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Switching control of quantum dynamics

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Abstract

This work introduces a solution based on switching techniques for controlling open quantum systems. Assuming the existence of a shared steady state for a set of marginally stable generators of quantum Markov semigroups, we propose algorithms for asymptotically stabilizing such state by suitably switching between the dynamical generators. The problem is of interest for state preparation protocols in experimental quantum physics and emerging quantum technologies. In this Thesis, after a brief introduction on quantum control and the motivation for this work, we first provide the necessary background on linear switching systems, as well as a concise presentation of the quantum systems and their dynamics. We focus in particular on Markovian models for open quantum systems, generated by equations in Lindblad form, following the analogy with classical Markov chain to better guide the reader unfamiliar with the quantum models. We next discuss new techniques for stabilizing quantum states by switching between Lindblad generators, and extend our results to the stabilization of common invariant subspaces of density operators. Promising directions for further developments of the results to state-based feedback strategies are suggested in the conclusions.

Sommario

Questo lavoro presenta una soluzione basata su tecniche di switching per il controllo di dinamiche di sistemi quantistici aperti. Assumendo l'esistenza di uno stato di equilibrio condiviso per un insieme di generatori di semigruppi di Markov quantistici marginalmente stabili, proponiamo algoritmi per stabilizzare asintoticamente tale stato attraverso uno switching opportuno tra i generatori della dinamica. Si tratta di un problema di interesse per la preparazione di stati nel campo della fisica quantistica sperimentale e di tecnologie quantistiche emergenti. In questa Tesi, dopo una breve introduzione sul controllo quantistico e sulle motivazioni di questo lavoro, forniamo le basi necessarie sui sistemi switching lineari, e successivamente una concisa presentazione riguardo ai sistemi quantistici e alla loro dinamica. Ci concentriamo in particolare su modelli Markoviani per sistemi quantistici aperti, generati da equazioni in forma di Lindblad, seguendo l'analogia con catene di Markov classiche in modo da agevolare un lettore poco familiare con i modelli quantistici. Discutiamo poi nuove tecniche per stabilizzare stati quantistici operando switching tra generatori di Lindblad, ed estendiamo i nostri risultati alla stabilizzazione di sottospazi invarianti di operatori densità. Sono infine suggerite nelle conclusioni direzioni promettenti per ulteriori sviluppi riguardo a strategie di switching basate su feedback dello stato.

Contents

Chapter 1

Introduction

Quantum control is a branch of control theory on which many researchers have worked in the last few decades. It is a complex and articulated area, which still needs to be examined in depth for being exploited in the best possible way. Actually, the growing interest on this research topic must be attributed to the powerful tools that quantum control provides to the experimentalists in several subfield of physics. Among the most important applications there is nuclear magnetic resonance [15], widely used in medical imaging and in spectroscopy. One of the typical tasks of this discipline is to describe the techniques to drive a quantum initial state to a predetermined target state. For classical systems many strategies have been developed and adopted for achieving this task, among which there are robust control [12], optimal control [10] techniques or even Lyapunov methods. Unfortunately, these methods cannot often be directly applied when dealing with quantum systems due to the troubles on exploiting measurements for quantum control. Indeed, such microscopic dynamics are characterized by behaviors which have no counterparts in classical physics.

One of the most important features of quantum mechanics is that it is difficult to acquire information about quantum states without disturbing them. On the other hand it is well known that the most effective strategy for controlling any classical system is by exploiting feedback information [11], that is by continuously observing state measurements. For this reason existing procedures must be adapted to the quantum frame in order to be fully applicable and others need to be created for fixing issues otherwise unsolvable. The difficulties that the control engineer has to face for overcoming these challenges are also linked to the fact that proper dynamical models on which conceiving control solutions are hard to define. This is mainly due to the interactions that can rise between a system and its environment [16], [18], [20]. Interpreting such unknown contributions as sources of noise is leading

to describe open quantum systems with stochastic models, able to take into account existing uncertainties but at the same time more difficult to handle.

The objective of this thesis is to present an open loop control technique which indiscriminately applies to pure and mixed states. The evolution of a quantum state is described by the generator of its dynamics, which is directly linked to the physical elements that compose the system (magnetic fields, laser...). Stability properties of such dynamics can be investigated using the same mathematical tools and definitions as for classical systems. Indeed, the most used class of models for quantum evolutions are actually linear systems for which spectral properties are well known [9]. Switching between marginally stable systems, we will show how to drive any initial state to a given shared steady state. Under few assumptions and using piecewise constant generators, we will thus manage to make a shared steady state globally asymptotically stable for a convenient switching system. The main issue for such technique is to find a stabilizable switching law between given generators, assuming that none of them is asymptotically stable.

Most of the definitions about switching systems will be given in Chapter 2, together with results about stability and algorithms for engineering stabilizable switching laws. In Chapter 3 we will introduce quantum formalism and derive Markovian Master Equations as suitable dynamical models. We will there moreover describe a procedure to express master equations as vector differential equations, in such a way to comfortably apply stability results of switching linear systems. Finally, in Chapter 4, we will show how to fully exploit switching theory in quantum field and some numerical examples will be provided.

Chapter 2

Switching systems

System theory deals both with continuous-time and discrete-time evolutions. Method for controlling them have been developed simultaneously, sometimes with specifities related to the particular kind of dynamics. Nevertheless, real systems are only rarely described by one or the other model and rather by an interaction of them. A clear example of this behavior is given by the motion of an automobile. The growth of its speed does not only depend on the acceleration input, but also on the gear shift position. Acceleration is a continuous input given by the driver while gear shift positions belong to a discrete set. A comprehensive model of motion should then consider both the dynamics and the influence of one on the other.

Such kind of system is called *hybrid system*, and this type of dynamics has attracted significant attention by control researchers. Moreover, when the real object of control is a continuous variable, for example because discrete contribution is sparse, we are used to talk about *switching systems*. Apart from being an intrinsic characteristic of many real systems, switching can also be engineered to drive the evolution of the variables of interest. This is the aspect we will focus on in the following sections, as we will later use such techniques to control quantum systems. More specifically, dealing with stabilization problems, we will identify proper time-based and state-based switching rules for making asymptotically stable steady states that would be unstable for individual subsystems.

2.1 Introduction to switching systems

The main components of a switching system are a set of dynamical systems and their switching signal. Each dynamical system can be expressed as a

generic function

$$f_p(x) : \mathbb{R}^N \longrightarrow \mathbb{R}^N,$$

associated to an index p that belongs to a set $\mathcal{P}$ which we will now on consider finite and discrete. As we will mostly deal with linear systems, we will then describe them as real matrices $A_p \in \mathbb{R}^{N\times N}$. We already anticipated that the switching signal can depend on the state of the system or on time.

In the first case the switching signal is a piecewise constant function

$$\sigma(x) : \mathbb{R}^N \longrightarrow \mathcal{P},$$

and the state space is subdivided into regions, to each of which is associated a value in $\mathcal{P}$. When the state crosses the boundary of a region the signal $\sigma(x)$ switches and the system associated to the selected index is chosen to drive the evolution of the state. In general, we could also suppose the existence of a reset map which specifies which must be the new state of the system after that it hurts a boundary. For our goals we will however consider the evolution of the state to be continuous.

In the second case, the switching signal is a piecewise constant function

$$\sigma(t) : [0, +\infty) \longrightarrow \mathcal{P},$$

where t denotes the evolution of the time. At given instants the signal $\sigma(t)$ switches to a value in $\mathcal{P}$ which must be determined by the controller. Again the system associated to the index selected is chosen to drive the state until the following switching.

The expression of a generic linear switching system is then the following:

$$\begin{cases} \dot{x} = A_p(x(t)), \quad \sigma(t) = p \\ x(0) = x_0 \end{cases} \tag{2.1}$$

Taking advantage of spectral theory for linear systems we are now able to examine some fundamental properties of switching systems stability.

2.2 Stability of switching systems

There are mainly two ways for studying the stability of switching systems:

- stability under arbitrary switching,
- stability under constrained switching.

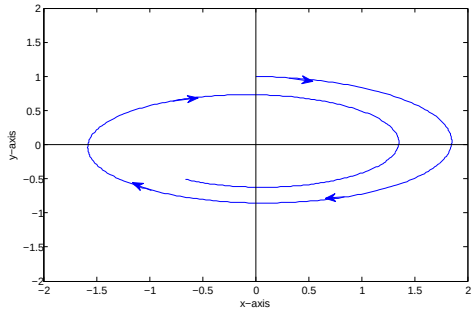
While the former describes which must be the features of a set of subsystems such that the switching system is stable for any switching signal, the latter, given a particular set of subsystems, tries to find out which must be the switching signal that makes the switching system stable. Before examining more attentively each of these ways we give the following fundamental definition:

Definition 2.1. *A switching system* (2.1) *is uniformly* ***asymptotically stable*** *if there exist a positive constant* δ *and a class* $\mathcal{KL}$ *function* β *such that for all switching signals* σ *the solutions of* (2.1) *with* $|x(0)| \leq \delta$ *satisfy the inequality*

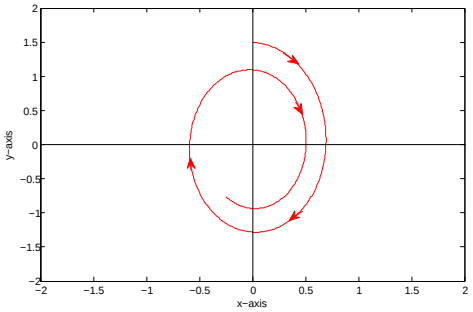
$$|x(t)| \leq \beta(|x(0)|, t) \qquad \forall t \geq 0. \tag{2.2}$$

If the inequality (2.2) *is valid for all switching signals and all initial conditions we obtain global uniform asymptotic stability (GUAS).*

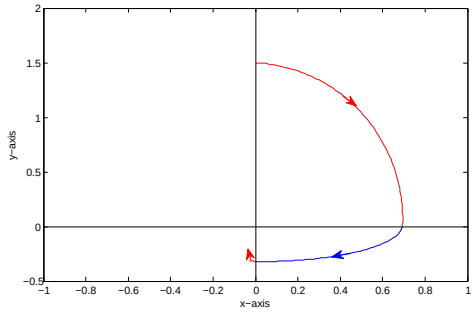
2.2.1 Stability under arbitrary switching



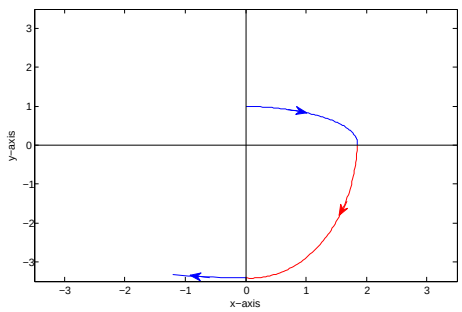
(a) Stable system



(b) Stable system



(c) Stable switching system



(d) Unstable switching system

Figure 2.1: Switching between stable dynamics

When dealing with stability under arbitrary switching a necessary condition for *GUAS* is certainly that each individual subsystem should be asymptotically stable. Indeed, if any subsystem p were unstable, then the switching system with $\sigma \equiv p$ would be unstable too. Nevertheless that is not a sufficient condition for *GUAS*. This fact can be illustrated by a simple example: consider two linear, stable dynamics on $\mathbb{R}^2$,

$$(a) = \begin{cases} \dot{x} = -0.1x + 4y \\ \dot{y} = -x - 0.1y \end{cases} \quad \text{and} \quad (b) = \begin{cases} \dot{x} = -0.1x + y \\ \dot{y} = -4x - 0.1y \end{cases} \tag{2.3}$$

whose trajectories are respectively of the form of those depicted in Figure 2.1a and Figure 2.1b. In Figure 2.1c the initial state of the system lies on the y-axis and its evolution begins according to the dynamics represented in a red line. Each time that the state crosses the x-axis, the system switches to the dynamics depicted in Figure 2.1a, while each time it crosses the y-axis it switches to that depicted in Figure 2.1b. This switching rule produces a contraction of the norm of the state and the resulting system is asymptotically stable. On the other hand, in Figure 2.1d, the initial state still lies on the y-axis but its evolution is initially described by the dynamics represented in a blue line. Now when the state crosses the x-axis, the system switches to the dynamics depicted in Figure 2.1b, while when it crosses the y-axis it switches to that depicted in Figure 2.1a. The norm of the state keeps then increasing and the switching system is unstable. It is therefore necessary to find additional conditions to assure asymptotic stability. These are generally derived by using Lyapunov techniques such as finding *common Lyapunov functions*.

Definition 2.2. *Given a positive definite continuously differentiable function $V : \mathbb{R}^N \to \mathbb{R}$, we will say that it is a* ***common Lyapunov function*** *for the set $\{f_p\}_{p=1,\dots,m}$ if there exists a positive definite continuous function $W : \mathbb{R}^N \to \mathbb{R}$ such that*

$$\frac{\partial V}{\partial x} f_p(x) \leq -W(x) \qquad \forall x, \quad \forall p \in \mathcal{P}.$$

This definition allows to formulate the following theorem:

Theorem 2.1. *If all the systems f_p share a common Lyapunov function, then the switched system (2.1) is GUAS.*

The proof of this theorem can be easily derived by extending that of Lyapunov's second method. It is worth to note that the existence of a common Lyapunov function is not a necessary condition for *GUAS*, as proved in [1].

