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The Leverage Effect in Density Forecasts of Equity Returns

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Abstract

The leverage effect refers to the well-known relationship between returns and volatility for an equity. When returns fall, volatility increases. In a recent study, L. Catania and N. Nonejad (2019) evaluate the role of the leverage effect in generating density forecasts of equity returns using well-known observation and parameter-driven conditional volatility models. These models differ in their assumptions regarding the parametric specification, the evolution of conditional volatility process, and how the leverage effect is specified. Catania and Nonejad's analysis investigates the ability of a model to generate accurate density forecasts whether the leverage effect is incorporated or not and compares different model-types using a large number of financial time series. As a result, for each model type, the specification with the leverage effect tends to generate more accurate density forecasts than its no-leverage counterpart. Moreover, among the specifications considered, the Beta-t-EGARCH model is the top performer, regardless of whether the same weight is attached to each region of the conditional distribution or the left tail is emphasized.

Contents

List of Tables

1 Introduction

1.1 Returns

The analysis of financial time series deals with the theory and practice of asset valuation over time. The three fundamental elements involved in financial time series analysis are prices, returns, and volatility. Prices represent the quotation of financial assets observed in official financial markets. Returns represent the relative change in the price of a financial asset over a given time interval. Volatility represents the most common measure of market uncertainty. Differently from the others, volatility is not directly observable (Tsay, 2010).

Many financial studies involve return time series instead of price time series. In fact, for average investors, return of an asset is a complete and scale-free summary of the investment opportunity. Furthermore, return series present more attractive statistical properties than price series, such as stationarity and ergodicity (Danielsson, 2011).

However, there are several definitions of an asset return. Let P_t be the price of an asset at time index t and assume the asset pays no dividends.

Definition 1 (One-period simple return). The *one-period simple net return* is the change in prices observed in time interval $[t-1, t]$ measured in percentage, indicated by R_t:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{1.1}$$

The corresponding *one-period simple gross return* is:

$$1 + R_t = \frac{P_t}{P_{t-1}}. \tag{1.2}$$

In order to provide a more complete set of information, it is useful to analyze different time intervals and horizons. Thus, it is necessary to convert daily returns to monthly or annual returns, or vice versa. A multi-period (n-period) return is given by:

$$\begin{aligned} R_t(n) &= (1+R_t)(1+R_{t-1})\dots(1+R_{t-n+1}) - 1 \\ &= \frac{P_t}{P_{t-1}}\frac{P_{t-1}}{P_{t-2}}\dots\frac{P_{t-n+1}}{P_{t-n}} - 1 \\ &= \frac{P_t}{P_{t-n}} - 1. \end{aligned} \tag{1.3}$$

where $R_t(n)$ is the return over the most recent n-periods from date $t-n$ to t.

An alternative return measure is continuously compounded returns.

Definition 2 (Continuously compounded returns). The *continuously compounded return* is the logarithm of gross return, indicated by Y_t:

$$Y_t = \log(1+R_t) = \log\frac{P_t}{P_{t-1}} = \log P_t - \log P_{t-1}. \tag{1.4}$$

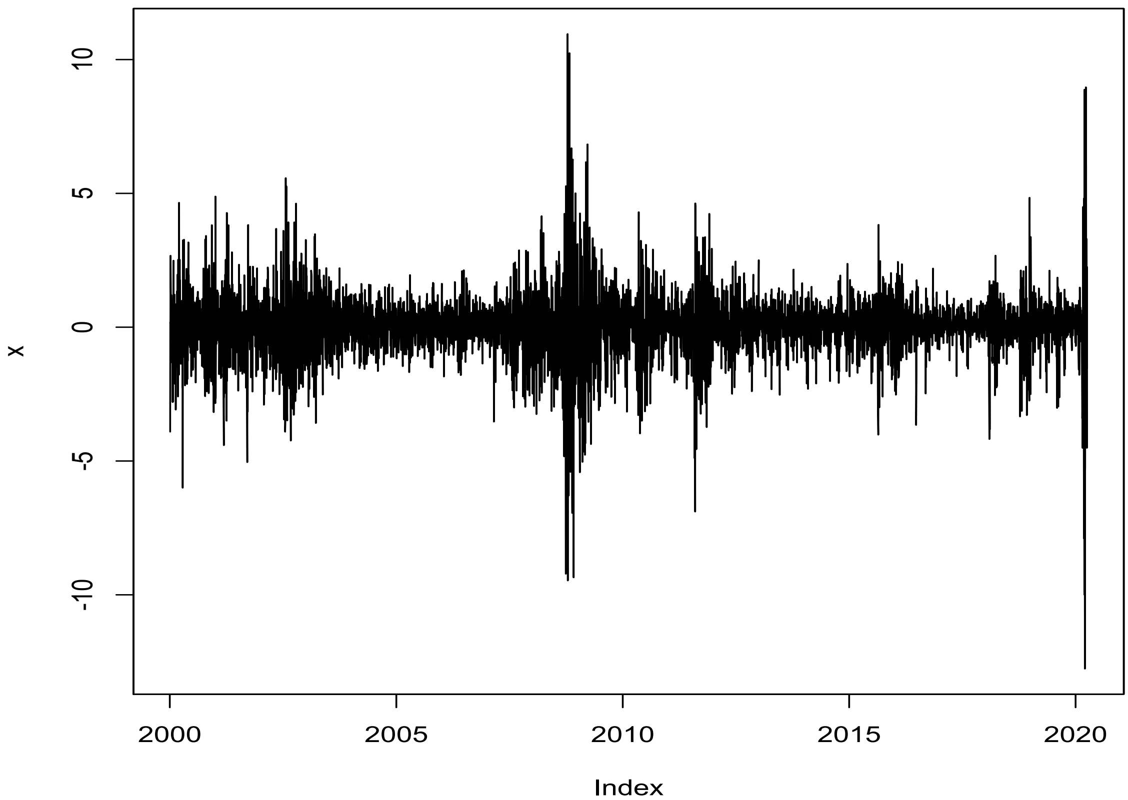

Figure 1.1: S&P 500 index returns form 01-01-2000 to 06-04-2020.

The advantages of logarithmic returns become clearer when considering multi-period return:

$$
\begin{aligned}
Y_t(n) &= \log\left(1 + R_t(n)\right) = \log\left((1 + R_t)(1 + R_{t-1})(1 + R_{t-2}) \ldots (1 + R_{t-n+1})\right) \\
&= \log\left(1 + R_t\right) + \log\left(1 + R_{t-1}\right) + \cdots + \log\left(1 + R_{t-n+1}\right) \\
&= Y_t + Y_{t-1} + \cdots + Y_{t-n+1}.
\end{aligned} \tag{1.5}
$$

Continuously compounded multi-period returns result from the sum of continuously compounded single-period returns. Conversely from simple returns, it is much easier to derive the time series properties of sums than of products. As a result, to introduce and model volatility, continuously compounded returns are undoubtedly preferable.

1.2 Volatility

There is no unique or universally accepted way to define volatility. Although it plays a very important role in financial econometrics, there is an inherent challenge in using it. Volatility is a latent factor, thus, can not be directly observed (Catania and Nonejad, 2019). From a statistics point of view, it is possible to introduce volatility using financial time series properties.

Figure 1.1 shows returns series for S&P500 index from 01-01-2000 to 06-04-2020. The *autocorrelation function* (ACF) is a standard graphical method to explore correlation in

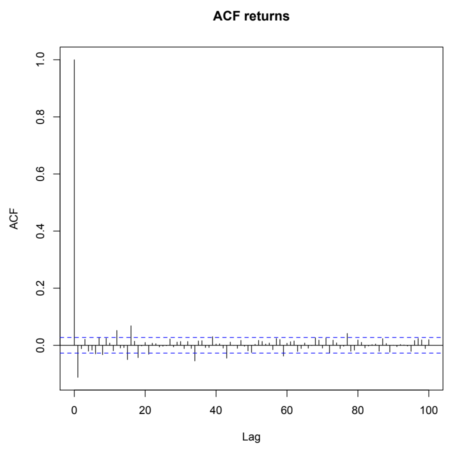

Figure 1.2: ACF for S&P 500 returns along with a 95% confidence interval.

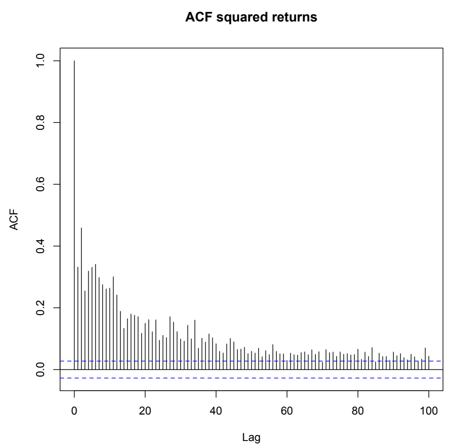

Figure 1.3: ACF for S&P 500 squared returns along with a 95% confidence interval.

statistical data. The ACF measures how returns on one day are correlated with returns on previous days. Evidence for predictability and linear dependence exist if such correlations are statistically significant.

Analyzing figures 1.2 and 1.3, a key feature of returns appears. Most autocorrelations of S&P 500 returns lie within the interval. Conversely, the ACF of squared returns is significant even at long lags. This behavior suggests the presence of heteroskedasticity and a form of dependence peculiar for the case. This dependence is the consequence of volatility. Considering the following heteroskedastic model for a return time series:

$$Y_t = \mu_t + \sigma_t \epsilon_t \tag{1.6}$$

where:

- Y_t represents the logarithmic return at time t;
- $\mu_t = E[Y_t|\mathcal{F}_{t-1}]$ represents the expected value of return at time t conditional to the information set at time $t-1$, $\mathcal{F}_{t-1}$;
- ϵ_t is a sequence of independent and identically distributed (iid) random variables with

mean zero and variance 1.

Definition 3 (Conditional volatility). The *conditional volatility* is the standard deviation of returns conditional to the information set, indicated by σ_t.

Considering model (1.6), $\sigma_t = \sqrt{V(Y_t|\mathcal{F}_{t-1})}$.

Consequently conditional volatility is a measure of return fluctuation over time. As a result, it is also a financial risk measure.

Unconditional volatility, indicated by σ, is defined as volatility over an entire time period. However, it can not be considered as relevant as conditional volatility in statistical and financial terms. First, it is recognized that squared returns (and also the absolute value of returns) are proxies for conditional volatility. Second, since conditional volatility changes over time, it is partially predictable. Finally, the behavior of conditional volatility process and its structure of dependence raise important questions regarding the stability of financial markets and the impact of price variations on the economy. Accordingly, conditional volatility process is what affects returns behavior over time and assets. Therefore, it is fundamental to model and predict conditional volatility in order to foster the knowledge of returns and enhance financial decisions.

The purpose of the study by Catania and Nonejad (2019) is to produce a much more thorough set of information about returns. Accordingly, they underline the lack of knowledge regarding the ability of conditional volatility models to generate a complete description of the conditional return distribution (return density). Thus, they provide a comparison of density forecasts among different volatility models.

1.3 The Stylized Facts of Financial Returns

Conditional volatility models theory is grounded on the financial returns statistical properties. Most financial returns exhibit three statistical properties. These are often called *the three stylized facts* of financial returns:

- Volatility clusters;
- Fat tails;
- Nonlinear dependence.

1.3.1 Volatility clusters

This feature of financial time series gained widespread recognition following Engle's publication "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation," in 1982. Traditional econometric models used to assume a constant one-period forecast variance. In order to generalize this implausible assumption, Engle introduced the definitions of conditional and unconditional volatility. Since conditional volatility changes over time, its cycles are widely studied. According to Danielsson (2011), the magnitudes of volatilities of financial returns tend to cluster together. As a result, financial market goes through periods when volatility is high and other periods when volatility is low. These phenomena are known as volatility clusters.

This property has massive implications in returns distribution. For example, as shown in figure 1.3 squared returns are highly correlated being proxies for conditional volatility. Moreover, volatility cycles generate returns changes. The relationship between returns and changes in volatility has been of great interest for researchers and risk managers in financial markets (Catania and Nonejad, 2019).

1.3.2 Fat tails

Most of the financial concepts developed during the first half of the 20th century rest upon the assumption that returns follow a normal distribution (Rachev, Menn, and Fabozzi, 2005). However, this hypothesis cannot be verified through empirical evidence. Accordingly, in his work, Mandelbrot (1963) underlines the presence of fat tails. He highlighted that returns series are characterized by the much higher probability of relatively large and small outcomes than normal distribution would predict. Consequently, following Mandelbrot statement, fat-tailed property has become one of the stylized facts of financial returns.

