2e^{2\varphi}\chi^2) \\
&+ 12e^{2\phi-\varphi}(1+e^{2\varphi}\chi^2) - 24e^{2\phi+\varphi}\chi^2 - 24e^{\varphi} + \frac{3}{4}e^{4\phi+\varphi}(16\rho^4)(1+3e^{2\varphi}\chi^2) \\
&+ \frac{9}{2}e^{4\phi+3\varphi}(16\rho^4)\chi^2 + 3e^{4\phi+\varphi}4\rho^2\chi^2(1+e^{2\varphi}\chi^2) + 6e^{4\phi+3\varphi}4\rho^2\chi^4 \\
&- 3e^{2\phi+\varphi}4\rho^2(1-3e^{2\varphi}\chi^2) + 18e^{2\phi+3\varphi}4\rho^2\chi^2] \\
&- \frac{3}{8}gm\chi e^{4\phi+3\varphi}[12\rho^2 + 2\chi^2] + \frac{3}{8}m^2 e^{4\phi+3\varphi} = 0
\end{aligned}
\tag{6.30}
$$

-

$$
\begin{aligned}
\partial_\rho V_g =& 3\rho e^{4\phi+\varphi}\Big[g^2[2\rho^2(1+3e^{2\varphi}\chi^2) + \chi^2(1+e^{2\varphi}\chi^2) \\
&- e^{-2\phi}(1-3e^{2\varphi}\chi^2)] - gm\chi e^{2\varphi}\Big] = 0
\end{aligned}
\tag{6.31}
$$

We need to solve this system of equation in order to find our vacua.
In the following, we will write

$$c = gm \tag{6.32}$$

To find the values we first noticed that taking $\rho = 0$ minimizes the potential along ρ itself (6.31), and thus we try to find vacua with such a value for the field.
With this choice also the other extremizations simplify a lot. For instance (6.28) is null if we take also $\chi = 0$. Thus we recover a potential in two free variables ϕ and φ. Our first approach in trying to find out whether such a potential holds some critical points has been visualizing it graphically. With this first method we've been able to spot indeed numerous minima and identify the values of the remaining free scalar fields in the potential at those critical points. We report our graphic result in (Figure 6.2):

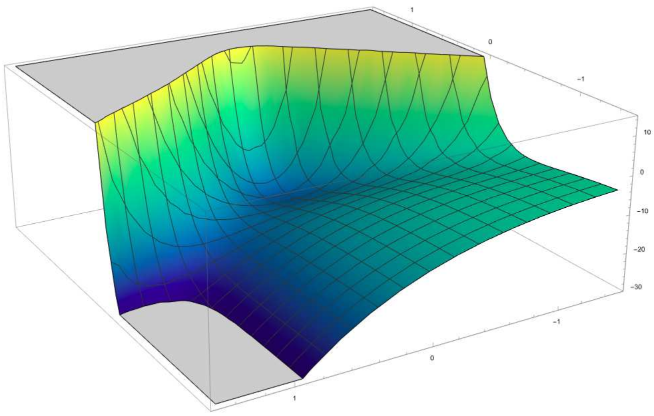

Figure 6.2: Potential in the full dyonically gauged theory with $g = m = 1$ for a slice with $\chi(r)$ and $\rho(r)$ consistently set to zero. The plot clearly shows a minimum close to the null value of ϕ and φ that we report in (Table 6.1). The difference in the presence of a critical point is glaring when we compare with (Figure 6.1).

We will need to check the consistency of the equations of motion when taking constant values of fields at the minima of the potential of course, and inserting these values in the BPS equations we'll find whether some supercharges are conserved or whether supersymmeytry is broken at the minima.
As we can see from (Figure 6.2) we have in fact some critical points (local minima) for small values of the two fields.

Now, plugging the null values for χ and ρ into (6.29) and (6.30) and summing the two equations, we can find two different set of values for ϕ and φ minimizing the potential, namely

$$e^{\phi_0^1} = c^{-1/3}5^{1/6} \qquad\qquad e^{\varphi_0^1} = c^{-1/3}5^{1/6} \tag{6.33}$$

$$e^{\phi_0^2} = c^{-1/3}2^{-1/6} \qquad\qquad e^{\varphi_0^2} = c^{-1/3}2^{5/6} \tag{6.34}$$

If we insert these values in the explicit expression of the potential (6.12) we find the two values of the potential at the minima:

$$V_0^1 = -g^2c^{-1/3}35^{7/6} \qquad\qquad V_0^2 = -g^2c^{-1/3}32^{17/6} \tag{6.35}$$

and as we see this gives us the expexted negative Λ need in AdS.

The last two solutions were just examples of the rich landscape of vacua we can find in our system. Therefore we used Mathematica to solve the system of equations consisting in setting ((6.28)-(6.31)) equa to zero. We obtained the following vacuum solutions:

$c^{-1/3}\chi$	$c^{-1/3}e-\varphi$	$c^{-1/3}\rho$	$c^{-1/3}e^{-\phi}$	$g^{-2}c^{1/3}V_0$
$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	0	$\frac{1}{\sqrt{2}}$	$-4(3^{3/2})$
$-\frac{1}{2^{7/3}}$	$\frac{\sqrt{15}}{2^{7/3}}$	$-\frac{1}{2^{7/3}}$	$\frac{\sqrt{15}}{2^{7/3}}$	$-\frac{2^{28/3}\sqrt{3}}{5^{5/2}}$
$\frac{1}{4}$	$\frac{\sqrt{15}}{4}$	$-\frac{\sqrt{3}}{4}$	$\frac{\sqrt{5}}{4}$	$-\frac{2^8 3^{3/2}}{5^{5/2}}$
0	$\frac{1}{5^{1/6}}$	0	$\frac{1}{5^{1/6}}$	$-35^{7/6}$
0	$2^{1/6}$	0	$\frac{1}{2^{5/6}}$	$-32^{17/6}$
$\frac{1}{2^{4/3}}$	$\frac{3^{1/2}}{2^{4/3}}$	$\frac{1}{2^{4/3}}$	$\frac{3^{1/2}}{2^{4/3}}$	$-\frac{2^{16/3}}{\sqrt{3}}$

Table 6.1: Vacuum expectation value for the scalar fields of the $SU(3)$-invariant theory. In the last column we report the value taken by the potential at the minimun, showing it always yields an asymptotically AdS space.

As we can see, all of the vacua yield a negative value of the potential, thus giving us the desired AdS solution.
In the Euclidea frame, some of the scalar fields will be rotated because of Wick rotation of the coordinates. Therefore, we report the Euclidean version of the scalar potential of the $SU(3)$-invariant subsector:

$$\begin{aligned} V_g^E =& \frac{1}{2}g^2[e^{4\phi-3\varphi}(1-e^{2\varphi}\chi^2)^3 - 12e^{2\phi-\varphi}(1-e^{2\varphi}\chi^2) - 24e^{\varphi}] \\ &+ igme^{4\phi+3\varphi}\chi^3 + \frac{1}{2}m^2e^{4\phi+3\varphi} \end{aligned} \tag{6.36}$$

As it is easy to observe, this potential will yield vacuum configuration with possibly complex extremal values of the field. We point out the fact that, in general, the results we will derive in the following will not depend on which particular vacuum we choose, therefore these complex vacua will not affect our calculations.

6.4 Supersymmetric Black-Hole

The $SU(3)$-invariant subsector Lagrangian (6.7) has been used to study the static supersymmetric black hole solution equations, imposing spherical or hyperbolic horizon symmetry ([60]-[61]). The purpose of this section is to briefly introduce the reader to the matter, as a final remark on the properties on the $SU(3)$-invariant subsector of our theory.

In the following we derive this geometry as arising from our Lagrangian.

The metric ansatz for a static spheric/hyperbolic geometry is:

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}dr^2 + e^{2(\psi(r)-U(r))}\left(d\theta^2 + (\frac{sin\sqrt{k}\theta}{\sqrt{k}})^2 d\phi^2\right) \tag{6.37}$$

The functions $\psi(r)$ and $U(r)$ depend only on the radial coordinate, assumption made valid also for the scalars in the theory.
We see from ζ's e.o.m.'s that it is possible to consistently set those fields to zero, keeping valid their own equation. Moreover, $\partial_{\zeta/\tilde{\zeta}}V_g(\zeta/\tilde{\zeta}=0) = 0$ setting the fields to 0 at the vacuum. Therefore in the following we will always consistently assume $\zeta = 0 = \tilde{\zeta}$.
The potential is independent on the axion field a, and we can set the field consistently to 0 as well.
The ansatz on the electric gauge fields are

$$\mathcal{A}^\Lambda = \mathcal{A}_t^\Lambda(r)dt - p^\Lambda \frac{cos\sqrt{k}\theta}{k}d\phi \tag{6.38}$$

where p^Λ are the constant magnetic charges of the electric gauge fields.
The ansatz for the magnetic field is

$$\tilde{\mathcal{A}}_0 = \tilde{\mathcal{A}}_{t0}(r)dt - e_0\frac{cos\sqrt{k}\theta}{k}d\phi \tag{6.39}$$

where e_0 is identified with the constant electric charge of $\mathcal{A}^0$ upon the duality relation with the electric field, while the one for the two-forms is

$$\mathcal{B}^0 = b_0(r)\frac{sin\sqrt{k}\theta}{\sqrt{k}}d\theta \wedge d\phi \tag{6.40}$$

Now, inserting these last two ansatz in (6.14), we get some constraints for a and b_0:

$$me_0 - gp^0 = 0 \tag{6.41}$$

$$b_0' = e^{4\phi+2\psi-4U}(g\mathcal{A}_t^0 - m\tilde{\mathcal{A}}_{t0}) \tag{6.42}$$

$$a' = 0 \tag{6.43}$$

We thus have a constraint on the charges, the eqations of motion for b_0 and set a to a constant.
We notice that with these choices the equation of motion (6.21) for ϕ reduces to:

$$2d * d\phi = \partial_\phi V_g * 1 \tag{6.44}$$

From (6.15) we derive the equations of motion at first order for the gauge fields, which are:

$$\begin{aligned}\mathcal{A}_t^{0'} =& e^{2U-2\psi-3\varphi}[(p^0 + \frac{1}{2}mb_0)e^{6\varphi}\chi^3 + 3p^1e^{2\varphi}\chi(1+e^2\varphi\chi^2)^2 \\ &- (e_0 + \frac{1}{2}gb_0)(1+e^{2\varphi}\chi^2)^3 - e_1e^{4\varphi}\chi^2(1+e^{2\varphi}\chi^2)]\end{aligned} \tag{6.45}$$

$$\begin{aligned}\mathcal{A}_t^{1'} =& e^{2U-2\psi+3\varphi}[(p^0 + \frac{1}{2}mb_0)\chi + 2p^1e^{-2\varphi}\chi(1+3e^{2\varphi}\chi^2) \\ &- (e_0 + \frac{1}{2}gb_0)e^{-2\varphi}\chi^2(1+e^{2\varphi}\chi^2) - \frac{1}{3}e_1e^{-2\varphi}(1+3e^{2\varphi}\chi^2)]\end{aligned} \tag{6.46}$$

$$\tilde{\mathcal{A}}_{t0}' = e^{2U-2\psi+3\varphi}[(p^0 + \frac{1}{2}mb_0) + 3p^1\chi^2 - (e_0 + \frac{1}{2}gb_0)\chi^3 - e_1\chi]. \tag{6.47}$$

and we are also able to write equations of motion for our metric componets $U(r)$ and $\psi(r)$, which turn out to be

$$\begin{aligned}&\psi'' - U'' + (\psi' - U')^2 + \phi'^2 + \frac{3}{4}(\varphi'^2 + e^{2\varphi}\chi'^2) + \frac{1}{4}e^{4\phi-4U}(\alpha_t^-)^2 = 0 \\ &\psi'' + 2\psi'^2 - e^{-2\psi} + 2e^{-2U}V_g - \frac{1}{2}e^{4\phi-4U}(\alpha_t^-)" = 0\end{aligned} \tag{6.48}$$

where we defined $\alpha_t^- = g\mathcal{A}_t^0 - m\tilde{\mathcal{A}}_{t0}$.
Upon solving these equations, we can recover a black hole geometry.
To impose the asymptotic AdS behaviour and define a proper spherical/hyperbolical geometry close to the horizon, we should set

$$e^{2U} = \frac{r^2}{L^2_{AdS_2}} \qquad\qquad e^{2(\psi-U)} = L^2_{\Sigma_2} \tag{6.49}$$

Furthermore, one can prove BPS equations can be satisfied, thus yielding a supersymmetric solution.

Chapter 7

Searching for Wormholes

We wonder whether equations of motion ((6.14)-(6.24)) can be consistently truncated to recover, in the case of no gauge fields, a solution of Giddings-Strominger Wormhole. The theory should grant us to keep a particle content akin to the one we have in Chapter 3, in order to yield the desired wormhole geometry solutions.
We would need, in order to have a proper GS solution, to keep an effective negative Cosmological Constant on, coming from a minimized potential, in order to recover an asymptotically AdS geometry hinted by the AdS vacua of the theory as in Chapter 6.3. It will turn out that the dependency of the potential on the axio-dilaton pair will prevent us from obtaining a real GS solution, which we will then try to approximate.

In Chapter 7.1, we try to give consistent truncations of our theory to a GS particle content, trying to interprete different particles in the $SU(3)$-invariant subsector of the theory as axions and dilatons.
In Chapter 7.2 we generalize our flat space Giddings Strominger wormhole solution to a non-flat space.
In Chapter 7.3 we apply to our model a method also used in [35] that approximates the solutions we have to GS ones. We will find new interesting relations that suggest the possibility of having GS-like solutions in our model. We will not only limit to the previously introduced $SU(3)$-invariant subsector, but we will also analyze the G_2 and $SO(4)$-invariant ones in our journey to find wormhole solutions.
In Chapter 7.4 a new model is introduced, keeping only vector fields and neglecting scalar fields. This will not yield a GS wormhole solution by definition, but by proving it is linked to the open wormhole solution we will try to embed it in our model.

7.1 Field content truncations

First we need to decide which fields to keep on in our theory and which fields to set to a constant. For such a purpose we need to compare the field content in (6.7) and the one we have in (4.16). The two theories stem from the dimensional compactification of two different frames in String theory, thus we do not expect to have the same solutions, especially because in one theory we have a Minkowski vacuum, while in the other one we have AdS_4.
We will therefore take the equations of motion found in Chapter 6.3 and write them for our purposes in the Euclidean frame.

7.1.1 Hypermultiplet

First we see that taking ζ and $\tilde{\zeta}$ as constants, as already noticed, solves (6.19) and (6.20), once one minimizes the potential along their directions in target space. Rewriting ζ and $\tilde{\zeta}$ using the combination ρ (see (6.27)), we see that $\partial_\rho V_g$ is null for the choice $\rho = 0$, thus also this choice is one that will often be made.

With this assumtpion also the equation of motion of a, (6.18) is satisfied once we set this field to a

constant as well. As we mentioned multiple times, the potential does not depend on a, thus such a field will often be put consistently to 0 in our calculations.
Nevertheless, we will occasionally be interested in leaving this field free and treat it as an axion field in full generality, thus we will extremize other fields in turn, as the above mentioned ζ and $\tilde{\zeta}$ and χ.

