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Trigonometric and tensor-product Floater-Hormann rational interpolant

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to my family

Thanks

This thesis represents the end of my university adventure, started some years ago. It is well known that the beginnings were so strong and I have sometimes vacillated but in my hearth I felt that could reach the final goal and graduate. It is useless to say that I could not have realized my dreams without the help of some special people.

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Chapter 1

Floater Hormann Rational Interpolant - FHRI

1.1 Generality

The Floater-Hormann rational interpolant, indicated as FHRI, is a family of interpolants, that depends on a parameter d. If we choose a function $f : [a, b] \longrightarrow \mathbb{R}$, a sequence of $n+1$ points, called *nodes*, $a = x_0 < ... < x_n = b$ and some values f_i=$f(x_i)$ $i = 0, ..., n$, each FHRI can be constructed as follow: choose any integer d with $0 \leq d \leq n$ and for each $i = 0, 1, ..., n-d$, let $p_i(x)$ denote the unique polynomial of degree at most d that interpolates f at the $d+1$ points x_i,...,x_{i+d}, then

$$r(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x) p_i(x)}{\sum_{i=0}^{n-d} \lambda_i(x)} \tag{1.1}$$

with

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i) \cdots (x - x_{i+d})}. \tag{1.2}$$

The functions $\lambda_i(x)$ only depends on the nodes x_i so that the rational interpolant $r(x)$ depends linearly on the data $f(x_i)$. As special cases for $d = 0$ we have the Berrut's interpolant and for $d = n$ the polynomial interpolant of total degree n.
In the following when we'll talk about FHRI, we'll focus our attention on an interpolant of that family with a particular d.
FHRI may be put in barycentric form [1]

$$r(x) = \frac{\sum_{k=0}^{n} \frac{w_k}{x - x_k} f(x_k)}{\sum_{k=0}^{n} \frac{w_k}{x - x_k}} \tag{1.3}$$

where

$$w_k = \sum_{i \in J_k} (-1)^i \prod_{j=i, j \neq k}^{i+d} \frac{1}{x_k - x_j} \tag{1.4}$$

with $J_k = \{i \in \{0, 1, 2, ..., n-d\} \text{ such that } k-d \leq i \leq k\}$.

Lemma 1.1.1 ([4]). *If the interpolation nodes are equidistant, then the weights in (1.4) satisfy*

$$\frac{1}{h^d d!} \sum_{i \in J_k} \binom{d}{k-i} = |w_k| \leq \frac{2^d}{h^d d!}. \tag{1.5}$$

In the special case of equispaced nodes, the weights can be written in another form, knowing that an uniform scaling does not change the interpolant, [3][1]

$$(-1)^d h^d d! w_k = (-1)^k \sum_{i \in J_k} \binom{d}{k-i} = (-1)^k \beta_k$$

where

$$\beta_k = \sum_{i=d}^{n} \binom{d}{i-k} = \begin{cases} \sum_{i=0}^{k} \binom{d}{k}, & \text{if } k \leq d, \\ 2^d, & \text{if } d \leq k \leq n-d, \\ \beta_{n-k}, & \text{if } k \geq n-d \end{cases} \tag{1.6}$$

So the rational interpolant becomes,

$$r(x) = \frac{\sum_{k=0}^{n} \frac{(-1)^k \beta_k}{x - x_k} f(x_k)}{\sum_{k=0}^{n} \frac{(-1)^k \beta_k}{x - x_k}} \tag{1.7}$$

If we rewrite it in a more suitable form, it has been proved the following

Theorem 1.1.2 (Absence of poles in $\mathbb{R}$). *[Th.1, [1]] For all d, $0 \leq d \leq n$, the rational function r in (1.1) has no poles in $\mathbb{R}$.*

Thus now is easy to show that:

- $r(x)$ interpolates the data $(x_i, f(x_i))$
- $r(x)$ reproduces polynomials of degree at most d

For what concerns the accuracy of FHRI in the function approximation, we define

$$h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i), \tag{1.8}$$

$$\|f\|_\infty = \max_{x \in [a,b]} |r(x) - f(x)|$$

and

$$\beta = \max_{1 \leq i \leq n-2} \min \left\{ \frac{x_{i+1} - x_i}{x_i - x_{i-1}}, \frac{x_{i+1} - x_i}{x_{i+2} - x_{i+1}} \right\},$$

the following results hold

Theorem 1.1.3 (Interpolation error - case $d = 0$)**.** *[Th.3,1] Suppose d=0, $f \in C^2([a,b])$ and let h be as in (1.8). If n is odd then*

$$\|r - f\|_\infty \leq h(1+\beta)(b-a)\frac{\|f''\|_\infty}{2}. \tag{1.9}$$

If n is even then

$$\|r - f\|_\infty \leq h(1+\beta)\left((b-a)\frac{\|f''\|_\infty}{2} + \left\|f'\right\|_\infty\right) \tag{1.10}$$

Theorem 1.1.4 (Interpolation error - case $d \geq 1$)**.** *[Th.2,1] Suppose $d \geq 1$, $f \in C^{d+2}([a,b])$ and let h be as in (1.8). If n-d is odd then*

$$\|r - f\|_\infty \leq h^{d+1}(b-a)\frac{\|f^{(d+2)}\|_\infty}{d+2}. \tag{1.11}$$

If n-d is even then

$$\|r - f\|_\infty \leq h^{d+1}\left((b-a)\frac{\|f^{(d+2)}\|_\infty}{d+2} + \frac{\|f^{(d+1)}\|_\infty}{d+1}\right) \tag{1.12}$$

The Lebesgue constant is an index of stability for the interpolant, i.e. represents how much errors on the data could be amplified. If we suppose that $f \in C([a,b])$ and that we don't know the exact values f_i but perturbated ones $\tilde{f}_i = f_i + \delta_i$, then

$$\|r_n - \tilde{r}_n\|_\infty \leq \Lambda_n \max_{0 \leq k \leq n} \left|f_k - \tilde{f}_k\right|.$$

where r_n,$\tilde{r}_n$ are FHRI, that interpolate (x_i, f_i) and $(x_i, \tilde{f}_i)$ respectively. If we define

$$b_j = \frac{\dfrac{w_j}{x - x_j}}{\displaystyle\sum_{k=0}^{n} \frac{w_k}{x - x_k}}, \tag{1.13}$$

they satisfy the *Lagrange property*

$$b_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{se } i = j, \\ 0 & \text{se } i \neq j \end{cases} \tag{1.14}$$

and therefore they form a cardinal base. The Lebesgue constant of this operator is

$$\Lambda_n = \max_{a \leq x \leq b} \Lambda_n(x),$$

where $\Lambda_n(x)$ is the Lebesgue function,

$$\Lambda_n(x) = \sum_{j=0}^{n} |b_j(x)|.$$

In the case of equidistant nodes, we cite

Theorem 1.1.5 (Bound Lebesgue constant at equidistant nodes - $d = 0$)**.** *[Ths.1-2,2] The Lebesgue constant associated with rational interpolation at equidistant nodes* $\{x_j = \frac{j}{n}\}_{j=0,\dots,n}$ *with the basis functions (1.13) satisfies*

$$c_n \ln(n+1) \leq \Lambda_n \leq (2 + \ln(n)) \tag{1.15}$$

where $c_n = \frac{2n}{4+n\pi}$ *with* $\lim_{n\to\infty} c_n = \frac{2}{\pi}$.

Theorem 1.1.6 (Bound Lebesgue constant at equidistant nodes - $d \geq 1$)**.** *[Ths 1-2,3] The Lebesgue constant associated with rational interpolation at equidistant nodes* $\{x_j = \frac{j}{n}\}_{j=0,\dots,n}$ *with the basis functions (1.13) and* $n \geq 2d$ *satisfies*

$$\frac{1}{2^{d+2}}\binom{2d+1}{d} \ln(\frac{n}{d} - 1) \leq \Lambda_n \leq 2^{d-1}(2 + \ln(n)). \tag{1.16}$$

Chapter 2

Trigonometric FHRI (TFHRI)

2.1 Trigonometric interpolation - Generality

Suppose $I = [0, b]$ with $b < 2\pi$ and be $f : I \longrightarrow \mathbb{R}$. For arbitrary $n + 1$ nodes $0 = x_0 < ... < x_n = b$ and some values $f_i := f(x_i)$ the trigonometric interpolant is

if $n = 2m$ with $m \in \mathbb{Z}$

$$p(x) = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos(kx) + b_k \sin(kx))$$

and if $n = 2m + 1$ with $m \in \mathbb{Z}$

$$p(x) = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos(kx) + b_k \sin(kx)) + a_{m+1} \cos(mx).$$

In the case $n = 2m$, the previous formulas tell us that the trigonometric interpolant is a linear combination of $2m{+}1$ functions, for instance $1, \cos(x), \sin(x), \ldots, \cos(mx), \sin(mx)$. They represent a Chebyshev system, that is for every set of data $(x_i, f_i)_{0 \leq 1 \leq n}$ an unique interpolant exists or equivalently

$$det(V) = \begin{vmatrix} 1 & \cos(x_0) & \sin(x_0) & \ldots & \cos(mx_0) & \sin(mx_0) \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & \cos(x_n) & \sin(x_n) & \ldots & \cos(mx_n) & \sin(mx_n) \end{vmatrix} \neq 0.$$

Instead in the case $n = 2m{+}1$, where p is a linear combination of $1, \cos(x), \sin(x), \ldots, \cos(mx), \sin(mx), \cos((m + 1)x)$, the uniqueness of the interpolant isn't directly garanteed and the non-vanishing determinant is to be checked case by case. Now the determinant has the following form [6]

$$det(V) = (-1)^{\frac{3m^2-m}{2}} 2^{2m^2-2m+1} \sin\left(\frac{\sum_{j=0}^{2m-1} x_j}{2}\right) \prod_{0 \leq q < p \leq 2m-1} \sin\left(\frac{x_p - x_q}{2}\right).$$

Knowing that x_k are distinct and $0 < \frac{x_p - x_q}{2} < \pi$, it follows

$$det(V) = 0 \iff \sin\left(\frac{\sum_{j=0}^{2m-1} x_j}{2}\right) = 0 \iff \sum_{j=0}^{2m-1} x_j = 2\nu\pi \ \exists \nu \in \mathbb{N}. \tag{2.1}$$

For every n, adding the condition $\sigma := \sum_{j=0}^{2m-1} x_j \neq 2\nu\pi \ \forall \nu \in \mathbb{Z}$ if n is odd, the trigonometric interpolant is given by [5]

$$p(x) = \sum_{k=0}^{n} \ell_k(x) f_k$$

where $\{\ell_k\}_{k=0,\ldots,n}$ are the Lagrange trigonometric polynomials,

$$\ell_k = a_k \ell(x)\left(\operatorname{cst}\left(\frac{x - x_k}{2}\right) + c\right), \tag{2.2}$$

$$a_k = \frac{1}{\prod_{j=0,\ j\neq k}^{n} \sin\left(\frac{x_k - x_j}{2}\right)}, \quad \ell(x) = \prod_{j=0}^{n} \sin\left(\frac{x - x_j}{2}\right)$$

and

$$\operatorname{cst}(x) := \begin{cases} \csc(x) = \frac{1}{\sin(x)} & \text{if } n \text{ even} \\ \cot(x) = \frac{\cos(x)}{\sin(x)} & \text{if } n \text{ odd.} \end{cases}, \quad c := \begin{cases} 0 & \text{if } n \text{ even} \\ \cot(\frac{\sigma}{2}) & \text{if } n \text{ odd.} \end{cases}$$

If we consider the general case $I = [a, b]$, p does not provide the formula of interpolant if for example thare are some nodes which difference is a multiple of 2π unequal to 0. With the aim to include also this case we introduce the pulsation $\omega = \frac{2\pi}{T}$ where T represents the period. So after choosing $T > b - a$ the trigonometric interpolant becomes

$$p(x) = \sum_{k=0}^{n} \ell_k(x) f_k$$

where $\{\ell_k\}_{k=0,\ldots,n}$ are the Lagrange trigonometric polynomials,

$$\ell_k = a_k \ell(x)\left(\operatorname{cst}\left(\frac{\omega}{2}(x - x_k)\right) + c\right), \tag{2.3}$$

$$a_k = \frac{1}{\prod_{j=0,\ j\neq k}^{n} \sin\left(\frac{\omega}{2}(x_k - x_j)\right)}, \quad \ell(x) = \prod_{j=0}^{n} \sin\left(\frac{\omega}{2}(x - x_j)\right),$$

$$c := \begin{cases} 0 & \text{if } n \text{ even} \\ \cot(\frac{\omega}{2}\sigma) & \text{if } n \text{ odd.} \end{cases}$$

2.2 Trigonometric FHRI (TFHRI)

Combining the results of Chapter 1 and of the previous section, we wish to extend FHRI to the trigonometric case. By noting the formulas of trigonometric interpolant, cited before and first mentioned by Gauss in [7], we think that the most suitable way to build the TFHRI, is replacing $\lambda_k(x)$ in (1.1) with

$$\lambda_k^t(x) := \frac{(-1)^k}{\sin(\frac{\omega}{2}(x-x_k))\cdots\sin(\frac{\omega}{2}(x-x_{k+d}))} \tag{2.4}$$

so TFHRI might be written as

$$r^t(x) = \frac{\sum_{i=0}^{n-d} \lambda_i^t(x)\Bigg(\sum_{k=i}^{i+d} a_{k,i}\ell^{(i)}(x)\Bigg(\operatorname{cst}\left(\frac{\omega}{2}(x-x_k)\right)+c_i\Bigg)f_k\Bigg)}{\sum_{i=0}^{n-d}\lambda_i^t(x)} \tag{2.5}$$

with

$$a_{k,i} = \frac{1}{\prod_{j=i,\ j\neq k}^{i+d}\sin\big(\frac{\omega}{2}(x_k-x_j)\big)}, \quad \ell^{(i)}(x) = \prod_{j=i}^{i+d}\sin\left(\frac{\omega}{2}(x-x_j)\right),$$

$$c_i = \cot\left(\frac{\omega}{2}\sum_{j=i}^{i+d}x_j\right)$$

Noting that,

$$1 = \sum_{k=i}^{i+d} a_{k,i}\ell^{(i)}(x)\Bigg(\operatorname{cst}\left(\frac{\omega}{2}(x-x_k)\right)+c_i\Bigg)$$

we write the interpolant as

$$r^t(x) = \frac{\sum_{k=0}^{n}\Bigg(\sum_{i\in J_k}(-1)^i a_{k,i}\operatorname{cst}\left(\frac{\omega}{2}(x-x_k)\right)+\sum_{i\in J_k}(-1)^i a_{k,i}c_i\Bigg)f_k}{\sum_{k=0}^{n}\Bigg(\sum_{i\in J_k}(-1)^i a_{k,i}\operatorname{cst}\left(\frac{\omega}{2}(x-x_k)\right)+\sum_{i\in J_k}(-1)^i a_{k,i}c_i\Bigg)} =$$

$$= \frac{\sum_{k=0}^{n}\Bigg(w_k^t\operatorname{cst}\left(\frac{\omega}{2}(x-x_k)\right)+\alpha_k\Bigg)f_k}{\sum_{k=0}^{n}\Bigg(w_k^t\operatorname{cst}\left(\frac{\omega}{2}(x-x_k)\right)+\alpha_k\Bigg)}.$$

where

$$w_k^t = \sum_{i\in J_k}(-1)^i a_{k,i} = \sum_{i\in J_k}(-1)^i\prod_{j=i,j\neq k}^{i+d}\frac{1}{\sin\left(\frac{\omega}{2}(x_k-x_j)\right)} \tag{2.6}$$

with $J_k = \{i \in \{0, 1, 2, ..., n-d\}$ *such that* $k-d \le i \le k\}$
and

$$\alpha_k := \begin{cases} 0 & \text{if } d \text{ even} \\ \sum_{i \in J_k} (-1)^i a_{k,i} c_i & \text{if } d \text{ odd} \end{cases}$$

2.3 Case d even

Now we consider the general case I=[a,b]. The TFHRI is

$$r^t(x) = \sum_{k=0}^{n} b_k^t(x) f_k, \tag{2.7}$$

where

$$b_k^t(x) = \frac{\dfrac{w_k^t}{\sin\left(\frac{\omega}{2}(x-x_k)\right)}}{\displaystyle\sum_{k=0}^{n} \frac{w_k^t}{\sin\left(\frac{\omega}{2}(x-x_k)\right)}} \tag{2.8}$$

With easy computations it may be verified that $r^t(x)$ indeed interpolates the data $\{x_k, f_k\}$.

Hypothesis 1 on ω

We assume that:

- $\frac{\omega}{2}|x-x_k| \le (b-a)\frac{\omega}{2} < \frac{\pi}{2} \Rightarrow \omega < \frac{\pi}{(b-a)}$

  So the sine is increasing and the functions $\sin(\frac{\omega}{2}|x-x_k|)$ do not vanish in I

Following the same proof of the corresponding theorem in [1] we may prove

Theorem 2.3.1 (Absence of poles). *For all even* $0 \le d \le n$ *and every pulsation* ω *that satisfies* ***Hyp 1****, the TFHRI* r^t *in (2.7) has no poles in [a,b].*

2.3.1 Lebesgue constant

Let I be as before, $z_i = \frac{(b-a)i}{n}$ the nodes and $y_i = \frac{bi}{n}$.We suppose that the pulsation ω satisfies the **Hyp 1**, so the Lebesgue constant is

$$\Lambda_n = \max_{z \in [a,b]} \sum_{k=0}^{n} |b_k| = \max_{y \in [0,b-a]} \frac{\displaystyle\sum_{k=0}^{n} \frac{|w_k^{t,y}|}{\sin\left(\frac{\omega}{2}|y-y_k|\right)}}{\left|\displaystyle\sum_{k=0}^{n} \frac{w_k^{t,y}}{\sin\left(\frac{\omega}{2}(y-y_k)\right)}\right|} = \max_{y \in [0,b-a]} \Lambda_n(y).$$

where $w_k^{t,y}$ are the weights in (2.20) with nodes y_i.

Due to the previous reasonings, we may assume that $I = [0, b-a]$ but we really focus our attention only $I = [0,1]$ because if $b \neq 1$ appear some constants that simplify each other. So the bounds in the following hold for every choice of I, only the pulsation ω takes care about what the actual interval I is.

We consider, without lost generality, the interval $I = [0,1]$ and a pulsation ω such that $\omega < \pi$. We define $g(x) := \frac{1}{\mathrm{sinc}(x)}$ and show its plot below.

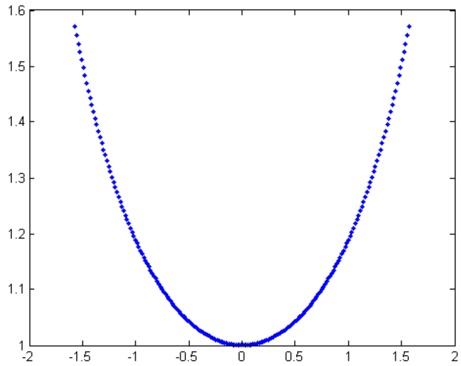

Figure 2.1: g-plot in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

It is evident that g is a positive function, is greater than 1 and is simmetric respect to the origin. As it is defined, g does not exist in $k\pi\ \forall k \in \mathbb{Z} \setminus \{0\}$. But for our purpouses, we consider it always in compact interval included in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then the following bounds hold

$$1 \leq g(\frac{\omega}{2}(x - x_k)) \leq \frac{\frac{\omega}{2}}{\sin(\frac{\omega}{2})} := M. \tag{2.9}$$

and

$$(x - x_k)(x_{k+1} - x) \leq \frac{1}{4n^2}. \tag{2.10}$$

We recall also the bounds for the partial sums of the Leibnitz series and the harmonic series, namely

$$\frac{\pi}{4} - \frac{1}{2n+3} \leq \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \leq \frac{\pi}{4} + \frac{1}{2n+3} \tag{2.11}$$

and

$$\ln(n+1) \leq \sum_{k=0}^{n} \frac{1}{k} \leq \ln(2n+1) \tag{2.12}$$

for any $n \in \mathbb{N}$. Moreover, it follows from (2.12) that

$$\sum_{k=0}^{n} \frac{1}{2k+1} = \sum_{k=1}^{2n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2k} \geq \ln(2n+2) - \frac{1}{2}\ln(2n+1) \geq \frac{1}{2}\ln(2n+3) \tag{2.13}$$

Theorem 2.3.2 (upper bound Lebesgue constant $d = 0$)**.** *The Lebesgue constant associated with rational interpolant in* $[0, 1]$*, with a pulsation* ω *according to Hyp 1, at equidistant nodes* $\{x_j = \frac{j}{n}\}_{j=0,\dots,n}$*, with the basis functions (2.8) satisfies*

$$\Lambda_n \leq \frac{M}{2-M}(2 + \ln(n)).$$

Proof. Let x be in I. If $x = x_k$ for any k then $\Lambda_n(x) = 1$. So let $x_k < x < x_{k+1}$ with $0 \leq k \leq n-1$ and consider

$$\Lambda_{n,k}(x) = \frac{(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n}\frac{1}{\sin(\frac{\omega}{2}|x-x_j|)}}{\left|(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}\right|} := \frac{N}{D}.$$

Our goal is to bound the numerator N from above and the denominator D from below. We first analyze the numerator,

$$N = (x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{1}{\sin(\frac{\omega}{2}|x-x_j|)} = (x-x_k)(x_{k+1}-x)\frac{2}{\omega}\sum_{j=0}^{n}\frac{g(\frac{\omega}{2}|x-x_j|)}{|x-x_j|} \leq$$

$$\leq \frac{2M}{\omega}(x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{1}{|x-x_j|} \leq \frac{2M}{\omega}\left(\frac{1}{n} + \frac{\ln(n)}{2n}\right)$$

where we use (2.9) and the last inequality is described in [2]. Now we focus on the denominator and split the proof into 4 cases.

