UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia ”Galileo Galilei”

Master Degree in Physics

Final Dissertation

Rich Structures of

Simple Supersymmetric Yang-Mills Theory

Thesis supervisor

Prof. Dmitri Sorokin

Candidate

Marco Zecchinato

Academic Year 2019/2020

Abstract

The aim of the thesis is to deepen the knowledge of general fundamental properties of classical and quantum field theories via the study of main features of the simplest supersymmetric gauge theory in four-dimensional spacetime, $\mathcal{N} = 1$ super Yang-Mills (SYM) theory. Starting with the study of symmetry properties of this theory, one proceeds to comprehend quantum anomalies affecting some of the symmetries and spontaneous breaking thereof, the phenomenon which leads to a rich structure of the SYM vacua. Then, one will study how different vacua are connected to each other via domain walls, which are examples of topologically-nontrivial solitonic objects appearing in many theoretical models. Finally, a generalization of the Veneziano-Yankielowicz lagrangian is discussed.

Contents

Introduction

In 2012, the detection of the long sought-after Higgs boson has provided the evidence of the last missing piece of the Standard Model of Particle Physics. Nevertheless, it is agreed that this is not the end of the story: new Physics is expected to show up at the TeV scale. Indeed, there are fundamental questions which do not find any answer within the SM, such as: which is the origin of dark matter and dark energy? How comes that neutrinos acquire such a tiny mass? Why is the Higgs boson mass protected by huge radiative corrections? These and many other issues have prompted the construction of many New Physics frameworks, and one of the most compelling thereof is *supersymmetry.* Given the importance of supersymmetry in modern theoretical Physics, in this thesis we have dealt with general properties of classical and quantum field theories in a supersymmetric scenario. In particular, the work focuses on the study of the supersymmetric generalisation of a Yang-Mils theory with gauge group $SU(N)$, analysing its symmetries and anomalies, as well as the N-fold degenerate vacuum structure and the peculiar field configuration, called domain wall, connecting these minima.

Supersymmetry is a spacetime symmetry, which, roughly speaking, *maps fermions into bosons and viceversa.* From an algebraic point of view, (rigid) supersymmetry is realized by a certain number of spinorial supercharges $\boldsymbol{Q}^i$: in flat space, these *supercharges* have trivial commutators with the momentum P_m,

$$[P_m, \boldsymbol{Q}^i] = 0\,,$$

but *not* with the generators of the Lorentz group M_{mn}, as one should expect from the fact that $\boldsymbol{Q}^i$ is a spinor,

$$[M_{mn}, \boldsymbol{Q}^i] \neq 0\,.$$

These facts have important phenomenological implications: any Standard Model particle should have a *superpartner*, i.e. a companion particle with the same quantum numbers as well as the same mass (as long as supersymmetry is not broken) but different spin, and thus in supersymmetry one better speaks of *superparticles* rather than particles, when referring to representations of the supersymmetry algebra on states. Actually, the latter sentence is not completely correct as it stands: in **chapter 1** we will render it more precise, by analysing the general structure of the supersymmetry algebra and of some of its standard representations.

Fundamental Theories of which supersymmetry could provide the completion are formulated in terms of fields, and therefore we should reformulate ordinary field theories in a language which is suited also for supersymmetric objects. We will see how this is possible in **chapter 2**, where the concept of *superspace* and *superfields* for $d = 4$, $\mathcal{N} = 1$ supersymmetry will be introduced in a pragmatic way. We will then study how a supersymmetry transformation is realised in terms of differential operators in superspace, and we will then put this formalism to work by studying how supersymmetric invariant lagrangians can be built. Starting from the simplest matter model, we will arrive at an $\mathcal{N} = 1$ SYM lagrangian.

In **chapter 3** we will momentarily put aside supersymmetry to introduce an important aspect of field theories: solitons. These are peculiar field configurations which arise as solutions of those field theories having non-linear equations of motion. In particular, static, i.e. time independent solutions will be our main concern. We will start by giving a possible definition of solitons in terms of *localised* energy density, which is particularly useful in field theory. We will provide some basic characterisation of soliton solutions, and we will analyse two examples of static soliton solutions in one spatial dimension: the $\mathbb{Z}_2$-*kink* in the $\phi^4_{d=2}$ theory, and the *sine-Gordon kink*. We will also illustrate the *Bogomol'nyi-Prasad-Sommerfield (BPS) method*, which is an alternative procedure by means of which we can derive the equation of motion for the soliton configuration. Moreover, the BPS method reveals that the energy $H[\phi]$ of the field model in which the soliton originates is subject to the condition

$$H[\phi] \geq C\,,$$

where C is a conserved quantity called *topological charge*: it is a peculiar conserved quantity, in that it is in general not related to any symmetry. The previous inequality, called *BPS bound*, is saturated by solitons, i.e. soliton solutions minimise the energy, and this minimal value equals the topological charge.

Then, we will consider the generalization of the kink solution from one to three spatial dimensions, studying the so-called *domain wall* configuration. In order to set the stage, we will begin with the analysis of the $\mathbb{Z}_2$-wall in the $\phi^4_{d=4}$ theory. After that, we will consider domain walls in $\mathcal{N} = 1$ supersymmetric Wess-Zumino model. There are two aspects which make topological solutions in supersymmetric field theories really special. First, they produce a modification of the supersymmetry algebra called *central extension*, even in the cases like $\mathcal{N} = 1$ supersymmtery in which central charges are *a priori* forbidded by group theoretical arguments; second, they preserve only half of the supersymmetry, that is, using the BPS equation of motion it is possible to define the parameters of an infinitesimal supersymmetry transformations in such a way that the transformation itself acts trivially on the wall. Supersymmetric solitons are called *critical* or 1/2-*BPS saturated* for this reason.

On the other hand, solitons are dynamical objects, and thus they should be described by some effective action. We will see that, in general, the action for p-dimensional topological defects coincides with the action describing a p-brane, a p-dimensional extended object which naturally arises in string theory. Both the bosonic and the supersymmetric case will be considered. In the supersymmetric case, it will turn out that the characteristic of preserving half of the supersymmetry limits the fermionic degrees of freedom of the membrane.

Finally, in **chapter 4**, we will put together what we have learned in the previous chapters to analyse in detail the $\mathcal{N} = 1$ SYM theory in $d = 4$, its anomalous symmetries wich lead ultimately to the formation of the gluino condensate $\langle\lambda\lambda\rangle$ and the formation of BPS configurations connecting the N vacua of the theory.

We will begin with the study of the R-symmetry and its anomaly, providing an explicit computation of the anomaly function. Actually, the theory has also an anomalous scale invariance. All the anomalous currents can be gathered to form a chiral supermultiplet, to be called $\mathcal{S}$. The dynamics of this supermultiplet, in turn, is described by the renown Veneziano-Yankielowicz (VY) effective action, whose construction based on purely symmetric grounds will be reviewed as well. The superfield $\mathcal{S}$ is special: its θ^2-component contains the instanton density term $\operatorname{tr} F_2 \wedge \widetilde{F}_2$. Thus, it is not an auxiliary field in the strictest sense, and it cannot be integrated out in the standard manner. Moreover, the VY potential is not single valued, as an identical transformation of the field $\mathcal{S}(x,\theta) \mapsto \mathcal{S}'(x, e^{\mathrm{i}\pi}\theta) = e^{2\pi\mathrm{i}}\mathcal{S}(x,\theta)$ shifts the potential by a term proportional to $\mathcal{S}$ itself. By augmenting the VY in an appropriate way, we will see how both these problems can be overcome. Thanks to the additional term, we can integrate out the θ^2-component, finding the effective scalar potential which reproduces the value of the gluino condensate. After that, we will consider the coupling of a dynamical membrane to the VY model. The presence of such membrane modifies the equations of motion of the auxiliary fields of the $\mathcal{S}$ superfield, leading to the formation of BPS-saturating domain wall configurations. We will then compute the tension of the system constituted by the wall and the membrane, showing that the presence of the membrane solves the mismatch between the tension of the BPS saturated domain wall configuration and the tension that one estimates by means of the scalar potential of the VY effective theory.

Finally, we discuss the consequences of the introduction of a new term in the VY lagrangian. This new contribution that we consider produces a mass term for the CP-odd glueball field C_m, which becomes a full-fledged dynamical field, and introduces a new dynamical scalar field which is dual to the CP-even glueball $F_{mn}F^{mn}$. In particular, the consequences of the presence of the new term on the degeneracy of the SYM vacua are analysed.

In conclusion, we mention some possible developments of the work presented in this thesis.

Chapter 1

Rigid Supersymmetry

This section is devoted to the introduction and the discussion of basic notions and properties of global supersymmetry and of some of its representations.

1.1 The Supersymmetry Algebra

The renown Coleman-Mandula theorem [15] states that, under a number of physically reasonable assumptions as locality, causality etc., the most general symmetry group that the S-matrix can enjoy is

$$G = ISO^{+}(1,3) \times L\,,$$

that is, the product of the Poincaré group generated by M_{mn} and P_m, and an internal symmetry group given by a Lie group whose generators are bosonic, Lorentz scalar hermitian operators B_a. The full symmetry algebra reads

$$[P_m, P_n] = 0\,, \tag{1.1a}$$

$$[M_{mn}, M_{pq}] = \mathrm{i}\eta_{mq}M_{np} + \mathrm{i}\eta_{np}M_{mq} - \mathrm{i}\eta_{mp}M_{nq} - \mathrm{i}\eta_{nq}M_{mp}\,, \tag{1.1b}$$

$$[M_{mn}, P_q] = \mathrm{i}\eta_{nq}P_m - \mathrm{i}\eta_{mq}P_n\,, \tag{1.1c}$$

$$[B_a, B_b] = \mathrm{i}f_{ab}{}^{c}B_c\,, \tag{1.1d}$$

$$[M_{mn}, B_a] = 0, \tag{1.1e}$$

$$[P_m, B_a] = 0\,. \tag{1.1f}$$

One can try to evade this *no-go theorem* by relaxing one – or more – of its assumptions. In particular, if one does not want to give up the physical assumptions, one may try to enlarge the allowed symmetries by modifying the algebraic structure. However, this seems to be unsuccessful, since the Coleman-Mandula theorem forbids non-trivial extensions of the Lorentz group by ordinary Lie algebras. On the other hand, one can notice that the theorem only deals with *commutators*: in fact, Haag, Lopuszański and Sohnius showed in [24] that the only possible consistent generalisation of the Lorentz algebra is that of a *graded* Lie algebra, i.e. an algebraic structure that allows for fermionic generators and anticommutators along with commutators and bosonic generators. More precisely:

Definition 1. A *graded Lie algebra* of grade n is a vector space

$$\mathfrak{G} = \bigoplus_{i=0}^{n} \mathfrak{G}_i \,,$$

where any of the $\mathfrak{G}_i$ is a vector space, and which is endowed with the product

$$\big[\cdot,\cdot\big\} : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$$

satisfying the following properties:

$$\begin{aligned}
&[\mathfrak{g}_i, \mathfrak{g}_j\} \in \mathfrak{G}_{i+j} \mod (n+1)\,, && \text{(grading)}\\
&[\mathfrak{g}_i, \mathfrak{g}_j\} = -(-1)^{ij}[\mathfrak{g}_j, \mathfrak{g}_i\}\,, && \text{(supersymmetrization)}\\
&[\mathfrak{g}_i, [\mathfrak{g}_j, \mathfrak{g}_k\}\}(-1)^{ik} + [\mathfrak{g}_j, [\mathfrak{g}_k, \mathfrak{g}_i\}\}(-1)^{ji} + [\mathfrak{g}_k, [\mathfrak{g}_i, \mathfrak{g}_j\}\}(-1)^{kj} = 0\,.
\end{aligned}$$

From the grading property, one can notice that only $\mathfrak{G}_0$ is a Lie algebra. The third relation is just the generalization of the usual Jacobi identity. The supersymmetry algebra is a particular graded Lie algebra:

Definition 2. The *supersymmetry algebra* is a graded Lie algebra of grade one, namely

$$\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1 \,,$$

where $\mathfrak{G}_0$ is the Poincaré algebra and $\mathfrak{G}_1 = \mathrm{span}\{Q^I_\alpha, \bar{Q}^I_{\dot\alpha}\}$ with $I = 1, 2, \ldots, \mathcal{N}$, where $(Q^I_\alpha, \bar{Q}^I_{\dot\alpha})$ is a set of $\mathcal{N} + \mathcal{N}$ anticommuting fermionic generators transforming in the representations $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ of the Lorentz group, respectively.

Besides the commutators in (1.1), the supersymmetry algebra contains also the following (anti-)commutators:

$$\big[P_m, Q^I_\alpha\big] = 0\,, \tag{1.2a}$$
$$\big[P_m, \bar{Q}^I_{\dot\alpha}\big] = 0\,, \tag{1.2b}$$
$$\big[M_{mn}, Q^I_\alpha\big] = \mathrm{i}(\sigma_{mn})_\alpha{}^\beta\, Q^I_\beta\,, \tag{1.2c}$$
$$\big[M_{mn}, \bar{Q}^{I\dot\alpha}\big] = \mathrm{i}(\bar\sigma_{mn})^{\dot\alpha}{}_{\dot\beta}\, \bar{Q}^{I\dot\beta}\,, \tag{1.2d}$$
$$\big\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\big\} = 2\,\sigma^m_{\alpha\dot\beta} P_m\, \delta^{IJ}\,, \tag{1.2e}$$
$$\big\{Q^I_\alpha, Q^J_\beta\big\} = \epsilon_{\alpha\beta}\, Z^{IJ}\,, \qquad Z^{IJ} = -Z^{JI}\,, \tag{1.2f}$$
$$\big\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\big\} = \epsilon_{\dot\alpha\dot\beta}\big(Z^{IJ}\big)^*\,. \tag{1.2g}$$

The objects denoted by Z^{IJ} are called *central charges*; they are Lorentz scalars, and should be a linear combination of the internal symmetry generators. With respect to the relations in (1.2), several remarks are in order [37].

- Equations (1.2c) and (1.2d) express the spinorial nature of the supersymmetry generators. Indeed $\mathrm{i}\sigma^{mn}$ and $\mathrm{i}\bar{\sigma}^{mn}$ are the Lorentz generators of the $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ representation, respectively. Furthermore, from these relations one can understand why it is said that supersymmetry transforms bosons into fermions and viceversa. Indeed, from the explicit expression of, for instance, $i\sigma_{mn}$, and recalling that $M_{ij}=\epsilon_{ijk}J_k$, it follows that

$$\begin{aligned}\left[M_{12},Q_1^I\right] &= \left[J_3,Q_1^I\right] = \mathrm{i}(\sigma_{12})_1{}^{\beta}Q^I_\beta = \frac{\mathrm{i}}{4}\Big[(\sigma_1)_{1\dot\gamma}(\bar\sigma_2)^{\dot\gamma\beta}-(\sigma_2)_{1\dot\gamma}(\bar\sigma)^{\dot\gamma\beta}\Big]Q^I_\beta \\ &= \frac{\mathrm{i}}{4}\Big[(\sigma_1)_{1\dot 2}(\bar\sigma_2)^{\dot 2 1}-(\sigma_2)_{1\dot 2}(\bar\sigma)^{\dot 2 1}\Big]Q^I_1 = \frac{1}{2}Q^I_1\,.\end{aligned}$$

  Proceeding along the same lines, we get

$$\left[J_3,Q^I_2\right]=-\frac{1}{2}Q^I_2 \quad \left[J_3,\bar{Q}^I_{\dot 1}\right]=-\frac{1}{2}\bar{Q}^I_{\dot 1} \quad \left[J_3,\bar{Q}^I_{\dot 2}\right]=\frac{1}{2}\bar{Q}^I_{\dot 2}\,,$$

  hence Q^I_1 and $\bar{Q}^I_{\dot 2}$ rise the z-component of the spin by one half unit, while $\bar{Q}^I_{\dot 1}$ and Q^I_2 lower it by half a unit.

- As far as equation (1.2e) is concerned, we notice that, given the transformation properties of Q^I_α and $\bar{Q}^I_{\dot\alpha}$ under the Lorentz group, their commutator has to be symmetric in the exchange $I\leftrightarrow J$ and should transform as a Lorentz vector, since

$$\left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right)=\left(\frac{1}{2},\frac{1}{2}\right)\,.$$

  The only object in the algebra with these transformation properties is indeed the momentum P_m, while the δ^{IJ} factor comes from an appropriate normalization of the supercharges Q^I_α and $\bar{Q}^I_{\dot\alpha}$.

- Equations (1.2a) and (1.2b) are very interesting. Compatibility with Lorentz symmetry demands the right hand side of, for instance, (1.2a) to transform as

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(0,\frac{1}{2}\right)=\left(0,\frac{1}{2}\right)\oplus\left(1,\frac{1}{2}\right)\,,$$

  and analogously for the hermitian conjugate relation. Therefore, discarding $\left(1,\frac{1}{2}\right)$ which is forbidden by Haag-Lopuszański-Sohnius theorem, the most general form one should expect would be

$$\left[P_m,Q^I_\alpha\right]=C^I{}_J(\sigma_{mn})_{\alpha\dot\alpha}\bar{Q}^{J\dot\alpha}\,,\quad \left[P_m,\bar{Q}^I_{\dot\alpha}\right]=(C^I{}_J)^*(\bar\sigma_{mn})_{\dot\alpha\alpha}Q^{J\alpha}\,,$$

  for $C^I{}_J$ undetermined matrix. Actually, it can be shown that this matrix vanishes: indeed, from the Jacobi identity applied to (Q,P,P) one gets $CC^*=0$. Nevertheless, this is still not enough to state that $C=0$, and we have to rely on equations (1.2f) and (1.2g).

- From representation theory, one expects for (1.2f) that

$$\left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right)=(0,0)\oplus(1,0)\,, \tag{1.3}$$

explicitly

$$\left\{Q^I_\alpha,Q^J_\beta\right\}=\epsilon_{\alpha\beta}Z^{IJ}+\epsilon_{\beta\gamma}(\sigma^{mn})_\alpha{}^\gamma\,M_{mn}Y^{IJ}\,, \tag{1.4}$$

for $Z^{IJ}=-Z^{JI}$ and $Y^{IJ}=Y^{JI}$. We can understand why the anticommutator takes that form by noting that:

  i. formally, in the left hand side the pairs of indices (α,I) and (β,J) are symmetric under the swapping, hence the anticommutator corresponds to a linear combination of terms where either α,β and I,J are both antisymmetric (hence the first term), or they are both symmetric (hence the second term);
  ii. as demanded by (1.3), the first term in the right hand side of (1.4) is invariant, being $\epsilon_{\alpha\beta}$ an invariant tensor and Z^{IJ} built as a linear combination of internal symmetry generators. The second term has to be a self-dual skew-symmetric tensor, and it can be constructed by means of σ_{mn}, which is self-dual, and M_{mn}, which is skew-symmetric, the whole combination being self-dual and antisymmetric.

  Nevertheless, M_{mn} does not commute with the four-momentum, whereas $\{Q,Q\}$ does, due to (1.2a) and (1.2b), hence the second term in (1.4) must vanish. The same holds for the conjugate relation. On the other hand, this commutation rule, together with the $\alpha\leftrightarrow\beta$ antisymmetric part of the generalized Jacobi identity for the system $[\{Q,Q\},P]$, implies that the $C^I{}_J$ matrices we encountered before are symmetric, hence $CC^*=CC^\dagger$ and this leads to $C=0$.

In general, Q and $\bar{Q}$ carry also a representation of the internal symmetry group G, so one expects that

$$\left[Q^I_\alpha,B_a\right]=(b_a)^I{}_J\,Q^J_\alpha,\quad\left[\bar{Q}_{I\dot\alpha},B_a\right]=-\bar{Q}_{J\dot\alpha}\,(b_a)^J{}_I\,,$$

with the second term obtained from the first one with hermitian conjugation, and assuming G to be compact so that we have unitary representations. The largest internal symmetry group which can act non-trivially on Q is called R-symmetry, and in the most general case it is $U(N)$. It has been said already that the central charges are Lorentz scalars which are linear combinations of the internal symmetry generators such as

$$Z^{IJ}=b^{IJ}_a B_a\,.$$

The name *central charge* stems from the fact that

$$\left[Z^{IJ},\text{any generator}\right]=0\,,$$

and together with the fact that

$$\left[Z^{IJ},Z^{KL}\right]=0\,,$$

we see that they form an invariant abelian subalgebra of the internal symmetry group. Furthermore, for $\mathcal{N} = 1$, $Z^{IJ} = 0$ given that the central charges are antisymmetric; for $\mathcal{N} > 1$ instead, the central charges do not necessarily vanish: for instance, massive supersymmetry representations are very differently realized depending on the case the central charges are trivially realized or not.

The $\mathcal{N} = 1$ supersymmetry is called simple or unextended: in this case, the only non-trivially acting internal symmetry is $U(1)$, which is generated by the R-charge in such a way that

$$[R, Q_\alpha] = -Q_\alpha, \quad \left[R, \bar{Q}_{\dot{\alpha}}\right] = +\bar{Q}_{\dot{\alpha}}\,.$$

This means that super-partners have different R-charge: in particular, these relations imply that if a particle has null R-charge, its super-partners have $R = \pm 1$.

1.2 Representations of the Supersymmetry Algebra

Before entering into the details, we can make some comments concerning general properties of representations of supersymmetry algebra on states.

First of all, since Poincaré algebra is a subalgebra of the full supersymmetry, irreducible representations of the supersymmetry algebra are also representations of the Poincaré algebra, though usually reducible. Consequently, a *super-particle* is a collection of particles which are related to each other by the generators Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$, and whose spins differ by one half unit. In fact, this is due to the fact that supersymmetry generators do not commute with M_{mn}, hence the square of the Pauli-Lubanski vector is no longer a Casimir operator, or, in other words, spin is not a good quantum number in the supersymmetry framework. Nevertheless, P^2 is still a Casimir, and therefore particles in the same supermultiplet have the same mass, but different spin. Such mass degeneracy has not been observed yet, hence, if supersymmetry is relized in Nature, it must be broken at a relatively high energy scale.

Secondly, one can show that the energy of any supersymmetric state is greater than or equal to zero. Let $|\phi\rangle$ be a generic state: from $\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\right\} = 2\,\sigma^m_{\alpha\dot{\alpha}} P_m \delta^{IJ}$ we have

$$\begin{aligned} 2\,\sigma^m_{\alpha\dot{\alpha}} \langle\phi| P_m |\phi\rangle &= \langle\phi| \left\{Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\right\} |\phi\rangle = \langle\phi| Q^I_\alpha (Q^I_\alpha)^\dagger + (Q^I_\alpha)^\dagger Q^I_\alpha |\phi\rangle \\ &= \|(Q^I_\alpha)^\dagger |\phi\rangle\|^2 + \|Q^I_\alpha |\phi\rangle\|^2 \geq 0\,, \end{aligned}$$

exploiting $(Q^I_\alpha)^\dagger = \bar{Q}^I_{\dot{\alpha}}$ and the positivity of the norm in the Hilbert space. After summing over $\alpha, \dot{\alpha} = 1, 2$, and remembering the identity $\operatorname{tr}\sigma^m = 2\delta^{m0}$, one gets

$$\langle\phi| P_0 |\phi\rangle \geq 0\,.$$

The final remark concerns the fact that any supermultiplet contains an equal number of bosonic and fermionic degrees of freedom. In order to prove that, let us introduce a fermion number operator

$$(-1)^{N_F} \quad | \quad (-1)^{N_F} |F\rangle = -|F\rangle\,,\ (-1)^{N_F} |B\rangle = |B\rangle\,.$$

What we need to show is that $\mathrm{tr}[(-1)^{N_F}] = 0$. To this end, we notice the relation between the supersymmetry generators and the fermion number operator

$$(-1)^{N_F} Q^I_\alpha = -Q^I_\alpha (-1)^{N_F} .$$

Therefore

$$\begin{aligned} 0 &= \mathrm{tr}\big[(-1)^{N_F} Q^I_\alpha \bar{Q}^J_{\dot\beta} + Q^I_\alpha (-1)^{N_F} \bar{Q}^J_{\dot\beta}\big] = \mathrm{tr}\big[(-1)^{N_F} \big\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\big\}\big] \\ &= 2\,\sigma^m_{\alpha\dot\alpha} \mathrm{tr}\big[(-1)^{N_F}\big] P_m \delta^{IJ} , \end{aligned}$$

and summing over I, J, for any $P_m \neq 0$ it follows that

$$\mathrm{tr}\big[(-1)^{N_F}\big] = 0 \qquad \Longrightarrow \qquad N_B = N_F .$$

Given that the mass is conserved within each supermultiplet, it is useful to distinguish massless representations and massive representations. According to this distinction, we will consider in detail some of the possibilities.

1.2.1 Massless Supermultiplets

In this case, all central charges vanish, hence from supersymmetry algebra we have

$$\big\{Q^I_\alpha, Q^J_\beta\big\} = 0 , \quad \big\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\big\} = 0 .$$

The construction of massless irreducible representations can be carried out with the following steps.

i. Since $P^2 = 0$, one can choose the frame where $P_m = (\omega, 0, 0, \omega)$, so that

$$\sigma^m P_m = \mathbb{1} P_0 - \sigma_3 P_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ,$$

and therefore

$$\big\{Q^I_\alpha, Q^J_{\dot\alpha}\big\} = 2\,\sigma^m_{\alpha\dot\alpha} P_m \delta^{IJ} = 4\omega \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot\alpha} \delta^{IJ} .$$

In particular, $\big\{Q^I_1, Q^J_{\dot 1}\big\} = 0$, and the positivity of the norm implies that these components annihilate particle states, namely

$$0 = \langle\phi|\, \big\{Q^I_1, Q^J_{\dot 1}\big\}\, |\phi\rangle = \| (Q^J_1)^\dagger\, |\phi\rangle\, \|^2 + \| Q^I_1\, |\phi\rangle\, \|^2 \qquad \Longrightarrow \qquad Q^I_1 = 0 = \bar{Q}^J_{\dot 1} .$$

ii. We are left with only half of the generators being non-trivial. They are used to define creation and annihilation operator as

$$a_I = \frac{1}{2\sqrt{\omega}} Q^I_2 , \quad a^\dagger_J = \frac{1}{2\sqrt{\omega}} \bar{Q}^J_{\dot 2} ,$$

which satisfy the harmonic oscillator algebra

$$\{a_I, a^\dagger_J\} = \delta_{IJ} , \quad \{a_I, a_J\} = 0 = \{a^\dagger_I, a^\dagger_J\} .$$

It is useful to recall that Q^I_2 lowers the helicity by one half unit, while $\bar{Q}^J_{\dot 2}$ rises it by half a unit.

iii. In order to build a representation, we have to choose a *vacuum state*, i.e. a state which *is destroied by all* a_I. This state carries a representation of the Poincaré group, hence, besides having mass $\mathsf{m} = 0$, it has helicity λ_0. This peculiar state is referred to as *Clifford vacuum*, and it is denoted by its helicity as $|\lambda_0\rangle$. This is not the vacuum of the theory, namely the stable configuration with minimum energy, but just a state with quantum numbers ω and λ_0, and such that

$$a_I\,|\lambda_0\rangle = 0\,, \quad \forall\, I = 1,\ldots,\mathcal{N}\,.$$

iv. We can now generate all the components of a supermultiplet by repeated applications of the operators $a_I^\dagger$, until the top state is reached:

$$\left\{|\lambda_0\rangle\,,\ a_I^\dagger\,|\lambda_0\rangle = |\lambda_0+\tfrac{1}{2}\rangle_I\,,\ \ldots,\ a_{I_1}^\dagger\cdots a_{I_\mathcal{N}}^\dagger\,|\lambda_0\rangle = |\lambda_0+\tfrac{\mathcal{N}}{2}\rangle_{I_1,\ldots,I_\mathcal{N}}\right\}.$$

Due to antisymmetry in the indices $I, J, \ldots$ there are $\binom{\mathcal{N}}{k}$ states with helicity equal to $\lambda = \lambda_0 + k/2$ with $k = 0, 1, \ldots, \mathcal{N}$, therefore

λ_0	$\lambda_0+\frac{1}{2}$	λ_0+1	$\cdots$	$\lambda_0+\dfrac{\mathcal{N}}{2}$
$\binom{\mathcal{N}}{0}=1$	$\binom{\mathcal{N}}{1}=1$	$\binom{\mathcal{N}}{2}=\dfrac{\mathcal{N}(\mathcal{N}-1)}{2}$	$\cdots$	$\binom{\mathcal{N}}{\mathcal{N}}=1$

Moreover, given that the helicity ranges from λ_0 to $\lambda_0+\mathcal{N}/2$, the total number of states will be

$$\sum_{k=0}^{\mathcal{N}}\binom{\mathcal{N}}{k} = 2^\mathcal{N} = \left(2^{\mathcal{N}-1}\right)_B + \left(2^{\mathcal{N}-1}\right)_F\,.$$

Usually, spectra of states derived from a Lorentz-covariant field theory exhibit PCT-symmetry, hence for any state with helicity λ there should be a P-reflected state with helicity $-\lambda$. However, supersymmetric spectra in general do not have this property. It follows that a given supermultiplet can be contained in a Lorentz-covariant field theory only jointly with its PCT-conjugate multiplet. A general criterium to thumb – with care, being not always satisfied – is that a supermultiplet is self-conjugate if $\lambda_0 = \mathcal{N}/4$.

$\mathcal{N} = 1$ supersymmetry. For unextended supersymmetry each massless supermultiplet contains two states, namely $|\lambda_0\rangle$ and $|\lambda_0+\frac{1}{2}\rangle$. The physically interesting cases are:

- *Matter* or *chiral multiplet*

$$\lambda_0 = 0 \quad\to\quad \left(0,\tfrac{1}{2}\right) \underset{PCT}{\oplus} \left(-\tfrac{1}{2},0\right)\,,$$

  where the degrees of freedom are those of a complex scalar and a Weyl fermion. It is where matter sits for $\mathcal{N} = 1$ supersymmetry.

- *Gauge* or *vector multiplet*

$$\lambda_0 = \frac{1}{2} \quad \rightarrow \quad \left(\frac{1}{2}, 1\right) \underset{PCT}{\oplus} \left(-1, -\frac{1}{2}\right) ,$$

  where the degrees of freedom are those of a vector field and a Weyl fermion, exactly those we need to build a supersymmetric gauge theory. Since supersymmetry generators commute with internal symmetry generators (but those of the R-symmetry), both vector field and Weyl fermion belong to the same representation of the gauge group, which is to say the adjoint representation.

- *Spin-$\frac{3}{2}$ multiplet*

$$\lambda_0 = 1 \quad \rightarrow \quad \left(1, \frac{3}{2}\right) \underset{PCT}{\oplus} \left(-\frac{3}{2}, -1\right) ,$$

  where the degrees of freedom are those of a spin-$\frac{3}{2}$ fermion and a vector boson.

- *Graviton multiplet*

$$\lambda_0 = \frac{3}{2} \quad \rightarrow \quad \left(\frac{3}{2}, 2\right) \underset{PCT}{\oplus} \left(-2, -\frac{3}{2}\right) ,$$

  whose degrees of freedom are the graviton and the gravitino.

$\mathcal{N} = 2$ supersymmetry. In the case at hand, any supermultiplet contains states with helicity $\left(\lambda_0,\, \lambda_0 + \frac{1}{2},\, \lambda_0 + \frac{1}{2},\, \lambda_0 + 1\right)$. The physically interesting cases are:

- *Vector multiplet*

$$\lambda_0 = 0 \quad \rightarrow \quad \left(0, \frac{1}{2}, \frac{1}{2}, 1\right) \underset{PCT}{\oplus} \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right) ,$$

  where the degrees of freedom are those of a complex scalar, two Weyl fermions and one vector boson.

- *Hypermultiplet*

$$\lambda_0 = -\frac{1}{2} \quad \rightarrow \quad \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) \underset{PCT}{\oplus} \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) ,$$

  the degrees of freedom being those of two complex scalars and two Weyl fermions. Though this representation fulfills the self-conjugation condition, it is actually not self-conjugate. Indeed, the way in which the states are constructed from the vacuum shows that the fermions are singlets under the $SU(2)$ R-symmetry group, while the scalars are doublets. The representation would be PCT-self conjugate if the two scalars were real; but this is not possible, because the two-dimensional representations of $SU(2)$ are pseudoreal, hence the two scalars have to be complex.

- *Gravitino multiplet*

$$\lambda_0 = -\frac{3}{2} \quad \rightarrow \quad \left(-\frac{3}{2}, -1, -1, -\frac{1}{2}\right) \underset{PCT}{\oplus} \left(\frac{1}{2}, 1, 1, \frac{3}{2}\right) ,$$

  containing one Weyl fermion, one gravitino and two vectors;

- *Graviton multiplet*

$$\lambda_0 = -2 \quad \rightarrow \quad \left(-2, -\frac{3}{2}, -\frac{3}{2}, -1\right) \underset{PCT}{\oplus} \left(1, \frac{3}{2}, \frac{3}{2}, 2\right) ,$$

whose degrees of freedom are those of one vector (often referred to as graviphoton), two gravitinos and one graviton.

1.2.2 Massive Supermultiplets

The logical step to build massive representations are similar to those needed for massless representations; nevertheless, now the number of supersymmetry generators does not diminish. Since $P_m P^m = \mathsf{m}^2 \neq 0$ we can choose a frame where $P_m = (\mathsf{m}, \mathbf{0})$, hence

$$\sigma^m P_m = \begin{pmatrix} \mathsf{m} & 0 \\ 0 & \mathsf{m} \end{pmatrix} \;\Rightarrow\; \{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\mathsf{m}\, \delta_{\dot{\alpha}\dot{\alpha}} \delta^{IJ} ,$$

and no generator is trivially realized on the states. Moreover, being the states massive, we better speak of spin rather than helicity. A given Clifford vacuum will thus be defined by mass m and spin j, and will have itself degeneracy $2j+1$, since j_z ranges from $-j$ to $+j$. Focusing on $\mathcal{N}=1$ supersymmetry, the creation and annihilation operators are

$$a_{1,2} = \frac{1}{\sqrt{2\mathsf{m}}} Q_{1,2} , \quad a^\dagger_{1,2} = \frac{1}{\sqrt{2\mathsf{m}}} \bar{Q}_{\dot{1},\dot{2}} ,$$

with a_1 and $a_2^\dagger$ rising the spin by half a unit, $a_1^\dagger$ and a_2 lowering the spin by half a unit. By defintion, $a_{1,2}\, |j\rangle = 0$, the other states being obtained applying $a^\dagger_{1,2}$. Some interesting cases are:

- *Matter multiplet*

$$j = 0 \quad \rightarrow \quad \left(-\frac{1}{2}, 0, 0', \frac{1}{2}\right) ,$$

and we see that the number of degrees of freedom is the same as that of $\mathcal{N}=1$ massless matter multiplet. The scalar dubbed with a prime has opposite parity with respect to the other state 0, and it is therefore a pseudoscalar.

- *Vector multiplet*

$$j = \frac{1}{2} \quad \rightarrow \quad \left(-1, \mathbf{2} \times -\frac{1}{2}, \mathbf{2} \times 0, \mathbf{2} \times \frac{1}{2}, 1\right) ,$$

where the degrees of freedom are those of one massive vector, one massive real scalar and one massive Dirac fermion, which are the same as those of a massless vector multiplet and a massless matter multiplet.

1.3 Representation on Fields: the Chiral Multiplet

We have just discussed supersymmetry representations on states. However, in what follows we will be interested in supersymmetric field theories, thus we have to understand how supersymmetry representations on fields can be constructed. The general strategy resembles that used previously to construct supermultiplets: again, it is a systematic procedure, hence it will be illustrated in one specific case, namely that of the chiral multiplet for $\mathcal{N} = 1$.

Let us consider a generic scalar field $\phi(x)$, that will serve us as a ground state for the representation, analogously to what we did previously starting from the Clifford vacuum. First, one imposes the constraint

$$\left[\bar{Q}_{\dot{\alpha}}, \phi(x)\right] = 0\,.$$

One can notice that $\phi(x)$ has to be complex. Indeed, if it were a real field, taking the hermitian conjugate of the above identity we would obtain $[Q, \phi(x)] = 0$, and the generalized Jacobi identity for $(\phi(x), Q, \bar{Q})$ would reduce to

$$\left[\phi(x), \left\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\right\}\right] = 2\,\sigma^m_{\alpha\dot{\alpha}}\,[\phi(x), P_m] = 0 \quad \rightsquigarrow \quad [\phi(x), P_m] = \mathrm{i}\partial_m\phi(x) = 0\,,$$

which means that the field $\phi(x)$ is actually a constant. Hence $\phi(x)$ has to be a complex scalar field. Secondly, one defines

$$[Q_\alpha, \phi(x)] \equiv \psi_\alpha(x)\,,$$

which is a new field belonging to the same representation. Then, the next step consists in acting on $\psi_\alpha(x)$ with the supersymmetry generators, so that

$$\{Q_\alpha, \psi_\beta(x)\} \equiv F_{\alpha\beta}(x)\,, \quad \left\{\bar{Q}_{\dot{\alpha}}, \psi_\beta\right\} \equiv X_{\beta\dot{\alpha}}(x)\,.$$

We have to ensure that these quantities are genuine new fields, i.e. that they depend neither on $\phi(x)$ or $\psi_\alpha(x)$, nor on their derivatives. From the generalized Jacobi identity for $(\phi(x), Q, \bar{Q})$ we find

$$\left\{\bar{Q}_{\dot{\alpha}}, \psi_\alpha(x)\right\} + 2\,\sigma^m_{\dot{\alpha}\alpha}\,[\phi(x), P_m] = 0\,,$$

hence

$$X_{\alpha\dot{\alpha}}(x) + 2\,\mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\,\partial_m\phi(x) = 0 \quad \rightsquigarrow \quad X_{\alpha\dot{\alpha}}(x) = -2\,\mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\,\partial_m\phi(x)\,.$$

On the other hand, applying the Jacobi identity to $(\phi(x), Q, Q)$ one gets

$$\{Q_\alpha, \psi_\beta(x)\} + \{Q_\beta, \psi_\alpha(x)\} = 0\,,$$

and therefore

$$F_{\alpha\beta}(x) + F_{\beta\alpha}(x) = 0 \quad \rightsquigarrow \quad F_{\beta\alpha}(x) = -F_{\alpha\beta}(x) \;\leftrightarrow\; F_{\alpha\beta}(x) = \epsilon_{\alpha\beta}F(x)\,.$$

Again, we can define

$$[Q_\alpha, F(x)] \equiv \lambda_\alpha\,, \quad \left[\bar{Q}_{\dot{\alpha}}, F(x)\right] \equiv \overline{\chi}_{\dot{\alpha}}\,,$$

and we have to check that these are new fields. In fact, form the Jacobi identity applied to $(\psi(x), Q, Q)$ and $(\psi(x), Q, \bar{Q})$, it turns out that λ_α vanishes and that $\bar{\chi}_{\dot{\alpha}}$ is proportional to a derivative of ψ_α. All in all, we have obtained the field multiplet $\big(\phi(x), \psi_\alpha(x), F(x)\big)$. Given that $\phi(x)$ and $F(x)$ are complex scalars while $\psi_\alpha(x)$ is a Weyl fermion, we have found the field counterpart of the or chiral supermultiplet. Actually, there seems to be a mismatch in the number of degrees of freedom. However, keeping in mind that the representation is *off-shell*, we have:

- four bosonic degrees of freedom, namely $\big(\mathbf{Re}\phi, \mathbf{Im}\phi;\ \mathbf{Re}F, \mathbf{Im}F\big)$;
- four fermionic degrees of freedom, i.e. $\big(\mathbf{Re}\psi_1, \mathbf{Im}\psi_1;\ \mathbf{Re}\psi_2, \mathbf{Im}\psi_2\big)$.

Imposing on-shellness, the degrees of freedom of ψ are reduced by two thanks to Dirac equation; on the other hand, Klein-Gordon equations for ϕ and F do not reduce the number of propagating degrees of freedom, hence we are left with $2_F + 4_B$, and the number of degrees of freedom does not match with that of the chiral supermultiplet. However, it turns out that F is not a dynamical field: in practice, this means that, on-shell, it is a function of ϕ and ψ thus its two degrees of freedom do not matter, and we have $2_F + 2_B$ as expected. The appearance of *auxiliary fields* is not peculiar of the chiral multiplet, rather it is a common feature in supersymmetric field theories.

By means of the general procedure we outlined here, it is possible to build multiplets with the appropriate field content to construct supersymmetric lagrangians. As usual, the action is $I = \int d^4x\, \mathcal{L}$, and, to see if a given theory is invariant under supersymmetry, we should act on any of its terms with a supersymmetry transformation, with the aim to show that the overall variation sums up to a total spacetime derivative. However, in practice this is really involved because we are using a formulation where supersymmetry is not manifest. As we will see in the next chapter, supersymmetric field theories are naturally formulated in *superspace*.

Chapter 2

Supersymmetric Field Theories

While ordinary field theories are defined in Minkowski space, it turns out that supersymmetric field theories are naturally defined in the so-called *superspace*: loosely speaking, it is an extension of the Minkowski spacetime, obtained by taking into account extra spacetime directions which are associated to supersymmetry generators. In this chapter, we will introduce the concept of superspace and superfield, and we will discuss how to construct lagrangians for supersymmetric field theories.

2.1 Superspace for $\mathcal{N} = 1$, $d = 4$ Supersymmetry

Before getting into the details of the superspace construction, let us stop for a moment to introduce a couple of basic definitions that will give a flavor of the underlying mathematical structure.

Coset Manifold The notion of coset manifold is a natural generalization of group manifold [14]. For a group G and $H < G$, the *coset* G/H is the set of equivalence classes of elements $g \in G$, where the equivalence relation is defined by right multiplication with elements $h \in H$,

$$\forall\, g,\, g' \in G \;:\; g \sim g' \;\leftrightarrow\; \exists\, h \in H \mid gh = g' \,.$$

The equivalence classes constituting the elements of G/H are denoted by gH, for g any representative of the class. Coset manifolds arise when G is a Lie group, and H is a Lie subgroup thereof. If this is the case, G/H inherits a manifold structure from G, and moreover, invariant metrics can be constructed in such a way that all $g \in G$ are isometries. The definitions we need are the following.

Definition 3. A (pseudo-)Riemannian manifold $\mathcal{M}$ is said *homogeneous* if it admits as an isometry the *transitive action* of a group G. A group is said to act transitively on $\mathcal{M}$ if any point of the manifold can be reached from any other by means of its action.

Definition 4. The subgroup $H_p < G$ which leaves fixed a point p of a homogeneous space (i.e. $\forall\, h \in H_p \Rightarrow hp = p$) is called *isotropy subgroup* of the point. Being the action of G transitive, any other point $q = gp$ for $g \in G$ and $g \notin H$ has an isotropy subgroup $H_q = gH_pg^{-1} < G$, which is conjugate to H_p, and hence isomorphic to it.

