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Abstract

A quantum mechanical system describing a particle swinging around a certain space might furnish us with useful insights on both the geometry and the topology of the space itself: throughout the thesis explicit examples of this peculiar relation are provided. Particularly, basing the discussion upon a celebrated article of Luis Alvarez-Gaumé, the exact Gauss-Bonnet formula is derived in a supersymmetric quantum mechanical system, representing the motion of a particle in a Riemannian manifold extended by additional Grassmann coordinates.

Lo studio del sistema quantistico che descrive una particella che si muove in un certo spazio può fornire indicazioni utili sia sulla geometria che sulla topologia dello spazio stesso: in questa tesi vengono studiati alcuni esempi di questa caratteristica relazione. In particolare, basandosi su un famoso articolo di Luis Alvarez-Gaumé, è stata derivata esplicitamente la formula di Gauss-Bonnet nel contesto di un sistema quantistico supersimmetrico, il quale descrive il moto di una particella su una varietà Riemanniana, generalizzata mediante l'aggiunta di coordinate di Grassmann.

Contents

1
Thesis overview

Along this thesis, we have studied a quantum mechanical system describing a unit-mass particle moving in a Riemannian manifold without constraints. Remarkably, from this basic physical information, we will be able to derive fundamental properties of the space the particle is living in.
To be more precise, we are going to derive, at first, the invariant Euler character of the manifold. Indeed, it turns out that this invariant characteristic coincides with the trace of a particular operator in the quantum mechanical system which is enumerating the difference between the number of bosonic states and fermionic ones at zero energy. Moreover, since the trace is invariant under continuous and small deformations, it can be computed in a convenient perturbation limit, therefore the evaluation of the corresponding Path Integral will end up precisely with the Gauss-Bonnet formula.
Furthermore, we have delved into the analysis of the Path Integral formulation of Quantum Mechanics and it was laid down the mathematical basis for approaching a broad understanding of the fermionic Path Integral, by the introduction of the Grassmann algebra and its integration theory. These tools were used to compute the above trace, or better, its regularized version, the Witten index.
From a wider perspective, it is worth noticing that this index is strictly connected with the supersymmetry breaking mechanism [10].
The thesis is eventually arranged as follows:

i. In the forthcoming chapter 2, an extensive review of formal definitions is provided, aiming at roughly explaining the space of the quantum mechanical system above presented. De Rham Cohomology and Riemannian geometry are discussed, with a brief introduction to the non-coordinate basis and Vielbeins.

ii. In the following chapter 3, the Path Integral formalization of Quantum Mechanics is introduced, starting from its derivation and then developing ways to tackle these functional integrals. This is done at first heuristically, secondly regarding it as a Gaussian integral and by the stationary phase expansion and finally, mentioning in a nutshell, the Wick rotation and the Generating functions as well.

iii. Next in the fourth chapter, 4, the reader is furnished with a broad introduction to anticommuting variables, by means of the Grassmann approach. Accordingly, a fermionic Path Integral formula is achieved by the usual method of splitting the time variable into chunks and integrating over the resolution of the unit.

iv. In chapter 5, the relevant non-linear sigma model is presented, commencing with the bosonic case and subsequently providing its fermionic extension. The supersymmetry charges Q and $\overline{Q}$ have been computed by the Noether procedure. Consequently, the relations between the differential structure of the manifold and the fermions are made clear, and the $\mathrm{Tr}\,(-1)^F$ operator and its physical and mathematical meaning are described. Furthermore, by means of the methods introduced in the previous chapters, the supertrace integral has been determined and therewith the Euler number is computed. In conclusion, the Gauss-Bonnet formula is obtained through a perturbative limit. Lastly, two examples are furnished, i.e. the 2-sphere S^2 and the torus T^2.

v. Finally, in the appendix A, the variation of the supersymmetric non-linear sigma model action under supersymmetry transformations is calculated, in the rather simple case of zero-curvature.

2

Geometry prelude

2.1 De Rham Cohomology

The main references for this chapter are [3] and [5].

Definition 2.1.1 (Smooth Manifold)**:**
*A **smooth manifold** of dimension m consists of:*

1. *M a Hausdorff topological space,*
2. *a family $\{(U_i, \phi_i)\}_{i\in I}$ where $\{U_i\}_{i\in I}$ is an open cover of M and, given the open subset $V_i \subset \mathbb{R}^m$, the maps $\phi_i : U_i \to V_i \subset \mathbb{R}^m$ are homeomorphisms such that for all the opens with $i, j \in I$ and with non-empty intersection $U_{ij} = U_i \cap U_j \neq \emptyset$, the following map $\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_{ij}) \to \phi_i(U_{ij})$ exists in $\mathcal{C}^\infty(\mathbb{R}^m)$.*

Definition 2.1.2 (Smooth Map)**:**
*Let M, N to be two smooth manifolds, let (U, φ) and (V, ϕ) to be two local charts respectively around $p \in U \subset M$ and $f(p) \in V \subset N$, then $f : M \to N$ is said to be a **smooth map** in $p \in M$ if*

$$\phi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \qquad \textit{is } \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^n)$$

A *curve* in M is a smooth map such that $c_1 : (a_1, b_1) \to M$ mapping t to $\big(x_1^1(t), \dots, x_1^n(t)\big)$ and given $c_2 : (a_2, b_2) \ni t \mapsto \big(x_2^1(t), \dots, x_2^n(t)\big)$ we say that they are equivalent, i.e. $c_1 \sim c_2$ if and only if

$$\frac{dx_1^i(t)}{dt}\Big|_{t=0} = \frac{dx_2^i(t)}{dt}\Big|_{t=0}$$

one might show that the above relation is transitive, symmetric and reflexive, thus it forms an equivalence relation and all the equivalent curves in M are denoted as $[c(t)]$.

Definition 2.1.3 (Tangent Space T_pM)**:**
*Let M to be a smooth manifold and $\forall\, p \in M$ we define the **tangent space T_pM** as the quotient:*

$$T_pM = \{\textit{all the curves } c(t) \textit{ taking values in } p \textit{ at } t = 0\}/\sim$$

where $\sim$ is the previous equivalence relation.

Furthermore, an element of this space could be regarded as a vector $[c(t)] = X \in T_pM$ acting as $X : \mathcal{C}^\infty(M) \to \mathbb{R}$ mapping $f \mapsto \frac{df(c(t))}{dt}_{|t=0} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{dt}_{|t=0}$.
Acting naturally on the basis of the vector space, $\left\langle dx^\mu, \frac{\partial}{\partial x^\nu}\right\rangle = \delta^\mu{}_\nu$, one could derive the dual space, T_p^*M, whose elements are said to be 1-forms on M at p, moreover by linearity one could define a *tensor* t of type (r, s) on $p \in M$ such that $t \in T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}$, furthermore we define:

Definition 2.1.4 (Tensor Field)**:**
*A **tensor field** is a map $t : M \to \bigcup_{p\in M} T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}$ such that at fixed $p \in M$ it's a (r, s) tensor*

like and its components $t^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s} : M \to \mathbb{R}$ are smooth, moreover the space of all tensor fields of type (r,s) is denoted by $\mathcal{T}^r_s M$ such that

$$\mathcal{T}^r_s M = \left\{ t : M \to \bigcup_{p\in M} T_pM^{\otimes r} \otimes T^*_pM^{\otimes s} =: \bigcup_{p\in M} T_p{}^r_s M \right\}$$

