UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Matematica "Tullio Levi-Civita"

CORSO DI LAUREA TRIENNALE IN MATEMATICA

CW-complexes and Free Groups

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Abstract

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by Pietro GREINER

The focus of this work is to show how some tools from algebraic topology can have useful applications to group theory. In order to do this in the first chapter we give a brief overview of some basic concepts that we are going to need in the subsequent discussion, like free groups, fundamental groups and covering spaces. In the second chapter we discuss graphs as topological spaces. In the third chapter we then proceed to introduce CW-complexes and we discuss some of their properties. In the last chapter we give some classical topological proofs of two important theorems on free groups and free products of groups.

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Introduction

The main goal of the work we present here is to provide an insightful view on how some basic concepts of algebraic topology can find an application in the field of group theory.

The main topological concept we are going to use in order to do so is the fundamental group. We recollect its definition in chapter 1 together with the most important tool at our disposal when dealing with it, the Seifert-Van Kampen theorem (Seifert, 1931, Van Kampen, 1933). Along side these two topics, we will also review some basic ideas of topology, mainly regarding the concept of covering spaces, and of key concepts of group theory with a focus on free groups and free products of groups.

After this first chapter, we will take care of some theory on graphs which will serve as an introduction to some of the concepts we will develop later on. More in detail, we will start by defining graphs as topological spaces and we will study some of their basic properties. Afterwards, we will dedicate some pages to dwell deeper into a special family of graphs, namely trees, which will serve as a base for all the question regarding paths and fundamental groups of graphs which will be the next thing we are going to investigate. We will then close this chapter with a characterization for the covering spaces of graphs.

With the end of the second chapter we also conclude the broader introduction and we begin approaching the main concepts and results present in this work. We will begin by introducing CW-complexes, where C stands for closure finite and W for weak topology, an idea from Whitehead, 1949, that gives us a class of spaces which, while still remaining quite general for the purposes of homotopy theory, has some nice properties that lead to easy computation. After getting an idea of some characteristic properties of these spaces form a topological standpoint we will delve into how this class of complexes will be useful to study ideas from group theory. In this sense we will apply Seifert-Van Kampen theorem to them and thanks to this we will find that, given any group G, it is quite easy to construct a CW-complex such that its fundamental group is isomorphic to G.

We will then get to the last chapter where by applying what we learned so far we will give two topological proofs of two theorems, the first is the following:

Theorem. *Let H be a subgroup of the free product $G = \ast_\lambda G_\lambda$. Then H is a free product itself*

$$H = F * (\mathop{\ast}_{\nu} H_\nu)$$

Where F is a free group and each H_ν is conjugate in G to a subgroup of one of the G_λ.

It was forulated first Kurosch, 1934 and the proof we give is modeled after the one in Baer and Levi, 1936.

The second is from Gruschko, 1940 and this is the statement:

Theorem. *Let $\varphi : F \to \ast_\lambda G_\lambda$ be an epimorphism of the free group F onto an arbitrary free product of groups. Then, there exists a decomposition of F as a free product, $\ast_\lambda F_\lambda$, such that $\varphi(F_\lambda) \subseteq G_\lambda$ for all $\lambda \in \Lambda$.*

The proof we give follows the one from Stallings, 1965.

Chapter 1

Prerequisites

1.1 Free product of groups and free groups

The need for this construction arises from the necessity of finding a group that contains a given collection of groups G_α as subgroups. Probably, the most natural way to find it is to consider the product group $\prod_\alpha G_\alpha$, whose elements are the functions $\alpha \to g_\alpha \in G_\alpha$ or the direct sum $\bigoplus_\alpha G_\alpha$ in which the elements are the functions taking on non-identity values at most finitely often. Both this constructions produce groups containing all the G_α as subgroups, but they also have the property that elements of different G_α commute with each other. This commutativity, when working with nonabelian groups, becomes unnatural, so we need to find a 'nonabelian' version of of $\prod_\alpha G_\alpha$ or $\bigoplus_\alpha G_\alpha$.

Definition 1.1. The precise construction of the **free product** $*_\alpha G_\alpha$ is as follows: as a set is the collection of the words $g_1 g_2 ... g_m$ of arbitrary finite length m where each g_i belongs to a G_{α_i} and it is not the identity element of G_{α_i}, and adjacent letters g_i and g_{i+1} belong to different G_α. Words satisfying this condition are called **reduced**, unreduced words can always be simplified to reduced word by writing adjacent letters of the same G_α as a single letter and canceling the trivial letters. The empty word is the identity element. The group operation is juxaposition: $(g_1 g_2 ... g_m)(h_1 h_2 ... h_n) = g_1 g_2 ... g_m h_1 h_2 ... h_n$, if the word is unreduced we than simplify it.
We call **free group** the free product of any number (finite or infinite) of copies of $\mathbb{Z}$

Lemma 1.1.1. *Any group is homomorphic image of a free group.*

1.2 The fundamental group

In this section, we will define the fundamental group of a topological space. In order to do so, we first need to establish some useful notation. We will let X and set $I = [0,1]$.
We define a **path** or an **arc** to be a continuous map from an interval into X. We say that the images of the end points of the interval are the **end points** of the path and, if these coincide, we call the path a **loop** based at the common end point.
Of the space X we say that is **arcwise-connected** or **pathwise-connected** if any two points can be joined by an arc. We call **arc components** the maximal subset of X to be arcwise-connected. We say that X is **locally arcwise connected** if every point has a basic family of arcwise-connected neighborhoods.

Definition 1.2. Let $f_0, f_1 : [a,b] \to X$ be two paths in X such that $f_0(a) = f_1(a), f_0(b) = f_1(b)$ we say that f_0 and f_1 are **homotopic** if there exists a continuous map

$$f : [a,b] \times I \to X$$

such that

$$t \in [a,b] \begin{cases} f(t,0) = f_0(t) \\ f(t,1) = f_1(t) \end{cases}$$

and

$$s \in I \begin{cases} f(a,s) = f_0(a) = f_1(a) \\ f(b,s) = f_0(b) = f_1(b) \end{cases}$$

We call f the **homotopy** between f_0 and f_1. It is readily seen that the relation of being homotopic is an equivalence on the set of path in X with the same terminal points.

At this point we shall note that for every pair of intervals $[a,b], [c,d]$ there are two linear isomorphisms between them, one mapping $a \to c, b \to d$ and one $a \to d, b \to d$, hence, from now on, we can consider just the paths from I to X without loss of generality.
In order to define the fundamental group we are going to need an operation between elements of such group. So we define the product of paths.

Definition 1.3. Let f, g be paths in X such that the terminal point of f is the initial point of g, then we define **the product of paths** $f \cdot g$ as follows:

$$f \cdot g = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \tag{1.1}$$

Next we need a couple of lemmas to show the properties of this product and the equivalence that we have defined.

Lemma 1.2.1. *The equivalence relation and the product are compatible, i.e., if $f_0 \sim f_1$ and $g_0 \sim g_1$ we have that $f_0 \cdot g_0 \sim f_1 \cdot g_1$.*

Lemma 1.2.2. *The multiplication of equivalence classes of paths is associative.*

For any point $x \in X$ we denote as ε_x the equivalence class of the constant map of I into x. This path class has the following fundamental property:

Lemma 1.2.3. *Let α be an equivalence class of paths with initial point x and terminal point y. Then $\varepsilon_x \cdot \alpha = \alpha \cdot \varepsilon_y = \alpha$.*

Now for any path f let $\bar{f}$ denote the path defined by

$$\bar{f}(t) = f(1-t)$$

Lemma 1.2.4. *Let α and $\bar{\alpha}$ denote the equivalence classes of f and $\bar{f}$, respectively. Then,*

$$\alpha \cdot \bar{\alpha} = \varepsilon_x, \qquad \bar{\alpha} \cdot \alpha = \varepsilon_y,$$

where x and y are the initial and terminal points of f.

Now we have all we need to define the fundamental point of of the space X.

Definition 1.4 (The fundamental Group)**.** Let x be any point of X; it is readily seen that the set of all loops based at x is a group with the product we have defined, we call it **fundamental group** of X at the base point x and it is denoted by $\pi_1(X, x)$.

The last result we need for this section in the following theorem.

Theorem 1.1 (Hatcher, 2000)**.** *If X is arcwise connected, the group $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic for any two points $x, y \in X$.*

1.3 Seifert-Van Kampen Theorem

Theorem 1.2 (Seifert-Van Kampen Hatcher, 2000)**.** *If X is the union of path-connected open sets A_α each containing the base point $x_0 \in X$ and if intersection $A_\alpha \cap A_\beta$ is path-connected, then the homorphism*

$$\Phi : \mathop{\ast}_{\alpha} \pi_1(A_\alpha) \to \pi_1(X)$$

is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected then the kernel is the normal subgroup N generated by all the elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$, and so Φ induces an isomorphism

$$\mathop{\ast}_{\alpha} \pi_1(A_\alpha)/N \approx \pi_1(X)$$

.

1.4 Covering spaces

1.4.1 Definition and basic properties

Definition 1.5. A **covering space** of a topological space X consists in a pair $(\tilde{X}, p)$, where $\tilde{X}$ is a space and $p : \tilde{X} \to X$ is a continuous map such that for each $x \in X$ exists an arcwise-connected open neighborhood U such that each of the components of $p^{-1}(U)$ is mapped topologically onto U by p. A neighborhood U that satisfies this condition is called **elementary neighborhood** and the map p is called **projection**.