1) k and n both even

$$D = \left|(x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}\right|.$$

Since by **Hyp 1** $-\frac{\pi}{2} < \frac{\omega}{2}(x-x_j) < \frac{\pi}{2}$, using the fact that sine is an increasing function in $[0, \frac{\pi}{2}]$ and $x > x_k$, we have

$$\sin\left(\frac{\omega}{2}(x-x_0)\right) > \sin\left(\frac{\omega}{2}(x-x_1)\right) > \dots > \sin\left(\frac{\omega}{2}(x-x_k)\right)$$

and

$$\sin\left(\frac{\omega}{2}(x-x_{k+1})\right) = -\sin(\frac{\omega}{2}(x_{k+1}-x)) > -\sin\left(\frac{\omega}{2}(x_{k+2}-x)\right) > \dots > -\sin\left(\frac{\omega}{2}(x_n-x)\right).$$

From these facts we get

$$\sum_{j=0}^{k}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} - \sum_{j=k+1}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))} = \tag{2.14}$$

$$= \frac{1}{\sin(\frac{\omega}{2}(x-x_0))} + \left(\frac{1}{\sin(\frac{\omega}{2}(x-x_2))} - \frac{1}{\sin(\frac{\omega}{2}(x-x_1))}\right) + ...+$$

$$+\left(\frac{1}{\sin(\frac{\omega}{2}(x-x_k))} - \frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))}\right) + \left(\frac{1}{\sin(\frac{\omega}{2}(x_{k+1}-x))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))}\right)+$$

$$+... + \left(\frac{1}{\sin(\frac{\omega}{2}(x_{n-1}-x))} - \frac{1}{\sin(\frac{\omega}{2}(x_n-x))}\right) > 0.$$

So we can ignore the absolute value in the denominator D. Following the idea of the proof in [2] and using also (2.9), we find

$$D = (x-x_k)(x_{k+1}-x)\sum_{j=0}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} = (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))}+\right.$$

$$\left. + \frac{g(\frac{\omega}{2}(x-x_k))}{\frac{\omega}{2}(x-x_k)} + \frac{g(\frac{\omega}{2}(x_{k+1}-x))}{\frac{\omega}{2}(x_{k+1}-x)} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\right) \geq$$

$$\geq (x-x_k)(x_{k+1}-x)\left(\frac{1}{\frac{\omega}{2}(x-x_k)} + \frac{1}{\frac{\omega}{2}(x_{k+1}-x)}\right)+$$

$$+(x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\right) = \frac{2}{\omega n}+(x-x_k)(x_{k+1}-x)S.$$

We study S and use the previous observation about the positiveness of the sum in the denominator,

$$S = \sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j)} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))} = \tag{2.15}$$

$$= \frac{1}{\sin(\frac{\omega}{2}(x-x_0))} + \sum_{s=1}^{\frac{k-2}{2}} \left(\frac{1}{\sin(\frac{\omega}{2}(x-x_{2s}))} - \frac{1}{\sin(\frac{\omega}{2}(x-x_{2s-1}))}\right)$$

$$-\frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))} + \sum_{s=2}^{\frac{n-k}{2}} \left(\frac{1}{\sin(\frac{\omega}{2}(x_{k+2s-1}-x))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2s}-x))}\right) \geq$$

$$\geq -\frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))} \geq -\frac{1}{\sin(\frac{\omega}{2}(x_k-x_{k-1}))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x_{k+1}))} =$$

$$= -g\left(\frac{\omega}{2n}\right)\frac{2}{\omega}\left(\frac{1}{x_k-x_{k-1}} + \frac{1}{x_{k+2}-x_{k+1}}\right) = g\left(\frac{\omega}{2n}\right)\frac{2(-2n)}{\omega} \geq \frac{2M(-2n)}{\omega}.$$

We return to D and using (2.10) we get

$$D = \frac{2}{\omega n}+(x-x_k)(x_{k+1}-x)S \geq \frac{2}{\omega n}-(x-x_k)(x_{k+1}-x)\frac{2M(2n)}{\omega} \geq \frac{2}{\omega n} - \frac{1}{4n^2}\frac{2M(2n)}{\omega} =$$

$$= \frac{2}{\omega n}\left(1-\frac{M}{2}\right) = \frac{2}{\omega n}\left(\frac{2-M}{2}\right) = \frac{2}{\omega}\left(\frac{2-M}{2n}\right) \tag{2.16}$$

2) k even and n odd

Now we add a single positive term $\frac{1}{x_n - x}$ to S and therefore all the previous bounds are still true. We prove again (2.16).

3) k odd and n even

With similar calculation as before, we show that

$$\sum_{j=0}^{k} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} + \sum_{j=k+1}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} < 0.$$

Now we consider $\hat{D}$, that is the denominator D without the absolute value. Then we construct a bound for D.

$$\hat{D} = (x-x_k)(x_{k+1}-x)\sum_{j=0}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} = (x-x_k)(x_{k+1}-x)\Bigg(\sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} -$$

$$-\frac{g(\frac{\omega}{2}(x-x_k))}{\frac{\omega}{2}(x-x_k)} - \frac{g(\frac{\omega}{2}(x_{k+1}-x))}{\frac{\omega}{2}(x_{k+1}-x)} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\Bigg) =$$

$$= -(x-x_k)(x_{k+1}-x)\Bigg(\frac{g(\frac{\omega}{2}(x-x_k))}{\frac{\omega}{2}(x-x_k)} + \frac{g(\frac{\omega}{2}(x_{k+1}-x))}{\frac{\omega}{2}(x_{k+1}-x)}\Bigg)+$$

$$+(x-x_k)(x_{k+1}-x)\Bigg(\sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))}\Bigg) =$$

$$= -\frac{2}{\omega}\Bigg((x_{k+1}-x)g(\frac{\omega}{2}(x-x_k)) + (x-x_k)g(\frac{\omega}{2}(x_{k+1}-x))\Bigg) + (x-x_k)(x_{k+1}-x)S.$$

We might estimate S and first write it as sum of negative and positive terms,

$$S = \sum_{j=0}^{k-1} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j))} - \sum_{j=k+2}^{n} \frac{(-1)^j}{\sin(\frac{\omega}{2}(x_j-x))} = \tag{2.17}$$

$$= \sum_{s=0}^{\frac{k-3}{2}} \overbrace{\Bigg(\frac{1}{\sin(\frac{\omega}{2}(x-x_{2s}))} - \frac{1}{\sin(\frac{\omega}{2}(x-x_{2s+1}))}\Bigg)}^{<0} + \frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))} + \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))} +$$

$$+ \sum_{s=2}^{\frac{n-k-1}{2}} \overbrace{\Bigg(\frac{1}{\sin(\frac{\omega}{2}(x_{k+2s}-x))} - \frac{1}{\sin(\frac{\omega}{2}(x_{k+2s-1}-x))}\Bigg)}^{<0} - \frac{1}{\sin(\frac{\omega}{2}(x_n-x))} \le$$

$$\le \frac{1}{\sin(\frac{\omega}{2}(x-x_{k-1}))} + \frac{1}{\sin(\frac{\omega}{2}(x_{k+2}-x))} \le \frac{g(\frac{\omega}{2}(x-x_k))}{\frac{\omega}{2}(x-x_{k-1})} + \frac{g(\frac{\omega}{2}(x_{k+2}-x))}{\frac{\omega}{2}(x_{k+2}-x)} \le$$

$$\le \frac{2M}{\omega}\Bigg(\frac{1}{x_k + x_{k-1}} + \frac{1}{x_{k+2}-x_{k+1}}\Bigg) = \frac{2M2n}{\omega}.$$

We return to $\hat{D}$ and use (2.10)

$$\hat{D} \le -\frac{2}{\omega}\Bigg((x_{k+1}-x)g(\frac{\omega}{2}(x-x_k))+(x-x_k)g(\frac{\omega}{2}(x_{k+1}-x))\Bigg)+(x-x_k)(x_{k+1}-x)S \le$$

$$\le -\frac{2}{\omega}\Bigg((x_{k+1}-x)g(\frac{\omega}{2}(x-x_k))+(x-x_k)g(\frac{\omega}{2}(x_{k+1}-x))\Bigg)+(x-x_k)(x_{k+1}-x)\frac{2M2n}{\omega} \le$$

$$\le \frac{2}{\omega}\Bigg(-\frac{1}{n}+\frac{M2n}{4n^2}\Bigg) = \frac{2}{\omega}\Bigg(\frac{-2+M}{2n}\Bigg).$$

Since $\omega \in [0,\pi)$, $M < g(\pi) < 2$ and so it implies that

$$D \ge \frac{2}{\omega}\Bigg(\frac{|-2+M|}{2n}\Bigg) = \frac{2}{\omega}\Bigg(\frac{2-M}{2n}\Bigg)$$

4) k and n both odd

In S the terms with $j \ge k+3$ are an even number so we can group them all in the second sum. So we get again the bounds of the case 3).

Finally we sum up the results and obtain

$$\Lambda_n = \max_{k=0,\dots,n-1}\Bigg(\max_{x_k<x<x_{k+1}} \Lambda_{n,k}(x)\Bigg) \le \frac{\frac{2M}{\omega}\Big(\frac{1}{n}+\frac{\ln(n)}{2n}\Big)}{\frac{2}{\omega}\Big(\frac{2-M}{2n}\Big)} = \frac{M}{2-M}\Bigg(2+\ln(n)\Bigg)$$

□

Theorem 2.3.3 (lower bound Lebesgue constant - d = 0)**.** *The Lebesgue constant associated with trigonometric rational interpolant in* $[0,1]$*, with a pulsation* ω *according to Hyp 1, at equidistant nodes* $\{x_j = \frac{j}{n}\}_{j=0,\dots,n}$ *with basis functions (2.8) satisfies*

$$\Lambda_n \ge \frac{2n}{M(4+n\pi)}\ln(n+1)$$

Proof. We will follow the proof in [2].

By general definition of the Lebesgue function we have

$$\Lambda_n(x) = \frac{\displaystyle\sum_{j=0}^{n}\frac{1}{\sin(\frac{\omega}{2}|x-\frac{j}{n}|)}}{\Bigg|\displaystyle\sum_{j=0}^{n}\frac{(-1)^j}{\sin(\frac{\omega}{2}(x-\frac{j}{n}))}\Bigg|} = \frac{\displaystyle\sum_{j=0}^{n}\frac{g(\frac{\omega}{2}|x-\frac{j}{n}|)}{\frac{\omega}{2}|x-\frac{j}{n}|}}{\Bigg|\displaystyle\sum_{j=0}^{n}\frac{g(\frac{\omega}{2}(x-\frac{j}{n}))(-1)^j}{\frac{\omega}{2}(x-\frac{j}{n})}\Bigg|} =$$

$$= \frac{\displaystyle\sum_{j=0}^{n}\frac{g(\frac{\omega}{2}|x-\frac{j}{n}|)}{|2nx-2j|}}{\Bigg|\displaystyle\sum_{j=0}^{n}\frac{g(\frac{\omega}{2}(x-\frac{j}{n}))(-1)^j}{(2nx-2j)}\Bigg|} =: \frac{N(x)}{D(x)}.$$

Our goal now is to bound the numerator $N(x)$ from below and the denominator $D(x)$ from above. We split the proof into 2 cases.
1) n even, $\exists$ k $\in \mathbb{N}$ such that n=2k We evaluate the numerator $N(x)$ at $x = \frac{n+1}{2n}$ and using (2.9), we obtain

$$N\left(\frac{n+1}{2n}\right) = \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|\frac{n+1-2j}{2n}|)}{|n+1-2j|} \geq \sum_{j}^{2k} \frac{1}{|2(k-j)+1|} \geq \ln(n+1).$$

The last inequality is described in [2]. Now we turn to the denominator

$$D\left(\frac{n+1}{2n}\right) = \left|\sum_{j=0}^{2k} \frac{g(\frac{\omega}{2}(\frac{2(k-j)+1}{2n}))(-1)^j}{(2(k-j)+1)}\right| \leq$$

$$\leq \left|\sum_{j=0}^{k} \frac{g(\frac{\omega}{2}(\frac{2(k-j)+1}{2n}))(-1)^j}{(2(k-j)+1)}\right| + \left|\sum_{j=k+1}^{2k} \frac{g(\frac{\omega}{2}(\frac{2(k-j)+1}{2n}))(-1)^j}{(2(k-j)+1)}\right| =$$

$$= \left|(-1)^k \sum_{j=0}^{k} \frac{g(\frac{\omega}{2}(\frac{2j+1}{4k}))(-1)^j}{2j+1}\right| + \left|(-1)^k \sum_{j=0}^{k-1} \frac{g(\frac{\omega}{2}(\frac{2j+1}{4k}))(-1)^j}{2j+1}\right| =$$

$$= \left|\sum_{j=0}^{k} \frac{g(\frac{\omega}{2}(\frac{2j+1}{4k}))(-1)^j}{2j+1}\right| + \left|\sum_{j=0}^{k-1} \frac{g(\frac{\omega}{2}(\frac{2j+1}{4k}))(-1)^j}{2j+1}\right| =: |A| + |B|$$

We define $y_j := \frac{\omega}{2}(\frac{2j+1}{4k})$, we focus on $|A|$ but the same reasoning holds for $|B|$. Since g is an increasing function in $[0, \pi)$, thanks to (2.11), we note that

$$A = (g(y_0) - \frac{1}{3}g(y_1)) + (\frac{1}{5}g(y_2) - \frac{1}{7}g(y_3)) + ... \leq g(y_1)(1 - \frac{1}{3}) + g(y_3)(\frac{1}{5} - \frac{1}{7}) + ... \leq$$

$$\leq M \sum_{j=0}^{k} \frac{1}{2j+1} \leq M\left(\frac{\pi}{4} + \frac{1}{2k+3}\right) = M\left(\frac{\pi}{4} + \frac{1}{n+3}\right).$$

A is the sum of positive terms. In fact, if k is odd, but this holds also if k is even,

$$A = \sum_{j=0}^{\frac{k-1}{2}} \left(\frac{g(y_{2j})}{4j+1} - \frac{g(y_{2j+1})}{4j+3}\right),$$

$$\frac{g(y_{2j})}{4j+1} - \frac{g(y_{2j+1})}{4j+3} \geq 0 \ \forall j \iff \sin\left(\frac{\omega}{2}\frac{(4j+1)}{4k}\right) \leq \sin\left(\frac{\omega}{2}\frac{(4j+2)}{4k}\right) \forall j.$$

But under our hypothesis on ω, this is true $\forall j$. Then we may ignore the absolute value and get

$$A \leq M\left(\frac{\pi}{4} + \frac{1}{n+3}\right).$$

With similar computations it may be seen that

$$|B| = B \leq M\left(\frac{\pi}{4} + \frac{1}{n+1}\right).$$

Finally we have

$$|A| + |B| \leq M\left(\frac{\pi}{4} + \frac{1}{n+3} + \frac{\pi}{4} + \frac{1}{n+1}\right) = M\left(\frac{\pi}{2} + \frac{2}{n+1}\right)$$

We sum up the results and obtain

$$\Lambda_n \geq \frac{2\ln(n+1)}{M\left(\pi + \dfrac{4}{n+1}\right)}$$

2) n odd, $\exists$ k such that n=2k+1 Now we consider $x = \frac{1}{2}$ and evaluate the numerator and the denominator both at this point. Let us start with the numerator. In the following sequence of inequalities we use (2.9) and (2.13) and the same proof in [2].

$$N\left(\frac{1}{2}\right) \geq \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|\frac{n-2j}{2n}|)}{|n-2j|} \geq \ln(2k+3) = \ln(n+2)$$

Now we focus on the denominator. As we have seen in the previous case, thanks to our hypothesis on ω, the sum is positive. So in the following the absolute value may be ignored.

$$D\left(\frac{1}{2}\right) = \left|\sum_{j=0}^{n} \frac{g(\frac{\omega}{2}(\frac{n-2j}{2n}))(-1)^j}{(2n\frac{1}{2} - 2j)}\right| = \left|\sum_{j=0}^{k} \frac{g(\frac{\omega}{2}(\frac{n-2j}{2n}))(-1)^j}{2(k-j)+1} + \right.$$

$$\left. + \sum_{j=k+1}^{2k+1} \frac{g(\frac{\omega}{2}(\frac{n-2j}{2n}))(-1)^j}{2(k-j)+1}\right| = \left|\sum_{j=0}^{k} \frac{(g(\frac{\omega}{2}(\frac{1-2j}{2n})) + g(\frac{\omega}{2}(\frac{-2j-1}{2n})))(-1)^j}{2j+1}\right|.$$

Defining

$$G_j = g\left(\frac{\omega}{2}\left(\frac{1-2j}{2n}\right)\right) + g\left(\frac{\omega}{2}\left(\frac{-2j-1}{2n}\right)\right)$$

it easy to see that $G_j > 0 \ \forall j$ and that it is an increasing sequence. Thanks to these observations and to the similar proof in [2], finally we have

$$D\left(\frac{1}{2}\right) = \sum_{j=0}^{k} G_j \frac{(-1)^j}{2j+1} \leq G_1(1 - \frac{1}{3}) + G_3(\frac{1}{5} - \frac{1}{7}) + ... \leq 2M \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \leq$$

$$\leq 2M\left(\frac{\pi}{4} + \frac{1}{2k+3}\right) = M\left(\frac{\pi}{2} + \frac{2}{n+2}\right).$$

We obtain

$$\Lambda_n\left(\frac{1}{2}\right) = \frac{N(\frac{1}{2})}{D(\frac{1}{2})} \geq \frac{\ln(n+2)}{M\left(\frac{\pi}{2}+\frac{2}{n+2}\right)} = \frac{2\ln(n+2)}{M\left(\pi+\frac{4}{n+2}\right)}.$$

and conclude

$$\Lambda_n \geq \frac{2\ln(n+2)}{M\left(\pi+\frac{4}{n+2}\right)} \geq \frac{2\ln(n+1)}{M\left(\pi+\frac{4}{n+1}\right)} \geq \frac{2\ln(n+1)}{M\left(\pi+\frac{4}{n}\right)} = \frac{2n}{M(4+n\pi)}\ln(n+1)$$

□

Proposition 2.3.4 (Bound weigths d even). *If the interpolation nodes are equispaced, then the weigths in 2.6 satisfy*

$$\frac{2^d}{\omega^d}\frac{1}{h^d d!}\sum_{i\in J_k}\binom{d}{k-i} \leq \frac{2^d}{\omega^d}|w_k| \leq |w_k^t| \leq \frac{2^d M^d}{\omega^d}|w_k| \leq \frac{2^d M^d}{\omega^d}\frac{2^d}{h^d d!}. \tag{2.18}$$

Proof.

$$|w_k^t| = \sum_{i\in J_k}\prod_{j=i,j\neq k}^{i+d}\frac{1}{\sin(\frac{\omega}{2}|x_k-x_j|)} \geq \frac{2^d}{\omega^d}\sum_{i\in J_k}\prod_{j=i,j\neq k}^{i+d}\frac{g(\frac{\omega}{2}|x_k-x_j|)}{|x_k-x_j|} \geq$$

$$\geq \frac{2^d}{\omega^d}\sum_{i\in J_k}\prod_{j=i,j\neq k}^{i+d}\frac{1}{|x_k-x_j|} = \frac{2^d}{\omega^d}|w_k|.$$

Now prove the bound from above.

$$|w_k^t| = \frac{2^d}{\omega^d}\sum_{i\in J_k}\prod_{j=i,j\neq k}^{i+d}\frac{g(\frac{\omega}{2}|x_k-x_j|)}{|x_k-x_j|} \leq \frac{2^d M^d}{\omega^d}\sum_{i\in J_k}\prod_{j=i,j\neq k}^{i+d}\frac{1}{|x_k-x_j|} = \frac{2^d M^d}{\omega^d}|w_k|.$$

Thanks to (1.5) the thesis follows easily.

□

Theorem 2.3.5 (Upper bound Lebesgue constant - d $\geq$ 2, even). *The Lebesgue constant associated with trigonometric rational interpolant in* $[0,1]$*, with a pulsation* ω *according to Hyp 1, at equidistant nodes* $\{x_j = \frac{j}{n}\}_{j=0,\dots,n}$ *with basis functions (2.8) satisfies*

$$\Lambda_n \leq M^{d+1}2^{d-1}\big(2+\ln(n)\big)$$

Proof. If $x = x_k$ for any k, then $\Lambda_n(x) = 1$. Otherwise, $x_k < x < x_{k+1}$ with $0 \leq k \leq n-1$ and we consider

$$\Lambda_{n,k}(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{|w_j^t|}{\sin(\frac{\omega}{2}|x-x_j|)}}{\left|\displaystyle\sum_{j=0}^{n} \frac{w_j^t}{\sin(\frac{\omega}{2}(x-x_j))}\right|} = \frac{h^d d!(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n} \frac{|w_j^t|}{\sin(\frac{\omega}{2}|x-x_j|)}}{\left|h^d d!(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n} \frac{w_j^t}{\sin(\frac{\omega}{2}(x-x_j))}\right|} := \frac{N_k(x)}{D_k(x)}.$$

First of all we try to bound the numerator from above.

$$N_k(x) = \frac{2}{\omega} h^d d!(x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|x-x_j|)|w_j^t|}{|x-x_j|} \leq$$

$$\leq \frac{2^{d+1}M^d}{\omega^{d+1}} h^d d!(x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}|x-x_j|)|w_j|}{|x-x_j|} \leq$$

$$\leq \frac{2^{d+1}M^{d+1}}{\omega^{d+1}}(x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{|w_j|h^d d!}{|x-x_j|} = \frac{2^{d+1}M^{d+1}}{\omega^{d+1}}(x-x_k)(x_{k+1}-x) \sum_{j=0}^{n} \frac{\beta_j}{|x-x_j|} \leq$$

$$\leq \frac{2^{d+1}}{\omega^{d+1}} M^{d+1} 2^d \left(\frac{1}{n} + \frac{1}{2n}\ln(n)\right)$$

where the last inequality holds thanks to [3].

Now we turn to the denominator. From the definition of β_j in (1.6) is easy to see that

$$(-1)^d d! h^d \sum_{j=0}^{n} \frac{w_j}{x-x_j} = (-1)^d d! h^d \sum_{j=0}^{n-d} \lambda_j(x) = \sum_{j=0}^{n} \frac{(-1)^j \beta_j}{x-x_j}$$

with $\lambda_j(x)$ as in (1.2). So $\forall j \in \{0, ..., n-d\}$

$$\lambda_j^t(x) = \frac{(-1)^j}{\sin(\frac{\omega}{2}(x-x_j)) \ldots \sin(\frac{\omega}{2}(x-x_{j+d}))} = \frac{2^d}{\omega^d} \prod_{i=j}^{j+d} g\left(\frac{\omega}{2}(x-x_i)\right) \lambda_j(x)$$

Assuming $k \leq n-d$ and keeping in mind the proof of Theorem 2.3.1 and (2.9), we have

$$D_k(x) = d! h^d (x-x_k)(x_{k+1}-x) \left| \sum_{j=0}^{n} \lambda_j^t(x) \right| \geq d! h^d (x-x_k)(x_{k+1}-x) |\lambda_k^t(x)| =$$

$$= \frac{d! h^d (x-x_k)(x_{k+1}-x)}{\sin(\frac{\omega}{2}(x-x_k)) \sin(\frac{\omega}{2}(x_{k+1}-x)) \cdots \sin(\frac{\omega}{2}(x_{k+d}-x))} =$$

$$= \frac{2^{d+1}}{\omega^{d+1}} \frac{d! h^d (x-x_k)(x_{k+1}-x) g(\frac{\omega}{2}(x-x_k)) \ldots g(\frac{\omega}{2}(x-x_{k+d}))}{(x-x_k)(x_{k+1}-x)\cdots(x_{k+d}-x)} \geq$$

$$\geq \frac{2^{d+1}}{\omega^{d+1}} \frac{d!h^d(x-x_k)(x_{k+1}-x)}{(x-x_k)(x_{k+1}-x)\cdots(x_{k+d}-x)} \geq$$

$$\geq \frac{2^{d+1}}{\omega^{d+1}} \frac{d!h^d}{(x_{k+2}-x_k)\cdots(x_{k+d}-x_k)} \geq \frac{2^{d+1}}{\omega^{d+1}} \frac{d!h^d}{d!h^{d-1}} = \frac{2^{d+1}}{\omega^{d+1}} \frac{1}{n}$$

If $k > n-d$ a similar reasoning leads to this lower bound for $D_k(x)$ by considering λ_{k-d+1} instead of λ_k. Finally we sum up the results and obtain

$$\Lambda_n = \max_{k=0,\dots,n-1} \left(\max_{x_k<x<x_{k+1}} \Lambda_{n,k}(x) \right) \leq M^{d+1} 2^{d-1} \big(2+\ln(n)\big)$$

□

Theorem 2.3.6 (Lower bound Lebesgue constant - d ≥ 2, even)**.** *The Lebesgue constant associated with trigonometric rational interpolant in* $[0,1]$*, with a pulsation* ω *according to Hyp 1, at equidistant nodes* $\{x_j = \frac{j}{n}\}_{j=0,\dots,n}$ *with basis functions (2.8) satisfies*

$$\Lambda_n \geq \frac{1}{M^{d+1}2^{d+2}} \binom{2d+1}{d} \ln(\frac{n}{d}-1)$$

Proof.

$$\Lambda_n(x) := \frac{h^d d! \displaystyle\sum_{j=0}^{n} \frac{|w_j^t|}{\sin(\frac{\omega}{2}|x-x_j|)}}{\left| h^d d! \displaystyle\sum_{j=0}^{n} \frac{w_j^t}{\sin(\frac{\omega}{2}(x-x_j))} \right|} := \frac{N(x)}{D(x)}.$$

We consider $x^* = \frac{x_1-x_0}{2} = \frac{1}{2n}$.