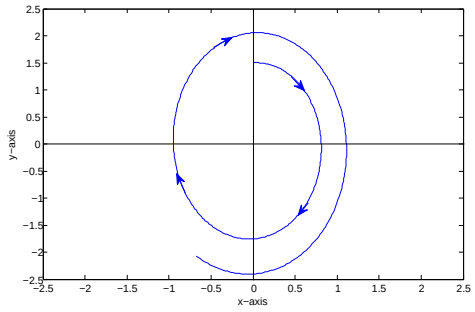
Moreover, finding a common Lyapunov function can be a difficult task. Anyway, for the objectives of this paper, stability under arbitrary switching is not a satisfiable requirement as our goal is to find proper switching signals to make asymptotically stable switching systems made of only marginally stable subsystems.

2.2.2 Stability under constrained switching

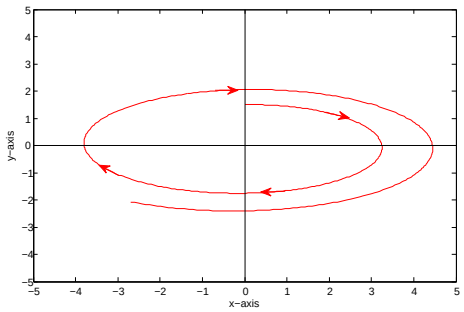
Even if some of the subsystems being switched are unstable, reaching asymptotic stability for the switching system can still be possible by appropriately choosing the switching signal. Let's suppose, for example, to dispose of two subsystems

$$(a) = \begin{cases} \dot{x} = 0.1x + 4y \\ \dot{y} = -x + 0.1y \end{cases} \quad \text{and} \quad (b) = \begin{cases} \dot{x} = 0.1x + y \\ \dot{y} = -4x + 0.1y \end{cases} \tag{2.4}$$

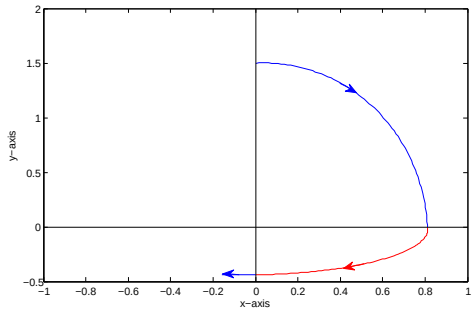
whose behavior is described in Figure 2.2a and Figure 2.2b.



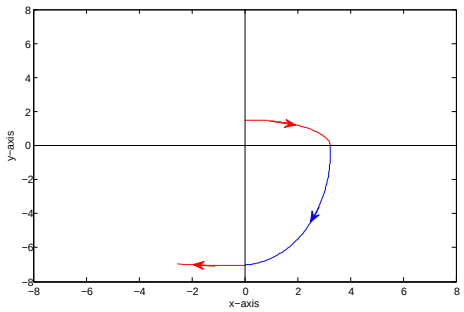
(a) Unstable system



(b) Unstable system



(c) Stable switching system



(d) Unstable switching system

Figure 2.2: Switching between stable dynamics

These subsystems are clearly unstable, and switching when the norm of the state starts decreasing keeps the switching system unstable. On the other

hand, if switching occurs when the norm of the state starts increasing, then the switching system gets asymptotically stable. This example shows that even if none of the individual subsystems are asymptotically stable a stabilizable switching signal can sometimes be found. In these cases Lyapunov's techniques can still be employed. Actually a common Lyapunov function cannot exist as if it would the switching system would be *GUAS* for Theorem 2.1 and we know it is not. Nevertheless, in particular cases of marginally stable subsystems, a stabilizing switching signal can be found by composing several Lyapunov functions [5]. We will not explore this way as more powerful tools exist to ensure asymptotic stability. In the next sections we will then examine two techniques to reach this aim. The first is a time-based switching while the second a state-based switching rule. Although they both found on the same assumptions, we will show that the first is much easier to apply in quantum field.

2.3 Periodic switching

We introduce now the mathematical background we will later base on.

Lemma 2.1. *Let $A_1, \ldots, A_m$ be matrices in $\mathbb{R}^{N\times N}$. Then there exists a positive real number ε such that*

$$exp(A_m t)exp(A_{m-1}t)\ldots exp(A_1 t) = exp\left(\left(\sum_{p=1}^{m} A_p\right)t + \Upsilon_c t^2\right),$$

for any $t \leq \varepsilon$, where entries of Υ_c are bounded.

Indeed, given two real square matrices A and B there always exists a complex matrix C such that

$$exp(A)exp(B) = exp(C).$$

Moreover, if $||A|| + ||B|| \leq ln(2)$ then C is real [6], and for Campbell-Baker-Hausdorff formula it is given by the convergent infinite expression

$$C = A + B + \frac{1}{2}[A, B] + \ldots,$$

where $[A, B] = AB - BA$ is called the commutator of A and B. We make now the following assumption.

Assumption 2.1. *There exists a convex combination A_c of A_p which is Hurwitz, that is*

$$\exists\ \alpha_1, \dots, \alpha_m \quad s.t \quad \sum_{p=1}^{m} \alpha_p A_p = A_c, \qquad \sum_{p=1}^{m} \alpha_p = 1.$$

and A_c is Hurwitz.

From Lemma 2.1 we can now prove the most important result of this section, as proposed in [2].

Theorem 2.2. *Suppose that Assumption 2.1 holds. Then the relative switching system is stabilizable.*

Proof. According to Lemma 2.1, if the length of the period ε is short enough,

$$exp(\alpha_m A_m \varepsilon) \dots exp(\alpha_1 A_1 \varepsilon) = exp\left(\left(\sum_{p=1}^{m} \alpha_p A_p\right)\varepsilon + \Upsilon_c \varepsilon^2\right) = exp((A_c + \Upsilon_c \varepsilon)\varepsilon),$$

and

$$\bar{A} := A_c + \Upsilon_c \varepsilon \tag{2.5}$$

is still Hurwitz. Indeed, the eigenvalues of a matrix continuously depend on its entries. As Υ_c is bounded, if $\varepsilon \to 0$, then the eigenvalues of $\bar{A}$ approch those of A_c and if A_c is Hurwitz there exists a ε for which $\bar{A}$ is Hurwitz too. Fixed such a ε, a periodic switching path can be defined as follows:

$$\sigma(t) = \begin{cases} 1 & mod(t, \varepsilon) \in [0, \alpha_1 \varepsilon) \\ \vdots & \\ m & mod(t, \varepsilon) \in [(\sum_{p=1}^{m-1} \alpha_p)\varepsilon, \varepsilon) \end{cases}$$

At the end of each switching period ε, a state vector v, whose dynamics is described by the above switching law, assumes the values

$$v(k\varepsilon) = exp(\bar{A}k\varepsilon)v(0), \qquad k = 1, 2, \dots,$$

where $\bar{A}$ was defined in (2.5). Moreover, let's define

$$\phi(s_2, s_1) := e^{A_p(s_2 - t_p)} e^{\alpha_{p-1} A_{p-1} \varepsilon} \dots e^{A_{k-1}(t_k - s_1)}, \quad s_1 \le s_2,$$

the transition matrix from the state at instant $s_1 \in (t_{k-1}, t_k)$ to that at instant $s_2 \in (t_p, t_{p+1})$ according to a given switching law. For any non-negative integers $l_1 \le l_2$ the evoution covers a finite number of cycles,

$$\phi(l_2\varepsilon, l_1\varepsilon) = e^{\bar{A}(l_2 - l_1)\varepsilon}.$$

As $\bar{A}$ is Hurwitz, there exist positive numbers κ and λ such that

$$||\phi(l_2\varepsilon, l_1\varepsilon)|| \leq \kappa e^{-\lambda(l_2-l_1)\varepsilon}.$$

For any $s_1 \leq s_2$, let l_1 and l_2 satisfy

$$l_1\varepsilon \leq s_1 < (l_1+1)\varepsilon, \quad (l_2-1)\varepsilon < s_2 \leq l_2\varepsilon.$$

Then

$$\begin{aligned} ||\phi(s_2, s_1)|| &\leq ||\phi(l_1\varepsilon, s_1)||||\phi(l_2\varepsilon, l_1\varepsilon)||||\phi(s_2, l_2\varepsilon)|| \\ &\leq \kappa e^{-\lambda(l_2\varepsilon - l_1\varepsilon)}||\phi(0, s_1 - l_1\varepsilon)||||\phi(0, l_2\varepsilon - s_2)||. \end{aligned}$$

Finally, denoting

$$\kappa_1 = \max_{0\leq t\leq\varepsilon} ||\phi(0,t)||,$$

which is always attainable because $\phi(0,t)$ is continuous in t, we get

$$||\phi(s_2, s_1)|| \leq \kappa_1^2\kappa e^{-\lambda(l_2-l_1)\varepsilon} \leq \kappa_1^2\kappa e^{-\lambda(s_2-s_1)}.$$

The transition matrix is exponentially convergent and the switching system is then stabilizable.

□

2.3.1 Finding a Hurwitz convex combination

The technique we have just presented does not require to find any Lyapunov function as it is based only on a time switching rule. Nevertheless, assuming that none of the individual subsystems is asymptotically stable, a Hurwitz convex combination of A_p does not always exist. When dealing with two subsystems a brute-force method can be used to find a couple of coefficients such to make the combination Hurwitz. In this case,

$$A_c = \alpha A_1 + (1-\alpha)A_2,$$

and, defining $\beta = \frac{1-\alpha}{\alpha} > 0$, then

$$\hat{A} := A_1 + \beta A_2.$$

Clearly, $\bar{A}$ is Hurwitz if and only if $\hat{A}$ is Hurwitz. As the value of β changes, the eigenvalues of $\hat{A}$ change too. A proper value of β is one for which all the eigenvalues of $\hat{A}$ have negative real part. In order to find it, one can determine the intervals of $0 < \beta < +\infty$ in which the sign of the eigenvalues is constant and pick any value of β for each interval to check if the relative

$\hat{A}$ is Hurwitz. Moreover, when the sign of the real part of an eigenvalue changes, that eigenvalue is purely imaginary. The endpoints of β, which delimit intervals where the sign of the eigenvalues of $\hat{A}$ does not change, are then those for which there exists a $\omega_i \in \mathbb{R}$ such that

$$det(A_1 + \beta_i A_2 + i\omega_i) = 0.$$

The algorithm for finding a Hurwitz convex combination of two unstable matrices can be thus synthesized:

1. Compute the set $\{\beta_i | \exists \omega_i : det(A_1 + \beta_i A_2 + i\omega_i) = 0\}$ and order β_i such that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n$ if n is the cardinality of the set.
2. Define a set of test points $\{\frac{\beta_1+\beta_2}{2}, \ldots, \frac{\beta_i+\beta_{i+1}}{2}, \ldots, 2\beta_n + 1\}$.
3. Calculate the eigenvalues of $\hat{A}$ with β equal to each test point to check if one makes $\hat{A}$ Hurwitz.

Even if this algorithm can be applied when switching between two subsystems, finding convex combinations for more than two matrices is NP-hard. Practical procedures for more complex switching systems are then unlikely to be found.

Nevertheless it is worth to note that the existence of a Hurwitz convex combination of A_p is not necessary for finding a stabilizing switching law. Let's consider for example the switching system whose subsystems are

$$A_1 = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix},$$

represented in Figure 2.3a and Figure 2.3b. Clearly, there cannot exist any Hurwitz convex combination of these matrices, as

$$\bar{A} = \alpha_1 A_1 + \alpha_2 A_2,$$

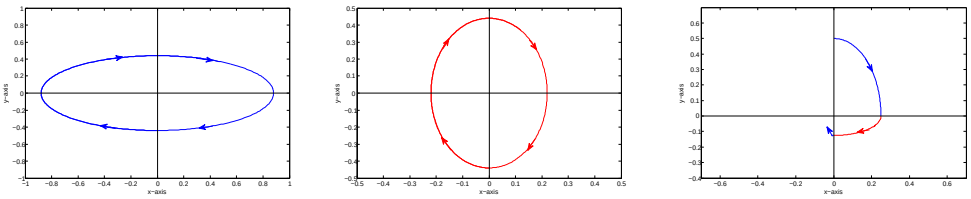
has imaginary eigenvalues

$$\lambda_{1,2} = \pm i\sqrt{(\alpha_1 + 4\alpha_2)(4\alpha_1 + \alpha_2)}.$$

However, the state-based switching law

$$\begin{cases} \sigma(x, y) = 1 & \text{if } xy \leq 0, \\ \sigma(x, y) = 2 & \text{if } xy > 0. \end{cases}$$

is stabilizing, as shown in Figure 2.3c.



(a) First system (b) Second system (c) Stable switching

Figure 2.3: Stable switching of marginally stable systems

2.4 State feedback switching

Supposing to exactly know the state of the system at each instant, a stabilizing switching law based on a state space partition can be easily engineered. Such technique again requires the validity of the Assumption 2.1, that is the existence of a Hurwitz convex combination A_c of the switching subsystems.