A typical example of homogeneous manifold is a two-dimensional sphere S_2, and the group acting transitively on it is $SO(3)$. Moreover, the north pole $(1, 0, 0)$ is invariant under that transformation of $SO(2) < SO(3)$ that rotates the sphere around the $\hat{z}$-axis.

The isotropy group of a homogeneous manifold is unique up to conjugation. Therefore it sufficies to calculate it for a properly chosen point and then all other follows. Any point p of a homogeneous manifold is naturally labelled by parameters describing the element of G which transports a conventional p_0 to p itself. On the other hand, g belongs to an equivalence class, thus if g carries p_0 to p, any other element of gH does the same, and one is led to characterise the points of a homogeneous space by the coset gH. It follows that a homogeneous manifold can be identified with the coset manifold G/H defined by the transitive group G divided by the isotropy group H. The equivalence classes constituting the points of G/H can be labelled by a set of d coordinates $y = (y^1, \ldots, y^d)$, where $d = \dim(G/H) = \dim G - \dim H$.

Let us apply what we have just seen to the Minkowski space. In this case, the group realising the transitive action is the Poincaré group. Moreover, noting that any Lorentz transformation leaves the origin unchanged, we can understand the $SO^+(1, 3)$ is the isotropy group. Therefore, it follows that Minkowski space is a 4-dimensional coset manifold which can be defined as

$$\mathcal{M}_{1,3} = \frac{ISO^+(1,3)}{SO^+(1,3)}\,.$$

Each coset – or, equivalently, each point in spacetime – has a unique representative which is a translation parametrized by a quadruplet $x^m = (x^0, x^1, x^2, x^3)$ labelling the coordinates of the point.

Superspace for $\mathcal{N} = 1$ rigid supersymmetry can be defined in an analogous way. First, we have to extend the Poincaré group to the *super-Poincaré* group, in such a way to contain the supersymmetry generators Q and $\bar{Q}$. But supersymmetry algebra is not a Lie algebra, and it cannot be exponentiated to produce the corresponding Lie group; in fact, it involves anticommutators. However, we can rewrite the algebra in terms of commutators by introducing a set of constant Grassmann numbers $(\theta_\alpha, \bar{\theta}_{\dot\alpha})$, which anti-commute with any fermionic object and commute with any bosonic object. By this strategy we can transform the anticommutators of the supersymmetry algebra into commutators, getting

$$\left[\theta Q, \bar{\theta}\bar{Q}\right] = -2\,\theta\sigma^m\bar{\theta}P_m\,, \quad \left[\theta Q,\, \theta Q\right] = \left[\bar{\theta}\bar{Q}, \bar{\theta}\bar{Q}\right] = 0\,.$$

The exponentiation procedure is now allowed, and a generic element of the super-Poincaré group $\mathsf{G}_{\text{s-P}}$ is given by

$$g(x,\omega,\theta,\bar{\theta}) = \exp\left\{\mathrm{i}\,x_m P^m + \mathrm{i}\,\theta Q + \mathrm{i}\,\bar{\theta}\bar{Q} + \frac{\mathrm{i}}{2}\omega_{mn}M^{mn}\right\}.$$

Given the generic element $g(x,\omega,\theta,\bar{\theta}) \in \mathsf{G}_{\text{s-P}}$, $\mathcal{N}=1$ superspace is defined as the $(4+4)$-dimensional coset

$$\mathcal{M}_{4|1} = \frac{\mathsf{G}_{\text{s-P}}}{SO^+(1,3)}\,.$$

A point in superspace, i.e. an equivalence class in the coset $\mathcal{M}_{4|1}$, is identified with the representative corresponding to a super-translation through the one-to-one map

$$\left(x^m,\theta_\alpha,\bar{\theta}_{\dot{\alpha}}\right) \;\leftrightarrow\; e^{x^m P_m} e^{\theta Q+\bar{\theta}\bar{Q}}\,.$$

2.1.1 Conventions for $\mathcal{N}=1$, $d=4$ Superspace

Along with the four commuting coordinates x^m, the superspace is endowed with anti-commuting coordinates θ_α and $\bar{\theta}_{\dot{\alpha}}$. The latter have peculiar properties which are worth to be recalled, with the aim to set some convention, too. One has to keep in mind that, given their anticommuting nature, $\theta_\alpha\theta_\beta = 0$ when $\alpha=\beta$: this implies that $\theta_\alpha\theta_\beta\theta_\gamma = 0$, because at least two indices are equal. Derivation is defined by

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}\,,\quad \bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial\theta^{\dot{\alpha}}}\,,\quad \partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha}\,,\quad \bar{\partial}^{\dot{\alpha}} \equiv \frac{\partial}{\partial\theta_{\dot{\alpha}}}\,,$$

with the conventions

$$\partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta\,,\quad \bar{\partial}^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\beta}}\,,$$

and

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta\,,\quad \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}}\,,\quad \partial_\alpha\bar{\theta}^{\dot{\beta}} = \bar{\partial}_{\dot{\alpha}}\theta^\beta = 0\,.$$

Let us now consider integration of anticommuting variables, sometimes called *Berezinian* integration. Starting from the case of a single Grassmann variable θ, one defines

$$\int \mathrm{d}\theta = 0\,,\quad \int \mathrm{d}\theta\;\theta = 1\,.$$

This implies that for a function $f(\theta) = f_0 + \theta f_1$ one has

$$\int \mathrm{d}\theta\; f(\theta) = \int \mathrm{d}\theta\;(f_0+\theta f_1) = f_1\,,\quad \int \mathrm{d}\theta\;\delta(\theta) f(\theta) = f_0\,,$$

and therefore Grassmann integration is equivalent to differentiation and $\delta(\theta)=\theta$. Moreover, Berezin integrals are naturally invariant under translation, namely

$$\int \mathrm{d}(\theta+\xi)\; f(\theta+\xi) = \int \mathrm{d}\theta\; f(\theta)\,.$$

As we will see later on, this property will prove important in the construction of supersymmetry invariant actions. The generalization of Berezin integrals to $\mathcal{N} = 1$, $d = 4$ superspace is straightforward. We will stick to the conventions

$$\mathrm{d}^2\theta = -\frac{1}{4}\mathrm{d}\theta^\alpha \mathrm{d}\theta^\beta \epsilon_{\alpha\beta} = \frac{1}{2}\mathrm{d}\theta^1 \mathrm{d}\theta^2\,, \quad \mathrm{d}^2\bar{\theta} = -\frac{1}{4}\mathrm{d}\bar{\theta}_{\dot{\alpha}} \mathrm{d}\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\mathrm{d}\bar{\theta}^{\dot{1}} \mathrm{d}\bar{\theta}^{\dot{2}}\,.$$

Finally, from the above conventions the following identities can be shown to hold:

$$\int \mathrm{d}^2\theta\ \theta\theta = \int \mathrm{d}^2\bar{\theta}\ \bar{\theta}\bar{\theta} = 1\,, \quad \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\ \theta\theta\,\bar{\theta}\bar{\theta} = 1\,,$$

$$\int \mathrm{d}^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta\,, \quad \int \mathrm{d}^2\bar{\theta} = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\partial}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}}\,.$$

2.2 Superfields

Roughly speaking, superfields are just fields in the superspace, i.e. functions of the superspace coordinates $(x^m, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. Given the anticommuting properties of the Grassmann coordinates, a generic superfield $\mathcal{Y}(x, \theta, \bar{\theta})$ can be expanded in terms of the *component fields* as

$$\begin{aligned}\mathcal{Y}(x, \theta, \bar{\theta}) = f(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}^{\dot{\alpha}}(x) + \theta^2\, \bar{m}(x) + \bar{\theta}^2\, n(x) + \theta\sigma^m\bar{\theta}\, A_m(x) \\ + \theta^2\, \bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\, \theta\rho(x) + \theta^2\bar{\theta}^2\, d(x)\,,\end{aligned}$$

where the shorthand notations $\theta^2 \equiv \theta\theta$ and $\bar{\theta}^2 \equiv \bar{\theta}\bar{\theta}$ are used. The component fields are all complex fields. In order to compute the effect of an infinitesimal supersymmetry transformation on a generic superfield, we need a realization of Q_α and $\bar{Q}_{\dot{\alpha}}$ as differential operators on superspace. For sake of clarity, we denote abstract operators by calligraphic letters, and their representations by latin letters. We begin by recalling that in the usual Minkowskian spacetime, translations are generated by the operator $\mathcal{P}_m$. For a generic field $\varphi(x)$, a translation by a constant vector a^m is given by

$$\varphi(x + a) = e^{-\mathrm{i}\, a\mathcal{P}}\varphi(x) e^{\mathrm{i}\, a\mathcal{P}} \approx \varphi(x) - \mathrm{i}a^m\, [\mathcal{P}_m, \varphi(x)]\,.$$

On the other hand, we can expand the translated field to linear order as

$$\varphi(x + a) = \varphi(x) + a^m\partial_m\varphi(x)\,,$$

so that, by comparing these equations, one gets

$$[\varphi(x), \mathcal{P}_m] = -\mathrm{i}\partial_m\varphi(x) \equiv P_m\varphi(x)\,.$$

Therefore, an infinitesimal translation of a field by a parameter a^m induces a variation

$$\delta_a\varphi(x) = \varphi(x + a) - \varphi(x) = \mathrm{i}a^m P_m\varphi(x)\,.$$

We will find how a generic superfield $\mathcal{Y}(x, \theta, \bar{\theta})$ behaves under a supersymmetry transformation. Calculations are straightforward though lengthy, and therefore we will summarise the main steps. Moreover, for sake of brevity we will omit writing the functional dependence of the superfield when unnecessary. One has

$$\mathcal{Y}(x+\delta x,\theta+\delta\theta,\bar\theta+\delta\bar\theta) \equiv e^{-\mathrm{i}(\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}})}\,\mathcal{Y}(x,\theta,\bar\theta)\,e^{\mathrm{i}(\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}})}\,, \tag{2.1}$$

and one defines also

$$\delta_{\epsilon,\bar\epsilon}\mathcal{Y}(x,\theta,\bar\theta) \equiv \mathcal{Y}(x+\delta x,\theta+\delta\theta,\bar\theta+\delta\bar\theta) - \mathcal{Y}(x,\theta,\bar\theta)\,. \tag{2.2}$$

As a first thing, we write

$$\mathcal{Y}(x,\theta,\bar\theta) = e^{-\mathrm{i}(x\mathcal{P}+\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}})}\,\mathcal{Y}(0,0,0)\,e^{\mathrm{i}(x\mathcal{P}+\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}})}\,,$$

and using the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^C\,,\quad C = A+B+[A,B]+\frac{1}{2}\big([A,[A,B]]-[B,[B,A]]\big)+\cdots\,,$$

it is possible to compute e.g. the leftmost exponential in (2.1), finding the infinitesimal variations of the coordinates

$$\delta x^m = \mathrm{i}\theta\sigma^m\bar\epsilon - \mathrm{i}\epsilon\sigma^m\bar\theta\,,\quad \delta\theta^\alpha = \epsilon^\alpha\,,\quad \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha}\,.$$

It is remarkable that a supersymmetry transformation includes a spacetime translation, though it should not be unexpected, for the commutator of two supercharges is proportional to the momentum. Secondly, we expand equation to linear order (2.2) as

$$\begin{aligned}\delta_{\epsilon,\bar\epsilon}\mathcal{Y}(x,\theta,\bar\theta) &= \mathcal{Y} + \delta x^m\partial_m\mathcal{Y} + \delta\theta^\alpha\frac{\partial}{\partial\theta^\alpha}\mathcal{Y} + \delta\bar\theta^{\dot\alpha}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\mathcal{Y} - \mathcal{Y}\\ &= (\mathrm{i}\theta\sigma^m\bar\epsilon - \mathrm{i}\epsilon\sigma^m\bar\theta)\partial_m\mathcal{Y} + \epsilon^\alpha\partial_\alpha\mathcal{Y} + \bar\epsilon^{\dot\alpha}\bar\partial_{\dot\alpha}\mathcal{Y}\,.\end{aligned} \tag{2.3}$$

On the other hand, from (2.1) we obtain

$$\delta_{\epsilon,\bar\epsilon}\mathcal{Y}(x,\theta,\bar\theta) = \mathcal{Y} - \mathrm{i}\left[\epsilon\mathcal{Q}+\bar\epsilon\bar{\mathcal{Q}},\mathcal{Y}\right] - \mathcal{Y} = -\mathrm{i}\epsilon^\alpha\left[\mathcal{Q}_\alpha,\mathcal{Y}\right] + \mathrm{i}\bar\epsilon^{\dot\alpha}\left[\bar{\mathcal{Q}}_{\dot\alpha},\mathcal{Y}\right]\,. \tag{2.4}$$

If we now define

$$Q_\alpha\mathcal{Y} \equiv \left[\mathcal{Y},\mathcal{Q}_\alpha\right]\,,\quad \bar Q_{\dot\alpha}\mathcal{Y} \equiv \left[\mathcal{Y},\bar{\mathcal{Q}}_{\dot\alpha}\right]\,,$$

by comparison with (2.3) one finds

$$\delta_{\epsilon,\bar\epsilon}\mathcal{Y}(x,\theta,\bar\theta) = \mathrm{i}(\epsilon Q + \bar\epsilon\bar Q)\mathcal{Y}(x,\theta,\bar\theta)\,,$$

and from equation (2.4) we finally obtain

$$Q_\alpha = -\mathrm{i}\partial_\alpha - \sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m\,,\quad \bar Q_{\dot\alpha} = \mathrm{i}\bar\partial_{\dot\alpha} + \theta^\alpha\bar\sigma^m_{\alpha\dot\alpha}\partial_m\,.$$

At this point, we can give a more precise definition of what a superfield is: it is a field in superspace which transforms according to (2.1) under a supersymmetry transformation.

We aim to build an action which is invariant under a transformation of the super-Poincaré group − up to total spacetime derivatives. The superspace formulation is by far much more convenient because the quantity

$$\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ \mathcal{Y}(x,\theta,\bar{\theta})$$

is invariant under supersymmetry provided that $\mathcal{Y}(x,\theta,\bar{\theta})$ is a superfield. In fact, we have seen above that Berezin integrals are translationally invariant: a supersymmetry transformation in superspace is nothing but a translation, and this implies that

$$\delta_{\epsilon,\bar{\epsilon}}\Big[\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ \mathcal{Y}(x,\theta,\bar{\theta})\Big] = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ \delta_{\epsilon,\bar{\epsilon}}\mathcal{Y}(x,\theta,\bar{\theta})\,,$$

i.e. there is no variation of the measure. On the other hand, we have seen that

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{Y} = \epsilon^{\alpha}\partial_{\alpha}\mathcal{Y} + \bar{\epsilon}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}\mathcal{Y} + \partial_m\left[\mathrm{i}(\theta\sigma^m\bar{\epsilon} - \epsilon\sigma^m\bar{\theta})\mathcal{Y}\right]\,,$$

and since the third term is a total derivative, and the first two terms are integrated to zero since there are not enough θ or $\bar{\theta}$ to make up for the measure, we end up with

$$\delta_{\epsilon,\bar{\epsilon}}\Big[\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ \mathcal{Y}(x,\theta,\bar{\theta})\Big] = 0\,.$$

Supersymmetric invariant actions are thus constructed by integrating in the superspace a properly defined superfield, say $\mathcal{S}$, which cannot be totally generic. Indeed, upon integration of the Grassmann coordinates, it has to produce a lagrangian density, namely a *real* and *Poincaré invariant* operator *of mass dimension four*. Summarising, we have

$$I = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\ \mathcal{S}(x,\theta,\bar{\theta}) = \int \mathrm{d}^4x\,\mathcal{L}\,.$$

2.2.1 Chiral Superfields

The generic superfield $\mathcal{Y}(x,\theta,\bar{\theta})$ we have dealt with up to now is not an irreducible representation of the supersymmetry algebra; in fact, its expansion shows that there are too many component fields. Supersymmetric constraints have to be imposed in order to obtain an irreducible representation: indeed, this reduces the number of components, but the superfield nature is not spoiled. One of such constraints can be imposed by means of the covariant derivatives, which are defined as

$$\mathcal{D}_{\alpha} \equiv \partial_{\alpha} + \mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m\,, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \equiv \bar{\partial}_{\dot{\alpha}} + \mathrm{i}\theta\sigma^m_{\alpha\dot{\alpha}}\partial_m\,,$$

and which obey the following algebra

$$\left\{\mathcal{D}_{\alpha},\bar{\mathcal{D}}_{\dot{\alpha}}\right\} = 2\mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\partial_m = -2\sigma^m_{\alpha\dot{\alpha}}P_m\,, \quad \left\{\mathcal{D}_{\alpha},\mathcal{D}_{\beta}\right\} = \left\{\bar{\mathcal{D}}_{\dot{\alpha}},\bar{\mathcal{D}}_{\dot{\beta}}\right\} = 0\,.$$

In fact, covariant derivatives can be used to put supersymmetry invariant constraints because for any superfield $\mathcal{Y}(x,\theta,\bar{\theta})$ it holds that

$$\delta_{\epsilon,\bar{\epsilon}}(\mathcal{D}_{\alpha}\mathcal{Y}) = \mathcal{D}_{\alpha}(\delta_{\epsilon,\bar{\epsilon}}\mathcal{Y})\,.$$

We can thus define *chiral superfields* as follows.

Definition 5. A *chiral superfield* is a superfield Φ such that $\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0$. Similarly, an *anti-chiral superfield* Ψ is such that $\mathcal{D}_\alpha\Psi = 0$.

Clearly, if Φ is a chiral superfield, its hermitian conjugate $\bar\Phi$ is anti-chiral. On the other hand, Φ cannot be both chiral and anti-chiral at the same time, otherwise it would be a constant because $\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\}\Phi = 0 = 2i\sigma^m_{\alpha\dot\alpha}\partial_m\Phi$. In order to find the expression of a chiral superfield in terms of its component fields, it is convenient to introduce new coordinates y^m and $\bar y^m$ as

$$y^m = x^m + \mathrm{i}\theta\sigma^m\bar\theta\,, \quad \bar y^m = x^m - \mathrm{i}\theta\sigma^m\bar\theta\,.$$

Noting that $\bar{\mathcal{D}}_{\dot\alpha}y^m = \bar{\mathcal{D}}_{\dot\alpha}\theta_\beta = 0$ and $\mathcal{D}_\alpha\bar y^m = \mathcal{D}_\alpha\bar\theta_{\dot\alpha} = 0$, the constraint of $\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0$ implies that Φ can only depend on y^m and θ, so that

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\,\theta\psi(y) - \theta^2 F(y)\,. \tag{2.5}$$

Analogously, $\bar\Phi$ can only depend on $\bar y^m$ and $\bar\theta$, and therefore

$$\bar\Phi(\bar y,\bar\theta) = \bar\phi(\bar y) + \sqrt{2}\,\bar\theta\bar\psi(\bar y) - \bar\theta^2\bar F(\bar y)\,. \tag{2.6}$$

The full expression in $(x^m, \theta_\alpha, \bar\theta_{\dot\alpha})$ coordinates can be obtained by Taylor-expanding (2.5) (and similarly (2.6)), finding

$$\Phi(x,\theta,\bar\theta) = \phi(x) + \sqrt{2}\theta\psi(x) - \theta^2F(x) + \mathrm{i}\theta\sigma^m\bar\theta\partial_m\phi(x) - \frac{\mathrm{i}}{\sqrt{2}}\theta^2\partial_m\psi(x)\sigma^m\bar\theta - \frac{1}{4}\theta^2\bar\theta^2\Box\phi(x)\,.$$

This expression can be conveniently rewritten as $\Phi(x,\theta,\bar\theta) = e^{\mathrm{i}\theta\sigma^m\bar\theta\partial_m}\Phi(x,\theta)$. Similarly, the corresponding expression for $\bar\Phi(x,\theta,\bar\theta)$ can be repackaged as $\bar\Phi(x,\theta,\bar\theta) = e^{-\mathrm{i}\theta\sigma^m\bar\theta\partial_m}\bar\Phi(x,\theta)$. From the expansion in terms of the component fields, we can see that the chiral superfield is worth its name: indeed, it contains exactly the degrees of freedom of the chiral super-multiplet. Let us now find how a chiral superfield behaves under a supersymmetry transformation, i.e. we want to compute $\delta_{\epsilon,\bar\epsilon}\Phi = \mathrm{i}(\epsilon Q + \bar\epsilon\bar Q)\Phi$. In particular, this is more conveniently done in the (y^m,θ) coordinate basis, where

$$Q_\alpha = -\mathrm{i}\partial_\alpha\,, \quad \bar Q_{\dot\alpha} = \mathrm{i}\bar\partial_{\dot\alpha} + 2\theta^\alpha\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial y^m}.$$

One has

$$\begin{aligned}
\delta_{\epsilon,\bar\epsilon}\Phi(y,\theta) &= \left[\epsilon^\alpha\partial_\alpha + 2\mathrm{i}(\theta\sigma^m\bar\epsilon)\frac{\partial}{\partial y^m}\right]\left(\phi + \sqrt{2}\theta\psi - \theta^2F\right)\\
&= \sqrt{2}\,\epsilon\psi - 2\,\epsilon\theta F + 2\mathrm{i}(\theta\sigma^m\bar\epsilon)\partial_m\phi + 2\mathrm{i}\sqrt{2}(\theta\sigma^m\bar\epsilon)\,\theta\partial_m\psi\\
&= \sqrt{2}\,\epsilon\psi + \sqrt{2}\theta^\alpha\left[\sqrt{2}\mathrm{i}(\sigma^m\bar\epsilon)_\alpha\partial_m\phi - \sqrt{2}\,\epsilon_\alpha F\right] + \mathrm{i}\sqrt{2}\theta^2(\bar\epsilon\bar\sigma^m)_\alpha\partial_m\psi^\alpha\,,
\end{aligned}$$

and therefore one ends up with

$$\delta\phi = \sqrt{2}\,\epsilon\psi\,, \quad \delta\psi_\alpha = -\sqrt{2}\,\epsilon_\alpha F + \sqrt{2}\,\mathrm{i}(\sigma^m\bar\epsilon)_\alpha\,\partial_m\phi\,, \quad \delta F = \sqrt{2}\,\mathrm{i}\partial_m\psi\,\sigma^m\bar\epsilon\,.$$

2.2.2 Real Scalar Superfields

The general expansion of $\mathcal{Y}(x, \theta, \bar{\theta})$ contains also a vector field $A_m(x)$. In order to describe gauge interactions in the superfield formalism, one has to find a new supersymmetric constraint which preserves the A_m component, and makes it real, too. All that we need to do is to define a *real scalar superfield* $\mathcal{V}$, is to impose the reality condition $\bar{\mathcal{V}} = \mathcal{V}$. The general expansion in terms of the component fields is

$$\begin{aligned}\mathcal{V}(x,\theta,\bar{\theta}) = {} & C(x) + \mathrm{i}\theta\chi(x) - \mathrm{i}\bar{\theta}\bar{\chi}(x) - \theta\sigma^m\bar{\theta}\, A_m(x) \\ & + \frac{\mathrm{i}}{2}\theta^2\big[M(x) + \mathrm{i}N(x)\big] - \frac{\mathrm{i}}{2}\theta^2\big[M(x) - \mathrm{i}N(x)\big] \\ & + \mathrm{i}\theta^2\bar{\theta}\Big[\bar{\lambda}(x) + \frac{\mathrm{i}}{2}\bar{\sigma}^m\,\partial_m\chi(x)\Big] - \mathrm{i}\bar{\theta}^2\theta\Big[\lambda(x) - \frac{\mathrm{i}}{2}\sigma^m\,\partial_m\bar{\chi}(x)\Big] \\ & + \mathrm{i}\theta^2\bar{\theta}^2\Big[D(x) - \frac{1}{2}\Box C(x)\Big]\,.\end{aligned}$$

One can notice that there are eight fermionic plus eight bosonic off-shell degrees of freedom. These can be reduced to four bosonic plus four fermionic by imposing appropriate gauge-fixing conditions. Finally, requiring on-shellness one ends up with two bosonic plus two fermionic degrees of freedom, which coincide with those of the massless vector super-multiplet.

We are interested in finding the supersymmetric generalization of gauge transformations. To this end, one can notice that, if Φ is a chiral superfield, then the combination $\Phi + \bar{\Phi}$ is still a superfield. It is not chiral or antichiral, but a real superfield. Moreover,

$$\mathcal{V} \mapsto \mathcal{V} + \Phi + \bar{\Phi} \qquad \Longrightarrow \qquad A_m \mapsto A_m + \partial_m(2\,\mathbf{Im}\phi)\,.$$

One can notice that this is just the way a vector field transforms under abelian gauge transformation, therefore $\mathcal{V} \mapsto \mathcal{V} + \Phi + \bar{\Phi}$ is the natural generalization of gauge transformations we are looking for. As far as the other components are concerned, one has

$$C \mapsto C + 2\,\mathbf{Re}\phi\,, \quad \chi \mapsto \chi - \mathrm{i}\sqrt{2}\psi\,, \quad M \mapsto M - 2\,\mathbf{Im}F\,, \quad N \mapsto N + 2\,\mathbf{Re}F\,,$$

while $\lambda(x)$ and $D(x)$ are left unchanged. Hence, choosing Φ in the proper way, it is possible to eliminate C, χ, M and N: this gauge is named after Wess and Zumino (WZ gauge in the following), and with this choice the vector superfield becomes

$$\mathcal{V}_{\mathrm{WZ}}(x,\theta,\bar{\theta}) = -\theta\sigma^m\bar{\theta}\,A_m(x) + \mathrm{i}\theta^2\,\bar{\theta}\bar{\lambda}(x) - \mathrm{i}\bar{\theta}^2\,\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2\,D(x)$$

We have ended up with four bosonic plus four fermionic degrees of freedom off-shell. Finally, imposing the equations of motion for $D(x)$, $A_m(x)$ and $\lambda(x)$, we get two fermionic plus two fermionic degrees of freedom on-shell. We can close this section with some remarks:

- in the WZ gauge no further constraint is put on A_m, thus we still have the freedom to perform ordinary gauge transformation;

- the WZ gauge breaks supersymmetry invariance, i.e. acting with a supersymmetry transformation on $\mathcal{V}_{\text{WZ}}$, we obtain a vector superfield which is no longer in the WZ gauge. Nevertheless, this gauge is useful in practical calculations, therefore, after a supersymmetry transformation, one has to perform a gauge transformation to compensate the new terms and go back to the WZ gauge;

- in the WZ gauge it holds that

$$\mathcal{V}_{\text{WZ}}^2 = \frac{1}{2}\theta^2\bar{\theta}^2\, A_m A^m \quad \rightarrow \quad \mathcal{V}_{\text{WZ}}^n = 0 \text{ for } n \geq 3\,.$$

2.3 Actions for $\mathcal{N} = 1$ Supersymmetry

We want to study the possible actions that one can build with chiral and vector superfields. These are the basic ingredients needed to build a unified theory of matter and gauge fields in a supersymmetric framework. We will first describe how supersymmetric matter lagrangians can be constructed, moving then to supersymmetric pure gauge field lagrangiangs and finally these two sectors will be coupled together.

2.3.1 Matter Fields Lagrangian

The simplest theory one can think of is that of a single chiral superfield Φ. Let us consider the quantity

$$\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \bar{\Phi}\Phi\,. \tag{2.7}$$

Besides being invariant under supersymmetry, it is real and scalar, as one can infer from the first component; in fact, this can be used as a reference because the only non-vanishing term in the integral, namely the $\theta^2\bar{\theta}^2$ component field, has the same structure: it is a scalar because there is no free index, and it is also real because $(\theta^2\bar{\theta}^2)^\dagger = \theta^2\bar{\theta}^2$. Moreover, the mass dimension is of the correct value: indeed, recalling that $\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi - \theta^2 F$ we see that $[\theta] = [\bar{\theta}] = -\frac{1}{2}$. Therefore, in general, if $[\mathcal{Y}]$ is the dimension of a generic superfield, the $\theta^2\bar{\theta}^2$ component has dimension $[\mathcal{Y}] + 2$. We can thus conclude that, since $[\bar{\Phi}\Phi] = 2$, then $[\bar{\Phi}\Phi\big|_{\theta^2\bar{\theta}^2}] = 4$. The integration in superspace is performed starting from $\Phi(y)$ and $\bar{\Phi}(\bar{y})$ and taking the product, expanding the result in $(x,\theta,\bar{\theta})$ and finally picking only the $\theta^2\bar{\theta}^2$ component. One ends up with

$$\mathcal{L}_k = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \bar{\Phi}\Phi = \partial_m\bar{\phi}\,\partial^m\phi + \frac{\mathrm{i}}{2}\big(\partial_m\psi\,\sigma^m\bar{\psi} - \psi\sigma^m\,\partial_m\bar{\psi}\big) + \bar{F}F\,. \tag{2.8}$$

The field F is not dynamical, as one can see from the absence of its kinetic term. Integrating out this field, we obtain a lagrangian describing physical degrees of freedom only. However, it is possible to obtain the equations of motion for ϕ, ψ and F directly from the superspace expression, rather than passing through the lagrangian (2.8). Indeed we have to remember that, being Φ a chiral superfield, equation (2.7) is a constrained integral. That integral can be rewritten as an unconstrained one, noting that the only difference between the covariant derivative $\mathcal{D}_\alpha$ and the usual derivative ∂_α is a total spacetime derivative hence

$$\int \mathrm{d}^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta = \frac{1}{4}\epsilon^{\alpha\beta}\mathcal{D}_\alpha\mathcal{D}_\beta\,,$$

which leads to

$$\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\bar{\Phi}\Phi = \frac{1}{4}\int \mathrm{d}^2\bar{\theta}\;\bar{\Phi}\mathcal{D}^2\Phi\,.$$

Varying the action with respect to $\bar{\Phi}$ one gets $\mathcal{D}^2\Phi = 0$, which furnishes the equations of motion for the component fields $-$ upon expanding in $(x, \theta, \bar{\theta})$. On the other hand, the action (2.7) can be generalized. Let us try to consider a generic function of Φ and $\bar{\Phi}$, say $K(\Phi, \bar{\Phi})$. The quantity

$$\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;K(\Phi, \bar{\Phi}) \tag{2.9}$$

is a viable matter action if the function $K(\Phi, \bar{\Phi})$:

- is a superfield of mass dimension $[K] = 2$;
- is real and scalar;
- depends only on Φ and $\bar{\Phi}$ and not on their covariant derivatives, because otherwise the $\theta^2\bar{\theta}^2$ component would give third (or higher) order derivative terms, that are usually not allowed in field theory, since they may lead to so-called ghosts whose kinetic energy is negative.

It turns out that the most general form of $K(\Phi, \bar{\Phi})$ satisfying the requirements above is the so-called Kähler potential

$$K(\Phi, \bar{\Phi}) = \sum_{i,j=1}^{\infty} c_{ij}\bar{\Phi}^i\Phi^j\,,$$

with $c_{ij}^* = c_{ij}$. Since $[\bar{\Phi}^i\Phi^j] = i + j$, the coefficients c_{ij} with i or j bigger than one have $[c_{ij}] < 0$, and (2.9) can produce non-renormalizable theories. It appears thus natural to associate c_{ij} with the cut-off of the theory Λ, in such a way that $c_{ij} \sim \Lambda^{2-(i+j)}$. Renormalizable theories instead require that the only non-vanishing coefficient is c_{11}. One can notice that the term $\Phi + \bar{\Phi}$ meets all the physical requirements to produce a supersymmetric action, but it gives vanishing contribution because the $\theta^2\bar{\theta}^2$ term is a total spacetime derivative. Thus two Kähler potentials K and K' such that $K'(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})$ (for Λ, $\bar{\Lambda}$ chiral and anti-chiral, respectively) give the same contribution to a lagrangian, despite being actually different.

The next step consists in finding a way to describe interactions between matter particles, as for instance scalar interactions or Yukawa couplings. There is another possibility to build supersymmetric invariant actions when dealing with chiral superfields.

Let Ψ be one such field: as we noted above, the integration of Ψ in full superspace gives null contribution, but what about an integration over half superspace? In fact, the integral

$$\int \mathrm{d}^4x\, \mathrm{d}^2\theta\ \Psi\,,$$

is not null because the θ^2 component is not a total derivative; moreover, concerning supersymmetry invariance it is perfectly legitimate, since the non-vanishing component transforms as a total spacetime derivative under supersymmetry transformations. Usually, integrals over the full superspace are called *D-terms*, while those over half superspace are named *F-terms*: the former are less general because one can play with derivatives and rewrite them as the latter, the contrary being generally not possible. By means of appropriate F-terms we can describe chiral superfields interactions. Let $W(\Phi)$ be a function of a chiral superfield Φ only: with this requirement, $W(\Phi)$ is a holomorphic function, i.e. $\frac{\partial W}{\partial \bar{\Phi}} = 0$, and it as also a chiral superfield, indeed

$$\bar{\mathcal{D}}_{\dot{\alpha}} W(\Phi) = \frac{\partial W}{\partial \Phi} \bar{\mathcal{D}}_{\dot{\alpha}} \Phi = 0\,.$$

By introducing the hermitian conjugate $\bar{W}(\bar{\Phi})$ the interaction lagrangian can be defined as

$$\mathcal{L}_{\text{int}} = \int \mathrm{d}^2\theta\ W(\Phi) + \int \mathrm{d}^2\bar{\theta}\ \overline{W}(\bar{\Phi})\,.$$

The function $W(\Phi)$ is the so-called *superpotential*, and apart from holomorphicity:

- it should not contain derivatives of Φ, because $\mathcal{D}_\alpha \Phi$ is no longer a chiral superfield;
- it should have mass dimension $[W] = 3$.

It follows that the superpotential has an expression as

$$W(\Phi) = \sum_{k=1}^{\infty} a_k \Phi^k\,,$$

and if one wants to keep the theory renormalizable then $k = 1, 2, 3$. On the other hand $W(\Phi)$ may be constrained also by the R-symmetry. Recalling that $[R, Q_\alpha] = -Q_\alpha$, we see that the R-charge is lowered by one unit each time Q_α acts. Moreover, we have seen that the chiral super-multiplet of fields is constructed basically acting with Q_α on a properly chosen complex scalar ϕ obtaining ψ_α, and then acting on it again with Q_α obtaining F. This means that if we assign $R(\phi) = r$, then $R(\psi) = r - 1$ and $R(F) = r - 2$. On the other hand, in building the superspace we have found that $\left[\theta Q, \bar{\theta}\bar{Q}\right] = -2\,\theta\sigma^m\bar{\theta}P_m$: since one should have $R(\theta) = -R(\bar{\theta})$ and the right hand side should have zero R-charge, it holds that $R(\theta) = -R(Q) = +1$, and conversely $R(\bar{\theta}) = -R(\bar{Q}) = -1$. Finally, Grassmann algebra is such that $\mathrm{d}\theta = \partial/\partial\theta$, and therefore $R(\mathrm{d}\theta) = -R(\mathrm{d}\bar{\theta}) = -1$. All in all, from the requirement $R(\mathcal{L}) = 0$ one has $R(\mathrm{d}^2\theta\, W) = 0$, hence $R(W) = +2$.

The integration of $W(\Phi)$ on superspace is easily performed recalling the expansion of the chiral-superfield in the (y^m, θ) coordinates, i.e. $\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \theta^2 F(y)$. It follows that

$$\begin{aligned} W(\Phi) &= W + \sqrt{2}\,\frac{\partial W}{\partial \phi}\theta\psi + \left(\frac{\partial^2 W}{\partial\phi\,\partial\phi}\theta\psi\,\theta\psi - \theta^2\frac{\partial W}{\partial \phi}F\right) \\ &= W + \sqrt{2}\,\frac{\partial W}{\partial \phi}\theta\psi - \theta^2\left(\tfrac{1}{2}\frac{\partial^2 W}{\partial\phi\,\partial\phi}\psi\psi + \frac{\partial W}{\partial \phi}F\right), \end{aligned}$$

using $\theta\psi\,\theta\psi = \theta^\alpha\psi_\alpha\,\theta^\beta\psi_\beta = -\frac{1}{2}\theta^2\psi\psi$. One should bear in mind that in the expansion the superpotential is evalueted on ϕ, namely $W = W(\phi)$. At this point the expression in $(x^m, \theta, \bar\theta)$ is obtained simply taking $y^m = x^m$, because the terms one misses, and which would have been obtained by means of the usual expansion procedure, are just total spacetime derivatives. One gets therefore

$$\mathcal{L}_{\text{int}} = \int \mathrm{d}^2\theta\; W(\Phi) + \int \mathrm{d}^2\bar\theta\; \overline{W}(\bar\Phi) = -\tfrac{1}{2}\frac{\partial^2 W}{\partial\phi\,\partial\phi}\psi\psi - \frac{\partial W}{\partial\phi}F + \text{h.c.}\,.$$

Gathering together what we have obtained up to now, we see that the most general supersymmetric invariant matter lagrangian reads

$$\mathcal{L}_{\text{m}} = \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\; K(\Phi,\bar\Phi) + \Big(\int \mathrm{d}^2\theta\; W(\Phi) + \text{h.c.}\Big)\,. \tag{2.10}$$

Supposing one wants to consider only renormalizable theories, the full expansion in terms of the component fields is

$$\mathcal{L}_{\text{m}} = \partial_m\bar\phi\,\partial^m\phi + \tfrac{\mathrm{i}}{2}\big(\partial_m\psi\,\sigma^m\bar\psi + \psi\sigma^m\,\partial_m\bar\psi\big) + F\bar F - \left(\frac{\partial W}{\partial\phi}F + \tfrac{1}{2}\frac{\partial^2 W}{\partial\phi\,\partial\phi}\psi\psi + \text{h.c.}\right),$$

with $W(\phi)$ at most cubic. The equations of motion for F and $\bar F$ are

$$\bar F = -\frac{\partial W}{\partial\phi}\,,\quad F = -\frac{\partial\overline{W}}{\partial\bar\phi}\,,$$

hence substituting above one gets the lagrangian

$$\mathcal{L}_{\text{m}} = \partial_m\bar\phi\,\partial^m\phi + \tfrac{\mathrm{i}}{2}\big(\partial_m\psi\,\sigma^m\bar\psi + \psi\sigma^m\,\partial_m\bar\psi\big) - \left|\frac{\partial W}{\partial\phi}\right|^2 - \tfrac{1}{2}\left(\frac{\partial^2 W}{\partial\phi\,\partial\phi}\psi\psi + \text{h.c.}\right), \tag{2.11}$$

and one can read-off the scalar potential $V(\phi,\bar\phi) = \left|\frac{\partial W}{\partial\phi}\right|^2$. The generalization to a set of n chiral-superfields is straightforward. In particular, for a renormalizable theory one has

$$K = \bar\Phi_i\Phi^i\,,\quad W = a_i\Phi^i + \frac{m_{ij}}{2}\bar\Phi^i\Phi^j + \frac{g_{ijk}}{3}\Phi^i\Phi^j\Phi^k\,,\quad V = \sum_{i=1}^n\left|\frac{\partial W}{\partial\phi}\right|^2\,.$$

2.3.2 Non-linear Sigma Model

Let us consider an important example of supersymmetric matter model in which renormalizability is not demanded. In this case, Kähler potential is no longer required to be quadratic, and the superpotential can be a polynomial of degree higher than three. For sake of simplicity we define the following quantities

$$K_i = \frac{\partial K}{\partial \phi^i}\bigg|_{\phi,\bar\phi}\,, \quad K^i = \frac{\partial K}{\partial \bar\phi_i}\bigg|_{\phi,\bar\phi}\,, \quad K_i{}^j = \frac{\partial^2 K}{\partial \phi^i \partial \bar\phi_j}\bigg|_{\phi,\bar\phi}\,,$$
$$W_i = \frac{\partial W}{\partial \phi^i}\bigg|_{\phi}\,, \quad W^i = \frac{\partial \overline{W}}{\partial \bar\phi}\bigg|_{\bar\phi}\,, \quad W_{ik} = \frac{\partial^2 W}{\partial \phi^i \partial \phi^j}\bigg|_{\phi}\,, \quad W^{ij} = \overline{W}_{ij}\,.$$

As a first thing, we want to extract the F-tem. The superpotential can be conveniently rewritten as

$$W(\Phi) = W(\phi) + W_i \Delta^i + \frac{1}{2} W_{ij}\Delta^i\Delta^j$$

for $\Delta^i(y,\theta) = \Phi^i(y,\theta) - \phi^i(y) = \sqrt{2}\theta\psi^i(y) - \theta^2 F(y)$, and therefore

$$\int \mathrm{d}^2\theta\; W(\Phi) + \text{h.c.} = -W_i F^i - \frac{1}{2} W_{ij}\psi^i\psi^j + \text{h.c.}\,.$$

As far as the D-term is concerned, the derivation is a bit more involved. In the $(x,\theta,\bar\theta)$ coordinates we introduce

$$\Delta^i(x,\theta,\bar\theta) = \Phi^i(x,\theta,\bar\theta) - \phi^i(x)\,, \quad \bar\Delta_i(x,\theta,\bar\theta) = \bar\Phi_i(x,\theta,\bar\theta) - \bar\phi_i(x)\,, \tag{2.12}$$

which explicitly read

$$\Delta^i(x,\theta,\bar\theta) = \sqrt{2}\theta\psi^i(x) - \theta^2 F^i(x) + \mathrm{i}\theta\sigma^m\bar\theta\partial_m\phi^i(x) - \frac{\mathrm{i}}{\sqrt{2}}\theta^2\partial_m\psi^i(x)\sigma^m\bar\theta - \frac{1}{4}\theta^2\bar\theta^2\Box\phi^i(x)\,,$$

and $\bar\Delta_i$ accordingly, being $\bar\Delta_i = (\Delta^i)^\dagger$. One can notice that, since $\Delta^i\Delta^j \sim \theta^2$, one has $\Delta^i\Delta^j\Delta^k = 0$. With the definitions in (2.12), the Kähler potential reads

$$\begin{aligned} K(\Phi,\bar\Phi) = K(\phi,\bar\phi) &+ \big(K_i\Delta^i + K^i\Delta_i\big) + \frac{1}{2}\big(K_{ij}\Delta^i\Delta^j + K^{ij}\bar\Delta_i\bar\Delta_j\big) + K_i{}^j\Delta^i\bar\Delta_j \\ &+ \frac{1}{2}\big(K^l_{ij}\Delta^i\Delta^j\bar\Delta_l + K^{ij}_l\Delta^l\bar\Delta_i\bar\Delta_j\big) + \frac{1}{4}K^{lm}_{ij}\Delta^i\Delta^j\bar\Delta_l\bar\Delta_m\,. \end{aligned}$$

Hence it follows that

$$\begin{aligned} \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\; K(\Phi,\bar\Phi) = &-\frac{1}{4}\big(K_i\Box\phi^i + K^i\Box\bar\phi_i\big) - \frac{1}{4}\big(K_{ij}\partial_m\phi^i\partial^m\phi^j + K^{ij}\partial_m\bar\phi_i\partial^m\bar\phi_j\big) \\ &+ K_i{}^j\Big(F^i\bar F_j + \frac{1}{2}\partial_m\phi^i\partial^m\bar\phi_j - \frac{\mathrm{i}}{2}\psi^i\sigma^m\partial_m\bar\psi_j + \frac{\mathrm{i}}{2}\partial_m\psi^i\,\sigma^m\bar\psi_j\Big) \\ &+ \frac{\mathrm{i}}{4}K^l_{ij}\big(\psi^i\sigma^m\bar\psi_l\,\partial_m\psi^j + \psi^j\sigma^m\bar\psi_l\,\partial_m\psi^i - 2\mathrm{i}\psi^i\psi^j\bar F\big) + \text{h.c.} \\ &+ \frac{1}{4}K^{lm}_{ij}\psi^i\psi^j\bar\psi_l\bar\psi_m + \text{total derivatives}\,. \end{aligned} \tag{2.13}$$

One can see that the first row of (2.13) is actually a total derivative, and hence can be dropped out of the lagrangian, finding

$$\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\ K(\Phi,\bar\Phi) = K^{j}_{\ i}\Big(F^i\bar F_j + \partial_m\phi^i\partial^m\bar\phi_j - \frac{\mathrm{i}}{2}\psi^i\sigma^m\partial_m\bar\psi_j + \frac{i}{2}\partial_m\psi^i\,\sigma^m\bar\psi_j\Big)$$
$$+\frac{\mathrm{i}}{4}K^l_{ij}\big(\psi^i\sigma^m\bar\psi_l\,\partial_m\psi^j + \psi^j\sigma^m\bar\psi_l\,\partial_m\psi^i - 2\mathrm{i}\psi^i\psi^j\bar F\big) + \text{h.c.}$$
$$+\frac{1}{4}K^{lm}_{ij}\psi^i\psi^j\bar\psi_l\bar\psi_m + \text{total derivatives}\,.$$

Since in the previous expression the Kähler potential shows up only as $K^i_{\ j}$, we see that

$$K(\phi,\bar\phi)\mapsto K(\phi,\bar\phi)+\Lambda(\phi)+\bar\Lambda(\bar\phi)$$

is a symmetry of the model. Moreover, $K_i^{\ j}$ is hermitian being $K(\phi,\bar\phi)$ real, and it is also positive definite, as one can see from the fact that the kinetic terms have positive sign. Thus $K_i^{\ j}$ possesses all the right properties to be interpreted as a metric on a manifol $\mathcal{K}$ of complex dimension, and whose coordinates are the scalar fields ϕ. In particular, one speaks of Kähler metric and Kähler manifold, respectively. However, one should prove that any term in the lagrangian can be written in terms of geometrical quantities defined on $\mathcal{K}$, before concluding that this is a sigma-model with Kähler manifold as target space. By means of the equations of motion of the auxiliary field

$$F^i = (K^{-1})^i_{\ j}W^j - \frac{1}{2}\Gamma^i_{lm}\psi^l\psi^m$$

with $\Gamma^i_{lm}\equiv (K^{-1})^i_{\ j}K^j_{lm}$, the lagrangian can be rewritten as

$$\mathcal{L} = K^j_{\ i}\Big[\partial_m\phi^i\,\partial^m\bar\phi^j + \Big(\frac{i}{2}\nabla_m\psi\,\sigma^m\bar\psi_j + \text{h.c.}\Big)\Big] - (K^{-1})^i_{\ j}W_iW^j$$
$$-\frac{1}{2}\big(W_{ij}-\Gamma^l_{ij}W_l\big)\psi^i\psi^j - \frac{1}{2}\big(W^{ij}-\Gamma^{ij}_l W^l\big)\bar\psi_i\bar\psi_j + \frac{1}{4}R^{lm}_{ij}\psi^i\psi^j\bar\psi_l\bar\psi_m\,,$$

where the covariant derivative is defined as

$$\nabla_m\psi = \psi^i\partial_m\psi^i + \Gamma^i_{jl}\,\partial_m\phi^i\,\psi^l\,,\quad \nabla_m\bar\psi_i = \partial_m\bar\psi_i + \Gamma^{jl}_i\,\partial_m\bar\phi_j\bar\psi_l\,,$$

and the curvature tensor is given by

$$R^{lm}_{ij} = K^{lm}_{ij} - K^r_{ij}(K^{-1})^s_{\ r}K^{lm}_s\,.$$

As one could expect, the component lagrangian and the relevant physical parameters such as masses and couplings depend on K in a geometrical fashion, namely through the connection Γ^i_{jk} and the curvature tensor R^{ij}_{kl}. There are infinitely many Kähler metrics corresponding to as many $\mathcal{N}=1$ supersymmetric sigma-models. The renormalizable case is recovered for $K_i^{\ j}=\delta^i_j$.

