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Boundary Terms in Supergravity

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Abstract

The supergravity action is generically invariant under local supersymmetry only up to boundary terms. In the context of the AdS/CFT correspondence, the latter play an essential role when implementing the technique of holographic renormalization. We consider minimal $\mathcal{N} = 2$ gauged supergravity in four dimensions and study the holographic counterterms that ensure supersymmetry of the bulk + boundary supergravity action. This provides an explicit proof that holographic renormalization preserves supersymmetry in the considered setup.

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Chapter 1

Introduction

One of the most important developments in the study of string theory in the last decades is represented by the AdS/CFT conjecture, proposed by Maldacena in 1997 [1] [2] [3]. It states that there exists a duality correspondence between a theory of quantum gravity, formulated in terms of string theory, defined on a background given by the product of an Anti de-Sitter space (AdS) and a compact manifold, and a particular supersymmetric quantum field theory called SuperConformal Field Theory (SCFT) living on the boundary of AdS.
The best studied example of Maldacena's conjecture was that type IIB superstring theory on the ten dimensional $\mathrm{AdS}_5 \times S^5$ curved background is dual to $\mathcal{N} = 4$ super Yang-Mills theory ($\mathcal{N} = 4$ SYM) with gauge group $SU(N)$.
A first basic check is the matching between the symmetry groups of the two theories. On the gravity side we have the isometry group of $\mathrm{AdS}_5 \times S^5$, on the gauge side there is the superconformal group of $\mathcal{N} = 4$ SYM: both of them are isomorphic to $SU(2, 2|4)$ [4].

The useful feature that makes this correspondence interesting is the fact that, in a certain regime which we explain better in the following, one has a duality between a strongly coupled (super)conformal field theory and a weakly coupled (super)gravity theory. That means we can study quantum field theories in strongly coupled regime via its dual, doing classical calculations in supergravity and using the rules of the correspondence to get information about the former. In order to explain better where this feature comes from, let us very briefly introduce the objects which live in type IIB string theory and then study how the parameters of the two theories are related. We will follow [4] and [5].
Originally type IIB string theory appeared to describe only closed strings. However it was found [6] that it also includes open strings whose endpoints lie on p-dimensional spatial hypersurfaces called Dp-branes. The D stands for Dirichlet, because the vibrating open strings have to satisfy a Dirichlet condition. The p-dimensional spatial hypersurfaces (without the open string) are called p-branes. They are extended mas-

sive and charged objects, namely interacting with the gravitational and gauge fields, and generalize the concepts of particles (0-branes) and strings (1-branes).
As in electro-magnetism, one can count the p-brane charges computing the flux of the gauge field, represented by a $(p+1)$-form A_{p+1}, on a sphere surrounding the source. In the case of an extended object with a $\mathbb{R}^{9-p}$ transverse space, the expression for the flux reads

$$\int_{S^{8-p}} *F_{p+2} = N, \tag{1.1}$$

where F_{p+2} is the field strength associated to the gauge field A_{p+1}, $*$ is the Hodge duality map, S^{8-p} is a $(8-p)$ dimensional sphere and N is the number of units of flux.

After this digression, we have all the tools to come back to the relations between the parameters of the two theories. On the gravity side we have the string coupling constant g_s, the N units of flux of a stack of $D3$-branes and a length scale α' proportional to (length)^2. Furthermore the gravitational coupling constant in ten dimensions is $\kappa_{10}^2 = 8\pi G_{10} = 64\pi^7 g_s^2 \alpha'^4$. In $\mathcal{N} = 4$ SYM there are the rank of the gauge group N and the coupling constant g_{YM} (or equivalently the t'Hooft parameter $\lambda = g_{YM}^2 N$). We used the same symbol N for the units of flux and the rank of the gauge group because it turns out that they have the same value. Furthermore the brane description in [7] implies the relation $g_s = \frac{g_{YM}^2}{4\pi}$.
The AdS/CFT duality is conjectured to hold for all values of the parameters, but it is difficult to formulate quantitative tests for the strong form of this correspondence. However, it is possible to find a regime where string theory can be well approximated by (classical) supergravity, where the calculations are more manageable.
Supergravity is a good approximation of string theory when the length scale ℓ associated to a field configuration in supergravity is long compared to the string length scale $\sqrt{\alpha'}$, namely $\ell^2 \gg \alpha'$. An equivalent condition is given by a very small value of the Riemann tensor $R_{MNPQ} \sim \frac{1}{\ell^2}$ (M, N, P and Q are ten dimensional indices), so that higher derivative corrections to the basic supergravity action are negligible. In [5] the authors show that $\ell^4 = 4\pi g_s \alpha'^2 N$. This means that the regime we are interested in is $g_s N \gg 1$.
Further supergravity theories suffer UV divergences. We don't know yet how to quantize the superstring in $\mathrm{AdS}_5 \times S^5$ and studying quantum string effects is very hard. In order to suppress them we need κ_{10}^2 to be very small, that implies the weakly interacting string condition $g_s \ll 1$. As a consequence of the condition $g_s N \gg 1$ one obtains $N \gg 1$, namely the large N limit.
As we mentioned before, g_s is related with the field theory parameters as

$$g_s = \frac{g_{YM}^2}{4\pi} = \frac{\lambda}{4\pi N} \ll 1. \tag{1.2}$$

Thus one gets $g_{YM} \ll 1$ and λ fixed. Further the previous equations provide

$$\frac{\ell^4}{\alpha'^2} = 4\pi g_s N = g_{YM}^2 N = \lambda \gg 1. \tag{1.3}$$

Thus we will choose a large value of λ, but fixed.
't Hooft argument states that the effective coupling of a Yang-Mills $SU(N)$ gauged theory in 't Hooft limit (that is large N and $\lambda \gg 1$ but fixed) is λ. In this limit calculations are simplified also for strongly coupled CFT, because only the planar diagrams are relevant.
Hence we found a regime in which a weakly coupled supergravity theory is dual to a strongly coupled gauge quantum field theory in planar limit.

The AdS/CFT correspondence is more general than its $AdS_5 \times S^5/\mathcal{N} = 4$ SYM incarnation: in fact it is possible to conjecture other AdS/CFT dualities. This is the case of the correspondence between the M-theory on AdS$_4 \times S^7/Z_k$ and a three-dimensional SCFT, which is the example considered in this work. The SCFT is represented by ABJM (Aharony, Bergman, Jafferis and Maldacena) theory [8], which is a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $U(N)_k \times U(N)_{-k}$, where the subscripts are the level of the Chern-Simons terms[1].
The ABJM theory has two parameters: the rank of the gauge group N and the Chern-Simons level k. In the large N limit with $\lambda = \frac{N}{k}$, the 't Hooft coupling of the planar diagrams is λ itself. Furthermore, in the regime $N^{1/5} \gg k$ one can prove that the duality is again of strong/weak type and we can use 11-dimensional supergravity instead of the entire M-theory defined on AdS$_4 \times S^7/Z_k$.

Working with supergravity in higher dimension can quickly become very complicated. It is in several situations convenient to perform a dimensional reduction of the higher-dimensional supergravity on the compact manifold so that a lower-dimensional supergravity is obtained. Thus we perform a Kaluza-Klein (KK) reduction (or truncation) on the compact manifold. In general this procedure leads to an infinite tower of fields, which can be split in a finite number of "light" modes and an infinite tower of "heavy" fields. The KK truncation is called *consistent* if one can set all the heavy modes to zero in the equations of motion leaving the field equations for the light modes only. This is possible only when the on-shell light modes don't source the heavy ones [9]. After one gets the solution for the equations of motion of the consistent truncated theory, one can uplift it on the compact manifold founding the solution for the entire

[1] The action S of a three-dimensional Chern-Simons theory is the integral of a Chern-Simons 3-form, namely

$$S = \frac{k}{4\pi} \int_{\mathcal{M}} Tr\big(\mathrm{d}A \wedge A + \frac{2}{3} A \wedge A \wedge A\big), \tag{1.4}$$

where $\mathcal{M}$ is a topological manifold, A is a gauge field and k is a constant called level of the theory.

starting space.

Hence by employing the consistent truncation procedure, one reduces to study the fields on a space of lower dimension. The eleven-dimensional supergravity which admits $\mathrm{AdS}_4 \times S^7$ as a solution has a consistent truncation in the minimal four dimensional $\mathcal{N}=2$ gauged supergravity theory [9], which correspondingly admits an AdS_4 vacuum. This is the theory which we are going to deal with on the gravity side. On the gauge side we have a three dimensional $\mathcal{N}=2$ superconformal field theory defined on the boundary of AdS space, that is conformally flat. Its action reads [8]

$$S_{CS}^{\mathcal{N}=2} = \frac{k}{4\pi}\int Tr\big(\mathrm{d}A\wedge A + \frac{2}{3}A\wedge A\wedge A - \bar{\chi}\chi + 2D\sigma\big), \tag{1.5}$$

where χ is the gaugino, D is the auxiliary field of the vector multiplet, and σ is the real scalar field in the vector multiplet.
The supergravity theory admits also other solutions, which are just Asymptotically locally AdS spaces (AlAdS): they look like AdS near-boundary, but are different in the interior (which we will call bulk) [10]. These correspond to deforming the dual SCFT in various ways as we will see in a moment.
Since the SCFT lives on the boundary of the bulk space, in order to apply the correspondence we need to study the near-boundary behaviour of the metric and fields in AlAdS. This issue imposes to use the asymptotic expansions of the metric and fields, which is known as the Fefferman-Graham expansions [10]. We will see how they work specifically in Chapter 2.

Finally we arrive to the AdS/CFT conjecture which can be expressed as two fundamental statements [4]. The first statement is that every bulk field is the source (or background field) of a boundary operator (with the same quantum numbers) and they are coupled on the boundary. For instance some standard couplings are

$$\int \mathrm{d}^d x\,\big(g_{\mu\nu}T^{\mu\nu} + A_\mu R^\mu\big), \tag{1.6}$$

where $g_{\mu\nu}$ and A^μ are the bulk metric and a gauge bulk field induced on the boundary, while $T_{\mu\nu}$ and R^μ are the stress-energy tensor and a conserved current of the CFT. This example shows how a conformal field theory (represented by the stress-energy tensor) couples to a generic curved spacetime $g_{\mu\nu}$.
The second statement is

$$W_{CFT} = -S^{os}_{SUGRA}, \tag{1.7}$$

where W_{CFT} is the generating functional of the connected correlation functions for the conformal field theory and S^{os}_{SUGRA} is the on-shell supergravity action, i.e. evaluated on a solution of the field equations. This holds in the regime we mentioned at the

beginning of the introduction, that is low energies and weak gravitational coupling.
These statements bring to light important features of the correspondence. A fact which will reveal to be fundamental for the present work is that global symmetries in CFT correspond to gauged symmetries of the off-shell gravity action. In the case of supersymmetry this means that if we want to preserve it in CFT, we need to ask a local supersymmetric off-shell gravity action, i.e. a theory of supergravity. Furthermore the second statement tells us that the global symmetries of the generating functional of the connected correlation functions are the same as those of the on-shell supergravity action. In fact local symmetry transformations in the bulk induce transformations of the bulk fields restricted to the boundary. From the CFT point of view, the latter are sources of the conserved currents and their transformations are global because the background fields are not dynamical fields of the CFT.
In the end let us notice that theoretically one can obtain all the correlators for CFT only from the knowledge of the supergravity action evaluated on solutions with general boundary conditions: this is an example of holography in physics.

We know that the correlation functions of a quantum field theory suffer UV divergences that have to be removed through a renormalization procedure in order to save its consistency. In a gauge/gravity theory correspondence there is a UV/IR connection, i.e. the UV divergences of the SCFT are related to the IR ones of the gravity theory. Hence we will explain the procedure to remove the UV divergences holographically through the elimination of the IR divergences, the so-called holographic renormalization procedure. In general it consists of regulating the bulk spacetime with a cut-off (in our case it will be a large, but finite value of the radial coordinate), adding the correct boundary counterterms that cancel exactly the divergences and removing the cut-off [10].
According to the features of the correspondence we mentioned above, in order to save the global supersymmetry in the SCFT, we need to preserve the local supersymmetry in the gravity theory. Usually the supergravity action is generically invariant under local supersymmetry up to boundary terms. In AdS/CFT correspondence we need an action that satisfies local supersymmetry including also the boundary terms. This is caused by the fact that we can't set to zero the value of the bulk fields at the boundary as they correspond to field theory sources, and we are interested in the field theory generating functional with generic sources switched on. Thus we must require that bulk action + boundary counterterms to respect local supersymmetry, namely the variation of counterterms needs to cancel exactly the boundary terms deriving from the bulk action variation. This corresponds to invariance of the SCFT generating functional under supersymmetry variations of the sources.
There are different techniques to construct the appropriate counterterms and we will analyse two of them: the standard [11] and the Hamiltonian [12] approaches. In particular the second one is used in [12], which is the article we will mainly refer to in the

present work. The Hamiltonian approach results more systematic than the standard one, even though sometimes it turns out to be mathematically more intricate. In the Chapters 2 and 3 we will explain the basics of both these methods.

This work stems out from the problem studied in [12]. In particular we focus on the renormalization problem of minimal $\mathcal{N} = 2$, $D = 4$ gauged supergravity, a theory that includes the gravity multiplet only, made up of the metric, one Dirac gravitino and a graviphoton.
The counterterms which preserve local supersymmetry also at the boundary were found by Papadimitriou in [12], using the Hamilton-Jacobi method. Our purpose will be to prove the accuracy of those counterterms computing the explicit supersymmetric variation of the bulk + boundary action.

In Chapter 2 we present the basic features of (super)conformal field theories, with particular attention to the three dimensional $\mathcal{N} = 2$ SCFT. Then we analyse the anti de-Sitter space and Asymptotically locally anti de-Sitter spaces, introducing the idea of Fefferman-Graham field expansion. After we talk about the AdS/CFT correspondence, focusing on the two statements of the conjecture and the role of the radial coordinate. In the end we explain how the standard approach to holographic renormalization works and we propose a simple explicative example.

In Chapter 3 we briefly introduce the concept of supergravity, focusing on supergravity theories which admit an (Al)AdS vacuum. We propose the simple case of $\mathcal{N} = 1$, $D = 4$ pure AdS supergravity as an extension of $\mathcal{N} = 1$, $D = 4$ pure supergravity. Then we study in detail the minimal $\mathcal{N} = 2$, $D = 4$ gauged supergravity theory. We compute its supersymmetric variation and find that it is a boundary term. Then we work out the supersymmetry variation of the counterterms obtained with the Hamilton-Jacobi method. In the end we compute the near-boundary expansion for the bulk + boundary action variation and we obtain that it vanishes, thus proving supersymmetry of the complete action.

In Chapter 4 we summarize the result obtained and we apply it to a concrete example, which is about the holographic relation between the partition function of certain superconformal field theories and the microscopic counting of the dual black holes entropy. In the end we present one possible development of our work.

Chapter 2

AdS/CFT correspondence

In the introduction we stated that AdS/CFT correspondence concerns the duality existing between a string theory defined on AdS times a compact manifold and a SCFT living in a conformally flat spacetime, corresponding to the boundary of AdS. The correspondence can be generalized so that AdS is deformed to an AlAdS space and SCFT is defined on a generic background.
Even though the duality holds for any value of the parameters that characterise the two theories, we will deal with the low energies and weak gravitational coupling regime, in which classical supergravity is valid. Furthermore we will omit the analysis of bulk fields on the entire manifold, because the consistent truncation procedure allows us to analyse their behaviour on a lower dimensional space by just keeping the degrees of freedom we need. If one is interested in the behaviour of the bulk fields on the entire manifold, the uplift procedure allows to obtain it.
In the case of our interest, the eleven-dimensional supergravity with an $\mathrm{AdS}_4 \times S^7$ vacuum admits a consistent truncation to minimal $D = 4$, $\mathcal{N} = 2$ gauged supergravity [9] with an Asymptotically locally AdS vacuum. Thanks to these arguments we will concentrate to theories with (Al)AdS vacua on the gravity side.

In this chapter we primarily give a look to the conformal theories and their supersymmetric extension, with particular regard to the $\mathcal{N} = 2$, $D = 3$ SCFT algebra and one of its representations, that is the conformal supercurrent multiplet. Then we will analyse the AdS space and the AlAdS space, introducing the useful tool of the Fefferman-Graham field expansion. We will provide an essential introduction to the AdS/CFT correspondence and its features, discussing separately the holographic renormalization procedure. As we mentioned in the introduction there are many different methods for constructing the boundary counterterms. In Section 2.6 we will present the standard approach applied to a pure gravity theory. Instead we postpone the explanation of the Hamiltonian approach to Chapter 3 because we need to introduce the framework of supergravity in order to discuss its functioning.

In this chapter we follow [4] [5] [10] [11] [13] [14] [15] [16] [17].

2.1 Conformal field theories

We will start discussing conformal field theories in flat space. Later we will discuss how the theory can be coupled to curved space.
The conformal symmetry is defined through the action of its transformation on the metric

$$\eta_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} = \Lambda(x)\eta_{\mu\nu} \tag{2.1}$$

with $\Lambda(x)$ an arbitrary positive function of the coordinates called scale factor. Note that $\Lambda(x) = 1$ corresponds to the Poincaré symmetry group and $\Lambda(x) = \Lambda$ represents a dilatation transformation.

Let us study an infinitesimal conformal transformation. It reads (up to the first order)

$$x'^{\alpha} = x^{\alpha} + \epsilon^{\alpha}(x) + \mathcal{O}(\epsilon^2) \tag{2.2}$$

where $\epsilon^{\alpha}(x) \ll 1$ and substituting this expression in (2.1) we obtain

$$\begin{aligned} &\eta_{\alpha\beta}\Big(\delta^{\alpha}_{\mu} + \partial_{\mu}\epsilon^{\alpha} + \mathcal{O}(\epsilon^2)\Big)\Big(\delta^{\beta}_{\nu} + \partial_{\nu}\epsilon^{\beta} + \mathcal{O}(\epsilon^2)\Big) = \\ =&\eta_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} + \mathcal{O}(\epsilon^2) = \Lambda(x)\eta_{\mu\nu}. \end{aligned} \tag{2.3}$$

We set $\Lambda(x) = 1 + K(x)$ and the equation becomes

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = K(x)\eta_{\mu\nu}. \tag{2.4}$$

It is possible to find the expression of $K(x)$ tracing the equation above

$$K(x) = \frac{2}{d}\partial_{\mu}\epsilon^{\mu}. \tag{2.5}$$

Thus we obtained an equation identifying the infinitesimal conformal transformation up to first order in the conformal Killing vector ϵ^{μ}

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} - \frac{2}{d}\partial_{\rho}\epsilon^{\rho}\eta_{\mu\nu} = 0. \tag{2.6}$$

There are infinite solutions for this equation in $d = 2$, while in $d \neq 2$ there is a finite number. In fact the possible solutions are $\epsilon^{\mu} = a^{\mu}$, $\omega^{\mu\nu}x_{\nu}$, Λx^{μ} and $b^{\mu}x^2 - 2b^{\nu}x_{\nu}x^{\mu}$ with the respective generators P^{μ}, $J^{\mu\nu}$, D and K^{μ}. Here $\omega^{\mu\nu}$ and $J^{\mu\nu}$ are antisymmetric tensors. Let us note that the first operator is associated to the traslations, the second to the Lorentz transformations, the third to the dilatations and the fourth to the special

conformal transformations.
If we have a d-dimensional theory with $d > 2$, the total number of solutions is

$$d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+1)(d+2)}{2}. \tag{2.7}$$

Realizing a rearrangement of the transformations generators, it can be proved that the conformal group is isomorphic to $SO(2, d)$.
Adding furthermore the discrete conformal symmetry

$$x_\mu \to \frac{x_\mu}{x^2}, \tag{2.8}$$

we obtain the full conformal group $O(2, d)$.

Under mild conditions we can prove that a theory invariant under Poincaré and scale transformations is also invariant under special conformal transformations (see [18] for details on this subtle issue). In fact, constructing the conformal currents as

$$J^\mu = T^{\mu\nu}\epsilon_\nu, \tag{2.9}$$

where ϵ_ν is each of the conformal Killing vectors, we immediately see that the conserved currents of translation, Lorentz and dilatation symmetries imply

$$\begin{aligned} &\partial_\mu\big(T^{\mu\nu}a_\nu\big) = 0 \to \partial_\mu T^{\mu\nu} = 0 \\ &\partial_\mu\big(T^{\mu\nu}\omega_{\nu\rho}x^\rho\big) = 0 \to T^{\mu\nu} = T^{\nu\mu} \\ &\partial_\mu\big(T^{\mu\nu}\Lambda x_\nu\big) = 0 \to T^\mu_\mu = 0. \end{aligned} \tag{2.10}$$

Now, studying the special conformal transformations current, we see that it is automatically conserved. In fact

$$\partial_\mu\big[T^{\mu\nu}\big(b_\nu x^2 - 2b^\rho x_\rho x_\nu\big)\big] = T^{\mu\nu}\big(2b_\nu x_\mu - 2b_\mu x_\nu - \eta_{\mu\nu}b^\rho x_\rho\big) = 0. \tag{2.11}$$

If the conformal theory also satisfies supersymmetry, we add to the $O(2, D)$ algebra the supercharges Q^a, the R-symmetry generators and the so-called conformal supercharges S^a to close the algebra. That enhanced algebra is called superconformal algebra and we have a superconformal theory. Those superconformal theories actually play a role in the AdS/CFT correspondence.

In the next section we will summarize some aspects of supersymmetry and provide the three dimensional $\mathcal{N} = 2$ superconformal algebra structure relations.

2.2 $\mathcal{N}=2$, $D=3$ superconformal algebra

According to the Coleman-Mandula theorem in the presence of massive particles, bosonic charges are limited to the Poincaré symmetry plus internal symmetry charges. Furthermore the symmetry algebra of the theory is a direct sum of the Poincaré algebra and a finite-dimensional compact Lie algebra for internal symmetry.
If we admit a graded algebra, the situation is governed by Haag-Łopuszański-Sohnius (HLS) theorem and we need to add spinor (super)charges Q^i_α to the symmetry algebra. Here α is a spinor spacetime index, $i = 1, \ldots, \mathcal{N}$ is the index labelling the number of supercharges (and supersymmetries).
Hence supersymmetry (SUSY) theories realize the most general symmetry possible within the framework of the few assumptions made in the hypotheses of the CM and HLS theorems. Further, because of the structure of supersymmetry transformations, in a certain way they unify bosons and fermions.
Moreover in theories that contain only massless fields and are scale invariant at the quantum level, there are the additional possibilities of conformal and superconformal symmetries.

The case of interest for this work is the three dimensional $\mathcal{N}=2$ superconformal algebra. It contains all the generators in the supersymmetry algebra, namely the bosonic generators of the Poincaré group P_μ and $M_{\mu\nu}$ and one (complex) spinor supercharge Q_α with its conjugate $\bar{Q}_\alpha$. The structure relations yield

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \mathrm{i}\big(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\big)\\
[M_{\mu\nu}, P_\rho] &= \mathrm{i}\big(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\big)\\
[P_\mu, P_\nu] &= [Q_\alpha, P_\mu] = \{Q_\alpha, Q_\beta\} = 0\\
\{Q_\alpha, \bar{Q}_\beta\} &= 2(\gamma^\mu)_{\alpha\beta}P_\mu + 2\mathrm{i}\epsilon_{\alpha\beta}Z,
\end{aligned} \tag{2.12}$$

where Z is a central charge[1] and we pick the gamma matrices to be real and symmetric. In addition, it contains two new bosonic generators, K_μ and D, the generators of special conformal transformations and dilatations, as well as the new fermionic generators S_α and $\bar{S}_\alpha$, called conformal supercharges, required to close the algebra.

[1]A central charge is an operator that commutes with all the other generators of the algebra.

The (anti)commutation relations between the generators read

$$\begin{aligned}
[M_{\mu\nu}, K_\rho] &= \mathrm{i}\big(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\big) \\
[K_\mu, P_\nu] &= -2\mathrm{i}M_{\mu\nu} - 2\mathrm{i}\eta_{\mu\nu}D \\
\{S_\alpha, \bar{S}_\beta\} &= 2(\gamma^\mu)_{\alpha\beta}K_\mu + 2\mathrm{i}\epsilon_{\alpha\beta}Z \\
[D, P_\mu] &= \mathrm{i}P_\mu \\
[D, K_\mu] &= -\mathrm{i}K_\mu \\
[D, Q_\alpha] &= \frac{\mathrm{i}}{2}Q_\alpha, \qquad [D, S_\alpha] = -\frac{\mathrm{i}}{2}S_\alpha \\
[D, \bar{Q}_\alpha] &= -\frac{\mathrm{i}}{2}\bar{Q}_\alpha, \qquad [D, \bar{S}_\alpha] = \frac{\mathrm{i}}{2}\bar{S}_\alpha \\
[M_{\mu\nu}, D] &= 0 \\
\{Q_\alpha, S_\beta\} &= M_{\mu\nu}[\gamma^\mu, \gamma^\nu]_{\alpha\beta} + 2D\epsilon_{\alpha\beta} + \mathrm{i}\epsilon_{\alpha\beta}R,
\end{aligned} \tag{2.13}$$

where R is the generator of a $U(1)$ R-symmetry which acts as an automorphism of this algebra, rotating the charges Q_α and $\bar{Q}_\alpha$ by opposite phases.

