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Tesi di Laurea

Forma lineare delle equazioni di Friedmann e probabilità quasi-classica

Linear form of Friedmann equations and quasi-classical probability

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Abstract

Following [1] and [2], the Friedmann equations are introduced as a pair of second-order differential equations that describe the expansion of the Universe. From the assumption of a Universe homogeneous in spacetime and isotropic in space, the Friedmann-Lemaître-Robertson-Walker metric is derived. Considering then the Universe as a perfect fluid, the Friedmann equations follow. A fundamental consequence of these equations is that, by analyzing the redshift effect in the FLRW metric, one obtains the relation $P \propto a^{-1}(t)$ with P being the momentum of a free falling particle and $a(t)$ the scale factor.
Once introduced the Friedmann equations, it is possible to linearize them by using the properties of the Schwarzian derivative. This is done first for vanishing curvature κ, obtaining the canonical eigenvalue form $O_\beta \phi_\beta = \beta^2 \Lambda c^2/3\, \phi_\beta$, where $O_\beta := \frac{d^2}{dt^2} + V_\beta$ is the space-independent Klein-Gordon operator. Then, the case with $\kappa \neq 0$ and $\beta = \frac{1}{2}$ is treated and the Schwarzian derivative is used to absorb the curvature term. The equations in this case are equivalent to the eigenvalues problems $O_{1/2}\psi = \Lambda c^2/12\, \psi$ and $O_1 a = \Lambda c^2/3\, a$, and the solutions are $\psi = \sqrt{a} \exp\left(\pm \frac{ic}{2} \sqrt{\kappa} \eta\right)$, where $\eta := \int_t^{t_o} a^{-1}(t')\, dt'$ is the conformal time. It turns out that it is possible to generalize the previous equations even for arbitrary β-times, without getting linear equations, using the relation that stands between the Schwarzian and the Riccati equations.
Remarkably, the comparison with a first order WKB approximation of the Schrödinger equation opens the possibility to a "quantum interpretation" of the Friedmann equations. In particular, starting from the solutions of the linear Friedmann equations it is possible to notice an analogy with the WKB approximate solutions of Quantum Mechanics. This makes it possible to consider them as the WKB approximation of the *quantum Friedmann equation*

$$\left(\frac{d^2}{dt^2} + \frac{kc^2}{4a_k^2}\right)\Psi_k = 0 \qquad \left(\frac{d^2}{dt^2} \pm \frac{c^2}{4a_0^2}\right)\Psi_0 = 0,$$

$k = \pm 1$. This leads to a quantum scale factor, which has the scale factor as classical approximation. This suggests the existence of some interesting similarities between General Relativity and Quantum Mechanics.

Seguendo [1] e [2], le equazioni di Friedmann sono introdotte come una coppia di equazioni differenziali del second'ordine che descrivono l'espansione dell'Universo. Assumendo un Universo omogeneo nello spaziotempo ed isotropo nello spazio, si può derivare la metrica di Friedmann-Lemaître-Robertson-Walker. Considerando poi l'Universo come un fluido perfetto, si ottengono le equazioni di Friedmann. Una conseguenza importante di queste equazioni è che, analizzando l'effetto redshift nella metrica di FLRW, si ottiene la relazione $P \propto a^{-1}(t)$ con P la quantità di moto di una particella libera e $a(t)$ il fattore di scala.
Una volta introdotte le equazioni di Friedmann, è possibile linearizzarle utilizzando le proprietà della derivata Schwarziana. Ciò viene fatto prima per curvatura κ nulla, ottenendo la forma canonica degli autovalori $O_\beta \phi_\beta = \beta^2 \Lambda c^2/3\, \phi_\beta$, dove $O_\beta := \frac{d^2}{dt^2} + V_\beta$ è l'operatore di Klein-Gordon indipendente dallo spazio. Quindi, viene trattato il caso in cui $\kappa \neq 0$ e $\beta = \frac{1}{2}$ e la derivata Schwarziana viene utilizzata per assorbire il termine di curvatura. Le equazioni in questo caso sono equivalenti ai problemi agli autovalori $O_{1/2}\psi = \Lambda c^2/12\, \psi$ e $O_1 a = \Lambda c^2/3\, a$, e le soluzioni sono $\psi = \sqrt{a} \exp\left(\pm \frac{ic}{2} \sqrt{\kappa} \eta\right)$, dove $\eta := \int_t^{t_o} a^{-1}(t')\, dt'$ è il tempo conforme. È inoltre possibile generalizzare le equazioni precedenti anche per β-tempi arbitrari, senza ottenere equazioni lineari, utilizzando la relazione tra l'equazione di Schwarz e quella di Riccati.
Sorprendentemente, il confronto con l'approssimazione WKB al prim'ordine dell'equazione di Schrödinger apre la possibilità ad una "interpretazione quantistica" delle equazioni di Friedmann. In particolare, partendo dalle soluzioni delle equazioni di Friedmann lineari è possibile notare un'analogia con le soluzioni approssimate WKB della Meccanica Quantistica. Ciò rende possibile considerarle come l'approssimazione WKB dell'*equazione di Friedmann quantistica*

$$\left(\frac{d^2}{dt^2} + \frac{kc^2}{4a_k^2}\right)\Psi_k = 0 \qquad \left(\frac{d^2}{dt^2} \pm \frac{c^2}{4a_0^2}\right)\Psi_0 = 0,$$

$k = \pm 1$. Viene quindi introdotto un fattore di scala quantistico, che ha il fattore di scala come approssimazione classica. Questo suggerisce l'esistenza di alcune analogie tra la Relatività Generale e la Meccanica Quantistica.

Contents

1 Introduction

Quantum cosmology is a branch of theoretical physics that aims to develop a quantum theory of the Universe. Its specific goal is to reconcile the two primary theories of the 20^{th} century: General Relativity and Quantum Mechanics. In particular, General Relativity is an highly non-linear theory, which finds its foundation in the Einstein's equation

$$\text{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1}$$

On the other hand, Quantum Mechanics is an intrinsically linear theory, based on the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(\mathbf{x},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\right)\psi(\mathbf{x},t). \tag{1.2}$$

This work is based on [1, 2] and it aims to reveal some intriguing aspects of General Relativity that could show a possible connection between the two theories.

The second section of this work is focused on the introduction of the Friedmann equations [3]. First, it is necessary to consider the Universe as a perfect fluid, homogeneous in spacetime and isotropic in space, described by the energy momentum tensor $T^{\mu\nu} = diag(\rho,p,p,p)$. Then, the FLRW metric is introduced in the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right]. \tag{1.3}$$

At this point it is possible to derive the Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{\kappa c^2}{a^2} \tag{1.4a}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{1.4b}$$

where the first one is linear while the second one is not.
In the context of the FLRW metric, it is possible to analyze also the redshift effect, which is typical of General Relativity. Furthermore, from the redshift relation, it follows that $P \propto a^{-1}(t)$ and also $a(t) \propto \lambda$, where λ is the de Broglie wavelength. As it will be shown, this suggests a possible relation between Quantum Mechanics and General Relativity.

In the third section a linear form of the Friedmann equations is derived using the invariance of the Schwarzian derivative under Möbius transformations. Starting with the $\kappa = 0$ case, the Friedmann equations are linearized in the form

$$O_\beta\phi_\beta = \beta^2\frac{\Lambda c^2}{3}\phi_\beta. \tag{1.5}$$

where the the space-independent Klein-Gordon operator is defined as $O_\beta := \frac{d^2}{dt^2} + V_\beta$. Introducing the eigenvalue problem

$$\begin{pmatrix} O_{-\beta} & 0 \\ 0 & O_\beta \end{pmatrix}\Psi_{-\beta\beta} = \beta^2\frac{\Lambda c^2}{3}\Psi_{-\beta\beta} \tag{1.6}$$

it is possible to group all the canonical forms of the equation. The solutions in this case are

$$\psi = a^\beta \qquad\qquad \psi^D = a^\beta t_\beta. \tag{1.7}$$

The case where $\kappa \neq 0$ is initially analyzed by setting $\beta = 1/2$, i.e. selecting the conformal time defined as

$$\eta(t) := \int_t^{t_o}\frac{dt'}{a(t')}, \tag{1.8}$$

where t is the initial time and t_o is the time of observation. The curvature term is subsequently absorbed through exponentiation, allowing the derivation of the equation

$$\left[\frac{d^2}{dt^2} + \frac{2\pi G}{3}\left(2\rho + \frac{3p}{c^2}\right)\right]\psi = \frac{\Lambda c^2}{12}\psi \tag{1.9}$$

that has the two linearly independent solutions

$$\psi = \sqrt{a}e^{-\frac{ic}{2}\sqrt{\kappa}\eta} \qquad \psi^D = \sqrt{a}e^{\frac{ic}{2}\sqrt{\kappa}\eta}. \tag{1.10}$$

In the last part of the section, this scheme is generalized even for arbitrary β-times when $k \neq 0$ using the connection between the Riccati and the Schwarzian equations. This will show that the $\beta = \frac{1}{2}$ case is the only one that linearizes the Friedmann equations. Finally, a scheme is presented on how to map solutions corresponding to different types of spatial geometries to each other.

In the last part, the WKB approximation of Quantum Mechanics is introduced in order to underline the analogy between the WKB approximate solutions of the Schrödinger equation and those of the linear form of the Friedmann equations. In particular, this analogy allows to formulate an equation which can be considered as the quantum version of the Friedmann equations

$$\left(\frac{d^2}{dt^2} + \frac{kc^2}{4a_k^2}\right)\Psi_k = 0 \qquad k = \pm 1 \tag{1.11a}$$

$$\left(\frac{d^2}{dt^2} \pm \frac{c^2}{4a_0^2}\right)\Psi_0 = 0 \qquad k = 0. \tag{1.11b}$$

The solutions in the $k = \pm 1$ cases are

$$\Psi_k = \sqrt{a_{qk}}e^{\pm\frac{ic}{2}\sqrt{k}\eta_{qk}}. \tag{1.12}$$

Note that the quantization of a_{qk} directly impacts on Ψ_k, which is in turn quantized.

The paper also includes a supplement at the end of the second section, intended to demonstrate the connection between the Newton's law of gravity and the Friedmann equations.

The Natural Units are used in sections 2 and 3 to simplify the notation, this means that $c = \hbar = 1$. In section 4 the Natural Units will be abandoned, since it will be necessary to restore the value and dimensionality of $\hbar$ and c.

2 Derivation of the Friedmann equations

The first section of the thesis focuses on the derivation of the Friedmann equations as shown in [4, ch. 8]. First, it is necessary to introduce the energy-momentum tensor ($T^{\mu\nu}$) that models the Universe, as done in [4, ch. 1,4]. In the second part, the Friedmann-Lemaître-Robertson-Walker (FLRW) metric is introduced. This is fundamental to proceed with the third part where, starting from Einstein's equations, the Friedmann equations will be derived. The following part concentrates in the explanation of the redshift, a phenomenon typical of General Relativity, and the conformal time is introduced. This part is taken from [5]. The last section of the chapter contains an additional introduction of the Friedmann equations starting from the Newton's law. This part is taken from [6] and it is a supplement, a curiosity aimed at connecting Newtonian gravity with General Relativity.

2.1 Mathematical prerequisites

To begin, it is fundamental to introduce some mathematical aspects which are presented in [4, Ch. 1, 4] and that will be used in the following sections. On this purpose, let (M, g) be a pseudo-Riemannian manifold.

Definition 2.1. *At any point $p \in M$ there exists a coordinate system $\hat{x}^\mu$ where the metric $g_{\hat{\mu}\hat{\nu}}$ takes its canonical form $diag(-1, -1, \dots, 1, 1, \dots)$ and the first derivatives vanish ($\partial_\alpha g_{\hat{\mu}\hat{\nu}} = 0$). These coordinates are called locally inertial coordinates and the associated basis vectors constitute a local Lorentz frame.*

Moreover, if the metric assumes its canonical form at a certain point $p \in M$ then, due to non degeneracy of g, the canonical form is the same in the whole manifold. In special relativity, following the "mostly positive" convention, the Minkowski metric is introduced as $\eta_{\hat{\mu}\hat{\nu}} = diag(-1, 1, 1, 1)$. This metric is invariant under Lorentz transformations in locally inertial coordinates and it is the only tensor $(0, 2)$ with this property.
In the next section, some other features of the pseudo-Riemannian manifolds will be considered. The most important ones are introduced here.

Definition 2.2. *Let $\phi : M \to M$ be a diffeomorphism, ϕ is a symmetry for the tensor $T \in T^0_r M$ if T is invariant after being pulled back under ϕ, namely*

$$\phi^* T = T. \tag{2.1}$$

In local coordinates $x = \{x^1, ..., x^m\}$ and $y = \{y^1, ..., y^m\} = \phi(x)$

$$\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}} (T_{|\phi(p)})_{\alpha_1 \dots \alpha_r} = (T_{|p})_{\mu_1 \dots \mu_r}. \tag{2.2}$$

It can be shown that the symmetry condition is equivalent to the requirement

$$\mathcal{L}_X T = 0 \tag{2.3}$$

where X is the vector field that generates the one-parameter family of symmetries ϕ and $\mathcal{L}$ is the Lie-derivative. This second definition underlines the fact that if X generates a symmetry for the tensor T, then the tensor results invariant under the displacement of ϵX (as ϵ approaches 0). This means that

$$\mathcal{L}_X T = 0 \iff \phi^* T = T \tag{2.4}$$

when ϕ is the one-parameter family of diffeomorphisms generated by X.