1.4.2 The fundamental group of a covering space

Theorem 1.3 (Massey, 1967)**.** *Let $(\tilde{X}, p)$ be a covering space of X and $x_0 \in X$. Then, the subgroups $p_*\pi_1(\tilde{X}, \tilde{x})$ for $\tilde{x} \in p^{-1}(x_o)$ are a conjugacy class of subgroups of $\pi_1(X, x_0)$.*

1.4.3 The action of $\pi_1(X, x)$ on $p^{-1}(x)$

Definition 1.6. An **automorphism** of a covering space $(\tilde{X}, p)$ of X is an invertible continuous map $\phi : \tilde{X} \to \tilde{X}$ such that for every $x \in \tilde{X}$ we have that $p(\phi(x)) = p(x)$. The set of all automorphisms of $(\tilde{X}, p)$ together with the composition of maps form a group for which we use the notation $A(\tilde{X}, p)$.

Definition 1.7. Let $\tilde{X}, p$ be a covering space of X and $x \in X$. For any $\tilde{x} \in p^{-1}(x)$ and any $\alpha \in \pi_1(X, x)$ we define $\tilde{x} \cdot \alpha$ as the terminal point of the unique path class $\tilde{\alpha} \in \tilde{X}$ with initial point $\tilde{x}$ and such that $p_*(\tilde{\alpha}) = \alpha$. With this definition of "$\cdot$" $\pi_1(X, x)$ is a group of right set operators on $p^{-1}(x)$.

Theorem 1.4 (Massey, 1967)**.** *Let $(\tilde{X}, p)$ be a covering space of X. Then, $A(\tilde{X}, p)$ is naturally isomorphic to the group of automorphisms of $p^{-1}(x)$ considered as a right $\pi_1(X, x)$-space.*

1.4.4 Existence theorem for covering spaces

Theorem 1.5 (Massey, 1967)**.** *Let X be a topological space which is connected, locally arcwise-connected and semilocally simply connected. Then, given any conjugacy class of subgroups of $\pi_1(X, x)$ there exists a covering space corresponding to the given conjugacy class.*

1.4.5 Induced covering spaces

Theorem 1.6 (Massey, 1967)**.** *Let $(\tilde{X}, p)$ be a covering space of X, let A be a connected subspace of X locally arc-wise connected and let $\tilde{A}$ an arc component of $p^{-1}(A)$. We use the notation: $\tilde{a} \in$*
$tildeA$, $a = p(\tilde{a})$, $p' = p|_A$, $i : A \to X$ the inclusion map. Then:

$$p'_*(\pi_1(\tilde{A}, \tilde{a})) = i^{-1}{}_* \left[p_*(\pi_1(\tilde{X}, \tilde{a}))\right].$$

1.4.6 Point set topology on covering spaces

Lemma 1.4.1. *Let $(\tilde{Y}, p)$ be a covering space of Y, both connected and locally arc-wise connected and we also assume Y to be regular Hausdorff. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of compact Hausdorff spaces which are locally arc-wise connected, simply connected, and let $\{f_\lambda : X_\lambda \to Y\}_{\lambda\in\Lambda}$ a family of continuous maps. Let now*

$$\{f_\lambda : X_\lambda \to \tilde{Y} \mid \lambda \in \Lambda, i \in M_\lambda\}$$

the family of liftings of f_λ such that $f_\lambda = pf_{\lambda i}$. Then if Y has the largest topology that makes f_λ continuous then $\tilde{Y}$ has the largest topology that makes $f_{\lambda i}$ continuous.

Chapter 2

Topology of graphs

2.1 Definition and basic properties

Definition 2.1. A **graph** is a pair consisting of an Hausdorf space X and a subset X^0, called **the set of vertices of** X, such that the following conditions hold:

1. X^0 is a discrete, closed subspace of X. Points of X^0 are called **vertices**.
2. $X \setminus X^0$ is the disjoint union of open subsets e_i, where each e_i is homeomorphic to an open interval of the real line. The set e_i are called edges.
3. For each e_i, its boundary $\overline{e_i} \setminus e_i$ is either one or two points. If $\overline{e_i} \setminus e_i$ consist in two points the pair $(\overline{e_i}, e_i)$ is homeomorphic to the pair $([0,1],(0,1))$; if $\overline{e_i} \setminus e_i$ consist in one point then $(\overline{e_i}, e_i)$ is homeomorphic to $(S^1, S^1 \setminus \{1\})$.
4. X has the **weak topology**: $A \subset X$ is closed (open) if and only if $A \cap \overline{e_i}$ is closed for every edge e_i.

Now that we have a formal definition of a graph as a topological space, it is useful to have some basic nomenclature for simple properties of graphs. We say that a graph is **finite** if it has only finite vertices and edges. A finite graph is compact as it is a finite union of compact subset. A graph is **locally finite** if each vertex is incident with a finite number of edges. A graph is locally compact if and only if is locally finite.
The next thing we need in order to make working with graphs easier we need the following lemma.

Lemma 2.1.1. *Every point of a graph has a basic family of contractible neighborhoods.*

Proof. It is clear that this property holds for all isolated vertices and for the interior points of an edge. So let v be a non isolated vertex and let U be an open set containing v, we need to show a contractible neighborhood V of v such that $V \subseteq U$. For every edge e incident in v we have that $U \cap \overline{e}$ is open. We choose V so that $V \cap \overline{e}$ is a contractible open neighborhood of v in $\overline{e}$, $V \cap \overline{e} \subseteq U \cap \overline{e}$ and so that for every non incident edge e', $V \cap \overline{e'} = \emptyset$. This choice is possible thanks to the third condition of the definition. From the fourth condition it follows that V is open, so we only need to prove that it is contractible.
For each edge e we now choose a contracting homotopy

$$\begin{aligned} \varphi_e :& V \cap \overline{e} \times I \to V \cap \overline{e} \\ & (x,0) \mapsto x \\ & (x,1) \mapsto v \\ & (v,t) \mapsto v. \end{aligned}$$

Then if we define $f : V \times I \to V$ so that $f|_{V \cap \bar{e} \times I} = \varphi_e$ we will have the desired contracting homotopy, whose continuity depends on the continuity of each φ_e and the weak topology. □

From this lemma follows that a graph is locally arc-wise connected and semilocally simply connected, hence the theory of covering spaces is applicable to graphs. As we covered the basic topological properties of graphs, we now want to give a precise definition of what paths are on a graph in order to study their fundamental group.
By definition, an edge e is homeomorphic to $(0,1)$, let $h_0, h_1 : e \to (0,1)$, we say that the two are equivalent if the composite homeomorphism $h_0 h_1^{-1} : (0,1) \to (0,1)$ is a monotone increasing map. It is clear that there are two equivalence classes and to **orient** an edge is to choose one of them.
If $(\bar{e}, e)$ is homeomorphic to $([0,1],(0,1))$, we may choose an homeomorphism $h : \bar{e} \to [0,1]$ such that $h|e$ is of the preferred equivalence class determined by the orientation of e. We call $h^{-1}(0), h^{-1}(1)$ **initial** and **terminal** vertices of e, respectively. In the case where $(\bar{e}, e)$ is homeomorphic to $(S^1, S^1 - \{1\})$ the one vertex is both initial and terminal.
We define an **edge path** to be a finite sequence of oriented edges $(e_1, e_2, ..., e_n)$ such that the terminal vertex of e_{i-1} is the initial vertex of e_i, for $i = 2, 3, ..., n$. The edge path is called **reduced** if it is not the case that edges e_i, e_{i-1} are the same edge with opposite orientations for any $i = 2, 3, ..., n$. We call the initial vertex of the path the initial vertex of e_1 and the terminal vertex of the path the terminal vertex of e_n.

2.2 Trees

A **tree** is a connected graph that contains no closed reduced edge paths. This kind of graphs will be really useful in the computation of the fundamental group of graphs in general. We will understand why with the following results.

Theorem 2.1 (Massey, 1967)**.** *Any tree is contractible.*

Proof. The first thing to do is to prove the statement for finite tree. We will do so by induction on the number of edges. For either 0 or 1 edges is obvious.
Now we assume to have proved the statement for every graph with less than n edges. In a connected finite graph T if every vertex is connected with two ore more edges we can find a closed loop, therefore in a tree there must be a vertex v connected with only one edge e. We can consider the subgraph given by

$$T' = T \setminus (e \cup \{v\}).$$

This subgraph is connected, hence is a tree itself and since it has one less edge is also contractible. Therefore, T in contractible since it deforms retracts onto T'.
Now let T be an arbitrary tree and let v_0 be one of its vertices, our goal will be to define an homotopy such that

$$\begin{aligned} \varphi_e : & T \times I \to T \\ & (x,0) \mapsto x \\ & (x,1) \mapsto v_0 \\ & (v_0,t) \mapsto v_0. \end{aligned}$$

First, for each vertex $v \in T$ we choose a finite subgraph $T(v)$ containing both v and v_0. For each such subgraph, we choose a path $t \to f(v,t)$ with initial point v and terminal point v_0 in $T(v)$. We also define $f(v_0, t) = v_0$ for all t. In this way we have defined a function

$$f : T^0 \times I \to T$$

where T^0 is the set of the vertices in T.
The next step is to extend f to the edges. Let v_1, v_2 be two vertices in T connected by e, then $T(V_1) \cup T(V_2) \cup e$ is a connected subgraph of T and therefore a tree. By the first part of the proof this tree is contractible and so it is simply connected. The set $\bar{e} \times I$ is homeomorphic to a square on which the map f is already defined on two sides, and on the other two the only choices are

$$f(x,0) = x$$
$$f(x,1) = v_0.$$

Since the four squares of the triangle are mapped to the simply connected $T(V_1) \cup T(V_2) \cup e$, the map can be extended to the interior of the square, and by doing this for every edge of T we get an extension of $f : T^0 \times I \to T$ to $f : T \times I \to T$. The fact that the map is continuous follows from the continuity on every edge and the weak topology. □

Every graph contains trees as subgraphs, for example subgraphs consisting of a single vertex, so we can order the set of this subgraphs contained in a given graph by inclusion.