Every local quantum field theory possesses a real, conserved, symmetric energy-momentum tensor $T_{\mu\nu}$ and every supersymmetric quantum field theory possesses a conserved supersymmetry current. In the case of a supersymmetric (conformal) field theory, the stress-energy tensor is embedded in the (conformal) supermultiplet. Furthermore the conserved conformal currents J^μ (see eq. (2.9)) turn into the conserved superconformal current embedded in the supermultiplet too.
By definition, the supercurrent is a supermultiplet containing the energy-momentum tensor and the (fermionic) supersymmetry current(s), along with some additional components such as the R-symmetry current. For the three dimensional $\mathcal{N}=2$ superconformal theory, the supercurrent reads

$$\big(T_{\mu\nu}, \mathcal{I}^\mu_\alpha, R^\mu\big), \tag{2.14}$$

where $\mathcal{I}^\mu_\alpha$ is the conserved superconformal current and R^μ is the conserved R-symmetry current.

In the following of this chapter and in the next one, we will analyse one of the fundamental statement of $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence: the supermultiplet of the three dimensional $\mathcal{N}=2$ SCFT exactly couples with the gravity multiplet of the minimal four dimensional $\mathcal{N}=2$ gauged supergravity theory.

2.3 Anti de Sitter space

Anti de Sitter space is a solution of the Einstein equations derived from the action

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{-g}(R - \Lambda), \tag{2.15}$$

where $D = d + 1$, $\kappa^2 = 8\pi G_D$ (G_D is the Newton's gravitational constant in D dimensions) and $\Lambda = -\frac{(D-1)(D-2)}{\ell^2}$ is the negative cosmological constant, function of D and the curvature radius of AdS space.
The equation of motion yields

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\frac{\Lambda}{2} g_{\mu\nu} \tag{2.16}$$

and tracing this equation we obtain $R = \frac{D}{D-2}\Lambda$. Substituting above we have

$$R_{\mu\nu} = -\frac{D-1}{\ell^2} g_{\mu\nu}. \tag{2.17}$$

Thus AdS_D is an Einstein space. Furthermore it is a maximally symmetric space and consequently the Riemann tensor takes the form

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \tag{2.18}$$

AdS_D can be embedded as an hyperboloid in a flat spacetime of dimension $D+1$ with metric $\eta_{\alpha\beta} = (-1, \underbrace{+1, +1, \cdots, +1}_{D-1}, -1)$. The hyperboloid is defined as

$$-(x^0)^2 + \sum_{i=1}^{D-1} (x^i)^2 - (x^D)^2 = -\ell^2 \tag{2.19}$$

with the $D+1$ coordinates x^0, x^i and x^D. It is clear from (2.19) that the isometry group of AdS_D space is $O(2, D-1)$.
The AdS_D line element can be obtained introducing the set of intrinsic coordinates

$$\begin{aligned} x^i &= r\bar{x}^i \quad \text{with} \quad \sum_{i=1}^{D-1} \bar{x}^2 = 1, \quad 0 \le r < \infty \\ x^0 &= \sqrt{r^2 + \ell^2} \sin\left(\frac{t}{\ell}\right), \quad 0 \le t < 2\pi\ell \\ x^D &= \sqrt{r^2 + \ell^2} \cos\left(\frac{t}{\ell}\right) \end{aligned} \tag{2.20}$$

and using them to express the line element of $\mathbb{R}^{2,D-1}$. The result is

$$\mathrm{d}s^2 = -\Big(1+\frac{r^2}{\ell^2}\Big)\mathrm{d}t^2 + \Big(1+\frac{r^2}{\ell^2}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}, \tag{2.21}$$

where $\mathrm{d}\Omega^2_{D-2}$ is the line element of a $(D-2)$-sphere.
This coordinate system is global because it covers the full AdS_D space. To avoid a periodic time-like coordinate, we extend his domain to $\mathbb{R}$. We will still refer to the universal cover of AdS_D as AdS_D.
Another set of global coordinates can be obtained defining a new radial coordinate y as $\cosh\Big(\frac{y}{\ell}\Big) = \sqrt{1+\frac{r^2}{\ell^2}}$, so that the line element becomes

$$\mathrm{d}s^2 = -\cosh^2\Big(\frac{y}{\ell}\Big)\mathrm{d}t^2 + \mathrm{d}y^2 + \ell^2\sinh^2\Big(\frac{y}{\ell}\Big)\mathrm{d}\Omega^2_{D-2}. \tag{2.22}$$

Later we also use the coordinate system called Poincaré patch coordinates. It doesn't cover all AdS space, but it has a useful feature: the slices at defined value of the radial coordinate are conformal to Minkowski space. The patch is defined by

$$\begin{aligned} x^0 &= \ell u \hat{x}^0 \\ x^i &= \ell u \hat{x}^i \quad \text{with} \quad i = 1, \cdots, D-2 \\ x^{D-1} &= \frac{1}{2u}\big(-1 + u^2(\ell^2 - \hat{x}^2)\big) \\ x^D &= \frac{1}{2u}\big(1 + u^2(\ell^2 + \hat{x}^2)\big) \end{aligned} \tag{2.23}$$

with $\hat{x}^2 = -(\hat{x}^0)^2 + \sum_{i=1}^{D-2}(\hat{x}^i)^2$, $0 < u < \infty$ and $-\infty < \hat{x}^0,\ \hat{x}^i < \infty$. The restriction to the domain of u is necessary to construct a single-valued map of the hyperboloid coordinates.
The line element becomes

$$\mathrm{d}s^2 = \ell^2\Big[\frac{\mathrm{d}u^2}{u^2} + u^2\Big(-(\mathrm{d}\hat{x}^0)^2 + \sum_{i=1}^{D-2}(\mathrm{d}\hat{x}^i)^2\Big)\Big]. \tag{2.24}$$

There is a particular surface in this coordinate system: $u = \infty$. It is, mathematically speaking, a conformal boundary[2] and it is referred as the boundary of AdS_D. It is a Minkowskian $\mathbb{R}^{1,D-2}$ plane.
Other two forms in which the Poincaré patch may be described are

$$\mathrm{d}s^2 = \ell^2\Big[\frac{\mathrm{d}z^2}{z^2} + \frac{1}{z^2}\big(-(\mathrm{d}\hat{x}^0)^2 + \sum_{i=1}^{D-2}(\hat{x}^i)^2\big)\Big] = \ell^2\Big[\mathrm{d}r^2 + \mathrm{e}^{2r}\big(-(\mathrm{d}\hat{x}^0)^2 + \sum_{i=1}^{D-2}(\hat{x}^i)^2\big)\Big], \tag{2.25}$$

[2]In the conformal compactification construction, one maps a manifold M onto the interior of a compact manifold $\tilde{M}$ and then call its boundary $\partial\tilde{M}$ the conformal boundary of the original manifold.

where we set $u = \frac{1}{z} = \mathrm{e}^r$.

2.4 AdS/CFT correspondence

After talking individually about the two actors of the correspondence, now we are ready to discuss it, exhibiting some general features of the duality. Let us take into account theories with AdS_{d+1} vacuum and CFTs living within a d-dimensional spacetime conformal to Minkowski space, represented by the AdS_{d+1} boundary.
Let us assume that the interactions between bulk fields (for instance metric, gauge and scalar fields) are described by the effective action S_{AdS}. Furthermore we assume a negative scalar potential required to have an AdS_{d+1} vacuum[3]. On the other side we introduce the Lagrangian L_{CFT} which rules the boundary fields dynamic.
The first statement of AdS/CFT is that every bulk field $\hat{h}$ in AdS is associated to a boundary operator $\mathcal{O}$ (with the same $O(2,d)$ quantum numbers) via their coupling on the boundary. In particular we have

$$L_{CFT} + \int \mathrm{d}^d x \, h\mathcal{O}. \tag{2.26}$$

Notice that for consistency we used h instead of $\hat{h}$, i.e. the value of the bulk field induced on the boundary.
One can obtain a unique $\hat{h}(x,z)$ from $h(x)$ demanding $\hat{h}$ solves the bulk field equation derived from S_{AdS} and imposing suitable boundary conditions for the solution. In terms of path integral, $h(x)$ represents the source of the operator $\mathcal{O}$. In fact

$$\mathrm{e}^{W[h]} = \int \mathcal{D}\mathcal{O} \, \mathrm{e}^{-\int \mathrm{d}^d x \left(\mathcal{L}_{CFT} - h\mathcal{O}\right)} \tag{2.27}$$

with $W[h]$ the generating functional of the connected correlation functions and $\mathcal{L}_{CFT}$ the Lagrangian density.
We can't say a priori which are the couplings between bulk fields and operators, but there are some of them that are standard

$$\int \mathrm{d}^d x \, \sqrt{g} \big(g_{\mu\nu} T^{\mu\nu} + A_\mu R^\mu + \Psi^\alpha_\mu \mathcal{I}^\mu_\alpha + \cdots \big), \tag{2.28}$$

where the ellipsis stands for other couplings beyond the first order in the fields. $g_{\mu\nu}$, A_μ and Ψ^α_μ are respectively the metric, a gauge field and a vector-spinor defined on the background. On the other side $T_{\mu\nu}$, R_μ and $\mathcal{I}^\alpha_\mu$ are the stress-energy tensor, a

[3] In the case of $\mathcal{N} = 2$ minimal supergravity, which we will study in the second part of this work, scalar fields are absent. Therefore a negative cosmological constant is provided by gauging the global subgroup $U(1)_R$.

conserved current and vector-spinor supercurrent of the CFT. Let us note that we introduced exactly the couplings we are interested in for our work. Indeed we will see that the gravity multiplet of the minimal four dimensional $\mathcal{N}=2$ gauged supergravity theory is composed of

$$\left(g_{\mu\nu}, \Psi_{\mu}^{\alpha}, A^{\mu}\right) \tag{2.29}$$

This example shows how a conformal field theory (represented by the stress-energy tensor) couples to a generic curved spacetime $g_{\mu\nu}$. Indeed in Chapter 3 we will be interested in a SCFT defined on a curved background, namely the boundary of an Asymptotically locally AdS space (see Section 2.5).
In the end we want to bring the attention of the reader on the fact that gauged symmetries of the off-shell gravity action correspond to global symmetries (therefore conserved currents) in the CFT. For this reason we will ask that the counterterms variation cancels exactly the boundary term deriving from the bulk action variation, so that we obtain a local supersymmetric gravity action.

The second statement of AdS/CFT is

$$W[h] = -S^{os}_{AdS}[\hat{h}], \tag{2.30}$$

where on the right hand side we have the on-shell gravity action, i.e. evaluated on a solution of the equation of motion $\hat{h}(x,z)$. This conjecture means we can theoretically obtain all the correlators for the CFT (that is the entire theory) from the knowledge of the gravity action evaluated on solutions with general boundary conditions, i.e. for generic values of boundary fields. Furthermore if we want to preserve supersymmetry in the CFT, we have to ask the on-shell supergravity action (plus boundary counterterms necessary to cancel the divergences) to be supersymmetric too.

Since the equations of motion in AdS are second order differential equations, we need to specify two boundary conditions to have a well-defined solution.
The first one concerns the form of $\hat{h}(z,x)$. In fact the fields, like the metric, blow up at the boundary and, for this reason, we can't simply set $h(x) = \hat{h}(z=0,x)$. On the contrary the right condition to require is that $\hat{h}$ can be factorized as $\hat{h}(z,x) = f(z)h(x)$, with $f(z)$ a function of z.
The second boundary condition to be imposed on the behaviour of the fields is their regularity in the centre of the bulk.

Eventually we want to draw the attention to the renormalization issue. Usually the correlation functions of a QFT suffer UV divergences. In the AdS/CFT correspondence such divergences are bound to IR ones on the gravity side, in a certain way due to the infinite volume of AdS. In order to cancel these divergences, we need to renormalize $S_{AdS}(\hat{h})$ through the so-called Holographic Renormalization procedure. In order to perform the latter there are many different methods. However we will take into account two of them, which will be explained in Sections 2.6 and 3.5.

A very interesting feature of AdS/CFT duality is the role that the radial coordinate of the gravity side plays. In fact it can be seen as an energy scale in the conformal theories and it is fundamental for the holographic interpretation.
Let us consider the AdS metric in the Poincaré patch (we will fix the AdS radius to $\ell = 1$ for simplicity)

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 + (\mathrm{d}x_\mu)^2}{z^2}. \tag{2.31}$$

A dilatation $x_\mu \to \lambda x_\mu$ in the CFT corresponds to the AdS isometry $z \to \lambda z$, $x_\mu \to \lambda x_\mu$. Since we know that the dilatation generator of the conformal algebra is related to the energy, we can identify $u = \frac{1}{z}$ as an energy scale. It means that the boundary of AdS ($u \to \infty$) is related to the UV regime in the CFT.
Thus we can better understand what we stated about the UV/IR connection in the introduction. Quantum effects and the renormalization procedure cause UV divergences in quantum field theories. For the reasoning we did before, the way to cancel them in a gravity/gauge duality is to eliminate the IR divergences (namely long distances) on the gravity side.

2.5 Asymptotically locally Anti de Sitter space

The AdS/CFT duality is a powerful tool to study physical theories through their dual ones. In order to allow for truly arbitrary sources, we need to extend its validity also to conformal field theories which couple with generally curved boundary, i.e. with an arbitrary source $g_{\mu\nu}$. Hence we are going to consider generalizations of AdS whose conformal boundary is generally curved, namely Asymptotically locally Anti de Sitter spaces (AlAdS). In the following of this section we try to understand better what they are.

First of all we define the concept of *conformally compact manifold.* Let us consider a manifold $\bar{M}$ with its interior M and the boundary ∂M. The metric G is *conformally compact* if it has a second order pole at ∂M and, defined a positive function $z(x)$ in M with a first order zero at ∂M, the function $g = z^2 G$ smoothly extends to $\bar{M}$. We call $g|_M = g_{(0)}$.
Another quantity that smoothly extends to $\bar{M}$ is

$$|\mathrm{d}z|_g^2 = g^{\mu\nu}\partial_\mu z \partial_\nu z \tag{2.32}$$

and after some calculations one can shows that the Riemann tensor of G is

$$R_{\mu\nu\rho\sigma}[G] = -|\mathrm{d}z|_g^2 \left(G_{\mu\rho}G_{\nu\sigma} - G_{\mu\sigma}G_{\nu\rho}\right) + \mathcal{O}(z^{-3}), \tag{2.33}$$

with the main term proportional to z^{-4}. Furthermore imposing the Einstein's metric condition we obtain $|\mathrm{d}z|^2_g|_M = \frac{1}{\ell^2}$, i.e. the curvature tensor of a conformally compact Einstein manifold is the same of an AdS one near-boundary.
We have finally arrived to the definition of Asymptotically locally AdS space: an AlAdS metric is a conformally compact Einstein metric. Its line element can be expressed near-boundary ($z \to 0$) through a Fefferman-Graham expansion as

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{z^2}\Big(\mathrm{d}z^2 + g_{ij}(x,z)\mathrm{d}x^i\mathrm{d}x^j\Big) \\ g_{ij}(x,z) &= g_{(0)ij}(x) + z g_{(1)ij}(x) + \cdots + z^d g_{(d)ij}(x) + z^d h_{(d)ij}(x)\log z^2 + \cdots \end{aligned} \tag{2.34}$$

where the metric $g_{(0)ij}$ is arbitrary. We can find the expression for the coefficients $g_{(k)ij}$, $k > 0$ solving the Einstein's equations iteratively by treating z as a small parameter. It can be proved that in a pure gravity theory all the coefficients associated to odd powers of z vanish until $g_{(d)ij}$. Further the coefficients $g_{(2)ij}, \ldots, g_{(d-2)ij}$ and the trace and covariant divergence of $g_{(d)ij}$ are fixed by the equations. On the contrary, the value of $g_{(d)ij}$ is related to the 1-point function expectation value of the CFT stress-energy tensor. $h_{(d)ij}$ is present only for even d.

We show the procedure to obtain explicitly the coefficients for the Fefferman-Graham metric expansion near-boundary in a simple example, namely a pure gravity theory. We will refer to [11] however with different convention on Riemann tensor[4]. Furthermore the authors of the article set $\ell = 1$. We will reinstate the factors ℓ at the end of Section 2.6 in the expression of counterterms, in order to compare it with the one found by Papadimitriou in [12] with the Hamilton-Jacobi method.
It will be useful to have the metric in the form

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho)\mathrm{d}x^i\mathrm{d}x^j \\ g_{ij}(x,\rho) &= g_{(0)ij} + \cdots + \rho^{\frac{d}{2}}\big(g_{(d)ij} + h_{(d)ij}\log\rho\big) + \cdots . \end{aligned} \tag{2.35}$$

The pure gravity theory is described by the Einstein-Hilbert action plus a cosmological constant term

$$S = \frac{1}{16\pi G_N}\int_M \mathrm{d}^{d+1}x\,\sqrt{\det G_{\mu\nu}}\big(R - 2\Lambda\big). \tag{2.36}$$

On a manifold M with boundary ∂M, we also have the Gibbons-Hawking term

$$S_{GH} = \frac{1}{16\pi G_N}\int_{\partial M} \mathrm{d}^d x\,\sqrt{\det g_{ij}}2K, \tag{2.37}$$

[4] Our convention on Riemann tensor is defined in Appendix A. It differs from the one in [11] for a minus sign. In fact their convention is $R_{\mu\nu\rho}{}^{\sigma} = \partial_\mu \Gamma_{\nu\rho}{}^{\sigma} - \Gamma_{\mu\tau}{}^{\sigma}\Gamma_{\nu\rho}{}^{\tau} - \mu \leftrightarrow \nu$.

where g_{ij} is the metric on the boundary and $K = g^{ij}K_{ij}$ is the trace of the extrinsic curvature K_{ij} of the boundary surface. In Section 3.5 we will discuss its expression in the framework of radial ADM decomposition formalism. This term is necessary to have a well posed Dirichlet problem[5].
Einstein's equation for the bulk action is

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} = -\Lambda G_{\mu\nu} \tag{2.38}$$

and inserting (2.35) in (2.38), we have

$$\begin{aligned} \rho\big(2g'' - 2g'g^{-1}g' + Tr(g^{-1}g')g'\big) - Ric(g) - (d-2)g' - Tr(g^{-1}g')g = 0 \\ (g^{-1})^{jk}(\nabla_i g'_{jk} - \nabla_k g'_{ij}) = 0 \\ Tr(g^{-1}g'') - \frac{1}{2}Tr(g^{-1}g'g^{-1}g') = 0, \end{aligned} \tag{2.39}$$

where the derivative with respect to ρ is denoted by the prime, ∇_i is the covariant derivative constructed from the metric g and $Ric(g)$ is the Ricci scalar of the boundary metric g.
One can solve this system of equations iteratively, differentiating successively with respect to ρ and setting $\rho = 0$. The first coefficient of the expansion results

$$g_{(2)ij} = -\frac{1}{d-2}\Big(R_{ij} - \frac{1}{2(d-1)}Rg_{(0)ij}\Big). \tag{2.40}$$

Beside the metric, also the other fields living in AlAdS can be expanded near-boundary through a Fefferman-Graham expansion. Let us call $\mathcal{F}$ a generic field with its spacetime and internal indices suppressed. The expansion is

$$\mathcal{F}(x,\rho) = \rho^m\Big[f_{(0)}(x) + \rho f_{(2)}(x) + \cdots + \rho^n\big(f_{(2n)}(x) + \tilde{f}_{(2n)}(x)\log\rho\big) + \cdots\Big] \tag{2.41}$$

As in the metric case, we can find the value of the coefficients $f_{(2m)}$, $m > 0$ solving iteratively the field equations for $\mathcal{F}$. Through this procedure one can fix all the coefficients apart from $f_{(0)}$ and another one that we call $f_{(2n)}$, where n depends on the field

[5] The variation of Einstein-Hilbert action yields the sum of two contributions. The former is proportional to Einstein tensor, while the latter is a boundary term which contains the variation of the metric $\delta g_{\mu\nu}$ and variations of the derivatives of the metric $\delta(\partial_\sigma g_{\mu\nu})$. Setting $\delta g_{\mu\nu} = 0$ on the boundary is not sufficient to kill all the surface contributions. However fixing both the metric and the derivatives of the metric on the boundary is uncomfortable.
For this reason Gibbons and Hawking (and York) proposed to add the boundary term S_{GH}, whose variation cancels the terms involving $\delta(\partial_\sigma g_{\mu\nu})$. So setting $\delta g_{\mu\nu} = 0$ becomes sufficient to make the action stationary.
More details about this issue can be found e.g. in [19].

considered. $f_{(0)}$ is the source function for the dual operator living in the conformal theory, while $f_{(2n)}$ is related to the 1-point function expectation value of the same operator.

2.6 Holographic Renormalization: Standard Approach

Although we relaxed the requests on the gravity side, the statements of AdS/CFT still apply. Every bulk field $\hat{h}$ is associated to a boundary operator $\mathcal{O}$ and its on-shell asymptotic value h represents the field theory source of the operator. The fundamental statement of AdS/CFT correspondence now reads

$$\mathrm{e}^{W_{CFT}[h]} \equiv \int \mathcal{D}\mathcal{O}\,\mathrm{e}^{-\int_{\partial AlAdS} h\mathcal{O}} = \mathrm{e}^{-S^{os}_{SUGRA}[\hat{h}]} \tag{2.42}$$

where $\partial AlAdS$ is the boundary of an asymptotic locally AdS and S^{os}_{SUGRA} is the on-shell supergravity action, namely evaluated on a solution.

After talking about these general features, we go into the substance of the issue. Let us take into account a generic bulk field $\mathcal{F}$, like the one we expanded in Section 2.5 (the metric is included too).
The first step to holographically renormalize an on-shell supergravity action is to regularize it, i.e. setting a cut-off at $\rho = \epsilon$ (with ϵ a small parameter) and evaluating the boundary terms at that value of ρ. In this way we can manipulate the divergent quantities at the boundary, taking the limit $\epsilon \to 0$ at the end of calculations. The regularized action takes the form

$$S_{reg}[f_{(0)}, \epsilon] = \int_{\rho=\epsilon} \mathrm{d}^d x \sqrt{g_{(0)}} \left[\epsilon^{-\nu} a_{(0)} + \epsilon^{-(\nu-1)} a_{(2)} + \cdots - a_{(2\nu)} \log\epsilon + \mathcal{O}(\epsilon^0)\right] \tag{2.43}$$

with ν a positive number depending on the conformal dimension of the dual operator and $a_{(2k)}$ local functions of the source $f_{(0)}$.
In order to cancel the divergences, we introduce the boundary counterterms defined as

$$S_{ct}[\mathcal{F}(x,\epsilon), \epsilon] = -\text{ divergent terms in } S_{reg}[f_{(0)}, \epsilon] \tag{2.44}$$

where, for a matter of covariance, the boundary counterterms are expressed as function of the field $\mathcal{F}(x,\epsilon)$ living on the surface $\rho = \epsilon$ with the induced metric $\gamma_{ij} = g_{ij}(x,\epsilon)/\epsilon$. To do that we need to express the source as $f_{(0)} = f_{(0)}(\mathcal{F}(x,\epsilon), \epsilon)$ from (2.41) and then we can evaluate the coefficients $a_{(2k)} = a_{(2k)}(f_{(0)}(\mathcal{F}(x,\epsilon), \epsilon))$.
Finally we define the subtracted action as

$$S_{sub}[\mathcal{F}(x,\epsilon), \epsilon] = S_{reg}[f_{(0)}, \epsilon] + S_{ct}[\mathcal{F}(x,\epsilon), \epsilon], \tag{2.45}$$

which is finite in the limit $\epsilon \to 0$. Hence we get the renormalized action

$$S_{ren}[f_{(0)}] = \lim_{\epsilon\to 0} S_{sub}[\mathcal{F}(x,\epsilon),\epsilon]. \tag{2.46}$$

We would like to focus the attention of the reader on two issues. Firstly we define a subtracted action because we need it to compute the correlators. Indeed the functional derivatives of the on-shell supergravity action must be taken before the evaluation of the limit $\epsilon \to 0$.
Secondly, if we want to keep a certain global symmetry in the conformal theory, for example supersymmetry, we need to request the sum $S_{ct} + S_{reg}$ to be invariant under the symmetry made local.

Now we want to show an explicit example of holographic renormalization. We will study the case of a pure gravity theory, which we analysed in Section 2.5, closely referring to [11].
The regulated action is

$$\begin{aligned} S_{reg} &= \frac{1}{16\pi G_N}\Big[\int_{\rho\geq\epsilon} \mathrm{d}^{d+1}x\,\sqrt{g}\big(R[g]-2\Lambda\big) + \int_{\rho=\epsilon} \mathrm{d}^d x\,\sqrt{\gamma}2K\Big] = \\ &= -\frac{1}{16\pi G_N}\int \mathrm{d}^d x\Big[\int_\epsilon \mathrm{d}\rho\,\frac{d}{\rho^{\frac{d}{2}}}\sqrt{\det g_{ij}(x,\rho)} + \\ &+ \frac{1}{\rho^{\frac{d}{2}}}\Big(-2d\sqrt{\det g_{ij}(x,\rho)} + 4\rho\partial_\rho\sqrt{\det g_{ij}(x,\rho)}\Big)\Big|_{\rho=\epsilon}\Big]. \end{aligned} \tag{2.47}$$

Evaluating S_{reg} for the solution we found in Section 2.5, we obtain

$$S_{reg} = \frac{1}{16\pi G_N}\int \mathrm{d}^d x\,\sqrt{\det g_{(0)}}\Big(\epsilon^{-\frac{d}{2}}a_{(0)} + \epsilon^{-\frac{d}{2}+1}a_{(2)} + \cdots\Big), \tag{2.48}$$

where

$$a_{(0)} = -2(1-d) \qquad a_{(2)} = \frac{(d-4)(d-1)}{(d-2)}Trg_{(2)} \qquad a_{(2m)} = \cdots . \tag{2.49}$$

Notice that the coefficients $a_{(0)}$ and $a_{(2)}$ are local functions of $g_{(0)}$ and its curvature tensor. This fact is true for all $a_{(2m)}$, $m \geq 0$.
In order to write the boundary counterterms in a covariant way, we need to express them in terms of the induced metric $\gamma_{ij} = \frac{1}{\epsilon}g_{ij}(x,\epsilon)$. Inverting the relation between γ_{ij} and $g_{(0)ij}$ perturbatively in ϵ, one finds

$$\begin{aligned} \sqrt{g_{(0)}} &= \epsilon^{\frac{d}{2}}\Big(1 - \frac{1}{2}Trg^{-1}_{(0)}g_{(2)} + \cdots\Big)\sqrt{\gamma} \\ Trg_{(2)} &= \frac{1}{2(d-1)}\frac{1}{\epsilon}\Big(-R[\gamma] + \cdots\Big) \end{aligned} \tag{2.50}$$

Putting all together and reinstating the factors ℓ by dimensional analysis, we have the covariant counterterms

$$S_{ct} = \frac{1}{16\pi G_N} \int_{\rho=\epsilon} \mathrm{d}^d x \sqrt{\gamma} \Big[\frac{2(1-d)}{\ell} - \frac{\ell}{d-2} R[\gamma] + \cdots \Big]. \tag{2.51}$$

Finally we have the renormalized action following (2.45) and (2.46).
A similar procedure can be adapted to remove the divergences due to other fields, such as a gauge field or fermion fields.

