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ModMax meets Cosmology

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Abstract

ModMax is the unique, non-linear, maximally symmetric one-parameter extension of classical Maxwell's electrodynamics. Shedding more light about the properties of ModMax is an intriguing goal. In this work we proposed a possible ModMax *precursor* Lagrangian theory in a 4-dimensional spacetime having linear combinations of Lorentz invariants and one more degree of freedom from a pseudoscalar, axion-like particle. We constrained the ModMax parameter γ requiring the ModMax precursor to be the principal player of the cosmic birefringence. We considered a sub-Planckian axion field and, using the recent bound on the birefringence angle $\beta \approx 0.342^\circ$ [1], we found $10^{-26} < \gamma < 10^{-19}$ for masses of the axion-like field $10^{-35}\,\mathrm{eV} \lesssim m \lesssim 10^{-30}\,\mathrm{eV}$. This thesis represents one of the first studies of the ModMax theory in a cosmological set-up.

Contents

Introduction

Maxwell's theory of electromagnetism was the first theory to describe electricity, magnetism and light as different manifestations of the same phenomenon: the electromagnetic field. This unification was possible introducing the Lorentz symmetry in a four-dimensional spacetime and writing the equation of electromagnetism in a manifestly covariant way. In addition to be relativistic invariant, Maxwell's theory is also gauge, conformal and $SO(2)$ invariant in the electric and magnetic space [2]. This last symmetry is called duality-symmetry. It plays an important role in various physical aspects of electrodynamics including Dirac's hypothesis on the existence of magnetically charged particles (monopoles).

However, Maxwell electrodynamics predicts a singularity: the divergence of the electric field of a point-like charge at its position. In order to solve this problem, in 1934 Born and Infeld proposed an electromagnetic theory being nonlinear but that reduces to Maxwell electrodynamics in the weak-field limit [3].

Few years later, another remarkable theory of nonlinear electrodynamics was proposed by Euler and Heisenberg. It was an effective theory of quantum electrodynamics updating the Maxwell's Lagrangian with the vacuum polarization effect [4]. By construction, also the Euler-Heisenberg theory reduces to Maxwell theory in the weak-field limit.

Years after a variety of non-linear electrodynamic (NED) models were proposed, such as the logarithmic, double-logarithmic, exponential, power-law, arcsin and others (see e.g. [5–9]). They have been created for applications in gravity, cosmology, condensed matter, optical materials and in general as possible guides to catch new physics [10, 11].

One can build a Lorentz-invariant NED theory in 4 dimension writing a Lagrangian density as a function of the two independent Lorentz invariants $S = -F_{\mu\nu}F^{\mu\nu}/4$ and $P = -F_{\mu\nu}\tilde{F}^{\mu\nu}/4$, where $\tilde{F}^{\mu\nu}$ is the Hodge dual of the field strength $F^{\mu\nu}$. A popular idea was that a valid candidate for a NED may reduce to Maxwell's theory in the low-energy limit, as is the case of Born-Infeld and Euler-Heisenberg theory, but usually some symmetries of Maxwell's theory are lost.

In 2020 Bandos, Lechner, Sorokin and Townsend discovered a *unique* NED theory being both conformal and duality invariant, it is a one-parameter generalization of Maxwell's theory dubbed ModMax [12], the parameter was denoted by γ and Maxwell's theory is recovered at $\gamma = 0$. In addition to the formal treatment of ModMax, its applications may open new results in phenomenology, since comparing its predictions with physical observations we can constrain the theory parameters.

An interesting property of almost all NEDs (with the exception of Born-Infeld theory [13]) is the phenomenon of birefringence which happens when light propagates in an electromagnetic background and interacts with the latter due to non-linearities. In a cosmological framework, the light propagating in the universe can produce birefringence if it couples to external fields, making them behave as a birefringent material. This phenomenon is called cosmic birefringence (it was also considered the scenario of photon self-interaction, see e.g [14]). The cosmic birefringence was initially predicted assuming a coupling between photons and a pseudoscalar field ϕ through a term $\propto \phi\tilde{F}^{\mu\nu}F_{\mu\nu}$, such a term is called Chern-Simons term [15]. The effect is a rotation of the linear polarization plane of photons during propagation. This phenomenology was largely studied in the last decades and in 2022 a joint analysis of polarization data from Planck and WMAP space missions has allowed the detection of a small rotation angle of the linear polarization plane of CMB photons in a propagation period running from their emission up to now (see [16–19] for recent works). The estimated angle is $\beta = 0.342^\circ\ ^{+0.094^\circ}_{-0.091^\circ}$ at 68% C.L. [1].

The purpose of the thesis is to derive the cosmic birefringence for the ModMax theory and use the observed value of the birefringence angle to constrain the ModMax parameter γ. Instead of working with the non-linear Lagrangian density of ModMax we follow the tool of auxiliary fields to obtain a formulation in which the Lagrangian density is quadratic in the electromagnetic fields and has an auxiliary scalar field [20]. The resulting Lagrangian is equivalent in form to a Maxwell-axion-dilaton theory and has similarities with other known theories [21, 22]. This result may be a hint on the existence of a more fundamental theory in which ModMax emerges (in a certain conformal limit) as an effective field theory. Supported by these observations, we promote the auxiliary field to be a physical field and we derive the cosmic birefringence from this new theory, that we called *ModMax precursor*. Then we numerically search an estimation of γ.

We consider a quadratic potential for the dynamical scalar field, that we treat as an axion-like particle (ALP) and we call it χ. We set our work in the conditions at which the quadratic potential is a good approximation of a more fundamental potential, as for example the cosine-type potential [23]. In order to estimate γ we have to solve the dynamics of the ALP and we do so in the assumptions of a negligible electromagnetic backreactions coming from the coupling between photon and the pseudoscalar field, and a sub-dominant contribution of the field to the universe expansion in a flat Friedmann-Lemaître-Robertson-Walker spacetime. We find the range $10^{-26} < \gamma < 10^{-19}$ for masses of ALP in the range $10^{-35}\,\text{eV} \lesssim m \lesssim 10^{-30}\,\text{eV}$ with an initial condition $\chi_{\text{in}} \sim 0.1M_{\text{Pl}}$ at the epoch of recombination, where M_{Pl} is the Planck mass. This is a reasonable range showing that the departure from the well established, low-energy theory of Maxwell electrodynamics towards the ModMax electrodynamics is small.

The Thesis is organized as follows.

1. In chapter 1 we present an overview of electrodynamics theories starting from the liner Maxwell theory, then we briefly introduce the Born-Infeld and Euler-Heisenberg theories as important examples of NED theories. A self-consistent part is dedicated to how to build a NED theory in general and the constraints it must respect in order to be relativistic, conformal and duality-invariant; this is discussed in both Lagrangian and Hamiltonian formalism. The chapter concludes with the exposition of ModMax theory.

2. In chapter 2 we present the phenomenon of the vacuum birefringence in NED theories. Then we introduce the polarization theory in order to set up the formalism that will be used along the text.

3. Chapter 3 is devoted to the cosmic version of the vacuum birefringence: the cosmic birefringence. We introduce the phenomenon considering the Maxwell-Chern-Simons theory. At the end of the chapter we discuss recent bounds on the photon-axion coupling of the Maxwell-Chern-Simons theory.

4. Chapter 4 contains the original part of the thesis. We start applying the tool of auxiliary fields in order to reformulate the ModMax Lagrangian in a quadratic form for the electromagnetic fields, then we discuss similarities of this new formulation with other known theories of electromagnetism. We promote the reformulated ModMax theory to a new theory of interacting electromagnetic and neutral pseudoscalar fields. Then we proceed by computing the cosmic birefringence angle in this new theory. We numerically estimate the ModMax parameter γ requiring that the new theory has produced the rotational angle observed in the polarization map of CMB.

Along the text we adopt natural units $c = \hbar = 1$ and a mostly positive metric $(-,+,+,+)$. We also use the following acronyms:

NED	non-linear electrodynamics
BI	Born-Infeld
BB	Bialynicki-Birula
EL	Eulero-Lagrange
EoM	equation of motion
EoS	equation of state
ALP	axion-like particle
CMB	cosmic microwave background
LSS	last-scattering surface
FE	Friedmann equation
FLRW	Friedmann-Lemaître-Robertson-Walker

Chapter 1

Overview of electrodynamics theories

Before introducing the ModMax theory we give a short review of how electrodynamics has developed along the years. During the second half of 1800 the Maxwell's equations were formulated in the vectorial form we nowadays know. Around the 1900 they were formulated in a relativistic covariant way based on the principle that physical laws must be invariant under the Poincarè group. The Poincarè group in the matrix representation over a Minkowski spacetime is:

$$\mathcal{P}(1,3) \equiv \{(\Lambda, a) \ : \ \Lambda \in O(1,3), a \in \mathbb{R}\} , \tag{1.1}$$

where Λ is a vector representation of the homogeneous Lorentz group:

$$O(1,3) \equiv \left\{\Lambda \in GL(4,\mathbb{R}) \ : \ \Lambda^{\top}\eta\Lambda = \eta\right\} , \tag{1.2}$$

and $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the (mostly positive) Minkowski metric.

1.1 Examples of electrodynamics theories

Maxwell theory

The Maxwell action of a source-free theory in a Minkowski spacetime written in a manifestly Lorentz-invariant and gauge-invariant form is [2]:

$$\mathcal{S}_{\mathrm{M}} = -\frac{1}{4}\int \mathrm{d}^4x\, F^{\mu\nu}F_{\mu\nu} , \tag{1.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and $A_\mu(x) = (A_0, \vec{A})$ is a gauge field called 4potential in the framework of electrodynamics.

The field strength $F_{\mu\nu}$ is gauge-independent, it is trivial to prove that it does not change under a gauge transformation of the 4-vector $A_\mu \to A_\mu + \partial_\mu\Omega(x)$, where Ω is a scalar function. Physical quantities, that must be gauge-independent, are the components of $F^{\mu\nu}$, electric and magnetic three-vector fields E^i and B^i are:

$$E^i = F^{0i} , \tag{1.4}$$

$$B^i = \frac{1}{2}\epsilon^{ijk}F_{jk} , \tag{1.5}$$

where ϵ^{ijk} is the Levi-Civita tensor.

Free Maxwell equations can be derived using the least action principle, and together with Bianchi identities are respectively [2]:

$$\partial_\mu F^{\mu\nu} = 0\,, \tag{1.6a}$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0\,, \tag{1.6b}$$

where $\tilde{F}^{\mu\nu}$ is the Hodge dual of the field strength:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,. \tag{1.7}$$

We point out that the Bianchi identity is due to the antisymmetric nature of $\tilde{F}^{\mu\nu}$ and so it will be valid for any theory of electromagnetism based on the field strength.

In terms of the electric and magnetic fields, the Maxwell action is:

$$\mathcal{S}_{\mathrm{M}} = \frac{1}{2}\int \mathrm{d}^4x \left(\vec{E}^2 - \vec{B}^2\right), \tag{1.8}$$

and equations (1.6) read:

$$\partial_t\vec{E} = \vec{\nabla}\times\vec{B}\,, \qquad \vec{\nabla}\cdot\vec{B} = 0\,, \tag{1.9a}$$
$$\partial_t\vec{B} = -\vec{\nabla}\times\vec{E}\,, \qquad \vec{\nabla}\cdot\vec{E} = 0\,. \tag{1.9b}$$

Maxwell theory is Lorentz and gauge invariant by construction, and one can show that it is also invariant under conformal transformations and $SO(2)$ duality rotation in the electromagnetic plane. We will write more about these symmetries in the next sections. Notice that the Maxwell action is quadratic in $F^{\mu\nu}$ thus the EoM are linear for electric and magnetic fields.

Born-Infeld theory

In 1934, Born and Infeld (BI) proposed a theory of NED with the aim to remove the divergence of the electric field of a point-like charge on its position [3], a problem which is present in the Maxwell electrodynamics. The BI theory is based on the Lagrangian density:

$$\mathscr{L}_{\mathrm{BI}} = T - T\sqrt{-\det\left(\eta_{\mu\nu} + T^{-1/2}F_{\mu\nu}\right)} \tag{1.10a}$$
$$= T - \sqrt{T^2 + \frac{T}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2}\,. \tag{1.10b}$$

Here T is the coupling parameter with the dimension of energy density, and $\eta_{\mu\nu}$ is the Minkowski metric. Since T is dimensional, the BI theory is not conformal, but is gauge and duality-invariant.

Let us expand the Lagrangian (1.10) in powers of FF:

$$\mathscr{L}_{\mathrm{BI}}\Big|_{T\to\infty} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{32T}\left((F_{\mu\nu}F^{\mu\nu})^2 + \left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right) + \mathcal{O}\left((FF)^2\right). \tag{1.11}$$

The weak-field limit is equivalent to take $T\to\infty$, in this limit Maxwell's theory is restored. On the other hand, the strong-field limit is equivalent to $T\to 0$ and it gives a purely imaginary total derivative:

$$\mathscr{L}_{\mathrm{BI}}\Big|_{T\to 0} = \frac{i}{4}\left|F_{\mu\nu}\tilde{F}^{\mu\nu}\right|, \tag{1.12}$$

so the corresponding action vanishes. The strong-field limit can be taken in the Hamiltonian formalism as pointed out by Bialynicki-Birula [24], taking the Legendre transform one gets the Hamiltonian density:

$$\mathcal{H}_{\mathrm{BI}}\left(\vec{D},\vec{B}\right) = \vec{D}\cdot\vec{E} - \mathscr{L}_{\mathrm{BI}}\left(\vec{B},\vec{E}\right) \tag{1.13a}$$

$$= \sqrt{T^2 + T\left(\vec{D}^2+\vec{B}^2\right) + \left|\vec{D}\times\vec{B}\right|^2} - T\,, \tag{1.13b}$$

where $\vec{D}$ is the conjugate momentum to the three-vector potential $\vec{A}$, called electric displacement vector field:

$$\vec{D} = \frac{\partial\mathscr{L}}{\partial\vec{E}}\,. \tag{1.14}$$

The strong-field limit ($T\to 0$) can now be computed and yields the Hamiltonian density of the so-called Bialynicki-Birula (BB) electrodynamics:

$$\mathcal{H}_{\mathrm{BB}} = \left|\vec{D}\times\vec{B}\right|. \tag{1.15}$$

BB theory contains all possible electromagnetic fields in the null field configuration, where both Lorentz scalars are null, i.e. $S=0$ and $P=0$. Examples of solutions in this configuration are plane waves.

The BB theory is naturally conformal invariant since no dimensional parameter is present in its Hamiltonian density to track a specific energy scale. Regarding the duality symmetry, BB theory is $SL(2,\mathbb{R})$ invariant in the plane $(\vec{D},\vec{B})$ rather than $SO(2)$. The duality symmetry is easily verifiable observing that the Hamiltonian density $\mathcal{H}_{\mathrm{BB}}$ does not change under a $SL(2,\mathbb{R})$ transformation:

$$\begin{pmatrix}\vec{D}\\ \vec{B}\end{pmatrix} \to \begin{pmatrix}\vec{D}'\\ \vec{B}'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\vec{D}\\ \vec{B}\end{pmatrix} \tag{1.16}$$

since we are considering special matrix with determinant equal to 1.

The BB theory possesses infinite number of conserved currents. However, it does not reduce to Maxwell's theory.

We will come back to the Hamiltonian formulation of electrodynamics in the next section.

Euler-Heisenberg theory

Another important theory was developed by Euler and Heisenberg (EH) few years after Born-Infeld theory [4]. The theory generalizes the Maxwell's Lagrangian with the vacuum polarization effects of QED, predicting the refraction of light in an electromagnetic field background, or in a particle perception the light by light scattering. However, both conformal and duality-invariance are lost. The effective Lagrangian density of EH theory is:[1]

$$\mathscr{L}_{\mathrm{EH}} = S - \frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}\exp\left(-m^2 s\right)\left[(es)^2 P\frac{\Re\cosh\left(es\sqrt{-S+iP}\right)}{\Im\cosh\left(es\sqrt{-S+iP}\right)} + \frac{2}{3}(es)^2 S - 1\right] \tag{1.17}$$

[1] This formulation of the Lagrangian was derived using the proper-time technique [25].

where we are using $4\pi\epsilon_0 = 1$. Here $\Re$ and $\Im$ stand for real and imaginary part, m is the electron mass and e is the electron charge.

In the weak-field limit EH Lagrangian takes the form:

$$\mathscr{L}_{\text{EH}} = S + \frac{2\alpha^2}{45m^4}\left(4S^2 + 7P^2\right) + \mathcal{O}\left((FF)^3\right), \tag{1.18}$$

where $\alpha = e^2$ is the fine structure constant in our units. The theory reduces to Maxwell electrodynamics in the weak-field limit by construction. The coefficient of the second-order term in the Lagrangian, written in the international system units, is very small:

$$\frac{2\hbar^2\alpha^2}{45\mu_0 m^4 c^5} = 1.32 \times 10^{-24}(\text{Tesla})^{-2} \tag{1.19}$$

making the quantum corrections hard to experimentally observe.

1.2 Non-Linear Electrodynamics theories

The electrodynamics theories reported above are examples of a class of non-linear theories describing self-interactions of electromagnetic fields. Let us now focus on NED theories and following article [24] and chapter 3 of [26] we propose a general treatment on how a NED theory should be in order to preserve some chosen symmetries. We will mainly focus on relativistic theories with conformal and duality symmetries.

The most general Lorentz-invariant NED in four-dimensions has a Lagrangian density constructed with the use of two Lorentz invariants, the scalar S and the pseudoscalar P [2]:

$$S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right), \tag{1.20}$$

$$P = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = \vec{E}\cdot\vec{B}\,. \tag{1.21}$$

Indeed, the NED action in a Minkowski spacetime is:

$$\mathcal{S}_{\text{NED}} = \int \mathrm{d}^4x\, \mathscr{L}_{\text{NED}}(S, P)\,. \tag{1.22}$$

Do note, in order to preserve parity-simmetry the NED Lagrangian should have only even powers of the pseudoscalar P.

Since Maxwell's theory successfully describes the classical electromagnetic phenomena, a natural assumption when building a NED theory is that a valid candidate may reduce to Maxwell's theory in the low-energy limit. Indeed, one assumes the Lagrangian density to be an analytic function that can be expanded in powers of S and P and that in the weak-field limit ($S^2, P^2 \ll S, P$) it reduces to Maxwell Lagrangian. However, not all proposed NED models, such as conformal NEDs, satisfy these assumptions.

Let us start a brief discussion of the equations of motion and stress-energy tensor of a generic NED. Then we will consider requirements for a NED be conformal and duality invariant.

[2] In $D = 4$ there exist only two independent Lorentz invariants (e.g. S and P) form which one can construct (pseudo)scalar products of any number of $F_{\mu\nu}$. For instance, $F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} = 8S^2 + 4P^2$.

Equation of motions

We assume the paradigm of least action principle as the tool to derive the EoM as the Eulero-Lagrange (EL) equations; varing the action with respect to the 4-potential we get:

$$
\begin{aligned}
0 &= \frac{\partial \mathscr{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} && (1.23a)\\
&= -\partial_\mu \left[\frac{\partial \mathscr{L}}{\partial S}\frac{\partial S}{\partial F_{\alpha\beta}}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} + (S \to P)\right] && (1.23b)\\
&= \partial_\mu \Big(\mathscr{L}_S F^{\mu\nu} + \mathscr{L}_P \tilde{F}^{\mu\nu}\Big)\,, && (1.23c)
\end{aligned}
$$

where by $(S \to P)$ we mean a term equivalent in form to the first term in parenthesis but with P instead of S. To work out the EoM we compute, one and for all, the following derivatives:[3]

$$
\begin{aligned}
\frac{\partial S}{\partial F_{\alpha\beta}} &= -\frac{1}{4}\frac{\partial}{\partial F_{\alpha\beta}}(\eta^{\rho\mu}\eta^{\sigma\nu}F_{\rho\sigma}F_{\mu\nu}) && (1.25a)\\
&= -\frac{1}{4}\eta^{\rho\mu}\eta^{\sigma\nu}\left[\Big(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\beta_\rho\delta^\alpha_\sigma\Big)F_{\mu\nu} + F_{\rho\sigma}\Big(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\Big)\right] && (1.25b)\\
&= -\frac{1}{4}\left[\Big(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\beta_\rho\delta^\alpha_\sigma\Big)F^{\rho\sigma} + F^{\mu\nu}\Big(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\Big)\right] && (1.25c)\\
&= -F^{\alpha\beta}\,, && (1.25d)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial P}{\partial F_{\alpha\beta}} &= -\frac{1}{4}\frac{\epsilon^{\mu\nu\rho\sigma}}{2}\frac{\partial}{\partial F_{\alpha\beta}}(F_{\mu\nu}F_{\rho\sigma}) && (1.26a)\\
&= -\frac{1}{4}\frac{\epsilon^{\mu\nu\rho\sigma}}{2}\left[\Big(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\Big)F_{\rho\sigma} + F_{\mu\nu}\Big(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\beta_\rho\delta^\alpha_\sigma\Big)\right] && (1.26b)\\
&= -\frac{1}{2}\frac{\epsilon^{\mu\nu\rho\sigma}}{2}\Big(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\Big)F_{\rho\sigma} && (1.26c)\\
&= -\frac{\epsilon^{\alpha\beta\rho\sigma}}{2}F_{\rho\sigma} && (1.26d)\\
&= -\tilde{F}^{\alpha\beta}\,, && (1.26e)
\end{aligned}
$$

in the third line we have used the property that $\epsilon^{\mu\nu\rho\sigma}$ is symmetric under the permutation of the couples of indices.

Then the EoM reads:

$$
\begin{aligned}
0 &= \partial_\mu\Big(\mathscr{L}_S F^{\alpha\beta}\Big(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\beta_\mu\delta^\alpha_\nu\Big) - \mathscr{L}_P\tilde{F}^{\alpha\beta}\Big(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\beta_\mu\delta^\alpha_\nu\Big)\Big) && (1.27a)\\
&= \partial_\mu\Big(\mathscr{L}_S F^{\mu\nu} + \mathscr{L}_P\tilde{F}^{\mu\nu}\Big)\,. && (1.27b)
\end{aligned}
$$

The Kronecker's deltas of the first line come from $\partial F_{\alpha\beta}/\partial(\partial_\mu A_\nu)$, and the subscripts S and P denote partial derivatives with respect to S and P.

Do note we can introduce the following tensor,

$$
\tilde{G}^{\mu\nu} \equiv -\frac{\partial \mathscr{L}}{\partial F_{\mu\nu}}\,, \tag{1.28}
$$

[3] Here we use the fact that the derivative of an antisymmetric tensor $T_{\mu\nu}$ by itself is:

$$
\frac{\partial T_{\mu\nu}}{\partial T_{\alpha\beta}} = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\,. \tag{1.24}
$$

in order to write the EL equations in the compact form:

$$\partial_\mu \tilde{G}^{\mu\nu} = 0\,. \tag{1.29}$$

Bianchi idenity still holds:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0\,. \tag{1.30}$$

Equations (1.29) and (1.30) are the dynamical equations for a NED written in tensor form. Let us write them in vectorial notation; we do so making an analogy with the Maxwell's theory where EoM are $\partial_\mu F^{\mu\nu} = 0$ and physical fields are $E^i = F^{i0}$ and $B^i = \epsilon^{ijk}F_{jk}/2$. For a NED we introduce the following variables:

$$D^i \equiv \tilde{G}^{0i} = -\frac{\partial \mathscr{L}}{\partial F_{0i}} = -\frac{\partial \mathscr{L}}{\partial E_i}\,, \tag{1.31a}$$

$$H^i \equiv \frac{1}{2}\epsilon^{ijk}\tilde{G}_{jk} = -\frac{1}{2}\epsilon^{ijk}\frac{\partial \mathscr{L}}{\partial F^{jk}} = -\frac{\partial \mathscr{L}}{\partial B_i}\,. \tag{1.31b}$$

D^i, H^i derive from the tensor $\tilde{G}^{\mu\nu}$ in the same way E_i, B_i derive from $F^{\mu\nu}$ in Maxwell electrodynamics. Using vectorial notation:

$$\vec{D} = -\frac{\partial \mathscr{L}}{\partial \vec{E}}\,, \tag{1.32a}$$

$$\vec{H} = -\frac{\partial \mathscr{L}}{\partial \vec{B}}\,. \tag{1.32b}$$

We call the vector $\vec{D}$ electric displacement and vector $\vec{H}$ magnetic field intensity. These equations are called constitutive relations in Lagrangian formalism: they are the link between $\vec{D}, \vec{H}$ and $\vec{E}, \vec{B}$. In fact $\vec{D}, \vec{H}$ are not independent variables, they depend on $\vec{E}, \vec{B}$ as the Lagrangian does. Then the EoM of a NED theory, $\partial_\mu \tilde{G}^{\mu\nu} = 0$, in vectorial form read:

$$\vec{\nabla} \cdot \vec{D} = 0\,, \tag{1.33a}$$

$$\partial_t \vec{D} = \vec{\nabla} \times \vec{H}\,. \tag{1.33b}$$

Together with the Bianchi identity:

$$\vec{\nabla} \cdot \vec{B} = 0\,, \tag{1.34a}$$

$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E}\,, \tag{1.34b}$$

they are the dynamical equations for a NED theory.

1.2.1 Stress-energy tensor

If we build a Poncarè invariant electromagnetic theory and we assume the EoM are the EL equations, the Noether theorem ensures the existence of a conserved tensor that we call the stress-energy tensor [26]:

$$T^{\mu\nu} = F^{\mu\alpha}\tilde{G}^{\nu}_{\alpha} - \eta^{\mu\nu}\mathscr{L}\,. \tag{1.35}$$

Do note this tensor is symmetric and conserved. Let us unpack the tensor $\tilde{G}^{\nu}_{\alpha}$:

$$T^{\mu\nu} = -F^{\mu\alpha}\frac{\partial \mathscr{L}}{\partial F^{\alpha}{}_{\nu}} - \eta^{\mu\nu}\mathscr{L} \tag{1.36a}$$

$$= -F^{\mu\alpha}\left(-\mathscr{L}_S F_\alpha{}^\nu - \mathscr{L}_P \tilde{F}^{\nu}_{\alpha}\right) - \eta^{\mu\nu}\mathscr{L}\,, \tag{1.36b}$$

now we use the identity

$$F^{\mu\alpha}\tilde{F}^{\nu}_{\alpha} = \eta^{\mu\nu}P \tag{1.37}$$

and the stress-energy tensor reads:

$$T^{\mu\nu} = \mathscr{L}_S F^{\mu\alpha}F_{\alpha}{}^{\nu} + \eta^{\mu\nu}(\mathscr{L}_P P - \mathscr{L})\,. \tag{1.38}$$

In this form the symmetry of the tensor is explicit. Regarding its conservation, we prove it using (1.35) since then it easily follows from the EL equations:

$$\partial_\nu T^{\mu\nu} = (\partial_\nu F^{\mu\alpha})\tilde{G}^{\nu}_{\alpha} + F^{\mu\alpha}\partial_\nu\tilde{G}^{\nu}_{\alpha} - \eta^{\mu\nu}\partial_\nu\mathscr{L}\,. \tag{1.39}$$

The first term is a contraction with the antisymmetric tensor $\tilde{G}^{\nu}_{\alpha}$ in which survives only the antisymmetric part of $\partial_\nu F^{\mu\alpha}$ for indices ν,α. The second term is zero because of EoM and the third term can be worked out using the chain rule. So:

$$\partial_\nu T^{\mu\nu} = \frac{1}{2}(\partial^\nu F^{\mu\alpha} - \partial^\alpha F^{\mu\nu})\tilde{G}_{\alpha\nu} + \frac{1}{2}\eta^{\mu\nu}\tilde{G}^{\alpha\beta}\partial_\nu F_{\alpha\beta} \tag{1.40a}$$

$$= \frac{1}{2}(\partial^\nu F^{\mu\alpha} + \partial^\alpha F^{\nu\mu} + \partial^\mu F^{\alpha\nu})\tilde{G}_{\alpha\nu} = 0\,. \tag{1.40b}$$

The result is zero because the quantity in square brackets is the Bianchi identity.

1.2.2 Conformal invariance

A property of relativistic conformal field theories is to have a traceless stress-energy tensor. Indeed if we set the trace of a stress-energy tensor for a NED equal to zero we obtain a constraint on the Lagrangian density: if the constraint is satisfied, the NED theory is conformal invariant. Let us take the trace of $T^{\mu\nu}$ given in equation (1.38)

$$T^{\mu}{}_{\mu} = \mathscr{L}_S F^{\mu\alpha}F_{\alpha\mu} + \eta^{\mu}{}_{\mu}(\mathscr{L}_P P - \mathscr{L}) \tag{1.41a}$$

$$= -\mathscr{L}_S F^{\mu\alpha}F_{\mu\alpha} + 4(\mathscr{L}_P P - \mathscr{L}) \tag{1.41b}$$

$$= 4(S\mathscr{L}_S + P\mathscr{L}_P - \mathscr{L})\,, \tag{1.41c}$$

where we used the definition of S. Hence, a conformal NED must have a Lagrangian density solving the partial differential equation:

$$\mathscr{L} = S\mathscr{L}_S + P\mathscr{L}_P\,, \tag{1.42}$$

which means that the Lagrangian density of a conformal theory must be a homogeneous function of S and P of order one, i.e. $\mathscr{L}(a^{-4}S, a^{-4}P) = a^{-4}\mathscr{L}(S,P)$.

As a proof of this statement let us consider a conformal NED theory with Lagrangian $\mathscr{L}(S,P)$ which, indeed, solves (1.42). Then take a scale transformation $x^\mu \to ax^\mu$ with a a constant parameter (this is the most intuitive conformal transformation). Using a dimensional analysis the 4vector transforms as $A_\mu \to a^{-1}A_\mu$, so the field-strength $F_{\mu\nu} \to a^{-2}F_{\mu\nu}$. Indeed Lorentz invariants transform as $S \to a^{-4}S$ and $P \to a^{-4}P$, and the Lagrangian density $\mathscr{L}(S,P) \to \mathscr{L}(a^{-4}S, a^{-4}P)$. We now show that the new Lagrangian density $\mathscr{L}(a^{-4}S, a^{-4}P)$ still solves equation (1.42) if it is an homogeneous function of S and P of degree one, proving explicitly that the theory is invariant under scale transformations. So:

$$\mathscr{L}(a^{-4}S, a^{-4}P) = a^{-4}S\frac{\partial\mathscr{L}(a^{-4}S, a^{-4}P)}{\partial(a^{-4}S)} + a^{-4}P\frac{\partial\mathscr{L}(a^{-4}S, a^{-4}P)}{\partial(a^{-4}P)} \tag{1.43a}$$

$$a^{-4}\mathscr{L}(S,P) = a^{-4}\mathscr{L}_S(S,P) + a^{-4}\mathscr{L}_P(S,P) \tag{1.43b}$$

$$\mathscr{L}(S,P) = S\mathscr{L}_S + P\mathscr{L}_P\,. \tag{1.43c}$$

The same result can be achieved with an intuitive approach at the level of the action: for a homogeneous Lagrangian the scaling factor a^{-4} is compensated by the rescaling of the integration measure in the action and the theory is scale invariant.

1.2.3 Duality invariance

The electric-magnetic duality symmetry may be a symmetry of the equations of motion and the Bianchi identities of a NED. This idea arose from the Maxwell's theory observing that we can linearly mix the electric and magnetic field and still obtaining the same EoMs and the Bianchi identities. Let us talk more about duality following ref. [27].