Theorem 2.2 (Massey, 1967)**.** *Let X be a graph; then any tree contained in X is contained in a maximal tree in X.*

Proof. If X is a finite graph then it is obvious. If X is not finite we consider a family of trees $\{T_\lambda\}_{\lambda \in \Lambda}$ linearly ordered by inclusion. Since

$$\bigcup_{\lambda \in \Lambda} T_\lambda$$

is also a tree we can conclude by applying Zorn's lemma. □

We can get an insight of the precise structure of maximal trees by the following condition.

Theorem 2.3 (Massey, 1967)**.** *Let X be a graph and let T be a a subgraph of X which is also a tree. Then, T is maximal if and only if contains all the vertices.*

Proof. Let T be a maximal tree that does not contain all the vertices. Since X is connected we find an edge path $e_1, ..., e_n$ from an initial vertex in T and its final one not in T. it is obvious that at least one of the edges is not in T, so we choose the minimum i for which this happens. Then $T \cup \bar{e}_i$ is a tree that strictly contains T, therefore T is not maximal.
For the other implication, let T be a tree containing all vertices, and let e be an edge connecting the vertices v_1, v_2. Since there is a unique path from v_2 to v_1 in T adjoining e would result in a closed loop, hence no edge can be adjoined and therefore T is maximal. □

2.3 Fundamental group of a graph

Let X be a connected graph, v_0 a vertex of X, and let T be a maximal tree of X, containing v_0. Let $\{e_\lambda : \lambda \in \Lambda\}$ be the set of edges of X not contained in T. Choose an orientation for each of the e_λ; let a_λ, b_λ denote the initial and terminal vertices of e_λ. To each e_λ, we associate a path class $\alpha_\lambda \in$ as follows: in T we have a unique reduced edge path A_λ from v_0 to a_λ and a unique edge path B_λ from b_λ to v_0. Then α_λ is the path class associated with the edge path $(A_\lambda, e_\lambda, B_\lambda)$. With the established notation we have the following theorem.

Theorem 2.4 (Massey, 1967). *The fundamental group $\pi_1(X, v_0)$ is the free group on the set of generators $\{\alpha_\lambda : \lambda \in \Lambda\}$.*

Proof. The first step will be the case where where Λ has only one element, i.e. There is only one edge not contained in T. We call this edge e_1. It is clear that every closed path in X must involve e_1. We now give e_1 an orientation. There must exists a reduced, closed edge path starting with e_1 which we indicate with $(e_1, ..., e_n)$. Of such paths we choose the shortest, and therefore there will be no repetition of edges or vertices, so it will be a simple path. We denote such path as

$$C = \bigcup_{i=1}^{n} \overline{e_i}.$$

C is a subgraph of X homeomorphic to a circle. Now we consider the complementary subgraph $X \setminus C$, each of its components $\overline{Y}_i$ is a subgraph of T, hence a tree. Each $\overline{Y}_i$ has exactly one vertex in common with C, if they had none X would be disconnected and if they they had more than one there we would find that Λ has more than one element. We can contract each $\overline{Y}_i$ to such vertex and therefore we obtain that C is a deformation retract of X hence the fundamental groups are isomorphic and it is clear that the generator of $\pi_1(C)$ is mapped by the inclusion in α_1.
The general case follows by applying Seifert Van Kampen theorem in the right way. Let x_λ be a point of $e - \lambda$ for each $\lambda \in \Lambda$, then the set $\{x_\lambda \mid \lambda \in \Lambda\}$ is closed and discrete since X has the weak topology. Let U be the complementary of this set, then U deforms retract onto T and therefore is contractible. For any index λ let $V_\lambda = U \cup x_\lambda$. We have that

$$V_\lambda \cap V_\mu = U$$

if $\lambda \neq \mu$. V_λ clearly deform retracts on $T \cup e_\lambda$ and therefore its fundamental group is the free group on the generator α_λ. We now apply Seifert Van Kampen to all V_λ and U which constitute an open covering of X and we obtain that

$$\pi_1(X) = \mathop{\ast}_{\lambda \in \Lambda} \pi_1(V_\lambda)$$

and therefore it is the free group on the generators α_λ. □

2.4 Covering space of a graph

Theorem 2.5 (Massey, 1967). *Let X be a connected graph with vertex set X^0 and let (Y, p) be a covering space of X. Then Y is a graph with vertex set $Y^0 = p^-1(X^0)$.*

Proof. We need to show the properties of the definition 2.1, so in order:

1. It is clear that Y^0 is a closed discrete subset of Y.

2. Each component of $p^{-1}(e)$ is a covering space of e and, since e is simply connected, each component is mapped homeomorphicly onto e. Also each component of $p^{-1}(e)$ is open in $p^{-1}(e)$ by local connectivity, and this gives us the second property.

3. If $\bar{e}$ is homeomorphic to $[0, 1,]$ than each component of $p^{-1}(\bar{e})$ is mapped homeomorphicly. If $\bar{e}$ is homeomorphic to S^1 then we apply the know results on the coverings of a circle.

4. The last property is a direct consequence of lemma 1.4.1.

$\square$

Chapter 3

CW-complexes

3.1 Basic definitions

After recalling the basic definitions and properties of free groups, we now need to introduce the other main mathematical object: the **CW-complexes** (where C and W refer respectivly to the closure-finiteness property and the weak topology that characterise them). They are constructed by the following procedure:

1. Start with a discrete space X^0, whose points are regarded as 0 cells.
2. Inductively, form the **n-skeleton** X^n from X^{n-1} by attaching n**-cells** e^n_α via maps $\varphi : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identification of $x \sim \varphi_\alpha(x)$ for every $x \in \partial D^n_\alpha$. Thus as a set$X^{n-1} \coprod_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
3. One can either stop the process at a finite stage setting $X = X^n$ or one can continue indefinitely, setting $X = \bigcup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (closed) iff $A \cap X^n$ is open (closed) for every n.

Now, is better to give some general definition and establish some notation that will result helpful in the following paragraphs.
Each cell e^n_α in a cell complex X has a **characteristic map** $\Phi : D^n_\alpha$ which extends the attaching map φ_α and is an homeomorphism of D^n_α onto e^n_α, namely we can take Φ_α to be the composition $D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha e^n_\alpha \to X^n \hookrightarrow X$ where the middle map is the quotient defining X^n.
A **subcomplex** of a cell complex X is a closed subspace of $A \subset X$ that is a union of cells of X. Since A is closed, the characteristic map of cell in A has image in A, in particular the attaching map of each cell in A has image in A, so A is a cell complex itself. A pair (X, A) consisting of a complex X and a subcomplex A is called a **CW pair**.
The unique cell e containing a given point $x \in X$ is called the **carrier** of x.

3.1.1 Collaring

Sometimes, when dealing with an adjunctions of n-cells (X, A), it is useful to be able to enlarge an open set of A to an open set of X, this can be done by **collaring**.
Let $\bar{\varphi} : \coprod B_\lambda \in X$ be a characteristic map and let φ be the corresponding attaching map. We assume that every ball B_λ is a copy of B^n so that for each point in $s \in B_\lambda$ can be multiplied for a scalar $t \in I$ (as a vector of $\mathbb{R}$) and the product ts is still a point of B_λ. The $\bar{\varphi}$ collar of set $V \subseteq A$ is defined to be

$$C_{\bar{\varphi}}(V) = V \cup \bar{\varphi}(\{ts : s \in \varphi^{-}1(V) \ \wedge \ \tfrac{1}{2} < t \leq 1\})$$

From this definition we can trivially deduce the consequences that we summarize in this lemma.

Lemma 3.1.1. *Let (X, A) be an adjunction of n-cells, let $\bar{\varphi}$ be a characteristic map for the adjunction and le V be a subset of A. Then*

1. $C_{\bar{\varphi}}(V) \cap A = V$.
2. $\bar{\varphi}^{-1}(C_{\bar{\varphi}}(V)) = \{ts : s \in \varphi^{-}1(V) \ \wedge \ \frac{1}{2} < t \leq 1\}$.
3. $C_{\bar{\varphi}}(V)$ *is open in* X *iff* V *is open in* A.
4. *If* V *is closed in* A*, then the closure of* $C_{\bar{\varphi}}(V)$ *is*
$$C_{\bar{\varphi}}(V) = V \cup \bar{\varphi}(\{ts : s \in \varphi^{-}1(V) \ \wedge \ \tfrac{1}{2} \leq t \leq 1\}).$$
5. *if* e *is an n-cell of* (X, A) *then*
$$e \cap \overline{C_{\bar{\varphi}}(V)} \neq \emptyset \Leftrightarrow e \cap C_{\bar{\varphi}}(V) \neq \emptyset \Leftrightarrow \bar{e} \cap V \neq \emptyset.$$
6. $C_{\bar{\varphi}}(V)$ *contains* V *as a strong deformation retract.*
7. *If* (V_λ) *is a locally finite family of subsets of* A *(respectively a family of pairwise disjoint subsets of* A*), then* $C_{\bar{\varphi}}(V_\lambda)$ *is a locally finite family of subsets of* X *(respectively a family of pairwise disjoint subsets of* X*).*

We now state a lemma that we are going to use later on to prove some topological properties of CW-complexes.