In the article [12] which we refer to in Chapter 3 for our calculations, the author uses a different approach to holographic renormalization, namely the Hamiltonian formalism approach. We will explain how this method works in the next chapter, making use directly of the case studied in [12].

Chapter 3

Holographic Renormalization in Supergravity

The parameters of global SUSY transformations are constant anti-commuting Majorana spinors ϵ_α. In supergravity SUSY is gauged, also with the Poincaré generators, since they are related to each other in the superalgebra. This means that gravity is included and the parameters of SUSY transformations become functions of spacetime coordinates.
We are interested in such theories because we saw that in order to preserve the global supersymmetry in a SCFT, we need to ask a theory (necessarily of gravity) that is locally supersymmetric. Further we are interested in theories which admit an (Al)AdS vacuum for our purposes: for this reason we will refer to these supergravities in this chapter.
The minimal content of a supergravity theory is represented by the gauge or gravity multiplet, which is composed by the metric $g_{\mu\nu}(x)$ (or equivalently a frame field $e^\alpha_\mu(x)$), $\mathcal{N}$ Majorana vector-spinor field $\Psi_\mu(x)$ and other fields, depending on the theory we are taking into account. In the basic case of $\mathcal{N} = 1$, $D = 4$ supergravity, the gauge multiplet does not contain additional fields. Instead the gauge multiplet of a minimal $\mathcal{N} = 2$, $D = 4$ supergravity (which is the case of interest in [12]) includes one gauge field, the graviphoton, in addition to the metric and two Majorana gravitinos.
The gravity multiplet of this theory precisely couples to the supercurrent of its dual theory, i.e. $\mathcal{N} = 2$ SCFT in three dimensions. In fact, as we mentioned in the previous chapter, its supermultiplet is composed by

$$\big(T_{ij}, \mathcal{I}^a_i, R^i\big). \tag{3.1}$$

We used the indices i and a instead of μ and α in order to make the notation compliant with the one used in this chapter. The boundary values of the gravity multiplet fields are the sources of the supercurrent components, namely

$$\int \mathrm{d}^3x \sqrt{g}\big(g_{ij}T^{ij} + A_i R^i + \Psi_i^a \mathcal{I}_a^i + \cdots\big). \tag{3.2}$$

where g is the determinant of the induced metric on the boundary.

In many cases, as the example provided in Chapter 4, one is eventually interested in evaluating to on-shell a bosonic solution. Thus we are interested in fixing the bosonic terms such that SUSY is preserved for this particular solution. In order to do this, we need to consider quadratic terms in the fermions as well as the pure bosonic terms and counterterms, because their supersymmetric variation talks each other. Thus our analysis will be restricted to quadratic order in fermions in the action.

We begin in Section 3.1 introducing the simplest example of supergravity theory, that is $\mathcal{N}=1$, $D=4$ pure supergravity.
In Section 3.2 we discuss the case of $\mathcal{N}=1$, $D=4$ pure AdS supergravity, which is an extension of the previous one.
In Section 3.3 we introduce the action of minimal $\mathcal{N}=2$, $D=4$ gauged AdS supergravity and the variations of the fields.
In Section 3.4 we compute in full details the supersymmetric variation of the bulk action, proving that it results in a boundary term.
In Section 3.5 we describe the Hamilton-Jacobi method for holographic renormalization and we find the covariant counterterms for $\mathcal{N}=2$, $D=4$ gauged AdS supergravity.
In Section 3.6 we study the near-boundary behaviour of bulk + boundary action variation.
In the end in Section 3.7 we prove that the bulk + boundary action variation exactly vanishes in the limit $r \to \infty$.

3.1 $\mathcal{N}=1$, $D=4$ pure supergravity

The simplest theory of supergravity, i.e. $\mathcal{N}=1$, $D=4$ pure supergravity, consists of the Hilbert and Rarita-Schwinger actions only. Its explicit form reads

$$S \equiv \frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\Big(R - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho\Big) \equiv S_2 + S_{3/2}, \tag{3.3}$$

where e stands for the determinant of the frame field e^a_μ, R is the Ricci scalar and the gravitino covariant derivative is given by

$$D_\mu\psi_\nu = \Big(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\Big)\psi_\nu. \tag{3.4}$$

We omitted the Christoffel symbols in the previous expression because their symmetric indices are contracted with $\gamma^{\mu\nu\rho}$ (which is completely antisymmetric) in $S_{3/2}$.
In order to prove the invariance of the action under local supersymmetry, we need the supersymmetric transformation rules. They read

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta e = \frac{1}{2}e\bar{\epsilon}\gamma^\rho\psi_\rho, \qquad \delta\psi_\mu = D_\mu\epsilon. \tag{3.5}$$

The prove of the invariance of the action under local supersymmetric transformations is really straightforward. Indeed the variations of the two pieces yield

$$\begin{aligned}\delta S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^4x\left[\left(2e\delta e^{a\mu}e^{b\nu}+\delta e e^{a\mu}e^{b\nu}\right)R_{\mu\nu ab}+ee^{a\mu}e^{b\nu}\delta R_{\mu\nu ab}\right]=\\ &=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\Big(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\Big)\left(-\bar{\epsilon}\gamma^\mu\psi^\nu\right),\\ \delta S_{3/2} &= -\frac{1}{\kappa^2}\int \mathrm{d}^4x\, e\bar{\epsilon}\overleftarrow{D}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\Big(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\Big)\left(\bar{\epsilon}\gamma^\mu\psi^\nu\right),\end{aligned} \tag{3.6}$$

where we omitted the boundary terms. The calculations in full details can be found in Chapter 9 of [5].
Notice that δS_2 is proportional to the geometric part of Einstein equation, as one could expect.

A very simple extension of this supergravity theory is $\mathcal{N}=1$, $D=4$ pure AdS supergravity. We are interested in this theory because it represents a close example to the supergravity theory we want to study, namely $\mathcal{N}=2$, $D=4$ gauged AdS supergravity. For this reason we will discuss it in the next section.

3.2 $\mathcal{N}=1$, $D=4$ pure AdS supergravity

$\mathcal{N}=1$ four dimensional pure AdS supergravity doesn't require new fields compared to the theory of the previous section. One starts defining the gravitino covariant derivative as

$$\hat{D}_\mu\psi_\nu \equiv \Big(D_\mu-\frac{1}{2\ell}\gamma_\mu\Big)\psi_\nu=\Big(\partial_\mu+\frac{1}{4}\omega_{\mu ab}\gamma^{ab}-\frac{1}{2\ell}\gamma_\mu\Big)\psi_\nu, \tag{3.7}$$

where the Christoffel symbols are omitted because they will be contracted with $\gamma^{\mu\nu\rho}$ in the action. From this definition we obtain the relation

$$[\hat{D}_\mu,\hat{D}_\nu]\epsilon=\frac{1}{4}\Big(R_{\mu\nu ab}+\frac{1}{\ell^2}\left(e_{a\mu}e_{b\nu}-e_{b\mu}e_{a\nu}\right)\Big)\gamma^{ab}\epsilon\equiv\hat{R}_{\mu\nu ab}\gamma^{ab}\epsilon, \tag{3.8}$$

which implies

$$\begin{aligned}\hat{R}_{\mu a} &\equiv R_{\mu\nu ab}e^{b\nu} = R_{\mu a} + \frac{3}{\ell^2}e_{a\mu},\\ \hat{R} &\equiv \hat{R}_{\mu a}e^{a\mu} = R + \frac{6}{\ell^2}.\end{aligned} \tag{3.9}$$

Substituting the quantities in the expression (3.3) with the "hat"-quantities, we obtain the action of the theory

$$\begin{aligned}S &\equiv \frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\Big(\hat{R} - \bar{\psi}_\mu\gamma^{\mu\nu\rho}\hat{D}_\nu\psi_\rho\Big) =\\ &= \frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\Big(R - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu - 2\Lambda\Big)\\ &\equiv S_2 + S_{3/2} + S_m + S_\Lambda,\end{aligned} \tag{3.10}$$

where the negative cosmological constant is $\Lambda = -\frac{3}{\ell^2}$.
As the previous case, we provide the supersymmetric transformation of the fields

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta e = \frac{1}{2}e\bar{\epsilon}\gamma^\rho\psi_\rho, \qquad \delta\psi_\mu = \hat{D}_\mu\epsilon. \tag{3.11}$$

The corresponding variation of the action

$$\begin{aligned}\delta S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\Big[\big(2e\delta e^{a\mu}e^{b\nu} + \delta e e^{a\mu}e^{b\nu}\big)R_{\mu\nu ab} + ee^{a\mu}e^{b\nu}\delta R_{\mu\nu ab}\Big] =\\ &= \frac{1}{2\kappa^2}\int \mathrm{d}^4x\, e\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big)\big(-\bar{\epsilon}\gamma^\mu\psi^\nu\big),\\[2ex] \delta S_{3/2} &= -\frac{1}{\kappa^2}\int \mathrm{d}^4x\, e\bar{\epsilon}\Big(\overleftarrow{D}_\mu + \frac{1}{2\ell}\gamma_\mu\Big)\gamma^{\mu\nu\rho}D_\nu\psi_\rho,\\[2ex] \delta S_m &= -\frac{1}{\kappa^2}\int \mathrm{d}^4x\, e\bar{\psi}_\mu\gamma^{\mu\nu}\Big(D_\nu - \frac{1}{2\ell}\gamma_\nu\Big)\epsilon,\\[2ex] \delta S_\Lambda &= \frac{3}{2\kappa^2\ell^2}\int \mathrm{d}^4x\,\bar{\epsilon}\gamma^\mu\psi^\nu g_{\mu\nu}.\end{aligned} \tag{3.12}$$

After some algebra and integrations by parts, one sees that δS is the integral of a total derivative. Thus the action S is locally supersymmetric up to boundary terms.

In the following section we will analyse the four dimensional $\mathcal{N}=2$ supergravity action, which is very similar to the $\mathcal{N}=1$, $D=4$ one. It is supersymmetric up to boundary terms too, but we want to employ it in the AdS/CFT correspondence. That means we can't simply set to zero the fields on the boundary of AdS in order to have a supersymmetric action as it would result in possibly unwanted boundary conditions, because we wish to keep arbitrary boundary values of the fields. Indeed, as mentioned in the previous chapter, from the AdS/CFT correspondence point of view we interpret the on-shell action as the generating functional of the connected functions and the boundary values of bulk fields as the sources in the dual CFT. Hence we desire to keep an arbitrary dependency of the generating functional from the sources in order to prove that the conformal quantum field theory generating functional is supersymmetric. Thus we need to add specific supersymmetric counterterms whose variation cancels the one of the bulk action.

3.3 Minimal $\mathcal{N}=2$ $D=4$ gauged AdS supergravity

We are going to study the minimal $\mathcal{N}=2$ gauged supergravity in four dimensions and verify that its action is invariant under supersymmetry transformations up to a boundary term. Then we will add the correct boundary counterterms to make the action supersymmetric invariant at the boundary too.
That theory includes the gravity multiplet only, made up of the metric $G_{\mu\nu}$, one Dirac gravitino (or equivalently two real Majorana gravitinos) Ψ_μ and a graviphoton A_μ, gauge field of the gauged $U(1)_R$ group[1].
This theory was found by Freedman and Das in 1977 [20]. There the rigid $SO(2)\sim U(1)_R$ symmetry rotating the two independent Majorana gravitinos present in the ungauged theory, is made local by introduction of a minimal gauge coupling $\mathfrak{g}=\frac{1}{\ell}$ between the graviphoton and the gravitino. Local supersymmetry then requires a negative cosmological constant and a gravitino mass term.
Thus the bulk action reads

$$\begin{aligned}S=&\frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-G}\Big[R-F^{\mu\nu}F_{\mu\nu}-2\Lambda-\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\Big(\overleftrightarrow{\nabla}_\nu+\frac{2\mathrm{i}}{\ell}A_\nu\Big)\Psi_\rho-\\&-\frac{2}{\ell}\bar{\Psi}_\mu\Big(\gamma^{\mu\nu}+\frac{\mathrm{i}\ell}{2}\big(\gamma^{\mu\nu\rho\sigma}F_{\rho\sigma}+2F^{\mu\nu}\big)\Big)\Psi_\nu\Big],\end{aligned} \tag{3.13}$$

where the gravitational constant κ and the cosmological constant Λ are defined as

$$\kappa^2=8\pi G_4,\qquad \Lambda=-\frac{3}{\ell^2}. \tag{3.14}$$

[1]The $U(1)_R$ gauged group is a subgroup of the complete R-symmetry group $U(2)_R$. We could only gauge this abelian subgroup because the gravity multiplet only contains one vector field.

The covariant derivative of the gravitino is

$$\nabla_\mu \Psi_\nu = \partial_\mu \Psi_\nu + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\Psi_\nu - \Gamma^\rho_{\mu\nu}\Psi_\rho, \tag{3.15}$$

where $\omega_{\mu\alpha\beta}$ is the spin connection and $\Gamma^\rho_{\mu\nu}$ is the Christoffel connection.
The supersymmetric variations are

$$\begin{aligned}
\delta e^\alpha_\mu &= \frac{1}{2}\big(\bar\epsilon\gamma^\alpha\Psi_\mu - \bar\Psi_\mu\gamma^\alpha\epsilon\big),\\
\delta A_\mu &= \frac{\mathrm{i}}{2}\big(\bar\Psi_\mu\epsilon - \bar\epsilon\Psi_\mu\big),\\
\delta\Psi_\mu &= \nabla_\mu\epsilon + \frac{\mathrm{i}}{4}\big(\gamma_\mu{}^{\nu\rho} - 2\delta^\nu_\mu\gamma^\rho\big)F_{\nu\rho}\epsilon - \frac{1}{2\ell}\big(\gamma_\mu - 2\mathrm{i}A_\mu\big)\epsilon,\\
\delta\bar\Psi_\mu &= \bar\epsilon\overleftarrow{\nabla}_\mu + \frac{\mathrm{i}}{4}\bar\epsilon\big(\gamma^{\rho\nu}{}_\mu - 2\delta^\nu_\mu\gamma^\rho\big)F_{\nu\rho} + \frac{1}{2\ell}\bar\epsilon\big(\gamma_\mu - 2\mathrm{i}A_\mu\big).
\end{aligned} \tag{3.16}$$

We can also obtain the variation of the inverse vielbein from (3.16)

$$\begin{aligned}
&e^\alpha_\mu e^\nu_\alpha \equiv \delta^\nu_\mu \quad \rightarrow \quad \delta(e^\alpha_\mu e^\nu_\alpha) = 0 = \delta e^\alpha_\mu e^\nu_\alpha + \delta e^\nu_\alpha e^\alpha_\mu\\
&\implies \delta e^\mu_\alpha = -\frac{1}{2}\big(\bar\epsilon\gamma^\mu\Psi_\alpha - \bar\Psi_\alpha\gamma^\mu\epsilon\big).
\end{aligned} \tag{3.17}$$

3.4 The variation of the bulk action

First of all we compute explicitly the variation of each term in the bulk action shown in Eq. (3.13), recalling that we work at quadratic order in the gravitino terms. The result already appeared in Appendix C of [12], but we have performed the computation and checked it in full detail.
Let us label the different pieces

$$\begin{aligned}
S =& \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-G}\Big[\underbrace{R}_{(a)} - \underbrace{F^{\mu\nu}F_{\mu\nu}}_{(b)} - \underbrace{2\Lambda}_{(c)} - \bar\Psi_\mu\gamma^{\mu\nu\rho}\Big(\underbrace{\overleftrightarrow{\nabla}_\nu}_{(d)} + \underbrace{\frac{2\mathrm{i}}{\ell}A_\nu}_{(e)}\Big)\Psi_\rho -\\
&- \frac{2}{\ell}\bar\Psi_\mu\Big(\underbrace{\gamma^{\mu\nu}}_{(f)} + \frac{\mathrm{i}\ell}{2}\big(\underbrace{\gamma^{\mu\nu\rho\sigma}F_{\rho\sigma}}_{(g)} + \underbrace{2F^{\mu\nu}}_{(h)}\big)\Big)\Psi_\nu\Big].
\end{aligned} \tag{3.18}$$

We will omit the Christoffel symbols in the covariant derivative of the gravitino because they are always contracted with the gamma matrices and there is no torsion.

Bosonic terms It is useful to remind the expression for the variation of $\sqrt{-G}$.
It is known that $\log(\det A) = Tr(\log A)$. Then

$$\delta\log(\det A) = \delta Tr(\log A) = Tr(\delta\log A) = Tr(A^{-1}\delta A) = \frac{\delta\det A}{\det A}. \tag{3.19}$$

Taking $A = G_{\mu\nu}$ in (3.19), we obtain

$$\delta G = G\cdot G^{\mu\nu}\delta G_{\mu\nu} = G\cdot\delta(G^{\mu\nu}G_{\mu\nu}) - G\cdot\delta G^{\mu\nu}G_{\mu\nu} = -G\cdot\delta G^{\mu\nu}G_{\mu\nu} \tag{3.20}$$

and using (3.20), the variation of $\sqrt{-G}$ is

$$\frac{\delta(\sqrt{-G})}{\delta G^{\mu\nu}}\delta G^{\mu\nu} = \frac{\sqrt{-G}}{2}(-G_{\mu\nu})\delta G^{\mu\nu}. \tag{3.21}$$

We start computing Einstein-Hilbert action

$$\begin{aligned}(a):\quad \delta(\sqrt{-G}R) &= -R\frac{\sqrt{-G}}{2}G_{\mu\nu}\delta G^{\mu\nu} + \sqrt{-G}\frac{\delta R}{\delta G^{\mu\nu}}\delta G^{\mu\nu} = \\ &= \sqrt{-G}\Big(R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} + G^{\rho\sigma}\frac{\delta R_{\rho\sigma}}{\delta G^{\mu\nu}}\Big)\delta G^{\mu\nu} = \\ &= -\Big[\sqrt{-G}\Big(R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu}\Big)\bar{\epsilon}\gamma^{\mu}\Psi^{\nu} + \text{h.c.}\Big] + \sqrt{-G}G^{\rho\sigma}\delta R_{\rho\sigma},\end{aligned} \tag{3.22}$$

where h.c. denotes the hermitian conjugate.
Let us study the variation of the Riemann tensor

$$\delta R_{\mu\nu\ \sigma}^{\ \ \rho} = 2\partial_{[\mu}\delta\Gamma^{\rho}_{\nu]\sigma} + 2\delta\Gamma^{\rho}_{[\mu|\tau|}\Gamma^{\tau}_{\nu]\sigma} + 2\Gamma^{\rho}_{[\mu|\tau|}\delta\Gamma^{\tau}_{\nu]\sigma} = 2\nabla_{[\mu}\delta\Gamma^{\rho}_{\nu]\sigma}, \tag{3.23}$$

with the last step justified by

$$\nabla_{[\mu}\delta\Gamma^{\rho}_{\nu]\sigma} = \partial_{[\mu}\delta\Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{[\mu|\tau|}\delta\Gamma^{\tau}_{\nu]\sigma} - \underbrace{\Gamma^{\tau}_{[\mu\nu]}\delta\Gamma^{\rho}_{\tau\sigma}}_{=0} - \Gamma^{\tau}_{[\mu|\sigma|}\delta\Gamma^{\rho}_{\nu]\tau}. \tag{3.24}$$

From (3.23) it follows that

$$\begin{aligned}\delta R_{\rho\sigma}G^{\rho\sigma} &= 2G^{\rho\sigma}\nabla_{[\tau}\delta\Gamma^{\tau}_{\rho]\sigma} = \nabla_{\tau}\big(G^{\rho\sigma}\delta\Gamma^{\tau}_{\rho\sigma}\big) - \nabla_{\rho}\big(G^{\rho\sigma}\delta\Gamma^{\tau}_{\tau\sigma}\big) = \\ &= \nabla_{\mu}\big(G^{\rho\sigma}\delta\Gamma^{\mu}_{\rho\sigma} - G^{\mu\sigma}\delta\Gamma^{\tau}_{\tau\sigma}\big).\end{aligned} \tag{3.25}$$

It is useful to prove that $\int \mathrm{d}^4x\,\sqrt{-G}\nabla_{\mu}j^{\mu} = \int \mathrm{d}^4x\,\partial_{\mu}(\sqrt{-G}j^{\mu})$.
Preliminarily we demonstrate the equivalence $\Gamma^{\rho}_{\mu\rho} = \partial_{\mu}(\log\sqrt{-G})$. Indeed

$$\Gamma^{\rho}_{\mu\rho} = \frac{1}{2}G^{\rho\sigma}(\partial_{\mu}G_{\rho\sigma} + \partial_{\rho}G_{\mu\sigma} - \partial_{\sigma}G_{\mu\rho}) = \frac{1}{2}G^{\rho\sigma}\partial_{\mu}G_{\rho\sigma},$$

$$\begin{aligned}\partial_{\mu}(\log\sqrt{-G}) &= \frac{1}{2}\partial_{\mu}\log(-\det G_{\rho\sigma}) = \frac{1}{2}\partial_{\mu}\log\big|\det G_{\rho\sigma}\big| = \\ &= \frac{1}{2}\partial_{\mu}Tr\log G_{\rho\sigma} = \frac{1}{2}Tr\,\partial_{\mu}\log G_{\rho\sigma} = \frac{1}{2}Tr\,G^{\rho\sigma}\partial_{\mu}G_{\nu\tau} = \\ &= \frac{1}{2}G^{\rho\sigma}\partial_{\mu}G_{\rho\sigma}.\end{aligned} \tag{3.26}$$

Thus we obtain that

$$\begin{aligned}\nabla_\mu j^\mu &= \partial_\mu j^\mu + \Gamma^\rho_{\rho\mu} j^\mu = \partial_\mu j^\mu + \frac{1}{2}\partial_\mu \log(-G) j^\mu = \partial_\mu j^\mu + \frac{\partial_\mu G}{2G} j^\mu = \\ &= \frac{1}{\sqrt{-G}}\partial_\mu(\sqrt{-G} j^\mu).\end{aligned} \tag{3.27}$$

Hence the last term in (3.22) reads

$$G^{\rho\sigma}\delta R_{\rho\sigma} = \frac{1}{\sqrt{-G}}\partial_\mu\big[\sqrt{-G}\big(G^{\rho\sigma}\delta\Gamma^\mu_{\rho\sigma} - G^{\mu\sigma}\delta\Gamma^\tau_{\tau\sigma}\big)\big]. \tag{3.28}$$

Now we need to write the explicit form of $\delta\Gamma^\mu_{\rho\sigma}$. Let us start from

$$\nabla_\lambda G_{\mu\nu} = 0 = \partial_\lambda G_{\mu\nu} - G_{\tau\nu}\Gamma^\tau_{\mu\lambda} - G_{\mu\tau}\Gamma^\tau_{\nu\lambda} \tag{3.29}$$

and differentiating it, we obtain

$$\begin{aligned}0 &= \partial_\lambda \delta G_{\mu\nu} - \delta G_{\tau\nu}\Gamma^\tau_{\mu\lambda} - \delta G_{\mu\tau}\Gamma^\tau_{\nu\lambda} - G_{\tau\nu}\delta\Gamma^\tau_{\mu\lambda} - G_{\mu\tau}\delta\Gamma^\tau_{\nu\lambda} = \\ &= \nabla_\lambda \delta G_{\mu\nu} - G_{\tau\nu}\delta\Gamma^\tau_{\mu\lambda} - G_{\mu\tau}\delta\Gamma^\tau_{\nu\lambda}.\end{aligned} \tag{3.30}$$

We sum (3.30) with itself where we change $\lambda \leftrightarrow \mu$ and we subtract itself with $\begin{cases}\mu \to \lambda \\ \nu \to \mu \\ \lambda \to \nu\end{cases}$; we get

$$\begin{aligned}&\nabla_\lambda \delta G_{\mu\nu} - G_{\tau\nu}\delta\Gamma^\tau_{\mu\lambda} - G_{\mu\tau}\delta\Gamma^\tau_{\nu\lambda} + \nabla_\mu \delta G_{\lambda\nu} - G_{\tau\nu}\delta\Gamma^\tau_{\lambda\mu} - G_{\lambda\tau}\delta\Gamma^\tau_{\nu\mu} - \\ &- \nabla_\nu \delta G_{\lambda\mu} + G_{\tau\mu}\delta\Gamma^\tau_{\lambda\nu} + G_{\lambda\tau}\delta\Gamma^\tau_{\mu\nu} = 0 \\ &\implies 2G_{\tau\nu}\delta\Gamma^\tau_{\mu\lambda} = \nabla_\lambda \delta G_{\mu\nu} + \nabla_\mu \delta G_{\lambda\nu} - \nabla_\nu \delta G_{\lambda\mu} \\ &\implies \delta\Gamma^\rho_{\mu\lambda} = \frac{1}{2}G^{\nu\rho}\big(\nabla_\lambda \delta G_{\mu\nu} + \nabla_\mu \delta G_{\lambda\nu} - \nabla_\nu \delta G_{\lambda\mu}\big).\end{aligned} \tag{3.31}$$

