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Stock price dynamics: an application of the Ehrenfest-Brillouin model to the Italian stock exchange

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Introduction

This thesis focuses on estimating the behaviour of a stock market, specifically the Italian one between June 1973 and April 1998, starting from empirical data. To achieve this, we employ a model inspired by quantum physics, known as the Ehrenfest-Brillouin model, which was developed by Paul and Tatiana Ehrenfest and later enhanced by Leon Brillouin. The original Ehrenfest urn model was developed in 1907 and, in 1927, Brillouin further enriched it by introducing Polya distributions. Then, in 1996, in his book titled *"New Approaches to Macroeconomic Modeling"*, physicist Masanao Aoki was the first one to approach the study of macroscopic variations in physical systems from an economic standpoint. He explored how the collective behaviour of interacting microeconomic entities could explain the macroscopic properties. Intriguingly, when certain variables are assigned specific values, the formulas used to model stock price dynamics demonstrate a remarkable resemblance to three important distributions observed in quantum physics. So we decide to use the Ehrenfest-Brillouin model because is a way to analyze the equilibrium distribution of the agents in the market, hence how the behaviour of the single agents can impact the behaviour of the market.

Chapter 1: Markov chains are introduced because the entities mentioned above (for example, in economics, we have agents, firms, and so on) change their state ruled by Markov-chain probabilistic dynamics. They also follow a Destruction-Creation process, where the first part, the destruction part, is called the "Ehrenfest term" and the creation one the "Brillouin term", the basis of the Ehrenfest-Brillouin model. Markovian dynamics are characterized by the property that the future state of the system depends solely on its current state, independent of its history, called the "lack of memory". In this thesis, Markov chains are useful to:

1. Describe the behaviour of systems that evolve over time.
2. Calculate the transition probability to a future state.

The concepts of Markov chains will then be applied to a practical example, namely 7 dice being rolled. This will be a valuable example to introduce the notations, but also to better understand the concept of destruction-creation, which is fundamental for understanding the Ehrenfest-Brillouin model. In Section 1.2, some properties of these chains will be explained furthermore. In particular, we focus on the finite Markov chain.

Chapter 2: We introduce the Polya distribution. The Polya distribution, also known as the Polya-Eggenberger distribution, is a probability distribution that describes the number of occurrences of different categories in a sequence of independent trials. It is useful mostly because the equilibrium distribution in the stock price dynamics (Chapter 3) is the 3-dimensional Polya. Then, the concept introduced in the previous chapter on the Destruction-Creation process is revisited and further explored. Specifically, the study of destruction and creation in a general case, such as in a Polya urn, is undertaken to establish a foundation of formulas and concepts for the rate of convergence.

Chapter 3: In this final chapter, we apply the mathematical knowledge covered in the previous chapters to the stock price dynamics in the Italian stock exchange between June 1973 and April 1998. Initially, the background in which the model will be applied is explained, demonstrating the final necessary formulas. In Section 3.1 all the previously introduced mathematical background is applied to the analysis of data from the Italian stock exchange. The goal is to estimate the number of active agents in the market (using the transition probabilities of a Markov chain), market intensity, and the rate of convergence to equilibrium.

Chapter 1

Markov chains

In this chapter, we introduce the definition of a Markov chain and how it can be essential in the Ehrenfest-Brillouin model.
This first chapter is based on [1], and the introduction of the terms of *destruction* and *creation* is from [4]

1.1 Basics on Markov chains

Consider, i.e. in economics, n elements (agents) in g categories (strategies). The state of the system is described by the equalities $X_1 = x_1, ..., X_n = x_n$, where x_i denotes the category to which the ith elements belongs, or by the vector shortcut $\mathbf{X}^{(n)} = \mathbf{x}^{(n)}$.

Definition 1.1. *We can now define a new vector* $\boldsymbol{Y}^{(n)}$*, that tells us the number of elements in each category and is written as:*

$$\boldsymbol{Y}^{(n)} = \boldsymbol{n} = (n_1, ., n_i, .., n_g), \qquad n_i \geq 0, \quad \sum_{i=1}^{g} n_i \ = n \tag{1.1}$$

and, of course, we must have the sum of every element in $Y^{(n)}$ *is* n*, because we have defined* $\boldsymbol{Y}^{(n)}$ *as the vector of the individual description of the numerousness of each category.*

Assume that, during a time interval, the ith element passes from the jth category to the kth category. How can we describe this movement? For starters, we think as the ith element abandons the jth population and returns after a 1-time interval to the kth population.
In Figure 1.1 we have the grey ball, initially in the first category, being moved to the fourth category. The initial occupation vector, in step one, is

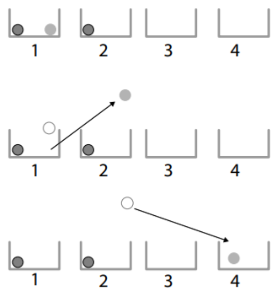

Figure 1.1: Visual representation of a destruction-creation process, from the book [1]. We have the grey ball, initially in the first category, being moved to the fourth category. In the initial occupation, we have 2 balls in the first box, 1 in the second and 0 in the others (we have 3 balls in the boxes). In the second step, we removed the grey ball (note that in this second step, we only have 2 balls in the boxes). Then, in the last step, we put the ball in the 4th box.

$\mathbf{n} = (2, 1, 0, 0)$ because we have 2 balls in the first box, 1 in the second and 0 in the others (in the general case we have n balls in the boxes). In the second step, the occupation vector is $\mathbf{n}_1 = (1, 1, 0, 0)$, because we removed the grey ball (note that in this second step, we only have 2 ($n-1$ in the general case) balls in the boxes). Then, in the last step, $\mathbf{n}_1^4 = (1, 1, 0, 1)$ because we put the ball on the 4th box. The notation on the vector $\mathbf{n}$ is the following:

- if the occupation vector is $\mathbf{n}$ the sum of the n_i must be n
- if it is $\mathbf{n}_j$ the sum of the n_i must be $n-1$ because we removed one elements from the jth category (called "the subscript")
- if it is $\mathbf{n}_j^k$ the sum of the n_i, must be n because we removed one element from the jth category ("the subscript") and we put it in the kth category ("the superscript"), thus we have the same number of elements as in the first step

If we want to see specifically the occupation vector in a general case:

1. STEP 1: $\mathbf{Y}_0^{(n)} = \mathbf{n} = (n_1, .., n_j, .., n_k, .., n_g)$
2. STEP 2: $\mathbf{Y}^{(n-1)} = \mathbf{n}_j = (n_1, .., n_j - 1, .., n_k, .., n_g)$
3. STEP 3: $\mathbf{Y}_1^{(n)} = \mathbf{n}_j^k = (n_1, .., n_j - 1, .., n_k + 1, .., n_g)$

This change of state could be viewed as the two-component transition:

- the *destruction* of the object on the jth category in the initial state $\mathbf{n}$ (called the "Ehrenfest's term"), resulting in the state vector $\mathbf{n}_j$ as seen in the second step, which happens with probability

$$\mathbb{P}(\mathbf{n}_j|\mathbf{n}) = \frac{n_j}{n}$$

- the *creation* of the object in the kth category given the new state vector $\mathbf{n}_j$, resulting in the final state vector $\mathbf{n}_j^k$, with probability

$$\mathbb{P}(\mathbf{n}_j^k|\mathbf{n}_j) = \frac{\alpha_k + n_k - \delta_{k,j}}{\alpha + n + 1}$$

  where $\alpha = (\alpha_1, ..., \alpha_g)$ and $\alpha = \sum_{i=1}^{g} \alpha_i$ is the vector of parameters and $\delta_{k,i}$ is the usual Kronecker's delta symbol, taking value 1 when $k = j$ and zero otherwise. The meaning of α_i's is tied to the probability of an accommodation to the ith cell, if empty.

Remark 1.1. *We have the following 2 properties:*

1. *The destruction probability is proportional to n_j*
2. *The creation probability is proportional to $\alpha_k + n_k - \delta_{k,j}$, where α_k is the initial weight of the category and $n_k - \delta_{k,j}$ is its current occupation number after destruction*

Definition 1.2. *If each change from $\mathbf{Y}_s$ to $\mathbf{Y}_{s+1}$ is probabilistic, and the transition probability only depends on the state $\mathbf{Y}_s$ and not on the full history $\mathbf{Y}_{s-1}, \mathbf{Y}_{s-2}, ..., \mathbf{Y}_0$, the probabilistic dynamics is said to be a* Markov chain*, whose transition probability is a matrix of elements*

$$\mathbb{P}(\,\boldsymbol{Y}_{s+1} = \boldsymbol{n}^{`}\,|\,\boldsymbol{Y}_t = \boldsymbol{n}) = \omega(\boldsymbol{n}, \boldsymbol{n}^{`}) \tag{1.2}$$

with $\mathbf{n}, \mathbf{n}^{`} \in S_g^n$.

Definition 1.3. *If the elements of the matrix $\omega(\boldsymbol{n}, \boldsymbol{n}^{`})$ do not depend on time* t*, the Markov chain is called* homogeneus.

Definition 1.4. *A* stochastic process *is a family of random variables*

$$\{X_i \in S, i \in I\}$$

where S is called "state space" and I is the "set of indices". X_t denotes the system's state at the discrete time t.