This property has important consequences in the field of finance. Many financial applications regarding portfolio theory and derivative pricing break down in the absence of normality. Moreover, the non-normality of returns is crucial in risk management. As explained in Danielsson (2011), an assumption of normal distribution for risk calculations leads to a gross underestimation of risk.

1.3.3 Nonlinear dependence

The final stylized fact of financial returns is nonlinear dependence. It consists on the observation that the dependence between different return series changes according to market conditions. In 1985, Hinich and Patterson were the first to provide evidence of non-linear dependence in NYSE stock returns. The market crash of 1987 shifted the paradigm. According to Hiremath (2014), the crash is the major event which influenced the role of nonlinearities in dynamics of stock returns.

This property plays a crucial role in many financial applications such as risk analysis. In fact, nonlinear dependence explains some financial series empirical characteristics. This can be demonstrated, for example, by empirical observations of asset prices and returns correlations. In the former case, most of the time, the prices of assets move relatively independently from each other, but in a crisis they all drop together. In the latter case, returns correlations are usually lower in bull markets than in bear markets.[1] Models of nonlinear dependence allow to capture such phenomena and to change dependence structure according to market conditions.

2 The Leverage Effect

This section focuses on the relationship between equity returns and volatility. According to financial literature, this relationship and its impacts on financial markets are known as the *leverage effect*. The leverage effect outlines two crucial empirical facts:

[1] A bull market is a market that is on the rise and where the economy is sound; while a bear market exists in an economy that is receding, where most stocks are declining in value (Kramer, 2020).

- When returns fall, volatility increases, thus, returns and volatility are negatively correlated;
- This correlation is asymmetric: it depends on whether returns are negative or positive.

According to Catania and Nonejad (2019), variations in equity prices and the affinity between price variations and volatility can imply huge losses or gains to investors involved in financial markets. Therefore, it is not surprising that since 1950's, the sources and the consequences of the leverage effect are topics of great interest for financial researchers.

2.1 The standard theory

The theoretical background on which the leverage effect is rooted is the Modigliani-Miller framework. In their work "The Cost of Capital, Corporation Finance and the Theory of Investment" (1958), they provide the principle for modern thinking on capital structure. This principle states that the fundamental asset of a corporation is the whole firm. (Figlewski and Wang, 2000). Thus, considering a firm with equity and debt in its capital structure, under the simplifying assumption that the debt is risk free, changes in firm value are entirely borne by the stock.

From this perspective, Black (1976) and Christie (1982) elaborate the most commonly recognized standard explanation for the leverage effect phenomenon. This theory is known as the leverage hypothesis. It suggests that a fall in equity value increases the debt to equity ratio. Consequently, the rise in riskiness of a firm generated translates into a higher volatility level (Catania and Nonejad, 2019).

A simpler version of this theory is presented in Figlewski and Wang (2000).

Consider a firm with equity and debt in its capital structure and assume the debt is risk free. Let $V = E + D$, represent the total firm value. $E = N * S$ denotes the total current market value of the firm's N outstanding shares of stock with current market price S. D is the value of the debt. Suppose there is a random change in overall firm value, ΔV.

All of the change in firm value will flow through the stock, so $\Delta E = \Delta V$. A percentage change in the stock price is produced as follows.

$$\frac{\Delta S}{S} = \frac{\Delta E}{E} = \frac{\Delta V}{V} * \frac{V}{E} = \frac{\Delta V}{V}\left(\frac{E+D}{E}\right) = \frac{\Delta V}{V}\left(1 + \frac{D}{E}\right) \tag{2.1}$$

The percentage change in the stock price equals the percentage change in firm value times one plus the debt / equity ratio. The more levered the firm is (high D/E), the more volatile the stock will be relative to the total firm. That is expressed in the following equation.

$$\sigma_S = \sigma_E = \sigma_V * L \tag{2.2}$$

where σ_S is the volatility of the return of the stock; σ_E is the volatility of total equity; σ_V is the volatility of the firm; and $L = (1 + D/E)$ is the measure of leverage. If σ_V is constant, the stock volatility σ_S will rise when the stock price goes down and fall when it goes up. Hence, the empirical observed connection between stock returns and volatility changes is understandable and consistent with the established principles of modern finance.

2.2 Alternative interpretations

The consequences provoked by the leverage effect on financial markets are plentiful. For this reason, trying to analyze them using solely the leverage hypothesis will lead to incomplete results. Consequently, in financial literature scholars are feeling the need for alternative theories.

In this context, Bekaert and Wu (2000) investigate the time-varying risk premiums theory, a different interpretation for the relationship between equity returns and volatility.

Time-varying risk premiums theory introduces a new point of view to study the leverage effect based on the concept of the volatility feedback. The volatility feedback is the set of consequences on stock prices caused by the volatility changes and the volatility clustering. Accordingly, since the volatility of the returns can be very different at different times, it seems plausible that changes in volatility may have important effects on required stock returns, thus, on the level of stock prices. As highlighted by Bekaert and Wu (2000), an anticipated increase in volatility raises the required return on equity, leading to an immediate stock price decline.

In their work "Asymmetric Volatility and Risk in Equity Markets" (2000) Bekaert and Wu find that is important to include leverage ratios in volatility dynamics, but that their economic effects are mostly dwarfed by the volatility feedback mechanism. In fact, volatility feedback is enhanced by a phenomenon called covariance asymmetry. This phenomenon consists on the increasing of the conditional covariances with the market, namely the firm risk measure. It is asymmetric because it is statistically significant only following negative market news.

Moreover, Bekaert and Wu's work is remarkable due to the stocks it focus on. Whereas most of the preceding empirical analysis has focused on U.S. stock returns, their empirical application uses Japanese stock returns from the Nikkei Index. Their results indicate that asymmetry is an important feature of stock market volatility in every market.

Thanks to Bekaert and Wu's work, volatility feedback theory has been categorized as one of the most accredited explanations for the negative relation between equity returns and volatility. As a result, its differences and advantages compared to the leverage hypothesis have been widely studied.

The main difference between the leverage hypothesis and the volatility feedback theory are the roles played by return shocks and volatility changes. The leverage hypothesis claims that return shocks lead to changes in conditional volatility. The volatility feedback hypothesis contends that return shocks are caused by changes in conditional volatility. Furthermore, the volatility feedback theory emphasizes the asymmetric behavior of the volatility. Which effect is the main determinant of the negative relationship between returns and volatility remains an open question. However, it appears clear that the leverage hypothesis cannot explain the full volatility response. Therefore, it is appropriate to analyze some of the most interesting explanations based on the volatility feedback.

French, Schwert and Stambaugh (1987) examine the intertemporal relation between risk and expected returns. In particular, they find evidence that the expected market risk premium, namely the expected return on a stock portfolio minus the risk-free interest rate, is positively related to the volatility of the stock market. Although their empirical result does not explain the leverage effect at the firm level (explained by Bekaert and Wu's covariance asymmetry), their work was fundamental to lighten up the attention on the volatility feedback theory.

Campbell and Hentschel (1992) underline that the volatility feedback has the potential to explain crucial empirical facts about stock returns. In particular, they find evidence that large negative stock returns are more common than large positive ones, so stock returns exhibit negative skewness. In fact, the volatility feedback amplifies large negative stock returns and dampens large positive returns, increasing the potential for large crashes. Moreover, they point out the necessity to modify the classic formulation of the conditional volatility models in order to include the asymmetric behavior of volatility.

Finally, Figlewski and Wang (2000) demonstrate the lack of empirical evidence of the leverage hypothesis. In fact, they argue that the asymmetry is the real nature of the phenomenon. Focusing on large stocks contained in the Standard and Poors 100 stock index (OEX) they observe that the leverage effect appears to be much more related to falling stock prices than to leverage per se. Consequently, the leverage effect should more properly be termed as a "down market effect".

In conclusion, even if the true explanation for the phenomenon is yet to be determined, the leverage effect has become one of the stylized facts that it is felt need to be incorporated into models of time-varying volatility (Figlewski and Wang, 2000).

3 Conditional Volatility Models

This section presents the classes of conditional volatility models used in this analysis. Due to conditional volatility nature, conditional volatility models are particular specifications of time-varying parameter models. Cox (1981) categorizes time series models with time-varying parameters into two classes: observation-driven models and parameter-driven models.

In an observation-driven model, current parameters are deterministic functions of lagged dependent variables as well as contemporaneous and lagged exogenous variables. In this approach, although parameters are stochastic, they are perfectly predictable one-step-ahead given past information. The likelihood function for observations-driven models is available in closed-form through the prediction error decomposition. This feature leads to simple estimation procedures and contributes to the popularity of this class of models in applied econometrics and statistics.

In parameter-driven models, parameter vary over time as dynamic processes with idiosyncratic innovations. Analytical expressions for the likelihood function are not available in closed-form for these models. Likelihood evaluation, therefore, becomes more involved for parameter-driven models, typically requiring the use of efficient simulation methods (Koopman, Lucas, and Schart, 2016).

As regards conditional volatility, in an observation-driven model conditional volatility is a deterministic function of lagged observations and conditional volatilities. In the parameter-driven specification, conditional log-volatility is modeled as an unobserved model with idiosyncratic innovations. (Catania and Nonejad, 2019).

3.1 Observation-driven Conditional Volatility Models

3.1.1 GARCH Models

By far, the most popular approach to model conditional volatility is the Generalized Autoregressive Conditional Heteroscedasticity specification, introduced in Engle (1982) and Bollerslev (1986).

As previously mentioned, Engle (1982) defines the distinction between the previous implausible assumption of a constant conditional volatility and the modern time-varying conditional volatility. Thus, Engle proposes a new class of stochastic processes called Autoregressive Conditional Heteroscedastic (ARCH) processes. These processes are mean zero, serially uncorrelated with nonconstant variances conditional on the past, but constant unconditional variance. For such processes, the recent past gives information about the one-period forecast variance (Engle, 1982).

In order to understand the reason for which ARCH framework represents the most attractive approach to model conditional volatility, it seems appropriate to analyze Engle's steps to formalize the ARCH model.

If a random variable y_t is drawn from the conditional density function $f(y_t|y_{t-1})$, the forecast of today's value based upon the past information, under standard assumptions, is simply $E[y_t|y_{t-1}]$, which depends upon the value of the conditioning variable y_{t-1}. The variance of this one-period forecast is given by $V(y_t|y_{t-1})$. This expression recognizes that the conditional forecast variance depends upon past information and may, therefore, be a random variable.

However, for econometric models previous to Engle's work, the conditional variance does not depend upon y_{t-1}. In particular, considering the first order autoregression:

$$y_t = \gamma y_{t-1} + \epsilon_t \qquad \epsilon_t \sim IID(0, \sigma^2)$$

$$\begin{aligned} E[y_t|y_{t-1}] &= \gamma y_{t-1} \qquad & E[y_t] &= 0 \\ V(y_t|y_{t-1}) &= \sigma^2 \qquad & V(y_t) &= \frac{\sigma^2}{1-\gamma^2} \end{aligned}$$

It is clear the necessity to introduce heteroscedasticity in the conditional variance term to allow the forecast variance to depend upon the past information. The standard approach of heteroscedasticity is to introduce an exogenous variable x_t, which predicts the variance. With a known zero mean, the model might be $y_t = \epsilon_t x_{t-1}$ and the conditional variance $\sigma^2 x_{t-1}$. However, this standard solution is unsatisfactory, because it requires the exogenous variable to be completely specified. Due to this difficulty, this correction is rarely considered in financial time series.

An alternative model that allows the conditional variance to depend upon the past realization of the series is the bilinear model described by Granger and Andersen (1978). In their formulation the model is $y_t = \epsilon_t y_{t-1}$ and the conditional variance is $\sigma^2 y_{t-1}^2$. However, the unconditional variance is either zero or infinity, which makes this an unattractive formulation.

It is Engle's solution the formulation that fills the previous gap, providing the generalization required.

$$
\begin{aligned}
y_t &= \epsilon_t h_t^{\frac{1}{2}} \\
h_t &= \alpha_0 + \alpha_1 y_{t-1}^2 \qquad V(\epsilon_t) = 1
\end{aligned}
$$

Adding the assumption of normality, it can be more directly expressed in terms of F_t, the information set available at time t:

$$y_t|F_{t-1} \sim N(0, h_t) \tag{3.1}$$

$$h_t = \alpha_0 + \alpha_1 y_{t-1}^2 \tag{3.2}$$

The variance function can be expressed more generally as

$$h_t = h(y_{t-1}, y_{t-2}, \ldots, y_{t-p}, \alpha) \tag{3.3}$$

where p is the order of the ARCH process and α is a vector of unknown parameters.