Plugging the null values for ρ and a that we found to be consistent with the theory, in (6.21), one gets the simplified equation of motion:

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}\partial_r\phi) = \frac{1}{4}\partial_\phi V_g \tag{7.1}$$

which again can be solved setting ϕ to a constant and minimizing the potential V_g along this field. Nevertheless, we find that setting ϕ to a defined constant in this case is more complicated than in the previous cases, because the potential has exponential terms in this field, multiplying the fields in the vector multiplet, which we'd like to leave funbounded. We see that, in the case in which $\rho = 0$,

$$\begin{aligned}\partial_\phi V_g^E =& \frac{1}{2}g^2[e^{4\phi-3\varphi}(1-e^{2\varphi}\chi^2)^3 - 6e^{2\phi-\varphi}(1-e^{2\varphi}\chi^2)\\ &+ igme^{4\phi+3\varphi}\chi^3 + \frac{1}{2}m^2e^{4\phi+3\varphi}\\ &\equiv 0\\ &\Rightarrow \text{a non trivial relation between } \phi, \varphi \text{ and } \chi.\end{aligned} \tag{7.2}$$

We therefore recover that all the hypermultiplet scalars cannot be easily set to constants consistently, and we will need to work on this issue.

7.1.2 Gauge fields

We can make some choices for our hypermltiplet scalars that allow us to consistently switch off all the gauge fields in our theory (see(6.14)-(6.17)). This is one of the multiple possible choices, but it allows us to recover equations of motion very similar to the ones we found in the non-massive Type IIA case of Chapter 3. In some cases we will choose to keep the gauge fields on, namely when we will treat a as our designated axion field.

7.1.3 Vector multiplet

Now that we established some relations among the hypermultiplet scalar fields and the gauge fields, we can turn to the vector multiplet scalars, which in most of the cases will be the ones of main interest.
The first thing we want to do is to select in our theory the fields that could possibly work as an axio-dilaton pair to derive Giddings-Strominger wormholes solutions:

• We use the results of the previous sections and we rewrite the newfound equations of motion for the fields φ and χ in the vector multiplet and the potential V_g in the theory where ρ is bounded to be $\rho = 0$.
First, we keep the gauge potentials off by assuming $a = \text{const}$.
The potential in the Euclidean frame therefore restricts to

$$\begin{aligned}V_g^E =& \frac{1}{2}g^2[e^{4\phi-3\varphi}(1-e^{2\varphi}\chi^2)^3 - 12e^{2\phi-\varphi}(1-e^{2\varphi}\chi^2) - 24e^{\varphi}]\\ &+ igme^{4\phi+3\varphi}\chi^3 + \frac{1}{2}m^2e^{4\phi+3\varphi}\end{aligned} \tag{7.3}$$

and it gives the same issues that we had with the Lorentzian frame potential, since it has interaction terms among ϕ, φ and χ that do not let us set one of the dilaton-like fields to a constant consistently.
In the Euclidean, (6.23) can be written as

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}\partial_r\varphi) = -e^{2\varphi}(\partial_r\chi)^2 + \frac{1}{3}\partial_\varphi V_g \tag{7.4}$$

while (6.24) takes the form

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}e^{2\varphi}\partial_r\chi) = -\frac{1}{3}\partial_\chi V_g \tag{7.5}$$

The Einstein equations instead have a scalar stress-energy tensor contribution written as

$$\begin{aligned}T^{\text{scalars}}_{\mu\nu} =& \frac{3}{2}(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi) - \frac{3}{2}e^{2\varphi}(\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}\partial_\rho\chi\partial^\rho\chi)\\ &+ 2(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi) - g_{\mu\nu}V_g\end{aligned} \tag{7.6}$$

while the vectorial part of the tensor is set to zero.

We can now compare (7.4)-(7.6) with (4.17) and see that, apart from a a factor $\frac{3}{2}$ that can be reabsorbed redefining the fields, they look alike once we pass to the Euclidean action and change signs to the axionic terms. The matter is, as anticipated in (7.2), we cannot set consistently ϕ to a constant, eliminating its interaction with the preferred axio-dilaton pair (φ,χ).
We can now try to find a solution for these fields such that we remain with one effective axio-dilaton pair. In this case, this would imply the identification of the vector multiplet field φ and the hypermultiplet scalar ϕ by. Introducing $\phi=\varphi=\Phi$, with the truncations made above, this would reduce the dilatons equation of motion to an effective

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}\partial_r\varphi) = -\frac{3}{7}e^{2\varphi}(\partial_r\chi)^2 + \frac{1}{7}\partial_\phi V_g^{\phi=\varphi} \tag{7.7}$$

We can also use the reasoning that we will explain in Chapter 7.3 and consider what we found as a "perturbation" around the exact AdS-GS wormhole solution and check the stability of the geometry applying the methods of Chapter 3. The kinetic term for the new dilaton would be

$$\mathcal{L}^{\Phi}_{\text{kin}} = -\frac{7}{2}d\Phi\wedge *d\Phi \tag{7.8}$$

which, upon redefinition yields a coupling constant

$$\Phi\to\tilde{\Phi}=\sqrt{7}\Phi \Rightarrow \beta = \frac{2}{\sqrt{7}} \tag{7.9}$$

therefore $\frac{1}{\beta^2}=\frac{7}{2}$ and the wormhole would be regular according to (3.18).

In order to give the two dilatons the same expression they need to solve the same equations of motion. Therefore we obtain

$$\begin{aligned}\frac{1}{4}\partial_\phi V_g \;&=\; -e^{2\varphi}(\partial_r\chi)^2 + \frac{1}{3}\partial_\varphi V_g, \qquad \phi(r)=\varphi(r)\\ \Rightarrow -e^{2\varphi}(\partial_r\chi)^2 &= e^{-3\varphi}g^2(4e^{4\varphi}+e^{4\phi}(1-e^{2\varphi}\chi^2)^2 - e^{2(\phi+\varphi)}(5-e^{2\varphi}\chi^2))\end{aligned} \tag{7.10}$$

and imposing ϕ=φ we find the condition

$$g^{rr}\partial_r\chi\partial_r\chi = g^2e^{-\varphi}\chi^2(1-e^{2\varphi}\chi^2) \tag{7.11}$$

so once this is satisfied the two dilaton-like fields can be thought to be equal.
If we take the usual ansatz for the metric

$$ds^2 = f(r)^2dr^2 + a(r)^2d\Omega_3^2 \tag{7.12}$$

we find an equation of the first grade

$$\partial_r\chi = gf(r)\chi e^{-\varphi/2}\sqrt{1-e^{2\varphi}\chi^2} \tag{7.13}$$

The other equation of motion for χ, when imposing our constraint on the dilatons, becomes

$$\frac{1}{\sqrt{g}}\partial_r(\sqrt{g}g^{rr}e^{2\varphi}\partial_r\chi) = e^{3\varphi}g\chi[g(e^{4\varphi}\chi^4 - 3 - 2e^{2\varphi}\chi^2) - e^{4\varphi}im\chi] \tag{7.14}$$

while the Einstein equation structure is clearly derived from (7.6):

$$T^{\text{scalars}}_{\mu\nu} = \frac{7}{2}(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi) - \frac{3}{2}e^{2\varphi}(\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}\partial_\rho\chi\partial^\rho\chi) - g_{\mu\nu}V_g \tag{7.15}$$

Using (7.13) in (7.14) we simplify to an equation of the first order:

$$\frac{1}{f(r)a(r)^3}\partial_r(a(r)^3e^{3\varphi/2}\chi\sqrt{1-e^{2\varphi}\chi^2}) = e^{3\varphi}\chi[g(1-e^{2\varphi}\chi^2)^2 - 4g - e^{4\varphi}im\chi] \tag{7.16}$$

The next step would be solving these first order differential equations numerically, since no analytical solution has turned out to be derivable, as we expected comparing our theory to [35].

• The second case we are interested in is the case of a charged wormhole with a dilaton ϕ and an axion a. We notice that a is the only particle in action (6.7) not to appear in the potential. Because of this, a shows a shift symmetry typical of axions. In this case the wormhole would need to be charged since we cannot switch off consistently the gauge fields (compare with equations of motion (6.14)-(6.17)). We find that keeping the gauging on, we cannot consistently put χ to a constant and thus we would end up with two seemingly unmatchable axio-dilaton pairs (different charges and different dilaton fields). We therefore try not to solve this system as an axio-dilaton one.

• The third case is the one in which we keep an axionic ζ or $\tilde{\zeta}$. This case yields particularly simple equations of motion compared to the previous ones, since we can now consistently put both a and χ to 0.
We would also like to have a single dilaton in our Lagrangian, but we once again have the problem of having both ϕ and φ appearing in the potential and inherently in the equations of motion. Therefore, we would like to identify the two as we did in the previous case. This turns out to yield a system of first order equations of motion that one could solve numerically.
We impose the two fields ϕ and φ solve the same equation of motion and thus have the same profile. This brings to equations of motion that mirror the ones we found in the first case we analyze, with ζ instead of χ.
In particular, requiring that the dilaton's equations of motion are solved by the same field reads

$$\begin{aligned} \frac{1}{4}\partial_\phi V_g &= +\frac{1}{4}e^{2\phi}(\partial_r\zeta)^2 + \frac{1}{3}\partial_\varphi V_g, \qquad \phi(r) = \varphi(r) \\ &\Rightarrow g^{rr}\partial_r\zeta\partial_r\zeta = g^2e^{\phi}\zeta^2(e^{2\phi}\zeta^2 - 1) \end{aligned} \tag{7.17}$$

or equivalently for $\tilde{\zeta}$.
The equation of motion for ζ instead reads

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}e^{2\phi}\partial_r\chi) = \frac{3}{2}e^{3\phi}g^2\zeta(e^{2\phi}\zeta^2 - 2) \tag{7.18}$$

which is even simpler than case (7.14).
The Einstein equations on the other hand display a much more involved scalar stress-energy tensor of the form:

$$\begin{aligned} T^{scalars}_{\mu\nu}/2 = &\frac{3}{2}(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi) + 2(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi) \\ &+\frac{1}{2}e^{2\phi}(\partial_\mu\zeta\partial_\nu\zeta - \frac{1}{2}g_{\mu\nu}\partial_\rho\zeta\partial^\rho\zeta) + +\frac{1}{2}e^{2\phi}(\partial_\mu\tilde{\zeta}\partial_\nu\tilde{\zeta} - \frac{1}{2}g_{\mu\nu}\partial_\rho\tilde{\zeta}\partial^\rho\tilde{\zeta}) \\ &+\frac{1}{2}e^{4\phi}(\xi_\mu\xi_\nu - \frac{1}{2}g_{\mu\nu}\xi_\rho\xi^\rho) - g_{\mu\nu}V_g \end{aligned} \tag{7.19}$$

As we anticipated, also in this case it is not possible to find an analytical solution for the field, and a more accurate numerical analysis would be needed., solving first order equations.

7.2 Exact AdS_4-GS Wormhole solutions

We wish to recover Giddings-Strominger solutions as in (3.2) with an explicit spectrum given by an axio-dilaton couple as in (4.16).
As we have seen in the previous paragraph, such a configuration is not practically achievable in our model, because we have a potential depending on the axio-dilaton paair(s), but we would like to see whether it gives simple analytic solutions of the metric and scalar fields.

The Euclidean action we would work with is

$$\frac{1}{2k^2}\int d^4x\sqrt{-g}\Big((\mathcal{R}-2\Lambda)-\frac{1}{2}(\partial\varphi)^2+\frac{1}{2}e^{2\varphi}(\partial\chi)^2\Big) \tag{7.20}$$

where we fixed the axio-dilaton coupling b to 2 as this is its value when such an interactive term is derived both from IIB (on S^5) and massive IIA in our reduction. Such an Euclidean action is already studied and the results we are seeking for have already been derived in 5 and 3 dimensions in [62]. We try and find new similar solutions in 4 dimensions, using the same line of reasoning.
We wish to use the results (4.18) and generalize them to a curved background space.
From the action, we recover once again the general equations of motion (3.5) and the Einstein equation (3.4), with the difference of having a cosmological constant-dependent terms as in the usual AdS case:

$$R_{\mu\nu}-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}e^{2\phi}\partial_\mu\chi\partial_\nu\chi-\Lambda g_{\mu\nu}=0 \tag{7.21}$$

with $\Lambda=-\frac{(D-1)(D-2)}{2l^2}$ as usual.
The solutions found in [62] for the 5 dimensions in the sub-extremal case are:

$$\begin{aligned}
ds^2 &= \frac{dr^2}{1+\frac{r^2}{l^2}-\frac{\tilde{q}^2}{r^6}}+r^2d\Omega_4^2\\
e^{b\phi(r)} &= \left(\frac{q_-}{\tilde{q}}\sin(\tilde{q}H(r))\right)^2\\
\chi(r) &= \frac{2}{bq_-}(\tilde{q}\cot(\tilde{q}H(r)-q_3))
\end{aligned} \tag{7.22}$$

and as we expected they mirror (4.18).
With this solution in mind, exploiting our knowledge of the sub-extremal instanton solution [49], corresponding to a wormhole one, in 4 dimensions, we guess a solution for the metric of the type

$$ds^2=\frac{dr^2}{1+\frac{r^2}{l^2}-\frac{\tilde{q}^2}{r^4}}+r^2d\Omega_3^2 \tag{7.23}$$

while the forms of the fields ϕ and χ are the same as in (7.22), but with a different shape for $H(r)$.
It is clear that once again this ansatz for the metric was chosen in order to recover, in the limit $r\to\infty$, the pure AdS solution (1.20). The $\tilde{q}$ dependent term instead is the one ensuring the wormhole-symmetric geometry.
It turns out the function $H(r)$ needs to be harmonic and, simply using our ansatz and trying to solve the e.o.m. for the field χ, ϕ and Einstein, one finds that it need to be shaped like

$$\partial_r H(r)=\frac{\sqrt{6}}{r^3}\sqrt{\frac{1}{1-\frac{q^2}{r^4}+\frac{r^2}{l^2}}} \tag{7.24}$$

When one tries and solve this analytically, no solution is found. Thus we'll leave the solution implicitly written as a function of $H(r)$, where the condition (7.24) is intended as a constraint.

For completeness, we give also the general result (which is anyway quite intuitive) for a solution of the GS wormhole geometry with AdS background in d dimensions:

$$ds^2 = \frac{dr^2}{1 + \frac{r^2}{l^2} - \frac{\tilde{q}^2}{r^{2(d-2)}}} + r^2 d\Omega_{d-1}^2 \qquad \partial_r H(r) = \frac{\sqrt{(d-1)!}}{r^{d-1}} \sqrt{\frac{1}{1 - \frac{q^2}{r^{2(d-2)}} + \frac{r^2}{l^2}}} \tag{7.25}$$

In this case, the symmetric structure of space-time due to wormhole geometry can appear more obscure, so we'll use a change of coordinates to make it clearer.
We want to write the metric with a new fitting effective radial coordinate $\rho(r)$ as

$$ds^2 = d\rho^2 + a(\rho)^2 d\Omega_3^2 \tag{7.26}$$

where ρ is a radial coordinate that goes from $-\infty$ to $+\infty$ (i.e. the two asymptotic AdS regions).