It is quite common also the following notation, thus $\mathcal{T}^1_0 = \mathfrak{X}(M)$ is the space of all the vector fields on M, similarly $\mathcal{T}^0_1 = \Omega^1(M)$ is the space of all the 1-forms and $\mathcal{T}^0_0 = \mathcal{C}^\infty(M,\mathbb{R})$ is the space of all the smooth function from M to $\mathbb{R}$.
Moreover, let M, N to be two smooth manifold and given a smooth function $f : M \to N$ exists the *push forward* map f_* defined by composition of the equivalence classes as $f_* : [c(t)] \to [f(c(t))]$ and therefore for $p \in M$ it such that :

$$f_* : T_pM \to T_{f(p)}N$$

Analogously for the dual, the *pullback* map f^* could be defined as $f^* : T^*_{f(p)}N \to T^*_pM$ such that the pairing between a vector $X \in T_pM$ and the 1-forms $\omega \in T^*_{f(p)}N$ reads:

$$\langle f^*\omega, X\rangle = \langle \omega, f_*X\rangle$$

Definition 2.1.5 (Antisymmetrizer $\mathcal{A}$)**:**
Let $\sigma \in S_r$ the symmetric group of the permutations of r elements, then it acts on the components of $\omega \in T_p{}^0_r M$ as $(\sigma\omega)_{\mu_1,\dots,\mu_r} = \omega_{\mu_{\sigma(1)}\cdots\mu_{\sigma(r)}}$ and therefore we define ω ***antisymmetrized*** *as:*

$$\mathcal{A}\omega = \frac{1}{r!}\sum_{\sigma\in S_r} \operatorname{sgn}(\sigma)\,\sigma\omega = \omega_{[\mu_1,\dots,\mu_r]}$$

Definition 2.1.6 (Differentiable r-Form)**:**
Let $\omega \in T_p{}^0_r M$ a $(0,r)$ tensor, then it is said to be a ***differentiable r-form*** *if it is fully antisymmetric such that $\mathcal{A}\omega = \omega$*

Definition 2.1.7 (Exterior Derivative)**:**
The ***exterior derivative*** *is the map $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$ defined as*

$$d\omega(V_1,\dots,V_{r+1}) = \sum_{i=1}^{r+1}(-1)^{i+1}V_i\left[\omega\left(V_1,\dots,\hat{V_i},\dots,V_{r+1}\right)\right] + \sum_{i<j}(-1)^{i+j}\omega\left([V_i,V_j],V_1,\dots,\hat{V}_i,\hat{V}_j,\dots,V_{r+1}\right)$$

Nonetheless, one might shift into a more convenient basis, defining the exterior product, denoted with the wedge $\wedge$, of two differentiable forms $\omega \in \mathcal{A}T_p{}^0_r M$ and $\xi \in \mathcal{A}T_p{}^0_s M$ such that

$$(\omega\wedge\xi)(V_1,\dots,V_{r+s}) = \frac{1}{r!s!}\sum_{\sigma\in S_{r+s}} \operatorname{sgn}(\sigma)\,\omega\left(V_{\sigma(1)},\dots,V_{\sigma(r)}\right)\xi\left(V_{\sigma(r+1)},\dots,V_{\sigma(r+s)}\right)$$

Thus,

$$d\omega = \frac{1}{r!}\frac{\partial\omega_{\mu_1\dots\mu_r}}{\partial x^\nu}dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}$$

and therefore, from that it could be derived the graded Leibniz rule:

$$d(\omega\wedge\xi) = d\omega\wedge\xi + (-1)^r\omega\wedge d\xi$$

One could also show that

$$d^2\omega = d\left(\frac{1}{r!}\frac{\partial\omega_{\mu_1\dots\mu_r}}{\partial x^\alpha}dx^\alpha\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}\right) = \frac{1}{r!}\frac{\partial^2\omega_{\mu_1\dots\mu_r}}{\partial x^\beta\partial x^\alpha}dx^\beta\wedge dx^\alpha\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r} = 0$$

This is due to the symmetry property of the double derivatives, commuting with each other, and the antisymmetry of the basis wedge products.
Moreover, it can be defined a sequence such that if $\dim(M) = m$ we read:

$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \to \ldots \xrightarrow{d_{m-1}} \Omega^m(M) \xrightarrow{d_m} 0 \tag{2.1}$$

the so-called **De Rham complex**, where the exterior derivative is rising the order of each component, next, a r-form $\omega \in \Omega^r(M)$ is said to be *closed* if $d\omega = 0$ and *exact* if $\omega = d\alpha$ with $\alpha \in \Omega^{r-1}(M)$, as a consequence of the previous result, $d^2 = 0$, it is clear that $\operatorname{Im} d_{r-1} \subset \ker d_r$:

Definition 2.1.8 (De Rham Cohomology)**:**
The ***r^{th} De Rham Cohomology group*** *is defined as*

$$H^r_{DR}(M) = \frac{\ker d_r}{\operatorname{Im} d_{r-1}}$$

Furthermore, we define[1]:

Definition 2.1.9 (Euler Characteristic)**:**
Let M smooth compact and oriented manifold of dimension m, then the ***Euler Characteristic $\chi(M)$*** *is such as*

$$\chi(M) = \sum_{r=0}^{m} (-1)^r \dim H^r_{DR}(M) \tag{2.2}$$

2.2 Riemannian Geometry

Definition 2.2.1 (Riemannian Metric g)**:**
Let M to be a smooth manifold, then a ***Riemannian metric g*** *on M is a tensor field of type $(0,2)$ such that satisfies*

1. *$g_p(X,Y) = g_p(Y,X)$ is symmetric for all $p \in M$ and all $X, Y \in T_pM$*
2. *$g_p(X,X) \geq 0$ is positive semi-definite[2] for all $p \in M$, $X \in T_pM$ where $g_p(X,X) = 0 \iff X = 0$.*

Let us recall that the set of all the vector fields in M is denoted with $\mathfrak{X}(M)$, then:

Definition 2.2.2 (Affine Connection)**:**
The ***affine connection ∇*** *is the map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ mapping $(X,Y) \mapsto \nabla_X Y$ such that satisfies:*

1. *$\mathbb{R}$- bilinearity*
2. *f-bilinearity in X for all $f \in \mathcal{C}^\infty(M)$ i.e. $\nabla_{fX} Y = f \nabla_X Y$*
3. *f-Leibniz bilinearity in Y for all $f \in \mathcal{C}^\infty(M)$ i.e. $\nabla_X (fY) = X(f)Y + f\nabla_X Y$*

When it is contracted such as $\nabla_X Y$ we say that it is the covariant derivative *of Y along X.*

[1] This is not an actual definition but rather a series of theorems, avoiding exceeding details, it could be just mentioned that the Euler Characteristic can be defined over the smooth compact and oriented manifold from the Euler-Poincaré theorem which claims that the dimension of the complex, i.e. the number of the r-simplexes in it, equals the dimension of the Simplicial Homology $H^{\text{simp}}_r(M)$ and finally, by De Rham duality it is possible to prove the isomorphism between $H^r_{DR}(M) \simeq H^{\text{simp}}_r(M)$ and therefore we have $\chi(M) := \sum_{r=0}^m (-1)^r \dim C_r(M) = \sum_{r=0}^m (-1)^r \dim H^{\text{simp}}_r(M) = \sum_{r=0}^m (-1)^r \dim H^r_{DR}(M)$