Let P be the positive definite solution of the Lyapunov equation

$$A_c^T P + P A_c = -I. \tag{2.6}$$

Let's moreover define

$$Q_p = A_p^T P + P A_p, \qquad p \in \mathcal{P},$$

and a set of arbitrary real numbers $r_p \in (0, 1)$ associated to each subsystem. We then choose the initial value of the switching signal as any of those for which

$$\sigma(t_0) = \arg\min_{p \in \mathcal{P}} \{x_0^T Q_1 x_0, \ldots, x_0^T Q_m x_0\}.$$

The first switching instant must then be chosen such that

$$t_1 = \inf\{t > t_0 : x^T(t) Q_{\sigma(t_0)} x(t) > -r_{\sigma(t_0)} x^T(t) x(t)\},$$

and the following active subsystem is that associated to

$$\sigma(t_1) = \arg\min_{p \in \mathcal{P}} \{x(t_1)^T Q_1 x(t_1), \ldots, x(t_1)^T Q_m x(t_1)\}.$$

Instant t_1 must exist because we supposed that none of each subsystem is asymptotically stable. The sequences of switching times and active systems can thus be recursively defined:

$$t_{k+1} = \inf\{t > t_k : x^T(t) Q_{\sigma(t_k)} x(t) > -r_{\sigma(t_k)} x^T(t) x(t)\}, \tag{2.7}$$

and

$$\sigma(t_{k+1}) = \arg\min_{p\in\mathcal{P}}\{x(t_{k+1})^T Q_1 x(t_{k+1}), \ldots, x(t_{k+1})^T Q_m x(t_{k+1})\}. \tag{2.8}$$

Before formally proving that it is a stabilizing switching law, we introduce the following lemma:

Lemma 2.2. *For each state $x \neq 0$ there exists a subsystem p for which*

$$x(t)^T Q_p x(t) \leq -x(t)^T x(t).$$

Proof. As A_c is a Hurwitz matrix, according to the definition (2.6) of P,

$$\begin{aligned} x(t)^T(A_c^T P + PA_c)x(t) &= x(t)^T\big((\sum_p \alpha_p A_p^T)P + P(\sum_p \alpha_p A_p)\big)x(t) \\ &= \alpha_1 x(t)^T(A_1^T P + PA_1)x(t) + \ldots + \alpha_m x(t)^T(A_m^T P + PA_m)x(t) \\ &= \alpha_1 x(t)^T Q_1 x(t) + \ldots + \alpha_m x(t)^T Q_m x(t) = -x(t)^T x(t) < 0. \end{aligned}$$

As the sum is negative, then there must exist at least one addend which is negative too. Considering that

$$\alpha_p \geq 0, \quad \forall p \in \mathcal{P},$$

then the lemma is proved. □

Given such a switching law we may wonder if the switching system is well-posed, that is if the set of jump times is finite for any finite interval.

Theorem 2.3. *Under the above switching law, the switching system is well-posed and asymptotically stable.*

Proof. We want to show now that such a switching law is acceptable, that is actually $t_{k+1} > t_k$. Let θ be any real number greater than 1. Consider the case

$$||x(t)|| \leq \theta||x_{k+1}|| \quad \forall t \in [t_k, t_{k+1}]. \tag{2.9}$$

and define

$$g(t) = x(t)^T(Q_p + I)x(t) \quad t \in [t_k, t_{k+1}],$$

where $p = \sigma(t_{k^+})$. From Lemma 2.2, being

$$x(t_k)^T Q_p x(t_k) \leq -x(t_k)^T x(t_k),$$

then

$$g(t_k) \leq 0, \tag{2.10}$$

and, from (2.7),

$$g(t_{k+1}) \geq (1 - r_p)x_{k+1}^T x_{k+1}. \tag{2.11}$$

Deriving the latter we get

$$\frac{dg}{dt} = x(t)^T(A_p^T(Q_p + I) + (Q_p + I)A_p)x(t).$$

Denoting

$$\eta_p := ||A_p^T(Q_p + I) + (Q_p + I)A_p||,$$

and using (2.9), we have

$$|\frac{dg}{dt}| \leq \theta^2 \eta_p x_{k+1}^T x_{k+1} \quad \forall t \in [t_k, t_k + 1]. \tag{2.12}$$

According to (2.10) and (2.11)

$$\frac{g(t_{k+1}) - g(t_k)}{t_{k+1} - t_k} \geq \frac{(1 - r_p)x_{k+1}^T x_{k+1}}{t_{k+1} - t_k}.$$

Remembering then (2.12),

$$\frac{(1 - r_p)x_{k+1}^T x_{k+1}}{t_{k+1} - t_k} \leq \theta^2 \eta_p x_{k+1}^T x_{k+1},$$

and, consequently,

$$t_{k+1} - t_k \geq \frac{(1 - r_p)}{\theta^2 \eta_p}.$$

If, on the other hand, (2.9) does not hold, then

$$\exists t^* \in [t_k, t_{k+1}) : ||x(t^*)|| > \theta ||x_{k+1}||.$$

As the system dynamics in this time interval is described by A_p, then

$$x(t^*) = exp(A_p(t^* - t_{k+1}))x_{k+1}.$$

From the latter, and remembering that

$$||exp(A_p(t^* - t_{k+1}))|| \leq exp(||A_p||(t_{k+1} - t^*)),$$

it follows that

$$t_{k+1} - t_k \geq t_{k+1} - t^* > \frac{ln(\theta)}{||A_p||}.$$

Finally, gathering both the cases, we can say that

$$t_{k+1} - t_k \geq \sup_{\theta>1} \min_{p\in\mathcal{P}} \left(\frac{1-r_p}{\theta^2 \eta_p}, \frac{ln(\theta)}{||A_p||} \right),$$

and that the switching signal is valid as the difference is always positive.

Choosing then $V(x) = x^T P x$ as Lyapunov function we get

$$\frac{dV}{dt} = x^T(t) Q_{\sigma(t)} x(t) \leq r_{\sigma(t)} x^T(t) x(t) \leq -r x^T(t) x(t),$$

where

$$r := \min\{r_1, \ldots, r_m\},$$

and the theorem is proved. □

We have presented two algorithms for making a switching system asymptotically stable. The first is based on a periodic sequence of switchings, while the second can be assimilated to a Lyapunov technique. Both of them are based on the only assumption that there exists a convex combination of the matrices which describe each subsystem. Moreover the second requires the knowledge of the state all along its evolution. That information can be easily obtained at any moment for classical systems by directly observing the value of the state. This is more difficult for quantum systems as any measurement of the state influences the dynamics of the system. In that case, switching instants must be determined outline from the exact knowledge of the initial state that is not always available. For these reasons, when applying switching techniques to quantum control, we will first consider the time-based algorithm, assuming that there actually exists a convex combination of the generators we can use.

Chapter 3

Quantum dynamics

3.1 Introduction to quantum dynamics

One of the main tasks of a control engineer is to drive the behavior of physical quantities related to dynamical systems, in order to make them assume determined values respecting given constraints. One of the typical examples of that could be to regulate the temperature of a room within a short time frame, or also to set the speed of a rotor by varying its power supply. The first step to reach those ends is always to build a mathematical model of the physical system. Such a model is generally intended as a set of equations which describe the evolution of relevant variables. For classical systems the knowledge of the initial values of those variables is a sufficient condition to determine their evolution and thus, if the model is accurate enough, to recover a valid estimation of the related physical quantities behavior.

For quantum systems this challenge is much more complex. In fact, according to quantum mechanics, all that is predictable is the probability to observe an outcome, rather than the outcome itself. This uncertainty is not due to a lack of quantum theory one could expect to overcome with further studies. It is on the contrary an essential characteristic of quantum systems, which dramatically distinguishes them from their classical counterparts. In this such particular frame a state should no more be interpreted as the set of variables which represent the attributes of a classical system but as the object that contains the information about the probability to measure an observable quantity.

3.1.1 Observables and states

In order to describe the evolution of quantum systems we need now to introduce the essential mathematical framework within which our analysis will

move. To any quantum system is associated a complex Hilbert space $\mathcal{H} \simeq \mathbb{C}^N$ which we will from now on suppose to be finite-dimensional. This assumption is not too restrictive for the objectives of this research and will allow us to use the powerful tools of linear algebra. In fact, bounded self-adjoint operators $\mathcal{B}(\mathcal{H})$ we will deal with, like observables and density operators, will mainly be represented as $N \times N$ complex matrices belonging to the set $\mathcal{M}_N$, relative to N-dimensional quantum systems. The set of complex valued matrices $\mathcal{M}_N$ can be itself intended as a Hilbert space equipped with the Hilbert-Schmidt scalar product:

$$\langle X, Y \rangle_{HS} = tr[X^\dagger Y]. \tag{3.1}$$

Definition 3.1. *An **observable** is a Hermitian operator $X \in \mathcal{B}(\mathbb{C})$ that is associated to a physical variable.*

Due to its hermiticity X can be diagonalized by an unitary matrix U, that is:

$$X = U X_d U^T,$$

where X_d is a diagonal matrix with real eigenvalues $\lambda_i \in \mathbb{R}$. Thus, for the spectral theorem, X can be expressed as the sum of orthogonal projections Π_i:

$$X = \sum_i \lambda_i \Pi_i,$$

where $\sum_i \Pi_i = I_N$ and I_N is the identity element of $\mathcal{M}_N$. The eigenvalues $\{\lambda_i\}$ are the possible outcomes of a measurement, while $\{\Pi_i\}$ are called *projection-valued measures* as they allow to compute quantum events probabilities as we will soon show.

Definition 3.2. *The most general expression of a **state** is a density operator, that is a square matrix $\rho \in \mathcal{M}_N$ with the following additional properties:*

- $\rho = \rho^\dagger \geq 0$;
- $tr[\rho] = 1$;
- $tr[\rho^2] \leq 1$.

We will now on denote the set of operators respecting those constraints $\mathcal{D}(\mathbb{C}) \subset \mathcal{B}(\mathbb{C})$. One important property of density matrices is that they generate a convex set, that is the convex combination of density matrices is still a density matrix:

$$\rho = \sum_j c_j \rho_j, \qquad \sum_j c_j = 1.$$

If a density matrix ρ does not admit a non-trivial convex decomposition then the state is said to be *pure*. This means that it can be expressed as

$$\rho = |\psi\rangle \langle\psi| ,$$

where $|\psi\rangle$ is a norm-1 state column vector in Dirac notation and $\langle\psi|$ is its adjoint representation. An inner product can be defined between vectors that belong to a Hilbert space $\mathcal{H}$:

$$\langle\psi, \phi\rangle = \langle\psi|\phi\rangle.$$

If ρ is the external product of vectors, the state is exactly known and ρ is a rank-one orthogonal projector. On the other side, if ρ can be expressed as a convex combination of pure states, that is:

$$\rho = \sum_j c_j |\psi_j\rangle \langle\psi_j| ,$$

then it is called a *mixed* state. This can be used to represent ensembles of identical systems prepared in different states or a single state prepared with classical uncertainty. In both cases coefficient c_j denotes respectively the fraction of population prepared in the *j-th* state and the probability of the state to be the *j-th* one.

The Hilbert-Schmidt scalar product allows to link the concepts of observable and state. In fact, given a system in the state ρ, the probability of obtaining λ_i as an outcome of a measurement on the observable X is given by:

$$p_i = tr[\rho\Pi_i].$$

Moreover, the expectation of the measurement on X can be easily derived:

$$\mathbb{E}_\rho[X] = \sum_i p_i\lambda_i = \sum_i tr[\rho\Pi_i]\lambda_i = tr[\rho \sum_i \lambda_i\Pi_i] = tr[\rho X].$$

According to Schrödinger picture, the evolution of a quantum system is entirely described by the evolution of its state. This means that while the state is time-varying, following the equations that describe the dynamical model, the observable is constant. Thus, the possible outcomes of a measurement keep unchanged while their probability varies as the state evolves.

3.1.2 Closed and open quantum systems

The most basic example of quantum dynamics is given by systems that are ideally isolated from their environment.

Definition 3.3. *A **closed quantum system** is a system which does not interchange information (energy or matter) with any other one.*

A model for this kind of evolution was first postulated in 1926 by Erwin Schrödinger. The *Schrödinger equation* for the state vector is the following *ODE*:

$$\hbar|\dot{\psi}\rangle = -iH\,|\psi\rangle\,, \tag{3.2}$$

where H is the Hamiltonian operator of the system and $\hbar$ is the Planck constant which we will hereafter suppose equal to 1. As the Hamiltonian is a Hermitian operator, the state vector undergoes an unitary evolution,

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle\,, \tag{3.3}$$

which preserves its norm. The dynamics of density operators can be easily derived from (3.3):

$$\rho(t) = |\psi(t)\rangle\,\langle\psi(t)| = U(t)\,|\psi(0)\rangle\,\langle\psi(0)|\,U(t)^\dagger = U(t)\rho(0)U(t)^\dagger, \tag{3.4}$$

whose infinitesimal version is given by quantum *Liouville-Von Neumann equation*:

$$\dot{\rho} = -i[H(t), \rho]. \tag{3.5}$$

If the system is non-conservative, and thus the Hamiltonian is time-dependent, the evolution must be written as a Dyson expansion:

$$U(t) = \mathcal{T} e^{\int_{t_0}^{t} H(s)ds}. \tag{3.6}$$

Definition 3.4. *An **open quantum system** is a quantum system that interacts with its environment.*

In this case the joint Hilbert space $\mathcal{H}$ that supports the whole dynamics can be factorized in the Hilbert space of the system of interest $\mathcal{H}_\mathcal{S}$ and that of the environment $\mathcal{H}_\mathcal{E}$. The mathematical operator which describes the interaction is the tensor product,

$$\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}.$$

When applied to matrices, such as density operators, tensor products can be interpreted as Kronecker products. If we suppose the coupled system driven by an Hamiltonian of the form

$$H_{tot} = H_\mathcal{S} \otimes I_\mathcal{E} + H_\mathcal{E} \otimes I_\mathcal{S} + H_{\mathcal{SE}},$$

where I denotes the identity on each subsystem, then the dynamics on $\mathcal{H}$ is still unitary and it can be studied as we previously did for closed quantum

systems. Nevertheless the object we want to observe is the evolution of the reduced state related to $\mathcal{S}$. Let's suppose for example the initial state of the joint system to be

$$\rho(0) = \rho_{\mathcal{S}}(0) \otimes \rho_{\mathcal{E}}(0).$$

Then its evolution is described by some unitary propagator in the following way:

$$\rho(t) = U_{\mathcal{SE}}(t)\rho(0)U_{\mathcal{SE}}(t)^{\dagger}. \tag{3.7}$$

The dynamics of the reduced state $\rho_{\mathcal{S}}$ can be derived from the joint one using the operation of *partial trace*, defined as:

$$tr_A[A \otimes B] = B \cdot tr[A]. \tag{3.8}$$

For density operators this naturally leads to

$$\rho_{\mathcal{S}}(t) = tr_{\mathcal{E}}[\rho(t)] = \rho_{\mathcal{S}}(t) \cdot tr[\rho_{\mathcal{E}}(t)]. \tag{3.9}$$

The evolutions thus obtained are in general non-Markovian and for this reason they are difficult to handle. We will show in the next section that under proper assumptions these dynamics can be made Markovian in such a way to provide mathematical models easier to control.