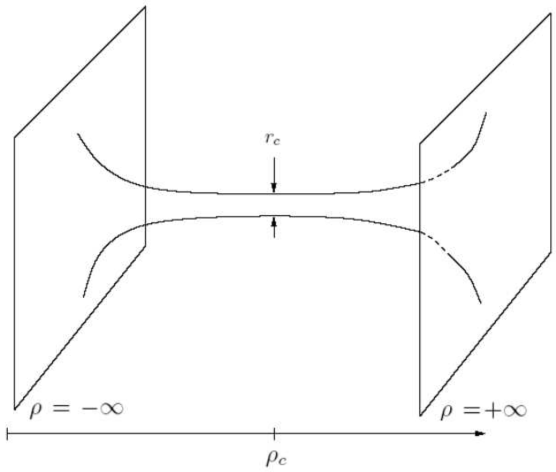

Figure 7.1: [62] Wormhole geometry arising from an asymptotically AdS space (the asymptotic spaces are at $\rho = \pm\infty$).

The function $a(\rho)$ can be viewed as a scale factor and is clearly equal to r in the old coordinate system. From the relation between the two coordinate systems we derive

$$\begin{aligned} d\rho^2 &= \left(\frac{\partial \rho}{\partial a}\right)^2 dr^2 \Rightarrow \\ &\Rightarrow \left(\frac{\partial a}{\partial \rho}\right)^2 = 1 - \frac{\tilde{q}^2}{a^4} + \frac{a^2}{l^2} \end{aligned} \tag{7.27}$$

which we are not interested to solve, but we notice that:

• the coordinate system doubles the geometry, with $a(\rho)$ going to $+\infty$ for both $\rho = \pm\infty$;

• we have a point named ρ_c, such that a reflection of $a(\rho)$ with respect to this point leaves the geometry unchanged. We do not calculate directly ρ_c, but we know in the old coordinates this corresponds to a derivable r_c such that

$$a(2\rho_c - \rho) = a(\rho) \tag{7.28}$$

where ρ_c is clearly a fixed point, thus corresponding to the middle of the throat of the wormhole.

7.3 Approximated solutions

If one wants to analytically find a wormhole geometry solution in the dyonically gauged $ISO(7)$ Supergravity model, it is no surprise by just looking at the complicated expression of Lagrangian (5.46) and the theory's plenty of fields, that this is a very hard task. We showed this in the previous sections.
This is why we restruct to subsectors of the main $ISO(7)$ theory, keeping a residual invariance under some symmetry.
What we try to do in the following is finding approximate GS wormhole solution in an AdS background by expanding the fields around the vacuum value they assume when minimizing the potential arising in these "sub-theories" by truncating potential (5.47) accordingly to the residual scalar field content.
This very useful tool to try and find asymptotic exact Giddings-strominger wormhole solutions has been previously used in [35] as a starting point for numerical solutions.
Approximation will be made, and we will stress the fact in the following every time it will be necessary.

7.3.1 The $SU(3)$ invariant subsector

The first thing we need to do is finding which ones of the fields appearing in the action (6.7) have a kinetic term swapping sing when taking the analytic continuation to Euclidean space-time. In order to do this, we refer to [63] and spot the scalar degrees of freedom coming from the R-R sector and the ones coming from the NS-NS sector. Since we have tensor fields in both the R-R and NS-NS sector, having a Levi-Civita tensor in front of their kinetic term, the fields appearing as scalar degrees of freedom stemming from the reduction of such kinetic terms will change sign in 4 dimensions, while this does not happen for the ones derived in the reduction from the internal degrees of freedom of scalar 10-dimensional fields. For a less String theory-dependent derivation of the swap in sign of the axion kinetical term, refer to Appendix A, where a path integral derivation of the rotating term is obtained. With the string-derived approach, we find that

- for the hypermultiplet, ζ and $\tilde{\zeta}$ derive from p-forms in the R-R sector compactification, while a comes from the 2-form B_2 of the NS-NS sector. On the other hand, ϕ comes from the metric degrees of freedom. Therefore ζ, $\tilde{\zeta}$ and a have sign-swapping kinetic terms and ϕ does not;
- for the vector multiplet, the field χ swaps sign, coming from the compactification of R-R d.o.f.s, while φ comes from scalar d.o.f.'s in the NS-NS sector.

We have seen that because of the potential V_g in (6.7) one cannot obtain Giddings-Strominger solutions. Anyway, we can derive an approximate solution working at a minimum of the potential and expanding some of the fields (the ones that are useful for our purposes) around the vacuum. There are several ways one can do this, choosing different fields in the role of the axio-dilaton pair(s). We try to test all these cases and check their stability using (3.18) as a criterion.
Notice that we are using an approximation and, in the case where we kept the field a, we also switched off the vector bosons which would give a charged wormhole solution instead of the usual neutral one. Thus, the following solutions are intended just as outlooks on interesting but asymptotical cases. Anyway, they can give much insight on what the regular wormhole solution actually looks like, being extrema of the solution. A numerical analysis of the full theory, in a similar fashion to what was done in [35], would offer a clearer sight on the way these asymptotic case evolve once we unbind the potential from its cosmological constant role.

We now review all of the choices for axio-dilaton pair made in Chapter 7.1 under this key, trying to find stable asymptotic wormhole solution under the criteria exposed in Chapter 3.2:

- (φ, χ): The most obvious case is the one where the vector multiplet fields are interpreted as the axio-dilaton pair. We need to choose now a vacuum of the potential. For simplicity, we could choose to use the first AdS_4 vacuum configuration in (Table 6.1). First, we set ρ and a to zero , then we also set the other scalar fields to the constants which will minimize the potential. The gauge fields will be all turned off (this time, since no a or ρ appears in the action, their disappearance is fully consistent). The potential takes its minimum value $V(\varphi) = -\Lambda$, thus yielding an Anti-deSitter vacuum solution. The action thus is

$$S = \frac{1}{2k^2} \int d^4x(\mathcal{R} + \Lambda) - \frac{1}{2} d\tilde{\varphi} \wedge *d\tilde{\varphi} + \frac{1}{2} e^{2/\sqrt{3}\tilde{\varphi}} d\tilde{\chi} \wedge *d\tilde{\chi} \tag{7.29}$$

  where we reabsorbed the coefficient to give the standard kinetic terms for the dilaton and the axion. In particular

$$\varphi \to \tilde{\varphi} = \sqrt{3}\varphi \qquad \chi \to \tilde{\chi} = \sqrt{3}\chi \tag{7.30}$$

  Thus we end up with the axio-dilaton pair $(\tilde{\varphi}, \tilde{\chi})$, with couling constant $\beta = \frac{2}{\sqrt{3}}$
  Checking the regularity of the solution, one finds

$$\frac{3}{4} = \left(\frac{D_4(0)}{2\pi}\right)^2 = \frac{1}{\beta^2} = \frac{3}{4} > \left(\frac{D_4(\infty)}{2\pi}\right)^2 = \frac{1}{3} \tag{7.31}$$

What does this mean? It signifies that our coupling saturates exactly the condition of stability for the wormhole. Thus, in this configuration the solution will be smooth and we can recover our approximated GS-AdS solution.
It is very important that such a result has been found, because, as already said, despite being a strong approximation, it tells us that in the proximity of a minimum of the potential such a solution must exist, but the full picture needs to be recovered numerically and not analytically. If we had a value of $\frac{1}{\beta^2}$ in between the two values for null and infinite curvature, we would have that for some values of a_0, the throat width of the wormhole, the solution would be unstable, and one would have to distinguish which ones would be the case. We also point out that, actually, the relevant quantity that separates stable and unstable solutions is $\frac{a_0}{l}$, relating the throat size and the typical scale of AdS. In the case of an ambiguous stability for the wormhole, we would be able to easily find which is the value of $\frac{a_0}{l}$ which saturates our condition using the formulae of Chapter 3.

- $(\phi, \zeta/\tilde{\zeta})$: One could now try to keep only ζ (or equivalently $\tilde{\zeta}$) as an axion-like particle inside the action, again switching off the gauge fields and choosing the same vacuum as before, keeping χ and φ at the bottom of the potential, setting a to zero with the equations of motion.
One would therefore recover the action

$$S = \frac{1}{2k^2} \int d^4x(\mathcal{R} + \Lambda) - \frac{1}{2} d\tilde{\phi} \wedge *d\tilde{\phi} + \frac{1}{2} e^{2\tilde{\phi}} d\hat{\zeta} \wedge *d\hat{\zeta} \tag{7.32}$$

where in this case we rescaled

$$\phi \to \tilde{\phi} = 2\phi \qquad \zeta \to \hat{\zeta} = \zeta \tag{7.33}$$

The regularity check, with $\beta = 1$ reads

$$\frac{1}{\beta^2} = 1 \; > \frac{3}{4} = \left(\frac{D_4(0)}{2\pi}\right)^2 > \left(\frac{D_4(\infty)}{2\pi}\right)^2 = \frac{1}{3} \tag{7.34}$$

Again, we found that the wormhole would be stable with this axio-dilaton couple for any value of $\frac{a_0}{l}$.

- (ϕ, a): We now want to use the fields a and ϕ as our axio-dilaton pair.
We set the gauge fields to zero and "artificially" set the scalar fields on the vacuum $V_g = -\Lambda$. We wish to keep the fields ϕ and a, setting all the other ones to constants. As we already pointed out, this is just an expansion around the vacuum and its usefulness has been probed in past works, but from (6.14)-(6.17) we can notice that a and the vector potentials are somehow bound by e.o.m.'s, therefore keeping the scalar field on and setting the gauge fields to constants is a particularly strong approximation.
The action we get is

$$S = \frac{1}{2} \int d^4x(\mathcal{R} + \Lambda) - \frac{1}{2} d\tilde{\phi} \wedge *d\tilde{\phi} + \frac{1}{2} e^{2\tilde{\phi}} d\tilde{a} \wedge *d\tilde{a} \tag{7.35}$$

where again we set

$$\phi \to \tilde{\phi} = 2\phi \qquad\qquad a \to \tilde{a} = a \tag{7.36}$$

In this case we have $\beta = 2$, thus

$$\left(\frac{D_4(\infty)}{2\pi}\right)^2 = \frac{1}{3} \; > \; \frac{1}{\beta^2} = \frac{1}{4} \tag{7.37}$$

Therefore, even if we were in the presence of a field that could generate a simple GS wormhole geometry without the presence of gauge fields, the solution would never yields a regular GS wormhole solution and we reject the possibility of having these two fields as our axio-dilaton pair.

In summary, what results have we obtained for the $SU(3)$-invariant subsector? We know that setting χ and ρ to zero, keeping a as an axion, yields a non regular solution, and no open wormhole can exist. On the other hand, putting a to zero and keeping χ will give an open wormhole geometry saturating the stability bound. Setting alternatively this fields to zero (and inherently decoupling from the theory its associated dilaton φ) and keeping either ζ or $\tilde{\zeta}$ and the dilaton ϕ, one recovers a fully stable GS womhole geometry. Most likely, the real coupling β will lie below the instability region in (3.15), yielding an open wormhole with $\frac{2}{\sqrt{3}} < \beta < 1$.

7.3.2 The G_2 invariant sector

If we expand the residual symmetry to

$$SO(7) \supset G_2 \tag{7.38}$$

we can write the invariant bosonic Lagrangian density, which turns out to have no vector or tensor fields in it, (see for example [57]) as:

$$\mathcal{L} = (\mathcal{R} - V)\text{vol}_4 - \frac{7}{2}[d\varphi \wedge *d\varphi + e^{2\varphi} d\chi \wedge *d\chi] \tag{7.39}$$

where the scalar potential is

$$V = \frac{7}{2} g^2 e^{\varphi}(1 + e^{2\varphi}\chi^2)^2(-5 + 7e^{2\varphi}\chi^2) - 7gme^{7\varphi}\chi^3 + \frac{1}{2}m^2 e^{7\varphi} \tag{7.40}$$

The only critical points of this potential are:

$g^{-2}c^{1/3}V_0$	$c^{-1/3}\chi$	$c^{1/3}e^{\phi}$
$-15(5^{1/6})$	0	$5^{\frac{1}{6}}$
$-\frac{32(2^{1/3})}{\sqrt{3}}$	$\frac{1}{2(2^{1/3})}$	$\frac{2(2^{1/3})}{\sqrt{3}}$

Table 7.1: Minima found for the Lorentzian scalar potential in the G_2-invariant subsector.

and as we can see, they are all AdS, just like in the previous analyzed sector. We can therefore try and apply our approximated calculations to this subsector as well.
The equations of motion associated to (7.39) are:

$$\begin{aligned}
(d * d\varphi) &= e^{2\varphi} d\chi \wedge *d\chi + \frac{1}{7}(\partial_\varphi V_g) \\
d(e^{2\varphi} * d\chi) &= \frac{1}{7}(\partial_\chi V_g)
\end{aligned} \tag{7.41}$$

The Einstein equations instead read

$$\begin{aligned}
&R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = T^{\text{scalars}}_{\mu\nu} \\
&T^{\text{scalars}}_{\mu\nu} = \frac{7}{2}(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi) \\
&+ \frac{7}{2}e^{2\varphi}(\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}\partial_\rho\chi\partial^\rho\chi) - \frac{1}{2}g_{\mu\nu}V_g
\end{aligned} \tag{7.42}$$

All the given equations are clearly defined in the Lorentzian frame. Once turning to the Euclidean theory, χ takes the role of the axion field and φ plays the part of the dilaton, as it should be clear from a quick comparison with previous subsectors.
We now want to apply the same reasoning as before, approximating the action with a curved-space GS one (minimizing the potential but keeping some fields on) in order to derive (hopefully) stable GS wormhole solutions. The vacuum configurations of the theory turn out to give a negative value of the minimized potential in both cases (see Table 7.1). In our approximation, it will be indifferent which one we'll pick. Therefore, we'll choose one and simply call (for instance the first one in Table 6) $V_0 = \Lambda$.

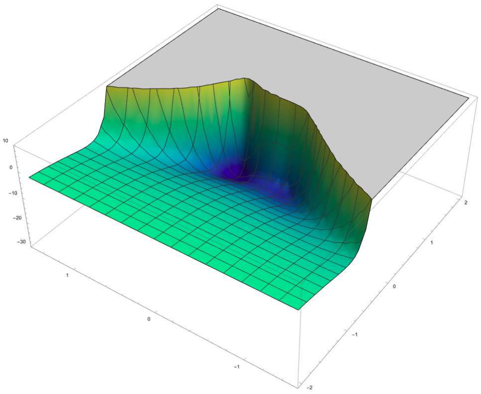

Figure 7.2: Minima of the full potential in the G_2-invariant subsector having set $m = g = 1$.

- (φ, χ): The only axio-dilaton pair in the subset is the one formed by (φ, χ) and we can rescale the fields as

$$\varphi \to \tilde{\phi} = \sqrt{7}\varphi \qquad \chi \to \tilde{\chi} = \sqrt{7}\chi \tag{7.43}$$

The coupling constant is then:

$$\beta = \frac{2}{\sqrt{7}} \;\Rightarrow\; \frac{1}{\beta^2} = \frac{7}{4} \tag{7.44}$$

Therefore this subsector of the $ISO(7)$ gauged theory always yields smooth AdS_4 GS approximated solutions with $\beta = \frac{2}{\sqrt{7}}$.