[2] The metric is said to be Psuedo-Riemannian if $g_p(X,Y) = 0$ for all $p \in M$ and if $\forall X \in T_pM$ implies $Y = 0$, $\forall Y \in T_pM$

In local coordinates if we define the Christoffel symbol $\Gamma^{\rho}_{\mu\nu}$ as $\nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu} = \Gamma^{\rho}_{\mu\nu} \frac{\partial}{\partial x^\rho}$ we have that the covariant derivative acts on the components of vector fields as

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^{\nu}_{\mu\rho} X^\rho$$

Next, we might define

Definition 2.2.3 (Torsion Tensor)**:**
Let M a smooth manifold and ∇ an affine connection, then $T : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$ mapping $(X, Y) \mapsto T(X, Y)$ is said to be the ***torsion tensor*** *where*[3]

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

And similarly for the curvature tensor or Riemann tensor:

Definition 2.2.4 (Riemann Tensor)**:**
The ***Riemann tensor*** *is defined by $R : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$ which maps $(X, Y, Z) \mapsto R(X, Y)Z$ as*

$$R(X, Y)Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) Z$$

In local coordinates is written as follow[4]:

$$R^{\mu}{}_{\nu\rho\sigma} = \Gamma^{\mu}{}_{\nu\sigma,\rho} - \Gamma^{\mu}{}_{\nu\rho,\sigma} + \Gamma^{\mu}{}_{\rho\tau}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\sigma\tau}\Gamma^{\tau}{}_{\nu\rho}$$

Example 2.2.0.1:
Let us take the S^2 sphere embedded in $\mathbb{R}^3$ with radius r.
The metric non-vanishing components are $g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2 \sin^2\theta$ therefore we read the non-null Christoffel symbols to be $\Gamma^{\theta}{}_{\phi\phi} = -\cos\theta\sin\theta$ and $\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$ and since the first two indices of the Riemann tensor are antisymmetric then they can only be $\theta\phi$ or vice versa, in a similar manner with the last two indices. Hence the only independent component is $R_{\theta\phi\theta\phi}$ or its permutations:

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\mu} R^{\mu}{}_{\phi\theta\phi} = g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} = r^2 R^{\theta}{}_{\phi\theta\phi} = r^2 \left[\Gamma^{\theta}{}_{\phi\phi,\theta} - \Gamma^{\theta}{}_{\phi\theta,\phi} + \Gamma^{\theta}{}_{\theta\mu}\Gamma^{\mu}{}_{\phi\phi} - \Gamma^{\theta}{}_{\phi\mu}\Gamma^{\mu}{}_{\phi\theta}\right] = \\ &= r^2 \partial_\theta \left(-\cos\theta\sin\theta\right) - 0 + 0 + r^2\cos^2\theta = r^2\sin^2\theta \end{aligned}$$

Example 2.2.0.2:
Let us take the $T^2 = S^1 \times S^1$ torus embedded in $\mathbb{R}^3$ with fixed rays $R > r$, the standard parametrization is for $\theta, \phi \in [0, 2\pi)$ such that $x = (R + r\cos\theta)\cos\phi$, $y = (R + r\cos\theta)\sin\phi$, $z = r\sin\theta$.
The metric non-vanishing components are $g_{\theta\theta} = r^2$ and $g_{\phi\phi} = (R + r\cos\theta)^2$ therefore we read the non-null Christoffel symbols to be $\Gamma^{\theta}{}_{\phi\phi} = \frac{(R+r\cos\theta)\sin\theta}{r}$ and $\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = -\frac{\sin\theta}{R+r\cos\theta}$ and thus, the only independent components are $R_{\theta\phi\theta\phi}$ and its permutations:

$$R_{\theta\phi\theta\phi} = r^2 \left(\frac{R + r\cos\theta}{r}\cos\theta - \sin^2\theta + \sin^2\theta\right) = r\cos\theta\,(R + r\cos\theta)$$

Moreover, one might say that an affine connection is *metric* with respect to the metric tensor g if and only if $\nabla g = 0$, then claiming without any proofs:

Theorem 2.2.1 (Levi-Civita)**:**
Let (M, g) a Riemannian manifold then it exists unique its affine connection ∇ metric with respect to g and with the torsion tensor vanishing, i.e. $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{(\mu\nu)}$

[3]The brackets are Lie Brackets, satisfying bilinearity, antisymmetry and Jacobi.
[4]It is been used the notation such that $,\rho$ means $\frac{\partial}{\partial x^\rho}$

Next let M to be a smooth manifold endowed with an affine connection ∇ and $c:(a,b)\to M$ to be a curve on it, then $X\in\mathfrak{X}(M)$[5] is said to be *parallelly transported* along c if

$$\nabla_{c_*\left(\frac{d}{dt}\right)}X=0$$

Definition 2.2.5 (Geodesic)**:**
A curve $c:(a,b)\to M$ is said to be a ***geodesic*** *with respect to an affine connection ∇ if its tangent vector field $c_*\left(\frac{d}{dt}\right)$ satisfies:*

$$\nabla_{c_*\left(\frac{d}{dt}\right)}c_*\left(\frac{d}{dt}\right)=0 \quad \textit{in local coordinates} \quad \frac{d^2X^\mu}{dt^2}+\Gamma^\mu_{\nu\lambda}\frac{dX^\nu}{dt}\frac{dX^\lambda}{dt}=0 \tag{2.3}$$

It is an *non-linear* ordinary differential equation of the second order, that is why the further model in the chapter 5 is dubbed *non-linear sigma model.*
Furthermore, given a point p in the Riemannian manifold (M,g), it exists an isomorphism between the tangent space T_pM and its dual T_p^*M by means of the following map, for all $X\in T_pM$ then $X\mapsto g(X,\cdot)$ and thus for all $v,w\in T_pM$ exist unique $\alpha=g(v,\cdot)$ and $\beta=g(w,\cdot)\in T_p^*M$, from that it is straightforwardly defined $\langle\alpha,\beta\rangle=g(v,w)$ extended to all the $\mathcal{A}T_p{}^0_rM$ as

$$\langle\alpha_1\wedge\cdots\wedge\alpha_r,\beta_1\wedge\cdots\wedge\beta_r\rangle=\det\left[\langle\alpha_i,\beta_j\rangle_{ij}\right]$$

Moreover let $m=\dim(M)$, then a top-form $\omega\in\Omega^m(M)$ everywhere non-vanishing could be defined as the *volume form* of the manifold, next, denoting with $\mathrm{Vol}\,(g)$ the volume form of the orientable Riemannian manifold (M,g), it is such that:

$$\mathrm{Vol}\,(g)=\sqrt{|\det g|}dx^1\wedge\cdots\wedge dx^m$$

Furthemore, through the metric g is possible to establish an isomorphism between $\Omega^k(M)$ and $\Omega^{m-k}(M)$