3.2 Markovian dynamics for open systems

3.2.1 Classical Markov semigroups

In order to highlight the importance of Markovianity assumptions for quantum dynamics analysis we first need to briefly recall what a stochastic process is and which of its properties are useful for our aims.

Definition 3.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family of random variables $\{X(t); t \in T\}$ defined on Ω is called a **stochastic process**. It is a continuous-time process if $T = [a, b], -\infty \leq a < b \leq -\infty$. It is a discrete-time process if $T \subseteq \mathbb{Z}$.*

As we already mentioned we are particularly interested in a specific set of stochastic processes, that is Markov processes.

Definition 3.6. *A discrete **Markov process** is a memoryless stochastic process, that is one for which:*

$$p_{t_1,\ldots,t_n}(x_n|x_{n-1};\ldots;x_1) = p_{t_{n-1},t_n}(x_n|x_{n-1}), \tag{3.10}$$

where $t_1, \ldots, t_n$ are picked from a countable set.

When the set of states $\mathbb{X} = \{1, 2, \ldots\}$ is countable too, this kind of process is called *Markov chain*. Let's denote, for simplicity,

$$p_{ij} := p(x_n = j | x_{n-1} = i), \tag{3.11}$$

that is the probability to step from the state i to the state j, supposing that this probability does not depend on time. By the law of total probability,

$$p(x_n = j) = \sum_{i \in \mathbb{X}} p_{ij} p(x_{n-1} = i). \tag{3.12}$$

This transition law can be expressed in a compact form,

$$p(x_n) = P^T p(x_{n-1}),$$

where

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix},$$

is a stochastic transition matrix. A stochastic matrix is one which can have only positive entries and whose rows sum to one. In this case all the entries are probabilities and each row sums to one as it represents the total probability of jumping to any state from a given one. A transition matrix is then a linear application describing an evolution which preserves the positivity and the total probability. Moreover, as we supposed that the transition matrix does not depend on time, the semigroup property holds. Indeed

$$p(x_{n+1}) = P^T p(x_n) = P^T P^T p(x_{n-1}) = (P^T)^2 p(x_{n-1}),$$

and by induction we get

$$p(x_{n+m}) = (P^T)^m (P^T)^n p(x_0) = (P^T)^{n+m} p(x_0). \tag{3.13}$$

Let's consider now a time-homogeneous continuous-time stochastic process $x(t)$. Defining

$$X_s^- = \{x(t), \quad \forall t \leq s\},$$

the past of $x(t)$ at time s, the Markov property can be expressed as follows:

$$p(x(t+s) = i | X_s^-) = p(x(t+s) = i | x(s)). \tag{3.14}$$

Supposing that $x(0) = i$, we define T_i the exact instant at which the process transitions away from i, that is

$$x(T_i) \neq i; \qquad x(t) = i, \quad \forall t < T_i.$$

We can easily observe that

$$\begin{aligned} p(T_i > s+t | T_i > s) &= p(x(r) = i, r \in [s, s+t] | x(r) = i, r \in [0, s]) \\ &= p(x(r) = i, r \in [s, s+t] | x(s) = i) \\ &= p(x(r) = i, r \in [0, t] | x(0) = i) \\ &= p(T_i > t). \end{aligned} \tag{3.15}$$

The last equation defines the loss of memory property, which completely characterizes exponential distributions. We can thus introduce a scalar $\lambda(i)$, which depends on state i, and according to which

$$p(T_i > t) = e^{-\lambda(i)t}.$$

Such parameter is fundamental, as it allows to derive a dynamical model for the evolution of transition probabilities as we will now show. Let's define

$$p_{ij} := p(x(T_i) = j | x(0) = i),$$

the probability of jumping from state i to state j and

$$\lambda(i, j) := \lambda(i) p_{ij}.$$

If time h is small enough, then

$$p(T_i < h) = \lambda(i)h + o(h), \tag{3.16}$$

and

$$\begin{aligned} p(x(h) = j | x(0) = i) &= p(T_i < h, x(T_i) = y | x(0) = i) + o(h) \\ &= \lambda(i) h p_{ij} + o(h) \\ &= \lambda(i, j) h + o(h), \end{aligned} \tag{3.17}$$

where $o(h)$ in the latter equation represents the probability of seeing two or more jumps in $[0, h]$. We are now able to derive the form of the generator of a Markov dynamics, as introduced in [24]. Denoting

$$P_{ij}(t) = p(x(t) = j | x(0) = i),$$

and remembering Markov property, (3.16), (3.17), we get:

$$\begin{aligned} \frac{dP_{ij}(t)}{dt} &= \lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \big[p(x(t+h) = j | x(0) = i) - p(x(t) = j | x(0) = i) \big] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{1}{h}\Big[\sum_{y\in\mathbb{X}}\big(p(x(t+h)=j|x(t)=y,x(0)=i)p(x(t)=y|x(0)=i) \\
&\quad - p(x(t)=j|x(0)=i)\big)\Big] \\
&= \lim_{h\to 0} \frac{1}{h}\big[(1-\lambda(j)h)P_{ij}(t) + \sum_{y\neq j}(\lambda(y,j)hP_{iy}(t)) - P_{ij}(t) + o(h)\big] \\
&= -\lambda(j)P_{ij}(t) + \sum_{y\neq j}\lambda(y,j)P_{iy}(t),
\end{aligned}
$$

which is the *Kolmogorov forward equation*. We will soon show that there exist equivalent equations for quantum dynamics. These equations will be the model we will base on for applying switching techniques.

3.2.2 Quantum Markov dynamics

We want now to show what kind of *dynamical map* $\mathcal{E}$ is apt to describe the evolution of a quantum state $\rho_{\mathcal{S}}$ between two given times t_0 and t_1:

$$\mathcal{E} : \rho_{\mathcal{S}}(t_0) \longrightarrow \rho_{\mathcal{S}}(t_1).$$

Let's suppose a quantum total initial state to be decomposed in the following way:

$$\rho(t_0) = \rho_{\mathcal{S}}(t_0) \otimes \rho_{\mathcal{E}}(t_0). \tag{3.18}$$

The propagator of the reduced initial state can be thus derived from that of the unitary complete one as

$$
\begin{aligned}
\mathcal{E}_{t,t_0}(\rho_{\mathcal{S}}(t_0)) &= tr_{\mathcal{E}}[U_{\mathcal{SE}}(t)[\rho_{\mathcal{S}}(t_0)\otimes\rho_{\mathcal{E}}(t_0)]U_{\mathcal{SE}}(t)^\dagger] \\
&= \sum_\alpha K_\alpha(t)\rho_{\mathcal{S}}(t_0)K_\alpha(t)^\dagger,
\end{aligned} \tag{3.19}
$$

where K_α are operators that depend only on the initial state of the environment. Indeed, let's suppose the initial state of the environment to be a pure state. The propagator of the reduced initial state can again be derived as

$$
\begin{aligned}
\mathcal{E}_{t,t_0}(\rho_{\mathcal{S}}(t_0)) &= tr_{\mathcal{E}}[U_{\mathcal{SE}}(t)[\rho_{\mathcal{S}}(t_0)\otimes|\psi\rangle_{\mathcal{E}}\langle\psi|]U_{\mathcal{SE}}(t)^\dagger] \\
&= \sum_\alpha {}_{\mathcal{E}}\langle\phi_\alpha|U_{\mathcal{SE}}(t)\,|\psi\rangle_{\mathcal{E}}\,\rho_{\mathcal{S}}(t_0)\,{}_{\mathcal{E}}\langle\psi|U^\dagger_{\mathcal{SE}}(t)\,|\phi_\alpha\rangle_{\mathcal{E}}\,,
\end{aligned}
$$

where $|\phi_\alpha\rangle_{\mathcal{E}}$ is a basis of $\mathcal{H}_{\mathcal{E}}$. Thus,

$$K_\alpha(t) = {}_{\mathcal{E}}\langle\phi_\alpha|U_{\mathcal{SE}}(t)\,|\psi\rangle_{\mathcal{E}}\,,$$

which is an operator acting on $\mathcal{H}_\mathcal{S}$. More precisely, supposing $\{|\gamma\rangle_\mathcal{S} \otimes |\phi\rangle_\mathcal{E}\}$ to be a basis on $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$, K_α is the operator whose matrix elements are

$$\{K_\alpha\}_{ij} = ({}_\mathcal{S}\langle\gamma_i| \otimes {}_\mathcal{E}\langle\phi_\alpha|)U_{\mathcal{SE}}(|\gamma_j\rangle_\mathcal{S} \otimes |\psi\rangle_\mathcal{E}).$$

It is worth to note that there is no loss of generality in assuming the initial state of the environment to be a pure state. Indeed if it is not, an extra system can be introduced to purify it. Let

$$\rho_\mathcal{E} = \sum_\alpha p_\alpha \, |\phi_\alpha\rangle \, {}_\mathcal{E}\langle\phi_\alpha|,$$

be any mixed state and $\mathcal{R}$ a system in the same state space as $\mathcal{E}$ with orthonormal basis $\{|\lambda\rangle_\mathcal{R}\}$. For Schmidt decomposition [17], a pure state in the extended system can be defined as

$$|\xi\rangle_{\mathcal{ER}} = \sum_\alpha \sqrt{p_\alpha} \, |\phi_\alpha\rangle_\mathcal{E} \otimes |\lambda_\alpha\rangle_\mathcal{R} \, .$$

The reduced state in $\mathcal{E}$ can be recovered via the partial trace operation

$$\begin{aligned} tr_\mathcal{R}[|\xi\rangle \, {}_{\mathcal{ER}}\langle\xi|] &= \sum_{\alpha\beta} \sqrt{p_\alpha p_\beta} \, |\phi_\alpha\rangle \, {}_\mathcal{E}\langle\phi_\beta| tr[|\lambda_\alpha\rangle \, {}_\mathcal{R}\langle\lambda_\beta|] \\ &= \sum_{\alpha\beta} \sqrt{p_\alpha p_\beta} \, |\phi_\alpha\rangle \, {}_\mathcal{E}\langle\phi_\beta|\delta_{\alpha\beta} \\ &= \sum_\alpha p_\beta \, |\phi_\alpha\rangle \, {}_\mathcal{E}\langle\phi_\alpha| \\ &= \rho_\mathcal{E}, \end{aligned}$$

which is exactly the initial state of the environment.

Definition 3.7. *A **Quantum Channel** is a positivity-preserving dynamical map that describes a physical evolution for any initial state $\rho_\mathcal{S}(t_0)$ and can be thus expressed as the Kraus decomposition*

$$\mathcal{E}_{t,t_0}(\rho_\mathcal{S}(t_0)) = \sum_\alpha K_\alpha(t)\rho_\mathcal{S}(t_0)K_\alpha(t)^\dagger, \tag{3.20}$$

where

$$\sum_\alpha K_\alpha(t)K_\alpha(t)^\dagger = I_\mathcal{S}. \tag{3.21}$$

Actually a quantum channel corresponds to the notion of transition matrix for a continuous-time Markov chain. The positivity-preserving property indicates that any density operator must evolve as a density operator, that is with positive (or null) eigenvalues. Moreover the property (3.21) is necessary for the evolution to be trace-preserving. These features are exactly the same as those of a stochastic matrix, whose entries must be positive and which must keep the norm of probability vectors unchanged.
By time continuity one expects that a dynamical map can be composed of two consecutive dynamical maps, that is:

$$\mathcal{E}_{t_2,t_0}(\rho) = \mathcal{E}_{t_2,t_1} \circ \mathcal{E}_{t_1.t_0}(\rho).$$

That is true, but let's suppose for a moment the initial state of the joint system to be a tensor product as in (3.18). Then, as we previously saw, the reduced evolution until t_2 can be obtained from:

$$\mathcal{E}_{t_2,t_0}(\rho_{\mathcal{S}}(t_0)) = tr_{\mathcal{E}}[U_{\mathcal{SE}}(t_2)[\rho_{\mathcal{S}}(t_0) \otimes \rho_{\mathcal{E}}(t_0)]U_{\mathcal{SE}}(t_2)^{\dagger}],$$

and, at the same way, the evolution until t_1 is:

$$\mathcal{E}_{t_1,t_0}(\rho_{\mathcal{S}}(t_0)) = tr_{\mathcal{E}}[U_{\mathcal{SE}}(t_1)[\rho_{\mathcal{S}}(t_0) \otimes \rho_{\mathcal{E}}(t_0)]U_{\mathcal{SE}}(t_1)^{\dagger}].$$

Both of these are clearly quantum channels but observing the following dynamics from t_1 to t_2,

$$\mathcal{E}_{t_2,t_1}(\rho_{\mathcal{S}}(t_1)) = tr_{\mathcal{E}}[U_{\mathcal{SE}}(t_2)\rho_{\mathcal{SE}}(t_1)U_{\mathcal{SE}}(t_2)^{\dagger}],$$

we notice that $\rho_{\mathcal{SE}}(t_1)$ is not necessarily factorisable as in (3.18). The last equation may therefore not be that of a quantum channel and thus not represent a physical evolution. This peculiarity is attributable to the irreversibility of non-unitary propagators, which is a characteristic of open systems dynamics.