The ARCH model has a variety of characteristics, which makes it attractive for econometric applications. For example, it allows to capture important stylized facts of financial time series, such as volatility clustering. Moreover, the likelihood function is available in closed form via the error decomposition. (Catania and Nonejad, 2019).

Conversely, the ARCH model presents also some weaknesses (Tsay, 2010):

- Although the ARCH model is simple, it often requires many parameters to adequately describe the volatility process for an asset return;

- The model assumes that positive and negative shocks have the same effects on volatility because it depends on the square of the previous shocks;

- The ARCH model is rather restrictive. For instance, α_1 of an ARCH(1) model must be in the interval $[0, \frac{1}{3}]$ if the series has a finite fourth moment. In practice, it limits the ability of ARCH models with Gaussian innovations to capture excess kurtosis;

- ARCH models are likely to overpredict the volatility because they respond slowly to large isolated shocks to the return series.

As a result, this model still needs some degree of specification. For this reason, Bollerslev notices that empirical applications of the ARCH model call for a more parsimonious specification, which keeps the attractive characteristics of Engle's framework. In his work "Generalized Autoregressive Conditional Heteroskedasticity" (1986) he introduces a generalization of the ARCH model. His GARCH model is:

$$\epsilon_t|F_{t-1} \sim N(0, h_t), \tag{3.4}$$

$$
\begin{aligned}
h_t &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \\
&= \alpha_0 + A(L)\epsilon_t^2 + B(L)h_t,
\end{aligned} \tag{3.5}
$$

where L is the lag operator, and

$$\begin{aligned} & & p \geq 0, \qquad & q > 0 \\ \alpha_0 > 0, \qquad & \alpha_i \geq 0, \qquad & i = 1, \ldots, q, & \\ & \beta_j \geq 0, \qquad & j = 1, \ldots, p. & \end{aligned}$$

Therefore, in the ARCH(q) process the conditional variance is specified as a linear function of past sample variances only, whereas the GARCH(p,q) process allows lagged conditional variances to enter as well.

Despite being more parsimonious, the standard GARCH model has the same weaknesses as the ARCH model. Modifications of the standard GARCH formulation allow the model to account for numerous stylized facts of financial returns. Financial returns often exhibit excess conditional kurtosis, which is not properly accounted for by the conditional Gaussian distribution in equation (3.4). Bollerslev (1987) introduces the Student-t-GARCH model, which substitutes the conditional Gaussian distribution in equation (3.4) with the Student-t distribution.

The conditional volatility equation (3.5) responds equally to positive and negative shocks. Namely, it does not account for the leverage effect. To address this issue, several researchers suggest different solutions, all introducing the asymmetric impact of negative and positive return innovations directly in the conditional volatility equation. One example is the Zakoian (1993) and Glosten, Jagannathan, and Runkle (1994) formulation of the T-GARCH or GJR-GARCH model. The T-GARCH (1,1) process with asymmetric response on volatility is defined by the following dynamic equation:

$$\epsilon_t \sim ID(0, h_t) \tag{3.6}$$

$$h_t = \gamma + \alpha\epsilon_{t-1}^2 + \rho D_{t-1}^- \epsilon_{t-1}^2 + \beta h_{t-1}, \tag{3.7}$$

where

$$D_{t-1}^- = \begin{cases} 1, & if \quad \epsilon_{t-1} < 0 \\ 0, & if \quad \epsilon_{t-1} \geq 0 \end{cases}$$

In this model positive innovations have an impact measured by the coefficient α, whereas negative ones have an impact measured by $\alpha + \rho$. If ρ is a positive value, the leverage effect is accounted for by the model. This model is one of the most popular asymmetric GARCH models.

An alternative GARCH model with asymmetric response to innovations is the E-GARCH (exponential GARCH) by Nelson (1991). E-GARCH (1,1) formulation is:

$$\epsilon_t = e^{h_t}\zeta_t \qquad \zeta_t \sim ID(0, 1) \tag{3.8}$$

$$h_t = \gamma + \alpha(|\zeta_{t-1}| - E[|\zeta_{t-1}|]) + \rho\zeta_{t-1} + \beta h_{t-1} \tag{3.9}$$

In this model γ is the level of conditional log-volatility, α and β determine the impact of past observations and conditional volatilities on h_t, and ρ controls the leverage effect. Due to the exponential link in function (5.1), conditional volatility is always positive and the only constraint imposed during the estimation procedure to ensure stationarity of h_t is $|\beta| < 1$. When the innovation $\zeta_t > 0$, then $\alpha + \rho$ determine the response to past observations. When $\zeta_t < 0$, then the magnitude of the response is $\alpha - \rho$. Evidently, when $\rho < 0$, the leverage effect is accounted for by the model (Catania and Nonejad, 2019). Due to the limited number of restrictions imposed by Nelson's formulation and its liability proved by its use in following literature (Harvey and Sucarrat (2014) and Harvey and Lange (2018)), this model is considered a useful tool to generate conditional volatility forecasts.

Furthermore, in an interesting study, Hansen and Lunde (2005) 330 GARCH-type models in terms of their ability to describe the conditional variance and to generate accurate conditional volatility forecasts. The out-of-sample comparison study finds no evidence that a simple GARCH(1,1) is outperformed by more sophisticated models in the context of exchange rate data. Conversely, the simple GARCH (1,1) model is inferior to models that incorporate the leverage effect in the context of equity return data. Confronting such a high number of models, this study demonstrates the importance and the diffusion of GARCH framework for the understanding of conditional volatility.

3.1.2 GAS models

Recently, Creal, Koopman, and Lucas (2013) and Harvey (2013) propose a new class of observation-driven models referred to as Generalized Autoregressive Score (GAS) or, equivalently, Dynamic Conditional Score (DCS) models. Similar to GARCH, estimation of GAS models is straightforward using maximum likelihood techniques. However, contrary to GARCH, the mechanism to update the parameters occurs through the scaled score of the conditional distribution for the observable variables (Catania and Nonejad, 2019).

Creal, Koopman, and Lucas (2013) argue that the score function is an effective choice for introducing a driving mechanism for time-varying parameters. In particular, by scaling the score function appropriately, standard observation-driven models, such as the GARCH, can be recovered. Consequently, the GAS model has the same advantages of the other observation-driven models. For this reason, extensions to asymmetric, long memory, and other more complicated dynamics can be considered without introducing further complexities. The main difference among GAS and the other observation-driven approaches is that the GAS model is based on the score, thus, it exploits the complete density structure and not just means and higher moments. Creal, Koopman, and Lucas explain the main GAS framework characteristics through their basic GAS model specification.

Let the $N \times 1$ vector y_t denote the dependent variable of interest, f_t the time-varying parameter vector, x_t a vector of exogenous variables (covariates), all at time t, and θ a vector of static parameters. Define $Y^t = (y_1, \ldots, y_t), F^t = (f_0, f_1, \ldots, f_t)$, and $X^t = (x_1, \ldots, x_t)$. The available information set at time t consists of $(f_t, \mathbf{F}_t)$, where

$$\mathbf{F}_t = (Y^{t-1}, F^{t-1}, X^{t-1}), \qquad t = 1, \ldots, n$$

The assumed conditional distribution that generates y_t is the observation density:

$$y_t \sim p(y_t | f_t, \mathbf{F}_t; \theta) \tag{3.10}$$

Furthermore, the assumed mechanism for updating the time-varying parameter f_t is given by the familiar autoregressive updating equation

$$f_{t+1} = \gamma + \sum_{i=1}^{p} A_i s_{t-i+1} + \sum_{j=1}^{q} B_j f_{t-j+1} \tag{3.11}$$

where γ is a vector of constants, coefficient matrices A_i and B_j have appropriate dimensions for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, while s_t is an appropriate function of past data, $s_t = s_t(y_t, f_t, F_t; \theta)$. The unknown coefficients in (3.11) are functions of θ; namely $\gamma = \gamma(\theta), A_i = A_i(\theta)$, and $B_j = B_j(\theta)$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. The main contribution of Creal, Koopman, and Lucas's work is the particular choice for the driving mechanism s_t that is applicable over a wide class of observation densities and nonlinear models.

Their approach is based on the observation density in function (3.10) for a given parameter f_t. When an observation y_t is realized, the time-varying parameter is updated to the next period $t + 1$ using (3.11) with

$$s_t = S_t \nabla_t, \qquad \nabla_t = \frac{\partial \ln p(y_t | f_t, \mathbf{F}_t; \theta)}{\partial f_t}, \qquad S_t = S(t, f_t, \mathbf{F}_t; \theta) \tag{3.12}$$

where $S(\cdot)$ is a matrix function.

The use of the score function for updating f_t is intuitive. It defines a steepest ascent direction for improving the model's local fit in terms of the likelihood or density at time t, given the current position of the parameter f_t. This provides the natural direction for updating the parameter. In addition, the score depends on the complete density, and not only on the first- or second-order moments of the observations y_t. This distinguishes the GAS framework from most of the other observation-driven approaches in the literature.

In addition, via its choice of the scaling matrix S_t, the GAS model allows for additional flexibility in how the score is used for updating f_t.

Considering the basic model $y_t = h_t \epsilon_t$ where the Gaussian disturbance ϵ_t has zero mean and unit variance, while h_t is a time-varying standard deviation. It is simple to show that the GAS(1,1) model with $S_t = \mathbf{I}_{t|t-1}^{-1}$ and $f_t = h_t^2$ reduces to:

$$f_{t+} = \gamma + A_1(y_t^2 - f_t) + B_1 f_t \tag{3.13}$$

which is equivalent to the standard GARCH(1,1) model as given by:

$$f_{t+1} = \gamma + \alpha_1 y_t^2 + \beta_1 f_t, \quad f_t = h_t^2 \tag{3.14}$$

As previously mentioned, the flexibility of GAS framework allows the model to account for numerous stylized facts of financial time series. As regards this analysis, a deeper look at how a GAS model specifies the leverage effect seems appropriate. Similar to GARCH, the leverage effect is incorporated directly in the conditional volatility equation. There will be an additional parameter (as ρ in GARCH model), which regulate the asymmetric response to negative and positive innovations.

As a result, GAS model is a class of observation-driven models with similar degree of generality as nonlinear non-Gaussian state space models, which, compared to GAS, are more complicated to estimate. In a recent and interesting study, Koopman, Lucas, and Schart (2016) have identified two main findings about GAS models using Monte Carlo simulations.

First, when the data generating process is a state space model, the predictive accuracy of a (misspecified) GAS model is similar to the one of a (correctly specified) state space model. This holds, in particular, when the conditional observation density for the GAS specification allows for heavy tails. For the nine model specifications considered, the loss in mean square error from using a GAS model instead of the correct state space specification is most of the time inferior to 1% and never higher than 2.5%.

Second, they have found that GAS models outperform many of the familiar observation-driven models in terms of generating accurate forecasts. By relying on the full density structure to update the time-varying parameters, GAS models capture additional information in the data that is not exploited by traditional observation-driven models.

They conclude that it is possible to obtain high predictive accuracy for many relevant time-varying models without the need to specify and estimate cumbersome and computationally brutal parameter-driven models. In most cases, a GAS model alternative is available, and it is both accurate and considerably easier to estimate. Therefore, GAS models are effective new tools for conditional volatility study that often lead to important forecasting gains.

3.2 Parameter-driven Conditional Volatility Models

3.2.1 Stochastic Volatility Models

An alternative to GARCH and GAS models is represented by the stochastic volatility (SV) model introduced in Taylor (1986), which is the example of parameter-driven volatility model in this analysis. In SV framework, conditional log-volatility is modeled as an unobserved process with idiosyncratic innovations. Typically, it is assumed that conditional log-volatility follows an autoregression of order one, AR(1), which is the simplest ARIMA model specification. (Catania and Nonejad, 2019).

A closed form likelihood function is not available for the SV framework. The difficulty in likelihood evaluation is the reason for the limited empirical applications of these models. However, scholars provide several methods for simulated maximum likelihood (SML) estimation in order to use SV models and analyze their advantages compared to GARCH-type models.

In SV models, unlike the GARCH framework, both the mean and the log-volatility equations have separate error terms. Accordingly, the basic SV model specification is (Malik and Pitt, 2011):

$$y_t = \epsilon_t e^{(\frac{h_t}{2})} \tag{3.15}$$

$$h_{t+1} = \mu(1-\phi) + \phi h_t + \sigma_\eta \eta_t, \quad t = 1, \ldots, T \tag{3.16}$$

where

$$\begin{pmatrix} \epsilon_t \\ \eta_t \end{pmatrix} \sim N(0, \Sigma), \qquad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.17}$$

Here y_t is the observed return, h_t are the unobserved time-varying log-volatilities, μ is the drift in the state equation, σ_η is the variance of log-volatility and ϕ is the persistence parameter.