Definition 2.2.6 (Hodge Star Operator $*$)**:**
Let $\omega,\eta\in\Omega^k(M)$ then $*\eta\in\Omega^{m-k}(M)$ is the only element such that:

$$\omega\wedge*\eta=\langle\omega,\eta\rangle\,Vol(g) \tag{2.4}$$

Finally, one could define the inner product on the space $\Omega^k(M)$ such that, given $\alpha,\beta\in\Omega^k(M)$:

$$(\alpha,\beta)_M=\int_M\langle\alpha,\beta\rangle\,Vol(g)=\int_M\alpha\wedge*\beta \tag{2.5}$$

And herewith, it could be defined the dual of the exterior derivative with respect to this inner product $d^*:\Omega^p(M)\to\Omega^{p-1}(M)$, in the succeeding way, for all $\alpha\in\Omega^{p-1}(M)$, $\beta\in\Omega^p(M)$:

$$\begin{aligned}(d\alpha,\beta)_M&=\int_M d\alpha\wedge*\beta\overset{\text{graded Leibniz}}{=}\int_M-(-1)^{p-1}\alpha\wedge d*\beta=\\&=(-1)^p\int_M\alpha\wedge*\left(*^{-1}d*\beta\right)=(-1)^p\left(\alpha,*^{-1}d*\beta\right)_M=(\alpha,d^*\beta)_M\end{aligned}$$

Then, by means of the identity $**\alpha=(-1)^{p(m-p)}\mathrm{id}$ and since $*^{-1}d*\beta$ is a $p-1$-form we read

$$d^*=(-1)^p**\left(*^{-1}d*\beta\right)=(-1)^p(-1)^{(p-1)(m-p+1)}*d*\beta \tag{2.6}$$

Definition 2.2.7 (Laplace Operator)**:**
Let M to be a smooth manifold of dimension m, $\Delta:\Omega^k(M)\to\Omega^k(M)$ is said to be the ***Laplace operator*** *for each $0\le k\le m$ by the formula*

$$\Delta=d\circ d^*+d^*\circ d$$

[5]This constraint could be relaxed by requiring that X is just defined in the image of $c(t)$.

Definition 2.2.8 (Harmonic Form)**:**
A form $\omega \in \Omega^k(M)$ is called ***harmonic*** *if*

$$\Delta\omega = 0$$

And we let $\mathrm{Harm}^k(M) = \ker\Delta$ *to be the space of all the harmonic forms.*

Furthermore, it is not difficult to prove after the fundamental theorem of elliptic operators [6] that every De Rham Cohomology class contains a unique harmonic form:

Theorem 2.2.2:
Let (M,g) be a compact and oriented Riemannian manifold, then the natural map

$$\mathrm{Harm}^k(M) \to H^k_{DR}(M,\mathbb{R})$$

is an isomorphism.

2.2.1 Vielbeins formalism

Let M to be a compact oriented Riemannian manifold of dimension m, with a metric form g as above, next we twist the canonical basis by a rotation matrix ${e_a}^\mu \in GL(m,\mathbb{R})$ which preserves the orientation as $\hat{e}_a = {e_a}^\mu \frac{\partial}{\partial x^\mu}$, moreover it has also to satisfy the orthonormality condition with respect to the metric components:

$$g(\hat{e}_a,\hat{e}_b) = {e_a}^\mu {e_b}^\nu g(e_\mu,e_\nu) = {e_a}^\mu {e_b}^\nu g_{\mu\nu} = \delta_{ab}$$

Thus, if we define as the inverse matrix ${e^a}_\mu$ such that ${e^a}_\mu {e_a}^\nu = {\delta_\mu}^\nu$ and vice versa ${e^a}_\mu {e_b}^\mu = {\delta^a}_b$ we can infer the important following relation:

$$g_{\mu\nu} = {e^a}_\mu {e^b}_\nu \delta_{ab} \tag{2.7}$$

The dual basis can be defined via $\left\langle \hat{\theta}^a, \hat{e}_b \right\rangle = {\delta^a}_b$ and therefore $\hat{\theta}^a = {e^a}_\mu dx^\mu$, we read

$$g_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{ab}\left({e^a}_\mu dx^\mu\right) \otimes \left({e^b}_\nu dx^\nu\right) = \delta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$$

Definition 2.2.9 (Vielbeins)**:**
The bases $\{\hat{e}_a\}$ and $\{\hat{\theta}^a\}$ are called the ***non-coordinate bases*** *and the coefficients ${e_a}^\mu$ are called, following the German language, in a one-dimensional space* einbein *and similarly up to 4-dimensions* vierbeins *and with many, i.e. more than four,* ***vielbeins****.*

Example 2.2.2.1:
Recalling the previous examples 2.2.0.1 and 2.2.0.2 of the 2-sphere S^2 and the $T^2 = S^1 \times S^1$ torus, let us compute the Riemann tensor components in the zweibeins basis respectively:

$$R^\theta{}_{\phi\theta\phi} = {e_a}^\theta {e^b}_\phi {e^c}_\theta {e^d}_\phi R^a{}_{bcd} = 1 \cdot \sin\theta \cdot 1 \sin\theta R^1{}_{212}$$

Which has to be $R^\theta{}_{\phi\theta\phi} = \sin^2\theta = \sin^2\theta R^1{}_{212}$ if and only if $R^1{}_{212} = 1$.
Analogously or by symmetry we get $R_{1212} = 1$, $R_{1221} = -1$, $R_{2121} = 1$ and $R_{2112} = -1$ for the 2-sphere S^2.
In a similar manner, we have the torus Riemann tensor components in the zweibeins basis:

$$R^\theta{}_{\phi\theta\phi} = \frac{(R + r\cos\theta)\cos\theta}{r} \quad \textit{and therefore} \quad R_{1212} = \frac{\cos\theta}{r\,(R + r\cos\theta)}.$$

3

Introduction to the Path Integral formalism

3.1 Quantum Mechanics Path Integral

Let us study a quantum mechanical system with n dynamical variables $\{q_1, \ldots, q_n\}$ and conjugate momenta $\{p_1, \ldots, p_n\}$, provided with a Hamiltonian $\hat{H}$ hermitian in the Hilbert space $\mathcal{H}$, describing the whole system.
Assume now this set-up is representing one single particle, then in such an arrangement we are implicitly hinting[1] a space-time formed by a one dimension compact and connected manifold M parameterized by a time coordinate t, e.g. if M is a circle S^1 then we would let $t \in [0, T)$ by identifying $t \simeq t + T$, whereas if it was an interval of length T with respect to the inner product of M we would let $t \in [0, T]$. Thus, if we take a target space N and a map $q : M \to N$, the **worldline** of the particle is said to be $q(M) \subset N$. For instance, for the case of the non-relativistic Quantum Mechanics one could consider $N = \mathbb{R}^n$ with the Euclidean metric δ or, as it will be useful for the following computations, take it to be a Riemannian manifold (N, g). By means of the map q, for each point t on M we have a point $q(t)$ on N, more in general we would cover just a patch of the space, $U \subset N$, and we will consider $q^i = \{q^1, \ldots, q^n\}$ as a local coordinate system.
At this point, let us consider a D-dimensional quantum system living in $\mathcal{H} = L^2(N)$, the space of square-integrable function on $N = \left(\mathbb{R}^D, \delta\right)$, then as usual, if we consider the unitary time evolution operator in natural unit $\hbar = 1$ and in Heisenberg picture $\hat{U}(t_f, t_i) = e^{-i\hat{H}(t_f - t_i)}$, the particle amplitude to travel from an initial position $\vec{q}_i \in N$ to a final $\vec{q}_f \in N$ is given by the **heat kernel**[2]:

$$K(\vec{q}_f, t_f; \vec{q}_i, t_i) = \langle q_f | \hat{U}(t_f, t_i) | q_i \rangle \tag{3.1}$$