By plugging the last equation into (3.28), we have

$$\sqrt{-G}G^{\rho\sigma}\delta R_{\rho\sigma} = \partial_\mu\Big(\sqrt{-G}\nabla_\nu\big(\bar\epsilon\gamma^{(\mu}\Psi^{\nu)}\big) - \sqrt{-G}\nabla^\mu\big(\bar\epsilon\gamma^\rho\Psi_\rho\big)\Big) + \text{h.c.}. \tag{3.32}$$

Thus the final form of (3.22) is

$$\begin{aligned}(a): \quad \delta(\sqrt{-G}R) = &-\sqrt{-G}\Big(R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu}\Big)\bar\epsilon\gamma^\mu\Psi^\nu + \\ &+ \partial_\mu\Big(\sqrt{-G}\nabla_\nu\big(\bar\epsilon\gamma^{(\mu}\Psi^{\nu)}\big) - \sqrt{-G}\nabla^\mu\big(\bar\epsilon\gamma^\rho\Psi_\rho\big)\Big) + \text{h.c.}.\end{aligned} \tag{3.33}$$

The variation of Maxwell and cosmological constant actions are straightforward. They read

$$\begin{aligned}(b): \quad \delta(\sqrt{-G}F^2) &= -F^2\frac{\sqrt{-G}}{2}G_{\mu\nu}\delta G^{\mu\nu} + 2\sqrt{-G}F_\mu{}^\rho F_{\nu\rho}\delta G^{\mu\nu} + \\ &+ 4\sqrt{-G}F^{\mu\nu}\nabla_\mu\delta A_\nu = \\ &= \Big(F^2\frac{\sqrt{-G}}{2}G^{\mu\nu} - 2\sqrt{-G}F^{\mu\rho}F^\nu{}_\rho\Big)\bar\epsilon\gamma_\mu\Psi_\nu - \\ &- 2\mathrm{i}\sqrt{-G}F^{\mu\nu}\nabla_\mu\big(\bar\epsilon\Psi_\nu\big) + \text{h.c.},\end{aligned} \tag{3.34}$$

$$(c): \quad \delta(\sqrt{-G}\cdot 2\Lambda) = 2\Lambda\frac{-\sqrt{-G}}{2}G_{\mu\nu}\delta G^{\mu\nu} = \Lambda\sqrt{-G}G_{\mu\nu}\bar\epsilon\gamma^\mu\Psi^\nu + \text{h.c.}. \tag{3.35}$$

Fermionic terms We are going to study the variation of the fermionic terms using the identities in Appendix A. The first one is the Rarita-Schwinger action

$$\begin{aligned}(d): \quad &\delta\big(\sqrt{-G}\bar\Psi_\mu\gamma^{\mu\nu\rho}\overleftrightarrow{\nabla}_\nu\Psi_\rho\big) = \\ &= \sqrt{-G}\delta\bar\Psi_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \sqrt{-G}\nabla_\nu\big(\delta\bar\Psi_\mu\big)\gamma^{\mu\nu\rho}\Psi_\rho + \text{h.c.} = \\ &= 2\sqrt{-G}\delta\bar\Psi_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \sqrt{-G}\nabla_\nu\big(\bar\Psi_\mu\gamma^{\mu\nu\rho}\Psi_\rho\big) + \text{h.c.} = \\ &= 2\sqrt{-G}\delta\bar\Psi_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \partial_\nu\big(\sqrt{-G}\bar\Psi_\mu\gamma^{\mu\nu\rho}\Psi_\rho\big) + \text{h.c.} = \\ &= \partial_\mu\big(2\sqrt{-G}\bar\epsilon\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \sqrt{-G}\delta\bar\Psi_\nu\gamma^{\nu\mu\rho}\Psi_\rho\big) - \underbrace{2\sqrt{-G}\bar\epsilon\gamma^{\mu\nu\rho}\nabla_\mu\nabla_\nu\Psi_\rho}_{(A)} + \\ &+ \frac{2}{\ell}\sqrt{-G}\bar\epsilon\gamma^{\mu\nu}\nabla_\mu\Psi_\nu - \frac{2\mathrm{i}}{\ell}\sqrt{-G}A_\mu\bar\epsilon\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho + \\ &+ \underbrace{\frac{\mathrm{i}}{2}\sqrt{-G}\bar\epsilon\big(\gamma^{\tau\sigma}{}_\mu - 2\delta^\sigma_\mu\gamma^\tau\big)F_{\sigma\tau}\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho}_{(B)} + \text{h.c.},\end{aligned} \tag{3.36}$$

where in the second equivalence we integrated by parts the second term and remembered $\nabla_\mu\gamma_\nu = 0$. In the last step we integrated by parts the first term and used (A.12).

Let us elaborate the terms (A) and (B). Remembering the equivalence $[\nabla_\mu, \nabla_\nu]\Psi_\rho = \frac{1}{4}R_{\mu\nu\alpha\beta}\gamma^{\alpha\beta}\Psi_\rho$, we get

$$\begin{aligned}(A) = -2\sqrt{-G}\bar\epsilon\gamma^{\mu\nu\rho}\nabla_\mu\nabla_\nu\Psi_\rho &= -2\sqrt{-G}\bar\epsilon\gamma^{\mu\nu\rho}\cdot\frac{1}{2}[\nabla_\mu, \nabla_\nu]\Psi_\rho = \\ &= -\frac{1}{4}\sqrt{-G}\bar\epsilon\gamma^{\mu\nu\rho}\gamma_{\alpha\beta}R_{\mu\nu}{}^{\alpha\beta}\Psi_\rho.\end{aligned} \tag{3.37}$$

Then using (A.13), the equation above becomes

$$
\begin{aligned}
-2\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\mu\nabla_\nu\Psi_\rho =& \frac{1}{2}\sqrt{-G}\bar{\epsilon}\Big(R_{\mu\nu\beta}{}^{\rho}\gamma^{\mu\nu\beta} + R_{\mu\nu\beta}{}^{\nu}\gamma^{\rho\mu\beta} - R_{\nu\mu\beta}{}^{\mu}\gamma^{\nu\rho\beta}\Big)\Psi_\rho - \\
&- \frac{1}{2}\sqrt{-G}\bar{\epsilon}\Big(R_{\mu\rho}\gamma^\mu\Psi^\rho + R_{\nu\rho}\gamma^\nu\Psi^\rho - RG_{\rho\sigma}\gamma^\rho\Psi^\sigma\Big),
\end{aligned}
\tag{3.38}
$$

where the first term is zero because $R_{[\mu\nu\beta]\rho} = 0$ and $R_{[\mu\nu]} = 0$. Thus we obtain

$$
-2\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\mu\nabla_\nu\Psi_\rho = -\sqrt{-G}\big(R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu}\big)\bar{\epsilon}\gamma^\mu\Psi^\nu. \tag{3.39}
$$

Now we deal with the term (B), using (A.14) and (A.15)

$$
\begin{aligned}
(B) =& \frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}\big(\gamma^{\tau\sigma}{}_{\mu} - 2\delta^\sigma_\mu\gamma^\tau\big)F_{\sigma\tau}\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho = \\
=& \frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}\big(\underbrace{4\gamma^\sigma{}_{[\rho}\delta^\tau{}_{\nu]}}_{(C)} + 4\delta^\sigma_{[\rho}\delta^\tau_{\nu]}\big)F_{\tau\sigma}\nabla^\nu\Psi^\rho - \mathrm{i}\sqrt{-G}\bar{\epsilon}F_\mu{}^\sigma\big(\gamma_\sigma{}^{\mu\nu\rho} + \underbrace{3\delta^{[\mu}_\sigma\gamma^{\nu\rho]}}_{(D)}\big)\nabla_\nu\Psi_\rho.
\end{aligned}
\tag{3.40}
$$

Writing explicitly (C) and (D), one sees that their sum vanishes.

The final form of (3.36) is

$$
\begin{aligned}
(d):\quad & \delta\big(\sqrt{-G}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\overleftrightarrow{\nabla}_\nu\Psi_\rho\big) = \partial_\mu\big(2\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \sqrt{-G}\delta\bar{\Psi}_\nu\gamma^{\nu\mu\rho}\Psi_\rho\big) + \\
&+ \frac{2}{\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}\nabla_\mu\Psi_\nu - 2\mathrm{i}\frac{1}{\ell}\sqrt{-G}A_\mu\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \\
&- \sqrt{-G}\big(R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu}\big)\bar{\epsilon}\gamma^\mu\Psi^\nu + \mathrm{i}\sqrt{-G}F^{\sigma\tau}\bar{\epsilon}\big(-\gamma_{\tau\sigma}{}^{\nu\rho} + 2\delta^{[\rho}_{[\tau}\delta^{\nu]}_{\sigma]}\big)\nabla_\nu\Psi_\rho + \text{h.c.}
\end{aligned}
\tag{3.41}
$$

There is a second piece in the Rarita-Schwinger action, due to the gauging of $U(1)_R$. Its variation reads

$$
\begin{aligned}
(e):\quad & \delta\big(\sqrt{-G}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}A_\nu\Psi_\rho\big) = \sqrt{-G}\delta\bar{\Psi}_\mu\gamma^{\mu\nu\rho}A_\nu\Psi_\rho + \text{h.c.} = \\
&= \sqrt{-G}\nabla_\mu\big(\bar{\epsilon}\gamma^{\mu\nu\rho}A_\nu\Psi_\rho\big) - \sqrt{-G}\bar{\epsilon}\nabla_\mu A_\nu\gamma^{\mu\nu\rho}\Psi_\rho - \\
&- \sqrt{-G}\bar{\epsilon}A_\nu\gamma^{\mu\nu\rho}\nabla_\mu\Psi_\rho + \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}\big(\gamma^{\sigma\tau}{}_{\mu} - 2\delta^\tau_\mu\gamma^\sigma\big)F_{\tau\sigma}\gamma^{\mu\nu\rho}A_\nu\Psi_\rho + \\
&+ \frac{1}{2\ell}\sqrt{-G}\bar{\epsilon}\big(\gamma_\mu - 2\mathrm{i}A_\mu\big)\gamma^{\mu\nu\rho}A_\nu\Psi_\rho + \text{h.c.} = \\
&= \partial_\mu\big(\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}A_\nu\Psi_\rho\big) - \frac{1}{2}\sqrt{-G}\bar{\epsilon}F_{\mu\nu}\gamma^{\mu\nu\rho}\Psi_\rho + \sqrt{-G}\bar{\epsilon}A_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho + \\
&+ \frac{1}{\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}A_\mu\Psi_\nu + \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}\big(\gamma^{\sigma\tau}{}_{\mu} - 2\delta^\tau_\mu\gamma^\sigma\big)F_{\tau\sigma}\gamma^{\mu\nu\rho}A_\nu\Psi_\rho + \text{h.c.}.
\end{aligned}
\tag{3.42}
$$

In the last step we used $-\frac{\mathrm{i}}{\ell}\sqrt{-G}A_\mu\gamma^{\mu\nu\rho}A_\nu\Psi_\rho = 0$ and $\nabla_\mu A_\nu\gamma^{\mu\nu\rho} = \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu\rho}$. We manipulate the last term in (3.42) using (A.14) and (A.15)

$$\begin{aligned}
&\frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}\big(\gamma^{\sigma\tau}{}_\mu - 2\delta^\tau_\mu\gamma^\sigma\big)F_{\tau\sigma}\gamma^{\mu\nu\rho}A_\nu\Psi_\rho = \\
&= \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}F_{\tau\sigma}A^\nu\big(4\gamma^\sigma{}_{[\rho}\delta^\tau_{\nu]} + 4\delta^\sigma{}_{[\rho}\delta^\tau_{\nu]}\big)\Psi^\rho - \frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F_\mu{}^\sigma\big(\gamma_\sigma{}^{\mu\nu\rho} + 3\delta^{[\mu}_\sigma\gamma^{\nu\rho]}\big)A_\nu\Psi_\rho = \\
&= -\frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\gamma_{\tau\sigma}{}^{\nu\rho}A_\nu\Psi_\rho + \mathrm{i}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\cdot 2\delta^{[\rho}_{[\tau}\delta^{\nu]}_{\sigma]}A_\nu\Psi_\rho.
\end{aligned} \tag{3.43}$$

The final form of (3.42) is

$$\begin{aligned}
(e):\quad &\delta\big(\sqrt{-G}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho\big) = \partial_\mu\big(\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}A_\nu\Psi_\rho\big) - \frac{1}{2}\sqrt{-G}\bar{\epsilon}F_{\mu\nu}\gamma^{\mu\nu\rho}\Psi_\rho + \\
&+ \sqrt{-G}\bar{\epsilon}A_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho + \frac{1}{\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}A_\mu\Psi_\nu - \\
&- \frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\gamma_{\tau\sigma}{}^{\nu\rho}A_\nu\Psi_\rho + 2\mathrm{i}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\delta^{[\rho}_{[\tau}\delta^{\nu]}_{\sigma]}A_\nu\Psi_\rho + \mathrm{h.c.}.
\end{aligned} \tag{3.44}$$

The variation of the gravitino mass term is

$$\begin{aligned}
(f):\quad &\delta\big(\sqrt{-G}\bar{\Psi}_\mu\gamma^{\mu\nu}\Psi_\nu\big) = \sqrt{-G}\delta\bar{\Psi}_\mu\gamma^{\mu\nu}\Psi_\nu + \mathrm{h.c.} = \\
&= \sqrt{-G}\Big[\nabla_\mu\bar{\epsilon}\gamma^{\mu\nu}\Psi_\nu + \frac{\mathrm{i}}{4}\bar{\epsilon}\big(\gamma^{\rho\sigma}{}_\mu - 2\delta^\sigma_\mu\gamma^\rho\big)F_{\sigma\rho}\gamma^{\mu\nu}\Psi_\nu + \\
&+ \frac{1}{2\ell}\bar{\epsilon}\big(\gamma_\mu - 2\mathrm{i}A_\mu\big)\gamma^{\mu\nu}\Psi_\nu\Big] + \mathrm{h.c.}.
\end{aligned} \tag{3.45}$$

Integrating by parts the first term and manipulating the second one through (A.16) and (A.17), we have

$$\frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}\big(\gamma^{\rho\sigma}{}_\mu - 2\delta^\sigma_\mu\gamma^\rho\big)F_{\sigma\rho}\gamma^{\mu\nu}\Psi_\nu = \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\gamma_{\sigma\tau}{}^\nu\Psi_\nu - \frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\gamma_\sigma\Psi_\tau. \tag{3.46}$$

The final form of (3.45) is

$$\begin{aligned}
(f):\quad &\delta\big(\sqrt{-G}\bar{\Psi}_\mu\gamma^{\mu\nu}\Psi_\nu\big) = \\
&= \partial_\mu\big(\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}\Psi_\nu\big) - \sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}\nabla_\mu\Psi_\nu + \frac{3}{2\ell}\sqrt{-G}\bar{\epsilon}\gamma^\nu\Psi_\nu - \\
&- \frac{\mathrm{i}}{\ell}\sqrt{-G}A_\mu\bar{\epsilon}\gamma^{\mu\nu}\Psi_\nu + \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\big(\gamma_{\sigma\tau}{}^\nu - 2\gamma_\sigma\delta^\nu_\tau\big)\Psi_\nu + \mathrm{h.c.}.
\end{aligned} \tag{3.47}$$

In the end we study the variation of (g) and (h) terms

$$
\begin{aligned}
(g):\quad & \delta\big(\sqrt{-G}\bar{\Psi}_\mu\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}\big) = \sqrt{-G}\delta\bar{\Psi}_\mu\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma} + \text{h.c.} = \\
& = \partial_\mu\big(\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}\big) - \sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho\sigma}\nabla_\mu\Psi_\nu F_{\rho\sigma} - \\
& - \sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho\sigma}\Psi_\nu\nabla_\mu F_{\rho\sigma} + \frac{1}{2\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\nu\rho\sigma}\Psi_\nu F_{\rho\sigma} - \frac{\mathrm{i}}{\ell}\sqrt{-G}\bar{\epsilon}A_\mu\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma} + \\
& + \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}\big(\gamma^{\tau\omega}{}_\mu - 2\delta^\omega_\mu\gamma^\tau\big)F_{\omega\tau}\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma} + \text{h.c.},
\end{aligned}
\tag{3.48}
$$

where we used (A.18) in the last step. Further we notice that the third term vanishes because of the Bianchi identity.
Finally we manipulate the last term of (3.48) remembering (A.19)

$$
\begin{aligned}
& \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}\big(\gamma^{\tau\omega}{}_\mu - 2\delta^\omega_\mu\gamma^\tau\big)F_{\omega\tau}\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma} = \\
& = \frac{3\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}\delta^{[\nu}_\omega\delta^\rho_\tau\gamma^{\sigma]}\Psi_\nu F_{\rho\sigma}F^{\omega\tau} - \frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F_\mu{}^\nu\gamma^{\mu\sigma\rho}\Psi_\nu F_{\rho\sigma} - \\
& - \frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F_\mu{}^\tau\big(\gamma^{\nu\sigma\mu}\Psi_\nu F_{\tau\sigma} + \gamma^{\nu\mu\rho}\Psi_\nu F_{\rho\tau}\big),
\end{aligned}
\tag{3.49}
$$

where the last line vanishes because

$$
F_\mu{}^\tau F_{\tau\sigma}\gamma^{\nu\sigma\mu} = F_\sigma{}^\tau F_{\tau\mu}\gamma^{\nu\mu\sigma} = -F_\mu{}^\tau F_{\tau\sigma}\gamma^{\nu\sigma\mu} = 0. \tag{3.50}
$$

The final form of (3.48) is

$$
\begin{aligned}
(g):\quad & \delta\big(\sqrt{-G}\bar{\Psi}_\mu\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}\big) = \\
& = \partial_\mu\big(\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}\big) - \sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho\sigma}\nabla_\mu\Psi_\nu F_{\rho\sigma} + \frac{1}{2\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\nu\rho\sigma}\Psi_\nu F_{\rho\sigma} - \\
& - \frac{\mathrm{i}}{\ell}\sqrt{-G}\bar{\epsilon}A_\mu\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma} + \frac{3\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}\delta^{[\nu}_\omega\delta^\rho_\tau\gamma^{\sigma]}\Psi_\nu F_{\rho\sigma}F^{\omega\tau} - \\
& - \frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F_\mu{}^\nu\gamma^{\mu\sigma\rho}\Psi_\nu F_{\rho\sigma} + \text{h.c.}.
\end{aligned}
\tag{3.51}
$$

The variation of (h) yields

$$
\begin{aligned}
(h):\quad & \delta\big(\sqrt{-G}F^{\mu\nu}\bar{\Psi}_\mu\Psi_\nu\big) = \sqrt{-G}F^{\mu\nu}\delta\bar{\Psi}_\mu\Psi_\nu + \text{h.c.} = \\
& = \sqrt{-G}\nabla_\mu\bar{\epsilon}F^{\mu\nu}\Psi_\nu + \frac{1}{2\ell}\bar{\epsilon}F^{\mu\nu}\big(\gamma_\mu - 2\mathrm{i}A_\mu\big)\Psi_\nu + \\
& + \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}\big(\gamma^{\sigma\tau}{}_\mu - 2\delta^\tau_\mu\gamma^\sigma\big)F_{\tau\sigma}F^{\mu\nu}\Psi_\nu + \text{h.c.}.
\end{aligned}
\tag{3.52}
$$

Adding up all the contributions, we obtain that the total variation of the action (3.13) is

$$
\begin{aligned}
\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x \Big\{ & \underbrace{-\sqrt{-G}\Big(R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu}\Big)\bar{\epsilon}\gamma^{\mu}\Psi^{\nu}}_{(a)} + \\
& + \partial_\mu\Big(\sqrt{-G}\nabla_\nu\big(\bar{\epsilon}\gamma^{(\mu}\Psi^{\nu)}\big) - \sqrt{-G}\nabla^{\mu}\big(\bar{\epsilon}\gamma^{\rho}\Psi_{\rho}\big)\Big) \underbrace{-\Lambda\sqrt{-G}G_{\mu\nu}\bar{\epsilon}\gamma^{\mu}\Psi^{\nu}}_{(b)} - \\
& - \Big[\underbrace{\Big(F^2\frac{\sqrt{-G}}{2}G^{\mu\nu} - 2\sqrt{-G}F^{\mu\rho}F^{\nu}{}_{\rho}\Big)\bar{\epsilon}\gamma_{\mu}\Psi_{\nu}}_{(c)} \underbrace{-2\mathrm{i}\sqrt{-G}F^{\mu\nu}\nabla_{\mu}(\bar{\epsilon}\Psi_{\nu})}_{(d)}\Big] - \\
& - \Big[\partial_\mu\big(2\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \sqrt{-G}\delta\bar{\Psi}_\nu\gamma^{\nu\mu\rho}\Psi_\rho\big) \underbrace{+\frac{2}{\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}\nabla_\mu\Psi_\nu}_{(e)} - \\
& \underbrace{-\frac{2\mathrm{i}}{\ell}\sqrt{-G}A_\mu\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho}_{(f)} \underbrace{-\sqrt{-G}\big(R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu}\big)\bar{\epsilon}\gamma^{\mu}\Psi^{\nu}}_{(a)} + \\
& + \mathrm{i}\sqrt{-G}F^{\sigma\tau}\bar{\epsilon}\big(\underbrace{-\gamma_{\tau\sigma}{}^{\nu\rho}}_{(g)} \underbrace{+2\delta^{[\rho}_{[\tau}\delta^{\nu]}_{\sigma]}}_{(d)}\big)\nabla_\nu\Psi_\rho\Big] - \\
& - \frac{2\mathrm{i}}{\ell}\Big[\partial_\mu\big(\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}A_\nu\Psi_\rho\big) - \underbrace{\frac{1}{2}\sqrt{-G}\bar{\epsilon}F_{\mu\nu}\gamma^{\mu\nu\rho}\Psi_\rho}_{(h)} \underbrace{+\sqrt{-G}\bar{\epsilon}A_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho}_{(f)} + \\
& \underbrace{+\frac{1}{\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}A_\mu\Psi_\nu}_{(i)} \underbrace{-\frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\gamma_{\tau\sigma}{}^{\nu\rho}A_\nu\Psi_\rho}_{(\ell)} \underbrace{+2\mathrm{i}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\delta^{[\rho}_{[\tau}\delta^{\nu]}_{\sigma]}A_\nu\Psi_\rho}_{(m)}\Big] - \\
& - \frac{2}{\ell}\Big[\partial_\mu\big(\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}\Psi_\nu\big) - \underbrace{\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}\nabla_\mu\Psi_\nu}_{(e)} \underbrace{+\frac{3}{2\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\nu}\Psi_\nu}_{(b)} - \\
& \underbrace{-\frac{\mathrm{i}}{\ell}\sqrt{-G}A_\mu\bar{\epsilon}\gamma^{\mu\nu}\Psi_\nu}_{(i)} + \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}F^{\sigma\tau}\big(\underbrace{\gamma_{\sigma\tau}{}^{\nu}}_{(h)} \underbrace{-2\gamma_\sigma\delta^{\nu}_{\tau}}_{(n)}\big)\Psi_\nu\Big] - \\
& - 2\mathrm{i}\Big[\underbrace{\sqrt{-G}\nabla_\mu\bar{\epsilon}F^{\mu\nu}\Psi_\nu}_{(d)} + \frac{1}{2\ell}\bar{\epsilon}F^{\mu\nu}\big(\underbrace{\gamma_\mu}_{(n)} \underbrace{-2\mathrm{i}A_\mu}_{(m)}\big)\Psi_\nu + \\
& + \frac{\mathrm{i}}{4}\sqrt{-G}\bar{\epsilon}\big(\underbrace{\gamma^{\sigma\tau}{}_{\mu}}_{(o)} \underbrace{-2\delta^{\tau}_{\mu}\gamma^{\sigma}}_{(c)}\big)F_{\tau\sigma}F^{\mu\nu}\Psi_\nu\Big] -
\end{aligned}
$$

$$
\begin{aligned}
&- \mathrm{i}\Big[\partial_\mu\big(\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}\big) - \underbrace{\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho\sigma}\nabla_\mu\Psi_\nu F_{\rho\sigma}}_{(g)} + \\
&\underbrace{+\frac{1}{2\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}}_{(h)} \underbrace{-\frac{\mathrm{i}}{\ell}\sqrt{-G}\bar{\epsilon}A_\mu\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}}_{(\ell)} + \\
&\underbrace{\frac{3\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}\delta^{[\nu}_\omega\delta^\rho_\tau\gamma^{\sigma]}\Psi_\nu F_{\rho\sigma}F^{\omega\tau}}_{(c)} \underbrace{-\frac{\mathrm{i}}{2}\sqrt{-G}\bar{\epsilon}F_\mu{}^\nu\gamma^{\mu\sigma\rho}\Psi_\nu F_{\rho\sigma}}_{(o)}\Big] + \text{h.c.}\Big\},
\end{aligned}
\tag{3.53}
$$

where we mark with the same letters pieces that cancel each others.