The impossibility to express a quantum channel as composition of infinitesimal quantum channels excludes the formulation of a quantum dynamical model by means of differential equations as for classical systems. Nevertheless, in most of the practical situations a Markovian model that allows such compositions may represent a good dynamical model.

Definition 3.8. *A system undergoes a* ***Markovian evolution*** *if the following composition law holds:*

$$\mathcal{E}_{t_2,t_0}(\rho) = \mathcal{E}_{t_2,t_1} \circ \mathcal{E}_{t_1,t_0}(\rho).$$

This property is analogue to the semigroup property expressed in (3.13) for classical evolutions, where conditional probabilities are interpreted as dynamical maps. We can thus finally give the following definition:

Definition 3.9. *A **Quantum Dynamical Semigroup** (QDS) is a family of quantum channels for which:*

1. $\mathcal{E}_t \circ \mathcal{E}_s = \mathcal{E}_{t+s}$,
2. $tr[\mathcal{E}(\rho)] = tr[\rho]$, *for each quantum state* ρ,
3. $\mathcal{E}(\rho) \geq 0$, *for each quantum state* ρ.

Supposing Markovianity of evolutions we will now be able to build easy differential models on which projecting effective control solutions.

3.2.3 Markovian Master Equations and Lindblad form

Assuming that Markov property holds, then the system dynamics can be described as a differential equation with the following form:

$$\frac{d\rho(t)}{dt} = \lim_{\varepsilon \to \infty} \frac{\rho(t+\varepsilon) - \rho(t)}{\varepsilon} = \lim_{\varepsilon \to \infty} \frac{[\mathcal{E}_{t+\varepsilon,t} - I]}{\varepsilon} \rho(t), \tag{3.22}$$

where the *generator* of the evolution is

$$\mathcal{L}_t = \lim_{\varepsilon \to \infty} \frac{[\mathcal{E}_{t+\varepsilon,t} - I]}{\varepsilon}.$$

The derivation of such a generator will be led in the same way as we did for Markovian classical dynamics with Kolmogorov forward equation. The most general expression for the generator of a QDS, that is the equation that makes Definiton 3.9 hold at each instant of the evolution of the state, is called *Markovian Master Equation* (MME).

Theorem 3.1. *The Markovian Master Equation of a time independent generator can be written in the form*

$$\mathcal{L}(\rho(t)) = -i[H, \rho(t)] + \sum_{i,j=1}^{N^2-1} a_{ij} \big[F_i \rho(t) F_j^\dagger - \frac{1}{2} \{ F_j^\dagger F_i, \rho(t) \} \big], \tag{3.23}$$

where H is self-adjoint, and the matrix $A = (a_{ij})$ is positive semidefinite.

Proof. Let's suppose to express Kraus decomposition (3.20) with respect to an orthonormal basis of N^2 operators $\{F_i\}$, one of these is proportional to the identity, for example $F_{N^2} = \frac{1}{\sqrt{N}} I$, and the others are traceless operators:

$$\rho(t) = \mathcal{E}_{t,0}(\rho(0)) = \sum_{i,j=1}^{N^2} c_{ij}(t) F_i \rho(0) F_j^\dagger, \tag{3.24}$$

In this case, thus,

$$\langle F_i, F_j \rangle = \delta_{ij}, \qquad c_{ij}(t) = \sum_\alpha \langle F_i, K_\alpha(t) \rangle \langle F_j, K_\alpha(t) \rangle^\dagger, \tag{3.25}$$

and matrix $[c_{ij}(t)]$ is Hermitian and positive for any t.

The form of the generator $\mathcal{L}$ can be obtained from (3.22), that is:

$$\begin{aligned}
\mathcal{L}(\rho(t)) &= \lim_{\varepsilon \to 0} \frac{\mathcal{E}_{t+\varepsilon,t}(\rho(t)) - \rho(t)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \Bigg\{ \frac{1}{N} \frac{c_{N^2 N^2}(\varepsilon) - N}{\varepsilon} \rho(t) \\
&+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} \left(\frac{c_{iN^2}(\varepsilon)}{\varepsilon} F_i \rho(t) + \frac{c_{N^2 i}(\varepsilon)}{\varepsilon} \rho(t) F_i^\dagger \right) \\
&+ \sum_{i,j=1}^{N^2-1} \frac{c_{ij}(\varepsilon)}{\varepsilon} F_i \rho(t) F_j^\dagger \Bigg\}
\end{aligned} \tag{3.26}$$

We define now for simplicity the following quantities:

$$\begin{aligned}
a_{N^2 N^2} &= \lim_{\varepsilon \to 0} \frac{c_{N^2 N^2}(\varepsilon) - N}{\varepsilon}, \\
a_{iN^2} &= \lim_{\varepsilon \to 0} \frac{c_{iN^2}(\varepsilon)}{\varepsilon}, \quad i = 1 \dots N^2 - 1, \\
a_{ij} &= \lim_{\varepsilon \to 0} \frac{c_{ij}(\varepsilon)}{\varepsilon}, \quad i, j = 1 \dots N^2 - 1, \\
F &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N^2-1} a_{iN^2} F_i, \\
G &= \frac{1}{2N} a_{N^2 N^2} I + \frac{1}{2}(F^\dagger + F), \\
H &= \frac{1}{2i}(F^\dagger - F),
\end{aligned}$$

where, again, the matrix $[a_{ij}]$ is Hermitian and positive and H is clearly self-adjoint. With the help of these definitions, equation (3.26) can be written

as

$$\mathcal{L}(\rho(t)) = -i[H, \rho(t)] + \{G, \rho(t)\} + \sum_{i,j=1}^{N^2-1} a_{ij} F_i \rho(t) F_j^\dagger. \tag{3.27}$$

As the evolution of a density operator must be trace preserving, then:

$$0 = tr[\mathcal{L}(\rho(t))] = tr\left[\left(2G + \sum_{i,j=1}^{N^2-1} a_{ij} F_j^\dagger F_i\right)\rho(t)\right], \tag{3.28}$$

from which we deduce that

$$G = -\frac{1}{2}\sum_{i,j=1}^{N^2-1} a_{ij} F_j^\dagger F_i. \tag{3.29}$$

Finally, replacing the latter in (3.28) we get the standard form of the generator

$$\mathcal{L}(\rho(t)) = -i[H, \rho(t)] + \sum_{i,j=1}^{N^2-1} a_{ij}\left[F_i \rho(t) F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_\cdot, \rho(t)\}\right], \tag{3.30}$$

□

The set of $\{F_i\}$ can be picked for example as the set of N-dimensional traceless extended Gell-Mann matrices, while the family of a_{ij} specifies the dissipative part of the generator. It is worth to note that (3.23) is a linear matrix ODE. This feature will be very useful in the rest of our work, as switching systems techniques are much more developed on linear systems. Moreover, choosing a different basis, a MME can be put in a symmetrized form called *Lindblad form*:

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_k \left[L_k \rho(t) L_k^\dagger - \frac{1}{2}\{L_k L_k^\dagger, \rho(t)\}\right], \tag{3.31}$$

where L_k are noise operators. The Lindblad form is easier to handle and will be then adopted from now on.

3.3 Distances and norms for density operators

When dealing with problems of stabilizability or controllability of dynamical systems it is fundamental to dispose of a measure such to quantify "how

close" is a given state from another. This happens for example when studying steady states properties or when trying to follow desired trajectories. Classical dynamics are generally described by the evolution of real vectors, on which distances can be measured by applying the well-known *Euclidean norm*. On the other side, a quantum state is represented as a density matrix, for which the familiar definition of distance is less intuitive.

Definition 3.10. *A* ***matrix norm*** $||\cdot||$*, is a vector norm on* $\mathcal{M}(\mathcal{H})$*, that is, for each* $A, B \in \mathcal{M}$*:*

1. $||A|| \geq 0, \quad ||A|| = 0$ *iff* $A = 0$,
2. $||\alpha A|| = |\alpha|||A||$ *for each* $\alpha \in \mathcal{H}$,
3. $||A + B|| \leq ||A|| + ||B||$.

Among the most used matrix norms there are the *Schatten p-norms*, that is, supposing to deal with N-dimensional square matrices:

$$||A||_p = \left(\sum_i^N \sigma_i^p\right)^{\frac{1}{p}},$$

where $\{\sigma_i\}$ is the set of singular values of A.

When $p = 1$, Schatten norm is called *trace norm*, that is:

$$||A||_{tr} = tr[\sqrt{A^\dagger A}].$$

Moreover, if matrix A is Hermitian, as happens for example for density operators, the trace norm is the sum of the absolute values of its eigenvalues.

When $p = 2$, Schatten norm is called *Frobenius norm*, that is

$$||A||_{Fr} = \sqrt{tr[A^\dagger A]} = \sqrt{\langle A, A\rangle_{HS}}.$$

Frobenius norm is also often called Hilbert-Schmidt norm, as it is directly associated to the Hilbert-Schmidt product defined in (3.1).

Although Frobenius norm appears to be the most immediate definition of norm as it corresponds to the Euclidean norm for vectors, we will mainly make use of trace norm as important results related to quantum dynamics are based on it.

3.3.1 Trace distance between quantum states

One of the most important tools provided by the trace norm is that of trace distance.

Definition 3.11. *The **trace distance** between two quantum states ρ and σ is:*

$$D(\rho,\sigma)=\frac{1}{2}tr[|\rho-\sigma|], \tag{3.32}$$

where the operator $|A|=\sqrt{A^\dagger A}$ denotes the positive square root of $A^\dagger A$.

We introduce now a property of the trace distance which will allow us to prove the main result of this section.

Lemma 3.1. *The following equation holds:*

$$D(\rho,\sigma)=\max_P tr[P(\rho-\sigma)], \tag{3.33}$$

where $\{P\leq I\}$ is the set of all the projectors.

Proof. First of all it is fundamental to show that the difference of two density operators, $\rho-\sigma$, can be expressed as the difference between two positive operators, $Q-S$, with orthogonal support. In fact, for spectral theorem, as $\rho-\sigma$ is still an Hermitian matrix, then:

$$\rho-\sigma=\sum_i \lambda_i |\psi_i\rangle\langle\psi_i| - (-\sum_j \lambda_j |\psi_j\rangle\langle\psi_j|)=Q-S,$$

where $\{\lambda_i\}$ is the set of positive eigenvalues, $\{\lambda_j\}$ the set of negative eigenvalues, and $\{|\psi_{i,j}\rangle\}$ the set of the relative orthogonal eigenvectors.

Now, $|\rho-\sigma|=Q+S$, and consequently,

$$D(\rho,\sigma)=\frac{1}{2}tr[|\rho-\sigma|]=\frac{1}{2}tr[Q+S]=tr[Q],$$

because $tr[Q-S]=0$. Choosing P as the projector on Q we get:

$$tr[P(\rho-\sigma)]=tr[P(Q-S)]=tr[Q]=D(\rho,\sigma).$$

Moreover, if P is any projector:

$$tr[P(\rho-\sigma)]=tr[P(Q-S)]\leq tr[PQ]\leq tr[Q]=D(\rho,\sigma),$$

which completes the proof. □

We saw in the last sections that the evolution of density matrices representing quantum states must be driven by trace-preserving maps. Exploiting this fact and Lemma 3.1, we can prove an interesting result related to the stability of quantum dynamics.

Theorem 3.2. *Suppose $\mathcal{E}$ is a QDS, and let ρ and σ be density operators. Then*

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma). \tag{3.34}$$

Proof. Using spectral decomposition, $\rho - \sigma = Q - S$, where Q and S are positive matrices with orthogonal support. Let's choose the projector P as that for which $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = tr[P(\mathcal{E}(\rho) - \mathcal{E}(\sigma)]$. Remembering that $tr[Q] = tr[S]$, and thus $tr[\mathcal{E}(Q)] = tr[\mathcal{E}(S)]$, we see that:

$$\begin{aligned}
D(\rho, \sigma) &= \frac{1}{2} tr[|\rho - \sigma] \\
&= \frac{1}{2} tr[|Q - S|] \\
&= \frac{1}{2} tr[Q] + \frac{1}{2} tr[S] \\
&= \frac{1}{2} tr[\mathcal{E}(Q)] + \frac{1}{2} tr[\mathcal{E}(S)] \\
&= tr[\mathcal{E}(Q)] \\
&\geq tr[P\mathcal{E}(Q)] \\
&\geq tr[P(\mathcal{E}(Q) - \mathcal{E}(S))] \\
&= tr[P(\mathcal{E}(\rho) - \mathcal{E}(\sigma))] \\
&= D(\mathcal{E}(\rho), \mathcal{E}(\sigma)).
\end{aligned}$$

□

This theorem shows that the trace distance between two states that undergo the same physical dynamics cannot increase. Even if we have not given yet a formal definition of stability for matrix dynamics, this means that there cannot exist unstable steady states with respect to the metric induced by the trace norm. Unfortunately this result generally does not hold for other norms such as Frobenius norm.