Notice that this coupling $\beta = \frac{2}{\sqrt{7}}$ was also present in our attempt at finding a proper axio-dilaton pair in the full theory by reducing ϕ and φ to the same field.

7.3.3 The $SO(4)$ invariant subsector

We now wish to analyze the particle content of the 4D theory displaying an invariance under the $SO(4)$ subgroup of $ISO(7)$.
Such a sub-theory contains only the metric and four real scalars (φ, χ, ρ and ϕ). From the coset representatives we recover as usual the scalar content of the action and one can prove that the action [57] in the Lorentzian signature turns out to be

$$\begin{aligned} S =& \frac{1}{2k^2}\int d^4x(\mathcal{R} - V)\mathrm{vol}_4 - \frac{6}{2}[d\varphi \wedge *d\varphi + e^{2\varphi}d\chi \wedge *d\chi] \\ & - \frac{1}{2}[d\phi \wedge *d\phi + e^{2\phi}d\rho \wedge *d\rho] \end{aligned} \tag{7.45}$$

where the potential is, notably, a function of all four scalar fields:

$$\begin{aligned} V =& \frac{1}{2}g^2e^{-\phi}(1 + e^{2\varphi}\chi^2)[-24e^{\phi+\varphi} - 8e^{2\phi} + e^{2\varphi}(-3 + (8\chi^2 - 3\rho^2)e^{2\phi})] \\ & + e^{4\varphi}\chi^2(9 + (3\rho + 4\chi)^2e^{2\phi})] - gm\chi^2(3\rho + 4\chi)e^{6\varphi+\phi} + \frac{1}{2}m^2e^{6\varphi+\phi} \end{aligned} \tag{7.46}$$

In order to perform our approximation, we need to make sure that our potential has some vacuum configurations, and we would like them to be AdS as well.

$g^{-2}c^{1/3}V_0$	$c^{-1/3}$	$c^{-1/3}\rho$	$c^{1/3}e^{\phi}$	$c^{1/3}e^{\varphi}$
$-15(5^{1/6})$	0	0	$5^{1/6}$	$5^{1/6}$
$-16(2^{1/3})$	0	0	$2^{1/3}$	$2^{-2/3}$

Table 7.2: Minima of the potential of the $SO(4)$-invariant subsector.

We therefore find that the solution has vacuum configurations, just like in the $SU(3)$, only when the magnetic charged $m \neq 0$. This once again proves the necessity of having a massive IIA theory for the embedding. All these vacua are moreover Anti-deSitter.
As one can easily see from the expession of the action, once we minimize the potential along one of the two axio-dilaton couples we have (namely along (φ, χ) or (ϕ, ρ)), we can set consistently those fields to constant values. Anyway, we do not recover a simple curved GS solution due to the dependence of

the potential on the remaining fields.
For this reason we once again use our approximation, setting the potential to a constant but keeping the fields of interest as an expansion around their minima.
Keeping one axio-dilaton pair we write

- (φ, χ): we rescale both the fields as to find the action

$$S = \frac{1}{2k^2} \int d^4(\mathcal{R} - \Lambda)\text{vol}_4 - \frac{1}{2} d\tilde{\varphi} \wedge *d\tilde{\varphi} + \frac{1}{2} e^{2/\sqrt{6}\tilde{\varphi}} d\tilde{\chi} \wedge *d\tilde{\chi} \tag{7.47}$$

  and the coupling $\beta = \frac{2}{\sqrt{6}}$ yields regular GS wormholes in Anti-deSitter since

$$\frac{1}{\beta^2} = \frac{3}{2} > \frac{3}{4} = \left(\frac{D_4(0)}{2\pi}\right)^2 \tag{7.48}$$

- (ϕ, ρ): in this case we have no regular wormhole. Indeed the action is already in the canonical form once we put φ and χ to constants, and the coupling constant is $\beta = 2$, which is always too big for our purposes.

These results tell us that in all likelihood, the real solution will interpolate between the open wormhole solution in which we set to a constant value the "axion-like" particle ρ, and a full-spectrum configuration. Because of this, we will have an actual solution which will have an axio-dilaton coupling with an upper bound of $\frac{1}{\beta^2} > \frac{3}{4}$.
On top of this, remembering (3.19), one can allow for a solution including both axio-dilaton pairs. The so-defined metric in the target space would yield a time-like geodesic with length

$$\sum_i \frac{1}{\beta_i^2} = \frac{7}{4} \tag{7.49}$$

The geometry would then be smooth and the fields regular.

7.4 Gauged model with no scalar fields

In this section, we would like to pass from a mainly scalar-driven wormhole solution to a frame where we make use of gauge fields to have a stable wormhole solution. Thus, we analyze some before known Supergravity theories in 4 dimensions, justifying their connection to the wormhole solution, and try and embed them in our dyonically gauged $ISO(7)$ model. Clearly, since the spectrum does not show any axion, our solutions are not of the Giddings-Strominger type.

7.4.1 The model

We wish to study the wormhole solution found in [36] and [37].

The model is much different from the ones we studied in previous chapters, but it still enables us to find a non-GS traversable wormhole solution in the Lorentzian frame.
First, we shortly review the subject outlining the structure of the model and its main features.

We build an asymptotically Anti-deSitter 4D wormhole metric in Lorentzian space-time starting from a 3-dimensional AdS metric and writing the 4D AdS metric in function of its 3D slices [37] (in an AdS remodeling of Boyer-Lindquist [64] coordinates for $J = 0$):

$$ds^2 = \frac{l^2 dr^2}{r^2+1} + a(r)^2 ds^2_{AdS_3} = \frac{l^2 dr^2}{r^2+1} + \frac{l^2}{4}(r^2+1)[-\cosh(\theta)^2 dt^2 + d\theta^2 + (du + \sinh(\theta)dt)^2] \tag{7.50}$$

where the $(r^2 + 1)$ factor determines the expansion (and contraction) of the 3-sphere section as a function of r. We see that it is symmetric with respect to $r_0 = 0$ (as well as for the radial metric

component), thus yielding an open wormhole geometry for the criteria we used above. For $r \to \infty$ instead we recover AdS vacuum geometry. Anyway, with a change of coordinates we can actually prove that this metric is simple AdS with a spherical boundary, as we can see from the fact that the metric proves to be simple Einstein.
Anyway, considering a generalization of this metric

$$ds^2 = \frac{4l^2 dr^2}{\sigma^2 f(r)} + g(r)(-\cosh^2(\theta)dt^2 + d\theta^2) + f(r)(du + \sinh(\theta)dt)^2 \tag{7.51}$$

which satisfies the AdS-behavior Einstein equation:

$$R_{\mu\nu} = -\frac{3}{l^2} g_{\mu\nu} \tag{7.52}$$

we find a solution

$$\begin{aligned} g(r) &= \frac{l^2}{\sigma}(r^2+1) \\ f(r) &= \frac{4l^2}{\sigma^2} \frac{r^4 + (6-\sigma)r^2 + lmr + \sigma - 3}{r^2+1} \end{aligned} \tag{7.53}$$

which is everywhere smooth iff $12 > \sigma > 3$, $|lm| < \frac{2}{3\sqrt{3}}(12-\sigma)\sqrt{\sigma-3}$ for $-\infty < r < \infty$ and the Kretschmann invariant is never ∞.
This metric cannot be rescaled to a simple AdS one and thus we interpret it as a wormhole solution.

We can generate such a solution in an $N = 2$ pure Supergravity model [36], with just a supermultiplet $(g_{\mu\nu}, \psi^\alpha_\mu, A_\mu)$, gauged under a subgroup $\mathcal{H}$ of the R-symmetry group of the theory SU(2).
If we restrict to the bosonic content of the Lagrangian, assuming that in the full action the Supersymmetry conditions are satisfied for a killing spinor ε^α, we write the remaining field content as $(M_p = 1)$ [65],[66]:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\Big(\mathcal{R} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 3q^2\Big) \tag{7.54}$$

where q is the charge of the gravitino under the gauge symmetry and, assuming $\mathcal{H}$ abelian, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
We derive the equations of motion with ease, which turn out to be

$$\begin{aligned} &\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0 \\ &R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = -\frac{1}{2}[F_{\mu\rho}F^\rho_\nu - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}] - 6q^2 g_{\mu\nu} \end{aligned} \tag{7.55}$$

and comparing the Einstein equation in (7.55) with that in (7.52), we substitute $l^2 \to \frac{1}{2q^2}$ and a "Reissner-Nordstrom"-like term accounting for the presence of gauge fields and find a solution for the metric:

$$\begin{aligned} g(r) &= \frac{1}{2q^2\sigma}(r^2+1) \\ f(r) &= \frac{2}{q^2\sigma^2}\frac{r^4 + (6-\sigma)r^2 + mr + \sigma - 3}{r^2+1} - \frac{Q^2+P^2}{r^2+1} \end{aligned} \tag{7.56}$$

Where P and Q are the magnetic and electric parameters determining the charges. The parameter m is the "mass" of the space-time, while σ is linked to the warping of the asymptotic region.
The solution for the gauge field we find is instead given by:

$$A = \Phi(r)(du + \sinh(\theta)dt), \qquad \Phi(r) = \frac{2Qr + P(1-r^2)}{1+r^2} \tag{7.57}$$

When translated to $F_{\mu\nu}$, it turns out that we will have both electric and magnetic components, thus justifying the presence in the solution of both Q and P.

7.4.2 Embedding within the dyonic $ISO(7)$ theory

We would like to recover the same particle content of this model by truncating some subsector of the main $ISO(7)$ theory.

The requests will be that this subsector needs to enjoy from the start (at least) an $\mathcal{N} = 2$ supersymmetry, have (at least) one gauged scalar and have (at least) one AdS vacuum.
We fist look at our $SU(3)$-invariant subsector described in detail in Chapter 6:

- The theory has an extended $\mathcal{N} = 2$ supersymmetry;
- The theory has a gauge invariance under $SO(1,1) \times U(1)$, thus showing two different gauge potentials. The further symmetry is gauged dyonically while the last one is gauged only electrically;
- As we pointed out in Chapter 6.3.2, we have several vacua upon setting our scalar fields to appropriate constant values, all of which are AdS as we desire.

Thus, we choose this subsector as a preferred frame in which to embed our theory. The other subsectors of the theory, such as the G_2 and $SO(4)$-invariant ones, do not show any supersymmetry left, and therefore cannot be truncated to a $\mathcal{N} = 2$ theory.

As we previously recovered, the scalar potential has minima only for a magnetic coupling constant different from zero. Therefore we will need to keep our magnetic fields on if we wish to have a symple AdS asymptotical behaviour.
Therefore the equations of motion simplify to

$$\begin{aligned} d\mathcal{B}^0 &= 0 \qquad & d\tilde{\mathcal{A}}_0 + \frac{1}{2}g\mathcal{B}^0 &= \mathcal{I}_{0\Lambda} * \mathcal{H}^\Lambda + \mathcal{R}_{0\Lambda}H^\Lambda \\ d(\mathcal{I}_{0\Lambda} * H^\Lambda + \mathcal{R}_{0\Lambda}H^\Lambda) &= 0 \qquad & d(\mathcal{I}_{1\Lambda} * H^\Lambda + \mathcal{R}_{1\Lambda}H^\Lambda) &= 0 \end{aligned} \tag{7.58}$$

as one can see by only looking at (6.7). Again, the first three equations are redundant and we can eliminate one without losing information.
The full Einstein equation reads:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = -\mathcal{I}_{\Lambda\Sigma}[\mathcal{H}^\Lambda_{\mu\rho}\mathcal{H}^{\Sigma\rho}_\nu - \frac{1}{4}g_{\mu\nu}\mathcal{H}^\Lambda_{\rho\sigma}\mathcal{H}^{\Sigma\rho\sigma}] - g_{\mu\nu}\Lambda \tag{7.59}$$

We notice that in the expression of the equations of motion, we need to take in account all three gauge fields of our theory (two electric fields A^0 and A^1 and one magnetic field $\tilde{A}_0$).
From the definition of the fluxes in the full theory (5.49), we recover the fluxes definitions in the $SU(3)$-invariant subsector as [57]:

$$\begin{aligned} \mathcal{H}^0 &= d\mathcal{A}^0 + m\mathcal{B}^0 \qquad & \mathcal{H}^1 &= d\mathcal{A}^1 \\ \tilde{\mathcal{H}}_0 &= d\tilde{\mathcal{A}}_0 + g\mathcal{B}^0 \qquad & \tilde{\mathcal{H}}_1 &= d\tilde{\mathcal{A}}_1 - 2g\mathcal{B}_2 \end{aligned} \tag{7.60}$$

From the expression of $\mathcal{M}_{MN}$ for the $SU(3)$ subsector (see Appendix B), we can recover the scalar-gauge matrix [57]:

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{R}_{\Lambda\Sigma} + i\mathcal{I}_{\Lambda\Sigma} = \frac{1}{2e^\varphi\chi + i}\begin{bmatrix} -\frac{e^{3\varphi}}{(e^\varphi\chi - i)^2} & \frac{3e^{2\varphi}\chi}{e^\varphi\chi - i} \\ \frac{3e^{2\varphi}\chi}{e^\varphi\chi - i} & 3(e^\varphi\chi^2 + e^{-\varphi}) \end{bmatrix} \tag{7.61}$$

from which one can get its (constant values) choosing an AdS vacuum.
For instance, from the fourth vacuum in (Table 6.1), one gets

$$\mathcal{R}_{\Lambda\Sigma} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathcal{I}_{\Lambda\Sigma} = \begin{bmatrix} -\frac{\sqrt{5}}{c} & 0 \\ 0 & -3\frac{c^{1/3}}{5^{1/6}} \end{bmatrix} \tag{7.62}$$

such that the equations of motion in (7.58) can be written in the more straightforward manner:

$$\begin{aligned} &d\mathcal{B}^0 = 0 \qquad && d\tilde{\mathcal{A}}_0 + \frac{1}{2} g \mathcal{B}^0 = -\frac{\sqrt{5}}{c} * \mathcal{H}^0 \\ &d * \mathcal{H}^0 = 0 \qquad && d * \mathcal{H}^1 = 0 \end{aligned} \tag{7.63}$$

we can therefore set consistently $\mathcal{B}^0$ to zero (we can easily verify it sits at a minimum of the gauge fields potential in (6.7)) and take the first electric gauge field $\mathcal{A}^0$ and $\tilde{\mathcal{A}}_0$ as constants, so that the first three equations of (7.63) are satisfied. We keep $\mathcal{A}^1$ on, satisfying its regular Maxwell equation, so that the only non-null flux is:

$$\mathcal{H} \equiv \mathcal{H}^1 = d\mathcal{A}^1 \equiv d\mathcal{A} \tag{7.64}$$

We did this for a particular choice of the vacuum, but this is obiously valid in general for a theory with constant scalar fields and null $\mathcal{R}_{\Lambda\Sigma}$ plus null $\mathcal{I}_{01} = \mathcal{I}_{10}$.
If only the first one of these two conditions is not satisfied, but we still have null off-diagonal components for $\mathcal{R}_{\Lambda\Sigma}$, we can still truncate all of the fields except for the usual $\mathcal{A}^1$, recovering the same case as the one we discussed, but with a "weighted" Maxwell equation, where terms $\mathcal{R}_{11}$ and $\mathcal{I}_{11}$ weight terms $\mathcal{H}$ and $*\mathcal{H}$ respectively.
If the second condition is not satisfied as well, then we have off-diagonal components for the scalar-gauge matrix real and imaginary parts, and to get our usual truncation we must impose a constraint on the components of the $\mathcal{A}^1$ flux:

$$\mathcal{I}_{01} * \mathcal{H}^1 = -\mathcal{R}_{01} \mathcal{H}^1 \tag{7.65}$$

If this constraint is satisfied, then we can safely set all our potentials to constants and $\mathcal{B}_0$ to zero and still have a consistent embedding. We will always assume equation (7.65) to be valid in the cases where $\mathcal{I}_{01}$ will be different from zero, and treat the theory as properly truncated. This will of course reflect on the structure of the Maxwell equation for $\mathcal{A}^1$ as well.