Proposition 3.1.1 (Fritsch and Piccinini, 1990)**.** *Let (X,A) be an adjunction of n-cells, V a closed subset of A and U be an open subset of X containing V. Then there is a characteristic map $\bar{\varphi}$ for the adjunction of A such that $\overline{C_{\bar{\varphi}}(V)}$ is still contained in U.*

Proof. Choose arbitrarily a characteristic map $\bar{f} : \coprod B_e \to X$ where the index e runs trough all the cells of the adjunction. The map $\bar{f}$ determines an attaching map $f : S_e \to A$ whose restriction to a sphere S_e will be denote by f_e. We now construct cellwise a 'transformation of coordinate' for which the attaching map is invariant. This transformation is needed only for cells such that

$$\bar{f}(\{ts : s \in B_e \ \wedge \ f(s) \in V \ \wedge \ \tfrac{1}{2} \leq t \leq 1\}) \not\subset U. \qquad (\star)$$

Let e be such cell. Then $V_e = f_e^{-1}(V)$ is non empty and $\bar{f}(B_e)$ is not fully contained in U. Hence the set $U_e = B_e \setminus f^{-1}(U)$ is a non-empty closed subset of B_e which does not meet the closed set V_e. The distance δ_e between U_e and B_e is well defined and greater than 0 as these two closed set are compact subsets of a metric space. From $(\star)$ we can conclude that $\delta_e \leq \frac{1}{2}$. Now we choose an homeomorphism $h_e : B_e \to B_e$ such that it coincide with the identity on the boundary and shrinks radially the ball $\{s \in B_e \mid |s| \leq 1 - \delta_e\}$ into $\{s \in B_e \mid |s| \leq \frac{1}{3}\}$. Now we define

$$\bar{\varphi}|_{B_e} = \bar{f}|_{B_e} \circ h_e$$

$\bar{\varphi}$ is characteristic map we needed. □

Since in general CW-complexes may not have a finite dimension, we need to generalize inductively the construction of collaring. In practice what we are going to do is to see the complex as iterated attachments of $(n+1)$-cells to its n-skeleton for each n creating a the infinite collar $C_\infty(V)$.

Definition 3.1. Let X be a CW-complex, let $\{X_n : n \in \mathbb{N}\}$ be the sequence its skeletons and let $\{\Phi_n : n \in \mathbb{N}\}$ be the sequence of characteristic maps of for the adjunctions (X^n, X^{n-1}). If $V_m \subset X^m$ we define its infinite collar as follows. For every $n \geq m$ we define

$$V_{n+1} = C_{\Phi^{n+1}}(V_n)$$

and then we take

$$C_\infty(V_m) = \bigcup_{n \geq m} V_n$$

The main properties of these objects are similar to the ones their finite dimensional counterpart and are reassumed in the following proposition.

Proposition 3.1.2 (Fritsch and Piccinini, 1990)**.** *Let X be a CW-complex, let $\{X_n : n \in \mathbb{N}\}$ be the sequence its skeletons, let $\{\Phi_n : n \in \mathbb{N}\}$ be the sequence of characteristic maps of for the adjunctions (X^n, X^{n-1}) and let V be an open or closed subset of an m-skeleton X^m. Then the infinite collar $C_\infty(V)$*

1. *Intersects X^m in V*
2. *Is open in X iff V is open in X^m*
3. *$C_{\bar{\varphi}}(V)$ is open in X iff V is open in A*
4. *Has as closure the union in X the union of the intermediate collars*
5. *Is the union space of the expanding sequence of intermediate collars*
6. *Contains V as a strong deformation retract*
7. *If (V_λ) is a locally finite family of subsets of X^m (respectively a family of pairwise disjoint subsets of X^m), then the family of their infinite collars is again locally finite (respectively a family of pairwise disjoint subsets of X)*

Proof. Properties 1, 2 and 6 are trivial. Properties 3, 4 follows from the normality of X. Property 5 follows from the fact that each V_n is a strong deformation retract of its succesor. □

3.2 Topological properties of CW-complexes

Proposition 3.2.1 (Whitehead, 1949)**.** *A CW-complex is a paracompact.*

Proof. Let X be a CW-complex and let $\{U_\lambda : \lambda \in \Lambda\}$ be an open covering. Our objective is to construct inductively a graded indexed set

$$\Gamma = \coprod_{n=0}^{\infty} \Gamma_n$$

and subsets $V_{\gamma,n}$ for every $\gamma \in \Gamma$ such that the family $\{V_\gamma : \gamma \in \Gamma\}$, where

$$V_\gamma = \bigcup_{n=0}^{\infty} V_{\gamma,n}$$

is an open, locally finite refinement of the covering $\{U_\lambda\}$. Moreover we will show that, for a fixed $m \in \mathbb{N}$ $\{V_{\gamma,m}\}$ is an open, locally finite refinement of the covering $U_\lambda \cap X^m$ of the m-skeleton X^m.
We say that an index γ has degree n if $\gamma\ in \Gamma$. As soon as we construct an index γ, we will also select an index $\lambda = \lambda(\gamma)$ and we will also construct $V_{\gamma,n}$ such that $V_{\gamma,n} \subseteq U_{\lambda(\gamma)}$. Furthermore the set $V_{\gamma,m}$ will be taken as a non empty subset of $X^m \setminus X^{m-1}$ for $m = \deg\gamma$, $V_{\gamma,m} = \emptyset$ for $m < \deg\gamma$ and $V_{\gamma,m} \supset V_{\gamma,\deg\gamma}$ for $m > \deg\gamma$.
We start the construction of Γ by taking $\Gamma_0 = X^0$; the for every $\gamma \in \Gamma$ we choose an index $\lambda(\gamma)$ such that $\gamma \in U_{\lambda(\gamma)} \cap X^0$ and for these γ we define

$$V_{\gamma,0} = \{\gamma\}.$$

Now we assume that we have constructed all the Γ_n up to Γ_{m-1} included, and all the corresponding $V_{\gamma,n}$ and the corresponding indexes $\lambda(\gamma)$. First we describe $V_{\gamma,m}$ for $\gamma \in \Gamma_1, \Gamma_2, \ldots, \Gamma m - 1$. We choose a characteristic map Φ for the adjunction (X^m, X^{m-1}) and for any γ we define

$$V_{\gamma,m} = C_\Phi(V_{\gamma,m-1}) \cap U_{\lambda(\gamma)}.$$

Because of the induction hypothesis the family $\{V_{\lambda,m-1}\}$ is open and locally finite, and so is $\{V_{\lambda,m}\}$ (Proposition 3.1.2 (3),(7)). Note that $V_m = \bigcup_\gamma V_{\gamma,m}$ for $\gamma \in \Gamma_1, \ldots \Gamma_{m-1}$ is an open set in X^m which contains X^{m-1}. With the right choice of coordinates (proposition 3.1.1) we can arrange so that also the closure of the collar of X^{m-1} is contained in V_m.
Let e be an m-cell and let $\Phi_e : B^m \to X$ be its characteristic map. The family $\{U'_\lambda : \Phi^{-1}(U_\lambda)\}$ covers the m-ball $B' = \{s \in B^m : |s| \leq \frac{3}{4}\}$; since B' is compact finitely many elements of this family, $U'_{\lambda_1}, U'_{\lambda_2}, \ldots, U'_{\lambda_k}$, suffice to cover it. So we define

$$\Gamma_e = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$$

and for every $\gamma \in \Gamma_e$ we define

$$\lambda(\gamma) = \gamma$$

and

$$V_{\gamma,m} = \Phi(\mathring{B}' \cap U'_\gamma)$$

Finally we define

$$\Gamma_m = \coprod_e \Gamma_e$$

where e runs over the set of all m-cells of X.
This completes our construction, now we need to verify that it behaves as it is supposed to.
First we prove that the family $\{V_{\gamma,m} : \deg\gamma \leq m$ is locally finite. In order to do this, we choose a point $x \in X$ whose carrier is the m-cell e. We know that x has a neighborhood U that meets only finitely many sets $V_{\gamma,m}$, with $\deg\gamma < m$ and we also know that e meets only finitely many sets $V_{\gamma,m}$, with $\deg\gamma = m$, the ones with index $\gamma \in \Gamma_e$; then $U \cap e$ is a neighborhood of x which meets only finitely many $V_{\gamma,m}$ with $\deg\gamma \leq m$.
Suppose now that $x \in X^{m-1}$. We need to show that any open neighborhood U of x in X^{m-1} meeting only finitely many $V_{\gamma,m-1}$ with $deg\gamma < m$ can be enlarged to a neighborhood U' of x in X^m that meets only finitely many $V_{\gamma,m}$ with $\deg\gamma \leq m$. We do this by analysing the various collaring processes.
Let Φ be one of the characteristic maps used in the construction of V_m; then for 3.1.1

$C_\Phi(U)$ intersects only finitely many $V_{\gamma,m}$ with $\deg \gamma < m$. Notice that $\tilde{B} = \bigcup_e \Phi_e(\mathring{B}')$ is a closed subset of X^m which does not meet X^{m-1}. Therefore, $U' = C_\Phi(U)\ \tilde{B}$ is a neighborhood of x in X^m having the desired property.
We also notice that U' intersects $V_{\gamma,m}$ iff U intersects $V_{\gamma,m-1}$, this will be useful later on.
The last step of the proof follows in this way. Fore every index γ

$$V_\gamma \cap X^m = V_{\gamma,m}$$

and therefore V_γ is open. Moreover the inclusion $V_\gamma \subset U_{\lambda(\gamma)}$ shows that the family $\{V_\gamma\}$ is a refinement of $\{U_\lambda\}$. For an arbitrary point x, take a non-negative integer m such that $x \in X^m$ and choose a neighborhood U_m that intersects only finitely many V_γ. As proved before U_m can be enlarged to neighborhoods of x U_n in X^n, all of which intersect a finite number of V_γ, which already intersect U_m. Thus

$$\bigcup_{n=m}^{\infty} U_n$$

is a neighborhood of x in X that intersects only finitely many V_γ; this proves that the family $\{V_\gamma\}$ is locally finite. □