Here we consider a general NED theory with Lagrangian density $\mathscr{L} = \mathscr{L}(F^{\mu\nu})$. Let us recall the EoM and Bianchi identity:

$$\partial_\mu \tilde{G}^{\mu\nu} = 0\,, \tag{1.44a}$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0\,, \tag{1.44b}$$

where

$$\tilde{G}^{\mu\nu} \equiv -\frac{\partial \mathscr{L}}{\partial F_{\mu\nu}}\,. \tag{1.45}$$

From here on we will adopt an abuse of notation for which tensors $F^{\mu\nu}, G^{\mu\nu}$ and their duals lose the indices, i.e. we write them easily as F, G and $\tilde{F}, \tilde{G}$, in order to make formulas look simpler.

Equations (1.44) are invariant under a general linear transformation $GL(2,\mathbb{R})$ in the plane $(\tilde{G}, \tilde{F})$ since the derivative is a linear operator, in fact if we consider an element of the group of general linear transformations 2×2 with real values:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})\,, \tag{1.46}$$

then tensors $(\tilde{G}, \tilde{F})$ transform as:

$$\begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} \to \begin{pmatrix} \tilde{G}' \\ \tilde{F}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} \tag{1.47}$$

and EoM for $\tilde{G}'$ and Bianchi identity for $\tilde{F}'$ are equivalent in form to (1.44):

$$\partial_\mu \tilde{G}'^{\mu\nu} = 0\,, \tag{1.48a}$$
$$\partial_\mu \tilde{F}'^{\mu\nu} = 0\,. \tag{1.48b}$$

Same result holds for G', F' since the Hodge dual is a linear transformation. This property is what we call "duality symmetry", however it is not a symmetry in the usual sense but refers to an ambiguity in our theoretical description: what one calls electric and magnetic is a matter of choice (at least in the absence of charged sources as we are doing here). For a NED theory one should ask that an element of $GL(2,\mathbb{R})$ leaves invariant in form also (1.45), meaning:

$$\tilde{G}' = -\frac{\partial \mathscr{L}(F')}{\partial F'}\,, \tag{1.49}$$

and this condition is a constraint for Lagrangians $\mathscr{L}(F)$ and parameters of the transformation. We can work out the constraint considering only NED theories that reduce to Maxwell's theory in the weak-field limit:

$$\mathscr{L}(F) \to \mathscr{L}_{\mathrm{M}} = -F^2/4\,. \tag{1.50}$$

Then the constraint which holds for $\mathscr{L}(F)$ must hold for $\mathscr{L}_{\mathrm{M}}(F)$ also. So let us apply a duality transformation to Maxwell's theory: first of all, do note for Maxwell's theory $\tilde{G} = F$, then applying a linear transformation we obtain a Lagrangian density $\mathscr{L}_{\mathrm{M}}(F') = -F'F'/4$ and a new tensor $\tilde{G}'$ and we require they satisfy the duality-invariant condition (1.49):

$$\tilde{G}' = -\frac{\partial(-F'F'/4)}{\partial F'} \tag{1.51a}$$
$$= F' \,. \tag{1.51b}$$

Recalling how tensor $\tilde{G}'$ and F' transform under (1.46) we work out the constrain:

$$\tilde{G}' = F' \tag{1.52a}$$
$$a\tilde{G} + b\tilde{F} = cG + dF \tag{1.52b}$$
$$a\tilde{G} = (b+c)G + dF \,, \tag{1.52c}$$

now we use the relation $F = \tilde{G}$ and its dual version $\tilde{F} = -G$:

$$a\tilde{G} - bG = cG + d\tilde{G} \tag{1.53a}$$
$$\tilde{G}(a-d) = G(b+c) \,. \tag{1.53b}$$

This system has a solution iff

$$a = d \,, \tag{1.54a}$$
$$b = -c \,. \tag{1.54b}$$

It means the duality group of a NED theory which reduces to Maxwell theory in the weak-field limit is a subgroup of $GL(2,\mathbb{R})$ having elements:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} . \tag{1.55}$$

Elements of this form are compositions of dilatations and $SO(2)$ rotations. Dilatations are scale-transformations performed multiplying $(\tilde{G}, \tilde{F})$ by a constant a using the matrix $a\mathbb{I}_{2\times 2}$; while $SO(2)$ rotations contain elements of the form (1.55) having determinant equal to 1, i.e. $a^2 + b^2 = 1$. A nontrivial solution of $\det = 1$ is given by trigonometric functions:

$$a = \cos\varphi \,, \tag{1.56a}$$
$$b = \sin\varphi \,, \tag{1.56b}$$

with $\varphi \in (0, 2\pi)$ (where we consider an open interval since at the boundary the transformation is an identity).

So, depending on the Lagrangian, one can have either an $SO(2)$-invariant theory or scale-invariant theory or both. Scale transformations are a subgroup of conformal transformations and we have seen there exist infinite conformal-invariant NEDs as solutions of the constraint (1.42). Regarding $SO(2)$-invariant NEDs, the constraint (1.49) must be satisfied by a transformation of the form

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \in SO(2) \,. \tag{1.57}$$

We consider an infinitesimal transformation for simplicity:

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \approx \begin{pmatrix} 1 & \varphi \\ -\varphi & 1 \end{pmatrix} \quad \text{for} \quad \varphi \ll 1 \,. \tag{1.58}$$

Then tensors $(\tilde{G}, \tilde{F})$ transform as:

$$\begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} \to \begin{pmatrix} \tilde{G}' \\ \tilde{F}' \end{pmatrix} = \begin{pmatrix} 1 & \varphi \\ -\varphi & 1 \end{pmatrix} \begin{pmatrix} \tilde{G} \\ \tilde{F} \end{pmatrix} = \begin{pmatrix} \tilde{G} + \varphi \tilde{F} \\ -\varphi \tilde{G} + \tilde{F} \end{pmatrix} . \tag{1.59}$$

Same form will hold for (G, F). Therefore condition (1.49) for the infinitesimal transformation reads:

$$\tilde{G} + \varphi \tilde{F} = -2 \frac{\partial \mathscr{L}(F - \varphi G)}{\partial (F - \varphi G)} \tag{1.60a}$$

$$= -2 \left[\frac{\partial \mathscr{L}(F)}{\partial F} + \frac{\mathrm{d}}{\mathrm{d}(F - \varphi G)} \left(\frac{\partial \mathscr{L}(F - \varphi G)}{\partial (F - \varphi G)} \right) \Bigg|_{\varphi G = 0} (-\varphi G) \right] \tag{1.60b}$$

$$= \tilde{G} - \varphi G \left. \frac{\mathrm{d}\tilde{G}'}{\mathrm{d}F'} \right|_{\varphi G = 0} \tag{1.60c}$$

$$= \tilde{G} - \varphi G \frac{\partial \tilde{G}}{\partial F} , \tag{1.60d}$$

where we have expanded the RHS near F up to first order. Simplifying terms we get:

$$\tilde{F} = -G \frac{\partial \tilde{G}}{\partial F} . \tag{1.61}$$

We can simplify more the equation noting we deal with tensors and their duals obtained using the total antisymmetric Levi-Civita tensor, so a derivative of terms like $\tilde{F}F$ and $\tilde{G}G$, with respect to the field strength, will give a sum of two terms because of the chain rule that should be equal because of antisymmetricity. In fact it holds:

$$\frac{\partial (\tilde{F}F)}{\partial F} = 2\tilde{F} , \tag{1.62a}$$

$$\frac{\partial (\tilde{G}G)}{\partial F} = 2G \frac{\partial \tilde{G}}{\partial F} , \tag{1.62b}$$

where factor 2 is what we expected. Now equation (1.61) can be written as a differential equation with respect to F on both sides:

$$\frac{\partial (\tilde{F}F)}{\partial F} = -\frac{\partial (\tilde{G}G)}{\partial F} . \tag{1.63}$$

Integrating out we find:

$$F\tilde{F} = -G\tilde{G} + \mathrm{const} , \tag{1.64}$$

where the integration constant can be set to zero since we are considering NED theories with Maxwell theory as low-energy limit, and equation (1.64) for Maxwell theory implies const $= 0$. So, NEDs with $SO(2)$-invariance and with Maxwell as low-energy limit must solve the constraint:

$$F\tilde{F} + G\tilde{G} = 0 . \tag{1.65}$$

Rewriting indices and recalling the definition of $\tilde{G}$:

$$F_{\mu\nu} \tilde{F}^{\mu\nu} - 2\epsilon^{\mu\nu\lambda\rho} \frac{\partial \mathscr{L}}{\partial F^{\mu\nu}} \frac{\partial \mathscr{L}}{\partial F^{\lambda\rho}} = 0 . \tag{1.66}$$

This result was firstly derived by Bialynicki-Birula [24] and then by Gibbons & Rasheed [28]. If we also assume the NED theory to be Poincarè-invariant, i.e. having a Lagrangian density $\mathscr{L} = \mathscr{L}(S, P)$, the constraint reads:

$$P\big(\mathscr{L}_P^2 - \mathscr{L}_S^2 + 1\big) + 2S\mathscr{L}_P\mathscr{L}_S = 0\,. \tag{1.67}$$

We have talked about the duality symmetry at the level of the EoM for a source-less theory, however it is not trivial to generalize it, both trying to extend the duality to the level of the action and considering sources.

1.2.4 Hamiltonian formalism

Up to now we have worked in a Lagrangian framework, obtaining the EoM as the EL equations of the theory. Our purpose now is to reformulate an electromagnetic theory with Lagrangian $\mathscr{L}$ in the Hamiltonian formalism. The first step is to change the independent variables we want to work with:

$$\left(\vec{E}, \vec{B}\right) \to \left(\vec{D}, \vec{B}\right), \tag{1.68}$$

where we recall $\vec{D}$ is the electric displacement. The electric field can be written as a function of new variables, i.e. $\vec{E} = \vec{E}(\vec{D}, \vec{B})$, inverting the constitutive relation (1.32a). Then we define the Hamiltonian density as the Legendre transform of the Lagrangian density:

$$\mathscr{H}\left(\vec{D}, \vec{B}\right) \equiv \vec{D}\cdot\vec{E} - \mathscr{L}\left(\vec{D}, \vec{B}\right).$$

The Legendre transform holds as long as $\mathscr{L}$ is a strictly convex function of $\vec{E}$. The analogue of the constitutive relations in the Hamiltonian formalism are:

$$\vec{E} = \frac{\partial\mathscr{H}}{\partial\vec{D}}\,, \tag{1.69}$$

$$\vec{H} = \frac{\partial\mathscr{H}}{\partial\vec{B}}\,. \tag{1.70}$$

They are the link between $\vec{D}, \vec{B}$ and $\vec{E}, \vec{H}$.

As an example, let us compute the Hamiltonian for the Maxwell's theory: from the constitutive relations (1.32) we find $\vec{D} = \vec{E}$ and $\vec{H} = \vec{B}$, then the Legendre transform of the Maxwell Lagrangian density reads:

$$\mathscr{H}_{\mathrm{M}} = \vec{D}\cdot\vec{B} - \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) \tag{1.71a}$$

$$= \vec{D}^2 - \frac{1}{2}\left(\vec{D}^2 - \vec{B}^2\right) \tag{1.71b}$$

$$= \frac{1}{2}\left(\vec{D}^2 + \vec{B}^2\right). \tag{1.71c}$$

Let us now review the stress-energy tensor and conformal and duality symmetries using the Hamiltonian formalism.

Stress-energy tensor

Let us recall the stress-energy tensor $T^{\mu\nu}$, using the constitutive relations we can write its components in a compact way:

$$T^{00} = -\vec{E}\cdot\vec{D} + \mathscr{L}\,, \tag{1.72a}$$
$$T^{0i} = \left(\vec{H}\times\vec{E}\right)^i, \tag{1.72b}$$
$$T^{i0} = \left(\vec{B}\times\vec{D}\right)^i, \tag{1.72c}$$
$$T^{ij} = E^iD^j + H^iB^j - \delta^{ij}\left(\mathscr{L} + \vec{H}\cdot\vec{B}\right). \tag{1.72d}$$

Do note the Hamiltonian density is exactly $\mathscr{H} = -T^{00}$. Since we are building a Poincarè-invariant theory with a symmetric stress-energy tensor, the following constraints must hold:

$$\vec{H}\times\vec{E} = \vec{B}\times\vec{D}\,, \tag{1.73a}$$
$$E^iD^j + H^iB^j = E^jD^i + H^jB^i\,. \tag{1.73b}$$

The first constraint comes from $T^{0i} = T^{i0}$ which holds because of Lorentz-boost invariance, the second constraint comes from $T^{ij} = T^{ji}$ which holds because of spatial-rotational invariance. Do note equation (1.73b) is an identity since for diagonal terms (i.e. $i = j$) the equation is identically zero, while for off-diagonal terms (i.e. $i \neq j$) one has to remember the definitions of $\vec{D} = \partial\mathscr{L}\big/\partial\vec{E}$ and $\vec{H} = -\,\partial\mathscr{L}\big/\partial\vec{B}$ implying $\vec{D}\parallel\vec{E}$ and $\vec{H}\parallel\vec{B}$ and so off-diagonal terms are zero by construction. Therefore, we can think to constraint (1.73a) to be the condition a NED must satisfy to be a relativistic theory.

The constraint (1.73a) can be restated using the Hamiltonian formalism, i.e. adopting the variables $\vec{D},\vec{B}$. The idea is to introduce the following functions:

$$s = \frac{1}{2}\left(\vec{D}^2 + \vec{B}^2\right), \tag{1.74}$$
$$\xi = \frac{1}{2}\left(\vec{D}^2 - \vec{B}^2\right), \tag{1.75}$$
$$\eta = \vec{D}\cdot\vec{B}\,, \tag{1.76}$$

and considering $\mathcal{H} = \mathcal{H}(s,\xi,\eta)$. Then if one compute:

$$\frac{\partial\mathscr{H}}{\partial\vec{D}} = \mathscr{H}_s\vec{D} + \mathscr{H}_\xi\vec{D} + \mathscr{H}_\eta\vec{B} = \vec{E}\,, \tag{1.77}$$
$$\frac{\partial\mathscr{H}}{\partial\vec{B}} = \mathscr{H}_s\vec{B} - \mathscr{H}_\xi\vec{B} + \mathscr{H}_\eta\vec{D} = -\vec{H}\,, \tag{1.78}$$

the quantity $\vec{H}\times\vec{E}$ reads:

$$\vec{H}\times\vec{E} = \left(\mathscr{H}_s^2 - \mathscr{H}_\xi^2 - \mathscr{H}_\eta^2\right)\vec{B}\times\vec{D} \tag{1.79}$$

and because of the constraint (1.73a), it must hold:

$$\mathscr{H}_s^2 - \mathscr{H}_\xi^2 - \mathscr{H}_\eta^2 = 1\,. \tag{1.80}$$

Conformal invariance

Having the components of the stress-energy tensor and the definition of the Hamiltonian density, we can write the condition for a conformal theory in the Hamiltonian formalism:

$$
\begin{aligned}
T^{\mu}{}_{\mu} &= T^{00}\eta_{00} + T^{ik}\delta_{ki} && (1.81a)\\
&= \mathscr{H} + \vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B} - 3\Big(\mathscr{L} + \vec{H}\cdot\vec{B}\Big) && (1.81b)\\
&= \mathscr{H} + \vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B} - 3\Big(\vec{D}\cdot\vec{E} - \mathscr{H} + \vec{H}\cdot\vec{B}\Big) && (1.81c)\\
&= 4\mathscr{H} - 2\vec{E}\cdot\vec{D} - 2\vec{H}\cdot\vec{B} && (1.81d)
\end{aligned}
$$

and the requirement of a traceless stress-energy tensor is satisfied if:

$$
\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B} = 2\mathscr{H}\,. \tag{1.82}
$$

Duality invariance

Let us now comment on the duality symmetry in the Hamiltonian formalism $\mathscr{H} = \mathscr{H}(\vec{D},\vec{B})$. One can observe that there are two $SO(2)$-invariant scalars:

$$
s = \frac{1}{2}\left(\left|\vec{D}\right|^2 + \left|\vec{B}\right|^2\right), \tag{1.83}
$$

$$
p = \left|\vec{D}\times\vec{B}\right|. \tag{1.84}
$$

Thus if the Hamiltonian $\mathscr{H}$ is a function of (s,p) the theory will be duality invariant.

Let us see when a theory with $\mathscr{H} = \mathscr{H}(s,p)$ is Poincarè-invariant. Recall the constraint to satisfy is (1.73a):

$$
\vec{H}\times\vec{E} = \vec{D}\times\vec{B}\,. \tag{1.85}
$$

Using the definitions of s,p and the constitutive relations in the Hamiltonian formalism,

$$
\vec{E} = \frac{\partial\mathscr{H}}{\partial\vec{D}} = \mathscr{H}_s\frac{\partial s}{\partial\vec{D}} + \mathscr{H}_p\frac{\partial p}{\partial\vec{D}}\,, \tag{1.86}
$$

$$
\vec{H} = \frac{\partial\mathscr{H}}{\partial\vec{H}} = \mathscr{H}_s\frac{\partial s}{\partial\vec{H}} + \mathscr{H}_p\frac{\partial p}{\partial\vec{H}}\,, \tag{1.87}
$$

the constraint (1.73a) implies:

$$
\mathscr{H}_s^2 + 2\frac{s}{p}\mathscr{H}_s\mathscr{H}_p + \mathscr{H}_p^2 = 1\,. \tag{1.88}
$$

Here subscripts denote partial derivatives. This is a rewriting of the condition that an $SO(2)$-invariant NED theory having Hamiltonian density $\mathscr{H} = \mathscr{H}(s,p)$ must satisfy in order to be Poincarè-invariant.

Do note p is $SL(2,\mathbb{R})$ electromagnetic duality-invariant, so a theory with Hamiltonian density $\mathscr{H} = \mathscr{H}(p)$ will be $SL(2,\mathbb{R})$-invariant. In this case the condition (1.88) reduces to $\mathscr{H}_p^2 = 1$ with solutions $\mathscr{H} = \pm p$. Choosing the positive sign, we have the Bialynicki-Birula electrodynamics.

An alternative basis to (s,p) for $SO(2)$ duality-invariance is (u,v) with:

$$
u = \frac{1}{2}\left(s + \sqrt{s^2 - p^2}\right), \tag{1.89}
$$

$$
v = \frac{1}{2}\left(s - \sqrt{s^2 - p^2}\right). \tag{1.90}
$$

Notice they are well defined since $s^2 - p^2 = \xi^2 + \eta^2 \geq 0$ holds everywhere [29]. Condition (1.88) in the new basis reads [29]:

$$\mathscr{H}_u \mathscr{H}_v = 1\,. \tag{1.91}$$

In this section we have found the condition for a relativistic theory to be conformal-invariant in equation (1.42), and condition to be $SO(2)$-invariant in equation (1.65). In principle there exist infinite number of Lagrangian densities solving either (1.42) or (1.65), but there exist only two NEDs being conformal and $SO(2)$ invariant *simultaneously*: the Bialynicki-Birula theory and the ModMax theory. We will tell more about ModMax in the next section.

1.3 ModMax theory

In 2020 Bandos, Lechner, Sorokin and Townsend discovered a NED theory being both conformal and duality invariant which they called ModMax. In the article [12] they took a generic NED Hamiltonian and imposed conditions for both conformal and duality invariance. They found two solutions: one yealds to the BB theory, the other yealds to ModMax. ModMax theory is remarkable since it is a one-parameter generalization of Maxwell electrodynamics and only when the parameter, denoted by γ, goes to zero Maxwell theory is restored. This is in contrast to the BB theory that does not reduce to Maxwell theory in any limit. When $\gamma \neq 0$ EoM of ModMax are nonlinear.

Let us now report the main results obtained in [12]. To have an $SO(2)$ electromagnetic duality invariant theory the Hamiltonian density was considered to be $\mathscr{H} = \mathscr{H}(s, p)$, or equivalently $\mathscr{H} = \mathscr{H}(u, v)$. To have a Poincarè-invariant theory, the Hamiltonian density must be a solution of (1.88) if we use (s, p) basis, or equivalently a solution of (1.91) in (u, v) basis. A solution of the Poincarè-invariance condition is an Hamiltonian density of the form:

$$\mathscr{H} = \sqrt{K} + \text{const}\,, \tag{1.92}$$

where K is a scalar function. E.g. considering $K = K(u, v)$ equation (1.91) reads $K_u K_v = 4K$.

Considering a quadratic K, the general solution for which K is non-negative for all (u, v) gives a Hamiltonian density depending on one parameter T with dimensions of an energy density and a dimensionless parameter γ. Assuming zero vacuum energy one fixes the additive constant, and the result Hamiltonian density is:

$$\mathscr{H} = \sqrt{T^2 + 2T[\exp(-\gamma)u + \exp(\gamma)v] + 4uv} - T\,, \tag{1.93}$$

A more detailed derivation is exposed in [29] section 4. Interesting limits of (1.93) are the following:

- when $\gamma = 0$ the Hamiltonian density reduces to the one of BI electrodynamics, indeed we can say (1.93) is a one-parameter generalization of the BI Hamiltonian density;
- the strong-field limit $T \to 0$ yields the Hamiltonian density $\mathscr{H} = p$ of BB electrodynamics;
- the weak-field limit $T \to \infty$ yields the so-called ModMax Hamiltonian density.

The ModMax Hamiltonian density ($T \to \infty$) is:

$$\begin{aligned} \mathscr{H}_{\text{MM}} &= (\cosh\gamma)s - (\sinh\gamma)\sqrt{s^2 - p^2} && (1.94a)\\ &= \frac{1}{2}\left(\cosh\gamma\left(\vec{D}^2 + \vec{B}^2\right) - \sinh\gamma\sqrt{\left(\vec{D}^2 + \vec{B}^2\right)^2 - 4\left(\vec{D}\times\vec{B}\right)^2}\right). && (1.94b) \end{aligned}$$

Do note for any value of γ the Hamiltonian density is conformal invariant. The Maxwell Hamiltonian density is recovered for $\gamma = 0$: ModMax electrodynamics is indeed the one-parameter extension of Maxwell electrodynamics being Lorentz, conformal and electric-magnetic duality invariant. ModMax and BB theories are the only electromagnetic theories with these symmetries. There may be other solutions of (1.88) (or equivalently (1.91)) corresponding to other duality and Lorentz invariant theories, but they are not conformal invariant. This is explicitly proved in [12].

Hamiltonian density (1.94) is well defined for null-field configurations, where it reduces to BB Hamiltonian density:

$$\mathscr{H}_{\text{MM}}\Big|_{\text{null-fields}} = \left|\vec{D}\times\vec{B}\right|. \tag{1.95}$$

Among these configurations plane waves [12] and topologically non-trivial knotted electromagnetic fields [30] are exact solutions of ModMax field equations. However, the linear superposition of plane waves is not a solution because of the non-linearity of the electrodynamics equations. The conditions of causality and unitarity require the parameter γ to be non-negative: $\gamma \geq 0$. The Hamiltonian density (1.94) for $\gamma > 0$ is not a convex function of $\vec{D}$ for all values of $(\vec{D}, \vec{B})$, so the Legendre transform of $\mathscr{H}(\vec{D}, \vec{B})$ with respect to $\vec{D}$ must be carried out only within the domain of $\mathscr{H}(\vec{D}, \vec{B})$ in which the Hamiltonian density is convex. This results in the following Lagrangian density of ModMax [12]:

$$\begin{aligned} \mathscr{L}_{\text{MM}} &= \frac{\cosh\gamma}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \frac{\sinh\gamma}{2}\sqrt{\left(\vec{E}^2 - \vec{B}^2\right)^2 + 4\left(\vec{E}\cdot\vec{B}\right)^2} && (1.96a)\\ &= \cosh\gamma S + \sinh\gamma\sqrt{S^2 + P^2}. && (1.96b) \end{aligned}$$

As it was for the ModMax Hamiltonian density, the Maxwell Lagrangian density is recovered when $\gamma \to 0$: there is no weak-field limit that reduces ModMax to Maxwell theory because of conformal invariance.

Using the variational principle, one obtains the Euler-Lagrange field equations:

$$\cosh\gamma\partial_\mu F^{\mu\nu} + \sinh\gamma\partial_\mu\left(\frac{SF^{\mu\nu} + P\tilde{F}^{\mu\nu}}{\sqrt{S^2 + P^2}}\right) = 0\,, \tag{1.97}$$

and together with Bianchi identitiy $\partial_\mu \tilde{F^{\mu\nu}} = 0$ they are a set of field equations for free ModMax theory. EoM are not well defined for null field configurations $S = P = 0$, while there are no porblems in the Hamiltonian formulation. Configurations of the type $|S| = c|P|$, where c is a constant, linearize the EoM allowing solutions of Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ with $|S| = c|P|$ be also solutions of ModMax equations.

If one takes the Legendre transform of the most-general Poincarè and $SO(2)$ invariant Hamiltonian density (1.93) with respect to $\vec{D}$ (under the assumption of a suitable domain where the Hamiltonian density is convex), one obtains the following Lagrangian density [29]:

$$\mathscr{L}_{\text{MM-BI}} = T - \sqrt{T^2 - 2T\mathscr{L}_{\text{MM}} - \frac{1}{16}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2}\,. \tag{1.98}$$

As was for the Hamiltonian density (1.93), the theory described by such Lagrangian is a generalized BI theory whose weak-field limit is ModMax (instead of Maxwell at which the BI theory (1.10) tends) and the strong-field limit is BB electrodynamics.

Chapter 2

Vacuum birefringence in NED theories

Almost all NEDs show birefringence property when ligth propagates in an electromagnetic background (in the following we will show why, see [9, 31–34] for some examples). In optical physics, birefringence is a double refraction of a light ray hitting a material: the ray is split by polarization, with respect to the optical axis of the material, into two rays taking different geodesics. In NED theories the electromagnetic background plays the role of the optical material and a small perturbation as an electromagnetic plane wave propagating in the background could split by polarization, this phenomenon is called (electromagnetic) vacuum birefringence. It is a general property of NED theories since their EoM is non-linear by construction and so superposition of waves is no more a solution as in the case of linear theories. Thus, when we have a plane wave on the top of a background we expect interactions to appear between the waves and the background in a NED theory [13, 35].

Different experiments were proposed and driven to observe the vacuum birefringence; we cite the PVLAS (Polarisation of Vacuum with LASER) collaboration in which a laser beam is shot into a Fabry-Perot cavity to increase the optical path, with a magnetic field background. PVLAS carried out 25 years of efforts in the search for the vacuum birefringence and dichroism, and it established the best limits on theories' parameters known so far [36]. In the astrophysical framework physicists take advantage of the strong magnetic field of neutron stars to catch vacuum birefringence, see e.g. [37].

We are going to open the chapter showing the effect of the nonlinearity of a classical NED in the dispersion relation of a plane wave propagating in an electromagnetic background. Then we will discuss the polarization theory in order to set up the formalism we will use in next chapters.

2.1 General treatment

Let us consider the general NED Lagrangian $\mathscr{L} = \mathscr{L}(S, P)$ in a Minkowski spacetime describing a system with an electromagnetic background and a small perturbation, as a wavefront, on top of the background; being general, we start considering a spacetime dependent background.

In terms of the 4vector potential, we consider:

$$A^\mu(x,t) = A^\mu_{\mathrm{B}}(x,t) + a^\mu(x,t)\,, \tag{2.1}$$

where A^μ_{B} refers to the electromagnetic background component and a^μ to the photon field perturbation. Given such a decomposition, the field-strength tensor has the following form:

$$F^{\mu\nu} = F_{\mathrm{B}}^{\mu\nu} + f^{\mu\nu} \tag{2.2a}$$
$$= \left(\partial^\mu A_{\mathrm{B}}^\nu - \partial^\nu A_{\mathrm{B}}^\mu\right) + \left(\partial^\mu a^\nu - \partial^\nu a^\mu\right). \tag{2.2b}$$

To see non-linear phenomena, it is sufficient to expand the NED Lagrangian density around the background up to the second order in a^μ. Doing so:

$$\mathscr{L}(S,P) = \mathscr{L}(S_{\mathrm{B}},P_{\mathrm{B}}) + \left.\frac{\partial \mathscr{L}}{\partial F_{\mu\nu}}\right|_{\mathrm{B}} f_{\mu\nu} + \frac{1}{2}\left.\frac{\partial^2 \mathscr{L}}{\partial F_{\mu\nu}\partial F_{\alpha\beta}}\right|_{\mathrm{B}} f_{\mu\nu} f_{\alpha\beta} + \mathcal{O}\left(f^3\right). \tag{2.3}$$

Quantities with subscript "B" are computed at the electromagnetic background. Let us use the notation of [38] to denote partial derivatives of the Lagrangian with respect to S, P:

$$c_1 = \left.\frac{\partial \mathscr{L}}{\partial S}\right|_{\mathrm{B}}, \quad c_2 = \left.\frac{\partial \mathscr{L}}{\partial P}\right|_{\mathrm{B}}, \qquad d_1 = \left.\frac{\partial^2 \mathscr{L}}{\partial S^2}\right|_{\mathrm{B}}, \qquad d_2 = \left.\frac{\partial^2 \mathscr{L}}{\partial P^2}\right|_{\mathrm{B}}, \qquad d_3 = \left.\frac{\partial^2 \mathscr{L}}{\partial S \partial P}\right|_{\mathrm{B}}, \tag{2.4}$$

also the following tensors are introduced in order to write the Lagrangian in a compact form:

$$G_{\mathrm{B}}^{\mu\nu} = c_1 F_{\mathrm{B}}^{\mu\nu} + c_2 \tilde{F}_{\mathrm{B}}^{\mu\nu}, \tag{2.5}$$
$$Q_{\mathrm{B}}^{\mu\nu\kappa\lambda} = d_1 F_{\mathrm{B}}^{\mu\nu} F_{\mathrm{B}}^{\kappa\lambda} + d_2 \tilde{F}_{\mathrm{B}}^{\mu\nu} \tilde{F}_{\mathrm{B}}^{\kappa\lambda} + d_3 F_{\mathrm{B}}^{\mu\nu} \tilde{F}_{\mathrm{B}}^{\kappa\lambda} + d_3 \tilde{F}_{\mathrm{B}}^{\mu\nu} F_{\mathrm{B}}^{\kappa\lambda}. \tag{2.6}$$

So the Lagrangian density reads:

$$\mathscr{L}(S,P) = \mathscr{L}(S_{\mathrm{B}},P_{\mathrm{B}}) - \frac{1}{2} G_{\mathrm{B}}^{\mu\nu} f_{\mu\nu} - \frac{1}{4}\left[c_1 f_{\mu\nu} f^{\mu\nu} + c_2 f_{\mu\nu} \tilde{f}^{\mu\nu} - \frac{1}{2} Q_{\mathrm{B}}^{\mu\nu\kappa\lambda} f_{\mu\nu} f_{\kappa\lambda}\right]. \tag{2.7}$$

Varying this Lagrangian with respect to a_μ and assuming that the background satisfies the EL equations of motion one gets the EL equations for the photon field:

$$0 = \frac{\partial \mathscr{L}}{\partial a_\nu} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu a_\nu)} \tag{2.8a}$$
$$= \partial_\mu \left[c_1 f^{\mu\nu} + c_2 \tilde{f}^{\mu\nu} - \frac{1}{2} Q_{\mathrm{B}}^{\mu\nu\kappa\lambda} f_{\kappa\lambda} + G_{\mathrm{B}}^{\mu\nu}\right]. \tag{2.8b}$$

The photon field also satisfies the Bianchi identity:

$$\partial_\mu \tilde{f}^{\mu\nu} = 0. \tag{2.9}$$

Do note for any theory depending on S only, coefficients c_2, d_2, d_3 are all vanishing. The simplest example is the Maxwell's theory for which $\mathscr{L}_{\mathrm{M}} = S$ and thus $c_1 = 1$ and $d_1 = 0$. If the Lagrangian depends also on P, one usually assumes the dependence on even powers of P in order to preserve the parity symmetry.