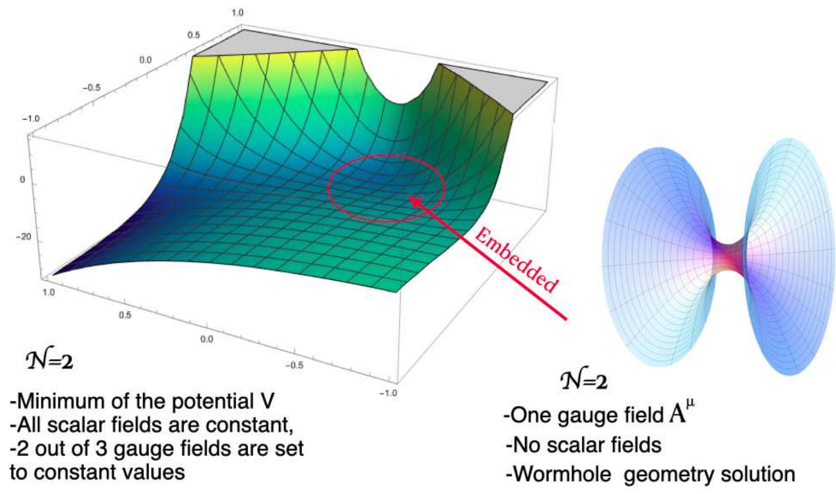

Figure 7.3: Visual depiction of the embedding of the only-gauge theory inside of the vacua of the gauge-scalar $ISO(7)$ theory. In the figure, we are embedding the theory inside a $\chi = 0 = \rho$ vacuum.

Going back to the choice of vacuum in (7.63), we notice that recover a cosmological constant $\Lambda = V_0 = -c^{-1/3} g^2 3(5^{7/6})$.
Einstein equation (7.59) can be recast as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 3 \frac{c^{1/3}}{5^{1/6}} [\mathcal{H}_{\mu\rho} \mathcal{H}^{\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} \mathcal{H}_{\rho\sigma} \mathcal{H}^{\rho\sigma}] - g_{\mu\nu} \Lambda \tag{7.66}$$

and matching this expression to (7.59) one finds

$$c = -\frac{1}{8}\frac{\sqrt{5}}{27} \qquad\qquad q^2 = \frac{1}{6}V_0 c^{-1/3} g^2 = 3\frac{5^{7/6}}{6} c^{-1/3} g^2 \tag{7.67}$$

We conclude that the theory of [36] can be succesfully embedded in our dyonically $ISO(7)$ gauged model choosing $\mathcal{A}^1$ as the $U(1)$ potential (as we expected, since this is the gauge grouped gauged by $\mathcal{A}^1$ in the full theory as well) and with an expression of the gravitino charge in terms of the electric charge of the $ISO(7)$ model given by:

$$q^2 = 15g^2 \Rightarrow q \approx 4g \tag{7.68}$$

In the following we report a table with all the possible scalar vacua choices in the $SU(3)$ that one can choose as suitable theories where to embed the theory of [36] in. We remind that for those vacua with an expectation value for the fields χ different from 0, we implied equation (7.65) to be valid. For every vacuum we find the value of $c = \frac{m}{g}$ and the relation between q and g as in (7.68):

$g^{-2}c^{1/3}V_0$	$\mathcal{I}_{00}$	$\mathcal{I}_{01} = \mathcal{I}_{10}$	$\mathcal{I}_{11}$	c	q/g
$-4(3^{3/2})$	$-\frac{3\sqrt{3}}{7c}$	$\frac{3\sqrt{3}}{14c^{1/3}}$	$-\frac{6\sqrt{3}c^{1/3}}{7}$	$-\left(\frac{7}{6}\right)^3\frac{1}{24\sqrt{3}}$	3,21
$-\frac{2^{28/3}\sqrt{3}}{5^{5/2}}$	$-\frac{9\sqrt{15}}{19c}$	$\frac{3\sqrt{15}}{38(2^{2/3})c^{1/3}}$	$-\frac{6(2^{2/3})\sqrt{15}c^{1/3}}{19}$	$-\left(\frac{19}{6}\right)^3\frac{1}{320\sqrt{15}}$	0,99
$-\frac{2^8 3^{3/2}}{5^{5/2}}$	$-\frac{9\sqrt{15}}{38c}$	$\frac{3\sqrt{15}}{76c^{1/3}}$	$-\frac{12\sqrt{15}c^{1/3}}{19}$	$-\left(\frac{19}{12}\right)^3\frac{1}{120\sqrt{15}}$	0,94
$-3(5^{7/6})$	$-\frac{\sqrt{5}}{c}$	0	$-\frac{3c^{1/3}}{5^{1/6}}$	$-\frac{\sqrt{5}}{216}$	3,87
$-3(2^{17/6})$	$-\frac{1}{\sqrt{2}c}$	0	$-3(2^{1/6})c^{1/3}$	$-\frac{1}{216\sqrt{2}}$	4,90
$-\frac{2^{16/3}}{\sqrt{3}}$	$-\frac{6\sqrt{3}}{7c}$	$\frac{3\sqrt{3}}{7(2^{2/3}c^{1/3}}$	$-\frac{3\sqrt{3}2^{2/3}c^{1/3}}{7}$	$-\left(\frac{7}{3}\right)^3\frac{1}{96\sqrt{3}}$	3,02

Table 7.3: In this table we summarize the value the magnetic-to-electric coupling ratio $c = \frac{m}{g}$ must take in order to be able to embed a residual $U(1)$ gauge invariant $\mathcal{N} = 2$ Supergravity theory with no scalars canonically into each vacuum of the $ISO(7)$ dyonically gauged model. Finally, we report the approximate relation between the electric coupling g and the gravitino electric charge q in the two models (we approximate it, reckoning it is a more meaningful and immediate value). We highlighted those vacua that yield a simple truncation process recovering standard Maxwell equations for the flux.

Therefore we recover that in order to be able to embed the theory, m and g must have opposite sign, while in each case both g and q are real quantities, as we desired for a physically meaningful theory.

We remark the fact that those that we found are not Giddings-Strominger wormhole solution, therefore we cannot recover a criterion for their stability based on the value of an axio-dilaton pair coupling constant.

Nevertheless, we proved in Chapter 7.4.1 that the particle content of the model makes it possible to have a theory with a traversable wormhole geometry solution for the Einstein equation. We proved that this theory can be embedded in a higher-dimensional massive Type IIA String Theory by first embedding it in the vacua of a four-dimensional Supergravity theory that was known to derive from a ten-dimensional Sting theory. This is the first time such a proof is given for the analyzed model.

Eventually we found the relation between the two Supergravity electric charges for every different embedding choice.

Chapter 8

Conclusions

This thesis work provides an insight on the possibility of obtaining a traversable Giddings Strominger wormhole solution in the $\mathcal{N} = 8$ dyonically-$ISO(7)$-gauged Supergravity model [57], sustained by an axio-dilaton pair. In particular, I first derived several vacua for an $SU(3)$-invariant subsector of the theory, pointing out the importance of having a dyonic gauging of the theory. Therefore, I carefully analyzed the particle content looking for a truncation that would yield a simple GS wormhole. Considering the AdS nature of the vacuum configurations of the theory, I derived an explicit solution for the fields in a four-dimensional Euclidean asymptotically AdS Giddings-Strominger wormhole geometry. I showed that the straightforward circumstance of having simple truncations to these AdS-GS solutions does not occur in the model. Thus, I detected, among the multiple fields that this Supergravity theory contemplates, some possible axio-dilaton pairs in the Euclidian action. I found many valuable candidates and used a previously established method [35] in order to approximate our framework to that of a GS theory by setting the potential to be a cosmological constant but still keeping some of the fields on. For every pair of fields we identified as an axio-dilaton couple, I computed the stability of the wormhole solution and found that in more than one case, taking in consideration numerous truncations of the full theory, a stable wormhole solution is possible in this asymptotic environment. This result hints that a Giddings-Stominger solution is possible in the full theory, still one would need to verify the evolution of all the fields along the wormhole throat and make sure their value does not explode. This should be done numerically. Moreover, I managed to embed a Supergravity theory only having a gauge bosonic particle content into the $ISO(7)$ model.

Given the truncation to the $SU(3)$ invariant subsector, I derived that one can truncate the theory in such a way to remain with a $(\phi,\, \varphi,\, \chi)$, $(\phi,\, \varphi,\, \zeta/\tilde{\zeta})$ or $(\phi,\, \varphi,\, a)$ particle content. In the first two cases, the gauge fields can be consistently turned down, while in the third case they cannot. Not being able to straightforwardly set either ϕ or φ to a constant keeping the other field free, I proposed an identification $\phi = \varphi$ in the first two cases, yielding first order differential equations that would need to be solved numerically.

I derived a solution, starting from the results in [62], for a hypothetical exact Giddings-Strominger truncation with asymptotic AdS behavior and found:

$$\begin{aligned} ds^2 &= \frac{dr^2}{1 + \frac{r^2}{l^2} - \frac{\tilde{q}^2}{r^4}} + r^2 d\Omega_4^2 \\ e^{2\phi(r)} &= \left(\frac{q_-}{\tilde{q}} \sin(\tilde{q} H(r))\right)^2 \\ \chi(r) &= \frac{1}{q_-}(\tilde{q} \cot(\tilde{q} H(r) - q_3)) \end{aligned} \tag{8.1}$$

where we have a constraint on the form of $H(r)$, which cannot be solved analytically:

$$\partial_r H(r) = \frac{\sqrt{6}}{r^3} \sqrt{\frac{1}{1 - \frac{q^2}{r^4} + \frac{r^2}{l^2}}} \tag{8.2}$$

By using the methods applied in [35], I found the asymptotic behavior of the theory for small variations of some chosen fields around their vacuum, such to have an approximate Giddings-Strominger AdS action with (8.1) as a solution. I verified the stability in different cases, using different perturbed axio-dilaton pairs, and found the results collected in the following table:

Dyonic $ISO(7)$ subsector symmetry	Axio-dilaton pair	Axio-dilaton coupling β	Stability of the solution
$SU(3)$	(χ, φ)	$\frac{2}{\sqrt{3}}$	✓
	$(\zeta/\tilde{\zeta}, \phi)$	1	✓
	(a, ϕ)	2	×
G_2	(χ, φ)	$\frac{2}{\sqrt{7}}$	✓
$SO(4)$	(χ, φ)	$\frac{2}{\sqrt{6}}$	✓
	(ρ, ϕ)	2	×

Table 8.1: Stable and unstable asymptotically AdS Giddings-Strominger approximated solutions. For every case we report the residual invariance the subsector of the main dyonic $ISO(7)$ theory holds, the axio-dilaton pair we have for each case and the effective coupling we obtain by setting the action in the canonical form. A yellow check in the last column means the axio-dilaton pair saturates the stability condition exactly.

Therefore many of these asymptotic cases provide the possibility of building a stable Giddings-Strominger solution. We would need to analyze the geometry in the full model with the fields in the potential free as well, using a numerical approach.
Finally, I passed to analyzing the gauge-on, scalar-off model of [36] and tried to embed it in our $ISO(7)$ model, specifically in the $SU(3)$-invariant subsector. I found that one can consistently truncate the vacua of the aforementioned subsector to the only-gauge model easily when the axion-like particle χ, appearing in the imaginary part of the scalar-gauge matrix $\mathcal{I}_{\Lambda\Sigma}$, has a null vacuum expectation value. On the other hand, when this is not true, we find that we must impose to the gauge field $\mathcal{A}^1$ the additional constraint:

$$\mathcal{I}_{01} * \mathcal{H}^1 = -\mathcal{R}_{01}\mathcal{H}^1 \tag{8.3}$$

Assuming this is always satisfied, I managed to embed the theory by matching the canonical form of the Einstein equations in the two models, with particular attention to the cosmological constant of [36], defined in terms of the gravitino charge q. For every $SU(3)$ vacuum, I found values for $c = \frac{m}{g}$ and q/g:

$g^{-2}c^{1/3}V_0$	$c = \frac{m}{g}$	q/g
$-4(3^{3/2})$	$-\left(\frac{7}{6}\right)^3 \frac{1}{24\sqrt{3}}$	3,21
$-\frac{2^{28/3}\sqrt{3}}{5^{5/2}}$	$-\left(\frac{19}{6}\right)^3 \frac{1}{320\sqrt{15}}$	0,99
$-\frac{2^8 3^{3/2}}{5^{5/2}}$	$-\left(\frac{19}{12}\right)^3 \frac{1}{120\sqrt{15}}$	0,94
$-3(5^{7/6})$	$-\frac{\sqrt{5}}{216}$	3,87
$-3(2^{17/6})$	$-\frac{1}{216\sqrt{2}}$	4,90
$-\frac{2^{16/3}}{\sqrt{3}}$	$-\left(\frac{7}{3}\right)^3 \frac{1}{96\sqrt{3}}$	3,02

Table 8.2: Values for the magnetic-to-electric coupling ratio $c = \frac{m}{g}$ and the approximate expression of q, charge of the gravitino, in terms of g, electric charge, in order to have an appropriate canonical embedding of the theory presented in [36] within the theory studied in [57].

Chapter 9

Outlooks on the traversable wormhole solution

Following our work on the embedding of the traversable wormhole solution into String theory, we would like to aknowledge some outlets that the topic of our research has in some different sectors of physics. We therefore present a glimpse into potential research topics concerning the traversable wormhole solution. These active research fields are considered intriguing by the author and suitable for prospective exploration, complementing the purpose of this thesis work.

Chapter 9.1 reviews multi-mouth traversable wormhole giving insight on their stability by means of quantum gravity effects. It explores the chance of having a second wormhole close to the troath of the first one, connecting different patches of the same space-time.
In Chapter 9.2 we give an overview on a well known multi-wormhole solution using a theory with gauge fields.
Chapter 9.3 is dedicated to the analysis of the GW signal coming from a hypotetical black hole orbiting an open wormhole solution.