Typically, it is imposed that $|\phi| < 1$, so that it results in a stationary process with the initial condition that

$$h_1 \sim N(0, \sigma_\eta^2/(1-\phi^2)). \tag{3.18}$$

In some regards, SV has proven to be more attractive than GARCH-type models. Jacquier, Polson, and Rossi (1994) find that compared to GARCH, SV yields a better and more robust description of the autocorrelation pattern of the squared returns. Kim, Shephard, and Chib (1998) demonstrate that a simple SV model typically fits data just as well as more heavily parameterized GARCH models. Both the studies provide also a practical likelihood-based framework to analyze inference and prediction for stochastic volatility models.

As a result, Malik and Pitt (2011) collect an high number of alternatives to provide a unified methodology for conducting likelihood-based inference on the unknown parameters of a general class of discrete-time stochastic volatility models. Moreover, they extend the basic formulation of the SV model in order to incorporate the leverage effect. Differently from observation-driven models, the leverage effect is expressed through a correlation coefficient between return and log-volatility innovations. Consequently, the leverage effect is introduced modifying the Σ matrix in equation (3.21). The SV model specification with leverage effect is, therefore:

$$y_t = \epsilon_t e^{(\frac{h_t}{2})} \tag{3.19}$$

$$h_{t+1} = \mu(1-\phi) + \phi h_t + \sigma_\eta \eta_t, \quad t = 1, \ldots, T \tag{3.20}$$

where

$$\begin{pmatrix} \epsilon_t \\ \eta_t \end{pmatrix} \sim N(0, \Sigma), \qquad \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{3.21}$$

and ρ measures the leverage effect.

Finally, Malik and Pitt (2011) propose a new hybrid volatility model, the SV-GARCH model, which attempts to bridge elements of SV and GARCH specifications. This model nests the standard GARCH model as a special case. It has the attractive feature of inheriting the same unconditional properties of the standard GARCH model and, adding a single parameter in the specifications, it also enables the consideration of the conditional heavy-tailed distribution of log-returns.

In conclusion, practical difficulties have kept financial researchers from using the SV model even though it offers a natural alternative to ARCH, and has some advantages in prediction of unobserved variances.

4 Forecast evaluation methodology

This section presents the forecast evaluation methodology used in the analysis by Catania and Nonejad (2019). According to Gneiting and Raftery (2007), with the continued proliferation of probabilistic forecasts in financial, environmental and meteorological applications, there is a critical need for principled techniques for the comparison and ranking of density forecasts.

In order to evaluate density forecasts from different volatility models, it is useful to follow Amisano and Giacomini's (2007) framework. In this framework, density forecasts are considered in a time series context, in which a rolling window consisting of the past m observations is used to fit a density forecast for the observation that is k time steps ahead. Specifically, suppose that $\mathbf{Z}_1, \ldots, \mathbf{Z}_T$ is a stochastic process, which can be partitioned as $\mathbf{Z}_t = (Y_t, \mathbf{X}_t)$ where Y_t is the variable of interest and $\mathbf{X_t}$ is a vector of predictors. Suppose that $T = m + n + k$. At times $t = m, \ldots, m + n$, density forecasts $\hat{f}_{t+k}$ and $\hat{g}_{t+k}$ for Y_{t+k} are generated, each of which depends only on $\mathbf{Z}_{t-m+1}, \ldots, \mathbf{Z}_t$. In this framework, the only requirement imposed on how the forecasts are produced is that they are measurable functions of data in the rolling estimation window. A forecast evaluation methodology is a tool that compares and ranks the competing density forecasting methods (Gneiting and Ranjan, 2011).

The comparison typically uses a proper scoring rule. A scoring rule is a loss function $S(f, y)$, whose arguments are the density forecast f and the realization y of the future observation Y. The density forecast is ideal if the sampling density of Y is indeed f. Hence, it is critically important that a scoring rule be proper, in sense that

$$\begin{aligned} E_f[S(f, Y)] &= \int f(y) S(f, y) dy \\ &\leq \int f(y) S(g, y) dy = E_f[S(g, Y)] \end{aligned} \tag{4.1}$$

for all density functions f and g. A scoring rule is strictly proper if (4.1) holds, with equality if and only if $f = g$ almost surely. Clearly, a strictly proper scoring rule prefers the ideal forecaster over any other. Scoring rules are taken to be negatively oriented penalties, therefore, the lower, the better.

Density forecast methods are then ranked by comparing their average scores. Specifically, if

$$\bar{S}_n^f = \frac{1}{n-k+1} \sum_{t=m}^{m+n-k} S(\hat{f}_t + k, y_{t+k}) \qquad \bar{S}_n^g = \frac{1}{n-k+1} \sum_{t=m}^{m+n-k} S(\hat{g}_t + k, y_{t+k}) \tag{4.2}$$

then f is preferred if $\bar{S}_n^f < \bar{S}_n^g$, and g is preferred otherwise. Amisano and Giacomini (2007) consider tests of equal forecast performance based on the test statistic

$$t_n = \sqrt{n} \frac{\bar{S}_n^f - \bar{S}_n^g}{\hat{\sigma}_n}, \tag{4.3}$$

where

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{j=-(k-1)}^{k-1} \sum_{t=m}^{m+n-|j|} \Delta_{t,k} \Delta_{t+|j|,k} \quad \text{and} \quad \Delta_{t,k} = S(\hat{f}_t + k, y_{t+k}) - S(\hat{g}_t + k, y_{t+k}), \tag{4.4}$$

as proposed by Diebold and Mariano (1995). Assuming suitable regularity conditions, the statistic t_n is asymptotically standard normal under the null hypothesis of equal forecast performance.

In Catania and Nonejad's analysis, density forecasts from different volatility models are evaluated based on the weighted Continuous Ranked probability Score (wCRPS), introduced in Gneiting and Ranjan (2011). The wCRPS method has advantages over the usually employed log-score criterion (the logarithm of the predictive density evaluated at the observed outcome),

Table 4.1: This table reports the weight functions for wCRPS. $\phi(z)$ and $\Phi(z)$ denote the pdf and cdf of a $N \sim (0,1)$ distribution.

Emphasis	Weight function
Uniform	$w(z) = 1$
Center	$w(z) = \phi(z)$
Tails	$w(z) = 1 - \phi(z)/\phi(0)$
Right tail	$w(z) = \Phi(z)$
Left tail	$w(z) = 1 - \Phi(z)$

namely, it is less sensitive to outliers and can better account for predictions that are close to but not equal to the outcome. Furthermore, Gneiting and Ranjan (2011), using a GARCH (1,1) specification, demonstrate that it is invalid to use weighted log-scores to emphasize certain areas of the distribution.

The wCRPS for a model i measures the average absolute distance between the empirical cumulative distribution function (CDF) of y_{t+h}, which is simply a step function in y_{t+k}, and the predicted CDF that is associated with model i's predictive density. Furthermore, the comparison between the empirical and the predicted CDF can also be weighed by a function that emphasizes particular regions of interest, for example, the center or the tails of the predictive density. We define wCRPS for model i at time $t+h$ conditional on the information set at time t as

$$wCRPS^{i}_{t+h|t} = \int_{-\infty}^{\infty} w(z)\Big(\hat{F}^{i}_{t+h|t}(z) - I_{(y_{t+h}<z)}\Big)^2 dz, \tag{4.5}$$

where $w(z)$ is the weight function and $\hat{F}^{i}_{t+h|t}(z)$ is the h-step ahead cumulative density function of model i, evaluated at z. The simplest case is $w(z) = 1$, in which the same amount of weight on each region of the predictive density is considered. Besides, different alternative formulations of $w(z)$ is considered, see Table 4.1. Through the weight function, a better understanding of where the eventual improvements from one specification relative to another come from is obtained. However, (4.5) is not available in closed form. Therefore, the following approximation is used:

$$wCRPS^{i}_{t+h|t} \sim \frac{(y_u - y_l)}{K-1}\sum_{k=1}^{K} w(y_k)\Big(\hat{F}^{i}_{t+h|t}(y_k) - I_{(y_{t+h}<y_k)}\Big)^2, \tag{4.6}$$

where

$$y_k = y_l + k\frac{y_u - y_l}{K}. \tag{4.7}$$

In (4.6), y_u and y_l are the upper and lower values, which define the range of integration. The accuracy of the approximation can be increased to any desired level by K. In this analysis, the set values are $y_l = -100, y_u = 100, K = 1000$, which work well for daily returns in percentage points. The model with lower average wCRPS, thus awCRPS, is always preferred. In other words, if the ratio of awCRPS for model i over j is greater than one, model j is preferred to model i and vice versa.

5 Data and models

A large number of financial time series are the analytical tool to study on the one hand the ability of a model to generate accurate density forecasts whether the leverage effect is incorporated or not, and, on the other hand, to compare different conditional volatility model-types. This section introduces data and models used in this analysis.

According to Catania and Nonejad (2019), too little is known about the ability of well-known observation and parameter-driven models to generate accurate density forecasts. Indeed, most research focuses on conditional volatility forecasts without a complete description of the conditional return distribution. Therefore, a large-scale analysis using a large number of different time series seems in order.

In particular, following Koopman, Lucas, and Schart (2016), four volatility models - three observation-driven and one parameter-driven - are considered. For each model-type the analysis will address a version with and a version without the leverage effect. An out-of-sample density forecast comparison is performed using more then four hundred financial time series, in order to achieve the following purposes:

- determine to what extent accounting for the leverage effect enables a model to generate accurate density forecasts;
- which model-type performs best at a given forecast horizon;
- how do results change across forecast horizons and returns.

5.1 Econometric framework

Let $y_1, \ldots, y_T$ denote a $T \times 1$ sequence of returns. It is assumed that the observed return at time t, thus y_t, is generated from $y_t = \Lambda(h_t)\epsilon_t$, where h_t is the conditional log-volatility at time t, $\Lambda(\cdot)$ is a nonlinear link function, and ϵ_t is a white-noise process. The $h(h \geq 1)$ step ahead conditional density of y_t is ,then, $p(y_{t+h}|F_t, \theta)$, where F_t denotes the information set at time t and θ is the vector of model parameters that govern h_t. Model comparison is performed using the weighted Continuous Ranked Probability Score (wCRPS) criterion of Gneiting and Ranjan (2011).

5.2 Data

The analysis is divided in two parts. First, the time series of daily Dow Jones (DJ) returns from March 3, 1902 to April 15, 2016 is considered, for a total of 30921 daily observations. The sample contains numerous historical events, such as World Wars (missing observations are removed), economic depressions, oil and financial crisis, the Arab-Israeli Wars and the Great Recession in 2008.

Second, shorter data sets from January 5, 2004 to December 31, 2014 are considered. These data sets consist of 2768 observations for a cross sectional dimension of 432 firms from the S&P500 index. The index composition considered is that of January 1, 2016. Using the Global Industry Classification Standard (GCIS) groups of firms are classified in 11 economic sectors, see

Table 5.1: Information regarding sectors in the S&P500 index.

Sector	Mcm	Total debt	Leverage	Nof
Consumer discretionary	12.41	3.15	77.85	70
Consumer staples	28.29	6.64	80.52	32
Energy	14.39	6.97	58.11	32
Financials	21.34	7.58	67.77	56
Health care	25.76	6.36	60.10	53
Industrials	15.46	3.41	75.59	57
Information technology	18.32	2.63	54.13	54
Materials	14.90	5.57	109.65	23
Real Estate	18.73	6.49	116.64	24
Telecommunication services	16.57	20.23	143.85	5
Utilities	19.41	13.88	119.72	26

Table 5.1. The number of firms - "Nof" in Table 5.1 - for each sector oscillates between twenty and sixty, apart from telecommunication services, which contains only five firms and consumer discretionary, which displays seventy firms.

Overall, the sample is heterogenous with respect to both the level of capitalization and the use of debt, as shown in Table where "Mcm" 5.1 stands for Median market capitalization. Price series are converted into the logarithmic percentage return series, using $100 \times \ln(P_t/P_{t-1})$, where P_t and P_{t-1} are the prices respectively at time t and $t-1$.