Definition 1.5. *A family of random variables $\{X_i \in S, i \in I\}$, where S is discrete and finite, and I is a subset of non-negative integers is called a* finite Markov chain *if:*

$$\begin{aligned}\mathbb{P}(X_0 =& x_0, X_1 = x_1, ..., X_s = x_s) = \\ &= \mathbb{P}(X_0 = x_0)\mathbb{P}(X_1 = x_1|X_0 = x_0)\cdots\mathbb{P}(X_s = x_s|X_{s-1} = x_{s-1})\end{aligned} \quad (1.3)$$

$\forall x_0, ..., x_s \in S$

1.1.1 A Markov chain for the seven dice

This subsection shows an example of a Markov chain for the seven dice. That can be very helpful for introducing the new notation and strengthening the concept of a Markov chain as I showed previously. This example is based on File [1].
Consider seven dice showing on the sequence

$$\mathbf{X}_0^{(7)} = (2, 6, 2, 1, 1, 5, 4)$$

Then, using the Definition (1.1) of the vector of individual description, we obtain $\mathbf{Y}_0^{(7)} = (2, 2, 0, 1, 1, 1)$. The first vector $\mathbf{X}_0^{(7)}$ tells us which number is on every dice, the second vector $\mathbf{Y}_0^{(7)}$ tells us the number of dices that shows the ith number. Assume now that, at each time step, a die is chosen at random. Then the die is cast, and the probability of any result is $\frac{1}{6}$. Let $\mathbf{Y}_0^{(7)} = (2, 2, 0, 1, 1, 1) = \mathbf{n}$. The probability of a die being chosen in the ith category is $\frac{n_i}{n}$ (using the notation introduced in Eq.(1.1)). The probability to get to the intermediate state

$$\mathbf{Y}^{(6)} = (2, 1, 0, 1, 1, 1) = \mathbf{n}_2 = (n_1, n_2 - 1, ..., n_6)$$

is $\frac{2}{7}$, because we had 2 dice with the number 2. Casting this die, the probability of obtaining 1 is $\frac{1}{6}$. Then

$$\mathbb{P}\big(\mathbf{Y}_1^{(7)} = (3, 1, 0, 1, 1, 1)|\mathbf{Y}_0^{(7)} = (2, 2, 0, 1, 1, 1)\big) = \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{21} \quad (1.4)$$

In general, we can compute the transition probabilities

$$\mathbb{P}\big(\mathbf{Y}_{t+1} = \mathbf{n}_i^j|\mathbf{Y}_t = \mathbf{n}\big) = \frac{n_i}{n} \cdot \frac{1}{6} \quad j \neq i \quad (1.5)$$

where $\mathbf{n}_i^j = (n_1, ..., n_i - 1, ..., n_j + 1, ..., n_6)$ because n=6 hence we are talking about the dice. If $j \neq i$, a true change takes place, and this can occur in only one way: selecting a die displaying an i then, after casting it, it displays a j. But the die could display the same number as before the casting. In this last case $\mathbf{n}^{`} = \mathbf{n}$, and this can occur in 6 alternative ways, each of them through a different destruction $\mathbf{n}_i$, with $i = 1, ..., 6$. So now the Eq. (1.5) can be rewritten in the following 2 different cases:

- $j \neq i$:

  then $\mathbb{P}\big(\mathbf{Y}_{t+1} = \mathbf{n}_i^j | \mathbf{Y}_t = \mathbf{n}\big) = \mathbb{P}(\mathbf{n}_i|\mathbf{n})\mathbb{P}(\mathbf{n}_i^j|\mathbf{n}_i)$ where $\mathbb{P}(\mathbf{n}_i|\mathbf{n}) = \frac{n_i}{n}$ is the destruction term, transforming a vector $\mathbf{Y}^{(n)}$ of size n into a vector $\mathbf{Y}^{(n-1)}$, and $\mathbb{P}(\mathbf{n}_i^j|\mathbf{n}_i)$ is the creation term that transforms a vector $\mathbf{Y}^{(n-1)}$ of size $n-1$ into a vector $\mathbf{Y}^{(n)}$ of size n.

- $j = i$:

  that's the probability of no change, so

$$\mathbb{P}\big(\mathbf{Y}_{t+1} = \mathbf{n} | \mathbf{Y}_t = \mathbf{n}\big) = \sum_{i=1}^{6} \frac{n_i}{n} \cdot \frac{1}{6} = \frac{1}{6} \tag{1.6}$$

In this case, all possible sequences of states are those sequences of consecutive occupation numbers differing by no more than 2:

$$|\mathbf{n}' - \mathbf{n}| = \sum_{i=1}^{g} |n_i' - n_i| \in \{0, 2\} \tag{1.7}$$

Therefore the difference is 0 if $j = i$, so no change, whereas is 2 if $j \neq i$. Given the initial state $\mathbf{Y}_{(0)}$, every other possible sequence $\mathbf{Y}_{(1)}, ..., \mathbf{Y}_{(t)}|\mathbf{Y}_{(0)}$ has probability:

$$\mathbb{P}\big(\mathbf{Y}_1 = \mathbf{y}_1 | \mathbf{Y}_0 = \mathbf{y}_0\big)\mathbb{P}\big(\mathbf{Y}_2 = \mathbf{y}_2 | \mathbf{Y}_1 = \mathbf{y}_1\big) \cdots \mathbb{P}\big(\mathbf{Y}_t = \mathbf{y}_t | \mathbf{Y}_{t-1} = \mathbf{y}_{t-1}\big)$$

where, if $|\mathbf{y}_t - \mathbf{y}_{t-1}| = 2$ we use the Eq.(1.5), while if $|\mathbf{y}_t - \mathbf{y}_{t-1}| = 0$ we are in a special case, so we use the Eq.(1.6), so the probability of every event can be determined.
The most relevant objects are the probabilities $\mathbb{P}\big(\mathbf{Y}_t = \mathbf{n}' | \mathbf{Y}_0 = \mathbf{n}\big)$ where t is very large, whose meaning is the probability of the states that can be observed after many moves. The expression $\mathbb{P}\big(\mathbf{Y}_t = \mathbf{n}' | \mathbf{Y}_0 = \mathbf{n}\big)$ can also be calculated by adding the probabilities of all sequences $\mathbf{Y}_1, ..., \mathbf{Y}_{t-1}$, connecting $\mathbf{Y}_0 = \mathbf{n}$ to $\mathbf{Y}_t = \mathbf{n}'$. After a number of trials bigger enough than the number of the dice, all dice will be selected at least one time, so the initial state ($\mathbf{Y}_0 = \mathbf{n}$) will be cancelled. So it can be defined as a "new" initial state, in this case,

the state at time $t-1$, so that the number of casting is now irrelevant. So $\mathbb{P}\big(\mathbf{Y}_t = \mathbf{n}'|\mathbf{Y}_0 = \mathbf{n}\big)$ does not depend by the initial state $\mathbf{Y}_0 = \mathbf{n}$, and

$$\lim_{t\to\infty} \mathbb{P}\big(\mathbf{Y}_t = \mathbf{n}'|\mathbf{Y}_0 = \mathbf{n}\big) = \pi(\mathbf{n}') \tag{1.8}$$

is the *multinomial distribution* which describes the casting of 7 fair dice. So the Markov chain reaches its equilibrium distribution $\pi(\mathbf{n}')$. Another interesting statistical propriety is the fraction of the time in which the system visits any given state as time goes by. Given, for example, the state $\mathbf{y} = (6,0,0,0,0,0)$, such the state where all the dice display a 1, its visiting fraction is the ratio between the number of times the state $\mathbf{y}$ appears and the total number of trials. If $t \to \infty$, with t fixed, the fraction of visiting any state converges to $\pi(\mathbf{n})$. This property is called *ergodicity*.

1.2 Finite Markov chains

In this section, dedicated to finite Markov chains, we present several properties of Markov chains that will prove valuable in the upcoming chapters.

1. Recalling the definition of a Markov chain, written in the Definition (1.5), is useful to stress that the typical Markovian property is that the predictive probability

   $$\mathbb{P}(X_s = x_s|X_{s-1} = x_{s-1}, ..., X_1 = x_1, X_0 = x_0)$$

   simplifies to

   $$\mathbb{P}(X_s = x_s|X_{s-1} = x_{s-1}, ..., X_1 = x_1, X_0 = x_0) = \mathbb{P}(X_s = x_s|X_{s-1} = x_{s-1})$$

   This means that the probability of a new (next) state depends on *only* the actual state and not on the history.

2. The following definition is given:

   Definition 1.6. *An* homogeneous Markov chain *is a Markov chain where* $\mathbb{P}(X_s = j|X_{s-1} = i)$ *depends only on* i, j *and not on* s.

   The notation is now

   $$\mathbb{P}(X_s = j|X_{s-1} = i) = w(i,j)$$

   The set of numbers $\{w(i,j)\}$ where $i, j \in \{1, ..., m\}$, and $m = \#S$, can be represented as a square matrix

   $$\mathbb{W} = \{w(i,j)\}_{i,j=1,...,m}$$

called the *transition matrix*. Each row of the matrix represents the probability of reaching the state $j = 1, 2, ..., m$ starting from the same state i. So $\mathbb{P}(X_s = j|X_{s-1} = i) = w(i, j)$ is non-negative and, for any i, $\sum_{j=1}^{m} w(i, j) = 1$, hence $\mathbb{W}$, on the rows, is a *stochastic matrix*.