3.4 Coherence-vector formulation

We have been dealing so far with density operators represented by complex Hermitian matrices. Moreover, under proper assumptions, we derived

dynamical models for quantum states expressed as matrices ODE. Such equations are quite difficult to handle, as most of the powerful tools of system theory, such as stability analysis and control techniques, were basically developed for real vectors evolutions. Fortunately, there exist methods to transform density matrices in vectors, and then to describe their dynamics with linear maps. We will therefore show a handy procedure to derive this kind of maps, and then use them for applying switching systems techniques that will be introduced in the next chapter.

3.4.1 Quantum states as real vectors

We described Hermitian operators as square matrices on the complex field. According to their definition these operators form a vector space on the real field:

$$\forall a, b \in \mathbb{R}, \quad \forall X, Y \in \mathcal{B}(\mathbb{C}), \qquad aX + bY \in \mathcal{B}(\mathbb{C}).$$

With respect to a given basis any Hermitian matrix can be univocally identified with the vector of coefficients relative to the elements that compose that basis.

Thus, if $\{F_i\}$ is any set of N^2 matrices that span the real space of complex $N \times N$ Hermitian matrices, then:

$$H = \alpha_0 F_0 + \alpha_1 F_1 + \ldots + \alpha_{N^2-1} F_{N^2-1}, \qquad \forall H \in \mathcal{B}(\mathbb{C}),$$

and we obtain the following bijective correspondence:

$$H \longleftrightarrow v_H = [\ \alpha_0 \quad \alpha_1 \quad \ldots \quad \alpha_{N^2-1}\ ]^T.$$

The value of the coefficients $\{\alpha_i\}$ is the usual scalar product between H and the elements of $\{F_i\}$:

$$\alpha_i = \langle H, F_i \rangle_{HS} = tr[H^\dagger F_i].$$

When choosing a basis for $\mathcal{D}(\mathbb{C})$, that is for matrices which all have the same trace, one can pick the identity and a set of $N^2 - 1$ orthonormal traceless matrices:

$$H = \frac{1}{N} I_N + \sum_{i=1}^{N^2-1} \alpha_i F_i.$$

This way, the first coefficient is the same for all the density operators, and can then be disregarded when studying vector dynamics. The other $N^2 - 1$ coefficients completely describe the evolution of the state and compose the so called *coherence vector*.

Example 3.1. *Coherence vector for 2-level states*

We show here how to obtain the vector form of a 2-level density operator. The most typical basis for such kind of systems is given by the set of Pauli matrices, usually denoted as:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

to which the identity, $\sigma_0 = I_2$, must be added. One can easily check that, opportunely scaled, Pauli matrices form an orthonormal basis:

$$\langle \frac{1}{\sqrt{2}}\sigma_i, \frac{1}{\sqrt{2}}\sigma_j \rangle_{HS} = \delta_{ij},$$

where δ_{ij} is the Kronecker product. Given any state ρ, we can easily decompose it:

$$\rho = v_0 \frac{1}{\sqrt{2}}\sigma_0 + v_x \frac{1}{\sqrt{2}}\sigma_x + v_y \frac{1}{\sqrt{2}}\sigma_y + v_z \frac{1}{\sqrt{2}}\sigma_z.$$

The values of v_i can be derived in the following way:

$$v_i = \langle \rho, \frac{1}{\sqrt{2}}\sigma_i \rangle_{HS} = \frac{1}{\sqrt{2}} tr[\rho\sigma_i],$$

and the vector becomes

$$v_\rho = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix},$$

where $x, y, z = tr[\sigma_{x,y,z}\rho]$. This representation corresponds to the density operator

$$\rho = \frac{1}{2} \begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}.$$

which, as any density operator, must be positive semidefinite. This means that the following relations hold:

- $z \geq -1$,
- $(1+z)(1-z) - (x-iy)(x+iy) \geq 0 \longrightarrow x^2 + y^2 + z^2 \leq 1$.

Thus, the free part of v_ρ belongs to the real sphere $\mathbb{S}^3$, often called *Bloch sphere*.

3.4.2 Linear and affine maps for vectorized dynamics

Given quantum states expressed in a vectorized representation we must now find the linear map $\hat{L}$ which describes the same evolution as $\mathcal{L}(\rho)$. With respect to the basis $\{F_i\}$, the density matrix ρ corresponds to the vector v_ρ according to the following relation:

$$v_\rho = \begin{bmatrix} \alpha_0^\rho \\ \alpha_1^\rho \\ \vdots \\ \alpha_{N^2-1}^\rho \end{bmatrix} = \begin{bmatrix} tr[\rho F_0] \\ tr[\rho F_1] \\ \vdots \\ tr[\rho F_{N^2-1}] \end{bmatrix}.$$

As a linear map we can find a vectorized representation of $\mathcal{L}(\rho)$ with respect to the same basis:

$$v_L = \begin{bmatrix} \alpha_0^L \\ \alpha_1^L \\ \vdots \\ \alpha_{N^2-1}^L \end{bmatrix}.$$

Exploiting the linearity of the trace we thus get:

$$\begin{aligned} \mathcal{L}(\rho) &= \sum_i \alpha_i^L F_i \\ &= \sum_i tr[\mathcal{L}(\rho)F_i]F_i \\ &= \sum_i tr[\mathcal{L}\big(\sum_j \alpha_j^\rho F_j\big)F_i]F_i \\ &= \sum_i tr[\big(\sum_j \alpha_j^\rho \mathcal{L}(F_j)\big)F_i]F_i \\ &= \sum_i \big(\sum_j \alpha_j^\rho tr[\mathcal{L}(F_j)F_i]\big)F_i. \end{aligned}$$

In matrix form the latter is equal to

$$\dot{v}_\rho = \hat{L}v_\rho = \begin{bmatrix} tr[\mathcal{L}(F_0)F_0] & \dots & tr[\mathcal{L}(F_{N^2-1})F_0] \\ tr[\mathcal{L}(F_0)F_1] & \dots & tr[\mathcal{L}(F_{N^2-1})F_1] \\ \vdots & \vdots & \vdots \\ tr[\mathcal{L}(F_0)F_{N^2-1}] & \dots & tr[\mathcal{L}(F_{N^2-1})F_{N^2-1}] \end{bmatrix} \begin{bmatrix} \alpha_0^\rho \\ \vdots \\ \alpha_{N^2-1}^\rho \end{bmatrix}. \quad (3.35)$$

If we suppose the first element of the basis to be the identity, then the first row of the matrix we have just derived must be equal to zero, as $\alpha_0^\rho = 1$ is

a constant of motion. Thus the map can be expressed as the following block matrix:

$$\hat{L} = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline b & & A & \end{array}\right].$$

The model which defines the evolution of the reduced state v_r^ρ, obtained from v_ρ by eliminating the first constant coefficient, is an affine equation:

$$\dot{v}_r^\rho = A v_r^\rho + b.$$

If $b = 0$ the identity is a steady state for $\mathcal{L}(\rho)$ and the dynamics is said to be *unital*. On the other side, if $b \neq 0$, then the generator is *non-unital*.

This kind of linear generator, which describes the evolution of vectorized quantum states, will be the object of our study in the next chapter. After having introduced switching systems theory we will apply it to sets of generators $\hat{L}_p$, and show how switching techniques can be efficiently exploited to control quantum systems.

Chapter 4

Switching control of quantum dynamics

In this chapter we will show how to control a quantum state by using switching systems techniques. In particular we will prove that, under certain conditions, a quantum state can be led to a stable equilibrium, even if Lindblad generators of its dynamics are only marginally stable and share that same equilibrium point. In the next sections the main part of our work will be described. Firstly a description of the problem and a definition of our aims will be given. After that we will provide a solution of the problem for a particular kind of dynamics, followed by a more complete and exhaustive analysis of the general case. Finally we will show how to make an invariant subspace of density operators shared by all the generators attractive.

4.1 Stability for Markovian Master Equations

As we saw in the previous sections, a quantum state can be represented as a density operator, that is a square $N \times N$ complex matrix ρ such that

$$1)\ \rho = \rho^\dagger; \quad 2)\ tr(\rho) = 1; \quad 3)\ tr(\rho^2) \leq 1.$$

Its dynamics in an open quantum system is described by the Markovian Master Equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)) = -i[H, \rho(t)] + \mathcal{L}_D(\rho(t)), \tag{4.1}$$

where H is the Hamiltonian of the open system and $\mathcal{L}_D$ describes the dissipative part of the generator. A set of density operators $\mathcal{S}$ is *invariant* if

$$\forall \rho(t_0) \in \mathcal{S} \longrightarrow \rho(t) \in \mathcal{S}, \quad \forall t \geq t_0.$$

Defining the distance of a state from a set

$$D(\rho, \mathcal{S}) := \inf_{\sigma \in \mathcal{S}} D(\rho, \sigma),$$

where the distance between two states is defined in (3.32), we say that a set $\mathcal{S}$ is *marginally stable* if it is invariant and

$$\forall \varepsilon > 0, \exists \delta \quad | \quad D(\rho(t_0), \mathcal{S}) \leq \delta \longrightarrow D(\mathcal{E}_t(\rho(t_0), \mathcal{S}) \leq \varepsilon, \forall t \geq t_0.$$

A set $\mathcal{S}$ is *globally asymptotically stable* if it is marginally stable and

$$\forall \rho(t_0) \longrightarrow \lim_{t \to +\infty} D(\mathcal{E}_t(\rho(t_0), \mathcal{S}) = 0.$$

The Lindblad equation is a linear matrix ODE, and can be therefore vectorized in such a way to obtain the following linear equation:

$$\frac{d}{dt} v(t) = \hat{L} v(t), \tag{4.2}$$

where $v(t)$ is a $N^2 \times 1$ column vector univocally associated with the density operator $\rho(t)$. Matrix $\hat{L}$ form depends on the way the quantum system interacts with its environment. In particular, as quantum channels are contractions in the trace norm (see Theorem 3.2), its eigenvalues lie on the imaginary axis if there is no dissipative part in the MME, while some of them are in the left complex half-plane if $\mathcal{L}_D \neq 0$. Moreover it can be proved [19] that if a set of density operators is invariant, then it is marginally stable.

As explained in Section 3.4, the vectorization of a linear matrix ODE can be obtained in several ways, for example by stacking the columns of the matrices on top of one another. Nevertheless, in the case of density operators, Hermitian and trace constant properties may be used for creating a smarter representation. By storing the trace value in the first element of the vector $v(t)$ we obtain a representation with the first element constant, and so whose first element derivative is equal to zero:

$$v(t) = \begin{bmatrix} 1 \\ * \\ \vdots \\ * \end{bmatrix} = \left[\begin{array}{c} 1 \\ \hline \\ v_r(t) \\ \\ \end{array}\right], \qquad \forall t \geq 0.$$

Consequently, matrix $\hat{L}$ first row must also be equal to zero,

$$\hat{L} = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline b & & A & \\ & & & \end{array}\right], \tag{4.3}$$

and the linear system (4.6) can be reduced in the affine form

$$\dot{v}_r(t) = Av_r(t) + b. \tag{4.4}$$

In this case, if matrix A is asymptotically stable, and therefore invertible, the unique steady state is

$$\bar{v}(t) = -A^{-1}b, \tag{4.5}$$

and it can be proved [13] it is globally asymptotically stable for (4.4). On the other side, if A is not asymptotically stable there must exist more than one, marginally stable, steady states.

4.2 Definition of the problem

Let's suppose now to be able to switch between m different environment configurations. This means that, for each of them, the quantum system evolution is described by

$$\frac{d}{dt}v(t) = \hat{L}_p v(t), \qquad p = 1, \dots, m. \tag{4.6}$$

Our aim is to show that, if each of these systems shares one equilibrium point with the others, it is possible, under certain conditions, to control any state to that equilibrium by appropriately switching between them, although none is asymptotically stable.

It is worth to note that finding a stabilizing switching law is possible only if there exists only one shared steady state and not more. Indeed, if there existed more, then it would be impossible to go from one to another and none would be asymptotically stable.

Even if there does not exist much literature on stabilizing quantum systems with switching techniques, their application in the quantum field is particularly interesting. Indeed, as quantum states cannot be directly observed without disturbing them, time-based switching laws allow to avoid issues created by the measurement procedures. Moreover, the control strategy we will present apply indiscriminately to pure and mixed states and to any kind of Lindblad generators. We will then begin from the particular case of unital generators, and then extend it to general of Lindblad ones.