Definition 2.3. *Let $\phi : M \to M$ be a diffeomorphism which maps M onto itself. ϕ is an isometry if it preserves the metric, which means*

$$\phi^* g_{|\phi(p)} = g_{|p}. \tag{2.5}$$

In local coordinates

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} (g_{|\phi(p)})_{\alpha\beta} = (g_{|p})_{\mu\nu} \qquad \forall p \in M. \tag{2.6}$$

Definition 2.4. A vector $X \in TM$ is called a *Killing vector* if

$$\mathcal{L}_X g = 0. \tag{2.7}$$

The vector field generated by X is called *Killing vector field*.

At this point, the following characterizations of a pseudo-Riemannian manifold (M, g) can be presented.

Definition 2.5. A manifold M is *isotropic* around a point $p \in M$ if for any two vectors $V, W \in T_pM$ there exists an isometry of M such that the push-forward of W under the isometry is parallel to V. In a less formal way: a manifold is *isotropic* if it appears the same regardless of the direction from which it is observed.

Definition 2.6. A manifold M is *homogeneous* if given $p, q \in M$ there is an isometry that takes p into q. In a less formal way: a manifold M is *homogeneous* if the metric is the same throughout the manifold.

Definition 2.7. A manifold with the maximum number of linear independent Killing vector fields (i.e. $\frac{1}{2}m(m+1)$)[1] is called *maximally symmetric space*.

2.2 The energy-momentum tensor of the Universe

In this section, the energy-momentum tensor of a perfect fluid is introduced to describe a Universe that, according to the *Cosmological Principle*, is homogeneous in space-time and isotropic in space. As a consequence, the Universe can be characterized by the following two parameters: ρ, the rest-frame energy density, and p, the isotropic rest-frame pressure. Isotropy implies that the energy-momentum tensor ($T^{\mu\nu}$) is diagonal in the rest-frame. Furthermore, the non-zero space-like components must be equal: $T^{11} = T^{22} = T^{33}$, as a consequence $T^{00} = \rho$ and $T^{ii} = p$. Therefore, the energy-momentum tensor of a perfect fluid takes the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{2.8}$$

This representation of $T^{\mu\nu}$ is only applicable in the rest frame. To generalize this to an arbitrary frame, the following equation must be used

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu} \tag{2.9}$$

where U^μ is the four-velocity of the fluid and $g^{\mu\nu}$ is the metric tensor of a pseudo-Riemannian manifold M ($g \in T^{0,2}M$). The convention employed is the “mostly positive” one, which states that the metric has signature $(-, +, +, +)$.

In particular, by expressing the Einstein’s equations in the form

$$G_{\mu\nu} = kT_{\mu\nu} \tag{2.10}$$

it is noticeable that the energy-momentum tensor is the cause of the curvature of spacetime; this opens up the possibility of an energy density characteristic of vacuum space, namely the *vacuum energy*. The energy-momentum tensor of the vacuum energy needs to be invariant under Lorentz transformations in locally inertial coordinates. As seen in the previous section $\eta_{\hat{\mu}\hat{\nu}}$ is the only tensor with this characteristics, so it is possible to deduce the following expression of the tensor associated to vacuum energy

$$T^{(vac)}_{\hat{\mu}\hat{\nu}} = -\rho_{(vac)}\eta_{\hat{\mu}\hat{\nu}} \tag{2.11}$$

that in arbitrary coordinates can be expressed as

$$T^{(vac)}_{\mu\nu} = -\rho_{(vac)}g_{\mu\nu}. \tag{2.12}$$

[1] In an m-dimensional manifold, there are m Killing vectors generating translations, $m-1$ boosts and $(m-1)(m-2)/2$ space rotations, for a total of $m(m+1)/2$.

Comparing this expression with (2.9) yields $(\rho_{(vac)} + p_{(vac)})(U_\mu U_\nu + g_{\mu\nu}) = 0$, which leads to the conclusion: $p_{vac} = -\rho_{vac}$. The decomposition of the energy-momentum tensor of Einstein's field equations

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.13}$$

in its "matter" and "vacuum" components, leads to

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^{(mat)}_{\mu\nu} \tag{2.14}$$

where $\Lambda = 8\pi G \rho_{(vac)}$ represents the *cosmological constant*, an alternative definition for the vacuum energy. By the second Bianchi identity in local coordinates $(\nabla_\alpha R)^\delta{}_{\epsilon\beta\gamma} + (\nabla_\beta R)^\delta{}_{\epsilon\gamma\alpha} + (\nabla_\gamma R)^\delta{}_{\epsilon\alpha\beta} = 0$ it is possible to show that the energy-momentum tensor is still conserved. In fact, contracting $\delta \leftrightarrow \beta$, then $\epsilon \leftrightarrow \gamma$ and remembering the definition of the Ricci tensor and the symmetries (or antisymmetries) of $R^\delta{}_{\epsilon\beta\gamma}$ one gets $\nabla_\alpha(\mathcal{R}\mathrm{Id} - 2\mathrm{Ric})^\alpha_\epsilon = -2\nabla_\alpha G^\alpha{}_\epsilon = 0$. Now remembering that ∇ is the Levi-Civita connection, so that $\nabla g = 0$, one obtains

$$\nabla_\mu(\mathrm{Ric} - \frac{1}{2}\mathcal{R}\mathrm{Id} + \Lambda g)^{\mu\nu} = 0, \tag{2.15}$$

this shows that the energy momentum tensor is conserved.

2.3 The Friedmann-Lemaître-Robertson-Walker (FLRW) metric

In this section, the FLRW metric will be introduced following [4, ch. 8]. In this regard, it is convenient to begin by considering the spacetime as homogeneous and isotropic. This hypothesis is advantageous because it makes the space maximally symmetric. Subsequently, the case where the space is homogeneous and isotropic and the time is homogeneous will be considered.
The Riemann tensor for a maximally symmetric m-dimensional manifold can be expressed as $R_{\rho\sigma\mu\nu} = k(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$ where $k = \frac{\mathcal{R}}{m(m-1)}$ is the normalized Ricci curvature and $\mathcal{R}$ is constant due to symmetry.

At any point in space the metric can be represented in its canonical form ($g_{\mu\nu} = \eta_{\mu\nu}$), so the maximally symmetric manifolds can be locally characterized by the signature of the metric $(-,+,+,+)$ and by the sign of k. In particular, the last parameter identifies three types of manifolds: if $k > 0$, the manifold is called *de Sitter space*; if $k < 0$, it is called *anti-de Sitter space*; if $k = 0$ it is called *flat space.* A flat space is a maximally symmetric space with zero curvature, which is the Minkowski space.

To verify the usefulness of this model it is necessary to demonstrate that it satisfies the Einstein's field equations. Considering that $Ric_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = 3kg_{\mu\nu}$ and $\mathcal{R} = Ric^\mu{}_\mu = 12k$ the Einstein's tensor becomes

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = -3kg_{\mu\nu}. \tag{2.16}$$

By (2.13) it is possible to deduce that $8\pi G T_{\mu\nu} = -3kg_{\mu\nu}$, so the energy-momentum tensor can be expressed as

$$T_{\mu\nu} = -\frac{3k}{8\pi G} g_{\mu\nu}. \tag{2.17}$$

Considering the components of the tensor $g_{\mu\nu}$

$$\rho = -p = \frac{3k}{8\pi G}. \tag{2.18}$$

Lastly, assuming that $T^{(mat)}_{\mu\nu}$ is not proportional to the metric gives

$$\Lambda = 3k. \tag{2.19}$$

This description is consistent only in maximally symmetric spaces, nevertheless the Universe is spatially homogeneous and isotropic, but evolving in time. For this reason, it is possible to consider the spacetime to be $\mathbb{R} \times \Sigma$, with $\mathbb{R}$ representing the time direction and Σ is a maximally symmetric three-manifold. The metric then takes the form

$$ds^2 = -dt^2 + R^2(t)d\sigma^2 \tag{2.20}$$

where t is the timelike coordinate that evolves, $R(t)$ is the *scale factor* and $d\sigma^2$ is the metric of Σ and it takes the form $d\sigma^2 = \gamma_{ij}du^i du^j$, where (u^1,u^2,u^3) are the coordinates of Σ and γ_{ij} is a maximally symmetric 3D metric. The *comoving coordinates* are those in which the metric is free of cross terms $dt \cdot du^i$, and the dt^2 term is independent of the u^i.

Since the space considered is maximally symmetric, then it will certainly be spherically symmetric, for this reason the spherical coordinates $(\xi,\ \theta,\ \varphi)$ will be introduced. In [4, ch. 5] a metric which is spherically symmetric and three dimensional is written in its most general form as

$$d\sigma^2 = e^{2\beta(\xi)}d\xi^2 + e^{2\gamma(\xi)}\xi^2 d\Omega^2 \tag{2.21}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the 3D solid angle. The choice of the exponential has been made in order to preserve the signature of the metric. Defining the radial coordinate as $\hat{r} = e^{\gamma(\xi)}\xi$, so that $d\hat{r} = \left(1 + \xi\frac{d\gamma(\xi)}{d\xi}\right)e^{\gamma(\xi)}d\xi$, and re-defining $e^{\alpha(\xi)} = \left(1 + \xi\frac{d\gamma(\xi)}{d\xi}\right)^{-1} e^{\beta(\xi)-\gamma(\xi)}$ the metric becomes, without loss of generality

$$d\sigma^2 = e^{2\alpha(\hat{r})}d\hat{r}^2 + \hat{r}^2 d\Omega^2. \tag{2.22}$$

The direct computation of the Ricci tensor's components gives the following results (where ${}^{(3)}$ indicates that it is associated to the 3D spatial component of the tensor)

$${}^{(3)}\mathrm{Ric}_{11} = \frac{2}{\hat{r}}\partial_1\alpha \tag{2.23a}$$

$${}^{(3)}\mathrm{Ric}_{22} = e^{-2\alpha}(\hat{r}\partial_1\alpha - 1) + 1 \tag{2.23b}$$

$${}^{(3)}\mathrm{Ric}_{33} = \left[e^{-2\alpha}(\hat{r}\partial_1\beta - 1) + 1\right]\ \sin^2(\theta) \tag{2.23c}$$

$${}^{(3)}\mathrm{Ric}_{ij} = 0 \qquad \forall i \neq j. \tag{2.23d}$$

The maximally symmetric metrics obey ${}^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$ and ${}^{(3)}\mathrm{Ric}_{ij} = 2k\gamma_{ij}$ with $k = \frac{{}^{(3)}\mathcal{R}}{6}$. Solving the equation obtained by considering $\gamma_{ij} = g_{ij}$ and setting ${}^{(3)}\mathrm{Ric}_{ij} = 2kg_{ij}$ equal to (2.23a) yields

$$\alpha(\hat{r}) = -\frac{1}{2}\ln(b - k\hat{r}^2) = -\frac{1}{2}\ln(1 - k\hat{r}^2) \tag{2.24}$$

where b was put equal to 1 since for flat space the metric has to be euclidean. With this result, considering (2.20) and (2.22), the *Robertson-Walker metric* can be defined as

$$ds^2 = -dt^2 + R^2(t)\left[\frac{d\hat{r}^2}{1 - k\hat{r}^2} + \hat{r}^2 d\Omega^2\right]. \tag{2.25}$$

In some cases it is useful to normalize the metric so that the curvature parameter k is equal to $\{-1, 0, 1\}$ in the case of open, flat and closed Universe respectively. Another possible solution is to work with a dimensionless scale factor $a(t)$. This can be achieved through the following substitutions

$$a(t) := \frac{R(t)}{R_0} \qquad [a] = \text{dimensionless} \tag{2.26a}$$

$$r := R_0\hat{r} \qquad [r] = \text{length} \tag{2.26b}$$

$$\kappa := \frac{k}{R_0^2} \qquad [\kappa] = \text{length}^{-2}, \tag{2.26c}$$

the sign of $\kappa \in \mathbb{R}$ characterizes the Universe. In these variables, the *FLRW metric* becomes

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2\right] \tag{2.27}$$

which is invariant under rescalings such as $r \to \lambda r$, $a \to \lambda^{-1} a$, $\kappa \to \lambda^{-2}\kappa$, where $\lambda \in \mathbb{R}$ is a dimensionless parameter.
Lastly, the non-zero Christoffel symbols associated to the FLRW metric are

$$\Gamma^0_{11} = \frac{a\dot{a}}{1-\kappa r^2} \qquad \Gamma^0_{22} = a\dot{a}r^2 \qquad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta$$
$$\Gamma^1_{11} = \frac{\kappa r}{1-\kappa r^2} \qquad \Gamma^1_{22} = -r(1-\kappa r^2) \qquad \Gamma^1_{33} = -r(1-\kappa r^2)\sin^2\theta$$
$$\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^2_{12} = \Gamma^3_{13} = r^{-1}$$
$$\Gamma^3_{23} = \cot\theta$$

where $(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$ and $\dot{a} = \frac{da}{dt}$. By definition, it is clear that $\Gamma^\alpha_{\beta\gamma}$ is symmetric in the indices β and γ. The non-zero components of the Ricci tensor are

$$\mathrm{Ric}_{00} = -3\frac{\ddot{a}}{a} \qquad \mathrm{Ric}_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1-\kappa r^2}$$
$$\mathrm{Ric}_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa) \qquad \mathrm{Ric}_{33} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\sin^2\theta$$

and the Ricci scalar is

$$\mathcal{R} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right].$$

2.4 The Friedmann equations

In this section, the Friedmann equations will be obtained starting from Einstein's field equations using the energy-momentum tensor $T_{\mu\nu} = diag(\rho, p, p, p)$, the Ricci tensor and the Ricci scalar previously introduced. Considering the first component of (1.1)

$$\mathrm{Ric}_{00} - \frac{1}{2}\mathcal{R}g_{00} + \Lambda g_{00} = 8\pi G T_{00}, \tag{2.28}$$

by direct substitution and rearranging the terms, the *first Friedmann equation* is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{\kappa}{a^2}. \tag{2.29a}$$