3. The probability of reaching state j from state i in r steps is:

$$w^{(r)}(i, j) = \mathbb{P}(X_r = j|X_0 = i)$$

From the "total probability theorem", all the possible states between 0 and r are considered, so in conclusion:

$$w^{(r)}(i, j) = \sum_{k=1}^{m} w(i, k)w^{(r-1)}(k, j) \tag{1.9}$$

which is a particular instance of the *Chapman - Kolmogorov equation*. From Eq.(1.9), using the formula of matrix multiplication:

$$\mathbb{W}^{(r)} \overset{*}{=} \mathbb{W}\times\mathbb{W}^{(r-1)} = \mathbb{W}\times\mathbb{W}\times\mathbb{W}^{(r-2)} = ... = \underbrace{\mathbb{W} \times \mathbb{W} \times \cdots \times \mathbb{W}}_{r\ times} = \mathbb{W}^r$$

where the first equation (*) has been iterated $r-2$ times. So, the r-step transition probabilities $w^{(r)}(i, j)$ are the entries of $\mathbb{W}^{(r)}$, or equivalently $\mathbb{W}^r$, the rth power of the transition probability matrix $\mathbb{W}$.

4. Consider the probability that the chain state is j after r steps: conditioned to the initial state i, $w^{(r)}(i, j) = \mathbb{P}(X_r = j|X_0 = i)$. If the initial condition is probabilistic and $P_i^{(0)} = \mathbb{P}(X_0 = i)$ as a consequence of the total probability theorem, it follows:

$$P_j^{(r)} = \mathbb{P}(X_r = j) = \sum_{k=1}^{m} P_k^{(0)} w^{(r)}(k, j) \tag{1.10}$$

From matrix theory, Eq.(1.10) is seen as the formula for the multiplication of a row vector with m components by an $m \times m$ matrix. In other words, if $\mathbf{P}^{(r)} = (P_1^{(r)}, ..., P_m^{(r)})$ represents the equations $\mathbb{P}(X_r = 1), ..., \mathbb{P}(X_r = m)$

then one can write

$$\mathbf{P}^{(r)} = \mathbf{P}^{(0)} \times \mathbb{W}^r$$

The simplest case of the Eq.(1.10) is called the *Markov equation* and is:

$$P_j^{(r+1)} = \sum_{k=1}^{m} P_k^{(r)} w(k, j) \tag{1.11}$$

where is written, in the matrix form as:

$$\mathbf{P}^{(r+1)} = \mathbf{P}^{(r)} \times \mathbb{W}$$

Chapter 2

Polya distributions and the Ehrenfest-Brillouin model

In this chapter, we are going to study the Ehrenfest-Brillouin model, such as the generalization of random “destruction” followed by “creations”, whose probability is no longer uniform over all the categories. Then we introduce the Polya process, which is useful in Chapter 3. This second chapter is based on [1].

2.1 Polya distributions

Definition 2.1. *The sequence $X_1, X_2, \ldots, X_n$ with $X_i \in \{1, \ldots, g\}$ is a* generized Polya process *if the predictive probability of the process is given by*

$$\mathbb{P}(X_{m+1} = j | X_1 = x_1, ..., x_m = x_m) = \frac{\alpha_j + m_j}{\alpha + m} \tag{2.1}$$

where

- $m_j = \#\{X_i = j, i = 1, ..., m\}$ *is the number of occurrences of the jth category in the evidence $(X_1 = x_1, ..., x_m = x_m)$*
- m *is the number of observations or trials*

As in the usual Polya process:

- α_j *is a positive integer, representing the number of balls of type (colour) in the auxiliary Polya urn.*
- α *is the total number of balls in the urn (or the total weight of the urn if $\alpha \in \mathbb{R}$), so $\alpha = \sum_{j=1}^{g} \alpha_j$*

Remark 2.1. *If α_j is positive and real (not an integer), it can be interpreted as the initial weight of the jth category*

Defining $p_j := \frac{\alpha_j}{\alpha}$, the Eq.(2.1) can be simply rewritten as:

$$\begin{aligned}\mathbb{P}(X_{m+1} = j|X_1 = x_1, ..., x_m = x_m) &= \frac{\alpha p_j + m_j}{\alpha + m} \\ &= \frac{\alpha}{\alpha + m} p_j + \frac{m}{\alpha + m}\frac{m_j}{m}\end{aligned} \tag{2.2}$$

where p_j has an interpretation as the initial probability for the jth category, so

$$p_j = \frac{\alpha_j}{\alpha} = \mathbb{P}(X_1 = j) \tag{2.3}$$

Considering an infinite number of balls in the urn, such as $\alpha \to \infty$, the Eq.(2.2) becomes:

$$\lim_{\alpha\to\infty}\left(\frac{\alpha}{\alpha + m} p_j + \frac{m}{\alpha + m}\frac{m_j}{m}\right) = p_j \tag{2.4}$$

so $\mathbb{P}(X_{m+1} = j|X_1 = x_1, ..., x_m = x_m) \overset{\alpha\to\infty}{=} p_j$ leading to the independent and identically distributed (i.i.d) multinomial distribution. The generalised Polya process's distribution converges to the multinomial process's distribution where the parameter α tends to infinity (α diverges). When α_j is negative, if $|\alpha_j| \in \mathbb{N}$ (is an integer) and $m_j \leq |\alpha_j| = |\alpha| p_j$, the Eq.(2.1) still represents a probability and can be rewritten as:

$$\mathbb{P}(X_{m+1} = j|X_1 = x_1, ..., x_m = x_m) = \frac{|\alpha| p_j - m}{|\alpha| - m} \tag{2.5}$$

but here the number of observations is limited by $|\alpha|$, and leads to the hypergeometric distribution, and it is proven in the subsection below.

In the upcoming pages, we will show the process of transitioning from Eq.(2.5) to the hypergeometric distribution. Let's start by having an urn with given composition $\mathbf{n} = (n_1, ..., n_g)$, with the state space $\{1, ..., g\}$, where g denotes different colours present in the urn. Consider a sequence of individual random variables $X_1, ..., X_n$ whose range is the label set $\{1, ..., g\}$ with the following predictive probability, as seen in Eq. (2.5):

$$\mathbb{P}_{\mathbf{n}}(X_{m+1} = j|X_1 = x_1, ..., X_m = x_m) = \mathbb{P}_{\mathbf{n}}(X_{m+1} = j|\mathbf{m}) = \frac{n_j - m_j}{n - m} \tag{2.6}$$

With the constraint $\sum_{i-=1}^{g} m_i = m$, where $m \leq n$ are the drawn balls, where the balls m_i are of the colour i, (The first m_1 is of colour 1, the following m_2

of colour 2, and so on) the probability of the fundamental sequence

$$\begin{aligned}\mathbf{X}_f^{(m)} =&(X_1 = 1, ..., X_{m_1} = 1, X_{m_1+1} = 2, ...,\\ &X_{m_1+m_2} = 2, ..., X_{m-m_g+1} = g, ..., X_m = g)\end{aligned} \tag{2.7}$$

can be find thanks to Eq.(2.6). By iterating Eq. (2.7), we obtain:

$$\begin{aligned}\mathbb{P}_{\mathbf{n}}(\mathbf{X}_f^{(m)}) =&\frac{n_1}{n}\frac{n_1-1}{n-1}\cdots\frac{n_1-m_1-1}{n-m_1-1}\\ &\times\frac{n_2}{n-m_1}\frac{n_2-1}{n-m_1-1}\cdots\frac{n_2-m_2-1}{n-m_1-m-2-1}\\ &\times\frac{n_g}{n-m+m_g}\frac{n_g-1}{n-m+m_g-1}\cdots\frac{n_g-m_g-1}{n-m-1}\end{aligned} \tag{2.8}$$

Proposition 2.1. *In a more compact form, Eq.(2.8) becomes:*

$$\mathbb{P}_{\boldsymbol{n}}(\boldsymbol{X}_f^{(m)}) = \frac{(n-m)!}{n!}\prod_{i=1}^{g}\frac{n_i!}{(n_i-m_i)!} \tag{2.9}$$

Proof. Multiplying all the denominators in the Eq.(2.8):

$$\frac{1}{n}\cdot\frac{1}{n-1}\cdots\frac{1}{n-m_1-1}\cdots\frac{1}{n-m-1} = \frac{(n-m)!}{n!} \tag{2.10}$$

In the same way, multiplying the numerators with m_1 in the Eq.(2.8):

$$n_1\cdot(n_1-1)\cdots(n_1-m_1-1) = \frac{n_1!}{(n_1-m_1)!} \tag{2.11}$$

Similarly, for the numerators with m_2:

$$n_2\cdot(n_2-1)\cdots(n_2-m_2-1) = \frac{n_2!}{(n_2-m_2)!} \tag{2.12}$$

So, by multiplying all the numerators we obtain:

$$\prod_{i=1}^{g}\frac{n_i!}{(n_i-m_i)!} \tag{2.13}$$

And the formula (2.9) is proven. □

The probability in Eq.(2.9) is the same for any individual sequence with the sampling vector $\mathbf{m}$, and the number of these sequences is given by the multinomial factor. Therefore:

$$\mathbb{P}_{\mathbf{n}}(\mathbf{m}) = \frac{m!}{\prod_{i=1}^{g} m_i!}\cdot\mathbb{P}_{\mathbf{n}}(\mathbf{X}_f^{(m)}) \tag{2.14}$$

leading to the *hypergeometric distribution*

$$\begin{aligned}
\mathbb{P}_{\mathbf{n}}(\mathbf{m}) =& \frac{m!}{\prod_{i=1}^{g} m_i!} \frac{(n-m)!}{n!} \prod_{i=1}^{g} \frac{n_i!}{(n_i - m_i)!} \\
=& \frac{m!(n-m)!}{n!} \prod_{i=1}^{g} \frac{n_i!}{m_i!(n_i - m_i)!} \\
=& \frac{\prod_{i=1}^{g} \binom{n_i}{m_i}}{\binom{n}{m}}
\end{aligned} \tag{2.15}$$