For simplicity we consider a uniform and constant external magnetic field $\vec{B}$ and no electric field. As a consequence, coefficients (2.4) are also uniform and constant in time; in addition whenever electric field is zero and we work with even powers of P, the coefficient $d_3 = 0$.

We unpack EoM for the perturbation (2.8) writing them in vectorial notation where we denote the photon electric and magnetic field as $\vec{e}, \vec{b}$ and the background as $\vec{B}$:

$$\vec{\nabla} \cdot \vec{e} + \frac{d_2}{c_1} \vec{B} \cdot \vec{\nabla}\left(\vec{B} \cdot \vec{e}\right) = 0, \tag{2.10a}$$
$$\vec{\nabla} \times \vec{b} + \frac{d_1}{c_1} \vec{B} \times \vec{\nabla}\left(\vec{B} \cdot \vec{b}\right) = \partial_t \vec{e} + \frac{d_2}{c_1} \vec{B} \partial_t \left(\vec{B} \cdot \vec{e}\right). \tag{2.10b}$$

These are a scalar EoM (2.10a) and a vectorial EoM (2.10b). The Bianchi identities are

$$\vec{\nabla} \cdot \vec{b} = 0\,, \tag{2.11a}$$
$$\vec{\nabla} \times \vec{e} + \partial_t \vec{b} = 0\,. \tag{2.11b}$$

Let us consider plane wave solutions:

$$\vec{e}(\vec{x}, t) = \vec{e}_0 \exp\Big[i\Big(\vec{k} \cdot \vec{x} - \omega t\Big)\Big]\,, \tag{2.12}$$
$$\vec{b}(\vec{x}, t) = \vec{b}_0 \exp\Big[i\Big(\vec{k} \cdot \vec{x} - \omega t\Big)\Big]\,, \tag{2.13}$$

where the components of the amplitude vectors $\vec{e}_0, \vec{b}_0$ are constant complex vectors. Inserting the plane wave ansatz into the EoM we derive the photon dispersion relation. From the Bianchi idensity we get:

$$-i\omega\vec{b} = i\vec{k} \times \vec{e}\,. \tag{2.14}$$

This means that the magnetic field $\vec{b}$ is orthogonal to $\vec{k}$, but we cannot say the same for the electric field $\vec{e}$ as normally happens in Maxwell electrodynamics. The non-orthogonality of $\vec{e}$ with respect to $\vec{k}$ is a consequence of the nonlinearity of the theory and it is explicit looking at (2.10a) where the divergence of $\vec{e}$ is different from zero because of $d_2 \neq 0$ in a NED theory.

We now substitute plane waves $\vec{e}$ and $\vec{b} = -(\vec{k} \times \vec{e})/\omega$ into the system (2.10). In this way we will have a system for the electric field only. The scalar EoM (2.10a) reads:

$$i\vec{k} \cdot \vec{e} + i\frac{d_2}{c_1}\vec{B} \cdot \vec{k}\Big(\vec{B} \cdot \vec{e}\Big) = 0\,. \tag{2.15}$$

While the vectorial EoM (2.10b) reads:

$$i\vec{k} \times \frac{\vec{k} \times \vec{e}}{\omega} + i\frac{d_1}{c_1}\vec{B} \times \vec{k}\left(\vec{B} \cdot \frac{\vec{k} \times \vec{e}}{\omega}\right) = -i\omega\vec{e} - i\omega\frac{d_2}{c_1}\vec{B}\Big(\vec{B} \cdot \vec{e}\Big)\,. \tag{2.16}$$

The LHS can be worked out using vectorial identities:

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)\,, \tag{2.17a}$$
$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)\,, \tag{2.17b}$$

where A, B, C are vectors. So the equation reads:

$$\Big[\vec{k} \cdot \Big(\vec{k} \cdot \vec{e}\Big) - k^2\vec{e}\Big] + \frac{d_1}{c_1}\vec{B} \times \vec{k}\Big[\Big(\vec{B} \times \vec{k}\Big) \cdot \vec{e}\Big] = -\omega^2\vec{e} - \omega^2\frac{d_2}{c_1}\vec{B}\Big(\vec{B} \cdot \vec{e}\Big)\,. \tag{2.18}$$

Thus, the system of EoM reads:

$$\Big[\vec{k} \cdot \Big(\vec{k} \cdot \vec{e}\Big) - k^2\vec{e}\Big] + \frac{d_1}{c_1}\vec{B} \times \vec{k}\Big[\Big(\vec{B} \times \vec{k}\Big) \cdot \vec{e}\Big] = -\omega^2\vec{e} - \omega^2\frac{d_2}{c_1}\vec{B}\Big(\vec{B} \cdot \vec{e}\Big)\,, \tag{2.19a}$$
$$\vec{k} \cdot \vec{e} + \frac{d_2}{c_1}\vec{B} \cdot \vec{k}\Big(\vec{B} \cdot \vec{e}\Big) = 0\,. \tag{2.19b}$$

Let us consider a configuration in which the wave-vector is orthogonal to the external magnetic field, namely $\vec{B} \cdot \vec{k} = 0$. This simplifies the computation since in this case the scalar EoM reads:

$$\vec{k} \cdot \vec{e} = 0\,, \tag{2.20}$$

namely the electric field spans together with $\vec{B}$ a plane orthogonal to $\vec{k}$. We can decompose the electric field into its parallel and orthogonal component with respect to the external magnetic field $\vec{B}$, respectively $\vec{e}_\parallel$ and $\vec{e}_\perp$:

$$\vec{e} = \vec{e}_\parallel + \vec{e}_\perp\,. \tag{2.21}$$

Let us choose a reference frame in order to assign labels to vectorial components: assume the external magnetic field $\vec{B}$ is oriented along z-axis and the wave vector $\vec{k}$ of the perturbation is along x-axis, then $\vec{e}_\parallel$ will be along z-axis and $\vec{e}_\perp$ will be along y-axis. In this reference frame the vectorial EoM reads:

$$\begin{pmatrix} 0 \\ (\omega_\perp^2 - k^2)e_\perp \\ \left(\omega_\parallel^2 - k^2\right)e_\parallel \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{d_1}{c_1}(Bk)^2 e_\perp \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \omega^2 \frac{d_2}{c_1} B^2 e_\parallel \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.22}$$

Indeed fields $\vec{e}_\parallel$ and $\vec{e}_\perp$ oscillate with different frequencies. The component of the electric field orthogonal to $\vec{B}$ oscillates with a frequency $\omega_\perp$ which is the solution of:

$$\left[\left(\omega_\perp^2 - k^2\right) + \frac{d_1}{c_1}(Bk)^2\right]e_\perp = 0 \implies \omega_\perp^2 = k^2\left(1 - \frac{d_1}{c_1}B^2\right). \tag{2.23}$$

While the component of the electric field parallel to $\vec{B}$ will oscillate with the frequency $\omega_\parallel$ which is the solution of:

$$\left[\left(\omega_\parallel^2 - k^2\right) + \omega_\parallel^2 \frac{d_2}{c_1}B^2\right]e_\parallel = 0 \implies \omega_\parallel^2 = k^2\left(\frac{c_1}{c_1 + d_2 B^2}\right). \tag{2.24}$$

So, for a NED theory the dispersion relations of a photon propagating in an external electromagnetic background are generically different from the relativistic ones and depend on the NED property.

Let us summarize the results: we approximated the nonlinearity expanding the Lagrangian around the background and keeping terms up to the second-order in the propagating field, we have considered the case of an external uniform magnetic field as a background and a plane wave perturbation propagating orthogonal to it, then the dispersion relations we have found are:

$$\omega_\parallel^2 = k^2\left(\frac{c_1}{c_1 + d_2 B^2}\right), \tag{2.25}$$

$$\omega_\perp^2 = k^2\left(1 - \frac{d_1}{c_1}B^2\right). \tag{2.26}$$

The components of the perturbation oscillate with different frequencies depending on their linear polaraization with respect to the external magnetic field: this produces birefringence.

Notice that in the limit $B \to 0$ or if we deal with the linear electrodynamics theory (i.e. $d_1 = d_2 = 0$) the usual Maxwellian dispersion relation $\omega^2 = k^2$ is recovered and no birefringence appears.

Having the photon dispersion relations, one can compute the refraction indices from the definition $n = v^{-1} = k/\omega$:

$$n_\parallel = \left(\frac{c_1 + d_2 B^2}{c_1}\right)^{1/2}, \tag{2.27}$$

$$n_\perp = \left(\frac{c_1}{c_1 - d_1 B^2}\right)^{1/2}, \tag{2.28}$$

and the birefringence condition $\omega_\parallel \neq \omega_\perp$ can be stated as $n_\parallel \neq n_\perp$, or

$$\Delta n = n_\parallel - n_\perp \,. \tag{2.29}$$

The PVLAS experiment has put an upper bound on vacuum birefringence in an external magnetic field $\vec{B} = 2.5\,\text{Tesla}$ [36]:

$$\Delta n_{\text{PVLAS}} \leq (12 \pm 17) \times 10^{-23} \,. \tag{2.30}$$

Although the sensibility of PVLAS did not reach the values predicted by the QED, computed from the EH Lagrangian $\Delta n_{\text{QED}} \approx 4 \times 10^{-24} (B/\text{Tesla})^2$ [39], its upper bound shows the smalness of vacuum birefringence phenomena and thus of nonlinear corrections.

2.1.1 Examples

Let us compute the refraction indices for some electrodynamics theories. We will always consider the configuration used above: $\vec{k}$ along x-axis and $\vec{B}$ along z-axis.

As we have already seen, in Maxwell electrodynamics only $c_1 \neq 0$ and any refraction index is equal to 1, so there is no birefringence as expected.

Euler-Heisenberg theory has an effective Lagrangian written in equation (1.17), and we report here its weak-field limit:

$$\mathscr{L}_{\text{EH}} \approx S + \frac{2\alpha^2}{45m^4}\left(4S^2 + 7P^2\right) . \tag{2.31}$$

Computing coefficients c_1, d_1 and d_2 from the weak-field Lagrangian one finds:

$$c_1^{\text{EH}} = 1 - \frac{8\alpha^2 B^2}{45m^4} \,, \qquad d_1^{\text{EH}} = \frac{16\alpha^2}{45m^4} \,, \qquad d_2^{\text{EH}} = \frac{28\alpha^2}{45m^4} \,. \tag{2.32}$$

Indeed, vacuum refraction indices in a magnetized environment for EH electrodynamics are:

$$n_\parallel^{\text{EH}} \approx 1 + \frac{14\alpha^2}{45m^4} B^2 \,, \tag{2.33}$$

$$n_\perp^{\text{EH}} \approx 1 + \frac{8\alpha^2}{45m^4} B^2 \,. \tag{2.34}$$

Birefringence is present but it is very small as anticipated above: $\Delta n \sim 10^{-24} B^2/\text{Tesla}^2$.

Generalized BI theory is a generalization of the BI Lagrangian (1.10) in which instead of the square root one takes the power p in the range $0 < p < 1$. Its Lagrangian is [38]:

$$\mathscr{L}_{\text{gBI}} = \beta^2 \left[1 - \left(1 - 2\frac{S}{\beta^2} - \frac{P^2}{\beta^4} \right)^p \right] , \tag{2.35}$$

where β^2 is a parameter with dimension of energy density. The usual BI theory is obtained for $p = 1/2$, and the Maxwell electrodynamics is restored in the weak-field limit $\beta^2 \to \infty$ with $p = 1/2$, as expected. Using $\mathscr{L}_{\text{gBI}}$, one computes the coefficients:

$$c_1^{\text{gBI}} = \frac{2p}{(1 + B^2/\beta^2)^{1-p}} \,, \quad d_1^{\text{gBI}} = \frac{4p(1-p)}{\beta^2(1 + B^2/\beta^2)^{2-p}} \,, \qquad d_2^{\text{gBI}} = \frac{2p}{\beta^2(1 + B^2/\beta^2)^{1-p}} \,. \tag{2.36}$$

So, the refraction indices are

$$n_{\parallel}^{\text{gBI}} = \left[1 - \frac{B^2}{B^2 + \beta^2}\right]^{-1/2}, \tag{2.37}$$

$$n_{\perp}^{\text{gBI}} = \left[1 - 2(1-p)\frac{B^2}{B^2 + \beta^2}\right]^{-1/2}. \tag{2.38}$$

Do note that in the case of BI theory ($p = 1/2$) the two refraction indices coincide and no birefringence is expected. It is the unique physically relevant NED theory having no birefringence [10].

ModMax theory has Lagrangian density (1.96), and hence

$$c_1^{\text{MM}} = \cosh\gamma + \left.\frac{\sinh\gamma\, S}{\sqrt{S^2 + P^2}}\right|_{\text{B}} = \exp(-\gamma)\,, \tag{2.39}$$

$$d_1^{\text{MM}} = \frac{\sinh\gamma}{\sqrt{S^2+P^2}}\left.\left[\frac{P^2}{S^2+P^2}\right]\right|_{\text{B}} = 0\,, \tag{2.40}$$

$$d_2^{\text{MM}} = \frac{\sinh\gamma}{\sqrt{S^2+P^2}}\left.\left[\frac{S^2}{S^2+P^2}\right]\right|_{\text{B}} = \frac{2\sinh\gamma}{B^2}\,, \tag{2.41}$$

where we used $S = -B^2/2$ and $P = 0$ for a magnetic background. Because of $d_1^{\text{MM}} = 0$, the component of the electric field orthogonal to $\vec{B}$ oscillates with the canonical dispersion relation $\omega^2 = k^2$. In fact the refraction indices are:

$$n_{\parallel}^{\text{MM}} = \exp\gamma\,, \tag{2.42}$$

$$n_{\perp}^{\text{MM}} = 1\,. \tag{2.43}$$

Notice there is no dependence on the magnitude B of the external field, in contrast to the cases of EH and BI theories. This is presumably a consequence of conformal invariance of ModMax [12]. The propagation of the parallel-mode is always sub-luminar as long as $\gamma > 0$. The magnitude of birefringence is:

$$\Delta n_{\text{MM}} = \exp\gamma - 1 \approx \gamma + \mathcal{O}(\gamma^2)\,. \tag{2.44}$$

Using the upper bound of PVLAS experiment, we can estimate an upper bound for the coupling constant γ of ModMax:

$$\gamma \leq 3 \times 10^{-22}\,. \tag{2.45}$$

This is a rough estimation obtained considering ModMax as the unique source of birefringence. In general birefringence may be a combined effect of classical and quantum non-linear contributions, indeed to be able to discriminate between NED theories one should also have experimental results for refraction indices themselves but this requires incredibly high sensitivity.

2.2 Polarization theory

In the previous section we have seen how a NED theory can produce a non-trivial dispersion relation for an electromagnetic plane wave because of the nonlinear terms; the result is a wave propagating with different frequency depending on its polarization. The polarization properties of electromagnetic waves are described by the polarization theory of electromagnetism. Let us introduce such theory following the Jackson's book "Classical Electrodynamics" [2] in order to set up the formalism we will use.

Consider an electric plane wave:

$$\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{pmatrix} \exp\left[i\left(\vec{k} \cdot \vec{x} - \omega t\right)\right], \tag{2.46}$$

where $E_i = E_i(x, t)$ are complex components of the plane wave, while $\mathcal{E}_i \in \mathbb{C}$ are amplitudes, they are spacetime independent in a plane wave. Thus, components of the wave are:

$$E_x = \mathcal{E}_x \exp\left[i\left(\vec{k} \cdot \vec{x} - \omega t\right)\right], \tag{2.47}$$

$$E_y = \mathcal{E}_y \exp\left[i\left(\vec{k} \cdot \vec{x} - \omega t\right)\right], \tag{2.48}$$

$$E_z = \mathcal{E}_z \exp\left[i\left(\vec{k} \cdot \vec{x} - \omega t\right)\right]. \tag{2.49}$$

From now on, let us consider a reference frame in which the propagating vector $\vec{k}$ is along z-axis, then the plane wave can be written in components as:

$$\vec{E} = \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ 0 \end{pmatrix} \exp[i(kz - \omega t)]. \tag{2.50}$$

Let us introduce constant real unit vectors $\vec{\epsilon}_x$ and $\vec{\epsilon}_y$ pointing respectively along x and y axis, then the wave can be written as:

$$\vec{E} = \left(\vec{\epsilon}_x \mathcal{E}_x + \vec{\epsilon}_y \mathcal{E}_y\right) \exp[i(kz - \omega t)]. \tag{2.51}$$

Vectors $\vec{\epsilon}_x$ and $\vec{\epsilon}_y$ indicate the direction of the electric field and are called the polarization vectors. The physical electric field is the real part of the wave $\vec{E}$, i.e. $\mathrm{Re}(\vec{E})$. It will span in general an ellipse in the (x, y) plane depending on the real part of E_x and E_y. Boundary configurations of the ellipse are the extremely flat ellipse, i.e. a straight line, and a circle. These configurations are respectively called linear polarization and circular polarization.

Linear polarization: it is obtained when the complex amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$ have the same phase δ. Calling with minor letters $e_x, e_y \in \mathbb{R}$ the modulus of $\mathcal{E}_x, \mathcal{E}_y$, the linear polarization requires $\mathcal{E}_x = e_x \exp(i\delta)$ and $\mathcal{E}_y = e_y \exp(i\delta)$. In this case the electric plane wave (2.51) reads:

$$\vec{E} = \left(\vec{\epsilon}_x e_x + \vec{\epsilon}_y e_y\right) \exp[i(kz - \omega t + \delta)]. \tag{2.52}$$

The quantity $\vec{\epsilon}_x e_x + \vec{\epsilon}_y e_y$ is therefore a real vector that can be written as a real-valued unit vector $\vec{\epsilon}$ times a real-valued magnitude e. Vector $\vec{\epsilon}$ is the new polarization vector of the linear plane wave; the magnitude is:

$$e = \sqrt{e_x^2 + e_y^2}. \tag{2.53}$$

The angle between $\vec{\epsilon}$ and $\vec{\epsilon}_x$ is:

$$\theta = \arctan\left(\frac{e_y}{e_x}\right). \tag{2.54}$$

Circular polarization: The circular polarization is obtained when $\mathcal{E}_x$ and $\mathcal{E}_y$ has the same modulus, that we call $e \in \mathbb{R}$, but differ by a phase of $\pi/2$, i.e. $\mathcal{E}_x = e\exp(i\delta)$ and $\mathcal{E}_y = e\exp(i\delta \pm i\pi/2) = \pm ie\exp(i\delta)$. Then the wave (2.51) reads:

$$\vec{E} = \left(\vec{\epsilon}_x \pm i\vec{\epsilon}_y\right) e\exp[i(kz - \omega t + \delta)]\,. \tag{2.55}$$

In this case the physical field $\mathrm{Re}(\vec{E})$ has components:

$$\begin{aligned} E_x^{\text{phys}} = \mathrm{Re}(E_x) &= \mathrm{Re}(e\exp[i(kz - \omega t + \delta)]) && (2.56)\\ &= e\cos(kz - \omega t + \delta)\,, && (2.57)\\ E_y^{\text{phys}} = \mathrm{Re}(E_y) &= \mathrm{Re}(\pm ie\exp[i(kz - \omega t + \delta)]) && (2.58)\\ &= \mp e\sin(kz - \omega t + \delta)\,, && (2.59) \end{aligned}$$

indeed the electric field spans a circle with frequency ω.

Polarization theory uses the Stokes parameters that are functions of the complex-valued electric field components E_i; again considering a wave propagating along z-axis, they are:

$$\begin{aligned} I &\equiv E_x^2 + E_y^2\,, && (2.60a)\\ Q &\equiv E_x^2 - E_y^2\,, && (2.60b)\\ U &\equiv 2\,\mathrm{Re}(E_x^* E_y)\,, && (2.60c)\\ V &\equiv 2\,\mathrm{Im}(E_x^* E_y)\,, && (2.60d) \end{aligned}$$

where $*$ indicates complex conjugate. Squared quantities are computed using the scalar product for complex numbers: $E_i^2 = E_i^* E_i$. Indeed Stokes parameters are real numbers. Here I is the intensity of the electric field, Q, U describe a linear polarization, and V describes a circular polarization. We convince ourself that such names are appropriate observing that if we perform a counterclockwise rotation of our reference frame by an angle θ around the z-axis (i.e. a counterclockwise rotation of the x, y-axes, or equivalently a clockwise rotation of electric field) the Stokes parameters that feel the rotation are Q and U only. Let us show it, such a rotation affects the component of the electric field:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} \to \begin{pmatrix} E'_x \\ E'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}\,, \tag{2.61}$$

Stokes parameters transform as:

$$\begin{aligned} &I \to I' = I\,, && (2.62)\\ &Q \to Q' = \cos(2\theta)Q + \sin(2\theta)U\,, && (2.63)\\ &U \to U' = -\sin(2\theta)Q + \cos(2\theta)U\,, && (2.64)\\ &V \to V' = V\,. && (2.65) \end{aligned}$$

Indeed parameters I and V remain untouched, only Q, U feel the rotation. If we define the quantity $Q \pm iU$ the rotation translates into an additional phase:

$$\begin{aligned} (Q \pm iU) \to \left(Q' \pm iU'\right) &= Q(\cos(2\theta) \mp i\sin(2\theta)) + U(\sin(2\theta) \pm i\cos(2\theta)) && (2.66a)\\ &= Q(\cos(2\theta) \mp i\sin(2\theta)) \pm iU(\cos(2\theta) \mp i\sin(2\theta)) && (2.66b)\\ &= (Q \pm iU)\exp(\mp i2\theta) && (2.66c) \end{aligned}$$

i.e. if the electric field rotates clockwise by an angle θ, the quantity $(Q \pm iU)$ rotates in the plane (Q, iU) by an angle $\mp 2\theta$.

Since $(Q \pm iU) \in \mathbb{C}$ we can write it using an amplitude P and a phase 2ψ:

$$(Q \pm iU) = P \exp(\pm i2\psi)\,, \tag{2.67}$$

where

$$P \equiv \left[Q^2 + U^2\right]^{1/2}\,, \tag{2.68}$$
$$2\psi \equiv \arctan(U/Q)\,. \tag{2.69}$$

In light of the relation between the electric field orientation and $Q \pm iU$, the quantity ψ is the initial angle of $\vec{E}$ with respect to x-axis in the (x, y) plane. Since ψ is the physical angle, we have defined the phase of $Q \pm iU$ with the factor 2.

The clockwise rotation by θ of the electric field performed in (2.61) changes the configuration to a new angle $\psi \to \Psi = \psi - \theta$. Consequently, the quantity (2.67) changes as follow:

$$P \exp(\pm i2\psi) \to P \exp(\pm i2\Psi) = P \exp[\pm i2(\psi - \theta)]\,. \tag{2.70}$$

The new angle Ψ can be computed as a function of the electric field components through the definition (2.69) (having $\psi \to \Psi$):

$$\Psi = \frac{1}{2}\arctan\left(\frac{U'}{Q'}\right) = \frac{1}{2}\arctan\left(\frac{2\,\mathrm{Re}(E_x'^{\,*}E_y')}{E_x'^{\,2} - E_y'^{\,2}}\right)\,, \tag{2.71}$$

where primes denote quantities after the rotation. So, if we know the evolution of electric field components of a plane wave we can compute the Stokes parameter Q', U' and obtain the orientation angle Ψ of the linear polarization.

Example. Let us make an example considering a linear-polarized plane wave propagating along z-axis:

$$\vec{E} = \vec{\mathcal{E}} \exp[i(kz - \omega t)] \tag{2.72}$$
$$= \vec{\epsilon}\, e \exp[i(kz - \omega t + \delta)]\,, \tag{2.73}$$

where $\vec{\epsilon}$ is the polarization vector, $e = (e_x^2 + e_y^2)^{1/2} \in \mathbb{R}$ is the amplitude. Stokes parameters Q, U are

$$Q = E_x^2 - E_y^2 = e_x^2 - e_y^2\,, \tag{2.74}$$
$$U = 2E_x^* E_y = 2e_x e_y\,. \tag{2.75}$$

They are constant, indeed the phase $2\psi = \arctan(U/Q)$ is constant during the propagation too. This means the wave starts with the electric field counterclockwise rotated by an angle ψ with respect to the x-axis and propagates without modifying it.

The example above allows us to predict that a rotation of the linear polarization will be possible when U and/or Q changes. Being more general as possible, the rotation happens in a plane wave where any component has different initial phase, frequency and wave-vector. For example, the electric field of such a plane wave propagating along z-axis is:

$$\vec{E}(\vec{x}, t) = \begin{pmatrix} \mathcal{E}_x \exp[i(k_x z - \omega_x t)] \\ \mathcal{E}_y \exp[i(k_y z - \omega_y t)] \\ 0 \end{pmatrix} = \begin{pmatrix} e_x \exp(i\delta_x) \exp[i(k_x z - \omega_x t)] \\ e_y \exp(i\delta_y) \exp[i(k_y z - \omega_y t)] \\ 0 \end{pmatrix}\,, \tag{2.76}$$

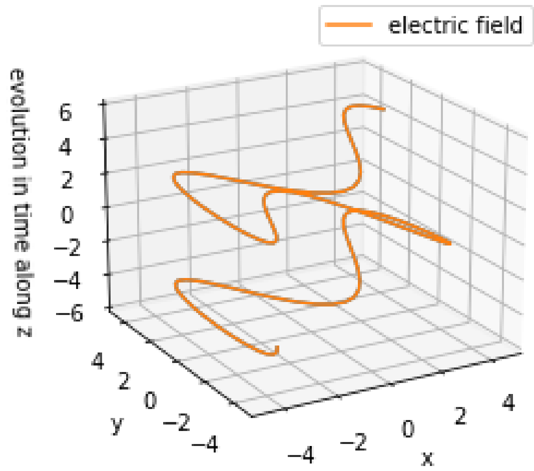

Figure 2.1: Example of the most-general plane-wave electric field (2.76). Here we used $(e_x, e_y) = (4, 3), (\delta_x, \delta_y) = (0.1, 0.3), (\omega_x, \omega_y) = (1, 5)$ but we kept constant the wave-vector to $k = 2$. We also chained z and t together, for simplicity. Notice the electric field is still periodic because we have chosen constant frequencies ω_x, ω_y. If they were not constant, the periodicity property would be lost.

where δ_x, δ_y are the initial phases and $e_x, e_y \in \mathbb{R}$. The physical electric field $\mathrm{Re}(\vec{E})$ of the most general plane-wave (2.76) will span a curve more complicated than an ellipse because of $k_x \neq k_y, \omega_x \neq \omega_y$ and $\delta_x \neq \delta_y$; look at figure 2.1 for an example. Stokes parameters can still be used with the same definitions given in (2.60). Even more, rotation properties we have obtained before are still valid, since they are obtained at the level of the components of the electric field $\vec{E}$ using only properties of complex numbers with no link to the values of the components of E_i.

2.2.1 CMB photons

Photons we are interested in the CMB photons, they have propagated from the last-scattering surface (LSS) up to the present time moving, indeed, toward us. Let us introduce right-handed coordinates with the z-axis taken in the direction of observer's lines of sight (these are the coordinates used by the CMB community [16]), the consequence is the replacement of $(Q \pm iU) \to (Q \mp iU)$, which translates into a minus sign in the angle ψ of (2.67):

$$\psi \to -\psi \equiv \beta \qquad \text{in CMB community}\,. \tag{2.77}$$

We called the new angle β as in [16]. Let us recall the definition of β using equation (2.69):

$$\beta = -\frac{1}{2}\arctan\left(\frac{U}{Q}\right)\,. \tag{2.78}$$

Notice that since $\psi > 0$ was a counterclockwise rotation of the electric field and $\psi < 0$ a clockwise rotation, in the coordinates of CMB community an angle $\beta > 0$ will be a clockwise rotation and $\beta < 0$ a counterclockwise rotation.

2.2.2 Helicity states

For a generic plane wave propagating along z-axis, instead of the Cartesian basis E_x, E_y we can use the helicity basis (also called helicity states):[1]

$$E_+ = E_x - iE_y \,, \tag{2.79}$$
$$E_- = E_x + iE_y \,. \tag{2.80}$$

The inverse relation (i.e. the Cartesian coordinates written as the function of the helicity ones) is:

$$E_x = \frac{1}{2}(E_+ + E_-) \,, \tag{2.81}$$
$$E_y = \frac{i}{2}(E_+ - E_-) \,. \tag{2.82}$$

We can write the Stokes parameters in the helicity states:

$$I = E_+^2 + E_-^2 \,, \tag{2.83}$$
$$Q = \mathrm{Re}(E_+^* E_-) \,, \tag{2.84}$$
$$U = \mathrm{Im}(E_+^* E_-) \,, \tag{2.85}$$
$$V = E_+^2 - E_-^2 \,. \tag{2.86}$$

The constancy of I is not a surprise: the intensity has no dependence on the direction, so a change of the basis cannot affect it.

We can still define the quantity $Q \pm iU$, and write it using the Euler formula as $P \exp(\pm i2\psi)$. Rotation properties, as (2.66), still hold using helicity states. Let us apply the CMB community notation, calling the rotation angle with β, its value is given in equation (2.78). In terms of the helicity states

$$\beta = -\frac{1}{2}\arctan\left(\frac{U}{Q}\right) = -\frac{1}{2}\arctan\left(\frac{\mathrm{Im}(E_+^* E_-)}{\mathrm{Re}(E_+^* E_-)}\right) . \tag{2.87}$$

In the following chapter we will present a cosmological framework in which we expect a rotation in time of the linear polarization of photons, i.e. a non-zero angle $\beta = \beta(t)$.

[1] In [2] coefficients are different but the idea is the same.

Chapter 3

Cosmic birefringence

From a cosmological point of view a birefringence phenomenon is a rotation of the linear polarization plane of photons during propagation in the universe. This phenomenon arises, for instance, if Maxwell electrodynamics includes an interaction between the electromagnetic field and an external field [16]; the coupling will produce a non-trivial dispersion relation for polarization modes resulting in a rotation of the polarization plane as photons propagate. The external field indeed behaves as a birefringent material. Usually the field is an axion-like particle (ALP) coupled to photons via the so-called Chern-Simons (CS) term (see e.g. [16–19, 40] for recent works):

$$\mathscr{L} \supset g_{a\gamma} a F_{\mu\nu} \tilde{F}^{\mu\nu} \, , \tag{3.1}$$

here $g_{a\gamma}$ is the common notation for the coupling constant between the ALP, denoted with a, and photons.

ALPs are pseudo Nambu-Goldstone bosons which arose in various extensions of the Standard Model, as in string theory where they might have a broad range of mass and couplings to gauge fields [23, 41, 42]. The idea of ALPs comes from a generalization of axions that are theoretical particles firstly proposed by Peccei and Quinn to solve the problem of strong CP-violation [43, 44]. ALPs are not yet observed, despite various experiments driven to catch them.

In cosmology, ALPs are candidates to the dark sector: depending on the ALP mass, they may be dark energy or dark matter (see [23] for a review). The signature of their coupling to photons, via a CS term, was sought in the CMB polarization maps collected by WMAP and Planck space missions. CMB photons were released at the surface of last scattering with an imprinted polarization marked by the scattering with electrons of the cosmic plasma, then they have propagated along 14 billions years. If they interacted with the ALP external field, they are the most natural target to observe the cosmic birefringence since the rotation of their polarization plane has accumulated during the propagation period. Thus, cosmic birefringence is an interesting test to go beyond the Standard Model of particle physics.