Proposition 3.2.2 (Whitehead, 1949)**.** *A CW-complex is locally contractible.*

Proof. Let X be a CW-complex, $x_0 \in X$ and U an open neighborhood of x_0 in X. Now let the m-cell e be the carrier of x_0 and let $\bar{c} : B^m \to X$ a characteristic map for e. We notice that the point $y = \bar{c}^{-1}(x_0)$ lies in the interior of B^m and that $\bar{c}^{-1}(U)$ is an open neighborhood of y in B^m. Now we choose a smaller m-ball $B \subseteq \mathring{B}^m$ such that $y \in B \subseteq \bar{c}^{-1}(U)$. Since the interior $\mathring{B} = B \setminus \partial B$ is a contractible neighborhood of y in $\mathring{B}^m$ the set $V_m = \bar{c}(\mathring{B})$ is a contractible open neighborhood of x_0 in X^m, whose closure $\overline{V_m}$ is contained in U. Selecting inductively the right coordinates (as in proposition 3.1.1) we obtain an infinite collar $V = C_\infty(V_m)$ that is still contained in U. By the properties (2) and (5) of proposition 3.1.2 we conclude that V is open in X and that V contracts to V_m hence to X_0. □

Proposition 3.2.3 (Whitehead, 1949)**.** *Any covering space $\tilde{K}$ of a CW-complex K is a CW-complex.*

Proof. Since $\tilde{K}$ is locally connected, each of its component will be both open and close, and is a covering of a component of K. Since $\tilde{K}$ will be a CW-complex if and only if all of its connected components are CW-complexes we can restrict our proof to connected complexes. We also assume $\tilde{K}$ to be a regular covering complex. We say that the open subset $U \subseteq K$ is an elementary neighborhood if p maps each component of $p^{-1}(U)$ topologically onto U. We call an elementary neighborhood basic if its closure is contained in an elementary neighborhood. We call $\tilde{U} \subseteq \tilde{K}$ a basic neighborhood in $\tilde{K}$ if it is a connected component of $p^{-1}(U)$ for some U basic neighborhood in K. Let G be the group of covering transformations and let $\tilde{U} \subseteq \tilde{K}$ be a basic neighborhood, then

$$p^{-1}(p(\tilde{U})) = \{T(\tilde{U}) \mid T \in G\}.$$

From the definition of $\tilde{K}$ and the normality of K it follows that basic neighborhoods are a basis for open sets both in $\tilde{K}$ and in K.
Now we consider a basic neighborhood U and a elemntary neighborhood V containing its closure, the connected components of $p^{-1}(V)$ are disjoint open sets in $\tilde{K}$.

Now let $Q \subseteq p^{-1}(U)$ be a set of points, such that there is at most one of them in each component of $p^{-1}(U)$. Q is closed and discrete because if there was a limit point $\overline{q}$ we would have $p(\overline{q}) \in V$ and therefore $\overline{q}$ would be in ove of the components of $p^{-1}(V)$ but there is at most one point of Q in each of them, absurd.
Now we show that for $\tilde{U} \subseteq \tilde{K}$ basic neighborhood, U^* its closure and $C \subseteq \tilde{K}$ compact the set

$$\{T \in G \mid U^* \cap T(C) \neq \emptyset\}$$

is finite. If $U^* \cap T(C) \neq \emptyset$ then $C \cap T^{-1}(U^*) \neq \emptyset$ and we choose $q_T \in C \cap T^{-1}(U^*)$. Since $T_1(U^*) \cap T_1(U^*) = \emptyset$ for $T_1 \neq T_2$ it follows from the previous paragraph that the set $\{q_T\}$ is discrete and closed and since C is compact it is also finite.
We now prove that $\tilde{K}$ has the weak topology. Let $\tilde{X} \subseteq \tilde{K}$ such that $\tilde{X} \cap e^*$ is closed for every cell $\tilde{e} \in \tilde{K}$. In order to prove that $\tilde{X}$ is closed it suffices to prove that $\tilde{X} \cap U^*$ is closed for any basic neighborhood $\tilde{U}$, since this would imply the openness of

$$\tilde{U} \setminus \tilde{X} = \tilde{U} \setminus (\tilde{X} \cap U^*)$$

and as a consequence the openness of $\tilde{K} \setminus \tilde{X}$. To simplify the notation we assume $\tilde{X} \subseteq U^*$. Let now $X = p(\tilde{X})$ and let e be a given cell in K. We have

$$X \cap e = p(\tilde{X} \cap p^{-1}(e)).$$

Let now $\tilde{e}$ be a cell in $\tilde{K}$ that covers e, then

$$p^{-1}(\overline{e}) = \bigcup_{T \in G} T(e^*).$$

Since e^* is compact it follows that only a finite number $\{T_i(e^*) \cap U^\star\}_{1 \leq i \leq k}$ are non empty. We call $P_i = \tilde{X} \cap T_i(e^*)$. We will then have

$$X \cap \overline{e} = p(\tilde{X} \cap p^{-1}(e)) = X \cap p\left(\bigcup_{i=1}^{k} P_i\right).$$

Now we observe that P_i is closed for the hypothesis on $\tilde{X}$ and therefore is compact thanks to the compactness of $T_i(e^*)$. It follows then that $\bigcup_{i=1}^{k} P_i$ is compact an so it is $X \cap \overline{e}$. Since e is arbitrary it follows that X is closed and therefore $p^{-1}(X)$ is closed too. Since $U^* \cap T(U^*) = \emptyset$ for $T \neq 1$ wi will have that:

$$\tilde{X} = U^* \cap \left(\bigcup_{T \in G} T(\tilde{X})\right) = U^* \cap p^{-1}(X)$$

hence $\tilde{X}$ is closed and $\tilde{K}$ has the weak topology.
Since K^0 is discrete it follows that $\tilde{K}_0 = p^{-1}(K_0)$ is a discrete set of points, and so it has the weak topology. If $n > 0$ than K^n is connected and $\tilde{K}^n$ is a covering complex. The injection homomorphism $\pi_1(K^n) \hookrightarrow pi_1(K)$ is onto whence $\tilde{K}^n$ is connected. Obviously $T(K^n) = K^n$ for any $T \in G$ and it follows that $\tilde{K}^n$ is a regular covering complex of K^n therefore $\tilde{K}^n$ has the weak topology and from this it follows that $\tilde{K}$ is a CW-complex.
Now assume that $\tilde{K}$ is not regular, and let $\widehat{K}$ be a universal covering complex of $\tilde{K}$ and therefore of K, let $p : \widehat{K} \to \tilde{K}$ be a covering map. Since p is open, it induces an identification topology on $\tilde{K}$. It easy to show that $\tilde{K}$ is closure finite and that from this follows that $\tilde{K}$ is a $CW - complex$. □

Proposition 3.2.4 (Whitehead, 1949). *Any compact subset of a CW-complex meets only a finite number of cells and it is contained in a finite subcomplex.*

Proof. Let X be a CW-complex and let K be a compact subset of X. Let E denote the set of all cells of X which intersects K. We choose a point $x_e \in K$ for each cell $e \in E$ and we denote the set of this points as Z. Now we show inductively that Z intersects any skeleton in a finite number of points, thus Z is a discrete closed subset of X and also of K. Then we recall that a discrete closed subset of a compact space is finite, and so Z is finite.
Clearly $Z \cap X^0 = K \cap X^0$ is a discrete and closed subset of K and so it is finite. Now we assume that $Z \cap X^{n-1}$ is finite. Since Z meets any n-cell just in a finite number of points, and X^n is determined by the family of X^{n-1} and all closed n-cells, $Z \cap X^n$ is discrete, hence it is finite.
We have now shown that E is finite, we just need to prove that for any cell e the subcomplex $X(e)$ is finite since it is clear that $X(E) \subseteq \bigcup_{e \in E} X(e)$ and the latter is finite if every X_e is.
We prove this part by induction on the dimension of the cells. If e is a 0-cell then $X(e) = e$. We now assume that $X(e)$ is finite for every cell of every dimension strictly less than n. Let e be an n-cell then $\bar{e} \setminus e$ is compact and contained in X^{n-1} and from the first part of the proof we deduce that it is contained in the union of a finite number of cells $e_1, e_2, \ldots, e_n$. From the induction hypothesis we know that each X_{e_i} is finite and moreover

$$\tilde{X} = e \cup \left(\bigcup_{i=1}^{n} X(e_i) \right)$$

is a finte subcomplex of X and $X(e) \subseteq \tilde{X}$. □

3.3 Operations on spaces

We will now describe some useful constructions that come in hand when dealing with cell complexes:

- Products.
  If X and Y are cell complexes then the space $X \times Y$ has the structure of a CW-complex with cells $e^m_\alpha \times e^n_\beta$, where e^m_α ranges over the cells of X and e^n_α ranges over Y. We shall notice that the topology of this space is slightly different from the product topology in general, we have a few more open cells. The two topologies are the same if X or Y has finitely many cells or if both X and Y have countably many cells.

- Quotients.
  Let (X, A) be a CW pair consisting of a cell complex X and a subcomplex A. Then, the quotient space Q/A has a natural cell structure. X/A consists of the cells $X \setminus A$ plus a new 0-cell that is the image of A. For a cell e^n_α attached by $\varphi_\alpha : S^{n-1} \to X^{n-1}$ we have that the corresponding cell in X/A is the composition $S^{n-1} \to X^{n-1} \to X/A$.

- Suspension.
  For a space X, the **suspension** SX is the quotient of $X \times I$ obtained by collapsing to one point $X \times \{0\}$ and $X \times \{1\}$.