The hypergeometric process explained is the simplest case of an *n-exchangeable process.*

Proposition 2.2. *Using Eq.*(2.1)*, we can find a similarity between Eq.*(2.15) *and the formula of the distribution of a Polya process, that is*

$$\mathbb{P}(\boldsymbol{m}) = Polya(\boldsymbol{m}; \alpha) = \frac{m!}{\prod_{i=1}^{g} m_i!} \cdot \frac{\prod_{i=1}^{g} \alpha_i^{[m_i]}}{\alpha^{[m]}} = \frac{m!}{\alpha^{[m]}} \cdot \prod_{i=1}^{g} \frac{\alpha_i^{[m_i]}}{m_i!} \tag{2.16}$$

Proof. Let's consider the fundamental sequence $\mathbf{X}_f^{(m)} = (1, ..., 1, ..., g, ..., g)$ consisting of m_1 labels 1 followed by m_2 labels 2 and so on, ending with m_g labels of g. The probability of this sequence is now

$$\begin{aligned}
\mathbb{P}_{\mathbf{n}}(\mathbf{X}_f^{(m)}) =& \frac{\alpha_1}{\alpha} \frac{\alpha_1 + 1}{\alpha + 1} \cdots \frac{\alpha_1 + m_1 - 1}{\alpha + m_1 - 1} \qquad (2.17) \\
\times& \frac{\alpha_2}{\alpha + m_1} \frac{\alpha_2 + 1}{\alpha + m_1 + 1} \cdots \frac{\alpha_2 + m_2 - 1}{\alpha + m_1 + m - 2 - 1} \\
\times& \frac{\alpha_g}{\alpha + m_1 + m_2 + \cdots + m_{g-1}} \cdot \frac{\alpha_g + 1}{\alpha + m_1 + m_2 + \cdots + m_{g-1} + 1} \\
\times& \frac{\alpha_g + m_g - 1}{\alpha + m_1 + m_2 + \cdots + m_g - 1} \\
=& \frac{\alpha_1^{[m_1]} \cdots \alpha_g^{[m_g]}}{\alpha^{[m]}} \qquad (2.18)
\end{aligned}$$

Where $\alpha^{[n]}$ is the *Pochhammer symbol* representing the upper factorial, and is defined by

$$\alpha^{[n]} = \alpha(\alpha + 1) \cdots (\alpha + n - 1) \tag{2.19}$$

□

Remark 2.2. *The formula in Eq.*(2.16) *does not depend on the order of appearance of the categories. Indeed, the probability of a different sequence with the same values of* $\boldsymbol{m} = (m_1, ..., m_g)$ *would have the same denominator and identical, yet permuted, terms in the numerator.*

Remark 2.3. *The Observation (2.8) means that the Polya process is exchangeable and that the finite-dimensional distributions depend only on the frequency vector* $\boldsymbol{m}$*.*

2.2 Destructions and creations

If we look into the economics subject, we can see that some entities (i.e. agents) are supposed to change their state unceasingly, following Markov-chain probabilistic dynamics. All these situations are characterized by a dynamical mechanism destroying an entity in a category and re-creating an entity in another category. Those events depend on the present state of the whole system. Looking back at the equation 1.5 in the first chapter, we can now generalize that *"destruction-construction mechanism"*. As seen in Chapter 1, we may consider a population made by n entities and g categories. the state of the system is described by the occupation number vector $\mathbf{n} = (n_1, ..., n_i, .., n_g), \qquad n_i \geq 0, \quad \sum_{i=1}^{g} n_i = n$. The sum is also denoted by

$$S_g^n = \{\mathbf{n} \mid n_i \geq 0, \sum_{i=1}^{g} n_i = n\} \tag{2.20}$$

The discrete-time evolution is given by the sequence of variables

$$\mathbf{Y}_0 = \mathbf{n}(0), \mathbf{Y}_1 = \mathbf{n}(1), ..., \mathbf{Y}_t = \mathbf{n}(t), ... \tag{2.21}$$

where $\mathbf{n}(\mathrm{t})$ belongs to S_g^n. We assumed that Eq.(2.21) describes the realization of a homogeneous Markov chain, whose one-step transition probability is a matrix whose entries are given by Eq.1.5

$$w(\mathbf{n}, \mathbf{n}_i^j) = \mathbb{P}\big(\mathbf{Y}_{t+1} = \mathbf{n}_i^j \mid \mathbf{Y}_t = \mathbf{n}\big)$$

with $\mathbf{n}_i^j$, $\mathbf{n} \in S_g^n$.

2.3 Occupation numbers as random variables and the approach to equilibrium

Let the initial state be $\mathbf{n}$. Let's define $\mathbf{D}^{(m)} = (d_1, .., d_g)$ as the vector that tells how many elements are selected (in order to be destructed) in any of the g categories, and let be $m = \sum_{i=1}^{g} d_i \ = m$ be the total number of the elements selected. After the destruction, the m elements are redistributed in the g categories, following the vector $\mathbf{C}^{(m)} = (c_1, .., c_g)$, and again

$\sum_{i=1}^{g} d_i = \sum_{i=1}^{g} c_i = m$. In other words, the vector $\mathbf{D}^{(m)}$ tells us how many elements we remove from each category (Destruction), and the vector $\mathbf{C}^{(m)}$ tells us how many elements we put in each category after the destruction (Creation).

Remark 2.4. *In the Ehrenfest-Brillouin model, the destruction always precedes the creation.*

So the process is now, by definition

$$\mathbf{Y}_{t+1} = \mathbf{Y}_t + \mathbf{I}_{t+1} \quad \text{where} \quad \mathbf{I}_{t+1} = -\mathbf{D}_{t+1} + \mathbf{C}_{t+1} \tag{2.22}$$

and where $\mathbf{I}_{t+1}$ is the vector of the increment.

$$\mathbb{P}(\mathbf{D}_{t+1} = \mathbf{d} \mid \mathbf{Y}_t = \mathbf{n}) = \frac{\prod_{i=1}^{g} \binom{n_i}{d_i}}{\binom{n}{m}} \tag{2.23}$$

$$\begin{aligned}\mathbb{P}(\mathbf{C}_{t+1} = \mathbf{c} \mid \mathbf{Y}_t = \mathbf{n}, \mathbf{D}_{t+1} = \mathbf{d}) &= \mathbb{P}(\mathbf{C}_{t+1} = \mathbf{c} \mid \mathbf{n}\text{-}\mathbf{d}) \\ &= \frac{\prod_{i=1}^{g} \binom{\alpha_i + n_i - d_i + c_i - 1}{c_i}}{\binom{\alpha + n - 1}{m}}\end{aligned} \tag{2.24}$$

The transition probability

$$\mathbb{P}(\mathbf{Y}_{t+1} = \mathbf{n}_i^j \mid \mathbf{Y}_t = \mathbf{n}) = \mathbb{P}(\mathbf{I}_{t+1} = \mathbf{n}_i^j - \mathbf{n} \mid \mathbf{Y}_t = \mathbf{n}) \tag{2.25}$$

is the sum of all paths of the kind

$$B_m = \{(\mathbf{d},\ \mathbf{c}) \mid -\mathbf{d} + \mathbf{c} = \mathbf{n}_i^j - \mathbf{n}\} \tag{2.26}$$

therefore one has

$$W_m(\mathbf{n}_i^j, \mathbf{n}) = \mathbb{P}(\mathbf{Y}_{t+1} = \mathbf{n}_i^j \mid \mathbf{Y}_t = \mathbf{n}) = \sum_{B_m} \mathbb{P}(\mathbf{c} \mid \mathbf{n} - \mathbf{d}) \mathbb{P}(\mathbf{d} \mid \mathbf{n}) \tag{2.27}$$

The transition matrix $W_m(\mathbf{n}, \mathbf{n}')$, given by Eq.(2.27) defines Markov chains with the Polya($\mathbf{n}$¸α) as invariant distribution. If the only constraint is $\sum_{i=1}^{g} n_i = n$ (see Section 2.5), given that for $m = 1$, the Markov chain defined by Eq.(2.27) are irreducible and aperiodic, and even more so is the same for any $m \leq n$; thus the invariant distribution Polya($\mathbf{n}$¸α) is also the limiting equilibrium distribution. Moreover, is expected that the rate with which equilibrium is reached is an increasing function of m. Let's consider the marginal description for a fixed category, whose initial weight is β:

$$\mathbf{Y}_{t+1} = \mathbf{Y}_t - \mathbf{D}_{t+1} + \mathbf{C}_{t+1} = \mathbf{Y}_t + \mathbf{I}_{t+1} \tag{2.28}$$