In the end, the variation of the action S results in a boundary term

$$
\begin{aligned}
\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x \Big\{&\partial_\mu\Big[\sqrt{-G}\nabla_\nu\big(\bar{\epsilon}\gamma^{(\mu}\Psi^{\nu)}\big) - \sqrt{-G}\nabla^\mu\big(\bar{\epsilon}\gamma^\rho\Psi_\rho\big) - \\
&- 2\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho + \sqrt{-G}\delta\bar{\Psi}_\nu\gamma^{\nu\mu\rho}\Psi_\rho - \frac{2\mathrm{i}}{\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho}A_\nu\Psi_\rho - \\
&- \frac{2}{\ell}\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu}\Psi_\nu - \mathrm{i}\sqrt{-G}\bar{\epsilon}\gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}\Big] + \text{h.c.}\Big\}.
\end{aligned}
\tag{3.54}
$$

3.5 Holographic Renormalization in the Hamiltonian formalism

Since the supersymmetric variation of S results in a boundary term, in order to make it vanish we need to add counterterms. We will need that the variation of such counterterms cancels the boundary term (3.54). Papadimitriou found them in the article [12].

In this section we will explain the procedure employed, i.e. the Hamilton-Jacobi method for holographic renormalization.

However let us preliminarily introduce the radial ADM (after Arnowitt, Deser and Misner) decomposition of the dynamical variables, necessary to express an AlAdS supergravity theory in the Hamiltonian formalism. According to that, we can see the bulk space as a foliation by r-slices Σ_r, where r is the radial coordinate[2] (see Appendix A of [12]).

We can decompose all the fields in the sum of the radial and the transverse components.

[2]To be more precise, in the case of AlAdS the radial coordinate may not be well-defined in the whole space, but it is at least in the neighbourhood of the boundary.

For example, in the specific case of [12]

$$\begin{aligned} \mathrm{d}s^2 &= G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (N^2 + N^i N_i)\mathrm{d}r^2 + 2N_i \mathrm{d}r \mathrm{d}x^i + g_{ij}\mathrm{d}x^i \mathrm{d}x^j \\ A &= A_r \mathrm{d}r + A_i \mathrm{d}x^i, \qquad \Psi = \Psi_r \mathrm{d}r + \Psi_i \mathrm{d}x^i, \\ \mu,\nu &= t,x,y,r \qquad i,j = t,x,y \end{aligned} \tag{3.55}$$

where g_{ij}, A_i and Ψ_i are dynamical fields on Σ_r, while the lapse function N, the shift functions N^i, A_r and Ψ_r are non dynamical fields. It means that computing the conjugate momenta of the last four fields we obtain as many constraints. Once we compute these constraints, we can fix the values of the four fields in a convenient way. A choice particularly useful for the holographic renormalization procedure is the so-called Fefferman-Graham gauge, which sets

$$N = 1, \qquad N^i = 0, \qquad A_r = 0, \qquad \Psi_r = 0, \qquad . \tag{3.56}$$

In the aforementioned gauge, the vielbeins and the inverse vielbeins become

$$\begin{aligned} &e_r^\alpha = (0,0,0,1), \qquad e_\alpha^r = (0,0,0,1), \qquad e_i^\alpha = (e_i^a, \mathbf{0}_3), \qquad e_\alpha^i = (e_a^i, \mathbf{0}_3) \\ &\alpha = 1,\dots,4 \qquad\qquad a = 1,2,3 \end{aligned} \tag{3.57}$$

with e_i^a the vielbeins on Σ_r. Furthermore the vielbeins allow to decompose the gamma matrices in the radial and the transverse components

$$\gamma^r = \gamma^\alpha e_\alpha^r = \gamma^3, \qquad \gamma^i = \gamma^\alpha e_\alpha^i = \gamma^a e_a^i. \tag{3.58}$$

Exactly like for the chirality projectors, we can define the radiality projectors as

$$\gamma_\pm = \frac{1}{2}(1 \pm \gamma^r) \quad \text{with} \quad \Psi_\pm = \gamma_\pm \Psi. \tag{3.59}$$

They are useful because the positive and negative radiality spinors have different Fefferman-Graham expansions.

A quantity that will play a role in the calculations is the extrinsic curvature K_{ij} of Σ_r and its trace K. Thus we report their expressions in the radial ADM decomposition formalism

$$K_{ij} = \frac{1}{2N}\big(\dot{g}_{ij} - D_i N_j - D_j N_i\big), \qquad K \equiv K_{ij} g^{ij}, \tag{3.60}$$

where the dot represents the radial derivative and D_i is the covariant derivative. In the Fefferman-Graham gauge they read

$$K_{ij} = \frac{1}{2}\dot{g}_{ij}, \qquad K = \frac{1}{2}\dot{g}_{ij} g^{ij} \tag{3.61}$$

which are the expressions we will use in the following.

We come back to the holographic renormalization issue. According to AdS/CFT postulates, one computes the stress-energy tensor, a conserved current and a scalar operator of the SCFT as

$$\begin{aligned} T_{ij} &= \frac{2}{\sqrt{g}}\frac{\delta S^{os}_{SUGRA}}{\delta g^{ij}}, \\ J^i &= \frac{1}{\sqrt{g}}\frac{\delta S^{os}_{SUGRA}}{\delta A_i}, \\ \mathcal{O} &= \frac{1}{\sqrt{g}}\frac{\delta S^{os}_{SUGRA}}{\delta \varphi}. \end{aligned} \tag{3.62}$$

These expressions remember the relation of the Hamilton-Jacobi theory (see Appendix B for more details)

$$p_\alpha = \frac{\partial \mathcal{S}}{\partial q^\alpha}, \tag{3.63}$$

where we identify Hamilton's principal function $\mathcal{S}$, the generalized coordinates q^α and the conjugate momenta p_α respectively with the supergravity action S^{os}_{SUGRA} evaluated on-shell, the sources J^α and the boundary operators $\mathcal{O}_\alpha$. Thus we can develop an analogy between the Hamilton-Jacoby theory in classical mechanics and the Holographic Renormalization in AdS/CFT correspondence.
We can also extend the coordinate space adding a new coordinate τ, an abstract time that is related with an energy scale μ in CFT. Exactly like in classical mechanics, we introduce the conjugate momentum to τ, namely the abstract Hamiltonian operator $\mathbb{H} = \int \mathrm{d}^d x\, \mathbb{h}(x)$, with $\mathbb{h}(x)$ the Hamiltonian density. The dynamics of the system is described by the Hamilton's equations

$$\dot{J}^\alpha = \frac{\delta \mathbb{H}}{\delta \mathcal{O}_\alpha}, \qquad \dot{\mathcal{O}}_\alpha = -\frac{\delta \mathbb{H}}{\delta J^\alpha}, \qquad \dot{\mathbb{H}} = \frac{\partial \mathbb{H}}{\partial \tau} \tag{3.64}$$

and the Hamilton's principal functional $\mathcal{S}$, like in the Hamilton-Jacobi theory, satisfies

$$\mathcal{O}_\alpha = \frac{\delta \mathcal{S}[J^\alpha, \tau]}{\delta J^\alpha}, \qquad \mathbb{H} = -\frac{\partial \mathcal{S}[J^\alpha, \tau]}{\partial \tau}. \tag{3.65}$$

Remembering the role of the radial coordinate r as an energy scale in the AdS/CFT correspondence, we can identify r with the time τ.

After this introduction, let us start to see how this method works. The action of the supergravity theory is (3.13). We need to add the standard Gibbons-Hawking counterterm fixed for the presence of the Dirac gravitino

$$S_{GH} = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}_\varepsilon} \mathrm{d}^3x\sqrt{-g}\big(2K + \bar{\Psi}_i\gamma^{ij}\Psi_j\big). \tag{3.66}$$

Firstly we express the action in Hamiltonian formalism, namely using the radial coordinate r. In order to do this, we plug the radial ADM decomposition of the fields in the action S and perform a radiality decomposition of the spinors. The result is expressed in (3.1) and (3.2) of [12]. The presence of the Gibbons-Hawking term ensures that we have a well-posed Dirichlet problem, as we explained in Section 2.6.

Secondly we obtain the conjugate momenta of the dynamical variables e_i^a, A_i, Ψ_{+i} and $\bar{\Psi}_{+i}$, which are shown in (3.4) of [12]. In the following we will call generically $\mathcal{F}$ the dynamical variables and $\pi_{\mathcal{F}}$ their conjugate momenta.
The radial Lagrangian doesn't present radial derivatives of the fields N, N^i, A_r and Ψ_r. Thus their conjugate momenta vanish identically and they are non-dynamical variables. They give rise to constraints, as we will see in the next step.
Then we can write down the radial Hamiltonian as the Legendre transform of the Lagrangian with respect to the dynamical variables (equation (3.8) of [12]). After some algebraic manipulations, the radial Hamiltonian can be written as a sum of the non-dynamical fields multiplied by some complicated functions. For instance

$$H = \int \mathrm{d}^3x \, (N\mathcal{H} + \cdots) \tag{3.67}$$

with $\mathcal{H}$ expressed in (3.12a) of [12].
Since the conjugate momentum of N vanishes identically, we have

$$\dot{\pi}_N = 0 = -\frac{\delta H}{\delta N} = \mathcal{H}. \tag{3.68}$$

This is exactly one of the constraints we mentioned. The same argument hold for the other non-dynamical variables.

Coming back to the radial Hamiltonian expressed in terms of $\mathcal{F}$, we can derive their radial evolution through half of the Hamilton's equations (see (3.14) and (3.15) of [12]) and the Hamilton-Jacobi expressions, summarized as

$$\pi_{\mathcal{F}} = \frac{\delta \mathbb{S}[\mathcal{F}]}{\delta \mathcal{F}}, \tag{3.69}$$

where $\mathbb{S}[\mathcal{F}]$ is the Hamilton's principal function. Indeed, putting the Hamilton-Jacobi expressions into the constraint equations, we obtain a set of Hamilton-Jacobi equations for $\mathbb{S}$. Following the analogy which we started the section with, notice that the functional $\mathbb{S}$ coincides with the on-shell action evaluated at $r = r_0$.
Once the solution $\mathbb{S}[\mathcal{F}]$ is found, we substitute the conjugate momenta $\pi_{\mathcal{F}}$ with (3.69) in the Hamilton's equations. Thus we have first order differential equations for the radial evolution of the dynamical variables, instead of the second order ones in the

standard approach.

Hence the issue reduces to find the functional $\mathbb{S}$. We show how to do this in the following procedure.
We introduce the dilatation operator δ_D, constructed knowing the leading asymptotic behaviour of $\mathcal{F}$. It is defined as

$$\delta_D = \int \mathrm{d}^3x \sum_{\mathcal{F}} c_{\mathcal{F}} \mathcal{F} \frac{\delta}{\delta \mathcal{F}} \tag{3.70}$$

with $c_{\mathcal{F}}$ the coefficient of the exponent of $\mathrm{e}^{r/\ell}$ in the leading asymptotic term of the field $\mathcal{F}$.
The dilatation operator δ_D allows us to find a solution $\mathbb{S} = \int \mathrm{d}^3x\, \mathbb{L}$ for the Hamilton-Jacobi equation in the expansion form

$$\mathbb{S} = \mathbb{S}_{(0)} + \mathbb{S}_{(1)} + \cdots = \int \mathrm{d}^3x \left(\mathbb{L}_{(0)} + \mathbb{L}_{(1)} + \cdots\right), \tag{3.71}$$

where all the terms of the expansion are eigenfunctions of δ_D (i.e. $\delta_D \mathbb{S}_{(n)} = (3-n)\mathbb{S}_{(n)}$) and covariant functionals of the induced fields. Higher orders of $\mathbb{S}$ are subleading relative to the lower ones.
From the relation (3.69), we obtain

$$\sum_{\mathcal{F}} \pi_{\mathcal{F}} \delta \mathcal{F} = \delta \mathbb{L} + \partial_i v^i \tag{3.72}$$

with $\partial_i v^i$ a generic total derivative. Applying the equation above to a local scaling transformation generated by δ_D, we have

$$\sum_{\mathcal{F}} c_{\mathcal{F}} \pi_{\mathcal{F}(n)} \mathcal{F} = (3-n)\mathbb{L}_{(n)} \tag{3.73}$$

where $\pi_{\mathcal{F}(n)}$ is the n-th term in the momentum expansion. $\mathbb{L}_{(n)}$ is defined up to a total derivative $\partial_i v^i_{(n)}$.
Then combining Hamilton's equations, Hamilton-Jacobi expressions, the expansion (3.71) and (3.73), we finally obtain $\mathbb{S}_{(0)}$

$$\mathbb{S}_{(0)} = \frac{2}{\kappa^2 \ell} \int \mathrm{d}^3x \sqrt{-g}. \tag{3.74}$$

Now plugging the expansion of $\mathbb{S}$ in the constraints derived from the non dynamical variables, we get a tower of linear equations for $\mathbb{L}_{(n)}$, $n > 0$. Papadimitriou finds the expressions for $\mathbb{L}_{(n)}$ up to the fourth order. We are interested only in $\mathbb{L}_{(2)}$ for our

purposes, because $\mathbb{L}_{(1)}$ and $\mathbb{L}_{(3)}$ vanish and $\mathbb{L}_{(4)}$ goes to zero in the limit $r \to \infty$.
The second term in the $\mathbb{S}$ expansion is

$$\mathbb{S}_{(2)} = \frac{\ell}{2\kappa^2} \int \mathrm{d}^3x \sqrt{-g}\Big(R[g] + \bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\gamma^{ijk}\Psi_{+k} - \bar{\Psi}_i\gamma^{ijk}\mathcal{D}_j\Psi_k\Big), \tag{3.75}$$

where $R[g]$ is the Ricci scalar constructed with the radial slice metric and

$$\mathcal{D}_i\Psi_j \equiv \nabla_i\Psi_j + \frac{\mathrm{i}}{\ell}A_i\Psi_j = \partial_i\Psi_j + \frac{1}{4}\omega_{iab}\gamma^{ab}\Psi_j - \Gamma^k_{ij}[g]\Psi_k + \frac{\mathrm{i}}{\ell}A_i\Psi_j \tag{3.76}$$

is the gauge covariant derivative on the radial slice.

Thus the covariant counterterms action is immediately determined

$$S_{ct} = -\big(\mathbb{S}_{(0)} + \mathbb{S}_{(2)}\big) + S_{GH}. \tag{3.77}$$

Notice that the purely geometric terms of the counterterms are the same of (2.51).

Now we can also determine the Fefferman-Graham expansion of the fields. Indeed by plugging the expression of $\mathbb{S}$ into (3.69) and these in the Hamilton's equation, we have a set of first order derivative equations for the fields, which can be solved order by order in powers of $\mathrm{e}^{r/\ell}$, starting from the higher power of the asymptotic behaviours of the fields. In the case of our interest, the Fefferman-Graham expansions result

$$\begin{aligned}
e_i^a &= \mathrm{e}^{r/\ell}e^a_{i(0)}(\vec{x}) + \mathrm{e}^{-r/\ell}e^a_{i(2)}(\vec{x}) + \mathrm{e}^{-2r/\ell}e^a_{i(3)}(\vec{x}) + \cdots \\
e_a^i &= \mathrm{e}^{-r/\ell}\tilde{e}^i_{a(0)}(\vec{x}) + \mathrm{e}^{-2r/\ell}\tilde{e}^i_{a(1)}(\vec{x}) + \mathrm{e}^{-3r/\ell}\tilde{e}^i_{a(2)}(\vec{x}) + \mathrm{e}^{-4r/\ell}\tilde{e}^i_{a(3)}(\vec{x}) + \cdots \\
A_i &= A_{(0)i}(\vec{x}) + \mathrm{e}^{-r/\ell}A_{(1)i}(\vec{x}) + \mathrm{e}^{-2r/\ell}A_{(2)i}(\vec{x}) + \cdots \\
\Psi_{+i} &= \mathrm{e}^{\frac{r}{2\ell}}\Psi_{(0)+i}(\vec{x}) + \mathrm{e}^{-\frac{3r}{2\ell}}\Psi_{(2)+i}(\vec{x}) + \mathrm{e}^{-\frac{5r}{2\ell}}\Psi_{(3)+i}(\vec{x}) + \cdots \\
\Psi_{-i} &= \mathrm{e}^{-\frac{r}{2\ell}}\Psi_{(1)-i}(\vec{x}) + \mathrm{e}^{-\frac{3r}{2\ell}}\Psi_{(2)-i}(\vec{x}) + \mathrm{e}^{-\frac{5r}{2\ell}}\Psi_{(3)-i}(\vec{x}) + \cdots,
\end{aligned} \tag{3.78}$$

where $e^a_{i(0)}$, $e^a_{i(3)}$, $A_{(0)i}$, $A_{(1)i}$, $\Psi_{(0)+i}$ and $\Psi_{(2)-i}$ are undetermined, while the other coefficients are determined by the procedure (see (4.27) of [12]). Let us note that we can switch between the expressions (2.41) and (3.78) thanks to the change of coordinate $\rho = \mathrm{e}^{-2r/\ell}$.
For the calculations of Section 3.7 we will only need one of the coefficients determined by the procedure, namely

$$\Psi_{(1)-i} = -\frac{\ell}{2}\big(\gamma^{\ell k}_{i(0)} + 2\delta_i^{[k}\gamma^{\ell]}_{(0)}\big)\mathcal{D}_{(0)k}\Psi_{(0)+\ell}, \tag{3.79}$$

where $\gamma^{\ell k}_{i(0)} \equiv \gamma_a{}^{bc}e^a_{i(0)}\tilde{e}^\ell_{b(0)}\tilde{e}^k_{c(0)}$, $\gamma^\ell_{(0)} \equiv \gamma^a\tilde{e}^\ell_{a(0)}$ and $\mathcal{D}_{(0)k}$ is the gauge covariant derivative constructed with $g_{(0)}$

$$\mathcal{D}_{(0)i}\Psi_j = \nabla_{(0)i}\Psi_j + \frac{\mathrm{i}}{\ell}A_i\Psi_j. \tag{3.80}$$

3.6 Near-boundary behaviour

In this section we are going to analyse the asymptotic behaviours of bulk action and boundary counterterm variations, keeping only the divergent and finite terms of the expansions. Then we will prove that their sum vanishes.
For this purpose we need to find out the supersymmetric variations of the fields at the boundary.

Asymptotic behaviour of supersymmetric fields variations First of all we decompose the fields through the radial ADM decomposition and we impose the Fefferman-Graham (FG) gauge. In order to keep the gauge conditions we need to set the supersymmetric variation of the radial components of the fields equal to zero.[3] Hence we obtain some useful relations

$$\begin{aligned} &\delta A_r = \frac{\mathrm{i}}{2}\bar{\Psi}_r \epsilon + \text{h.c.} = 0 \ \ \forall \epsilon \qquad \Longrightarrow \qquad \Psi_r = 0, \\ &\delta \Psi_r = 0 = \nabla_r \epsilon + \frac{\mathrm{i}}{4}\big(\gamma_r{}^{\nu\rho} - 2\delta_r^{\nu}\gamma^{\rho}\big) F_{\nu\rho}\epsilon - \frac{1}{2\ell}\big(\gamma_r - 2\mathrm{i}A_r\big)\epsilon. \end{aligned} \tag{3.81}$$

In order to solve this equation iteratively, we make an ansatz for the ϵ

$$\epsilon = \mathrm{e}^{\frac{r}{2\ell}}\eta_{(0)+}(\vec{x}) + \mathrm{e}^{-\frac{r}{2\ell}}\eta_{(1)-}(\vec{x}) + \mathrm{e}^{-\frac{3r}{2\ell}}\big(\eta_{(2)+}(\vec{x}) + \eta_{(2)-}(\vec{x})\big) + \cdots \tag{3.82}$$

with $\eta_{(n)+}(\vec{x})$ and $\eta_{(n)-}(\vec{x})$ of definite "radiality", i.e.

$$\gamma^3\eta_{\pm} = \pm\eta_{\pm} \qquad \bar{\eta}_{\pm}\gamma^3 = \mp\bar{\eta}_{\pm}. \tag{3.83}$$

Therefore expanding the second equation of (3.81) and making use of the expressions in (3.78) and Appendix C, at $\mathrm{e}^{\frac{r}{2\ell}}$ and $\mathrm{e}^{-\frac{r}{2\ell}}$ orders we get

$$\partial_r \epsilon - \frac{1}{2\ell}\gamma_r \epsilon = 0, \tag{3.84}$$

whose solution is

$$\epsilon = \mathrm{e}^{\frac{r}{2\ell}}\eta_{(0)+}(\vec{x}) + \mathrm{e}^{-\frac{r}{2\ell}}\eta_{(1)-}(\vec{x}), \tag{3.85}$$

where $\eta_{(0)+}(\vec{x})$ and $\eta_{(1)-}(\vec{x})$ are arbitrary spinors.
At $\mathrm{e}^{-\frac{3r}{2\ell}}$ order, the starting equation yields

$$\begin{aligned} -\frac{3}{2\ell}\eta_{(2)+} - \frac{3}{2\ell}\eta_{(2)-} + \frac{1}{4}\omega_{(2)rab}\gamma^{ab}\eta_{(0)+} + \frac{\mathrm{i}}{4}\gamma^{3ij}_{(0)}F_{(0)ij}\eta_{(0)+} + \frac{\mathrm{i}}{2\ell}\gamma^{i}_{(0)}A_{(1)i}\eta_{(0)+} - \\ -\frac{1}{2\ell}\eta_{(2)+} + \frac{1}{2\ell}\eta_{(2)-} = 0 \end{aligned} \tag{3.86}$$

[3] In a more complete view of this problem, we might impose that all transformations under which the action is symmetric have to preserve the FG gauge, as in Appendix B of [12].

which implies

$$\begin{aligned}\Longrightarrow\ \eta_{(2)+} &= \frac{\ell}{8}\omega_{(2)rab}\gamma^{ab}\eta_{(0)+} + \frac{\mathrm{i}\ell}{8}\gamma^{3ij}_{(0)}F_{(0)ij}\eta_{(0)+}\\ \Longrightarrow\ \eta_{(2)-} &= \frac{\mathrm{i}}{2}\gamma^{i}_{(0)}A_{(1)i}\eta_{(0)+}.\end{aligned} \tag{3.87}$$

At $\mathrm{e}^{-\frac{5r}{2\ell}}$ order we get

$$\begin{aligned}&-\frac{5}{2\ell}\eta_{(3)+} - \frac{5}{2\ell}\eta_{(3)-} + \frac{1}{4}\omega_{(3)rab}\gamma^{ab}\eta_{(0)+} + \frac{1}{4}\omega_{(2)rab}\gamma^{ab}\eta_{(1)-} + \frac{\mathrm{i}}{4}\gamma^{3ij}_{(0)}F_{(1)ij}\eta_{(0)+} +\\ &+ \frac{\mathrm{i}}{4}\gamma^{3ij}_{(0)}F_{(0)ij}\eta_{(1)-} + \frac{\mathrm{i}}{\ell}\gamma^{i}_{(0)}A_{(2)i}\eta_{(0)+} + \frac{\mathrm{i}}{2\ell}\gamma^{i}_{(0)}A_{(1)i}\eta_{(1)-} - \frac{1}{2\ell}\eta_{(3)+} + \frac{1}{2\ell}\eta_{(3)-} = 0,\end{aligned} \tag{3.88}$$

which yields

$$\begin{aligned}\Longrightarrow \eta_{(3)+} &= \frac{\ell}{12}\omega_{(3)rab}\gamma^{ab}\eta_{(0)+} + \frac{\mathrm{i}\ell}{12}\gamma^{3ij}_{(0)}F_{(1)ij}\eta_{(0)+} + \frac{\mathrm{i}}{6}\gamma^{i}_{(0)}A_{(1)i}\eta_{(1)-}\\ \Longrightarrow \eta_{(3)-} &= \frac{\ell}{8}\omega_{(2)rab}\gamma^{ab}\eta_{(1)-} + \frac{\mathrm{i}\ell}{8}\gamma^{3ij}_{(0)}F_{(0)ij}\eta_{(1)-} + \frac{\mathrm{i}}{2}\gamma^{i}_{(0)}A_{(2)i}\eta_{(0)+}.\end{aligned} \tag{3.89}$$

Now we can find the asymptotic behaviour for the variation of the fields (remember that a term composed by two parts of opposite "radiality" vanishes)

$$\begin{aligned}\delta e^a_i \simeq& \,\delta\big(\mathrm{e}^{r/\ell}e^a_{i(0)} + \mathrm{e}^{-r/\ell}e^a_{i(2)} + \mathrm{e}^{-2r/\ell}e^a_{i(3)}\big) \simeq\\ \simeq& \frac{1}{2}\big(\mathrm{e}^{\frac{r}{2\ell}}\bar{\eta}_{(0)+} + \mathrm{e}^{-\frac{r}{2\ell}}\bar{\eta}_{(1)-} + \cdots\big)\gamma^a\big(\mathrm{e}^{\frac{r}{2\ell}}\Psi_{(0)+i} + \mathrm{e}^{-\frac{r}{2\ell}}\Psi_{(1)-i} + \cdots\big) + \text{h.c.},\end{aligned}$$

where the symbol $\simeq$ means that we stop the Fefferman-Graham expansion at the order we will need in the calculations. The expression yields

$$\begin{aligned}\Longrightarrow\ \delta e^a_{i(0)}(\vec{x}) =& \frac{1}{2}\bar{\eta}_{(0)+}\gamma^a\Psi_{(0)+i} + \text{h.c.}\\ \delta e^a_{i(2)}(\vec{x}) =& \frac{1}{2}\big(\bar{\eta}_{(0)+}\gamma^a\Psi_{(2)+i} + \bar{\eta}_{(1)-}\gamma^a\Psi_{(1)-i} + \bar{\eta}_{(2)+}\gamma^a\Psi_{(0)+i}\big) + \text{h.c.}\\ \delta e^a_{i(3)}(\vec{x}) =& \frac{1}{2}\big(\bar{\eta}_{(0)+}\gamma^a\Psi_{(3)+i} + \bar{\eta}_{(2)-}\gamma^a\Psi_{(1)-i} + \bar{\eta}_{(1)-}\gamma^a\Psi_{(2)-i} +\\ &+ \bar{\eta}_{(3)+}\gamma^a\Psi_{(0)+i}\big) + \text{h.c.}\end{aligned} \tag{3.90}$$