- Join.
  Given two spaces X and Y we can define the **join** $X * Y$ as the quotient of the

space $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$.
The first useful case of this is the cone CX that arise from $X \times \{v\} \times I$ where v becomes the vertex of the cone. The second is the join of n points which in general is a convex polyhedron of dimension $n-1$ called a **simplex**.

- Wedge Sum.
  Given two spaces X, Y and one point in each of them, respectively x_0, y_0 we define the **wedge sum** $X \vee Y$ as the quotient of the disjoint union $X \amalg Y$ under the identification of x_0 and y_0.

- Smash Product.
  The smash product is the quotient of the product where we collapse thw wedge sum: $X \wedge Y = X \times Y / X \amalg Y$.

3.4 Application of SVK to CW-Complexes

Suppose we attach a collection of 2-cells e^2_α, to the path-connected space X via maps $\varphi_\alpha : S^1 \to X$. If s_0 is a base point of S^1 then φ_α determines a loop at $\varphi_\alpha(s_0)$ that we shall call φ_α. Since $\varphi_\alpha(s_0)$ may change for different α we choose a base point $x_0 \in X$ and a path γ_α from x_0 to $\varphi_\alpha(s_0)$ for every α. Then $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$ is a loop in x_0 for every α. Let N be the normal subgroup of $\pi_1(X)$ generated by $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$.

Theorem 3.1 (Massey, 1967)**.** *Let Y be a CW-complex obtained from X by attaching n-cells. Then*

1. *If Y is obtained from X by attaching 2-cells as described above then, the inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is N. Thus $\pi_1(Y) \approx \pi_1(X)/N$.*

2. *If Y is obtained from X by attaching n-cells with $n > 2$ then, the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X, x_0) \to \pi_1(Y, x_0)$.*

3. *For a CW-complex X the inclusion of the 2-skeleton $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X, x_0) \to \pi_1(Y, x_0)$.*

Proof.

1. Let us expand Y to a slightly larger space Z that deformation retracts onto Y and is more convenient for applying SVK. The space Z is obtained from Y by attaching rectangular strips $S_\alpha = I \times I$, with the lower edge $I \times \{0\}$ attached along γ_α, the right edge $\{1\} \times I$ attached to an arc of e^2_α and the left edges $\{0\} \times I$ identified together. In each cell e_α we choose a point y_α not in the arc along which S_α is attached. Let $A = Z \setminus \bigcup_\alpha \{y_\alpha\}$ and let $B = Z \setminus X$. Then A retracts onto X and B is contractible. Since $\pi_1(B) = 0$, SVK applied to the cover $\{A, B\}$ says that $\pi_1(Z)$ is isomorphic to the quotient of $\pi_1(A)$ by the normal subgroup generated by the image of $\pi_1(A \cap B) \to \pi_1(A)$.
   So it remains to prove that $\pi_1(A \cap B)$ is generated by $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$. This can be shown by another application of SVK, this time to the cover of $A \cap B$ by the open sets $A_\alpha = A \cap B \setminus \bigcup_{\beta \neq \alpha} e^2_\beta$. Since A_α deformation retracts onto a circle in e^2_α, we have that $\pi_1(A_\alpha) \approx \mathbb{Z}$ generated by a loop homotopic to $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$, and the result follows.

2. The proof follows the same steps of the previous point, the only difference is that A_α deformation retracts onto S^{n-1} and for $n > 2$ we have that $\pi_1(A_\alpha) = 0$ hence $\pi_1(A \cap B) = 0$.

3. If X is finite dimensional we simply proceed by induction using the previous point. If X is not finite dimensional let $f : I \to X$ be a loop at basepoint $x_0 \in X^2$. $f(I)$ is compact, hence, for proposition 3.2.4 lies in some X^n. This, because of point (2), means that f is homotopically equivalent to a loop in X^2 hence we have the surjection of $\pi_1(X^2)$ onto $\pi_1(X)$.
To prove it is also injective let take a loop f in X^2 which is nullhomotopic in X via homotopy $F : I \times I \to X$. This has compact image in X hence is contained in some X^n, with $n > 2$ otherwise the result follows trivially. Since from point (2) we know that $\pi_1(X^2)$ is isomorphic to $\pi_1(X^n)$ we conclude that f is nullhomotopic in X^2.

□

Corollary 3.1.1. *For every group G there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \approx G$.*

Proof. Choose a presentation of $G = \langle g_\alpha \mid r_\beta \rangle$, this exists since every group is a quotient of a free group, so the g_α's can be taken to be the generators of this free group with the r_β being the generators of the kernel of the map from the free group to G. Now construct X_G from $\bigvee_\alpha S^1_\alpha$ by attaching 2-cells e_β by the loops specified by the words r_β.

□

Lemma 3.4.1. *Let X be a connected CW-complex which is the union a the collection of subcomplexes $\{A_\lambda : \lambda \in \Lambda\}$. Suppose that exists a non-empty tree T which is a subcomplex of the 1-skeleton X^1 and that for every $\lambda \neq \mu$ we have $A_\lambda \cap A_\mu = T$. Then, for any vertex $v \in T$ we have that $\pi_1(X, v)$ is the free product of the groups $\pi_1(A_\lambda, v)$ with respect to the homomorphism $\varphi_\lambda : \pi_1(A_\lambda, v) \to \pi_1(X, v)$ induced by the inclusion $A_\lambda \hookrightarrow X$.*

Proof. If X, and consequently every A_λ, are of dimension 1, this follows from theorem 2.4 applied to a maximal tree in X containing T and to each A_λ.
Now we consider the case where X is 2-dimensional. We must prove that given any group H and any collection of homomorphism $\psi_\lambda : \pi_1(A_\lambda) \to H$, there exists a unique homomorphism $\sigma : \pi_1(X) \to H$ such that $\sigma\varphi_\lambda = \psi_\lambda$ for all λ. We denote with $j_\lambda : \pi_1(A^1_\lambda) \to \pi_1(A_\lambda)$ and $j : \pi_1(X^1) \to \pi_1(X)$ the homomorphisms induced by the respective inclusion maps. Then for each λ we have the following commutative diagram:

$$\begin{tikzcd}
\pi_1\left(A^1_\lambda\right) \arrow[r, "j_\lambda"] \arrow[d, "\varphi^1_\lambda"] & \pi_1\left(A^1_\lambda\right) \arrow[r, "\psi_\lambda"] \arrow[d, "\varphi_\lambda"] & H \\
\pi_1\left(X^1\right) \arrow[r, "j"] & \pi_1(X) &
\end{tikzcd}$$

We know from what we said before that there exists a unique homomorphism $\sigma' : \pi_1(X^1) \to H$ such that

$$\psi_\lambda j_\lambda = \sigma' \varphi^1_\lambda \qquad (\star)$$

for all $\lambda \in \Lambda$.
From theorem 3.1 that j (respectively, j_λ) is an epimorphism and the generators of its kernel are in one to one correspondence with the 2-cells of X (respectively, A_λ). Let e^2_i be any 2-cell of X and let γ_i the corresponding generator of the kernel of j.

We choose an index λ such that $e_i^2 \subseteq A_\lambda$ then γ_i is also a generator of the kernel of j_λ. From the equation $(\star)$ and the fact that φ_λ^1 is a monomorphism it follows that $\sigma'(\gamma_i) = 0$. Because this is true for every 2-cell e_i^2, it follows that exists a unique homomorphism $\sigma : \pi_1(X) \to H$ such that $\sigma' = \sigma j$. it is clear that σ has the required properties. In a more general case we can apply what we have just shown to the 2-skeletons of X and A_λ thanks to theorem 3.1. $\square$

Chapter 4

Kurosh subgroup theorem and Grushko's theorem

4.1 Kurosh subgroup theorem

4.1.1 The theorem

Theorem 4.1 (Kurosch, 1934)**.** *Let H be a subgroup of the free product $G = *_\lambda G_\lambda$. Then H is a free product itself*

$$H = F * (\mathop{\Huge *}_{\nu} H_\nu)$$

Where F is a free group and each H_ν is conjugate in G to a subgroup of one of the G_λ.

Proof. For each $\lambda \in \Lambda$ we choose a 2-dimensional CW-complex X_λ be with a single vertex v_λ such that its fundamental group is:

$$\pi_1(X_\lambda, v_\lambda) = G_\lambda.$$

Then we choose a point v_0 such that for all $\lambda \in \Lambda$ $v_0 \notin X_\lambda$ and we join v_0 to each v_λ via a path e_λ. If we call X the union of of all X_λ, all e_λ and v_0, and we give this set a weak topology we obtain a connected 2-dimensional CW-complex and

$$\pi_1(X, v_0) = G$$

as a consequence of 3.4.1. Now, we denote with $(\tilde{X}, p)$ a covering space of (X, v_0) associated to the subgroup H which, thanks to 3.2.3, we know to be a CW-complex itself. We can now choose a vertex $\tilde{v}_0 \in p^{-1}(v_0)$ such that

$$\pi_1(\tilde{X}, \tilde{v}_0) = H.$$

For each $\lambda \in \Lambda$ we write the set of components of $p^-1(X_\lambda)$ in this way:

$$\left\{ \tilde{X}_{\lambda\mu} \mid \mu \in M_\lambda \right\}.$$

For all $\lambda \in \Lambda$ we will have that $(\tilde{X}_{\lambda\mu}, p|_{\tilde{X}_{\lambda\mu}})$ is a covering space of (X_λ) hence it is also a $2-$dimensional CW-complex. We now choose $T_{\lambda\mu}$ maximal tree in the 1-skeleton of $\tilde{X}_{\lambda\mu}$.
The union of this trees, together with $p^{-1}(e_\lambda)$ for all $\lambda \in \Lambda$ gives us the connected graph Y which is contained in the 1-skeleton of X. Now let T be a maximal tree in Y containing each $T_{\lambda\mu}$.
Now, we are almost ready to apply the lemma 3.4.1 to conclude the proof. In order to do so, lets consider the covering of X given by the subcomplexes Y, $\tilde{X}_{\lambda\mu} \cup T$ for each pair λ, μ. Each of this subcomplexes is connected, contains v_0 and the intersection of

any two of them is T. From lemma 3.4.1 we can conclude that

$$\pi_1(\tilde{X}, v_0) = \pi_1(Y, v_0) * \left(\mathop{\ast}_{\lambda,\mu} \pi_1(\tilde{X}_{\lambda\mu} \cup T, v_0) \right).$$