3.1.1 Path Integral formula

Once defined the heat kernel as in (3.1) the continuous time variable could be fit in a lattice, i.e. $t_n = t_i + n\Delta t$ where each step would increase of the amount $\Delta t = \frac{t_f - t_i}{N}$ with $t_0 = t_i$ and $t_N = t_f$ and therefore we read:

$$\hat{U}(t_f, t_i) = \left(e^{-i\hat{H}\Delta t}\right)^N$$

Thus, the transition amplitude can be computed between an initial and final coordinate eigenstate, $|q_0\rangle$ and $|q_N\rangle$ respectively, by performing a Fourier transformation to the following amplitude, where $|p_0\rangle$ represents an initial momentum eigenstate, after having inserted two $N-1$ completeness relations $1 = \int d^D\vec{q}\, |q\rangle \langle q|$ and $1 = \int \frac{d^D\vec{p}}{(2\pi)^D} |p\rangle \langle p|$ in the following way, as was done in [9]:

$$\begin{aligned}
\langle q_N | \hat{U}(t_f, t_i) | p_0 \rangle &= \langle q_N | e^{-i\hat{H}\Delta t} \ldots e^{-i\hat{H}\Delta t} | p_0 \rangle = \\
&= \int \prod_{i=1}^{N-1} \frac{d^D\vec{q}_i d^D\vec{p}_i}{(2\pi)^D} \langle q_N | e^{-i\hat{H}\Delta t} | p_{N-1} \rangle \langle p_{N-1} | q_{N-1} \rangle \langle q_{N-1} | e^{-i\hat{H}\Delta t} | p_{N-2} \rangle \ldots \langle q_1 | e^{-i\hat{H}\Delta t} | p_0 \rangle = \\
&= \int \mathcal{D}\Omega \prod_{i=0}^{N-1} \langle q_{i+1} | e^{-i\hat{H}\Delta t} | p_i \rangle \prod_{i=1}^{N-1} \langle p_i | q_i \rangle = \int \mathcal{D}\Omega \prod_{i=0}^{N-1} \langle q_{i+1} | e^{-i\hat{H}\Delta t} | p_i \rangle e^{-i \sum_{k=1}^{N-1} \vec{p_k} \cdot \vec{q_k}} =
\end{aligned}$$

[1] See [7] for more details.

[2] It originally has been considered to solve heat conduction problems given some differential operator $\mathcal{M}$ and some initial conditions: $(\partial_t - \mathcal{M}) K(\vec{x}, t; \vec{x}_0, t_0) = 0$ and $\lim_{t \to 0} K(\vec{x}, t; \vec{x}_0, t_0) = \delta(\vec{x} - \vec{x}_0)$, it is therefore describing the heat distribution over the space at a given time.

Where for simplicity sake it is been defined the measure

$$\int \mathcal{D}\Omega = \int \prod_{i=1}^{N-1} \frac{d^D\vec{q}_i d^D\vec{p}_i}{(2\pi)^D}$$

Consequently, without loss of generality,[3] let us consider the simplest case when $\hat{H} = H(\hat{\vec{q}}, \hat{\vec{p}}) = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{q}})$:

$$\begin{aligned}
&= \int \mathcal{D}\Omega \prod_{i=0}^{N-1} \langle q_{i+1}| \left(1 - i\hat{H}\Delta t + \dots\right) |p_i\rangle \, e^{-i\sum_{k=1}^{N-1} \vec{p}_k\cdot\vec{q}_k} = \\
&= \int \mathcal{D}\Omega \prod_{i=0}^{N-1} \left(1 - i\Delta t \left(\frac{\vec{p}_i^2}{2m} + V\left(\vec{q}_{i+1}\right)\right) + \dots\right) \langle q_{i+1}|p_i\rangle \, e^{-i\sum_{k=1}^{N-1} \vec{p}_k\cdot\vec{q}_k} = \\
&= \int \mathcal{D}\Omega \prod_{i=0}^{N-1} \left(e^{-i\Delta t H(\vec{q}_{i+1}, \vec{p}_i)}\right) e^{i\vec{p}_i\cdot\vec{q}_{i+1}} e^{-i\sum_{k=1}^{N-1} \vec{p}_k\cdot\vec{q}_k}
\end{aligned}$$

Next, let us conclude the whole derivation by a Fourier transformation, having noticed that $\Delta t \to 0$ if and only if $N \to \infty$

$$\begin{aligned}
\langle q_N| \hat{U}(t_f, t_i) |q_0\rangle &= \int \frac{d^D\vec{p}_0}{2\pi} e^{i\vec{q}_0\cdot\vec{p}_0} \langle q_N| \hat{U}(t_f, t_i) |p_0\rangle = \\
&= \int \frac{d^D\vec{p}_0}{2\pi} e^{i\vec{q}_0\cdot\vec{p}_0} \int \mathcal{D}\Omega \prod_{i=0}^{N-1} \left(e^{-i\Delta t H(\vec{q}_{i+1}, \vec{p}_i)}\right) e^{i\vec{p}_i\cdot\vec{q}_{i+1}} e^{-i\sum_{k=1}^{N-1} \vec{p}_k\cdot\vec{q}_k} = \\
&= \int \frac{d^D\vec{p}_0}{2\pi} \int \mathcal{D}\Omega e^{i\sum_{i=0}^{N-1} \vec{p}_i\cdot(\vec{q}_{i+1} - \vec{q}_i) - \Delta t H(\vec{q}_{i+1}, \vec{p}_i)} = \\
&= \int \frac{d^D\vec{p}_0}{2\pi} \int \mathcal{D}\Omega e^{i\Delta t\sum_{i=0}^{N-1} \vec{p}_i\cdot\frac{(\vec{q}_{i+1} - \vec{q}_i)}{\Delta t} - H(\vec{q}_{i+1}, \vec{p}_i)}
\end{aligned} \tag{3.2}$$

the exponent might be interpreted as the discretized action which in the continuous limit for $N \to \infty$ (and therefore in the $\Delta t \to 0$ limit) and after having set $\vec{q}(t)$ such that $\vec{q}(t_n) = \vec{q}_n$ (and analogously $\vec{p}(t)$ such that $\vec{p}(t_n) = \vec{p}_n$). It tends to:

$$\Delta t \sum_{i=0}^{N-1} \vec{p}_i \cdot \frac{\left(\vec{q}_{i+1} - \vec{q}_i\right)}{\Delta t} - H(\vec{q}_{i+1}, \vec{p}_i) = \Delta t \sum_{i=0}^{N-1} \vec{p}_i \cdot \frac{d\vec{q}_{i+1}}{dt} - H(\vec{q}_{i+1}, \vec{p}_i) \longrightarrow S[\vec{q}, \vec{p}] = \int_{t_0}^{t_N} dt L(\vec{q}, \vec{p})$$

Therefore if we define the integral measures

$$\int \mathcal{D}\vec{q} = \lim_{N\to\infty} \int \prod_{i=1}^{N-1} d^D\vec{q}_i \quad \text{and} \quad \int \mathcal{D}\vec{p} = \lim_{N\to\infty} \int \prod_{i=0}^{N-1} \frac{d^D\vec{p}_i}{(2\pi)^D} \tag{3.3}$$

we end up with:

$$K(\vec{q}_f, t_f; \vec{q}_i, t_i) = \int_{t_i}^{t_f} \mathcal{D}\vec{q} \int \mathcal{D}\vec{p} e^{iS[\vec{q}, \vec{p}]} \tag{3.4}$$

In this way the reader can interpret the transition amplitude

$$K(\vec{q}_f, t_f; \vec{q}_i, t_i) = \sum_{\text{Histories } \gamma} e^{iS[\gamma]}$$

as a sum over all the possible histories i.e. a functional integral over all the phase space functions with some boundary conditions which sort out the histories that are going to be added up in the sum.