Let us keep the notation β for the rotation angle, as used in the Polarization theory section 2.2. From the third public realise of Planck maps the estimated value of β is $\beta = 0.35^\circ \pm 0.14^\circ$ (68% C.L.), which excludes a null angle at 99.2% C.L. [45]. Time after, the same treatment was extended to the fourth public release of Planck maps estimating $\beta = 0.30^\circ \pm 0.11^\circ$ (68% C.L.) [46]. The latest measurements of cosmic birefringence came from a joint analysis of polarization data from Planck and WMAP giving

$$\beta = 0.342^\circ \ {}^{+0.094^\circ}_{-0.091^\circ} \quad (68\%\text{C.L.}) \tag{3.2}$$

and excluding $\beta = 0$ at 99.987% C.L. In addition, they found no evidence of an angle β being dependent on photon frequency [1].

Such precise estimations were possible by correlating the CMB measurements with measurements of the Galactic foregrounds emission. This analysis was introduced by Komatsu and Minami [45] and allowed to mitigate the systematic uncertanty coming from the miscalibration of the Planck polarization angle detectors which was a limit in the previous analysis.

From the experimental estimation of β it was possible to constrain the coupling constant of ALP-photons [17]. Physicists also carried out astrophysical observations and terrestrial experiments in order to detect these particles, for instance let us mention the Chandra and CAST (CERN Axion Solar Telescope) collaborations in this respect, and we will say more about them at the end of the chapter [47, 48].

An interesting thing to extrapolate from the ALPs phenomenology is their allowed huge mass range.

3.1 Axion-induced cosmic birefringence

The simplest way to predict cosmic birefringence is implementing Maxwell electrodynamics with a CS term considering an external pseudoscalar ALP field. Let us call it by $\chi(t, \vec{x})$ and we denote the resulting theory as axion-Maxwell electrodynamics, its action is [49]:

$$\mathcal{S} = \mathcal{S}_{\mathrm{M}} + \mathcal{S}_{\mathrm{CS}} = \int \mathrm{d}^4x\, \sqrt{-g}S + \int \mathrm{d}^4x\, \sqrt{-g}\frac{\alpha}{2f}\chi P\,, \tag{3.3}$$

where g is the determinant of the metric, α is a dimensionless coupling constant, f is the decay-constant and has the dimension of energy. Here we have parametrized the coupling $g_{a\gamma}$ of the CS term with α and f.

In the paradigm of ΛCDM model of cosmology we consider a flat-FLRW spacetime in conformal coordinates [50]:

$$\mathrm{d}s^2 = a^2(\eta)\big[-\mathrm{d}\eta^2 + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\big]\,, \tag{3.4}$$

where $\mathrm{d}\eta = \mathrm{d}t\,/a(t)$ is the conformal time and t is the cosmic time. The determinant of this metric is $g = -a^8$. Such framework was treated by Komatsu in [16].

Note that $P = -\frac{1}{4}F\tilde{F}$ violates parity symmetry, therefore χ must be a pseudoscalar field in order for the term $\chi F\tilde{F}$ to be parity-invariant.

Let us derive the EoM for the 4-potential as the EL equations:

$$0 = \frac{\partial\mathscr{L}}{\partial A_\alpha} - \nabla_\mu\left[\frac{\partial\mathscr{L}}{\partial(\nabla_\mu A_\alpha)}\right], \tag{3.5}$$

where the Lagrangian density is:

$$\mathscr{L} = S + \frac{\alpha}{2f}\chi P\,. \tag{3.6}$$

Indeed the EoM are:

$$0 = \nabla_\mu\left[F^{\mu\nu} + \frac{\alpha}{2f}\chi\tilde{F}^{\mu\nu}\right] \tag{3.7a}$$

$$= \nabla_\mu F^{\mu\nu} + \frac{\alpha}{2f}\nabla_\mu\chi\tilde{F}^{\mu\nu}\,, \tag{3.7b}$$

where the covariant derivative of $\tilde{F}^{\mu\nu}$ was set to zero because of the Bianchi identity. Opening the Faraday tensor:

$$0 = \nabla_\mu(\nabla^\mu A^\nu - \nabla^\nu A^\mu) + \frac{\alpha}{2f}\nabla_\mu\chi\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\nabla_\rho A_\sigma - \nabla_\sigma A_\rho) \tag{3.8a}$$

$$= \nabla_\mu\nabla^\mu A^\nu - (\nabla^\nu\nabla_\mu A^\mu + R^\nu{}_\mu A^\mu) + \frac{\alpha}{2f}\nabla_\mu\chi\epsilon^{\mu\nu\rho\sigma}\nabla_\rho A_\sigma\,. \tag{3.8b}$$

In the last line we have recalled that covariant derivatives of a vector do not commute introducing the Ricci tensor.

The effect of the CS term vanishes if χ is constant. This can be seen also at the level of the action since $\sqrt{-g}F\tilde{F}$ is a total derivative: χ must depend on spacetime. In general χ depends on both time and space, however in order to keep the coarse grain description of the universe as homogeneous and isotropic let a field depending only on time: $\chi = \chi(\eta)$. We can think of this assumption as if we were using a space-averaged field i.e. its homogeneous part. Another approximation we use is the geometric-optic approximation, it is valuable since we are studying light propagating in cosmological scales, then we can drop out terms containing Ricci and Riemann tensors with respect to terms having the derivative of the 4-potential [51]. With these assumptions the EoM reads:

$$0 = \Box A^\nu - \nabla^\nu(\nabla_\mu A^\mu) + \frac{\alpha}{2f}\chi'\epsilon^{0\nu\rho\sigma}\nabla_\rho A_\sigma\,, \tag{3.9}$$

where $\Box \equiv \nabla_\mu\nabla^\mu$, and prime denotes a conformal time derivative.

Let us impose on the four-potential the Coulomb gauge: $A_0 = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$ (namely we deal with 3-vector transverse waves). Thus, EoM are:

$$\vec{A}'' - \left[\nabla^2\vec{A} + \frac{\alpha}{f}\chi'\vec{\nabla}\times\vec{A}\right] = 0\,. \tag{3.10}$$

As expected they are a generalization of the EoM for the free vector-field A^μ minimally coupled with gravity, i.e. the EL equation obtained varying the action $\mathcal{S}_\mathrm{M}$ alone with respect to the 4potential would be

$$\vec{A}'' - \nabla^2\vec{A} = 0\,. \tag{3.11}$$

The EoM takes the same form as in Minkowski spacetime, this should not be a surprise since a massless vector field, as A^μ, is conformally coupled to gravity.

Let us expand the 4potential with respect to a basis:

$$\vec{A}(\eta,\vec{x}) = \int \mathrm{d}^3k\, T_k(\eta)\vec{S}_k(\vec{x})\,, \tag{3.12}$$

where $\vec{k}$ is the comoving wavevector as $\vec{x}$ is the comoving spatial coordinate. The physical wavevector is $\vec{k}_\mathrm{phys} = \vec{k}/a$, thus the physical wavelength is $\lambda_\mathrm{phys} \propto 1/k_\mathrm{phys} = a/k$. Here T stands for time and S for space, recalling what their dependencies are. We choose a set of plane waves as a basis for the spatial part:

$$\vec{S}_k(\vec{x}) = \vec{N}_k \exp\!\left(i\vec{k}\cdot\vec{x}\right), \tag{3.13}$$

where $\vec{N}_k$ is a normalization vector with no dependence on spacetime. Then:

$$\vec{A}(\eta,\vec{x}) = \int \mathrm{d}^3k\, T_k(\eta)\vec{N}_k \exp\!\left(i\vec{k}\cdot\vec{x}\right) \equiv \int \mathrm{d}^3k\, \vec{\mathcal{A}}_k(\eta)\exp\!\left(i\vec{k}\cdot\vec{x}\right), \tag{3.14}$$

where we have collected the time component $T_k(\eta)$ and the normalization vector $\vec{N}_k$ into a unique term called $\vec{\mathcal{A}}_k$.

Let us consider a single wavevector k and the corresponding mode to the 4-potential:

$$\vec{A}_k(\eta,\vec{x}) = \vec{\mathcal{A}}_k(\eta)\exp\left(i\vec{k}\cdot\vec{x}\right). \tag{3.15}$$

Inserting it into the EoM (3.10) we find the EoM for $\vec{\mathcal{A}}_k(\eta)$ in the Fourier space:

$$0 = \vec{\mathcal{A}}_k'' + k^2\vec{\mathcal{A}}_k - i\frac{\alpha}{f}\chi'\vec{k}\times\vec{\mathcal{A}}_k\,. \tag{3.16}$$

Recall the gauge $\vec{\nabla}\cdot\vec{A} = 0$, which means we are dealing with waves whose polarization is orthogonal to the propagation direction $\vec{k}$. With no loss of generality, let us consider a reference frame in which $\vec{k}$ is oriented along z-axes, then $A_z = 0$ and the third component of $\vec{\mathcal{A}}_k$ is also zero. EoM in the Fourier space reduces to a system of two equations for first and second components of $\vec{\mathcal{A}}_k$. In order to keep a light notation, we drop the subscript $_k$ and use subscripts $_{x,y}$ to denote the first and second component of $\vec{\mathcal{A}}_k$. So, the EoM in Fourier space can be written as follows:

$$\mathcal{A}_x'' + k^2\mathcal{A}_x = -i\frac{\alpha}{f}\chi' k\mathcal{A}_y\,, \tag{3.17}$$

$$\mathcal{A}_y'' + k^2\mathcal{A}_y = i\frac{\alpha}{f}\chi' k\mathcal{A}_x\,. \tag{3.18}$$

Let us introduce new helicity states, the circular helicity states:

$$\begin{pmatrix}\mathcal{A}_x\\ \mathcal{A}_y\end{pmatrix} \to \begin{pmatrix}\mathcal{A}_+\\ \mathcal{A}_-\end{pmatrix} = \begin{pmatrix}\mathcal{A}_x - i\mathcal{A}_y\\ \mathcal{A}_x + i\mathcal{A}_y\end{pmatrix}. \tag{3.19}$$

Inserting them into the system we find:

$$\mathcal{A}_+'' + k^2\mathcal{A}_+ = \frac{\alpha}{f}\chi' k\mathcal{A}_+\,, \tag{3.20}$$

$$\mathcal{A}_-'' + k^2\mathcal{A}_- = -\frac{\alpha}{f}\chi' k\mathcal{A}_-\,, \tag{3.21}$$

that in a compact form can be written as:

$$\mathcal{A}_\pm'' + \left[k^2 \mp k\frac{\alpha}{f}\chi'\right]\mathcal{A}_\pm = 0\,. \tag{3.22}$$

Let us call the quantity in the square brackets as:

$$\omega_\pm^2 \equiv k^2 \mp k\frac{\alpha}{f}\chi'\,. \tag{3.23}$$

If we assume that $\omega_\pm$ is a constant quantity, a solution of (3.22) is:

$$\mathcal{A}_\pm = C_\pm\exp(-i\omega_\pm\eta + \delta_\pm)\,, \tag{3.24}$$

where $C_\pm \in \mathbb{R}$ is an amplitude and $\delta_\pm$ is an initial phase; note ω is a comoving quantity as also k is.

Now we can rewrite the single k-mode (3.15) in the helicity basis:

$$A_\pm(\eta,\vec{x}) = \mathcal{A}_\pm\exp\left(i\vec{k}\cdot\vec{x}\right) \tag{3.25a}$$

$$= C_\pm\exp[i(-\omega_\pm\eta + kz + \delta_\pm)]\,. \tag{3.25b}$$

In other words, a plane wave with a pulsation equal to $\omega_\pm = \text{const}$ (by hypotesis) is an exact solution of the EoM (3.22) for the 4potential in the helicity basis.

However, the hypothesis of a constant $\omega_\pm$ implies $\chi' = \text{const}$ which is a stringent requirement. We can relax the initial assumption allowing $\omega_\pm$ of equation (3.23) to change, but slowly; this implies that also the field χ' changes slowly. To be more precise, we assume that the effective angular velocity of (3.23) varies slowly with time within a period, i.e. $|\omega'|/\omega^2 \ll 1$ [16]. It is the assumption of the WKB method to find an approximated solution of a differential equation of the type (3.22). So, using the WKB method we find the approximated solution [16]:

$$\mathcal{A}_\pm \approx (2\omega_\pm)^{-1/2} \exp\left[-i \int \mathrm{d}\eta\, \omega_\pm + i\delta_\pm\right], \tag{3.26}$$

here the integral ranges from an initial time η_i up to a time η_0, the term $\delta_\pm$ is the initial phase that the helicity state $\mathcal{A}_\pm$ had at the initial time η_i. So the WKB solution considers the evolutionary history of the slowly-varying $\omega_\pm$ through the integral, where $\omega_\pm$ is defined in equation (3.23). Let us take the square root of (3.23) and expand it up to the linear term in χ' (it is sufficient since it is small by assumption):

$$\omega_\pm = k\left[1 \mp \frac{\alpha\chi'}{fk}\right]^{1/2} \approx k\left[1 \mp \frac{1}{2}\frac{\alpha\chi'}{fk}\right]. \tag{3.27}$$

When we took the square root we considered only the plus solution, then we expanded the square root since $\chi'/k \ll 1$: the conformal time derivative is of the order of the visible universe $' \sim \eta^{-1}$ while k^{-1} is of the order of the photon wavelength, so the overall term is small. The CS coupling produces indeed non-trivial dispersion relations, modifying the Maxwell one $\omega_\pm = k$ adding the term $\mp\alpha\chi'/(2f)$.

Since the additional term is small and we are interested more in the variation in phase than a variation in amplitude, we keep $\omega_\pm$ in the exponent of the WKB solution (3.26) while we set $\omega_\pm \sim k$ (constant by hypothesis) in the amplitude [16]. So the dependence in the WKB solution becomes:

$$\mathcal{A}_\pm \propto \exp\left[-i \int_{\eta_i}^{\eta_0} \mathrm{d}\eta\, \omega_\pm + i\delta_\pm\right]. \tag{3.28}$$

Group and phase velocity From (3.27) we compute the corresponding phase velocity:

$$v_{\mathrm{p},\pm} \equiv \frac{\omega_\pm}{k} \approx 1 \mp \frac{1}{2}\frac{\alpha\chi'}{fk}. \tag{3.29}$$

It can be bigger or smaller than 1, i.e. greater than the speed of light in vacuum. However this is not problematic.

Regarding the group velocity, we see from equation (3.27) that the CS coupling does not contribute at first order: computing $v_{\mathrm{g},\pm} \equiv \mathrm{d}\omega_\pm/\mathrm{d}k$ from (3.27) we would obtain a unitary group velocity, as in Maxwell theory. Therefore, let us expand $\omega_\pm$ up to the second order in χ':

$$\omega_\pm \approx k\left[1 \mp \frac{1}{2}\frac{\alpha\chi'}{fk} - \frac{1}{8}\left(\frac{\alpha\chi'}{fk}\right)^2\right]. \tag{3.30}$$

Now the group velocity reads:

$$v_{\mathrm{g},\pm} \approx 1 + \frac{1}{8}\left(\frac{\alpha\chi'}{fk}\right)^2 \tag{3.31}$$

and the CS coupling modifies it adding a tiny positive contribution, making the group velocity bigger than one.[1]

3.1.1 Birefringence angle

As we can see in (3.27) the Maxwell-Chern-Simons theory breaks parity predicting different pulsations for the helicity states of a plane wave: $\omega_+ \neq \omega_-$. In light of what polarization theory tells us it migth be that in the Maxwell-CS theory the polarization plane of a linearly-polarized light gets rotated during its propagation. Let us compute the rotation angle from (2.69). We consider CMB photons, using the CMB community convention for coordinates: right-handed coordinates with the z-axis in the direction of observer's line of sight. Since we worked in helicity states to solve the EoM, we will use the definition (2.87) where the rotation angle β is given with helicity states; we recall here its formula:

$$\beta = -\frac{1}{2}\arctan\left(\frac{\mathrm{Im}(E_+^* E_-)}{\mathrm{Re}(E_+^* E_-)}\right). \tag{3.32}$$

The electric field components are:

$$E_\pm = \nabla_0 A_\pm \propto \nabla_0\Big(\mathcal{A}_\pm(\eta)\exp\Big(i\vec{k}\cdot\vec{x}\Big)\Big) \tag{3.33a}$$

$$\propto \exp\left[-i\int \mathrm{d}\eta\, \omega_\pm + i\delta_\pm\right], \tag{3.33b}$$

where we did not write the spatial derivative of A^0 since we are in the Coulomb gauge choice. In the second line we have used equation (3.28) neglecting the amplitude: the rotation angle β is defined as a ratio of electric field components thus the amplitude does not contribute. In fact:

$$\beta = -\frac{1}{2}\arctan\left(\frac{\mathrm{Im}(\exp\big[i\int \mathrm{d}\eta\, \omega_+ - i\delta_+\big]\exp\big[-i\int \mathrm{d}\eta\, \omega_- + i\delta_-\big])}{\mathrm{Re}(\exp\big[i\int \mathrm{d}\eta\, \omega_+ - i\delta_+\big]\exp\big[-i\int \mathrm{d}\eta\, \omega_- + i\delta_-\big])}\right) \tag{3.34a}$$

$$= -\frac{1}{2}\arctan\left(\frac{\sin\big(\int \mathrm{d}\eta\,(\omega_+ - \omega_-) - (\delta_+ - \delta_-)\big)}{\cos\big(\int \mathrm{d}\eta\,(\omega_+ - \omega_-) - (\delta_+ - \delta_-)\big)}\right) \tag{3.34b}$$

$$= -\frac{1}{2}\int \mathrm{d}\eta\,(\omega_+ - \omega_-) + \frac{1}{2}(\delta_+ - \delta_-). \tag{3.34c}$$

The plane of linear polarization is rotated relative to the initial angle $(\delta_+ - \delta_-)/2$ at the surface of last scattering by a quantity $-\int \mathrm{d}\eta\,(\omega_+ - \omega_-)/2$. Without loss of generality, we set $\delta_+ - \delta_- = 0$ from now on, so the the polarization plane rotates by an angle:

$$\beta = -\frac{1}{2}\int \mathrm{d}\eta\,(\omega_+ - \omega_-). \tag{3.35}$$

The integration window is the conformal time interval of propagation, that for the oldest observable photons, the CMB photons, goes from the time of last-scattering surface to today. It is a range of 14 billions years, so even if the difference between ω_+ and ω_- is small the effect accumulates because of the presence of the integral.

[1] The meaning of having a group velocity bigger than c is a long historical debate, see e.g. [52].

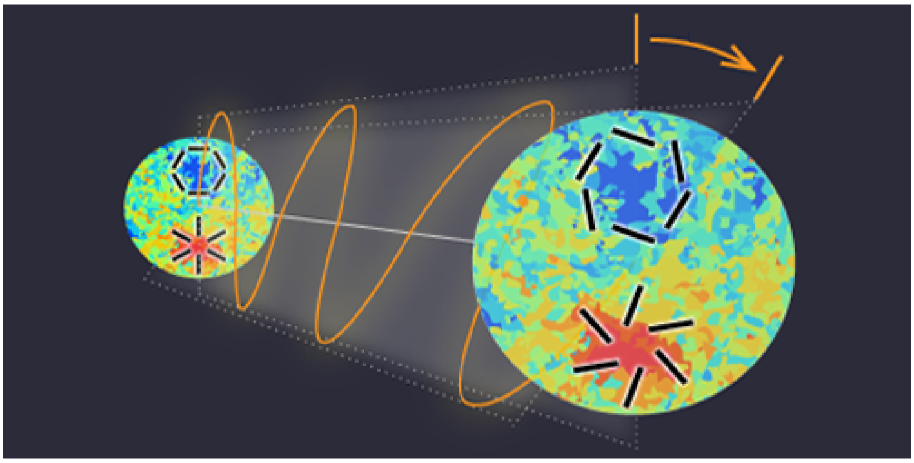

Figure 3.1: The picture shows a pictorial idea of a rotation angle for linear polarization of photons that are propagating in the universe and interacting with a background field. The orange arrow on the rigth-top of the figure represents the rotation angle β. Credits: Y. Minami.

We now insert the frequency of helicity states in Maxwell-CS theory (3.27):

$$\beta = -\frac{1}{2}\int_i^0 \mathrm{d}\eta \left[\left(k - \frac{\alpha\chi'}{2f}\right) - \left(k + \frac{\alpha\chi'}{2f}\right)\right] \tag{3.36a}$$

$$= \frac{1}{2}\int_i^0 \mathrm{d}\eta \, \frac{\alpha}{f}\chi' \tag{3.36b}$$

$$= \frac{\alpha}{2f}[\chi(\eta_0) - \chi(\eta_i)] \,. \tag{3.36c}$$

Therefore in order to quantify the cosmic birefringence through the rotation angle β we must know the evolution of the ALP field χ. For a pictorial visualization of the cosmic birefringence angle we report the representation made by Y. Minami in figure 3.1.

This was the general description of cosmic birefringence proposed in literature [16, 18, 49, 53–55]. If we were to consider the most general pseudoscalar field $\chi = \chi(\eta, \vec{x})$, it would produce a rotation angle dependent also on the direction of observation giving rise to an anisotropic cosmic birefringence, see e.g. [56].

If the field χ is the real player on the observed cosmic birefringence the consequences are remarkable: first of all χ will be a new ingredient beyond SM, then it might play the role of dark energy or dark matter as an axion-like particle [23].

3.1.2 Bounds on parameters

A coupling between photons and ALPs can give rise to different phenomena and it is a while people try to observe them experimentally, thus constraining the axion-photon coupling $g_{a\gamma} \equiv \alpha/(2f)$.

The CERN Axion Solar Telescope (CAST) collaboration searched for axions coming from the solar core: a 9 T refurbished Large Hadron Collider test magnet is directed towards the Sun with the purpose to observe solar axions converting to X-ray photons in the strong magnetic field and then record the photon by X-ray detectors. They claimed the upper bound $g_{a\gamma} < 0.66 \times 10^{-10}\,\mathrm{GeV}^{-1}$ for $m \lesssim 0.02\,\mathrm{eV}$ at 95% C.L. [47].

Chandra collaboration had observed the active galactic nucleus NGC 1275 at the center of the Perseus cluster and from the data analysis it has been provides the limit

$g_{a\gamma} < 10^{-13}\,\mathrm{GeV}^{-1}$ for $m < 10^{-12}\,\mathrm{eV}$ depending on the magnetic field, at 99.7% C.L. [48].

Fedderke et al. [57] proposed two observables from low-mass axions, playing the role of dark matter, using CMB polarization measurements. The first is an overall suppression of CMB polarization, which is referred to as the washout effect. It is due to early-time oscillations of the axion field that wash out the polarization produced at last scattering, reducing the polarized fraction with respect to the standard prediction. The second is due to late-time oscillations of the axion field which cause the CMB polarization to oscillate in phase, it is a local effect since it depends on how the local ALP oscillates. The effect is called AC oscillation (or time-variable cosmic birefringence) and makes distant static polarized sources to appear to oscillate. Since the oscillation is in phase, the frequency is proportional to the ALP mass. Both effects are sensitive to the axion-photon coupling constant. The BICEP/Keck collaboration [58] focused on AC oscillation using data from the 2012-2015 observing seasons of the Keck Array, they found $g_{a\gamma} < 4.5 \times 10^{-12}\,\mathrm{GeV}^{-1} \times (m/(10^{-21}\,\mathrm{eV}))$ for a mass range $10^{-23}\,\mathrm{eV} \leq m \leq 10^{-18}\,\mathrm{eV}$, in the hypothesis that ALPs covers the whole dark matter.

The SPT-3G Collaboration (third survey receiver operating on the South Pole Telescope) set a similar bound $g_{a\gamma} < 1.18 \times 10^{-12}\,\mathrm{GeV}^{-1} \times (m/(10^{-21}\,\mathrm{eV}))$ for a mass range $10^{-22}\,\mathrm{eV} \leq m \leq 10^{-20}\,\mathrm{eV}$, still assuming dark matter comprises a single ALP [59].

In figure 3.2 we report the plot elaborated in [58] where it is shown the parameter space of $g_{a\gamma}(m)$ for axion-like dark matter from Keck and other collaborations.

The analysis performed in [60] uses the CMB cosmic birefringence phenomenon and assumes ultra-light ALP with $m_a \sim (10^{-27} - 10^{-24})$eV, it was found a coupling $g_{a\gamma} \lesssim (10^{-17} - 10^{-12})\mathrm{GeV}^{-1}$.

The experimental observation of a cosmic birefringence angle $\beta_{\mathrm{exp}} \approx 0.342\,\mathrm{deg}$ [1] is a new channel to constrain axion-photon coupling and other parameters of ALPs. In [17] authors investigate the possibility that (ALPs) with various potentials account for the isotropic birefringence. They also study the case of ALPs working as early dark energy[2] and simultaneously producing the observed isotropic cosmic birefringence. Same investigation is carried on in [55]. The cosmic birefringence opens a window in the search of ultra-light ALP field, in fact we expect axions to have a small enough mass not to oscillate until the last scattering epoch.[3] Otherwise, if the field rapidly oscillate during the photon decoupling epoch, its effective background value during this epoch is an average over oscillations and gets suppressed; in this case, as shown in (3.36), what mainly contributes to birefringence angle is the ALP field value today, and hence the isotropic birefringence is significantly suppressed [17].

[2]Early dark energy is a possible solution to the Hubble tension problem based on a dynamical dark energy field that can be a scalar field [63].

[3]The dynamics of a free scalar field in FLRW predicts that the field starts oscillate when $m \sim H$, we will justify this claim in section 4.4.

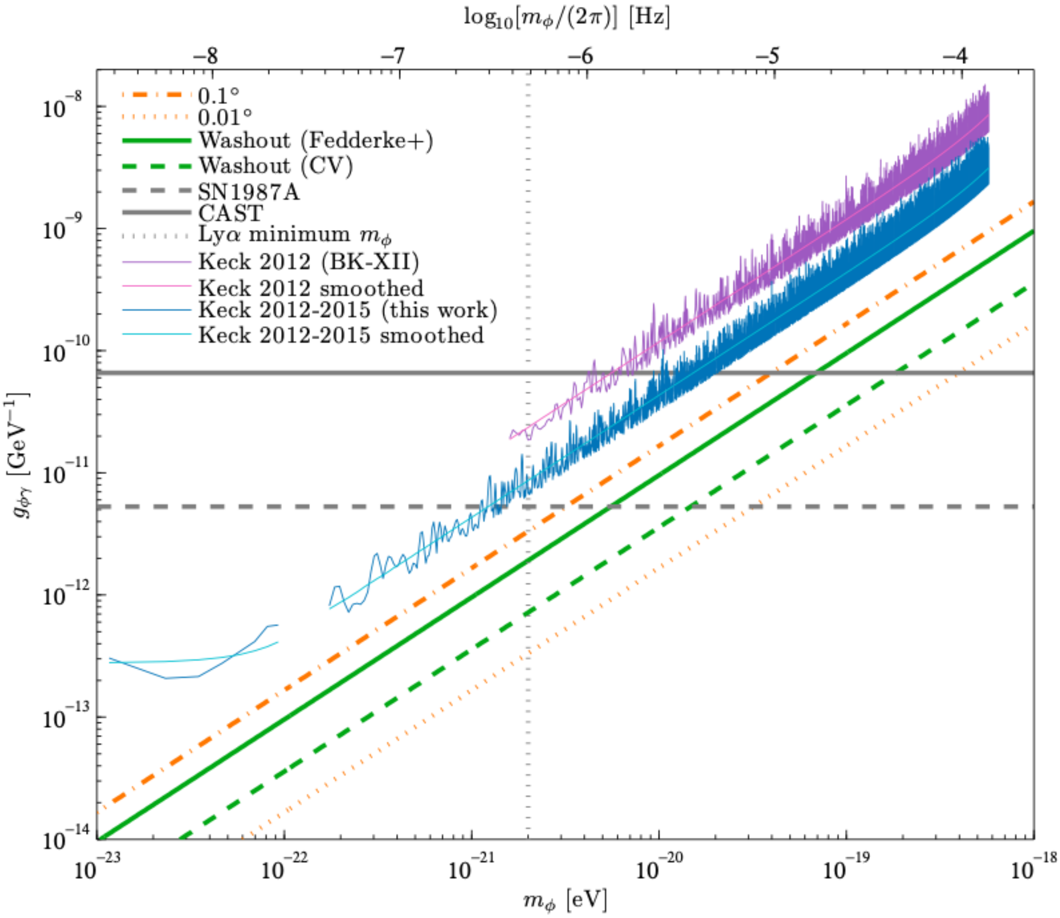

Figure 3.2: Parameter space for the axion-photon coupling $g_{\phi\gamma}$ as a function of the axion mass m_ϕ, assuming ALPs are dark matter. Allowed values are below the curves, where the allowed region is restricted to larger masses and smaller coupling constants, i.e. toward the bottom right of the figure. The blue line is the constraint found in [58] from which we have taken the figure; a smoothed approximation is shown in cyan color. The results of the same analysis performed to BK-XII are shown in purple and magenta. The orange dot-dashed and dotted lines show the constraints that would be achieved if the rotation amplitude were constrained to 0.1 deg and 0.01 deg, respectively. The green solid line shows the constraint set by Fedderke et al. [57] from the washout effect in Planck power spectra. The dashed green line shows the cosmic-variance limit for the washout effect. The dashed grey horizontal line shows the limit from the lack of a gamma-ray excess from SN1987A [61]. The solid grey horizontal line is the limit set by the CAST experiment [47]. The dotted grey vertical line is a constraint on the minimum axion mass (with axion as fuzzy dark matter) from observations of small-scale structure in the Lyman-α forest [62]. The figure (and almost all of the caption) is taken from [58].

Chapter 4

Reformulation of ModMax Lagrangian

We have seen how a CS coupling in the Maxwell's theory can produce a rotation of the polarization plane. What happens if we consider a NED theory coupled to external fields? Refs. [21],[22] show a procedure of building a general Lagrangian of electromagnetism coupled to a scalar and a pseudoscalar field. The authors of those papers have called the resulting theory as nonlinear axion-dilaton electrodynamics. Surprisingly the ModMax Lagrangian has similarities with this general theory.

4.1 ModMax and other fields

The idea of the nonlinear axion-dilaton electrodynamics in 4-dimension is to start from the building blocks of a Lorentz-invariant NED, S and P, and couple them to new fields. The electromagnetic field can be coupled to a scalar field φ in a easy way through a term $I = f(\varphi)S$ where the function $f(\varphi)$ depends on the model; this term is linear in S and may be nonlinear in the scalar field (depending on f). Theories with $\mathscr{L} = \mathscr{L}(I)$ have been denoted as dilaton electrodynamics (see e.g. [64–66]). To add also a pseudoscalar field ϕ, the easiest term to introduce is a CS coupling ϕP, which can be generalized to $J = g(\phi)P$ where $g(\phi)$ is a function depending on the model. Electromagnetic theories having pseudoscalar fields are called axion electrodynamics [67–69].

A unification of the dilaton and axion electrodynaimcs can be described by a Lagrangian which is a function of the four invariants S, P, I and J. This unification is a trend started some decades ago when people tried to unify the Maxwell-dilaton theory and axion electrodynamics [70–72], and then extended the procedure to formulate nonlinear versions of this unified theory [28, 73–75].