9.1 Multi-Mouth Traversable Wormholes

In this section, we refer to [67] and [68] for our dissertation and [69] for the concepts we will use.
Our focus lies in the creation of traversable wormholes featuring multiple mouths within a four-dimensional space. This discussion also delves into the correlated quantum entanglement.
Typically, the principle of topological censorship enforces the non-traversability of wormholes due to the null-energy condition. However, recent developments have revealed that finely controlled quantum effects could defy this condition in a manner permitting the establishment of traversable wormholes.
The management of these quantum effects on a smaller scale is imperative for ensuring control. To address this, we necessitate a foundational solution that characterizes the configuration of interest by introducing slight perturbations. It is customary to disallow acausal shadows, thereby often restricting our consideration to wormholes connecting distinct regions. Nevertheless, there exist more intriguing scenarios that necessitate the existence of these shadows. Our initial focus is on the conventional wormhole structure with two mouths. By incorporating a count of N_f four-dimensional massless fermions, we can generate the requisite negative energy.
One of our primary concerns pertains to the inherent delicacy of the wormhole's structural equilibrium, which could be compromised by the insertion of another black hole. To mitigate this issue, we rely on the substantial redshift discrepancy between the central region and the opening of the initial passage. We further extend this notion by encompassing smaller black holes, each harboring an additional connection to distant regions within the same spacetime.
Initially, classical matter falls short of supplying the negative energy imperative for sustaining the wormhole structure. However, alternative methods are available to circumvent this limitation. No-

tably, we can consider the Casimir energy as a potential source.
Our journey commences with a near-extremal Reissner-Nordstrom black hole, approached within its proximity to the event horizon. The associated metric is parameterized as follows:

$$ds^2 = r_e^2\left(-(\rho_r^2 - 1)d\tau_r^2 + \frac{d\rho_r^2}{\rho_r^2 - 1} + d\Omega^2\right) \tag{9.1}$$

where r_e is the horizon radius of the BH. We employ the Rindler coordinates ρ and τ and leverage the $AdS_2 \times S^2$ symmetry within close proximity to the horizon. A crucial aspect of achieving a wormhole geometry involves the angular component of the metric expanding as we move away from the throat of the wormhole. To account for deviations from the original symmetry and the alteration in angular circumference, we introduce the parameter set ϕ and γ into the metric. It is essential for ϕ to exhibit growth in both the positive and negative radial directions. This expansion necessitates the presence of negative energy.
To generate this negative energy, we capitalize on the Casimir effect facilitated by a magnetic field within a black hole, coupled with a massless charged fermion field. This combination yields a considerable number of effective massless fermions (for more details, see [68]). Notably, this approach has been shown to produce negative energy, a prerequisite for our endeavor.
It has been demonstrated that the energy of these wormholes deviates from the energy of two disconnected extremal black holes by an amount that can be expressed as:

$$E = \frac{r_e^3}{Gl^2} - \frac{N_f Q}{6}\left(\frac{\pi}{\pi l + d_{out}} - \frac{1}{4l}\right) \tag{9.2}$$

Where πl equals the length of the throat in an appropriate metric rescaled conformally, d_{out} is the gap between the mouths in the outer region and Q is the magnetic charge of the black holes. We have that the first contribution to this energy is nothing but the energy of a Reissner-Nordström black hole over extremality.
Looking for a minimum in the energy, we can thus calculate the length of the wormhole at equilibrium and from this derive the binding energy of the wormhole. We do this assuming that the wormhole mouths are way closer to each other in asymptotically flat space-time than they are crossing the wormhole ($d_{out} \ll l$)

$$l \approx \frac{Q^2 l_P}{N_f} \tag{9.3}$$

We can explore various scenarios based on the relative lengths of the wormhole compared to the distance between its two mouths. It's essential to recall that the Casimir effect necessitates closed field lines, which in turn requires the two ends of the wormhole to be within the same region of spacetime. This mutual gravitational attraction between the two mouths leads to a finite period before one collapses onto the other, rendering the wormhole traversable during this interval. To prevent this collapse, several strategies can be employed. One involves introducing an external magnetic field to separate the mouths, or alternatively placing them in orbit around each other, albeit this approach would lead to the emission of gravitational and electromagnetic waves from the mouths. Another avenue of exploration involves considering cosmic strings [70].
Moving forward, we aim to incorporate a nearly miniature black hole within the throat of the primary wormhole. This new insertion is treated as a perturbation to the throat, while the other end of the wormhole is treated as a metric perturbation extending towards infinity. Throughout the calculation, we'll omit the backreaction from the mouths.
The challenges at hand revolve around establishing the stability of this new configuration and determining how to attain the requisite negative energy to render the passage viable. Addressing these queries necessitates selecting a specific model that encompasses particles within or beyond the standard model.
In one approach, the black hole inserted into the throat experiences a uniform static magnetic field, propelling it towards an exit and establishing an equilibrium configuration when counterbalanced by

the strong gravitational potential within the throat. Alternatively, the black hole can be modeled as a charged Reissner-Nordström black hole under a $U(1)$ gauge distinct from electromagnetism, operating within a non-standard model. Just as with the two mouths of the initial wormhole, the stability concerns of the new mouth can be addressed, utilizing methods analogous to those employed with the first wormhole, thus extending its lifetime.
Regarding the realization of negative energy, crucial for binding the traversable wormhole, this can be achieved through the presence of massless fermions that follow closed loops along the field lines. As these fermions traverse the third gorge, they maintain its openness. Alternatively, the fermions could be replaced by cosmic strings, serving the same purpose.
Constraints on the size of the new mouth can also be derived. Seminal analysis indicates that its size must exceed the Planck length, particularly greater than $\sqrt{N_f} l_P$. Furthermore, a limit arises from considerations of the null energy condition, implying that the mass must not exceed the magnitude of the Casimir energy. This ensures a total negative energy, thereby guaranteeing the wormhole's traversability. To derive this limit, we solve Einstein's equations with terms from the energy-momentum tensor contributed by fermionic Casimir energy, Maxwell's field, and the mass of the central wormhole in the first gorge.
We approximate the wormhole mouth's location as δ, considerably smaller than the gorge's length. As our interest lies in the deformation of the S^2 scale along the gorge, we can treat the black hole as a wall defect, spread across the gorge's width. This allows us to formulate the $G_{\tau\tau}$ component of the Einstein tensor and incorporate the energy-momentum tensor components according to:

$$G_{\tau\tau} = \gamma - (1+\rho^2)(-1+\rho\phi' + (1+\rho^2)\phi'') - (1+\rho^2)\phi \tag{9.4}$$

and the three components of $T_{\tau\tau}$ are

$$\begin{aligned} T^{B}_{\tau\tau} &= -\frac{1}{4g^2} g_{\tau\tau} F^2 \\ T^{Ferm}_{\tau\tau} &= -\frac{\alpha}{8\pi G} G \\ T^{mass}_{\tau\tau} &= \frac{\beta}{4\pi G}\delta(\rho) \end{aligned} \tag{9.5}$$

where α and β are two constants, in particular β is proportional to the mass of the black hole we set in the throat.
Solving the Einstein equation we can obtain the solution for the ϕ parameter that governs the expansion of S^2:

$$\phi(\rho) = \alpha(1+\rho\arctan\rho) - \beta|\rho| \tag{9.6}$$

We observe how increasing the mass enhances the radius while simultaneously reducing the width of the throat. This reduction is to the extent that, when the energy contribution from the black hole (BH) within the throat surpasses that from Casimir, the throat closes. By enforcing a requirement for the total energy to be negative, a limit for β emerges, setting an upper bound on the mass. Once normalized by the redshift outside the throat, this bound on the mass provides a limit on the energy of the small black hole. Remarkably, this limiting energy aligns with the energy difference between a traversable and an untraversable wormhole, as previously identified. Consequently, this insight implies that introducing a third mouth inside the wormhole results in a traversable solution only when the total energy is less than what would yield a sickly wormhole without the third mouth. In general, wormholes featuring multiple mouths are expected to exhibit lower energy than an assembly of disconnected black holes.
Exploring further, we might aim to ascertain whether our solution can be systematically constructed by inserting the small black hole from an initial configuration where it resides outside the throat into a situation where it is within the primary wormhole. This inquiry concerns the maintenance of sufficiently low total energy throughout the entire insertion process of the black hole into the throat. To this end, we extend the term associated with the momentum-energy tensor linked to the black

hole's mass to encompass a radial position ρ_0 for the black hole, not necessarily situated at the origin:

$$T^{mass}_{\tau\tau} = \frac{\beta}{4\pi G}(1+\rho_0^2)\delta(\rho-\rho_0) \tag{9.7}$$

Subsequently, we proceed to re-solve the equation of motion as prescribed by Einstein's equation to determine the shape of the function $\phi(\rho)$. To achieve this, we introduce an additional component to the Energy-Momentum tensor attributed to cosmic strings, characterized by a tension denoted as T. This tension exerts a force that endeavors to displace our black hole from its position within the throat. By integrating the boundary conditions, we can derive an equation that governs the behavior of the tension. Specifically, the tension diminishes to zero as the radial position of the black hole approaches infinity, corresponding to the black hole residing outside the throat. Additionally, the tension also vanishes as the radius approaches zero, signifying an equilibrium state when the black hole is optimally positioned within the throat.
The solution is a step function:

$$\phi(\rho) = \alpha(1+\rho\arctan\rho) - \frac{\beta}{1+\rho_0^2}(\Theta(\rho-\rho_0)(\rho-\rho_0)-k) \tag{9.8}$$

The key parameters within the scope of the ϕ field, as a perturbation of the angular component of the metric, include the constant integration factor 'k,' ranging between zero and 0.5, and, of course, β. By stipulating that ϕ experiences growth for both large positive and negative values of the radius, we can establish an upper bound on β and, consequently, on the mass. Adherence to this constraint ensures the feasibility of introducing our black hole into the throat, originating from an exterior position.
An intriguing phenomenon to investigate involves the exchange of signals between the mouths of the initial wormhole and the subsequent absorption of these signals by the third mouth. This third mouth can be likened to a 'leaky pipe,' with signal absorption characteristics proportional to the mouth's area. A second noteworthy effect pertains to time and Shapiro delay, which are anticipated to exhibit their customary forms and be explained by familiar underlying causes.

9.2 Multi-centered black holes in gauged D = 5 Supergravity

This review of multi-centered black holes solution, which we wish to expand in the future, to a possible multi-wormhole solution, summarizes the works in from [71], [72], [73], [74] and [75].
The multi-center MP Reissner-Norstrom solution, contemplating a cosmological Einstein/Maxwell GR theory, can be thought of as a restriction to the bosonic sector of D=5, N=2 gauged Supergravity.
We couple N=2 Supergravity to n-1 abelian vector multiplets.
The fields we get are

- a metric $g_{\mu\nu}$
- a gravitino ψ_μ
- n vector potentials A^I_μ
- n-1 gauginos λ_i
- n-1 scalars ϕ^i

The bosonic part of the Lagrangian turns out to be

$$e^{-1}\mathcal{L} = \frac{1}{2}R + g^2 V - \frac{1}{4}G_{IJ}F^I_{\mu\nu}F^{\mu\nu J} - \frac{1}{2}\mathcal{G}_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \frac{e^{-1}}{48}\epsilon^{\mu\nu\rho\sigma\lambda}C_{IJK}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda \tag{9.9}$$

with

$$V(X) = V_I V_J\left(6X^I X^J - \frac{9}{2}\mathcal{G}^{ij}\partial_i X^I \partial_j X^J\right) \tag{9.10}$$

where X^I are the real scalar fields, functions of ϕ^i and V_I's are constants by means of which we define the N=2 "graviphoton".
We define $\mathcal{V} = \frac{1}{6}C_{IJK}X^I X^J X^K$ setting it to 1, defining a so called "very special geometry".
C_{IJK} is defined in such a way that, if we get our 5D theory compactifying M-theory on a Calabi-Yau 3-fold, $X_I = \frac{1}{6}C_{IJK}X^J X^K$ corresponds to the size of 4-cycles on the manifold. The metrics G_{IJ} and $\mathcal{G}_{ij}$ are functions of $\mathcal{V}$.
The gauged-SUSY transformations for fermions are

$$\begin{aligned}
\delta\psi_\mu &= \left[\mathcal{D}_\mu + \frac{i}{8}X_I(\Gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\Gamma^\rho)F^I_{\nu\rho} + \frac{1}{2}g\Gamma_\mu X^I V_I\right]\epsilon, \\
\delta\lambda_i &= \left[-\frac{1}{4}G_{IJ}\Gamma^{\mu\nu}F^J_{\mu\nu} + \frac{3i}{4}\Gamma^\mu\partial_\mu X_I + \frac{3i}{2}gV_I\right]\partial_i X^I\epsilon.
\end{aligned}$$

where we defined the covariant derivative.

9.2.1 Ungauged case

It was found that the solutions for such a particle content in the action is

$$ds^2 = -e^{-4U}dt^2 + e^{2U}d\vec{x}d\vec{x} \tag{9.11}$$

$$A^I_t = a^{-2U}X^I \tag{9.12}$$

$$X_I = \frac{1}{3}e^{-2U}H_I \tag{9.13}$$

where H_I are harmonic functions $h_I + \Sigma_{j=1,N}\frac{q_{Ij}}{|\vec{x}-\vec{x_j}|^2}$ where we summed over the black holes positions. The constants h_i are related to the values of X_I at infinity and q_I are the charges.
If we try and solve SUSY conditions for fermions in the absence of gauging, g=0, we get a solution for the killing spinor $\epsilon = e^{-U}\epsilon_0$.
Considering pure SUGRA, thus setting the scalars to 0, we get the solutions

$$ds^2 = -H^{-2}dt^2 + Hd\vec{x}d\vec{x} \tag{9.14}$$

$$A_t = 3H^{-1} \tag{9.15}$$

9.2.2 Single black hole

For a single BH $H = 1 + \frac{q}{r^2}$, so the metric is

$$ds^2 = -(1+\frac{q}{r^2})^{-2}dt^2 + (1+\frac{q}{r^2})(dr^2 + r^2 d\Omega_3^2) \tag{9.16}$$

and defining $\rho^2 = r^2 + q$ we get a Schwarzschild solution

$$ds^2 = -(1-\frac{q}{\rho^2})^2 dt^2 + (1-\frac{q}{\rho^2})^{-2}(d\rho^2 + \rho^2 d\Omega_3^2) \tag{9.17}$$

$$F_{t\rho} = -3\partial_\rho(1-\frac{q}{\rho^2}) \tag{9.18}$$

9.2.3 Guaged, single BH case

Turning on the gauging introduces a scalar potential generating $\Lambda < 0$ and defining

$$f = 1 + g^2 r^2 e^{6U} \tag{9.19}$$

we find

$$ds^2 = -e^{-4U}f dt^2 + e^{2U}(f^{-1}dr^2 + r^2 d\Omega_3^2) \tag{9.20}$$

$$A_t^I = a^{-2U} X^I \tag{9.21}$$

$$X_I = \frac{1}{3} e^{-2U} H_I \tag{9.22}$$

similar to the multi-center ungauged solution except for the function f and, obviously, the non-assumption of isotropy.
We find that the killing spinor solution yields a killing spinor which satisfies a projection condition on the starting one. Thus, our theory preserves only half of the original supersymmetries.