To obtain the second equation, instead, it is necessary to multiply by $g^{\nu\mu}$ both terms of (1.1). This gives (remembering that $g^\nu{}_\nu = 4$ and $T^\nu{}_\nu = 3p - \rho$)

$$-\mathcal{R} + 4\Lambda = 8\pi G(3p - \rho).$$

By substituting the terms calculated in the previous section and (2.29a), the *second Friedmann equation* follows

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.29b}$$

Sometimes the parameter $H := \frac{\dot{a}}{a}$, called *Hubble parameter*, is introduced as it characterizes the rate of expansion of the Universe. The second equation (2.29b) is a linear differential equation, while the first one (2.29a) is highly non linear.
This section is concluded deriving the continuity equation. Starting from (2.29a) and taking the time derivative, one gets

$$2\frac{\dot{a}\ddot{a}a}{a^3} - 2\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\dot{\rho} + \frac{2\kappa}{a^2}\frac{\dot{a}}{a}. \tag{2.30}$$

Hence, replacing $\frac{\ddot{a}}{a}$ with (2.29b) and $\left(\frac{\dot{a}}{a}\right)^2$ with (2.29a) gives

$$2H\left[-\frac{4\pi G}{3}(\rho+3p)+\frac{\Lambda}{3}\right]-2H\left[\frac{8\pi G}{3}\rho+\frac{\Lambda}{3}-\frac{\kappa}{a^2}\right]=\frac{8\pi G}{3}\dot{\rho}+\frac{2\kappa}{a^2}H \tag{2.31}$$

$$\implies \frac{8\pi G}{3}\left(\dot{\rho}+3H(\rho+p)\right)=0. \tag{2.32}$$

Considering only the part in brackets one obtains the *continuity equation*

$$\dot{\rho}+3H(\rho+p)=0. \tag{2.33}$$

2.5 Redshift and conformal time

The redshift is a typical phenomenon observed in General Relativity. In this section, this phenomenon will be introduced and an analogy with the Quantum Mechanics will emerge. This part is based on [5] except for the part regarding the massive particles, where [7] is taken into account.
In General Relativity, the trajectory of a photon (which is massless) is described by $x^\mu(s)$, with s as the affine parameter. The momentum of the photon is

$$P^\mu=\frac{dx^\mu}{ds}, \tag{2.34}$$

hence the energy of the photon measured by an observer with four-velocity u is $E=-u_\mu P^\mu$. The equations of the geodesics where the photon travels in vacuum are

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0 \tag{2.35a}$$

$$\frac{dP^\mu}{ds}+\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta=0 \tag{2.35b}$$

and the photon travels on null geodesics $g_{\mu\nu}P^\mu P^\nu=0$. The FLRW metric (2.27) can be reparameterized as

$$ds^2=-dt^2+a^2(t)d\sigma^2 \tag{2.36}$$

with

$$d\sigma^2=d\chi^2+f^2(\chi)d\Omega^2$$

$$f(\chi)=\begin{cases}\sqrt{\kappa^{-1}}\sin(\sqrt{\kappa}\chi) & \kappa>0\\ \chi & \kappa=0\\ \sqrt{-\kappa^{-1}}\sinh(\sqrt{\kappa}\chi) & \kappa<0.\end{cases}$$

If the observer is at the origin ($\chi=0$) at t_o looking in the direction $(\tilde{\theta},\tilde{\varphi})$ and a photon is emitted with energy E_i at $(t_i,\chi_i,\tilde{\theta},\tilde{\varphi})$, and directed towards the origin, it will arrive with energy E_o. The photon is travelling radially ($P^\theta=0, P^\varphi=0$), so $g_{\mu\nu}P^\mu P^\nu=0$ becomes (the sign "$-$" in the second equation is chosen considering the direction of the photon)

$$-E^2+a^2(t)(P^\chi)^2=0\implies P^\chi=-\frac{E}{a(t)} \tag{2.37}$$

By (2.35b), considering the non-zero Christoffel symbols and the non-zero coefficients, it is possible to find $\frac{\dot{E}}{E}=-\frac{\dot{a}}{a}$, that is $E=\frac{c}{a}$, $c\in\mathbb{R}$. To obtain this result it is necessary to remember that $\frac{dE}{ds}=\frac{dE}{dt}\frac{dt}{ds}=\dot{E}\frac{dt}{ds}=\dot{E}\frac{dx^0}{ds}=\dot{E}P^0=\dot{E}E$. A photon is a massless particle and this implies that $E=|\vec{P}|$, as a consequence $P\propto a^{-1}$. In terms of wavelength ($E=\frac{hc}{\lambda}=h\nu$) this relation can be expressed as $\lambda\propto a\ \forall t$. From this the redshift relation follows

$$\frac{\lambda(t_o)}{a(t_o)}=\frac{\lambda(t)}{a(t)}. \tag{2.38}$$

The *redshift* z, a parameter used to quantify the expansion of the Universe, is defined as

$$z := \frac{\lambda(t_o) - \lambda(t)}{\lambda(t)} = \frac{a(t_o) - a(t)}{a(t)} \iff a(t) = \frac{a(t_o)}{1+z} = \frac{1}{1+z} \tag{2.39}$$

where in the last passage $a(t_o)$ was set equal to 1.
This result can be generalized also for massive particles, as done in [7]. In this regard, it is necessary to consider the momentum

$$P := m_0 \sqrt{g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}}. \tag{2.40}$$

From the geodesics equation (2.35a) when the particle is near the origin ($\chi = 0$) it is possible to obtain for $\mu = 1$

$$\frac{d^2\chi}{ds^2} = -\kappa\chi \left(\frac{d\chi}{ds}\right)^2 - 2\frac{\dot{a}}{a} \frac{dt}{ds} \frac{d\chi}{ds}. \tag{2.41}$$

Here the relations $\frac{dx^2}{ds} = 0$ and $\frac{dx^3}{ds} = 0$ hold. Since $\chi = 0$

$$\frac{d^2\chi}{ds^2} = -2\frac{\dot{a}}{a} \frac{dt}{ds} \frac{d\chi}{ds} \tag{2.42}$$

$$\frac{ds}{dt} \frac{d^2\chi}{ds^2} = -2\frac{\dot{a}}{a} \frac{d\chi}{ds} \tag{2.43}$$

$$\frac{d}{dt}\left(\frac{d\chi}{ds}\right) = -2\frac{\dot{a}}{a} \frac{d\chi}{ds} \tag{2.44}$$

that once solved gives

$$\frac{d\chi}{ds} \propto \frac{1}{a^2(t)}. \tag{2.45}$$

From (2.40), considering that $g = -dt^2 + a^2(t)d\chi^2 + 0$ because $\chi = 0$

$$P_m = m\sqrt{a^2(t)\left(\frac{d\chi}{ds}\right)^2} = ma(t)\frac{d\chi}{ds} \Longrightarrow \frac{P_m}{ma(t)} = \frac{d\chi}{ds} \propto a^{-2}(t) \tag{2.46}$$

$$P_m \propto \frac{1}{a(t)} \tag{2.47}$$

$$a(t) \propto \lambda \tag{2.48}$$

The last result is extremely important because starting from a gravitational context, without considering quantum aspects, the relation found says that the de Broglie wavelength $\lambda_m = \frac{h}{P_m}$, which satisfies the redshift relation, can be associated with a massive particle as well.

To conclude it is important to introduce, following [5], the conformal time which will be fundamental in the following sections. Continuing the discussion about the photon previously initiated, it was said that its trajectory was null: $u^\mu u_\mu = 0$ ($u^\theta = 0$ and $u^\varphi = 0$). In particular

$$-(u^t)^2 + a^2(t)(u^\chi)^2 = 0 \Rightarrow \frac{u^\chi}{u^t} = \pm\frac{1}{a} \tag{2.49}$$

$$\frac{d\chi}{dt} = \frac{d\chi}{ds}\frac{ds}{dt} = \frac{u^\chi}{u^t} = \pm\frac{1}{a}. \tag{2.50}$$

Now, since the photon is moving towards the origin $\frac{d\chi}{dt} = -\frac{1}{a}$, this gives

$$\int_{x_i}^{x_o=0} d\chi = \int_{t_i}^{t_o} -\frac{dt}{a(t)} \tag{2.51}$$

and by redefining the variable t so that t_o is the time of the observer and t is the initial time, the *conformal time* is

$$\eta(t) := \int_t^{t_o} \frac{dt'}{a(t')} \tag{2.52}$$

This definition implies

$$\dot{\eta} = a^{-1}(t) \tag{2.53}$$

that explains the relation between the derivative of the conformal time and the scale factor.

2.6 Newtonian cosmology: from Newton to Friedmann equations

The final part of this section aims to connect, following [6], the Newtonian gravitation with the Friedmann equations. Although the derivation is not entirely self-contained as it relies on a result from General Relativity (the Birkhoff's theorem), it nevertheless offers some interesting insights.
Starting from an inertial frame with origin O and choosing two points A and B, the velocity at which they move away from O is given by $\mathbf{v}_A = H_0\mathbf{r}_A$ and $\mathbf{v}_B = H_0\mathbf{r}_B$, where H_0 represents the Hubble parameter at the present time while $\mathbf{v}$ and $\mathbf{r}$ are respectively the velocity and the position vectors. By the vector addition rule, the recession velocity of B as seen by an observer in A is

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A = H_0\left(\mathbf{r}_B - \mathbf{r}_A\right)$$

so the observer in A sees all the other points in the Universe receding with velocities described by the same Hubble law. In a homogeneous universe every particle moving with the substratum has a purely radial velocity proportional to its distance from the observer. Considering the comoving coordinates, which are coordinates that expand with the Universe, the distance $\mathbf{r}$ can be expressed as the product of the comoving distance $\mathbf{x}$ and a term $a(t)$, the scale factor, which is a function of time only

$$\mathbf{r}_{AB} = a(t)\mathbf{x}_{AB}.$$

The derivation of an equation describing the universal expansion thus reduces to determining $a(t)$.
Let now A, B, C, D be four masses lying on a sphere of radius $\mathbf{r}$ centered at O, the Birkhoff's theorem states that the net gravitational effect of a uniform environment on a spherical cavity is zero, i.e. the force acting on A, B, C, D is the gravitational attraction from the matter internal to $\mathbf{r}$ acting as if it were a massive point located at O. The energy of a particle of mass m resting in one of the four points is

$$U = T + V = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}Gm\rho r^2$$

where ρ is the density of matter within the sphere of radius $\mathbf{r}$. Substituting the equation for $\mathbf{r}$ previously introduced gives

$$U = T + V = \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi}{3}Gm\rho a^2x^2$$

which, replacing $\kappa = -\frac{2U}{mx^2}$ and rearranging the terms, reveals the *first Friedmann equation*

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}.$$

Due to the independence of the other terms of the equation from x, κ must be independent of x, thus $U \propto x^2$. On the other hand, the conservation of the total energy implies that κ is independent of t, this leads to the conclusion that κ is just a constant, unchanging in both space and time. The relation between κ and U reveals:

- $\kappa > 0 \Rightarrow U < 0$, so that $T < V$: the expansion of the Universe will at some time t stop and reverse itself.

- $\kappa < 0 \Rightarrow U > 0$, so that $T > V$: the expansion of the Universe will continue forever.
- $\kappa = 0 \Rightarrow U = 0$: the expansion of the Universe will slow down and stop for $t \to \infty$.

Furthermore, combining the previous equations for $\mathbf{r}$ and $\mathbf{v}$, one obtains $\mathbf{v} = \frac{|\dot{\mathbf{r}}|}{|\mathbf{r}|}\mathbf{r} = \frac{\dot{a}}{a}\mathbf{r}$, where the Hubble parameter $H = \frac{\dot{a}}{a}$ can be identified.

Using the relation of thermodynamics $dE + pdV = TdS$, applying it to an expanding volume V of unit comoving radius and using $E = mc^2$, the energy within the volume becomes

$$E = \frac{4\pi}{3}a^3\rho$$

whilst the energy exchange and the volume derivative in time are (without considering the x term since it will be canceled in the following passage)

$$\frac{dE}{dt} = 4\pi\rho a^2\dot{a} + \frac{4\pi}{3}a^3\dot{\rho} \qquad\qquad \frac{dV}{dt} = 4\pi a^2\dot{a}.$$

Assuming a reversible expansion, i.e. $dS = 0$, the continuity equation is obtained

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0.$$

At this point, after the derivation with respect to the time of the first Friedmann equation

$$2\frac{\dot{a}}{a}\frac{a\ddot{a} - \dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{\kappa\dot{a}}{a^3},$$

the substitution of $\dot{\rho}$ and the addiction of the first Friedmann equation at both sides gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

which is the *second Friedmann equation*. It is important to note that the pressure here acts to increase the gravitational force, thereby further decelerating the expansion. Additionally, κ cancelled out in the derivation.