The same argument holds for the other fields

$$
\begin{aligned}
\delta\tilde{e}^i_{a(0)}(\vec{x}) =& -\frac{1}{2}\bar{\eta}_{(0)+}\gamma^i_{(0)}\tilde{e}^j_{a(0)}\Psi_{(0)+j} + \text{h.c.}\\
\delta\tilde{e}^i_{a(2)}(\vec{x}) =& -\frac{1}{2}\bar{\eta}_{(0)+}\big(\tilde{e}^j_{a(0)}\tilde{e}^i_{b(2)} + \tilde{e}^j_{a(2)}\tilde{e}^i_{b(0)}\big)\gamma^b\Psi_{(0)+j} - \frac{1}{2}\bar{\eta}_{(1)-}\gamma^i_{(0)}\tilde{e}^j_{a(0)}\Psi_{(1)-j}-\\
&-\frac{1}{2}\big(\bar{\eta}_{(0)+}\gamma^i_{(0)}\tilde{e}^j_{a(0)}\Psi_{(2)+j} + \bar{\eta}_{(2)+}\gamma^i_{(0)}\tilde{e}^j_{a(0)}\Psi_{(0)+j}\big) + \text{h.c.}\\
\delta\tilde{e}^i_{a(3)}(\vec{x}) =& -\frac{1}{2}\bar{\eta}_{(0)+}\big(\tilde{e}^j_{a(0)}\tilde{e}^i_{b(3)} + \tilde{e}^j_{a(3)}\tilde{e}^i_{b(0)}\big)\gamma^b\Psi_{(0)+j} - \frac{1}{2}\bar{\eta}_{(2)-}\gamma^i_{(0)}\tilde{e}^j_{a(0)}\Psi_{(1)-j}-\\
&-\frac{1}{2}\bar{\eta}_{(1)-}\gamma^i_{(0)}\tilde{e}^j_{a(0)}\Psi_{(2)-j} - \frac{1}{2}\big(\bar{\eta}_{(0)+}\gamma^i_{(0)}\tilde{e}^j_{a(0)}\Psi_{(3)+j} + \bar{\eta}_{(3)+}\gamma^i_{(0)}\tilde{e}^j_{a(0)}\Psi_{(0)+j}\big)+\\
&+\text{h.c.}
\end{aligned}
\tag{3.91}
$$

$$
\begin{aligned}
\delta A_{(0)i} =& -\frac{\mathrm{i}}{2}\big(\bar{\eta}_{(0)+}\Psi_{(1)-i} + \bar{\eta}_{(1)-}\Psi_{(0)+i}\big) + \text{h.c.}\\
\delta A_{(1)i} =& -\frac{\mathrm{i}}{2}\big(\bar{\eta}_{(0)+}\Psi_{(2)-i} + \bar{\eta}_{(2)-}\Psi_{(0)+i}\big) + \text{h.c.}\\
\delta\Psi_{(0)+i} =& \mathcal{D}_{(0)i}\eta_{(0)+} - \frac{1}{\ell}\gamma_{i(0)}\eta_{(1)-}\\
\delta\Psi_{(1)-i} =& \mathcal{D}_{(0)i}\eta_{(1)-} + \frac{1}{2}\omega_{(0)ia3}\gamma^{a3}\eta_{(2)+} + \frac{1}{2}\omega_{(2)ia3}\gamma^{a3}\eta_{(0)+} + \frac{\mathrm{i}}{4}\gamma^{jk}_{i(0)}F_{(0)jk}\eta_{(0)+}-\\
&-\frac{\mathrm{i}}{2}\gamma^j_{(0)}F_{(0)ij}\eta_{(0)+} - \frac{1}{2\ell}\gamma_{i(0)}\eta_{(2)+} - \frac{1}{2\ell}e^a_{i(2)}\gamma_a\eta_{(0)+},
\end{aligned}
\tag{3.92}
$$

where we used the expressions of Appendix C.

Asymptotic behaviour of the bulk action variation In order to analyse the asymptotic behaviour of bulk action variation, we need to remind the Stokes' theorem

$$
\int_{\mathcal{M}} \mathrm{d}^4x\,\partial_\mu j^\mu = \int_{\partial\mathcal{M}} \mathrm{d}^3x\, n_\mu j^\mu, \tag{3.93}
$$

where $n_\mu = (0,0,0,1)$ is the normalized vector orthogonal to the radial slices.

Combining (3.93) and the expressions of Appendix C, we can compute the asymptotic expansion of the bulk action SUSY variation δS, which has been found to be expression (3.54). It reads

$$
\begin{aligned}
\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^3x\, n_r\Big\{\sqrt{-g}\Big[&\underbrace{\frac{1}{2}\nabla_i\big(\bar\epsilon\gamma^r\Psi_j g^{ij}\big)}_{(A)} + \underbrace{\frac{1}{2}\nabla_i\big(\bar\epsilon\gamma^i\Psi^r\big)}_{(B)} - \underbrace{\partial^r\big(\bar\epsilon\gamma^i\Psi_i\big)}_{(C)} - \\
&- 2\,\underbrace{\bar\epsilon\gamma^{rij}\nabla_i\Psi_j}_{(D)} + \underbrace{\delta\bar\Psi_i\gamma^{irj}\Psi_j}_{(E)} - \underbrace{\frac{2\mathrm{i}}{\ell}\bar\epsilon\gamma^{rij}A_i\Psi_j}_{(F)} - \\
&- \underbrace{\frac{2}{\ell}\bar\epsilon\gamma^{ri}\Psi_i}_{(G)} - \underbrace{\mathrm{i}\bar\epsilon\gamma^{rijk}\Psi_i F_{jk}}_{(H)}\Big]\Big\} + \mathrm{h.c.},
\end{aligned}
\tag{3.94}
$$

where we mark the different pieces in order to analyse them separately. Each expansion will be stopped to the finite term because at the end of calculations we will take the limit $r\to\infty$.

Let us start from the asymptotic behaviour of (A). It yields

$$
\begin{aligned}
(A):\qquad &\frac{1}{2}\sqrt{-g}\nabla_i\big(\bar\epsilon\gamma^r\Psi_j g^{ij}\big) \simeq -\frac{1}{2\ell}\mathrm{e}^{3r/\ell}\sqrt{-\det(g_{(0)})}\bar\eta_{(0)+}\gamma^i_{(0)}\Psi_{(0)+i}+ \\
&+ \frac{1}{2}\mathrm{e}^{r/\ell}\sqrt{-\det(g_{(0)})}\Big[-\frac{1}{2\ell}Tr(g^{-1}_{(0)}g_{(2)})\bar\eta_{(0)+}\gamma^i_{(0)}\Psi_{(0)+i}+ \\
&+ \nabla_{(0)i}\big(-\bar\eta_{(0)+}\Psi_{(1)-j}g^{ij}_{(0)} + \bar\eta_{(1)-}\Psi_{(0)+j}g^{ij}_{(0)}\big)- \\
&- \frac{1}{\ell}\bar\eta_{(0)+}\Big(\tilde e^i_{a(2)}\gamma^a\Psi_{(0)+i} + \gamma^i_{(0)}\Psi_{(2)+i} + \gamma^k_{(0)}\Psi_{(0)+j}g_{(0)ik}g^{ij}_{(2)}\Big)- \\
&- \frac{1}{\ell}\bar\eta_{(2)+}\gamma^i_{(0)}\Psi_{(0)+i} - \frac{1}{\ell}\bar\eta_{(1)-}\gamma^i_{(0)}\Psi_{(1)-i}\Big]+ \\
&+ \frac{1}{2}\sqrt{-\det(g_{(0)})}\Big[\nabla_{(0)i}\big(-\bar\eta_{(0)+}\Psi_{(2)-j}g^{ij}_{(0)} + \bar\eta_{(2)-}\Psi_{(0)+j}g^{ij}_{(0)}\big)- \\
&- \frac{1}{\ell}\bar\eta_{(0)+}\Big(\tilde e^i_{a(3)}\gamma^a\Psi_{(0)+i} + \gamma^i_{(0)}\Psi_{(3)+i} + \gamma^k_{(0)}\Psi_{(0)+j}g_{(0)ik}g^{ij}_{(3)}\Big)+ \\
&+ \frac{1}{2\ell}\bar\eta_{(0)+}\gamma^k_{(0)}\Psi_{(0)+j}g_{(3)ik}g^{ij}_{(0)} - \frac{1}{\ell}\bar\eta_{(3)+}\gamma^i_{(0)}\Psi_{(0)+i} - \frac{1}{\ell}\bar\eta_{(2)-}\gamma^i_{(0)}\Psi_{(1)-i}- \\
&- \frac{1}{\ell}\bar\eta_{(1)-}\gamma^i_{(0)}\Psi_{(2)-i}\Big],
\end{aligned}
\tag{3.95}
$$

where we used $\Gamma^r_{ik} = -\frac{1}{2}\partial_r g_{ik} \simeq -\frac{1}{\ell}\mathrm{e}^{2r/\ell}g_{(0)ik} + \frac{1}{2\ell}\mathrm{e}^{-r/\ell}g_{(3)ik}$.

The near-boundary behaviour of (B) reads

$$
\begin{aligned}
(B):\qquad & \frac{1}{2}\sqrt{-g}\nabla_i\big(\bar{\epsilon}\gamma^i\Psi^r\big) \simeq -\frac{1}{2\ell}\mathrm{e}^{3r/\ell}\sqrt{-\det(g_{(0)})}\bar{\eta}_{(0)+}\gamma^i_{(0)}\Psi_{(0)+i}- \\
& -\frac{1}{2\ell}\mathrm{e}^{r/\ell}\sqrt{-\det(g_{(0)})}\Big[\frac{1}{2}Tr(g^{-1}_{(0)}g_{(2)})\bar{\eta}_{(0)+}\gamma^i_{(0)}\Psi_{(0)+i}+ \\
& +\bar{\eta}_{(0)+}\Big(\tilde{e}^i_{a(2)}\gamma^a\Psi_{(0)+i}+\gamma^i_{(0)}\Psi_{(2)+i}+\gamma^i_{(0)}\Psi_{(0)+k}g_{(0)ij}g^{jk}_{(2)}\Big)+ \\
& +\bar{\eta}_{(2)+}\gamma^i_{(0)}\Psi_{(0)+i}+\bar{\eta}_{(1)-}\gamma^i_{(0)}\Psi_{(1)-i}\Big]- \\
& -\frac{1}{2\ell}\sqrt{-\det(g_{(0)})}\Big[\bar{\eta}_{(0)+}\Big(\tilde{e}^i_{a(3)}\gamma^a\Psi_{(0)+i}+\gamma^i_{(0)}\Psi_{(3)+i}+\gamma^i_{(0)}\Psi_{(0)+k}g_{(0)ij}g^{jk}_{(3)}\Big)- \\
& -\frac{1}{2}\bar{\eta}_{(0)+}\gamma^i_{(0)}\Psi_{(0)+k}g_{(3)ij}g^{jk}_{(0)}+\bar{\eta}_{(3)+}\gamma^i_{(0)}\Psi_{(0)+i}+ \\
& +\bar{\eta}_{(2)-}\gamma^i_{(0)}\Psi_{(1)-i}+\bar{\eta}_{(1)-}\gamma^i_{(0)}\Psi_{(2)-i}\Big].
\end{aligned}
\tag{3.96}
$$

The (C) term expansion is

$$
\begin{aligned}
(C):\qquad & \sqrt{-g}\partial^r\big(\bar{\epsilon}\gamma^i\Psi_i\big) \simeq -\frac{2}{\ell}\mathrm{e}^{r/\ell}\sqrt{-\det(g_{(0)})}\Big(\bar{\eta}_{(0)+}\gamma^i_{(0)}\Psi_{(2)+i}+ \\
& +\bar{\eta}_{(0)+}\tilde{e}^i_{a(2)}\gamma^a\Psi_{(0)+i}+\bar{\eta}_{(1)-}\gamma^i_{(0)}\Psi_{(1)-i}+\bar{\eta}_{(2)+}\gamma^i_{(0)}\Psi_{(0)+i}\Big)- \\
& -\frac{3}{\ell}\Big(\bar{\eta}_{(0)+}\tilde{e}^i_{a(3)}\gamma^a\Psi_{(0)+i}+\bar{\eta}_{(3)+}\gamma^i_{(0)}\Psi_{(0)+i}+\bar{\eta}_{(0)+}\gamma^i_{(0)}\Psi_{(3)+i}+ \\
& +\bar{\eta}_{(2)-}\gamma^i_{(0)}\Psi_{(1)-i}+\bar{\eta}_{(1)-}\gamma^i_{(0)}\Psi_{(2)-i}\Big).
\end{aligned}
\tag{3.97}
$$

The asymptotic behaviour of (D) is

$$
\begin{aligned}
(D):\qquad & 2\sqrt{-g}\bar{\epsilon}\gamma^{rij}\nabla_i\Psi_j \simeq 2\mathrm{e}^{3r/\ell}\sqrt{-\det(g_{(0)})}\bar{\eta}_{(0)+}\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(0)-}+ \\
& +2\mathrm{e}^{r/\ell}\sqrt{-\det(g_{(0)})}\Big\{\frac{1}{2}Tr(g^{-1}_{(0)}g_{(2)})\big(\bar{\eta}_{(0)+}\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(0)-}\big)+ \\
& +\bar{\eta}_{(0)+}\Big[\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(2)-}+\big(\tilde{e}^i_{a(0)}\tilde{e}^j_{b(2)}+\tilde{e}^i_{a(2)}\tilde{e}^j_{b(0)}\big)\gamma^{3ab}(\nabla_i\Psi_j)_{(0)-}\Big]+ \\
& +\bar{\eta}_{(1)-}\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(1)+}+\bar{\eta}_{(2)+}\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(0)-}\Big\}+ \\
& +2\sqrt{-\det(g_{(0)})}\Big\{\bar{\eta}_{(0)+}\Big[\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(3)-}+ \\
& +\big(\tilde{e}^i_{a(0)}\tilde{e}^j_{b(3)}+\tilde{e}^i_{a(3)}\tilde{e}^j_{b(0)}\big)\gamma^{3ab}(\nabla_i\Psi_j)_{(0)-}\Big]+\bar{\eta}_{(1)-}\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(2)+}+ \\
& +\bar{\eta}_{(2)-}\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(1)+}+\bar{\eta}_{(3)+}\gamma^{3ij}_{(0)}(\nabla_i\Psi_j)_{(0)-}\Big\},
\end{aligned}
\tag{3.98}
$$

where we used (C.2) and (C.13) to simplify the expression above.

The (E) term reads

$$(E): \qquad \sqrt{-g}\delta\bar{\Psi}_i\gamma^{irj}\Psi_j. \tag{3.99}$$

However δS includes the hermitian conjugate of every terms we marked. Further in Eq. (3.92) we computed the near-boundary expression for $\delta\Psi_i$. For this reason we prefer to study the asymptotic behaviour of the hermitian conjugate of (E) (which we call $(\bar{E})$), that is

$$\begin{aligned}(\bar{E}): \qquad & \sqrt{-g}\bar{\Psi}_j\gamma^{i3j}\delta\Psi_i \simeq \\ & \simeq \mathrm{e}^{r/\ell}\sqrt{-\det(g_{(0)})}\Big(\bar{\Psi}_{(0)+j}\gamma^{i3j}_{(0)}\delta\Psi_{(1)-i} + \bar{\Psi}_{(1)-j}\gamma^{i3j}_{(0)}\delta\Psi_{(0)+i}\Big) + \\ & + \sqrt{-\det(g_{(0)})}\Big(\bar{\Psi}_{(0)+j}\gamma^{i3j}_{(0)}\delta\Psi_{(2)-i} + \bar{\Psi}_{(2)-j}\gamma^{i3j}_{(0)}\delta\Psi_{(0)+i}\Big).\end{aligned} \tag{3.100}$$

The asymptotic expansions of (F), (G) and (H) terms read

$$\begin{aligned}(F): \qquad & \frac{2\mathrm{i}}{\ell}\sqrt{-g}\bar{\epsilon}\gamma^{rij}A_i\Psi_j \simeq \\ & \simeq \frac{2\mathrm{i}}{\ell}\mathrm{e}^{r/\ell}\sqrt{-\det(g_{(0)})}\Big(\bar{\eta}_{(0)+}\gamma^{3ij}_{(0)}A_{(0)i}\Psi_{(1)-j} + \bar{\eta}_{(1)-}\gamma^{3ij}_{(0)}A_{(0)i}\Psi_{(0)+j}\Big) + \\ & + \frac{2\mathrm{i}}{\ell}\sqrt{-\det(g_{(0)})}\Big[\bar{\eta}_{(0)+}\gamma^{3ij}_{(0)}A_{(1)i}\Psi_{(1)-j} + \bar{\eta}_{(1)-}\gamma^{3ij}_{(0)}A_{(1)i}\Psi_{(0)+j} + \\ & + \bar{\eta}_{(0)+}\gamma^{3ij}_{(0)}A_{(0)i}\Psi_{(2)-j} + \bar{\eta}_{(2)-}\gamma^{3ij}_{(0)}A_{(0)i}\Psi_{(0)+j}\Big],\end{aligned} \tag{3.101}$$

$$\begin{aligned}(G): \qquad & \frac{2}{\ell}\sqrt{-g}\bar{\epsilon}\gamma^{ri}\Psi_i \simeq \\ & \simeq -\frac{2}{\ell}\mathrm{e}^{3r/\ell}\sqrt{-\det(g_{(0)})}\bar{\eta}_{(0)+}\gamma^{i}_{(0)}\Psi_{(0)+i} - \\ & - \frac{2}{\ell}\mathrm{e}^{r/\ell}\sqrt{-\det(g_{(0)})}\Big[\bar{\eta}_{(0)+}\tilde{e}^{i}_{a(2)}\gamma^{a}\Psi_{(0)+i} + \bar{\eta}_{(0)+}\gamma^{i}_{(0)}\Psi_{(2)+i} + \\ & + \bar{\eta}_{(2)+}\gamma^{i}_{(0)}\Psi_{(0)+i} - \bar{\eta}_{(1)-}\gamma^{i}_{(0)}\Psi_{(1)-i} + \frac{1}{2}Tr(g^{-1}_{(0)}g_{(2)})\bar{\eta}_{(0)+}\gamma^{i}_{(0)}\Psi_{(0)+i}\Big] - \\ & - \frac{2}{\ell}\sqrt{-\det(g_{(0)})}\Big[\bar{\eta}_{(0)+}\tilde{e}^{i}_{a(3)}\gamma^{a}\Psi_{(0)+i} + \bar{\eta}_{(0)+}\gamma^{i}_{(0)}\Psi_{(3)+i} + \\ & + \bar{\eta}_{(3)+}\gamma^{i}_{(0)}\Psi_{(0)+i} - \bar{\eta}_{(2)-}\gamma^{i}_{(0)}\Psi_{(1)-i} - \bar{\eta}_{(1)-}\gamma^{i}_{(0)}\Psi_{(2)-i}\Big],\end{aligned} \tag{3.102}$$

$$\begin{aligned}(H): \qquad & \mathrm{i}\sqrt{-g}\bar{\epsilon}\gamma^{rijk}\Psi_iF_{jk} \simeq \\ & \simeq \mathrm{i}\sqrt{-\det(g_{(0)})}\Big(\mathrm{e}^{r/\ell}\bar{\eta}_{(0)+}\gamma^{rijk}_{(0)}\Psi_{(0)+i}F_{(0)jk} + \bar{\eta}_{(0)+}\gamma^{rijk}_{(0)}\Psi_{(0)+i}F_{(1)jk}\Big).\end{aligned} \tag{3.103}$$

Asymptotic behaviour of counterterms variation We are going to study the asymptotic behaviour of the counterterms action variation. As in the bulk action case, we will stop the expansion to the finite term. Let us remember the boundary action

$$\begin{aligned} S_{ct} = S_{GH} + S_{(0)} + S_{(2)} = \frac{1}{\kappa^2}\int_{\partial\mathcal{M}_\varepsilon} \mathrm{d}^3x\,\sqrt{-g}\Big[\frac{1}{2}\big(2K + \bar{\Psi}_i\gamma^{ij}\Psi_j\big) - \frac{2}{\ell} - \\ - \frac{\ell}{2}\big(R[g] + \bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\hat{\gamma}^{ijk}\Psi_{+k} - \bar{\Psi}_{+i}\hat{\gamma}^{ijk}\mathcal{D}_j\Psi_{+k}\big)\Big], \end{aligned} \tag{3.104}$$

where $\partial\mathcal{M}_\varepsilon$ is a finite cutoff surface, g is the determinant of the induced metric g_{ij}, $K \equiv g^{ij}K_{ij}$ is the trace of the extrinsic curvature K_{ij} of the radial slice. As we did in all this work, we set the Fefferman-Graham gauge.

Let us examine the variation of (3.104) term by term, making use of the expansions in Appendix C. We start from $S_{(0)}$ variation

$$\begin{aligned} \delta S_{(0)} =& -\frac{2}{\kappa^2\ell}\int \mathrm{d}^3x\frac{\sqrt{-g}}{2}g^{ij}\delta g_{ij} \simeq \\ \simeq& -\frac{1}{\kappa^2\ell}\int \mathrm{d}^3x\,\sqrt{-\det(g_{(0)})}\Big[\mathrm{e}^{3r/\ell}g^{ij}_{(0)}\delta g_{(0)ij} + \\ &+ \mathrm{e}^{r/\ell}\Big(g^{ij}_{(0)}\delta g_{(2)ij} + g^{ij}_{(2)}\delta g_{(0)ij} + \frac{1}{2}Tr(g^{-1}_{(0)}g_{(2)})g^{ij}_{(0)}\delta g_{(0)ij}\Big) + \\ &+ g^{ij}_{(0)}\delta g_{(3)ij} + g^{ij}_{(3)}\delta g_{(0)ij}\Big]. \end{aligned} \tag{3.105}$$

The near-boundary behaviour of δS_{GH} reads

$$\begin{aligned} \delta S_{GH} =& \frac{1}{2\kappa^2}\int \mathrm{d}^3x\,\delta\big[\sqrt{-g}\big(2K + \bar{\Psi}_i\gamma^{ij}\Psi_j\big)\big] \simeq \\ \simeq& \frac{1}{\kappa^2}\int \mathrm{d}^3x\,\sqrt{-g}\Big(\frac{1}{2}g^{ij}\delta g_{ij}K + \frac{1}{2}\big(\bar{\Psi}_i\gamma^{ij}\delta\Psi_j + \mathrm{h.c.}\big) + \delta K\Big) \simeq \\ \simeq& \frac{1}{\kappa^2}\int \mathrm{d}^3x\,\sqrt{-\det(g_{(0)})}\Big\{\mathrm{e}^{3r/\ell}\Big(\frac{1}{2}g^{ij}_{(0)}\delta g_{(0)ij}K_{(0)} + (\delta K)_{(0)}\Big) + \\ &+ \mathrm{e}^{r/\ell}\Big[\frac{1}{2}Tr(g^{-1}_{(0)}g_{(2)})\Big(\frac{1}{2}g^{ij}_{(0)}\delta g_{(0)ij}K_{(0)} + (\delta K)_{(0)}\Big) + \\ &+ \frac{1}{2}\big(g^{ij}_{(2)}\delta g_{(0)ij} + g^{ij}_{(0)}\delta g_{(2)ij}\big)K_{(0)} + \frac{1}{2}g^{ij}_{(0)}\delta g_{(0)ij}K_{(2)} + \\ &+ \frac{1}{2}\Big(\bar{\Psi}_{(0)+i}\gamma^{ij}_{(0)}\delta\Psi_{(1)-j} + \bar{\Psi}_{(1)-i}\gamma^{ij}_{(0)}\delta\Psi_{(0)+j} + \mathrm{h.c.}\Big) + (\delta K)_{(2)}\Big] + \\ &+ \frac{1}{2}\big(g^{ij}_{(3)}\delta g_{(0)ij} + g^{ij}_{(0)}\delta g_{(3)ij}\big)K_{(0)} + \frac{1}{2}g^{ij}_{(0)}\delta g_{(0)ij}K_{(3)} + \\ &+ \frac{1}{2}\Big(\bar{\Psi}_{(0)+i}\gamma^{ij}_{(0)}\delta\Psi_{(2)-j} + \bar{\Psi}_{(2)-i}\gamma^{ij}_{(0)}\delta\Psi_{(0)+j} + \mathrm{h.c.}\Big) + (\delta K)_{(3)}\Big\}. \end{aligned} \tag{3.106}$$

The asymptotic expansion of $\delta S_{(2)}$ is

$$\begin{aligned}
\delta S_{(2)} =& -\frac{\ell}{2\kappa^2}\int \mathrm{d}^3x\,\delta\Big[\sqrt{-g}\Big(R[g]+\big(\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\gamma^{ijk}\Psi_{+k}+\text{h.c.}\big)\Big)\Big]\simeq\\
\simeq& -\frac{\ell}{2\kappa^2}\int \mathrm{d}^3x\,\sqrt{-g}\Big\{\frac{1}{2}g^{ij}\delta g_{ij}R[g]+\delta R+\\
&+\Big[\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\gamma^{ijk}\delta\Psi_{+k}+\delta\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\gamma^{ijk}\Psi_{+k}+\text{h.c.}\Big]\Big\}\simeq\\
\simeq& -\frac{\ell}{2\kappa^2}\int \mathrm{d}^3x\,\sqrt{-\det(g_{(0)})}\Big\{\mathrm{e}^{r/\ell}\Big\{\frac{1}{2}g^{ij}_{(0)}\delta g_{(0)ij}R_{(2)}+(\delta R)_{(2)}+\\
&+\Big[\big(\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(0)}\gamma^{ijk}_{(0)}\delta\Psi_{(0)+k}+\big(\delta\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(0)}\gamma^{ijk}_{(0)}\Psi_{(0)+k}+\text{h.c.}\Big]\Big\}+\\
&+\Big[\big(\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(1)}\gamma^{ijk}_{(0)}\delta\Psi_{(0)+k}+\big(\delta\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(1)}\gamma^{ijk}_{(0)}\Psi_{(0)+k}+\text{h.c.}\Big]\Big\}.
\end{aligned}\tag{3.107}$$