Moreover we know that $\pi_1(Y, v_0)$ is a free group and

$$\pi_1(\tilde{X}_{\lambda\mu} \cup T, v_0) = \pi_1(\tilde{X}_{\lambda\mu}, v_0)$$

since Y is a graph and $\tilde{X}_{\lambda\mu}$ is a deformation retract of $\tilde{X}_{\lambda\mu} \cup T$. We can conclude that under the monomorphism

$$p_* : \pi_1(\tilde{X}, v_0) \to \pi_1(X, v_0)$$

$\pi_1(\tilde{X}_{\lambda\mu} \cup T, v_0)$ maps onto a conjugate of a subgroup $\pi_1(X_\lambda \cup e_\lambda, v_0) = G_\lambda$ which conjugate it is depends on the choice of the maximal tree T. □

As it is stated, the theorem does not give to us the necessary insight in the structure of the subgroups H_ν nor on to which extent they are uniquely determined by H. In order to prove a more detailed version of the theorem, the notion of double cosets come in handy. For any $g \in G$ the **double coset** of the subgroups H and G_λ is the set:

$$HgG_\lambda = \{hgx \mid h \in H\,,\, x \in G_\lambda\}$$

any two of this double cosets are either disjoint or identical.

Theorem 4.2 (Kurosch, 1934)**.** *Assume that the hypotheses of 4.1. Then, for each index $\lambda \in \Lambda$, there exists a set of rapresentatives $\{\beta_{\lambda\mu} \mid \mu \in M_\lambda\}$ one from each double coset of H and G_λ such that*

$$H = F * \left[\mathop{\ast}_{\lambda\in\Lambda} \mathop{\ast}_{\mu\in M_\lambda} \left(H \cap \beta_{\lambda\mu} G_\lambda \beta_{\lambda\mu}^{-1} \right) \right]$$

where F is a free group.

Proof. While keeping the notation established in the previous proof, we also introduce, for each λ, $Y_\lambda = X_\lambda \cup \bar{e}_\lambda$. We observe that any Y_λ is a subcomplex of X containing v_0.
We denote as

$$\{\tilde{Y}_{\lambda\mu} \mid \mu \in M_\lambda\}$$

the set of connected components of $p^{-1}(Y_\lambda)$ indexed so that $\tilde{X}_{\lambda\mu} \subseteq \tilde{Y}_{\lambda\mu}$ for all μ.
We can think of $\tilde{Y}_{\lambda\mu}$ as obtained from $\tilde{X}_{\lambda\mu}$ by the adjunction "tails" in the same number as the covering sheets of $(\tilde{X}_{\lambda\mu}, p|_{\tilde{X}_{\lambda\mu}})$ onto X_λ.
For each $\tilde{Y}_{\lambda\mu}$ we now choose a vertex $\bar{v}_{\lambda\mu}$ such that $p(\bar{v}_{\lambda\mu}) = v_0$. The proof of the theorem follows from the commutativity, for every pair of indexes λ, μ, of the following

diagram

$$\begin{tikzcd}
 & \pi_1(\tilde{Y}_{\lambda\mu} \cup T, \overline{v}_{\lambda\mu}) \arrow[r, "u_{\lambda\mu}"] \arrow[d, "\varphi_{\lambda\mu}"] & \pi_1(\tilde{Y}_{\lambda\mu} \cup T, \overline{v}_0) \arrow[d, "\varphi_{\lambda\mu}"] \\
\pi(\tilde{Y}_{\lambda\mu}, \overline{v}_{\lambda\mu}) \arrow[ur, bend left, "j_{\lambda\mu}"] \arrow[r, "i_{\lambda\mu}"] \arrow[d, "\varphi_{\lambda\mu_*}"] & \pi_1(\tilde{X}_{\lambda\mu}, \overline{v}_{\lambda\mu}) \arrow[r, "u_{\lambda\mu}"] \arrow[d, "p_*"] & \pi_1(\tilde{X}, \overline{v}_0) \arrow[d, "p_*"] \\
\pi_1(Y_\lambda, v_0) \arrow[r, "i_\lambda"] & \pi_1(X, v_0) \arrow[r, "w_{\lambda\mu}"] & \pi_1(X, v_0)
\end{tikzcd}$$

where $p_{\lambda\mu} = p|_{Y_{\lambda mu}}$ and $i_\lambda, i_{\lambda\mu}, j_{\lambda\mu}, \varphi_{\lambda\mu}$ are all induced by inclusions maps.
For each vertex $\overline{v}_{\lambda\mu}$ we call $\alpha_{\lambda\mu}$ the path class in T from $\overline{v}_0$ to $\overline{v}_{\lambda\mu}$ and we define

$$\beta_{\lambda\mu} = p_*(\alpha_{\lambda\mu}) \in \pi_1(X, v_0).$$

The isomorphisms $u_{\lambda\mu}, \beta_{\lambda\mu}$ are defined as:

$$u_{\lambda\mu}(x) = \alpha_{\lambda\mu} x {\alpha_{\lambda\mu}}^{-1}$$
$$w_{\lambda\mu}(y) = \beta_{\lambda\mu} x {\beta_{\lambda\mu}}^{-1}.$$

It is clear that $j_{\lambda\mu}$ is an isomorphism, $w_{\lambda\mu}$ is an inner automorphism and all the homorphisms in the diagram are monomorphisms.
By construction we have

$$\pi_1(X, v_0) = G$$
$$p_*\pi_1(\tilde{X}, v_0) = H$$
$$i_\lambda \pi_1(Y_\lambda, v_0) = G_\lambda.$$

If we now call $p_*(\varphi_{\lambda\mu}(\pi_1(\tilde{Y}_{\lambda\mu} \cup T, v_0))) = H_{\lambda\mu}$ we can apply theorem 4.1 and to obtain that H is the free product of F and all the groups $H_{\lambda\mu}$.
We now apply the theorem 1.6 to the diagram and we obtain:

$$i_\lambda(p_{\lambda\mu_*}(\pi_1(\tilde{Y}_{\lambda\mu}, \overline{v}_{\lambda\mu}))) = [p_*(pi_1(\tilde{X}, \overline{v}_{\lambda\mu}))] \cap G_\lambda.$$

We procede to apply the isomorphisms $u_{\lambda\mu}, w_{\lambda\mu}$ to this equality and we use the commutativity of the graph to get

$$H_{\lambda\mu} = H \cap \left(\beta_{\lambda\mu} G_\lambda {\beta_{\lambda\mu}}^{-1} \right).$$

The last thing we need to prove is that $\{\beta_{\lambda\mu} \mid \mu \in M_\lambda\}$ is a set of representatives of HxG_λ.
In order to do this, we first consider the action of $G = \pi_1(X, v_0)$ on the set $p^{-1}(v_0)$.
We have the subgroup H is the isotropy subgroup corresponding to $\overline{v}_0$ and so we can identify $p^{-1}(v_0)$ with the cosets Hx.
Then we consider the action of G_λ on $p^{-1}(v_0)$ or equivalently on on the coset space G/H. For any $\mu \in M_\lambda$, G_λ permutes the points of $Y_{\lambda\mu} \cap p^{-1}(v_0)$ transitively, therefore the set of components $\{\tilde{Y}_{\lambda\mu} \mid \mu \in M_\lambda\}$ is in one to one correspondence with the set of double cosets HxG_λ and any choice of paths $\beta_{\lambda\mu}$ such that $v_0 \cdot \beta_{\lambda\mu} = \overline{v}_{\lambda\mu} \in \tilde{Y}_{\lambda\mu}$ is a choice oof representatives for these double cosets. $\square$

4.2 Grushko's theorem

Theorem 4.3 (Gruschko, 1940)**.** *Let $\varphi : F \to \ast_\lambda G_\lambda$ be an epimorphism of the free group F onto an arbitrary free product of groups. Then, there exists a decomposition of F as a free product, $\ast_\lambda F_\lambda$, such that $\varphi(F_\lambda) \subseteq G_\lambda$ for all $\lambda \in \Lambda$.*

Before giving a proof of the theorem, which is from Stallings, 1965, we want to first sketch out what our strategy will be and then introduce some technical tool that we are going to need.

The first thing we have to define is a topological equivalent of φ. We will start by introducing the CW-complexes B_λ with a single vertex v_λ and such that $\pi_(B_\lambda, v_\lambda) = G_\lambda$. These complexes can be assumed to be pairwise disjoint without loss of generality. We define Y as the quotient space of $\bigcup_\lambda B_\lambda$ under the identification of all the vertices v_λ to a single vertex v. Y with the weak topology is itself a CW-complex with a single vertex and

$$\pi_1(Y, v) = \mathop{\ast}_{\lambda} G_\lambda.$$

We now le $\{y_\tau\}$ be a basis for F. For each index τ we can represent φy_τ as a unique reduced word of $\ast\, G_\lambda$

$$\varphi y_\tau = a_1, ..., a_n.$$

We now choose an associated circle S_τ for every index τ and we divide it into n segments $W_1, ..., W_n$ creating a graph with n edges and n vertices.