[3] Actually this is an approximation which could be evaluated by the Campbell-Baker-Haussdorf formula as a series of homogeneous polynomials in the potential V and its derivatives and by following [4] it can be shown its convergence to zero.

3.2 Solving techniques for the Path Integral

3.2.1 Path Integral in Configuration Space

The Path Integral, as prescribed above, is the most general expression. One could however re-arrange the exponent in 3.2 in such a way that, if the Hamiltonian is quadratic in the momenta, for instance as before, $H(\vec{q}_{i+1}, \vec{p}_i) = \frac{\vec{p}_i^2}{2m} + V(\vec{q}_{i+1})$, it would then give us a standard Gaussian integral:

$$\begin{aligned}
K(\vec{q}_f, t_f; \vec{q}_i, t_i) &= \int \frac{d^D\vec{p}_0}{2\pi} \int \mathcal{D}\Omega e^{i\Delta t \sum_{i=0}^{N-1} \vec{p}_i \cdot \frac{(\vec{q}_{i+1}-\vec{q}_i)}{\Delta t} - \frac{\vec{p}_i^2}{2m} - V(\vec{q}_{i+1})} = \\
&= \int \left(\prod_{i=1}^{N-1} d^D\vec{q}_i \right) e^{-i\Delta t \sum_{i=0}^{N-1} V(\vec{q}_{i+1})} \int \left(\prod_{i=0}^{N-1} \frac{d^D\vec{p}_i}{(2\pi)^D} \right) e^{-i\frac{\Delta t}{2m} \sum_{i=0}^{N-1} \vec{p}_i^2 - 2m\vec{p}_i \frac{(\vec{q}_{i+1}-\vec{q}_i)}{\Delta t}} = \\
&= \int \left(\prod_{i=1}^{N-1} d^D\vec{q}_i \right) e^{-i\Delta t \sum_{i=0}^{N-1} V(\vec{q}_{i+1}) - \frac{m}{2}\left(\frac{(\vec{q}_{i+1}-\vec{q}_i)}{\Delta t}\right)^2} \times \\
&\times \int \left(\prod_{i=0}^{N-1} \frac{d^D\vec{p}_i}{(2\pi)^D} \right) e^{-i\frac{\Delta t}{2m} \sum_{i=0}^{N-1} \left(\vec{p}_i - \frac{m(\vec{q}_{i+1}-\vec{q}_i)}{\Delta t}\right)^2} = \\
&= \int \left(\prod_{i=1}^{N-1} d^D\vec{q}_i \right) e^{-i\Delta t \sum_{i=0}^{N-1} V(\vec{q}_{i+1}) - \frac{m}{2}\left(\frac{(\vec{q}_{i+1}-\vec{q}_i)}{\Delta t}\right)^2} \prod_{n=0}^{N-1} \frac{1}{(2\pi)^D} \left(\frac{2m\pi}{i\Delta t}\right)^{\frac{D}{2}}
\end{aligned}$$

for $N \to +\infty$ and by recognizing the action in configuration space $S[q] = \int_{t_i}^{t_f} L(q, \dot{q})$ and naming the overall constant $C = \lim_{N\to\infty} \prod_{n=0}^{N-1} \left(\frac{m}{2\pi i \Delta t}\right)^{\frac{D}{2}}$, clearly diverging, which could be set out by normalization, we read:

$$K(\vec{q}_f, t_f; \vec{q}_i, t_i) = C \int_{\vec{q}_i}^{\vec{q}_f} \mathcal{D}\vec{q} e^{iS[\vec{q}]} \tag{3.5}$$

3.2.2 Stationary Phase approximation

Trying to work out the configuration space heat kernel $K(\vec{q}_f, t_f; \vec{q}_i, t_i)$, we might grasp its behaviour by means of the following considerations, first let us re-introduce the $\hbar$ in the exponent argument as $\frac{i}{\hbar}S[\vec{q}]$.
Thus, since $\hbar \ll 1$, the phase which is given by the action, will oscillate wildly running over all the paths $\vec{q}_i(t)$.
There will be a stationary phase of constructive interference around the peculiar path $\vec{q}_*(t)$, which is defined under the constraint $\delta S = 0$. It demands that the differences of the phases $\delta\vec{q}_*(t)$ of the paths close to $\vec{q}_*(t)$ are vanishing.
Vice versa, away from this condition, the several paths will destructively interfere.
In this way, we could explain the principle of least action which implies all the classical physics, indeed the path $\vec{q}_*(t) \equiv \vec{q}_{\rm cl}(t)$ is regarded as the constructive interference of all the neighborhood trajectories.
Hence, by considering the expansion around the stationary phase path $\vec{q}(t) = \vec{q}_{\rm cl}(t) + \vec{y}(t)$ with boundary conditions $\vec{y}(t_i) = \vec{y}(t_f) = 0$, up to the second order we read:

$$S[\vec{q}_f, t_f; \vec{q}_i, t_i] = S_{|\vec{q}_{\rm cl}} + \delta S_{|\vec{q}_{\rm cl}} + \frac{1}{2}\delta^2 S_{|\vec{q}_{\rm cl}} + \dots = S_{|\vec{q}_{\rm cl}} + \frac{1}{2}\delta^2 S_{|\vec{q}_{\rm cl}} + \dots$$

Thus:

$$K(\vec{q}_f, t_f; \vec{q}_i, t_i) = C \int_{\vec{q}_i}^{\vec{q}_f} \mathcal{D}\vec{q} e^{\frac{i}{\hbar}\left(S_{|\vec{q}_{\rm cl}} + \frac{1}{2}\delta^2 S_{|\vec{q}_{\rm cl}} + \dots\right)} = C e^{\frac{i}{\hbar}S[\vec{q}_{\rm cl}(t_f), \vec{q}_{\rm cl}(t_i)]} \int_{\vec{y}_i=0}^{\vec{y}_f=0} \mathcal{D}\vec{y} e^{\frac{i}{\hbar}\left(\frac{1}{2}\delta^2 S_{|\vec{q}_{\rm cl}} + \dots\right)} \tag{3.6}$$

For instance, given the Lagrangian of the one dimensional harmonic oscillator $L = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2}$ which by the classical equation of motion $\ddot{q}_{cl} = -\omega^2 q_{cl}$ and without any need of a Taylor expansion:

$S[q(t)] = S[q_{cl}(t)] + S[y(t)] + \int_{t_i}^{t_f} dt \frac{d}{dt}(\dot{q}_{cl} y) = S[q_{cl}(t)] + S[y(t)]$ where in the last step the latter term vanishes because of the boundary conditions on the fluctuations, thus:

$$K(q_f, t_f; q_i, t_i) = C e^{iS[q_{cl}]} \int_{y_i=0}^{y_f=0} \mathcal{D}y e^{i\int_{t_i}^{t_f} dt \left(\frac{m\dot{y}^2}{2} - \frac{m\omega^2 y^2}{2}\right)}$$

One could compute $e^{iS[q_{cl}]}$ by means of the classical solution $q_{cl}(t) = q_i \cos\left[\omega\left(t - t_i\right)\right] + B \sin\left[\omega\left(t - t_i\right)\right]$, whereas for the remaining integral we write the Fourier series expansion with $\tau = t_f - t_i$ of the fluctuations:

$$y = \sum_{p=1}^{\infty} \tilde{y}_p \sqrt{\frac{2}{\tau}} \sin\left(\frac{p\pi\left(t - t_i\right)}{\tau}\right) \tag{3.7}$$

The action is therefore written through its truncated Fourier series and by the time lattice of before $t_n = t_i + n\Delta t$ we recognize that $y_0 = y_N = 0$ are the boundary conditions

$$S[y] = \int_{t_i}^{t_f} dt \left(\frac{m\dot{y}^2}{2} - \frac{m\omega^2 y^2}{2}\right) = \sum_{p=1}^{N-1} \tilde{y}_p^2 \int_{t_i}^{t_f} dt \frac{m}{\tau}\left(\frac{p\pi}{\tau}\right)^2 \cos^2\left(\frac{p\pi}{\tau}(t - t_i)\right) - \frac{m\omega^2}{\tau} \sin^2\left(\frac{p\pi}{\tau}(t - t_i)\right) =$$

$$= \sum_{p=1}^{N-1} \tilde{y}_p^2 \frac{m}{\tau}\left(\frac{p\pi}{\tau}\right)^2 \left[\frac{\tau}{2} + \sin\left(\frac{2p\pi}{\tau}\tau\right)\frac{\tau}{4p\pi}\right] - \frac{m\omega^2}{\tau}\left[\frac{\tau}{2} - \sin\left(\frac{2p\pi}{\tau}(\tau)\right)\frac{\tau}{4p\pi}\right] = \sum_{p=1}^{N-1} \tilde{y}_p^2 \frac{m}{2}\left[\left(\frac{p\pi}{\tau}\right)^2 - \omega^2\right]$$

The Path Integral will be henceforth a Gaussian integral over the functional measure of the non-constant modes $\int_{y_i=0}^{y_f=0} \mathcal{D}y = \lim_{N\to\infty} \det\left(\frac{\partial y_n}{\partial \tilde{y}_p}\right) \prod_{p=1}^{N-1} \int d\tilde{y}_p$ i.e.

$$K(q_f, t_f; q_i, t_i) = C e^{iS[q_{cl}]} \det\left(\frac{\partial y_n}{\partial \tilde{y}_p}\right) \prod_{p=1}^{N-1} \int d\tilde{y}_p e^{i\tilde{y}_p^2 \frac{m}{2}\left[\left(\frac{p\pi}{\tau}\right)^2 - \omega^2\right]} = C e^{iS[q_{cl}]} \det\left(\frac{\partial y_n}{\partial \tilde{y}_p}\right) \prod_{p=1}^{N-1} \sqrt{\frac{2\pi i}{m\left[\left(\frac{p\pi}{\tau}\right)^2 - \omega^2\right]}} =$$

$$= C e^{iS[q_{cl}]} \det\left(\frac{\partial y_n}{\partial \tilde{y}_p}\right) \left(\frac{2i\tau^2}{m\pi}\right)^{\frac{N-1}{2}} \frac{1}{(N-1)!} \prod_{p=1}^{N-1} \frac{1}{\sqrt{1 - \left(\frac{\omega\tau}{p\pi}\right)^2}} =$$

$$= C' e^{iS[q_{cl}]} \sqrt{\frac{m}{2\pi i\tau}} \prod_{p=1}^{N-1} \frac{1}{\sqrt{1 - \left(\frac{\omega\tau}{p\pi}\right)^2}} = C'' e^{iS[q_{cl}]} \sqrt{\frac{m\omega}{2\pi i \sin\left(\omega\tau\right)}}$$

Where the identity $\prod_{p=1}^{\infty} 1 - \left(\frac{\omega\tau}{p\pi}\right)^2 = \frac{\sin(\omega\tau)}{\omega\tau}$ has been used.
We have assembled the former constant C' in such a way that the remaining c-number is the Path Integral of the action given $V = 0$, i.e. the free field and the latter constant such that it could be set to $C'' = 1$ since the heat kernel at $\tau = 0$ gives exactly the above delta function.
In order to compute such free field integral, one would have to evaluate the following integral, first solving it by parts then recognizing the Gaussian integral in the succeeding form, given M a $D \times D$ real symmetric matrix:

$$\int_{\mathbb{R}^d} d^D x e^{-x^t M x} = \pi^{\frac{D}{2}} \det\left(M\right)^{-\frac{1}{2}} \tag{3.8}$$

Thus, in our case:

$$\int_{y_i=0}^{y_f=0} \mathcal{D}y e^{i\frac{m}{2}\int_{t_i}^{t_f} dt \dot{y}^2} = \int_{y_i=0}^{y_f=0} \mathcal{D}y e^{i\frac{m}{2}\int_{t_i}^{t_f} dt y\left(-\frac{d^2}{dt^2}\right)y} = \sqrt{\frac{m}{\pi i}} \det\left[-\frac{d^2}{dt^2}\right]^{-\frac{1}{2}}$$

Here it comes a subtlety, i.e. the functional determinant, to tackle it we have first to study its eigenvalues $\det\left[-\frac{d^2}{dt^2}\right] = \prod_{n=1}^{+\infty} \lambda_n$ with:

$$-\frac{d^2}{dt^2} y_n(t) = \lambda_n y_n(t) \quad \text{such that } y_n(t_i) = y_n(t_f) = 0 \quad \Longleftrightarrow$$

$$y_n(t) = \sin\left(\frac{n\pi(t - t_i)}{\tau}\right) \quad \Longleftrightarrow \quad \lambda = \left(\frac{n\pi}{\tau}\right)^2$$

Moreover, we have the succeeding formal equivalence, given a positive-definite operator $\hat{O}$, from the infinitesimal expansion of the determinant $\det\left(1+\epsilon\hat{O}\right) = 1+\epsilon\,\mathrm{Tr}\left(\hat{O}\right)+\mathcal{O}(\epsilon^2)$ one could show that:

$$\det\hat{O} = e^{\mathrm{Tr}\left(\ln\hat{O}\right)}. \quad \text{Hence, we deduce:} \quad \ln\left[\det\hat{O}\right] = \mathrm{Tr}\left(\ln\hat{O}\right) = \sum_{n=1}^{+\infty}\ln\lambda_n \tag{3.9}$$