4.3 Special case: unital generators

Let's suppose for now that, for each generator L_p,

$$b_p = [0 \quad \dots \quad 0]^T, \qquad p = 1, \dots, m;$$

and that each of them shares the same equilibrium point $\bar{v}$. Matrices $\hat{L}_p$ are then block diagonal:

$$\hat{L}_p = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & A_p & \\ 0 & & & \end{array}\right], \qquad p = 1, \dots, m.$$

Moreover,

$$\bar{v} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

is clearly a common steady state for such kind of systems. From a physical point of view these dynamics are called unital as the identity (opportunely scaled to be a valid density operator) is a fixed point for each of them. Stabilizing the state of a system to the identity means generating a completely random state. As block diagonal matrices, the relative dynamics can be easily described by exponentiating the blocks on the diagonal, that is

$$\left[\begin{array}{c} 1 \\ \hline v_r(t) \end{array}\right] = \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & e^{A_p t} & \\ 0 & & & \end{array}\right] \left[\begin{array}{c} 1 \\ \hline v_r(0) \end{array}\right]. \tag{4.7}$$

In order to guarantee the asymptotic stability of the switching system we have to suppose that there exists a convex combination A_c of the different A_p, that is that Assumption 2.1 holds. Choosing a period ε and dwelling on each system A_p for a time proportional to its coefficient $\alpha_p \varepsilon$, the evolution of the reduced state v_r can be described as

$$v_r(t) = e^{A_p(t-t_p)} e^{\alpha_{p-1} A_{p-1} \varepsilon} \cdots e^{\alpha_1 A_1 \varepsilon} v_r(t_0), \qquad \forall t \in [t_p, t_{p+1}]. \tag{4.8}$$

At the end of each switching period ε the evolution expressed in (4.8) can also be written as

$$v_r(k\varepsilon) = e^{(A_c + \varepsilon \Upsilon_c) k\varepsilon} v_r(0), \qquad k = 0, 1, 2 \dots \tag{4.9}$$

If ε is small enough,

$$\bar{A} := A_c + \varepsilon \Upsilon_c,$$

is still Hurwitz and

$$v_r(t) \xrightarrow{t \to +\infty} \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}^T.$$

Finally, remembering that the first component of $v(t)$ is constant

$$v(t) \xrightarrow{t\to+\infty} \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T.$$

Obviously from the vectorized noted $\bar{v}$ it is possible to derive the equivalent density operator $\bar{\rho}$ to which the corresponding switching MME converges.

Proposition 4.1. *Let's suppose to dispose of a set of m unital generators and that the Assumption 2.1 holds. Then, periodically applying to any initial state each generator $\hat{L}_p$ for a small enough time proportional to the relative coefficient α_p, the switching system asymptotically converges to the completely mixed state $\rho = \frac{1}{N} I$.*

4.3.1 Symmetric unital generators

We have just showed that the existence of a Hurwitz convex combination of the generators $\hat{L}_p$ is enough to prove the stabilizability of the completely mixed state. Actually, formulating Assumption 2.1 is not necessary if all the generators $\hat{L}_p$ are unital, symmetric and share only one steady state. In this case, it is sufficient to suppose that each of them is employed enough times, for example by periodically repeating a sequence that involves all the generators or by selecting them randomly. If such matrices are marginally stable, as we suppose, there exists a change of basis that makes them describe the evolution of vectors whose norm can never increase, and switching between them must lead to the shared steady state. We will now formally prove this conjecture summed up in the following proposition.

Proposition 4.2. *Let's consider m unital, symmetric, marginally stable generators $\hat{L}_p$, which share only one steady state $\bar{v}$. Any switching law that employs them all often enough is stabilizable.*

Proof. According to the hypotheses, we are dealing with generators which have all the following form:

$$L_p = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & A_p & \\ 0 & & & \end{array}\right], \qquad A_p = (A_p)^T, \qquad p = 1, \dots, m.$$

As the shared steady is unique it must be that whose coherent part is

$$\bar{v}_r = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}^T.$$

As a symmetric matrix, any matrix A_p is orthogonally diagonalizable:

$$A_p^d = U_p^T A_p U_p,$$

where U_p is an orthogonal matrix. Moreover, for marginal stability, A_p^d has real negative or null eigenvalues on its diagonal. Let's take the Lyapunov function

$$V = \frac{1}{2} v_r^T v_r.$$

which is positive definite. Clearly that Lyapunov function can also be expressed as

$$V = \frac{1}{2}(\tilde{v}_r^T U_p^T)(U_p \tilde{v}_r) = \frac{1}{2}\tilde{v}_r^T \tilde{v}_r,$$

where we denote $\tilde{v}_r$ the vector v_r in the orthogonal basis of the system p which drives its evolution. The derivative of the Lyapunov function is

$$\dot{V} = \tilde{v}_r^T \frac{\partial \tilde{v}_r}{\partial t} = \tilde{v}_r^T A_p \tilde{v}_r \leq 0,$$

as A_p is negative semidefinite. If $\tilde{v}_r$ belongs to the eigenspace relative to an eigenvalue equal to zero, then that vector keeps unchanged. If it does not, and v_r is not the origin, then its norm decreases. Clearly, if all the generators are periodically employed or they are randomly selected, there exists at least one generator for which

$$\dot{V} = \tilde{v}_r^T \frac{\partial \tilde{v}_r}{\partial t} = (\tilde{v}_r)^T A_p \tilde{v}_r < 0,$$

for each $\tilde{v}_r$. Indeed, if it wouldn't, there would be a state for which

$$\frac{\partial \tilde{v}_r}{\partial t} = A_p \tilde{v}_r = 0, \qquad p = 1, 2, \ldots, m.$$

That is impossible as the origin is supposed to be the only shared steady state. Thus the norm of the reduced vector $\tilde{v}_r$ keeps decreasing and the density operator $\bar{\rho} = \frac{1}{N} I_N$ is asymptotically stable for the switching dynamics made by the Lindblad generators $\mathcal{L}_p$ that correspond to $\hat{L}_p$. □

4.4 General case: non-unital generators

Non-unital generators are clearly those for which the identity is not a steady state. When dealing with vectorized evolutions they assume the form of block triangular matrices, as expressed in (4.3). These cases, where the reduced dynamics is described by an affine model, are slightly more complex.

Nevertheless we will show that, without making any supplementary assumption with respect to the case discussed in the last section, we will manage to make any shared fixed point asymptotically stable by switching between the generators.

The dynamics of each reduced vectorized equation can be thus expressed as

$$\dot{v}_r(t) = A_p v_r(t) + b_p, \qquad p = 1, \dots, m. \tag{4.10}$$

We will suppose from now on that there exists a common steady state $\bar{v}_r$ for each generator and that none of A_p matrices are Hurwitz. In this case we cannot directly use the theory on linear switching systems because equations like (4.10) have an affine form. Nevertheless, the following proposition will allow us to overcome this problem.

Proposition 4.3. *If all the generators $\hat{L}_p$ share the same steady state $\bar{v}$, then there exists a change of basis matrix*

$$T^{-1} = \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline T_Q & & T_R & \end{array}\right], \tag{4.11}$$

that removes the affine component b_p in (4.10), that is that makes all the generator matrices $\hat{L}_p^d$ block diagonal:

$$\hat{L}_p^d = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \tilde{A}_p & \\ 0 & & & \end{array}\right] = T^{-1} \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline b_p & & A_p & \end{array}\right] T, \qquad p = 1, \dots, m. \tag{4.12}$$

For T^{-1} to be such a matrix, defining as usual $\bar{v}_r$ as the coherent part of $\bar{v}$, the following constraint must be respected:

$$T_Q + T_R \bar{v}_r = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proof. Let's suppose the block structure of a matrix T^{-1} to be

$$T^{-1} = \left[\begin{array}{c|c} T_S & T_P \\ \hline T_Q & T_R \end{array}\right], \tag{4.13}$$

where $T_S \in \mathbb{R}$, $T_P \in \mathbb{R}^{1\times(N^2-1)}$, $T_Q \in \mathbb{R}^{(N^2-1)\times 1}$, $T_R \in \mathbb{R}^{(N^2-1)\times(N^2-1)}$. For that matrix to be the requested change of basis, the steady state $\bar{v}_r$ must turn to a steady state for all the block diagonalized generators $\hat{L}_p^d$. As the only equilibrium point for all the possible generators $\hat{L}_p^d$ is the vector that corresponds to the purely mixed state, the change of basis must be such that

$$\left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] = \left[\begin{array}{c|c} T_S & T_P \\ \hline T_Q & T_R \end{array}\right] \left[\begin{array}{c} 1 \\ \hline \bar{v}_r \end{array}\right], \tag{4.14}$$

and thus the following equations must hold:

- $T_S + T_P\bar{v}_r = 1$;
- $T_Q + T_R\bar{v}_r = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}^T$.

Moreover, from (4.12) we have

$$T^{-1}\hat{L}_p = \hat{L}_p^d T^{-1}.$$

Developing the last equation we get

$$\left[\begin{array}{c|c} T_S & T_P \\ \hline T_Q & T_R \end{array}\right] \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline b_p & & A_p & \end{array}\right] = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline T_R b_p & & T_R A_p & \end{array}\right], \tag{4.15}$$

and

$$\left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \tilde{A}_p & \\ 0 & & & \end{array}\right] \left[\begin{array}{c|c} T_S & T_P \\ \hline T_Q & T_R \end{array}\right] = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline \tilde{A}_p T_Q & & \tilde{A}_p T_R & \end{array}\right]. \tag{4.16}$$

As (4.15) and (4.16) must give the same result, then:

$$\begin{aligned} T_R b_p &= \tilde{A}_p T_Q, \\ T_R b_p &= T_R A_p T_R^{-1} T_Q, \\ b_p - A_p T_R^{-1} T_Q &= \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}^T. \end{aligned}$$

If we define from the latter $\bar{v}_r$ as a fixed point,

$$\bar{v}_r = -T_R^{-1} T_Q,$$

then we obtain

$$T_Q + T_R \bar{v}_r = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}^T,$$

which is a condition we had already imposed. Choosing for simplicity

- $T_S = 1$,
- $T_P = \begin{bmatrix} 0 & \dots & 0 \end{bmatrix}$,

we finally get

$$T^{-1} = \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline T_Q & & T_R & \end{array}\right], \tag{4.17}$$

which is actually a change of basis matrix. □

If Assumption 2.1 holds, then the steady state $\bar{v}$ can be made asymptotically stable for a switching system. Indeed, periodically switching between generators as described in 4.3, the evolution of the state is given by

$$\begin{aligned} v(t) &= e^{L_p(t-t_p)} e^{\alpha_{p-1} L_{p-1} \varepsilon} \cdots e^{\alpha_1 L_1 \varepsilon} v(0) \\ &= T e^{L_p^d (t-t_p)} T^{-1} T e^{\alpha_{p-1} L_{p-1}^d \varepsilon} T^{-1} \cdots T e^{\alpha_1 L_1^d \varepsilon} T^{-1} v(0) \\ &= T e^{L_p^d (t-t_p)} e^{\alpha_{p-1} L_{p-1}^d \varepsilon} \cdots e^{\alpha_1 L_1^d \varepsilon} T^{-1} v(0). \end{aligned} \tag{4.18}$$

At the end of each period ε,

$$v_r(k\varepsilon) = T e^{(\tilde{A}_c + \Upsilon_\varepsilon) k\varepsilon} T^{-1} v_r(0), \qquad k = 0, 1, 2, \dots \tag{4.19}$$

and if ε is small enough

$$v(t) \xrightarrow{t \to +\infty} \bar{v}.$$

It is finally worth to note that the considerations made in Section 4.3.1 for unital generators are fully extendable to the case of non-unital generators. That is, if after having changed basis all the generators are described by matrices $\tilde{A}_p$ that are symmetric, then any switching law is stabilizing.

Proposition 4.4. *Let's suppose to dispose of a set of m non-unital generators with an unique shared steady state $\bar{\rho}$ and that the Assumption 2.1 holds. Then, periodically applying to any initial state each generator $\mathcal{L}_p$ for a small enough time proportional to the relative coefficient α_p, the switching system asymptotically converges to $\bar{\rho}$.*

4.5 Examples of stabilization by switching generators

We will now show two easy numerical examples of stabilization of quantum systems with switching techniques. While the first one concerns the stabilization to a pure steady state, the second proves the applicability of this control strategy to a generic mixed state.

Example 4.1. *Stabilization of a pure steady state.*

Let's suppose to dispose of two marginally stable generators. The first describes an unitary evolution driven by the Hamiltonian

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

while the second describes a dissipative evolution driven by the only Lindblad generator

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Those two generators both share the steady state

$$\bar{\rho} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

as can be clearly seen by replacing the values of H and L in (3.31). One possible choice of orthonormal basis for 3-level systems is given by the set of Gell-Mann matrices, which represent the natural extension of Pauli matrices for 2-level systems. We will refer to them as $\{\sigma_i\}_{i=0,\dots,8}$, where again $\sigma_0 = \frac{1}{\sqrt{3}}I$, while σ_i, $i = 1,\dots,8$, are orthonormal traceless Hermitian matrices. The vectorized formulation $\bar{v}$ for the steady state $\bar{\rho}$ can be easily derived by the Hilbert-Schmidt product between $\bar{\rho}$ and each element of the basis σ_i:

$$\bar{v}_i = \langle \bar{\rho}, \sigma_i \rangle_{HS} = trace[\bar{\rho}\sigma_i] \longrightarrow \bar{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1.22 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.71 \end{bmatrix}.$$

The corresponding generators for the vectorized evolution can be obtained as explained in (3.35):

$$\hat{L}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1.73 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.73 & 0 \end{bmatrix},$$

and

$$\hat{L}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.82 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0.58 \\ 0 & 0 & 0 & 0 & -0.5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We must now find a change of basis such to make those generators unital. A possible choice for that is

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -0.82 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.41 \end{bmatrix}.$$

Applying this change of basis we find

$$\hat{L}_1^d = T^{-1}\hat{L}_1 T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.81 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.22 & 0 & 0 & 0 & 0 & 1.22 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.45 & 0 \end{bmatrix},$$

and

$$\hat{L}_2^d = T^{-1}\hat{L}_2 T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0 & -0.5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of these generators are

$$\alpha_1 = \begin{bmatrix} 0 & 2i & -2i & i & -i & 0 & i & -i & 0 \end{bmatrix},$$

and

$$\alpha_2 = \begin{bmatrix} 0 & -1 & 0 & -0.5 & -0.5 & -0.5 & -0.5 & -1 & -1 \end{bmatrix}.$$

Excepted the eigenvalue relative to the evolution of the trace of the density operator we find out that each generator has at least one more eigenvalue equal to zero. This means that using only one generator would make impossible to stabilize the desired steady state. Nevertheless the convex combination

$$\hat{L}_{conv} = \frac{1}{2}\hat{L}_1 + \frac{1}{2}\hat{L}_2,$$

has the following eigenvalues:

$$\alpha_{conv} = \begin{bmatrix} 0 \\ -0.38 + 0.96i \\ -0.38 - 0.96i \\ -0.23 \\ -0.25 + 0.5i \\ -0.25 - 0.5i \\ -0.25 + 0.5i \\ -0.25 - 0.5i \\ -0.5 \end{bmatrix}.$$

Now all the eigenvalues except for one have negative real part. Quickly switching between both the systems, and dwelling on each one for the same time, brings then the state to the equilibrium, which is nothing but $\bar{\rho}$ in a different basis.