Suppose that $Y_t = k$, and merge the other $n-k$ elements into a single category, whose initial weight is $\alpha - \beta$. The destruction move chooses m objects from the occupation vector $(k, n-k)$ without replacement, and d units are removed from the fixed category, whereas $m-d$ are taken away from the remaining part of the system. The resulting occupation vector after the destruction move is $(k-d, n-k-m+d)$.
The creation move consists of m extraction from a Polya urn with initial composition $(\beta + k - d, \alpha - \beta + n - k - m + d)$, and c elements are created in the category with initial weight β and $m-c$ elements are created in the category with initial weight $\alpha - \beta$. Given the initial state $(k, n-k)$, the expected value of D is a function of the starting state $Y_t = k$:

$$\mathbb{E}(D|Y_t = k) = m\frac{k}{n} \tag{2.29}$$

As for creations, adding to the evidence the destroyed state $(d, m-d)$, one gets

$$\mathbb{E}(C|Y_t = k, D_{t+1} = d) = m\frac{\beta + k - d}{\alpha + n - m} \tag{2.30}$$

To eliminate d, we have to use the equation

$$\begin{aligned}\mathbb{E}(C|Y_t = k) &= \mathbb{E}(\mathbb{E}(C|Y_t = k, D_{t+1} = d)) \\ &= m\frac{\beta + k - \mathbb{E}(D|Y_t = k)}{\alpha + n - m} \\ &= m\frac{\beta + k - m\frac{k}{n}}{\alpha + n - m}\end{aligned} \tag{2.31}$$

Proposition 2.3. *Remembering the definition of I_{t+1} in the Eq.(2.22), using Eq.(2.29) and Eq.(2.31):*

$$\mathbb{E}(I_{t+1}|Y_t = k) = -\frac{m\alpha}{n(\alpha + n - m)}\left(k - \frac{n\beta}{\alpha}\right) \tag{2.32}$$

Proof. In the following lines, we present proof of the previously stated proposition, specifically emphasizing the truth of Eq.(2.32).

$$\begin{aligned}
\mathbb{E}(I_{t+1}|Y_t = k) &= -\mathbb{E}(D_{t+1}|Y_t = k) + \mathbb{E}(C_{t+1}|Y_t = k) \\
&= -\mathbb{E}(D|Y_t = k) + \mathbb{E}(C|Y_t = k) \\
&= -m\frac{k}{n} + m\frac{\beta + k - m\frac{k}{n}}{\alpha + n - m} \\
&= m\left(\frac{n\beta + nk - mk - k\alpha - nk + mk}{n(\alpha + n - m)}\right) \\
&= m\left(\frac{n\beta + \cancel{nk} - \cancel{mk} - k\alpha - \cancel{nk} + \cancel{mk}}{n(\alpha + n - m)}\right) \\
&= m\left(\frac{n\beta - k\alpha}{n(\alpha + n - m)}\right) \\
&= -m\left(\frac{k\alpha - n\beta}{n(\alpha + n - m)}\right) \\
&= -\frac{m}{n(\alpha + n - m)}\left(k\alpha - n\beta\right) \\
&= -\frac{m\alpha}{n(\alpha + n - m)}\left(k - \frac{n\beta}{\alpha}\right) \qquad (2.33)
\end{aligned}$$

□

Remark 2.5. *The average increment* $\mathbb{E}(I_{t+1}|Y_t = k) = 0$ *if* $k = \frac{n\beta}{\alpha} := \mu$, *which is the* equilibrium value *for* k*, and it is meanreverting: it reduces the deviations away from the equilibrium expected value.*

Definition 2.2. *The rate of approach to equilibrium is the coefficient*

$$r = \frac{m\alpha}{n(\alpha + n - m)} \qquad (2.34)$$

and it depends on the size n*, the total initial weight* α *and the number of changes* m*.*

Remark 2.6. *Setting* $m = 1$ *and* α *that tends to infinity, then* $r = \frac{1}{n}$*, the value for the Ehrenfest aperiodic model.*

Remark 2.7. *The rate in Definition (2.2) is a linear function of* m *if* $m \ll n$*, and it grows to 1 for* $m = n$*.*

Remark 2.8. *For* $m = n$ *and* $r = 1$*, the destruction completely empties the system, so that the subsequent creation probability is already the equilibrium one.*

The following paragraph is not so important per se but is a useful tool that is used to estimate the weekly and monthly autocorrelation of the daily returns in Chapter 3. In the limiting case of Remark(2.8), Y_{t+1} does not depend on Y_t, so the correlation $\mathbb{C}(Y_{t+1}, Y_t)$ is expected to be 0. Therefore one can derive the equation

$$\mathbb{E}(I_{t+1}|Y_t = k) = -r(k - \mu)$$

or, equivalently

$$\mathbb{E}(I_{t+1}|Y_t = k) - \mu = (1 - r)(k - \mu) \tag{2.35}$$

Iterating the Eq.(2.35), we obtain

$$\mathbb{E}(Y_{t+1}|Y_t = k) - \mu = (1 - r)^s(k - \mu) \tag{2.36}$$

Definition 2.3. *The autocorrelation function of the process, known as the* Bravais-Pearson autocorrelation function*, once it reaches stationarity is.:*

$$\rho(Y_{t+s}, Y_t) = \rho(s) = (1 - r)^s \tag{2.37}$$

2.4 A consequence of the hard constraint on the number of elements

Consider the occupation number random variables $Y_1, .., Y_g$, such that $y_i = n_i$. The hard constraint $\sum_{i=1}^{g} n_i = n$ implies that

$$\mathbb{V}\left(\sum_{i=1}^{g} Y_i\right) = 0$$

Therefore one gets:

$$0 = \mathbb{V}\left(\sum_{i=1}^{g} Y_i\right) = \sum_{i=1}^{g} \mathbb{V}(Y_i) + 2\sum_{i=1}^{g} \sum_{j=i+1}^{g} \mathbb{C}(Y_i, Y_j) \tag{2.38}$$

If the random variables Y_i are equidistributed, it turns out that

$$g\mathbb{V}(Y_i) + g(g - 1)\mathbb{C}(Y_i, Y_j) = 0 \tag{2.39}$$

leading to

$$\mathbb{C}(Y_i, Y_j) = -\frac{1}{g - 1}\mathbb{V}(Y_i) \tag{2.40}$$

Given that $\mathbb{V}(Y_i)+\mathbb{V}(Y_j)$, the Bravais-Pearson correlation coefficient becomes

$$\rho_{Y_i,Y_j} = -\frac{1}{g - 1} \tag{2.41}$$

In particular, for a dichotomous Polya distribution, one finds that $\rho_{Y_i,Y_j} = -1$, and the correlation coefficient vanishes for $g \to \infty$.

2.5 The negative binomial distribution as a marginal distribution of the general multivariate Polya distribution

In the general case, the chosen category has weight α_1 and the *thermostat's weight* is $\alpha - \alpha_1$. Therefore the term $\frac{\alpha_1^{[k]}}{k!}$ does not simplify, $g-1$ is replaced by $\alpha - \alpha_1$, and the thermodinamic limit[1] with the hypotesis of $n, \alpha \gg 1$ becomes $\chi = \frac{n}{\alpha}$. Considering that,for definition

$$x^{[m]} = \frac{\Gamma(x+m)}{\Gamma(x)} = x \cdot (x+1) \cdots (x+m-1)$$

one has

$$\begin{aligned}\frac{(\alpha-\alpha_1)^{[n-k]}}{\alpha^{[n]}} =& \frac{\Gamma(\alpha-\alpha_1+n-k)}{\Gamma(\alpha-\alpha_1)} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+n)} \\ =& \frac{\Gamma(\alpha+n-\alpha_1-k)}{\Gamma(\alpha+n)} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha-\alpha_1)}\end{aligned} \tag{2.42}$$

In the limit x much greater than m, $\frac{\Gamma(x-m)}{\Gamma(x)} \simeq x^{-m}$. Therefore one finds

$$\frac{(\alpha-\alpha_1)^{[n-k]}}{\alpha^{[n]}} = \frac{\alpha^{\alpha_1}}{(\alpha+n)^{\alpha_1+k}} \tag{2.43}$$

and multiplying by $\frac{n!}{(n-k)!} \simeq n^k$ one has

$$\begin{aligned}\mathbb{P}(k) \simeq & NegBin(k; \alpha_1, \chi) \\ & \frac{\alpha_1^{[k]}}{k!} \left(\frac{1}{1+\chi}\right)_1^{\alpha} \left(\frac{\chi}{1+\chi}\right)^k \qquad k = 0, 1, 2, ...\end{aligned} \tag{2.44}$$

This distribution is called the *negative binomial distribution.* If $\alpha_1 = 1$ and $\alpha = g$, we find a special case, and it is the *geometric distribution.* If α_1 is an integer, the usual interpretation of the negative binomial random variable is the description of the (discrete) waiting time of the first α_1th success in a binomial process with parameter $p = \frac{1}{(1+\chi)}$, so

$$\mathbb{P}(k) \simeq NegBin(k; \alpha_1, \chi) \simeq Bin(p)$$

[1]We call it the "thermodynamic limit" because it recalls the "thermodynamic limit" in physics, where $n = N \to \infty$ and $\alpha = V \to \infty$ as well, and $\frac{N}{V}$ is constant

The moments of the negative binomial distribution can be obtained from the corresponding moments of the $Polya(n_1, n - n_1; \alpha_1, \alpha - \alpha_1)$ in the limit $n, \alpha \gg 1$, with $\chi = \frac{n}{\alpha}$. We put here some examples that can be useful on Eq.(3.10) and Eq.(3.11):

$$\mathbb{E}(n_1 = k) = n\frac{\alpha_1}{\alpha} \rightarrow \alpha_1\chi \tag{2.45}$$

and

$$\mathbb{V}(n_1 = k) = n\frac{\alpha_1}{\alpha}\frac{\alpha - \alpha_1}{\alpha}\frac{\alpha + n}{\alpha + 1} \rightarrow \alpha_1\chi(1 + \chi) \tag{2.46}$$

where we suppose that both α and n tend to infinity. Note that if α_1 is an integer, k can be interpreted as the sum of α_1 independent and identically distributed variables, and the expected value and the variance are simply the sums.