3.7 Supersymmetric invariance of bulk + boundary action

Finally we have all the tools to prove the supersymmetric invariance of the total action up to subleading terms that vanish when we take the limit $r \to \infty$. The variation of the total action can be summarized as

$$\delta S + \delta S_{ct} = \frac{1}{\kappa^2} \int \mathrm{d}^3 x \sqrt{-\det(g_{(0)})} \big(\mathrm{e}^{3r/\ell} A + \mathrm{e}^{r/\ell} B + C + \mathcal{O}(\mathrm{e}^{-r/\ell}) \big) \tag{3.108}$$

with the three terms defined as

$$\begin{aligned} A =& \Big(-\frac{1}{2\ell} \bar{\eta}_{(0)+} \gamma^{i}_{(0)} \Psi_{(0)+i} + \bar{\eta}_{(0)+} \gamma^{ij}_{(0)} (\nabla_i \Psi_j)_{(0)-} + \frac{1}{\ell} \bar{\eta}_{(0)+} \gamma^{i}_{(0)} \Psi_{(0)+i} + \text{h.c.} \Big) - \\ & - \frac{1}{\ell} g^{ij}_{(0)} \delta g_{(0)ij} + \frac{1}{2\ell} g^{ij}_{(0)} \delta g_{(0)ij} g^{k\ell}_{(0)} g_{(0)k\ell} + \frac{1}{\ell} \Big(g^{ij}_{(0)} \delta g_{(0)ij} + \delta g^{ij}_{(0)} g_{(0)ij} \Big), \end{aligned} \tag{3.109}$$

$$\begin{aligned} B =& \Big\{ \frac{1}{4} \nabla_{(0)i} \big(-\bar{\eta}_{(0)+} \Psi_{(1)-j} g^{ij}_{(0)} + \bar{\eta}_{(1)-} \Psi_{(0)+j} g^{ij}_{(0)} \big) + \frac{3}{2\ell} \bar{\eta}_{(0)+} \tilde{e}^{i}_{a(2)} \gamma^{a} \Psi_{(0)+i} + \\ & + \frac{3}{2\ell} \bar{\eta}_{(0)+} \gamma^{i}_{(0)} \Psi_{(2)+i} - \frac{1}{2\ell} \bar{\eta}_{(0)+} \gamma^{k}_{(0)} \Psi_{(0)+j} g_{(0)ik} g^{ij}_{(2)} + \frac{3}{2\ell} \bar{\eta}_{(2)+} \gamma^{i}_{(0)} \Psi_{(0)+i} - \\ & - \frac{1}{2\ell} \bar{\eta}_{(1)-} \gamma^{i}_{(0)} \Psi_{(1)-i} + \bar{\eta}_{(0)+} \Big[\gamma^{ij}_{(0)} (\nabla_i \Psi_j)_{(2)-} + \big(\tilde{e}^{i}_{a(0)} \tilde{e}^{j}_{b(2)} + \tilde{e}^{i}_{a(2)} \tilde{e}^{j}_{b(0)} \big) \gamma^{ab} (\nabla_i \Psi_j)_{(0)-} \Big] - \\ & - \bar{\eta}_{(1)-} \gamma^{ij}_{(0)} (\nabla_i \Psi_j)_{(1)+} + \bar{\eta}_{(2)+} \gamma^{ij}_{(0)} (\nabla_i \Psi_j)_{(0)-} + \bar{\Psi}_{(1)-i} \gamma^{ij}_{(0)} \delta \Psi_{(0)+j} + \\ & + \frac{\mathrm{i}}{\ell} \Big(\bar{\eta}_{(0)+} \gamma^{ij}_{(0)} A_{(0)i} \Psi_{(1)-j} - \bar{\eta}_{(1)-} \gamma^{ij}_{(0)} A_{(0)i} \Psi_{(0)+j} \Big) + \frac{\mathrm{i}}{2} \bar{\eta}_{(0)+} \gamma^{ijk}_{(0)} \Psi_{(0)+i} F_{(0)jk} + \text{h.c.} \Big\} - \\ & - \frac{1}{\ell} \Big(g^{ij}_{(0)} \delta g_{(2)ij} + g^{ij}_{(2)} \delta g_{(0)ij} \Big) + \frac{1}{2} \big(g^{ij}_{(2)} \delta g_{(0)ij} + g^{ij}_{(0)} \delta g_{(2)ij} \big) K_{(0)} + \\ & + \frac{1}{2} g^{ij}_{(0)} \delta g_{(0)ij} K_{(2)} + (\delta K)_{(2)} - \frac{\ell}{2} \Big\{ \frac{1}{2} g^{ij}_{(0)} \delta g_{(0)ij} R_{(2)} + (\delta R)_{(2)} + \\ & + \Big[\big(\bar{\Psi}_{+i} \overleftarrow{\mathcal{D}}_j \big)_{(0)} \gamma^{ijk}_{(0)} \delta \Psi_{(0)+k} + \big(\delta \bar{\Psi}_{+i} \overleftarrow{\mathcal{D}}_j \big)_{(0)} \gamma^{ijk}_{(0)} \Psi_{(0)+k} + \text{h.c.} \Big] \Big\} + \\ & + \frac{1}{2} Tr(g^{-1}_{(0)} g_{(2)}) \Big[\Big(-\frac{1}{2\ell} \bar{\eta}_{(0)+} \gamma^{i}_{(0)} \Psi_{(0)+i} - \bar{\eta}_{(0)+} \gamma^{3ij}_{(0)} (\nabla_i \Psi_j)_{(0)-} + \\ & + \frac{1}{\ell} \bar{\eta}_{(0)+} \gamma^{i}_{(0)} \Psi_{(0)+i} + \text{h.c.} \Big) - \frac{1}{\ell} g^{ij}_{(0)} \delta g_{(0)ij} + \frac{1}{2} g^{ij}_{(0)} \delta g_{(0)ij} K_{(0)} + (\delta K)_{(0)} \Big] \end{aligned} \tag{3.110}$$

and

$$
\begin{aligned}
C =&\Big\{\frac{1}{4}\nabla_{(0)i}\big(-\bar\eta_{(0)+}\Psi_{(2)-j}g^{ij}_{(0)}+\bar\eta_{(2)-}\Psi_{(0)+j}g^{ij}_{(0)}\big)+\\
&+\frac{2}{\ell}\bar\eta_{(0)+}\tilde e^i_{a(3)}\gamma^a\Psi_{(0)+i}+\frac{2}{\ell}\bar\eta_{(0)+}\gamma^i_{(0)}\Psi_{(3)+i}+\frac{2}{\ell}\bar\eta_{(3)+}\gamma^i_{(0)}\Psi_{(0)+i}-\\
&-\frac{1}{2\ell}\bar\eta_{(0)+}\gamma^i_{(0)}\Psi_{(0)+k}g_{(0)ij}g^{jk}_{(3)}+\frac{1}{4\ell}\bar\eta_{(0)+}\gamma^i_{(0)}\Psi_{(0)+k}g_{(3)ij}g^{jk}_{(0)}+\\
&+\bar\eta_{(0)+}\Big[\gamma^{ij}_{(0)}(\nabla_i\Psi_j)_{(3)-}+\big(\tilde e^i_{a(0)}\tilde e^j_{b(3)}+\tilde e^i_{a(3)}\tilde e^j_{b(0)}\big)\gamma^{ab}(\nabla_i\Psi_j)_{(0)-}\Big]-\\
&-\bar\eta_{(1)-}\gamma^{ij}_{(0)}(\nabla_i\Psi_j)_{(2)+}-\bar\eta_{(2)-}\gamma^{ij}_{(0)}(\nabla_i\Psi_j)_{(1)+}+\bar\eta_{(3)+}\gamma^{ij}_{(0)}(\nabla_i\Psi_j)_{(0)-}+\\
&+\bar\Psi_{(2)-i}\gamma^{ij}_{(0)}\delta\Psi_{(0)+j}+\frac{\mathrm{i}}{\ell}\Big[\bar\eta_{(0)+}\gamma^{ij}_{(0)}A_{(1)i}\Psi_{(1)-j}-\bar\eta_{(1)-}\gamma^{ij}_{(0)}A_{(1)i}\Psi_{(0)+j}+\\
&+\bar\eta_{(0)+}\gamma^{ij}_{(0)}A_{(0)i}\Psi_{(2)-j}-\bar\eta_{(2)-}\gamma^{ij}_{(0)}A_{(0)i}\Psi_{(0)+j}\Big]+\frac{\mathrm{i}}{2}\bar\eta_{(0)+}\gamma^{ijk}_{(0)}\Psi_{(0)+i}F_{(1)jk}+\text{h.c.}\Big\}-\\
&-\frac{1}{\ell}\Big(g^{ij}_{(0)}\delta g_{(3)ij}+g^{ij}_{(3)}\delta g_{(0)ij}\Big)+\frac{1}{2}\Big(g^{ij}_{(3)}\delta g_{(0)ij}+g^{ij}_{(0)}\delta g_{(3)ij}\Big)K_{(0)}+\frac{1}{2}g^{ij}_{(0)}\delta g_{(0)ij}K_{(3)}+\\
&+(\delta K)_{(3)}-\frac{\ell}{2}\Big[\big(\bar\Psi_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(1)}\gamma^{ijk}_{(0)}\delta\Psi_{(0)+k}+\big(\delta\bar\Psi_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(1)}\gamma^{ijk}_{(0)}\Psi_{(0)+k}+\text{h.c.}\Big].
\end{aligned}
\tag{3.111}
$$

After some algebraic calculations one obtains

$$
\begin{aligned}
A =&0\\
B =&\nabla_{(0)i}\Big(-\frac{1}{4}\bar\eta_{(0)+}\Psi_{(1)-j}g^{ij}_{(0)}+\frac{1}{4}\bar\eta_{(1)-}\Psi_{(0)+j}g^{ij}_{(0)}+\bar\eta_{(0)+}\gamma^{ij}_{(0)}\Psi_{(1)-j}-\\
&-\frac{1}{2}\bar\eta_{(1)-}\gamma^{ij}_{(0)}\Psi_{(0)+j}-\frac{\ell}{2}\bar\Psi_{(0)+i}\gamma^{ijk}_{(0)}\mathcal{D}_{(0)k}\eta_{(0)+}+\text{h.c.}\Big)-\\
&-\frac{\ell}{2}\nabla_{(0)i}\Big(g^{jk}_{(0)}\delta\Gamma^i_{\ (0)jk}-g^{ik}_{(0)}\delta\Gamma^j_{\ (0)jk}\Big)\\
C =&\nabla_{(0)i}\Big(-\frac{1}{4}\bar\eta_{(0)+}\Psi_{(2)-j}g^{ij}_{(0)}+\frac{1}{4}\bar\eta_{(2)-}\Psi_{(0)+j}g^{ij}_{(0)}+\bar\eta_{(0)+}\gamma^{ij}_{(0)}\Psi_{(2)-j}+\\
&+\mathrm{i}\bar\eta_{(0)+}\gamma^{ijk}_{(0)}A_{(1)j}\Psi_{(0)+k}+\text{h.c.}\Big).
\end{aligned}
\tag{3.112}
$$

Hence the variation of the total action results in a covariant derivative on a radial slice. In order to make it vanish, we can set the fields to zero in the limit $(x^i)^2\to\infty$. Otherwise, in Euclidean signature, we can assume the boundary is compact, hence the boundary term obtained by integrating (3.112) vanishes.

By imposing one of these two conditions, we can safely take the limit $r \to \infty$ and finally achieve our purpose: we proved that the counterterms cancel the divergences of the bulk action and we obtained a local supersymmetric theory also at the boundary.

Chapter 4

Conclusions and applications

In Chapter 3 we showed explicitly how the counterterms found through the Hamilton-Jacobi method cancel the infrared divergences and the finite terms of the bulk action supersymmetric variation. In order to obtain our result, we needed to fix an additional condition. In Lorentzian signature we imposed the requirement on the boundary fields behaviour, i.e. their values vanish in the limit $(x^i)^2 \to \infty$. Alternatively, working in Euclidean signature, we assumed the boundary is compact.
This argument proves the correctness of the holographic renormalization procedure employed in Section 3.5 and the invariance under local supersymmetry of the bulk + boundary action of minimal $\mathcal{N}=2$ four dimensional gauged supergravity.
Furthermore, according to what we said in Chapter 2 about the AdS/CFT correspondence, the generating functional of connected functions of the dual theory (namely $\mathcal{N}=2$ three dimensional SCFT), which coincides with minus the on-shell supergravity action, results UV renormalized after the holographic renormalization procedure.

The case studied from the theoretical point of view has important applications, for instance in the field of black holes and microscopic counting of their entropy. In particular we have an example in [21]. The authors find a holographic relation between the partition function Z of $\mathcal{N}=2$ three dimensional SCFTs compactified on a Riemann surface $\Sigma_{\mathfrak{g}}$ of genus $\mathfrak{g} > 1$ and the entropy of a supersymmetric magnetically charged AdS_4 black hole S_{BH}, which is a bosonic solution for the minimal $\mathcal{N}=2$ four dimensional gauged supergravity action.
Our work result is important for [21] because we proved the correctness of the counterterms found in [12], which holographically renormalize the minimal $\mathcal{N}=2$ gauged four dimensional supergravity action preserving invariance under supersymmetry variations of the boundary fields, to be identified with the field theory sources. Because of this latter property and since the symmetries between the two theories match, it means that the correct dual theory for $\mathcal{N}=2$ three dimensional SCFT is the one represented by the bulk action (3.13) + the boundary terms (3.104). Furthermore, since we renor-

malized the action, we can obtain finite quantities from the supergravity theory and compare them with their holographic dual realization in quantum field theory.

We will retrace the steps made in [21] in order to find the holographic relation between Z and S_{BH}. All of the following results are valid in the large N limit, that we briefly discussed in the introduction of our work.
The action is the one in (3.13) with the gravitino sets to zero, namely

$$I_{\text{Einst+Max}} = -\frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-G}\Big(R+6-\frac{1}{4}F^2\Big) \tag{4.1}$$

Notice we have the relations $\frac{1}{2}F_{here} = F_{there}$, which is an irrelevant rescaling of the field strength, and $\ell = 1$. The magnetically charged AdS_4 black hole solution is given by

$$\begin{aligned} \mathrm{d}s_4^2 &= -\Big(\rho-\frac{1}{2\rho}\Big)^2\mathrm{d}t^2+\Big(\rho-\frac{1}{2\rho}\Big)^{-2}\mathrm{d}\rho^2+\rho^2\mathrm{d}s^2_{\mathbb{H}^2},\\ F &= \frac{\mathrm{d}x_1\wedge\mathrm{d}x_2}{x_2^2}, \end{aligned} \tag{4.2}$$

where $\mathrm{d}s^2_{\mathbb{H}^2} = \frac{1}{x_2^2}\big(\mathrm{d}x_1^2+\mathrm{d}x_2^2\big)$. The entropy of this extremal black hole is given in terms of the horizon area by the standard Bekenstein-Hawking formula

$$S_{BH} = \frac{\text{Area}}{4G_N^{(4)}} = \frac{\big(\mathfrak{g}-1\big)\pi}{2G_N^{(4)}} \tag{4.3}$$

where $G_N^{(4)}$ is the four dimensional Newton's constant. One of the result obtained in [21] is that the on-shell supergravity action coincides with minus the black hole entropy. In order to evaluate the action on the solution (4.2), we need to add the counterterms that cancel the divergences. The correct ones are

$$I_{\text{count}} = \frac{1}{\kappa^2}\int \mathrm{d}^3x\,\sqrt{-g}\Big(2+\frac{1}{2}R[g]-K\Big), \tag{4.4}$$

that are the terms we found in Chapter 3 (see Equation (3.104)) with the gravitino sets to zero. However the integral $I_{\text{Eucl}} = I_{\text{Einst+Max}} + I_{\text{count}}$ is not well-defined when explicitly evaluated on the extremal solution (4.2), because the integrand vanishes, while the integration over the time leads to infinity. In order to solve this issue, one needs to consider a non-extremal deformation of the solution and evaluate the regulated action on it. There are two non-extremal deformations: one amounts to allowing for a generic magnetic charge Q under the graviphoton, and the other to adding a mass η.

The non-extremal solution, discussed in [22] and [23], reads

$$
\begin{aligned}
\mathrm{d}s^2 &= -V(\rho)\mathrm{d}t^2 + \frac{1}{V(\rho)}\mathrm{d}\rho^2 + \rho^2 \mathrm{d}s^2_{\mathbb{H}^2}, \\
F &= 2Q\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{x_2^2}, \\
V(\rho) &= -1 - \frac{2\eta}{\rho} + \frac{Q^2}{\rho^2} + \rho^2,
\end{aligned}
\tag{4.5}
$$

with the extremal solution restored for $Q \to \frac{1}{2}$ and $\eta \to 0$. Evaluating the action on this generic solution they obtain

$$
I^{os}_{\mathrm{Eucl}} = \frac{2\pi\rho_0\big(\mathfrak{g}-1\big)}{2\rho_0 G_N^{(4)}\left|\rho_0^2 + \frac{\eta}{\rho_0} - \frac{Q^2}{\rho_0^2}\right|}\Big(Q^2 - \rho_0^4 + \eta\rho_0\Big), \tag{4.6}
$$

where ρ_0 is the horizon radius, i.e. the value of ρ which solves the equation $V(\rho) = 0$. Notice that for the extremal black hole solution $\rho_0 \to \frac{1}{\sqrt{2}}$.
Taking the extremal limit for I^{os}_{Eucl} they have

$$
I_{\mathrm{Extr}} = -\frac{\pi\big(\mathfrak{g}-1\big)}{2G_N^{(4)}} + \mathcal{O}\Big(\big(Q-\frac{1}{2}\big)^{1/2}\Big) + \mathcal{O}\big(\eta^{1/2}\big). \tag{4.7}
$$

Comparing this result with the entropy formula (4.3), they find

$$
S_{BH} = -I_{\mathrm{Extr}}. \tag{4.8}
$$

On the other hand, according to AdS/CFT conjecture, the gravitational on-shell action is related to the partition function of the dual CFT theory as $I_{\mathrm{Extr}} = -\log Z$. Therefore they obtain the sought holographic relation

$$
\log Z = S_{BH}, \tag{4.9}
$$

which shows that black hole entropy can be identified with the logarithm of the partition function of the CFT theory defined on the boundary of the black hole itself.
Although the black hole is a bosonic solution, the fact that the holographic counterterms are such that supersymmetry of the bulk+boundary action is preserved is crucial for the supergravity and field theory results to match.

In the end we briefly discuss a natural development of the present work. It consists to work out the holographic counterterms in matter coupled $\mathcal{N} = 2$ gauged supergravity, whose fermionic part is only partially known, and to check again supersymmetry of the bulk + boundary on-shell action. In this case, there would be radical differences with respect to the case of minimal $\mathcal{N}=2$ gauged supergravity. Indeed, while for the latter there are not ambiguities related to the finite counterterms, for the former there are ambiguities. Holographic renormalization has been discussed for matter coupled $\mathcal{N}=2$ gauged supergravity with vacua that have conformally flat boundaries (see [24], [25], [26], [27], [28], [29]), but no one has discussed yet vacua with arbitrary curved boundaries.

Appendix A

Conventions

In this appendix we briefly enumerate some conventions employed in this work. The metric is "mostly positive", i.e. $(-,+,\dots,+)$. The Riemann tensor is

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= g_{\rho\rho'}\big(\partial_\mu\Gamma^{\rho'}_{\nu\sigma} - \partial_\nu\Gamma^{\rho'}_{\mu\sigma} + \Gamma^{\rho'}_{\mu\tau}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\rho'}_{\nu\tau}\Gamma^{\tau}_{\mu\sigma}\big) = \\ &= e^a_\rho e^b_\sigma\big(\partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b\big) \end{aligned} \tag{A.1}$$

with the affine connection $\Gamma^\rho_{\mu\nu}$ and the spin connection $\omega_\mu{}^{ab}$ defined by

$$\begin{aligned} \Gamma^\rho_{\mu\nu} &= \frac{1}{2}g^{\rho\sigma}\big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\big) \\ \omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e^{b]}_{\nu]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e^c_\sigma. \end{aligned} \tag{A.2}$$

The Ricci tensor and scalar are obtained contracting the Riemann tensor with the metric, namely

$$R_{\mu\nu} = R_\mu{}^\rho{}_{\nu\rho}, \qquad R = R_{\mu\nu}g^{\mu\nu}. \tag{A.3}$$

The covariant derivatives for a spinor and a vector read

$$\begin{aligned} \nabla_\mu\lambda &= \partial_\mu\lambda + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\lambda \\ \nabla_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho}V^\rho. \end{aligned} \tag{A.4}$$

The Dirac spinor conjugate is defined as

$$\bar\lambda = \mathrm{i}\lambda^\dagger\gamma^0. \tag{A.5}$$

Let us introduce the Clifford algebra for the gamma matrices in four dimensions. It is defined by

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}. \tag{A.6}$$

We will employ a representation of gamma matrices which satisfies

$$\left(\gamma^0\right)^2 = -\mathbb{1}_4, \qquad \left(\gamma^i\right)^2 = \mathbb{1}_4 \quad i = 1,2,3. \tag{A.7}$$

Further we remember the well known relation

$$\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{A.8}$$

In the context of supersymmetry and supergravity theories, it is useful to rearrange the gamma matrices in the following way

$$\gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right] \tag{A.9}$$

$$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]} = \frac{1}{2}\left\{\gamma^\mu, \gamma^{\nu\rho}\right\} \tag{A.10}$$

and more generally,

$$\gamma^{\mu_1\cdots\mu_n} = \gamma^{[\mu_1\cdots\mu_n]} = \begin{cases} \frac{1}{2}\left[\gamma^{\mu_1}, \gamma^{\mu_2\cdots\mu_n}\right] & \text{for even } n \\ \frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\cdots\mu_n}\right\} & \text{for odd } n. \end{cases} \tag{A.11}$$

We enumerate some useful expressions for the contraction of two gamma matrices in four dimensions

$$\gamma_\mu\gamma^{\mu\nu\rho} = 2\gamma^{\nu\rho} \tag{A.12}$$

$$\gamma^{\mu\nu\rho}\gamma_{\alpha\beta} = 6\gamma^{[\mu\nu}{}_{[\beta}\delta^{\rho]}_{\alpha]} + 6\gamma^{[\mu}\delta^{\nu}_{[\beta}\delta^{\rho]}_{\alpha]} \tag{A.13}$$

$$\gamma_\sigma\gamma^{\mu\nu\rho} = \gamma_\sigma{}^{\mu\nu\rho} + 3\delta^{[\mu}_\sigma\gamma^{\nu\rho]} \tag{A.14}$$

$$\gamma^{\sigma\tau\mu}\gamma_{\mu\nu\rho} = 4\gamma^{[\sigma}{}_{[\rho}\delta^{\tau]}_{\nu]} + 4\delta^{[\sigma}_{[\rho}\delta^{\tau]}_{\nu]} \tag{A.15}$$

$$\gamma^{\rho\sigma\mu}\gamma_{\mu\nu} = \gamma^{\rho\sigma}{}_\nu + 4\gamma^{[\rho}\delta^{\sigma]}_\nu \tag{A.16}$$

$$\gamma^\rho\gamma_{\mu\nu} = \gamma^\rho{}_{\mu\nu} + 2\delta^\rho_{[\mu}\gamma_{\nu]} \tag{A.17}$$

$$\gamma_\mu\gamma^{\mu\nu\rho\sigma} = \gamma^{\nu\rho\sigma} \tag{A.18}$$

$$\gamma^{\tau\omega\mu}\gamma_{\mu\nu\rho\sigma} = 6\gamma_{[\nu}\delta^{[\omega}_\rho\delta^{\tau]}_{\sigma]} \tag{A.19}$$

Appendix B

Hamiltonian mechanics

In this appendix we remind some aspects of the Hamilton-Jacobi theory for classical system. It will turn out to be useful when we will explain the holographic renormalization procedure in terms of Hamiltonian formalism in Section 3.5.

Let us consider the action

$$S = \int^{t} \mathrm{d}t' L(q^{\alpha}, \dot{q}^{\alpha}, t') \tag{B.1}$$

where L is the Lagrangian of the system, q^{α} are generalized coordinates in the configuration space and $\dot{q}^{\alpha}$ are the respective velocities.
In the Hamiltonian formalism, we define the conjugate momenta p_{α} of q^{α} as

$$p_{\alpha} = \frac{\partial L}{\partial \dot{q}^{\alpha}} \tag{B.2}$$

The evolution of a Hamiltonian system is described by the Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{B.3}$$

with the so-called Hamilton's function or Hamiltonian H, which is achieved from the Lagrangian performing a Legendre transform, i.e. $H = \sum p_{\alpha} \dot{q}^{\alpha} - L$.
The difficulty of solving an Hamiltonian system is located in the integration of (B.3).

We know that a canonical transformation, which we write as

$$Q = f(q, p, t) \qquad P = g(q, p, t) \tag{B.4}$$

preserves the formalism of Hamilton's equations, conjugating the Hamiltonian H to another one K.

The Hamilton-Jacobi method is a procedure that exploits a canonical transformation to simplify the integrating issue. Indeed it conjugates H to another Hamiltonian K easier to integrate. Therefore it moves the problem to the research of the generating function of the canonical transformation that makes this work.