5.3 Models

5.3.1 GARCH model

As regards observation-driven conditional volatility models, the conditional volatility h_t depends on its own lagged values and the lagged values of the observed return y_t in a deterministic way. In this analysis, the simplest specification considered is the t-EGARCH(p,q) model introduced in Nelson (1991). In particular, Catania and Nonejad find that specifying $p = 1$ and $q = 1$ generally works well, and increasing the number of parameters does not add any significant improvement in terms of generating density forecasts. Thus, the t-EGARCH(1,1) model is given by:

$$y_t = e^{(h_t/2)}\epsilon_t \qquad \epsilon_t \sim St(\nu), \tag{5.1}$$

$$h_{t+1} = \gamma + \alpha(|\epsilon_t| - E[|\epsilon_t|]) + \rho\epsilon_t + \beta h_t, \tag{5.2}$$

where $St(\nu)$ stands for the Student-t distribution with $\nu > 2$ degrees of freedom. The choice of the Student's t distribution in equation (5.1) follows Bollerslev's intent of capturing the excess conditional kurtosis exhibited by financial returns. In equation (5.2), γ is the level of conditional log-volatility, α determines the impact of past observations, β determines the impact of past conditional volatilities and ρ regulates the leverage effect. The inclusion of the term $E[|\epsilon_t|]$ implies that $|\epsilon_t| - E[|\epsilon_t|]$ is a Martingale difference sequence with respect to F_{t-1}. Thus, the unconditional log-volatility level, $\bar{h}$, is given by $\frac{\gamma}{1-\beta}$. Moreover, as previously mentioned, the exponential link in equation (5.1) ensures that the only constraint imposed in the estimation procedure to provide stationarity of h_t is $|\beta| < 1$. Clearly, if ρ is negative the leverage effect is accounted for by the model, as the magnitude of the response to a negative innovation is $\alpha - \rho$. The t-EGARCH (1,1) specification that does not incorporate the leverage effect has $\rho = 0$.

5.3.2 GAS model

The example of a GAS model in this analysis is the Beta-t-EGARCH(p, q) model introduced by Harvey (2013). This model specification is based on the previously introduced t-EGARCH model and shares its most attractive features, such as the exponential link function (see equation (5.1)) and the heavy-tailed Student's t distribution. Moreover, the Beta-t-EGARCH model allows the variance, or scale, to be driven by an equation that depends on the conditional score of the last observation. As a result, Beta-t-EGARCH specification belongs to the DCS or GAS framework, and represents a model that is more flexible and practically useful (Harvey and Sucarrat, 2014).

Accordingly, the classic EGARCH specification in Nelson (1991) is sensitive to outliers and has the unfortunate theoretical property that unconditional moments do not exist when the conditional distribution is Student's t. The GAS model resolves these problems and, in doing so, yields a specification, which is open to the development of a full asymptotic theory for the distribution of the maximum likelihood estimator. (Harvey and Lange, 2018).

Similar to t-EGARCH, p and q are set as 1, thus, the Beta-t-EGARCH(1,1) model is given by:

$$y_t = e^{(h_t)}\epsilon_t \qquad \epsilon_t \sim St(\nu) \tag{5.3}$$

$$h_{t+1} = \gamma + \alpha\mu_t + sgn(-\epsilon_t)\rho(\mu_t + 1) + \beta h_t, \tag{5.4}$$

where in (5.4) $sgn(x)$ returns the sign of the variable x and μ_t is the score of the distribution of y_t with respect to h_t given as $\mu_t = ((\nu + 1)\epsilon_t^2/((\nu - 2) + \epsilon_t^2)) - 1$. According to Harvey (2013), the model reported in (5.3) and (5.4) has the nice property of being more robust to extreme observations compared to the simpler t-EGARCH. The inclusion of the leverage effect for the Beta-t-EGARCH model is also more intuitive then for t-EGARCH. Indeed, since $\mu_t + 1$ is always positive if $\epsilon_t < 0$, then the volatility level at time $t + 1$ is increased by an amount $\rho(\mu_t + 1)$ if $\rho > 0$.

The robustness property of Beta-t-EGARCH relative to t-EGARCH can be easily seen by comparing the response of h_t to ϵ_t. Indeed, taking apart the leverage effect controlled by ρ, the response of h_t for t-EGARCH is piece-wise linear in ϵ_t, while for Beta-t-EGARCH is a smooth function bounded by ν.

Furthermore, as reported in Harvey and Sucarrat (2014), an announcement made by the computer firm Apple illustrates the robustness of Beta-t-EGARCH. On Thursday, September 28, 2000 a profit warning was issued (CNN Money, see http://money.cnn.com/200/09/29/markets/techwrap/, retrieved November 1, 2011), which led the value of the stock to plunge from an end-of-trading value of \$26.75 to \$12.88 on the subsequent day. In terms of volatility, this fall was a one-off event, since it apparently had no effect on the variability of the price changes of the following days. Firstly, volatility forecasts were performed using the t-GARCH model and the Beta-t-EGARCH model. Following this process, the forecasts were compared to observed absolute returns in order to verify their accuracy. As a result, the t-GARCH forecast of one-step volatility exceeded absolute returns for almost two months after the event. Thus, a clear-cut example of forecast failure. Conversely, the Beta-t-EGARCH forecasts remained in the same range of variation as the absolute returns. Thus, a robust prediction.

5.3.3 Semi-parametric GARCH model

The t-EGARCH and Beta-t-EGARCH models both assume the same parametric specification for the innovation ϵ_t, namely $\epsilon_t \sim St(\nu)$. In order to investigate the role of the leverage effect in a semi-parametric framework, the semi-parametric EGARCH model is also considered. Particularly, the SPEGARCH(1,1) is given by:

$$y_t = e^{(h_t/2)}\epsilon_t \qquad \epsilon_t \sim IID(0,1) \tag{5.5}$$

$$h_{t+1} = \gamma + \alpha|\epsilon_t| + \rho(\epsilon_t) + \beta h_t, \tag{5.6}$$

where $\epsilon_t \sim IID(0,1)$ means that ϵ_t is an IID sequence of white-noise shocks with mean 0 and variance 1. It is important to notice that, contrary to (5.1) - (5.2), (5.6) does not include $E[|\epsilon_t|]$ due to the obvious lack of parametrical assumption of ϵ_t in (5.5).

5.3.4 SV model

In the context of parameter-driven models, the stochastic volatility (SV) model is considered. The SV model is given by:

$$y_t = e^{(h_t/2)}\epsilon_t, \qquad \epsilon_t \sim N(0,1) \tag{5.7}$$

$$h_{t+1} = \mu + \phi h_t + \sigma\eta_t, \qquad \eta_t \sim N(0,1), \tag{5.8}$$

where μ is the level of conditional log-volatility, ϕ and σ, respectively denote the persistence and the conditional volatility of volatility. The leverage effect is incorporated assuming correlation between ϵ_t and η_t, i.e. $E[\eta_t\epsilon_t] = \rho$ and $|\rho| < 1$. Thus, a negative shock at time t increases volatility at time $t+1$.

For each model presented in this section, a version with leverage and a version without are considered, as described in Table 5.2.

6 Empirical results

This section presents the empirical results obtained by the evaluation of the density forecasts produced by the models in Table 5.2. In this analysis, the models are used to obtain and evaluate $h = 1, 5$, and 20 days ahead forecasts. The multi-step ahead distribution ($h > 1$) is estimated by standard Monte Carlo simulation techniques.

According to Catania and Nonejad, 10000 observations from the one step ahead distribution, thus $y_{t+h}|F_t$, are initially sampled. Subsequently, for each $l = 2, \ldots, h$ the time-varying parameters are updated using the variance updating equations and the sampling from the one-step ahead distribution, conditional on previous simulated draws, is iterated until the end of the forecast horizon. Accordingly, last simulated observations are draws from the distribution

Table 5.2: This table lists the model labels together with a brief description of the models. The acronym 'NL' denotes 'no leverage'.

Model	Description
t-EGARCH	t-EGARCH model
t-EGARCH-NL	t-EGARCH without the leverage effect, i.e. $\rho = 0$ in the estimation procedure
Beta-t-EGARCH	Beta-t-EGARCH model
Beta-t-EGARCH-NL	Beta-t-EGARCH without the leverage effect, i.e. $\rho = 0$ in the estimation procedure
SPEGARCH	semi-parametric EGARCH model
SPEGARCH-NL	SPEGARCH without the leverage effect, i.e. $\rho = 0$ in the estimation procedure
SV	Stochastic Volatility model
SV-NL	Stochastic volatility model without the leverage effect, i.e. $\rho = 0$ in the estimation procedure

$y_{t+h}|F_t$. In the SPEGARCH case, in which there is no parametrical assumption on ϵ_t, random draws are sampled from past observations.

Each forecast is based on the re-estimation of the underlying model using a rolling window of 1000 observations, which corresponds to roughly 4 years of data. At each step, as a new observation arrives, the model is re-estimated and a density forecast h periods ahead is computed using the recursive method of forecasting.

As regards DJ data, the out-of-sample period consists of 29921 observations running from July 7, 1905 until April 15, 2016.

In the context of S&P500 equity returns (given the remarkable number of return series considered), the models are re-estimated every 40 days instead of each day in order to reduce the computational burden. Accordingly, the parameters are fixed within the 40 days window, and only data are updated. The out-of-sample period consists of 1768 observations running from December 24, 2007 until December 31, 2014.

6.1 Dow Jones

6.1.1 Leverage effect

The analysis starts from a pairwise model comparison, where the version of the models that considers the leverage effect is compared with its non-leverage counterpart. The results in Table 6.1 report the ratios of awCRPS for the model with the leverage effect over the version without the leverage effect. For instance, the column "t-EGARCH" reports the average wCRPS of t-EGARCH over t-EGARCH-NL for various choices of $w(z)$, see Table 4.1, at $h = 1, 5$ and 20. The apexes a, b, and c indicate rejection of the null-hypothesis of equal predictive ability according to the Diebold and Mariano (1995) test at 1%, 5%, and 10%, respectively.

Considering the uniform case in which $w(z) = 1$ weights equally across the conditional distribution, at $h = 1$ for each model-type, the version that accounts for the leverage effect is able to generate statistically significant more accurate density forecasts than the version without leverage effect. On average, reductions in awCRPS around 3 to 5 percent are obtained, in fact

Table 6.1: Pairwise density forecast comparison using daily DJ returns.

Model	t-EGARCH	Beta-t-EGARCH	SPEGARCH	SV
$h = 1$				
Uniform	$0.997^{(a)}$	$0.997^{(a)}$	$0.997^{(a)}$	$0.995^{(a)}$
Center	$0.998^{(a)}$	$0.998^{(a)}$	$0.998^{(a)}$	$0.997^{(a)}$
Tails	$0.994^{(a)}$	$0.995^{(a)}$	$0.994^{(a)}$	$0.992^{(a)}$
Tail-r	$0.996^{(a)}$	$0.996^{(a)}$	$0.995^{(a)}$	$0.994^{(a)}$
Tail-l	$0.998^{(a)}$	$0.998^{(a)}$	$0.998^{(a)}$	$0.997^{(a)}$
$h = 5$				
Uniform	$0.996^{(a)}$	$0.999^{(b)}$	$0.995^{(a)}$	$0.999^{(a)}$
Center	$0.997^{(a)}$	1.000	$0.997^{(a)}$	$0.999^{(a)}$
Tails	$0.993^{(a)}$	$0.998^{(a)}$	$0.992^{(a)}$	$0.997^{(a)}$
Tail-r	$0.995^{(a)}$	$0.999^{(c)}$	$0.995^{(a)}$	$0.998^{(a)}$
Tail-l	$0.996^{(a)}$	$0.999^{(c)}$	$0.995^{(a)}$	$0.999^{(a)}$
$h = 20$				
Uniform	$0.986^{(a)}$	$1.001^{(a)}$	$0.985^{(a)}$	$0.999^{(a)}$
Center	$0.987^{(a)}$	$1.001^{(a)}$	$0.986^{(a)}$	$0.999^{(a)}$
Tails	$0.983^{(a)}$	$1.001^{(a)}$	$0.983^{(a)}$	0.999
Tail-r	$0.986^{(a)}$	$1.001^{(a)}$	$0.986^{(a)}$	$0.999^{(b)}$
Tail-l	$0.986^{(a)}$	$1.001^{(a)}$	$0.984^{(a)}$	$1.000^{(c)}$

the ratio awCRPS is 0.997 for t-EGARCH, SPEGARCH and Beta-t-EGARCH and 0.995 for SV. At $h = 5$ and $h = 20$, incorporating the leverage effect does not result in any major change for Beta-t-EGARCH and SV. Conversely, Beta-t-EGARCH-NL statistically outperforms Beta-t-EGARCH at $h = 20$. Differently, t-EGARCH and SPEGARCH dominate their no leverage counterpart by more than 10 percent at $h = 20$, with a ratio awCRPS 0.986 and 0.985, respectively.