2.3.3 Gauge Field Lagrangian

The basic quantity we have to consider to find a generalization of YM theories is the real superfield $\mathcal{V}$.

Abelian Gauge Field Theories

Let us begin with the simplest case of abelian gauge groups. We recall that the vector field A_m naturally appears as a component of $\mathcal{V}$, thus all that we need to do is finding a proper generalization of the field-strength tensor. Let us introduce the objects

$$\mathcal{W}_\alpha = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha \mathcal{V}\,, \quad \overline{\mathcal{W}}_{\dot\alpha} = -\frac{1}{4}\mathcal{D}\mathcal{D}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{V}\,,$$

which are clearly superfields because $\mathcal{V}$ is a superfield and $\mathcal{D}$, $\bar{\mathcal{D}}$ commute with supersymmetry transformations. Moreover, $\mathcal{W}_\alpha$ is a chiral superfield since $\bar{\mathcal{D}}^3 = 0$. Recalling now that abelian gauge transformations are given by $\mathcal{V} + \Phi + \bar\Phi$, one has

$$\mathcal{W}_\alpha \mapsto \mathcal{W}_\alpha - \frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha\left(\Phi + \bar\Phi\right) = \mathcal{W}_\alpha + \frac{1}{4}\bar{\mathcal{D}}^{\dot\beta}\underbrace{\{\bar{\mathcal{D}}_{\dot\beta}, \mathcal{D}_\alpha\}}_{2\mathrm{i}\,\sigma^m_{\alpha\dot\beta}}\Phi = \mathcal{W}_\alpha\,,$$

hence one can take advantage of gauge invariance to work in the WZ-gauge and simplify the calculations.

In order to find the component fields of $\mathcal{W}_\alpha$, it is convenient to use (y^m, θ) coordinates. One has

$$\mathcal{V}_{\mathrm{WZ}} = -\theta\sigma^m\bar\theta\, A_m(y) + \mathrm{i}\theta^2\,\bar\theta\bar\lambda(y) - i\bar\theta^2\,\theta\lambda(y) + \frac{1}{2}\theta^2\bar\theta^2\big[D(y) + \mathrm{i}\partial_m A^m(y)\big]\,,$$

and recalling that $\mathcal{D}_\alpha = \partial_\alpha + 2i\,\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}(\partial/\partial y^m)$, after a lenghty calculation one ends up with

$$\mathcal{W}_\alpha = -\mathrm{i}\lambda_\alpha + \theta_\alpha D - \mathrm{i}(\sigma^{mn}\theta)_\alpha F_{mn} + \theta^2(\sigma^m\partial_m\bar\lambda)_\alpha\,,$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$ is the usual field-strength for abelian gauge theories. The supersymmetric field-strength is a chiral superfield whose lowest component is the fermion λ_α, called *gaugino*, hence $\mathcal{W}_\alpha$ is also termed *gaugino superfield.* Since $[\mathcal{W}_\alpha] = \frac{3}{2}$, we see that a possible lagrangian is given by a scalar term constructed with the use of two gaugino superfields, namely

$$\int \mathrm{d}^2\theta\ \mathcal{W}^\alpha\mathcal{W}_\alpha = -\frac{1}{2}F_{mn}F^{mn} + \frac{\mathrm{i}}{4}\epsilon^{mnpq}F_{mn}F_{pq} - 2\mathrm{i}\,\lambda\sigma^m\partial_m\bar\lambda + D^2\,. \tag{2.14}$$

However, in order to get a real object, we should consider also the hermitian conjugate, so that

$$\mathcal{L}_{\mathrm{gauge}} = \tfrac{1}{4}\int \mathrm{d}^2\theta\ \mathcal{W}^\alpha\mathcal{W}_\alpha + \mathrm{h.c.} = -\tfrac{1}{4}F_{mn}F^{mn} - \mathrm{i}\lambda\sigma^m\partial_m\bar\lambda + \tfrac{1}{2}D^2\,.$$

Non-Abelian Gauge Field Theories

We now turn our attention to non-abelian gauge theories. The first thing one needs to do is to promote the vector superfield to an object living in the Lie algebra of the group G. In particular, we take $\mathcal{V} = \mathcal{V}^a t^a$, $a = 1, \ldots, \mathrm{dim}G$, for t^a hermitian generators and with the normalization $\mathrm{tr}\big[t^a t^b\big] = \delta^{ab}$. Secondly, we have to define the finite version of the gauge transformation $\mathcal{V} \mapsto \mathcal{V} + \Phi + \bar{\Phi}$. This is given by

$$e^{\mathcal{V}} \mapsto e^{\mathrm{i}\bar{\Lambda}} e^{\mathcal{V}} e^{-\mathrm{i}\Lambda} \,, \tag{2.15}$$

and it reduces to the abelian transformation to leading order in Λ, and identifying $\Phi = -\mathrm{i}\Lambda$. Still, we use the WZ gauge, and recalling that $\mathcal{V}^n_{\mathrm{WZ}} = 0$ for $n \geq 3$ one gets

$$e^{\mathcal{V}} = 1 + \mathcal{V} + \tfrac{1}{2}\mathcal{V}^2 \,.$$

The generalization of the gaugino superfield is provided by

$$\mathcal{W}_\alpha = -\tfrac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\big[e^{-\mathcal{V}}\mathcal{D}_\alpha e^{\mathcal{V}}\big] \,, \quad \overline{\mathcal{W}}_\alpha - \tfrac{1}{4}\mathcal{D}\mathcal{D}\big[e^{\mathcal{V}}\bar{\mathcal{D}}_{\dot{\alpha}} e^{-\mathcal{V}}\big] \,.$$

Applying the transformation (2.15) one gets

$$\begin{aligned}
\mathcal{W}_\alpha \mapsto -\tfrac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\big[e^{\mathrm{i}\Lambda}e^{-\mathcal{V}}e^{-\mathrm{i}\bar{\Lambda}}\mathcal{D}_\alpha\big(e^{\mathrm{i}\bar{\Lambda}}e^{\mathcal{V}}e^{-\mathrm{i}\Lambda}\big)\big] &= -\tfrac{1}{4}e^{\mathrm{i}\Lambda}\bar{\mathcal{D}}\bar{\mathcal{D}}\big[e^{-\mathcal{V}}\bar{\mathcal{D}}_\alpha\big(e^{\mathcal{V}}e^{-\mathrm{i}\Lambda}\big)\big] \\
&= -\tfrac{1}{4}e^{\mathrm{i}\Lambda}\bar{\mathcal{D}}\bar{\mathcal{D}}\big[e^{-\mathcal{V}}\big(\bar{\mathcal{D}}_\alpha e^{\mathcal{V}}\big)e^{-\mathrm{i}\Lambda} + \mathcal{D}_\alpha e^{-\mathrm{i}\Lambda}\big] \\
&= e^{\mathrm{i}\Lambda}\mathcal{W}_\alpha e^{-\mathrm{i}\Lambda} + \tfrac{1}{4}e^{\mathrm{i}\Lambda}\bar{\mathcal{D}}^{\dot{\beta}}\underbrace{\{\bar{\mathcal{D}}_{\dot{\beta}}, \mathcal{D}_\alpha\}}_{2\mathrm{i}\,\sigma^m_{\alpha\dot{\alpha}}\partial_m} e^{-\mathrm{i}\Lambda} = e^{\mathrm{i}\Lambda}\mathcal{W}_\alpha e^{-\mathrm{i}\Lambda} \,,
\end{aligned}$$

and likewise $\overline{\mathcal{W}}_{\dot{\alpha}} \mapsto e^{\mathrm{i}\bar{\Lambda}}\overline{\mathcal{W}}_{\dot{\alpha}}e^{-\mathrm{i}\bar{\Lambda}}$, i.e. $\mathcal{W}_\alpha$ and $\overline{\mathcal{W}}_{\dot{\alpha}}$ transform covariantly as they should do. In order to find the component fields, we make use of the (y^m, θ) coordinates. Expanding $e^{\mathcal{V}}$ in the WZ gauge one gets

$$\begin{aligned}
\mathcal{W}_\alpha &= -\tfrac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\Big[\Big(1 - \mathcal{V} + \tfrac{1}{2}\mathcal{V}^2\Big)\mathcal{D}_\alpha\Big(1 + \mathcal{V} + \tfrac{1}{2}\mathcal{V}^2\Big)\Big] \\
&= -\tfrac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha\mathcal{V} - \tfrac{1}{8}\bar{\mathcal{D}}\bar{\mathcal{D}}\big[\big(\mathcal{D}_\alpha\mathcal{V}\big)\mathcal{V} + \mathcal{V}\big(\mathcal{D}_\alpha\mathcal{V}\big)\big] + \tfrac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\big(\mathcal{V}\mathcal{D}_\alpha\mathcal{V}\big) \\
&= -\tfrac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}_\alpha\mathcal{V} + \tfrac{1}{8}\bar{\mathcal{D}}\bar{\mathcal{D}}[\mathcal{V}, \mathcal{D}_\alpha\mathcal{V}] \,.
\end{aligned}$$

The first term is already known from the discussion of the abelian case. Concerning the second term, one has

$$\tfrac{1}{8}\bar{\mathcal{D}}\bar{\mathcal{D}}\,[\mathcal{V}, \mathcal{D}_\alpha\mathcal{V}] = -\tfrac{1}{2}(\sigma^{mn}\theta)_\alpha\,[A_m, A_n] - \tfrac{\mathrm{i}}{2}\theta^2\sigma^m_{\alpha\dot{\alpha}}\,\big[A_m, \bar{\lambda}^{\dot{\alpha}}\big] \,,$$

and finally one arrives at

$$\mathcal{W}_\alpha = -\mathrm{i}\lambda_\alpha(y) + \theta_\alpha D(y) - \mathrm{i}(\sigma^{mn}\theta)_\alpha F_{mn} + \theta^2\left[\sigma^m\nabla_m\bar{\lambda}(y)\right]_\alpha \,,$$

with non-abelian field-strength and gauge covariant derivative respectively given by

$$F_{mn} = \partial_m A_n - \partial_n A_m - \tfrac{\mathrm{i}}{2}\,[A_m, A_n] \,, \quad \nabla_m \bullet = \partial_m \bullet - \tfrac{\mathrm{i}}{2}\,[A_m, \bullet] \,.$$

One can also introduce a coupling constant g through the redefinitions

$$\mathcal{V} \mapsto 2g\mathcal{V} \;\leftrightarrow\; A_m \mapsto 2gA_m \,, \quad \lambda \mapsto 2g\lambda \,, \quad D \mapsto 2gD \,,$$

which in turn imply

$$F_{mn} = \partial_m A_n - \partial_n A_m - \mathrm{i}g\,[A_m, A_n]\;, \quad \nabla_m \bullet = \partial_m \bullet - \mathrm{i}g\,[A_m, \bullet]\;.$$

This results in $\mathcal{W}_\alpha \mapsto 2g\mathcal{W}_\alpha$, therefore, in order that the kinetic terms be canonically normalized, the superspace integral defining the lagrangian has to be multiplied by factor $1/4g^2$. In this way, one gets

$$\mathcal{L}_{\mathrm{SYM}} = \frac{1}{16g^2}\int \mathrm{d}^2\theta\;\operatorname{tr}\mathcal{W}^\alpha\mathcal{W}_\alpha + \text{h.c.} = \operatorname{tr}\Big[-\frac{1}{4}F_{mn}F^{mn} - \mathrm{i}\lambda\sigma^m\nabla_m\bar{\lambda} + \frac{1}{2}D^2\Big]\;. \quad (2.16)$$

On the other hand, a topological term can be included in a straightforward way if we take a complexified coupling. In particular, we can define $\tau = \frac{2\pi}{g^2} + \mathrm{i}\frac{\Theta_{\mathrm{YM}}}{2\pi}$, so that the lagrangian becomes now

$$\begin{aligned}\mathcal{L}_{\mathrm{SYM}} &= \frac{\tau}{32\pi}\int \mathrm{d}^2\theta\;\operatorname{tr}\mathcal{W}^\alpha\mathcal{W}_\alpha + \text{h.c.}\\ &= \operatorname{tr}\Big[-\frac{1}{4}F_{mn}F^{mn} - \mathrm{i}\lambda\sigma^m\nabla_m\bar{\lambda} + \frac{1}{2}D^2\Big] + \frac{\Theta_{\mathrm{YM}}}{8\pi^2}g^2\operatorname{tr}\Big[\partial_m(\bar{\lambda}\sigma^m\lambda) + \frac{1}{4}\epsilon_{mnpq}F^{mn}F^{pq}\Big]\;.\end{aligned}$$

2.3.4 Gauge-Matter Lagrangian

We can finally see how to couple the matter and gauge sectors in a supersymmetric framework.

Let us consider a chiral superfield Φ transforming in a given representation of the gauge group, say $\mathcal{R}$, so that $T^a \to (T^a_{\mathcal{R}})_{ij}$, with $i,j = 1,\ldots,\dim\mathcal{R}$. For sake of clarity, in the following the subscript $\mathcal{R}$ will be omitted when unnecessary. The action of the gauge group on the matter superfield is defined as

$$\Phi \mapsto \Phi' = e^{\mathrm{i}\Lambda}\Phi\;,\;\; \Lambda = \Lambda^a T^a\;,$$

for Λ again a chiral superfield to ensure Φ' be such. However, the kinetic term we considered so far $\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\bar{\Phi}\Phi$ is not gauge invariant, indeed

$$\bar{\Phi}\Phi \mapsto \bar{\Phi}e^{-\mathrm{i}\bar{\Lambda}}e^{\mathrm{i}\Lambda}\Phi \neq \bar{\Phi}\bar{\Phi}\;.$$

Recalling that $e^{\mathcal{V}} \mapsto e^{-i\bar{\Lambda}}e^{\mathcal{V}}e^{-i\Lambda}$, we see that we can construct a gauge invariant kinetic term as $\bar{\Phi}e^{\mathcal{V}}\Phi$. On the other hand, also the superpotential has to be compatible with gauge symmetry. This means that terms like $t_{i_1\ldots i_n}\Phi^{i_1}\cdots\Phi^{i_n}$ are allowed if and only of $t_{i_1\ldots i_n}$ is an invariant tensor for the gauge group, and if $\mathcal{R}\times\cdots\times\mathcal{R}$ contains the singlet representation. Keeping this in mind, we can write the complete lagrangian for charged matter as

$$\mathcal{L}_{\mathrm{m}} = \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\bar{\Phi}e^{\mathcal{V}}\Phi + \int \mathrm{d}^2\theta\;W(\Phi) + \int \mathrm{d}^2\bar{\theta}\;\overline{W}(\bar{\Phi})\,.$$

Working in the WZ gauge, it is possible to find the D-term of this lagrangian. One has $\bar{\Phi}e^{\mathcal{V}}\Phi = \bar{\Phi}\Phi + \bar{\Phi}\mathcal{V}\Phi + \frac{1}{2}\bar{\Phi}\mathcal{V}^2\Phi$, and while the first term is already known, we have for the second and the third contribution

$$\bar{\Phi}\mathcal{V}\Phi\big|_{\theta^2\bar{\theta}^2} = \frac{\mathrm{i}}{2}\bar{\phi}A^m\,\partial_m\phi - \frac{\mathrm{i}}{2}\partial_m\bar{\phi}\,A^m\phi - \frac{1}{2}\bar{\psi}\bar{\sigma}^m A_m\psi + \frac{\mathrm{i}}{\sqrt{2}}\bar{\phi}\lambda\psi - \frac{\mathrm{i}}{\sqrt{2}}\bar{\psi}\bar{\lambda}\psi + \frac{1}{2}\bar{\phi}D\phi\,,$$
$$\bar{\Phi}\mathcal{V}^2\Phi\big|_{\theta^2\bar{\theta}^2} = \frac{1}{2}\bar{\phi}A^m A_m\phi\,.$$

All in all, one gets

$$\bar{\Phi}e^{\mathcal{V}}\Phi_{|_{\theta^2\bar{\theta}^2}} = \overline{\nabla_m\phi}\,\nabla^m\phi - \mathrm{i}\bar{\psi}\bar{\sigma}^m\nabla_m\psi + \bar{F}F + \frac{\mathrm{i}}{\sqrt{2}}\bar{\phi}\lambda\psi - \frac{\mathrm{i}}{\sqrt{2}}\bar{\psi}\bar{\lambda}\phi + \frac{1}{2}\bar{\phi}D\phi\,,$$

with

$$\nabla_m = \partial_m - \frac{\mathrm{i}}{2}A_m^a T_{\mathcal{R}}^a\,,$$

and $\bar{\phi}\lambda\psi = \bar{\phi}_i(T^a_{\mathcal{R}})_{ij}\lambda^a\psi_j$, similarly for the complex conjugate and terms alike. Introducing a coupling g as we did previously, and using the identity $\bar{\psi}\bar{\sigma}^m\nabla_m\psi = \psi\sigma^m\nabla_m\bar{\psi}$ we get

$$\bar{\Phi}e^{\mathcal{V}}\Phi\big|_{\theta^2\bar{\theta}^2} = \overline{\nabla_m\phi}\,\nabla^m\phi - \mathrm{i}\psi\sigma^m\nabla_m\bar{\psi} + \bar{F}F + \mathrm{i}g\sqrt{2}\,\bar{\phi}\lambda\psi - \mathrm{i}g\sqrt{2}\,\bar{\psi}\bar{\lambda}\phi + g\bar{\phi}D\phi\,,$$

with $\nabla_m = \partial_m - \mathrm{i}gA^a_m T^a_{\mathcal{R}}$. Actually, there is still a term that we can add to the lagrangian: this is the so-called *Fayet-Iliopoulos term.* Suppose that the gauge group contains some $U(1)$ factor. Let $\mathcal{V}^A$ be the vector superfield corresponding to one of these factors. Its D-term transforms as a total spacetime derivative under $\mathcal{V}^a \mapsto \mathcal{V}^a - \mathrm{i}\Lambda + \mathrm{i}\bar{\Lambda}$, hence a lagrangian like

$$\mathcal{L}_{\text{F-I}} = \sum_{A=1}^{M}\xi_A\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\mathcal{V}^A = \frac{1}{2}\sum_{A=1}^{M}\xi_A D^A$$

is both susy invariant and gauge invariant.

All in all, the most general supersymmetric invariant, renormalizable, gauge invariant lagrangian describing the coupling of matter and gauge fields is

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{SYM}} + \mathcal{L}_{\text{m}} + \mathcal{L}_{\text{F-I}}\\ &= \frac{\tau}{32\pi}\int \mathrm{d}^2\theta\;\operatorname{tr}\mathcal{W}^\alpha\mathcal{W}_\alpha + \text{h.c.}\\ &\qquad + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\bar{\Phi}e^{2g\mathcal{V}}\Phi + \Big(\int \mathrm{d}^2\theta\;W(\Phi) + \text{h.c.}\Big) + \sum_A \xi_A\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\mathcal{V}^A\,. \end{aligned}\tag{2.17}$$

2.3.5 Supersymmetric Vacua

From the explicit expressions in terms of the component fields that we have found before, one can notice that (2.17) contains the auxiliaty fields D^a and F^j. These can be integrated out by means of their equations of motion

$$\bar{F}_j = \frac{\partial W}{\partial\phi_j}\,,\quad D^a = -g\big(\bar{\phi}T^a\phi + \xi^a\big)\,,\tag{2.18}$$

with the condition $\xi^a = 0$ for $a \notin A$. Plugging (2.18) into (2.17) one obtains the scalar potential

$$V(\phi,\bar{\phi}) = \frac{\partial W}{\partial \phi^j}\frac{\partial \overline{W}}{\partial \bar{\phi}^j} + \frac{1}{2}g^2\sum_a \left|\bar{\phi}_j (T^a)^i_j \phi^j + \xi^a\right|^2 = \left(\bar{F}F + \frac{1}{2}D^2\right)\Big|_{\text{on-shell}} .$$

The scalar potential is a semi-definite positive quantity, namely

$$V(\phi,\bar{\phi}) \geq 0\,,$$

the equality being saturated when the D-term equation and the F-term equation are simultaneously satisfied, i.e. when

$$\bar{F}_j(\phi) = 0 \quad \text{and} \quad D^a(\phi,\bar{\phi}) = 0\,. \tag{2.19}$$

The configurations satisfying (2.19) are supersymmetric vacua. This can be understood noting that

- a vacuum has to be Lorentz invariant, hence all field derivatives as well as any field apart from scalar fields have to be null on the vacuum;
- the only allowed non-trivial term in the hamiltonian which is non-vanishing has to be scalar, and thus it coincides with the scalar potential.

It follows that the vacua are in one-to-one correspondence with the minima (either global or local) of the scalar potential. On the other hand, we know that in supersymmetry the energy is non-negative. This holds in particular for susy vacua $|\Omega\rangle$, namely

$$\langle \Omega\,| P^0 \,|\Omega\rangle \sim \sum_\alpha \left(\|Q_\alpha\,|\Omega\rangle\,\|^2 + \|(Q_\alpha)^\dagger\,|\Omega\rangle\,\|^2\right) \geq 0\,.$$

The vacuum energy is null if and only if $|\Omega\rangle$ is a suspersymmetric state, which is to say $Q_\alpha\,|\Omega\rangle = 0 = Q^\dagger_\alpha\,|\Omega\rangle$ for all α. On the contrary, susy is broken if and only if the vacuum energy is positive. One can conclude that susy vacua are in one-to-one correspondence with the zeros of the scalar potential. In order to find the zeros of $V(\phi,\bar{\phi})$, one first looks for the socalled *D-flat directions*, namely the space of scalar field vevs such that

$$D^a(\phi,\bar{\phi}) = 0\,. \tag{2.20}$$

The subset of the D-flat directions which are also *F-flat directions*, i.e. which satisfies

$$F_j(\phi) = 0 \tag{2.21}$$

is called *(classical) moduli space*: this is the space of the classical susy vacua. In solving for (2.20) and (2.21) one should eliminate any gauge ambiguity, because solutions related by a gauge transformation are physically equivalent. The moduli space represents the space of fields the scalar potential does not depend on. Each of the flat directions has an associated massless scalar particle, called modulus.

Let us suppose that the scalar potential has several degenerate discrete (gauge inequivalent) minima. These minima, which coincide with the vacua of the theory, are all physically equivalent. However, the fields are continuous functions and hence they cannot jump from one minimum to the other, because the minima form a discrete set. Novertheless, as we will see in the following chapter, when the vacuum manifold displays such a disconnected topology, there are peculiar field configurations which can interpolate between different vacua. These configurations are generally called *solitons*, and can be present in a wide variety of theories, ranging from high energy physics to condensed matter physics.

Chapter 3

Solitary Waves, Solitons, and Domain Walls

In this chapter we want to recall some basic properties of solitons and solitary waves [32], moving then to the discussion of domain walls, a particular case of soliton solution. To begin with, we consider the simplest wave equation $\Box\phi(t, x) = 0$ for a scalar field in $(1 + 1)$-dimensions. This equation is dispersionless, which means that its solutions are localised wave-packets which propagate undistorted in shape. The equation is also linear, hence the sum of two or more localised wave packets is still a solution. For $t \to -\infty$ different wave packets are separated in space, but as the time goes by, they approach each other and come to collide at a certain $t = t_C$. After that, however, linearity guarantees that for $t \to +\infty$ the wave packets return widely separated, and each of them retains its original shape. Clearly, these features are due to the extreme simplicity of the equation we are dealing with. Is it possible to have solutions with at least one of these properties, even in more complicated cases?

3.1 Solitary Waves vs. Solitons

Roughly speaking, a *solitary wave* is a field configuration which moves without distortion in a non-linear and dispersive set-up; if, additionally, these configurations are not modified by scattering, then they are called *solitons*. Actually, in the literature there is no standard definition of what a solitary wave or a soliton is. However, we can consider a working definition which is well suited for many practical cases, and which allows us to underline, in a rather simple way, which are the fundamental features of these peculiar solutions. We can quantify the requirements of absence of dispersion and retainment of the original shape after scattering as follows. Let us consider a system of R coupled fields $\{\phi_i\}_{i=1}^{R}$, with an associated energy density $\mathcal{H}(t, \mathbf{x})$ – whose space integral is the conserved total energy functional $H[\phi]$. We can suppose that the minimal value of $H[\phi]$ be zero. Solutions of non-linear field equations are called *localised* configurations if, at any time t, they have finite energy density in a limited spatial region, and if $\mathcal{H}(t,\mathbf{x}) \xrightarrow{|\mathbf{x}|\to\infty} 0$ rapidly enough so as to be an integrable function. We are finally ready to provide a definition for solitary waves.

Definition 6. A *solitary wave* is a non-singular localised solution of non-linear field equations having energy density whose spacetime dependence is $\mathcal{H}(t,\mathbf{x}) = \mathcal{H}(\mathbf{x}-\mathbf{u}t)$, for some velocity vector $\mathbf{u}$.

We notice that the definition holds irrespectively to the space dimension. Moreover, one can see that any static localised solution is a solitary wave with velocity $\mathbf{u} = \mathbf{0}$. We can now define solitons, which are nothing but solitary waves that are stable under scattering. This feature makes such a big difference that the vast majority of solutions of non-linear field equations are indeed solitary waves.

Definition 7. Assume to have a system of coupled non-linear field equations, having a solitary wave solution with energy density $\mathcal{H}_0(\mathbf{x}-\mathbf{u}t)$. Suppose also that these equations admit a solution for $t \to -\infty$ consisting of n solitary waves, which have arbitrary initial postions and velocities. This solution is characterised by an energy density $\mathcal{H}(t,\mathbf{x})$, which is such that

$$\mathcal{H}(t,\mathbf{x}) \quad \xrightarrow{t\to-\infty} \quad \sum_{i=1}^{n} \mathcal{H}_0(\mathbf{x}-\mathbf{r}_i-\mathbf{u}_i t)\,.$$

If the evolution governed by the non-linear equations is such that

$$\mathcal{H}(t,\mathbf{x}) \quad \xrightarrow{t\to+\infty} \quad \sum_{i=1}^{N} \mathcal{H}_0(\mathbf{x}-\mathbf{r}_i-\mathbf{u}_i t+\boldsymbol{\epsilon}_i)\,,$$

for $\boldsymbol{\epsilon}_i$ a constant vector, then such a solitary wave is a *soliton.*

This definition implies that, whatever occurs, the solitary waves will end up recovering their initial shape, and the only reminiscence of their interaction is a little deviation $\boldsymbol{\epsilon}_i$ from the initial trajectories. Solitons are solitary waves, but the converse does not hold, usually. The fact that the definition of solitons is particularly involved reflects the fact that it is usually very difficult to show whether a solitary wave is also a soliton.

3.2 Solitary Waves in Two-dimensional Scalar Theories

We will now consider the simplest possible case, namely static solutions for a theory of a single real scalar field in a $(1+1)$-dimensional spacetime. Keeping the potential $V(\phi)$ as general as possible, the lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\phi')^2 - V(\phi)\,, \tag{3.1}$$

where the dot and the prime denote derivatives with respect to the time and the spatial coordinate, respectively. The non-linear field equation stemming from the lagrangian is

$$\Box\phi = \ddot{\phi} - \phi'' = -\frac{\partial V}{\partial \phi} \,,$$

and we see that the potential represents the non linear term. The important quantity we have to consider is the conserved total energy functional

$$H[\phi] = \int_{-\infty}^{+\infty} \mathrm{d}x \, \left[\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\phi')^2 + V(\phi)\right] .$$

We suppose that the potential has $M \geq 1$ distinct absolute minima, and without loss of generality we can assume that the minima coincide with the zeros of the potential. In this setting, we see that necessarily $H[\phi] \geq 0$. In particular, if we let $\phi_0^{(j)}$ $(j = 1, \dots, M)$ be the value of the field at which the potential reaches its j-th minimum, we have

$$H[\phi]\big|_{\phi_0^{(j)}} = 0 \,, \quad \forall\, j = 1, \dots, M \,,$$

i.e. the energy functional is minimised whenever the potential is minimised. Let us focus on static solutions. Since a solitary wave has, by definition, localised energy density, it is clear that it must approach one of the minima of the potential as $x \to \pm\infty$. If there is a unique minimum, then $\phi(x) \to \phi_0$ for $x \to \pm\infty$; instead, in case $V(\phi)$ has several degenerate minima, the field can approach a certain $\overline{\phi}_0^j$ for $x \to -\infty$, and either the same or a different one for $x \to +\infty$. These are the boundary conditions which we have to impose on the solutions of $\phi'' = \frac{\partial V}{\partial \phi}$. It is interesting to note that the system we are considering has a mechanical analogue. Let us identify the space coordinate with a ficticious time variable, and the field with the spatial coordinate of a unit-mass particle, i.e. $\phi(x) \to X(\tau)$. Correspondingly, for the the field equation

$$\phi'' = \frac{\partial V}{\partial \phi} \quad \to \quad \frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} = \frac{\partial V}{\partial X} \,, \tag{3.2}$$

thus we have obtained the Newton's second law of dynamics for a pointlike unit-mass particle in a conservative potential $-V(X(\tau))$, with conserved mechanical energy $E = \frac{1}{2}(\mathrm{d}X/\mathrm{d}\tau)^2 - V(X)$. This is not to be confused with the total energy functional, which, with the identifications above, reads

$$H[X] = \int_{-\infty}^{+\infty} \mathrm{d}x \, \left[\frac{1}{2}\left(\frac{\mathrm{d}X}{\mathrm{d}\tau}\right)^2 + V(X)\right] ,$$

and represents the total action of the motion of the particle. We can see that the static solitary wave solutions correspond to finite-action, zero-energy trajectory of the particle. Integrating the expression obtained by multiplying the rightmost equation in (3.2) with $X' \equiv \frac{\mathrm{d}X}{\mathrm{d}\tau}$, we obtain

$$\frac{1}{2}(X')^2 = V(X) \,, \tag{3.3}$$

with null integration constant due to the boundary conditions. We can now understand the qualitative behavior of the particle:

- if $V(\phi)$ has a single minimum at $\phi = \phi_0$, the particle feels a potential $-V(X(\tau))$ which is everywhere negative but in correspondence with the absolute maximum at $X = X_0$, where it is null. The boundary conditions demand that the particle starts at X_0 for $\tau \to -\infty$, and comes back there again for $\tau \to +\infty$. However, non-trivial solutions cannot satisfy these requirements. Indeed, after the particle has left the top of the potential, it will never be able to change direction and come back where it started, because the kinetic energy becomes larger and larger;

- suppose now that the potential $V(\phi)$ has several degenerate minima. For concreteness, we can suppose that there are three minima, and correspondingly that $-V(X)$ has three maxima at $X_0^{(1)}$, $X_0^{(2)}$ and $X_0^{(3)}$. It is possible now to satisfy the boundary conditions, e.g. taking solutions which start from $X_0^{(1)}$ at $t \to -\infty$ and arrive at $X_0^{(2)}$ for $t \to +\infty$. There are actually four possibilities, namely $1 \to 2$, $2 \to 3$, $3 \to 2$ and $2 \to 1$, where the number $j = 1, 2, 3$ stands for the maximum of $V(X)$ under consideration. However, it is not possible to have a solution as $1 \to 3$, because at the $X_0^{(j)}$'s the velocity, the acceleration and all the successive derivatives vanish. Hence, when the particle has reached a maximum it remains trapped.

We can conlude that, if the potential has just one minimum, non-trivial static solitary wave solutions are not present. Conversly, if the potential has n degenerate minima, there are $2(n-1)$ such solutions, connecting two neighbouring minima. Finally, we can integrate explicitly (3.3) by quadratures. Indeed, we have $\mathrm{d}\phi/\mathrm{d}x = \pm[2V(\phi)]^{1/2}$, and therefore we arrive at

$$x - \bar{x} = \pm \int_{\phi(\bar{x})}^{\phi(x)} \frac{\mathrm{d}z}{\sqrt{2V(z)}}\,, \tag{3.4}$$

for $\bar{x}$ an arbitrary point in space. Since $\phi(x)$ approaches any two neighbouring minima of $V(\phi)$ for $x \to \pm\infty$ only, at any finite value of x the potential is non-vanishing. Hence the integrand is non-singular and positive everywhere but at the end-points if $x \to +\infty$ and $\bar{x} \to -\infty$.

3.2.1 Coupled Scalar Fields

Let us consider now a system of R coupled real scalar fields, described by the lagrangian

$$\mathcal{L} = \sum_{i=1}^{R} \frac{1}{2}\left[(\dot{\phi}_i)^2 - (\phi_i')^2\right] - V(\{\phi_i\})\,,$$

where the potential $V(\{\phi_i\})$ is supposed to vanish in correspondence of its minima. The field equations for static configurations are a set of R coupled equations

$$\phi_i'' = \frac{\partial V}{\partial \phi_i}\,.$$

In this framework, no general method exist to find all localised solutions. Nevertheless, we can still make use of the mechanical analogue that proved very useful previously. In the present case the particle moves in an R-dimensional space under the influence of the potential $-V(\{X_i(\tau)\})$. The solutions we are looking for require the boundary conditions

$$V(\{X_i\}) = 0 \quad \text{and} \quad X_i' = 0 \quad \forall\ i = 1, \dots, R\,,$$

and this corresponds to a motion between two neighbouring minima (maxima) of $V(\{X\})$ (of $-V(\{X\})$). There are however two major differences with respect to the single field case:

- if the potential $V(\{\phi_i\})$ has a unique minimum, it is still true that the particle cannot have zero velocity after leaving the extremal point. However, the particle could move along a closed path, so that it returns to the starting point. This implies that a solitary wave solution is indeed possible;
- even in the simplified setting provided by the mechanical analogue there is no simple way to integrate the coupled equations of motion.

There are particular cases where it is possible to find solutions by using the following strategy. Suppose we can guess an orbit for the analogue particle respecting the boundary conditions. We have thus a one-dimensional curve, described by $R-1$ relations, in a R-dimensional spacetime. With the equation for the orbit at hand, we can obtain by quadratures the x dependence of $\phi_i(x)$ *along this curve.* For concreteness, we can e.g. consider the case of two scalar fields ϕ and ψ. These fields are coupled by a potential $V(\phi, \psi)$, which we assume to have two degenerate minima at the points P_1 and P_2. The analogue particle moves on the plane $\{\phi, \psi\}$. Suppose to have an orbit $\gamma(\phi, \psi) = 0$. It is possible to relate directly the orbit to the potential proceeding as follows. First, we eliminate the x dependence:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}x} = 0 = \frac{\partial\gamma}{\partial\phi}\phi' + \frac{\partial\gamma}{\partial\psi}\psi' \quad \Longrightarrow \quad \left(\frac{\partial\gamma}{\partial\phi}\right)^2 (\phi')^2 = \left(\frac{\partial\gamma}{\partial\psi}\right)^2 (\psi')^2\,. \tag{3.5}$$

Then, we integrate the equations of motion in such a way to obtain two relations like (3.3), i.e.

$$\tfrac{1}{2}(\phi')^2 = \int \mathrm{d}\phi\ \frac{\partial V}{\partial\phi} + c_\phi\,, \quad \tfrac{1}{2}(\psi')^2 = \int \mathrm{d}\psi\ \frac{\partial V}{\partial\psi} + c_\psi\,,$$

for c_ϕ and c_ψ integration constants, and finally we can substitute back in (3.5), obtaining

$$\left(\frac{\partial\gamma}{\partial\phi}\right)^2 \left(\int \mathrm{d}\phi\ \frac{\partial V}{\partial\phi} + c_\phi\right) = \left(\frac{\partial\gamma}{\partial\psi}\right)^2 \left(\int \mathrm{d}\psi\ \frac{\partial V}{\partial\psi} + c_\psi\right).$$

We have thus obtained a relationship between $\gamma(\phi, \psi)$ and $V(\phi, \psi)$ which does not involve the coordinate x. It has to be kept in mind that the itegrals above are to be evaluated along the orbit, namely ϕ and ψ are related by $\gamma(\phi, \psi) = 0$ in the integrals.

The above procedure suffers some drawbacks: it is a trial and error approach, and there is no mean by which we can derive the equation of the orbit, rather than guess it. Secondly, we are forced to keep some free parameters which are fixed a posteriori by requiring the solution to work. In doing this, however, we are constraining the solution, usually in a quite tighten way. Nonetheless, this strategy can be straightforwardly generalised to the case of $R > 2$ fields, though the computations will inevitably become lengthier.

3.2.2 Topological Indices

In this framework, it is possible to define peculiar conserved quantities called *topological charges*. The key difference between these objects and other more conventional conserved charges is their origin, because topological indices are not (in general) related to the presence of symmetries.

Let us consider for simplicity the case of a single scalar field (3.1), with a potential having a discrete number of degenerate minima where it vanishes. We consider non-singular finite-energy solutions of the equation of motion, not necessarily being solitary waves or solitons. In any case, the field has to tend to one of the minima of the potential at any time and at any point on spatial infinity, irrespectively of the fact that it is static or time-dependent. In the $(1+1)$-dimensional case, the spatial infinity corresponds to the points $\pm\infty$. Let ϕ_0 be one of the minima of the potential, and suppose e.g. that at some instant $\bar{t}$ we have

$$\lim_{x\to+\infty} \phi(x, \bar{t}) = \phi_0^{(+)} .$$

As time goes by, the field varies in a continuous fashion as dictated by the equation of motion. In particular, $\phi(+\infty, t)$ is a continuous function of time. Since the energy of the solution is finite and conserved, the field $\phi(+\infty, t)$ has to be one of the minima of $V(\phi)$ for all t. However, the minima of the potential constitute a *discrete set*, hence $\phi(+\infty, t)$ remains stuck at the initial minimum forever, for the field is a continuous function and cannot make discrete jumps. The same applies also if we consider the minimum $\phi_0^{(-)}$ at which the field tends for $x \to -\infty$. The space of all non-singular, finite-energy solutions can be divided in sectors characterised by two time independent indices: $\phi(-\infty)$ and $\phi(+\infty)$. Different regions are topologically disconnected, which is to say that the configurations we are considering cannot go from one sector to the other through continuous deformations. A *topological charge* is just (proportional to) the difference of the topological indices characterising a given topological sector. As any other conserved quantity, it can be obtained as the space integral of the time component of a divergenceless current.