We first investigate the numerator, using (2.9) and the bounds of weights w_j^t,

$$N(x^*) = d!h^d \sum_{j=0}^{n} \frac{|w_j^t|}{\sin(\frac{\omega}{2}|x^*-x_j|)} \geq d!h^d \frac{2^{d+1}}{\omega^{d+1}} \sum_{j=0}^{n} \frac{g(\frac{\omega}{2}(x^*-x_j))|w_j|}{|x^*-x_j|} \geq \frac{2^{d+1}}{\omega^{d+1}} \sum_{j=0}^{n} \frac{\beta_j}{|x^*-x_j|} \geq$$

$$\geq \frac{2^{d+1}}{\omega^{d+1}} n 2^d \ln(\frac{n}{d}-1)$$

where the last inequality has been taken from [2].

Now we turn to the denominator.

$$D(x^*) = h^d d! \left| \sum_{j=0}^{n-d} \lambda_j^t(x^*) \right|.$$

We notice that $\lambda_0^t(x^*)$ e $\lambda_1^t(x^*)$ have the same sign and that the following $\lambda_j^t(x^*)$ oscillate in sign and decrease in absolute value. So we get, using (2.9),

$$D(x^*) \leq h^d d! (|\lambda_0^t(x^*)| + |\lambda_1^t(x^*)|) = d!h^d \frac{2^{d+1}}{\omega^{d+1}} \Bigg(|\lambda_0(x^*)| \prod_{i=0}^{d} g(\frac{\omega}{2}(x^*-x_i)) +$$

$$+|\lambda_1(x^*)|\prod_{i=1}^{d+1} g(\frac{\omega}{2}(x^*-x_i))\Big) \le \frac{2^{d+1}}{\omega^{d+1}} M^{d+1}\Big(|\lambda_0(x^*)|+|\lambda_1(x^*)|\Big) \le \frac{2^{d+1}}{\omega^{d+1}} M^{d+1} n \frac{2^{2d+2}}{\binom{2d+1}{d}}$$

where the last inequality follow from the proof in [2].

Finally we obtain

$$\Lambda_n \ge \frac{1}{M^{d+1}2^{d+2}}\binom{2d+1}{d}\ln(\frac{n}{d}-1)$$

$\square$

2.3.2 Numerical Experiments

How to choose ω

The pulsation ω allows the costruction of TFHRI for every functions (peiodic and non periodic). The problem is how to choose it. In fact the **Hypothesis 1** does not provide an unique value of ω. The most reasonable way of choosing ω is to minimize the relative error of interpolation.

The relative error is given by

$$Err_{rel} = \frac{\|r^t - f\|_\infty}{\|f\|_\infty}$$

and we approximate it in the following way

$$Err_{rel} \approx \frac{\max_{x \in \hat{I}} |r^t(x) - f(x)|}{\max_{x \in \hat{I}} |f(x)|}$$

where $\hat{I}$ is a discretization of I.

In the numerical experiments the test interval is $I = [0, \pi]$, so $\omega \in (0, 1)$. First of all non periodic functions have been taken in consideration. The plots of error as a function of ω, Figure 2.2, reveal that $\omega_{opt} \approx 0$. It is not unexpected because, recalling the definition of the pulsation, if $\omega \to 0$ then $T \to +\infty$, so it seems to be right since the function f is non periodic as supposed.

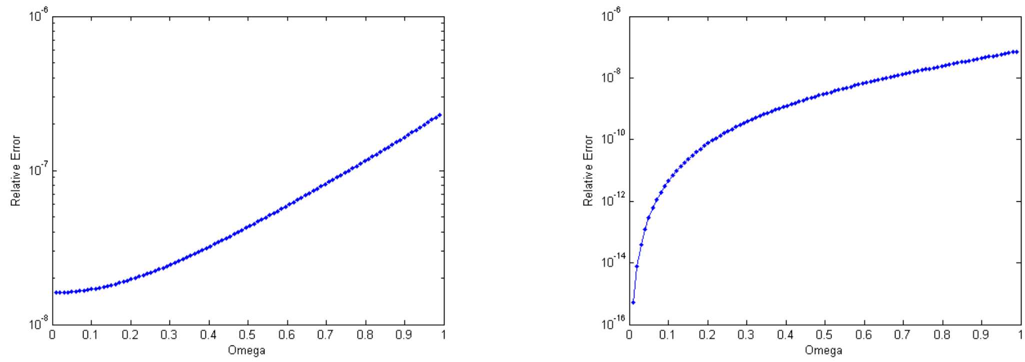

Figure 2.2: ω-Error plot with $I = [0, \pi]$, $n = 40$ and $d = 4$. The test functions are $\exp(x)$ (left) and x^2 (right).

It is clear that the choice of ω_{opt} is strictly related to the number of nodes, because the relative error depends on it, the value of d and function to approximate. If the number of nodes grows for fixed d, the behaviour of the error remains the same while if the function changes for fixed n, d the error may present important changes.

In particular the numerical results show that:

- $d = 0$: the choice of ω does not produce a relevant reduction of the error no matter what the function f is

- $d > 0$: according to the specific f, ω affects more or less the error. In Figure 2.2 the test functions are e^x and x^2 with $d = 4$. On the right the decrease of the error is indeed marked and the plot reveals that the error goes from 10^{-8} to 10^{-16}, therefore the correct choice of the pulsation allows a better approximation than in the second case on the left.

On the other hand if the test function is periodic with $T = \pi$ or a submultiple, the Figures 2.3 show that $\omega_{opt} \approx 1$. Knowing that f is π periodic, it is reasonable that the best choice of ω is such that the period is $\approx \pi$. No matter what f is, for different values of d the decrease of the error is not so remarked as before. In Figure 2.3 on the left the ω_{opt} brings a decrease of error from 10^{-6} to 10^{-7}, while if $f(x) = \sin(8x)$, on the right, each values of $\omega \in (0, 1)$ are equivalent.

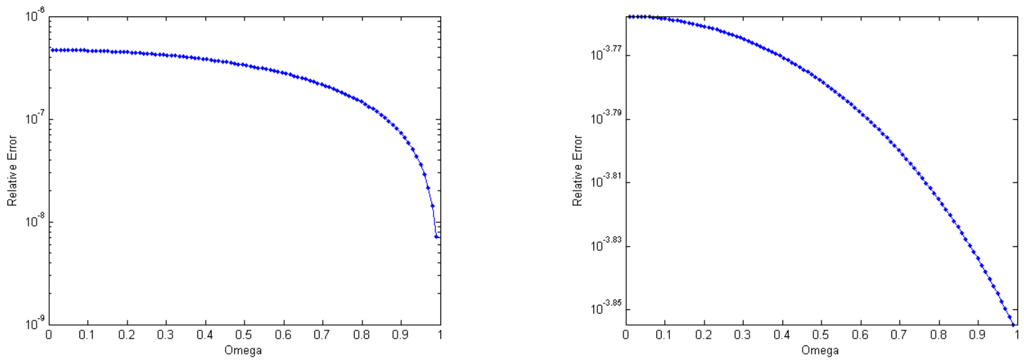

Figure 2.3: ω-Error plot with $I = [0, \pi]$, $n = 40$ and $d = 4$. The test function are $\sin(2x)$ (left) and $\sin(8x)$ (right).

Approximation and error

Using our matlab codes, we have tested TFHRI first in some regular cases. In the first one TFHRI approximates a non periodic function, for instance $f_1 = x^2$ in $[0, \pi]$, with different numbers of nodes. In the second case we try to approximate a periodic function, for example $f_2 = \sin(2x)$ in $[0, \pi]$ and in the third case a function, like $f_3 = |x|$, which has a singularity in 0 so here we consider $[-\frac{\pi}{2}, \frac{\pi}{2}]$ as interval in such way to include 0.

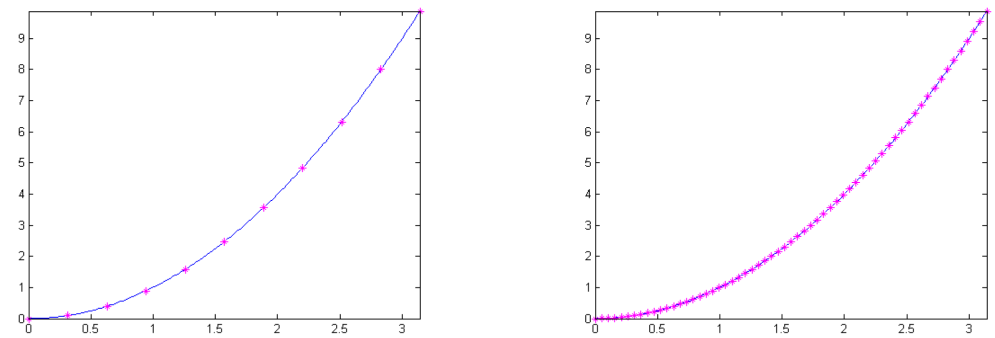

Figure 2.4: TFHRI - Test function $f(x) = x^2$ with $n+1$ equispaced nodes in $[0, \pi]$, $n = 10, 60$, $\omega = 0.1$ and $d = 2$

According to the previous analysis, we have chosen $\omega = 0.1$ with f_1, f_2 and $\omega = 0.5$ with f_3. These seem good choices of the pulsation, especially in the first two cases, because ω is not too close to zero but so small to realize a significant reduction of the relative error.

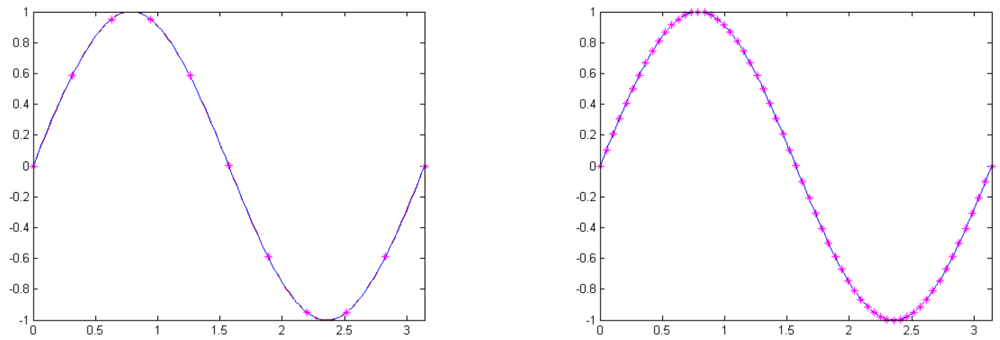

Figure 2.5: TFHRI - Test function $f(x) = \sin(2x)$ with $n+1$ equispaced nodes in $[0, \pi]$, $n = 10, 60$, $\omega = 0.1$ and $d = 2$

The accuracy of the interpolant is good also with few nodes, as may be seen in Figures 2.4, 2.5, 2.6 and expecially Figure 2.6 reveals that TFHRI has not any problem in approximating f_3 also in 0. This aspect is remarked in Table 2.3.2, where we have listed the errors and in particular with $n = 60$ TFHRI can interpolate f_3 with accuracy of 10^{-3} and there are no oscillations in any of the three cases.

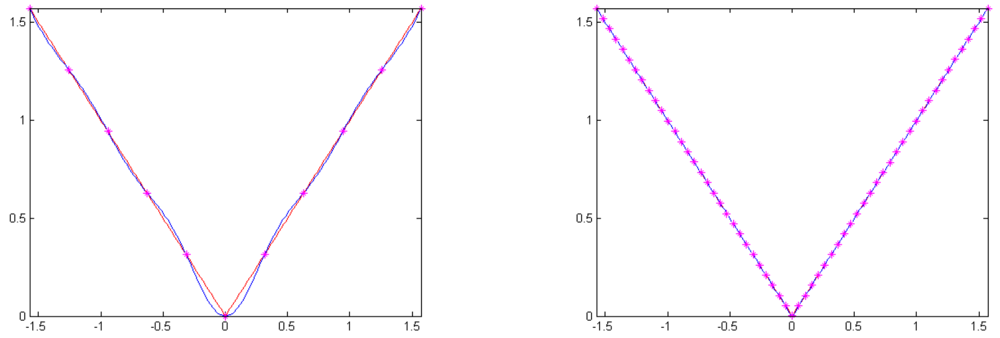

Figure 2.6: TFHRI - Test function $f(x) = |x|$ with $n+1$ equispaced nodes in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $n = 10, 60$, $\omega = 0.5$ and $d = 2$

n	x^2	$\sin(2x)$	$\|x\|$	f_4
10	8.2385e-09	2.2994e-04	3.6496e-02	2.9614e-01
20	1.9024e-10	1.1549e-05	1.0019e-02	1.9753e-01
40	4.5348e-12	3.3554e-07	3.2860e-03	1.3781e-01
60	5.1995e-13	4.0640e-08	1.7667e-03	1.1211e-01

Table 2.1: Trigonometric interpolation error - $d = 4$

Then we have tested trigonometric rational interpolant with particular functions, for instance with

$$f_4 = \begin{cases} -1, & \text{if } x \in [0, \frac{\pi}{2}], \\ 1, & \text{if } x \in (\frac{\pi}{2}, \pi] \end{cases}$$

and $f_5 = \operatorname{erf}(50(x - \frac{\pi}{2}))$ in $[0, \pi]$ both with $\omega = 0.1$, where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

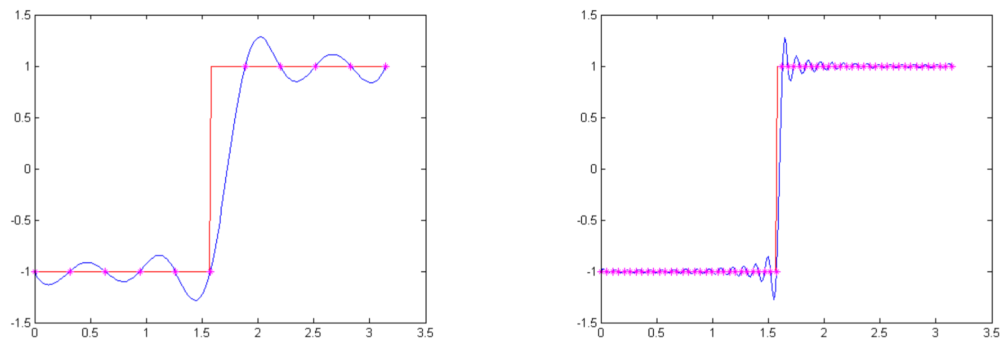

Figure 2.7: TFHRI - Test function f_4 with $n+1$ equispaced nodes in $[0, \pi]$, $n = 10, 60$, $\omega = 0.1$ and $d = 2$

In Figure 2.7 TFHRI tries to approximate a function with a jump-discontinuity. With few nodes the oscillations are marked while they attenuate with $n = 60$. Table 2.3.2 reports that the order of the error is 10^{-1}. f_5 represents an approximation of f_4. Instead of having a discontinuity, it connects values $-1, 1$ in a smooth way. Of course the earned smoothness has relevant consequences in the approximation as we see in Figure 2.8, where the oscillations near $\frac{\pi}{2}$ are less marked.

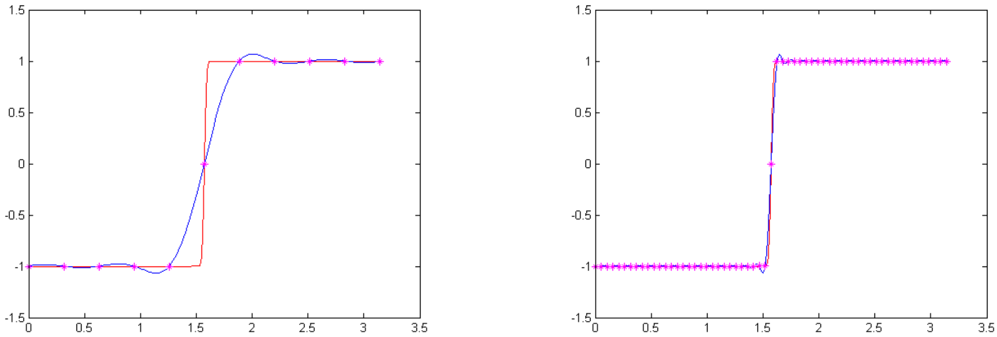

Figure 2.8: TFHRI - Test function f_5 with $n + 1$ equispaced nodes in $[0, \pi]$, $n = 10, 60$, $\omega = 0.1$ and $d = 2$

Then we have compared the interpolation errors of FHRI and TFHRI approximating f_5 and the results have grouped in Table 2.3.2. It may be seen that the TFHRI is a little more accurate than FHRI with the same number of nodes but the order of precision is the same.

n	$FHRI$	$TFHRI$
10	2.0988e-01	2.0722e-01
20	1.3539e-01	1.3072e-01
40	7.8745e-02	7.1333e-02
60	5.1243e-02	4.2108e-02

Table 2.2: Comparison TFHRI error and FHRI error - $d = 4$

Finally we consider the Runge function, $f_6 = \frac{1}{1+25x^2}$ in $[-\pi, \pi]$ with $\omega = 0.05$. We know that classical Lagrange interpolant may not approximate f_6 with equispaced nodes, due to the marked oscillations closed to the endpoints ot the interval. Instead we underline that FHRI is accurate in approximating f_6 as we describe in [11]. Now we investigate the TFHRI with f_6 as test function and compare the performances of it with $d < n$ and $d = n$, that is the classical trigonometric interpolant.

The Table 2.3.2 reveals that also in the trigonometric case Runge phenomenon appears and the classical interpolant is inaccurate and unusable. Instead TFHRI provides a good approximation with order 10^{-2}.

n	$d = 2$	$d = n$
10	6.3845e-01	6.9782e+00
20	1.9567e-01	8.4798e+02
40	2.5088e-02	3.6907e+07

Table 2.3: Comparison TFHRI error with $d = 2$ and $d = n$

Lebesgue function and constant

We focus our attention on the growth of the Lebesgue constant and on the goodness of its theoretical bounds.

We have compute the Lebesgue contants associated with the TFHRI with $d = 0$ and $d = 2$ and $4 \leq n \leq 40$, n only even bacause in this way the behaviour of constant is clear and the "zig-zag" movement does not happen. We have used again our matlab codes, in which we approximate the values of Lebesgue constants, evaluating

$$\Lambda_n \approx \max_{0 \leq k \leq n} \Lambda_n \left(\frac{k}{N} \right)$$

with $N = 1000$.

We suppose that $I = [0, 1]$ and $\omega \in (0, \pi)$. But recalling the theoretical bounds for $d \geq 0$, d even and noticing that if $\omega \to 0$ then $M \to 1$, ω must be chosen in such a way that $M \approx 1$.

Therefore with the aim of making accurate upper and lower bound, ω must be taken close to 0. We suppose $\omega = 0.1$, as in the previous experiments, and in this case $M \approx 1.0041$, so it seems to be a good choice.

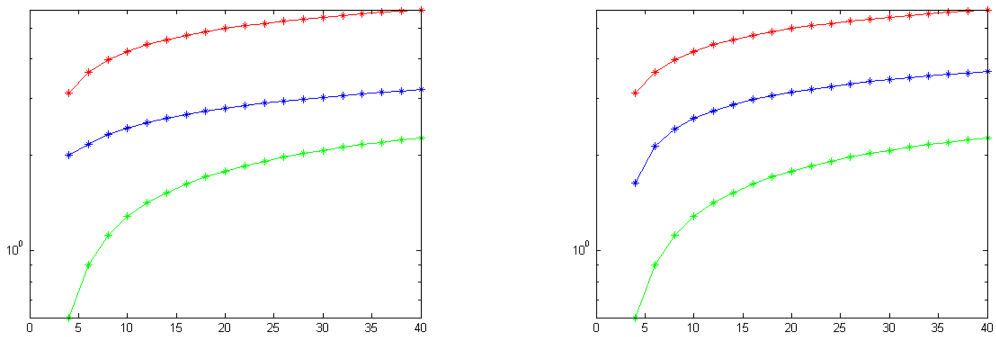

Figure 2.9: TFHRI - Lebesgue constants with $n + 1$ nodes in $[0, 1]$, $4 \leq n \leq 40$ even, $\omega = 0.1$, $d = 0$ (left) and $d = 2$ (right). The plots shows the Lebesgue constants (blue), the upper bounds (red) and the lower bounds (green)

Figure 2.9 shows the comparison between the Lebesgue constant (blue) and two bounds (upper red, lower green) with $d = 0$ (left) and $d = 2$ (right) in semilogarithmic scale because in this way we can appreciate better the behaviour of three quantities. The plots reveal that bounds are a little bit accurate and that Lebesgue constant satisfies them in both cases.

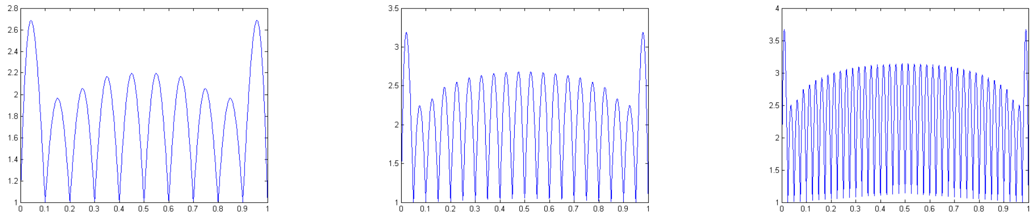

Figure 2.10: Lebesgue functions - TFHRI with $n+1$ nodes in $[0, 1]$, $n = 10, 20, 40$, $\omega = 0.1$ and $d = 2$

Then we plot the Lebesgue funtions $\Lambda_n(x)$ with different values of n and we report them in Figure 2.10. We compare these results with the corresponding ones of the FHRI and we notice that in both cases the Lebesgue functions appear to be simmetric respect to the middle of the interval and that the maximum is close to its endpoints.