Referring to the results obtained with different choices of $w(z)$, it is possible to highlight the role of the leverage effect. The models that incorporate the leverage effect tend to predict the tails of the conditional distribution better than the models without the leverage effect. In particular, at $h = 1$ all levered versions outperform their no levered counterpart by 5 percent or more. Conversely, fewer improvements are obtained once the center of the predictive density is emphasized. This is particularly notable for SV at $h = 1$ as compared to SV-NL, where the improvements are around 8 percent at the tails, 5 percent in the uniform case and 3 percent when the center is focused.

Increasing the forecast horizon, there are major differences between the different model-types. SPEGARCH and t-EGARCH are able to predict the tails and the center of the conditional distribution significantly better than their non-leverage counterpart. At $h = 20$ the two levered GARCH specifications outperform t-EGARCH-NL and SPEGARCH-NL by 15 percent on average, with a reduction of 17 percent if the tails are emphasized. Conversely, the trend is reversed for Beta-t-EGARCH and SV. As h is increased, the improvements are of smaller magnitudes and Beta-t-EGARCH-NL outperforms its levered counterpart at $h = 20$.

6.1.2 Business cycle

According to Catania and Nonejad, the connection between the business cycle and relative forecast performance can potentially shed light on the sources of the predictive gains provided by the model with the leverage effect. Therefore, it is useful to use the long DJ returns series to understand the relationship between the historical events and the predictive gains furnished by the models that account the leverage effect.

Catania and Nonejad find that the gains from the model with the leverage effect are typically concentrated near the peak, the highest point between the end of an economic expansion and the start of a contraction, and the trough, the period marking the end of declining economic activity and the transition to expansion. The former is very evident in the context of the Great Recession, where notable gains are obtained in favor of the model with the leverage effect. The latter is very evident, with regards to the period after the recession in the early 1980s and during the 2000s. Differently both specification (i.e. with and without the leverage) generate similar density forecasts during calm periods.

Moreover, Catania and Nonejad underline the differences between models that account for the leverage effect and their relative performance. Beta-t-EGARCH and Beta-t-EGARCH-NL generate very similar density forecasts, except for the Great Recession period where the levered version is better. Conversely, SV gains over SV-NL are concentrated during the 1950s and 1960s.

6.1.3 Models comparison

Other interesting results are obtained from the comparison between different model-types. In particular, Table 6.2 reports the ratios of awCRPS for the range of the models considered over the t-EGARCH-NL. Different forecast horizons, $h = 1, 5, 20$ and also different weight functions, $w(z)$, are considered. The apexes a, b, and c indicate rejection of the null-hypothesis of equal predictive ability relative to t-EGARCH-NL according to the Diebold and Mariano (1995) test at 1%, 5%, and 10%, respectively. In order to save space Beta-t-EGARCH-NL and Beta-t-EGARCH are indicated by β-t-EG-NL and β-t-EG, respectively.

At $h = 1$, all models except SV-NL and SPEGARCH-NL generate statistically significant more accurate density forecasts than the benchmark t-EGARCH-NL model. Beta-t-EGARCH shows the highest reductions in awCRPS, followed by SV and t-EGARCH. In particular, Beta-t-EGARCH outperforms t-EGARCH-NL by 9 percent when the weight function emphasizes the tails of the conditional distribution.

At $h = 5$, Beta-t-EGARCH is again the top performer, followed by Beta-t-EGARCH-NL and SV. The two versions of the GAS model outperform the benchmark model by at least 9 percent for each weight function considered. Moreover, Beta-t-EGARCH-NL shows higher reductions in awCRPS compared to t-EGARCH and SPEGARCH and very similar density forecasts as SV. Furthermore, both the SV versions outperform the GARCH specifications. Accordingly, compared to Beta-t-EGARCH and SV, t-EGARCH and SPEGARCH are not able to generate more accurate density forecasts.

At $h = 20$, Beta-t-EGARCH-NL is now the top performer. As previously observed (see Table 6.1), the levered version of Beta-t-EGARCH is outperformed by Beta-t-EGARCH-NL at the furthest forecast horizon. Beta-t-EGARCH and Beta-t-EGARCH-NL show reductions in

Table 6.2: Density forecast comparison using Dow Jones returns.

Model	t-EGARCH	β-t-EG-NL	β-t-EG	SPEGARCH-NL	SPEGARCH	SV-NL	SV
$h = 1$							
Uniform	$0.997^{(a)}$	$0.998^{(a)}$	$0.995^{(a)}$	$1.001^{(a)}$	$0.998^{(a)}$	$1.001(a)$	$0.997^{(a)}$
Center	$0.998^{(a)}$	$0.998^{(a)}$	$0.996^{(a)}$	$1.001^{(b)}$	$0.999^{(a)}$	$1.001(a)$	$0.998^{(a)}$
Tails	$0.994^{(a)}$	$0.997^{(a)}$	$0.991^{(a)}$	$1.002^{(b)}$	$0.996^{(a)}$	1.000	$0.993^{(a)}$
Tail-r	$0.996^{(a)}$	$0.998^{(a)}$	$0.994^{(a)}$	$1.002^{(a)}$	$0.997^{(a)}$	$1.001(a)$	$0.996^{(a)}$
Tail-l	$0.998^{(a)}$	$0.998^{(a)}$	$0.995^{(a)}$	1.001	$0.998(c)$	$1.001^{(c)}$	$0.997^{(a)}$
$h = 5$							
Uniform	$0.996^{(a)}$	$0.991^{(a)}$	$0.990^{(a)}$	$1.002^{(a)}$	$0.997^{(a)}$	$0.994^{(a)}$	$0.992^{(a)}$
Center	$0.997^{(a)}$	$0.994^{(a)}$	$0.994^{(a)}$	$1.001^{(a)}$	$0.998^{(a)}$	$0.996^{(a)}$	$0.995^{(a)}$
Tails	$0.993^{(a)}$	$0.985^{(a)}$	$0.983^{(a)}$	$1.002^{(a)}$	$0.994^{(a)}$	$0.988^{(a)}$	$0.985^{(a)}$
Tail-r	$0.995^{(a)}$	$0.991^{(a)}$	$0.991^{(a)}$	$1.001^{(b)}$	$0.996^{(a)}$	$0.993^{(a)}$	$0.992^{(a)}$
Tail-l	$0.996^{(a)}$	$0.991^{(a)}$	$0.990^{(a)}$	$1.002^{(a)}$	$0.998^{(a)}$	$0.994^{(a)}$	$0.993^{(a)}$
$h = 20$							
Uniform	$0.986^{(a)}$	$0.968^{(a)}$	$0.969^{(a)}$	$1.002^{(a)}$	$0.987^{(a)}$	$0.973^{(a)}$	$0.973^{(a)}$
Center	$0.987^{(a)}$	$0.973^{(a)}$	$0.974^{(a)}$	$1.002^{(a)}$	$0.988^{(a)}$	$0.977^{(a)}$	$0.976^{(a)}$
Tails	$0.983^{(a)}$	$0.958^{(a)}$	$0.959^{(a)}$	$1.002^{(a)}$	$0.984^{(a)}$	$0.965^{(a)}$	$0.964^{(a)}$
Tail-r	$0.986^{(a)}$	$0.968^{(a)}$	$0.969^{(a)}$	1.000	$0.986^{(a)}$	$0.973^{(a)}$	$0.972^{(a)}$
Tail-l	$0.986^{(a)}$	$0.968^{(a)}$	$0.969^{(a)}$	$1.003^{(a)}$	$0.988^{(a)}$	$0.973^{(a)}$	$0.973^{(a)}$

awCRPS around 40 percent when the tails are emphasized and around 30 percent in the other cases. SV and SV-NL have a predictive gain around the 35 percent in awCRPS relative to the tails and around the 25 percent in the other cases compared to t-EGARCH-NL. As regards t-EGARCH and SPEGARCH, they outperform their non-leverage counterparts with gains that increase relative to farther forecast horizons.

6.1.4 Frequency

The precedent analysis is repeated changing the data frequency. Accordingly, weekly DJ returns from 1921 to 2016 are considered and using the models in Table 5.1 density forecasts $h = 1$ (a week), $h = 4$ (a month) and, $h = 12$ (a quarter) step-ahead are generated. The models comparison is reported in Table 6.3. The apexes a, b, and c indicate rejection of the null-hypothesis of equal predictive ability relative to t-EGARCH-NL according to the Diebold and Mariano (1995) test at 1%, 5%, and 10%, respectively. In order to save space Beta-t-EGARCH-NL and Beta-t-EGARCH are indicated by β-t-EG-NL and β-t-EG, respectively.

At $h = 1$, Beta-t-EGARCH is the top performer, however, all models that account for the leverage effect produce very similar results and outperform their non-leverage counterpart. When the weight function emphasizes the tails of the conditional distribution, Beta-t-EGARCH-NL is able to generate density forecasts as accurate as ones generated by levered models.

At $h = 4$, t-EGARCH and SPEGARCH and their non-leverage counterpart generate similar density forecasts, while Beta-t-EGARCH-NL and SV-NL outperform their levered specifications. Beta-t-EGARCH-NL is also the top performer providing the highest reductions in awCRPS, in particular when the tails are emphasized.

At $h = 12$, the same pattern is observed. Beta-t-EGARCH-NL is again the top performer, outperforming Beta-t-EGARCH by about 2 percent.

In conclusion, it appears that the leverage model's ability to outperform its non-leverage

Table 6.3: Density forecast comparison using Dow Jones returns.

Model	t-EGARCH	β-t-EG-NL	β-t-EG	SPEGARCH-NL	SPEGARCH	SV-NL	SV
$h = 1$							
Uniform	$0.997^{(a)}$	$0.998^{(a)}$	$0.996^{(a)}$	1.000	$0.997^{(b)}$	1.000	$0.997^{(a)}$
Center	$0.999^{(b)}$	1.000	0.999	$0.999^{(a)}$	0.998	1.001	$0.999^{(b)}$
Tails	$0.996^{(a)}$	$0.996^{(a)}$	$0.994^{(a)}$	1.001	$0.997^{(a)}$	1.000	$0.996^{(a)}$
Tail-r	$0.996^{(a)}$	$0.998^{(b)}$	$0.995^{(a)}$	1.001	$0.997^{(b)}$	$1.001^{(c)}$	$0.997^{(a)}$
Tail-l	0.998	$0.997^{(b)}$	$0.997^{(b)}$	1.000	0.998	0.999	$0.997^{(b)}$
$h = 4$							
Uniform	1.000	$0.991^{(a)}$	$0.993^{(a)}$	$1.003^{(a)}$	1.000	$0.995^{(a)}$	$0.996^{(a)}$
Center	1.001	$0.998^{(a)}$	$0.999^{(b)}$	1.000	0.999	$0.999^{(b)}$	1.000
Tails	0.999	$0.986^{(a)}$	$0.988^{(a)}$	$1.005^{(a)}$	1.001	$0.992^{(a)}$	$0.993^{(a)}$
Tail-r	1.000	$0.991^{(a)}$	$0.992^{(a)}$	$1.003^{(a)}$	1.001	$0.996^{(a)}$	$0.997^{(b)}$
Tail-l	1.000	$0.992^{(a)}$	$0.993^{(a)}$	$1.003^{(a)}$	1.000	$0.994^{(a)}$	$0.996^{(a)}$
$h = 12$							
Uniform	$0.994^{(a)}$	$0.972^{(a)}$	$0.975^{(a)}$	$1.004^{(a)}$	$0.996^{(a)}$	$0.978^{(a)}$	$0.980^{(a)}$
Center	$0.995^{(a)}$	$0.987^{(a)}$	$0.988^{(a)}$	1.001	$0.995^{(a)}$	$0.989^{(a)}$	$0.990^{(a)}$
Tails	$0.993^{(a)}$	$0.961^{(a)}$	$0.965^{(a)}$	$1.007^{(a)}$	$0.996^{(a)}$	$0.969^{(a)}$	$0.972^{(a)}$
Tail-r	$0.994^{(a)}$	$0.973^{(a)}$	$0.975^{(a)}$	$1.003^{(a)}$	$0.996^{(a)}$	$0.979^{(a)}$	$0.980^{(a)}$
Tail-l	$0.994^{(a)}$	$0.971^{(a)}$	$0.974^{(a)}$	$1.005^{(a)}$	$0.996^{(a)}$	$0.977^{(a)}$	$0.979^{(a)}$

counterpart depends on the choice of data frequency.