Actually, these equations are incomplete since the vacuum energy density Λ is omitted. This factor can be introduced with the addiction of the following potential term to the energy expression

$$V_\Lambda = -\frac{1}{6}\Lambda m r^2 = -\frac{1}{6}\Lambda m a^2 x^2$$

that corresponds to a new force term, namely

$$\mathbf{F}_\Lambda = -\frac{\partial V_\Lambda}{\partial r}\hat{\mathbf{r}} = \frac{1}{3}\Lambda m\mathbf{r}$$

which is radial and depends on the sign of Λ. It is interesting to note that a positive value of Λ corresponds to a repulsive force counteracting the conventional attractive gravitation while a negative value of Λ represents an additional attractive force. Including the new potential in the previous analysis yields

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} + \frac{\Lambda}{3}$$

that is the first Friedmann equation. Similarly, considering the time derivative of this equation, substituting $\dot{\rho}$ and adding the first equation as done before, it is possible to obtain the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}.$$

3 Linear form of Friedmann equations

In this section, the Möbius transformations and the Schwarzian derivative will be briefly introduced following [8, ch. 5] and [9, sec. 4]. Then, as done in [1, 2], the Friedmann equations will be put in linear form first in the case of flat space ($\kappa = 0$), secondly the curvature will be absorbed through exponentiation and even the $\kappa \neq 0$ case will be treated selecting, among all the possible solutions, those which include the conformal time (η). Finally, the solutions of the linear Friedmann equations will be generalized to arbitrary β and some maps between the three different geometries ($k = 0, \pm 1$) will be shown.[2]

3.1 The Möbius transformations and the Schwarzian derivative

In this part, the Möbius transformations and the Schwarzian derivative will be introduced according to the information found in [8]. To start, some definitions are needed in order to introduce the Schwarzian derivative.

Definition 3.1. The group of $(n \times n)$-dimensional invertible matrices on the field V is called the *General Linear Group* of degree n, $GL_n(V)$.

Definition 3.2. The group of $(n \times n)$-dimensional matrices on the field V with $det = 1$ is called the *Special Linear Group* of degree n, $SL_n(V)$.

Definition 3.3. The *Projective General Linear Group* is defined as $PGL_n(\mathbb{C}) := \frac{GL_n(\mathbb{C})}{\gamma \mathbb{I}_n}$ with $\gamma \in \mathbb{C} \setminus \{0\}$. The *Projective Special Linear Group* is defined as $PSL(n, \mathbb{C}) = PSL_n(\mathbb{C}) := \frac{SL_n(\mathbb{C})}{\pm \mathbb{I}_n}$.

This means that the PGL is the quotient group where GL_n is the general linear group and $\mathbb{I}_n$ is the identity matrix. The PSL contains only those matrices with $det = 1$.
Let $\mathbb{C}_\infty$ be $\mathbb{C} \cup \{\infty\}$.

Definition 3.4. A *Möbius transformation* is a map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ so that

$$f(z) := \frac{az + b}{cz + d} \tag{3.1}$$

where $(a, b, c, d) \in \mathbb{C}$ and $ad - bc \neq 0$.

The request $ad - bc \neq 0$ is necessary to ensure the invertibility of the map, which otherwise would map $\mathbb{C}_\infty$ to a single point.

Preposition 3.1. A Möbius transformation is an homeomorphism.

To proceed, $\mathcal{M}(\mathbb{C}_\infty)$ can be defined as the following set of Möbius transformations

$$\mathcal{M}(\mathbb{C}_\infty) := \left\{ f : \mathbb{C}_\infty \to \mathbb{C}_\infty \ \middle| \ f(z) = \frac{az + b}{cz + d}, \ \ ad - bc \neq 0 \right\}. \tag{3.2}$$

This set has the following additional structures.

Preposition 3.2. $\mathcal{M}(\mathbb{C}_\infty)$ is a group in the composition of functions.

Preposition 3.3. $GL_2(\mathbb{C}) \cong \mathcal{M}(\mathbb{C}_\infty)$: it exists a surjective group homomorphism Υ: $GL_2(\mathbb{C}) \to \mathcal{M}(\mathbb{C}_\infty)$ with kernel the scalar matrices ($Ker\Upsilon = \gamma \mathbb{I}$, $\gamma \in \mathbb{R}$).

From the definition of the Möbius transformation it is evident that the set $(a, b, c, d) \in \mathbb{C}$, $ad - bc \neq 0$, does not define a unique transformation, but it identifies a group of maps due to the invariance under rescalings $(a, b, c, d) \to \lambda(a, b, c, d)$, $\lambda \in \mathbb{C} \setminus \{0\}$, of (3.1). This result, together with the preposition 3.3, leads to the important conclusion

$$\frac{GL_2(\mathbb{C})}{\sim} \cong \mathcal{M}(\mathbb{C}_\infty) \tag{3.3}$$

[2] In the last part of the section it is adopted the convention where k is dimensionless and equal to $\{0, \pm 1\}$ and a has the units of a length.

where $\sim$ is the equivalence relation previously anticipated which states that if $A, B \in GL_2(\mathbb{C})$, then $A \sim B \iff A = \lambda B$, $\lambda \in \mathbb{C}$. By choosing the matrices with $det = 1$ as representative of the equivalence classes, the preposition 3.3 becomes

$$SL_2(\mathbb{C}) \cong \mathcal{M}(\mathbb{C}_\infty). \tag{3.4}$$

To conclude, according to the invariance under rescalings

$$PSL(2, \mathbb{C}) \cong \mathcal{M}(\mathbb{C}_\infty). \tag{3.5}$$

The last result implies that the invertibility condition $ad - bc \neq 0$ can be expressed, without loss of generality, as $ad - bc = 1$. The last part of this section aims to define the Schwarzian derivative and its properties, making immediately clear the strong relation with the Möbius transformations.

Definition 3.5. Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic map such that $f' = \frac{df}{dz} \neq 0$, the *Schwarzian derivative* of f in the variable z is defined as

$$(Sf)(z) = \{f, z\} := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \left(\frac{f'''(z)}{f'(z)}\right) - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{3.6}$$

Lastly, it is important to underline some properties of the Schwarzian derivative which can be trivially proved by direct computation:

- If $g(z)$ is a Möbius transformation, then
$$\{g, z\} = 0 \tag{3.7}$$
- If $f(z)$ is a function and $g(z)$ a Möbius transformation, then
$$\{g \circ f, z\} = \{f, z\} \tag{3.8}$$
- The *chain rule* of the Schwarzian derivative states
$$\{f(x), x\} = \{f(y), y\} \left(\frac{\partial y(x)}{\partial x}\right)^2 + \{y(x), x\} \tag{3.9}$$
where $x, y \in \mathbb{C}$.

The last propriety of the Schwarzian derivative which will be useful in the next sections is the following one, taken from [9, sec. 4]. Considering the second order linear differential equation

$$\frac{d^2 f(t)}{dt^2} + Q(t) f(t) = 0 \tag{3.10}$$

with two linearly independent solutions $f_1(t)$ and $f_2(t)$, the equation $g(t) = \frac{f_1(t)}{f_2(t)}$, with $f_2 \neq 0$, solves the Schwarzian equation

$$\{g(t), t\} = 2Q(t). \tag{3.11}$$

The converse is also true: if such $g(t)$ exists, then $f_1(t)$ and $f_2(t)$ are unique up to a scale factor.

3.2 Linear form of Friedmann equations

By the end of this section, following [1], the Friedmann equations will be linearized, this will reveal an underlying analogy with the Schrödinger equation.

In order to obtain the linear form of (2.29), the first step is to consider the linear combination

$$X_\beta(a) = \frac{\ddot{a}}{a} + (\beta - 1)\left(\frac{\dot{a}}{a}\right)^2. \tag{3.12}$$

Introducing the β-times (t_β) as

$$\dot{t}_\beta := a^{1/\delta(\beta)} \tag{3.13}$$

and requesting

$$X_\beta(a) = \delta\left[\frac{\dddot{t}_\beta}{\dot{t}_\beta} + (\delta\beta - 1)\left(\frac{\ddot{t}_\beta}{\dot{t}_\beta}\right)^2\right] \tag{3.14}$$

to be proportional to the Schwarzian derivative of t_β

$$\{t_\beta, t\} = \frac{\dddot{t}_\beta}{\dot{t}_\beta} - \frac{3}{2}\left(\frac{\ddot{t}_\beta}{\dot{t}_\beta}\right)^2 \tag{3.15}$$

it is possible to find the condition $\delta = -1/(2\beta)$, which implies

$$t_\beta = \int_t^{t_o} dt' a^{-2\beta} \qquad\qquad \dot{t}_\beta = a^{-2\beta}. \tag{3.16}$$

The substitution of δ and (3.13) in (3.14) gives

$$X_\beta(a) = \frac{\ddot{a}}{a} + (\beta - 1)\left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{2\beta}\{t_\beta, t\}. \tag{3.17}$$

From the invariance of the Schwarzian derivative under Möbius transformations it follows that $X_\beta(a) = X_\beta(a')$. Moreover, the linear fractional transformations $t_\beta \to t'_\beta = \frac{At_\beta+B}{Ct_\beta+D}$ imply

$$\dot{t}'_\beta = \frac{d}{dt}\left(\frac{At_\beta + B}{Ct_\beta + D}\right) = \frac{\cancel{A\dot{t}_\beta C t_\beta} + AD\dot{t}_\beta - \cancel{CA\dot{t}_\beta t_\beta} - BC\dot{t}_\beta}{(Ct_\beta + D)^2} = \frac{\dot{t}_\beta(AD - BC)}{(Ct_\beta + D)^2} \underset{AD-BC=1}{=} \dot{t}_\beta(Ct_\beta + D)^{-2}$$

$$\Longrightarrow (\dot{t}'_\beta)^{-\frac{1}{2\beta}} = (\dot{t}_\beta)^{-\frac{1}{2\beta}}(Ct_\beta + D)^{\frac{1}{\beta}}$$

so that from the equality $X_\beta(a) = X_\beta(a')$ it follows that

$$a' = a(Ct_\beta - D)^{\frac{1}{\beta}}, \tag{3.18}$$

which is the Möbius transformation of a induced by the transformation of t_β. At this point, it is important to consider the following identity verifiable by direct computation

$$\dot{t}_\beta^{1/2}\frac{d}{dt}\frac{1}{\dot{t}_\beta}\frac{d}{dt}\dot{t}_\beta^{1/2}\psi_\beta = \left(\frac{d^2}{dt^2} + \frac{1}{2}\{t_\beta, t\}\right)\psi_\beta. \tag{3.19}$$

One can compute the kernel of the left-hand side of the equation, obtaining the span of the two linearly-independent functions $\psi_\beta = \dot{t}_\beta^{-1/2}$ and $\psi_\beta^D = \dot{t}_\beta^{-1/2} t_\beta$. Obviously, this is also the kernel of the right-hand side. From the Schwarzian equation (3.11), one obtains that solving $\{t_\beta, t\} = 2U_\beta$ is equivalent to solve

$$\left(\frac{d^2}{dt^2} + U_\beta\right)\phi_\beta = 0 \tag{3.20}$$

where, in this case, U_β can be computed substituting the Friedmann equations in (3.17) and considering that $\{t_\beta, t\} = 2U_\beta$. It follows that $U_\beta = V_\beta(\rho, p) - \beta^2\frac{\Lambda}{3} + \beta(\beta-1)\frac{\kappa}{a^2}$ and $V_\beta(\rho, p) = -\frac{4}{3}\pi G\beta\left[(2\beta-3)\rho - 3p\right]$. In particular, since the kernel of (3.19) contains the equations $\psi_\beta = a^\beta$ and $\psi_\beta^D = a^\beta t_\beta$, for the propriety previously introduced the solution of the Schwarzian equation is

$$t_\beta = \frac{\psi_\beta^D}{\psi_\beta} \tag{3.21}$$

which is invariant when replacing ψ_β and ψ_β^D with their arbitrary linearly independent combinations

$$\begin{pmatrix} \psi'^D_\beta \\ \psi'_\beta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi^D_\beta \\ \psi_\beta \end{pmatrix}. \tag{3.22}$$

This feature comes from the invariance of the Schwarzian derivative under Möbius transformations. In fact, the matrix above is in the set $SL_n(\mathbb{C})$ and, since $PSL(2,\mathbb{C}) \cong \mathcal{M}(\mathbb{C}_\infty)$, this replacement is equivalent to a Möbius transformation. But, as said, the Schwarzian derivative is invariant under this kind of transformations, so the propriety above derives. Finally, from (2.29) and (3.17) it is possible to obtain

$$\frac{1}{2}\{t_\beta, t\} = V_\beta(\rho, p) - \beta^2 \frac{\Lambda}{3} + \beta(\beta - 1)\frac{\kappa}{a^2} \tag{3.23}$$

$$V_\beta(\rho, p) = -\frac{4}{3}\pi G \beta \left[(2\beta - 3)\rho - 3p\right]. \tag{3.24}$$

3.2.1 Linear form in the case of flat space

Here, a linear form of the Friedmann equations for flat space ($\kappa = 0$) will be presented, revealing an interesting new symmetry. In the case of flat space (3.20) becomes

$$\left(\frac{d^2}{dt^2} + V_\beta(\rho, p) - \beta^2 \frac{\Lambda}{3}\right)\phi_\beta = 0. \tag{3.25}$$

This, defining the space-independent Klein-Gordon operator as $O_\beta := \frac{d^2}{dt^2} + V_\beta(\rho, p)$, gives

$$O_\beta \phi_\beta = \beta^2 \frac{\Lambda}{3}\phi_\beta, \tag{3.26}$$

which is a linear eigenvalues equation. Now, for any pair of $(\alpha, \beta) \in \left\{\mathbb{C}^2 \smallsetminus (0,0) | \alpha \neq \beta\right\}$, so that $X_\alpha(a) \not\propto X_\beta(a)$, the Friedmann equations in the case of flat space have the *canonical eigenvalue form*

$$\begin{pmatrix} O_\alpha & 0 \\ 0 & O_\beta \end{pmatrix} \Psi_{\alpha\beta} = \frac{\Lambda}{3} \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix} \Psi_{\alpha\beta} \tag{3.27}$$

with four linearly independent solutions

$$\Psi_{\alpha\beta} = \begin{pmatrix} a^\alpha t_\alpha \\ a^\alpha \\ a^\beta t_\beta \\ a^\beta \end{pmatrix}. \tag{3.28}$$