The generalization to the case of R coupled scalar fields is straightforward. Now, we suppose that the potential $V(\{\phi_i\})$ has M degenerate minima at the points $\{P_1, \ldots, P_M\}$. The fields have to approach one of these points for $x \to -\infty$, and either the same or another one for $x \to +\infty$. Then, the pair (P_j, P_k) characterises the topological sector where a particular solution lives. If $j = k$, the solution is termed *non-topological*, otherwise it is said *topological*. As we have seen above, for the case of only one scalar field static solutions are necessarily topological, while, if there are several fields, static non-topological solutions can exist.

Solitary wave solutions have the remarkable feature of stability. This can be understood on the basis that, within each topological sector, these peculiar solutions have the minimal energy possible, therefore, there is no lower energy state to which solitary waves can decay.

3.2.3 The $\mathbb{Z}_2$ Kink

Basically, the only complications in the case of a single real scalar field in $(1+1)$-dimensions can arise in (3.4), if the form of the potential is too involved. A particular case in which the integration is feasible is that of the ϕ^4 theory, namely for

$$V(\phi) = \frac{1}{4}\lambda^2\Big(\phi^2 - \frac{\mathrm{m}^2}{\lambda^2}\Big)^2 .$$

The model enjoys the $\mathbb{Z}_2$ symmetry $\phi \mapsto -\phi$, hence the two minima $\phi_0 = \pm\mathrm{m}/\lambda \equiv \pm v$ are physically equivalent. Nonetheless, the choice of either $+v$ or $-v$ leads to a spontaneous symmetry breaking. By choosing $\phi(\bar{x}) = 0$ in equation (3.4), we get the explicit solution

$$\phi_k(x) = +v \,\tanh\Big[\frac{\mathrm{m}}{\sqrt{2}}(x-\bar{x})\Big]\,, \quad \phi_{ak}(x) = -v \,\tanh\Big[\frac{\mathrm{m}}{\sqrt{2}}(x-\bar{x})\Big]\,. \tag{3.6}$$

The plus sign corresponds to the so-called *kink*, while the solution with the minus sign is called *antikink*. A constant shift $\bar{x} \mapsto \bar{x} + a$ only moves the solution in space: this reflects the translational invariance of the theory. However, one can notice that, once a choice of $\bar{x}$ is made, translational symmetry is spontaneously broken. In fact, we have found a family of solutions, rather than a single one, and only by considering the family as a whole we can preserve translational invariance. The symmetries under $x \mapsto -x$ and $\phi \mapsto -\phi$ imply (taking $\bar{x} = 0$)

$$\phi_k(x) = -\phi_{ak}(x) = \phi_{ak}(-x)\,,$$

where $\phi_k(x)$ stands for the kink and $\phi_{ak}(x)$ for the antikink. Moreover, the fact that we started with a relativistic lagrangian means that we can obtain a moving solution just by applying a Lorentz transformation to (3.6). Focusing on the kink, we see that the energy density of this solutions is $\mathcal{H}(x) = (m^4/2\lambda^2)\,\mathrm{sech}^4\big[m(x-\bar{x})/\sqrt{2}\big]$, and the total conserved energy, called *kinkmass*, is

$$M_k = \int_{-\infty}^{+\infty} \mathrm{d}x\ \mathcal{H}(x) = \frac{2\sqrt{2}}{3}\mathrm{m}v^2\,.$$

It is interesting to notice that, applying a Lorentz boost with velocity u, the kinkmass changes according to $M_k' = M_k/\sqrt{1-u^2}$, exactly as the mass of a relativistic particle changes under a Lorentz transformation. This suggests the interpretation of the kink as a *lump*, namely a localised, self-supporting packet of energy. According to our previous definition, a kink is not a soliton, but only a solitary wave. This is easily understood from the fact that two kinks cannot even exist at the same time. Indeed, if the first begins at $\phi_1(-\infty) = -v$ and ends at $\phi_1(+\infty) = +v$, the second one will start at $\phi_2(-\infty) = 0$, ending at $\phi_2(+\infty) = 2v$. This will then produce a constant non-vanishing energy density, invalidating the locality, which is actually the fundamental requirement for soliton solutions to exist. It has been shown that also two antikinks and a kink-antikink system cannot produce localised solutions.

Also, we can notice that (3.6) is singular as $\lambda \to 0$: this signals the non-perturbative nature of the kink solution, which cannot be obtained from a perturbative expansion of the linear equation.

Furthermore, this model displays four topological sectors: two of them are characterised by $(\pm v, \pm v)$, and correspond to the trivial solution; then we have $(v, -v)$ and $(-v, v)$ which correspond to the kink and the antikink, respectively. The conserved topological charge and the associated current are given by

$$C = \int_{-\infty}^{+\infty} \mathrm{d}x \; k^0 = v \left[\phi(+\infty) - \phi(-\infty)\right] \quad \Leftrightarrow \quad k^m(x) = v \left[\epsilon^{mn}\partial_n \phi(x)\right] .$$

We see that $\partial_m k^m = 0$ identically: as we mentioned above, the presence of such conserved charges does not follow from any continuous symmetry: rather, it is due to the finiteness of energy.

3.2.4 The Bogomol'nyi Method

There is another general method by which we can find the first-order equation kink solution (3.3), as an alternative to considering the mechanical analogue. Let us consider the total-energy functional for static configurations. Through some manipulations we can rewrite

$$\begin{aligned} H[\phi] &= \int_{\mathbb{R}} \mathrm{d}x \, \left[\frac{1}{2}(\phi')^2 + V(\phi)\right] = \int_{\mathbb{R}} \mathrm{d}x \, \left[\frac{1}{2}\left(\phi' \mp \sqrt{V(\phi)}\right)^2 \pm \sqrt{2V(\phi)}\phi'\right] \\ &= \int_{\mathbb{R}} \mathrm{d}x \, \left[\frac{1}{2}\left(\phi' \mp \sqrt{V(\phi)}\right)^2\right] \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \mathrm{d}z \; \sqrt{2V(z)} \, . \end{aligned} \tag{3.7}$$

Since the first integral in (3.7) is positive definite, we see that, necessarily, the so-called *Bogomol'nyi-Prasad-Sommerfield (BPS) bound* holds

$$H[\phi] \geq \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \mathrm{d}z \; \sqrt{2V(z)} \, .$$

The bound is saturated by the configurations which satisfy the BPS equation

$$\phi' \mp \sqrt{V(\phi)} = 0 \, ,$$

which is just the equation (3.3). In particular, we know that for the solutions satisfying the boundary conditions $\phi(-\infty) = -v$ and $\phi(+\infty) = +v$, the energy is minimal, and it is equal to

$$H_{\min} = \pm \int_{\phi(-\infty)}^{\phi(+\infty)} \mathrm{d}z \; \sqrt{2V(z)} = \pm \frac{2\sqrt{2}}{3} \mathrm{m} v^2 \, ,$$

and this coincides with the (anti)kinkmass we encountered above.

3.2.5 The Sine-Gordon Model

As we stated already, it is usually complicated to find out whether a given solitary wave solution is also a soliton. The so-called *sine-Gordon model* is one of those cases in which a soliton solution has been found explicitly.

The model consists of a single real scalar field in $(1+1)$-dimension, whose dynamics is ruled by the lagrangian

$$\mathcal{L}_{\rm sG} = \frac{1}{2}\partial_m\phi\,\partial^m\phi - \frac{\mathrm{m}^4}{\lambda^2}\Big[1-\cos\Big(\frac{\lambda}{\mathrm{m}}\phi\Big)\Big]\,.$$

We can simplify the lagrangian by applying the redefinitions

$$y = \mathrm{m}x\,,\quad \tau = \mathrm{m}t\,,\quad \varphi = \frac{\lambda}{\mathrm{m}}\phi\,,$$

in such a way that the equation of motion reads

$$\frac{\partial^2\varphi}{\partial\tau^2} - \frac{\partial^2\varphi}{\partial y^2} + \sin\varphi = 0\,,$$

and the energy functional can be written as

$$H[\varphi] = \frac{\mathrm{m}^3}{\lambda^2}\int \mathrm{d}y\,\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial y}\right)^2 - (1-\cos\varphi)\right].$$

The lagrangian and the equation of motion are invariant under the discrete symmetries

$$\varphi\mapsto-\varphi\,,\quad \varphi\mapsto\varphi+2n\pi\,,$$

for $n\in\mathbb{Z}$, and consistently with these symmetries the energy functional vanishes at the minima of the potential $\varphi_n = 2n\pi$. We have thus an infinite number of topological sectors characterised by the indices (n_j, n_k), corresponding to the asymptotic values approached by the field for $x\to\pm\infty$. Without loss of information, one can assume that only φ modulo 2π be meaningful. Specialising the formula (3.4) to the present case, we can obtain the explicit solutions

$$\varphi_s(y) = 4\arctan\big[\exp(y-y_0)\big]\,, \tag{3.8}$$

$$\varphi_{as}(y) = -4\arctan\big[\exp(y-y_0)\big] = -\varphi_s(y)\,. \tag{3.9}$$

The first solution connects the minima $j\to j+1 \mod 2\pi$, and has topological charge $Q=+1$; the second one, instead, proceeds in the opposite sense $j+1\to j \mod 2\pi$ and has topological charge $Q=-1$. Unlike the case of the solutions for the ϕ^4 model, equations (3.8) and (3.9) are genuine solitons. The second one is often referred to as *antisoliton*, to underline that it is obtained by the soliton via a $\mathbb{Z}_2$ transformation, and it moves backwards. Actually, there is a third solution, called *doublet* or *breather*, which is given by

$$\varphi_b(\tau,y) = 4\arctan\left[\frac{\sin\big(v\tau/\sqrt{1+v^2}\big)}{v\,\cosh\big(y/\sqrt{1+v^2}\big)}\right].$$

Without entering into the details, we mention here only that the breather can be thought of as a bound state of a soliton-antisoliton pair. It is a periodic solution, where the soliton and the antisoliton oscillate with respect to one another with period $T_b = 2\pi\sqrt{1+v^2}/v$. One has to keep in mind that the solutions of the sine-Gordon model as we have reported above describe the field configuration in its own rest frame. It is interesting to notice that, differently from the soliton and the antisoliton, the doublet depends on time, this being due to the fact that it represents an oscillating system.

Many authors have show that φ_s, φ_{as} and φ_b are genuine solitons. Remarkably, one can show that a soliton and an antisoliton attract each other, while two solitons (or antisolitons) repel each other. However, despite the mutual attraction, a soliton and an antisoliton will separate after they have collided. In contrast, the breather system will never blow up, and the two waves will keep oscillating confined within $\pm 4 \arctan[(1/v)\,\mathrm{sech}(y/\sqrt{1+v^2})]$.

3.2.6 A No-Go Argument for Scalar Theories in $d > 1+1$

We have seen that, even in two dimensions, static solutions obey partial differential equations. Moreover, as soon as we consider more than two scalar fields, there is no general method we can employ to solve the equations of motion. In addition, there is an argument which states that it is not possible to find (non-trivial) static solitary wave solutions in three or more spatial dimensions. Indeed, let us consider a generic scalar theory in $(d+1)$-dimensions, comprised of R coupled real scalar fields $\phi \equiv \{\phi_i\}_{i=1}^{R}$

$$\mathcal{L} = \frac{1}{2}\partial_m \phi\, \partial^m \phi - V(\phi)\,.$$

We suppose that $V(\phi)$ is non-negative, and that it is null only at its absolute minima. The equation of motion for a static configuration is

$$\triangle \phi = \frac{\partial V}{\partial \phi}\,,$$

with $\triangle = \partial_i \partial^i$ denoting the Laplacian operator in d-dimensions. The above equation can be derived applying the principle of least action to the static energy functional

$$W[\phi] = \int \mathrm{d}^d x\, \left[\frac{1}{2}\partial_i \phi\, \partial^i \phi + V(\phi)\right] = W_1[\phi] + W_2[\phi]\,.$$

Suppose to have a static solution $\widehat{\phi}(\mathbf{x})$, and consider the rescaled (static) configuration $\phi_\alpha(\mathbf{x}) = \widehat{\phi}(\alpha \mathbf{x})$. The energy of the rescaled configuration is

$$H[\phi_\alpha] = W_1[\phi_\alpha] + W_2[\phi_\alpha] = \alpha^{2-d}\, W_1[\widehat{\phi}] + \alpha^{-d}\, W_2[\widehat{\phi}]\,, \tag{3.10}$$

and since $\widehat{\phi}(\mathbf{x})$ makes $W[\phi]$ stationary, it must in particular extremise $W[\phi_\alpha]$ with respect to variations in α, namely

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} W[\phi_\alpha]\bigg|_{\alpha=1} = 0\,. \tag{3.11}$$

By differentiating (3.10) and using (3.11) one gets

$$(2-d) W_1[\widehat{\phi}] = d\, W_2[\widehat{\phi}]\,. \tag{3.12}$$

From the explicit expression of $W[\phi]$, we see that it is non-negative, and such are also $W_1[\phi]$ and $W[\phi_2]$. Therefore, the last identity cannot be satisfied for a number of spatial dimensions $d \geq 3$, unless $W_1[\widehat{\phi}] = W_2[\widehat{\phi}] = 0$. Hence this argument forbids non-trivial static solutions for more than two spatial dimensions.

3.3 Domain Walls

We consider now a special class of soliton solutions called *domain walls*. They are the generalization of the kink solution to more than one spatial dimension: then, roughly speaking, they are smooth static configurations interpolating between the discrete minima of a potential. However, unlike the kink, domain walls are extended objects.

Given what we have seen in the previous section, why are we talking about a *static* configuration? Actually, the above argument forbis *finite energy* solutions in more than $d = 2$ spatial dimensions, and nothing is said about *infinite energy* solutions. In fact, domain walls are infinitely extended, and as such they have infinite energy: however, now it is not the energy *per se* to be meaningful, but the energy per unit area.

3.3.1 Non-Supersymmetric Walls

We begin with the simplest case possible, namey by generalizing the kink to $d = 4$ spacetime dimensions [34]. The lagrangian of the model reads

$$\mathcal{L} = \frac{1}{2}\partial_m\phi\,\partial^m\phi - \frac{\lambda^2}{4}\left(\phi^2 - v^2\right)^2 . \tag{3.13}$$

The potential has two discrete degenerate minima at $\phi_0 = \pm v$, and the choice of either of the two (physically equivalent) vacua produces the spontaneous breaking of the $\mathbb{Z}_2$ symmetry of the theory. As we noted already, the field cannot pass abruptly from $-v$ to $+v$, for it is a continuous function: there must be a transition region connecting the vacua, and this is indeed the domain wall. Therefore, a domain wall flags the breaking of some discrete symmetry. The energy density in the transition region is clearly larger than that in the vacua, but the wall arranges itself in order that its energy per unit surface is minimised. In other words, the domain wall is a configuration which minimises the *wall tension* functional

$$T_w \equiv \frac{H[\phi]}{\text{Area}} = \int \mathrm{d}z\,\left\{\frac{1}{2}\left[\frac{\mathrm{d}\phi(z)}{\mathrm{d}z}\right]^2 + \frac{\lambda^2}{4}\left[\phi^2(z) - v^2\right]^2\right\} .$$

We notice that the wall is considered in its rest frame, and we have assumed that the wall profile lies in the z-direction so that the boundary conditions are e.g. $\phi_w(z \to -\infty) = -v$ and $\phi_w(z \to +\infty) = +v$. The minimization condition leads to the equation of motion

$$-\frac{\mathrm{d}^2\phi_w}{\mathrm{d}z^2} + \lambda\phi_w(\phi_w^2 - v^2) = 0 .$$

As we have seen previously, we can consider the mechanical analogue, replacing $\phi_w(z)$ with $X(\tau)$, and thus rewriting the previous equation as $X'' - \lambda^2\, X(X^2 - v^2) = 0$. Adapting the steps we followed in section 3.2 to the present case, we arrive at the first order equation

$$\frac{\mathrm{d}\phi_w}{\mathrm{d}z} = \pm\frac{\lambda}{\sqrt{2}}(\phi_w^2 - v^2) , \tag{3.14}$$

where the minus sign corresponds to the wall solution and the plus sign to the antiwall solution, the ambiguity being solved thanks to the boundary conditions. The solution of the equations in (3.14) is

$$\phi_w(z) = \pm v \tanh\Big[\frac{\mathrm{m}}{\sqrt{2}}(z-\bar{z})\Big] \tag{3.15}$$

in agreement with (3.6).

Let us now apply the Bogomol'nyi method. We introduce a quantity $W(\phi)$ such that

$$V(\phi) = \tfrac{1}{2}\Big(\frac{\mathrm{d}W}{\mathrm{d}\phi}\Big)^2 ,$$

so that the minima of $W(\phi)$ coincide with those of the potential $V(\phi)$. In our case

$$W(\phi) = \frac{\lambda}{\sqrt{2}}\Big(\frac{1}{3}\phi^3 - v^2\phi\Big) ,$$

and wall tension can be rewritten as

$$T_w = \frac{1}{2}\int \mathrm{d}z\,\Big[\Big(\frac{\mathrm{d}\phi}{\mathrm{d}z}\Big)^2 + \Big(\frac{\mathrm{d}W}{\mathrm{d}\phi}\Big)^2\Big] = \int \mathrm{d}z\,\Big[\frac{1}{2}\Big(\frac{\mathrm{d}\phi}{\mathrm{d}z} \pm \frac{\mathrm{d}W}{\mathrm{d}\phi}\Big)^2 \mp \frac{\mathrm{d}\phi}{\mathrm{d}z}\frac{\mathrm{d}W}{\mathrm{d}\phi}\Big] . \tag{3.16}$$

Choosing the plus sign inside the square brackets and the minus sign outside corresponds to selecting the wall solution, conversely we would select the antiwall solution. In any case, the second term in the braces of the rightmost equation in (3.16) is a total derivative, and therefore it can be written as

$$\int_{-\infty}^{+\infty} \mathrm{d}z\,\frac{\mathrm{d}\phi}{\mathrm{d}z}\frac{\mathrm{d}W}{\mathrm{d}\phi} = \int_{-\infty}^{+\infty} \mathrm{d}z\,\frac{\mathrm{d}W(\phi(z))}{\mathrm{d}z} = W(v) - W(-v) \equiv \Delta W .$$

This is a boundary or topological term, and does not depend on any detail of the function $\phi(z)$, i.e. it is the same for any field satisfying the boundary conditions $\phi_w(z\to\pm\infty) = \pm v$. If we now focus on the wall solution, we have

$$T_w = -\Delta W + \frac{1}{2}\int \mathrm{d}z\,\Big(\frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{\mathrm{d}W}{\mathrm{d}\phi}\Big)^2 , \tag{3.17}$$

and since the integrand is positive definite, it follows that for any configuration interpolating between the vacua $\pm v$ the following condition holds:

$$T_w \geq -\Delta W .$$

This relation becomes an equality only when the integrand in equation (3.17) vanishes, namely when the BPS equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{\mathrm{d}W}{\mathrm{d}\phi} = 0$$

is satisfied. Noting that this is just equation (3.14), we conclude that domain walls are the configurations with boundary conditions $\phi_w(+\infty) = +v$ and $\phi_w(-\infty) = -v$, which minimise the tension functional, or, equivalently, which saturate the BPS bound.

3.4 Domain Walls in Supersymmetric Field Theory

In section 1.1 we have seen how central charges appear in the supersymmetry algebra. On the other hand, we are considering an $\mathcal{N} = 1$ supersymmetry framework, where *a priori* central charges are null. However, this is not the end of the story. Indeed, as noted first in [44] in supersymmetric theories where solitons are present, the algebra is centrally extended.

3.4.1 Central Charges in Minimal Supersymmetry

Consider a d-dimensional spacetime, and let n_Q be the minimal number of supersymmetry generators, which we gather in an n_Q dimensional vector as

$$Q_i = \begin{pmatrix} Q_1 & \dots & Q_{n_Q/2} & Q_1^\dagger & \dots & Q_{n_Q/2}^\dagger \end{pmatrix}\,, \quad Q^i = (Q_i)^{\mathrm{T}}\,.$$

Then, the number of central charges corresponds to the number of independent components of the $n_Q \times n_Q$ symmetric matrix $\{Q^i, Q_j\}$. We have therefore

$$N_{cc} = \frac{1}{2} n_Q (n_Q + 1)\,,$$

which is actually the *maximal* number of central charges. Indeed, from equation (1.2e) page 6, one sees that actually d central charges can be reabsorbed in the momentum by a redefinition of the latter. Nevertheless, there are particular situations in which (some of) the central charges that may appear in equation (1.2e) are dynamically distinguishable from the momentum, an thus cannot be reabsorbed therein. Nonetheless, further symmetry and dynamical constraints as well can diminish the number of central charges. On general grounds, central charges can be classified depending on their algebraic structure.

The $d = 2$ and $d = 3$ cases. For bidimensional (non-chiral) supersymmetric theories there are $n_Q = 2$ supersymmetry generators. Therefore, one expects at most $N_{cc} = 3$. In particular, two central charges are the component of a two-vector, while the other one is a scalar, and we have

$$\{Q_\alpha,\, Q_\beta\} = 2(\gamma^m\gamma^0)_{\alpha\beta}(P_m + Z_m) + \mathrm{i}(\gamma^5\gamma^0)_{\alpha\beta} Z\,,$$

where we have used gamma matrices in two dimensions. Actually, if we want to stick to the case of unbroken supersymmetry, we have to discard the vectorial central charge. Indeed, if it were present, there would be a vectorial order parameter which breaks Lorentz invariance as well as supersymmetry of the vacuum. Hence, we can conclude that, within this framework, only one central charge is possible.

In three dimensions we still have $n_Q = 2$ and hence $N_{cc} = 3$, and all the charges are collected in a three-components vector, so that

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^m\gamma^0)_{\alpha\beta}(P_m + Z_m)\,,$$

where now we have the gamma matrices in three dimensions. The central charge Z_m is just the generalization of the scalar central charge discussed above, indeed by an appropriate choice of the reference frame it can be reduced to a real number times the vector $(\,0\;0\;1\,)$. This charge is associated with a domain line directed along the second axis.

The $d = 4$ case. In this framework, we have $n_Q = 4$ wich implies $N_{cc} = 10$, and these charges show up in the anticommutators as

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^m_{\alpha\dot\alpha}(P_m + Z_m)\,, \quad \{Q_\alpha, Q_\beta\} = (\Sigma^{mn})_{\alpha\beta} Z_{[mn]}\,, \quad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = (\bar{\Sigma}^{mn})_{\dot\alpha\dot\beta} \bar{Z}_{[mn]}\,, \tag{3.18}$$

where $(\Sigma^{mn})_{\alpha\beta} = \sigma^m_{\alpha\dot\alpha}(\bar\sigma^n)^{\dot\alpha}{}_\beta$. The antisymmetric tensors $Z_{[mn]}$ and $\bar{Z}_{[mn]}$ are associated with domain walls, and reduce to a complex number and a three-component spatial vector orthogonal to the wall. Instead, Z_m is a four vector orthogonal to P_m wich is associated with strings, and reduces to a real number and three-dimensional unit spatial vector parallel to the string.

3.4.2 Domain Walls in the Wess-Zumino Model

Domain walls exist also in the simplest supersymmetric models, such as the minimal Wess-Zumino (WZ) model. The model is given by the lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{WZ}} &= \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\; \bar\Phi\Phi + \Big(\int \mathrm{d}^2\theta\; W(\Phi) + \mathrm{h.c.}\Big) \\
&= \partial_m\bar\phi\,\partial^m\phi - \mathrm{i}\psi\sigma^m\partial_m\bar\psi + F\bar F - \left[\frac{\partial W(\phi)}{\partial\phi}F + \frac{1}{2}\frac{\partial^2 W(\phi)}{\partial\phi\partial\phi}\psi\psi + \mathrm{h.c.}\right],
\end{aligned}$$

with renormalizable superpotential

$$W(\Phi) = \lambda\Big(\frac{\mathrm{m}^2}{\lambda^2}\Phi - \frac{1}{3}\Phi^3\Big)\,,$$

with m and λ not necessarily real parameters, since ϕ is complex. However, the phase of m and λ can be chosen at will thanks to the R-symmetry. We can eliminate the auxiliary fields F and $\bar F$ by means of their equations of motion, finding the scalar potential. One finds that there are two degenerate classical vacua at $\phi_0 = \pm\mathrm{m}/\lambda$: they are physically equivalent because of the $\mathbb{Z}_2$ symmetry, which is however spontaneously broken once either of the two vacua is chosen.

We want to find the BPS equation for purely bosonic wall configurations: hence, we put $\psi = 0$ in the lagrangian. We suppose that the domain wall profile is along the z-direction, and we assume that we are in the wall rest frame. Then, we can manipulate the wall tension functional as

$$\begin{aligned} T_w &= \int_{-\infty}^{+\infty} \mathrm{d}z \left(\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}z}\frac{\mathrm{d}\phi}{\mathrm{d}z} + \bar{F}F \right) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}z \left[\left(e^{-\mathrm{i}\eta} \bar{F} \frac{\mathrm{d}\phi}{\mathrm{d}z} + \mathrm{h.c.} \right) + \left| \frac{\mathrm{d}\phi}{\mathrm{d}z} + e^{\mathrm{i}\eta} F \right|^2 \right] \\ &= \int_{-\infty}^{+\infty} \mathrm{d}z \left[\left(e^{-\mathrm{i}\eta} \frac{\mathrm{d}W}{\mathrm{d}z} + \mathrm{h.c.} \right) + \left| \frac{\mathrm{d}\phi}{\mathrm{d}z} + e^{\mathrm{i}\eta} F \right|^2 \right], \end{aligned} \tag{3.19}$$

using $\bar{F} = -\mathrm{d}W/\mathrm{d}\phi$, with η an *a priori* arbitrary phase. Now, we notice that the second term in (3.19) is non-negative, while the first term is a total derivative and depends only on the boundary conditions. Hence, we have

$$T_w \geq 2\,\mathbf{Re}\big(e^{-\mathrm{i}\eta}\Delta W\big)\,.$$

The procedure we have followed, sometimes referred to as *Bogomoln'yi completion*, can be performed for any value of η. Nevertheless, the strongest bound is obtained when $e^{-\mathrm{i}\eta}\Delta W$ is real, and this, in turn, happens if $\eta = \arg\left(\mathrm{m}^3/\lambda\right)$. As far as the solution to the BPS equation is concerned, the situation seems more complicated with respect to what we have seen above, because now

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = e^{\mathrm{i}\eta}\frac{\mathrm{d}\overline{W}}{\mathrm{d}\bar{\phi}}\,,$$

entails actually two equations: one for the real part and one for the imaginary. However, the following identity holds

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathbf{Im}\big(e^{-\mathrm{i}\eta}W\big)\right] = 0\,,$$

as one can prove taking the derivative of $e^{-\mathrm{i}\eta}W$ with respect to z, applying the BPS equation and taking the imaginary part. This implies the constraint

$$\mathbf{Im}\big(e^{-\mathrm{i}\eta}W\big) = \text{constant}\,,$$

which means that in the complex W plane the domain wall trajectory is a straight line. The explicit solution for a wall configuration is $\phi_w(z) = (\mathrm{m}/\lambda)\,\tanh(|\mathrm{m}|z)$, and is very similar to the solution (3.15). If we assume now that $\eta = \arg\left(\mathrm{m}^3/\lambda\right)$, when the BPS bound is saturated we have

$$T_w = 2|\Delta W| = \frac{8}{3}\left|\frac{m^3}{\lambda^2}\right| \equiv |Z|\,.$$

The central charge Z corresponds to the central extension of supersymmetry algebra given in (3.18). In particular, we have

$$\{Q_\alpha, Q_\beta\} = -4\,\Sigma_{\alpha\beta}\bar{Z}\,, \quad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = -4\,\bar{\Sigma}_{\dot\alpha\dot\beta}Z\,, \tag{3.20}$$

where

$$\Sigma_{\alpha\beta} = -\frac{1}{2}\int \mathrm{d}x_{[m}\mathrm{d}x_{n]}\,\sigma^m_{\alpha\dot\alpha}(\bar\sigma^n)^{\dot\alpha}{}_\beta$$

defines the wall area tensor. The algebra (3.20) can be obtained by calculating the Noether currents J^m_α and $J^m_{\dot\alpha}$ associated with the invariance under supersymmetry transformations, computing the appropriate anti-commutators and finally taking the integral of the zero-component. The wall is not annihilated by the usual central charges Q_α and $\bar Q_{\dot\alpha}$. However, it is possible to define a combination thereof, say $\widetilde{Q}_\alpha$, such that $\widetilde{Q}_\alpha\,|\mathrm{wall}\rangle = 0$. These new supercharges satisfy the subalgebra

$$\{\widetilde{Q}_\alpha, \widetilde{Q}_\beta\} = 8\,\Sigma_{\alpha\beta}(T_w - |Z|)\,,$$

and this means that $T_w = |Z|$ on the wall, i.e. the fact that the wall is annihilated by a supercharge does not imply the vanishing of its energy, but rather that its tension equals the central charge.

Another remarkable characteristic of the wall solution is that it preserves half of the supersymmetry, namely two out four supercharges annihilate the wall when act on it. This is why domain walls (and other solitons of this kind) are called *1/2-BPS saturated*. Let us see where this comes from. First, we recall that we are considering a purely bosonic configuration, and thus we have taken $\psi_\alpha = 0$. Then, we recall also that the action of an infinitesimal supersymmetry transformation on the field components of the chiral superfield is

$$\delta\phi = \sqrt{2}\,\epsilon\psi\,, \quad \delta\psi_\alpha = \mathrm{i}\sqrt{2}(\sigma^m\bar\epsilon)_\alpha\partial_m\phi - \sqrt{2}\epsilon_\alpha F\,. \tag{3.21}$$

Applying the BPS equation

$$F\big|_{\bar\phi=\phi^*_w} = -e^{-\mathrm{i}\eta}\frac{\mathrm{d}\phi_w}{\mathrm{d}z}\,,$$

for $\eta = \arg\big(m^3/\lambda\big)$, to the rightmost relation in (3.21), we see that the latter becomes

$$\delta\psi_\alpha = \sqrt{2}\big[\mathrm{i}(\sigma^3)_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha} + \epsilon_\alpha e^{-\mathrm{i}\eta}\big]\frac{\mathrm{d}\phi_w}{\mathrm{d}z}\,,$$

and vanishes provided that

$$\epsilon_a = -\mathrm{i}e^{\mathrm{i}\eta}(\sigma^3)_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\,.$$

Hence two out of four supertransformations act trivially on the domain wall.

3.5 Wall Dynamics and Membranes

Domain walls and solitons are dynamical objects, and thus it is natural to think that they are described by some effective action. Interestingly enough, in the long-wavelength limit, the dynamics of p-dimensional topological defects is governed by the action describing a p-brane, which is a p-dimensional extended object which generalises the notion of point particle. This remarkable feature has been first discovered by Nielsen and Olesen (NO) [31], who showed that the effective action of a single vortex solution of the abelian Higgs model is the Nambu-Goto action (to be defined below) governing the dynamics of a relativistic string[1]. The question of the long-wavelength effective action of topological defects in supersymmetry was first addressed in [26]: in particular, it was shown that in the supersymmetric generalization of the abelian Higgs model, the NO-vortex solution is described by the 4-dimensional Green-Schwartz superstring [22, 23]. Moreover, it was shown in [25] that the 4-dimensional vortex solution can be extended to six dimensions, and there it represents a 3-brane. This observation prompted the construction of an action for a supermembrane in an 11-dimensional spacetime [9]. A systematic study of the conditions under which supersymmetric p-dimensional extended objects exist in a d-dimensional spacetime was first performed in [1]. This work brought to the first classification[2] of the allowed pairs (p, d) in four sequences denoted by the letters **R**, **H**, **C** and **O**, which correspond to the four composition-division algebras. It has been also estabilished that the number of supersymmetries is $\mathcal{N} = 1$ for $p > 1$, while $\mathcal{N} = 1, 2$ are possible for $p = 1$. In [7], it was shown that the p-branes belonging to the **R**, **C** and **H** sequences are afflicted by gravitational anomalies. Therefore, only those of the **O** sequence can be regarded as fundamental objects. Instead, all the other quantum inconsistent super p-branes are to be interpreted as effective actions for p-dimensional topological defects in supersymmetric field theories [39].

3.5.1 Free Bosonic Membrane

Now we want to see how it is possible to obtain the effective action for a domain wall in the so-called thin wall approximation, i.e. supposing that the thickness is much smaller than the other length scales characterising the wall.

Let us first reason by analogy. As a (relativistic) particle sweeps a worldline, we expect a p-dimensional object to sweep a $(p+1)$-dimensional worldvolume: a string outlines a world*sheet*, a membrane a world*volume*. Introducing an arbitrary parameter τ, so as to parametrise the motion of a (single) massive[3] relativistic particle with four functions $X^m(\tau)$, we have

$$I_{p=0} = -\mathsf{m} \int \mathrm{d}\tau \sqrt{\eta_{mn} \frac{\mathrm{d}X^m}{\mathrm{d}\tau} \frac{\mathrm{d}X^n}{\mathrm{d}\tau}} \, .$$

The choice of the parameter τ is arbitrary, i.e. different parameterizations of the same path are physically equivalent, and any physical quantity must be independent of this choice: that is, for any monotonic function $\tau'(\tau)$, the paths $X'^m(\tau'(\tau))$ and $X^m(\tau)$ are the same. We have introduced a redundant – though more symmetric – notation in order to have time and space on the same footing, i.e. to render Lorentz invariance manifest. Parametrization invariance has to be mantained consistently also in the case of the generic

[1] A particle is also called 0-brane, a string – 1-brane, a membrane – 2-brane, while for $p > 2$ the object is referred to as simply p-brane.

[2] Further studies have expanded the set of known p-branes, e.g bringing to the identification of *Dirichlet p-branes* in string theory.

[3] This action is not suited for massless particles. The issue can be overcome introducing a new field $e(\tau)$ defined along the worldline, and using the action $I = -\frac{1}{2}\int \mathrm{d}\tau \left[e^{-1}(\tau)\dot{X}^2 + e(\tau)\mathsf{m}^2\right]$, for $\dot{X}^2 = \eta_{mn}\frac{\mathrm{d}X^m}{\mathrm{d}\tau}\frac{\mathrm{d}X^n}{\mathrm{d}\tau}$. In this action one can consistently set $\mathsf{m} = 0$. The field $e(\tau)$ is often referred to as *einbein*, and transforms in an appropriate way in order to guarantee parametrization invariance. The einbein can be interpreted as the square root of the metric of the worldline.

p-brane action. Keeping this in mind, we can postulate the action for a string and a membrane. Let $\sigma^a = (\tau, \sigma)$ and $\xi^i = (\xi^0, \xi^1, \xi^2)$ be the coordinates parametrizing the worldsheet and the worldvolume of the string and the membrane, respectively. Then, the string defines a surface $X^m(\sigma^a)$, and the membrane a volume $X^m(\xi^i)$. The obvious generalizations of the action for the relativistic particle are the Nambu-Goto actions

$$I_{p=1} = -T_s \int \mathrm{d}^2\sigma \sqrt{-\det \mathrm{h}}\,, \quad I_{p=2} = -T_m \int \mathrm{d}^3\xi \sqrt{-\det \mathrm{g}}\,, \tag{3.22}$$

where T_s denotes the tension of the string, and T_m the tension of the membrane, and

$$\mathrm{h}_{ab} = \eta_{mn} \frac{\partial X^m}{\partial \sigma^a} \frac{\partial X^n}{\partial \sigma^b}\,, \quad \mathrm{g}_{ij} = \eta_{mn} \frac{\partial X^m}{\partial \xi^i} \frac{\partial X^n}{\partial \xi^j}$$

are, respectively, the induced metric on the worldsheet and the worldvolume, obtained by the pull-back of the Minkowski metric. On physical grounds, these actions seem reasonable. However, if they arise as effective actions for topological defects of some model, there should be some reminiscence of the original field theory in the final result. At this stage, we cannot tell where the link with the underlying model is: we can only suppose that all the information inherited from the field model should be in the tension. This is indeed true, as we will see in a moment. Let us first outline the general strategy to deduce the long wavelength effective action for a soliton. In a semi-classical approach:

i. expand a generic perturbation around the soliton solution in terms of normal modes of non-zero frequency and collective coordinates for the zero-frequency modes;

ii. integrate out the non-zero modes.

One ends up with an effective action describing the zero-modes, which, in turn, are determined by the symmetries which are spontaneously broken by the soliton. As we discussed in section 3.2.3, once we pick one soliton out of the whole family of solutions, translational symmetry is spontaneously broken. By introducing collective coordinates, translational symmetry is recovered as non-linearly realized symmetry, with the collective coordinates as Goldstone modes. Thus, if we let $\mathbf{X}(t)$ be the collective coordinate associated with translational symmetry, it can only appear through $\dot{\mathbf{X}}$. The soliton is static to lowest order in the semi-classical approximation, thus the kinetic energy shows up only at the next order, and it will take the non-relativistic form $\frac{1}{2}\mathsf{m}\dot{\mathbf{X}}$. How do we cope with the lack of Lorentz symmetry? We can introduce a new variable $X^0(t)$ and describe the soliton with four functions $X^m = (X^0, \mathbf{X})$, provided that the action is invariant under world-p-volume reparametrizations: this invariance allows us to remove X^0 by means of a gauge choice.

In order to see where is the link between the effective action of the p-dimensional topological defect and the underlying field theory, let us focus on the case of membranes in d=4. We consider a generic bosonic field theory $\mathcal{L}(\phi, \partial_m\phi)$ having a domain wall solution $\phi_w(z) \equiv \phi_0(z)$. As usual, we consider the wall in its rest frame, and we assume that it lies in the $z = 0$ plane. Following the afore outlined strategy, we separate the field as $\phi(t, \mathbf{x}) = \phi_0(z) + \delta\phi(\xi)$, with $\xi = (t, x, y)$. To lowest order in the semi-classical approximation, the lagrangian becomes $\mathcal{L}_0 \equiv \mathcal{L}(\phi_0, \partial_z\phi_0)$. Now, we want to specify the position of the membrane parametrically through the functions $X^m(\xi)$, which describe the embedding of the membrane worldvolume in spacetime. Introducing a time-like vector $N^m(\xi)$ orthogonal to the membrane, the position of any point having position ξ^i on the membrane and displaced orthogonally to the membrane by a little amount z is specified by

$$x^m = X^m(\xi) + zN^m(\xi)\,.$$

This constitutes a change of coordinates, and, ignoring $o(z)$ terms, the new metric reads

$$\mathrm{g}_{IJ} = \eta_{mn}\frac{\partial x^m}{\partial \xi^I}\frac{\partial x^n}{\partial \xi^J} = \begin{pmatrix} \mathrm{g}_{ij}(\xi) & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \partial_i X \cdot \partial_j X(\xi) & 0 \\ 0 & -1 \end{pmatrix}\,,$$

with $\xi^I = (\xi^i, z)$, and we used a short-hand notation for scalar product $A \cdot B = \eta_{mn}A^m B^n$. The fact that $\mathrm{g}_{zz} = -1$ is due to N^m being a timelike vector, i.e. $N \cdot N = -1$. It follows

$$I = \int \mathrm{d}^4x\ \mathcal{L}(\phi, \partial_m\phi) \approx \int \mathrm{d}z\,\mathrm{d}^3\xi\ \sqrt{-\mathrm{g}}\,\mathcal{L}(\phi_0, \partial_z\phi_0) = \Big(\int \mathrm{d}z\ \mathcal{L}_0\Big)\int \mathrm{d}^3\xi\ \sqrt{-\mathrm{g}}\,,$$

and identifying $-T_m = \int \mathrm{d}z\ \mathcal{L}_0$, we end up with

$$I_{\text{eff}} = -T_m \int \mathrm{d}^3\xi\ \sqrt{-\mathrm{g}}\,, \tag{3.23}$$

which coincides with the membrane action in equation (3.22). As we guessed above, the tension of the wall contains the contribution of the underlying field model.

3.5.2 Free Supermembrane and κ-symmetry

Let us now consider supersymetric membranes. In particular, we want to underline which are the consequences of the presensence of supersymmetry.

We have seen in section 3.4.2 that domain walls which may form in supersymmetric field theories are rather peculiar, in that they break half supersymmetry. Therefore, a spinor collective coordinate, say $\theta^\alpha(t)$, is also needed for the effective description of the supermembrane. Only half of the supersymmetry is broken, hence only *half* of the component of $\theta^\alpha(t)$ are needed, but at the same time, to enforce manifest Lorentz invariance, *all* the components of $\theta^\alpha(t)$ are to be included. These requirements are compatible only if we arrange the effective action to possess a fermionic gauge symmetry which allows us to sweep away the redundant components of $\theta(t)$: this is the so-called *κ-symmetry.* Therefore, the effective action of a super membrane:

- Must be a Lorentz-invariant functional of $\mathcal{Z}^M = (X^m, \theta^\alpha)$;
- Must be invariant under worldvolume reparametrizations and κ-symmetry;
- Must be invariant under a non-linearly realized translation and supersymmetry invariance, for which $d-3$ coordinates $\mathbf{X}$ and half of θ in $\mathcal{Z}$ are Goldstone bosons and fermions;

- Must reduce to the (standard) bosonic membrane action (3.23) for $\theta = 0$.