6.2 S&P 500

6.2.1 Leverage effect

The same models are used to generate density forecasts at $h = 1$, $h = 5$, and $h = 20$ using daily S&P500 equity returns. Differently from DJ analysis, Catania and Nonejad report the results as in Table 6.4, which indicate the percentages of times where the model with the leverage effect generates more accurate density forecasts than the model without the leverage effect for each model-type. According to Table 4.1, different weight functions, $w(z)$, are considered. The change of methodology for the representation of the results is not explicitly described by the authors.

For each model-type, the specification that accounts for the leverage effect tends to generate more accurate density forecasts than the model without the leverage effect. Considering the uniform case, thus $w(z) = 1$, the improvements from considering the leverage effect follow the same patterns observed with DJ returns series. Accordingly, the improvements decrease with regards to Beta-t-EGARCH and SV as h increases, whereas the opposite is true for t-EGARCH and SPEGARCH.

The most interesting results are obtained when $w(z) = 1 - \Phi(z)$, namely the left-tail is emphasized by the weight function. Indeed, the percentages in this case are considerably higher than the remaining cases. This is primarily due to the fact that compared to DJ, equity return series contain more frequent negative extreme observations.

Lastly, in the remaining specifications of the weight function, results obtained are very similar as the uniform case.

Table 6.4: Pairwise density forecast comparison using daily S&P500 equity returns.

Model	t-EGARCH	Beta-t-EGARCH	SPEGARCH	SV
$h = 1$				
Uniform	82	89	74	89
Center	77	79	69	85
Tails	82	89	75	89
Tail-r	46	57	39	61
Tail-l	91	96	86	94
$h = 5$				
Uniform	88	70	84	76
Center	78	62	77	72
Tails	90	73	84	77
Tail-r	71	59	72	66
Tail-l	94	79	90	82
$h = 20$				
Uniform	99	51	94	45
Center	99	33	95	38
Tails	99	54	93	47
Tail-r	97	44	92	41
Tail-l	100	53	95	53

6.2.2 Models comparison

Given that models that account for the leverage effect pairwise outperform their non-leverage counterparts, it appears appropriate to compare density forecasts between the levered versions of the models. Table 6.5 reports the percentages of time where each leverage model is able to generate more accurate density forecasts than the other leverage models when $w(z) = 1$.

Here, we find that Beta-t-EGARCH is the clear top performer. Regardless the forecast horizon, Beta-t-EGARCH is able to generate more accurate density forecasts then the other leverage models for at least the 98 percent of time. Selecting the second best model is more difficult. t-EGARCH slightly outperforms SV at $h = 1$, whereas it is outperformed by SV at $h = 5$ and $h = 20$. SPEGARCH is always outperformed by the other models, except at $h = 20$ where it outperforms t-EGARCH.

Table 6.6 reports the percentages of time where each leverage model is able to generate more accurate density forecasts than the other leverage models when $w(z) = 1 - \Phi(z)$, namely the left tail is emphasized. Analyzing the left tail, which is of interest for risk managers, results again confirm the superior performance of Beta-t-EGARCH. The same pattern previously observed occurs for the the second best model. Increasing the forecast horizon, SV outperforms the two GARCH models.

6.2.3 Economic sectors

Next, the difference between the model that accounts for the leverage effect and the model without the leverage effect is analyzed by decomposing equities according to sectors reported in Table 5.1. Tables 6.7 and 6.8 report the results of the pairwise comparison in the uniform case and when the left tail is emphasized respectively.

Table 6.5: Density forecast comparison using daily S&P500 equity returns, $w(z) = 1$.

Model	t-EGARCH	Beta-t-EGARCH	SPEGARCH	SV
$h = 1$				
t-EGARCH		1	81	54
Beta-t-EGARCH	99		99	98
SPEGARCH	19	1		33
SV	46	2	67	
$h = 5$				
t-EGARCH		0	66	43
Beta-t-EGARCH	100		99	99
SPEGARCH	34	1		34
SV	57	1	66	
$h = 20$				
t-EGARCH		1	45	24
Beta-t-EGARCH	99		98	100
SPEGARCH	55	2		27
SV	76	0	73	

Table 6.6: Density forecast comparison using daily S&P500 equity returns, $w(z) = 1 - \Phi(z)$.

Model	t-EGARCH	Beta-t-EGARCH	SPEGARCH	SV
$h = 1$				
t-EGARCH		4	82	59
Beta-t-EGARCH	96		98	93
SPEGARCH	18	2		36
SV	41	7	64	
$h = 5$				
t-EGARCH		0	65	41
Beta-t-EGARCH	100		100	98
SPEGARCH	35	0		36
SV	59	2	64	
$h = 20$				
t-EGARCH		0	42	19
Beta-t-EGARCH	100		100	100
SPEGARCH	58	0		23
SV	81	0	77	

Table 6.7: Density forecast comparison using daily S&P500 equity returns by sectors, $w(z) = 1$.

Model	t-EGARCH	Beta-t-EGARCH	SPEGARCH	SV
$h = 1$				
Consumer discretionary	76	89	80	89
Consumer staples	84	91	66	84
Energy	100	81	84	78
Financials	77	80	71	88
Health care	91	92	68	94
Industrials	91	98	88	98
Information technology	81	100	70	94
Materials	83	91	78	91
Real Estate	67	71	62	67
Telecommunication services	40	60	20	80
Utilities	65	92	73	88
$h = 5$				
Consumer discretionary	84	67	83	81
Consumer staples	78	75	78	78
Energy	84	38	84	41
Financials	95	70	93	79
Health care	92	89	83	89
Industrials	95	82	86	89
Information technology	80	83	74	80
Materials	83	74	78	74
Real Estate	88	29	79	38
Telecommunication services	80	80	80	100
Utilities	100	54	96	65
$h = 20$				
Consumer discretionary	99	40	94	47
Consumer staples	97	56	88	59
Energy	100	38	97	22
Financials	100	55	98	61
Health care	98	66	85	47
Industrials	98	53	95	47
Information technology	100	65	89	37
Materials	100	26	100	30
Real Estate	100	25	96	21
Telecommunication services	100	60	100	60
Utilities	100	58	100	62

Table 6.8: Density forecast comparison using daily S&P500 equity returns by sectors, $w(z) = 1 - \Phi(z)$.

Model	t-EGARCH	Beta-t-EGARCH	SPEGARCH	SV
$h = 1$				
Consumer discretionary	86	96	90	91
Consumer staples	91	97	78	88
Energy	100	91	91	81
Financials	88	91	84	98
Health care	94	98	81	96
Industrials	93	98	89	96
Information technology	93	98	89	96
Materials	87	96	83	96
Real Estate	88	96	79	96
Telecommunication services	60	80	60	100
Utilities	100	100	100	96
$h = 5$				
Consumer discretionary	94	76	90	83
Consumer staples	91	78	84	78
Energy	97	62	91	66
Financials	96	80	98	80
Health care	100	92	91	92
Industrials	96	89	95	93
Information technology	91	85	81	87
Materials	87	78	87	87
Real Estate	88	50	83	50
Telecommunication services	80	80	80	80
Utilities	100	77	100	81
$h = 20$				
Consumer discretionary	100	49	97	51
Consumer staples	97	56	91	53
Energy	100	44	97	31
Financials	100	55	98	73
Health care	98	74	85	57
Industrials	100	53	95	49
Information technology	100	59	93	46
Materials	100	30	100	43
Real Estate	100	25	96	33
Telecommunication services	100	60	100	80
Utilities	100	62	100	69

Table 6.9: Density forecast comparison from S&P500 equity returns, $h = 1$, $w(z) = 1$.

Model	t-EGARCH	β-t-EG-NL	β-t-EG	SPEGARCH-NL	SPEGARCH	SV-NL	SV
Alcoa	1.001	$0.993^{(a)}$	$0.993^{(a)}$	$1.004^{(a)}$	$1.003^{(a)}$	$0.997^{(b)}$	0.998
American Express	1.001	$0.991^{(a)}$	$0.992^{(a)}$	$1.002^{(b)}$	1.002	$0.992^{(a)}$	$0.996^{(b)}$
Boeing	$0.997^{(a)}$	$0.995^{(a)}$	$0.993^{(a)}$	1.000	$0.996^{(a)}$	$0.998^{(c)}$	1.001
Caterpillar	$0.998^{(c)}$	$0.993^{(a)}$	$0.991^{(a)}$	$1.004^{(a)}$	1.000	$0.998^{(c)}$	1.001
Chevron	$0.997^{(a)}$	$0.992^{(a)}$	$0.993^{(a)}$	0.999	$0.995^{(b)}$	$0.998^{(c)}$	1.000
Walt Disney	$0.996^{(a)}$	$0.992^{(a)}$	$0.989^{(a)}$	0.999	$0.995^{(a)}$	$0.998^{(c)}$	1.000
General Electric	$0.998^{(b)}$	$0.990^{(a)}$	$0.987^{(a)}$	1.000	$0.998^{(c)}$	$0.992^{(a)}$	$0.996^{(b)}$
IBM	$0.998^{(b)}$	$0.995^{(a)}$	$0.992^{(a)}$	1.001	$0.998^{(c)}$	$0.998^{(c)}$	1.000
Intel	$1.003^{(a)}$	$0.993^{(a)}$	$0.993^{(a)}$	$1.001^{(b)}$	$1.002^{(b)}$	1.001	$1.002^{(b)}$
Johnson & Johnson	$0.996^{(a)}$	0.999	$0.993^{(a)}$	0.999	$0.996^{(a)}$	$0.995^{(a)}$	1.001
JPMorgan	$1.005^{(b)}$	$0.993^{(a)}$	$0.991^{(a)}$	$1.005^{(a)}$	$1.003^{(c)}$	$0.991^{(a)}$	$0.995^{(a)}$
Coca-cola	$0.996^{(a)}$	$0.997^{(b)}$	$0.993^{(a)}$	$1.002^{(b)}$	0.999	$0.997^{(b)}$	$1.002^{(b)}$
McDonald's	0.999	$0.995^{(a)}$	$0.994^{(a)}$	1.000	0.999	1.000	1.001
Merck	$0.998^{(a)}$	$0.994^{(a)}$	$0.990^{(a)}$	$1.008^{(a)}$	$1.004^{(b)}$	0.999	1.000
Microsoft	1.000	$0.994^{(a)}$	$0.993^{(a)}$	$1.005^{(a)}$	$1.006^{(a)}$	1.001	1.002
Pfizer	$0.998^{(b)}$	$0.995^{(a)}$	$0.994^{(a)}$	1.000	1.036	1.000	1.001
Procter & Gamble	$0.997^{(a)}$	$0.993^{(a)}$	$0.991^{(a)}$	$1.003^{(a)}$	1.039	1.000	$1.002^{(b)}$
AT&T	1.001	$0.994^{(a)}$	$0.993^{(a)}$	$1.003^{(a)}$	$1.004^{(a)}$	$0.996^{(a)}$	0.998
Walmart	1.001	$0.998^{(c)}$	0.998	1.000	1.000	$1.003^{(b)}$	$1.003^{(a)}$
ExxonMobil	$0.997^{(a)}$	$0.998^{(b)}$	$0.997^{(a)}$	$1.002^{(b)}$	0.999	0.999	$1.004^{(a)}$

In the former case, generally, for each model, the specification with the leverage effect tends to outperform the one without the leverage effect. Moreover, similar trends as Table 6.4 are observed even when equities at sector-level are considered. Beta-t-EGARCH and SV generate more accurate density forecasts than their non-leverage counterparts, but their gains decrease as h increases. For t-EGARCH and SPEGARCH the opposite is the case. Furthermore, there are interesting variations within sector and forecast horizons. For instance, the percentages in favor of Beta-t-EGARCH and SV are higher in the financial and healthcare sectors and lower in real estate and telecommunications.

In the latter case, the pattern is very similar. The models that account for the leverage effect are able to predict the left tail of the conditional distribution better than their non-leverage counterparts.

6.2.4 Equities returns

Following Koopman, Lucas, and Schart (2016), density forecasts from twenty equities returns series are considered and the awCRPS ratios over the t-EGARCH-NL at $h = 1$, $h = 5$, and $h = 20$ are reported respectively in Tables 6.9, 6.10, 6.11. The apexes a, b, and c indicate rejection of the null-hypothesis of equal predictive ability according to the Diebold and Mariano (1995) test at 1%, 5% and 10 %, respectively. The weight function specification considered is $w(z) = 1$, thus the uniform case, with similar results for the other cases. In order to save space Beta-t-EGARCH-NL and Beta-t-EGARCH are indicated by β-t-EG-NL and β-t-EG, respectively.