We can define a map $f : S_\tau \to Y$ so that $f|_{W_i}$ is a closed path in some B_λ representing a_i. After we do this for every τ we can identify all the starting points of all the S_τ and by doing so we obtain a graph X with a single vertex which is finite, connected and for which we have $\pi_1(X) = F$. We have that the maps $S_\tau \to Y$ give rise to a continuous maps $f : X \to Y$ such that the induced morphism $f_* : \pi_1(X) \to \pi_1(Y)$ is equivalent to φ. Now we can associate every edge W_i of every S_τ to a unique index λ such that $f(W_i) \subseteq B_\lambda$. We than denote the subgraph associated to each index λ as A_λ. Then f maps A_λ into B_λ and $\bigcup_\lambda A_\lambda$ consists of all the vertices of X.

Now that we laid the groundwork we describe what the core strategy of the proof is. The general idea is to create a connected 2-dimesional CW-complex X' such that deforms retact onto X and a map $f' : X' \to Y$ that extends f to X', so that $\pi_1(X') \approx \pi_1(X) = F$ and f' is still equivalent to φ. Moreover we will construct X' in such a way that it is the union of connected subcomplexes A'_λ such that $A_\lambda \subseteq A'_\lambda$ and they fit the hypothesis of 3.4.1. In order to ensure that this is always possible we will need some technical tools. First of all we need a definition, going forward we keep the notation already established.

Definition 4.1 (Stallings system)**.** A **Stallings system** $(K, \{C_\lambda\}_{\lambda\in\Lambda}, f)$ is a triplet consisting of a finite 2-dimensional CW-complex K, a family of subcoplexes $\{C_\lambda\}_{\lambda\in\Lambda}$ and a continuous map $f : K \to Y$ such that:

1. The complex K is the union of C_λ

$$K = \bigcup_{\lambda\in\Lambda} C_\lambda.$$

2. For any $\mu, \nu \in \Lambda$ we have

$$C_\mu \cap C_\nu = \bigcap_{\lambda\in\Lambda} C_\lambda.$$

3. For any index λ, $f(C_\lambda) \subseteq B_\lambda$.

4. f maps the n-skeleton of K into the n-skeleton of Y.

Now we give some additional conventions and nomenclature regarding Stallings systems.

- For any Stallings system the base point is always in $\bigcup_{\lambda\in\Lambda} C_\lambda$.
- An intuitive way to think about different λ as different colors, so we will say that a path in K is **monochromatic** if it lies entirely in C_λ.
- A path in the 1-skeleton of K is called a **loop** if both of its end-points coincide in a vertex.
- A path in the 1-skeleton of K is called a **tie** if its end-points are vertices in different component of $\bigcup_{\lambda\in\Lambda} C_\lambda$.
- We say that a tie $g : I \to K$ is a **binding tie** if there exists a λ such that $g(I) \subseteq C_\lambda$ and $f \circ g(I)$ is equivalent to the constant path in B_λ. Another equivalent approach to define a binding tie is to denote with η the equivalence class of g in C_λ and with f_λ the restriction $f|_{C_\lambda}$ and to require $f_*(\eta) = 1 \in \pi_1(B_\lambda)$. it is also important to observe that a binding tie is always monochromatic.

Now we need to introduce a special construction that for stallings systems that will be key in our proof. Let $(K, \{C_\lambda\}_{\lambda\in\Lambda}, f)$ be a Stallings system and $g : I \to C_\mu$ a binding tie of color μ. Now we take a 2-dimensional closed disk D and we split its boundary into two two segments c_1, c_2 which intersect only at their endpoints. By identifying c_1 with the unit interval we have that g act as an adjunction map of D. By adjoining D via g we get a CW-complex K' which has one more 2-cell D and one additional edge c_2 than K and that deforms retract onto it. Let's now denote with C'_μ the union of C_μ and D and for any $\lambda \neq \mu$ we denote with C'_λ the union of C_λ with c_2. Clearly then we will have that

$$\bigcap_{\lambda\in\Lambda} C'_\lambda = \left(\bigcap_{\lambda\in\Lambda} C_\lambda\right) \cup c_2.$$

Now we need to extend f to a map $f' : K' \to Y$, and we are going to do it this way: f' maps c_2 onto the unique vertex v of Y and then is extended to a continuous map of D into B_μ. This extension can always be achieved thanks to the equivalence condition on $f \circ g(I)$. So we have built a new Stallings system $(K', \{C'_\lambda\}_{\lambda\in\Lambda}, f')$.

This construction gives us the possibility of getting a new Stallings system, for which the connected components of $\bigcap_{\lambda\in\Lambda} C'_\lambda$ are one fewer than $\bigcap_{\lambda\in\Lambda} C_\lambda$ when we have a binding tie, is therefore natural to ask ourself if a binding tie always exist. We show with the following lemma that this is always the case.

Lemma 4.2.1. *Let $(K, \{C_\lambda\}_{\lambda\in\Lambda}, f)$ be a Stallings system such that $f_* : \pi_1(K) \to \pi_1(Y)$ is an epimorphism. If*

$$\bigcap_{\lambda\in\Lambda} C_\lambda$$

is not connected, then there exists a binding tie.

Proof. Consider a base point for each of the connected components of $\bigcap_{\lambda\in\Lambda} C_\lambda$ and any loop or tie g whose initial and terminal points are such base points. From what we know about graphs follows that any such loop or ties is equivalent to a product of paths, each of which runs along an edge, therefore we can think of them as a product of monochromatic paths. It follows that is possible to write

$$g = g_1 ... g_n$$

where for all $1 \leq i < n$ the paths g_i, g_{i+1} are of different colors. Hence the end points of all g_i will lie in $\bigcap_{\lambda \in \Lambda} C_\lambda$.
For each g_i we now define h_i as a path joining the end point of g_i to the base point of its connected component of $\bigcap_{\lambda \in \Lambda} C_\lambda$. We will have

$$g \sim (g_1 h_1)({h_1}^{-1} g_2 h_2)...({h_{n-1}}^{-1} g_n).$$

In this form each term is monochromatic and its end points are among the chosen base points. This means that each loop or tie in a Stallings system is equivalent to a product of monochromatic loops and ties each with ends points among the base points and two consecutive terms are of different colors.
Next we show that there exists a tie g such that its class η is mapped onto 1 by f_*. Since K is connected and $\bigcap_{\lambda \in \Lambda} C_\lambda$ is not, there exists a tie h such that it is end point are in two different components of $\bigcap_{\lambda \in \Lambda} C_\lambda$. Let now θ denote the equivalence class of h. Since f_* is an epimorphism, there will be a loop k in K based at the initial point of of h whose equivalence class ζ satisfies $f_*(\theta) = f_*(\zeta)$. The desired tie will be $k^{-1}h = g$.
It follows that we can assume

$$g \sim g_1 ... g_n$$

to be a product of monochromatic loops and ties, and for each g_i we denote with η_i its equivalence class. Now we can omit any loop g_1 such that $f_*(\eta_i) = 1$ and if after doing this we have two consecutive ties or loops of the same color we can lump them together. Notice that since the end-points of g are distinct this process will still result in a tie with at least one factor.
So far we have shown that exists a tie $g \sim g_1 ... g_n$, of equivalence class $\eta = \eta_1 ... \eta_n$ for which these conditions hold:

- $f_*(\eta) = 1$.
- For all $1 \leq i < n$ the ties or loops g_i, g_{i+1} are monochromatic of different colors.
- For any i for which g_i is a loop, $f_*(\eta_i) \neq 1$.

If we consider such tie we have that

$$1 = f_*(\eta) = f_*(\eta_1) ... f_*(\eta_n)$$

where each couple of terms $f_*(\eta_i), f_*(\eta_{i+1})$ belongs to a different free factors $\pi_1(\lambda)$. Therefore there must be a g_i such that $f_*(\eta_i) = 1$, otherwise we would have a reduced word equivalent to 1 in the free product

$$\mathop{\ast}_{\lambda \in \Lambda} B_\lambda.$$

Since we made sure to not have such loops in our construction g_i must be a tie, therefore binding tie since $f_*(\eta_i) = 1 \in \pi_1(Y)$, moreover it is monochromatic by construction. Since $\pi_1(Y)$ is a free product, $f f_*(\eta_i) = 1 \in \pi_1(B_\lambda)$ for the right λ. □

Now everything is in place to complete the proof of the main theorem 4.3.

Proof of theorem 4.3. Let now $(K, \{C_\lambda\}_{\lambda\in\Lambda}, f)$ be a Stallings system be the Stallings system we constructed at the beginning of the section so that f_* represent the epimorphism φ. All the vertices of X will are in

$$\bigcap_{\lambda\in\Lambda} A_\lambda.$$

If this intersection is disconnected we can find a binding tie thanks to 4.2.1 and then apply the construction we have seen before to get a new Stalling system $(K^1, \{C^1{}_\lambda\}_{\lambda\in\Lambda}, f^1)$ that deforms retract onto $(K, \{C_\lambda\}_{\lambda\in\Lambda}, f)$, with f^1 extending f and with one less connected component in

$$\bigcap_{\lambda\in\Lambda} A^1{}_\lambda.$$

Since there are a finite number of components in

$$\bigcap_{\lambda\in\Lambda} A_\lambda$$

after a finite number of steps n we will get $(K^n, \{C^n{}_\lambda\}_{\lambda\in\Lambda}, f^n)$ for which

$$\bigcap_{\lambda\in\Lambda} A^n{}_\lambda.$$

is connected. $\square$

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