In the non supersymmetric case, we have described the membrane via the embedding $\xi^i \to X^m(\xi)$, with the coordinates $\xi^i = (\xi^0, \xi^1, \xi^2)$ parametrising the worldvolume. Similarly, we now think of supermembranes as objects living in superspace. In particular, in what follows we will concentrate on the $\mathcal{N} = 1$, $d = 4$ superspace $\mathcal{M}_{4|1}$. Using ξ^i to parametrise the worldvolume, we describe the membrane via the super-embedding

$$\xi^\mu \;\mapsto\; \mathcal{Z}(\xi) = \left(X^m(\xi), \theta^\alpha(\xi), \bar{\theta}^{\dot{\alpha}}(\xi)\right) .$$

The action of a $\mathcal{N} = 1$, $d = 4$ supermembrane reads [2]

$$I_{p=2} = \int_{W_3} \mathrm{d}^3\xi \sqrt{-\mathrm{g}} + \int_{W_3} \mathcal{B}_3 \,, \tag{3.24}$$

where the induced metric on the 2-brane is

$$\mathrm{g}_{ij} = \eta_{ab}\, E_i{}^a(\xi) E_j{}^b(\xi) \,,$$

with $E_i{}^a(\xi) \equiv \partial_j \mathcal{Z}^M(\xi)\, E_M{}^a(\mathcal{Z}(\xi))$, and

$$E^a(\xi) = \mathrm{d}\mathcal{Z}^M(\xi)\, E_M{}^a(\mathcal{Z}(\xi)) \equiv \mathrm{d}\xi^i E_i{}^a(\xi) = \mathrm{d}X^a + \mathrm{i}\theta\sigma^a \mathrm{d}\bar{\theta} - \mathrm{i}\mathrm{d}\theta\, \sigma^a \bar{\theta}$$

denotes the worldvolume pull-back of the flat superspace bosonic vielbein. The real 3-form $\mathcal{B}_3(\xi)$ is given by

$$\mathcal{B}_3 = \frac{\mathrm{i}}{2} E^b \wedge E^a \wedge E^\alpha (\sigma_{ab}\, \theta)_\alpha - \frac{\mathrm{i}}{2} E^b \wedge E^a \wedge \bar{E}_{\dot{\alpha}} (\bar{\sigma}_{ab}\, \bar{\theta})^{\dot{\alpha}} + \frac{1}{2} E^a \wedge E^\alpha \wedge \bar{E}^{\dot{\alpha}} (\sigma_a)_{\alpha\dot{\alpha}} (\theta^2 + \bar{\theta}^2) \,,$$

and we can compute

$$\mathcal{F}_4 \equiv \mathbf{d}\mathcal{B}_3 = \frac{\mathrm{i}}{2} E^b \wedge E^a \wedge E^\alpha \wedge E_\beta (\sigma_{ab})_\alpha{}^\beta - \frac{\mathrm{i}}{2} E^b \wedge E^a \wedge \bar{E}_{\dot{\alpha}} \wedge \bar{E}^{\dot{\beta}} (\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \,.$$

The action (3.24) is invariant under the local fermionic κ-symmetry transformation, that is under

$$\delta_\kappa X^m(\xi) = \mathrm{i}\kappa\sigma^m\bar{\theta} - \mathrm{i}\theta\sigma^m\bar{\kappa} \,, \quad \delta_\kappa \theta^\alpha(\xi) = \kappa^\alpha(\xi) \,, \quad \delta_\kappa \bar{\theta}^{\bar{\alpha}}(\xi) = \bar{\kappa}^{\dot{\alpha}}(\xi) \,,$$

which we can rewrite in a more compact fashion as

$$\delta_\kappa \mathcal{Z}^M(\xi) = \kappa^\alpha(\xi) E_\alpha{}^M(\xi) + \bar{\kappa}^{\dot{\alpha}}(\xi) E_{\dot{\alpha}}{}^M(\xi) \,. \tag{3.25}$$

Actually, the action is κ-symetric provided that κ^α and $\bar{\kappa}^{\dot{\alpha}}$ are related by

$$\kappa_\alpha = (\Gamma\bar{\kappa})_\alpha \,, \quad \bar{\kappa}^{\dot{\alpha}} = (\bar{\Gamma}\kappa)^{\dot{\alpha}} \,, \tag{3.26}$$

where

$$\Gamma_{\alpha\dot{\alpha}} = \frac{\mathrm{i}\,\epsilon^{ijk}}{3!\sqrt{-\mathrm{g}}}\, \epsilon_{abcd} E_i{}^b E_j{}^c E_k{}^d (\sigma^a)_{\alpha\dot{\alpha}} \,, \quad \bar{\Gamma}^{\dot{\alpha}\alpha} = \frac{\mathrm{i}\,\epsilon^{ijk}}{3!\sqrt{-\mathrm{g}}}\, \epsilon_{abcd} E_i{}^b E_j{}^c E_k{}^d (\bar{\sigma}^a)^{\dot{\alpha}\alpha} \,, \tag{3.27}$$

with $\bar{\Gamma}^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\Gamma_{\beta\dot{\beta}}$. One can easily see that $\mathrm{tr}\,\Gamma = \mathrm{tr}\,\bar{\Gamma} = 0$ and that $\Gamma_{\alpha\dot{\alpha}}\bar{\Gamma}^{\dot{\alpha}\beta} = \delta_\alpha^\beta$. Hence, equation (3.26) defines two constraints, which are just projection condition which halves the number of independent components of the fermionic κ-symmetry.

We will now demonstrate that (3.24) is κ-symmetric. First, we review a number of useful results. Even though the κ-symmetry transformation acts on the coordinates $\mathcal{Z}^M(\xi)$, it is more convenient to consider how it acts on the super-vielbein. In such case we have [6]

$$\begin{aligned}
&\iota_\kappa E^a(\xi) \equiv \delta_\kappa \mathcal{Z}^M E_M{}^a(\mathcal{Z}(\xi)) = 0\,,\\
&\iota_\kappa E^\alpha(\xi) \equiv \delta_\kappa \mathcal{Z}^M E_M{}^\alpha(\mathcal{Z}(\xi)) = \kappa^\alpha\,, \quad \iota_\kappa E^{\dot{\alpha}}(\xi) \equiv \delta_\kappa \mathcal{Z}^M E_M{}^{\dot{\alpha}}(\mathcal{Z}(\xi)) = \bar{\kappa}^{\dot{\alpha}}\,,
\end{aligned}$$

where $\iota_\kappa\Omega^{(p)}$ denotes the interior product of the p-form $\Omega^{(p)}$ with the vector field $\delta_\kappa\mathcal{Z}^M$, i.e. the contraction between $\Omega^{(p)}$ and $\delta_\kappa\mathcal{Z}^M$

$$\iota_\kappa\Omega^{(p)} = E^{M_p}\wedge E^{M_{p-1}}\wedge\cdots\wedge E^{M_2}\wedge\iota_\kappa E^{M_1}\,\Omega_{M_1\ldots M_p}\,.$$

Then, thanks to the *Cartan formula* $\delta_\kappa\bullet = \mathbf{d}(\iota_\kappa\bullet) + \iota_\kappa(\mathbf{d}\bullet)$, we find

$$\delta_\kappa E^a(\xi) = \mathbf{d}(\iota_\kappa E^a) + \iota_\kappa(\mathbf{d}E^a) = -2\mathrm{i}\,\iota_\kappa(E\wedge\sigma^a\bar{E}) = 2\mathrm{i}(\kappa\sigma^a)_{\dot{\alpha}}\bar{E}^{\dot{\alpha}} - 2\mathrm{i}E^\alpha(\sigma^a\bar{\kappa})_\alpha\,. \tag{3.28}$$

It is useful to consider the worldvolume Hodge dual of the bosonic super-vielbein, which is defined as

$$\star E^a(\xi) \equiv \frac{1}{2}\mathrm{d}\xi^i\wedge\mathrm{d}\xi^j\sqrt{-\mathrm{g}}\,\epsilon_{ijk}\mathrm{g}^{kl}E_l{}^a\,.$$

Adopting the convention $\mathrm{d}\xi^i\wedge\mathrm{d}\xi^j\wedge\mathrm{d}\xi^k = \mathrm{d}^3\xi\,\epsilon^{ijk}$ for the integration on the worldvolume, we have

$$\mathrm{d}^3\xi\,\sqrt{-\mathrm{g}} = \frac{1}{3}\star E_a\wedge E^a\,, \tag{3.29}$$

and thus we can rewrite the Nambu-Goto term in the action directly in terms of the vielbein. Moreover, we can see that

$$\delta_\kappa(\star E_a\wedge E^a) = 3\,\star E_a\wedge\delta_\kappa E^a\,. \tag{3.30}$$

Proof. Let us demonstrate (3.29) and (3.30). In order to avoid confusion, target space indices will be underlined. For equation (3.29) we have

$$\begin{aligned}
\star E_{\underline{a}}\wedge E^{\underline{a}} &= \left(\frac{1}{2}\mathrm{d}\xi^i\wedge\mathrm{d}\xi^j\sqrt{-\mathrm{g}}\,\epsilon_{ijk}\,\mathrm{g}^{kl}E_l{}^{\underline{b}}\,\eta_{\underline{b}\underline{a}}\right)\wedge\mathrm{d}\xi^r E_r{}^{\underline{a}}\\
&= \frac{1}{2}\mathrm{d}^3\xi\,\sqrt{-\mathrm{g}}\,\epsilon^{ijr}\epsilon_{ijk}\,\mathrm{g}^{kl}E_l{}^{\underline{b}}E_r{}^{\underline{a}}\eta_{\underline{b}\underline{a}} = \mathrm{d}^3\xi\,\sqrt{-\mathrm{g}}\;\mathrm{g}^{kr}\left(E_k{}^{\underline{b}}E_r{}^{\underline{a}}\eta_{\underline{b}\underline{a}}\right)\\
&= \mathrm{d}^3\xi\,\sqrt{-\mathrm{g}}\,\mathrm{g}^{kr}\mathrm{g}_{kr} = 3\,\mathrm{d}^3\xi\,\sqrt{-\mathrm{g}}\,.
\end{aligned}$$

For equation (3.30) instead, we have

$$\star E_{\underline{a}}\wedge\delta E^{\underline{a}} = \mathrm{d}^3\xi\,\sqrt{-\mathrm{g}}\,g^{ij}E_{i\underline{a}}\,\delta E_j{}^{\underline{a}}\,,$$

and from the variation of the determinant

$$\delta\sqrt{-\mathrm{g}} = \frac{1}{2}\sqrt{-\mathrm{g}}\,\mathrm{g}^{ij}\,\delta\mathrm{g}_{ij} = \sqrt{-\mathrm{g}}\,\mathrm{g}^{ij}\,E_{i\underline{a}}\,\delta E_j{}^{\underline{b}}\,,$$

we see that

$$\mathrm{d}^3\xi\;\delta(\sqrt{-\mathrm{g}}) = \star E_{\underline{a}} \wedge \delta E^{\underline{a}}\,.$$

Therefore, applying (3.29) we get (3.30). □

Finally, the last relations we need are

$$\frac{1}{2}E^b \wedge E^a \wedge E^\alpha(\sigma_{ab})_\alpha{}^\beta = \star E_a \wedge E^\alpha(\sigma^a)_{\alpha\dot\alpha}\bar\Gamma^{\dot\alpha\beta}\,, \tag{3.31a}$$

$$\frac{1}{2}E^b \wedge E^a \wedge \bar E_{\dot\alpha}(\bar\sigma_{ab})^{\dot\alpha}{}_{\dot\beta} = -\star E_a \wedge \bar E_{\dot\alpha}(\bar\sigma^a)^{\dot\alpha\beta}\Gamma_{\beta\dot\beta}\,. \tag{3.31b}$$

Proof. To prove that (3.24) is κ-symmetric, we have to compute the variation of

$$I_{p=2} = \frac{1}{3}\int_{W_3}\star E_a \wedge E^a + \int_{W_3}\mathcal{B}_3 \equiv I_{\mathrm{NG}} + I_{\mathrm{WZ}}\,,$$

and show that it is null. From equations (3.28) and (3.30) the variation of the first term is

$$\delta_\kappa I_{\mathrm{NG}} = \int_{W_3}\star E_a \wedge \delta_\kappa E^a = \int_{W_3}\star E_a \wedge \left[2\mathrm{i}(\kappa\sigma^a)_{\dot\alpha}\bar E^{\dot\alpha} - 2\mathrm{i}E^\alpha(\sigma^a\bar\kappa)_\alpha\right]\,. \tag{3.32}$$

Appliying Cartan formula to the second term we have

$$\delta_\kappa I_{\mathrm{WZ}} = \int_{W_3}\left[\mathbf{d}(\iota_\kappa\mathcal{B}_3) + \iota_\kappa(\mathbf{d}\mathcal{B}_3)\right] = \int_{\partial W_3}\iota_\kappa\mathcal{B}_3 + \int_{W_3}\iota_\kappa\mathcal{F}_4\,, \tag{3.33}$$

where we have used Stoke's theorem, which we assume to be defined as in the bosonic case (see appendix B.1). At this pont, we suppose that membrane is closed so that $\partial W_3 = \varnothing$ and the boundary term in (3.33) vanishes. By a direct computation we get

$$\begin{aligned}\iota_\kappa\mathcal{F}_4 &= \mathrm{i}\left[E^b \wedge E^a \wedge E^\alpha(\sigma_{ab}\kappa)_\alpha - E^b \wedge E^a \wedge \bar E_{\dot\alpha}(\bar\sigma_{ab}\bar\kappa)^{\dot\alpha}\right]\\ &= 2\mathrm{i}\left[\star E_a \wedge E^\alpha(\sigma^a)_{\alpha\dot\alpha}(\bar\Gamma\kappa)^{\dot\alpha} + \star E_a \wedge \bar E_{\dot\alpha}(\bar\sigma^a)^{\dot\alpha\alpha}(\Gamma\bar\kappa)_\alpha\right]\,,\end{aligned}$$

where we have used equations (3.31a) and (3.31b) in passing from the first to the second line. Putting (3.32) and (3.33) together we get

$$\begin{aligned}\delta_\kappa I_{p=2} &= 2\mathrm{i}\int_{W_3}\star E_a \wedge \left[(\kappa\sigma^a)_{\dot\alpha}\bar E^{\dot\alpha} - E^\alpha(\sigma^a\bar\kappa)_\alpha + E^\alpha(\sigma^a)_{\alpha\dot\alpha}(\bar\Gamma\kappa)^{\dot\alpha} + \bar E_{\dot\alpha}(\bar\sigma^a)^{\dot\alpha\alpha}(\Gamma\bar\kappa)_\alpha\right]\\ &= -2\mathrm{i}\int_{W_3}\star E_a \wedge \left[\bar E_{\dot\alpha}(\bar\sigma^a\kappa)^{\dot\alpha} + E^\alpha(\sigma^a\bar\kappa)_\alpha - E^\alpha(\sigma^a)_{\alpha\dot\alpha}(\bar\Gamma\kappa)^{\dot\alpha} - \bar E_{\dot\alpha}(\bar\sigma^a)^{\dot\alpha\alpha}(\Gamma\bar\kappa)_\alpha\right]\end{aligned}$$

and provided that (3.26) holds, we end up with

$$= -2\mathrm{i}\int_{W_3} \star E_a \wedge \left[\bar{E}_{\dot{\alpha}}(\bar{\sigma}^a\kappa)^{\dot{\alpha}} + E^{\alpha}(\sigma^a\bar{\kappa})_{\alpha} - E^{\alpha}(\sigma^a)_{\alpha\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}} - \bar{E}_{\dot{\alpha}}(\bar{\sigma}^a)^{\dot{\alpha}\alpha}\kappa_{\alpha}\right] = 0\,.$$

Therefore, the action (3.24) is κ-symmetric. □

We want to see now, more precisely, which are the physical degrees of freedom of the membrane. Let us start with the bosonic degrees of freedom. We have described the volume swept by the membrane in spacetime by means of four functions $X^m(\xi)$ of the worldvolume coordinates (ξ^0, ξ^1, ξ^2). We can fix the gauge of the reparametrization invariance by choosing, for instance, the *static gauge*, i.e. imposing the condition $X^m(\xi) = (\xi^0, \xi^1, \xi^2, \phi(\xi))$. Therefore, we end up with only one physical bosonic degree of freedom, which is described by the function $\phi(\xi)$. The fermionic degrees of freedom, instead, are halved by the κ-symmetry. Indeed, a κ-symmetry transformation is given by

$$\begin{aligned} \theta_{\alpha} \quad &\mapsto \quad \theta'_{\alpha} = \theta_{\alpha} + \kappa_{\alpha}\,, \\ \bar{\theta}^{\dot{\alpha}} \quad &\mapsto \quad \bar{\theta}'^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{\kappa}^{\dot{\alpha}}\,, \end{aligned}$$

where the parameters of the transformation are related one another by the relations in equation (3.26). From these transformation laws, we see that the κ-symmetry can eliminate those components of θ and $\bar{\theta}$ which satisfy the same projection relations as (3.26); the remaining independent component are orthogonal to those that have been gauged away, and satisfy the same constraint but with opposite sign. All in all, we are left with two real independent degrees of freedom, which we can identify with the two degrees of freedom of a two-component Majorana spinor $\psi_{\alpha}(\xi)$. All in all, the physical degrees of freedom of the membrane are $\{\phi(\xi), \psi_{\alpha}(\xi)\} \equiv \mathcal{M}$: the first represents the displacement of the membrane from its rest position, and we have identified it with the Goldstone boson associated to the spontaneous breaking of translational symmetry; the second one is a Goldstino, which signals the (partial) spontaneous breaking of global supersymmetry [25] due to the presence of the membrane. Therefore, $\mathcal{M}$ is a Goldstone supermultiplet which lives on the membrane worldvolume.

At this point, it is natural to ask how to couple the supermembrane to other physical objects as, in particular, superfields. We will see this in the next chapter, in the case of an $\mathcal{N} = 1$, $SU(N)$ Super Yang-Mills theory: interestingly enough, this coupling sources BPS domain walls configurations.

Chapter 4

$SU(N)$ SYM Theory

In section 2.3.3 we have seen which is the generic lagrangian of a pure $\mathcal{N} = 1$ supersymmetric non-abelian gauge field theory. In this chapter we want to consider in some more detail the structure of these theories, focusing on the gauge group $SU(N)$.

The first appearance of pure $\mathcal{N} = 1$ SYM theories dates back to the '70s [19, 33, 43]. Though at a first glance this kind of theories seems rather simple, extensive studies of its structure carried on over the years have pointed out a complicated quantum behaviour. In particular, it is known since the '80s that $SU(N)$ SYM has N degenerate supersymmetric vacua, each of which is characterised by a different vev of the gluino condensate [35]

$$\langle \lambda^{a,\alpha}\lambda^a_\alpha \rangle = \Lambda^3 \exp\Big\{2\pi \mathrm{i}\frac{\mathrm{n}}{N}\Big\}\,, \quad \mathrm{n} = 0, 1, \ldots, N-1\,, \tag{4.1}$$

where Λ is a dynamically generated scale at which the condensate forms due to non-perturbative effects. Later on, in [16], it was suggested that there shoud exist BPS domain walls interpolating between different vacua, say e.g. j and k, whose BPS saturating tension is

$$T_w = \frac{N}{8\pi^2}\big|\langle\lambda\lambda\rangle_{\mathrm{j}} - \langle\lambda\lambda\rangle_{\mathrm{k}}\big|\,. \tag{4.2}$$

An explicit solitonic solution describing domain walls in an effective theory of SYM has been found only recently [5].

4.1 Symmetries and Anomalies

We begin with the analysis of the $SU(N)$ SYM theory

$$\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}\Big[-\frac{1}{4}F_{mn}F^{mn} - \mathrm{i}\lambda\sigma^m\nabla_m\bar\lambda + \frac{1}{2}D^2\Big] + \frac{\Theta_{\mathrm{YM}}}{8\pi^2}g^2\,\mathrm{tr}\Big[\partial_m(\bar\lambda\sigma^m\lambda) + \frac{1}{4}\epsilon_{mnpq}F^{mn}F^{pq}\Big]\,, \tag{4.3}$$

starting from its classical symmetries. We recall that both the gauge field and the gaugino transform in the adjoint representation of $SU(N)$, and we choose the generators as $(t^a_{\mathrm{Ad}})^{bc} = -\mathrm{i}f^{abc}$. The parameter Θ_{YM} denotes the usual theta angle of YM theory.

The gaugino sector of the SYM lagrangian

$$\mathcal{L} = -\mathrm{i}\,\mathrm{tr}\big(\lambda\sigma^m\nabla_m\bar{\lambda}\big) \tag{4.4}$$

is invariant under chiral rotations

$$\lambda_\alpha \mapsto e^{\mathrm{i}\alpha}\lambda_\alpha\,, \quad \bar{\lambda}_{\dot{\alpha}} \mapsto e^{-\mathrm{i}\alpha}\bar{\lambda}_{\dot{\alpha}}\,.$$

This is the R-symmetry of the theory under consideration. As usual, the invariance under a continuous transformation is related to the presence of a conserved current. In our case, such current, often dubbed *R-current*, is given by

$$R^m = \mathrm{tr}\big(\lambda\sigma^m\bar{\lambda}\big)\,,$$

and can be obtained by applying the Noether theorem. Alternatively, one can consider a local parameter $\alpha(x)$ rather than simply α: as this local transformation is a symmetry of the lagrangian if the parameter is taken constant, we can define a conserved current according to

$$\delta_\alpha I = -\int \mathrm{d}^4x\ R^m(x)\,\partial_m\alpha(x) = \int \mathrm{d}^4x\ \big[\partial_m R^m(x)\big]\,\alpha(x)\,.$$

Provided that the equations of motion are satisfied, then $\delta I = 0$ for any variation of the fields included those induced by $\alpha(x)$, hence $\partial_m R^m(x) = 0$. This second point of view will be useful later on. Together with the R-current, there are two other (classically) conserved quantities. One is the superconformal current $J^m_\alpha = \frac{1}{2}J_{\beta\alpha\dot{\alpha}}(\bar{\sigma}^m)^{\dot{\alpha}\alpha}$, the other is the stress-energy tensor T_{mn}. Moreover, the trace of the stress-energy tensor vanishes, $T^m{}_m = 0$: this is due to the scale-invariance of the theory, namely invariance under the transformations

$$x^m \mapsto e^{-\sigma}x'^m\,, \quad A_m \mapsto e^{\sigma}A'_m\,, \quad \lambda_\alpha \mapsto e^{\frac{3}{2}\sigma}\lambda'_\alpha\,.$$

The presence of $U(1)_R$ symmetry, Poincarè invariance, scale invariance and supersymmetry implies that the theory is superconformal: consequently, we have the following identities

$$\partial_m R^m = 0\,, \quad \partial_m(x_n T^{mn}) = T^m{}_m = 0\,, \quad (\bar{\sigma}^m)^{\dot{\alpha}\alpha}J^m_\alpha = 0\,. \tag{4.5}$$

The classically conserved currents belong to a current supermultiplet [20], which, up to a multiplicative numeric constant, is defined as

$$\mathcal{J}_{\alpha\dot{\alpha}} = g^2\ \mathrm{tr}\big(e^{2g\mathcal{V}}\mathcal{W}_\alpha e^{-2g\mathcal{V}}\overline{\mathcal{W}}_{\dot{\alpha}}\big) \ni (R_m;\, J^m_\alpha;\, T_{mn})\,.$$

In this framework, the continuity equations (4.5) become

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = 0\,.$$

Yet, this is true *only at the classical level*: chiral symmetry and scale invariance are broken at the quantum level. These are *anomalous symmetries.*

Anomalies: some general remarks. Before going on, it is useful to stress some general features of anomalies. A *classical* symmetry is termed anomalous when it is broken at the quantum level. Given a (compact) Lie group G, the occurrence of an anomaly entirely depends on the matter content of the theory under consideration, or, more precisely, on the representation of G where matter transforms. The immediate consequence is that the classical continuity equation for the current associated to the symmetry under G is spoiled. In particular, if J^m is the classically conserved current which obeys $\partial_m J^m = 0$, quantum effects modify this relation in such a way that $\partial_m J^m(x) = -\mathcal{A}(x)$, where $\mathcal{A}(x)$ is the anomaly function. If G is a global symmetry, this means that a classical selection rule is not respected at the quantum level, i.e. classically forbidden processes can indeed take place due to quantum effects. On the other hand, when the anomaly resides in the gauge group the corresponding quantized theory is not consistent, for it is not renormalizable and it could contain states of negative norm which violate unitarity.

Let us focus in particular on the anomaly of the R-symmetry. In order to see how the chiral $U(1)_R$ is broken, let us rewrite (4.4) in terms of Majorana (bi)spinors

$$\lambda = \begin{pmatrix} \lambda_\alpha \\ \bar\lambda^{\dot\alpha} \end{pmatrix} , \quad \lambda^{\mathrm{T}} = \begin{pmatrix} \lambda^\alpha & \bar\lambda_{\dot\alpha} \end{pmatrix} .$$

Recalling that we chose the gamma matrices as

$$\gamma^m = \begin{pmatrix} \mathbb{0} & \sigma^m \\ \bar\sigma^m & \mathbb{0} \end{pmatrix} , \quad \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix} ,$$

the gaugino lagrangian can be rewritten as

$$\mathcal{L} = -\frac{\mathrm{i}}{2}\,\mathrm{tr}\big(\lambda^{\mathrm{T}}\gamma^m\nabla_m\lambda\big) .$$

This lagrangian is invariant under chiral rotations $\lambda \mapsto e^{\mathrm{i}\alpha\gamma^5}\lambda$, and the corresponding conserved current is $R^m = -\frac{1}{2}\lambda^{\mathrm{T}}\gamma^m\gamma^5\lambda$, where the trace over the color indices is understood.

In the path integral formalism the anomaly is due to the determinant of the jacobian of the transformation which is not unity. In particular, we consider

$$\int \mathfrak{D}\lambda\, \exp\Big\{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}\Big\} .$$

By applying the chiral rotation $U_5(x) = e^{\mathrm{i}\alpha(x)\gamma^5}$ to the fields, the measure changes according to $\mathfrak{D}\lambda \mapsto [\mathrm{Det}\,\mathcal{U}_5(x)]^{-1}\,\mathfrak{D}\lambda$, with $\mathcal{U}_5(x)$ the operator satisfying $\langle x|\mathcal{U}_5|y\rangle = U_5(x)\delta^{(4)}(x-y)$. In order to compute the anomaly, we have to compute this determinant[1] [11]. However, the determinant is ill-defined as it stands. Indeed, we have

$$\begin{aligned}\mathrm{Tr}\log\mathcal{U}_5 &= \int \mathrm{d}^4x\,\langle x|\mathrm{Tr}\log\mathcal{U}_5|x\rangle = \int \mathrm{d}^4x\; \mathrm{tr}\log U_5(x)\delta^4(x-x) \\ &= \mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\,\mathrm{tr}\,\gamma^5\delta^4(0) ,\end{aligned}$$

[1]Here and in what follows Tr and Det denote functional trace and determinant, while tr and det denote matrix trace and determinant.

and from

$$[\mathrm{Det}\,\mathcal{U}_5]^{-1} = e^{-\mathrm{Tr}\log\mathcal{U}_5} = e^{\mathrm{i}\int \mathrm{d}^4x\,\alpha(x)\mathcal{A}(x)}\,,$$

follows that the anomaly function $\mathcal{A}(x) = -\,\mathrm{tr}\,\gamma^5\delta^{(4)}(0)$ is the product of the divergent $\delta^{(4)}(0)$ and the vanishing $\mathrm{tr}\,\gamma^5$. It is possible to regularize the integral by cutting-off the large momenta contributions, namely by introducing a function $f(s)$ such that

$$f(s) = \begin{cases} 1 \text{ for } s = 0 \\ 0 \text{ for } s = \infty \end{cases} \qquad \text{and} \qquad sf'(s) = \begin{cases} 0 \text{ for } s = 0 \\ 0 \text{ for } s = \infty \end{cases}\,,$$

and replacing $\int \mathrm{d}^4x\,\alpha(x)\mathcal{A}(x) \equiv -\mathrm{Tr}\,\mathcal{C}$ for $\mathcal{C} = \alpha(\hat{x})\gamma^5$ with

$$\int \mathrm{d}^4\,x\,\alpha(x)\mathcal{A}(x) \equiv -\lim_{\Lambda\to\infty}\mathrm{Tr}\,\mathcal{C}_\Lambda\,, \quad \mathcal{C}_\Lambda = \alpha(\hat{x})\gamma_5 f\Big(\Big(\mathrm{i}\frac{\widehat{\nabla\!\!\!/}}{\Lambda}\Big)^2\Big)\,.$$

The quantities with a hat denote quantum-mechanical operators. In particular, the Dirac operator $\widehat{\nabla\!\!\!/}$ is such that $\langle\phi|\widehat{\nabla\!\!\!/}|x\rangle = \nabla\!\!\!/\,\langle\phi|x\rangle$. Let us compute the trace:

$$\begin{aligned}
\mathrm{Tr}\,\mathcal{C}_\Lambda &= \int \mathrm{d}^4x\ \mathrm{Tr}\Big\langle x\,\Big|\alpha(\hat{x})\gamma_5 f\Big(\Big(\mathrm{i}\frac{\widehat{\nabla\!\!\!/}}{\Lambda}\Big)^2\Big)\Big|\,x\Big\rangle \\
&= \int \mathrm{d}^4x\ \alpha(x)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\ e^{\mathrm{i}qx}\,\mathrm{Tr}\Big[\gamma^5\Big\langle q\,\Big|f\Big(\Big(\mathrm{i}\frac{\widehat{\nabla\!\!\!/}}{\Lambda}\Big)^2\Big)\Big|\,x\Big\rangle\Big]e^{-\mathrm{i}qx} \\
&= \int \mathrm{d}^4x\ \alpha(x)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\ e^{\mathrm{i}qx}\ \mathrm{tr}\Big[\gamma_5 f\Big(\Big(\mathrm{i}\frac{\nabla\!\!\!/}{\Lambda}\Big)^2\Big)\Big]e^{-\mathrm{i}qx} \\
&= \int \mathrm{d}^4x\ \alpha(x)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\ e^{\mathrm{i}qx}\ \mathrm{tr}\Big[\gamma_5 f\Big(-\frac{1}{\Lambda^2}(\partial\!\!\!/ - \mathrm{i}A\!\!\!/)^2\Big)\Big]e^{-\mathrm{i}qx} \\
&= \int \mathrm{d}^4x\ \alpha(x)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\ \mathrm{tr}\Big[\gamma_5 f\Big(-\frac{1}{\Lambda^2}(-\mathrm{i}q\!\!\!/ + \nabla\!\!\!/)^2\Big)\Big]\,.
\end{aligned}$$

With a change of variable $q \mapsto \Lambda p$ one gets

$$\mathrm{Tr}\,\mathcal{C}_\Lambda = \int \mathrm{d}^4x\ \alpha(x)\int \Lambda^4\frac{\mathrm{d}^4p}{(2\pi)^4}\ \mathrm{tr}\Big[\gamma_5 f\Big(-\Big(-\mathrm{i}p\!\!\!/ + \frac{\nabla\!\!\!/}{\Lambda}\Big)^2\Big)\Big].$$

The only way to get rid of the Λ^4 multiplying the measure, is having another Λ^4 at the denominator. Moreover, the trace of the Dirac matrices is non-vanishing if and only if there are four gammas together with γ_5. Both these problems are solved at the same time picking the second order term of the expansion of the regulator function around p. In particular, we get

$$\mathrm{Tr}\,\mathcal{C}_\Lambda = \int \mathrm{d}^4x\ \alpha(x)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\ \frac{1}{2}f''(p^2)\ \mathrm{tr}\big[\gamma_5(\nabla\!\!\!/\nabla\!\!\!/)^2\big]\,.$$

On one side, using $[\nabla_m, \nabla_n] = -\mathrm{i}\, gF_{mn}$, we obtain

$$\not{\nabla}\not{\nabla} = \gamma^m\gamma^n\nabla_m\nabla_n = \nabla^2 - \frac{\mathrm{i}g}{4}[\gamma^m, \gamma^n]F_{mn}\,,$$

from which

$$\begin{aligned}\mathrm{tr}\Big[\gamma_5\Big(-\frac{\mathrm{i}g}{4}[\gamma^m, \gamma^n]F_{mn}\Big)^2\Big] &= -\frac{g^2}{16}\,\mathrm{tr}\big(\gamma_5[\gamma^m, \gamma^n][\gamma^p, \gamma^q]\big)\,\mathrm{tr}\,F_{mn}F_{pq}\\ &= -\frac{g^2}{4}\,\mathrm{tr}\big(\gamma_5\gamma^m\gamma^n\gamma^p\gamma^q\big)\,\mathrm{tr}\,F_{mn}F_{pq} = -\mathrm{i}\,2Ng^2\epsilon^{mnpq}F^a_{mn}F^b_{pq}\,,\end{aligned}$$

using the ciclicity of the trace and $\mathrm{tr}\big(\gamma_5\gamma^m\gamma^n\gamma^p\gamma^q\big) = -4\mathrm{i}\epsilon^{mnpq}$, as well as the identity $f^a_{cd}f^b_{cd} = 2N\delta^{ab}$ for the structure constants of $SU(N)$. On the other side, Wick rotating the momentum integral we have

$$\begin{aligned}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{1}{2}f''(p^2) &= \frac{\mathrm{i}}{16\pi^4}\int\mathrm{d}^4p_E\;f''(-p_E^2) = \frac{\mathrm{i}}{32\pi^4}V(S_3)\int_0^\infty\mathrm{d}p_E\;p_E^3\,f''(-p_E^2)\\ &= \frac{\mathrm{i}}{32\pi^2}\int_0^{-\infty}\mathrm{d}y\;y\,f''(y) = \frac{\mathrm{i}}{16\pi^2}\big[yf'(y) - f(y)\big|_0^{-\infty} = \frac{\mathrm{i}}{32\pi^2}\,,\end{aligned}$$

using $V(S_3) = 2\pi^2$ and changing variable as $y = -p_E^2$. In the very last step the defining properties of the function $f(s)$ have been used. Putting everything together we arrive at

$$\mathrm{Tr}\,\mathcal{C}_\Lambda = \int\mathrm{d}^4x\;\alpha(x)\Big[\frac{N}{16\pi^2}g^2\,\epsilon^{mnpq}F^a_{mn}F^a_{pq}\Big]\,,$$

which defines the anomaly function as

$$\mathcal{A}(x) = -\frac{N}{16\pi^2}g^2\,\epsilon^{mnpq}F^a_{mn}F^a_{pq}\,.$$

The anomaly spoils the conservation of the R-current. This is easily seen from the path integral, indeed

$$\begin{aligned}\int\mathfrak{D}\lambda\;e^{\mathrm{i}\int\mathrm{d}^4x\,\mathcal{L}} &= \int\mathfrak{D}\lambda\;e^{\mathrm{i}\int\mathrm{d}^4x\,\mathcal{L}'} = \int\mathfrak{D}\lambda\;e^{\mathrm{i}\int\mathrm{d}^4x\;\alpha(x)\mathcal{A}(x)}e^{\mathrm{i}\int\mathrm{d}^4x\;\left(\mathcal{L}+\alpha(x)\partial_mR^m(x)\right)}\\ &\approx\int\mathfrak{D}\lambda\;e^{\mathrm{i}\int\mathrm{d}^4x\,\mathcal{L}}\Big[1 + \mathrm{i}\int\mathrm{d}^4x\;\alpha(x)\big(\mathcal{A}(x) + \partial_mR^m(x)\big)\Big]\,,\end{aligned}$$

from which[2]

$$\partial_m\,\langle R^m(x)\rangle = -\mathcal{A}(x)\qquad\Longrightarrow\qquad\partial_m\,\langle\lambda\sigma^m\bar{\lambda}\rangle = \frac{N}{16\pi^2}g^2\,\epsilon^{mnpq}F^a_{mn}F^a_{pq}\,.$$

Nevertheless, the chiral symmetry is not fully broken: there is still a residual $\mathbb{Z}_{2N}$ symmetry. We can see this by noting that in the full SYM lagrangian (4.3) the $U(1)_R$

[2]The precise expression involves the vacuum expectation value $\langle\ldots\rangle$; however, the brakets are usually omitted for sake of simplicity.

group becomes an actual symmetry (loosely speaking, we can compensate the anomaly) if the phase transformation is accompanied with a shift of the theta angle as

$$\Theta_{\mathrm{YM}} \mapsto \Theta_{\mathrm{YM}} + 2N\alpha \,.$$

Since $\Theta_{\mathrm{YM}} \mapsto \Theta_{\mathrm{YM}} + 2\pi k$ for $k \in \mathbb{Z}$ is a symmetry by itself, it follows that, whenever $\alpha = \pi k/N$, the symmetry is preserved *despite* the anomaly. Ultimately, even the $\mathbb{Z}_{2N}$ symmetry is broken: this time the break up is due to non-perturbative effects which produce the gaugino condensate (4.1). One can easily see that the remaining symmetry is just $\mathbb{Z}_2$.

Keeping into account also the anomaly of the scale invariance, the relations in (4.5) become [27]

$$\partial_m R^m = \frac{N}{16\pi^2} g^2 \, \epsilon^{mnpq} F^a_{mn} F^a_{pq} \,, \tag{4.6a}$$

$$T^m{}_m = -3\frac{N}{16\pi^2} g^2 \, F^a_{mn} F^{a,mn} \,, \tag{4.6b}$$

$$(\bar{\sigma}^m J_m)^{\dot{\alpha}} = -\mathrm{i}\, 3\frac{N}{4\pi^2} g^2 \, (\bar{\sigma}^{mn}\bar{\lambda})^{a,\dot{\alpha}} F^a_{mn} \,. \tag{4.6c}$$

The anomalies, in turn, can be repackaged in a chiral superfield $\mathcal{S}$ [20], which is defined as

$$\mathcal{S}(y,\theta) = \mathrm{tr}\, \mathcal{W}^\alpha \mathcal{W}_\alpha = s + \sqrt{2}\theta^\alpha \, \chi_\alpha + \theta^2 F \,, \tag{4.7}$$

and whose components are readily found by the definition of the gaugino superfield to be

$$s = -\,\mathrm{tr}\, \lambda^\alpha \lambda_\alpha \,, \tag{4.8a}$$

$$\chi_\alpha = \sqrt{2}\; \mathrm{tr}\Big(F_{mn} (\sigma^{mn})_\alpha{}^\beta \lambda_\beta - \mathrm{i}\lambda_\alpha D \Big) \,, \tag{4.8b}$$

$$F = \mathrm{tr}\Big(-2\mathrm{i}\lambda\sigma^m \nabla_m \bar{\lambda} - \frac{1}{2} F_{mn}F^{mn} + D^2 - \frac{\mathrm{i}}{4}\epsilon_{mnpq}F^{mn}F^{pq} \Big) \,. \tag{4.8c}$$

In the superfield formalism the set of equations (4.6) gets repackaged into [27]

$$\bar{\mathcal{D}}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = \frac{\beta(g)}{2g} \mathcal{D}_\alpha \mathcal{S} \,, \tag{4.9}$$

and the conjugate equation, where we have introduced $\beta(g)/2g = -3Ng^2/(16\pi^2)$.

4.2 The Veneziano-Yankielowicz Effective Lagrangian

The Veneziano-Yankielowicz (VY) lagrangian is an effective theory of colourless degrees of freedom of the $\mathcal{N} = 1$ SYM multiplet associated with (4.8), which describes the N-fold degeneracy of the SYM vacuum, and demonstrates the formation of the gluino condensate as well. It was first derived in [40] in a rather heuristical way; soon after it was systematically shown in [36] how its form is (almost) completely fixed by anomalous superconformal Ward identities. We will now review the main steps outlined in [36] to build the VY-lagrangian.