Using the spirit of Occam's razor concept, as done in [21], one can define the generalized invariant $\mathcal{I} = \mathcal{I}(S, P, I, J)$ and build a nonlinear axion-dilaton electrodynamics having a Lagrangian $\mathscr{L} = \mathscr{L}(\mathcal{I})$. In [21], the authors have used the internal symmetry of classical electrodynamics to motivate the following proposal for $\mathcal{I}$:

$$\mathcal{I} = S\cos\chi + P\sin\chi\,, \tag{4.1}$$

where $\chi(\mathbf{x}, t)$ is a pseudoscalar field. They argued that a single field is sufficient to mimic both axion and dilaton contributions into $\mathcal{I}$ because:

1. using trigonometric functions the theory will be invariant under the discrete symmetry $\chi \to \chi + 2\pi n$, where n is an integer, as required for axion electrodynamics;

2. the cosine is an even function of χ, so $S\cos\chi$ can be interpreted as a dilaton-like coupling to the electromagnetic field;

3. the sine is an odd function and so $P\sin\chi$ is a true invariant, it describes a nonlinear photon-axion interaction.

Finally, it is reasonable to assume that for small $\mathcal{I}$ the Lagrangian tends to $\mathscr{L}(\mathcal{I}) \to \mathcal{I}$, so that for $\chi \to 0$ one recovers the classical Lagrangian of axion electrodynamics: $\mathscr{L} = S + \chi P$. at first order in χ.

ModMax theory can be reformulated in a form similar to the axion-dilaton electrodynamics. In [20] it was proposed a Lagrangian density classically equivalent to ModMax obtained through the tool of auxiliary fields. The aim of the reformulation was to take out the square root in (1.96). The reformulated Lagrangian density is:

$$\mathscr{L} = \cosh\gamma S + \sinh\gamma(S\psi_1 + P\psi_2) - \frac{1}{2}\rho^2\big(\psi_1^2 + \psi_2^2 - 1\big)\,, \tag{4.2}$$

where ψ_1, ψ_2 and ρ are three auxiliary scalar fields. The equivalence of this Lagrangian with (1.96) is explicit computing the Eulero-Lagrange EoM for the auxiliary fields:

$$\sinh\gamma S = \rho^2\psi_1\,, \tag{4.3a}$$

$$\sinh\gamma P = \rho^2\psi_2\,, \tag{4.3b}$$

$$\rho\big(\psi_1^2 + \psi_2^2 - 1\big) = 0\,. \tag{4.3c}$$

For $\rho = 0$, equations (4.3a), (4.3b) implies $S = P = 0$: the null configuration. In this case the Lagrangian density reduces to the BB Lagrangian density.

In the case of $\rho \neq 0$, we insert equations (4.3a), (4.3b) into the Lagrangian density obtaining:

$$\mathscr{L} = \cosh\gamma S + \frac{1}{2}\big(\rho^{-2}\sinh^2\gamma\big(S^2 + P^2\big) + \rho^2\big) \tag{4.4}$$

and into the EoM for ρ (4.3c) finding:

$$\rho^4 = \sinh^2\gamma\big(S^2 + P^2\big)\,. \tag{4.5}$$

Substituting this back into equation (4.4) one gets the original ModMax Lagrangian:

$$\mathscr{L}_{\mathrm{MM}} = \cosh\gamma S + \sinh\gamma\sqrt{S^2 + P^2} \tag{4.6}$$

Using the square of ρ into equation (4.2) ensures that equation (4.5) has the only positive solution for ρ^2, assuming $\gamma > 0$.

An alternative solution for equation (4.3c) is given by $\psi_1 = \cos\chi$ and $\psi_2 = \sin\chi$, where χ is a single auxiliary field. Substituting these solutions into (4.2) one finds:

$$\mathscr{L} = S\cosh\gamma + \sinh\gamma(S\cos\chi + P\sin\chi)\,. \tag{4.7}$$

To go back to the original ModMax Lagrangian, one computes the EoM for χ and substitutes it into (4.7). The formulation (4.7) is interesting and it will be the building block of the work we are going to present in the next pages.

Based on its derivation, (4.7) is an equivalent, classical reformulation of ModMax Lagrangian, where χ is an auxiliary pseudoscalar field. The parenthesis in (4.7) are equivalent to the invariant $\mathcal{I}$ introduced in [21, 22]; the difference is that in an axion-dilaton electrodynamics the fields are dynamical. Therefore let us promote the auxiliary field χ to be an external pseudoscalar field and look what are the consequences. Doing so, we assume that ModMax might be an effective field theory originating from the γ-parametrized axion-dilaton electrodynamics theory (4.7), with the nonlinearity features of ModMax being symptoms of the axion-dilaton-photon coupling.

The invariant $\mathcal{I}$ of [21, 22] was built having in mind the $SO(2)$ duality of Maxwell. Thus we also expect similarities between (4.7) and other duality-invariant electrodynamics.

4.1.1 Duality symmetry and other fields

As a refresh, we recall that duality in electromagnetism was first observed in the Maxwell theory as the invariance of the EoM under a $SO(2)$ rotation of the doublet $(F^{\mu\nu}, \tilde{F}^{\mu\nu})$ [2].

Let us consider the linearized-ModMax Lagrangian density (4.7) and apply the following field redefinition:

$$a = -\sinh\gamma\sin\chi\,, \tag{4.8a}$$
$$e^{-\phi} = \cosh\gamma + \sinh\gamma\cos\chi\,. \tag{4.8b}$$

Then the Lagrangian (4.7) becomes:

$$\mathscr{L} = e^{-\phi}S - aP\,. \tag{4.9}$$

This is formally the Lagrangian density of the Maxwell's theory coupled to an axion a and a dilaton ϕ [20]. In such a theory the fields a, ϕ represent two degrees of freedom, thus the redefinition of (4.8) would be correct if we promote also γ to be a dynamical scalar field $\gamma = \gamma(t, \vec{x})$, in addition to the promotion done for the field χ.

The axion-dilaton Maxwell electrodynamics (4.9) is interesting because its EoM are $SL(2,\mathbb{R})$ invariant rather than $SO(2)$. This generalization is possible thanks to the coupling with the scalar field ϕ and the pseudoscalar field a. In [76] the authors found that the $SL(2,\mathbb{R})$ duality-invariance is achieved, at the level of the EoM, for the action

$$\mathcal{S} = \int \mathrm{d}^4x\,\sqrt{-g}\Big[R - 2(\partial\phi)^2 - \frac{1}{2}e^{4\phi}(\partial a)^2 - e^{-2\phi}F^2 - aF\tilde{F}\Big]\,, \tag{4.10}$$

which includes the kinetic terms for the axion and dilaton:

$$\mathscr{L}_{\text{kin-ax-dil}} = -\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}e^{2\phi}(\nabla a)^2\,. \tag{4.11}$$

Kinetic terms are themselves $SL(2,\mathbb{R})$ invariant. The Ricci scalar is not a problem as long the $SL(2,\mathbb{R})$ transformation of the electromagnetic and scalar fields does not produces a variation of the stress-energy tensor since it would be a source of variations of the metric. It can be shown that the stress-energy tensor derived from (4.10) is $SL(2,\mathbb{R})$ invariant [74, 76].

From these results we can see an analogy between the ModMax reformulation (4.7) and dilaton-axion Maxwell electrodynamics from the point of view of the duality symmetry. Dilaton-axion Maxwell electrodynamics has $SL(2,\mathbb{R})$-invariant EoM and this symmetry holds also for the theory (4.7) since it is preserved by the field redefinition (4.8), as long as γ is a dynamical field. But, in the ModMax theory γ is assumed to be a constant and this breaks $SL(2,\mathbb{R})$ to $SO(2)$.

For the sake of completeness we also introduce the history of the duality symmetry for NED theories. Few years after the result of [76], the $SO(2)$ duality was extended to

$SL(2,\mathbb{R})$ for NED theories by Gibbons & Rasheed [74].[1] The main result of their work is that, given a NED theory with Lagrangian density $\mathscr{L}_{\text{NED}} = \mathscr{L}_{\text{NED}}(g, F)$ with $SO(2)$ duality EoM (here g indicates the metric and F the field strength of a gauge vector field), one can build a new theory with Lagrangian density:

$$\mathscr{L} = R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}e^{2\phi}(\nabla a)^2 - aP + \mathscr{L}_{\text{NED}}\Big(g, e^{-\frac{1}{2}\phi}F\Big) + \text{const} \tag{4.12}$$

and the new theory has $SL(2,\mathbb{R})$ duality EoM. Here, as before, ϕ is a scalar field and a is a pseudoscalar field. Notice that the NED Lagrangian $\mathscr{L}_{\text{NED}}$ has a rescaled Faraday tensor as an argument: $F^{\mu\nu} \to e^{-\frac{1}{2}\phi}F^{\mu\nu}$.

As one would expect the two results (4.10) and (4.12) are analogous in form.

Our discussion concerns a duality at the level of the EoM. Having a theory duality-invariant at the level of the action is nontrivial. $SL(2,\mathbb{R})$ duality-invariant actions for an electromagnetic theory were constructed e.g. in [75, 77].

On the dilaton field We now give a very introductory comment about dilation in the context of high-energy physics and extra-dimensional theories, the aim to get an idea about one possible origin of these degrees of freedom.

In string theory axions and dilatons arise as low-energy degrees of freedom of the string [64]. The exponential coupling of the dilaton ϕ with a gauge field introduced above appears in a critical superstring theory in the low energy limit and after reduction from ten to four space-time dimensions. The effective action contains the following terms [64, 78]:

$$\mathcal{S} \supset \int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\,e^{-2\phi}\Big[R + 4(\partial\phi)^2 - F^2\Big]\,. \tag{4.13}$$

The 4-dimensional dilaton field ϕ is defined by a combination of its 10-dimensional counterpart and the volume of the 6-dimensional compact internal space [64].

The above action is written with respect to the string metric $\tilde{g}_{ab}$ (also referred to as the Jordan frame [78]). If one applies a Weyl transformation $g_{ab} = e^{-2\phi}\tilde{g}_{ab}$ at the level of the action, moving to the so-called Einstein frame, then [78–80]:

$$\mathcal{S} \supset \int \mathrm{d}^4x\,\sqrt{-g}\Big[R - 2(\partial\phi)^2 - e^{-2\phi}F^2\Big]\,. \tag{4.14}$$

If we consider the gauge vector field to be the photon, the term $e^{-2\phi}F^2$ defines the dilaton-Maxwell theory. Thus the exponential coupling of the dilaton has an origin from compactification of string theories.

4.1.2 The ModMax precursor and its limits

In light of the above discussion, we will work with the "upgraded" ModMax theory with Lagrangian density (4.7) in which we have introduced a new degree of freedom: the pseudoscalar field $\chi(t,\vec{x})$. Since here we consider χ as a physical field, we add its kinetic term and a potential to the Lagrangian density. Then the (new) theory is described by:

$$\mathscr{L} = S\cosh\gamma + \sinh\gamma\left(S\cos\left(\frac{\chi}{g}\right) + P\sin\left(\frac{\chi}{g}\right)\right) - \frac{1}{2}g_{\mu\nu}\nabla^\mu\chi\nabla^\nu\chi - V(\chi)\,. \tag{4.15}$$

The γ parameter still parametrizes the family of ModMax-like theories, with Maxwell electrodynamics at $\gamma = 0$. Do note the argument of trigonometric functions in (4.18) must

[1] Their work was published some months later their results on the $SO(2)$ duality of a NED theory [28].

be dimensionless, thus we introduced a rescaled field χ/g, where g is a quantity with the dimension of an energy and so the overall term χ/g is dimensionless.

In [20] it was pointed out that the "original" Lagrangian density of ModMax, as stated in (1.96), predicts a rescaled electric charge by a factor $e^{-\gamma}$ with respect to the one of Maxwell's theory. Consequences are, for example, a screening of the Coulomb field of an electric particle at rest [20],

$$\vec{E} = e^{-\gamma} \frac{e}{4\pi} \frac{\vec{r}}{r^3} \,, \tag{4.16}$$

and thus of the Coulomb force. Here e is the electric charge and $\vec{r}$ is the position vector in space with the particle placed at $\vec{r} = 0$, r is the modulus.

This difference is a matter of convention on the overall coefficient of ModMax Lagrangian (1.96), if one rescales it by a factor $e^{-\gamma}$ then the field equations, the charge and Coulomb law will be the same as in Maxwell's theory [20]. This tuning on the electric sector is balanced out by the magnetic sector which acquires a rescaling by e^{γ} [20].[2] Therefore, we choose to work with the rescaled ModMax Lagrangian:

$$\mathscr{L}_{\text{MM}} = e^{-\gamma} \Big[\cosh\gamma S + \sinh\gamma \sqrt{S^2 + P^2} \Big] \,, \tag{4.17}$$

then going through the same machinery that furnish us the Lagrangian (4.15), we obtain its rescaled formulation:

$$\mathscr{L} = e^{-\gamma} \left[S\cosh\gamma + \sinh\gamma \left(S\cos\left(\frac{\chi}{g}\right) + P\sin\left(\frac{\chi}{g}\right) \right) \right] - \frac{1}{2} g_{\mu\nu} \nabla^{\mu}\chi \nabla^{\nu}\chi - V(\chi) \,. \tag{4.18}$$

We call this theory *ModMax precursor*. Let us now see if there is some interesting limits of this theory.

Limits of the ModMax precursor Let us consider the electrodynamics part of the ModMax precursor (4.18). Interesting limits of this theory are achieved by $\chi/g \to 0$ and $\gamma \to 0$. The former limit should give an electrodynamics theory with polynomial couplings $P\chi/g, S(\chi/g)^2, \ldots$ because of the expansion of the trigonometric functions. The latter limit should drive the theory towards axion-dilaton Maxwell electrodynamics since ModMax parameter γ going to zero points to Maxwell but external scalar fields are kept. Finally taking both limits together we should find the Maxwell's theory.

In figure 4.1 we sketch these limits approximating up to first order in χ and γ, namely neglecting any contribution $\mathcal{O}(\gamma^2, (\chi/g)^2, \gamma\chi/g)$. These limits reflect exactly the expectations. From the figure we can see the role of γ: it parametrizes families of electrodynamics theories and, depending on the value of χ/g, the theory shows an axion or an axion-dilaton behavior. Only for $\gamma = 0$ Maxwell's theory is restored.

From figure 4.1 we see the convenience of using the rescaled Lagrangian (4.18): in any limit the theory reduces to a Lagrangian with the first term that is exactly the Maxwell Lagrangian plus corrections at first order.

Taking $\gamma, \chi/g \to 0$ we see that the theory (4.18) reduces to Maxwell with no corrections up to first order; so let us see what is the first signature of ModMax going up to *second*

[2] The idea is to fix the coefficients such that the electric sector equal to the Maxwell one and this is balanced by the magnetic sector which acquires the opposite rescaling. In fact the rescaled theory has a modified duality-invariance condition, $G_{\mu\nu}\tilde{G}^{\mu\nu} = -e^{-2\gamma} F_{\mu\nu}\tilde{F}^{\mu\nu}$, and so the doublet which maintains the $SO(2)$ duality will be $(\tilde{G}^{\mu\nu}, e^{-\gamma}\tilde{F}^{\mu\nu})$ instead of $(\tilde{G}^{\mu\nu}, \tilde{F}^{\mu\nu})$ [20] (notice our formulas are lightly different with respect to [20] because of we used a different definition for $\tilde{G}$ and the doublet).

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{ModMax precursor ansatz}\\ \mathscr{L} = e^{-\gamma}\left[S\cosh\gamma + \sinh\gamma\left(S\ \cos\frac{\chi}{g} + P\ \sin\frac{\chi}{g}\right)\right]\end{array} \arrow[r, "\gamma\ \to\ 0"] \arrow[d, "\chi/g\ \ll 1"'] & \begin{array}{c}\text{axion-dilaton ED}\\ \mathscr{L} \approx S - \gamma\left[S - \left(S\cos\frac{\chi}{g} + P\ \sin\frac{\chi}{g}\right)\right]\end{array} \arrow[d, "\chi/g\ \ll 1"] \\
\begin{array}{c}\gamma\text{-family of axion ED}\\ \mathscr{L} \approx S + e^{-\gamma}\sinh(\gamma)\ P\ \frac{\chi}{g}\end{array} \arrow[r, "\gamma\ \to\ 0"'] & \begin{array}{c}\text{Maxwell ED}\\ \mathscr{L} \approx S\end{array}
\end{tikzcd}
$$

Figure 4.1: Relavant limits of the ModMax precursor theory (4.18). Lagrangians are expanded up to first order in γ and χ/g, neglecting $\mathcal{O}(\gamma^2, (\chi/g)^2, \gamma\chi/g)$.

order:

$$
\begin{aligned}
\mathscr{L} &\approx \left(1 - \gamma + \frac{\gamma^2}{2}\right)\left[\left(1 + \frac{\gamma^2}{2}\right)S + \gamma\left(S\left(1 - \frac{1}{2}\left(\frac{\chi}{g}\right)^2\right) + P\frac{\chi}{g}\right)\right] && \text{(4.19a)}\\
&\approx \left(1 - \gamma + \frac{\gamma^2}{2}\right)\left[\left(1 + \gamma + \frac{\gamma^2}{2}\right)S + \gamma\frac{\chi}{g}P\right] && \text{(4.19b)}\\
&\approx S\left(1 - \gamma + \frac{\gamma^2}{2}\right)\left(1 + \gamma + \frac{\gamma^2}{2}\right) + \gamma\frac{\chi}{g}P && \text{(4.19c)}\\
&\approx S + \gamma P\frac{\chi}{g}\,. && \text{(4.19d)}
\end{aligned}
$$

This is exactly the Lagrangian of the Maxwell-CS theory, that has usually the following Lagrangian density:

$$
\mathscr{L}_{\text{M-CS}} = S + \frac{\alpha}{2f}P\chi\,. \tag{4.20}
$$

We conclude that the ModMax precursor theory (4.18) mimics the Maxwell-CS electrodynamics in the limits of small γ and χ/g. In these limits:

$$
\frac{\gamma}{g} \sim \frac{\alpha}{2f}\,. \tag{4.21}
$$

In Section 4.3 we will study the cosmic birefringence in the theory (4.18) make some comments about the well-known results of the cosmic birefringence in the Maxwell-CS theory.

4.2 Stress-energy tensor of the theory

Let us compute the stress-energy tensor of the theory (4.18). We can identify three Lagrangian densities: the Maxwell Lagrangian times a function of the ModMax parameter, an interaction Lagrangian and a free-scalar field Lagrangian:

$$
\mathscr{L} = \mathscr{L}_{\gamma\text{M}} + \mathscr{L}_{\text{int}} + \mathscr{L}_{\chi}\,, \tag{4.22}
$$

where

$$\mathscr{L}_{\gamma\mathrm{M}} = e^{-\gamma}\cosh\gamma S\,, \tag{4.23}$$

$$\mathscr{L}_{\mathrm{int}} = e^{-\gamma}\sinh\gamma\left(S\cos\left(\frac{\chi}{g}\right) + P\sin\left(\frac{\chi}{g}\right)\right), \tag{4.24}$$

$$\mathscr{L}_{\chi} = -\frac{1}{2}g_{\mu\nu}\nabla^{\mu}\chi\nabla^{\nu}\chi - V(\chi)\,. \tag{4.25}$$

From the definition of the stress-energy tensor we expect that the total stress-energy tensor of the theory will be the sum of three terms, one per each Lagrangian density:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{S}}{\delta g^{\mu\nu}} = T^{(\gamma\mathrm{M})}_{\mu\nu} + T^{(\mathrm{int})}_{\mu\nu} + T^{(\chi)}_{\mu\nu} \tag{4.26a}$$

Let us compute each stress-energy tensor individually:

$$\begin{aligned}
T^{(\gamma\mathrm{M})}_{\mu\nu} &= e^{-\gamma}\cosh\gamma\left(-\frac{2}{\sqrt{-g}}\right)\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^4x\,\sqrt{-g}S &&(4.27\mathrm{a})\\
&= e^{-\gamma}\cosh\gamma\left(-\frac{2}{\sqrt{-g}}\right)\int \mathrm{d}^4x\left[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}S + \sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}\left(-\frac{1}{4}g^{\alpha\sigma}g^{\beta\rho}F_{\alpha\beta}F_{\sigma\rho}\right)\right] &&(4.27\mathrm{b})\\
&= e^{-\gamma}\cosh\gamma\left(-\frac{2}{\sqrt{-g}}\right)\int \mathrm{d}^4x\left[-\frac{\sqrt{-g}}{2}g_{\mu\nu}S + \sqrt{-g}\left(-\frac{1}{2}F_{\mu}{}^{\alpha}F_{\nu\alpha}\right)\right]\delta(x-y) &&(4.27\mathrm{c})\\
&= e^{-\gamma}\cosh\gamma(g_{\mu\nu}S + F_{\mu}{}^{\alpha}F_{\alpha\nu}) &&(4.27\mathrm{d})
\end{aligned}$$

that is the Maxwell's stress-energy tensor in curved spacetime with a factor $\cosh\gamma$, which keep tracks of the ModMax model. The stress-energy tensor coming from the interaction part of the Lagrangian density is:

$$\begin{aligned}
T^{(\mathrm{int})}_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^4x\,\sqrt{-g}e^{-\gamma}\sinh\gamma\left(S\cos\left(\frac{\chi}{g}\right) + P\sin\left(\frac{\chi}{g}\right)\right) &&(4.28\mathrm{a})\\
&= -\frac{2e^{-\gamma}\sinh\gamma}{\sqrt{-g}}\int \mathrm{d}^4x\left[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\left(S\cos\left(\frac{\chi}{g}\right) + P\sin\left(\frac{\chi}{g}\right)\right)\right.\\
&\qquad\left. + \sqrt{-g}\left(\frac{\delta S}{\delta g^{\mu\nu}}\cos\left(\frac{\chi}{g}\right) + \frac{\delta P}{\delta g^{\mu\nu}}\sin\left(\frac{\chi}{g}\right)\right)\right]. &&(4.28\mathrm{b})
\end{aligned}$$

Let us evaluate the variation of the Lorentz-invariant $P = -\tilde{F}^{\alpha\beta}F_{\alpha\beta}/4$. We recall that in a general spacetime with metric $g^{\mu\nu}$, the Levi-Civita tenosr is $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}$ with $\varepsilon_{\mu\nu\rho\sigma}$ the Levi-Civita symbol, i.e. the total antisymmetric object so that $\varepsilon_{0123} = 1$. Thus:

$$\begin{aligned}
\frac{\delta P}{\delta g^{\mu\nu}} &= -\frac{1}{4}\frac{\delta}{\delta g^{\mu\nu}}\left[\frac{1}{2}\frac{\varepsilon^{\alpha\beta\rho\sigma}}{\sqrt{-g}}F_{\rho\sigma}F_{\alpha\beta}\right] &&(4.29\mathrm{a})\\
&= -\frac{1}{4}\frac{\delta(\sqrt{-g})^{-1}}{\delta g^{\mu\nu}}\left[\frac{1}{2}\varepsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma}F_{\alpha\beta}\right] &&(4.29\mathrm{b})\\
&= -\frac{1}{4}\left(-\frac{1}{2\sqrt{-g}}\right)g_{\mu\nu}\varepsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma}F_{\alpha\beta}\delta(x-y) &&(4.29\mathrm{c})\\
&= \frac{1}{2}g_{\mu\nu}P\delta(x-y)\,. &&(4.29\mathrm{d})
\end{aligned}$$

The stress-energy tensor from the interaction part reads:

$$T^{(\mathrm{int})}_{\mu\nu} = -\frac{2e^{-\gamma}\sinh\gamma}{\sqrt{-g}}\int \mathrm{d}^4x\left[-\frac{\sqrt{-g}}{2}g_{\mu\nu}\left(S\cos\left(\frac{\chi}{g}\right)+P\sin\left(\frac{\chi}{g}\right)\right)\right. \tag{4.30a}$$

$$\left.+\sqrt{-g}\left(-\frac{1}{2}F_\mu{}^\alpha F_{\nu\alpha}\cos\left(\frac{\chi}{g}\right)+\frac{1}{2}g_{\mu\nu}P\sin\left(\frac{\chi}{g}\right)\right)\right]\delta(x-y) \tag{4.30b}$$

$$= e^{-\gamma}\sinh\gamma\left[g_{\mu\nu}\left(S\cos\left(\frac{\chi}{g}\right)+P\sin\left(\frac{\chi}{g}\right)\right)+F_\mu{}^\alpha F_{\nu\alpha}\cos\left(\frac{\chi}{g}\right)-g_{\mu\nu}P\sin\left(\frac{\chi}{g}\right)\right] \tag{4.30c}$$

$$= e^{-\gamma}\sinh\gamma\left[g_{\mu\nu}S\cos\left(\frac{\chi}{g}\right)+F_\mu{}^\alpha F_{\nu\alpha}\cos\left(\frac{\chi}{g}\right)\right] \tag{4.30d}$$

$$= e^{-\gamma}\sinh\gamma[g_{\mu\nu}S+F_\mu{}^\alpha F_{\nu\alpha}]\cos\left(\frac{\chi}{g}\right). \tag{4.30e}$$

The last contribution is

$$T^{(\chi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^4x\,\sqrt{-g}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi - V(\chi)\right) \tag{4.31a}$$

$$= -\frac{2}{\sqrt{-g}}\int \mathrm{d}^4x\left[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi - V(\chi)\right)-\frac{1}{2}\sqrt{-g}\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}}\partial_\alpha\chi\partial_\beta\chi\right] \tag{4.31b}$$

$$= \partial_\mu\chi\partial_\nu\chi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi + V(\chi)\right), \tag{4.31c}$$

where in the second line we have assumed that the potential does not depend on the metric. The result is the stress-energy tensor of a scalar field in a spacetime with metric $g_{\mu\nu}$.

Along these computations we have used the well-known results:

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = \frac{\partial\sqrt{-g}}{\partial g^{\alpha\beta}}\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta(x-y)\,, \tag{4.32}$$

$$\frac{\delta g^{\rho\sigma}(x)}{\delta g^{\mu\nu}(y)} = \delta^\rho{}_\mu\delta^\sigma{}_\nu\delta(x-y)\,, \tag{4.33}$$

the first two deltas are Kronecker type, while the third one is a 4-dimensional Dirac delta.

Putting all together, the stress-energy tensor for the theory (4.18) reads:[3]

$$T_{\mu\nu} = e^{-\gamma}\cosh\gamma(g_{\mu\nu}S - F_\mu{}^\alpha F_{\alpha\nu}) \tag{4.35a}$$

$$+e^{-\gamma}\sinh\gamma\left(g_{\mu\nu}S+F_\mu{}^\alpha F_{\nu\alpha}\right)\cos\left(\frac{\chi}{g}\right) \tag{4.35b}$$

$$+\partial_\mu\chi\partial_\nu\chi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi + V(\chi)\right). \tag{4.35c}$$

[3] Same result can be obtained using the definition

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}}{\delta g^{\mu\nu}} \tag{4.34}$$

of the stress-energy tensor. This formulation does not require to pass through the integral.

4.3 ModMax cosmic birefringence

The theory (4.18) is linear in the Lorentz-invariant electromagnetic scalars S and P and they are coupled to an ALP, then we saw that it is able to mimic the Maxwell-CS theory. These are hints that (4.18) might generate cosmic birefringence. We investigate this scenario with the same prescription used to derive β for the Maxwell-CS theory.

Let us consider a flat-FLRW spacetime and start computing EoM for the 4vector A^μ:

$$0 = \frac{\partial \mathscr{L}}{\partial A_\nu} - \nabla_\mu \left[\frac{\partial \mathscr{L}}{\partial(\nabla_\mu A_\nu)} \right] \tag{4.36a}$$

$$= -\nabla_\mu \left[\cosh\gamma \frac{\partial S}{\partial F_{\alpha\beta}} \frac{\partial F_{\alpha\beta}}{\partial(\nabla_\mu A_\nu)} + \sinh\gamma \left(\frac{\partial S}{\partial F_{\alpha\beta}} \frac{\partial F_{\alpha\beta}}{\partial(\nabla_\mu A_\nu)} \cos\left(\frac{\chi}{g}\right) + \frac{\partial P}{\partial F_{\alpha\beta}} \frac{\partial F_{\alpha\beta}}{\partial(\nabla_\mu A_\nu)} \sin\left(\frac{\chi}{g}\right) \right) \right] \tag{4.36b}$$

$$= -\nabla_\mu \left[-\cosh\gamma F^{\mu\nu} + \sinh\gamma \left(-F^{\mu\nu} \cos\left(\frac{\chi}{g}\right) - \tilde{F}^{\mu\nu} \sin\left(\frac{\chi}{g}\right) \right) \right] , \tag{4.36c}$$

then differentiating:

$$0 = \left[\cosh\gamma + \sinh\gamma \cos\left(\frac{\chi}{g}\right) \right] \nabla_\mu F^{\mu\nu} + \sinh\gamma \left[\left(\nabla_\mu \cos\left(\frac{\chi}{g}\right) \right) F^{\mu\nu} + \left(\nabla_\mu \sin\left(\frac{\chi}{g}\right) \right) \tilde{F}^{\mu\nu} \right] , \tag{4.37}$$

where we used the Bianchi identity $\nabla_\mu \tilde{F}^{\mu\nu} = 0$. Let us now define the following 4-vectors:

$$C_\mu \equiv \nabla_\mu \cos\left(\frac{\chi}{g}\right) , \tag{4.38a}$$

$$S_\mu \equiv \nabla_\mu \sin\left(\frac{\chi}{g}\right) , \tag{4.38b}$$

and the following scalar quantity:

$$D(\gamma, \chi/g) \equiv \cosh\gamma + \sinh\gamma \cos\left(\frac{\chi}{g}\right) . \tag{4.39}$$

Then the EoM reduces to the more readable form:

$$D(\gamma, \chi/g) \nabla_\mu F^{\mu\nu} = -\sinh\gamma \left[C_\mu F^{\mu\nu} + S_\mu \tilde{F}^{\mu\nu} \right] . \tag{4.40}$$

As done for the Maxwell-CS theory, let us use the Coulomb gauge $\nabla_\mu A^\mu = 0$ together with $A^0 = 0$, and consider a pseudoscalar field $\chi = \chi(\eta)$ which depends only on the conformal time. The 4vectors C_μ, S_μ have only the zeroth component different from zero. The EoM of the 4potential reads:

$$D(\gamma, \chi/g) \Box \vec{A} = \sinh\gamma \left[C_0 \vec{A}' - 2 S_0 \vec{\nabla} \times \vec{A} \right] , \tag{4.41}$$

where $\Box \equiv \nabla^\mu \nabla_\mu$. Here we have a new contribution $C_0 \vec{A}'$ with respect to the Maxwell-CS theory due to the dilaton-like coupling, some coefficients that depend on γ coming from the ModMax theory and the function $D(\gamma, \chi/g)$.

Consistency check Let us make a comparison between EoM (4.40) and the Maxwell-CS EoM (3.7), with no gauge fixing chosen yet. First of all χ must be a dynamical (i.e. non-constant) field, otherwise EoM reduces to Maxwell equations. When the field χ is small, EoM (4.40) reduces to:

$$e^{\gamma}\nabla_{\mu}F^{\mu\nu} = -\sinh\gamma\nabla_{\mu}\left(\frac{\chi}{g}\right)\tilde{F}^{\mu\nu} \tag{4.42}$$

that is a ModMax (i.e. γ-dependent) generalization of standard Maxwell-axion electrodynamics $\nabla_{\mu}F^{\mu\nu} = -\nabla_{\mu}a\tilde{F}^{\mu\nu}$ where a is the axion field [81]. Now, if we take the limit $\chi/g \ll 1$ we find the EoM of the Maxwell-CS theory:

$$(1+\gamma)\nabla_{\mu}F^{\mu\nu} = -\gamma\nabla_{\mu}\left(\frac{\chi}{g}\right)\tilde{F}^{\mu\nu}\,. \tag{4.43}$$

For $\gamma = 0$ we get the free-Maxwell equations, no χ-photon coupling survives as expected.