2.4 Case d odd

The TFHRI has the following form,

$$r^t(x) = \frac{\displaystyle\sum_{k=0}^{n} \left(w_k^t \cot\left(\frac{\omega}{2}(x - x_k)\right) + \alpha_k \right) f_k}{\displaystyle\sum_{k=0}^{n} w_k^t \cot\left(\frac{\omega}{2}(x - x_k)\right) + \sum_{k=0}^{n} \alpha_k} \tag{2.19}$$

where

$$w_k^t = \sum_{i \in J_k} (-1)^i a_{k,i} = \sum_{i \in J_k} (-1)^i \prod_{j=i, j \neq k}^{i+d} \frac{1}{\sin\left(\frac{\omega}{2}(x_k - x_j)\right)} \tag{2.20}$$

with $J_k = \{i \in \{0, 1, 2, ..., n-d\} \text{ such that } k - d \leq i \leq k\}$.

In the previous paragraph we have underlined the fact that uniqueness of the interpolant isn't always garanteed if the number n is odd. It implies that if d is odd the construction of all the interpolants of $d+1$ adiacent nodes may be not always garanteed, and so we can't build TFHRI. So after choosing $n+1$ nodes and d odd, we can construct the TFHRI if and only if every sequence of $d+1$ adiacent nodes is such that

$$\frac{\omega}{2} \sum_{j=k}^{k+d} x_j \neq 2\pi\nu \;\forall \nu \in \mathbb{Z} \;\forall k \in \{0, 1, \ldots, n-d\}. \tag{2.21}$$

If there is an index $\bar{j}$ such that

$$\sum_{j=\bar{j}}^{\bar{j}+d} x_j = 0$$

then for this set of nodes the construction of the interpolant is impossible.

Otherwise, we must choose ω in such way that the previous sums are different from multiple of π.

In the following we'll suppose that d is such that (2.21) holds.

Recalling (2.19), we analyze the denominator.

$$\sum_{k=0}^{n} \alpha_k = \sum_{k=0}^{n} \left(\sum_{i \in J_k} (-1)^i a_{k,i} c_i \right) = \sum_{i=0}^{n-d} (-1)^i c_i \left(\sum_{k=i}^{i+d} a_{k,i} \right).$$

We may prove the following

Proposition 2.4.1. *If n is odd, $I = [a, b]$ and the nodes are $\{x_k := a + \frac{(b-a)k}{n}\}_{k=0,\ldots,n}$, then*

$$\sum_{k=0}^{n} \prod_{j=0\; j \neq k}^{n} \sin\left(\frac{\omega}{2}(x_k - x_j)\right) = 0$$

or equivalently

$$\prod_{j=0\ j\neq k}^{n} \sin\left(\frac{\omega}{2}(x_k - x_j)\right) = -\prod_{j=0\ j\neq n-k}^{n} \sin\left(\frac{\omega}{2}(x_{n-k} - x_j)\right)\ \forall k \in 0,\ldots,\frac{n+1}{2}-1.$$

Proof. Since we evaluate the sine function olny in points like $\frac{\omega}{2}(x_k - x_j)$ without lose of generalization, we can restrict the proof to $I = [0,1]$. We fix k and define $h := |x_0 - x_1|$.

$$\prod_{j=0\ j\neq k}^{n} \sin\left(\frac{\omega}{2}(x_k - x_j)\right) = \prod_{j<k} \sin\left(\frac{\omega}{2}(x_k - x_j)\right) \prod_{j>k} \sin\left(\frac{\omega}{2}(x_k - x_j)\right) =$$

$$= \prod_{j<k} \sin\left(\frac{\omega}{2}|x_k - x_j|\right)(-1)^{n-k} \prod_{j>k} \sin\left(\frac{\omega}{2}|x_k - x_j|\right) = (-1)^{n-k} \prod_{j=1}^{k} \sin\left(\frac{\omega h j}{2}\right) \prod_{j=1}^{n-k} \sin\left(\frac{\omega h j}{2}\right)$$

Instead for $n-k$,

$$\prod_{j=0\ j\neq n-k}^{n} \sin\left(\frac{\omega}{2}(x_{n-k} - x_j)\right) = \prod_{j<n-k} \sin\left(\frac{\omega}{2}(x_{n-k} - x_j)\right) \prod_{j>n-k} \sin\left(\frac{\omega}{2}(x_{n-k} - x_j)\right) =$$

$$= \prod_{j<n-k} \sin\left(\frac{\omega}{2}|x_{n-k} - x_j|\right)(-1)^{k} \prod_{j>n-k} \sin\left(\frac{\omega}{2}|x_{n-k} - x_j|\right) =$$

$$= (-1)^{k} \prod_{j=1}^{n-k} \sin\left(\frac{\omega h j}{2}\right) \prod_{j=1}^{k} \sin\left(\frac{\omega h j}{2}\right).$$

Since n is odd and $k + n - k = n$, it follows that k and $n - k$ may be not even or odd both, so one is even and the other odd. Then $(-1)^k = -(-1)^{n-k}$ and the thesis holds. □

Thanks to the previous result, we conclude that $\sum_{k=i}^{i+d} a_{k,i} = 0\ \forall i$ and so TFHRI can be written as

$$r^t(x) = \sum_{k=0}^{n} b_k^t(x) f_k,$$

where

$$b_k^t(x) = \frac{w_k^t \cot\left(\frac{\omega}{2}(x - x_k)\right) + \alpha_k}{\sum_{k=0}^{n} w_k^t \cot\left(\frac{\omega}{2}(x - x_k)\right)} \tag{2.22}$$

With easy computations it may be verified that $r^t(x)$ indeed interpolates the data $\{x_k, f_k\}$.

In general the control in (2.21) may be little bit complicated, so in the following we consider only $I = [0,1]$ as interpolation interval and in addition of **Hyp 1** in the previous section, we also ask that

Hypothesis 2 on ω

- $\frac{\omega}{2}\sum_{j=k}^{k+d} x_j < \frac{\pi}{2} \ \forall k \in 0, ..., n-d$

Following the same proof of the corresponding Theorem in [1] we can prove

Theorem 2.4.2 (Absence of poles)**.** *For all odd* $0 \leq d \leq n$ *and for every pulsation* ω *that satisfies* ***Hyp 1-2****, the TFHRI* r^t *in (2.19) has no poles in [0,1].*

2.4.1 Lebesgue constant

Recalling the expression of TFHRI and

$$\cot(x) + \cot(y) = \frac{\sin(x+y)}{\sin(x)\sin(y)} \quad \forall x, y \neq k\pi \ k \in \mathbb{Z}, \tag{2.23}$$

we may prove

Theorem 2.4.3 (Upper bound Lebesgue constant - d odd)**.** *The Lebesgue constant associated with trigonometric rational interpolant in* $[0,1]$*, with a pulsation* ω *according to Hyp 1-2, at equidistant nodes* $\{x_j = \frac{j}{n}\}_{j=0,...,n}$ *with basis function (2.22) satisfies*

$$\Lambda_n \leq \bar{C}_1 M^{d+1} 2^{d-1}\big(2 + \ln(n)\big).$$

Proof. Let x be in I. If $x = x_k$ for any k then $\Lambda_n(x) = 1$. So let $x_k < x < x_{k+1}$ with $0 \leq k \leq n-1$.

$$\Lambda_{n,k}(x) := \frac{\sum_{s=0}^{n}\left| w_j^t \cot\left(\frac{\omega}{2}(x - x_s)\right) + \alpha_s\right|}{\left|\sum_{s=0}^{n} w_s^t \cot\left(\frac{\omega}{2}(x - x_s)\right)\right|} = \frac{N_k(x)}{D_k(x)}.$$

We introduce the following notation

$$\tilde{z}_{s,i}(x) = \frac{\sin(\frac{\omega}{2}(x + \sum_{j=i,\ j\neq k}^{i+d} x_j))}{\sin(\frac{\omega}{2}\sum_{j=i}^{i+d} x_j)}.$$

Since the terms $a_{s,i}$ oscillate in sign, and under the assumptions on ω for fixed s the terms in parentheses have the same sign and $\tilde{z}_{s,i}(x)$ is greater than 0, using (2.23) we can rewrite N_k

$$N_k(x) = \sum_{s=0}^{n}\left|\sum_{i\in J_s}(-1)^i a_{s,i}\left(\cot\left(\frac{\omega}{2}(x - x_s)\right) + c_i\right)\right| =$$

$$= \sum_{s=0}^{n} \left(\sum_{i \in J_s} |a_{s,i}| \tilde{z}_{s,i}(x) \right) \frac{1}{\sin(\frac{\omega}{2}|x - x_s|)} \leq Z_k(x) \sum_{s=0}^{n} \frac{|w_s^t|}{\sin(\frac{\omega}{2}|x - x_s|)}$$

with $Z_k(x) = \max_{(s,i) \in S} \tilde{z}_{s,i}(x),\ S = \{0 \leq s \leq n,\ i \in J_s\}$.
It easy to see that $Z_k(x) = \tilde{z}_{0,0}(x)\ \forall k$ and $\forall x \in (x_k, x_{k+1})$.
Then

$$\Lambda_{n,k}(x) \leq \frac{d!h^d (x - x_k)(x_{k+1} - x)\tilde{z}_{0,0}(x) \sum_{s=0}^{n} \frac{|w_s^t|}{\sin(\frac{\omega}{2}|x - x_s|)}}{d!h^d (x - x_k)(x_{k+1} - x) \left| \sum_{j=0}^{n} w_j^t \cot\left(\frac{\omega}{2}(x - x_j)\right) \right|}.$$

Using the same proof of the d even case we get

$$\Lambda_{n,k}(x) \leq \tilde{z}_{0,0}(x) M^{d+1} 2^{d-1} (2 + \log(n)).$$

Taking the maximum and recalling that $\tilde{z}_{0,0}(x)$ is an increasing function in x, we can conclude

$$\Lambda_n = \max_{k=0,\ldots,n-1} \left(\max_{x_k < x < x_{k+1}} \Lambda_{n,k}(x) \right) \leq \bar{C}_1 M^{d+1} 2^{d-1} (2 + \log(n)).$$

where $\bar{C}_1 := \tilde{z}_{0,0}(1)$. □

Theorem 2.4.4 (Lower bound Lebesgue constant - d odd)**.** *The Lebesgue constant associated with trigonometric rational interpolant in* $[0, 1]$*, with a pulsation* ω *according to Hyp 1-2, at equidistant nodes* $\{x_j = \frac{j}{n}\}_{j=0,\ldots,n}$ *with basis functions (2.22) satisfies*

$$\Lambda_n \geq \frac{\bar{C}_2}{M^{d+1} 2^{d+2}} \binom{2d+1}{d} \ln(\frac{n}{d} - 1).$$

Proof.

$$\Lambda_n(x) = \frac{d!h^d \sum_{k=0}^{n} \left| w_k^t \cot\left(\frac{\omega}{2}(x - x_k)\right) + \alpha_k \right|}{\left| d!h^d \sum_{k=0}^{n} w_k^t \cot\left(\frac{\omega}{2}(x - x_k)\right) \right|} := \frac{N(x)}{D(x)}.$$

We consider $x^* = \frac{x_1 - x_0}{2} = \frac{1}{2n}$.
We first investigate the numerator, using (2.9) and the bounds of weights w_j^t, we have as in the previous proof

$$N(x^*) = d!h^d \sum_{k=0}^{n} \left(\sum_{i \in J_k} |a_{k,i}| \frac{\tilde{z}_{k,i}}{\sin(\frac{\omega}{2}|x^* - x_k|)} \right) \geq C d!h^d \sum_{k=0}^{n} \frac{|w_k^t|}{\sin(\frac{\omega}{2}|x^* - x_k|)}$$

with $\tilde{z}_{k,i} := \dfrac{\sin(\frac{\omega}{2}(x^* + \sum_{j=i,\ j\neq k}^{i+d} x_j))}{\sin(\frac{\omega}{2}\sum_{j=i}^{i+d} x_j)}$
and $C := \min_S \tilde{z}_{r,s}$ with S as in the previous proof.

Using the same inequality in the prove of the lower bound d even case, finally we have

$$N(x^*) \geq C\frac{2^{d+1}}{\omega^{d+1}} n2^d \ln(\frac{n}{d} - 1).$$

Indeed the minimum C may be taken in a smaller set than S. In fact it may be proved that

$$\tilde{z}_{k,i} > \tilde{z}_{k+1,i}$$

k such that $i \in J_k$. In fact, if fix k and $i \in J_k$, recalling that under our assumpions on ω the sine is an increasing function, noting that

$$\frac{\omega}{2}(x^* + \sum_{j=i,\ j\neq k}^{i+d} x_j) > \frac{\omega}{2}(x^* + \sum_{j=i,\ j\neq k+1}^{i+d} x_j),$$

we get

$$\tilde{z}_{k,i} = \frac{\sin(\frac{\omega}{2}(x^* + \sum_{j=i,\ j\neq k}^{i+d} x_j))}{\sin(\frac{\omega}{2}\sum_{j=i}^{i+d} x_j)} > \frac{\sin(\frac{\omega}{2}(x^* + \sum_{j=i,\ j\neq k+1}^{i+d} x_j))}{\sin(\frac{\omega}{2}\sum_{j=i}^{i+d} x_j)} = \tilde{z}_{k+1,i}$$

Thanks to the previous statement, if we define $\bar{S} = \{(r,s) \in S | \max\{r | s \in J_r\}\}$

$$C = \bar{C}_2 = \min_{\bar{S}} \tilde{z}_{r,s}.$$

The inequality becomes

$$N(x^*) \geq \bar{C}_2\frac{2^{d+1}}{\omega^{d+1}} n2^d \ln(\frac{n}{d} - 1). \tag{2.24}$$

As for the denominator, the inequality of the case d even is still true,

$$D(x^*) = h^d d! \left|\sum_{j=0}^{n-d} \lambda_j^t(x^*)\right| \leq \frac{2^{d+1}}{\omega^{d+1}} M^{d+1} n \frac{2^{2d+2}}{\binom{2d+1}{d}}.$$

Finally we sum up the results and obtain

$$\Lambda_n \geq \frac{\bar{C}_2}{M^{d+1}2^{d+2}} \binom{2d+1}{d} \ln(\frac{n}{d} - 1).$$

□

2.5 Numerical experiments

How to choose ω

As in the d even case, there is no unique admissible value of ω, such that satisfies **Hyp 1-2**. Thus we want to choose the pulsation, ω_{opt}, that minimize the interpolation relative error. Since the construction of the interpolant with odd d is more hard-working than the other one, we focus only on the interval $I = [0, 1]$. Due to **Hyp 1-2**, ω must take values in $(0, \min(\frac{\pi}{x_{n-d}+\ldots+x_n}, \frac{\pi}{2}))$.

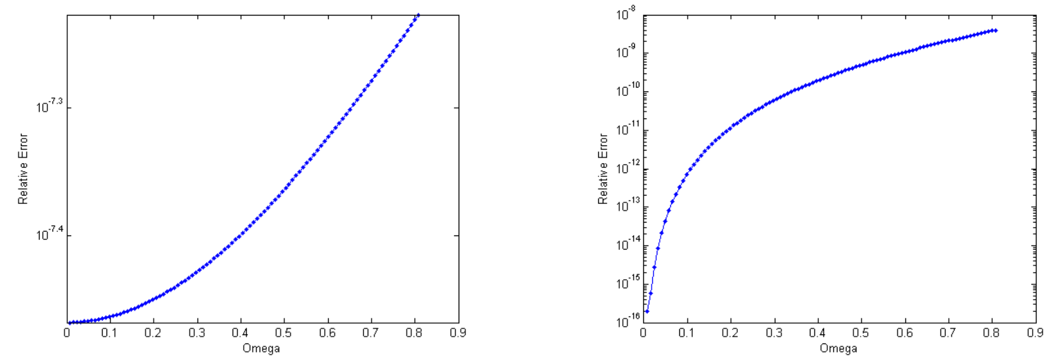

Figure 2.11: ω-Error plot with $I = [0, 1]$, $n = 40$ and $d = 3$. The test functions are e^x (left) and x^2 (right).

First the test functions are non periodic, for instance x^2 and e^x. Obviously the optimal value of the pulsation depends on n, d and the test function but our numerical experiments show in particular that the test function plays the most important role. In Figure 2.11 we notice that on the right the choice of ω may produce a relevant reduction of the error, from 10^{-8} to 10^{-16}, instead on the left the reduction is not evident. Finally, in this case $\omega_{opt} \approx 0$, but it is not unexpected thanks to the reasonings made in the d even case.

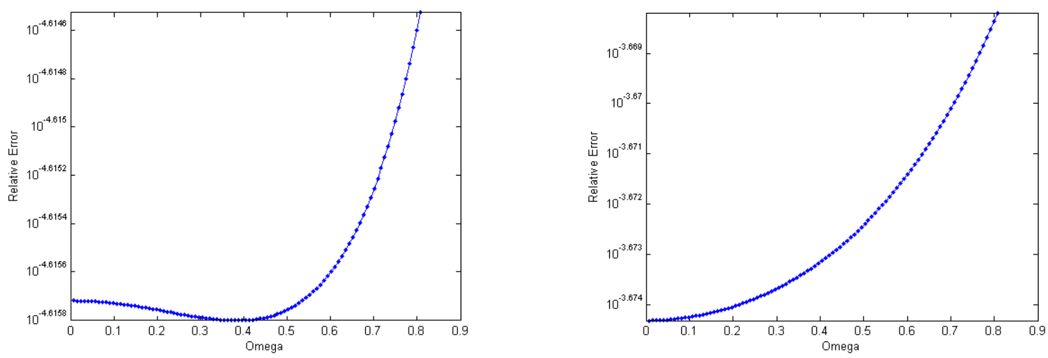

Figure 2.12: ω-Error plot with $I = [0, 1]$, $n = 40$ and $d = 3$. The test functions are $\sin(2\pi x)$ (left) and $\sin(4\pi x)$ (right).

Instead if the test function is periodic with $T = 1$ or a submultiple, for example $\sin(2\pi x), \sin(4\pi x)$, the Figure 2.12 shows that $\omega_{opt} \approx 0$. But the reduction of the error is almost absent so there are infinity admissible values of ω no matter what f is.

Approximation and error

Using our matlab codes, we have tested TFHRI in some particular cases and we restict the analysis on $I = [0, 1]$, as said in the previous section. In the d even case, the choice of ω was indeed indipendent from n but now these variables are related due to the **Hyp 2**. So for each choice of n we have seen the ω-Error plot and then we have chosen a suitable pulsation. In the first case TFHRI approximates a non periodic function, for instance $f_1 = x^2$ in $[0, 1]$. Our ω-tests reveal that a suitable pulsation is $\omega = 0.1$ because represents the right compromise between the reduction of the error and the distance to zero. Figure 2.13 show that TFHRI is good in approximating the test function also with few nodes.

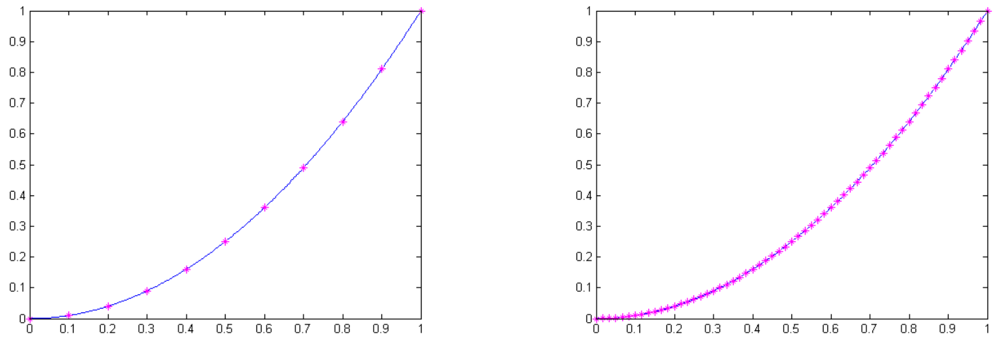

Figure 2.13: TFHRI - Test function $f_1(x) = x^2$ with $n + 1$ equispaced nodes in $[0, 1]$, $n = 10, 60$, $\omega = 0.1$ and $d = 1$

In the second case we try to approximate a periodic function, for example $f_2 = \sin(2\pi x)$ in $[0, 1]$. As before we have to control what may be a good value of ω. From our tests it appears that, as said before, different pulsations do not produce a significant reduction of the error. So we choose $\omega = 0.1$ as before.

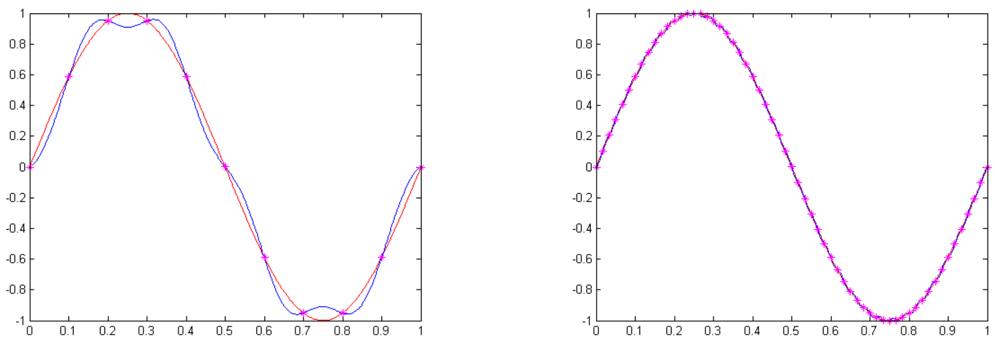

Figure 2.14: TFHRI - Test function $f_2(x) = \sin(2\pi x)$ with $n+1$ equispaced nodes in $[0, 1]$, $n = 10, 60$, $\omega = 0.1$ and $d = 1$

TFHRI seems to have some problems in the approximation, in fact with $n = 10$ the oscillations are really marked. This may be seen in Figure 2.14 (left) and on the right we notice that the interpolant improves its performances.

In the third case we want to approximate a function, like $f_3 = |x|$, which has a singularity in 0 so here we consider $[-1, 1]$ as interval in such way to include 0 in the interpolation interval.