One can also note that for each β there is a dual canonical equation with $\alpha = -\beta$ that has the same eigenvalue $\beta^2 \frac{\Lambda}{3}$. It follows that the set of all the possible canonical forms of (3.27) can be grouped in the following eigenvalue problem

$$\begin{pmatrix} O_{-\beta} & 0 \\ 0 & O_\beta \end{pmatrix} \Psi_{-\beta\beta} = \beta^2 \frac{\Lambda}{3} \Psi_{-\beta\beta}. \tag{3.29}$$

As indicated, the solutions required to make this eigenvalue problem equivalent to the Friedmann equations are

$$\psi = a^\beta \qquad\qquad \psi^D = a^\beta t_\beta. \tag{3.30}$$

This shows that, in the case of flat space, there are infinitely main pairs of linear differential equations equivalent to the Friedmann equations. Since these equations are satisfied by different powers of a, they are related in a non linear way. In particular the map $a \mapsto a^\alpha$ corresponds to $t_\beta \mapsto t_{\alpha\beta}$, so that $\dot{t}_\beta = a^{-2\beta} \mapsto$

$\dot{t}_{\alpha\beta} = (\dot{t}_\beta)^\alpha$. This implies that

$$\begin{aligned}\{t_{\alpha\beta}, t\} &= \frac{\dddot{t}_{\alpha\beta}}{\dot{t}_{\alpha\beta}} - \frac{3}{2}\left(\frac{\ddot{t}_{\alpha\beta}}{\dot{t}_{\alpha\beta}}\right)^2 \\ &= \alpha(\alpha-1)\left(\frac{\ddot{t}_\beta}{\dot{t}_\beta}\right)^2 + \alpha\frac{\dddot{t}_\beta}{\dot{t}_\beta} - \frac{3}{2}\alpha^2\left(\frac{\ddot{t}_\beta}{\dot{t}_\beta}\right)^2 \\ &= \alpha\{t_\beta, t\} - \frac{1}{2}\alpha(\alpha-1)\left(\frac{\ddot{t}_\beta}{\dot{t}_\beta}\right)^2 .\end{aligned} \tag{3.31}$$

Even $V_\beta(\rho, p)$ transforms into

$$V_{\alpha\beta} - (\alpha\beta)^2\frac{\Lambda}{3} = \alpha V_\beta - \alpha\beta^2\frac{\Lambda}{3} - \frac{1}{3}\alpha(\alpha-1)\beta^2(8\pi G\rho + \Lambda). \tag{3.32}$$

This can be derived from

$$V_\beta = \frac{1}{2}\{t_\beta, t\} + \beta^2\frac{\Lambda}{3} - \beta(\beta-1)\frac{\kappa}{a^2} \longmapsto V_{\alpha\beta} = \frac{1}{2}\{t_{\alpha\beta}, t\} + \alpha^2\beta^2\frac{\Lambda}{3} - \alpha\beta(\alpha\beta-1)\frac{\kappa}{a^2} \tag{3.33}$$

by substituting $\{t_{\alpha\beta}, t\}$ and the first Friedmann equation. In addition, one can consider

$$V_\beta(\rho, p) = -\frac{4}{3}\pi G\beta\left[(2\beta-3)\rho - 3p\right] \longmapsto \tag{3.34}$$

$$\longmapsto V_{\alpha\beta}(\rho, p) = -\frac{4}{3}\pi G\alpha\beta\left[(2\beta-3)\rho - 3p\right] + \alpha(\alpha-1)\beta^2\frac{\Lambda}{3} - \frac{1}{3}\alpha(\alpha-1)\beta^2\left(8\pi G\rho + \Lambda\right)$$

from which it follows that

$$V_{\alpha\beta}(\rho, p) = -\frac{4\pi G\beta}{3}\left[(2\beta-3)\alpha^2\rho - 3(\alpha p - \alpha(\alpha-1)\rho)\right], \tag{3.35}$$

so that $V_{\alpha\beta}(\rho, p) = V_\beta(\alpha^2\rho, \alpha p - \alpha(\alpha-1)\rho)$. From the previous substitutions the following transformations of (3.27) yield:

$$\begin{cases} a \to a^\alpha \\ \rho \to \alpha^2\rho \\ p \to \alpha p - \alpha(\alpha-1)\rho \\ \Lambda \to \alpha^2\Lambda \end{cases} \qquad \begin{cases} a \to a^\alpha \\ \rho \to \alpha^2\rho + (\alpha^2-1)\dfrac{\Lambda}{8\pi G} \\ p \to \alpha p - \alpha(\alpha-1)\rho + (1-\alpha^2)\dfrac{\Lambda}{8\pi G} \\ \Lambda \to \Lambda \end{cases}$$

The invariance of the Friedmann equations under these transformations shows an hidden symmetry, which is non-trivial due to the non linearity of the transformations.

3.2.2 Absorbing the curvature by exponentiation

Continuing with the discussion following [1, sec. 4], the curvature factor κ will be absorbed through exponentiation. This makes it possible to write (2.29a) in a linear form even in the case of arbitrary curvature. Furthermore, the solutions of the Friedmann equations written in this way reveal a new analogy with the WKB solutions of the Schrödinger equation, this will be discussed in the next section.

In the previous part, the relation $\frac{1}{2}\{t_\beta, t\} = V_\beta(\rho, p) - \beta^2\frac{\Lambda}{3} + \beta(\beta-1)\frac{\kappa}{a^2}$ was set out. The choice of $\beta = \frac{1}{2}$ identifies the following equation that contains the conformal time $\eta = t_{1/2}$

$$\{\eta, t\} = \frac{4}{3}\pi G(2\rho + 3p) - \frac{\Lambda}{6} - \frac{\kappa}{2}\dot{\eta}^2. \tag{3.36}$$

The chain rule of the Schwarzian derivative (3.9) applied on $\{e^{i\sqrt{\kappa}\eta}, t\}$ implies that $\{e^{i\sqrt{\kappa}\eta}, t\} = \dot{\eta}^2\{e^{i\sqrt{\kappa}\eta}, \eta\} + \{\eta, t\}$. Computing the first Schwarzian derivative[3], the following relation is obtained

$$\{e^{i\sqrt{\kappa}\eta}, t\} = \{\eta, t\} + \frac{\kappa}{2}\dot{\eta}^2. \tag{3.37}$$

This means that the spatial curvature term in (3.36) can be absorbed through the exponentiation $\eta \mapsto e^{\pm i\sqrt{\kappa}\eta}$, and the equation becomes

$$\{e^{i\sqrt{\kappa}\eta}, t\} = \frac{4}{3}\pi G(2\rho + 3p) - \frac{\Lambda}{6} \tag{3.38}$$

which is invariant under Möbius transformations. This substitution is the same as

$$a \to \frac{a}{\sqrt{\kappa}} e^{\mp i\sqrt{\kappa}\eta} \tag{3.39}$$

that can be obtained from the previous one and (3.16). With this procedure, a non linear problem has been converted to a linear equation. The solution of the non linear problem (3.36) can be obtained by taking the logarithm of the function that solves (3.38) and also a can be computed. From the propriety of the Schwarzian equation, (3.38) is equivalent to the eigenvalue problem

$$\left(\frac{d^2}{dt^2} + \frac{2}{3}\pi G(2\rho + 3p)\right)\psi = \frac{\Lambda}{12}\psi. \tag{3.40}$$

In the case of flat space, this equation and (2.29b) are the unique solutions that allow the linearization of the Friedmann equations since any other linear form consists in a linear combination of these two expressions, process that breaks the linearity. Moreover, it is important to highlight the fact that the chain rule of the Schwarzian derivative and the exponentiation of the β-times give a term proportional to $\dot{t}_\beta^2$, so that the curvature can be obtained simply through the exponentiation of η. Hence the solution of the problem naturally selects the conformal time η among all the β-times. Two linearly independent solutions of (3.40) are

$$\psi = \sqrt{a}e^{-\frac{i}{2}\sqrt{\kappa}\eta} \qquad\qquad \psi^D = \sqrt{a}e^{\frac{i}{2}\sqrt{\kappa}\eta} \tag{3.41}$$

and by replacing these functions in (3.40) one obtains a linear combination of the Friedmann equations. Since the $\kappa \to 0$ limit must give the solutions of (3.19) in the $\beta = \frac{1}{2}$ case, i.e. $\psi_{\frac{1}{2}} = \sqrt{a}$ and $\psi^D_{\frac{1}{2}} = \sqrt{a}\eta$, the following linear combinations of (3.41) should be considered

$$\psi = \sqrt{a}\cos\left(-\frac{\sqrt{\kappa}}{2}\eta\right) \qquad\qquad \psi^D = 2\frac{\sqrt{a}}{\sqrt{\kappa}}\sin\left(\frac{\sqrt{\kappa}}{2}\eta\right) \tag{3.42}$$

which satisfy

$$\psi^2 + \frac{\kappa}{4}{\psi^D}^2 = a. \tag{3.43}$$

At this point, it is important to focus on the selection of the conformal time. In fact, in order to find a linear solution of the Friedmann equations it was crucial to select, among all the β-times, the only physically relevant one: the conformal time η.

The next step consists in introducing the wave-function

$$\tilde{\Psi} = \sqrt{a}e^{\frac{i}{2}\sqrt{\kappa}\eta} \tag{3.44}$$

which, if $\kappa \geq 0$, yields

$$|\tilde{\Psi}|^2 = a. \tag{3.45}$$

[3]Note that the derivative $\{e^{i\sqrt{\kappa}\eta}, \eta\}$ in the chain rule is computed with respect to η.

This function enables the identification of the following space-independent Klein-Gordon eigenvalue problems

$$O_{1/2}\tilde{\Psi} = \frac{\Lambda}{12}\tilde{\Psi} \tag{3.46}$$

$$O_1 a = \frac{\Lambda}{3}a. \tag{3.47}$$

The second equation, which can be explicitly written as

$$\left[\frac{d^2}{dt^2} + \frac{4\pi G}{3}(\rho + 3p)\right] a = \frac{\Lambda}{3}a, \tag{3.48}$$

is equivalent to the second Friedmann equation (2.29b). On the other side, it has been proved that the first equation is equivalent to the first Friedmann equation. Depending on the boundary conditions, the solution of this equation is a linear combination of $\tilde{\Psi}$ and another linearly independent solution: $\tilde{\Psi}^D = \sqrt{a}e^{-\frac{i}{2}\sqrt{\kappa}\eta}$. Lastly, the dependence on κ of a is determined by the initial conditions on $\tilde{\Psi}$, according to equation (3.43).

To conclude, after having put (2.29) in linear form, it is important to connect these results with what was anticipated in section 2.5. In particular, it was stated in (2.47) that $a \propto P^{-1}$. By substituting this relation in the solution of ψ previously found one obtains

$$\psi = \frac{1}{\sqrt{P}}e^{\pm\frac{i\sqrt{\kappa}}{2}\int P(t')\,dt'} \tag{3.49}$$

which is similar to the WKB approximation of the solution of the one dimensional time-independent Schrödinger equation (see section 4.1).

3.2.3 Generalization to arbitrary β-times

It is possible to extend what was just observed for arbitrary pairs of β's even in the $\kappa \neq 0$ case. However, this will result in a non linear equation, meaning that the case $\beta = \frac{1}{2}$ is the only one that allows to obtain a linear form of the Friedmann equations. In this regard, it is necessary to exploit the connection that exists between the Schwarzian and the Riccati equations. All the theory concerning the Riccati equation is taken from [10].

In order to extend (3.20) to an arbitrary β it is necessary to introduce the relation between the Schwarzian equation $\{w, t\} = f$ and the Riccati equation. In particular, defining $y = w''/w'$, one obtains the following Riccati equation

$$y' = \frac{y^2}{2} + f. \tag{3.50}$$

Furthermore, if $y = -2u'/u$, the following linear ODE yields

$$u'' + \frac{1}{2}fu = 0 \tag{3.51}$$

and, since $w''/w' = -2u'/u$, the solution is

$$w = \frac{U}{u} \tag{3.52}$$

where u and U are two independent solutions of the linear ODE above and w is the solution of the Schwarzian equation. Set

$$\phi_\beta = a^\beta \exp\left(\int f_{\kappa,\beta}(t)\,dt\right), \tag{3.53}$$

with $f_{\kappa,\beta}$ a solution of the Riccati equation

$$\frac{d}{dt}f_{\kappa,\beta}(t) = -\left(f_{\kappa,\beta}(t)\right)^2 - 2\beta\frac{\dot{a}}{a}f_{\kappa,\beta}(t) + \beta(\beta-1)\frac{\kappa}{a^2}. \tag{3.54}$$

Any pair of equations (3.20) with different β's is equivalent to the Friedmann equations with nonzero k. In fact, one can compute the second derivative of (3.53) obtaining

$$\ddot{\phi}_\beta = \left(\beta\frac{\ddot{a}a - \dot{a}^2}{a^2} + \frac{d}{dt}f_{\kappa,\beta}(t) + \left(\beta\frac{\dot{a}}{a} + f_{\kappa,\beta}\right)^2\right)\phi_\beta \tag{3.55}$$

and replacing this equation in (3.26) gives

$$\left[\beta\frac{\ddot{a}}{a} - \beta\left(\frac{\dot{a}}{a}\right)^2 + \beta^2\left(\frac{\dot{a}}{a}\right)^2 + \frac{d}{dt}f_{\kappa,\beta} + f_{\kappa,\beta}^2 + 2\beta\frac{\dot{a}}{a}f_{\kappa,\beta} + V_\beta\right]\phi_\beta = \beta^2\frac{\Lambda}{3}\phi_\beta. \tag{3.56}$$

Recognising the Riccati equation one gets

$$\left[\beta\left(\frac{\ddot{a}}{a} + (\beta-1)\left(\frac{\dot{a}}{a}\right)^2\right) + \beta(\beta-1)\frac{\kappa}{a^2} + V_\beta\right]\phi_\beta = \left(\beta X_\beta(a) + \beta(\beta-1)\frac{\kappa}{a^2} + V_\beta\right)\phi_\beta = \beta^2\frac{\Lambda}{3}\phi_\beta \tag{3.57}$$

that is equivalent to (3.20) for arbitrary curvature and β. Note that if $\beta = \frac{1}{2}$ this last equation is equivalent to (3.36), while for arbitrary β's the equation is nonlinear, it is sufficient to remember that $X_\beta(a) \propto \{t_\beta, t\}$ and observe that the equation depends on a^{-2}. Furthermore, from [10] one can derive that given a particular solution $\tilde{f}_{\kappa,\beta}(t)$ of the Riccati equation, the general solution can be written in the following form

$$f_{\kappa,\beta} = \tilde{f}_{\kappa,\beta}(t) + \Theta(t)\cdot\left(\int^t \Theta(t')\,dt'\right)^{-1} \tag{3.58}$$

where

$$\Theta(s) = a^{-2\beta}\exp\left(-2\int \tilde{f}_{\kappa,\beta}(s)\,ds\right). \tag{3.59}$$

At this point, it is important to notice that the values $\beta = 1/2$ and $\beta = 1$ are of particular interest, since they allow $\tilde{f}_{\kappa,\beta}$ to be expressed as an elementary function of a, resulting in the functions seen in the previous section.