Now, let us use the same decomposition introduced in section 3, namely we consider a k-mode of the 4potential of the form:

$$\vec{A}_k(\eta,\vec{x}) = \vec{\mathcal{A}}_k(\eta)\exp\left(i\vec{k}\cdot\vec{x}\right). \tag{4.44}$$

Thus, EoM in Fourier space reads:

$$D(\gamma,\chi/g)\left[\vec{\mathcal{A}}'' + k^2\vec{\mathcal{A}}\right] = -\sinh\gamma\left[C_0\vec{\mathcal{A}}' - 2iS_0\vec{k}\times\vec{\mathcal{A}}\right]. \tag{4.45}$$

It takes the same form as in Minkowski spacetime because of the 4potential is massless.

Assume a propagation along z-axis and let us forget the pedix $_k$, even if we are still working with a single mode. Recall we are in the Coulomb gauge, thus components of $\vec{\mathcal{A}}$ different from zero are $\mathcal{A}_x, \mathcal{A}_y$ only. So the EoM by components is:

$$D(\gamma,\chi/g)\left[\mathcal{A}_x'' + k^2\mathcal{A}_x\right] = -\sinh\gamma\left[C_0\mathcal{A}_x' - 2iS_0(-k\mathcal{A}_y)\right], \tag{4.46a}$$
$$D(\gamma,\chi/g)\left[\mathcal{A}_y'' + k^2\mathcal{A}_y\right] = -\sinh\gamma\left[C_0\mathcal{A}_y' - 2iS_0(k\mathcal{A}_x)\right]. \tag{4.46b}$$

As done before, we introduce the helicity states $\mathcal{A}_+, \mathcal{A}_-$. In this new basis, the system reads:

$$D(\gamma,\chi/g)\left[\mathcal{A}_+'' + k^2\mathcal{A}_+\right] + \sinh\gamma C_0\mathcal{A}_+' - 2\sinh\gamma S_0 k\mathcal{A}_+ = 0\,, \tag{4.47a}$$
$$D(\gamma,\chi/g)\left[\mathcal{A}_-'' + k^2\mathcal{A}_-\right] + \sinh\gamma C_0\mathcal{A}_-' + 2\sinh\gamma S_0 k\mathcal{A}_- = 0\,. \tag{4.47b}$$

that we write in the following compact form:

$$D(\gamma,\chi/g)\left[\mathcal{A}_\pm'' + k^2\mathcal{A}_\pm\right] + \sinh\gamma C_0\mathcal{A}_\pm' \mp 2\sinh\gamma S_0 k\mathcal{A}_\pm = 0\,. \tag{4.48}$$

Equation (4.48) is the EoM for the k-mode of the Fourier decomposition of the 4potential written with helicity states in the Coulomb gauge for the ModMax precursor theory (4.18).

4.3.1 Solution with constant frequency

A linear polarized plane-wave of the form

$$\mathcal{A}_\pm(\eta) = C_\pm\exp[i(-\omega\eta + \delta_\pm)] \tag{4.49}$$

with constant pulsation $\omega_\pm = \text{const}$ is a simple solution of the EoM (4.48). To find the pulsation we insert our ansatz into the EoM. From now on we just write $D(\gamma, \chi)$ as D to simplify the notation:

$$0 = D\big[(-i\omega)^2 A_\pm + k^2 A_\pm\big] + \sinh\gamma C_0(-i\omega)A_\pm \mp 2\sinh\gamma S_0 k A_\pm \tag{4.50a}$$
$$= \big[-\omega^2 D - i\omega\sinh\gamma C_0 \mp 2k\sinh\gamma S_0 + k^2 D\big]A_\pm \,. \tag{4.50b}$$

Indeed the pulsation ω must solve:

$$\omega^2 D + i\omega\sinh\gamma C_0 \pm 2k\sinh\gamma S_0 - k^2 D = 0\,. \tag{4.51}$$

Solutions to this equation are complex numbers and this is a consequence of the presence of a first-derivative in the EoM. We consider an ansatz $\omega = A + iB$ where A, B are real functions. To simplify notations we use the following definitions:

$$\sinh\gamma C_0 \equiv C\,, \tag{4.52}$$
$$2\sinh\gamma S_0 \equiv Z\,. \tag{4.53}$$

Then the equation for ω reads:

$$0 = D(A+iB)^2 + iC(A+iB) \pm Zk - Dk^2 \tag{4.54a}$$
$$= D(A^2 + 2iAB - B^2) + iCA - CB \pm Zk - Dk^2 \tag{4.54b}$$
$$= (DA^2 - DB^2 - CB \pm Zk - Dk^2) + i(2DAB + CA)\,. \tag{4.54c}$$

The equation is satisfied when both the real and imaginary contributions are zero:

$$2DAB + CA = 0\,, \tag{4.55}$$
$$DA^2 - DB^2 - CB \pm Zk - Dk^2 = 0\,. \tag{4.56}$$

Solutions of the system are:

$$B = -\frac{C}{2D}\,, \tag{4.57}$$
$$A^2 = k^2 \mp \frac{Z}{D}k - \frac{C^2}{4D^2}\,. \tag{4.58}$$

Let us take the square root and consider only the plus solution for A^2:

$$A = k\left[1 \mp \frac{1}{D}\left(\frac{Z}{k}\right) - \frac{1}{4D^2}\left(\frac{C}{k}\right)^2\right]^{1/2} \approx k\left[1 \mp \frac{1}{2D}\left(\frac{Z}{k}\right) - \frac{1}{8D^2}\left(\frac{C}{k}\right)^2\right], \tag{4.59}$$

where we have approximated the square root since $Z/k \ll 1$ and $C/k \ll 1$. The reason is the same exposed in [16]: the conformal time derivative present in Z and C is of the order of the visible universe while k^{-1} is of the order of the photon wavelength.

Putting all together we obtain that the pulsation of the states $\mathcal{A}_+$ and $\mathcal{A}_-$ is:

$$\omega_\pm = A + iB = k\left[1 \mp \frac{1}{2D}\left(\frac{Z}{k}\right) - \frac{1}{8D^2}\left(\frac{C}{k}\right)^2\right] - i\frac{C}{2D}\,. \tag{4.60}$$

Recalling how we defined C, D, Z, we have

$$\begin{aligned}\omega_\pm =& k\left[1 \mp \frac{1}{2(\cosh\gamma + \sinh\gamma\cos(\chi/g))}\left(\frac{2\sinh\gamma\nabla_0\sin(\chi/g)}{k}\right)\right.\\ &\left. - \frac{1}{8(\cosh\gamma + \sinh\gamma\cos(\chi/g))^2}\left(\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{k}\right)^2\right]\\ &- i\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{2(\cosh\gamma + \sinh\gamma\cos(\chi/g))}\,.\end{aligned} \tag{4.61}$$

These are the pulsations for a linear polarized plane-wave A^μ written in the $+$ and $-$ polarization basis. Each mode k has a different pulsation ω_+ and ω_- respectively, and as a consequence the condition for cosmic birefringence is achieved.

The imaginary part $-iC/(2D)$ corresponds to an exponential modulation of the amplitude of the 4potential. Notice the denominator D is always positive by definition, since $\gamma > 0$ and $\cosh\gamma \geq 1$ and $\cosh\gamma \geq \sinh\gamma$; the numerator will determine the sign of the exponential, i.e. the dynamics of χ.

This contribution is due to the first-derivative in the EoM, which comes from the dilaton-like photon coupling. This does not happen for the axion-photon coupling because of the antisymmetric nature of $F_{\mu\nu}$ (i.e. Bianchi identity). It is interesting to notice that the imaginary part does not depend on the wavelength $\sim k^{-1}$.

If we observe the result (4.61) we see that the only difference between the pulsation of the $+$ and $-$ polarization state comes from the $\pm$ sign in front of the second term of (4.61). It is due to the axion-photon coupling (in fact it is $\propto \nabla_0\sin(\chi/g)$) that contains the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ into P, it mixes cartesian components A_i in an antisymmetric way using a vector product, as shown in equation (4.45), and the result is the $\mp$ sign. On the other hand the dilaton-like photon coupling does not distinguish between $+$ and $-$ polarization.

To conclude, we recall that equation (4.61) holds with the stringent assumption of $\omega_\pm = \text{const}$. This assumption binds the dynamics of χ since it constraints its first derivative making the ansatz of a plane wave with constant frequency unrealistic.

Consistency check It is interesting to expand the pulsation $\omega_\pm$ of equation (4.61) in the cases of small χ/g and γ, up to first order for example. Assuming a small field χ/g in the Lagrangian density (4.7) means to consider a theory with a linear axion-photon interaction $P\sin(\chi/g) \sim P\chi/g$ and a negligible dilaton-like photon interaction $S\cos(\chi/g) \sim S$. In this case the pulsation (4.61) reads:

$$\omega_\pm \approx k\left[1 \mp \frac{1}{2\exp(2\gamma)}\left(2\frac{\chi'}{g}\frac{\sinh\gamma}{k}\right)\right], \tag{4.62}$$

where $\nabla_0\sin(\chi/g) \sim \partial_\eta(\chi/g) \equiv \chi'/g$ and $\nabla_0\cos(\chi/g) \sim 0$ in our expansion. It is easy to see that when $\gamma = 0$ Maxwell dispersion relation $\omega = k$ is restored, as expected.

In the extreme limit of no pseudoscalar field, namely $\chi = 0$, we find the usual dispersion relation $\omega_\pm^2 = k^2$.

We now compute the phase and group velocity from equation (4.61). The phase velocity is:

$$
\begin{aligned}
v_{\mathrm{p},\pm} \equiv \frac{\omega_\pm}{k} = \Bigg[1 &\mp \frac{1}{2(\cosh\gamma + \sinh\gamma\cos(\chi/g))}\left(\frac{2\sinh\gamma\nabla_0\sin(\chi/g)}{k}\right) \\
&- \frac{1}{8(\cosh\gamma + \sinh\gamma\cos(\chi/g))^2}\left(\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{k}\right)^2\Bigg] \\
&- \frac{i}{k}\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{2(\cosh\gamma + \sinh\gamma\cos(\chi/g))}\,,
\end{aligned} \tag{4.63}
$$

where a complex part was expected because of $\omega_\pm \in \mathbb{C}$. However, the imaginary part of the pulsation does not depend on k and so we foresee a real group velocity. Computing it:

$$
v_{\mathrm{g},\pm} \equiv \frac{\mathrm{d}\omega_\pm}{\mathrm{d}k} = 1 + \frac{1}{8(\cosh\gamma + \sinh\gamma\cos(\chi/g))^2}\left(\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{k}\right)^2 . \tag{4.64}
$$

Let us now make a comment regarding the comparison between the group velocity (4.64), which holds for the ModMax precursor theory, and the group velocity (3.31) that we have computed from the Maxwell-CS theory. In the Maxwell-CS theory we had to expand the dispersion relation up to second order in χ' to find the first signature from the axion in the group velocity. Now, in the ModMax precursor theory, we have found a group velocity deriving the pulsation (4.61) that was obtained expanding up to *first* order in χ' the dispersion relation: into (4.64) we used the ansatz $\omega_\pm = A + iB$ where A is an expansion up to first order.

The reason of this difference is the following: in the Maxwell-CS theory the first non-trivial contribution to the group velocity comes from the axion and it is of second order; in the ModMax precursor theory the first non-trivial contribution comes from the dilaton and it is of first order. If we had expanded up to second order (4.61), we would have seen an axion correction too.

In addition, the group velocity (4.64) up to first order is bigger than 1.

4.3.2 Solution with slowly varying frequency

Up to now we have considered linearly polarized plane wave solutions of the EoM (4.48),

$$
D\big[\mathcal{A}''_\pm + k^2\mathcal{A}_\pm\big] + \sinh\gamma C_0\mathcal{A}'_\pm \mp 2\sinh\gamma S_0 k\mathcal{A}_\pm = 0\,, \tag{4.65}
$$

with a pulsation $\omega_\pm = \text{const}$. The aim of this section is to give a more physical ansatz allowing a slowly-varying frequency. We will look for an approximated solution as done in the Maxwell-CS theory. Firstly, let us use the definitions (4.52) in order to ease the notation:

$$
\sinh\gamma C_0 \equiv C\,, \tag{4.66}
$$

$$
2\sinh\gamma S_0 \equiv Z\,. \tag{4.67}
$$

Then the differential equation (4.48) reads:

$$
D\big[\mathcal{A}''_\pm + k^2\mathcal{A}_\pm\big] + C\mathcal{A}'_\pm \mp ZkA_\pm = 0\,. \tag{4.68}
$$

This equation is of the form:

$$
y'' + f(\eta)y' + g_\pm(\eta)y = 0\,, \tag{4.69}
$$

where:

$$y = \mathcal{A}_\pm \,, \tag{4.70a}$$
$$f = C/D \,, \tag{4.70b}$$
$$g_\pm = k^2 \mp kZ/D \,. \tag{4.70c}$$

An approximated solution for this equation is suggested in [82]: The idea is to assume that

$$y = v(\eta)p(\eta) \,, \tag{4.71}$$

and to separate the differential equation for y into two equations for v and p. In fact, inserting $y = vp$ into the differential equations one obtains:

$$0 = \left[v'p + vp'\right]' + f\left[v'p + vp'\right] + g_\pm vp \tag{4.72a}$$
$$= v''p + v'\left[2p' + pf\right] + v\left[p'' + fp' + g_\pm p\right] \tag{4.72b}$$

and, under the assumption that $p \neq 0$, we can divide by it finding:

$$0 = v'' + v'\left[2\frac{p'}{p} + f\right] + v\left[\frac{p''}{p} + f\frac{p'}{p} + g_\pm\right] . \tag{4.73}$$

If the function p is such that the first square bracket is set to zero, equation (4.73) reduces to a wave equation $v'' + \omega_\pm^2 v = 0$, where

$$\omega_\pm^2 \equiv \frac{p''}{p} + f\frac{p'}{p} + g_\pm \,, \tag{4.74}$$

that has the form needed to apply the WKB approximation. In our case the frequency depends on $g_\pm$ and so it also carries the index $\pm$.[4]

Going step by step, we first have to find a solution of:

$$2\frac{p'}{p} + f = 0 \,, \tag{4.75}$$

which is

$$p = \exp\left[-\frac{1}{2}\int f(\eta)\,\mathrm{d}\eta\right] . \tag{4.76}$$

Now we can assume that $\left|\omega_\pm'\right| \ll \omega_\pm^2$ and use the WKB approximation considering an approximated solution of the form:

$$v_\pm \approx (2\omega_\pm)^{-1/2} \exp\left[-i\int \mathrm{d}\eta\,\omega_\pm + i\delta_\pm\right] . \tag{4.77}$$

We can rewrite the frequency using (4.75) to substitute p' and p'':

$$\omega_\pm^2 = \frac{(-pf/2)'}{p} + \frac{(-pf/2)f}{p} + g_\pm = -\frac{1}{4}f^2 + f' + g_\pm \,. \tag{4.78}$$

[4]We point out that (4.69) is the equation of a dampled harmonic oscillator with time-dependent friction and time-dependent frequency. The assumption we took allows us to find a solution that is consistent with the ansatz generally proposed for the cosmic birefringence in Maxwell-CS [16], however a general solution of (4.69) can be found.

Putting all together, a solution of the differential equation (4.69) is:

$$y_\pm = v_\pm p \approx (2\omega_\pm)^{-1/2} \exp\left[-i\int \mathrm{d}\eta\,\omega_\pm + i\delta_\pm\right] \exp\left[-\frac{1}{2}\int f(\eta)\,\mathrm{d}\eta\right], \tag{4.79}$$

and it holds under the assumptions that the integral of f does not diverge and $\omega_\pm$ changes slowly in time.

Let us use this method to solve the ModMax EoM (4.48). The solution will be

$$\mathcal{A}_\pm \approx (2\omega_\pm)^{-1/2} \exp\left[-i\int \mathrm{d}\eta\,\omega_\pm + i\delta_\pm\right] \exp\left[-\frac{1}{2}\int \frac{C}{D}\,\mathrm{d}\eta\right], \tag{4.80}$$

with a pulsation:

$$\begin{aligned}\omega_\pm^2 &= -\frac{1}{4}\left(\frac{C}{D}\right)^2 + \left(\frac{C}{D}\right)' + \left(k^2 \mp k\frac{Z}{D}\right) &(4.81a)\\ &= k^2\left[\left(1 \mp \frac{Z}{kD}\right) - \frac{1}{4}\left(\frac{C}{kD}\right)^2 + \left(\frac{C'D - CD'}{k^2D^2}\right)\right], &(4.81b)\end{aligned}$$

where we used the definitions (4.70). We take the square root of the pulsation and expand recalling that $\mathrm{d}\eta \sim$ visible universe and $k^{-1} \sim$ photon wavelength, so $k^{-1}/\,\mathrm{d}\eta \ll 1$. This means:

$$\frac{Z}{k}, \frac{C}{k}, \frac{C'}{k^2}, \frac{CD'}{k^2} \ll 1\,. \tag{4.82}$$

So we are authorized to expand the square root. Then the pulsation reads:

$$\omega_\pm \approx k\left[1 \mp \frac{1}{2D}\left(\frac{Z}{k}\right) - \frac{1}{8D^2}\left(\frac{C}{k}\right)^2 + \frac{1}{2D}\left(\frac{C'}{k^2}\right) - \frac{1}{2D^2}\left(\frac{D'C}{k^2}\right)\right]. \tag{4.83}$$

Substituting the definitions of C, D, Z in (4.80) and (4.83) (and computing the conformal time derivatives) we get:

$$\mathcal{A}_\pm \approx (2\omega_\pm)^{-1/2} \exp\left[-i\int \mathrm{d}\eta\,\omega_\pm + i\delta_\pm\right] \exp\left[\int \left(-\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{2(\cosh\gamma + \sinh\gamma)}\right)\mathrm{d}\eta\right], \tag{4.84}$$

with a pulsation:

$$\begin{aligned}\omega_\pm = k\Bigg[1 &\mp \frac{1}{2(\cosh\gamma + \sinh\gamma\cos(\chi/g))}\left(\frac{2\sinh\gamma\nabla_0\sin(\chi/g)}{k}\right)\\ &- \frac{1}{8(\cosh\gamma + \sinh\gamma\cos(\chi/g))^2}\left(\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{k}\right)^2\\ &+ \frac{1}{2(\cosh\gamma + \sinh\gamma\cos(\chi/g))}\left(\frac{\sinh\gamma\nabla_0^2\cos(\chi/g)}{k^2}\right)\\ &- \frac{1}{2(\cosh\gamma + \sinh\gamma\cos(\chi/g))^2}\left(\frac{\sinh\gamma(\nabla_0\cos(\chi/g))^2}{k^2}\right)\Bigg]. \end{aligned}\tag{4.85}$$

This is our result for the pulsation of a linear-polarized plane wave for the ModMax reformulated theory (4.7) in the WKB approximation.

Consistency check Notice the first derivative in the EoM (4.48) gives arise to the p modulation of the wave solution $y_\pm$, i.e. the dilaton-photon coupling implies a correction to the solution for an axion-photon theory. The modulation is related to the amplitude of the solution which is multiplied by an exponential factor:

$$p = \exp\left[-\frac{1}{2}\int \frac{C}{D}\,\mathrm{d}\eta\right] = \exp\left[\int\left(-\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{2(\cosh\gamma+\sinh\gamma)}\right)\mathrm{d}\eta\right]. \tag{4.86}$$

The integrand is exactly the imaginary part of the pulsation (4.61) that we found for a canonical plane-wave solution with $\omega_\pm = \mathrm{const}$ (an imaginary pulsation implies an exponential modulation of the amplitude). However now we integrate over conformal time because we assumed $\omega'_\pm$ small but different from zero.

The dilaton-like photon coupling $S\cos(\chi/g)$ modifies the pulsation $\omega^2_\pm$ adding all terms proportional to C, however these terms do not track the $\pm$ distinction: the dilaton-like coupling does not distinguish the photon polarizations. The first term proportional to C of (4.85) is:

$$-\frac{k}{8D^2}\left(\frac{C}{k}\right)^2 = -\frac{k}{8(\cosh\gamma+\sinh\gamma\cos(\chi/g))^2}\left(\frac{\sinh\gamma\nabla_0\cos(\chi/g)}{k}\right)^2 \tag{4.87}$$

that is exactly the third term of (4.61). The fourth and fifth term of (4.85) are new features coming from the dilaton-like coupling, in fact they are respectively proportional to:

$$\text{4th term} \propto \nabla_0^2\cos\left(\frac{\chi}{g}\right) = -\left[\left(\frac{\chi'}{g}\right)^2\cos\left(\frac{\chi}{g}\right)+\frac{\chi''}{g}\sin\left(\frac{\chi}{g}\right)\right], \tag{4.88}$$

$$\text{5th term} \propto \left[\nabla_0\cos\left(\frac{\chi}{g}\right)\right]^2 = \left(\frac{\chi'}{g}\right)^2\sin^2\left(\frac{\chi}{g}\right), \tag{4.89}$$

and both come from the C term.

As expected, in the limit of $\omega_\pm \to \mathrm{const}$ (which implies a slow dynamical field χ) we can discard $\mathcal{O}((\chi'/g)^2)$ and $\mathcal{O}(\chi''/g)$ terms and pulsation (4.85) tends to (4.61). If we further assume $\chi/g \ll 1$ and $\gamma \ll 1$ we recover the Maxwell theory.

4.3.3 Computing the birefringence angle

We have found the time-dependent contribution to the k-mode of the 4potential, $\mathcal{A}_k(\eta)$, in an axion dilaton-like photon theory with a FLRW metric: it is given by equation (4.84). Then the full k-mode is:

$$A_\pm(\eta,\vec{x}) = \mathcal{A}_\pm(\eta)\exp\left(i\vec{k}\cdot\vec{x}\right). \tag{4.90}$$

Our purpose now is to compute the electric field and then the birefringence angle predicted by this theory, according to definition (2.87). So let us proceed computing the electric field:

$$E_\pm = \nabla_0 A_\pm \propto \exp\left[-i\int \mathrm{d}\eta\,\omega_\pm + i\delta_\pm\right], \tag{4.91}$$

where we are neglecting any term contributing to the amplitude since it does not contribute to the birefringence angle. The electric field in this theory is equivalent in form to the electric field of a Maxwell-CS theory, indeed the birefringence angle will be also equal in form:

$$\beta = -\frac{1}{2}\int \mathrm{d}\eta\,(\omega_+ - \omega_-) \tag{4.92}$$

setting $\delta_+ - \delta_- = 0$. The pulsation is defined in equation (4.85). So the birefringence angle reads:

$$\beta = \int \mathrm{d}\eta \, \frac{\sinh\gamma \nabla_0 \sin(\chi/g)}{\cosh\gamma + \sinh\gamma\cos(\chi/g)} \,. \tag{4.93}$$

Terms proportional to C do not contribute since the dilaton-like coupling is insensible to $+$ and $-$ polarizations.

On the generality of our result We never took into account the specific behaviour of $\chi(\eta)$. Indeed, what we have found up to now, holds in general for a theory with Lagrangian density:

$$\mathscr{L} = \cosh\gamma S + \sinh\gamma(Sf(\chi) + Ph(\chi)) \,. \tag{4.94}$$

The results that we obtained hold in form with the substitution:

$$C = \sinh\gamma \nabla_0 f(\chi) \,, \tag{4.95}$$
$$Z = 2\sinh\gamma \nabla_0 h(\chi) \,. \tag{4.96}$$

Functions $f(\chi)$ and $h(\chi)$ must respect some constraints in order to deal with a physical theory: $f(\chi)$ should be an even function of χ and $h(\chi)$ an odd one. In addition they should reduce to the Maxwell theory in the limits of small χ and γ.

The generality of our results allow us to make the following *claim*: since a dilaton-like coupling $Sf(\chi)$ does not contribute to the modification of the pulsation of one polarization mode with respect to the other, it is not a player in the cosmic birefringence phenomenon. This means that the ALP-photon interaction is the only term one should focus on in computing the birefringence angle of an electromagnetic theory. The dilaton-like coupling modifies the amplitude of a photon, and how this may be important for observations is left for future discussions.

Let us now go on with the computation of the birefringence angle (4.93), deriving the sine:

$$\beta = \int \mathrm{d}\eta \, \frac{\sinh\gamma \, \cos(\chi/g)}{\cosh\gamma + \sinh\gamma\cos(\chi/g)} \, \frac{\chi'}{g} \tag{4.97a}$$
$$= \int \mathrm{d}\left(\frac{\chi}{g}\right) \frac{\sinh\gamma\cos(\chi/g)}{\cosh\gamma + \sinh\gamma\cos(\chi/g)} \,. \tag{4.97b}$$

The integral ranges from an initial value χ_i/g to a final value χ_0/g, therefore we must know the dynamics of the field χ in order to calculate the birefringence angle.

The integral can be written as follows:

$$\beta = \int \mathrm{d}\left(\frac{\chi}{g}\right) \frac{\tanh\gamma\cos(\chi/g)}{1 + \tanh\gamma\cos(\chi/g)} \tag{4.98a}$$
$$= \int \mathrm{d}\left(\frac{\chi}{g}\right) \frac{\tanh\gamma\cos(\chi/g) + (1-1)}{1 + \tanh\gamma\cos(\chi/g)} \tag{4.98b}$$
$$= \int \mathrm{d}\left(\frac{\chi}{g}\right) - \int \mathrm{d}\left(\frac{\chi}{g}\right) \frac{1}{1 + \tanh\gamma\cos(\chi/g)} \,. \tag{4.98c}$$

The first term is formally equal to birefringence angle that one obtains in a Maxwell-CS theory; however, while in Maxwell-CS the field is modulated by the coupling $\alpha/(2f)$, here it is dimensionally normalized by g. Then the magnitude of the first term is "corrected" by the second term which we expect to be similar to the first since γ should be small.

There is an analytic solution for the undetermined integral:

$$\int \mathrm{d}\left(\frac{\chi}{g}\right)\left[1-\frac{1}{1+\tanh\gamma\cos(\chi/g)}\right]=\frac{\chi}{g}-2\cosh(\gamma)\mathrm{arccot}\left[\exp(\gamma)\cot\left(\frac{\chi}{2g}\right)\right]+\mathrm{const}\,. \tag{4.99}$$

The procedure to obtain such result uses Weierstrass' substitutions. So before using solution (4.99) to evalute the integrand in a range $[\chi_i/g, \chi_0/g]$ we should verify that the Weierstrass' substitutions are well defined in the domain [83].

Consistency check At the beginning of the chapter we talked about the limits of ModMax precursor and we found that when $\gamma \ll 1$ and $\chi/g \ll 1$ we recover the Maxwell-CS theory up to second order. Let us see whether we recover the usual formulation for the cosmic birefringence angle,

$$\beta_{\mathrm{CS}}=\frac{\alpha}{2f}\int \mathrm{d}\chi\ , \tag{4.100}$$

taking these limits into our result (4.98c). We will expand the integrand up to first order such that the overall integral is of second order because of the measure:

$$\beta=\int \mathrm{d}\left(\frac{\chi}{g}\right)\left[1-\frac{1}{1+\tanh\gamma\cos(\chi/g)}\right] \tag{4.101a}$$
$$\approx\int \mathrm{d}\left(\frac{\chi}{g}\right)\left[1-\frac{1}{1+(\gamma-\gamma^3/3)(1-(\chi^2/g^2)/2)}\right] \tag{4.101b}$$
$$\approx\int \mathrm{d}\left(\frac{\chi}{g}\right)\left[1-\frac{1}{1+\gamma}\right] \tag{4.101c}$$
$$\approx\int \mathrm{d}\left(\frac{\chi}{g}\right)\left[1-(1-\gamma)\right] \tag{4.101d}$$
$$\approx\gamma\int \mathrm{d}\left(\frac{\chi}{g}\right)\ . \tag{4.101e}$$

It is exactly what we hoped to find, thus the ModMax cosmic birefringence angle is consistent with the limit of its Lagrangian.

4.4 Evolution of the pseudoscalar field

In order to compute the birefringence angle we must know the evolution of the pseudoscalar field χ. We recall the Lagrangian (4.18) we are working with:

$$\mathscr{L}=e^{-\gamma}\left[S\cosh\gamma+\sinh\gamma\left(S\cos\left(\frac{\chi}{g}\right)+P\sin\left(\frac{\chi}{g}\right)\right)\right]-\frac{1}{2}g_{\mu\nu}\nabla^\mu\chi\nabla^\nu\chi-V(\chi)\,. \tag{4.102}$$

The dynamics of χ will depend on the model chosen, i.e. on the potential. For the time being we do not specify it and we just compute the EoM as the EL equations:

$$0=\frac{\partial\mathscr{L}}{\partial\chi}-\nabla_\mu\left[\frac{\partial\mathscr{L}}{\partial(\nabla_\mu\chi)}\right]. \tag{4.103}$$

Using the identity that holds for a generic 4vector V^μ:

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}V^\mu\right), \tag{4.104}$$

we rewrite the EL equations as:

$$0 = \frac{\partial \mathscr{L}}{\partial \chi} - \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\frac{\partial \mathscr{L}}{\partial(\nabla_\mu \chi)}\right]. \tag{4.105}$$

Then for our Lagrangian density (4.18) they read:

$$0 = e^{-\gamma}\sinh\gamma\left[-S\sin\left(\frac{\chi}{g}\right) + P\cos\left(\frac{\chi}{g}\right)\right] - \frac{\partial V}{\partial \chi} - \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}(-g^{\mu\nu}\nabla_\nu\chi)\right]. \tag{4.106}$$

We are working with a flat-FLRW metric written in conformal coordinates $(\eta, \vec{x})$, so $\sqrt{-g} = a^4$, and we keep the assumption to deal with a homogeneous field $\chi = \chi(\eta)$. Then the only non-vanishing coordinate derivatives are with respect to the conformal time. So EoM reads:

$$0 = e^{-\gamma}\sinh\gamma\left[-S\sin\left(\frac{\chi}{g}\right) + P\cos\left(\frac{\chi}{g}\right)\right] - \frac{\partial V}{\partial \chi} + a^{-4}\partial_0\left[a^4\left(-a^{-2}\right)\partial_0\chi\right] \tag{4.107a}$$

$$= e^{-\gamma}\sinh\gamma\left[-S\sin\left(\frac{\chi}{g}\right) + P\cos\left(\frac{\chi}{g}\right)\right] - \frac{\partial V}{\partial \chi} - 2a'a^{-3}\chi' - a^{-2}\chi''. \tag{4.107b}$$

We recall that prime denotes a derivative with respect to the conformal time. Defining the conformal Hubble parameter as:

$$\mathcal{H}(\eta) \equiv \frac{a'(\eta)}{a(\eta)} \tag{4.108}$$

and rearranging terms, EoM is:

$$\chi'' + 2\mathcal{H}\chi' - a^2 e^{-\gamma}\sinh\gamma\left[-S\sin\left(\frac{\chi}{g}\right) + P\cos\left(\frac{\chi}{g}\right)\right] + a^2\frac{\partial V}{\partial \chi} = 0. \tag{4.109}$$

The time window on which we focus on is from the recombination era up to today, assuming a simplified universe where all CMB photons were emitted simultaneously at recombination time. If we woule be more accurate, we should consider the photon emission with a statistical distribution (called photon visibility function in literature, see e.g. [17, 19]). Such a distribution will peak at the recombination and reionization era, but for the time being we consider a simultaneous photon emission at the recombination era, with redshift $z_{\text{rec}} \sim 1090$ [50].