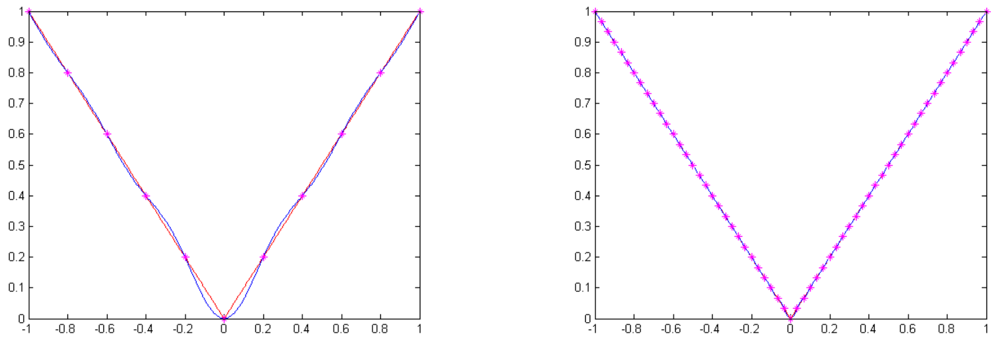

Figure 2.15: TFHRI - Test function $f_3(x) = |x|$ with $n+1$ equispaced nodes in $[-1, 1]$, $n = 10, 60$, $\omega = 1$ and $d = 1$

Unlike before, here we must consider an interval simmetric with respect to the origin. First of all since the number of nodes is odd, we have no problem with the vanishing of the sum of adjacent groups. Then in this case the **Hyp 2** turns to be **Hyp 2 bis**

- $\left|\frac{\omega}{2}\sum_{j=k}^{k+d} x_j\right| < \frac{\pi}{2}\ \forall k \in 0, ..., n-d.$

If $n = 10$ the Hyp 1-2 bis tell us that $\omega < \frac{\pi}{2} \approx 1.57$ and $\omega < \frac{\pi}{x_{n-1}+x_n} \approx 1.74$ that is $\omega < 1.57$. Numerical experiments show that a good pulsation is $\omega = 1.4$ that satisfies the constrains. The results are reported in Figure 2.15 (left) and it may be noticed that the interpolation is not so bad. Instead if $n = 60$ the constrains become $\omega < 1.57$ and $\omega < \frac{\pi}{x_{n-1}+x_n} \approx 1.60$ that is $\omega < 1.57$ and the tests reveal that ω could be equal to 1. The interpolation has improved in Figure 2.15 (right).

n	x^2	$\sin(2\pi x)$	$\|x\|$	f_4
10	2.3704e-10	3.3161e-03	2.8209e-02	3.9915e-01
20	1.3215e-11	2.5948e-04	9.1889e-03	2.7664e-01
40	7.3003e-13	2.4242e-05	3.2108e-03	1.9461e-01
60	1.3579e-13	6.4106e-06	1.7483e-03	1.5851e-01

Table 2.4: Trigonometric interpolation error - $d = 3$

The we take the test function f_4, previously introduced with $\omega = 0.1$. Figure 2.16 represent the rational interpolant with some values of n. The results are

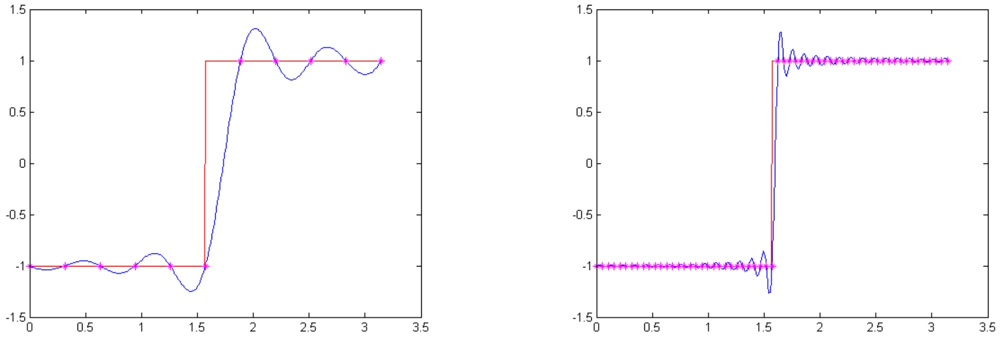

Figure 2.16: TFHRI - Test function f_4 with $n+1$ equispaced nodes in $[0,\pi]$, $n = 10, 60$, $\omega = 0.1$ and $d = 1$

similar to ones obtained in the d even case and some oscillations appear close to $\frac{\pi}{2}$.

We compare the relatives error with $d = 3$, reported in Table 2.5. Thanks to ω-tests we consider $\omega = 0.1$ with f_1, $\omega = 0.5$ with f_2 and $\omega = 0.8$ with f_3. It appears evident that the interpolant has not any problem in approximating smooth functions but we underline also the fact that it is good also in the third case where the error is of the order 10^{-3}. Regarding f_4, the approximation does not improve one in the d even case and the order of precision remains 10^{-1}. As in previous case, now we analyze the test function f_5, that is a smooth approximation of f_4 with $\omega = 0.1$.

If the test function is f_5 and $\omega = 0.1$ numerical results are the same as in the previous section and the oscillations attenuate, Figure 2.17.

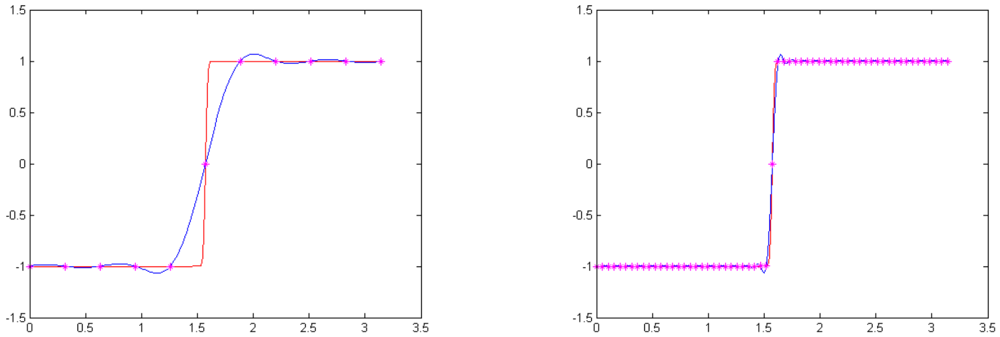

Figure 2.17: TFHRI - Test function f_5 with $n+1$ equispaced nodes in $[0,\pi]$, $n = 10, 60$, $\omega = 0.1$ and $d = 1$

We compare the performance of FHRI and TFHRI approximating f_5 and Table 2.5 confirms the results of the previous section: the precision is the same but TFHRI is little more accurate.

Finally we take in exam Runge function f_6 in $[-\pi,\pi]$ with $\omega = 0.1$, compare performances of TFHRI and classical trigonometric interpolant and report the results in Table 2.5. It is evident that FHRI has problems in approximating

n	$FHRI$	$TFHRI$
10	2.0972e-01	2.0646e-01
20	1.3536e-01	1.3059e-01
40	7.8744e-02	7.1312e-02
60	5.1242e-02	4.2099e-02

Table 2.5: Comparison TFHRI error and FHRI error - $d = 3$

n	$d = 3$	$d = n$
10	1.3865e-01	6.9782e+00
20	9.1543e-02	8.4798e+02
40	3.8680e-02	3.6907e+07

Table 2.6: Comparison TFHRI error with $d = 3$ and $d = n$

the test function due to the fact that the Runge phenomenon appears. Instaed TFHRI may overcomes this problem and its precision is of 10^{-2}.

Lebesgue function and constant

We compute an approximation of the Lebesgue constant as we have explained in the d even case. With the aim of making the theoretical bounds accurate, we must choose ω such that the terms $\bar{C}_1 M^{d+1}$ and $\dfrac{\bar{C}_2}{M^{d+1}}$ are as small as possible. After our analysis, we conclude that $\omega = 0.1$ may be a good choice for every n in both cases.

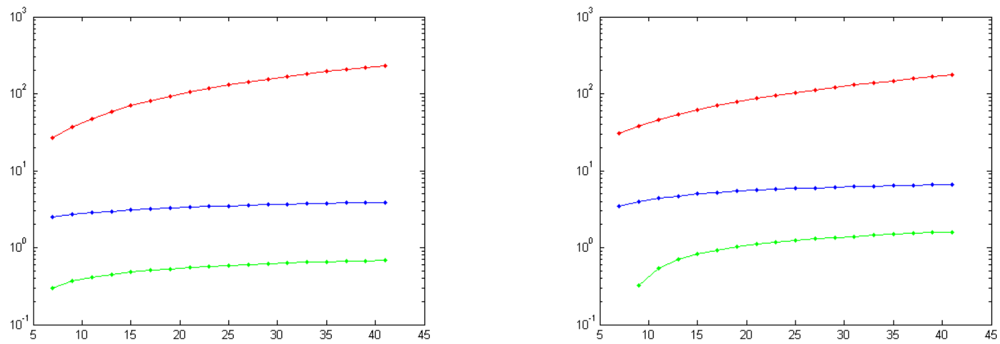

Figure 2.18: TFHRI - Lebesgue constants with $n + 1$ nodes, $6 \leq n \leq 40$ even, $d = 1$ (left) and $d = 3$ (right) and $\omega = 0.1$. The plots show the Lebesgue constants (blue), the upper bounds (red) and the lower bounds (green)

Figure 2.18 show the Lebesgue constants and two bounds in semilogarithmic scale. We can appreciate the accuracy of bounds and the fact that the Lebesgue constant satisfies them as in d even case.

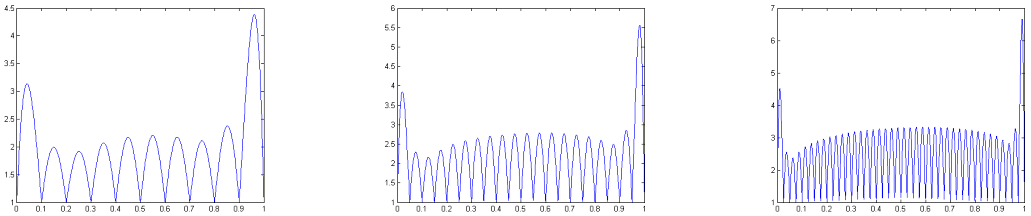

Figure 2.19: Lebesgue functions - TFHRI with $n+1$ nodes with $n = 10, 20, 40$ and $d = 3$

Finally we analyze the Lebesgue function with $\omega = 0.1$ in Figure 2.19. Differently from the d even case the Lebesgue function is not simmetric with respect to the middle of the interval and the maximum is reached near its extreme on the right.

Chapter 3

Tensor Product FHRI (TPFHRI)

Suppose $[a_x, b_x]$, $[a_y, b_y] \subset \mathbb{R}$, $n, m \in \mathbb{N}$, $a_x < x_0 < \cdots < x_n < b_x$, $a_y < y_0 < \cdots < y_m < b_y$, $X := \{x_i\}_i$, $Y := \{y_j\}_j$, the grid $\mathbf{G} = X \times Y$ and $d_1, d_2 \in \mathbb{N},\ 0 \leq d_1 \leq n,\ 0 \leq d_2 \leq m$, then for each x_i and y_j define the 1d base functions

$$b_i^x(x) = \frac{\dfrac{w_i^x}{x - x_i}}{\displaystyle\sum_{k=0}^{n} \frac{w_k^x}{x - x_k}}, \qquad b_j^y(y) = \frac{\dfrac{w_j^y}{y - y_j}}{\displaystyle\sum_{h=0}^{m} \frac{w_h^y}{y - y_h}}. \tag{3.1}$$

where the weights are

$$w_i^x = \sum_{s \in J_i^x} (-1)^s \prod_{t=s, t \neq k}^{s+d_1} \frac{1}{x_i - x_t}, \qquad w_j^y = \sum_{s \in J_j^y} (-1)^s \prod_{t=s, t \neq k}^{s+d_2} \frac{1}{y_j - y_t}$$

and $J_k^x = \{s \in \{0, 1, 2, ..., n - d_1\}\ \textit{such that}\ k - d_1 \leq s \leq k\}$ and $J_k^y = \{s \in \{0, 1, 2, ..., m - d_2\}\ \textit{such that}\ k - d_2 \leq s \leq k\}$

Given a function $f \in C([a_x, b_x] \times [a_y, b_y])$, the **tensor product FHRI**, indicated in the following as TPFHRI, is

$$r(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_i^x(x) b_j^y(y) f(x_i, y_j) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{b}_{ij}(x, y) f(x_i, y_j).$$

We can rewrite it in a more compact form,

$$r(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{K}} \mathbf{b_k}(\mathbf{x}) f(\mathbf{x_k}), \tag{3.2}$$

with $\mathcal{K} = \{(i, j) | 0 \leq i \leq n,\ 0 \leq j \leq m\}$, $\mathbf{k} = (k_1, k_2)$ and $\mathbf{x_k} = (x_{k_1}, y_{k_2})$.

3.1 TPFHRI original form

In 1d case, FHRI may be written in form (1.1), so we try to define the TPFHRI with an analogous formula. It is the following one

$$r(x,y) = \frac{\sum_{i=0}^{n-d_1}\sum_{j=0}^{m-d_2} \lambda_i(x)\lambda_j(y)p_{ij}(x,y)}{\sum_{i=0}^{n-d_1}\lambda_i(x)\sum_{j=0}^{m-d_2}\lambda_j(y)}. \tag{3.3}$$

where p_{ij} is the tensor product interpolant on the nodes $\{(x_s, y_t)\}_{i\le s\le i+d_1 \;\; j\le t\le j+d_2}$, i.e.

$$p_{i,j}(x,y) = \sum_{p=i}^{i+d_1}\sum_{q=j}^{j+d_2} \ell_p(x)\ell_q(y)f(x_p,y_q)$$

where $\ell_p(x)$ and $\ell_q(y)$ are the Lagrange polynomials relate on nodes $x_i, ..., x_{i+d_1}$ and $y_j, ..., y_{j+d_2}$ respectively.

Now we put it in barycentric form an see that the result coincides with (3.2).

First we concentrate on the numerator in (3.3),

$$\sum_{i=0}^{n-d_1}\sum_{j=0}^{m-d_2} \lambda_i(x)\lambda_j(y)\left(\sum_{p=i}^{i+d_1}\sum_{q=j}^{j+d_2} \ell_p(x)\ell_q(y)f(x_p,y_q))\right) =$$

$$= \sum_{i=0}^{n-d_1}\lambda_i(x)\left(\sum_{j=0}^{m-d_2}\lambda_j(y)\left(\sum_{p=i}^{i+d_1}\ell_p(x)\left(\sum_{q=j}^{j+d_2}\ell_q(y)f(x_p,y_q)\right)\right)\right) =$$

$$= \sum_{i=0}^{n-d_1}\lambda_i(x)\left(\sum_{p=i}^{i+d_1}\ell_p(x)\left(\sum_{j=0}^{m-d_2}\lambda_j(y)\left(\sum_{q=j}^{j+d_2}\ell_q(y)f(x_p,y_q)\right)\right)\right)$$

If we consider f as a function of only one variable y then, recalling the reasonings in [1], it becomes

$$\sum_{i=0}^{n-d_1}\lambda_i(x)\left(\sum_{p=i}^{i+d_1}\ell_p(x)\underbrace{\left(\sum_{j=0}^{m}\frac{w_j^y}{y-y_j}f(x_p,y_j)\right)}_{g(x_p)}\right) =$$

$$= \sum_{i=0}^{n-d_1}\lambda_i(x)\left(\sum_{p=i}^{i+d_1}\ell_p(x)g(x_p)\right) = \sum_{i=0}^{n}\frac{w_i^x}{x-x_i}g(x_i) =$$

$$= \sum_{i=0}^{n}\frac{w_i^x}{x-x_i}\sum_{j=0}^{m}\frac{w_j^y}{y-y_j}f(x_i,y_j)$$

Then we analyze the denominator in (3.3). In the 1d case [1] holds

$$\sum_{i=0}^{n-d}\lambda_i(x) = \sum_{k=0}^{n}\frac{w_k}{x-x_k},$$

and so

$$\sum_{i=0}^{n-d_1} \lambda_i(x) \sum_{j=0}^{m-d_2} \lambda_j(y) = \sum_{k=0}^{n} \frac{w_k^x}{x - x_k} \sum_{h=0}^{m} \frac{w_h^y}{y - y_h}$$

The TPFHRI becomes

$$r(\mathbf{x}) = \frac{\sum_{\mathbf{k} \in \tilde{\mathcal{K}}} \lambda_{\mathbf{k}}(\mathbf{x}) p_{\mathbf{k}}(\mathbf{x})}{\sum_{\mathbf{k} \in \tilde{\mathcal{K}}} \lambda_{\mathbf{k}}(\mathbf{x})}. \tag{3.4}$$

where $\tilde{\mathcal{K}} = \{(i,j) \mid 0 \leq i \leq n - d_1, 0 \leq j \leq m - d_2\}$.

3.1.1 Properties

From the fact that the TPFHRI have an analogous structure of the 1d case, the following result holds

Corollario 3.1.1 (Absence of poles in $\mathbb{R}^2$)**.** *For all $0 \leq d_1 \leq n$ and $0 \leq d_2 \leq m$, the TPFHRI r has no poles in $\mathbb{R}^2$.*

The thesis follows directly showing that the denominator in (3.4) is the product of two factors and we can apply the Theorem 1.0.1 to each of them. We conclude that they are strictly positive and so the thesis holds.

The TPFHRI interpolates the data $(\mathbf{x_k}, f(\mathbf{x_k}))_{\mathbf{k} \in \mathcal{K}}$. We multiply the numerator and the denominator in (3.4) by

$$\mu_{\mathbf{k}} = (-1)^{n-d_1}(-1)^{m-d_2}(x - x_0)\dots(x - x_n)(y - y_0)\dots(y - y_m)\lambda_{\mathbf{k}}(\mathbf{x}) = \mu_{k1}\mu_{k2}$$

From the 1d case [1] we know that $\mu_k(x_i) > 0$ if $k \in J_i$, otherwise is equal to zero. So $\mu_{\mathbf{k}}(x_s, y_t) > 0$ if $k_1 \in J_s$ and $k_2 \in J_t$, zero otherwise.

Let $\mathbf{x_h}$ be a node, then

$$r(\mathbf{x_h}) = \frac{\sum_{\mathbf{k} \in \mathcal{K}} \mu_{\mathbf{k}}(\mathbf{x_h}) p_{\mathbf{k}}(\mathbf{x_h})}{\sum_{\mathbf{k} \in \mathcal{K}} \mu_{\mathbf{k}}(\mathbf{x_h})} = \frac{\sum_{k_1 \in J_{h_1}, k_2 \in J_{h_2}} \mu_{\mathbf{k}}(\mathbf{x_h}) p_{\mathbf{k}}(\mathbf{x_h})}{\sum_{k_1 \in J_{h_1}, k_2 \in J_{h_2}} \mu_{\mathbf{k}}(\mathbf{x_h})} =$$

$$= \frac{\sum_{k_1 \in J_{h_1}, k_2 \in J_{h_2}} \mu_{\mathbf{k}}(\mathbf{x_h}) f(\mathbf{x_h})}{\sum_{k_1 \in J_{h_1}, k_2 \in J_{h_2}} \mu_{\mathbf{k}}(\mathbf{x_h})} = f(\mathbf{x_h})$$

3.1.2 Error estimates

Let $a \leq x_0 < \cdots < x_n \leq b$ be a sequence of $n+1$ distinct points. and $g : [a, b] \longrightarrow \mathbb{R}$.We define the divided differences as

$$g[x_0, ..., x_n] := \frac{g[x_1, ..., x_n] - g[x_0, ..., x_{n-1}]}{x_n - x_0},$$
$$g[x_0] = g(x_0).$$

Then the following result holds

Theorem 3.1.2 (Theorem 1 in [10])**.** *Let $x, x_0, ..., x_n$ be $n + 2$ distinct points and $g \in C^{n+1}(I)$ where $I = [\min(x, x_0, ..., x_n), \max(x, x_0, ..., x_n)]$. Then for some point $\xi = \xi(x) \in I$ holds*

$$g[x_0, ..., x_n, x] = \frac{g^{n+1}(\xi)}{(n+1)!}. \tag{3.5}$$

Now we try to extend previous result and definition to the bivariate case. First of all we introduce the bivariate divided differences.

Let us consider $f : [a_x, b_x] \times [a_y, b_y] \longrightarrow \mathbb{R}$ and $a_x \leq x_0 < \cdots < x_s \leq b_x$, $a_y \leq y_0 < \cdots < y_t \leq b_y$ two sequences of $s + 1$ and $t + 1$ distinct points respectively. The bivariate divided differences are

$$f[x_0][y_0, ..., y_t] = \frac{f[x_0][y_1, ..., y_t] - f[x_0][y_0, ..., y_{t-1}]}{y_t - y_0}, \tag{3.6}$$

$$f[x_0, ..., x_s][y_0] = \frac{f[x_1, ..., x_s][y_0] - f[x_0, ..., x_{s-1}][y_0]}{x_s - x_0}, \tag{3.7}$$

$$f[x_0, ..., x_s][y_0, ..., y_t] = \frac{f[x_0, ..., x_s][y_1, ..., y_t] - f[x_0, ..., x_s][y_0, ..., y_{t-1}]}{y_t - y_0} = \tag{3.8}$$

$$= \frac{f[x_1, ..., x_s][y_0, ..., y_t] - f[x_0, ..., x_{s-1}][y_0, ..., y_t]}{x_s - x_0}, \tag{3.9}$$

$$f[x_0][y_0] = f(x_0, y_0).$$

We indicate with $C^{n,m}$ the space of functions of two variables that are C^n in the first variable and C^m in the second one.

Theorem 3.1.3. *Let $x, x_0, ..., x_s \in [a_x, b_x]$ be $s+2$ distinct points and $y, y_0, ..., y_t \in [a_y, b_y]$ be $t+2$ distinct points and $f \in C^{s+1,t+1}([a_x, b_x] \times [a_y, b_y])$. Then for some points $\xi_x^1 \in [a_x, b_x], \xi_y^1 \in [a_y, b_y]$ and $(\xi_x^2, \xi_y^2) \in [a_x, b_x] \times [a_y, b_y]$ holds*

$$f[x_0, ..., x_s, x][y] = \frac{1}{(s+1)!} \frac{\partial^{s+1} f(x, y)}{\partial x^{s+1}}\bigg|_{x=\xi_x^1}, \tag{3.10}$$

$$f[x][y_0, ..., y_t, y] = \frac{1}{(t+1)!} \frac{\partial^{t+1} f(x, y)}{\partial y^{t+1}}\bigg|_{y=\xi_y^1}, \tag{3.11}$$

$$f[x_0, ..., x_s, x][y_0, ..., y_t, y] = \frac{1}{(t+1)!(s+1)!} \frac{\partial^{s+t+2} f(x, y)}{\partial y^{t+1} \partial x^{s+1}}\bigg|_{x=\xi_x^2, y=\xi_y^2}. \tag{3.12}$$

Proof. From (3.8) and (3.9) we notice that if we fix $x \in [a_x, b_x]$, $x \neq x_i$, $y \in [a_y, b_y]$, $y \neq y_j$ and

$$g(y) := f[x_0, ..., x_s, x](y) \;\; h(x) := f[y_0, ..., y_t, y](x),$$

$$\hat{g}(y) := f[x](y) \;\; \hat{h}(x) := f[y](x),$$

then

- $g, \hat{g} \in C^{t+1}([a_x, b_x])$, $h, \hat{h} \in C^{s+1}([a_y, b_y])$
- $g[y_0, ..., y_t, y] = f[x_0, ..., x_s, x][y_0, ..., y_t, y]$, $\hat{g}[y_0, ..., y_t, y] = f[x][y_0, ..., y_t, y]$
- $h[x_0, ..., x_s, x] = f[x_0, ..., x_s, x][y_0, ..., y_t, y]$, $\hat{h}[x_0, ..., x_s, x] = f[y][x_0, ..., x_s, x]$.