The following considerations apply to a generic supersymmetric theory having anomaly structure

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{G}_{\alpha\dot{\alpha}} + 2\mathcal{D}_\alpha X = 0\,. \tag{4.10}$$

The superfield $\mathcal{G}_{\alpha\dot{\alpha}}$ is a general superfield containing the conserved currents, while X is some chiral superfield containing the anomalies of the theory. These superfields do not coincide necessarily with the $\mathcal{J}_{\alpha\dot{\alpha}}$ and $\mathcal{S}$ of the previous sections. First of all, we assume that $\mathcal{G}_{\alpha\dot{\alpha}}$ and X correspond to the low-energy degrees of freedom of the theory under consideration. Therefore, the low-energy dynamics is determined by the Green functions of these fields stemming from the generating functional

$$e^{\mathrm{i}\mathcal{W}} = \int \mathfrak{D}\varphi\, \exp\Big\{\mathrm{i}I[\varphi] + \mathrm{i}\int \mathrm{d}^4x\, \Big(\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; J_{\mathcal{G}}^{\alpha\dot{\alpha}}\mathcal{G}_{\alpha\dot{\alpha}} + \int \mathrm{d}^2\theta\; J_X X + \int \mathrm{d}^2\bar{\theta}\; J_{\bar{X}}\bar{X}\Big)\Big\}\,,$$

for $\mathcal{W} = \mathcal{W}[J_{\mathcal{G}}, J_X, J_{\bar{X}}]$ and φ denoting the generic field content of the theory. The anomalous Ward identities arising from (4.10) take the form

$$\Big[\bar{\mathcal{D}}^{\dot{\alpha}}\frac{\delta}{\delta J_{\mathcal{G}}^{\alpha\dot{\alpha}}} + 2\mathcal{D}^\alpha\frac{\delta}{\delta J_X} + 2\omega_\alpha(J_{\mathcal{G}}) + 2\omega_\alpha(J_X)\Big]\mathcal{W}[J_{\mathcal{G}}, J_X, J_{\bar{X}}] = 0\,,$$

where $\omega_\alpha(J_{\mathcal{G}})$ and $\omega_\alpha(J_X)$ are the contact terms produced by the source couplings accounting for the transformation properties of $\mathcal{G}_{\alpha\dot{\alpha}}$ and X. We now introduce the effective action in the standard way as the Legendre transform of $\mathcal{W}[J_{\mathcal{G}}, J_X, J_{\bar{X}}]$, i.e.

$$\Gamma[\mathcal{G}, X, \bar{X}] = \mathcal{W} - \int \mathrm{d}^4x\, \Big(\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; J_{\mathcal{G}}^{\alpha\dot{\alpha}}\mathcal{G}_{\alpha\dot{\alpha}} + \int \mathrm{d}^2\theta\; J_X X + \int \mathrm{d}^2\theta\; J_{\bar{X}}\bar{X}\Big)\,.$$

Our goal is to find the explicit expression for this effective action. To this end, we use the identities $J_{\mathcal{G}}^{\alpha\dot{\alpha}} = -\frac{\delta\Gamma}{\delta\mathcal{G}^{\alpha\dot{\alpha}}}$, $J_X = -\frac{\delta\Gamma}{\delta X}$ (similarly for $J_{\bar{X}}$), ending up with the superconformal Ward identity

$$\bar{\mathcal{D}}^{\dot{\alpha}}\mathcal{G}_{\alpha\dot{\alpha}} + 2\mathcal{D}_\alpha X + 2[\omega_\alpha(\mathcal{G}) + \omega_\alpha(X)]\Gamma = 0\,. \tag{4.11}$$

This identity is of great help to find the expression of $\Gamma[\mathcal{G}, X, \bar{X}]$. With this information at hand, it is possible to obtain the corresponding identities obeyed by the current superfields of chiral, dilatation and conformal transformations[3] $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{K}$. The aforementioned identites are

$$\partial^m\mathcal{R}_m - \mathrm{i}(\mathcal{D}^2X - \bar{\mathcal{D}}^2\bar{X}) - \mathrm{i}[\omega^R(\mathcal{G}) + \omega^R(X,\bar{X})]\Gamma = 0\,, \tag{4.12a}$$

$$\partial^m\mathcal{D}_m + \frac{3}{2}(\mathcal{D}^2X + \bar{\mathcal{D}}^2\bar{X}) - \mathrm{i}[\omega^D(\mathcal{G}) + \omega^D(X,\bar{X})]\Gamma = 0\,, \tag{4.12b}$$

$$\partial^m\mathcal{K}_{mn} + 3x_n(\mathcal{D}^2X + \bar{\mathcal{D}}^2\bar{X}) - \mathrm{i}\omega_n^K\Gamma = 0\,, \tag{4.12c}$$

[3] Knowing the precise expression these superfields is not necessary for our purposes. All that we need to know is that they can be defined starting from $\mathcal{G}_{\alpha\dot{\alpha}}$, and that the lowest component is the current corresponding to R-symmetry, dilatation and conformal symmetry, respectively.

where ω^R, ω^D and ω^K are local symmetry operators, whose explicit expression is not needed here. Actually, the effective theory we are looking for involves the chiral anomaly superfield X (and its conjugated) only: indeed, on physical grounds, it seems more reasonable that the low-energy degrees of freedom are those encoded in X. Let us therefore focus on X and $\bar{X}$. Luckily enough, it turns out that, in this setting, computations are simpler. First of all, we have to eliminate the field $\mathcal{G}_{\alpha\dot{\alpha}}$. This is obtained by considering the zero-momentum version of equations (4.12), namely by integrating these equations over the spacetime. Indeed, the first term vanishes being a total spacetime derivative, and one is left with

$$\widehat{\Omega}^R(X,\bar{X})\Gamma + \mathrm{i}\int \mathrm{d}^4x\,\left(\mathcal{D}^2X - \bar{\mathcal{D}}^2\bar{X}\right) = 0\,, \tag{4.13a}$$

$$\widehat{\Omega}^D(X,\bar{X})\Gamma + \mathrm{i}\frac{3}{2}\int \mathrm{d}^4x\,\left(\mathcal{D}^2X + \bar{\mathcal{D}}^2\bar{X}\right) = 0\,, \tag{4.13b}$$

$$\widehat{\Omega}^K_n(X,\bar{X})\Gamma + 3\mathrm{i}\int \mathrm{d}^4x\; x_n\left(\mathcal{D}^2X + \bar{\mathcal{D}}^2\bar{X}\right) = 0\,. \tag{4.13c}$$

Here, $\widehat{\Omega}^G(X,\bar{X})$ denotes an operator whose components are differential functional operators defined as

$$\Omega^G \equiv -\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\;(\delta_G X)\frac{\delta}{\delta X}\,,$$

where $\delta_G X = \mathrm{i}[G,X]$ represents the variation of X under the action of the generator G of the superconformal group. Note that $\widehat{\Omega}^G$ is a (coordinate-independent) superfield, and the index G refers to its lowest component, namely

$$\widehat{\Omega}^R(\theta,\bar{\theta}) = \Omega^R + \dots\,, \quad \widehat{\Omega}^D(\theta,\bar{\theta}) = \Omega^D + \dots\,, \quad \widehat{\Omega}^K_n(\theta,\bar{\theta}) = \Omega^K_n + \dots\,.$$

The non-homogeneity of equations (4.13) suggests to write the effective action as $\Gamma = \Gamma_0 + \Gamma_1$, where Γ_0 is any particular solution of the complete equation, and Γ_1 is the general solution of the homogeneous equation. At this point, the strategy is to solve one of the (4.13), and then substitute the solution in the other two equations to check that it is indeed a legitimate solution. For instance, one could start form (4.13a). In particular, we can find rather easily the solution of the full equation Γ_0, observing that the anomalous term does not depend on $(\theta,\bar{\theta})$: this means that to find a solution it suffices to consider the lowest component of the superfield operator $\widehat{\Omega}^G$. Recalling that $4\int \mathrm{d}^2\theta\; X = \mathcal{D}^2X$ up to total spacetime derivatives, and imposing that $\Gamma_0 = 4\int \mathrm{d}^4x\,\mathrm{d}^2\theta\; h(X) + \text{h.c.}$, using the explicit expression for $\delta^R X$ one arrives at

$$X\frac{\mathrm{d}h(X)}{\mathrm{d}X} - h(X) - \frac{1}{2}X = 0\,,$$

and a similar equation holds for $\bar{X}$. This equation is solved by

$$h(X) = \frac{1}{2}X\left(\log\frac{X}{\mu^3} - 1\right)\,,$$

and similarly for the conjugate. By introducing this function in the other equations (after some manipulations) one can realise that it is a valid solution. As far as the homogeneous term is concerned, we can still focus on the lowest component of $\widehat{\Omega}^G$ only, thanks to the superfield nature of this object. Therefore, we have:

$$\Omega^R \Gamma_1 = 0\,, \quad \Omega^D \Gamma_1 = 0\,, \quad \Omega_n^K \Gamma_1 = 0\,.$$

Requiring that $\Gamma_1 = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K(X, \bar{X})$ for a generic Kähler potential $K(X, \bar{X})$, we observe that:

- in order that the first relation be satisfied, K must have null chiral weight;
- in order that the second relation be satisfied, K must have mass dimension equal to two;
- in order that the third relation be satisfied, K must have null chiral weight *and* mass dimension equal to two.

These constraints ensure chiral, dilatation and conformal invariance, respectively. All in all, one gets

$$\Gamma[X, \bar{X}] = \int \mathrm{d}^4x \, \left\{\gamma \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K(X, \bar{X}) + \left[\int \mathrm{d}^2\theta\; X\left(\log\frac{X}{\mu^3} - 1\right) + \text{h.c.}\right]\right\},$$

with K constrained as stated above, and γ a dimensionless constant. Superconformal invariance has been tacitly assumed: this restricts further the form of $K(X, \bar{X})$, for it implies mass dimension $\mathbf{d}$ and chiral weight $\mathbf{k}$ of a chiral superfield to be related by $\mathbf{d} = \frac{3}{2}\mathbf{k}$ (for an antichiral superfield $\mathbf{d} = -\frac{3}{2}\mathbf{k}$ instead). Given all the properties $K(X, \bar{X})$ has to satisfy, it can be shown that the most general form it can assume is

$$K(X, \bar{X}) = (\bar{X}X)^{1/3} f(\bar{Z}, Z)\,, \quad \text{for} \;\; Z = X^{1/3}(\bar{\mathcal{D}}^2 \bar{X}^{1/3})^{-1/2}\,,$$

with the function $f(x, y)$ subject to the condition $f^*(x, y) = f(y, x)$ [36]. This general term gives rise to an effective potential whose bosonic component is unbounded from below. In the simplest case in which $f(x, y) = 1$, the potential is bounded from below (as we will see later), and we arrive at

$$\mathcal{L}_{\text{VY}} = \gamma \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; (\bar{X}X)^{\frac{1}{3}} + \left[\int \mathrm{d}^2\theta\; X\left(\log\frac{X}{\mu^3} - 1\right) + \text{h.c.}\right], \tag{4.14}$$

which is very similar to the Veneziano-Yankielowicz lagrangian.

4.3 The Special Chiral Superfield

By comparing (4.9) and (4.10), we observe that the superfield X in equation (4.14) is actually $\mathcal{S}$ defined in (4.8) up to a rescaling. We can then rewrite

$$K(\bar{\mathcal{S}}, \mathcal{S}) = \frac{1}{16\pi^2\rho}(\bar{\mathcal{S}}\mathcal{S})^{1/3}\,, \quad W(\mathcal{S}) = \frac{N}{16\pi^2}\mathcal{S}\left(\log\frac{\mathcal{S}}{\Lambda^3} - 1\right),$$

for ρ a dimensionless positive constant. Recalling equation (4.8c), we see

$$\begin{aligned}
\mathbf{Re}F &= \frac{1}{2}\,\mathrm{tr}\Big(-\frac{1}{2}F_{mn}F^{mn} + D^2 - \mathrm{i}\lambda\sigma^m\nabla_m\bar{\lambda}\Big) \equiv A\,,\\
\mathbf{Im}F &= \mathrm{tr}\Big[-\partial_m(\lambda\sigma^m\bar{\lambda}) - \frac{1}{4}\epsilon_{mnpq}F^{mn}F^{pq}\Big]\\
&= \partial_m\Big[-(\lambda\sigma^m\bar{\lambda}) + \epsilon^{mnpq}\Big(A_n\,\partial_p A_q - \frac{2\mathrm{i}}{3}A_nA_pA_q\Big)\Big] \equiv \partial_m C^m\,.
\end{aligned}$$

In particular, the imaginary part of F is an instanton density, and thus it is (locally) the exterior derivative of a Chern-Simons 3-form (see appendix B.1). Indeed, noting that

$$\mathrm{d}\Big[\frac{1}{3!}\,\mathrm{d}x^q\mathrm{d}x^p\mathrm{d}x^n\,\epsilon_{npqm}\,\mathrm{tr}(\lambda\sigma^m\bar{\lambda})\Big] = \mathrm{d}^4x\;\mathrm{tr}\big[\partial_m(\lambda\sigma^m\bar{\lambda})\big]\,,$$

and recalling that $\mathrm{tr}\,F_2\wedge F_2 = \mathrm{tr}\big(A\mathrm{d}A - \frac{2i}{3}A^3\big)$, we end up with

$$\begin{aligned}
F_4 \equiv \mathrm{d}^4x\;\mathbf{Im}F &= -\mathrm{d}^4x\;\;\mathrm{tr}\big[\partial_m(\lambda\sigma^m\bar{\lambda})\big] - \mathrm{tr}\,F_2\wedge F_2\\
&= -\mathrm{d}\;\mathrm{tr}\Big[\Big(A\,\mathrm{d}A - \frac{2i}{3}A^3\Big) + \frac{1}{3!}\mathrm{d}x^n\mathrm{d}x^p\mathrm{d}x^q\epsilon_{npqm}(\lambda\sigma^m\bar{\lambda})\Big] = \mathrm{d}C_3\,. \qquad (4.15)
\end{aligned}$$

The 1-form $C_1 \equiv C_m\mathrm{d}x^m$ is the Hodge-dual of C_3, i.e. $C_1 = *C_3$, and therefore

$$F = A + \mathrm{i}\partial_m C^m = A + \mathrm{i}\,*\mathrm{d}C_3\,.$$

It has been shown in [21] that chiral superfields enjoying such a peculiar structure for the F-term can always be expressed as

$$\mathcal{S} = -\frac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}U\,,\quad \bar{\mathcal{S}} = -\frac{1}{4}\mathcal{D}\mathcal{D}U\,, \qquad (4.16)$$

for $U(x,\theta,\bar{\theta})$ being a real scalar superfield in the case at hand. The latter superfield has C_1 among its independent bosonic components. In particular, one has

$$\frac{1}{8}\bar{\sigma}_m^{\dot{\alpha}\alpha}[\mathcal{D}_\alpha,\bar{\mathcal{D}}_{\dot{\alpha}}]U\big|_{\theta,\bar{\theta}=0} = C_m\,,\quad \frac{1}{4}\mathcal{D}^2U\big|_{\theta,\bar{\theta}=0} = -\bar{s}\,,\quad \frac{1}{16}\mathcal{D}^2\bar{\mathcal{D}}^2U\big|_{\theta,\bar{\theta}=0} = F\,. \qquad (4.17)$$

The F-component is invariant under the gauge transformation $C_3 \mapsto C_3 + \mathrm{d}\Lambda_2$, where $\Lambda_2(x)$ is a 2-form gauge parameter. This symmetry property holds also at the superfield level: indeed, $\mathcal{S}$ remains unchanged under the shift $U \mapsto U + L$ for L such that $\mathcal{D}^2L = \bar{\mathcal{D}}^2L = 0$; the superfield L is termed *linear superfield*, and plays the role of the gauge 2-form $\Lambda_2(x)$.

4.3.1 The Veneziano-Yankielowicz Effective Scalar Potential

We are interested in finding the scalar effective potential to identify the vacua of the theory. As pointed out in [5], there are some issues concerning how to integrate out the F-component field of $\mathcal{S}$: indeed, we have just seen that it is not an auxiliary complex field in the strictest sense, for it contains the dual 4-form of $\partial_m C^m$ associated with the SYM instanton density $\mathrm{tr}\,F_2\wedge F_2$. On the other hand, the superpotential is not single-valued under identical R-symmetry phase transformations of the field $\mathcal{S}(x,\theta) \mapsto \mathcal{S}'(x, e^{i\pi}\theta) = e^{2\pi i}\mathcal{S}(x,\theta)$. Indeed, we have

$$W(\mathcal{S}) \mapsto W(\mathcal{S}) + i\frac{N}{8\pi}\mathcal{S}\,,$$

and therefore

$$\mathcal{L}_{\mathrm{VY}} \mapsto \mathcal{L}_{\mathrm{VY}} - \frac{N}{4\pi}\mathbf{Im}F = \mathcal{L}_{\mathrm{VY}} - \frac{N}{4\pi}\partial_m C^m\,.$$

According to [4, 5], we can augment the VY lagrangian with the boundary term

$$\mathcal{L}_{\mathrm{bd}} = -\frac{1}{128\pi^2}\Big(\int \mathrm{d}^2\theta\,\bar{\mathcal{D}}^2 - \int \mathrm{d}^2\bar{\theta}\,\mathcal{D}^2\Big)\Big[\Big(\frac{1}{12\rho}\bar{\mathcal{D}}^2\frac{\bar{\mathcal{S}}^{1/3}}{\mathcal{S}^{2/3}} + \log\frac{\Lambda^{3N}}{\mathcal{S}^N}\Big)U\Big] + \mathrm{h.c.}\,, \tag{4.18}$$

so that the shift of the superpotential is compensated, and considering U as an independent superfield we can eliminate in a consistent way the auxiliary fields within F by solving their equations of motion. Actually, we can see that $\mathcal{L} = \mathcal{L}_{\mathrm{VY}} + \mathcal{L}_{\mathrm{bd}}$ is not only invariant under the identical phase transformation we considered before, but also under a generic $U(1)$ R-symmetry transformation. Indeed, for $\mathcal{S}(x,\theta) \mapsto \mathcal{S}'(x, e^{-i\alpha/2}\theta) = e^{i\alpha}\mathcal{S}(x,\theta)$ one has

$$\begin{aligned}
\mathcal{L}_{\mathrm{VY}} &\mapsto \mathcal{L}_{\mathrm{VY}} - \frac{N}{8\pi^2}\alpha\,\partial_m C^m\,,\\
\mathcal{L}_{\mathrm{bd}} &\mapsto \mathcal{L}_{\mathrm{bd}} - \frac{1}{128\pi^2}(-2iN\alpha)\Big(\int \mathrm{d}^2\theta\,\bar{\mathcal{D}}^2 - \int \mathrm{d}^2\bar{\theta}\,\mathcal{D}^2\Big)U = \mathcal{L}_{\mathrm{bd}} + \frac{N}{8\pi^2}\alpha\,\partial_m C^m\,.
\end{aligned}$$

Nevertheless, we know that the R-symmetry is broken down to a discrete $\mathbb{Z}_{2N}$. We can recover this feature with the requirement

$$\frac{1}{16\pi^2}\left(\frac{1}{12\rho}\bar{\mathcal{D}}^2\frac{\bar{\mathcal{S}}^{1/3}}{\mathcal{S}^{2/3}} + \log\frac{\Lambda^{3N}}{\mathcal{S}^N}\right)\Bigg|_{\mathrm{bd}} = -i\frac{\mathrm{n}}{8\pi}\,, \tag{4.19}$$

where $\mathrm{n} = 0, 1, \ldots, N-1 \mod N$ characterizes the asymptotic vacua of the theory. This choice of the boundary makes the augmented lagrangian $\mathcal{L}$ invariant under the transformation $U \mapsto U + L$, because L is a linear superfield. Since our goal is to find the explicit expression of the scalar potential, we put to zero the fermionic component, namely $\chi_\alpha = 0$. According to what we have seen in section 2.3.2, we find

$$\begin{aligned}
\mathcal{L}^B_{\mathrm{VY}} &= K_{s\bar{s}}\big[\partial_m s\,\partial^m\bar{s} + (\partial_m C^m)^2 + A^2\big] + \big[W_s(A + i\partial_m C^m) + \mathrm{h.c}\big]\\
&= K_{s\bar{s}}\,\partial_m s\partial^m\bar{s} + \big\{K_{s\bar{s}}\big[(\partial_m C^m)^2 + A^2\big] + 2A\,\mathbf{Re}W_s - 2(\partial_m C^m)\,\mathbf{Im}W_s\big\}\,.
\end{aligned} \tag{4.20}$$

where

$$K_{s\bar{s}} \equiv \frac{\partial K(s,\bar{s})}{\partial s\,\partial\bar{s}} = \frac{1}{9}\frac{1}{16\pi^2\rho}(\bar{s}s)^{-\frac{2}{3}}\,, \quad W_s \equiv \frac{\partial W(s)}{\partial s} = \frac{N}{16\pi^2}\log\frac{s}{\Lambda^3}\,. \tag{4.21}$$

Moreover, after a lengthy calculation one finds

$$\mathcal{L}^B_{\mathrm{bd}} = -2\partial_m\big[C^m\big(K_{s\bar{s}}\,\partial_n C^n - \mathbf{Im}W_s\big)\big]\,. \tag{4.22}$$

Now we integrate out the fields A and C^m from the lagrangian by solving their equations of motion. We do not need to consider (4.22), because it is a total derivative, and therefore it gives no contribution to the equation of motion. From (4.20) we have, instead,

$$K_{s\bar{s}}A + \mathbf{Re}W_s = 0 \quad \rightsquigarrow \quad A = -K_{s\bar{s}}^{-1}\left(\mathbf{Re}W_s\right)\,, \tag{4.23}$$

$$\partial_m\big(K_{s\bar{s}}\partial_n C^n - \mathbf{Im}W_s\big) = 0 \quad \rightsquigarrow \quad \partial_m C^m = K_{s\bar{s}}^{-1}\Big(\mathbf{Im}W_s - \frac{\mathrm{n}}{8\pi}\Big)\,, \tag{4.24}$$

where in solving the latter equation we have chosen an integer integration constant which is compatible with the condition (4.19). Now we can plug equations (4.23) and (4.24) into $\mathcal{L}^B = \mathcal{L}^B_{\mathrm{VY}} + \mathcal{L}^B_{\mathrm{bd}}$ to find the scalar potential. Despite the boundary term is unimportant for the derivation of the equations of motion, it is actually essential to ensure that the scalar potential is always non-negative. We have

$$\begin{aligned} V(s,\bar{s}) = -\big\{&K_{s\bar{s}}\big[(\partial_m C^m)^2 + A^2\big] + 2A\,\mathbf{Re}W_s - 2\partial_m C^m\,\mathbf{Im}W_s \\ &+ 2K_{s\bar{s}}(\partial_m C^m)^2 + 2(\partial_m C^m)\mathbf{Im}W_s\big\}\,, \end{aligned}$$

hence, one ends up with [29]

$$V(s,\bar{s}) = \frac{9\rho}{16\pi^2}N^2|s|^{\frac{4}{3}}\left[\log^2\frac{|s|}{\Lambda^3} + \Big(\arg s - 2\pi\frac{\mathrm{n}}{N}\Big)^2\right]. \tag{4.25}$$

The variable n is discrete, hence the potential is single-valued as well as multi-branched, being periodic in n with period N. Moreover, $V(s,\bar{s})$ vanishes for

$$\langle s\rangle = \Lambda^3\exp\Big\{2\pi\mathrm{i}\,\frac{\mathrm{n}}{N}\Big\}\,, \quad \mathrm{n} = 0,1,\ldots,N-1\,,$$

which reproduces the gluino condensate in equation (4.1). One can notice also that the scalar potential presents cusps at $\arg s = \pi(\mathrm{k}+1)/N$, where n changes value from k to $\mathrm{k}+1$. Finally, we notice that the potential is invariant under the simultaneous shifts

$$\mathrm{n}\mapsto\mathrm{n}+\mathrm{k}\,, \quad \arg s\mapsto\arg s + 2\pi\frac{\mathrm{k}}{N}\,,$$

which correspond to the $\mathbb{Z}_N$ symmetry.

4.4 Coupling the Supermembrane to the VY Model

We are finally ready to consider the issue we mentioned at the end of section 3.5.2, namely the coupling of a supermembrane and a bulk superfield. In particular, thanks to the results of [6, 3], we will couple a membrane to the $\mathcal{N}=1$ SYM theory and its VY effective description.

In a theory involving only a chiral 3-form superfield as that in (4.16), the most general action describing its coupling to a membrane in $\mathcal{N}=1$, 4-dimensional flat superspace is

$$I_2 = -\frac{1}{4\pi}\int_{W_3}\mathrm{d}^3\xi\sqrt{-\mathrm{g}}\,|k\mathcal{S}+c| - \frac{k}{4\pi}\int_{W_3}\mathcal{C}_3 - \Big(\frac{\bar{c}}{4\pi}\int_{W_3}\widehat{\mathcal{B}}_3 + \mathrm{h.c.}\Big)\,, \tag{4.26}$$

where $c = k_1 + \mathrm{i}k_2$ and k are real constant charges, which characterise the coupling of the membrane to a 3-form gauge superfield $\mathcal{C}_3$ and a complex super 3-form $\widehat{\mathcal{B}}_3$ [12, 21]. These superforms are defined as [6, 5]

$$\mathcal{C}_3 = 2\mathrm{i}E^a \wedge E^\alpha \wedge \bar{E}^{\dot\alpha}(\sigma_a)_{\alpha\dot\alpha}U + \frac{1}{4}E^b \wedge E^a\big[E^\alpha(\sigma_{ab})_\alpha{}^\beta\mathcal{D}_\beta U - \bar{E}^{\dot\beta}(\bar\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\bar{\mathcal{D}}_{\dot\alpha}U\big]$$
$$- \frac{1}{48}E^c \wedge E^b \wedge E^a\epsilon_{abcd}(\bar\sigma^d)^{\dot\beta\alpha}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}]U\,,$$

and

$$\widehat{\mathcal{B}}_3 = -\frac{1}{2}E^b \wedge E^a \wedge E^\alpha(\sigma_{ab}\theta)_\alpha - \mathrm{i}E^a \wedge E^\alpha \wedge \bar{E}^{\dot\alpha}(\sigma_a)_{\alpha\dot\alpha}\theta^2\,.$$

Moreover, the associated field strength 4-forms are given by

$$\mathcal{G}_4 \equiv \mathbf{d}\mathcal{C}_3 = -\frac{1}{2}E^b \wedge E^a \wedge \big[E^\alpha \wedge E_\beta(\sigma_{ab})_\alpha{}^\beta\bar{\mathcal{S}} + \bar{E}_{\dot\alpha} \wedge \bar{E}^{\dot\beta}(\bar\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\mathcal{S}\big]$$
$$- \frac{1}{12}E^c \wedge E^b \wedge E^a \wedge \big[\epsilon_{abcd}E^\alpha(\sigma^d)_{\alpha\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{S}} - \epsilon_{abcd}\bar{E}_{\dot\alpha}(\bar\sigma^d)^{\dot\alpha\alpha}\mathcal{D}_\alpha\mathcal{S}\big]$$
$$+ \frac{\mathrm{i}}{96}E^d \wedge E^c \wedge E^b \wedge E^a\epsilon_{abcd}\big(\mathcal{D}^2\mathcal{S} - \bar{\mathcal{D}}^2\bar{\mathcal{S}}\big) \qquad (4.27)$$

and by

$$\widehat{\mathcal{F}}_4 \equiv \mathbf{d}\widehat{\mathcal{B}}_3 = -\frac{1}{2}E^b \wedge E^a \wedge E^\alpha \wedge E_\beta(\sigma_{ab})_\alpha{}^\beta\,.$$

In (4.27) we have substituted U with $\mathcal{S}$ and $\bar{\mathcal{S}}$ using the identities in equation (4.16). The bulk superfield $\mathcal{S}$ is evaluated on the membrane worldvolume $\mathcal{Z}^M = \mathcal{Z}^M(\xi)$, which in turn is parametrised by three coordinates (ξ^1, ξ^2, ξ^3), and exactly like in section 3.5.2 we have

$$\mathrm{g}_{ij}(\xi) = \eta_{ab}E_i{}^a(\xi)E_j{}^b(\xi)\,, \quad E^a(\xi) = \mathrm{d}\xi^i E_i{}^a(\xi)\,, \quad E_i{}^a = \partial_i\mathcal{Z}^M(\xi)E_M{}^a(\xi)\,,$$

for the induced metric on the membrane and the (pull-back of the) bosonic vielbein.

At this point, one could ask why it is necessary to include the constant c and the 3-form $\widehat{\mathcal{B}}_3$, rather then considering only $|\mathcal{S}|$ and the 3-form $\mathcal{C}_3$. It has to be noticed that, in the SYM case, $\mathcal{S} = \mathrm{tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha$ is a nilpotent superfield. It is thus essential to consider the modulus $|k\mathcal{S} + c|$ rather than simply $|\mathcal{S}|$, in that the latter is not well defined. On the other hand, the action must be κ-symmetric: since we have included the constant c, the mere presence of $\mathcal{C}_3$ is not enough to fullfill this requirement, and hence the action must comprise also $\widehat{\mathcal{B}}_3$ to be κ-symmetric.

We note also that the real part of $\widehat{\mathcal{B}}_3$ has already appeared in section 3.5.2 as part of the free supermembrane action.

4.4.1 Worldvolume Symmetries

By construction, the action in equation (4.26) is invariant under worldvolume diffeomorphisms $\xi^i \mapsto \zeta^i(\xi)$ and κ-symmetry transformations

$$\delta_k x^m(\xi) = \mathrm{i}\kappa\sigma^m\bar\theta - \mathrm{i}\theta\sigma^m\bar\kappa\,, \quad \delta_\kappa\theta^\alpha = \kappa^\alpha\,, \quad \delta_\kappa\bar\theta^{\dot\alpha} = \bar\kappa^{\dot\alpha}\,, \qquad (3.25)$$

which induce the following transformation law of the bosonic supervielbein

$$\delta_\kappa E^a = -2\mathrm{i}(\kappa\sigma^a)_{\dot\alpha}\bar E^{\dot\alpha} + 2\mathrm{i}E^\alpha(\sigma^a\bar\kappa)_\alpha\,. \tag{3.28}$$

Moreover, a generic chiral superfield Φ and its conjugate $\bar\Phi$ transform as

$$\delta_\kappa\Phi = \kappa^\alpha\mathcal{D}_\alpha\Phi\,,\quad \delta_\kappa\bar\Phi = \bar\kappa_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi\,.$$

Relying on the formulae we have seen in section 3.5.2 we will now show that I_2 is indeed κ-symmetric.

Proof. As we did previously, we assume that the membrane is closed, so that $\partial W_3 = \varnothing$ and the variation of any of the two superform is given by $\delta_\kappa\int_{W_3}\bullet = \int_{W_3}\iota_\kappa\,\mathbf{d}\bullet$. From what we have already seen, it is easy to find

$$\bar c\,(\iota_\kappa\mathcal{F}_4) + c\,(\iota_\kappa\mathcal{F}_4) = -2\star E_a\wedge\left[\bar c\,E^\alpha(\sigma^a)_{\alpha\dot\alpha}(\bar\Gamma\kappa)^{\dot\alpha} - c\,\bar E_{\dot\alpha}(\bar\sigma^a)^{\dot\alpha\alpha}(\Gamma\bar\kappa)_\alpha\right], \tag{4.28}$$

where we have applied the identities in equation (3.31). Then we have

$$\begin{aligned}\iota_\kappa\mathcal{G}_4 = &-E^b\wedge E^a\wedge\left[E^\alpha(\sigma_{ab}\kappa)_\alpha\,\bar{\mathcal{S}} - \bar E_{\dot\alpha}(\bar\sigma_{ab}\bar\kappa)^{\dot\alpha}\mathcal{S}\right]\\ &-\frac{1}{12}E^c\wedge E^b\wedge E^a\epsilon_{abcd}\left[(\kappa\sigma^d)_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{S}} + (\bar\kappa\bar\sigma^d)^\alpha\mathcal{D}_\alpha\mathcal{S}\right]\\ =&-2\star E_a\wedge\left[E^\alpha(\sigma^a)_{\alpha\dot\alpha}(\bar\Gamma\kappa)^{\dot\alpha}\bar{\mathcal{S}} - \bar E_{\dot\alpha}(\bar\sigma^a)^{\dot\alpha\alpha}(\Gamma\bar\kappa)_\alpha\mathcal{S}\right] && (4.29a)\\ &+\frac{1}{12}E^c\wedge E^b\wedge E^a\epsilon_{abcd}\left[\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{S}}\,(\bar\sigma^d\kappa)^{\dot\alpha} + \mathcal{D}^\alpha\mathcal{S}\,(\sigma^d\bar\kappa)_\alpha\right], && (4.29b)\end{aligned}$$

using again (3.31). Finally, identifying $|k\mathcal{S}+c| = |\Phi|$, we have

$$\begin{aligned}\delta_\kappa\left(\mathrm{d}^3\xi\sqrt{-\mathrm{g}}\,|\Phi|\right) &= (\star E_a\wedge\delta_\kappa E^a)|\Phi| + \mathrm{d}^3\xi\,\sqrt{-\mathrm{g}}\,\frac{(\delta_\kappa\bar\Phi)\Phi + \bar\Phi(\delta_\kappa\Phi)}{2|\Phi|}\\ &= \star E_\alpha\wedge\left[-2\mathrm{i}(\kappa\sigma^a)_{\dot\alpha}\bar E^{\dot\alpha} + 2\mathrm{i}E^\alpha(\sigma^a\bar\kappa)_\alpha\right]|\Phi| && (4.30a)\\ &\quad+\mathrm{d}^3\xi\sqrt{-\mathrm{g}}\,\frac{(\bar\kappa^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi)\Phi + \bar\Phi(\kappa^\alpha\mathcal{D}_\alpha\Phi)}{2|\Phi|}\,. && (4.30b)\end{aligned}$$

Now, we put together (4.28), (4.29a) (4.30a), finding

$$\begin{aligned}(4.28) + k\,(4.29a) + (4.30a) = &-2\star E_a\wedge E^\alpha(\sigma^a)_{\alpha\dot\alpha}\left[(\bar\Gamma\kappa)^{\dot\alpha}(\bar c + k\bar{\mathcal{S}}) + \mathrm{i}\bar\kappa^{\dot\alpha}|\Phi|\right]\\ &-2\star E_a\wedge\bar E_{\dot\alpha}(\bar\sigma^a)^{\dot\alpha\alpha}\left[(\Gamma\bar\kappa)_\alpha(c + k\mathcal{S}) - \mathrm{i}\kappa_\alpha|\Phi|\right],\end{aligned}$$

and thus this variation vanishes if and only if κ_α and $\bar\kappa^{\dot\alpha}$ obey the conditions

$$\kappa_\alpha = -\mathrm{i}\frac{k\mathcal{S}+c}{|k\mathcal{S}+c|}(\Gamma\bar\kappa)_\alpha\,,\quad \bar\kappa^{\dot\alpha} = \mathrm{i}\frac{k\bar{\mathcal{S}}+\bar c}{|k\mathcal{S}+c|}(\bar\Gamma\kappa)^{\dot\alpha}\,, \tag{4.31}$$

which are, in some sense, reminiscent of those in equation (3.26). At this point, we have to check that the constraints (4.31) allow the mutual cancellation of (4.29b) with (4.30b). To this aim, we notice that the identities

$$\mathrm{d}^3\xi\sqrt{-\mathrm{g}}\,\Gamma = \frac{\mathrm{i}}{3!}E^b\wedge E^c\wedge E^d\epsilon_{abcd}\,\sigma^d\,,\quad \mathrm{d}^3\xi\sqrt{-\mathrm{g}}\,\bar\Gamma = \frac{\mathrm{i}}{3!}E^b\wedge E^c\wedge E^d\epsilon_{abcd}\,\bar\sigma^d\,,$$

follow from the very definitions of $\Gamma_{\alpha\dot\alpha}$ and $\bar\Gamma^{\dot\alpha\alpha}$ in (3.27). Paying attention to the indices, which, when ordered correctly, produce an overall minus sign, we obtain

$$(4.29\text{b}) = \frac{\mathrm{i}}{2}\mathrm{d}^3\xi\sqrt{-\mathrm{g}}\left[\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{S}}(\bar\Gamma\kappa)^{\dot\alpha} + \mathcal{D}^\alpha\mathcal{S}(\Gamma\kappa)_\alpha\right].$$

Since $\mathcal{D}_\alpha\Phi = k\,\mathcal{D}_\alpha\mathcal{S}$ and similarly for its conjugate, and ignoring the common factor which is unnecessary, we find

$$\begin{aligned}(4.29\text{b}) + (4.30\text{b}) &= \left[\mathrm{i}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{S}}(\bar\Gamma\kappa)^{\dot\alpha} + \mathrm{i}\mathcal{D}^\alpha\mathcal{S}(\Gamma\kappa)_\alpha + (\bar\kappa^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{S}})\frac{\Phi}{|\Phi|} + \frac{\bar\Phi}{|\Phi|}(\kappa^\alpha\mathcal{D}_\alpha\mathcal{S})\right]\\ &= -\left[\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{S}}\,\bar\kappa^{\dot\alpha}\frac{\Phi}{|\Phi|} - \mathrm{i}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{S}}\,(\bar\Gamma\kappa)^{\dot\alpha}\right] + \left[\frac{\bar\Phi}{|\Phi|}(\mathcal{D}^\alpha\mathcal{S})\kappa_\alpha + \mathrm{i}(\mathcal{D}^\alpha\mathcal{S})(\Gamma\bar\kappa)_\alpha\right],\end{aligned}$$

and the two terms are (separetely) null provided that the conditions in (4.31) are satisfied. □

We have seen already that the degrees of freedom of the membrane are those of a supermultiplet $\mathcal{M} = \{\phi(\xi), \psi(\xi)\}$, where the scalar $\phi(\xi)$ describes the fluctuations of the membrane in the transverse direction, and $\psi(\xi)$ is a two-component $SL(2,\mathbb{R})$ Majorana spinor. The $\mathcal{N}=1$, $d=3$ Goldstone supermultiplet $\mathcal{M}$ is associated with the spontaneous breaking of the translational symmetry in the direction transverse to the membrane, and to the halving of the bulk $\mathcal{N}=1$, $d=4$ supersymmetry due to the membrane itself. The broken symmetries are non-linearly realized on the Goldstone supermultiplet, whose interaction with the chiral superfield $\mathcal{S}$ is described by the action (4.26).

4.4.2 Dynamic Membrane as a Source of BPS Domain Walls

Now, we want to see how the presence of a membrane modifies the equations of motion of the auxuliary fields in the VY effective model, inducing a BPS domain wall solution. The insertion of the membrane solves the discrepancy between the exact value of the tension of the BPS saturating wall, and that estimated by means of the potential (4.25), which appears to be much smaller. This issue was pointed out long ago in [28], where it was also suggested that at the cusps of the potential there should be an object accounting for the missing contribution to the tension. This object, in fact, could be identified with a membrane [5].

We consider a static membrane located at $x^3 = 0 = \theta^\alpha = \bar\theta^{\dot\alpha}$, and whose worldvolume extends along the directions $\xi^i = x^i$, for $i = 0,1,2$. In these conditions, the induced metric on the membrane g_{ij} reduces to the flat three dimensional metric[4] η_{ij}. Moreover,

[4]We suppose that the metrics g_{ij} and η_{ij} inherit the moslty-minus signature of the metric η_{mn} of the target space $\mathcal{M}_4$.

the 3-form $\widehat{\mathcal{B}}_3$ (and its conjugate) is null, while for $\mathcal{C}_3$ we have

$$\mathcal{C}_3\big|_{\theta,\bar\theta=0} = \mathrm{d}^3\xi\, C^3,$$

according to equation (4.17). Therefore, the action describing the coupling between VY model and a static membrane is

$$I = \int \mathrm{d}^3\xi\, \mathrm{d}x^3\ \left(\mathcal{L}^B_{\mathrm{VY}} + \mathcal{L}_{\mathrm{m}}\right),$$

where, explicitly, the lagrangian is

$$\begin{aligned}\mathcal{L} &\equiv \mathcal{L}^B_{\mathrm{VY}} + \mathcal{L}_{\mathrm{m}}\\ &= K_{s\bar s}\big(\partial_m s\,\partial^m \bar s + F\bar F\big) + \big(W_s F + \mathrm{h.c.}\big) - 2\partial_m\big[C^m(K_{s\bar s}\,\partial_n C^n - \mathbf{Im}W_s)\big]\\ &\quad - \frac{1}{4\pi}\big(|ks+c| + kC^3\big)\delta(x^3)\,. \end{aligned} \tag{4.32}$$

Let us then find the equations of motion for the auxiliary fields A and C_m. Actually, the equation of motion of A coincides with that we have found previously

$$A = -K^{-1}_{s\bar s}\big(\mathbf{Re}W_s\big)\,. \tag{4.23}$$

On the contrary, the equation of motion of C_m is amended by the presence of the membrane term. Indeed, we have

$$\partial_m \frac{\partial\mathcal{L}}{\partial\,\partial_n C^n} - \frac{\partial\mathcal{L}}{\partial C^m} = 0 \quad \rightsquigarrow \quad \partial_m\big(K_{s\bar s}\,\partial_n C^n - \mathbf{Im}W_s\big) = -\frac{k}{8\pi}\delta_{m3}\,\delta(x^3)\,,$$

and therefore

$$\partial_m C^m = K^{-1}_{s\bar s}\left[\mathbf{Im}W_s - \frac{\mathrm{n}'}{8\pi}\right], \quad \mathrm{n}' \equiv \mathrm{n} + k\mathsf{H}(x^3)\,, \tag{4.33}$$

where $\mathsf{H}(x^3)$ denotes the Heaviside step function. With these solutions at hand, we can find the on-shell value of F and its conjugate

$$F = -\frac{1}{K_{s\bar s}}\left[\overline{W}_{\bar s} + \frac{\mathrm{i}\mathrm{n}'}{8\pi}\right], \quad \bar F = -\frac{1}{K_{s\bar s}}\left[W_s - \frac{\mathrm{i}\mathrm{n}'}{8\pi}\right]. \tag{4.34}$$

These equations prompt the introduction of the following modified superpotential

$$\widehat{W}(s) \equiv W(s) - \frac{\mathrm{i}\mathrm{n}'}{8\pi}s\,,$$

and similarly for the conjugate. We see that the modified superpotential has a jump at the position of the membrane, hence its local minima describe two SYM vacua: one, say on the left of the membrane, labeled by n; the other, on the right, labeled by $\mathrm{n}+k$. Furthermore, together with the bulk field equations, we have to consider also the equations of motion of the membrane field $x^3(\xi)$, which, for a static membrane, reduce to

$$\big(\partial_3|ks+c| + k\,\partial_m C^m\big)\big|_{x^3=0} = 0\,. \tag{4.35}$$

We focus on the BPS configurations interpolating between the vacua at $x^3 \to \pm\infty$ where the vev of the component field s is, respectively

$$\langle s\rangle_{-\infty} = \Lambda^3\, e^{2\pi \mathrm{i} \mathrm{n}/N} \quad \text{and} \quad \langle s\rangle_{+\infty} = \Lambda^3\, e^{2\pi \mathrm{i}(\mathrm{n}+k)/N}\,.$$

As we have already seen several times, the wall profile is determined by the x^3-dependence of the scalar field $s = s(x^3)$, which is assumed to be constant along the other directions. Since we are considering a bosonic configuration, we set the fermionic component χ_α of $\mathcal{S}$ to zero. Moreover, we have to require that the variation of χ_α is zero under 1/2 supersymmetry preserved by the membrane supporting the wall solution. That is, we enforce

$$\delta\chi_\alpha = \sqrt{2}\big[\mathrm{i}(\sigma^3)_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\dot s + \epsilon_\alpha F\big] \stackrel{!}{=} 0\,, \tag{4.36}$$

where $\dot s \equiv \frac{\mathrm{d}s}{\mathrm{d}x^3}$. On the other hand, for the static membrane we have $\theta^\alpha = \bar\theta^{\dot\alpha} = 0$: these conditions are preserved only by a cobination of supersymetry and κ-symmetry transformation, that is

$$\delta\theta^\alpha = \epsilon^\alpha + \kappa^\alpha = 0 \qquad \Longleftrightarrow \qquad \epsilon^\alpha = -\kappa^\alpha.$$

Now, we have to recall that the κ-symmetry parameters are subject to the constraints in equation (4.31). Therefore, the previous equation implies that *on the membrane* the supersymmetry parameter ϵ^α – and $\bar\epsilon^{\dot\alpha}$ as well – has to satisfy an analogous condition to that in (4.31), namely

$$\epsilon_\alpha = e^{\mathrm{i}\eta}(\sigma^3)_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\,, \tag{4.37}$$

where $\eta = \arg(ks + c)|_{x^3=0}$ and it is constant on the bulk. If we plug equation (4.37) in (4.36) we find the BPS equation

$$\dot s = \mathrm{i}\, e^{\mathrm{i}\eta} F = -\mathrm{i}\, e^{\mathrm{i}\eta}\frac{\widehat{W}_s}{K_{s\bar s}}\,. \tag{4.38}$$

It is possible to assume that, on the membrane, $ks(0)+c$, $ks(0)$ and c have the same phase η. This particular choice is convenient in that it makes equations (4.35) and (4.38) mutually consistent. Moreover, we can easily show that, similarly to what we have seen in section 3.4.2, the following identity holds:

$$\frac{\mathrm{d}}{\mathrm{d}x^3}\mathbf{Re}(\widehat{W}e^{-\mathrm{i}\eta}) = 0\,. \tag{4.39}$$

Indeed

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x^3}\widehat{W} &= W_s\dot s - \frac{\mathrm{i}\mathrm{n}'}{8\pi}\dot s - \frac{\mathrm{i}k}{8\pi}\delta(x^3)s = \widehat{W}_s\dot s - \frac{\mathrm{i}k}{8\pi}\delta(x^3)s \\ &= \mathrm{i}\widehat{W}_s e^{\mathrm{i}\eta}F - \frac{\mathrm{i}k}{8\pi}\delta(x^3)s = -\mathrm{i}\frac{\overline{\widehat{W}}_{\bar s}\widehat{W}_s}{K_{s\bar s}} - \frac{\mathrm{i}}{8\pi}\delta(x^3)|ks|e^{\mathrm{i}\eta}\,,\end{aligned}$$

and this, upon multiplication of the phase factor $e^{-\mathrm{i}\eta}$, leads to (4.39). Clearly, the identity (4.39) implies

$$\mathbf{Re}(\widehat{W}e^{-\mathrm{i}\eta}) = \text{constant}\,. \tag{4.40}$$

Let us finally compute the tension of the configuration. The first step consists in substituting the on-shell value of the auxiliary fields A and C_m in the lagrangian (4.32). The computations are straightforward but lengthy and we will therefore skip them. All in all, taking into account that s depends only on x^3, one ends up with the action

$$I = -\int \mathrm{d}^3\xi\,\mathrm{d}x^3\; K_{s\bar{s}}\Big[\dot{s}\dot{\bar{s}} + \frac{\widehat{\overline{W}}_{\bar{s}}\widehat{W}_s}{K^2_{s\bar{s}}}\Big] - \int \mathrm{d}^3\xi\,\mathrm{d}x^3\;\frac{|ks+c|}{4\pi}\,\delta(x^3)\,.$$

This action can be neatly rewritten in a BPS-like fashion as

$$\begin{aligned} I = &-\int \mathrm{d}^3\xi\,\mathrm{d}x^3\; K_{s\bar{s}}\Big(\dot{s} \pm \mathrm{i}e^{\mathrm{i}\beta}\frac{\widehat{\overline{W}}_{\bar{s}}}{K_{s\bar{s}}}\Big)\Big(\dot{\bar{s}} \mp \mathrm{i}e^{-\mathrm{i}\beta}\frac{\widehat{W}_s}{K_{s\bar{s}}}\Big) \\ &\mp \mathrm{i}\int \mathrm{d}^3\xi\,\mathrm{d}x^3\;\big(\dot{s}e^{-\mathrm{i}\beta}\widehat{W}_s - \dot{\bar{s}}e^{\mathrm{i}\beta}\widehat{\overline{W}}_{\bar{s}}\big) - \int \mathrm{d}^3\xi\; T_m\,, \end{aligned}$$

where β is an arbitrary phase and $T_m \equiv |ks(0)+c|/4\pi$ is the membrane tension. Now, taking $\beta = \eta$ and selecting the upper sign, thanks to equation (4.38) the first line vanishes, and we are left with the on-shell value of the action

$$I = \int \mathrm{d}^3\xi\,\mathrm{d}x^3\; 2\,\mathbf{Im}\big(\dot{s}e^{-\mathrm{i}\eta}\widehat{W}_s\big) - \int \mathrm{d}^3\xi\; T_m\,.$$

The integration of the first term on the variable x^3 is easily performed noting that, by definition

$$\widehat{W}_s\dot{s} = \frac{\mathrm{d}}{\mathrm{d}x^3}\widehat{W} + \frac{\mathrm{i}}{8\pi}\delta(x^3)\,ks\,.$$

Thus, on one hand we have

$$\int_{\mathbb{R}} \mathrm{d}x^3\; 2\,\mathbf{Im}\Big(\frac{\mathrm{d}\widehat{W}}{\mathrm{d}x^3}e^{-\mathrm{i}\eta}\Big) = 2\,\mathbf{Im}\big[(\widehat{W}_{+\infty} - \widehat{W}_{-\infty})e^{-\mathrm{i}\eta}\big] = -2\,\mathbf{Im}\big[(\widehat{W}_{+\infty} - \widehat{W}_{-\infty})e^{-\mathrm{i}(\eta-\pi)}\big]\,.$$

On the other hand, instead, we have

$$2\,\mathbf{Im}\Big(\frac{\mathrm{i}}{8\pi}\delta(x^3)\,ks\,e^{-\mathrm{i}\eta}\Big) = \frac{1}{4\pi}\delta(x^3)\Big(\frac{\mathrm{i}kse^{-\mathrm{i}\eta} + \mathrm{i}k\bar{s}e^{\mathrm{i}\eta}}{2\mathrm{i}}\Big) = \frac{1}{4\pi}\mathbf{Re}\big(\delta(x^3)\,ks\,e^{-\mathrm{i}\eta}\big)\,.$$

Therefore, we arrive at

$$I = -\int \mathrm{d}^3\xi\; 2\,\mathbf{Im}\big[(\widehat{W}_{+\infty} - \widehat{W}_{-\infty})e^{-\mathrm{i}(\eta-\pi)}\big] - \int \mathrm{d}^3\xi\;\Big[T_m - \frac{1}{4\pi}\mathbf{Re}\big(ks(0)e^{-\mathrm{i}\eta}\big)\Big]\,.$$

Now, taking into account (4.40), and requiring

$$\mathbf{Im}\big[(\widehat{W}_{+\infty} - \widehat{W}_{-\infty})e^{-\mathrm{i}(\eta-\pi)}\big] \geq 0\,,$$

one finds that the phase of $(\widehat{W}_{+\infty} - \widehat{W}_{-\infty})$ is given by $\eta - \frac{\pi}{2}$. Moreover, recalling that on the membrane $\arg(ks(0)+c) = \arg(ks(0)) = \arg(c) = \eta$ so that $4\pi T_m - \mathbf{Re}\big(ks(0)e^{-\mathrm{i}\eta}\big) = |c|$, we get that the energy density per unit surface of the system is

$$T = 2|\widehat{W}_{+\infty} - \widehat{W}_{-\infty}| + \frac{|c|}{4\pi} = T_w + T_0\,.$$

The first term in the expression is the tension of the domain wall saturating the BPS bound, while the second coincides with the tension of a free membrane. For $|c| = 0$ we have $T_0 = 0$: this means that the contribution of the membrane to the overall tension exactly cancels the jump of the superpotential at $x^3 = 0$. On the other hand, if the membrane were not there we would have obtained $T' = T_w - |ks(0)|/4\pi$, which is indeed less than that of the BPS bound: this is the discrepancy pointed out in [28] which we mentioned at the beginning of this section. The contribution of the membrane tension is thus of fundamental importance in order to ensure the saturation of the BPS bound. In particular, for $|c| = 0$ the value of the wall tension turns out to be

$$T_w = \frac{N}{8\pi^2}\Lambda^3 \left| e^{2\pi \mathrm{i}\frac{\mathrm{n}+k}{N}} - e^{2\pi \mathrm{i}\frac{\mathrm{n}}{N}} \right| ,$$

i.e. exactly the one predicted in [16].