An alternative way to write equation (4.109) is changing variable from the conformal time η to the scale factor a. The measure changes as follows:

$$\mathrm{d}a = \frac{\partial a}{\partial \eta}\mathrm{d}\eta \implies \frac{\mathrm{d}}{\mathrm{d}\eta} = \frac{\partial a}{\partial \eta}\frac{\mathrm{d}}{\mathrm{d}a} = a'\frac{\mathrm{d}}{\mathrm{d}a} = a\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}a}. \tag{4.110}$$

Using it we are able to find how each term of the EoM changes under the change of variable from η to a:

$$\chi'(\eta) = \frac{\mathrm{d}\chi}{\mathrm{d}\eta} \to \chi'(a) = a\mathcal{H}\frac{\mathrm{d}\chi}{\mathrm{d}a}, \tag{4.111}$$

$$\chi''(\eta) = \frac{\mathrm{d}^2\chi}{\mathrm{d}\eta^2} \to \chi''(a) = a\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}a}\left(a\mathcal{H}\frac{\mathrm{d}\chi}{\mathrm{d}a}\right) \tag{4.112}$$

$$= a\mathcal{H}^2\frac{\mathrm{d}\chi}{\mathrm{d}a} + a^2\mathcal{H}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}a}\frac{\mathrm{d}\chi}{\mathrm{d}a} + a^2\mathcal{H}^2\frac{\mathrm{d}^2\chi}{\mathrm{d}a^2}, \tag{4.113}$$

and so equation (4.109) parameterized with the scale factor reads:

$$a^2\mathcal{H}^2\frac{\mathrm{d}^2\chi}{\mathrm{d}a^2}+a^2\mathcal{H}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}a}\frac{\mathrm{d}\chi}{\mathrm{d}a}+3a\mathcal{H}^2\frac{\mathrm{d}\chi}{\mathrm{d}a}-a^2e^{-\gamma}\sinh\gamma\left[-S\sin\left(\frac{\chi}{g}\right)+P\cos\left(\frac{\chi}{g}\right)\right]+a^2\frac{\partial V}{\partial\chi}=0\,. \tag{4.114}$$

Let us write the differential equation in the form $y''=f(y',y)$ that is useful for numerical purpose:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}a^2}=-\left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}a}+3\frac{1}{a}\right)\frac{\mathrm{d}\chi}{\mathrm{d}a}+\frac{1}{\mathcal{H}^2}e^{-\gamma}\sinh\gamma\left[-S\sin\left(\frac{\chi}{g}\right)+P\cos\left(\frac{\chi}{g}\right)\right]-\frac{1}{\mathcal{H}^2}\frac{\partial V}{\partial\chi}\,. \tag{4.115}$$

To solve this equation we must know the dynamics of the conformal Hubble parameter and the dynamics of electromagnetic fields since they appear through the Lorentz-invariants S and P.

The range of the scale factor we are interested in goes from the scale factor value at recombination epoch up to the today scale factor. The redshift at recombination era is $z_{\rm rec}\sim 1090$ that corresponds to a scale factor $a_{\rm rec}=a_0/(1+z_{\rm rec})\sim 1/1091$ since $a_0=1$ is the scale factor today. Therefore we will study the dynamics of the field χ in a range $[a_{\rm rec},a_0]=[1/1091,1]$.

4.4.1 Dynamics of electromagnetic fields

EoM (4.115) contains the Lorentz-electromagnetic invariants:

$$S=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}=\frac{1}{2}\left(\vec{E}^2-\vec{B}^2\right), \tag{4.116}$$

$$P=-\frac{1}{4}F^{\mu\nu}\tilde{F}_{\mu\nu}=\vec{E}\cdot\vec{B}\,. \tag{4.117}$$

How is their dynamics in function of the scale factor a? Let us recall the definitions of electric and magnetic fields (using Coulomb gauge):

$$E^i=F^{0i}=\partial^0A^i\,, \tag{4.118}$$

$$B^i=\frac{1}{2}\epsilon^{ijk}F_{jk}=\epsilon^{ijk}\partial_jA_k\,. \tag{4.119}$$

In a FLRW spacetime $\mathrm{d}s^2=a^2(\eta)(-\mathrm{d}\eta^2+\mathrm{d}x^2)$, the 4-potential is conformally coupled to gravity, i.e. it does not feel the universe expansion; however the physical electromagnetic fields $\vec{E}$ and $\vec{B}$ do:

$$E^i=\partial^0A^i=g^{00}\partial_0A^i=-\frac{A^{i\prime}}{a^2}\,, \tag{4.120}$$

$$B^i=\epsilon^{ijk}\partial_jA_k=\sqrt{-g}\varepsilon^{ijk}\partial_jA_k=\sqrt{-g}g^{ip}\varepsilon_{pqr}\left(g^{qj}\partial_j\right)\left(g^{rk}A_k\right) \tag{4.121}$$

$$=\frac{1}{a^2}\delta^{ip}\varepsilon_{pqr}\delta^{qj}\delta^{rk}\partial_jA_k=\frac{(\vec{\nabla}\times\vec{A})^i}{a^2}\,. \tag{4.122}$$

They go as $\propto a^{-2}$. We can define $-A^{i\prime}\equiv\mathcal{E}^i$ and $(\vec{\nabla}\times\vec{A})^i\equiv\mathcal{B}^i$ as the electric and magnetic field measured in a Minkowski framework respectively, so:

$$E^i=\frac{\mathcal{E}^i}{a^2}\,, \tag{4.123a}$$

$$B^i=\frac{\mathcal{B}^i}{a^2}\,. \tag{4.123b}$$

Now the problem moves to how do $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ evolve? Before proceeding, let us point out what exactly we need into the EoM (4.115): it is a differential equation for a homogeneous field χ in a FLRW spacetime, therefore we are interested in any homogeneous contribution from the cosmic species. In order to respect this, we consider electric and magnetic fields averaged in space[5] and their value is:

$$\langle E_i \rangle = 0\,,\ \langle B_i \rangle = 0\,,\ \langle E_i B_j \rangle = 0\,, \tag{4.125}$$

because of a lack of phase relation between waves that makes equally probable to have a positive or negative magnitude. This is an old approach to vector field in cosmology (for example we find it into the paper [85] from the '30). While electric and magnetic fields squared are nonvanishing:

$$\langle E_i E_j \rangle = \frac{\delta_{ij}}{3}\left\langle E^2 \right\rangle = \frac{\delta_{ij}}{3}\frac{\left\langle \mathcal{E}^2 \right\rangle}{a^4}\,, \tag{4.126}$$

$$\langle B_i B_j \rangle = \frac{\delta_{ij}}{3}\left\langle B^2 \right\rangle = \frac{\delta_{ij}}{3}\frac{\left\langle \mathcal{B}^2 \right\rangle}{a^4}\,, \tag{4.127}$$

where $E^2 = \eta_{ij} E^i E^j$ and analogously for B^2, with η_{ij} the Minkowski metric because of our parametrization (4.123).

In light of these results, we see that the averaged value for S can be different from zero, while for P is vanishing:

$$\langle S \rangle = \frac{1}{2}\left(\frac{\left\langle \mathcal{E}^2 \right\rangle}{3a^4} - \frac{\left\langle \mathcal{B}^2 \right\rangle}{3a^4} \right)\,, \tag{4.128}$$

$$\langle P \rangle = 0\,. \tag{4.129}$$

Therefore the electromagnetic configuration P and its couplings, i.e. what we have called axion-photon coupling, do not contribute to the dynamics of the background field $\chi(\eta)$ (i.e. to the coarse grained homogeneous and isotropic universe). If one relaxes the isotropy assumption, interactions of the axion-dilaton-photon type are allowed producing anisotropic effects.[6]

Regarding the configuration $\langle S \rangle$, if we allow a small degree of anisotropy we should consider a 4potential of the form $A^\mu = A^\mu_{(0)} + \delta A^\mu$, where $A^\mu_{(0)} = 0$ is the background 4potential which must be zero since the observable anisotropy is small [88], and δA^μ is a small correction. Then, any EM field that can arise from the 4potential will be proportional to $\partial(\delta A^\mu)$ contributing to the invariants S, and also P, with $(\partial(\delta A^\mu))^2$. So $\left\langle \mathcal{E}^2 \right\rangle$ and $\left\langle \mathcal{B}^2 \right\rangle$ will be very small and their difference in $\langle S \rangle$ even more. In addition, its value dilutes along the cosmic time as $\propto a^{-4}$. Therefore it is negligible for our purpose and we set:

$$\langle S \rangle = 0\,. \tag{4.130}$$

With this line of reasoning, we have lost any trace of electromagnetic fields in the EoM for the field χ. Proceeding as we are doing is equivalent to neglect the electromagnetic

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[5] Given a quantity X, its average $\langle X \rangle$ at a time t can be assumed to be as defined in [84]:

$$\langle X \rangle \equiv \lim_{V \to V_0} \frac{1}{V} \int \mathrm{d}^3x\, \sqrt{-g}\; X\,, \tag{4.124}$$

where $V = \int \mathrm{d}^3x\, \sqrt{-g}$ and V_0 is a sufficiently large time-dependent volume.

[6] In the inflationary scenario anisotropic signature of couplings $f(\chi)S$ and $f(\chi)P$ were extensively studied, see e.g. [86, 87].

backreaction on the dynamics of χ. However the ModMax theory has left its signature in the dynamics of 4potential (4.48) and in its phenomenology of the birefringence angle (4.97) through the parameter γ.

Therefore, we reduced the EoM of the pseudoscalar field χ to the well-known Klein-Gordon equation in a FLRW metric:[7]

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}a^2} = -\left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}a} + 3\frac{1}{a}\right)\frac{\mathrm{d}\chi}{\mathrm{d}a} - \frac{1}{\mathcal{H}^2}\frac{\partial V}{\partial \chi}\,. \tag{4.133}$$

Qualitative description of the dynamics Equation (4.133) describes the dynamics of a field under the influence of a friction term ($\propto \mathrm{d}\chi/\mathrm{d}a$) and a classic force ($\propto \partial V/\partial\chi$). The first term on the RHS is called Hubble friction since it is due to the universe expansion, and brakes the field producing the damping. If the potential is a parabola, then (4.133) describes a damped harmonic oscillator equation.

let us suppose the field χ is near the minimum of the potential that has a parabola shape. χ seeks to roll towards the minimum, at early times the friction term dominates the dynamics and the field slowly rolls down the potential suffering a damping. Then because of the Hubble parameter is a decreasing function, there will be a time, and equivalently a scale factor, at which the slope starts to dominate, the field runs to the potential minimum and starts oscillate around it.

Equation (4.133) derives from the Lagrangian density of a free field, $\mathscr{L}_\chi = -(\partial\chi)^2/2 - V(\chi)$, propagating in a FLRW spacetime; in fact we assumed to neglect any backreaction. From the Lagrangian density we can compute the stress-energy tensor:[8]

$$T_{\mu\nu} = \partial_\mu\chi\partial_\nu\chi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi + V(\chi)\right), \tag{4.134}$$

and obtain the EoS of χ:

$$w_\chi \equiv \frac{P_\chi}{\rho_\chi} = \frac{T^{ii}}{T^{00}} = \frac{\chi'^2/(2a^2) - V(\chi)}{\chi'^2/(2a^2) + V(\chi)} = \frac{\mathcal{H}^2(\mathrm{d}\chi/\mathrm{d}a)^2/2 - V(\chi)}{\mathcal{H}^2(\mathrm{d}\chi/\mathrm{d}a)^2/2 + V(\chi)}\,, \tag{4.135}$$

where we switched from conformal time derivative to scale factor derivative using (4.111).

Going on with the qualitative discussion on the dynamics of χ from the EoM (4.133), we note that as long as the friction term dominates the dynamics and the field slowly rolls down the potential with a small χ' its EoS reduces to $w_\chi \sim -1$. If this condition holds up to today, the ALP behaves as dark energy and may comprise all or part of it; this scenario is called thawing quintessence [89] since the EoS may move from -1 by today times.

On the other hand, when the slope of the potential becomes bigger than the friction term and the field oscillates its EoS is averaged over oscillations and evolves to a value ~ 0, there the ALP behaves as dust and becomes a component of the dark matter sector.

[7] Do note the second term of EoM can be rewritten as $-\frac{\mathrm{d}}{\mathrm{d}a}\log(\mathcal{H}a^3)\frac{\mathrm{d}\chi}{\mathrm{d}a}$. Just for completeness, we report the formulation of (4.133) parameterized by the conformal time and cosmic time, where the EoM assumes the friendly formulations:

conformal time
$$\chi'' + 2\mathcal{H}\chi' + a^2\frac{\partial V}{\partial\chi} = 0\,, \tag{4.131}$$

cosmic time
$$\ddot{\chi} + 3H\dot{\chi} + \frac{\partial V}{\partial\chi} = 0\,. \tag{4.132}$$

[8] It is the third line of (4.35), namely neglecting coupling.

4.4.2 Dynamics of Hubble parameter

The dynamics of the Hubble parameter is determined by the first Friedmann equation (FE) that in a flat-FLRW spacetime reads:

$$H^2 = \frac{8\pi G}{3}\rho\,, \tag{4.136}$$

here ρ is the sum of energy densities of cosmic species; in our model, the universe is filled by non-relativistic matter, radiation, a cosmological constant Λ and the field χ. Introducing the reduced Planck mass $m_{\rm Pl} \equiv (8\pi G)^{-1/2}$ we rewrite the first FE as:

$$H^2 = \frac{1}{3m_{\rm Pl}^2}(\rho_m + \rho_r + \rho_\Lambda + \rho_\chi)\,. \tag{4.137}$$

Using the assumption of describing the cosmic species as barotropic fluids with linear EoS $P_i = w\rho_i$, with w constant, together with the third Friedmann equation, one finds a power-law dependence $\rho_i \propto a^{-\alpha}$, with $\alpha > 0$. Thus, introducing the critical energy density:

$$\rho_c(a) \equiv \frac{3}{8\pi G}H^2(a) = 3m_{\rm Pl}^2 H^2(a) \tag{4.138}$$

and the energy budget $\Omega \equiv \rho/\rho_c$ one can write the Hubble parameter in the following common way:

$$H^2 = H_0^2\left[\Omega_{0m}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{0r}\left(\frac{a}{a_0}\right)^{-4} + \Omega_\Lambda + \Omega_\chi\right]. \tag{4.139}$$

The pedix $_0$ indicates quantities computed at today time. Our range of interest starts at a redshift $z_{\rm rec} \sim 1090$ at which the universe is yet matter dominated, thus we neglect the radiation contribution to the cosmic energy budget during all the time interval we are considering.

The Hubble parameter today is [90]:

$$H_0 = (67.4 \pm 0.5)\,{\rm km\,s^{-1}\,Mpc^{-1}} \approx 1.4 \times 10^{-33}\,{\rm eV} \tag{4.140}$$

at 68% C.L. and thus the critical energy density today is:

$$\rho_{0c} = 3m_{\rm Pl}^2 H_0^2 \approx 3.7 \times 10^{-11}\,{\rm eV}^4\,. \tag{4.141}$$

where we used $m_{\rm Pl} \approx 2.4 \times 10^{27}\,{\rm eV}$. The value of today energy budget of matter and cosmological constant is [90]:

$$\Omega_{0m} = (0.315 \pm 0.007) \quad 68\%\ \text{C.L.}\,, \tag{4.142a}$$

$$\Omega_\Lambda = (0.679 \pm 0.013) \quad 68\%\ \text{C.L.}\,. \tag{4.142b}$$

Regarding the energy budget from the axion field, Ω_χ, we take the strong assumption that it is *sub-dominant* with respect to the energy densities of the other cosmic species during the time we are interested in. So we neglect it from the energy budget that fuels the evolution of $H(a)$. This approximation is realizable if the field has a small enough mass or an adequate initial condition $\chi_{\rm in} = \chi(a_{\rm rec})$, we will come back to this later. With this assumption the Hubble parameter simplifies to:

$$H = H_0\left[\Omega_{0m}\left(\frac{a}{a_0}\right)^{-3} + \Omega_\Lambda\right]^{1/2}, \tag{4.143}$$

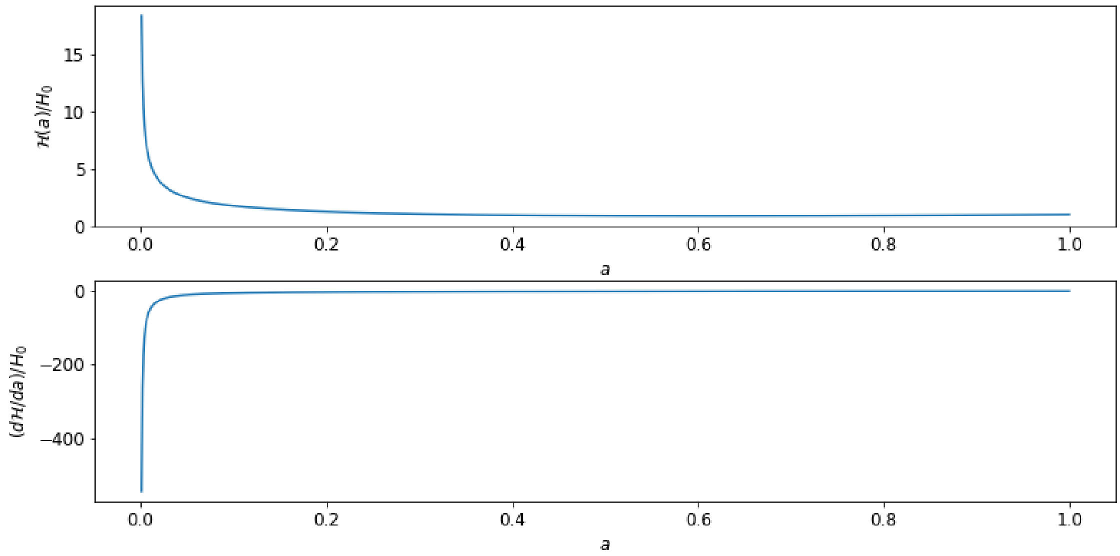

Figure 4.2: *Top*: the conformal Hubble parameter normalized to the Hubble constant, $\mathcal{H}(a)/H_0$. *Bottom*: the derivative of $\mathcal{H}(a)/H_0$ with respect to the scale factor. Recall $H_0 \approx 1.4 \times 10^{-33}$ eV, the range goes from the recombination time $a_{\rm rec} \approx 1/1091$ to today $a_0 = 1$. The only relevant energy budgets considered are matter and dark energy in the assumption that the ALP is sub-dominant.

and so the procedure to solve the dynamics of χ simplifies a lot, in fact this assumption decouples the ALP EoM from the evolution of the Hubble parameter. This simplification is often taken in literature [19, 56, 91], see [53] for a more detailed treatment of the axion abundance.

The first Friedmann equation furnishes an Hubble parameter $H = H(a)$, however we used the conformal Hubble parameter $\mathcal{H} = \mathcal{H}(a)$ into EoM (4.115). Let us obtain it from the definition:

$$H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{a^2}\frac{\mathrm{d}a}{\mathrm{d}\eta} = \frac{1}{a}\mathcal{H} \implies \mathcal{H} = aH\,. \tag{4.144}$$

The today scale factor is $a_0 = 1$, then the conformal Hubble parameter reads:

$$\mathcal{H}(a) = aH_0\big(\Omega_{0m}a^{-3} + \Omega_\Lambda\big)^{1/2}\,. \tag{4.145}$$

Since we also need its derivative $\mathrm{d}\mathcal{H}/\mathrm{d}a$, let us compute it now:

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{H}(a)}{\mathrm{d}a} &= H_0\big(\Omega_{0m}a^{-3} + \Omega_\Lambda\big)^{1/2} + aH_0\frac{\big(-3\Omega_{0m}a^{-4}\big)}{2\big(\Omega_{0m}a^{-3} + \Omega_\Lambda\big)^{1/2}} && (4.146\text{a})\\
&= \frac{\mathcal{H}}{a} - \frac{3}{2}\big(\Omega_{0m}a^{-4}\big)aH_0\frac{\big(\Omega_{0m}a^{-3} + \Omega_\Lambda\big)^{1/2}}{\big(\Omega_{0m}a^{-3} + \Omega_\Lambda\big)} && (4.146\text{b})\\
&= \frac{\mathcal{H}}{a} - \frac{3}{2}\mathcal{H}\left(\frac{\Omega_{0m}}{\Omega_{0m}a + \Omega_\Lambda a^4}\right) && (4.146\text{c})\\
&= \frac{\mathcal{H}}{a}\left[1 - \frac{3}{2}\left(\frac{\Omega_{0m}}{\Omega_{0m} + \Omega_\Lambda a^3}\right)\right]. && (4.146\text{d})
\end{aligned}$$

We plot the conformal Hubble parameter and its derivative in figure 4.2.

A different assumption on the ALP energy budget

There are some papers citing the following upper bounds for the ALP parameter density [17, 19, 40, 53, 92]:

$$\Omega_\chi \leq \Omega_{\chi,\max} = \begin{cases} 0.69 & \text{for } m \leq 8.5 \times 10^{-34}\,\text{eV} \\ 0.006h^{-2} & \text{for } 10^{-32}\,\text{eV} \leq m \leq 10^{-25.5}\text{eV} \end{cases} \tag{4.147}$$

Let us see from where they come. Consider a general potential near its minimum, thus approximated with a parabola $V \sim m^2\chi^2/2$. The quantities that determine the dynamics of χ are the mass m (which contributes to the slope) and the Hubble parameter H (which contributes to the friction). Requiring to be consistent with the Planck 2018 data on CMB, it was found that for masses

$$m \leq 8.5 \times 10^{-34}\,\text{eV} \tag{4.148}$$

the energy budget Ω_χ has an upper bound of 0.69 i.e. for these masses the ALP can cover the whole DE energy budget $\Omega_\Lambda \approx 0.69$ without spoiling out observations. While observations of CMB and large scale structure put the constraint $\Omega_\chi h^2 \leq 0.006$ (where $h = 0.677$ is the dimensionless Hubble constant [17]) if one requires that the EoS of χ evolves toward 0 and the mass is in the range

$$10^{-32}\,\text{eV} \leq m \leq 10^{25.5}\text{eV}\,. \tag{4.149}$$

In other words, in this mass range χ is a tiny fraction of dark matter. From these reasonings the upper bound (4.147) comes out.

Works cited at the beginning of the section assume to deal with an axion model having a mass contained into the ranges of (4.147), with a field near the minimum and with negligible backreactions from other fields. Then they assume an ALP abundance which saturates the bound (4.147). Doing so, χ will play the role of the whole DE in the mass range indicated in the first line, whereas it will play the role of a spectator field, covering a tiny fraction of dark matter, in the mass range indicated in the second line. In the intermediate region of masses the value of $\Omega_{\chi,\max}$ is usually obtained interpolating the two maximum values in the plane $\log(\Omega_\chi) - \log(m)$ [17]. This is another approach to simplify the dynamics of $H(a)$ since its formula reduces to:

$$H = H_0\left[\Omega_{0m}\left(\frac{a}{a_0}\right)^{-3} + \Omega_\Lambda\right]^{1/2}, \tag{4.150}$$

where $\Omega_\Lambda \approx \Omega_\chi$ or $\Omega_\chi \subset \Omega_{0m}$ but negligible for the right mass ranges.

However we will not follow this road, we assume Ω_χ is sub-dominant being either a part of dark matter or a dark-energy like component.

Complement: on the general evolution of the ALP

In this paragraph we set up the system of equations which describes the evolution of χ in general, namely with no assumption on its abundance Ω_χ. We just keep the assumption that there are no important backreactions to the dynamics of χ. Thus the EoM is the one already computed in (4.133) and we recall it here below:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}a^2} = -\left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}a} + 3\frac{1}{a}\right)\frac{\mathrm{d}\chi}{\mathrm{d}a} - \frac{1}{\mathcal{H}^2}\frac{\partial V}{\partial \chi}\,. \tag{4.151}$$

It must be coupled to the first Friedmann equation:

$$\mathcal{H}(a) = aH_0[\Omega_m(a) + \Omega_r(a) + \Omega_\Lambda + \Omega_\chi(a)]^{1/2}\,, \tag{4.152}$$

where as anticipated we set no limits on the energy budget $\Omega_\chi(a)$. The abundance Ω_χ is by definition:

$$\Omega_\chi(a) \equiv \frac{\rho_\chi}{\rho_c} = \frac{\chi'^2/(2a^2) + V(\chi)}{3(\mathcal{H}/a)^2 m_{\rm Pl}^2} = \frac{a^2}{3\mathcal{H}^2 m_{\rm Pl}^2}\left[\frac{\mathcal{H}^2}{2}\left(\frac{{\rm d}\chi}{{\rm d}a}\right)^2 + V(\chi)\right]\,, \tag{4.153}$$

where we used the component $T^{00} = \rho_\chi$ of the stress-energy tensor (4.134). All these equations together compose a system that describes the dynamics of χ in general.

4.5 Numerical solutions

Summing up what we said above, we want to solve the following second-order differential equation for $\chi(a)$:

$$\frac{{\rm d}^2\chi}{{\rm d}a^2} = -\left(\frac{1}{\mathcal{H}}\frac{{\rm d}\mathcal{H}}{{\rm d}a} + 3\frac{1}{a}\right)\frac{{\rm d}\chi}{{\rm d}a} - \frac{1}{\mathcal{H}^2}\frac{\partial V}{\partial\chi}\,. \tag{4.154}$$

We assume the ALP has a density parameter Ω_χ sub-dominant with respect to the other cosmic species, thus the Hubble parameter and its derivative are:

$$\mathcal{H}(a) = aH_0\left(\Omega_{0m}a^{-3} + \Omega_\Lambda\right)^{1/2}\,, \tag{4.155}$$

$$\frac{{\rm d}\mathcal{H}}{{\rm d}a} = \frac{\mathcal{H}}{a}\left[1 - \frac{3}{2}\left(\frac{\Omega_{0m}}{\Omega_{0m} + \Omega_\Lambda a^3}\right)\right]\,. \tag{4.156}$$

Inserting them into the differential equation:

$$\frac{{\rm d}^2\chi}{{\rm d}a^2} = -\frac{1}{a}\left(4 - \frac{3}{2}\frac{\Omega_{0m}}{\Omega_{0m} + \Omega_\Lambda a^3}\right)\frac{{\rm d}\chi}{{\rm d}a} - \frac{1}{\mathcal{H}^2}\frac{\partial V}{\partial\chi}\,. \tag{4.157}$$

The range of interest is:

$$[a_{\rm rec}\ ,\ a_0] = [1/1091\ ,\ 1]\,. \tag{4.158}$$

Having the dynamics of the field we can estimate the birefringence angle (4.97) as a function of the ModMax parameter γ which is unknown.

To proceed we must specify the potential to work with. We get inspired by the usual axion-potential of QCD [23]:[9]

$$V(\chi) = m^2 f^2\left[1 - \cos\left(\frac{\chi}{f}\right)\right]\,, \tag{4.159}$$

where f and m have the dimension of a mass, f is the scale of the shift symmetry breaking. This potential is not unique and without detailed knowledge of the non-perturbative physics it cannot be predicted. For example, considering "higher order instanton corrections" the potential evolves to $V \propto [1 - \cos(\chi/f)]^n$ with $n > 1$ [23]. These potentials have a plateau at $\chi \sim \pi f$ and a power-law behavior $V \propto \chi^{2n}$ for $\chi \ll f$, citing [93] they are potentials with power-law minima and flattened "wings".

[9] However we point out the axion field we are considering is not necessary the one of QCD.

We choose to work near the minimum of the potential, assuming $\chi/f \ll 1$, i.e. Taylor expanding the potential and neglecting higher order of axion self-interactions which would be mediated by powers of $1/f$. In this limit the dominant term is the mass term, namely the quadratic potential:

$$V(\chi) \approx \frac{1}{2}m^2\chi^2 \,. \tag{4.160}$$

We consider:

$$f \sim M_{\rm Pl} \approx 1.2 \times 10^{28}\,\text{eV}\,, \tag{4.161}$$

of the order of the Planck mass. Then the initial value of χ to allow an effective description of the model through a mass potential (4.160) must be $\chi_{\rm in} \ll M_{\rm Pl}$.

4.5.1 Dynamics in a quadratic potential

We work with the quadratic potential:

$$V(\chi) = \frac{1}{2}m^2\chi^2 \,. \tag{4.162}$$

In SI units there should be a factor c^2 multiplying everything, the mass should be in kg and the field should have the dimension of meter$^{-3/2}$ $\times$ kg$^{-1/2}$. While in natural units the mass is $[m] = [\chi] = \text{eV}$.

The EoM for the pseudoscalar field in such a potential is:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}a^2} = -\left(\frac{1}{\mathcal{H}}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}a} + 3\frac{1}{a}\right)\frac{\mathrm{d}\chi}{\mathrm{d}a} - \frac{m^2}{\mathcal{H}^2}\chi \,. \tag{4.163}$$

Let us refresh the qualitative dynamics of the field and give some estimates: the first term on the RHS is the Hubble friction while the second term is the potential slope. In this model the mass m plays an important role: given a value for m, in the time interval in which $\mathcal{H}/a \gg m$ the friction term dominates the dynamics and the field slowly rolls down the potential suffering a damping. Then as the universe expands there will be a time at which $\mathcal{H}/a \approx m$ and the second term in the RHS starts to dominate, consequently the field starts oscillating and continues as long as $\mathcal{H}/a < m$.

Considering our "time" interval $[a_{\rm rec}, a_0]$ if $m \gg \mathcal{H}_{\rm rec}/a_{\rm rec}$ the field started oscillating before the recombination epoch and keeps oscillating during its evolution, the contributions to the birefringence angle β (4.97) get compensated resulting in a small angle. On the other hand, if $m \ll \mathcal{H}_0/a_0$ the Hubble friction dominates the dynamics and the field evolves only a little, then the birefringence angle that depends on the field dynamics will be small. Therefore the cosmic birefringence effect is most sensitive to the intermediate mass region:[10]

$$\frac{\mathcal{H}_0}{a_0} \lesssim m \lesssim \frac{\mathcal{H}_{\rm rec}}{a_{\rm rec}} \tag{4.164a}$$

$$H_0 \lesssim m \lesssim H_{\rm rec} \tag{4.164b}$$

$$10^{-33}\,\text{eV} \lesssim m \lesssim 10^{-29}\,\text{eV}\,. \tag{4.164c}$$

[10] Same discussion was proposed in [17] and [19] where authors considered a Maxwell-CS theory, however the idea holds for out theory too since it is based on the EoM of χ and as long as we neglect the photon backreaction the EoM is a Klein-Gordon-like equation.

Notice that for $m \lesssim \mathcal{H}_0/a_0 \sim 10^{-33}\,\mathrm{eV}$, since the field does not roll down the potential, it would be dark energy since its EoS approaches -1. On the other hand, for higher values of the mass the field oscillates and its EoS moves toward an averaged value of zero and χ would constitute a fraction of dark matter in the universe today.