At $h = 1$, both Beta-t-EGARCH versions consistently outperform the benchmark t-EGARCH-NL for all equities considered. Generally, Beta-t-EGARCH is the top performer, but for American Express and Chevron Beta-t-EGARCH-NL outperforms its levered version. As a result, compared to t-EGARCH, SPEGARCH and SV, Beta-t-EGARCH delivers the most

Table 6.10: Density forecast comparison from S&P500 equity returns, $h = 5$, $w(z) = 1$.

Model	t-EGARCH	β-t-EG-NL	β-t-EG	SPEGARCH-NL	SPEGARCH	SV-NL	SV
Alcoa	$1.002^{(b)}$	$0.986^{(a)}$	$0.987^{(a)}$	$1.003^{(a)}$	$1.002^{(b)}$	$0.994^{(a)}$	$0.993^{(a)}$
American Express	$0.995^{(c)}$	$0.979^{(a)}$	$0.983^{(a)}$	$1.004^{(a)}$	$0.994^{(b)}$	$0.984^{(a)}$	$0.984^{(a)}$
Boeing	$0.994^{(a)}$	$0.989^{(a)}$	$0.989^{(a)}$	1.000	$0.993^{(a)}$	$0.995^{(a)}$	0.998
Caterpillar	$0.997^{(b)}$	$0.990^{(a)}$	$0.988^{(a)}$	1.000	$0.996^{(b)}$	0.999	1.002
Chevron	$0.998^{(b)}$	$0.988^{(a)}$	$0.992^{(a)}$	0.998	$0.996^{(a)}$	1.000	1.000
Walt Disney	$0.991^{(a)}$	$0.984^{(a)}$	$0.982^{(a)}$	$0.995^{(a)}$	$0.992^{(a)}$	$0.993^{(a)}$	$0.997^{(c)}$
General Electric	$0.994^{(a)}$	$0.978^{(a)}$	$0.976^{(a)}$	$0.998^{(b)}$	$0.994^{(a)}$	$0.982^{(a)}$	$0.986^{(a)}$
IBM	$0.996^{(a)}$	$0.989^{(a)}$	$0.987^{(a)}$	$1.005^{(a)}$	$0.997^{(b)}$	$0.994^{(a)}$	$0.997^{(b)}$
Intel	$1.002^{(a)}$	$0.989^{(a)}$	$0.989^{(a)}$	0.999	1.001	0.999	0.999
Johnson & Johnson	$0.995^{(a)}$	$0.988^{(a)}$	$0.984^{(a)}$	1.000	$0.994^{(a)}$	$0.986^{(a)}$	$0.991^{(a)}$
JPMorgan	$1.006^{(a)}$	$0.978^{(a)}$	$0.979^{(a)}$	$1.009^{(a)}$	1.001	$0.983^{(a)}$	$0.984^{(a)}$
Coca-cola	$0.993^{(a)}$	$0.985^{(a)}$	$0.983^{(a)}$	$1.008^{(a)}$	$1.003^{(b)}$	$0.990^{(a)}$	$0.993^{(a)}$
McDonald's	1.000	$0.993^{(a)}$	$0.993^{(a)}$	0.999	1.000	1.002	1.001
Merck	$0.995^{(a)}$	$0.989^{(a)}$	$0.982^{(a)}$	$1.002^{(c)}$	$0.995^{(a)}$	$0.997^{(b)}$	$0.998^{(c)}$
Microsoft	1.001	$0.988^{(a)}$	$0.988^{(a)}$	$1.003^{(a)}$	$1.004^{(b)}$	$0.997^{(c)}$	0.998
Pfizer	$0.998^{(b)}$	$0.989^{(a)}$	$0.990^{(a)}$	$1.004^{(a)}$	1.039	0.998	1.000
Procter & Gamble	$0.993^{(a)}$	$0.983^{(a)}$	$0.980^{(a)}$	$1.002^{(b)}$	1.031	$0.995^{(a)}$	0.998
AT&T	$0.997^{(b)}$	$0.988^{(a)}$	$0.987^{(a)}$	$1.002^{(b)}$	1.001	$0.990^{(a)}$	$0.992^{(a)}$
Walmart	1.000	$0.997^{(c)}$	$0.997^{(c)}$	$1.002^{(b)}$	$1.002^{(c)}$	$1.008^{(a)}$	$1.008^{(a)}$
ExxonMobil	0.999	$0.995^{(a)}$	$0.996^{(b)}$	1.000	1.000	1.000	1.001

Table 6.11: Density forecast comparison from S&P500 equity returns, $h = 20$, $w(z) = 1$.

Model	t-EGARCH	β-t-EG-NL	β-t-EG	SPEGARCH-NL	SPEGARCH	SV-NL	SV
Alcoa	$0.994^{(a)}$	$0.953^{(a)}$	$0.953^{(a)}$	1.000	$0.988^{(a)}$	$0.970^{(a)}$	$0.969^{(a)}$
American Express	$0.959^{(a)}$	$0.929^{(a)}$	$0.934^{(a)}$	1.001	$0.959^{(a)}$	$0.942^{(a)}$	$0.941^{(a)}$
Boeing	$0.981^{(a)}$	$0.962^{(a)}$	$0.967^{(a)}$	$0.997^{(a)}$	$0.978^{(a)}$	$0.975^{(a)}$	$0.975^{(a)}$
Caterpillar	$0.983^{(a)}$	$0.969^{(a)}$	$0.970^{(a)}$	$0.991^{(a)}$	$0.974^{(a)}$	$0.985^{(a)}$	$0.984^{(a)}$
Chevron	$0.987^{(a)}$	$0.971^{(a)}$	$0.976^{(a)}$	0.999	$0.986^{(a)}$	$0.984^{(a)}$	$0.981^{(a)}$
Walt Disney	$0.977^{(a)}$	$0.960^{(a)}$	$0.960^{(a)}$	$0.994^{(a)}$	$0.982^{(a)}$	$0.975^{(a)}$	$0.974^{(a)}$
General Electric	$0.984^{(a)}$	$0.936^{(a)}$	$0.934^{(a)}$	$0.993^{(a)}$	$0.982^{(a)}$	$0.952^{(a)}$	$0.956^{(a)}$
IBM	$0.982^{(a)}$	$0.963^{(a)}$	$0.962^{(a)}$	$1.007^{(a)}$	$0.986^{(a)}$	$0.975^{(a)}$	$0.974^{(a)}$
Intel	$0.998^{(a)}$	$0.973^{(a)}$	$0.973^{(a)}$	$0.998^{(a)}$	$0.992^{(a)}$	$0.986^{(a)}$	$0.986^{(a)}$
Johnson & Johnson	$0.991^{(a)}$	$0.963^{(a)}$	$0.961^{(a)}$	$0.997^{(a)}$	$0.987^{(a)}$	$0.963^{(a)}$	$0.963^{(a)}$
JPMorgan	$0.989^{(a)}$	$0.920^{(a)}$	$0.919^{(a)}$	$1.005^{(a)}$	$0.978^{(a)}$	$0.928^{(a)}$	$0.932^{(a)}$
Coca-cola	$0.983^{(a)}$	$0.950^{(a)}$	$0.949^{(a)}$	$1.006^{(a)}$	$0.993^{(a)}$	$0.961^{(a)}$	$0.962^{(a)}$
McDonald's	$0.995^{(a)}$	$0.979^{(a)}$	$0.981^{(a)}$	1.000	$0.997^{(b)}$	$0.993^{(b)}$	$0.992^{(a)}$
Merck	$0.989^{(a)}$	$0.977^{(a)}$	$0.969^{(a)}$	$0.992^{(a)}$	$0.976^{(a)}$	$0.982^{(a)}$	$0.983^{(a)}$
Microsoft	$0.995^{(a)}$	$0.961^{(a)}$	$0.961^{(a)}$	$0.988^{(a)}$	$0.987^{(a)}$	$0.976^{(a)}$	$0.976^{(a)}$
Pfizer	$0.990^{(a)}$	$0.969^{(a)}$	$0.970^{(a)}$	$1.007^{(a)}$	1.037	$0.982^{(a)}$	$0.983^{(a)}$
Procter & Gamble	$0.985^{(a)}$	$0.960^{(a)}$	$0.955^{(a)}$	$0.987^{(a)}$	0.994	$0.974^{(a)}$	$0.975^{(a)}$
AT&T	$0.990^{(a)}$	$0.965^{(a)}$	$0.964^{(a)}$	0.999	$0.991^{(a)}$	$0.974^{(a)}$	$0.975^{(a)}$
Walmart	$0.995^{(a)}$	$0.985^{(a)}$	$0.985^{(a)}$	$1.004^{(a)}$	0.999	1.000	1.000
ExxonMobil	$0.990^{(a)}$	$0.975^{(a)}$	$0.977^{(a)}$	$0.996^{(a)}$	$0.990^{(a)}$	$0.983^{(a)}$	$0.982^{(a)}$

consistent pattern. Furthermore, SV-NL outperform SV in the majority of the equities.

At $h = 5$, Beta-t-EGARCH and Beta-t-EGARCH-NL are again the top performers. They outperform the other models by at least 5 percent for each equity. The magnitude of improvements of the GAS specification over t-EGARCH and SV increase as h increases. For instance, referring to Chevron, Beta-t-EGARCH outperform SV by 8 percent and t-EGARCH by 6 percent. For General Electric, the gains are about 10 percent and 14 percent over SV and t-EGARCH.

At $h = 20$, the pattern is very similar. Indeed, among the twenty returns series and the three forecast horizons considered, Beta-t-EGARCH-NL outperforms t-EGARCH, SPEGARCH and SV models. t-EGARCH is able to generate more accurate forecasts than t-EGARCH-NL, whereas SV and SV-NL generate similar forecasts. Finally, as h increases, SV and SV-NL outperform the GARCH models.

6.3 Summary

The empirical results reported provide the following acknowledgements:

- For each model-type, the specification with the leverage effect is able to generate more accurate density forecasts than the specification without the leverage effect. In particular, Beta-t-EGARCH and SV perform relatively better than their non-leverage counterparts at $h = 1$ and $h = 5$ compared to $h = 20$, whereas the opposite trend is observed for t-EGARCH and SPEGARCH. Thus, the parametric specification and how the leverage effect is incorporated affect the out-of-sample performance at different forecast horizons. Moreover, the specification that accounts for the leverage effect is able to predict the tails and, in some cases, also the center of the conditional distribution significantly better than the non-leverage model. The last point is very evident in the context of the S&P500 index when the left tail is emphasized.

- The business cycle and the choice of data frequency have relationships with the leverage effect. Indeed, predictive gains from the model with the leverage effect are concentrated near the peaks and the troughs. Conversely, when the data frequency decreases from daily to weekly observations, adding the leverage effect has a negative impact during quiet periods. Moreover, the non-leverage model is able to predict the left tail just as well as the model with the leverage effect at the weekly sampling frequency.

- Results indicate that Beta-t-EGARCH is the top performer. Furthermore, Beta-t-EGARCH-NL outperforms t-EGARCH, SPEGARCH and SV in some cases. Accordingly, the choice of the evolution of the conditional volatility process can play just as an important role as incorporating the leverage effect with regards to generating accurate density forecasts.

- Beta-t-EGARCH is the preferred model taking into consideration the following two aspects. First it is the model that generates the most accurate density forecasts. Second it is a model that guarantees recursive model estimation in a rather parsimonious way, that is able to obtain parameters estimates for every out-of-sample observation, while maintaining a reasonable computation time.

7 Conclusions

This study describes the work by Catania and Nonejad (2019), who examine the role of the leverage effect with regards to generating density forecasts of equity returns using well-known observation and parameter-driven conditional volatility models. The conditional volatility models considered differ in the parametric specification, the evolution of the conditional volatility process, and how the leverage effect is specified.

Considering daily Dow Jones and more than four hundred equities from S&P500 index returns series, the main finding obtained is that the models with the leverage effect tend to generate statistically significant more accurate density forecasts compared to their non-leverage counterparts. Predictive gains from models with the leverage effect tend to mainly concentrate on the onset of recessions.

A comparison between volatility models demonstrates that Beta-t-EGARCH is the top performer. Moreover, Beta-t-EGARCH-NL outperforms t-EGARCH, SPEGARCH and SV. Therefore, besides accounting for the leverage effect, the choice of the parametric specification and evolution of the conditional volatility process is important with regards to generating accurate density forecasts.

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