4.5 Introducing Dynamical Glueballs

In this last section we proceed along an unexplored direction, analysing the modifications of the VY theory produced by introducing a mass term for the CP-odd glueball field C_m, and also a new scalar field u which is related to the CP-even glueball $F_{mn}F^{mn}$. To this end, we consider the term proposed in [18], that is

$$\mathcal{L}' = -\frac{1}{\delta} \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; \frac{U^2}{(\mathcal{S}\bar{\mathcal{S}})^{1/3}} , \tag{4.41}$$

where δ is a dimensionless positive constant. This modification is worth to be considered because, in so doing, the field C_m acquires one degrees of freedom, and therefore becomes a propagating field. We focus on the bosonic components only: in such case we shall ignore the χ_α term of $\mathcal{S}$ in (4.7), while the component expansion of the superfield U reads

$$U = u + \theta^2 \bar{s} + \bar{\theta}^2 s - 2\,\theta\sigma^m\bar{\theta}\, C_m + \theta^2\bar{\theta}^2 \Big(A + \frac{1}{4}\Box u \Big) . \tag{4.42}$$

The component u is a real scalar field, which, following [18], describes a CP-even glueball. We notice that the new term is not invariant under the shift $U \mapsto U + L$, for L a linear superfield[5]; however, it is necessary to break this symmetry if one wants to retain the fields C_m and u as dynamical variables.

In order to obtain the explicit component expansion of the new term, we define

$$\widehat{G}(\mathcal{S}, \bar{\mathcal{S}}) \equiv (\mathcal{S}\bar{\mathcal{S}})^{-\frac{1}{3}} ,$$

thus we interpret the denominator in (4.41) as a new "Kähler potential". Now, by expanding $\widehat{G}(\mathcal{S}, \bar{\mathcal{S}})$ as we have seen in section 2.3.2, and then multiplying the resulting

[5] Alternatively, if we do not want to give up gauge symmetry, following [5] we could introduce a Stückelberg linear superfield $\mathbb{L}$ with transformation properties $\mathbb{L} \mapsto \mathbb{L} + L$, and consider $(U - \mathbb{L})^2$ rather than simply U^2.

expression by (4.42), we find the component lagrangian

$$\mathcal{L}' = -\frac{1}{\delta}u^2\Big[-\frac{1}{4}\Box\widehat{G} + \widehat{G}_{s\bar{s}}\big(\partial_m s\,\partial^m\bar{s} + F\bar{F}\big)\Big] - \frac{2}{\delta}u\big(\widehat{G}_s\,sF + \widehat{G}_{\bar{s}}\,\bar{s}F\big)$$
$$+ \frac{2}{\delta}u\,C_m\mathsf{q}^m - \frac{2}{\delta}\widehat{G}\Big[u\Big(A + \frac{1}{4}\Box u\Big) + s\bar{s} + C_mC^m\Big]\,,$$

where now $\widehat{G} = \widehat{G}(s,\bar{s})$, and we have defined

$$\widehat{G}_s \equiv \frac{\partial\widehat{G}}{\partial s}\,,\quad \widehat{G}_{\bar{s}} \equiv \frac{\partial\widehat{G}}{\partial\bar{s}}\,,\quad \widehat{G}_{s\bar{s}} \equiv \frac{\partial^2\widehat{G}}{\partial s\,\partial\bar{s}}\,,$$

and also

$$\mathsf{q}^m \equiv \mathrm{i}(\widehat{G}_s\,\partial^m s - \widehat{G}_{\bar{s}}\,\partial^m\bar{s})\,.$$

Noting that $-2u\,\Box u = 2\partial_m u\,\partial^m u - \Box u^2$, which implies

$$-\frac{1}{2}\widehat{G}u\,\Box u = \frac{1}{2}\widehat{G}\,\partial_m u\,\partial^m u - \frac{1}{4}\widehat{G}\,\Box u^2\,,$$

we see that

$$\frac{1}{4}u^2\,\Box\widehat{G} + \frac{1}{2}\widehat{G}\,\partial_m u\,\partial^m u - \frac{1}{4}\widehat{G}\,\Box u^2 = \frac{1}{2}\widehat{G}\,\partial_m u\,\partial^m u$$

up to a total spacetime derivative[6] which we will ignore. Therefore, the full modified VY lagrangian reads

$$\begin{aligned}\mathcal{L} = {} & K_{s\bar{s}}\big(\partial_m s\,\partial^m\bar{s} + F\bar{F}\big) + \big(W_sF + \text{h.c.}\big) - 2\partial_m\big[C^m\big(K_{s\bar{s}}\partial_nC^n - \mathbf{Im}W_s\big)\big]\\ & - \frac{1}{4\pi}\big(|ks| + kC_3\big)\delta(x^3) + \frac{1}{2\delta}\widehat{G}\,\partial_m u\,\partial^m u - \frac{1}{\delta}\widehat{G}_{s\bar{s}}\,u^2\big(\partial_m s\,\partial^m\bar{s} + F\bar{F}\big)\\ & - \frac{2}{\delta}u\big(\widehat{G}_s\,sF + \widehat{G}_{\bar{s}}\,\bar{s}F\big) - \frac{2}{\delta}\widehat{G}\big(s\bar{s} + C_mC^m + uA\big) + \frac{2}{\delta}u\,C_m\mathsf{q}^m\,.\end{aligned} \tag{4.43}$$

For simplicity, we have taken immediately $c = 0$. We can notice that for $\delta \to \infty$ the lagrangian reduces to (4.32).

Before we proceed further, an important comment is in order. In the previous sections, in order to make the elimination of the auxiliary field F a consistent procedure, we have introduced the independent superfield U accompanied by a boundary term. Now, we are adding a new contribution, which contains U. Therefore, it is natural to expect that the boundary terms (4.22) gets modified. This peculiarity has not been noticed before. Anyway, we will not be able to give the full superfield expression for the modified boundary term. However, by inspection of the full bosonic lagrangian (4.43) we can see which is the bosonic component of the term amending the boundary contribution. Indeed, we observe that the first term in the first line and the last term in the second line of (4.43) have the same structure, apart from the overall factor: therefore, it is convenient

[6] Indeed, given two arbitrary functions $f(x)$ and $g(x)$, we have:

$$(\Box f)g - f\,\Box g = \partial_m\big[(\partial^m f)g\big] - \partial_m f\,\partial^m g - \partial_m\big(f\,\partial^m g\big) + \partial_m f\,\partial^m g = \partial_m\big[(\partial^m f)g - f\,\partial^m g\big]\,.$$

to repackage these two terms in a single one by formally defining a new "Kähler potential" $\widetilde{K}(s, \bar{s}, u)$ such that

$$\widetilde{K}_{s\bar{s}} \equiv K_{s\bar{s}} - \frac{1}{\delta}\widehat{G}_{s\bar{s}}\, u^2\,.$$

On the other hand, we observe that the term $K_{s\bar{s}}(\partial_m C^m)^2$ wich appears in the first term of the first line of (4.43), shows up in the boundary term, too. Hence, the bosonic component of new contribution which modifies the boundary term must involve $\widetilde{K}$ rather than K, and thus we have to substitute $K_{s\bar{s}}$ with $\widetilde{K}_{s\bar{s}}$ in the boundary term. Let us be more precise now. Consider the variation of the kinetic term $I_k = \int \mathrm{d}^4x\ \widetilde{K}_{s\bar{s}}(\partial_m C^m)^2$ with respect to δC_m:

$$\delta I_k = \int \mathrm{d}^4x \left\{ -\big[\partial_m\big(2\widetilde{K}_{s\bar{s}}\,\partial_n C^n\big)\big]\,\delta C^m + \partial_m\big[2\widetilde{K}_{s\bar{s}}(\partial_n C^n)\,\delta C^m\big]\right\}. \tag{4.44}$$

The second term is a total derivative, and it vanishes imposing that the variation of C_m vanishes on the boundary, i.e. $\delta C_m\big|_{\mathrm{bd}}= 0$. However, this procedure is not well-defined, because C_m is a gauge field *defined up to a total derivative.* Instead, one should impose that the variation of the field strength vanishes on the boundary, namely $\delta(\partial_m C^m)\big|_{\mathrm{bd}}= 0$. This is achieved by adding the total derivative term $-2\partial_m\big[\widetilde{K}_{s\bar{s}}\,C^m(\partial_n C^n)\big]$, whose variation cancels the second term in (4.44) and modifies the boundary term in the lagrangian.

All in all, after some other manipulations, we end up with

$$\begin{aligned}\mathcal{L} = {}& \widetilde{K}_{s\bar{s}}\big(\partial_m s\,\partial^m \bar{s} + F\bar{F}\big) + \big(W_s F + \text{h.c.}\big) - 2\partial_m\big[C^m\big(\widetilde{K}_{s\bar{s}}\,\partial_n C^n - \mathbf{Im}W_s\big)\big] \\ & - \frac{1}{4\pi}\big(|ks| + kC_3\big)\delta(x^3) + \frac{1}{2\delta}\widehat{G}\partial_m u\,\partial^m u - \frac{2}{3\delta}\widehat{G}\,uA \\ & - \frac{2}{\delta}\widehat{G}\big(s\bar{s} + C_m C^m\big) + \frac{2}{\delta}u\,C_m \mathsf{q}^m\,.\end{aligned}$$

4.5.1 What Happens to the Degeneracy of the SYM Vacua?

The new lagrangian turns out to be very complicated. As a first thing, we want to see if the several new terms modify the vacuum structure: in particular, we are interested in understanding if the new contributions spoil the N-fold degeneracy of the vacua, because, if this were the case, domain wall solutions would not arise. As a first thing, we want to solve the equation of motion for the field A only; this is still an easy task because the equation of motion for A is an algebraic equation. On the contrary, the equation of motion for C_m is more complicated, in that it contains also the divergence $\partial_m C^m$. Then, by plugging the explicit expression for A back in the lagrangian (4.43), we want to see how the fiel C_m and its field-strength enter the scalar potential. Finally, by choosing the vev of any field but that of s and $\bar{s}$ to be zero, we want to check if we can reproduce the value of the gaugino condensate that we have obtained in the VY theory.

The equation of motion for A turns out to be

$$\frac{\partial\mathcal{L}}{\partial A} = 0 \qquad \Longrightarrow \qquad \widetilde{K}_{s\bar{s}}A + \mathbf{Re}W_s - \frac{1}{3\delta}\widehat{G}u = 0\,,$$

and the solution is easily obtained:

$$A = -\frac{\mathbf{Re}W_s}{\widetilde{K}_{s\bar{s}}}\Big(1 - \tfrac{1}{3\delta}\frac{\widehat{G}\,u}{\mathbf{Re}W_s}\Big)\,.$$

Then, we have

$$\widetilde{K}_{s\bar{s}}A^2 = \frac{(\mathbf{Re}W_s)^2}{\widetilde{K}_{s\bar{s}}}\Big(1 - \tfrac{1}{3\delta}\frac{\widehat{G}\,u}{\mathbf{Re}W_s}\Big)^2\,, \tag{4.45a}$$

$$2(\mathbf{Re}W_s)A = -2\frac{(\mathbf{Re}W_s)^2}{\widetilde{K}_{s\bar{s}}}\Big(1 - \tfrac{1}{3\delta}\frac{\widehat{G}\,u}{\mathbf{Re}W_s}\Big)\,, \tag{4.45b}$$

$$-\tfrac{2}{3\delta}\widehat{G}\,uA = \tfrac{2}{3\delta}\frac{\mathbf{Re}W_s}{\widetilde{K}_{s\bar{s}}}\widehat{G}\,u\Big(1 - \tfrac{1}{3\delta}\frac{\widehat{G}\,u}{\mathbf{Re}W_s}\Big)\,, \tag{4.45c}$$

and putting these terms together we arrive at

$$(4.45a) + (4.45b) + (4.45c) = -\frac{1}{\widetilde{K}_{s\bar{s}}}\Big(\mathbf{Re}W_s - \tfrac{1}{3\delta}\widehat{G}\,u\Big)^2\,.$$

Actually, it is also useful to consider the equation of motion for C_m

$$\partial_m\big(\widetilde{K}_{s\bar{s}}\,\partial_n C^n - \mathbf{Im}W_s\big) = -\tfrac{2}{\delta}\widehat{G}\,C_m + \tfrac{1}{\delta}u\mathsf{q}_m - \frac{k}{8\pi}\delta_{3m}\delta(x^3)\,.$$

Even if we cannot find an explicit solution of this equation, it proves useful to write the on-shell lagrangian (and the potential, too) in a cleaner way. Ultimately, the lagrangian (4.43) reads

$$\begin{aligned}\mathcal{L} =&\widetilde{K}_{s\bar{s}}\big[\partial_m s\,\partial^m\bar{s} - (\partial_m C^m)^2\big] + \tfrac{1}{2\delta}\widehat{G}\,\partial_m u\,\partial^m u - \tfrac{1}{4\pi}|ks|\delta(x^3)\\ &-\tfrac{2}{\delta}\widehat{G}\big(s\bar{s} - C_m C^m\big) - \frac{1}{\widetilde{K}_{s\bar{s}}}\Big(\mathbf{Re}W_s - \tfrac{1}{3\delta}\widehat{G}\,u\Big)^2\,.\end{aligned}$$

At this point, – ignoring the presence of the membrane for the time being – we can find the "potential"[7]

$$V = \widetilde{K}_{s\bar{s}}(\partial_m C^m)^2 + \tfrac{2}{\delta}\widehat{G}\big(s\bar{s} - C_m C^m\big) + \frac{1}{\widetilde{K}_{s\bar{s}}}\Big(\mathbf{Re}W_s - \tfrac{1}{3\delta}\widehat{G}\,u\Big)^2\,.$$

We are finally ready to consider $\langle V\rangle$, which is meant to be the above potential with all the arguments evalued at their vevs. We put all vevs but $\langle s\rangle$ and $\langle\bar{s}\rangle$ to zero, which implies $\widetilde{K}_{s\bar{s}} \to K_{s\bar{s}}$, and (writing s and $\bar{s}$ instead of $\langle s\rangle$ and $\langle\bar{s}\rangle$ for sake of simplicity) we find

$$\begin{aligned}\langle V\rangle &= \frac{(\mathbf{Re}W_s)^2}{K_{s\bar{s}}} + \tfrac{2}{\delta}\widehat{G}\,s\bar{s}\\ &= \frac{9\rho N^2}{16\pi^2}|s|^{\frac{4}{3}}\Big(\log^2\frac{|s|}{\Lambda^3} + \frac{32\,\pi^2}{9\rho N^2\,\delta}\Big)\,.\end{aligned}$$

[7]This is not a scalar potential in the strictest sense as we have defined it in section 2.3.5: indeed, there should not be derivatives, and we should have scalar fields only.

There is a single value of $|s|$ which produces $\langle V\rangle = 0$, that is

$$\langle s\rangle = 0\,, \quad \langle \bar{s}\rangle = 0\,.$$

Thus, we observe that the introduction of the new term (4.41):

- spoils the N-fold degeneracy of the SYM vacua;
- allows for only one supersymmetric minimum in which the gaugino condensate does not form.

However, we can see that it is possible to restore the degeneracy of the SYM vacua. Following [18], we modify the relation (4.16) between $\mathcal{S}$ and U as

$$\mathcal{S} - \langle\mathcal{S}\rangle = -\tfrac{1}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}U\,, \tag{4.46}$$

and similarly for the conjugate. This implies that in the expression (4.42) the composite fields s and $\bar{s}$ are to be substituted by $\boldsymbol{\Delta} \equiv s - \langle s\rangle$ and $\overline{\boldsymbol{\Delta}} \equiv \bar{s} - \langle\bar{s}\rangle$, respectively, and this in turn means that the lagrangian (4.43) becomes now

$$\begin{aligned}\mathcal{L} = {}& \widetilde{K}_{s\bar{s}}\big(\partial_m s\,\partial^m\bar{s} + F\bar{F}\big) + \big(W_s F + \text{h.c.}\big) - 2\partial_m\big[C^m\big(\widetilde{K}_{s\bar{s}}\partial_n C^n - \mathbf{Im}W_s\big)\big] \\ & - \tfrac{1}{4\pi}\big(|ks| + kC_3\big)\delta(x^3) + \tfrac{1}{2\delta}\widehat{G}\,\partial_m u\,\partial^m u + \tfrac{2}{\delta}u\,C_m\mathsf{q}^m \\ & + \tfrac{4}{3\delta}\widehat{G}\,u\Big[A\,\mathbf{Re}(\omega) - (\partial_m C^m)\,\mathbf{Im}(\omega)\Big] - \tfrac{2}{\delta}\widehat{G}\Big[\boldsymbol{\Delta}\overline{\boldsymbol{\Delta}} + C_m C^m + uA\Big]\,,\end{aligned} \tag{4.47}$$

where we have used the relation

$$-\tfrac{2}{\delta}u\big(\widehat{G}_s\,\boldsymbol{\Delta}\,F + \text{h.c.}\big) = \tfrac{2}{3\delta}\widehat{G}\,u\Big(\tfrac{\boldsymbol{\Delta}}{s}F + \text{h.c.}\Big) = \tfrac{4}{3\delta}\widehat{G}\,u\Big[A\,\mathbf{Re}(\omega) - (\partial_m C^m)\,\mathbf{Im}(\omega)\Big]\,,$$

with $\omega \equiv \boldsymbol{\Delta}/s$. Then, solving the equation of motion for A we obtain

$$A = -\frac{\mathbf{Re}W_s}{\widetilde{K}_{s\bar{s}}}\Big(1 - \frac{1}{3\delta}\frac{\widehat{G}'\,u}{\mathbf{Re}W_s}\Big)\,,$$

where $\widehat{G}' \equiv \widehat{G}\big(3 - 2\mathbf{Re}(\omega)\big)$. The equation of motion for C_m is instead

$$\partial_m\Big[\widetilde{K}_{s\bar{s}}\,\partial_n C^n - \mathbf{Im}W_s - \tfrac{2}{3\delta}\widehat{G}\,u\,\mathbf{Im}(\omega)\Big] = -\tfrac{2}{\delta}\widehat{G}\,C_m + \tfrac{1}{\delta}u\mathsf{q}_m - \frac{k}{8\pi}\delta_{3m}\delta(x^3)\,.$$

Luckily enough, there is little difference with repsect to what we have seen before. Therefore, proceeding in the same way as we did previously[8], we arrive at the potential

$$\langle V\rangle = \frac{(\mathbf{Re}W_s)^2}{K_{s\bar{s}}} + \tfrac{2}{\delta}\widehat{G}'\,\boldsymbol{\Delta}\overline{\boldsymbol{\Delta}}\,.$$

[8]We can notice that the first and the last term in the last line of (4.47) can be easily recombined to give $-\frac{2}{3\delta}\widehat{G}'\,uA$.

However, we have to remember that s is actually $\langle s\rangle$ (similarly for the conjugate) hence the second contribution above vanishes (since $\langle\boldsymbol{\Delta}\rangle = \langle\overline{\boldsymbol{\Delta}}\rangle = 0$) and we are left with only

$$\langle V\rangle = \frac{(\mathbf{Re}W_s)^2}{K_{s\bar{s}}} = \tfrac{9\rho N^2}{16\pi^2}|s|^{\frac{4}{3}}\log^2\tfrac{|s|}{\Lambda^3}\,,$$

which in fact can reproduce the value of the gaugino condensates

$$\langle s\rangle = \Lambda^3\, e^{2\pi \mathrm{i}\frac{\mathrm{n}}{N}} \quad \text{and} \quad \langle\bar{s}\rangle = \Lambda^3\, e^{-2\pi \mathrm{i}\frac{\mathrm{n}}{N}}\,.$$

This conclusion need to be treated with care. In fact, we have obtained the structure of vacua described by the original VY theory only thanks to the shift of the superfield $\mathcal{S}$ in (4.46), which is something that we have imposed by hand. In other words, the conclusion seems artificial, because the N-fold degeneracy of the vacua does not emerge naturally from the theory, as it does, conversely, in the VY theory. However, we have to keep in mind that we are dealing with a low-energy effective field theory, and making aprioristic assumptions with the aim to reproduce some characteristic of the underlying high-energy "parent" theory is not actually problematic.

Anyway, further studies are necessary to clarify the ambiguous situation described here.

Conclusion and Outlook

In this thesis we have reviewed some fundamental features of classical and quantum field theories, developing general tools that we have used to analyse, in particular, the classical and quantum structure of $\mathcal{N}=1$, $SU(N)$ super Yang-Mills (SYM) theory in $d=4$. We have used the first three chapters to set the stage, recalling the fundamental aspects of supersymmetry, supersymmetric field theories and soliton theory.

Then, in the fourth chapter, we have studied the symmetries of the $\mathcal{N}=1$, $d=4$, $SU(N)$ SYM theory, and how they are broken by anomalies: in particular, we have focused on the anomalous $U(1)$ R-symmetry, providing also an explicit computation of the anomaly function; despite the result is well known, we have not been able to find in the literature any explicit derivation thereof, and thus we have decided to include it here. We have then reviewed the construction of the renown Veneziano-Yankielowicz (VY) effective lagrangian, which is the low-energy approximation of the SYM lagrangian. We have shown that this lagrangian reproduces the value of the gaugino condensate, and it estabilishes that the SYM vacuum is comprised of N distinct vacua, which differ one from the other by the value of the gaugino condensate. Actually this is true only after an appropriate modification of the lagrangian. This consists in the introduction of a boundary term and a new independent superfield U; this adjustment is necessary to make the VY lagragian single valued and to integrate-out the auxiliary fields consistently. Then, we have seen that the degenerate vacua are connected to each other by means of domain walls, and we have shown that the BPS tension is saturated only if we include the presence of a supermembrane in the effective theory.

Finally, in the last section of chapter four we have studied the extension of the VY lagrangian with the term $\mathcal{L}' = -\frac{1}{\delta}\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \frac{U^2}{(\mathcal{S}\bar{\mathcal{S}})^{1/3}}$, which was first considered in [18]. Thanks to this new term, the CP-odd glueball C_m acquires one degrees of freedom and becomes a propagating field: in fact, in the VY theory C_m disappears because it is integrated out; however, on physical grounds, there is no reason to think that the CP-odd glueball field should be absent in the low-energy theory. Moreover, $\mathcal{L}'$ introduces also a new propagating massive degree of freedom, which is dual to a CP-even glueball field $F_{mn}F^{mn}$. This modification brought us to two *original observations*:

i. the new term $\mathcal{L}'$ includes the superfield U, which whose consistent variation requires the presence of the boundary term as mentioned earlier. Therefore, even the boundary term has to be modified in an appropriate way. The superfield expression of the modified boundary term has not been computed here, but, at least, we have found which should be the bosonic component. The construction of the full modified boundary term in superfield formalism could be one of the future developments of this thesis;

ii. the new term spoils the N-fold degenerate vacuum structure of the theory. In particular, we are left with only one supersymmetric vacuum, obtained when the vev of all fields is null. However, the degenerate structure of the vacua is recovered if we assume that the relation between $\mathcal{S}$ and U is $\mathcal{S} - \langle\mathcal{S}\rangle = -\frac{1}{4}\bar{\mathcal{D}}^2 U$, rather than $\mathcal{S} = -\frac{1}{4}\bar{\mathcal{D}}^2 U$. On one hand, this seems artificial, because the degeneracy of the vacua does not pop out spontaneously as it does in VY theory. On the other hand, however, we are working in an effective field-theoretical framework, where aprioristic assumptions are made in order to make the effective theory more adherent to the underlying "parent" theory. The situation is ambiguous, and needs to be clarified in further studies.

Together with the construction of the modified boundary term in superfied formalism and the clarification of the issue that we have mentioned just above, the work presented in this thesis can be developed in another way. Indeed, rather than pure SYM theories, one could consider $\mathcal{N} = 1$, $d = 4$ super QCD theories containing matter in the fundamental representation of $SU(N)$ within generalized Wess-Zumino models such as that proposed by Taylor, Veneziano and Yankielowicz in [38]. On one hand, the presence of matter ensures the existence of *canonical* BPS domain walls – i.e. domain walls which saturate the BPS bound without the need of other dynamical objects as e.g. membranes. Explicit solutions for these walls and their features are known (see e.g. [8]). On the other hand, instead, *non-canonical* walls should also form; it could then be meaningful to understand if this is actually so, and if it were the case, under which conditions these walls form and which dynamical object sources the configuration.

Appendix **A**

Lorentz and Poincaré Group: Some Reminders

In this appendix some basic notions of Lorentz and Poincaré group as well as of their representations are recalled.

We will use dotted and undotted greek letters of the beginning of the alphabet for spinorial indices. Moreover, (lower case) latin letters of the middle of the alphabet are denote four-vector indices, while those at the beginning of the alphabet are reserved for gauge indices. The metric of the Minkowski space $\mathcal{M}_{1,3}$ is chosen to have mostly minus signature, namely

$$\eta_{mn} = \mathrm{diag}(+,-,-,-)\,.$$

The conventions on the spinorial notation are those of [10]. They will be pointed out in what follows.

A.1 Lorentz and Poincaré Group

The Lorentz group is the set of linear transformations $x^m \mapsto x'^m = \Lambda^m{}_n x^n$ leaving the quadratic form $x^2 = \eta_{mn}x^m x^n$ invariant. The matrices Λ have to satisfy the condition

$$\Lambda^{\mathrm{T}}\eta\Lambda = \eta \;\leftrightarrow\; \Lambda^m{}_p\Lambda^n{}_q\eta_{mn} = \eta_{pq}\,. \tag{A.1}$$

This defines the constraints

$$\det\Lambda = \pm 1\,, \quad |\Lambda^0{}_0| \geq 1\,,$$

which divide the parameters space in four disconnected pieces:

- $\det\Lambda = +1$ and $\det\Lambda = -1$ are the proper and improper Lorentz transformations respectively, and only the former subset enjoys a subgroup structure;
- $\Lambda^0{}_0 \geq +1$ and $\Lambda^0{}_0 \leq -1$ are the orthochronous and non-orthocronous Lorentz transformations respectively, and only the first constitute a subgroup.

Lorentz group is a non-compact Lie group, denoted by

$$O(1,3) = \left\{\Lambda \in GL(4,\mathbb{R}) \,|\, \Lambda^{\mathrm{T}}\eta\Lambda = \eta\right\} .$$

However, we are interested in its subgroup

$$SO^{+}(1,3) = \left\{\Lambda \in O(1,3) \,|\, \det\Lambda = +1,\, \Lambda^{0}{}_{0} \geq +1\right\} ,$$

called special Lorentz group. An in important relation that is worth to keep in mind is the *homomorphism* between $SO^{+}(1,3)$ and $SL(2,\mathbb{C}) = \{M \in GL(2,\mathbb{C}) \,|\, \det M = +1\}$. In particular, for any $A, B \in SL(2,\mathbb{C})$ there exists an associated Lorentz matrix Λ so that

$$\Lambda(A)\Lambda(B) = \Lambda(AB) .$$

Proof. Let us start by introducing a set of four matrices $\sigma_m = (\mathbb{1}, \sigma_k)$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.2}$$

are the usual Pauli matrices. We can also introduce $\sigma^m = (\mathbb{1}, -\sigma_k)$. The matrices σ_m are a complete set, which is to say that any 2×2 matrix can be written as a linear combination thereof. Given a four-vector x^m we can build a map from Minkowski space $\mathcal{M}_4$ to the set of 2×2 hermitian complex matrices $\mathbb{H}_2$ as

$$\rho \,:\, x^m \mapsto x^m\sigma_m \equiv X .$$

in fact, the matrix X is hermitian thanks to the hermiticity of the Pauli matrices. Now we consider $A \in SL(2,\mathbb{C})$, and we act on it with X in such a way that

$$A \,:\, X \mapsto AXA^{\dagger} \equiv X' .$$

The new matrix X' is still hermitian, hence we have realized a mapping $\mathbb{H}_2 \to \mathbb{H}_2$. The final step consists in the application of the inverse map ρ^{-1} to X' to get back a four-vector x'^m. The inverse map is defined as

$$\rho^{-1} = \frac{1}{2}\mathrm{tr}\left[\bullet\,\bar{\sigma}^m\right] ,$$

for $\bar{\sigma}^m = (\mathbb{1}, \sigma_k)$. Indeed, one has

$$\rho^{-1}(X) = \frac{1}{2}\mathrm{tr}\left[X\bar{\sigma}^m\right] = \frac{1}{2}\mathrm{tr}\left[\sigma^n\bar{\sigma}^m\right]x_n = \eta^{mn}x_n = x^m .$$

We have thus realized a map from Minkowski space into itself

$$x^m \xrightarrow{\rho^{-1}\circ A\circ\rho} \frac{1}{2}\mathrm{tr}\left[A\sigma_n A^{\dagger}\bar{\sigma}^m\right]x^m = x'^n ,$$

which is nothing else but a Lorentz transformation obtained via $A \in SL(2,\mathbb{C})$ as

$$\Lambda^m{}_n(A) = \frac{1}{2}\mathrm{tr}\left[\bar{\sigma}^m A\sigma_n A^{\dagger}\right] . \tag{A.3}$$

Moreover, one can notice that the map is two-to-one, since both A and $-A$ give the same Lorentz transformation. On the other hand, an isomorphism holds between the Lorentz group and $SL(2,\mathbb{C})/\mathbb{Z}_2$. □

Given that $SO^+(1,3)$ and $SL(2,\mathbb{C})$ are homomorphic, they are also related as topological spaces. Their topology can be determined by studying them as spaces of matrices. In particular, we can get the topology of the Lorentz group by that of $SL(2,\mathbb{C})$ and identifying matrices with opposite sign. First, we can notice that any complex matrix can be written as the product of a unitary matrix, say U, and the exponential of a hermitian one, say H; thus

$$A = U\, e^{H} \, .$$

For $A \in SL(2,\mathbb{C})$ in particular, one has

$$\det A = 1 = (\det U)\, e^{\mathrm{tr} H} \qquad \Longrightarrow \qquad \det U = 1\, , \;\; \mathrm{tr} H = 0\, .$$

It follows that any $A \in SL(2,\mathbb{C})$ can be written as the product of a matrix $U \in SU(2) \simeq S^3$, and the exponential of a traceless hermitian 2×2 matrix H, which is parametrized by three real numbers. One can conclude that, topologically,

$$SL(2,\mathbb{C}) \simeq \mathbb{R}^3 \times S^3 \qquad \Longrightarrow \qquad SO(1,3) \simeq \mathbb{R}^3 \times S^3/\mathbb{Z}_2\, .$$

This shows that the Lorentz group is indeed non-compact.

Lorentz Algebra and its Representations

As it is often the case, it is more convenient to determine the representations of the algebra rather than those of the group directly. By linearising the relation (A.1) around the identity, one gets that the Lorentz algebra is Lorentz algebra is

$$\mathfrak{so}(1,3) = \left\{ \omega \in M(4,\mathbb{R}) \,|\, \eta\omega = -(\eta\omega)^{\mathrm{t}} \right\} \, .$$

This implies that $\dim(\mathfrak{so}(1,3)) = 6$, because this is the number of independent components in a 4×4 antisymmetric matrix. The six generators are $\mathbf{J} = (J_1, J_2, J_3)$ and $\mathbf{K} = (K_1, K_2, K_3)$, and they satisfy the following commutation relations:

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k\, , \quad [J_i, K_j] = \mathrm{i}\epsilon_{ijk} K_k\, , \quad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk} J_k\, . \tag{A.4}$$

While the J_j are hermitian, the K_i are anti-hermitian, this being due to the non compactness of the group. Moreover, the first commutator in (A.4) shows that the J_i are the generators of the rotation group $SO(3) < SO^+(1,3)$, while the second relation means that the boosts K_i are spatial vectors. Nevertheless, to build non-unitary finite dimensional irreducible representations of this algebra, it is more useful to introduce the complex linear combinations of $\mathbf{J}$ and $\mathbf{K}$

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{J} + \mathrm{i}\mathbf{K}\right)\, , \quad \mathbf{T} = \frac{1}{2}\left(\mathbf{J} - \mathrm{i}\mathbf{K}\right)\, ,$$

which are both hermitian. These combinations satisfy the commutation relations

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k\, , \quad [T_i, T_j] = \mathrm{i}\epsilon_{ijk} T_k\, , \quad [S_i, T_j] = 0 \tag{A.5}$$

which are the commutators for two $SU(2)$ algebras. This means that the complexified Lorentz algebra splits into two commuting $\mathfrak{su}(2)$. However, in order that all rotations and boost parameters be real, one takes all J_i and K_i to be imaginary, and this means that

$$(S_i)^* = -T_i\,, \quad (T_i)^* = -S_i\,.$$

In terms of algebras, this can be summarised by writing

$$\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{su}(2) \oplus \mathfrak{su}^*(2)\,. \tag{A.6}$$

This identification implies that we can classify irreducible representations of the Lorentz algebra in term of those of the special unitary group algebra, since there is a one-to-one correspondence between the irreducibe representations of a complex Lie algebra and those of any of its real forms. We have a Casimir for each of the $\mathfrak{su}(2)$ component, which is

$$\sum_{i=1}^{3} S_i S_i\,, \quad \sum_{i=1}^{3} T_i T_i\,,$$

respectively, and they have eigenvalues $s(s+1)$ and $t(t+1)$, with $s, t \in \mathbb{N}/2$. All in all, each representation of $\mathfrak{so}(1,3)$ is labeled by the pair (s,t), has dimension $(2s+1)(2t+1)$, and since $J_3 = S_3 + T_3$, the spin of the representation is given by $j = s + t$.

It is useful to gather all Lorentz generators in an antisymmetric tensor M_{mn} with

$$M_{0i} = K_i\,, \quad M_{ij} = \epsilon_{ijk} J_k\,,$$

so that the commutators in (A.4) become

$$[M_{mn}, M_{pq}] = \mathrm{i}\eta_{mq} M_{np} + \mathrm{i}\eta_{np} M_{mq} - \mathrm{i}\eta_{mp} M_{nq} - \mathrm{i}\eta_{nq} M_{mp}\,.$$

Poincarè Group, its Algebra and its Representations

According to Einstein's principle of special relativistic covariance, the most general transformations leaving all relativistic observables invariant are Lorentz transformations and spacetime translations. These symmetries form a group, which is named after the mathematician Henri Poincaré. In group theoretical language, the proper Poincaré group is given by

$$ISO(1,3) = \mathbb{R}^{1,3} \rtimes SO^+(1,3)\,,$$

and its algebra reads

$$\begin{aligned}
[P_m, P_n] &= 0\,,\\
[M_{mn}, P_p] &= \mathrm{i}\eta_{np} P_m - \mathrm{i}\eta_{mp} P_n\,,\\
[M_{mn}, M_{pq}] &= \mathrm{i}\eta_{mq} M_{np} + \mathrm{i}\eta_{np} M_{mq} - \mathrm{i}\eta_{mp} M_{nq} - \mathrm{i}\eta_{nq} M_{mp}\,,
\end{aligned}$$

where P_m is the generator of spacetime translations, that is the four-momentum.

Finite dimensional non-unitary irreducible representations of the Poincaré group are organised according to the classification of those of $\mathfrak{so}(1,3)$ as we have seen before. Without entering into the detail, we will quote here the main results [30, 41]. There are two quadratic Casimir's operators, namely:

- one is $P_m P^m$;
- the other one is provided by the square of the Pauli-Lubanski vector $W_m = \frac{1}{2}\epsilon_{mnpq}P^n M^{pq}$, i.e. $W_m W^m$.

According to the first of the two Casimirs, we can identify two kind of irreducible representations.

Massive representations. They are characterised by $P_m P^m = \mathsf{m}^2 > 0$, where m is the mass of the representation. Given that $W_m W^n$ is a scalar, it can be computed in any frame. In particular, it is convenient to consider the rest frame of the particle, namely $P_m = (\mathsf{m}, \mathbf{0})$, so that

$$W^m = \frac{\mathsf{m}}{2}\epsilon^{0mpq}M_{pq} \;\Rightarrow\; W_0 = 0\,,\; W^i = \frac{\mathsf{m}}{2}\epsilon^{0ijk}M_{jk} = \mathsf{m}J^j\,.$$

Hence $W_m W^m = -\mathbf{W}^2 = -\mathsf{m}^2\mathbf{J}^2$, which means that its eigenvalues are given by $-\mathsf{m}^2 j(j+1)$, where j is the spin of the representation. On the other hand, one can notice that the previous choice of the four-momentum still leaves the freedom to perform spatial rotations; in other words, the space of one particle states with momentum given above is a basis of the representation of spatial rotations. The group of transformations which leave invariant a given choice of P^m is called little group; since we want to include spinor representations, the little group is $SU(2)$, so that $j \in \mathbb{N}/2$. All in all, this means that each massive representation is distinguished by its mass m and its spin j, and the states within are labeled by $j_z = j, j-1, \dots, -j$. This in turn implies that massive particles fall into $(2j+1)$-dimensional multiplets.