Time-independent system For a time-independent system one can prove that there exists a generating function $\mathcal{W}(q^\alpha, P^\alpha)$ which satisfies

$$p_\alpha = \frac{\partial \mathcal{W}}{\partial q^\alpha}(q^\alpha, P_\alpha) \qquad Q^\alpha = \frac{\partial \mathcal{W}}{\partial P_\alpha}(q^\alpha, P_\alpha) \tag{B.5}$$

and $\mathcal{W}(q^\alpha, P^\alpha)$ is a solution for the (reduced) Hamilton-Jacobi equation

$$H\Big(q^\alpha, \frac{\partial \mathcal{W}}{\partial q^\alpha}(q^\alpha, P_\alpha)\Big) = E \tag{B.6}$$

with E some constant.

Time-dependent system For a time-dependent system we add to Hamilton's equations the following one

$$\dot{H} = \frac{\partial H}{\partial t} \tag{B.7}$$

Furthermore there is a generating function $\mathcal{S}(q^\alpha, P_\alpha, t)$ which satisfies

$$p_\alpha = \frac{\partial \mathcal{S}}{\partial q^\alpha}(q^\alpha, P_\alpha) \qquad Q^\alpha = \frac{\partial \mathcal{S}}{\partial P_\alpha}(q^\alpha, P_\alpha) \tag{B.8}$$

and $\mathcal{S}(q^\alpha, P_\alpha, t)$ is a solution for the Hamilton-Jacobi equation

$$H\Big(q^\alpha, \frac{\partial \mathcal{S}}{\partial q^\alpha}(q^\alpha, P_\alpha, t), t\Big) + \frac{\partial \mathcal{S}}{\partial t} = 0 \tag{B.9}$$

Notice that the Hamilton-Jacobi formalism for time-dependent system reduces to the time-independent one setting

$$\mathcal{S}(q^\alpha, P_\alpha, t) = \mathcal{W}(q^\alpha, P_\alpha) - Et \tag{B.10}$$

where the function $\mathcal{S}$ is known as Hamilton's principal function, while $\mathcal{W}$ is called the characteristic function.

Appendix C

Fefferman-Graham expansions

We compute and report the Fefferman-Graham expansions of different quantities valid for the $\mathcal{N}=2$ gauged four dimensional supergravity theory. We will stop the asymptotic expansions to the needed order with regard to the calculations in Chapter 3. We will work in Fefferman-Graham gauge.

Vielbeins and inverse vielbeins The following tautologies

$$\begin{aligned}\delta^i_j &\equiv \tilde{e}^i_a e^a_j \simeq \left(\mathrm{e}^{-r/\ell}\tilde{e}^i_{a(0)} + \mathrm{e}^{-2r/\ell}\tilde{e}^i_{a(1)} + \cdots\right)\left(\mathrm{e}^{r/\ell}e^a_{j(0)} + \mathrm{e}^{-r/\ell}e^a_{j(2)} + \cdots\right),\\ \delta^b_a &\equiv \tilde{e}^i_a e^b_i \simeq \left(\mathrm{e}^{-r/\ell}\tilde{e}^i_{a(0)} + \mathrm{e}^{-2r/\ell}\tilde{e}^i_{a(1)} + \cdots\right)\left(\mathrm{e}^{r/\ell}e^b_{i(0)} + \mathrm{e}^{-r/\ell}e^b_{i(2)} + \cdots\right)\end{aligned} \tag{C.1}$$

yield the expressions

$$\begin{aligned}&e^a_{j(0)}\tilde{e}^i_{a(0)} = \delta^i_j, \qquad e^b_{i(0)}\tilde{e}^i_{a(0)} = \delta^b_a,\\ &\tilde{e}^i_{a(1)}e^a_{j(0)} = 0 \quad \Longrightarrow \quad \tilde{e}^i_{a(1)} = 0,\\ &e^a_{j(2)}\tilde{e}^i_{a(0)} = -\tilde{e}^i_{a(2)}e^a_{j(0)}, \qquad e^b_{i(2)}\tilde{e}^i_{a(0)} = -e^b_{i(0)}\tilde{e}^i_{a(2)},\\ &e^a_{j(3)}\tilde{e}^i_{a(0)} = -\tilde{e}^i_{a(3)}e^a_{j(0)}, \qquad e^b_{i(3)}\tilde{e}^i_{a(0)} = -e^b_{i(0)}\tilde{e}^i_{a(3)}.\end{aligned} \tag{C.2}$$

These equivalences are widely used to simplify the expressions of Chapter 3.

Spin connection The expressions of the spin connection read

$$\begin{aligned}\omega_i{}^{a3} &= 2e^{\nu[a}\partial_{[i}e^{3]}_{\nu]} - e^{\nu[a}e^{3]\sigma}e_{i\gamma}\partial_\nu e^\gamma_\sigma =\\ &= e^{\nu[a}\partial_i e^{3]}_\nu - e^{\nu[a}\partial_\nu e^{3]}_i - \frac{1}{2}e^{ja}e^{3r}e_{i3}\partial_j e^3_r + \frac{1}{2}e^{r3}e^{aj}e_{ic}\partial_r e^c_j =\\ &= \frac{1}{2}\partial_r e^a_i + \frac{1}{2}e^{aj}e_{ib}\partial_r e^b_j,\end{aligned} \tag{C.3}$$

$$\begin{aligned} {\omega_i}^{ab} &= 2e^{j[a}\partial_{[i}e^{b]}_{j]} - e^{j[a}e^{b]k}e_{ic}\partial_j e^c_k, \\ {\omega_r}^{ab} &= e^{i[a}\partial_r e^{b]}_i, \qquad {\omega_r}^{a3} = 0. \end{aligned} \tag{C.4}$$

Their asymptotic expansions are

$$\begin{aligned}
\omega_{ia3} &\simeq \frac{1}{\ell}\mathrm{e}^{r/\ell}\eta_{ab}e^b_{i(0)} + \frac{1}{\ell}\mathrm{e}^{-r/\ell}\eta_{bc}\tilde{e}^j_{a(2)}e^b_{i(0)}e^c_{j(0)} + \frac{1}{2\ell}\mathrm{e}^{-2r/\ell}\Big(3\eta_{bc}\tilde{e}^j_{a(3)}e^b_{i(0)}e^c_{j(0)} - \eta_{ab}e^b_{i(3)}\Big) = \\
&= \mathrm{e}^{r/\ell}\omega_{(0)ia3} + \mathrm{e}^{-r/\ell}\omega_{(2)ia3} + \mathrm{e}^{-2r/\ell}\omega_{(3)ia3}, \\
\\
\omega_{rab} &\simeq \frac{1}{\ell}\mathrm{e}^{-2r/\ell}\eta_{c[b}\big(\tilde{e}^i_{a](2)}e^c_{i(0)} - \tilde{e}^i_{a](0)}e^c_{i(2)}\big) + \frac{1}{\ell}\mathrm{e}^{-3r/\ell}\eta_{c[b}\big(\tilde{e}^i_{a](3)}e^c_{i(0)} - 2\tilde{e}^i_{a](0)}e^c_{i(3)}\big) = \\
&= \mathrm{e}^{-2r/\ell}\omega_{(2)rab} + \mathrm{e}^{-3r/\ell}\omega_{(3)rab}, \\
\\
\omega_{iab} &\simeq 2\eta_{c[b}\tilde{e}^j_{a](0)}\partial_{[i}e^c_{j](0)} - \tilde{e}^j_{[a|(0)|}\tilde{e}^k_{b](0)}\eta_{cd}e^c_{i(0)}\partial_j e^d_{k(0)} + \\
&+ \mathrm{e}^{-2r/\ell}\Big[2\eta_{c[b}\Big(\tilde{e}^j_{a](0)}\partial_{[i}e^c_{j](2)} + \tilde{e}^j_{a](2)}\partial_{[i}e^c_{j](0)}\Big) - \\
&- \big(\tilde{e}^j_{[a|(2)|}\tilde{e}^k_{b](0)} + \tilde{e}^j_{[a|(0)|}\tilde{e}^k_{b](2)}\big)\eta_{cd}e^c_{i(0)}\partial_j e^d_{k(0)} - \tilde{e}^j_{[a|(0)|}\tilde{e}^k_{b](0)}\eta_{cd}\big(e^c_{i(0)}\partial_j e^d_{k(2)} + e^c_{i(2)}\partial_j e^d_{k(0)}\big)\Big] = \\
&= \omega_{(0)iab} + \mathrm{e}^{-2r/\ell}\omega_{(2)iab}.
\end{aligned} \tag{C.5}$$

Metric and its variation In order to study the variation of bulk and boundary actions, we need to employ the asymptotic expansion of the metric. Its explicit expression is

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}r^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + e^a_i e^b_j\eta_{ab}\mathrm{d}x^i\mathrm{d}x^j \equiv \\
&\equiv \mathrm{d}r^2 + \big(\mathrm{e}^{2r/\ell}g_{(0)ij}(\vec{x}) + g_{(2)ij}(\vec{x}) + \mathrm{e}^{-r/\ell}g_{(3)ij}(\vec{x}) + \cdots\big)\mathrm{d}x^i\mathrm{d}x^j \simeq \\
&\simeq \mathrm{d}r^2 + \big(\mathrm{e}^{r/\ell}e^a_{i(0)} + \mathrm{e}^{-r/\ell}e^a_{i(2)} + \cdots\big)\big(\mathrm{e}^{r/\ell}e^b_{j(0)} + \mathrm{e}^{-r/\ell}e^b_{j(2)} + \cdots\big)\eta_{ab}\mathrm{d}x^i\mathrm{d}x^j
\end{aligned} \tag{C.6}$$

which implies

$$\begin{aligned}
\Longrightarrow\ g_{(0)ij} &= \eta_{ab}e^a_{i(0)}e^b_{j(0)} \\
g_{(2)ij} &= \eta_{ab}\big(e^a_{i(0)}e^b_{j(2)} + e^a_{i(2)}e^b_{j(0)}\big) \\
g_{(3)ij} &= \eta_{ab}\big(e^a_{i(0)}e^b_{j(3)} + e^a_{i(3)}e^b_{j(0)}\big).
\end{aligned} \tag{C.7}$$

We obtain the asymptotic expansions of g^{ij} with the same argument

$$\begin{aligned}
g^{ij}_{(0)} &= \eta^{ab}\tilde{e}^i_{a(0)}\tilde{e}^j_{b(0)} \\
g^{ij}_{(2)} &= \eta^{ab}\big(\tilde{e}^i_{a(0)}\tilde{e}^j_{b(2)} + \tilde{e}^i_{a(2)}\tilde{e}^j_{b(0)}\big) \\
g^{ij}_{(3)} &= \eta^{ab}\big(\tilde{e}^i_{a(0)}\tilde{e}^j_{b(3)} + \tilde{e}^i_{a(3)}\tilde{e}^j_{b(0)}\big).
\end{aligned} \tag{C.8}$$

Hence the variation of the metric δg_{ij} and its inverse δg^{ij} read

$$\begin{aligned}
\delta g_{(0)ij} &= \eta_{ab}\big(\delta e^a_{i(0)} e^b_{j(0)} + e^a_{i(0)} \delta e^b_{j(0)}\big) \\
\delta g_{(2)ij} &= \eta_{ab}\big(\delta e^a_{i(0)} e^b_{j(2)} + e^a_{i(0)} \delta e^b_{j(2)} + \delta e^a_{i(2)} e^b_{j(0)} + e^a_{i(2)} \delta e^b_{j(0)}\big) \\
\delta g_{(3)ij} &= \eta_{ab}\big(\delta e^a_{i(0)} e^b_{j(3)} + e^a_{i(0)} \delta e^b_{j(3)} + \delta e^a_{i(3)} e^b_{j(0)} + e^a_{i(3)} \delta e^b_{j(0)}\big), \\
\\
\delta g^{ij}_{(0)} &= \eta^{ab}\big(\delta \tilde{e}^i_{a(0)} \tilde{e}^j_{b(0)} + \tilde{e}^i_{a(0)} \delta \tilde{e}^j_{b(0)}\big) \\
\delta g^{ij}_{(2)} &= \eta^{ab}\big(\delta \tilde{e}^i_{a(0)} \tilde{e}^j_{b(2)} + \tilde{e}^i_{a(0)} \delta \tilde{e}^j_{b(2)} + \delta \tilde{e}^i_{a(2)} \tilde{e}^j_{b(0)} + \tilde{e}^i_{a(2)} \delta \tilde{e}^j_{b(0)}\big) \\
\delta g^{ij}_{(3)} &= \eta^{ab}\big(\delta \tilde{e}^i_{a(0)} \tilde{e}^j_{b(3)} + \tilde{e}^i_{a(0)} \delta \tilde{e}^j_{b(3)} + \delta \tilde{e}^i_{a(3)} \tilde{e}^j_{b(0)} + \tilde{e}^i_{a(3)} \delta \tilde{e}^j_{b(0)}\big).
\end{aligned} \tag{C.9}$$

Determinant of the metric In order to study the expansion of the determinant of the metric we manipulate it in an useful way

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}r^2 + \mathrm{e}^{2r/\ell}\big[g_{(0)}\big(1 + \mathrm{e}^{-2r/\ell}(g^{-1}_{(0)} g_{(2)}) + \cdots\big)\big]_{ij} \mathrm{d}x^i \mathrm{d}x^j \\
G &= g = \mathrm{e}^{6r/\ell} \det(g_{(0)ij}) \det\big[1 + \mathrm{e}^{-2r/\ell}(g^{-1}_{(0)} g_{(2)}) + \cdots\big]_{k\ell} \simeq \\
&\simeq \mathrm{e}^{6r/\ell} \det(g_{(0)ij})\big(1 + \mathrm{e}^{-2r/\ell} Tr[(g^{-1}_{(0)} g_{(2)})_{k\ell}] + \cdots\big).
\end{aligned} \tag{C.10}$$

Thus we obtain

$$\begin{aligned}
\Longrightarrow \sqrt{-G} = \sqrt{-g} &\simeq \mathrm{e}^{3r/\ell}\sqrt{-\det(g_{(0)})}\sqrt{1 + \mathrm{e}^{-2r/\ell} Tr(g^{-1}_{(0)} g_{(2)})} \simeq \\
&\simeq \mathrm{e}^{3r/\ell}\sqrt{-\det(g_{(0)})} + \frac{1}{2}\mathrm{e}^{r/\ell} Tr(g^{-1}_{(0)} g_{(2)})\sqrt{-\det(g_{(0)})},
\end{aligned} \tag{C.11}$$

where we have used the two equivalences valid for small A: $\det(1 + A) \simeq 1 + TrA$ and $\sqrt{1 + A} \simeq 1 + \frac{1}{2}A$.

Covariant derivative of the supersymmetric spinor parameter and gravitino In order to study the variation of bulk + boundary action, we need to compute the near-boundary expansions of $\nabla_i \epsilon$ and $\nabla_i \Psi_j$. The first reads

$$\begin{aligned}
\nabla_i \epsilon &= \partial_i \epsilon + \frac{1}{4}\omega_{i\alpha\beta}\gamma^{\alpha\beta}\epsilon \simeq \\
&\simeq \frac{1}{2}\mathrm{e}^{\frac{3r}{2\ell}}\omega_{(0)ia3}\gamma^{a3}\eta_{(0)+} + \\
&+ \mathrm{e}^{\frac{r}{2\ell}}\Big(\nabla_{(0)i}\eta_{(0)+} + \frac{1}{2}\omega_{(0)ia3}\gamma^{a3}\eta_{(1)-}\Big) + \mathrm{e}^{-\frac{r}{2\ell}}\Big[\Big(\frac{1}{2}\omega_{(0)ia3}\gamma^{a3}\eta_{(2)-}\Big) + \\
&+ \Big(\nabla_{(0)i}\eta_{(1)-} + \frac{1}{2}\omega_{(0)ia3}\gamma^{a3}\eta_{(2)+} + \frac{1}{2}\omega_{(2)ia3}\gamma^{a3}\eta_{(0)+}\Big)\Big] \equiv \\
&\equiv \mathrm{e}^{\frac{3r}{2\ell}}(\nabla_i \epsilon)_{(0)-} + \mathrm{e}^{\frac{r}{2\ell}}(\nabla_i \epsilon)_{(1)+} + \mathrm{e}^{-\frac{r}{2\ell}}\big((\nabla_i \epsilon)_{(2)+} + (\nabla_i \epsilon)_{(2)-}\big),
\end{aligned} \tag{C.12}$$

with $\nabla_{(0)i}$ the covariant derivative defined through $g_{(0)}$.

The asymptotic expansion of the gravitino covariant derivative yields

$$\begin{aligned}
\nabla_i\Psi_j \equiv \partial_i\Psi_j + \frac{1}{4}\omega_{i\alpha\beta}\gamma^{\alpha\beta}\Psi_j \simeq& \\
\simeq \frac{1}{2}\mathrm{e}^{\frac{3r}{2\ell}}\omega_{(0)id3}\gamma^{d3}\Psi_{(0)+j} + \mathrm{e}^{\frac{r}{2\ell}}\Big(\nabla_{(0)i}\Psi_{(0)+j} + \frac{1}{2}\omega_{(0)id3}\gamma^{d3}\Psi_{(1)-j}\Big)+& \\
+\,\mathrm{e}^{-\frac{r}{2\ell}}\Big[\Big(\nabla_{(0)i}\Psi_{(1)-j} + \frac{1}{2}\omega_{(0)id3}\gamma^{d3}\Psi_{(2)+j} + \frac{1}{2}\omega_{(2)id3}\gamma^{d3}\Psi_{(0)+j}\Big)+& \\
+\Big(\frac{1}{2}\omega_{(0)id3}\gamma^{d3}\Psi_{(2)-j}\Big)\Big]+& \\
+\,\mathrm{e}^{-\frac{3r}{2\ell}}\Big[\Big(\nabla_{(0)i}\Psi_{(2)+j} + \frac{1}{4}\omega_{(2)ide}\gamma^{de}\Psi_{(0)+j} + \frac{1}{2}\omega_{(0)id3}\gamma^{d3}\Psi_{(3)-j} + \frac{1}{2}\omega_{(2)id3}\gamma^{d3}\Psi_{(1)-j}\Big)+& \\
+\Big(\nabla_{(0)i}\Psi_{(2)-j} + \frac{1}{2}\omega_{(3)id3}\gamma^{d3}\Psi_{(0)+j} + \frac{1}{2}\omega_{(0)id3}\gamma^{d3}\Psi_{(3)+j}\Big)\Big] \equiv& \\
\equiv \mathrm{e}^{\frac{3r}{2\ell}}(\nabla_i\Psi_j)_{(0)-} + \mathrm{e}^{\frac{r}{2\ell}}(\nabla_i\Psi_j)_{(1)+} + \mathrm{e}^{-\frac{r}{2\ell}}\big((\nabla_i\Psi_j)_{(2)+} + (\nabla_i\Psi_j)_{(2)-}\big)+& \\
+\,\mathrm{e}^{-\frac{3r}{2\ell}}\big((\nabla_i\Psi_j)_{(3)+} + (\nabla_i\Psi_j)_{(3)-}\big).&
\end{aligned} \tag{C.13}$$

We intentionally omitted the Christoffel symbols in $\nabla_i\Psi_j$ because its symmetric indices are always contracted with two antisymmetric indices of the gamma in the bulk and boundary actions.

Extrinsic curvature, Ricci scalar and gauge covariant derivative of gravitino
In the variation of counterterms action, a number of quantities and their variations appear. We report their near-boundary expansions, remembering that we are interested in terms with up to two fermionic fields.

Let us start from the trace of the extrinsic curvature of a radial slice and its supersymmetric variation

$$
\begin{aligned}
K &= \frac{1}{2} g^{ij} \partial_r g_{ij} \simeq \\
&\simeq \frac{1}{\ell} g^{ij}_{(0)} g_{(0)ij} + \frac{1}{\ell} \mathrm{e}^{-2r/\ell} g^{ij}_{(2)} g_{(0)ij} + \frac{1}{\ell} \mathrm{e}^{-3r/\ell} \Big(g^{ij}_{(3)} g_{(0)ij} - \frac{1}{2} g^{ij}_{(0)} g_{(3)ij} \Big) \equiv \\
&\equiv K_{(0)} + \mathrm{e}^{-2r/\ell} K_{(2)} + \mathrm{e}^{-3r/\ell} K_{(3)},
\end{aligned}
$$

$$
\begin{aligned}
\delta K &= \frac{1}{2} \delta g^{ij} \partial_r g_{ij} + \frac{1}{2} g^{ij} \partial_r \delta g_{ij} \simeq \\
&\simeq \frac{1}{\ell} \Big(\delta g^{ij}_{(0)} g_{(0)ij} + g^{ij}_{(0)} \delta g_{(0)ij} \Big) + \frac{1}{\ell} \mathrm{e}^{-2r/\ell} \Big(\delta g^{ij}_{(2)} g_{(0)ij} + g^{ij}_{(2)} \delta g_{(0)ij} \Big) + \\
&+ \frac{1}{\ell} \mathrm{e}^{-3r/\ell} \Big[\Big(\delta g^{ij}_{(3)} g_{(0)ij} + g^{ij}_{(3)} \delta g_{(0)ij} \Big) - \frac{1}{2} \Big(\delta g^{ij}_{(0)} g_{(3)ij} + g^{ij}_{(0)} \delta g_{(3)ij} \Big) \Big] \equiv \\
&\equiv (\delta K)_{(0)} + \mathrm{e}^{-2r/\ell} (\delta K)_{(2)} + \mathrm{e}^{-3r/\ell} (\delta K)_{(3)}.
\end{aligned}
\tag{C.14}
$$

In $S_{(2)}$ term of Section 3.6 there are the Ricci scalar of a radial slice $R[g]$, the gravitino gauge covariant derivative $\nabla_i \Psi_j$ and their variations.
The expansion of the former yields

$$
\begin{aligned}
R[g] &= \big(\partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^i_{i\ell} \Gamma^\ell_{jk} - \Gamma^i_{j\ell} \Gamma^\ell_{ik} \big) g^{jk} \simeq \\
&\simeq \mathrm{e}^{-2r/\ell} \big(\partial_i \Gamma^i_{(0)jk} - \partial_j \Gamma^i_{(0)ik} + \Gamma^i_{(0)i\ell} \Gamma^\ell_{(0)jk} - \Gamma^i_{(0)j\ell} \Gamma^\ell_{(0)ik} \big) g^{jk}_{(0)} \equiv \\
&\equiv \mathrm{e}^{-2r/\ell} R_{(2)},
\end{aligned}
\tag{C.15}
$$

where Γ^k_{ij} near-boundary behaviour is

$$
\Gamma^k_{ij}[g] \simeq \frac{1}{2} g^{k\ell}_{(0)} \big(\partial_i g_{(0)j\ell} + \partial_j g_{(0)i\ell} - \partial_\ell g_{(0)ij} \big) \equiv \Gamma^k_{(0)ij}[g]. \tag{C.16}
$$

Its variation reads

$$
\begin{aligned}
\delta R &= R_{ij}[g] \delta g^{ij} + \nabla_i \big(g^{jk} \delta \Gamma^i_{jk} - g^{ik} \delta \Gamma^j_{jk} \big) \simeq \\
&\simeq \mathrm{e}^{-2r/\ell} \big[-\eta_{(0)+} \gamma^i_{(0)} \Psi_{(0)+k} R_{i(0)}^{\ k} + \nabla_{(0)i} \big(g^{jk}_{(0)} \delta \Gamma^i_{(0)jk} - g^{ik}_{(0)} \delta \Gamma^j_{(0)jk} \big) \big] \equiv \\
&\equiv \mathrm{e}^{-2r/\ell} (\delta R)_{(2)}
\end{aligned}
\tag{C.17}
$$

In the end the expansions of the gauge covariant derivative of gravitino and its variation yield

$$\begin{aligned}
\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j &= \partial_j\bar{\Psi}_{+i} - \frac{1}{4}\bar{\Psi}_{+i}\omega_{jab}\gamma^{ab} - \Gamma^k_{ij}[g]\bar{\Psi}_{+k} - \frac{\mathrm{i}}{\ell}A_j\bar{\Psi}_{+i} \simeq \\
&\simeq \mathrm{e}^{\frac{r}{2\ell}}\mathcal{D}_{(0)j}\bar{\Psi}_{(0)+i} - \frac{\mathrm{i}}{\ell}\mathrm{e}^{-\frac{r}{2\ell}}A_{(1)j}\bar{\Psi}_{(0)+i} \equiv \\
&\equiv \mathrm{e}^{\frac{r}{2\ell}}\big(\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(0)} + \mathrm{e}^{-\frac{r}{2\ell}}\big(\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(1)}, \\
\delta\big(\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big) &\simeq \partial_j\delta\bar{\Psi}_{+i} - \frac{1}{4}\delta\bar{\Psi}_{+i}\omega_{jab}\gamma^{ab} - \Gamma^k_{ij}[g]\delta\bar{\Psi}_{+k} - \frac{\mathrm{i}}{\ell}A_j\delta\bar{\Psi}_{+i} \simeq \\
&\simeq \mathrm{e}^{\frac{r}{2\ell}}\mathcal{D}_{(0)j}\delta\bar{\Psi}_{(0)+i} - \frac{\mathrm{i}}{\ell}\mathrm{e}^{-\frac{r}{2\ell}}A_{(1)j}\delta\bar{\Psi}_{(0)+i} \equiv \\
&\equiv \mathrm{e}^{\frac{r}{2\ell}}\big(\delta\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(0)} + \mathrm{e}^{-\frac{r}{2\ell}}\big(\delta\bar{\Psi}_{+i}\overleftarrow{\mathcal{D}}_j\big)_{(1)}.
\end{aligned} \tag{C.18}$$

where $\mathcal{D}_{(0)i}$ is the gauge covariant derivative constructed with $g_{(0)}$.

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