Now we may apply the previous theorem to $\hat{h}, \hat{g}, g$ and h. Then consider f as a function of only one variable x. We use twice the previous theorem. So we get

$$f[x_0, ..., x_s, x][y] = \hat{h}[x_0, ..., x_s, x] = \frac{1}{(s+1)!} \frac{d^{s+1}\hat{h}(x)}{dx^{s+1}}\bigg|_{x=\xi_x^1} = \frac{1}{(s+1)!} \frac{\partial^{s+1} f(x,y)}{\partial x^{s+1}}\bigg|_{x=\xi_x^1}.$$

$$f[x][y_0, ..., y_t, y] = \hat{g}[y_0, ..., y_t, y] = \frac{1}{(t+1)!} \frac{d^{t+1}\hat{g}(y)}{dy^{t+1}}\bigg|_{y=\xi_y^1} = \frac{1}{(t+1)!} \frac{\partial^{t+1} f(x,y)}{\partial y^{t+1}}\bigg|_{y=\xi_y^1}.$$

With similar reasonings, we might obtain

$$f[x_0, ..., x_s, x][y_0, ..., y_t, y] = g[y_0, ..., y_t, y] = \frac{1}{(t+1)!} \frac{d^{t+1} g(y)}{dy^{t+1}}\bigg|_{y=\xi_y^2} =$$

$$= \frac{1}{(t+1)!} \frac{\partial^{t+1}}{\partial y^{t+1}} (f[x_0, ..., x_s, x][y])\bigg|_{y=\xi_y^2} = \frac{1}{(t+1)!} \underbrace{(\frac{\partial^{t+1} f(x,y)}{\partial y^{t+1}})\bigg|_{y=\xi_y^2}}_{\hat{f}(x) \in C^{s+1}([a_x, b_x])} [x_0, ..., x_s, x] =$$

$$= \frac{1}{(t+1)!} \hat{f}[x_0, ..., x_s, x] = \frac{1}{(t+1)!(s+1)!} (\frac{d^{s+1}\hat{f}(x)}{dx^{s+1}})\bigg|_{x=\xi_x^2} =$$

$$= \frac{1}{(t+1)!(s+1)!} \frac{\partial^{s+t+2} f(x,y)}{\partial y^{t+1} \partial x^{s+1}}\bigg|_{x=\xi_x^2, y=\xi_y^2}.$$

□

After defining $h_x := \max_{0 \leq i \leq n-1}(x_{i+1} - x_i)$, $h_y := \max_{0 \leq j \leq m-1}(y_{j+1} - y_j)$ and $h := \max\{h_x, h_y\}$ we have all the ingredients to prove

Theorem 3.1.4. *Let $n, m, d_1, d_2 \in \mathbb{N}$ be numbers such that $\leq d_1 \leq n$, $1 \leq d_2 \leq m$ and $f \in C^{d_1+2,d_2+2}([a_x, b_x] \times [a_y, b_y]; \mathbb{R})$.*
*Then if **m-d_2** is **odd***
***n-d_1* odd**

$$\|r-f\|_\infty \leq \frac{(b_x-a_x)}{(d_1+2)}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty h^{d_1+1} + \frac{(b_y-a_y)}{(d_2+2)}\left\|\frac{\partial^{d_2+2}f}{\partial y^{d_2+2}}\right\|_\infty h^{d_2+1} +$$

$$+\left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{(d_1+2)(d_2+2)} h^{d_1+d_2+2}$$

***n-d_1* even**

$$\|r-f\|_\infty \leq \left(\frac{(b_x-a_x)}{(d_1+2)}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty + \frac{1}{(d_1+1)}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\|_\infty\right) h^{d_1+1} + \frac{(b_y-a_y)}{(d_2+2)}\left\|\frac{\partial^{d_2+2}f}{\partial y^{d_2+2}}\right\|_\infty h^{d_2+1} +$$

$$+\left(\left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{(d_1+2)(d_2+2)} + \left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+2}\partial x^{d_1+1}}\right\|_\infty \frac{(b_y-a_y)}{(d_2+2)(d_1+1)}\right) h^{d_1+d_2+2}$$

*Otherwise, if **m-d_2** is **even***
***n-d_1* odd**

$$\|r-f\|_\infty \leq \frac{(b_x-a_x)}{(d_1+2)}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty h^{d_1+1} + \left(\frac{(b_y-a_y)}{(d_2+2)}\left\|\frac{\partial^{d_2+2}f}{\partial y^{d_2+2}}\right\|_\infty + \frac{1}{(d_2+1)}\left\|\frac{\partial^{d_2+1}f}{\partial y^{d_2+1}}\right\|_\infty\right) h^{d_2+1} +$$

$$+\left(\left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{(d_1+2)(d_2+2)} + \left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)}{(d_1+2)(d_2+1)}\right) h^{d_1+d_2+2}$$

***n-d_1* even**

$$\|r-f\|_\infty \leq \left(\frac{(b_x-a_x)}{(d_1+2)}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty + \frac{1}{(d_1+1)}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\|_\infty\right) h^{d_1+1} + \left(\frac{(b_y-a_y)}{(d_2+2)}\left\|\frac{\partial^{d_2+2}f}{\partial y^{d_2+2}}\right\|_\infty +\right.$$

$$\left.+\frac{1}{(d_2+1)}\left\|\frac{\partial^{d_2+1}f}{\partial y^{d_2+1}}\right\|_\infty\right) h^{d_2+1} + \left(\left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_y-a_y)(b_x-a_x)}{(d_1+2)(d_2+2)} +\right.$$

$$\left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_y-a_y)}{(d_1+1)(d_2+2)} + \left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)}{(d_1+1)(d_2+2)} +$$

$$\left.+\frac{1}{(d_1+1)(d_2+1)}\left\|\frac{\partial^{d_1+d_2+2}f}{\partial y^{d_2+1}\partial x^{d_1+1}}\right\|_\infty\right) h^{d_1+d_2+2}$$

Proof. Let us consider the difference $r-f$. From [10] we known that

$$p_{ij} - f = \omega_{i,d_1+1}(x) f[x_i, ..., x_{i+d_1}, x][y] + \omega_{j,d_2+1}(y) f[x][y_j, ..., y_{j+d_2}, y] +$$

$$-\omega_{i,d_1+1}(x)\omega_{j,d_2+1}(y) f[x_i, ..., x_{i+d_1}, x][y_j, ..., y_{j+d_2}, y].$$

where

$$\omega_{i-1,d+1}(z) \equiv 1,\ \omega_{s,d+1} = \omega_{s-1,d+1}(z)(z-z_s),\ s = i, ..., i+d \ \ (d = d_1, d_2)$$

Therefore

$$r(x,y)-f(x,y)=\frac{\displaystyle\sum_{i=0}^{n-d_1}\sum_{j=0}^{m-d_2}\lambda_i(x)\lambda_j(y)(p_{ij}(x,y)-f(x,y))}{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)\sum_{j=0}^{m-d_2}\lambda_j(y)}=$$

$$=\frac{\displaystyle\sum_{i=0}^{n-d_1}\sum_{j=0}^{m-d_2}\lambda_i(x)\lambda_j(y)\omega_{i,d_1+1}(x)f[x_i,...,x_{i+d_1},x][y]}{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)\sum_{j=0}^{m-d_2}\lambda_j(y)}+$$

$$+\frac{\displaystyle\sum_{i=0}^{n-d_1}\sum_{j=0}^{m-d_2}\lambda_i(x)\lambda_j(y)\omega_{j,d_2+1}(y)f[x][y_j,...,y_{j+d_2},y]}{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)\sum_{j=0}^{m-d_2}\lambda_j(y)}+$$

$$-\frac{\displaystyle\sum_{i=0}^{n-d_1}\sum_{j=0}^{m-d_2}\lambda_i(x)\lambda_j(y)\omega_{i,d_1+1}(x)\omega_{j,d_2+1}(y)f[x_i,...,x_{i+d_1},x][y_j,...,y_{j+d_2},y]}{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)\sum_{j=0}^{m-d_2}\lambda_j(y)}=$$

$$=: A+B-C$$

We analyze individually the three terms.

A

$$A=\frac{\displaystyle\sum_{i=0}^{n-d_1}\sum_{j=0}^{m-d_2}\lambda_i(x)\lambda_j(y)\omega_{i,d_1+1}(x)f[x_i,...,x_{i+d_1},x][y]}{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)\sum_{j=0}^{m-d_2}\lambda_j(y)}=$$

$$=\frac{(\displaystyle\sum_{j=0}^{m-d_2}\lambda_j(y))(\sum_{i=0}^{n-d_1}\lambda_i(x)\omega_{i,d_1+1}(x)f[x_i,...,x_{i+d_1},x][y])}{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)\sum_{j=0}^{m-d_2}\lambda_j(y)}=$$

$$=\frac{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)\omega_{i,d_1+1}(x)f[x_i,...,x_{i+d_1},x][y]}{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)}=\frac{\displaystyle\sum_{i=0}^{n-d_1}(-1)^i f[x_i,...,x_{i+d_1},x][y]}{\displaystyle\sum_{i=0}^{n-d_1}\lambda_i(x)}=\frac{N_A}{D_A}$$

In the following we use the definition of the bivariate divided differences and an inequality, taken from [1], for instance

$$\sum_{i=0}^{n-d_1-1}(x_{i+d_1+1}-x_i) \le (d_1+1)(b_x-a_x), \quad \sum_{j=0}^{m-d_2-1}(y_{j+d_2+1}-y_j) \le (d_2+1)(b_y-a_y)$$

From the proof of theorem 2 in [1], follows that

$$|D_A| = |\sum_{i=0}^{n-d_1}\lambda_i(x)| \ge \frac{1}{d_1!h_x^{d_1+1}} \ge \frac{1}{d_1!h^{d_1+1}}$$

n-d$_1$ odd

$$N_A = -\sum_{i=0,\ i\ even}^{n-d_1-1}(x_{i+d_1+1}-x_i)f[x_i,...,x_{i+d_1+1},x][y] =$$

$$= -\sum_{i=0,\ i\ even}^{n-d_1-1}(x_{i+d_1+1}-x_i)f[x_i,...,x_{i+d_1+1},x][y] =$$

$$= -\sum_{i=0,\ i\ even}^{n-d_1-1}(x_{i+d_1+1}-x_i)\ \frac{1}{(d_1+2)!}\frac{\partial^{d_1+2}f(x,y)}{\partial x^{d_1+2}}\bigg|_{x=\xi^1_{x,i}}.$$

Then

$$|N_A| = \frac{1}{(d_1+2)!}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty \sum_{i=0,\ i\ even}^{n-d_1-1}(x_{i+d_1+1}-x_i) \le \frac{(d_1+1)(b_x-a_x)}{(d_1+2)!}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty$$

$$|A| = \frac{N_A}{D_A} \le \frac{(d_1+1)(b_x-a_x)d_1!h^{d_1+1}}{(d_1+2)!}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty = \frac{(b_x-a_x)}{(d_1+2)}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty h^{d_1+1} \tag{3.13}$$

n-d$_1$ even

$$N_A = -\sum_{i=0,\ i\ even}^{n-d_1-2}(x_{i+d_1+1}-x_i)f[x_i,...,x_{i+d_1+1},x][y] + f[x_{n-d_1},...,x_n,x][y] =$$

$$= -\sum_{i=0,\ i\ even}^{n-d_1-2}(x_{i+d_1+1}-x_i)\frac{1}{(d_1+2)!}\ \frac{\partial^{d_1+2}f(x,y)}{\partial x^{d_1+2}}\bigg|_{x=\xi^1_{x,i}} + \frac{1}{(d_1+1)!}\ \frac{\partial^{d_1+1}f(x,y)}{\partial x^{d_1+1}}\bigg|_{x=\xi^1_x}$$

Then

$$|N_A| \le \frac{1}{(d_1+2)!}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty \sum_{i=0,\ i\ even}^{n-d_1-2}(x_{i+d_1+1}-x_i) + \frac{1}{(d_1+1)!}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\|_\infty \le$$

$$\le \frac{(d_1+1)(b_x-a_x)}{(d_1+2)!}\left\|\frac{\partial^{d_1+2}f}{\partial x^{d_1+2}}\right\|_\infty + \frac{1}{(d_1+1)!}\left\|\frac{\partial^{d_1+1}f}{\partial x^{d_1+1}}\right\|_\infty$$

Finally

$$|A| = \frac{N_A}{D_A} \leq \frac{(d_1+1)(b_x-a_x)}{(d_1+2)!} \left\| \frac{\partial^{d_1+2} f}{\partial x^{d_1+2}} \right\|_\infty d_1! h^{d_1+1} + \frac{d_1! h^{d_1+1}}{(d_1+1)!} \left\| \frac{\partial^{d_1+1} f}{\partial x^{d_1+1}} \right\|_\infty =$$

$$= \frac{(b_x-a_x)}{(d_1+2)} \left\| \frac{\partial^{d_1+2} f}{\partial x^{d_1+2}} \right\|_\infty h^{d_1+1} + \frac{h^{d_1+1}}{(d_1+1)} \left\| \frac{\partial^{d_1+1} f}{\partial x^{d_1+1}} \right\|_\infty =$$

$$= \left(\frac{(b_x-a_x)}{(d_1+2)} \left\| \frac{\partial^{d_1+2} f}{\partial x^{d_1+2}} \right\|_\infty + \frac{1}{(d_1+1)} \left\| \frac{\partial^{d_1+1} f}{\partial x^{d_1+1}} \right\|_\infty \right) h^{d_1+1}. \quad (3.14)$$

B

Following the computations about A, is easy to verify similar bounds for B.

m-d$_2$ odd

$$|B| \leq \frac{(b_y-a_y)}{(d_2+2)} \left\| \frac{\partial^{d_2+2} f}{\partial y^{d_2+2}} \right\|_\infty h^{d_2+1} \quad (3.15)$$

m-d$_2$ even

$$|B| \leq \left(\frac{(b_y-a_y)}{(d_2+2)} \left\| \frac{\partial^{d_2+2} f}{\partial y^{d_2+2}} \right\|_\infty + \frac{1}{(d_2+1)} \left\| \frac{\partial^{d_2+1} f}{\partial y^{d_2+1}} \right\|_\infty \right) h^{d_2+1} \quad (3.16)$$

C

From the proof of theorem 2 in [1], follows that

$$|D_C| \geq \frac{1}{d_1! d_2! h_x^{d_1+1} h_y^{d_2+1}} \geq \frac{1}{d_1! d_2! h^{d_1 d_2+2}}.$$

m-d$_2$ odd

$$C = \frac{\sum_{i=0}^{n-d_1} \sum_{j=0}^{m-d_2} (-1)^i (-1)^j f[x_i, ..., x_{i+d_1}, x][y_j, ..., y_{j+d_2}, y]}{\sum_{i=0}^{n-d_1} \lambda_i(x) \sum_{j=0}^{m-d_2} \lambda_j(y)} =: \frac{N_C}{D_C}.$$

$$N_C = \sum_{i=0}^{n-d_1} (-1)^i \sum_{j=0}^{m-d_2} (-1)^j f[x_i, ..., x_{i+d_1}, x][y_j, ..., y_{j+d_2}, y] =$$

$$= \sum_{i=0}^{n-d_1} (-1)^i \left((-1) \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \left. \frac{\partial^{d_2+2} f[x_i, ..., x_{i+d_1}, x](y)}{\partial y^{d_2+2}} \right|_{y=\xi_j^y} \right) =$$

$$= \sum_{i=0}^{n-d_1} (-1)^{i+1} \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \underbrace{\left. \frac{\partial^{d_2+2} f(x,y)}{\partial y^{d_2+2}} \right|_{y=\xi_j^y}}_{\tilde{f}_j(x) \in C^{d_1+2}([a_x,b_x])} [x_i, ..., x_{i+d_1}, x] =$$

$$= \sum_{i=0}^{n-d_1} (-1)^{i+1} \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \tilde{f}_j[x_i,...,x_{i+d_1},x]$$

n-d_1 odd

$$N_C = \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \sum_{i=0}^{n-d_1} (-1)^{i+1} \tilde{f}_j[x_i,...,x_{i+d_1},x] =$$

$$= \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \left((-1)^2 \sum_{i=0,\ i\ even}^{n-d_1-1} (x_{i+d_1+1}-x_i) \tilde{f}_j[x_i,...,x_{i+d_1+1},x] \right) =$$

$$= \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \sum_{i=0,\ i\ even}^{n-d_1-1} \frac{(x_{i+d_1+1}-x_i)}{(d_1+2)!} \left. \frac{\partial^{d_1+2} \tilde{f}_j(x)}{\partial x^{d_1+2}} \right|_{x=\xi^x_{i,j}}$$

Then

$$|N_C| \le \left\| \frac{\partial^{d_1+d_2+4} f}{\partial y^{d_2+2} \partial x^{d_1+2}} \right\|_\infty \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \sum_{i=0,\ i\ even}^{n-d_1-1} \frac{(x_{i+d_1+1}-x_i)}{(d_1+2)!} \le$$

$$\le \left\| \frac{\partial^{d_1+d_2+4} f}{\partial y^{d_2+2} \partial x^{d_1+2}} \right\|_\infty \frac{(d_1+1)(d_2+1)(b_x-a_x)(b_y-a_y)}{(d_1+2)!(d_2+2)!}.$$

Finally

$$|C| \le \left\| \frac{\partial^{d_1+d_2+4} f}{\partial y^{d_2+2} \partial x^{d_1+2}} \right\|_\infty \frac{(d_1+1)(d_2+1)(b_x-a_x)(b_y-a_y) d_1! d_2! h^{d_1+d_2+2}}{(d_1+2)!(d_2+2)!} =$$

$$= \left\| \frac{\partial^{d_1+d_2+4} f}{\partial y^{d_2+2} \partial x^{d_1+2}} \right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{(d_1+2)(d_2+2)} h^{d_1+d_2+2} \tag{3.17}$$

n-d_1 even

$$N_C = \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \left(\sum_{i=0,\ i\ even}^{n-d_1-2} (x_{i+d_1+1}-x_i) \tilde{f}_j[x_i,...,x_{i+d_1},x] - \tilde{f}_j[x_{n-d_1},...,x_n,x] \right) =$$

$$= \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \left(\sum_{i=0,\ i\ even}^{n-d_1-2} \frac{(x_{i+d_1+1}-x_i)}{(d_1+2)!} \left. \frac{\partial^{d_1+2} \tilde{f}_j(x)}{\partial x^{d_1+2}} \right|_{x=\xi^x_i} - \right.$$

$$\left. - \frac{1}{(d_1+1)!} \left. \frac{\partial^{d_1+1} \tilde{f}_j(x)}{\partial x^{d_1+1}} \right|_{x=\hat{\xi}^x} \right) =$$

$$= \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \sum_{i=0,\ i\ even}^{n-d_1-2} \frac{(x_{i+d_1+1}-x_i)}{(d_1+2)!} \left.\frac{\partial^{d_1+2}\tilde{f}_j(x)}{\partial x^{d_1+2}}\right|_{x=\xi_i^x} -$$

$$- \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \frac{1}{(d_1+1)!} \left.\frac{\partial^{d_1+1}\tilde{f}_j(x)}{\partial x^{d_1+1}}\right|_{x=\hat{\xi}^x}$$

Then

$$|N_C| \leq \left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \sum_{i=0,\ i\ even}^{n-d_1-2} \frac{(x_{i+d_1+1}-x_i)}{(d_1+2)!} +$$

$$+ \left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+2}\partial x^{d_1+1}}\right\|_\infty \sum_{j=0,\ j\ even}^{m-d_2-1} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \frac{1}{(d_1+1)!} \leq$$

$$\leq \left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(d_2+1)(b_x-a_x)(b_y-a_y)}{(d_1+2)!(d_2+2)!} +$$

$$+ \left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+2}\partial x^{d_1+1}}\right\|_\infty \frac{(b_y-a_y)(d_2+1)}{(d_2+2)!(d_1+1)!}.$$

Finally

$$|C| \leq \left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(d_2+1)(b_x-a_x)(b_y-a_y)d_1!d_2!h^{d_1+d_2+2}}{(d_1+2)!(d_2+2)!} +$$

$$+ \left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+2}\partial x^{d_1+1}}\right\|_\infty \frac{(b_y-a_y)(d_2+1)d_1!d_2!h^{d_1+d_2+2}}{(d_2+2)!(d_1+1)!} =$$

$$= \left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)h^{d_1+d_2+2}}{(d_1+2)(d_2+2)} + \left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+2}\partial x^{d_1+1}}\right\|_\infty \frac{(b_y-a_y)h^{d_1+d_2+2}}{(d_2+2)(d_1+1)} =$$

$$= \left(\left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{(d_1+2)(d_2+2)} + \left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+2}\partial x^{d_1+1}}\right\|_\infty \frac{(b_y-a_y)}{(d_2+2)(d_1+1)}\right) h^{d_1+d_2+2} \tag{3.18}$$

m-d$_2$ even

$$N_C = \sum_{i=0}^{n-d_1} (-1)^i \sum_{j=0}^{m-d_2} (-1)^j f[x_i,...,x_{i+d_1},x][y_j,...,y_{j+d_2},y] =$$

$$= \sum_{i=0}^{n-d_1} (-1)^i \Bigg((-1) \sum_{j=0,\ j\ even}^{m-d_2-2} (y_{j+d_2+1}-y_j) f[x_i,...,x_{i+d_1},x][y_j,...,y_{j+d_2+1},y] +$$

$$+f[x_i,...,x_{i+d_1},x][y_{m-d_2},...,y_m,y]\Big) =$$

$$= \sum_{i=0}^{n-d_1}(-1)^i\Bigg((-1)\sum_{j=0,\ j\ even}^{m-d_2-2}\frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}\left.\frac{\partial^{d_2+2}f[x_i,...,x_{i+d_1},x](y)}{\partial y^{d_2+2}}\right|_{y=\xi_j^y}+$$

$$+\frac{1}{(d_2+1)!}\left.\frac{\partial^{d_2+1}f[x_i,...,x_{i+d_1},x](y)}{\partial y^{d_2+1}}\right|_{y=\hat{\xi}^y}\Bigg) =$$

$$= \sum_{i=0}^{n-d_1}(-1)^i\Bigg((-1)\sum_{j=0,\ j\ even}^{m-d_2-2}\frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}\underbrace{\left.\frac{\partial^{d_2+2}f(x,y)}{\partial y^{d_2+2}}\right|_{y=\xi_j^y}}_{\tilde{f}_j(x)}[x_i,...,x_{i+d_1},x]+$$

$$+\frac{1}{(d_2+1)!}\underbrace{\left.\frac{\partial^{d_2+1}f(x,y)}{\partial y^{d_2+1}}\right|_{y=\hat{\xi}^y}}_{\bar{f}(x)\in C^{d_1+2}([a_x,b_x])}[x_i,...,x_{i+d_1},x]\Bigg) =$$

$$= \sum_{i=0}^{n-d_1}(-1)^i\Bigg((-1)\sum_{j=0,\ j\ even}^{m-d_2-2}\frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}\tilde{f}_j[x_i,...,x_{i+d_1},x]+\frac{1}{(d_2+1)!}\bar{f}[x_i,...,x_{i+d_1},x]\Bigg) =$$

$$= \sum_{i=0}^{n-d_1}(-1)^{i+1}\sum_{j=0,\ j\ even}^{m-d_2-2}\frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}\tilde{f}_j[x_i,...,x_{i+d_1},x]+\sum_{i=0}^{n-d_1}(-1)^i\frac{1}{(d_2+1)!}\bar{f}[x_i,...,x_{i+d},x] =$$

$$= \sum_{j=0,\ j\ even}^{m-d_2-2}\frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}\sum_{i=0}^{n-d_1}(-1)^{i+1}\tilde{f}_j[x_i,...,x_{i+d_1},x]+\sum_{i=0}^{n-d_1}(-1)^i\frac{1}{(d_2+1)!}\bar{f}[x_i,...,x_{i+d},x]$$

n-d$_1$ odd

$$N_C = \sum_{j=0,\ j\ even}^{m-d_2-2}\frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}(-1)^2\sum_{i=0,\ i\ even}^{n-d_1-1}\frac{(x_{i+d_1+1}-x_i)}{(d_1+2)!}\left.\frac{\partial^{d_1+2}\tilde{f}_j(x)}{\partial x^{d_1+2}}\right|_{x=\xi_i^x}+$$

$$+(-1)\sum_{i=0,\ i\ even}^{n-d_1-1}\frac{(x_{i+d_1+1}-x_i)}{(d_2+1)!(d_1+2)!}\left.\frac{\partial^{d_1+2}\bar{f}(x)}{\partial x^{d_1+2}}\right|_{x=\hat{\xi}_i^x}$$