Massless representations. They are characterised by $P_m P^m = 0$. In this case there is no rest frame, but we can still perform a Lorentz transformation to the frame where e.g. $P_m = (\omega, 0, 0, \omega)$, with ω the energy of the particle. The little group is now[1] $SO(2)$, i.e. the group of rotations in the $z = 0$ plane generated by J_3. This is an abelian group and therefore its irreducible representations are one-dimensional: indeed, states are distinguished by the eigenvalue of J_3, and they coincide with the helicity λ of the particle, for we have chosen $\mathbf{P}$ along the direction 3. Moreover, it can be shown that λ is quantized, which is to say $\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots$. All in all, this shows that massless particles have only one degree of freedom. From the point of view of the representations of the Poincaré group, a massles particle with helicity $+\lambda$ is different from one whose helicity is $-\lambda$; nevertheless, it holds that in this case P_m and W_m are linearly dependent, the costant of proprotionality being the helicity, namely $W_m = \lambda P_m$. On the other hand, $W^0 = \frac{1}{2}\epsilon^{0ijk}P_i M_{jk} = \mathbf{P}\cdot\mathbf{J}$, and finally $\lambda = \mathbf{P}\cdot\mathbf{J}/P^0$, hence given that $\mathbf{P}$ and $\mathbf{J}$ are, respectively, a vector and a pseudo-vector under parity, we understand that the representations (i.e. states) with $+\lambda$ and $-\lambda$ are related by a parity transformation. Electromagnetism and Gravity are parity-invariant interactions, thus it is more natural to

[1] More precisely, the little group is now $ISO(2, \mathbb{R})$. However, the translations are associated with states of continuous spin: they are not present in Nature, hence we neglect them.

define photons and gravitons to be at the same time irreducible representations of both the Poincaré group and parity.

A.2 Spinorial Representations of the Lorentz Group

We can conveniently introduce spinors as those objects carrying the basic representations of $SL(2,\mathbb{C})$. We have two such representations:

- the *fundamental* $D(\mathrm{M}) = \mathrm{M}\ \forall\ \mathrm{M} \in SL(2,\mathbb{C})$. Spinors trasforming in this representation are two components objects
$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \quad \text{such that} \quad \psi_\alpha \mapsto \psi'_\alpha = \mathrm{M}_\alpha{}^\beta \psi_\beta\,,$$
for $\alpha, \beta = 1, 2$, where $\psi_{1,2}$ are complex Grassmann numbers;
- the *complex conjugate* $D(\mathrm{M}) = \mathrm{M}^*\ \forall\ \mathrm{M} \in SL(2,\mathbb{C})$. Spinors in this representation are instead
$$\bar\psi = \begin{pmatrix}\bar\psi_{\dot 1} \\ \bar\psi_{\dot 2}\end{pmatrix} \quad \text{such that} \quad \bar\psi_{\dot\alpha} \mapsto \bar\psi_{\dot\alpha} = \mathrm{M}^*{}_{\dot\alpha}{}^{\dot\beta} \bar\psi_{\dot\beta}\,,$$
with $\bar\psi_{\dot 1,\dot 2}$ complex Grassmann numbers.

For a generix matrix $\mathrm{M} \in SL(2,\mathbb{C})$ it holds that

$$\mathrm{M} = \exp\left\{(u_j + \mathrm{i} v_j)\sigma_j\right\}\,, \quad \mathrm{M}^* = \exp\left\{(u_j - \mathrm{i} v_j)\sigma_j^*\right\}\,.$$

This shows that $SL(2,\mathbb{C})$ matrices can be expressed in terms of the generators of the spin-$\frac{1}{2}$ representation of the complexified $\mathfrak{su}(2)_\mathbb{C}$ in accordance with (A.5). More precisely, M is built through the exponentiation of $\mathbf{S}$, while M^* through exponentiation of $\mathbf{T}$: this means that ψ_α transforms in the representation $\left(\frac{1}{2}, 0\right)$ and $\bar\psi^{\dot\alpha}$ in the $\left(0, \frac{1}{2}\right)$ of the Lorentz group. Stated differently, due to the homomorphism between $SO^+(1,3)$ and $SL(2,\mathbb{C})$ the spinor representations ψ_α and $\bar\psi^{\dot\alpha}$ are also representations of the Lorentz group, and given the isomorphism $\mathfrak{so}(1,3)_\mathbb{C} \simeq \mathfrak{su}(2) \oplus \mathfrak{su}^*(2)$, they can be labeled in terms of $SU(2)$ representations as $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$, respectively.

The representations M and M^* are not equivalent, i.e. there is no similarity matrix which relate them. Let us now introduce the antisymmetric tensors

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\,, \quad \epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\,.$$

These are invariant tensors of $SL(2,\mathbb{C})$. Indeed, if we take for instance $\epsilon_{\alpha\beta}$ we have

$$\epsilon_{\alpha\beta} \mapsto \epsilon'_{\alpha\beta} = \mathrm{M}_\alpha{}^\gamma\, \mathrm{M}_\beta{}^\delta \epsilon_{\gamma\delta} = (\det \mathrm{M})\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}\,,$$

and similarly for the others. The epsilon tensors can be viewed as a metric in spinor space, in the sense that we can use them to build higher spin representations as well as to rise and lower spinor indices with the following conventions:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta\,, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta\,, \quad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}\,, \quad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}\,.$$

These tensors are the similarity matrices by means of which we can build the representation $\mathrm{M}^{-1\mathrm{T}}$ and $\mathrm{M}^{*-1\mathrm{T}}$, which are equivalent to M and M^*, respectively. Indeed, we have

$$(\mathrm{M}^{-1\mathrm{T}})^{\delta}{}_{\gamma} = \epsilon^{\delta\alpha}\mathrm{M}_{\alpha}{}^{\beta}\epsilon_{\beta\gamma} \quad (\mathrm{M}^{*-1\mathrm{T}})^{\dot\delta}{}_{\dot\gamma} = \epsilon^{\dot\delta\dot\alpha}\mathrm{M}^{*}{}_{\dot\alpha}{}^{\dot\beta}\epsilon_{\dot\beta\dot\gamma}\,.$$

From our conventions ψ_α are row vectors and ψ^α are column vectors, while undotted indices follow the opposite convention. Moreover, we have $(\mathrm{M}_{\alpha}{}^{\beta})^* = (\mathrm{M}^{*-1\mathrm{T}})^{\dot\alpha}{}_{\dot\beta}$, hence one can see that $(\psi_\alpha)^* = \bar\psi^{\dot\alpha}$ and $(\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha}$. Since ψ_α transforms with M and ψ^α with $\mathrm{M}^{-1\mathrm{T}}$ we can build invariant quantities contracting an upper index with a lower index − the same *mutatis mutandis* applies for $\bar\psi$. In particular, we choose the convention for the scalar product of spinors to be the so-called "ten to four" and "eight to two" for undotted and dotted indices, respectively. Namely, we contract spinor indices according to what follows:

$$\begin{aligned}
\psi\chi &\equiv \psi^\alpha\chi_\alpha = \epsilon^{\beta\alpha}\psi_\beta\chi_\alpha = -\epsilon^{\beta\alpha}\chi_\alpha\psi_\beta = \epsilon^{\alpha\beta}\chi_\alpha\psi_\beta = \chi^\alpha\psi_\alpha = \chi\psi\,,\\
\bar\psi\bar\chi &\equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \epsilon_{\dot\beta\dot\alpha}\bar\psi^{\dot\beta}\bar\chi^{\dot\alpha} = -\epsilon_{\dot\beta\dot\alpha}\bar\chi^{\dot\beta}\bar\psi^{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta}\bar\psi^{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi\,.
\end{aligned}$$

Finally, recalling that a hermitian matrix X transforms as $\mathrm{M}X\mathrm{M}^\dagger$ under $SU(2)$, and that the index structure of M and $\mathrm{M}^\dagger$ is $\mathrm{M}_\alpha{}^\beta$ and $\mathrm{M}^*{}^{\dot\beta}{}_{\dot\alpha}$, respectively, one can see that σ_m naturally has a dotted and an undotted index, namely $\sigma^m_{\alpha\dot\alpha}$. On the other hand $\bar\sigma^m = (\bar\sigma^m)^{\dot\alpha\alpha}$ and the two sets are related by

$$(\bar\sigma^m)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^m_{\beta\dot\beta} = -\epsilon^{\alpha\beta}\sigma^m_{\beta\dot\beta}\epsilon^{\dot\beta\dot\alpha}\,.$$

Moreover, we have the following properties:

$$\begin{aligned}
\mathrm{tr}[\sigma^m\bar\sigma^n] &= 2\eta^{mn}\,,\\
\sigma^m\bar\sigma^n + \sigma^n\bar\sigma^m &= 2\eta^{mn}\mathbb{1}\,, \quad \bar\sigma^m\sigma^n + \bar\sigma^n\sigma^m = 2\eta^{mn}\mathbb{1}\\
(\sigma^m)^{\alpha\dot\alpha}(\bar\sigma_m)_{\dot\beta\beta} &= 2\delta^\alpha_\beta\delta^{\dot\alpha}_{\dot\beta}\,, \quad (\sigma^m)^{\alpha\dot\alpha}(\bar\sigma_m)^{\dot\beta\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\,.
\end{aligned}$$

The four-components spinor notations. It may be useful to keep in mind the relation with Dirac spinors. In the Weyl representation the gamma matrices are

$$\gamma^m = \begin{pmatrix} \mathbb{0} & \sigma^m \\ \bar\sigma^m & \mathbb{0}\end{pmatrix}\,, \quad \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1}\end{pmatrix}\,,$$

and a Dirac spinor is defined as

$$\Psi = \begin{pmatrix}\psi_\alpha \\ \bar\chi^{\dot\alpha}\end{pmatrix} \sim \left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\,,$$

hence, we can see that

$$\gamma^5\Psi = \begin{pmatrix}\psi_\alpha \\ 0\end{pmatrix} \quad \gamma^5\Psi = -\begin{pmatrix}0 \\ \bar\chi^{\dot\alpha}\end{pmatrix}\,,$$

which means that they are left-handed and right-handed Weyl spinors, respectively. Moreover, Lorentz generators are

$$\Sigma^{mn} = \frac{i}{2}\gamma^{mn} = \frac{\mathrm{i}}{4}\begin{pmatrix} \sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m & \\ & \bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m \end{pmatrix} ,$$

with the two-indices Pauli matrices being defined as

$$(\sigma^{mn})_\alpha{}^\beta \equiv \frac{1}{4}\left[\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m\right]_\alpha{}^\beta , \tag{A.8a}$$

$$(\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{1}{4}\left[\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m\right]^{\dot{\alpha}}{}_{\dot{\beta}} , \tag{A.8b}$$

which act respectively as Lorentz generators on ψ_α and $\bar{\psi}^{\dot{\alpha}}$.

Appendix B

Differential Forms and Vielbeins: Working Definitions

In this appendix some basic features of differential forms and the vielbein formalism are recalled [11, 17]. These notions will be also generalised to flat superspace [13, 42]. Mathematical rigour is put aside in favour of collecting practical tools for manipulations.

B.1 Differential Forms

Differential forms are totally antisymetric covariant tensor fields. Introducing the symbol $\mathrm{d}x^m$, one can define the antisymmetric tensor product, dubbed *wedge product*, as

$$\mathrm{d}x^n \wedge \mathrm{d}x^m = \mathrm{d}x^n \otimes \mathrm{d}x^m - \mathrm{d}x^m \otimes \mathrm{d}x^n \,.$$

It is clear that $\mathrm{d}x^m \wedge \mathrm{d}x^n = -\mathrm{d}x^n \wedge \mathrm{d}x^m$ and $\mathrm{d}x^m \wedge \mathrm{d}x^m = 0$. In an analogous way one can define higher order wedge products, which are totally antisymmetric tensors. A p-form is an object defined as

$$\xi^{(p)} = \frac{1}{p!}\,\mathrm{d}x^{m_p} \wedge \cdots \wedge \mathrm{d}x^{m_1}\,\xi_{m_1\ldots m_p}\,,$$

whose coefficients $\xi_{m_1\ldots m_p}$ are totally antisymmetric tensors. Antisymmetry implies that, for a d-dimensional spacetime, the maximum degree a differential form can have is $d = p$. Given a p-form and a q-form we can build a $(p+q)$-form by means of the wedge product as

$$\begin{aligned}\xi^{(p)} \wedge \zeta^{(q)} &= \frac{1}{p!q!}\,\mathrm{d}x^{m_p} \wedge \cdots \wedge \mathrm{d}x^{m_1} \wedge \mathrm{d}x^{n_q} \wedge \cdots \wedge \mathrm{d}x^{n_1}\,\zeta_{n_1\ldots n_q}\xi_{m_1\ldots m_p} \\ &= \frac{1}{p!q!}\,\mathrm{d}x^{m_p} \wedge \cdots \wedge \mathrm{d}x^{n_1}\,\zeta_{[n_1\ldots n_q}\xi_{m_1\ldots m_p]}\,.\end{aligned}$$

With our conventions one has

$$\xi^{(p)} \wedge \zeta^{(q)} = (-1)^{pq}\zeta^{(q)} \wedge \xi^{(p)}\,,$$

i.e. odd forms anticommute. The advantage of dealing with differential forms is that they are scalars under coordinate transformations, for $\xi_{m_1\dots m_p}$ transforms as a covariant tensor and $\mathrm{d}x^{m_p}\wedge\dots\wedge\mathrm{d}x^{m_1}$ transforms as a contravariant tensor. There is a number of operations on differential forms one can define. The first one we are interested in is the *exterior derivative*, which acts on a p-form as

$$\mathrm{d}\xi^{(p)} = \frac{1}{p!}\,\mathrm{d}x^{m_p}\wedge\dots\wedge\mathrm{d}x^{m_1}\wedge\mathrm{d}x^{m}\,\partial_m\xi_{m_1\dots m_p}\,.$$

One can notice that, with our choice of conventions, we have

$$\mathrm{d}\left(\xi^{(p)}\wedge\zeta^{(q)}\right) = \xi^{(p)}\wedge\mathrm{d}\zeta^{(q)} + (-)^q\mathrm{d}\xi^{(p)}\wedge\zeta^{(q)}\,.$$

We see that the exterior derivative maps a p-form into a $(p+1)$-form. Moreover, it is nilpotent, namely it satisfies the property d^2: this is due to the fact that the derivatives $\partial_m\partial_n$ are symmetric and the wedge product is antisymmetric. A p-form which vanishes upon acting with the exterior derivative, $\mathrm{d}\xi^{(p)}=0$, is termed *closed*. If, instead, a globally well defined $(p-1)$-form $\zeta^{(p-1)}$ exists, such that $\xi^{(p)}=\mathrm{d}\zeta^{(p-1)}$, then $\xi^{(p)}$ is said *exact*. Interestingly enough, differential forms automatically provide an invariant integration measure. Since $\mathrm{d}x^{m_d}\wedge\dots\wedge\mathrm{d}x^{m_1}$ is antisymmetric in all its indices, on a d-dimensional (sub-)manifold it has to be proportional to the Levi-Civita symbol $\boldsymbol{\varepsilon}^{m_d\dots m_1}$. This is referred to as *symbol* because it is not a true tensor on a curved manifold, but a tensor-density: in such a case the genuine Levi-Civita tensor is $\epsilon^{m_1\dots m_d}$, for which we have

$$\boldsymbol{\varepsilon}^{m_d\dots m_1} = \sqrt{-g}\,\epsilon^{m_d\dots m_1}\,,\qquad \boldsymbol{\varepsilon}_{m_d\dots m_1} = \frac{1}{\sqrt{-g}}\epsilon_{m_d\dots m_1}\,.$$

We have therefore $\mathrm{d}x^{m_d}\wedge\dots\wedge\mathrm{d}x^{m_1} = \sqrt{-g}\,\mathrm{d}^dx\,\epsilon^{m_d\dots m_1}$, hence for any d-form

$$\xi^{(d)} = \frac{1}{d!}\,\mathrm{d}x^{m_d}\cdots\mathrm{d}x^{m_1}\,\xi_{m_1\dots m_d} = \frac{1}{d!}\,\sqrt{-g}\,\mathrm{d}^dx\,\epsilon^{m_d\dots m_1}\xi_{m_d\dots m_1}\,,$$

where the wedge product is understood. It will be omitted from here on for sake of brevity. We can thus directly integrate any d-form on the d-dimensional manifold $\mathcal{M}_d$ or, for $p<d$, any p-form on a p-dimensional submanifold $\mathcal{M}_p$. In this framework, Stokes' theorem reads

$$\int_{\mathcal{M}_p}\mathrm{d}\xi^{(p-1)} = \int_{\partial\mathcal{M}_p}\xi^{(p-1)}\,.$$

Another interesting operation in this framework is the Hodge-dual, which transforms p-forms into $(d-p)$-forms. Its action is defined by

$$*(\mathrm{d}x^{m_p}\cdots\mathrm{d}x^{m_1}) = \frac{1}{(d-p)!}\,\mathrm{d}x^{n_d}\cdots\mathrm{d}x^{n_{p+1}}\,\epsilon_{n_{p+1}\dots n_d}{}^{m_p\dots m_1}\,,$$

hence, the dual of a p-form is

$$*\xi^{(p)} = \frac{1}{p!(d-p)!}\,\mathrm{d}x^{n_d}\cdots\mathrm{d}x^{n_{p+1}}\,\epsilon_{n_{p+1}\dots n_d}{}^{m_p\dots m_1}\xi_{m_1\dots m_p}\,.$$

Gauge Fields as Differential Forms

Non-abelian gauge theories can be written in a rather elegant way by means of differential forms. The basic objects one deals with are the gauge-connection 1-form and the field-strength 2-form

$$A = \mathrm{d}x^m A_m\,, \quad F_2 = \frac{1}{2}\mathrm{d}x^n \mathrm{d}x^m F_{mn}\,,$$

for $A_m = A_m^a t^a$ and $F_{mn} = F_{mn}^a t^a$ with t^a generator of the gauge group. Now the forms are matrix valued, hence their commutation properties are less trivial than above. In particular $A^2 = \frac{1}{2}\mathrm{d}x^n \mathrm{d}x^m [A_m, A_n]$, and one can write

$$F_2 = \frac{1}{2}\mathrm{d}x^n \mathrm{d}x^m F_{mn} = \frac{1}{2}\mathrm{d}x^n \mathrm{d}x^m \left(\partial_m A_n - \partial_n A_m - \mathrm{i}[A_m, A_n]\right) = \mathrm{d}A - \mathrm{i}A^2\,.$$

The covariant derivative is defined as $\nabla = \mathrm{d}x^m \nabla_m = \mathrm{d} - \mathrm{i}A$: when applied to a p-form in the adjoint representation, the A acts as a commutator if p is even, as an anticommutator otherwise. Thus the Bianchi identity $\nabla_{[m}F_{pq]} = 0$ can be written as

$$\nabla F_2 = \mathrm{d}F_2 - \mathrm{i}(AF_2 - F_2 A) = 0\,.$$

We can now get rid of the imaginary unity with the redefinition $\mathrm{T}^a = -\mathrm{i}t^a$, so that

$$\mathsf{A} = -\mathrm{i}A\,, \quad \mathsf{F} = -\mathrm{i}F_2 \leftrightarrow \mathsf{F} = \mathrm{d}\mathsf{A} + \mathsf{A}^2\,, \quad \nabla = \mathrm{d} + \mathsf{A}\,.$$

When spacetime is topologically non-trivial, the associated manifold is composed of a finite number of domains U_j which locally look like $\mathbb{R}^d$ or open subsets thereof. Tensor fields are defined on each region separately, along with transition maps in the intersections $U_j \cup U_k$. For a non-abelian gauge theory, in the overlapping region different gauge connections and field-strength tensors are related by a finite gauge transformation as

$$\mathsf{A}_{(i)} = b_{ij}^{-1}\left(\mathrm{d} + \mathsf{A}_{(j)}\right) b_{ij}\,, \quad \mathsf{F}_{(i)} = b_{ij}^{-1}\mathsf{F}_{(j)} b_{ij}\,.$$

These transformation laws define the *gauge bundle*, and the group valued transition function b_{ij} contains the topological information of the gauge bundle itself.

Chern-Simons Form

Let us consider gauge-invariant polinomials of the field strength tensor defined as

$$P_r(\mathsf{F}) = \mathrm{tr}\,\mathsf{F}^r\,,$$

where r denotes the number of F in the trace. This object is a closed form, as one can prove through the Bianchi identity. Moreover, integrals of $P_r(\mathsf{F})$ are topologically invariant, i.e. they are invariant under continuous deformations of A which leave the transition functions unchanged. This second property can be also equivalently rephrased, stating that the difference of two P_r is exact

$$P_r(\mathsf{F}_1) - P_r(\mathsf{F}_0) = \mathrm{d}Q_{2r-1}(\mathsf{F}_1, \mathsf{F}_0)\,.$$

On the other hand, the $P_r(\mathsf{F})$ are locally exact by themselves, because they are closed. Therefore, on each patch U_j one has $P_r(\mathsf{F}) = \mathrm{d}C_{2r-1}(\mathsf{A}_{(j)}, \mathsf{F}_{(j)})$, with

$$C_{2r-1} = r \int_0^1 \mathrm{d}t \; \mathrm{tr}\Big[\mathsf{A}\Big(\mathrm{d}\mathsf{A} + t\mathsf{A}^2\Big)^{r-1}\Big] t^{r-1} = r \int_0^1 \mathrm{d}t \; \mathrm{tr}\Big[\mathsf{A}\Big(\mathsf{F} + (t-1)\mathsf{A}^2\Big)^{r-1}\Big] t^{r-1} \, .$$

The form $C_{2r-1}(\mathsf{A}, \mathsf{F})$ is called *Chern-Simons form.* If we are e.g. interested in the 2-form $P_2(\mathsf{F}) = \mathrm{tr}\,\mathsf{F}\mathsf{F}$, one can straightforwardly compute

$$C_3(\mathsf{A}, \mathsf{F}) = \mathrm{tr}\Big(\mathsf{A}\mathrm{d}\mathsf{A} + \tfrac{2}{3}\mathsf{A}^3\Big) = \mathrm{tr}\Big(\mathsf{A}\mathsf{F} - \tfrac{1}{3}\mathsf{A}^3\Big) \, ,$$

or, equivalently

$$C_3(A, F_2) = -\,\mathrm{tr}\Big(A\mathrm{d}A - \tfrac{2}{3}\mathrm{i}A^3\Big) = -\,\mathrm{tr}\Big(AF_2 + \tfrac{1}{3}\mathrm{i}A^3\Big) \, .$$

B.2 Vielbein Formalism

Suppose to have a four-dimensional non necessarily flat spacetime $\mathcal{M}$, endowed with a metric $g_{mn}(x)$ in local coordinates x^m. At some point $P \in \mathcal{M}$, we introduce an orthonormal frame $\{e^a\}$, $e^a = \mathrm{d}x^m e^a_m\big|_P$, for $a = 0, \ldots, 3$, in such a way that we can decompose the metric as

$$g_{mn}(x) = e^a_m(x) e^b_n(x) \eta_{ab} \, , \quad \eta^{ab} = e^a_m(x) e^b_n(x) g^{mn}(x) \, , \tag{B.1}$$

where $\eta_{ab} = \eta^{ab} = \mathrm{diag}(+,-,-,-)$ is the usual Minkowski metric. The uppercase indices a, b, ... are sometimes called *Lorentz indices*, while lower case ones are referred to as *world indices*. The set $\{e^a\}$ is called *vielbein*: its elements can be interpreted as matrix valued 1-forms, despite the fact that they carry indices which are conceptually different. The matrices e^a_m can be seen as the similarity matrices providing the trasformation from the coordinate basis $\{\mathrm{d}x^m\}$ of the cotangent space at P, $T^*_P(\mathcal{M})$, to the orthonormal basis $\{e^a\}$ of $T^*_P(\mathcal{M})$. Clearly, we can define the inverse relations to (B.1) introducing at some $P \in \mathcal{M}$ the inverse vielbein $\{e_a\}$, $e_a = e^m_a \partial_m\big|_P$, so that

$$g_{mn}(x) = e^m_a(x) e^n_b(x) \eta^{ab} \, , \quad \eta_{ab} = e^m_a(x) e^n_b(x) g_{mn}(x) \, .$$

Analogously to e^a_m, the matrices e^m_a provide the similarity transformation from a coordinate basis $\{\partial_m\}$ of the tangent space in P, $T_P(\mathcal{M})$, to the orthonormal basis $\{e_a\}$ of $T_P(\mathcal{M})$. Using the fact that the matrix e^m_a is the inverse of e^a_m, i.e. $e^a_m e^M_b = \delta^a_b$ and $e^a_m e^n_a = \delta^n_m$, and by means of equation (B.1) we obtain the relations

$$e^m_a = \eta_{ab}\, g^{mn} e^b_n \, , \quad e^a_m = \eta^{ab}\, g_{mn} e^n_b \, .$$

We notice also that the vielbein and its inverse can be used to translate Lorentz indices into world indices an viceversa. Therefore we can write e.g. $V^m = V^a e^m_a$ or $V_m = V_a e^a_m$. Recalling the transformation law of the metric under diffeomorphisms $x^m \mapsto y^m$

$$g_{mn}(x) \mapsto g'_{mn}(y) = \frac{\partial x^p}{\partial y^m} \frac{\partial x^q}{\partial y^n} g_{pq}(x) \, ,$$

one can verify that the vielbein changes according to

$$e'^{a}_{m}(y) = \frac{\partial x^p}{\partial y^m} e^a_p(x) \,.$$

B.3 Differential Forms and Vielbeins in Flat Superspace, a Brief Analysis

The extension of the objects we introduced above to the superspace is rather straightforward, at least at the level of complexity at which we are considering them.

We denote elements of the superspace by $\mathcal{Z}^M \sim (x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. The nature of z^M can be either bosonic ($M = m$) or fermionic ($M = \alpha$, $\dot{\alpha}$), hence one has to keep in mind that

$$\mathcal{Z}^M \, \mathcal{Z}^N = (-)^\sigma \, \mathcal{Z}^N \, \mathcal{Z}^M \,,$$

for $\sigma = 0$ when at least one index is bosonic, and $\sigma = 1$ otherwise. The wedge product is defined in the same way as in ordinary space, with added anticommutation properties depending on the nature of the index, namely

$$\mathrm{d}\mathcal{Z}^M \wedge \mathrm{d}\mathcal{Z}^N = -(-)^\sigma \, \mathrm{d}\mathcal{Z}^N \wedge \mathrm{d}\mathcal{Z}^M \,, \quad \mathrm{d}\mathcal{Z}^M \, \mathcal{Z}^N = (-)^\sigma \, \mathcal{Z}^N \, \mathrm{d}\mathcal{Z}^M \,.$$

We can thus define 1-form in superspace as

$$\Upsilon = \mathrm{d}\mathcal{Z}^M \Upsilon_M = \mathrm{d}x^m \Upsilon_m(z) + \mathrm{d}\theta^\alpha \Upsilon_\alpha(z) + \mathrm{d}\bar{\theta}^{\dot{\alpha}} \Upsilon_{\dot{\alpha}}(z) \,,$$

and, in general, p-forms are defined as

$$\Xi^{(p)} = \mathrm{d}\mathcal{Z}^{M_p} \wedge \cdots \wedge \mathrm{d}\mathcal{Z}^{M_1} \, \Xi_{M_1 \ldots M_p} \,.$$

We notice that, by definition, the coefficient functions of p-forms in superspace have mixed symmetry, hence there is no value of p above which all forms vanish. This is in contrast to what happens in the usual d-dimensional spacetime, where any p-form with $p > d$ vanish because of antisymmetry. All the features we have seen in appendix B.1 are easily generalised to the present case. In particular, one has

$$\Xi^{(p)} \wedge \Upsilon^{(q)} = (-)^{pq} \, \Upsilon^{(q)} \wedge \Xi^{(p)} \,,$$

and one can introduce the exterior derivative of a p-form as

$$\mathrm{d}\Xi^{(p)} = \mathrm{d}\mathcal{Z}^{M_p} \wedge \cdots \wedge \mathrm{d}\mathcal{Z}^{M_1} \wedge \mathrm{d}\mathcal{Z}^M \, \partial_M \Xi_{M_1 \ldots M_p} \,,$$

and which enjoys the properties

$$\begin{aligned} \mathrm{d}\left(\Xi^{(p)} \wedge \Upsilon^{(q)}\right) &= \Xi^{(p)} \wedge \mathrm{d}\Upsilon^{(q)} + (-)^q \mathrm{d}\Xi^{(p)} \wedge \Upsilon^{(q)} \,, \\ \mathrm{d}^2 &= 0 \,. \end{aligned}$$

Differential forms in superspace, like those in the usual spacetime, are invariant under coordinate transformations $\mathcal{Z}^M \mapsto \mathcal{Y}^M$.

We considered differential forms using the superspace differential $\mathrm{d}\mathcal{Z}^M$ as a natural basis. However, this is not a particularly meaningful choice in supersymmetry: indeed, the exterior derivative does not map superfields into superfields, because $\partial/\partial\mathcal{Z}^M$ does not commute with supersymmetry generators. On the other hand, we know that the supercovariant derivatives $\mathcal{D}_\alpha$ and $\mathcal{D}_{\dot\alpha}$ do commute with Q_α and $\bar{Q}_{\dot\alpha}$. Hence, roughly speaking, we want a new basis where we can trade d with $\mathcal{D}_A = (\partial_m, \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha})$. Let us be more precise. We introduce the (flat) supervielbein basis $E^A = (E^m, E^\alpha, \bar{E}^{\dot\alpha}) = \mathrm{d}\mathcal{Z}^M E_M{}^A(\mathcal{Z})$, with $E_M{}^A$ an invertible function of the superspace coordinates. Therefore, we can introduce the inverse $E_M{}^A$ so that

$$E_M{}^A E_A{}^N = \delta^N{}_M \,, \quad E_M{}^A E_B{}^M = \delta^A{}_B \,. \tag{B.2}$$

We thus define the differential

$$\mathbf{d} \equiv \mathrm{d}\mathcal{Z}^M \frac{\partial}{\partial \mathcal{Z}^M} = E^A \mathcal{D}_A = \mathrm{d}\mathcal{Z}^M E_M{}^A E_A{}^N \frac{\partial}{\partial \mathcal{Z}^N} \,,$$

so that

$$\mathcal{D}_A = E_A{}^M \frac{\partial}{\partial \mathcal{Z}^M} \,.$$

From the explicit expression of the super-covariant derivatives

$$\mathcal{D}_\alpha = \partial_\alpha + \mathrm{i}\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m \,, \quad \bar{\mathcal{D}}_{\dot\alpha} = \bar\partial_{\dot\alpha} + \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m \,,$$

we can find the explicit expression of the matrix E^A_M, and then, from (B.2), we can obtain the expression of $E_M{}^A$. It turns out that

$$E_A{}^M = \begin{pmatrix} \delta_a{}^m & 0 & 0 \\ \mathrm{i}\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} & \delta_\alpha{}^\mu & 0 \\ \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot\alpha} & 0 & \delta_{\dot\alpha}{}^{\dot\mu} \end{pmatrix} , \quad E_M{}^A = \begin{pmatrix} \delta_m{}^a & 0 & 0 \\ -\mathrm{i}\sigma^a_{\mu\dot\mu}\bar\theta^{\dot\mu} & \delta_\mu{}^\alpha & 0 \\ -\mathrm{i}\theta^\mu\sigma^a_{\mu\dot\mu} & 0 & \delta_{\dot\mu}{}^{\dot\alpha} \end{pmatrix} .$$

From the defining expression $E^A = \mathrm{d}\mathcal{Z}^M E_M{}^A$ and the explicit matricial expression of $E_A{}^M$ we can find

$$E^\alpha = \mathrm{d}\theta^\alpha \,, \quad \bar{E}^{\dot\alpha} = \mathrm{d}\bar\theta^{\dot\alpha} \,, \quad E^a = \mathrm{d}x^a + \mathrm{i}\theta\sigma^a\mathrm{d}\bar\theta - \mathrm{i}\mathrm{d}\theta\sigma^a\bar\theta \,.$$

Thus, we have

$$\mathbf{d}E^\alpha = 0 \,, \quad \mathrm{d}\bar{E}^{\dot\alpha} = 0 \,, \quad \mathbf{d}E^a = -2\mathrm{i}\,\mathrm{d}\theta \wedge \sigma^a \mathrm{d}\bar\theta = -2\mathrm{i}\,E \wedge \sigma^a \bar{E} \,.$$

Similarly to the usual vielbein, general coordinate transformations $\mathcal{Z}^M \mapsto \mathcal{Y}^M$ acts on the supervielbein according to the law

$$E'_M{}^A(\mathcal{Y}) = \frac{\partial \mathcal{Z}^N}{\partial \mathcal{Y}^M} E_N{}^A(\mathcal{Z}) \,.$$

Also in this case, we can use the supervielbein to convert indices of type A (tangent space indices) into indices of type M (world indices) and viceversa.

Bibliography

[1] A. Achúcarro, J. M. Evans, P. K. Townsend, and D. L. Wiltshire. "Super p-Branes". In: *Phys. Lett. B* 198 (1987), p. 441. DOI: 10.1016/0370-2693(87)90896-3.

[2] A. Achúcarro, J. P. Gauntlett, K. Itoh, and P. K. Townsend. "World Volume Supersymmetry From Space-time Supersymmetry of the Four-dimensional Supermembrane". In: *Nucl. Phys. B* 314 (1989), p. 129. DOI: 10.1016/0550-3213(89)90115-6.

[3] I. Bandos, F. Farakos, S. Lanza, L. Martucci, and D. Sorokin. "Three-forms, Dualities and Membranes in Four-dimensional Supergravity". In: *JHEP* 07 (2018), pp. 0 – 28. DOI: 10.1007/JHEP07(2018)028. arXiv: 1803.01405 [hep-th].

[4] I. Bandos, S. Lanza, and D. Sorokin. "How $\mathcal{N} = 1$, $D = 4$ SYM domain walls look like". In: *19th Hellenic School and Workshops on Elementary Particle Physics and Gravity*. Apr. 2020. arXiv: 2004.11232 [hep-th].

[5] I. Bandos, S. Lanza, and D. Sorokin. "Supermembranes and domain walls in $\mathcal{N} = 1$, $D = 4$ SYM". In: *JHEP* 12 (2019). [Erratum: JHEP 05, 031 (2020)], p. 021. DOI: 10.1007/JHEP12(2019)021. arXiv: 1905.02743 [hep-th].

[6] I. Bandos and C. Meliveo. "Superfield Equations for the Interacting System of $D = 4$ $\mathcal{N} = 1$ Supermembrane and Scalar Multiplet". In: *Nucl. Phys. B* 849 (2011), p. 1. DOI: 10.1016/j.nuclphysb.2011.03.010. arXiv: 1011.1818 [hep-th].

[7] I. Bars. "First Massive Level and Anomalies in the Supermembrane". In: *Nucl. Phys. B* 308 (1988), p. 462. DOI: 10.1016/0550-3213(88)90573-1.

[8] V. Bashmakov, F. Benini, S. Benvenuti, and M. Bertolini. "Living on the walls of super-QCD". In: *SciPost Phys.* 6.4 (2019), p. 044. DOI: 10.21468/SciPostPhys.6.4.044. arXiv: 1812.04645 [hep-th].

[9] E. Bergshoeff, E. Sezgin, and P. K. Townsend. "Supermembranes and Eleven-Dimensional Supergravity". In: *Phys. Lett. B* 189 (1987), p. 75. DOI: 10.1016/0370-2693(87)91272-X.

[10] M. Bertolini. *Lectures on Supersymmetry*. SISSA Lecture Notes. 2019. URL: https://people.sissa.it/~bertmat/susycourse.pdf.

[11] A. Bilal. *Lectures on Anomalies*. 2008. arXiv: 0802.0634 [hep-th].

[12] P. Binetruy, F. Pillon, G. Girardi, and R. Grimm. "The Three form Multiplet in Supergravity". In: *Nucl. Phys. B* 477 (1996), p. 175. DOI: 10.1016/0550-3213(96)00370-7. arXiv: hep-th/9603181.

[13] I. L. Buchbinder and S. M. Kuzenko. *Ideas and methods of Supersymmetry and Supergravity: A Walk through Superspace*. 1st ed. IOP Publishing, 1995.

[14] L. Castellani, R. D'Auria, and P. Fré. *Supergravity and superstrings*. Vol. 1. World Scientific Publishing, 1991.

[15] S. R. Coleman and J. Mandula. "All Possible Symmetries of the S-Matrix". In: *Phys. Rev.* 159 (1967), p. 1251. DOI: 10.1103/PhysRev.159.1251.

[16] G. R. Dvali and M. Shifman. "Domain Walls in Strongly Coupled Theories". In: *Phys. Lett. B* 396 (1997). [Erratum: Phys.Lett.B 407, 452 (1997)], p. 64. DOI: 10.1016/S0370-2693(97)00131-7. arXiv: hep-th/9612128.

[17] T. Eguchi, P. B. Gilkey, and A. J. Hanson. "Gravitation, Gauge Theories and Differential Geometry". In: *Phys. Rept.* 66 (1980), p. 213. DOI: 10.1016/0370-1573(80)90130-1.

[18] G. R. Farrar, G. Gabadadze, and M. Schwetz. "On the Effective Action of $\mathcal{N} = 1$ supersymmetric Yang-Mills Theory". In: *Phys. Rev. D* 58 (1998), p. 015009. DOI: 10.1103/PhysRevD.58.015009. arXiv: hep-th/9711166.

[19] S. Ferrara and B. Zumino. "Supergauge Invariant Yang-Mills Theories". In: *Nucl. Phys. B* 79 (1974), p. 413. DOI: 10.1016/0550-3213(74)90559-8.

[20] S. Ferrara and B. Zumino. "Transformation Properties of the Supercurrent". In: *Nucl. Phys. B* 87 (1975), p. 207. DOI: 10.1016/0550-3213(75)90063-2.

[21] S. J. Gates Jr. "Super p-form Gauge Superfields". In: *Nucl. Phys. B* 184 (1981), p. 381. DOI: 10.1016/0550-3213(81)90225-X.

[22] M. B. Green and J. H. Schwarz. "Covariant Description of Superstrings". In: *Phys. Lett. B* 136 (1984), p. 367. DOI: 10.1016/0370-2693(84)92021-5.

[23] M. B. Green and J. H. Schwarz. "Properties of the Covariant Formulation of Superstring Theories". In: *Nucl. Phys. B* 243 (1984), p. 285. DOI: 10.1016/0550-3213(84)90030-0.

[24] R. Haag, J. T. Lopuszański, and M. Sohnius. "All Possible Generators of Supersymmetries of the S-Matrix". In: *Nucl. Phys. B* 88 (1975), p. 257. DOI: 10.1016/0550-3213(75)90279-5.

[25] J. Hughes, J. Liu, and J. Polchinski. "Supermembranes". In: *Phys. Lett. B* 180 (1986), p. 370. DOI: 10.1016/0370-2693(86)91204-9.

[26] J. Hughes and J. Polchinski. "Partially Broken Global Supersymmetry and the Superstring". In: *Nucl. Phys. B* 278 (1986), p. 147. DOI: 10.1016/0550-3213(86)90111-2.

[27] Ö. Kaymakcalan and J. Schechter. “Superconformal Anomalies and the Effective Lagrangian for Pure Supersymmetric QCD”. In: *Nucl. Phys. B* 239 (1984), p. 519. DOI: 10.1016/0550-3213(84)90261-X.

[28] I. I. Kogan, A. Kovner, and M. Shifman. “More on Supersymmetric Domain Walls, N Counting and Glued Potentials”. In: *Phys. Rev. D* 57 (1998), p. 5195. DOI: 10.1103/PhysRevD.57.5195. arXiv: hep-th/9712046.

[29] A. Kovner and M. Shifman. “Chirally symmetric phase of supersymmetric gluodynamics”. In: *Phys. Rev. D* 56 (1997), p. 2396. DOI: 10.1103/PhysRevD.56.2396. arXiv: hep-th/9702174.

[30] M. Maggiore. *A Modern Introduction to Quantum Field Theory*. Oxford University Press, 2005.

[31] H. B. Nielsen and P. Olesen. “Vortex Line Models for Dual Strings”. In: *Nucl. Phys. B* 61 (1973). Ed. by J.C. Taylor, p. 45. DOI: 10.1016/0550-3213(73)90350-7.

[32] R. Rajaraman. *Solitons and Instantons*. Elsevier Science Publishers, 1989.

[33] A. Salam and J. A. Strathdee. “Supersymmetry and Nonabelian Gauges”. In: *Phys. Lett. B* 51 (1974), p. 353. DOI: 10.1016/0370-2693(74)90226-3.

[34] M. Shifman. *Advanced topics in quantum field theory: A lecture course*. Cambridge Univ. Press, 2012.

[35] M. Shifman and A. I. Vainshtein. “On Gluino Condensation in Supersymmetric Gauge Theories. $SU(N)$ and $O(N)$ Groups”. In: *Sov. Phys. JETP* 66 (1987), p. 1100. DOI: 10.1016/0550-3213(88)90680-3.

[36] G. M. Shore. “Constructing Effective Actions for $\mathcal{N} = 1$ Supersymmetry Theories. 1. Symmetry Principles and Ward Identities”. In: *Nucl. Phys. B* 222 (1983), p. 446. DOI: 10.1016/0550-3213(83)90544-8.

[37] M. F. Sohnius. “Introducing Supersymmetry”. In: *Phys. Rept.* 128 (1985), p. 39. DOI: 10.1016/0370-1573(85)90023-7.

[38] T. R. Taylor, G. Veneziano, and S. Yankielowicz. “Supersymmetric QCD and Its Massless Limit: An Effective Lagrangian Analysis”. In: *Nucl. Phys. B* 218 (1983), p. 493. DOI: 10.1016/0550-3213(83)90377-2.

[39] P. K. Townsend. “Supersymmetric Extended Solitons”. In: *Phys. Lett. B* 202 (1988), p. 53. DOI: 10.1016/0370-2693(88)90852-0.

[40] G. Veneziano and S. Yankielowicz. “An Effective Lagrangian for the Pure $\mathcal{N} = 1$ Supersymmetric Yang-Mills Theory”. In: *Phys. Lett. B* 113 (1982), p. 231. DOI: 10.1016/0370-2693(82)90828-0.

[41] S. Weinberg. *The Quantum Theory of Fields*. 1st ed. Vol. 1. Cambridge University Press, 1995.

[42] J. Wess and J. Bagger. *Supersymmetry and Supergravity*. Princeton, NJ, USA: Princeton University Press, 1992.

[43] J. Wess and B. Zumino. “Supergauge Invariant Extension of Quantum Electrodynamics”. In: *Nucl. Phys. B* 78 (1974), p. 1. DOI: 10.1016/0550-3213(74)90112-6.

[44] E. Witten and D. I. Olive. “Supersymmetry Algebras That Include Topological Charges”. In: *Phys. Lett. B* 78 (1978), p. 97. DOI: 10.1016/0370-2693(78)90357-X.