Chapter 3

An application to stock price dynamics

In this last chapter, we will discuss an application of the previous model firstly to the price increments in a stock market and secondly to the Italian stock exchange between 1973 and 1998. This last chapter is based on [1] and [3].

We consider a stock market with n agents, each labelled from 1 to n, trading a single risky asset, which log-price at each time step t is denoted by the random variable $X(t)$. During each time period, from $t-1$ to t, an agent can decide either to buy, to sell or not to trade.

Definition 3.1. *We define the variable Φ_i that represents the number of shares requested by the* i*-th agent. The variable Φ_i can assume 3 values:*

$$\begin{cases} +1 : \text{bullish } \textit{behaviour} \\ 0 : \text{neutral } \textit{behaviour} \\ -1 : \text{bearish } \textit{behaviour} \end{cases}$$

where:

- bullish *behaviour happens when agent i wants to buy (+1) hoping that the price of the risky assets will increase*
- neutral *behaviour happens when agent i decides not to trade (0), thus not to sell and not to buy*
- bearish *behaviour happens when agent i wants to sell (-1) hoping that the price of the risky assets will decrease*

Definition 3.2. *The aggregate excess demand for the risky asset at time t, is defined by the sum of Φ_i at the time t:*

$$D(t) = \sum_{i=1}^{n} \Phi_i(t) \tag{3.1}$$

Remark 3.1. *Since $\Phi_i = 0$ if the i-th agent chooses a* neutral *behaviour, $D(t)$ becomes*

$$D(t) = n_+ - n_- \tag{3.2}$$

Remark 3.2. *Assuming that the price log-return is proportional to $D(t)$, then we have*

$$\Delta X(t) = X(t+1) - X(t) = \frac{1}{\eta} D(t) = \frac{1}{\eta} \sum_{i=1}^{n} \Phi_i(t) \tag{3.3}$$

where on the last equality we used the Eq.(3.1) and η is defined as the excess demand needed to move the percentage return by one unit.

Without loss of generality, we assume $\eta = 1$, so $\Delta X(t) = \sum_{i=1}^{n} \Phi_i(t)$. If we want to evaluate the distribution of the return, we have to describe the joint distribution of $\{\Phi_i(t)\}_{i=1,..,n}$. As mentioned before, n agents and 3 different strategies (bullish, bearish, neutral) are needed in this model.

Definition 3.3. *The state of the system is written as the vector*

$$\boldsymbol{n}(t) = (n_+(t), n_-(t), n_0(t))$$

called the occupation vector*, which denotes the number of agents who choose to buy, sell, and not trade. Particularly, the agents n_+ and n_- are called* active agents*, and thanks to the occupation vector $\boldsymbol{n}(t)$ can be treated as a random variable.*

The excess demand $D(t)$ in Eq.(3.2), has now a new interpretation as the difference between those who want to sell and those who want to buy

$$D(t) = n_+(t) - n_-(t)$$

so the effective demand (3.1) is a function of the occupation vector $\mathbf{n}(t)$ defined in Definition 3.3. To determine the transition probabilities of a Markov chain is necessary to define the parameters: $\alpha_+, \quad \alpha_-, \quad \alpha_0$,
$\alpha = \alpha_+ + \alpha_- + \alpha_0$ associated with the three strategies.

Remark 3.3. *Let m be the number of active agents, i.e. the agents who choose a* bullish *behaviour or a* bearish *behaviour, then the percentage of the agents who decide to trade (either sell or buy) is the ratio between m and the total number of agents, n:*

$$P = \frac{m}{n} = \frac{n_+ + n_-}{n} \tag{3.4}$$

For positive α's each chosen agent tends to join the majority(this is known as *herding behaviour*). For negative weights ($\alpha < 0$) the selected agent tends to behave at odds with the actual majority's behaviour (known as *contrarian behaviour*). For very large weights ($|\alpha| \to \infty$) agents are not influenced by their environment. If the probabilistic dynamics follows (2.22) and (2.23), (2.24) and (2.27) the equilibrium distribution is the *3-dim-Polya* ($i \in \{+, -, 0\}$):

$$\pi(\mathbf{n}) = \pi(n_+, n_-, n_0) = \frac{n!}{\theta^{[n]}} \prod_i \frac{\alpha_i^{[n_i]}}{n_i!} \tag{3.5}$$

with the constraint $n = n_+ + n_- + n_0$, n the number of agents.

Definition 3.4. *With the hypothesis*

$$\alpha_0 \to \infty, \ n \to \infty,$$

the "thermodynamic limit" *is defined as the constant χ:*

$$\chi = \frac{n}{\alpha} \tag{3.6}$$

So the Polya distribution in the Eq.(3.5) factories:

$$\pi(n_+, n_-, n_0) \to \mathbb{P}(n_+)\mathbb{P}(n_-) \tag{3.7}$$

where

$$\mathbb{P}(n_+) = \frac{\alpha_+^{[n_+]}}{n_+!} \left(\frac{1}{1+\chi}\right)^{\alpha_+} \left(\frac{\chi}{1+\chi}\right)^{n_+} \sim NegBin(\alpha_+, \chi) \tag{3.8}$$

and

$$\mathbb{P}(n_-) = \frac{\alpha_-^{[n_-]}}{n_-!} \left(\frac{1}{1+\chi}\right)^{\alpha_-} \left(\frac{\chi}{1+\chi}\right)^{n_-} \sim NegBin(\alpha_-, \chi) \tag{3.9}$$

Remark 3.4. *The thermodynamic limit corresponds to increasing the number of agents and the initial propensity to be "neutral", conserving the average number of "bulls" and "bears". So, in formulas, using the Eq.(3.6):*

$$\mathbb{E}(n_+) = (n_+ + n_- + n_0)\frac{\alpha_+}{\alpha_+ + \alpha_- + \alpha_0} = n\frac{\alpha_+}{\alpha} \to \alpha_+\chi$$

$$\mathbb{E}(n_-) = (n_+ + n_- + n_0)\frac{\alpha_-}{\alpha_+ + \alpha_- + \alpha_0} = n\frac{\alpha_-}{\alpha} \to \alpha_-\chi$$

that is surrounded by a "reservoir" of neutral agents, that can provide new active agents, or absorb them. In order to get the moments for the equilibrium distribution of the excess demand note that:

$$\mathbb{E}(n_+) = \alpha_+\chi \tag{3.10}$$

$$\mathbb{V}(n_+) = \alpha_+\chi(1+\chi) \tag{3.11}$$

$$\mathbb{K}^*(n_+) = \frac{1}{\alpha_+}\left(6 + \frac{1}{\chi(1+\chi)}\right) \tag{3.12}$$

where $\mathbb{K}^*$ is the kurtosis for the negative binomial distribution, which is large for α_+ small. The same formulas hold true for n_-, as long as we replace α_- with α_+.

Proposition 3.1. *Using the Eq.(3.2) on the second passage:*

$$\mathbb{E}(\Delta X) = \mathbb{E}(n_+ - n_-) = (\alpha_+ - \alpha_-)\chi \tag{3.13}$$

Given the independence of n_+ and n_- we obtain:

$$\mathbb{V}(\Delta X) = \mathbb{V}(n_+ - n_-) = (\alpha_+ + \alpha_-)\chi(1+\chi) \tag{3.14}$$

and the kurtosis:

$$\begin{aligned}\mathbb{K}^*(\Delta X) =& \frac{\mathbb{V}(n_+)^2\mathbb{K}(n_+) + \mathbb{V}(n_-)^2\mathbb{K}(n_-)}{\big(\mathbb{V}(n_+) + \mathbb{V}(n_-)\big)^2} \\ =& \frac{1}{\alpha_+ + \alpha_-}\left(6 + \frac{1}{\chi(1+\chi)}\right)\end{aligned} \tag{3.15}$$

The Equations (3.13), (3.14) and (3.15) are proven at the end of this chapter, in Section 3.2. In general, assuming that the skewness of the empirical distribution is negligible, 3 equations connect the moments of the excess demand distribution to the three parameters α_+, α_- and χ, that specify the model. The parameters α_+, α_- and χ can now be estimated from the mean (3.13), the standard deviation (3.14) and the *excess* kurtosis (3.15) of the empirical distribution of the data. Moreover, the *intensity of the market* $\frac{m}{n}$, namely the fraction of active agents (ratio in the Eq.(3.4)) can be determined from the daily autocorrelation.