Then

$$|N_C| \le \left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty\sum_{j=0,\ j\ even}^{m-d_2-2}\frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}\sum_{i=0,\ i\ even}^{n-d_1-1}\frac{(x_{i+d_1+1}-x_i)}{(d_1+2)!}+$$

$$+\left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \sum_{i=0,\ i\ even}^{n-d_1-1} \frac{(x_{i+d_1+1}-x_i)}{(d_2+1)!(d_1+2)!} \leq$$

$$\leq \left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(d_2+1)(b_x-a_x)(b_y-a_y)}{(d_1+2)!(d_2+2)!}+$$

$$+\left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(b_x-a_x)}{(d_2+1)!(d_1+2)!}$$

Finally

$$|C| \leq \left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(d_2+1)(b_x-a_x)(b_y-a_y)d_1!d_2!h^{d_1+d_2+2}}{(d_1+2)!(d_2+2)!}+$$

$$+\left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(b_x-a_x)d_1!d_2!h^{d_1+d_2+2}}{(d_2+1)!(d_1+2)!} \leq$$

$$\leq \left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)(2-c)h^{d_1+d_2+2}}{(d_1+2)(d_2+2)}+\left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)h^{d_1+d_2+2}}{(d_2+1)(d_1+2)} =$$

$$= \left(\left\|\frac{\partial^{d_1+d_2+4}f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{(d_1+2)(d_2+2)}+\left\|\frac{\partial^{d_1+d_2+3}f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)}{(d_1+2)(d_2+1)}\right)h^{d_1+d_2+2} \tag{3.19}$$

n-d_1 even

$$|N_C| = \sum_{j=0,\ j\ even}^{m-d_2-2} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}\Bigg((-1)^2 \sum_{i=0,\ i\ even}^{n-d_1-2} (x_{i+d_1+1}-x_i)\tilde{f}_j[x_i,...,x_{i+d_1+1},x]-$$

$$-\tilde{f}_j[x_{n-d_1},...,x_n,x]\Bigg) + (-1)\sum_{i=0,\ i\ even}^{n-d_1-2} \frac{(x_{i+d_1+1}-x_i)}{(d_1+1)!}\bar{f}[x_i,...,x_{i+d_1+1},x]+$$

$$+\frac{1}{(d_1+1)!}\bar{f}[x_{n-d_1},...,x_n,x] =$$

$$= \sum_{j=0,\ j\ even}^{m-d_2-2} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!}\Bigg(\sum_{i=0,\ i\ even}^{n-d_1-1} \frac{(x_{i+d_1+1}-x_i)}{(d_2+2)!}\left.\frac{\partial^{d_1+2}\tilde{f}_j(x)}{\partial x^{d_1+2}}\right|_{x=\xi_i^x}+$$

$$-\frac{1}{(d_1+1)!}\left.\frac{\partial^{d_1+1}\tilde{f}_j(x)}{\partial x^{d_1+1}}\right|_{x=\hat{\xi}^x}\Bigg)+(-1)\sum_{i=0,\ i\ even}^{n-d_1-2} \frac{(x_{i+d_1+1}-x_i)}{(d_1+1)!(d_2+2)!}\left.\frac{\partial^{d_1+2}\bar{f}(x)}{\partial x^{d_1+2}}\right|_{x=\hat{\xi}_i^x}+$$

$$+\frac{1}{(d_1+1)!(d_1+1)!}\left.\frac{\partial^{d_1+1}\bar{f}(x)}{\partial x^{d_1+1}}\right|_{x=\tilde{\xi}^x} =$$

$$= \sum_{j=0,\ j\ even}^{m-d_2-2} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \sum_{i=0,\ i\ even}^{n-d_1-2} \frac{(x_{i+d_1+1}-x_i)}{(d_1+2)!} \left.\frac{\partial^{d_1+2}\tilde{f}_j(x)}{\partial x^{d_1+2}}\right|_{x=\xi_i^x} -$$

$$- \sum_{j=0,\ j\ even}^{m-d_2-2} \frac{(y_{j+d_2+1}-y_j)}{(d_2+2)!} \frac{1}{(d_1+1)!} \left.\frac{\partial^{d_1+1}\tilde{f}_j(x)}{\partial x^{d_1+1}}\right|_{x=\hat{\xi}^x} +$$

$$+(-1) \sum_{i=0,\ i\ even}^{n-d_1-2} \frac{(x_{i+d_1+1}-x_i)}{(d_1+1)!(d_2+2)!} \left.\frac{\partial^{d_1+2}\bar{f}(x)}{\partial x^{d_1+2}}\right|_{x=\hat{\xi}_i^x} + \frac{1}{(d_1+1)!(d_1+1)!} \left.\frac{\partial^{d_1+1}\bar{f}(x)}{\partial x^{d_1+1}}\right|_{x=\tilde{\xi}^x} =$$

Then

$$|N_C| \le \left\|\frac{\partial^{d_1+d_2+4} f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(d_2+1)(b_y-a_y)(b_x-a_x)}{(d_1+2)!(d_2+2)!} +$$

$$+ \left\|\frac{\partial^{d_1+d_2+3} f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(d_2+1)(b_y-a_y)}{(d_2+2)!(d_1+1)!} +$$

$$+ \left\|\frac{\partial^{d_1+d_2+3} f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(b_x-a_x)}{(d_1+1)!(d_2+2)!} + \frac{1}{(d_1+1)!(d_2+1)!} \left\|\frac{\partial^{d_1+d_2+2} f}{\partial y^{d_2+1}\partial x^{d_1+1}}\right\|_\infty$$

Finally

$$|C| \le \left\|\frac{\partial^{d_1+d_2+4} f}{\partial y^{d+2}\partial x^{d+2}}\right\|_\infty \frac{(d_1+1)(d_2+1)(b_y-a_y)(b_x-a_x)d_1!d_2!h^{d_1+d_2+2}}{(d_1+2)!(d_2+2)!} +$$

$$+ \left\|\frac{\partial^{d_1+d_2+3} f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(d_2+1)(b_y-a_y)d_1!d_2!h^{d_1+d_2+2}}{(d_2+2)!(d_1+1)!} +$$

$$+ \left\|\frac{\partial^{d_1+d_2+3} f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(d_1+1)(b_x-a_x)d_1!d_2!h^{d_1+d_2+2}}{(d_1+1)!(d_2+2)!} +$$

$$+ \frac{d_1!d_2!h^{d_1+d_2+2}}{(d_1+1)!(d_2+1)!} \left\|\frac{\partial^{d_1+d_2+2} f}{\partial y^{d_2+1}\partial x^{d_1+1}}\right\|_\infty \le$$

$$= \left\|\frac{\partial^{d_1+d_2+4} f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_y-a_y)(b_x-a_x)h^{d_1+d_2+2}}{(d_1+2)(d_2+2)} +$$

$$+ \left\|\frac{\partial^{d_1+d_2+3} f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_y-a_y)h^{d_1+d_2+2}}{(d_1+1)(d_2+2)} + \left\|\frac{\partial^{d_1+d_2+3} f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)h^{d_1+d_2+2}}{(d_1+1)(d_2+2)} +$$

$$+ \frac{h^{d_1+d_2+2}}{(d_1+1)(d_2+1)} \left\|\frac{\partial^{d_1+d_2+2} f}{\partial y^{d_2+1}\partial x^{d_1+1}}\right\|_\infty =$$

$$= \left(\left\|\frac{\partial^{d_1+d_2+4} f}{\partial y^{d_2+2}\partial x^{d_1+2}}\right\|_\infty \frac{(b_y-a_y)(b_x-a_x)}{(d_1+2)(d_2+2)} + \left\|\frac{\partial^{d_1+d_2+3} f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_y-a_y)}{(d_1+1)(d_2+2)} +\right.$$

$$\left. + \left\|\frac{\partial^{d_1+d_2+3} f}{\partial y^{d_2+1}\partial x^{d_1+2}}\right\|_\infty \frac{(b_x-a_x)}{(d_1+1)(d_2+2)} + \frac{1}{(d_1+1)(d_2+1)} \left\|\frac{\partial^{d_1+d_2+2} f}{\partial y^{d_2+1}\partial x^{d_1+1}}\right\|_\infty \right) h^{d_1+d_2+2}$$

□

Recalling the proof in [1] with $d = 0$, we define

$$\beta_x := \max_{1\le i\le n-2} \min\left\{\frac{x_{i+1}-x_i}{x_i-x_{i-1}}, \frac{x_{i+1}-x_i}{x_{i+2}-x_{i+1}}\right\}$$

$$\beta_y := \max_{1\le j\le m-2} \min\left\{\frac{y_{j+1}-y_j}{y_j-y_{j-1}}, \frac{y_{j+1}-y_j}{y_{j+2}-y_{j+1}}\right\}.$$

Theorem 3.1.5. *Suppose $d = 0$ and $f \in C^2([a_x, b_x] \times [a_y, b_y]; \mathbb{R})$.*
Then if ***m*** *is* ***odd***

n odd

$$\|r-f\|_\infty \le \left(\frac{(b_x-a_x)(1+\beta_x)}{2}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_\infty + \right.$$
$$\left.+\frac{(b_y-a_y)(1+\beta_y)}{2}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_\infty\right)h + \frac{(b_x-a_x)(b_y-a_y)}{4}(1+\beta_x)(1+\beta_y)\left\|\frac{\partial^4 f}{\partial y^2\partial x^2}\right\|_\infty h^2$$

n even

$$\|r-f\|_\infty \le \left(\frac{(b_x-a_x)(1+\beta_x)}{2}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_\infty + (1+\beta_x)\left\|\frac{\partial f}{\partial x}\right\|_\infty + \frac{(b_y-a_y)(1+\beta_y)}{2}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_\infty\right)h+$$
$$+\left(\left\|\frac{\partial^4 f}{\partial y^2\partial x^2}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{4} + \left\|\frac{\partial^3 f}{\partial y^2\partial x}\right\|_\infty \frac{(b_y-a_y)}{2}\right)(1+\beta_x)(1+\beta_y)h^2$$

Then if ***m*** *is* ***even***

n odd

$$\|r-f\|_\infty \le \left(\frac{(b_x-a_x)(1+\beta_x)}{2}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_\infty + (1+\beta_y)\frac{(b_y-a_y)}{2}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty\right)h+$$
$$+\left(\left\|\frac{\partial^4 f}{\partial y^2\partial x^2}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{4} + \left\|\frac{\partial^3 f}{\partial y^1\partial x^2}\right\|_\infty \frac{(b_x-a_x)}{2}\right)(1+\beta_x)(1+\beta_y)h^2$$

n even

$$\|r-f\|_\infty \le \left((1+\beta_x)\left(\frac{(b_x-a_x)}{2}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_\infty + \left\|\frac{\partial f}{\partial x}\right\|_\infty\right) + (1+\beta_y)\left(\frac{(b_y-a_y)}{2}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty\right)\right)h+$$
$$+\left(\left\|\frac{\partial^4 f}{\partial y^2\partial x^2}\right\|_\infty \frac{(b_y-a_y)(b_x-a_x)}{4} + \left\|\frac{\partial^3 f}{\partial y\partial x^2}\right\|_\infty \frac{(b_y-a_y)}{2} + \left\|\frac{\partial^3 f}{\partial y\partial x^2}\right\|_\infty \frac{(b_x-a_x)}{2} + \right.$$
$$\left.+\left\|\frac{\partial^2 f}{\partial y\partial x}\right\|_\infty\right)h^2(1+\beta_x)(1+\beta_y)$$

Proof. We repeat the same reasonings as before with $d = 0$ and use the lower bound 17 in [1]. For instance, as above,

$$r(x,y) - f(x,y) = \frac{\sum_{i=0}^{n}(-1)^i f[x_i,x][y]}{\sum_{i=0}^{n}\lambda_i(x)} + \frac{\sum_{j=0}^{m}(-1)^j f[x][y_j,y]}{\sum_{j=0}^{m}\lambda_j(y)} -$$

$$-\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^i(-1)^j f[x_i,x][y_j,y]}{\sum_{i=0}^{n}\lambda_i(x)\sum_{j=0}^{m}\lambda_j(y)} =: A + B - C.$$

From [1] we deduce that

$$\left|\sum_{i=0}^{n}\lambda_i(x)\right| \geq \frac{1}{h_x(1+\beta_x)} \geq \frac{1}{h(1+\beta_x)}, \qquad \left|\sum_{j=0}^{m}\lambda_j(y)\right| \geq \frac{1}{h_y(1+\beta_y)} \geq \frac{1}{h(1+\beta_y)},$$

$$\left|\sum_{i=0}^{n}\lambda_i(x)\sum_{j=0}^{m}\lambda_j(y)\right| \geq \frac{1}{h^2(1+\beta_x)(1+\beta_y)}$$

A

n odd

$$|A| \leq \frac{(b_x-a_x)h(1+\beta_x)}{2}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_\infty$$

n even

$$|A| \leq h(1+\beta_x)\left(\frac{(b_x-a_x)}{2}\left\|\frac{\partial^2 f}{\partial x^2}\right\|_\infty + \left\|\frac{\partial f}{\partial x}\right\|_\infty\right)$$

B

m odd

$$|B| \leq \frac{(b_y-a_y)h(1+\beta_y)}{2}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_\infty$$

m even

$$|B| \leq h(1+\beta_y)\left(\frac{(b_y-a_y)}{2}\left\|\frac{\partial^2 f}{\partial y^2}\right\|_\infty + \left\|\frac{\partial f}{\partial y}\right\|_\infty\right)$$

C

m odd

n odd

$$|C| \leq \frac{(b_x-a_x)(b_y-a_y)}{4}(1+\beta_x)(1+\beta_y)\left\|\frac{\partial^4 f}{\partial y^2\partial x^2}\right\|_\infty h^2$$

n even

$$|C| \leq \left(\left\|\frac{\partial^4 f}{\partial y^2\partial x^2}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{4} + \left\|\frac{\partial^3 f}{\partial y^2\partial x}\right\|_\infty \frac{(b_y-a_y)}{2}\right)(1+\beta_x)(1+\beta_y)h^2$$

m even

n odd

$$|C| \leq \left(\left\|\frac{\partial^4 f}{\partial y^2\partial x^2}\right\|_\infty \frac{(b_x-a_x)(b_y-a_y)}{4} + \left\|\frac{\partial^3 f}{\partial y^1\partial x^2}\right\|_\infty \frac{(b_x-a_x)}{2}\right)(1+\beta_x)(1+\beta_y)h^2$$

n even

$$|C| \leq \left(\left\|\frac{\partial^4 f}{\partial y^2\partial x^2}\right\|_\infty \frac{(b_y-a_y)(b_x-a_x)}{4} + \left\|\frac{\partial^3 f}{\partial y\partial x^2}\right\|_\infty \frac{(b_y-a_y)}{2} + \left\|\frac{\partial^3 f}{\partial y\partial x^2}\right\|_\infty \frac{(b_x-a_x)}{2} + \right.$$
$$\left. + \left\|\frac{\partial^2 f}{\partial y\partial x}\right\|_\infty\right)h^2(1+\beta_x)(1+\beta_y)$$

We sum up the results and the thesis follows. □

3.2 Lebesgue constant

We know that in 1d case (1.16) holds. So let $\Lambda_{nm}(x,y)$ be the Lebesgue function associated with $\mathbf{G}$ as before

$$\Lambda_{nm}(x,y) = \sum_{\mathbf{k}\in\mathcal{K}} |\mathbf{b_k}| = \sum_{k_1=0}^{n}\sum_{k_2=0}^{m} |b^x_{k_1}||b^y_{k_2}| = \left(\sum_{k_1=0}^{n} |b^x_{k_1}|\right)\left(\sum_{k_2=0}^{m} |b^y_{k_2}|\right) = \Lambda_n(x)\Lambda_m(y), \tag{3.20}$$

and in particular case $m = n$ we have

$$\Lambda_{nn}(x,y) = \big(\sum_{k_1=0}^{n} |b^x_{k_1}|\big)^2$$

Thanks to (1.15) and (1.16) it is easy to prove

Proposition 3.2.1. *The Lebesgue constant associated with TPFHRI at equispaced grid* $\{(x_i,y_j)\}_{0\le i\le n,\ 0\le j\le m}$ *with basis function* $\boldsymbol{b_k} = b^x_{k_1} b^y_{k_2}$ *in (3.1) satisfies*
if $d_1 = d_2 = 0$

$$c^x_n c^y_n \ln(n+1)\ln(m+1) \le \Lambda_{nm}(x,y) \le (2+\ln(n))(2+\ln(m)) \tag{3.21}$$

where $c^x_n = \frac{2n}{4+n\pi}, c^y_n = \frac{2m}{4+m\pi}$,
if $d_1, d_2 \ge 1$

$$\frac{1}{2^{d_1+d_2+4}}\binom{2d+1}{d}^2 \ln(\frac{n}{d_1}-1)\ln(\frac{m}{d_2}-1) \le \Lambda_{nm}(x,y) \le 2^{d_1+d_2}(2+\ln(n))(2+\ln(m)) \tag{3.22}$$

if $d_1 = 0$ *and* $d_2 \ge 1$ *(or equivalently viceversa)*

$$\frac{c^x_n}{2^{d_2+2}}\binom{2d+1}{d} \ln(n-1)\ln(\frac{m}{d_2}-1) \le \Lambda_{nm}(x,y) \le 2^{d_2}(2+\ln(n))(2+\ln(m)). \tag{3.23}$$

3.3 Numerical experiments

Approximation and error

Using our matlab codes, we have tested TPFHRI in three particular cases. Now we decide to focus our analysis on the approximation of the interpolant with $\mathbf{d_2}$=$\mathbf{d_1}$, **m=n**, $I = [0,1]^2$, unless otherwise specified, and in the following parameters d_1, d_2, n, m will be chosen in different way.

The first test function is Franke's one, that is $f_1 = \frac{3}{4}e^{-\frac{1}{4}((9x-2)^2+(9y-2)^2)} + \frac{3}{4}e^{-(\frac{(9x+1)^2}{49}+\frac{(9y+1)}{10})} + \frac{1}{2}e^{-\frac{1}{4}((9x-7)^2+(9y-3)^2)} - \frac{1}{5}e^{-((9x-4)^2+(9y-7)^2)}$.

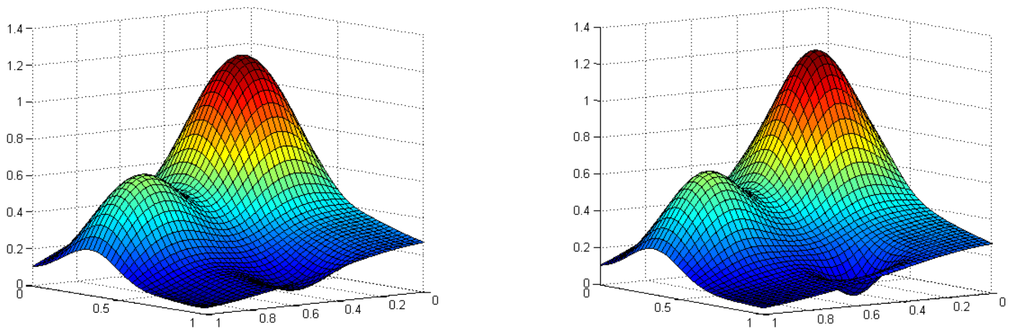

Figure 3.1: TFHRI - Test function f_1 with $(n+1)^2$ equispaced nodes, $n = 6, 10$, and $d = 2$

It is infinitly smooth and the Figure 3.1 reveals in fact that the interpolant has no problems in approximating it neither with $n = 6$ (left) nor with $n = 10$ (right). The approximation is good and with $n = 10$ the minimum point is more evident than in the case on the left. The interpolant is accurate and the order of precision is 10^{-3}, Table 3.1.

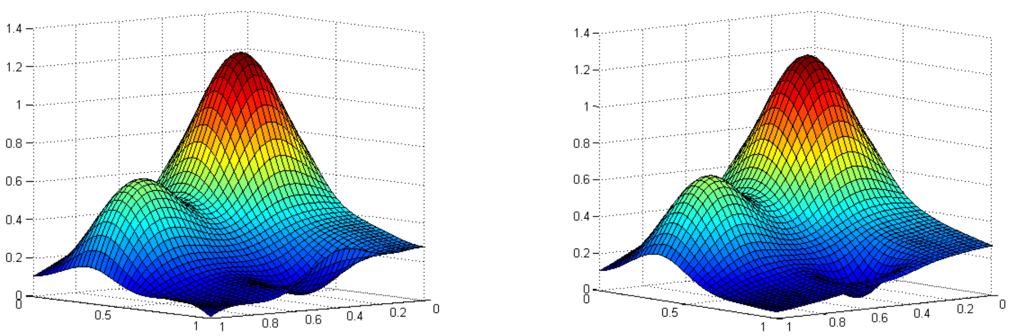

Figure 3.2: TFHRI - Test function f_1 with $(n+1)^2$ Chebyshev nodes, $n = 6, 10$, and $d = 2$

We also consider **Chebyshev nodes**, that is the tensor product of Chebyshev points in two directions. In this case the approximation seems to be little bit good but on the boundary of the domain some defects happen, see Figure 3.2 on the left but they totally disappear on the right. The approximation seems to be precise also in this case and the errors, reported in Table 3.2, come to order 10^{-3} with $n = 14$.