9.2.4 Finding the metric ansatz

If we turn on the gauging, we get the previously defined term f which mimics a cosmological constant Λ. Therefore, the new found vacuum AdS solution is, considering a single BH

$$ds^2 = -(1 + g^2 r^2)dt^2 + \frac{dr^2}{1 + g^2 r^2} + r^2 d\Omega_3^2 \tag{9.23}$$

Still we are searching for a isotropic solution for multiple BH's, so we assume a more generic ansatz

$$ds^2 = -e^{2A} dt^2 + e^{2B} d\vec{x} d\vec{x} \tag{9.24}$$

where the exponentials are function of t and $\vec{x}$.
Assuming the asymptotic (BHs far away from each other) solution where we normalize the first exponential factor and write

$$ds^2 = -dt^2 + e^{-2gt} d\vec{x} d\vec{x} \tag{9.25}$$

we get a positive value of R, which we do not want. In order to bypass this matter, we need to have either a complex coupling constant or complex time, thus going to Euclidean metric.

9.2.5 Solutions

We set SUSY invariance for the gravitino transformations and group them into the ones that remain valid also in the non-gauged limit and the ones exclusive to the gauged case. From this we obtain

$$X_I F_{tm}^I = -\partial_m e^A \tag{9.26}$$

for the fields,

$$B(t, \vec{x}) = -\frac{1}{2} A(t, \vec{x}) + f(t) \tag{9.27}$$

for the metric ansatz and

$$\epsilon(t, \vec{x}) = e^{1/2 A(t \vec{x})} \hat{\epsilon}(t) \tag{9.28}$$

for the killing spinor.
Focusing on imposing the same conditions for the gaugino instead we find that gauge fields and scalar are related through

$$A_t^I = e^A X^I \tag{9.29}$$

which is the same relation we had in the ungauged multi-center solution and in the gauged single BH one.
The next equation we need to use is the EOM for the gauge fields and we find two relations, stating

$$e^{-A} X_I = \frac{1}{3} H_I(t, \vec{x}) \tag{9.30}$$

and setting the shape of H as

$$H_I = H_I(e^{f(t)} \vec{x}) = h_I + \Sigma_{j=1,N} \frac{q_{Ij}}{e^{2f(t)} |\vec{x} - \vec{x_j}|^2} \tag{9.31}$$

and using the gauged teory gaugino SUSY conditions we find the shape of $f(t)$ as $f(t) = -igt$, in such a way that

$$ds^2 = -e^{2A}dt^2 + e^{-A}e^{-2igt}d\vec{x}d\vec{x} \tag{9.32}$$

and the function A can be derived from the "very special geometry" assumption we made earlier.
Using the gauged gravitino SUSY conditions we can derive the shape of the killing spinor as

$$\epsilon = e^{2A}e^{igt}\epsilon_0 \tag{9.33}$$

9.2.6 Summary

In summary, we found that a D=5, N=2 Supergravity theory yields a Wick rotated solution breaking half of the original supersymmetries. The solutions are

$$ds^2 = -e^{2A} + e^{-A}e^{-2igt}d\vec{x}d\vec{x} \tag{9.34}$$

$$A_t^I = e^A X^I \tag{9.35}$$

$$X_I = \frac{1}{3}e^A H_I \tag{9.36}$$

and A is set by the very special geometry condition we chose.
$H_I(t)$ is a harmonic function written as $h_I + e^{2igt}\Sigma_{j=1,N}\frac{q_{Ij}}{|\vec{x}-\vec{x_j}|^2}$.
For each and everyone of these solutions we can recover the ungauged limit by setting $g \to 0$.

9.3 Gravitational Waves from a black hole orbiting in a wormhole geometry

The analysis of gravitational waves has yielded insights into various systems, including the merging of black holes, black holes and neutron stars, as well as neutron star mergers. Additionally, fractional data from gravitational waves can be used to identify cosmic defects and exoplanets. More intriguingly, it is even possible to detect more exotic phenomena such as a black hole spiraling towards the center of a wormhole.
In the realm of wormhole gravitational waves analysis [76], [77] researchers predominantly focus on two phases: the ringdown and the production of echoes. The paper under consideration delves into the scenario where a black hole descends in circular orbits within a wormhole until its passage results in a distinct signal before the ringdown phase. While the study does not delve into echo effects, it lays the foundational groundwork for investigating the behavior of a simple black hole that undergoes a collapse within a wormhole's throat.
Our trajectory follows a black hole as it traverses the gorge, potentially oscillating back and forth to create a temporal gap between the chirp signal and the subsequent anti-chirp signal. It's noteworthy that if a black hole emerges from another universe into our own, we will observe solely an anti-chirp signal.
The wormhole in consideration comprises the fusion of two Schwarzschild black hole spacetimes, with an additional black hole possessing a mass at least ten times smaller than the apparent mass of the wormhole. The metric adopted is the standard Schwarzschild metric expressed in Gaussian coordinates. Each of the two black holes has a sphere of radius 'a' removed from it, lying outside the regular event horizon at 2M, with this region hosting exotic matter violating the Null Energy Condition.
The model encompasses three parts: two external regions with known energy-momentum tensors and a boundary. Introducing the Gaussian coordinate time variable, we formulate the energy-momentum tensor as:

$$T^{\mu\nu} = S^{\mu\nu}\delta(\eta) + T_1^{\mu\nu}\Theta(\eta) + T_2^{\mu\nu}\Theta(-\eta) \tag{9.37}$$

We do not analyze the nature of the exotic matter used to keep the wormhole open, nor the possible alternatives to it. Choosing an exotic matter momentum energy tensor equal to

$$S^i_j = \mathrm{diag}(\sigma, \tau, \tau) \tag{9.38}$$

, where sigma is a negative energy density, as required, we get that the throat radius is equal to three times the mass of the wormhole from Einstein's equation. When a black hole approaches this throat, the process closely resembles the collapse of two black holes until it reaches a radius of three times the mass, at which point it interacts with the exotic matter on the throat. If the interaction is solely gravitational, the black hole will shift as it passes the throat. By confining this shift to the throat's region, it will eventually revert to its initial configuration. A similar process occurs when the black hole retreats to its initial universe, effectively leading to a back-and-forth motion between universes.
Assuming sufficient distance between the two throats to avoid signal interference between universes, a strong signal in one universe will diminish as the black hole emerges in the other universe. The observer in the second universe detects a pronounced anti-chirping signal, decreasing in frequency. The signal reaches zero when the black hole attains its maximum radius in the second universe and rapidly increases in frequency as it nears the throat. Subsequently, it crosses back into the first universe, thus repeating the cycle until the black hole resides within the throat.
Distinguishing our process from the merger of two black holes involves the echo phenomenon of gravitational waves and the emission of gravitational waves due to coupling with particles beyond the standard model.
In the simplified scenario, we treat the black hole's mass as much smaller than the wormhole's apparent mass, permitting a perturbation-based approach using post-Newtonian techniques. This approach remains valid as long as the black hole remains outside the false event horizon of the wormhole's mouth, which is situated beyond the throat.
Exploring the types of orbits the black hole can follow on its trajectory to the throat, we find four cases:

- Bound orbits
- Two-world bound orbits
- Two-world escape orbits
- Escape orbits

Analyzing the gravitational waves from this process, we identify a frequency inversely proportional to the apparent mass of the wormhole. The initial phase of the process bears resemblance to the merger of two black holes. However, a notable deviation occurs as we encounter a gap in the frequency spectrum, followed by periodic cycles of gravitational waves gradually decreasing in amplitude. These gaps, resulting from the current model that neglects secondary wormhole structure effects, could potentially exhibit echo effects.

Appendix A

The axion term in Euclidean path integral formulation

We wish to give a physical explanation for the rotation of the axionic term in the action when we perform our Wick rotation in space time $t \to ix_4$. This will not follow the String theory dimensional compactification approach, which of course leads to the rotation of kinetic terms deriving from dimensional reduction on p-forms, but we will try to focus directly on the 4D Supergravity theory and see how we get this rotated term without the need of higher dimensional justifications.
We will follow the dissertiation of [62] on the topic, though the topic is more widely treated in [78] and [79].

The path integral We wish to discuss the path integral formulation of a theory containing an axio-dilaton couple and gravity.
We are interested in the partition function

$$Z = \langle \phi_F, \chi_F | e^{-HT} | \phi_I, \chi_I \rangle \,, \qquad T \to \infty \tag{A.1}$$

where the ϕ_I, χ_I and ϕ_F, χ_F are evaluated at the space-like (in Euclidean space) surfaces Σ_I and Σ_F. We can write Z explicitly as

$$Z = \int_{\text{b.c.}} d[\phi] d[\chi] \; e^{-\frac{1}{2}\int_{\mathcal{M}} (d\phi \wedge * d\phi + e^{b\phi} d\chi \wedge * d\chi)} \tag{A.2}$$

where we impose Dirichlet boundaries for the fields.
We wish to calculate the partition function to obtain the quantum effects in the theory, but this is, as usual, possible only in the semi-classical approach where we will get instanton contributions, which will however be subdominant with respect to the perturbative terms.

Boundary terms and momentum space We define a set of momentum eigenstates

$$|\pi\rangle = \int d[\chi] e^{i \int_\Sigma \pi \chi} \, |\chi\rangle \tag{A.3}$$

with a definition similar to the position and coordinate eigenstates one in quantum mechanics.
Therefore we can write

$$\begin{aligned} Z &= \int d[\pi_I] d[\pi_F] \, \langle \chi_F | \pi_F \rangle \, \langle \pi_F | e^{-HT} | \pi_I \rangle \, \langle \pi_I | \chi_I \rangle \\ &= \int d[\pi_I] d[\pi_F] \;\; e^{\left(i \int_{\Sigma_F} \pi_F \chi_F - i \int_{\Sigma_I} \pi_I \chi_I\right)} \, \langle \pi_F | e^{-HT} | \pi_I \rangle \end{aligned} \tag{A.4}$$

We want to rewrite Z again in terms of the proper field variable, thus we go back to field space, using an auxiliary variable $\tilde{\chi}$, which has not the physical field meaning held by χ and therefore has no defined

boundary conditions.
We can write:

$$K_E(\pi_F, \pi_I, T) \equiv \langle \pi_F | e^{-HT} | \pi_I \rangle = \int d[\tilde{\chi_I}] d[\tilde{\chi_F}] \langle \pi_F | \tilde{\chi_F} \rangle \langle \tilde{\chi_F} | e^{-HT} | \tilde{\chi_I} \rangle \langle \tilde{\chi_I} | \pi_I \rangle \tag{A.5}$$

and interpreting $\langle \tilde{\chi}_F | e^{-HT} | \tilde{\chi}_I \rangle$ as a path integral, we can rewrite, combining integration over χ and $\tilde{\chi}_{I,F}$:

$$K_E = \int_{\text{no b.c.}} d[\chi] e^{-\frac{1}{2}\int_{\mathcal{M}} e^{b\phi} d\chi \wedge * d\chi - i\int_{\Sigma_F} \pi_F \chi + i\int_{\Sigma_I} \pi_I \chi} \tag{A.6}$$

We can now use the semi-classical approximation with no Dirichlet boundaries for the fields, since we integrated over them.

The dualization We now try and recover the instanton solution as usual solving the equations of motion in our action:

$$d(e^{b\phi} * d\chi) = 0 \qquad\qquad e^{b\phi} * d\chi - i\pi|_{\Sigma_{I,F}} = 0 \tag{A.7}$$

As we can see from these equations, the second equation imposes a boundary condition on the field χ. It states that our fields χ and π are not real, thus we recover a non acceptable saddle point approximation following this approach.
What we want to do now is change our strategy: we will dualize the χ field to a (D-1)-form F.
The dual path integral is defined as:

$$\int d[F] d[\chi] e^{\int_{\mathcal{M}} (-\frac{1}{2} e^{-b\phi} F \wedge * F + i\chi dF)} \tag{A.8}$$

where the second term imposes our duality relation. One can easily eliminate one of the variables by a simple integration by parts.
We can now easily derive the equation of motion for F

$$d(e^{-b\phi} * F) = 0 \qquad \Rightarrow \qquad F = e^{b\phi} d\lambda \tag{A.9}$$

Therefore F must be an exact form.
We now find the equations of motion for the entire system, including ϕ

$$d * d\phi + \frac{b}{2} e^{-b\phi} F \wedge * F = 0 \xRightarrow{(A.9)} d * d\phi + \frac{b}{2} e^{b\phi} d\lambda \wedge * d\lambda = 0 \tag{A.10}$$

where we notice that we have the "wrong" sign in front of the axionic term with respect to the Lorentzian frame equations of motion.
Thereore, we recovered the equations of motion and the action we used all along this thesis work by simple considerations on the instantonic nature of our solution using the path integral formulation.
We recovered a saddle point approximation of an imaginary field λ with an action:

$$S = \int_{\mathcal{M}} \frac{1}{2} [d\phi \wedge * d\phi - e^{b\phi} d\lambda \wedge * d\lambda + 2d(\lambda e^{b\phi} * d\lambda)] \tag{A.11}$$

and real boundary conditions

$$e^{b\phi} d\lambda|_{\Sigma_{I,F}} = \pi_{I,F} \tag{A.12}$$

Appendix B

Scalar content of $SU(3)$ and G_2 subsectors of dyonically gauged $ISO(7)$ Supergravity

We want to derive the explicit form for the scalar matrix $\mathcal{M}_{MN}$ in (5.48) for the $SU(3)$ and the G_2 subsectors. In order to do this, we will need to write for each case the coset representatives of the target space with residual symmetries, writing the result with an appropriate choice of covariant indices. We will then contract the indices of the coset representative with its transposed to find $\mathcal{M}_{MN}$ as written in (B.7). Using this matrix it is not only possible to derive standard kinetic terms for the scalars but also deriving the gauge kinetic matrix

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{R}_{\Lambda\Sigma} + i\mathcal{I}_{\Lambda\Sigma} \tag{B.1}$$

again from equation (B.7).
We derive again the results already found in [57] and therefore give a generic derivation of the scalar kinetic terms in (6.7) and (7.39).

B.1 $SU(3)$ invariant subsector

We embedded $SU(3)$ in SO(7)$\subset$ $ISO(7)$ in such a way that $\mathbf{7}\to\mathbf{1}+\mathbf{3}+\bar{\mathbf{3}}$. In terms of the $SL(8)$ indices, we split $A \to (a \oplus 8) \oplus (1 \oplus \hat{a})$ with $a = 2, 4, 6$ and $\hat{a} = 3, 5, 7$, followed by a complexification of the form

$$z_0 = x_1 + ix_8 \qquad z_1 = x_2 + ix_3 \qquad z_2 = x_4 + ix_5 \qquad z_3 = x_6 + ix_7 \tag{B.2}$$

When we restrict to this sector, the fields we keep take value along three invariant tensors,
- the $SU(3)$-invariant metric δ_{ij}
- the two-form $J_{ij}, i = 2, ..., 7$
- the complex totally antisymmetric tensor of $SU(3)$.