3.1 The Italian stock exchange between 1973 and 1998

In this section, we analyze some real data from the Italian stock exchange between June 1973 and April 1998. The data are taken from [1] and [3]. In the following table we reported the mean $\widehat{\mu}$, the standard deviation $\widehat{\sigma}$ and the kurtosis $\widehat{\kappa}$ for the Italian Stock Exchange as well as daily (d), weekly (w) and monthly (m) autocorrelation estimates, from June 1973 to April 1998, divided into a further 6 subperiods:

- I, III, V: normal periods
- II, IV, VI: transition periods

Definition 3.5. *We call* normal periods *the periods characterized by bounded variation of prices, in other words they correspond to long stable motions. We call* transition periods *the ones characterized by strong movements of stock prices, corresponding to short transitions to a new band.*

	I	II	III	IV	V	VI
	6.73-8.80	8.80-6.81	6.81-5.85	5.85-5.86	5.86-6.97	6.97-4.98
n.of obs	1870	206	1028	266	2892	187
$\widehat{\mu}$	-0.03	0.43	0.00	0.43	0.00	0.25
$\widehat{\sigma}$	1.50	3.81	2.30	1.98	1.50	2.26
$\widehat{\kappa}$	6.85	5.98	13.48	6.22	14.31	9.27
d autoc	0.14	0.09	0.17	0.06	0.15	-0.08
w autoc	0.02	-0.08	-0.01	-0.05	0.00	-0.03
m autoc	0.02	0.21	0.03	-0.10	-0.03	-0.08

Table 3.1: Italian stock exchange of the daily percentage returns for the market index from June 1973 to April 1998

In this second table, the estimated values for the parameters α_+, α_-, χ and $f := \frac{m}{n}$, are shown. We recall being:

- α_+, α_- : transitions probability of a Markov chain
- $\chi = \frac{n}{\alpha_+ + \alpha_- + \alpha_0}$ constant
- $f := \frac{m}{n}$ is the ratio of active agents

Period	Observed				Estimated			
	$\widehat{\mu}$	$\widehat{\sigma}$	$\widehat{\kappa}$	d	α_+	α_-	χ	f
I	-0.03	1.50	6.85	0.14	0.47	0.50	0.84	0.92
II	0.43	3.81	5.98	0.09	0.67	0.38	1.47	0.96
III	0.00	2.30	13.48	0.17	0.23	0.23	1.79	0.93
IV	0.43	1.98	6.22	0.06	0.75	0.30	0.96	0.97
V	0.00	1.50	14.31	0.15	0.22	0.22	1.41	0.93
VI	0.25	2.26	9.27	-0.08	0.43	0.25	1.39	1.03

Table 3.2: Parameter estimates for the model described, based on the data from (3.1)

Using the estimated values of α_+, α_- and χ, one gets the exact equilibrium distribution for *bulls* and *bears*. The exact equilibrium distribution of returns is the distribution of $\Delta X = n_+ + n_-$ where n_+ and n_- are independent random variables distributed according to Equations (3.8) and (3.9). This distribution does not depend on m, or the intensity $f = \frac{m}{n}$. However, the latter quantities are essential for the time evolution of the process. The parameter m can be derived from the daily autocorrelation (d) in the six periods. The rate of convergence to equilibrium does depend on f and on $\chi = \frac{n}{\alpha}$, where we recall that $\alpha = \alpha_+ + \alpha_- + \alpha_0$. So, from the Eq.(2.34) of the rate of convergence, in the present case, we can write:

$$r = \frac{f}{1 + \chi(1 - f)} \tag{3.16}$$

leading to

$$f = \frac{r + r\chi}{1 + r\chi} \tag{3.17}$$

The rate is estimated from the empirical value of the daily autocorrelation, based on Bravais-Pearson autocorrelation, one can write $r = 1 - d$ and obtain an estimate value of f. The weekly and monthly autocorrelations of the daily returns can be estimated from Eq.(2.37) for $s = 5$ and $s = 20$ (where $s = 5$ are the working days in a week and $s = 20$ are the working days in a month), and then we can compare the results to the last rows of the first Table.

Definition 3.6. *The* Bravais-Pearson autocorrelation coefficient *between 2 random variables is defined as*

$$\rho_{X,Y} = \frac{\mathbb{C}(X, Y)}{\sigma_X \sigma_Y} \tag{3.18}$$

where σ_X and σ_Y are the standard deviation of X and Y respectively.

Remark 3.5. *Some proprieties of the Bravais-Pearson autocorrelation coefficient are:*

- *The Bravais-Pearson autocorrelation coefficient is symmetric:*

$$\rho_{X,Y} = \frac{\mathbb{C}(X,Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{C}(Y,X)}{\sigma_Y \sigma_X} = \rho_{Y,X}$$

- *If $\rho_{X,Y} = -1$ the random variables X and Y are called* perfectly anti-correlated
- *If $\rho_{X,Y} = 1$ the random variables X and Y are called* perfectly correlated
- *If $\rho_{X,Y} = 0$ the random variables X and Y are called* uncorrelated*, meaning that $\mathbb{C}(X,Y) = 0$ and that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$*

Proposition 3.2. *The Bravais-Pearson autocorrelation coefficient is the ratio between the covariance of two variables and the product of their standard deviations; thus, it is essentially a normalized measurement of the covariance, such that the result always has a value between -1 and 1.*

Proof. Let W and Z be 2 random variables with moment 2. Then

$$\mathbb{E}(XY)^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$$

and can be proved by considering the positive mean

$$0 \leq \mathbb{E}[(\theta X + Y)^2] \qquad \forall \theta \in \mathbb{R}$$

expanding the mean value

$$0 \leq \theta^2 \mathbb{E}(X^2) + 2\theta \mathbb{E}(XY) + \mathbb{E}(Y^2) \qquad \forall \theta \in \mathbb{R}$$

and solving for the variable θ

$$\frac{\Delta}{4} = \mathbb{E}(XY)^2 - \mathbb{E}(X^2)\mathbb{E}(Y^2) \leq 0$$

Let's now define 2 new variables, W and Z as

$$\begin{cases} W = X - \mathbb{E}(X) \\ Z = Y - \mathbb{E}(Y) \end{cases}$$

Replacing W and Z in the equation $\mathbb{E}(WZ)^2 \leq \mathbb{E}(W^2)\mathbb{E}(Z^2)$, we obtain

$$\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y)))^2 \leq \mathbb{E}((X-\mathbb{E}(X))^2)\mathbb{E}((Y-\mathbb{E}(Y))^2)$$

where $\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y)))^2 = \mathbb{C}(X,Y)^2$
and $\mathbb{E}((X-\mathbb{E}(X))^2)\mathbb{E}((Z-\mathbb{E}(Z))^2) = \sigma_X^2\sigma_Y^2$
so

$$|\mathbb{C}(X,Y)| \leq \sigma_X\sigma_Y$$

and for the definition

$$\rho_{X,Y} = \frac{\mathbb{C}(X,Y)}{\sigma_X\sigma_Y}$$

so we have proven

$$|\rho_{X,Y}| \leq 1 \quad \longleftrightarrow \quad -1 \leq \rho_{X,Y} \leq 1$$

□

3.1.1 Excess kurtosis for the sum of independent random variables

Definition 3.7. *The kurtosis of a random variable X is defined as*

$$\mathbb{K}(X) = \frac{\mathbb{E}[(X-\mathbb{E}(X))^4]}{\mathbb{V}^2(X)}$$

Let's define a normally distributed random variable Y. Then the probability density for Y is:

$$p(x) = N(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Definition 3.8. *The kurtosis for a normally distributed random variable is given by:*

$$\mathbb{K}_N(X) = 3$$

Remark 3.6. *The value $\mathbb{K}_N(X) = 3$ is taken as reference for the excess kurtosis, defined as:*

$$\mathbb{K}^*(X) = \mathbb{K}(X) - 3$$

Definition 3.9. *We call a random variable* leptokurtic *when it has $\mathbb{K}^*(X) > 0$, such as a positive excess kurtosis. We call a random variable* mesokurtic *when it has $\mathbb{K}^*(X) = 0$, such as excess kurtosis equals zero. We call a random variable* platykurtic *when it has $\mathbb{K}^*(X) < 0$, such as a negative excess kurtosis*

Remark 3.7. *The tails of a leptokurtic random variable are heavier than the tails of a normal distribution, meaning that extreme events are more likely to occur than in the normal case.*

In the following we now prove Proposition 3.1.

Proof. For simplicity, let's rename: $\alpha_+ = a$ and $\alpha_- = b$.

Proof of the Eq.(3.13):

$$\mathbb{E}(\Delta X) = \mathbb{E}(n_+ - n_-) = (\alpha_+ - \alpha_-)\chi$$

Recalling the Equation (3.10): $\mathbb{E}(n_+) = a\chi$, we have, doing the same procedure, $\mathbb{E}(n_-) = b\chi$ So, for the linearity of the mean:

$$\mathbb{E}(\Delta X) = \mathbb{E}(n_+ - n_-) = \mathbb{E}(n_+) - \mathbb{E}(n_-) = a\chi + b\chi = (a+b)\chi$$

Proof of the Eq.(3.14) :

$$\mathbb{V}(\Delta X) = \mathbb{V}(n_+ - n_-) = (a+b)\chi(1+\chi)$$

We recall the Eq.(3.11): $\mathbb{V}(n_+) = \alpha_+\chi(1+\chi)$

$$\begin{aligned}
\mathbb{V}(\Delta X) = \mathbb{V}(n_+ - n_-) &= \mathbb{E}\big((n_+ - n_-)^2\big) - \mathbb{E}\big((n_+ - n_-)\big)^2 = \\
&= \mathbb{E}(n_+^2 - 2n_+n_- + n_-^2) - \mathbb{E}\big((n_+ - n_-)\big)^2 \\
&= \mathbb{E}(n_+^2) - 2\mathbb{E}(n_+)\mathbb{E}(n_-) + \mathbb{E}(n_-^2) - \mathbb{E}\big((n_+ - n_-)\big)^2 \\
&= \mathbb{E}(n_+^2) - 2ab\chi^2 + \mathbb{E}(n_-^2) - (a-b)^2\chi^2 \\
&= \mathbb{E}(n_+^2) - \cancel{2ab\chi^2} + \mathbb{E}(n_-^2) - a^2\chi^2 + \cancel{2ab\chi^2} - b^2\chi^2 \\
&= \mathbb{V}(n_+) + \mathbb{E}(n_+)^2 + \mathbb{V}(n_-) + \mathbb{E}(n_-)^2 - a^2\chi^2 - b^2\chi^2 \\
&= a\chi(1+\chi) + \cancel{a^2\chi^2} + b\chi(1+\chi) + \cancel{b^2\chi^2} - \cancel{a^2\chi^2} - \cancel{b^2\chi^2} \\
&= (a+b)\chi(1+\chi)
\end{aligned}$$