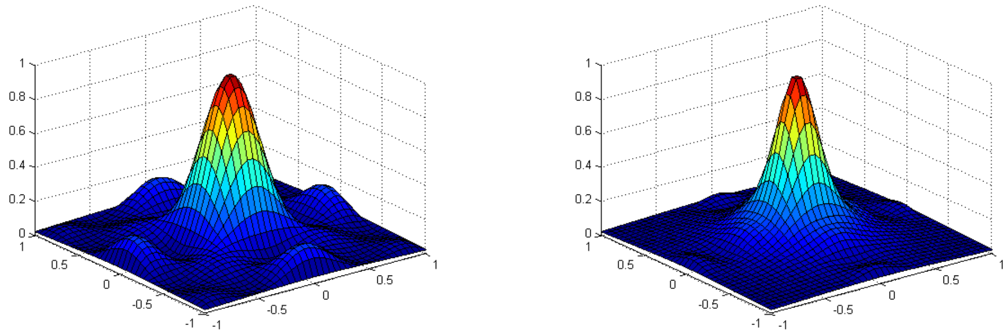

Figure 3.3: TFHRI - Test function f_2 with $(n+1)^2$ equispaced nodes, $n = 6, 10$, and $d = 2$

In the second case we try to approximate the bivariate Runge's function $f_2 = \frac{1}{1+25(x^2+y^2)}$. With $n = m = 6$ some oscillations appear all around the center of the square as expression of some numerical instabilities but if n increases, Figure 3.3 on the right, the oscillatory phenomenon disappears at all. The error, as listed in Table 3.1, is not so small and its order is 10^{-2}.

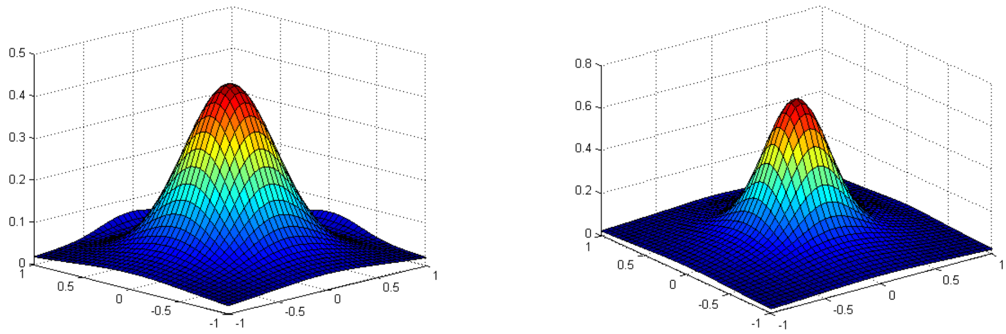

Figure 3.4: TFHRI - Test function f_2 with $(n+1)^2$ Chebyshev nodes, $n = 6, 10$, and $d = 2$

If the nodes are Chebyshev ones, with $n = 6$ oscillations do not occur. On the other hand as said before, the distribution of the nodes does not allow a good approximation of the maximum point in the middle of the square. Also in this case the equispaced nodes are to be preferred because the errors, listed in Table 3.2, are higher than the errors with equispaced nodes where the order of precision is 10^{-2}.

The last test function is the norm, $f_3 = \sqrt{x^2 + y^2}$, that is not smooth in $(0, 0)$. As before the Figure 3.5 reveals that the approximation is apparently good with different number of nodes and the interpolant with the growth of n try to reproduce the pick. The precision never goes below 10^{-2}.

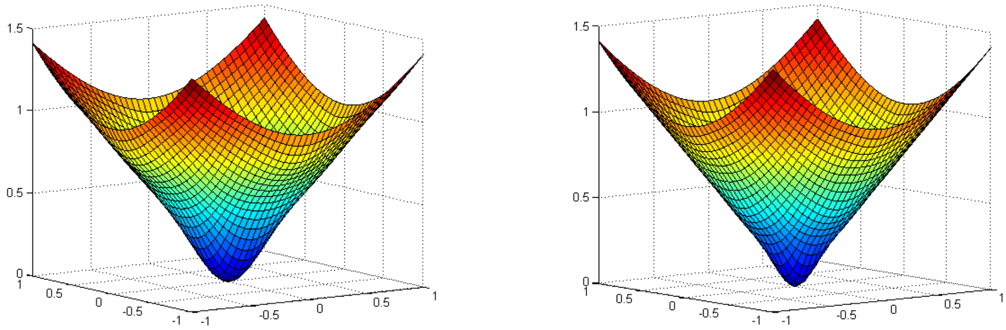

Figure 3.5: TFHRI - Test function f_1 with $(n+1)^2$ equispaced nodes in $[-1,1]^2$, $n = 6, 10$ and $d = 2$

Considering the Chebyshev nodes, Figure 3.6 shows that the interpolant can't reproduce the pick as good as in the previous case. Chebyshev points concentrate on the endpoints of the interval and so the error goes to 10^{-2} with $n = 14$ only.

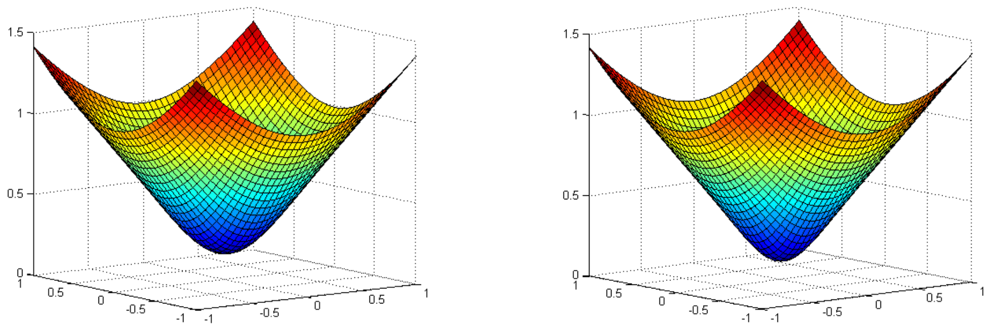

Figure 3.6: TFHRI - Test function f_1 with $(n+1)^2$ Chebyshev nodes in $[-1,1]^2$, $n = 6, 10$ and $d = 2$

n	*Franke*	*Runge*	$\sqrt{x^2+y^2}$
6	9.4968e-02	2.6486e-01	7.3468e-02
8	2.9503e-02	1.2947e-01	4.7397e-02
10	2.4679e-02	6.4220e-02	3.7900e-02
14	1.0439e-03	1.9142e-02	2.6930e-02

Table 3.1: TPFHRI error with $(n+1)^2$ equispaced nodes - $d = 3$

n	$Franke$	$Runge$	$\sqrt{x^2+y^2}$
6	1.8569e-01	5.7585e-01	1.6496e-01
8	4.2389e-02	4.1838e-01	1.2710e-01
10	2.2181e-02	3.0831e-01	1.0699e-01
14	3.0878e-03	1.5609e-01	7.9833e-02

Table 3.2: TPFHRI error with $(n+1)^2$ Chebyshev nodes - $d = 3$

Now our analysis focuses on the case $\mathbf{d}_1 \neq \mathbf{d}_2$.

With f_1 as test function, for $n = m = 15$ the small error occurs with $d_1 = 3$ and $d_2 = 3$, as it may be seen in Table 3.3. In the case in exam, the experiments reveal also that if $n \neq m$ less or equal 15, the interpolation error increases and the optimal values of d_1 and d_2 change.

$d_1 - d_2$	0	1	2	3	4
0	3.0377e-02	2.6207e-02	2.6137e-02	2.5826e-02	2.5893e-02
1	1.5843e-02	3.5168e-03	3.2423e-03	3.2180e-03	3.9903e-03
2	1.4671e-02	2.9402e-03	2.2292e-03	1.5610e-03	3.8251e-03
3	1.4707e-02	2.5326e-03	1.8469e-03	1.0439e-03	3.8194e-03
4	1.4737e-02	2.5544e-03	1.8792e-03	1.0456e-03	3.8194e-03

Table 3.3: TPFHRI error with different d_1 and d_2 and equispaced nodes - Franke's function

In the case of Chebyshev points our numerical tests suggest that the chioce $n = m$ is to be preferred and that the optimal values for the parameters are $d_1 = 4$ and $d_2 = 3$ that are different from before.

$d_1 - d_2$	0	1	2	3	4
0	3.5844e-02	2.7782e-02	2.7255e-02	2.7985e-02	2.7455e-02
1	2.0489e-02	5.3281e-03	5.2965e-03	4.2832e-03	4.3006e-03
2	2.0523e-02	5.4305e-03	4.5228e-03	3.1983e-03	3.3785e-03
3	2.0742e-02	5.5770e-03	4.5151e-03	3.0878e-03	3.2696e-03
4	2.0606e-02	5.3663e-03	4.5567e-03	2.8108e-03	2.9945e-03

Table 3.4: TPFHRI error with different d_1 and d_2 and Chebyshev nodes - Franke's function

If consider the norm function, the optimal values, related to the reduction of the error, are $d_1 = 2$ and $d_2 = 0$ or viceversa since the function is simmetric, Table 3.3. Also in this case, if $n \neq m$ the error does not decrease and therefore the best choice is $n = m$. Again similar reasonings hold for the case of Chebyshev nodes and it is better to take $n = m$. With this set of nodes the optimal values, as reported in Table 3.3 are $d_1 = d_2 = 4$ unlike before.

$d_1 - d_2$	0	1	2	3	4
0	3.1154e-02	2.4053e-02	2.3793e-02	2.3842e-02	2.3825e-02
1	2.4053e-02	2.4643e-02	2.4597e-02	2.4605e-02	2.4602e-02
2	2.3793e-02	2.4597e-02	2.4337e-02	2.4386e-02	2.4369e-02
3	2.3842e-02	2.4605e-02	2.4386e-02	2.4394e-02	2.4391e-02
4	2.3825e-02	2.4602e-02	2.4369e-02	2.4391e-02	2.4374e-02

Table 3.5: TPFHRI error with different d_1 and d_2 and equispaced nodes - norm

It is good to emphazize that the specific test function will suggest how to choose d_1, d_2, n, m. Theoretically one may consider them to make the error bound in the previous section as small as possible. For example if a supnorm of a derivative is too big, these variables may be chosen in such way that the power of h, that multiplies the supnorm, oppose to it and keeps the quantity under control.

$d_1 - d_2$	0	1	2	3	4
0	8.1594e-02	8.1317e-02	8.1004e-02	8.0719e-02	8.0389e-02
1	8.1317e-02	8.1039e-02	8.0725e-02	8.0439e-02	8.0107e-02
2	8.1004e-02	8.0725e-02	8.0411e-02	8.0123e-02	7.9790e-02
3	8.0719e-02	8.0439e-02	8.0123e-02	7.9833e-02	7.9499e-02
4	8.0389e-02	8.0107e-02	7.9790e-02	7.9499e-02	7.9164e-02

Table 3.6: TPFHRI error with different d_1 and d_2 and Chebyshev nodes - norm

Now we are interested in testing the theoretical bounds of the interpolation error both with $d = 0$ and $d > 0$. The function to approximate is $f(x, y) = cos(x + y)$ because in this way all the supnorms of the derivatives are bounded from above by 1. Thanks to the previous reasonings, we take in consideration only the case $m = n$.

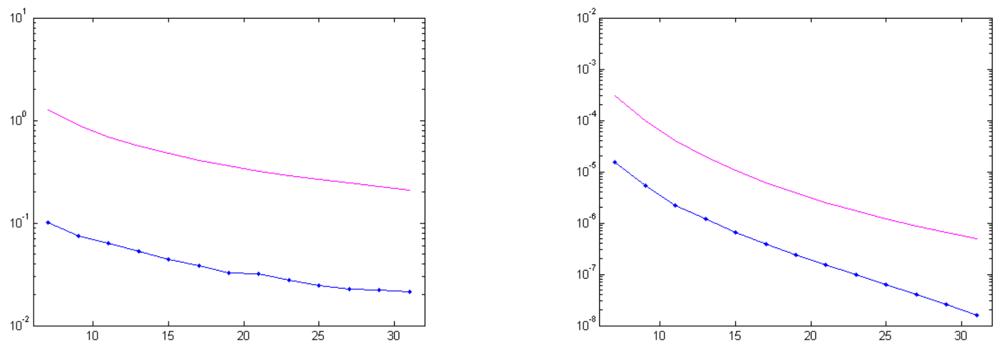

Figure 3.7: TFHRI - Interpolation error of $f(x, y) = \cos(x + y)$ with $(n + 1)^2$ equispaced nodes in $[0, 1]^2$, $n = 6, ..., 30$, $d = 0$ (left) and $d = 3$ (right)

Figure 3.7 shows that our bound is correct in both cases. With $d = 3$ we notice that the error is smaller than in the other one. Then we control that the bounds of the error are correct also if $d_1 \neq d_2$. The results show that the bounds are right and that the precision is high, in particular in the plot in Figure 3.8 on the right, in which the order of precision is in semilogaritmic scale 10^{-8}.

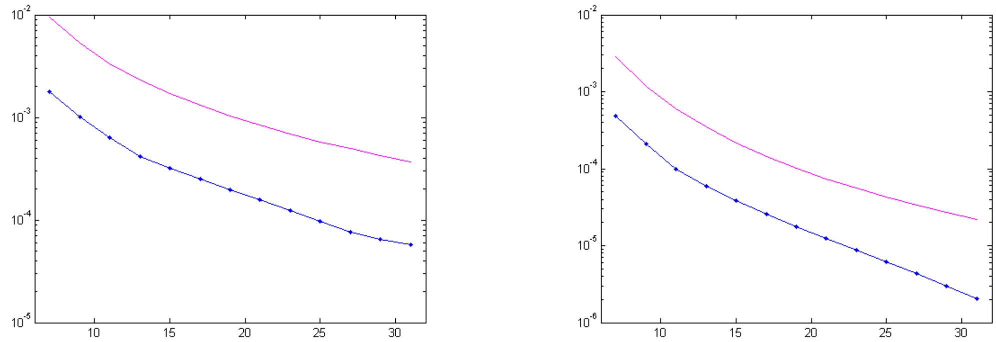

Figure 3.8: TFHRI - Interpolation error of $f(x, y) = \cos(x + y)$ with $(n + 1)^2$ equispaced nodes in $[0, 1]^2$, $n = 6, ..., 30$, $d_1 = 1, d_2 = 3$ (left) and $d_1 = 2, d_2 = 3$ (right)

Lebesgue constant and function

We choose $m = n$ and with different n compute the Lebesgue constant. Figure 3.9 offers the plot of the Lebesgue constants compared with the upper and the lower bounds. We notice that in both cases the bounds are correct but if $d = 0$ the upper bound is closer to the constant than one in $d = 1$ on the right.

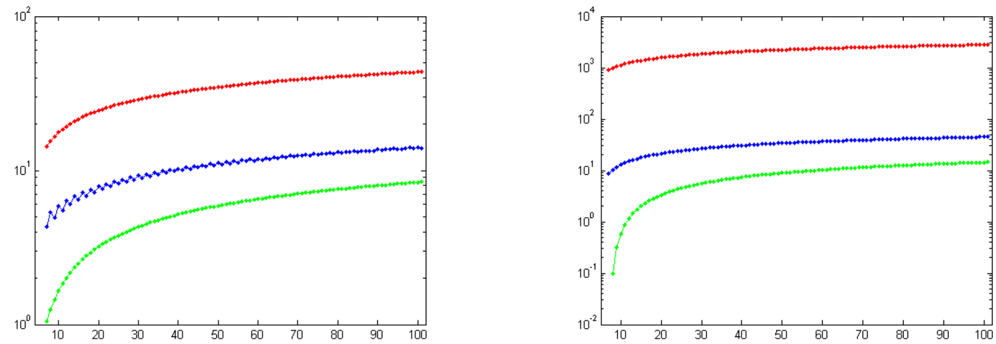

Figure 3.9: Lebesgue constant (blue) and theoretical bounds: upper bound (red) and lower bound (green) with $N = (n + 1)^2$ equispaced nodes in $[0, 1]^2$ with $n = m = 6, ..., 100$, $d = 0$ (left) and $d = 3$ (right)

Since the Lebesgue function associated with TPFHRI is the product of two 1-dimensional Lebesgue functions, one in x-direction and the other one in y-direction, and knowing the effects to the function if d changes, we expect that the actual Lebesgue function with some d_1 and d_2 presents a mix of the behaviours of the 1-dimensional ones. For instance from [11] we know that, in case of equispaced nodes, if $d = 0$ the maximum occurs in the middle of the interval and that it moves towards the endpoints when d increases. Instead in the Chebyshev case, the maximum occurs again in the middle of the interval and while d grows the Lebesgue function becomes smaller and smaller going from the middle to each endpoints.

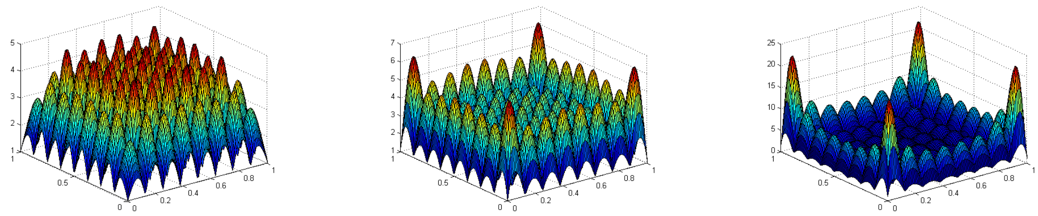

Figure 3.10: Lebesgue functions - TPFHRI with $(n+1)^2$ equispaced nodes in $[0,1]^2$, $n = 8$ and $d_1 = d_2 = 0, 2, 4$

We investigate the Lebesgue function of TFHRI. First we take $d_1 = d_2$ and we want to observe what happend if we change the parameter $d = d_1 = d_2$ with equisaced nodes and Chebyshev ones.

It is clear from Figure 3.10 that the maximum of the Lebesgue function, when d grows, occurs from the center of the center of the square ($d = 0$) to the boundery of it ($d = 2$) and finally to the corners of the square but as well as the maximum moves, it also increases from 6 to 20.

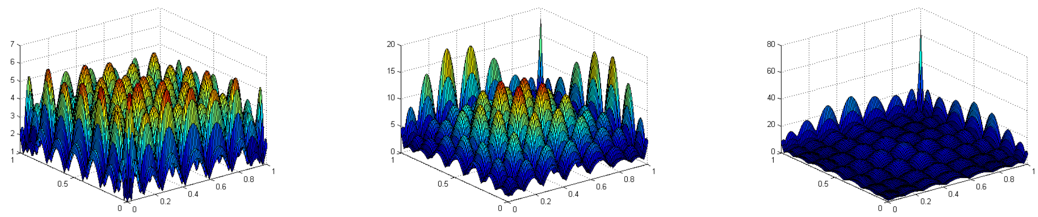

Figure 3.11: Lebesgue functions - TPFHRI with $(n+1)^2$ Chebyshev nodes in $[0,1]^2$, $n = 8$ and $d_1 = d_2 = 0, 2, 4$

Instead in the case of Chebyshev nodes, the maximum is in the center of the square $d = 0$, $d = 2$ and finally with $d = 4$ it is on one corner, see Figure 3.11.

Now we consider $d_1 \neq d_2$. In particular $d_1 = 3$ and $d_2 = 0, 1, 2$.

With equispaced nodes, Figure 3.12, the higher picks distribute on two sides of the square and finally the maximum are on the four corners.

In the case of Chebyshev nodes, the Lebesgue function reaches values bigger than in the case of equispaced points. First the maximum points are in the center of the square and on one side but finally the Lebesgue constant, that is obviously the higher pick, is on one corner, Figure 3.13.

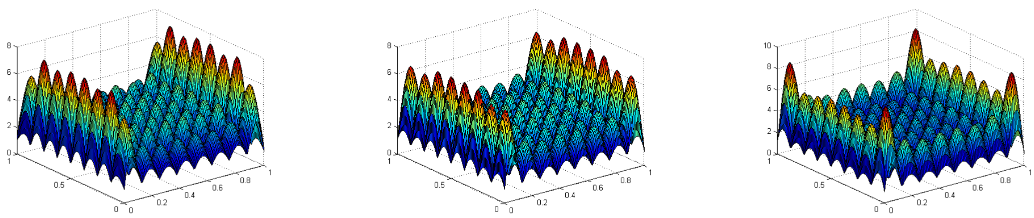

Figure 3.12: Lebesgue functions - TPFHRI with $(n+1)^2$ equispaced nodes in $[0,1]^2$, $n=8$ and $d_1=3$, $d_2=0,1,2$

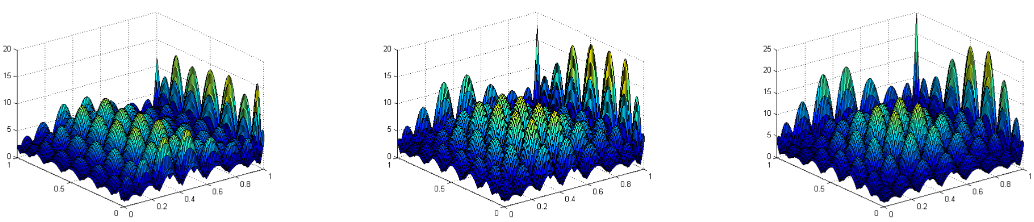

Figure 3.13: Lebesgue functions - TPFHRI with $(n+1)^2$ Chebyshev nodes in $[0,1]^2$, $n=8$ and $d_1=3$, $d_2=0,1,2$

Chapter 4

Software

As a completion of this work, we have grouped our main Matlab codes into a CD. Using them, we have obtained all the results, explained in the previous chapters.

For each routine we propose also a demo, in which everyone may see how to use the routine itself in a concrete example. We realize all the operations by sums, ratios, multiplications of matrixes and vectors. In this way we have used the potentiality of Matlab at all and the codes are more compact and short.

Regarding trigonometric interpolation, we have written specific codes for d even and odd. If d is even, the main function is **eval_bi_trig_deven.m** that returns a $n \times m$ matrix with the evaluations of the basis functions at some points. With rotine **weights_trig_deven.m**, that returns a vector of trigonometric weights, TFHRI may be compute and **demo_TFHRI_deven.m** allows to appreciate its performances in approximating $x^2, \sin(2x)$ and Runge function. Functions **leb_const_trig_deven.m** and **leb_fun_trig_deven.m** return Lebesgue constant and an approximation of Lebesgue function respectively. In the first one we use **eval_bi_trig_deven.m** with $N = 1000$ evaluation points because in such way the approximation is good and times of computation not too long. Some experiments have been made in **demo_leb_const_trig_deven.m** and **demo_leb_fun_trig_deven.m**. In the d odd case there are the same routine as before, so we have **eval_bi_trig_dodd.m**, **weights_trig_dodd_plus_constant.m**, **leb_const_trig_dodd.m**, **leb_fun_trig_dodd.m** and everybody can make experiments using the relative demo, **demo_TFHRI_dodd.m**, **demo_leb_const _trig_dodd.m**, **demo_leb_fun_trig_dodd.m**. The only difference is the presence of a routine, called **find_omega.m**, that given d,x and the endpoints of the interval, returns a pulsation, which satisfies **Hyp 1-2**.

For what tensor product interpolant concerns, we have taken the codes of one dimensional FHRI in [11], for instance **eval_bi.m** and **weights.m**. In **demo_TP FHRI.m** and **demo_TPFHRI_cheb.m** we test the interpolant in approximating some functions with equispaced grid and tensor product Chebyshev grid respectively. In **leb_const_grid.m** we use the one dimensional function to compute the Lebesgue constant in the case of tensor product. **demo_leb_const_tensor.m** computes the Lebesgue constant with $n = m$ and returns its plot. Similarly **leb_fun_grid.m** use the one dimensional code and compute the Lebesgue function of the tensor case. One can see the plot of that function using **demo_fun_ten**

sor.m with equispaced grid and **demo_fun_tensor_cheb.m** with the tensor product of Chebyshev points.

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