Example 4.2. *Stabilization of a mixed steady state.*

Let's suppose now to dispose of two marginally stable unital generators, that is generators for which the identity is a common steady state. The relative Lindblad operators are

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Choosing the same basis as before, that is the set of Gell-Mann matrices, we naturally get, for the vectorized steady state $\bar{v}$,

$$\bar{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Again, the evolution corresponding to the Lindblad operators G_1 and G_2 are respectively

$$\hat{L}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & -2.5 & 0 & 0 & 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & -2.5 & -1.5 & 0 & 0 & 0 & 0.87 \\ 0 & 0 & 0 & -1.5 & -1 & 0 & 0 & 0 & 0.87 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 & 0 & 0 & -2.5 & 0 \\ 0 & 0 & 0 & 0.87 & 0.87 & 0 & 0 & 0 & -1.5 \end{bmatrix},$$

and

$$\hat{L}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of L_1 are

$$\alpha_1 = \begin{bmatrix} 0 & 0 & -1 & -4 & -1 & 0 & -1 & -4 & -1 \end{bmatrix},$$

while those of L_2 are

$$\alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -0.5 & -0.5 & -0.5 & -0.5 & -2 & -2 \end{bmatrix}.$$

Each generators has more than one eigenvalue equal to zero but if we define like in the previous example

$$\hat{L}_{conv} = \frac{1}{2}L_1 + \frac{1}{2}L_2,$$

we find a convex combination which eigenvalues have all negative real part except for that relative to the trace evolution:

$$\alpha_{conv} = \begin{bmatrix} 0 & -0.5 & -0.75 & -2.37 & -0.63 & -0.5 & -0.75 & -2.25 & -1.5 \end{bmatrix},$$

Switching between the generators we can now bring any state to the completely mixed state $\bar{\rho} = \frac{1}{3}I$.

4.6 Stability of subspaces

When stabilization of a given state is not a pressing constraint, or cannot be reached by switching systems, convergence to invariant subspaces of the Hilbert space $\mathcal{D}(\mathbb{C})$, of density operators, can be an interesting challenge. The expression of that space as a direct sum of orthogonal subspaces,

$$\mathcal{D} = \mathcal{H}_S \oplus \mathcal{H}_R,$$

leads to the following block representation of any density operator ρ:

$$\rho = \left[\begin{array}{c|c} \rho_S & \rho_P \\ \hline \rho_Q & \rho_R \end{array}\right].$$

The block ρ_S is an invariant set for the generator $\mathcal{L}$ if

$$\mathcal{L}\left(\left[\begin{array}{c|c} \rho_S & 0 \\ \hline 0 & 0 \end{array}\right]\right) = \left[\begin{array}{c|c} \mathcal{L}_S(\rho_S) & 0 \\ \hline 0 & 0 \end{array}\right].$$

Our aim is to make such kind of set attractive, that is, supposing that there exists an invariant set shared by every generator $\mathcal{L}_p$, to find a switching law for which

$$\rho(t) = \left[\begin{array}{c|c} \rho_S & \rho_P \\ \hline \rho_Q & \rho_R \end{array}\right] \xrightarrow{t\to+\infty} \left[\begin{array}{c|c} \rho_S & 0 \\ \hline 0 & 0 \end{array}\right] \in \mathcal{H}_S. \tag{4.20}$$

The density matrix can be vectorized in order to obtain an intuitive representation of the state:

$$v = \left[\begin{array}{c} 1 \\ \hline v_S \\ \hline v_R \end{array}\right], \tag{4.21}$$

where v_S is the invariant part of the vector, and v_R is the part that must be controlled to zero for making v_S attractive. We saw in the previous sections that an admissible generator for this kind of vector must be such to keep the first element constant:

$$\hat{L} = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline b & & A & \\ & & & \end{array}\right].$$

More precisely, for the vectorization realized in (4.21), generators must have the specific block structure

$$\hat{L} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline b_S & L_S & L_P \\ \hline 0 & 0 & L_R \end{array}\right],$$

in order to keep invariant states dynamics inside the invariant set. It is worth to note that if the square block L_R is not Hurwitz, then the invariant set is not attractive. Moreover, as the dynamics of v_R is driven only by L_R block, there is only need to focus on that block to design a switching law that makes the invariant set attractive.

Let's suppose to dispose of m generators $\hat{L}_p$ with the same invariant set and that there exists a convex combination of the blocks L_R^p, that is

$$\exists\, \alpha_1 \dots \alpha_m \quad s.t \quad \sum_{p=1}^{m} \alpha_p L_R^p = L_R^c, \qquad \sum_{p=1}^{m} \alpha_p = 1, \tag{4.22}$$

where L_R^c is Hurwitz. Switching generators in a period ε accordingly to the coefficients α_p, as described in the previous sections, we obtain the following evolution

$$v(t) = e^{\hat{L}_p(t-t_{p-1})} e^{\alpha_{p-1}\hat{L}_{p-1}\varepsilon} \cdots e^{\alpha_1 \hat{L}_1 \varepsilon} v(t_0), \tag{4.23}$$

If ε is small enough, at the end of each period this equation can be approximated with

$$v(k\varepsilon) = exp\left(\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline b_S^c & L_S^c & L_P^c \\ \hline 0 & 0 & L_R^c \end{array}\right] \right) v(t_0), \qquad k = 0, 1, 2, \dots \tag{4.24}$$

It is clear from the last relation that the components of $v(t)$ belonging to $v_R(t)$ decrease exponentially, and the invariant part $v_S(t)$ becomes attractive.

Proposition 4.5. *Let's suppose to dispose of a set of m generators with a common invariant set and that equation (4.22) holds. Then, periodically applying to any initial state each generator L_p for a small enough time proportional to the relative coefficient α_p, the switching system asymptotically converges to the common invariant set, which thus becomes attractive.*

Chapter 5

Conclusion

Our aim was to investigate how to exploit switching systems techniques for making a given quantum state globally asymptotically stable. We therefore chose to restrict our analysis to Lindblad dynamics which all share one same steady state but which are not asymptotically stable. Recalling results presented in [2], we proved that a linear switching system can be made asymptotically stable, even if none of the individual subsystems is stable itself. Indeed this happens if there exists a Hurwitz convex combination of the matrices which describe the subsystems. Unfortunately, finding such a convex combination, if it exists, is NP-hard unless there are only two subsystems, case for which we provided an easy algorithm. As a linear map acting on density operators ρ, Lindblad equations $\mathcal{L}$ can be expressed as matrices $\hat{L}$ which apply a linear transformation on quantum states v_ρ formulated in a vectorized notation. For actually describing the dynamics of a quantum state, any transformation $\hat{L}$ must have a defined structure. Its first row must thus be made by zeros for preserving the trace component and it cannot have positive eigenvalues. These constraints lead each generator $\hat{L}_p$ to have the following form:

$$\hat{L}_p = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline b_p & & A_p & \end{array}\right], \qquad p = 1, \dots, m, \tag{5.1}$$

where A_p must have at least one eigenvalue with real part equal to zero in order to be only marginally stable as we requested. If all the generators are unital the column vectors b_p are null. That means that finding a Hurwitz convex combination of A_p is enough for making the complete switching system asymptotically stable, as the first component is equal to one for all the acceptable vectors. On the other hand, if the generators $\hat{L}$ are not unital,

than the challenge is slightly more complex as the dynamics of the reduced vectors v_r^ρ are affine:

$$\dot{v}_r^\rho = Av_r^\rho + b.$$

Nevertheless, as we assumed that all the generators share the same steady state, there exists a change of basis T that makes all the switching dynamics unital. Asymptotic stability can again be reached by switching between the new submatrices $\tilde{A}_p$. We also showed that if all the submatrices A_p of unital generators, or $\tilde{A}_p$ of non-unital generators after changing basis, are symmetric, then the switching system is uniformly asymptotically stable, that is any switching law is stabilizing. Finally we proved that switching techniques can be used for making shared invariant subspaces attractive too. In this case the only requirement is to find a Hurwitz convex combination of the submatrices which describe the dynamics of the components which do not belong to the invariant sets.

As we showed in Chapter 2, state-based switching laws can be studied for stabilizing classical systems. Their application to quantum generators is obstructed by the difficulty of exploiting feedback information by observing states. Nevertheless, there exist methods for approximately recovering the state of a quantum system, for example by performing measures on light fields which have interacted with it. This kind of operation requires the use of stochastic models we did not mention in this work. State-based switching is however a valuable alternative, as it allows to increase convergence rate by applying optimization algorithms. Adapting this technique to quantum field is thus an interesting challenge, that deserves to be taken up.

Bibliography

[1] Liberzon, Daniel. *Switching in systems and control*. Springer, 2003.

[2] Sun, Zhendong. *Switched linear systems: control and design*. Springer, 2006.

[3] Wicks, Mark A., Philippos Peleties, and Raymond A. DeCarlo. *"Construction of piecewise Lyapunov functions for stabilizing switched systems."* Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on. Vol. 4. IEEE, 1994.

[4] Wicks, M., P. Peleties, and R. DeCarlo. *"Switched controller synthesis for the quadratic stabilisation of a pair of unstable linear systems."* European Journal of Control 4.2 (1998): 140-147.

[5] Branicky, Michael S. *"Multiple Lyapunov functions and other analysis tools for switched and hybrid systems."* IEEE Transactions on Automatic Control 43.4 (1998): 475-482.

[6] Blanes, Sergio, and Fernando Casas. *"On the convergence and optimization of the Baker–Campbell–Hausdorff formula."* Linear algebra and its applications 378 (2004): 135-158.

[7] Hespanha, Joao P. *"Uniform stability of switched linear systems: extensions of LaSalle's invariance principle."* IEEE Transactions on Automatic Control 49.4 (2004): 470-482.

[8] Li, Zheng Guo, C. Y. Wen, and Yeng Chai Soh. *"Stabilization of a class of switched systems via designing switching laws."* IEEE Transactions on Automatic Control 46.4 (2001): 665-670.

[9] Fornasini, Ettore, and Giovanni Marchesini. *Appunti di teoria dei sistemi*. Libreria Progetto, 1988.

[10] Fleming, Wendell H., and R. W. Rishel. *Deterministic and Stochastic Optimal Control (Stochastic Modelling and Applied Probability)*. Springer, 1982.

[11] Franklin, Gene F., et al. *Feedback control of dynamic systems*. Vol. 3. Reading: Addison-Wesley, 1994.

[12] Zhou, Kemin, and John Comstock Doyle. *Essentials of robust control*. Vol. 104. Upper Saddle River, NJ: Prentice hall, 1998.

[13] Schirmer, S. G., and Xiaoting Wang. *"Stabilizing open quantum systems by Markovian reservoir engineering."* Physical Review A 81.6 (2010): 062306.

[14] Bergholm, Ville, and Thomas Schulte-Herbrüggen. *"How to Transfer between Arbitrary n-Qubit Quantum States by Coherent Control and Simplest Switchable Noise on a Single Qubit."* arXiv preprint arXiv:1206.4945 (2012).

[15] D'Alessandro, Domenico. *Introduction to quantum control and dynamics*. CRC press, 2007.

[16] Alicki, R., and K. Lendi. *"Quantum Dynamical Semigroups and Applications"*, 1987.

[17] Nielsen, Michael A., and Isaac L. Chuang. *Quantum computation and quantum information*. Cambridge University Press, 2010.

[18] Petruccione, Francesco, and Heinz-Peter Breuer. *The theory of open quantum systems*. Oxford University Press, 2002.

[19] Ticozzi, Francesco, and Claudio Altafini. *"Modeling and control of quantum systems: an introduction."* IEEE Transactions on Automatic Control 57.8 (2012): 1898-1917.

[20] Rivas, Ángel, and Susana F. Huelga. *Open Quantum Systems: An Introduction*. Springer, 2011.

[21] Wang, Xiaoting, and S. G. Schirmer. *"Contractivity of the Hilbert-Schmidt distance under open-system dynamics."* Physical Review A 79.5 (2009): 052326.

[22] Perez-Garcia, David, et al. *"Contractivity of positive and trace-preserving maps under L_p norms."* Journal of Mathematical Physics 47.8 (2006): 083506.

[23] Wolf, Michael M. *"Quantum Channels and Operations."* Niels-Bohr Institute in Copenhagen, 2008/2009.

[24] Anderson, David F. *"Introduction to Stochastic Processes with Applications in the Biosciences."* University of Wisconsin at Madison, 2013.