The splitting of indices is thus $[AB] \to [ij] \oplus [1j] \oplus [i8] \oplus [18]$.
With the choice of parametrization (B.2), the $SU(3)$ invariant tensors can be written as

$$\begin{aligned} J &= e^2 \wedge e^3 + e^4 \wedge e^5 + e^6 \wedge e^7 \\ \Omega &= (e^2 + ie^3) \wedge (e^4 + ie^5) \wedge (e^6 + ie^7) \end{aligned} \tag{B.3}$$

satisfying

$$J \wedge \Omega = 0 \qquad\qquad \Omega \wedge \Omega = -\frac{4}{3} iJ \wedge J \wedge J \tag{B.4}$$

and we would like to express our $SU(3)$-invariant scalar matrix in terms of these forms.
We obtain the coset representatives, choosing the 6 combinations of generators of $E_{7(7)}$ that are invariant under $SU(3)$ [57],

$$\mathcal{V}_{SK} = e^{-12g_4^{(+)}} e^{\frac{1}{4}\varphi g_1} \qquad\qquad \mathcal{V}_{QK} = e^{ag_2^{(+)}} e^{-6\zeta g_5^{(+)}} e^{-6\tilde{\zeta} g_6^{(+)}} e^{\phi g_3} \tag{B.5}$$

and we introduce

$$\begin{aligned} X = 1 + e^{2\varphi}\chi^2 \qquad & Y = 1 + \frac{1}{4}e^{2\phi}(\zeta^2 + \tilde{\zeta}^2) \qquad Z = e^{2\phi}a \\ j_1 = \zeta Z + \tilde{\zeta} Y \qquad & \qquad j_2 = \tilde{\zeta} Z - \zeta Y \end{aligned} \tag{B.6}$$

to write

$$\mathcal{M}_{MN} = 2\mathcal{V}^{ij}_{(M}\mathcal{V}_{N)ij} = \begin{bmatrix} \mathcal{M}_{\Lambda\Sigma} & \mathcal{M}^{\Sigma}_{\Lambda} \\ \mathcal{M}^{\Lambda}_{\Sigma} & \mathcal{M}^{\Lambda\Sigma} \end{bmatrix} = \begin{bmatrix} -(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & (\mathcal{R}\mathcal{I}^{-1})^{\Sigma}_{\Lambda} \\ (\mathcal{I}^{-1}\mathcal{R})^{\Lambda}_{\Sigma} & -(\mathcal{I}^{-1})^{\Lambda\Sigma} \end{bmatrix} \tag{B.7}$$

So, the scalar matric can be written (in a SU(6) covariant formalism), that is particularly easy for the residual symmetry frame.

• The "electric" part of the scalar tensor is:

$$\begin{aligned} \mathcal{M}_{[18][18]} &= e^{3\varphi} \\ \mathcal{M}_{[i8][k8]} &= e^{2\phi+\varphi}\delta_{ik} \\ \mathcal{M}_{[18][kl]} &= e^{3\varphi}\chi^2 J_{kl} \\ \mathcal{M}_{[i8][kl]} &= -\frac{1}{2}e^{2\phi+\varphi}\chi[\tilde{\zeta}(Re\Omega)_{ikl} - \zeta(Im\Omega)_{ikl}] \\ \mathcal{M}_{[i8][1l]} &= e^{\varphi}[Z\delta_{il} + (Y-1)J_{il}] \\ \mathcal{M}_{[1j][1l]} &= e^{-2\phi+\varphi}(Y^2+Z^2)\delta_{jl} \\ \mathcal{M}_{[1j][kl]} &= -\frac{1}{2}e^{\varphi}\chi[j_2(Re\Omega)_{jkl} - j_1(Im\Omega)_{jkl}] \\ \mathcal{M}_{[ij][kl]} &= e^{-\varphi}X(X-Y)J_{ij}J_{kl} + 3e^{-\varphi}X(Y-1)J_{[ij}J_{kl]} + 2e^{-\varphi}XY\delta_{k[i}\delta_{j]l} \end{aligned} \tag{B.8}$$

• The "magnetic" part of the matrix on the other hand turns out to be:

$$\begin{aligned} \mathcal{M}^{[18][18]} &= e^{-3\varphi}X^3 \\ \mathcal{M}^{[i8][k8]} &= e^{-(2\phi+\varphi)}X(Y^2+Z^2)\delta^{ik} \\ \mathcal{M}^{[18][kl]} &= e^{\varphi}\chi^2 X J^{kl} \\ \mathcal{M}^{[i8][kl]} &= -\frac{1}{2}e^{\varphi}\chi[j_1(Re\Omega)^{ikl} + j_2(Im\Omega)^{ikl}] \\ \mathcal{M}^{[i8][1l]} &= -e^{-\varphi}X[Z\delta^{il} + (Y-1)J^{il}] \\ \mathcal{M}^{[1j][1l]} &= e^{2\phi-\varphi}X\delta^{jl} \\ \mathcal{M}^{[1j][kl]} &= \frac{1}{2}e^{2\phi+\varphi}\chi[\zeta(Re\Omega)^{jkl} + \tilde{\zeta}(Im\Omega)^{jkl}] \\ \mathcal{M}^{[ij][kl]} &= e^{\varphi}(X-Y)J^{ij}J^{kl} + 3e^{\varphi}(Y-1)J^{[ij}J^{kl]} + 2e^{\varphi}Y\delta^{k[i}\delta^{j]l} \end{aligned} \tag{B.9}$$

• Finally, the "mixed" components read:

$$\begin{aligned} \mathcal{M}^{[18]}_{[18]} &= e^{-3\varphi}\chi^3 \\ \mathcal{M}^{[i8]}_{[k8]} &= e^{\varphi}\chi[ZJ^i_k - (Y-1)\delta^i_k] \\ \mathcal{M}^{[18]}_{[kl]} &= -e^{-\varphi}\chi X^2 J_{kl} \\ \mathcal{M}^{[i8]}_{[kl]} &= \frac{1}{2}e^{-\varphi}X[j_1(Re\Omega)^i_{kl} + j_2(Im\Omega)^i_{kl}] \\ \mathcal{M}^{[i8]}_{[1l]} &= e^{-2\phi+\varphi}\chi(Y^2+Z^2)J^i_l \\ \mathcal{M}^{[ij]}_{[18]} &= -e^{3\varphi}\chi J^{ij} \\ \mathcal{M}^{[ij]}_{[k8]} &= \frac{1}{2}e^{2\phi+\varphi}[\tilde{\zeta}(Re\Omega)^{ij}_k - \zeta(Im\Omega)^{ij}_k] \\ \mathcal{M}^{[1j]}_{[k8]} &= -e^{2\phi+\varphi}\chi J^j_k \end{aligned} \tag{B.10}$$

and

$$\begin{aligned}
\mathcal{M}^{[1j]}_{[1l]} &= -e^{\varphi}\chi[ZJ^{j}_{l} + (Y-1)\delta^{j}_{l}] \\
\mathcal{M}^{[ij]}_{[1l]} &= \frac{1}{2}e^{\varphi}[j_2(Re\Omega)^{ij}_{l} - j_1(Im\Omega)^{ij}_{l}] \\
\mathcal{M}^{[1j]}_{[kl]} &= -\frac{1}{2}e^{2\phi-\varphi}X[\zeta(Re\Omega)^{j}_{kl} + \tilde{\zeta}(Im\Omega)^{j}_{kl}] \\
\mathcal{M}^{[ij]}_{[kl]} &= e^{\varphi}\chi(Y-X)J^{ij}J_{kl} - 3e^{\varphi}\chi YJ^{[ij}J^{rs]}\delta_{r[k}\delta_{l]s} - 2e^{\varphi}\chi(Y-1)\delta^{ij}_{kl}
\end{aligned} \tag{B.11}$$

The mirror sector of the mixed scalar matrix, $\mathcal{M}^{[CD]}_{[AB]}$, is equal to the one we found since the matrix is symmetric by definition.

B.2 G_2 invariant subsector

We split the $SU(8)$ representation indices as $\mathbf{8}\rightarrow\mathbf{7}+\mathbf{1}$ so that the index A will be written as $I \oplus 8$, $I = 1, ..., 7$ and the **28** can be written as $[\mathbf{AB}] \rightarrow [\mathbf{IJ}] \oplus [\mathbf{I8}]$.
With this parametrization of the indices, the G_2 invariant forms can be written as

$$\begin{aligned}
&\delta_{IJ}, \\
&\psi_{IJK} = e_{123} + e_{145} + e_{167} - e_{246} + e_{257} + e_{473} + e_{635}, \\
&\tilde{\psi}_{IJKL} = e_{4567} + e_{6723} + e_{2345} - e_{1357} + e_{1346} + e_{1562} + e_{1724}
\end{aligned} \tag{B.12}$$

The coset representative, as usual written by writing G_2 generators in terms of the ones of the **133** fundamental representation of $E_{7(7)}$, is

$$\mathcal{V} = e^{-12\chi g_2^{(+)}} e^{\frac{1}{4}\varphi g_1} \tag{B.13}$$

We also define the combination

$$X = 1 + e^{2\varphi}\chi^2 \tag{B.14}$$

the relevant combinations of indices in our sectors (the other ones can be obtained by symmetricity of the matrix) are:

$$\begin{aligned}
\mathcal{M}^{[IJ][KL]} &= 2e^{\varphi}X\delta^{K[I}\delta^{J]L} + e^{3\varphi}\chi^2\tilde{\psi}^{IJKL} \\
\mathcal{M}^{[IJ][K8]} &= e^{\varphi}\chi^2X\psi^{IJK} \\
\mathcal{M}^{[I8][K8]} &= e^{-3\varphi}X^3\delta^{IK} \\
\mathcal{M}^{[IJ]}_{[KL]} &= -2e^{3\varphi}\chi^3\delta^{IJ}_{KL} - e^{\varphi}\chi X\tilde{\psi}^{IJ}_{KL} \\
\mathcal{M}^{[IJ]}_{[K8]} &= -e^{3\varphi}\chi\psi^{IJ}_{K} \\
\mathcal{M}^{[I8]}_{[KL]} &= -e^{-\varphi}\chi X^2\psi^{I}_{KL} \\
\mathcal{M}^{[I8]}_{[K8]} &= -e^{3\varphi}\chi^3\delta^{I}_{K} \\
\mathcal{M}_{[IJ][KL]} &= 2e^{-\varphi}X^2\delta_{K[I}\delta_{J]L} + e^{\varphi}\chi^2X\tilde{\psi}_{IJKL} \\
\mathcal{M}_{[IJ][K8]} &= e^{3\varphi}\chi^2\psi_{IJK} \\
\mathcal{M}_{[I8][K8]} &= e^{3\varphi}\delta_{IK}
\end{aligned} \tag{B.15}$$

Appendix C

Equations of motion of Type IIA String Theory and consistency of the truncation

In the following, we review the equations of motion obtained by the massive Type IIA String Theory Lagrangian (see [58], with conventions from [80]) , considering as usual just the massless modes. We do not divide internal and external space field-components, in order to give a more polished form for our equations.
We check the consistency of the truncation ansatz of (4.2) and (4.3) and find the equations of motion fro the 4-dimensional theory (4.17) reducing the 10-dimensional ones.

C.1 Equations of motion

The equation of motion for the NS-NS sector fields B_2 and Φ and for the R-R sector tensor fields C_1 and C_3 are [58]:

$$\begin{aligned} & d * d\Phi = -\frac{1}{2}e^{-\Phi}H_3 \wedge *H_3 + \sum_{p=2,4}\frac{5-p}{4}e^{(5-p)\Phi/2}F_p \wedge *F_p \\ & d(e^{-\Phi} * H_3) = \frac{1}{2}F_4 \wedge F_4 + e^{\Phi/2}F_2 \wedge *F_4 \\ & d(e^{3\Phi/2} * F_2) = -e^{\Phi/2}H_3 \wedge *F_4 \\ & d(e^{\Phi/2} * F_4) = -H_3 \wedge F_4 \end{aligned} \tag{C.1}$$

while the Einstein equation is given by

$$\begin{aligned} 2R_{MN} =& \partial_M\Phi\partial_N\Phi + e^{-\Phi}(\frac{1}{2}(H_3)^2_{MN} - \frac{1}{24}g_{MN}H_3^2) \\ & + \sum_{p=2,4} e^{(5-p)\Phi/2}\left(\frac{1}{(p-1)!}(F_p)^2_{MN} - \frac{p-1}{8(p!)}g_{MN}F_p^2\right) \end{aligned} \tag{C.2}$$

and upon truncation of the F_0 form identified with the Romans mass m, we recover as we expected equations (5.3) and (5.4).
The Bianchi identities read

$$dH_3 = 0 \qquad dF_2 = 0 \qquad dF_4 = H_3 \wedge F_2 \tag{C.3}$$

and we can see that these two sets of equations can be derived again from (6.7) upon Romans mass truncation.
These equations can be readily integrated to find an expression of the fluxes H_3, F_2 and F_4 as a function of the potentials B_2, C_1 and C_3 (see (C.7).

C.2 Bounds on the 10D String theory for consistency of the reduced 4D Supergravity theory

We need to check whether our ansatz:

$$ds_{10}^2 = e^{2a\varphi}ds_4^2 + e^{2b\varphi}\mathcal{M}_{\alpha\beta}d\theta^\alpha d\theta^\beta \tag{C.4}$$

$$\begin{aligned} &C_3 = \chi_1 d\theta^1 \wedge d\theta^2 \wedge d\theta^3 + \chi_2 d\theta^1 \wedge d\theta^4 \wedge d\theta^5 + \chi_3 d\theta^2 \wedge d\theta^5 \wedge d\theta^6 + \chi_4 d\theta^3 \wedge d\theta^4 \wedge d\theta^6 \\ &H_3 = 0 = F_2 \end{aligned} \tag{C.5}$$

violate some of the constraints coming from the full 10-dimensional theory.
First we consider (C.1) and (C.2) without H_3 and F_2 tensor fields to find the equations

$$\begin{aligned} &(A) \quad d * d\Phi = \frac{1}{4}e^{\Phi/2}F_4 \wedge *F_4 \\ &(B) \quad 0 = \frac{1}{2}F_4 \wedge F_4 \\ &(C) \quad d(e^{\Phi/2} * F_4) = 0 \\ &(D) \quad 2R_{MN} = \partial_M\Phi\partial_N\Phi + e^{\Phi/2}\left(\frac{1}{6}(F_4)^2_{MN} - \frac{1}{64}g_{MN}F_4^2\right) \end{aligned} \tag{C.6}$$

with Bianchi

$$dF_4 = 0 \quad \Rightarrow \quad F_4 = dC_3 \tag{C.7}$$

Since we assumed all the axions χ_i to depend only on the radial coordinate, equation (B) is satisfied. It is really easy to check that, upon identification of all the axions $\chi_i = \chi/2$ and thus setting all moduli fields Φ to zero, equations $(A), (C)$ and (D) instead put constraints on our field that read as (4.17):

$$\begin{aligned} &(A) \quad \Rightarrow \quad d * d\phi = \frac{1}{2}e^{\phi/2}\sqrt{g}(\partial_r\chi)^2 \\ &(C) \quad \Rightarrow \quad 0 = \partial_r(e^{\phi/2}\partial_r\chi * dr) \\ &(D) \quad \Rightarrow \quad R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - e^{b\phi}\partial_\mu\chi\partial_\nu\chi \end{aligned} \tag{C.8}$$

Therefore everything is consistent and we recover our 4D theory without any issue.

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