Proof of the Equation (3.15) :

$$\mathbb{K}^*(\Delta X) = \frac{1}{a+b}\left(6 + \frac{1}{\chi(1+\chi)}\right)$$

$$
\begin{aligned}
\mathbb{K}^*(\Delta X) = \mathbb{K}^*(n_+ - n_-) &\overset{(*)}{=} \frac{\mathbb{V}(n_+)^2\mathbb{K}(n_+) + \mathbb{V}(n_-)^2\mathbb{K}(n_-)}{\big(\mathbb{V}(n_+) + \mathbb{V}(n_-)\big)^2} = \\
&= \frac{a^{\cancel{2}}\chi^2(1+\chi)^2\frac{1}{\cancel{a}}\big(6+\frac{1}{\chi(1+\chi)}\big) + b^{\cancel{2}}\chi^2(1+\chi)^2\frac{1}{\cancel{b}}\big(6+\frac{1}{\chi(1+\chi)}\big)}{\big(a\chi(1+\chi) + b\chi(1+\chi)\big)^2} \\
&= \frac{6a\chi^2(1+\chi)^2 + a\frac{\chi^{\cancel{2}}(1+\chi)^{\cancel{2}}}{\cancel{\chi(1+\chi)}} + 6b\chi^2(1+\chi)^2 + b\frac{\chi^{\cancel{2}}(1+\chi)^{\cancel{2}}}{\cancel{\chi(1+\chi)}}}{a^2\chi^2(1+\chi)^2 + 2ab\chi^2(1+\chi)^2 + b^2\chi^2(1+\chi)^2} \\
&= \frac{6a\chi^2(1+\chi)^2 + a\chi(1+\chi) + 6b\chi^2(1+\chi)^2 + b\chi(1+\chi)}{\chi^2(1+\chi)^2(a^2+b^2+2ab)} \\
&= \frac{6a\chi^2(1+\chi)^2 + a\chi(1+\chi) + 6b\chi^2(1+\chi)^2 + b\chi(1+\chi)}{\chi^2(1+\chi)^2(a+b)^2} \\
&= \frac{6\cancel{\chi^2(1+\chi)^2}\cancel{(a+b)}}{\cancel{\chi^2(1+\chi)^2}(a+b)^{\cancel{2}}} + \frac{\cancel{(a+b)}\cancel{\chi(1+\chi)}}{\chi^{\cancel{2}}(1+\chi)^{\cancel{2}}(a+b)^{\cancel{2}}} \\
&= \frac{1}{a+b}\left(6 + \frac{1}{\chi(1+\chi)}\right)
\end{aligned}
$$

The equation is now being proved:

$$
\mathbb{K}^*(\Delta X) = \mathbb{K}^*(n_+ - n_-) = \frac{\mathbb{V}(n_+)^2\mathbb{K}(n_+) + \mathbb{V}(n_-)^2\mathbb{K}(n_-)}{\big(\mathbb{V}(n_+) + \mathbb{V}(n_-)\big)^2}
$$

We call $Z = X - Y$, where $Z = \Delta X, X = n_+$ and $Y = n_-$. Without loss of generality, we assume that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$. Then we have

$$
(X-Y)^4 = X^4 - 4X^3Y + 6X^2Y^2 - 4XY^3 + Y^4
$$

and, if we average, the terms $-4X^3Y - 4XY^3$ vanish because $\mathbb{E}(X) = \mathbb{E}(Y) = 0$, (for the linearity of the mean)

$$
\mathbb{E}(4X^3Y) = 4\mathbb{E}(X)\mathbb{E}(Y^3) = 0
$$

and we have

$$
\begin{aligned}
&\mathbb{E}[(X-Y)^4] = \mathbb{E}(X^4) + 6\mathbb{V}(X)\mathbb{V}(Y) + \mathbb{E}(Y^4) \\
\text{and}\quad &\mathbb{V}[(X-Y)^4] = (\mathbb{V}(X) + \mathbb{V}(Y))^2
\end{aligned}
$$

Now, thanks to the definition of kurtosis, we have that

$$
\mathbb{E}(X^4) = \mathbb{K}(X)\mathbb{V}^2(X)
$$

and

$$\mathbb{E}(Y^4) = \mathbb{K}(Y)\mathbb{V}^2(Y)$$

So we have

$$\mathbb{K}(Z) = \frac{\mathbb{V}^2(X)\mathbb{K}(X) + 6\mathbb{V}(X)\mathbb{V}(Y) + \mathbb{V}^2(Y)\mathbb{K}(Y)}{(\mathbb{V}(X) + \mathbb{V}(Y))^2}$$

Recalling the Observation (3.6):

$$\begin{aligned}
\mathbb{K}^*(Z) &= \mathbb{K}(Z) - 3 \\
&= \frac{\mathbb{V}^2(X)\mathbb{K}(X) + 6\mathbb{V}(X)\mathbb{V}(Y) + \mathbb{V}^2(Y)\mathbb{K}(Y)}{(\mathbb{V}(X) + \mathbb{V}(Y))^2} - 3 \\
&= \frac{\mathbb{V}^2(X)(\mathbb{K}(X) - 3) + \mathbb{V}^2(Y)(\mathbb{K}(Y) - 3)}{(\mathbb{V}(X) + \mathbb{V}(Y))^2} \\
&= \frac{\mathbb{V}(X)^2\mathbb{K}^*(X) + \mathbb{V}(Y)^2\mathbb{K}^*(Y)}{\big(\mathbb{V}(X) + \mathbb{V}(Y)\big)^2}
\end{aligned}$$

$\square$

Conclusion

At the outset of this thesis, we introduce a simplified problem known as the Ehrenfest-Brillouin model for 4 urns (refer to Figure 1.1). Subsequently, we extend this model to a more elaborate scenario involving 7 dice being rolled on a table (Subsection 1.1.1). The theory of Polya distributions and Markov chains can be employed in both cases, as the processes exhibit the property of "lack of memory". However, these examples serve as a preliminary taste of the thesis, where a significantly more expansive and intricate model, the Ehrenfest-Brillouin model takes centre stage. This thesis aims to estimate, within a stock market, the variables α_+ and α_-, χ, and f, given the information about daily autocorrelation and mean, variance and kurtosis. Each of these variables represents something within the financial market:

- m represents the number of active agents, and $f = \frac{m}{n}$ represents the ratio of active agents to the total number of agents.

- α_+ and α_- represent the transition probabilities of a Markov chain, respectively, for buying and selling.

- $\chi = \frac{n}{\alpha}$ is a constant, called the "thermodynamic limit" if both n and α tend to infinity

The mean, variance and kurtosis analyzed are $\mathbb{E}(\Delta X)$, $\mathbb{V}(\Delta X)$, $\mathbb{K}(\Delta X)$, where ΔX represents the subtraction between the agents n_+ with a *bullish behaviour* and the agents n_- with a *bearish behaviour*. In our framework, we have formulas (see Proposition 3.1) that relate $\mathbb{E}(\Delta X)$, $\mathbb{V}(\Delta X)$, $\mathbb{K}(\Delta X)$ (observable data), and χ, α_+, α_- (data that we need to estimate). On the other hand, we can estimate $f = \frac{m}{n}$, where m is the total number of active agents, from the daily autocorrelation. The variable f can be estimated starting from the rate of convergence

$$r = \frac{f}{1 + \chi(1 - f)}$$

and by setting $r = 1 - d$ (an estimation based on the Bravais-Pearson autocorrelation). By inverting the formula for r, we can derive a formula for the value of f (see Equation 3.17)

Our models are developed within the theory of Polya distribution and Markov chain particularly in the context of the Ehrenfest-Brillouin model, and remain essential for developing this thesis as they provide the mathematical and statistical framework to be applied in the specific case I am addressing, which is an economic-financial application of the model. However, it is interesting to note that initially the Ehrenfest-Brillouin model was applied in physics. In fact, the equilibrium distribution

$$\pi(\mathbf{n}) = \pi(n_+, n_-, n_0) = \frac{n!}{\theta^{[n]}} \prod_i \frac{\alpha_i^{[n_i]}}{n_i!}$$

can be seen as some multivariate distributions of quantum physics:

- If all $\alpha_i > 0$, the special case of equilibrium distribution for $\alpha_i = 1$ and $\alpha = g$ is the *Bose-Einstein distribution*

- If all $\alpha_i < 0$, the equation is the g-dimensional hypergeometric distribution, and for $\alpha_i = -1$ and $\alpha = -g$ is the *Fermi-Dirac distribution*

- As $|\alpha| \to \infty$, the limit is the multinomial distribution whose symmetric case is known as the *Maxwell-Boltzmann distribution*

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