To verify the consistency of this generalization, it is possible to consider the $\kappa = 0$ case. With this hypothesis, the Riccati equation becomes

$$\frac{d}{dt}f_{0,\beta}(t) + f_{0,\beta}(t)^2 + 2\beta\frac{\dot{a}}{a}f_{0,\beta}(t) = 0 \tag{3.60}$$

which has as trivial particular solution $\tilde{f}_{0,\beta}(t) = 0$, while the general solution (3.58) becomes

$$f_{0,\beta} = \dot{t}_\beta t_\beta^{-1}. \tag{3.61}$$

The corresponding linearly independent functions ϕ_β are given by $\phi_\beta = a^\beta = \psi_\beta$ for the particular solution, and $\phi_\beta^D = a^\beta t_\beta = \psi_\beta^D$ for the general solution. These solutions coincide with those previously identified in the $\kappa = 0$ case.
It is important to underline that among all the β-times, the ones that simplify the expression are those for $\beta = 1/2$ and $\beta = 1$, so the conformal time is, once again, naturally selected.

Lastly, it is interesting to solve the Riccati equation rewriting the solution ϕ_β in a more familiar way, since it is similar to equation (3.30). In particular, it is possible to satisfy equation (3.54) in the case where $\beta \neq 1$ and $\kappa \neq 0$ by parameterizing $a(t)$ and $f_{\kappa,\beta}(t)$ through an intermediate function $s(t)$ and defining

$$a(t) = \sqrt{(\beta-1)\kappa}\,s(t)\left[\int^t\left(s^{2(\beta-1)}(t')\int^{t'} s^{-2\beta}(t'')\,dt''\right)dt'\right]^{\frac{1}{2}} \tag{3.62}$$

$$f_{\kappa,\beta}(t) = \frac{s^{-2\beta}(t)}{\int^t s^{-2\beta}(t')\,dt'} = \frac{d}{dt}\log\left(\int^t s^{-2\beta}(t')\,dt'\right). \tag{3.63}$$

In this way, the solution (3.53) can be written as

$$\phi_\beta = a^\beta \int^t s^{-2\beta}(t')\, dt' \tag{3.64}$$

that resembles the one of the $k = 0$ case.

3.3 Mappings amongst different geometries

As previously mentioned, this section focuses on presenting a scheme to map solutions corresponding to different types of spatial geometries. On this purpose, the convention where $\kappa \in \mathbb{R}$ has the dimension of $lenght^{-2}$ is abandoned in favor of the one where $k \in \{-1, 0, 1\}$ is dimensionless. Here the three different values of the spatial curvature will be connected with simple mappings. To simplify the notation, since $\beta = \frac{1}{2}$, the β subscript will be replaced with the k subscript in this section.
Starting from the $k = 0$ case, the two linearly independent solutions are $\psi_0 = \sqrt{a_0}$ as well as $\psi_0^D = \sqrt{a_0}\eta_0$, with $\eta_0 = \int a_0^{-1}\, dt$ the conformal time. On the other hand, the $k = -1$ solutions are $\psi_{-1} = \sqrt{a_{-1}}e^{\eta_{-1}/2}$ and $\psi_{-1}^D = \sqrt{a_{-1}}e^{-\eta_{-1}/2}$, with $\eta_{-1} = \int a_{-1}^{-1}\, dt$. The approach consists in defining a relation between a_0 and a_{-1} so that one of the two scale factors solves the Friedmann equation of the other one. Starting from this relation, the next step is to find the mappings of the pressure and the energy density. Considering the two products

$$\psi_0\psi_0^D = a_0\eta_0 \qquad\qquad \psi_{-1}\psi_{-1}^D = a_{-1} \tag{3.65}$$

and setting them equal, one obtains the relations

$$a_{-1} = a_0\eta_0 \tag{3.66}$$

$$\dot{a}_{-1} = \dot{a}_0\eta_0 + \cancel{a_0}\cancel{\dot{\eta}_0} = \dot{a}_0\eta_0 + 1. \tag{3.67}$$

Substituting these relations in the first Friedmann equation when $k = -1$ yields

$$\frac{\Lambda}{3} + \frac{1}{a_{-1}^2} + \frac{8\pi G}{3}\rho_{-1} - \left(\frac{\dot{a}_{-1}}{a_{-1}}\right)^2 = \frac{\Lambda}{3} + \cancel{\frac{1}{(a_0\eta_0)^2}} + \frac{8\pi G}{3}\rho_{-1} - \cancel{\frac{1}{(a_0\eta_0)^2}} - \frac{2\dot{a}_0}{a_0^2\eta_0} - \left(\frac{\dot{a}_0}{a_0}\right)^2 = 0 \tag{3.68}$$

$$\frac{\Lambda}{3} + \frac{8\pi G}{3}\left(\rho_{-1} - \frac{3}{4\pi G}\frac{\dot{a}_0}{a_0^2\eta_0}\right) - \left(\frac{\dot{a}_0}{a_0}\right)^2 = 0 \tag{3.69}$$

which is the first Friedmann equation in the $k = 0$ case where $\rho_0 = \rho_{-1} - \frac{3}{4\pi G}\frac{\dot{a}_0}{a_0^2\eta_0}$. This condition means that there is a shift in the energy density of $\delta\rho_0 = \rho_{-1} - \rho_0 = \frac{3}{4\pi G}\frac{\dot{a}_0}{a_0^2\eta_0}$.
Taking into account the second Friedmann equation in the $k = -1$ case and the relation

$$\ddot{a}_{-1} = \frac{\dot{a}_0}{a_0} + \ddot{a}_0\eta_0 \tag{3.70}$$

one obtains

$$\frac{\ddot{a}_{-1}}{a_{-1}} = \frac{\Lambda}{3} - 4\pi G\left(p_{-1} + \frac{\rho_{-1}}{3}\right) = \frac{\dot{a}_0}{a_0^2\eta_0} + \frac{\ddot{a}_0\cancel{\eta_0}}{a_0\cancel{\eta_0}} \tag{3.71}$$

$$\cancel{\frac{\Lambda}{3}} - \cancel{4\pi G}\left(p_{-1} + \frac{\rho_{-1}}{3}\right) = (\rho_{-1} - \rho_0)\frac{\cancel{4\pi G}}{3} + \cancel{\frac{\Lambda}{3}} - \cancel{4\pi G}\left(p_0 + \frac{\rho_0}{3}\right) \tag{3.72}$$

$$p_{-1} + \frac{\rho_{-1}}{3} = -\frac{1}{3}(\rho_{-1} - \rho_0) + p_0 + \frac{\rho_0}{3} \tag{3.73}$$

$$p_{-1} = p_0 - \frac{2}{3}(\rho_{-1} - \rho_0) = p_{-1} = p_0 - \frac{2}{3}\delta p_0 \tag{3.74}$$

So even for the pressure there is a shift of $\delta p_0 = -\frac{2}{3}\delta\rho_0$.

The case where $k = +1$ is equivalent to the previously discussed case where $k = -1$, since the first Friedmann equation can be rendered as the second one through the mapping $a_1 = ia_{-1}$. In this case, however, it is necessary to make a modification to the original functions. Indeed, up to this point, the three cases have been treated separately, so it was possible to choose ψ_k and ψ_k^D with null phase. However, to map the solutions with a certain curvature to another, it is necessary to consider also the phase of the function. In particular, to find the relations that connect the $k = 0$ case with the $k = 1$ case it is necessary to consider that $a_1 = ia_{-1}$. On this purpose, the solution of the $k = 1$ case must be written as

$$\psi_1 = e^{-i\frac{\pi}{4}}\sqrt{a_1}e^{\frac{i}{2}\eta_1} \qquad \psi_1^D = e^{-i\frac{\pi}{4}}\sqrt{a_1}e^{-\frac{i}{2}\eta_1} \tag{3.75}$$

so that, using the relation (3.66), one gets

$$a_1 = ia_0\eta_0 \tag{3.76}$$

which is consistent with the conditions $a_1 = ia_{-1}$ and $a_{-1} = a_0\eta_0$ previously introduced. At this point, one can directly compute the shifts of ρ and p.

A summary of the mappings just presented is

$$k = 0 \to k = -1 \qquad\qquad k = 0 \to k = 1$$

$$\begin{cases} a_{-1} = a_0\eta_0 \\ \rho_{-1} = \rho_0 + \delta\rho_0 \\ \quad \delta\rho_0 = \dfrac{3}{4\pi G}\dfrac{\dot{a}_0}{a_0^2\eta_0} \\ p_{-1} = p_0 + \delta p_0 \\ \quad \delta p_0 = -\dfrac{2}{3}\delta\rho_0 = -\dfrac{1}{2\pi G}\dfrac{\dot{a}_0}{a_0^2\eta_0} \end{cases} \qquad \begin{cases} a_1 = ia_0\eta_0 \\ \rho_1 = \rho_0 + \delta\rho_0 \\ \quad \delta\rho_0 = \dfrac{3}{4\pi G}\dfrac{\dot{a}_0}{a_0^2\eta_0} \\ p_1 = p_0 + \delta p_0 \\ \quad \delta p_0 = -\dfrac{2}{3}\delta\rho_0 = -\dfrac{1}{2\pi G}\dfrac{\dot{a}_0}{a_0^2\eta_0} \end{cases}$$

$$k = -1 \to k = 1$$
$$a_1 = ia_{-1}$$

Note that the transformations of p_k and ρ_k from the $k = 0$ to the $k = \pm1$ cases are the same. This is coherent with the fact that the map from the $k = -1$ to the $k = +1$ case affects only the scale factor and not the other terms.

4 The quantum Friedmann equation

In this section, the WKB approximation is introduced as a method to solve the time-independent Schrödinger equation focusing in particular on those aspects that will be relevant in the development of the second part, where from the Friedmann equations an analogous quantum equation will be introduced. In this way, the intriguing underlying analogy between Quantum Mechanics and General Relativity will be revealed.

As anticipated, in this section the Natural Units are abandoned to restore the values of c and $\hbar$, as this will be necessary for some key steps.

4.1 The WKB approximation

The first part of this section focuses on the WKB approximation, which provides an estimate of the eigenfunctions and eigenvalues of the one-dimensional stationary Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E\right)\psi(x) = 0. \tag{4.1}$$

This part is presented following [11]. To apply this approximation, it is necessary to hypothesize that the potential $V(x)$ in the Hamiltonian H of the system is slowly varying, so that the solutions of the Schrödinger equation can be thought as the results of an equation with a locally constant potential.
First, it is necessary to consider a particle with one degree of freedom and a level set $\mathcal{B}$ in the phase space

$$\mathcal{B} = \{(x, P) \in \mathbb{R}^2 | H(x, P) = E\}. \tag{4.2}$$

If a wave moves on $\mathcal{B}$, due to the de Broglie hypothesis the wave has $P = \mathsf{k}\hbar$ where k is the local frequency, so that its phase is given by the integration of $\frac{1}{\hbar}dx$ along the curve. Thus, the wave can be expressed as

$$\cos\left(\frac{1}{\hbar}\int_{x_0}^{x} P\,dx - \delta\right) \tag{4.3}$$

with x_0 an arbitrary point on $\mathcal{B}$ and δ an arbitrary phase. The Bohr-Sommerfeld condition implies that

$$\frac{1}{\hbar}\oint_{\mathcal{B}} P\,dx = 2n\pi \qquad\qquad n \in \mathbb{R}. \tag{4.4}$$

In order to correctly predict the energy for complex systems, it is necessary to introduce the Maslov correction, which consists in replacing $n \to n + \frac{1}{2}$, thus obtaining

$$\frac{1}{\hbar}\oint_{\mathcal{B}} P\,dx = 2\left(n + \frac{1}{2}\right)\pi \qquad\qquad n \in \mathbb{R}. \tag{4.5}$$

Now, using Green's theorem ($\oint_{\mathcal{B}} P\,dx = \int_{\tilde{\mathcal{B}}} \frac{\partial P}{\partial P}\,dPdx = \int_{\tilde{\mathcal{B}}} dPdx$, with $\partial\tilde{\mathcal{B}} = \mathcal{B}$) and calling $\mathcal{A}$ the area of the region enclosed in $\mathcal{B}$, it is possible to rewrite the previous equation as

$$\frac{1}{2\pi\hbar}\mathcal{A} = n + \frac{1}{2}. \tag{4.6}$$

The Maslov correction does not give exactly the energy levels, but only the leading order of the semi-classical approximation to the energy levels.
The WKB solutions aim to approximate the wave function $\psi(x)$ in (4.1) for small values of $\hbar$. Their behavior will be analyzed in three different regions: the classically allowed region, where $V(x) < E$, the classically forbidden region, where $V(x) > E$, and the "Turning points", where $V(x) = E$.