Let us insert the Hubble parameter into the EoM, then getting:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}a^2} = -\frac{1}{a}\left(4 - \frac{3}{2}\frac{\Omega_{0m}}{\Omega_{0m} + \Omega_\Lambda a^3}\right)\frac{\mathrm{d}\chi}{\mathrm{d}a} - \frac{m^2}{aH_0(\Omega_{0m}a^{-3} + \Omega_\Lambda)^{1/2}}\chi\,. \tag{4.165}$$

We now proceed to numerically solve EoM (4.165). A nice way to plot the solution is considering a normalized field $X \equiv \chi/\chi_{\mathrm{in}}$, where $\chi(a_{\mathrm{rec}}) \equiv \chi_{\mathrm{in}}$. Now X is adimensional and the initial values of the Cauchy problem are $X(0) = 1$ and $\mathrm{d}X(0)/\mathrm{d}a = 0$. The choice $\mathrm{d}X(0)/\mathrm{d}a = 0$ is a natural choice if we want the field χ to start oscillate after the initial time; this corresponds to a field behaving as dark energy at the beginning [56] (see it from the EoS (4.135)). We show numerical solutions in figure 4.3 with different values of the mass. We have chosen the initial value:

$$\chi_{\mathrm{in}} = 0.1f \sim 0.1M_{\mathrm{Pl}} \approx 1.2 \times 10^{27}\,\mathrm{eV}\,. \tag{4.166}$$

The dynamics were obtained using the fourth-order Runge-Kutta method. The masses considered are:

$$m = \left[10^{-35},\ 10^{-34},\ 10^{-33},\ 10^{-32},\ 10^{-31},\ 10^{-30}\right]\,\mathrm{eV}\,, \tag{4.167}$$

thus taking values into the interval $10^{-33}\,\mathrm{eV} \lesssim m \lesssim 10^{-29}\,\mathrm{eV}$ from which we expect the highest contribution to the birefringence angle because of the dynamics of χ. For masses $m > 1 \times 10^{-30}\,\mathrm{eV}$ the field oscillates too much and our numerical solver is not able to track the field evolution.[11]

From the dynamics of χ we also calculate its EoS $w_\chi(a)$ (4.135) and its energy budget $\Omega_\chi(a)$ (4.153), both are shown together with the dynamics in figure 4.3.

From the plot of Ω_χ we see its value never exceeds the 15% of the whole energy budget. The initial condition used to find Ω_χ is $\chi_{\mathrm{in}} = 0.1M_{\mathrm{Pl}}$, however lowering it will lower also the energy budget since $\Omega_\chi \propto \chi_{\mathrm{in}}^2$. We understand it from the definition:[12]

$$\Omega_\chi(a) \equiv \frac{\rho_\chi}{\rho_c} = \frac{a^2}{3\mathcal{H}^2 m_{\mathrm{Pl}}^2}\left[\frac{\mathcal{H}^2}{2}\left(\frac{\mathrm{d}\chi}{\mathrm{d}a}\right)^2 + \frac{1}{2}m^2\chi^2\right] \tag{4.168a}$$

$$= \frac{a^2\chi_{\mathrm{in}}^2}{3\mathcal{H}^2 m_{\mathrm{Pl}}^2}\left[\frac{\mathcal{H}^2}{2}\left(\frac{\mathrm{d}X}{\mathrm{d}a}\right)^2 + \frac{1}{2}m^2X^2\right] \propto \chi_{\mathrm{in}}^2\,, \tag{4.168b}$$

where we have introduced the normalized field X that has a dynamics independent on the initial condition since $X(a_{\mathrm{rec}}) = 1$ by definition (see first plot of fig. 4.3).

This is not a direct proof about the sub-dominance of χ, it is more a consistency check about our assumption to neglect the ALP from the cosmic energy budget.

[11] See [94] (figure 1) for the dynamics of χ with mass $m > 10^{-30}\,\mathrm{eV}$, up to $m = 10^{-27}\,\mathrm{eV}$. Notice they used a log-scale and they parameterized by the redshift instead of the scale factor as we did.

[12] This does not hold on general, a relation $\Omega_\chi \propto \chi_{\mathrm{in}}^n$ holds as long as $V \propto \chi^n$. For example, for the cosine-type potential (4.159) this is not true; however, if we lower the initial condition, then the field starts even closer to the potential minimum where the cosine potential is well described by the quadratic one, thus Ω_χ from a cosine-type potential is similar to the energy budget from a quadratic potential in the range $0 \leq \chi_{\mathrm{in}} \ll f$.

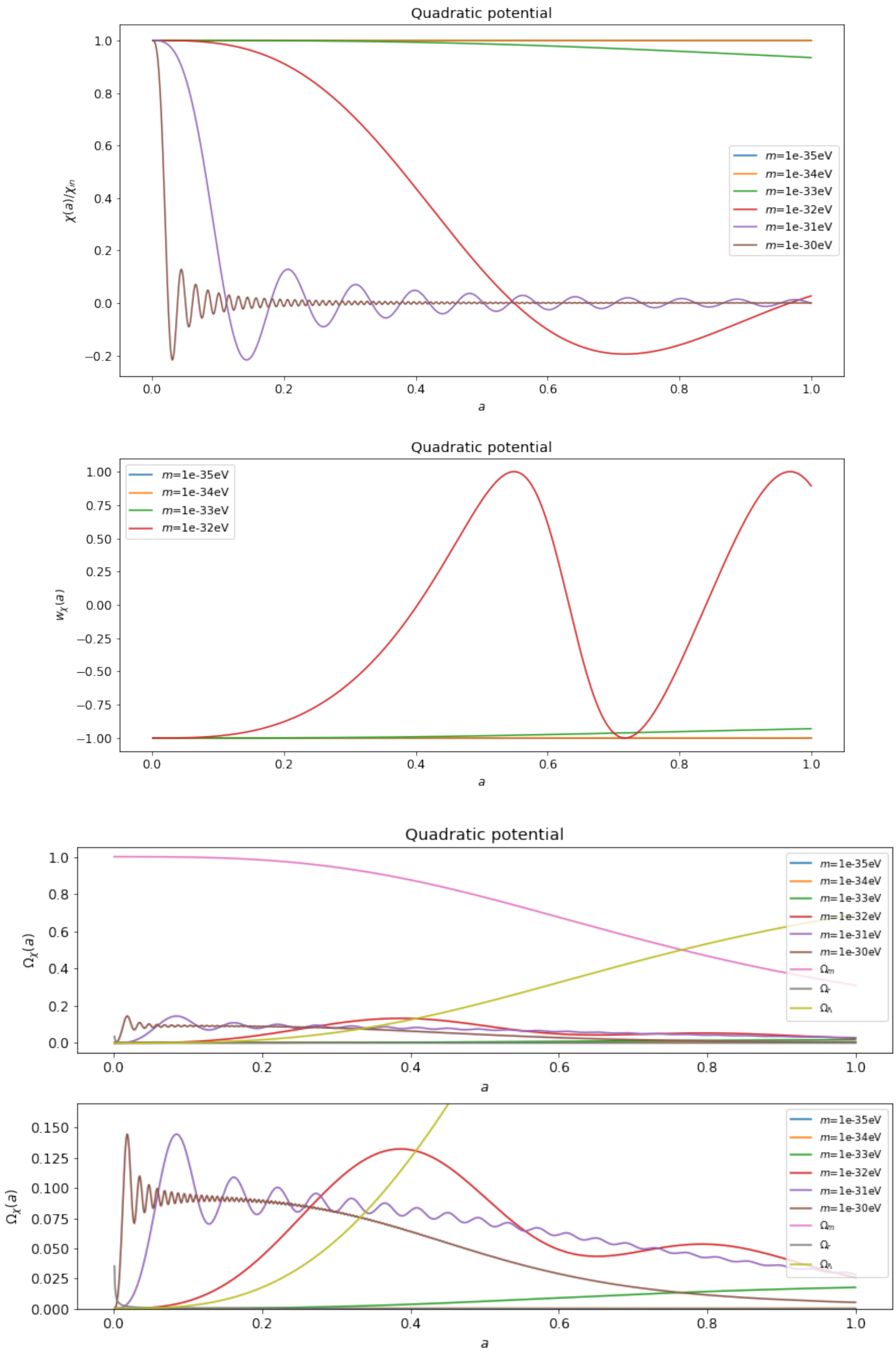

Figure 4.3: *First*: the dynamics of the normalized field $X(a) = \chi(a)/\chi_{\rm in}$ with $\chi_{\rm in} \sim 0.1 M_{\rm Pl}$ and χ is solution of EoM (4.165) where a quadratic potential was considered. Notice that changing the mass of the field, changes the time at which the field starts oscillating, as expected. The blue line is covered by the orange one, for both cases the field is lighter than H_0 thus it does not oscillate. *Second*: the EoS of the ALP field, the initial point is $w_\chi = -1$ for any mass because of the initial condition $d\chi_{\rm in}/da = 0$. For light masses the field does not oscillate and its EoS remains at -1. For heavy fields the EoS evolves because of the oscillations; we do not plot fields with $m > 1 \times 10^{-32}$ eV because of the oscillation of w_χ is so fast to cover the other mass cases. *Third* and *fourth*: the energy budget of χ, the second picture zooms the y-axis in the range of values $[0, 0.170]$.

4.6 Birefringence angle and other constraints

In the previous section we have computed the dynamics of the field χ in a quadratic potential with an initial condition $\chi_{\text{in}} = 0.1 M_{\text{Pl}}$. Now we can use this result to compute the cosmic birefringence angle β from equation (4.97), from which we get β as a function of γ, the ModMax parameter. We can get an estimate of γ using the observational result $\beta_{\text{exp}} \approx 0.342°$ [1]. The idea is to seek the root of the following equation:

$$\beta_{\text{exp}} - \beta = \beta_{\text{exp}} - \int_{\chi_{\text{in}}/g}^{\chi_0/g} \mathrm{d}\left(\frac{\chi}{g}\right) \frac{\sinh\gamma\cos(\chi/g)}{\cosh\gamma + \sinh\gamma\cos(\chi/g)} = 0\,. \tag{4.169}$$

Since we got χ as $\chi = X\chi_{\text{in}}$, where X has discrete values coming from our numerical solution, also χ is an array of discrete values and the integral is approximated to a sum:

$$\beta_{\text{exp}} - \sum_{\chi=\chi_{\text{in}}/g}^{\chi_0/g} \delta\chi \frac{\sinh\gamma\cos(\chi/g)}{\cosh\gamma + \sinh\gamma\cos(\chi/g)} = 0\,, \tag{4.170}$$

where $\delta\chi$ is the step into the array of χ's values. Notice angles must be expressed in radiants: $\beta_{\text{exp}} \approx 0.005\,97\,\text{rad}$; and the sum ranges from the initial value of the dimensionless field $\chi(a_{\text{rec}})/g = \chi_{\text{in}}/g$ up to today value $\chi(a_0)/g = \chi_0/g$.

As one can see from (4.97), changing g is equivalent to change the initial condition χ_{in}: they participate together as χ/g into the definition of β, therefore as long as we change g and χ_{in} keeping their ratio constant we get the same amount of β. Because of this reasoning, we think g cannot have a physical meaning.

To proceed with the computation of (4.97) we need to fix the dynamics of χ/g. We propose the following guess: we choose χ/g such that it evolves equal to the dynamics of the ALP χ, that is described by the Lagrangian density $\mathscr{L}_\chi = -(\partial\chi)^2/2 - V(\chi)$. With this choice, g is of the order of unit and has the same dimension of χ. Since along the text we have used natural units writing the dimensionality in electronvolt, we guess $g \sim \mathcal{O}(1)$ eV. We recall it is just a trial based on the idea of inserting the same dynamics of χ as the argument of the trigonometric functions of (4.97), with no rescaling.

Since we have guessed what we want into the cosine of (4.170), let us continue finding its roots. We solve the equation numerically using the `fsolve` function imported from `scipy.optimize` package. We report the roots with the initial value $\chi_{\text{in}} = 0.1 M_{\text{Pl}}$ in table 4.1. We should be worried about the negative values in table 4.1 since the ModMax theory predicts $\gamma \geq 0$, however this is a result on how the numerical solver, `fsolve`, works: it accepts a root up to a certain tolerance, proposing values $|\gamma| <$ tolerance.

Probably, equation (4.170) is too smooth around the true root $\gamma_{\text{true}} \geq 0$ that is near zero. Then, `fsolve` iterates roots around zero jumping from negative to positive values up to having found a proposal that is lower than the tolerance. Thus also negative values are accepted by `fsolve` and their outcome is quite random. So let us focus on the order of magnitude of the estimates and not on the sign: the idea is to consider the modulo $|\gamma|$ of any proposal from `fsolve`, namely the third column of table 4.1.

We have found that ModMax precursor is able to produce the observed cosmic birefringence angle with a $\gamma < 10^{-20}$ for a mass $m > 10^{-35}\,\text{eV}$, lowering down to $\gamma < 10^{-25}$ for a bigger mass $m > 10^{-32}\,\text{eV}$. We see a predictable patter: for smaller masses γ must be bigger in order to balance the slow evolution of the field (see EoM (4.163): having a small mass the oscillating time comes later).

To get a more robust estimate for γ and to study its dependence on χ_{in}, we scan over the initial value for each mass, i.e. we find the roots of (4.170) for different initial conditions

mass (eV)	estimation for γ	modulo $\lvert\gamma\rvert$
10^{-30}	4×10^{-27}	10^{-27}
10^{-31}	1×10^{-25}	10^{-25}
10^{-32}	-1×10^{-25}	10^{-25}
10^{-33}	3×10^{-25}	10^{-25}
10^{-34}	-1×10^{-22}	10^{-22}
10^{-35}	-7×10^{-20}	10^{-20}

Table 4.1: Estimates of γ with the ALP in a quadratic potential and initial value $\chi_{\text{in}} = 0.1 M_{\text{Pl}}$.

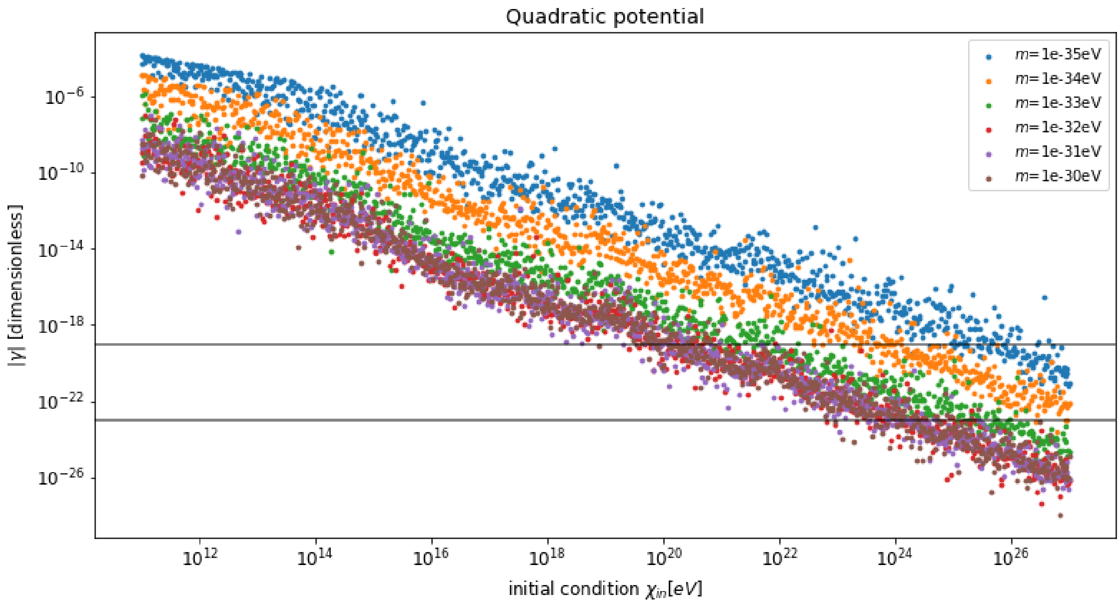

Figure 4.4: Values of γ for different initial conditions χ_{in} in a quadratic potential. We plotted the modulus of the root proposed by `fsolve|`.

in the range:

$$10^{-17} M_{\text{Pl}} \leq \chi_{\text{in}} \leq 0.1 M_{\text{Pl}} \,. \tag{4.171}$$

We take 1000 values in this interval logarithmically separated in order to explore each order of magnitude, and we do so for each mass considered. We plot the estimates of γ as $\gamma = \gamma(\chi_{\text{in}})$ in figure 4.4, where initial conditions are sampled from (4.171) as explained above. We see that the higher χ_{in} is and the lower γ is; this is a consequence of the integral in β (see (4.97)).

In addition, from figure 4.4 we recognize the pattern that at lower masses we get higher γ. To confirm this trend we plot the estimate of γ for different masses in figure 4.5. To do so we must choose the initial conditions; we take the initial, final and median value of the interval (4.171) namely we consider $\chi_{\text{in}} = \left(10^{-16}, 10^{-9}, 0.1\right) M_{\text{Pl}}$. Regarding the choice of the mass, up to now we have worked with 6 different masses that are reported in (4.167). To get a more robust result we now extend the analysis to 1000 masses taken logarithmically from the range $[10^{-35}, 10^{-30}]$eV. We expect a decreasing γ as long as m increases, as already realized observing figure 4.4. Looking at figure 4.5 we confirm all the hypothesis we had.

Another interesting behavior that we infer from figure 4.5 is an apparent flatness of γ in the mass range $10^{-33}\,\mathrm{eV} \lesssim m \lesssim 10^{-29}\,\mathrm{eV}$. The reason is that in this mass range the ALP field starts oscillating after the recombination epoch and its amplitude dilutes. In this case the field today, χ_0, has a negligible value with respect to the initial one, $\chi_{\rm in}$. The effective result is a contribution to the birefringence angle, and thus to γ, coming only from $\chi_{\rm in}/g$. In fact, we do not see any flatness in 4.4 for the mass range $10^{-33}\,\mathrm{eV} \lesssim m \lesssim 10^{-30}\,\mathrm{eV}$. Even more, we see that for this mass range the points in figure 4.4 are superimposed (especially for $m > 10^{-32}\,\mathrm{eV}$) and this is a symptom of the "independence" of γ from the mass in this range.

Our final estimation In light of what we have found, we give the following conclusion: the ModMax precursor theory (4.18) with a dynamical, background ALP $\chi(t)$ in a quadratic potential is able to rotate the linear polarization plane of CMB photons, thanks to the coupling between them and the field χ.

In order to reproduce the observed rotation angle of $\beta_{\rm exp} \approx 0.342°$ [1], the ModMax parameter γ takes a value fixed by field dynamics normalized by a factor that we have called g; as a consequence γ depends on the initial condition of the ALP field $\chi_{\rm in}$ and its mass m.

If we consider the initial condition $\chi_{\rm in} \approx 0.1 M_{\rm Pl} \sim 10^{27}\,\mathrm{eV}$ and we normalize the field dynamics just taking out the eV dimension, our analysis furnishes the range:

$$10^{-26} < \gamma < 10^{-19}\,, \tag{4.172}$$

for masses $10^{-35}\,\mathrm{eV} \lesssim m \lesssim 1 \times 10^{-30}\,\mathrm{eV}$.

In the particular range $10^{-33}\,\mathrm{eV} \lesssim m \lesssim 10^{-30}\,\mathrm{eV}$ where we expect the maximum contribution to the cosmic birefringence angle from the dynamics of the ALP (see eq. (4.164)), the range on γ shrinks to:

$$10^{-26} < \gamma < 10^{-23}\,. \tag{4.173}$$

Lowering the initial condition predicts an higher γ, as shown in figure 4.4. On the other hand, exploring the range of bigger initial conditions $\chi_{\rm in} \geq M_{\rm Pl}$ will probably require a different potential that considers the non-perturbative contributions. For this reason, we do not extend our result (4.173) to ALPs with initial conditions bigger than the ones considered here.

The dependence of γ from the ALP mass is shown in 4.5.

What about Maxwell-Chern-Simons? From [17] we report the authors' estimate for $g_{a\gamma}$ obtained analyzing the cosmic birefringence with a Maxwell-CS theory:

$$g_{a\gamma} > 10^{-29}\,\mathrm{eV}^{-1} \tag{4.174}$$

for masses $10^{-35}\,\mathrm{eV} \lesssim m \lesssim 10^{-30}\,\mathrm{eV}$ and initial condition $\chi_{\rm in} < M_{\rm Pl}$ (from figure 1 of [17]).

Can we compare their estimate with our result (4.172)? It will be possible if and only if the ModMax precursor theory is in the limits by which it mimics Maxwell-CS. We recall the result (4.19d) showing that these limits are $\gamma \ll 1$ and $\chi/g \ll 1$.

Since we have obtained the range (4.172) with the guess that χ/g goes as the axion field χ with an initial value $\chi_{\rm in} \sim 0.1 M_{\rm Pl}$, the quantity χ/g is not small at all, and the second limit is not achieved. So, bounds (4.174) and (4.172) cannot be compared.

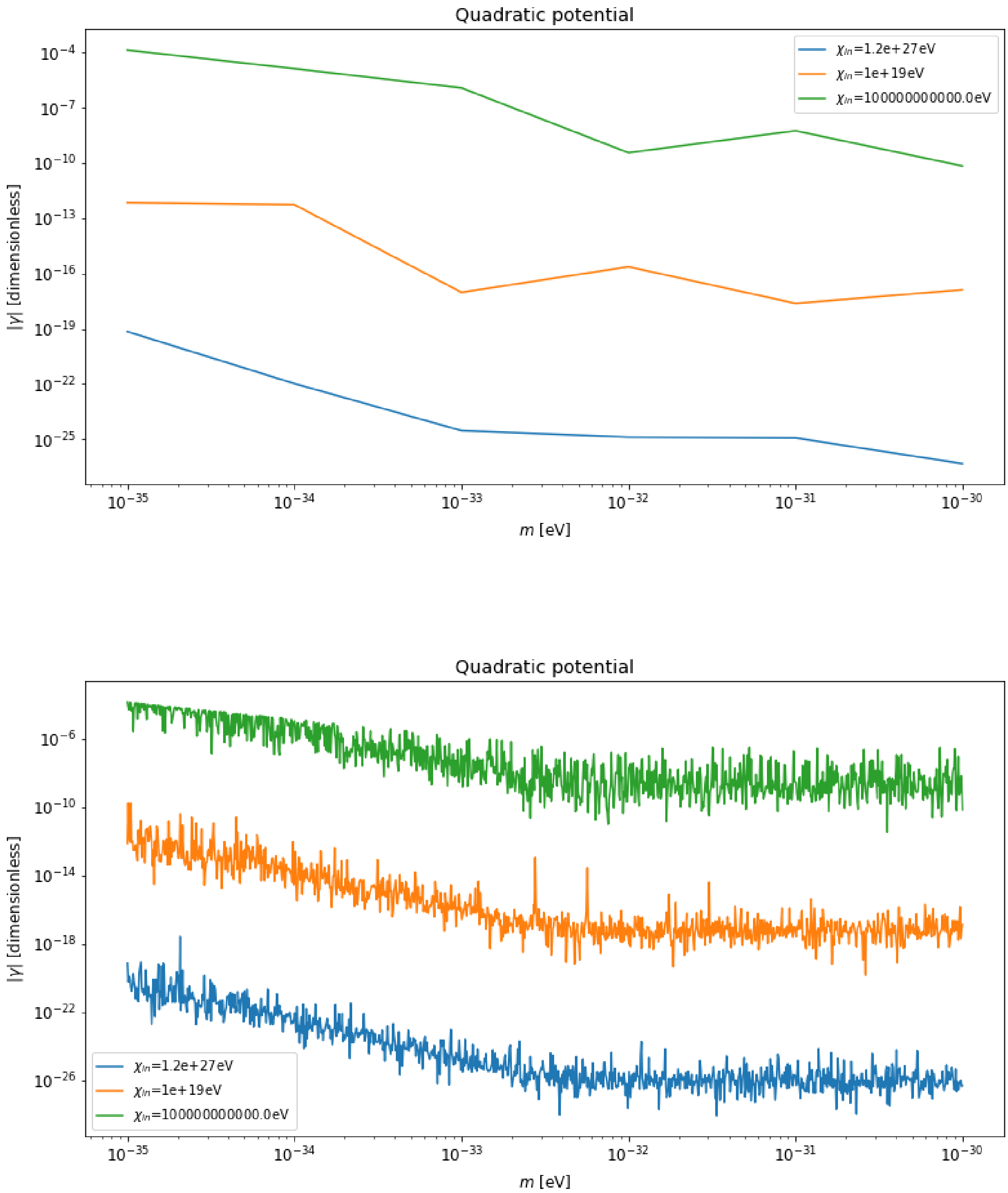

Figure 4.5: *Top*: the estimate value of γ as a function of the 6 masses considered along the text, their value is shown in (4.167). *Bottom*: a more robust result where it is shown the value of γ for 1000 different masses taken from the interval $[10^{-35}, 10^{-30}]$eV with log steps (base 10) in order to uniformly sample the masses at each order of magnitude. We see the trend that for heavier masses we get smaller γ.

We are in a region of the space of parameters in which ModMax precursor is *not* mimicking Maxwell-CS but the phenomenon of cosmic birefringence is however produced. As equation (4.97) tells us, we are looking at a rotation angle produced by the dynamics of an axion, background field. The dynamics is "elaborated" by a the periodic function of cosine, instead of linearly participate as predicts by the CS term (3.1).

We conclude making in evidence that the coupling $g_{a\gamma}$ of CS has some behaviors in common with γ (see figures of [17]): they both have a dependence on the initial condition (fig. 4.4) and show a flatness in the mass range $10^{-33}\,\mathrm{eV} \lesssim m \lesssim 10^{-29}\,\mathrm{eV}$ (fig. 4.5), however they are consequences of the integral over χ in the definition of β and no claims between our result and $g_{a\gamma}$ should be driven.

One of the last assumptions we took to derive γ was to consider the dynamics of the dimensionless quantity χ/g as the one of the field χ with $\chi_{\rm in} \sim 0.1 M_{\rm Pl}$. This was a guess that takes us away from the region in parameter space where ModMax precursor mimics Maxwell-CS. The true role of g and how it should link the dynamics of χ to the adimensional argument of ModMax-precursor trigonometric functions, that we simply wrote as χ/g, is left for future study.

Conclusions

ModMax theory (1.96) is a classical, non-linear theory of electromagnetism which has the same maximal symmetries as Maxwell's theory, i.e. Lorentz invariance extended to the full conformal symmetry and electric-magentic duality. It has the dimensionless parameter γ and reduces to Maxwell's theory when $\gamma = 0$. It was shown to be the unique non-linear theory with all these properties [12]. In an attempt to quantize the theory, it was proposed a reformulation of the ModMax Lagrangian density (see equation (4.7)) having quadratic form for the electromagnetic fields and introducing an auxiliary field [20].

In this thesis we pointed out some similarities between the reformulated ModMax Lagrangian (4.7) and other electrodynamics theories, based on the duality symmetry. The reformulated ModMax theory is similar in form to the axion-dilaton electrodynamics [21, 22, 67–69] (see (4.9) for the Lagrangian density of axion-dilaton-Maxwell theory). Based on these similarities, we promoted the reformulated ModMax theory to a new theory of interacting electromagnetic and neutral pseudoscalar fields, that we have called ModMax precursor.

The interaction between pseudoscalar fields and photons produces a rotation of the linear polarization plane of photons as they propagate in the universe. This phenomenon is called cosmic birefringence and is deserving attention after the claim of a detection of a nonzero rotation angle at 99.987% C.L. of the polarization plane of CMB photons [1].

Usually the pseudoscalar-photon coupling is defined through the Chern-Simons term (3.1) [15–19, 49]. We reviewed its phenomenology deriving the equation of motion for the Maxwell electrodynamics together with the Chern-Simons term and we computed the expression for the cosmic birefringence angle. Then we passed to the ModMax precursor theory since it contains the ingredients to predict cosmic birefringence. Using the same formalism adopted for the Maxwell-Chern-Simons theory we computed the equation for the rotation angle, see (4.97). It depends on the dynamics of the pseudoscalar field, thus we derived its equation of motion and solved it numerically considering a FLRW metric. We focused on the time interval from the recombination epoch up to now. We assigned a quadratic potential (4.160) to the pseudoscalar field and we used the assumption that the energy density of the pseudoscalar field is sub-dominant with respect to the energy densities of the other cosmic species. This allowed us to split the evolution of the Hubble parameter from the evolution of the pseudoscalar field. We also neglected the backreactions of photons to the dynamics of the pseudoscalar field imposing the requirement of homogeneity and isotropy. Chosen the initial condition of the field and its mass we have been able to solve its dynamics.

We analyzed the parameter space of the theory to reproduce the observed amount of cosmic birefringence and estimated γ.

We found ranges on γ by which the ModMax precursor is able to reproduce the observed amount of cosmic birefringence. Our result is:

$$10^{-26} < \gamma < 10^{-19}\,, \tag{4.175}$$

for masses of the pseudoscalar field $10^{-35}\,\mathrm{eV} \lesssim m \lesssim 1\times10^{-30}\,\mathrm{eV}$. The range shrinks to $^{-26} < \gamma < 10^{-23}$, for masses in $10^{-33}\,\mathrm{eV} \lesssim m \lesssim 1\times10^{-30}\,\mathrm{eV}$ (see figure 4.4). Both estimates were obtained considering the initial condition $\chi_{\rm in} \approx 0.1 M_{\rm Pl}$ for the pseudoscalar field; we studied the dependence of γ on the initial condition and the mass in figures 4.4 and 4.5.

Our results are based on the hypothesis that the arguments of the non-linear functions that define the rotation angle (4.97) are equal to the dynamics of the pseudoscalar field; using our parametrization we mean $g \sim \mathcal{O}(1)$ eV (see (4.18)).

Our estimates bound γ to have tiny values, somehow similar to the values of the Maxwell-Chern-Simons coupling constant obtained with similar hypothesis [17]. However our analysis was not performed in the region of parameter space where ModMax precursor mimics Maxwell-Chern-Simons (condition $\chi/g \ll 1$ is not achieved – (4.101)). Thus, we developed a new way to predict the phenomenon of cosmic birefringence based on the ModMax precursor theory, that generalizes the Chern-Simons term.

Future perspectives In this thesis we performed one of the first studies of ModMax in Cosmology. Focusing on the phenomenon of isotropic cosmic birefringence we were able to constraint the ModMax parameter γ. Keep investigating the interplay between different theories and frameworks, as ModMax and Cosmology, will furnish new understanding of the Universe.

An immediate upgrade of our work that one could think about is the studying of anisotropic cosmic birefringence which allows the rotation angle β to depend on photon's direction (see e.g. [56]).

In this work we have chosen different assumptions and approximations, so a natural way to improve our results is by generalizing such assumptions. Starting from technical ones, we could improve our description of the photon emissions: we assumed a simultaneous emission at a redshift $z_{\rm rec} \sim 1090$, while a more physical description should use a statistical distribution peaked at both recombination and ionization epochs [17, 19]. We also recall that we assumed an axion energy budget Ω_χ being sub-dominant with respect to the other cosmic species. A more general treatment would require to solve equations (4.151), (4.152) and (4.153) all together. Doing so, we will be able to study whether the pseudoscalar field of the ModMax precursor theory might play the role of early dark energy [55, 63]. Also a more comprehensive discussion of the electromagnetic backreaction on the dynamics of χ should be done.[13]

As pointed out in section 4.3.3, the dilaton-like coupling does not participate on the cosmic birefringence angle (we recall in the ModMax precursor theory it is $S\cos(\chi/g)$). Therefore it would be interesting to investigate effects related to such a coupling in other phenomena and compare it with the axion-photon coupling.

The aim of these perspectives is to keep investigating the parameter space of the ModMax theory and precursors in regions in which the theories wander away from the commonly used Chern-Simons coupling.

[13] An example in a inflationary scenario is [95], where the authors considered a generic $U(1)$ gauge field.

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