Classically allowed region Given a potential $V(x)$ and an energy level E, it is possible to express the momentum as a function of x. Choosing the plus sign one obtains

$$P(x) = \sqrt{2m(E - V(x))}. \tag{4.7}$$

The approximate solutions searched have the form

$$\psi(x) = A(x)e^{\pm i\frac{S_0(x)}{\hbar}} = A(x)e^{\pm\frac{i}{\hbar}\int P(x)\,dx} \tag{4.8}$$

where $S_0'(x) = P(x)$. The amplitude $A(x)$ is a smooth function independent of $\hbar$ and thus "slowly varying" for small $\hbar$ compared to $\frac{S_0(x)}{\hbar}$. The next step is to show that for every $E \in \mathbb{R}$ that admits a classically allowed region, it is possible to build an approximate eigenfunction.

Preposition 4.1. For any two numbers E_1, E_2 (with $\inf_{x\in\mathbb{R}} V(x) < E_1 < E_2$) there exists a constant C as well as a non zero function $A(x) \in C^\infty(\mathbb{R})$ so that $\forall E \in [E_1, E_2]$, the support of $A(x)$ is contained in the classically allowed region at energy E and the function $\psi(x)$, given by

$$\psi(x) = A(x)e^{\pm\frac{i}{\hbar}\int P(x)\,dx}, \tag{4.9}$$

satisfies

$$\|H\psi - E\psi\| \leq C\hbar\|\psi\|. \tag{4.10}$$

Proof. For any $E \in [E_1, E_2]$ the classically allowed region for energy E contains the classically allowed region for energy E_1. So $A(x)$ is chosen as a non-zero element of $C^\infty(\mathbb{R})$ with support in the classically allowed region for the energy value E_1. At this point it is possible to evaluate $H\psi - E\psi$ by direct computation

$$\begin{aligned}
(H - E)\psi(x) &= -\frac{\hbar^2}{2m}\left\{A''(x) \pm \frac{2i}{\hbar}A'(x)P(x) \pm \frac{i}{\hbar}A(x)P'(x) - \frac{1}{\hbar^2}P^2(x)A(x)\right\}e^{\pm\frac{i}{\hbar}\int P(x)\,dx} + (V(x) - E)\psi(x)\\
&= -\frac{\hbar^2}{2m}\left\{A''(x) \pm \frac{2i}{\hbar}A'(x)P(x) \pm \frac{i}{\hbar}A(x)P'(x)\right\}e^{\pm\frac{i}{\hbar}\int P(x)\,dx} + A(x)e^{\pm\frac{i}{\hbar}\int P(x)\,dx}(E - V(x)) +\\
&\quad + A(x)e^{\pm\frac{i}{\hbar}\int P(x)\,dx}(V(x) - E)\\
&= -\frac{\hbar^2}{2m}\left\{A''(x) \pm \frac{2i}{\hbar}A'(x)P(x) \pm \frac{i}{\hbar}A(x)P'(x)\right\}e^{\pm\frac{i}{\hbar}\int P(x)\,dx}\\
\Longrightarrow &\|H\psi - E\psi\| \leq \frac{\hbar^2}{2m}\|A''(x)\| + \frac{\hbar}{2m}\|2A'(x)P(x) + A(x)P'(x)\|
\end{aligned}$$

Since $\|\psi\|$ is independent of $\hbar$, the right-hand side of the inequality is of order $\hbar\|\psi\|$. Furthermore $\|2A'(x)P(x) + A(x)P'(x)\|$ is a bounded function of E in $[E_1, E_2]$, so the result follows.

□

The preposition shows that for every E that admits a classically allowed region and for every reasonable choice of $A(x)$, it is possible to obtain an approximate solution of the time independent Schrödinger equation with an error of order $\hbar$. This means that $\forall E > \min V(x)$, there exists an $\tilde{E}$ in the spectrum of H such that

$$|E - \tilde{E}| \leq C\hbar. \tag{4.11}$$

If $\lim_{x\to\pm\infty} V(x) = \infty$, then H has a discrete spectrum and $\tilde{E}$ is an eigenvalue for H. So, for such potentials, given any number $E \in [E_1, E_2]$ there is an eigenvalue of H within $c\hbar$ of E. If $\hbar \to 0$, the spectrum of H "fills up" the entire range of values approximating the classical energy spectrum: this is a manifestation of the classical limit of Quantum Mechanics.

The WKB approximation operates, in the classically allowed region, away from the turning points. As anticipated, the potential $V(x)$ needs to be smooth and real-valued. It is also requested that $V(x) \to \infty$ for $x \to \pm\infty$ and that $\frac{V'(x)}{V(x)}$, $\frac{V''(x)}{V(x)}$ are bounded if $x \to \pm\infty$. For any $E \in [E_1, E_2]$ there are exactly two *turning* *points* $a < b$ (properly $a(E) < b(E)$) where $V(x) = E$ and $V'(x) \neq 0$. Proving preposition (4.1) it was shown that

$$(H - E)\psi(x) = -\frac{\hbar^2}{2m}\left\{A''(x) \pm \frac{2i}{\hbar}A'(x)P(x) \pm \frac{i}{\hbar}A(x)P'(x)\right\}e^{\pm\frac{i}{\hbar}\int P(x)\,dx}. \tag{4.12}$$

Since the request is to have an error smaller than $\hbar$, the following condition must be set

$$2A'(x)P(x) - P'(x)A(x) = 0 \tag{4.13}$$

$$A(x) = \frac{cnst}{\sqrt{P(x)}}. \tag{4.14}$$

Introducing this condition in (4.12) gives

$$(H - E)\psi(x) = -\frac{\hbar^2}{2m}\frac{A''(x)}{A(x)}\psi(x) \tag{4.15}$$

which has an error of order $\hbar^2$. However, since $P(x) \to 0$ approaching to the turning points, $A(x)$ becomes unbounded, this complication will be solved later.
Setting a as the basepoint of integration and, since $H = H^\dagger$, taking the real part of the solution, the *oscillatory WKB function* is

$$\psi(x) = \frac{R}{\sqrt{P(x)}}\cos\left(\frac{1}{\hbar}\int_a^x P(y)\,dy - \delta\right) \tag{4.16}$$

with amplitude R and phase δ in $\mathbb{R}$.
Introducing the Born postulate, the probability density function related to the wave function ψ is $\mathcal{P}(x) = \frac{|\psi|^2}{\|\psi\|^2}$, so that

$$\mathcal{P}(x) \propto \frac{1}{P(x)} \propto \frac{1}{v(x)} \tag{4.17}$$

where $v(x)$ is the velocity of the particle. This means that if $[c, d] \in [a, b]$, the probability of finding the particle in the smaller interval is proportional to

$$\int_c^d \frac{dx}{v(x)} \tag{4.18}$$

that is equivalent to, considering $x(t)$ as the trajectory of the particle

$$\int_{t_c}^{t_d} \frac{v(x(t))}{v(x(t))}\,dt = \int_{t_c}^{t_d} dt = \Delta t_{cd}. \tag{4.19}$$

In conclusion, the probability of finding a particle described by the WKB approximation in the interval $[c, d]$ is proportional to Δt_{cd}. This concept is defined as *quasi-classical probability.*

Classically forbidden region To analyze the WKB approximation in the classically forbidden region, it is convenient to introduce the quantity

$$Q(x) := \sqrt{2m(V(x) - E)}. \tag{4.20}$$

The approximate solutions searched have the form

$$\psi(x) = A(x)e^{\pm\frac{1}{\hbar}\int_{x_0}^x Q(x)\,dx}. \tag{4.21}$$

Following the previous analysis

$$A(x) = \frac{cnst}{\sqrt{Q(x)}}. \tag{4.22}$$

In order to calculate the integral of $|\psi|^2$, it is necessary to choose the "$-$" sign in the interval $(b, +\infty)$ and the "$+$" sign in $(-\infty, a)$ to preserve the integrability of the function. It is convenient to take a as basepoint of the integral and to reverse the direction of integration. This results, away from turning points, in the *exponential WKB functions*, which are expressed as

$$\psi_1(x) = \frac{c_1}{\sqrt{Q(x)}} e^{-\frac{1}{\hbar}\int_x^a Q(t)\,dt} \qquad (-\infty, a) \tag{4.23a}$$

$$\psi_2(x) = \frac{c_2}{\sqrt{Q(x)}} e^{-\frac{1}{\hbar}\int_b^x Q(t)\,dt} \qquad (b, \infty). \tag{4.23b}$$

According with the ODE theory any solution of a smooth potential is smooth, so the singularities of the solutions around the turning points come from the approximation. For small $\hbar$, the exact solutions follow the WKB approximation until x goes close to the turning points, where the exact solutions become large, but finite.

Near the turning points The last part of this section is focused on finding the WKB approximation in a region close to the turning points following [12]. Let b be a turning point, so that for $x < b$ there is the classically allowed region. In a region sufficiently distant from the turning point, the approximation takes the form

$$\psi(x) = \frac{c_1}{\sqrt{P(x)}} e^{\frac{i}{\hbar}\int_b^x P(y)\,dy} + \frac{c_2}{\sqrt{P(x)}} e^{-\frac{i}{\hbar}\int_b^x P(y)\,dy} \qquad x < b \tag{4.24a}$$

$$\psi(x) = \frac{c}{2\sqrt{|P(x)|}} e^{-\frac{1}{\hbar}\int_b^x P(y)\,dy} \qquad x > b. \tag{4.24b}$$

Since $H = H^\dagger$, it is possible to consider only the real-linear combination of the first solution. In order to determine the coefficients, it is necessary to follow the variation of $\psi(x)$ from $x < b$ to $x > b$.
For small $|x - b|$, expanding in Taylor series $E - V(x)$ one gets

$$E - V(x) \approx F_0(x - b) + \mathcal{O}(2) \qquad F_0 = -\left.\frac{\partial V}{\partial x}\right|_{x=b} < 0. \tag{4.25}$$

To avoid the singularity, $\psi(x)$ must be extended to the complex plane with the introduction of the complex function $\psi(z)$ where $z \in \mathbb{C}$ so that $\psi(z)|_{z\in\mathbb{R}} = \psi(x)$. It is necessary to move around the singularity $z = b$ following a semicircle that maintains the quasi-classical condition. Under these requests, the above equation becomes

$$\psi(x) = \frac{c}{2\sqrt[4]{2m|F_0|}} \frac{1}{\sqrt[4]{z-b}} \exp\left(-\frac{1}{\hbar}\int_b^z \sqrt{2m|F_0|(z-b)}\,dz\right) \tag{4.26}$$

where a factor $\frac{1}{2}$ has been added, its utility will be understood later.
The next step consists in considering the path followed moving from the right-hand side to the left-hand side of $x = b$, following the semicircle $\gamma : z - b = \rho e^{i\varphi}$ with $\varphi \in [0, \pi]$ in the superior complex half-plane. At the end of the path ($\varphi = \pi$) the exponent of the previous equation becomes

$$-\frac{1}{\hbar}\int_b^z \sqrt{2m|F_0|(x-b)}\,dx = -\frac{i}{\hbar}\int_b^z \sqrt{2mF_0(x-b)}\,dx = -\frac{i}{\hbar}\int_b^z P(x)\,dx \tag{4.27}$$

which is equivalent to the one of $\psi(x)$ in the $x < b$ case. Analysing the coefficients, one gets that

$$c_2 = \frac{c}{2} e^{-i\frac{\pi}{4}} \tag{4.28}$$

while considering the semicircle in the lower-half plane directed from the left-hand side to the right-hand side of b, the condition on the second coefficient can be obtained

$$c_1 = \frac{c}{2} e^{i\frac{\pi}{4}}. \tag{4.29}$$

In conclusion, if $x < b$ one obtains

$$\psi(x) = \frac{c}{\sqrt{P(x)}} \cos\left(\frac{1}{\hbar}\int_b^x P(y)\,dy + \frac{\pi}{4}\right) \tag{4.30}$$

which is equivalent to

$$\psi(x) = \frac{c}{\sqrt{P(x)}} \cos\left(\frac{1}{\hbar}\int_x^b P(y)\,dy - \frac{\pi}{4}\right). \tag{4.31}$$

This solution must be equal, in the classically allowed region, to the one obtained starting from the other turning point, meaning that the sum of the phases must be equal to $n\pi$, $n \in \mathbb{Z}$. This implies

$$\frac{1}{\hbar}\int_a^b P(x)\,dx - \frac{\pi}{2} = n\pi \tag{4.32}$$

$$\frac{1}{2\hbar}\oint P(x)\,dx = \left(n + \frac{1}{2}\right)\pi \tag{4.33}$$

which is the Maslov condition.

At this point, it is important to stress out that if $\psi_{WKB}(x) = \frac{1}{\sqrt{P}} e^{\pm i\frac{S_0}{\hbar}}$ and $E = T + V(x) = \frac{P^2}{2m} + V(x)$, with $S_0 \in \mathbb{R}$ so that the exponential in ψ_{WKB} is a phase, then the Hamiltonian can be obtained as

$$\frac{1}{2m|\psi_{WKB}(x)|^4} = E - V(x). \tag{4.34}$$

This will be useful for reconstructing the quantum version of the Friedmann equations. To conclude this part, referring to what is presented in [2], it should be highlighted that the approximate function $\psi_{WKB}(x)$ itself satisfies a Schrödinger equation. In particular, starting from

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V - E + X\right)\psi_{WKB}(x) = 0 \tag{4.35}$$

one would find the X term. The direct computation of the derivative yields

$$\psi''_{WKB} = -\left(\frac{S_0'^2}{\hbar^2} + \frac{1}{2}\{S_0, x\}\right)\psi_{WKB}(x) \tag{4.36}$$

and by substituting this in the above equation gives

$$X(x) = -\frac{\hbar^2}{4m}\{S_0, x\} \tag{4.37}$$

where $S_0 = \pm\int P(x)\,dx$ is the Hamilton characteristic function. This result gives the equivalent classical equation of motion

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V - E - \frac{\hbar^2}{4m}\{S_0, x\}\right)\psi_{WKB}(x) = 0 \tag{4.38}$$

while the exact wave equation can be written as

$$\left(-\frac{d^2}{dx^2} + \frac{\psi''_{WKB}(x)}{\psi_{WKB}(x)} + \frac{1}{2}\{S_0, x\}\right)\psi(x) = 0. \tag{4.39}$$

Note that if $V \leq E$, starting from (4.34) it is possible to write the one dimensional stationary Schrödinger equation as

$$\left(-\hbar^2\frac{d^2}{dx^2} + \frac{1}{|\psi_{WKB}|^4}\right)\psi(x) = 0. \tag{4.40}$$

4.2 The quantum Friedmann equation

In the previous section, it was shown that the Friedmann equations admit a linear form expressed as

$$\left[\frac{d^2}{dt^2}+\frac{2\pi G}{3}\left(2\rho+\frac{3p}{c^2}\right)\right]\psi=\frac{\Lambda c^2}{12}\psi \tag{4.41a}$$

$$\left[\frac{d^2}{dt^2}+\frac{4\pi G}{3}\left(\rho+\frac{3p}{c^2}\right)\right]a_k=\frac{\Lambda c^2}{3}a_k. \tag{4.41b}$$

The ψ in equation (4.41a) is a linear combination of the two linearly independent solutions

$$\psi_k=\sqrt{a_k}e^{-\frac{ic}{2}\sqrt{k}\eta_k} \qquad \psi_k^D=\sqrt{a_k}e^{\frac{ic}{2}\sqrt{k}\eta_k} \tag{4.42}$$

where $\eta_k=\int_t^{t_o}a_k^{-1}(t')\,dt'$ in the $k=\pm1$ cases. Considering then the following two linear combinations of (4.42)

$$\phi_k=\sqrt{a_k}\cos\left(\frac{c\sqrt{k}}{2}\eta_k\right) \qquad \phi_k^D=2\sqrt{\frac{a_k}{k}}\sin\left(\frac{c\sqrt{k}}{2}\eta_k\right) \tag{4.43}$$

and taking the $k\to0$ limit one obtains

$$\psi_0=\sqrt{a_0} \qquad \psi_0^D=c\sqrt{a_0}\eta_0, \tag{4.44}$$

which are the solutions in the $k=0$ case.

In order to continue the analysis of the Friedmann equations, in this section a differential equation that can be interpreted as the quantum version of the linear form of the Friedmann equation (4.41a), namely the quantum Friedmann equation, will be formulated. This part is presented following [2].
To begin, it is necessary to resemble some important aspects presented in [9, 13]. In particular, one can write the solution of the one dimensional stationary Schrödinger equation $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V-E\right)\psi=0$ using the polar decomposition as

$$\psi=Re^{i\frac{S}{\hbar}}. \tag{4.45}$$

From the stationary continuity equation in one dimension $\frac{d}{dx}\left(R^2\frac{dS}{dx}\right)=0$, one gets $R=1/\sqrt{S'}$. Considering now a free falling particle, the momentum is given by $P=\sqrt{2m(E-V)}$ and, from the WKB approximation of the stationary Schrödinger equation, it was shown that up to order $\hbar$ the relation $S_0'=P$ yields, where S_0 is the Hamiltonian characteristic function in one dimension. Therefore, the approximate WKB solution of the one dimensional stationary Schrödinger equation is $\psi_{WKB}=\frac{1}{\sqrt{S_0'}}e^{\frac{i}{\hbar}S_0}$. It can be demonstrated that an exact solution of the stationary Schrödinger equation can always be expressed as

$$\psi=\frac{1}{\sqrt{S'}}e^{\frac{i}{\hbar}S}, \tag{4.46}$$

where S is the quantum stationary Hamilton-Jacobi characteristic function that satisfies the quantum stationary Hamilton-Jacobi equation $\frac{S'^2}{2m}+V-E+\frac{\hbar^2}{4m}\{S,x\}=0$.

At this point, it is necessary to resume the redshift relation introduced in section 2.5. The relation reads

$$\frac{\lambda(t)}{a(t)}=\frac{\lambda(t_o)}{a(t_o)} \tag{4.47}$$

where $\lambda(t)$ is the wavelength emitted by a source at time t and $\lambda(t_o)$ is the wavelength observed at t_o. The substitution of the de Broglie wavelength for a massive particle ($\lambda_m=h/P_m$) gives

$$P_k(t)a_k(t)=P_k(t_o)a_k(t_o). \tag{4.48}$$

It follows from (4.42) that the ψ in (4.41a) are the wave functions in the WKB approximation of a free falling particle which read

$$\psi_{-1} = \frac{1}{\sqrt{P_{-1}}} e^{\frac{c}{\hbar}\int P_{-1}\, dt'} \qquad \psi_{-1}^D = \frac{1}{\sqrt{P_{-1}}} e^{-\frac{c}{\hbar}\int P_{-1}\, dt'} \tag{4.49}$$

$$\psi_{1} = \frac{1}{\sqrt{P_{1}}} e^{-i\frac{c}{\hbar}\int P_{1}\, dt'} \qquad \psi_{1}^D = \frac{1}{\sqrt{P_{1}}} e^{i\frac{c}{\hbar}\int P_{1}\, dt'} \tag{4.50}$$

$$\psi_0 = \frac{1}{\sqrt{P_0}} \qquad \psi_0^D = \frac{1}{\sqrt{P_0}} \int P_0\, dt', \tag{4.51}$$

where the normalization $a_k(t_o) = \frac{\hbar}{2P_k(t_o)}$ was chosen. Note that for $k=1$ one has $|\psi_1|^2 = |\psi_1^D|^2 = \frac{1}{P_1}$, reminding of the probabilistic interpretation of the QM-WKB approximation in the classically allowed region. This, together with (4.49) and (4.50), is the first analogy between General Relativity and Quantum Mechanics. Resembling the one dimensional Hamilton-Jacobi theory with the substitution $x \mapsto t$ one can obtain

$$P(t) = \frac{dS_0(t)}{dt}, \tag{4.52}$$

that integrated gives the analogous of the Hamiltonian characteristic function in one dimension

$$S_0(t) = \frac{\hbar\sqrt{k}}{2P_k(t_o)} \int_t^{t_o} \frac{dt'}{a_k(t')}, \tag{4.53}$$

note that the conformal time plays the role of the Hamiltonian characteristic function multiplied for some constants. Consequently, based on what has been seen so far, it is possible to obtain the quantum Friedmann equation. In fact, the solutions (4.42) of (4.41a) can be interpreted as the WKB approximation of a broader case. In particular from (4.34), with the substitution $x \mapsto t$, the *quantum Friedmann equation* for $k = \pm 1$ is

$$\left(\frac{d^2}{dt^2} + \frac{kc^2}{4a_k^2}\right)\Psi_k = 0. \tag{4.54}$$

This equation has (4.41a) as WKB approximation and, using the equation corresponding to (4.39), one gets

$$\left[\frac{d^2}{dt^2} + \frac{2\pi G}{3}\left(2\rho + \frac{3p}{c^2}\right) - \frac{\Lambda c^2}{12} - \frac{1}{2}\{\eta_k, t\}\right]\Psi_k = 0. \tag{4.55}$$

Using the observation that led to (4.46), it is possible to express the exact solutions of (4.54) for $k = \pm 1$ from the WKB approximate solutions ψ_k. In fact, one can notice that the structure of the cosmological WKB is the same of the WKB of Quantum Mechanics, this makes it possible to write the solutions

$$\Psi_k = \sqrt{a_{qk}}\, e^{-\frac{ic}{2}\sqrt{k}\eta_{qk}} \qquad \Psi_k^D = \sqrt{a_{qk}}\, e^{\frac{ic}{2}\sqrt{k}\eta_{qk}}, \tag{4.56}$$

where $k = \pm 1$ and

$$\eta_{qk} = \int_t^{t_o} \frac{dt'}{a_{qk}}. \tag{4.57}$$

The new term introduced is the *quantum scale factor* a_{qk}, which is a quantum version of a_k. The quantum scale factor is defined in analogy with the quantum S previously introduced.

To proceed, it is interesting to demonstrate the following relation starting from (4.54). In particular, by substituting the solution Ψ_k expressed in (4.56) and computing the second derivative, one gets

$$\{\eta_{qk}, t\} + \frac{c^2 k}{2a_{qk}^2} - \frac{c^2 k}{2a_k^2} = 0. \tag{4.58}$$

In this last equation, it is possible to rewrite the term $-\frac{c^2 k}{2a_k^2}$ as $\{\eta_k, t\} - \frac{4}{3}\pi G(2\rho + \frac{3p}{c^2}) + \frac{\Lambda c^2}{6}$, thus obtaining

$$-\frac{1}{2a_{qk}^2}\left[\frac{c^2 k}{2} + a_{qk}^2\bigg(\{\eta_{qk}, t\} + \{\eta_k, t\}\bigg)\right] + \frac{2\pi G}{3}\left(2\rho + \frac{3p}{c^2}\right) - \frac{\Lambda c^2}{12} = 0 \tag{4.59}$$

in the $k = \pm 1$ cases.

So far, only a part of the problem has been addressed; however, when considering the case $k = 0$, the solution differs since the functions (4.44) diverge from those of the WKB approximation. Nevertheless, it is possible to explicit the quantum Friedmann equation even in this case just by identifying the equation solved by

$$\chi_0 = \sqrt{a_0}e^{b\eta_0}, \tag{4.60}$$

with b as a constant. This equation has the same form as ψ_{WKB}. In particular, by direct computation one gets

$$\ddot{\chi}_0 = \left(\frac{\ddot{a}_0}{2a_0} - \frac{\dot{a}_0^2}{4a_0^2} + \frac{b^2}{a_0^2}\right)\chi_0, \tag{4.61}$$

so that equation (4.41a) differs by the term $-\frac{b^2}{a_0^2}$ from the other cases, namely

$$\left[\frac{d^2}{dt^2} + \frac{2\pi G}{3}\left(2\rho + \frac{3p}{c^2}\right) - \frac{b^2}{a_0^2} - \frac{\Lambda c^2}{12}\right]\chi_0 = 0. \tag{4.62}$$

Now, since the Friedmann equations are invariant under rescalings, b has to be equal to either $\frac{c}{2}$ or $\frac{ic}{2}$. In conclusion, following the same procedure as before, the quantum Friedmann equation for $k = 0$ is

$$\left(\frac{d^2}{dt^2} \pm \frac{c^2}{4a_0^2}\right)\Psi_0 = 0. \tag{4.63}$$

4.2.1 Scale factor as probability density

To conclude, one can note that the scale factor can be interpreted as a *probability density* when $k = 1$. Recalling the normalization $a_k(t_o)P(t_o) = \frac{\hbar}{2}$ previously introduced, the quantum Friedmann equation can be rewritten as

$$\left(\hbar^2\frac{d^2}{dt^2} + kP_k^2(t)\right)\Psi_k = 0. \tag{4.64}$$

Its solution gives the probability amplitude of finding a particle in a measurement at given time. In analogy with the WKB of the Quantum Mechanics, one can express the solutions of the quantum Friedmann equation expanding in power of $\hbar$

$$\Psi_k = \Psi_{WKB} + \mathcal{O}\left(\hbar^2\right) \tag{4.65}$$

which, since $k = 1$, yields

$$|\Psi_k|^2 = |\Psi_{WKB}|^2 + \mathcal{O}\left(\hbar^2\right) \tag{4.66}$$

or equivalently

$$a_{qk}(t) = a_k(t) + \mathcal{O}\left(\hbar^2\right). \tag{4.67}$$

This means that the scale factor a_k can also be interpreted as the first order approximation of a more general quantum scale factor a_{qk} and the linear first Friedmann equation (4.41a) can be interpreted as a quasi-classical approximation of (4.54). Now, considering the results proposed up to this point, it is evident that, in analogy with what has been defined as "quasi-classical probability", the scale factor $a_k(t)$ can be interpreted as the analogous in t-space of a probability density obtained through the WKB approximation.

It is also possible to express the scale factor starting from the solution of the quantum Friedmann equation Ψ_q, in fact one can rewrite it as

$$a_k^2 = -\frac{k}{4}\frac{\Psi_q}{\ddot{\Psi}_q}, \tag{4.68}$$

so that, deriving and substituting a_k, it is possible to obtain

$$\dot{a}_k = \frac{\sqrt{-k}}{8}\sqrt{\frac{\ddot{\Psi}_q}{\Psi_q}}\frac{\Psi_q\dddot{\Psi}_q - \dot{\Psi}_q\ddot{\Psi}_q}{\ddot{\Psi}_q^2}. \tag{4.69}$$

In conclusion, this work has shown that it is possible to linearize the Friedmann equations through the use of the Schwarzian derivative. This leads flat and non flat cases to a quantum version of the Friedmann equations in analogy to the WKB approximation of Quantum Mechanics. In this new context, the Friedmann equations can be interpreted as the WKB approximation of their more general quantum version.

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