Definition 3.2.1 (Spectral ζ-Function)**:**
*Given $\hat{O}$ positive-definite operator, the **spectral ζ-function** is defined as*

$$\zeta_{\hat{O}}(s) = \sum_{s=1}^{+\infty}\frac{1}{\lambda_n^s} < \infty \qquad \textit{for } \mathrm{Re}\ s \gg 1$$

It is analytic with respect to $s \gg 1$ and such that $\frac{d}{ds}\zeta_{\hat{O}}(s)\Big|_{s=0} = -\sum_{n=1}^{+\infty}\ln\lambda_n$

Hence, we deduce:

$$\det\hat{O} = e^{-\frac{d}{ds}\zeta_{\hat{O}}(s)|_{s=0}}$$

Particularly for our case, recalling that the Riemann-ζ-function is defined as $\zeta(s) = \sum_{n=0}^{\infty}\frac{1}{n^s}$:

$$\zeta_{-\hat{\frac{d^2}{dt^2}}}(s) = \sum_{n=1}^{+\infty}\left(\frac{n\pi}{\tau}\right)^{-2s} = \left(\frac{\tau}{\pi}\right)^2 s\sum_{n=1}^{\infty}\frac{1}{n^{2s}} = \left(\frac{\tau}{\pi}\right)^{2s}\zeta(2s)$$

Therefore

$$\frac{d}{ds}\zeta_{-\hat{\frac{d^2}{dt^2}}}(s)\Big|_{s=0} = 2\ln\left(\frac{\tau}{\pi}\right)\zeta(0)+\zeta'(0) = -\ln(2\tau)$$

and finally $\det\left[-\frac{d^2}{dt^2}\right] = 2\tau$.

3.2.3 Generating function and Time Rotation

We can characterize a quantum mechanical system by means of its **generating function**, also known as *partition function* for its similarities with statistical mechanics, it is defined as:

$$\mathcal{Z}(\tau) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-i\left(t_f-t_i\right)\hat{H}}\right) \tag{3.10}$$

Implementing in it the formula for the heat kernel (3.4) and taking $|y\rangle$ to form a basis for the Hilbert space $\mathcal{H} = L^2\left(\mathbb{R}^n, d\vec{y}^n\right)$:

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-i\left(t_f-t_i\right)\hat{H}}\right) = \int_{\mathbb{R}^n} d^n\vec{y}\,\langle y|\,e^{-i\left(t_f-t_i\right)\hat{H}}\,|y\rangle = \int_{\mathbb{R}^n} d^n\vec{y}\int_{\vec{q}_i=\vec{q}_f=\vec{y}}\mathcal{D}\vec{q}\int\mathcal{D}\vec{p}e^{iS[\vec{q},\vec{p}]}$$

The larger the real part $\mathrm{Re}\{iS\}$ becomes, the quicker the oscillation of the integral will be, thus a common way to fix this is to rotate time into the complex plane as $t = e^{-i\epsilon}\tau$, for instance the action would read:

$$S_E = \int_{\tau_i}^{\tau_f} d\tau\left(p\frac{dq}{dt} - e^{-i\epsilon}H(q,p)\right) \qquad \text{if } H \text{ is positive definite} \quad \mathrm{Re}\{iS\} = -\sin(\epsilon)\int_{\tau_i}^{\tau_f} d\tau H(q,p) < 0$$

Hence, the Path Integral will converge for large oscillation, at this purpose sometimes this rotation is done from the scratch, by taking the time such that $\epsilon = \frac{\pi}{2}$, in the textbooks is well known as **Wick rotation** and in this case times becomes solely imaginary or Euclidean and for instance, given the path integral in configuration space (3.5):

$$K(\vec{q}_f, t_f; \vec{q}_i, t_i) = C\int_{\vec{q}_i}^{\vec{q}_f}\mathcal{D}\vec{q}e^{-S_E[\vec{q}]}$$

4

Introduction to Grassmann variables and Fermionic Path Integral

4.1 Grassmann numbers

Parallelly as were defined the complex numbers, by introducing the imaginary unit i such that $i^2 = -1$, we can define the so-called **Grassmann variables** or **anticommuting number** η by the requirement that

$$\eta^2 = 0.$$

Thus we define a supernumber

Definition 4.1.1 (Supernumber)**:**
*A **supernumber** is defined as $Z = a + b\eta = a + \eta b$ where a, b are commuting numbers.*

We can define consequently, a function of supernumber by its Taylor expansion, $f(Z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(b\eta)^n$ and from the Grassmann parameter definition we have for $n \geq 2$ that $(b\eta)^n = 0$, thus any function could be written down as

$$f(Z) = f(a) + bf'(a)\eta$$

i.e. a supernumber itself. Moreover, we can differentiate it as follow

$$\frac{\partial}{\partial\eta} Z = b \quad \text{and therefore} \quad \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta} Z = 0 \tag{4.1}$$

Furthermore, if we define the integral requiring, as usual, the linearity under non-integration parameters and the invariance under linear shifts of the integration variable, we have:

$$\int d\eta f(a+\eta) = \int d\eta f(\eta)$$

which could be expanded considering a infinitesimal as $f(\eta + a) \simeq f(\eta) + a\partial_\eta f(\eta)$, thus:

$$\int d\eta f(a+\eta) = \int d\eta f(\eta) + \int d\eta a \partial_\eta f(\eta) \overset{\text{measure invariance}}{=} \int d\eta f(\eta) \quad \Longleftrightarrow \quad \int d\eta \partial_\eta f(\eta) = 0$$

comparing it to (4.1) and recalling that a function of a supernumber is a supernumber itself, we have the formal definition of the integral [1]:

$$\int d\eta = \partial_\eta$$

Increasing the number of anticommuting variables requires a more sophisticated analysis, thus:

Definition 4.1.2 (Real Grassmann Algebra)**:**
*A **real Grassmann algebra** Λ_N is defined by introducing a set N hermitian like Grassmann parameters η_i satisfying:*

$$\{\eta_i, \eta_j\} = 0 \,\forall\, i, j = 1, \ldots N \quad \text{and } \eta_i^\dagger = \eta_i$$

[1] It is a Berezin integration.

Definition 4.1.3 (Element of Λ_N)**:**
An arbitrary element of the algebra is defined from its expansion

$$Z(\eta) = Z_0 + \sum_{i=1}^{N} b_i\eta_i + \sum_{i<j} b_{ij}\eta_i\eta_j + \cdots = \sum_{0\le k\le N} \frac{1}{k!}\sum_{\{i\}} b_{i_1\ldots i_k}\eta_{i_1}\ldots\eta_{i_k}$$

where $b_{i_1\ldots i_k}$ are c-number antisymmetric under the exchange of any two indices.

If $\epsilon_{k_1k_2\ldots k_N}$ is the Levi-Civita symbol, we have that the elements of the real Grassmann algebra also satisfy:

$$\begin{cases} \eta_{k_1}\eta_{k_2}\ldots\eta_{k_N} & = \epsilon_{k_1k_2\ldots k_N}\eta_1\eta_2\ldots\eta_N \\ \eta_{k_1}\eta_{k_2}\ldots\eta_{k_M} & = 0 \quad \text{ for } M > N \end{cases}$$

If $N = 2$ we have $Z = a + b_1\eta_1 + b_2\eta_2 + c\eta_1\eta_2$ we can vary it with respect to η_1 in two ways:

$$\delta Z = \delta\eta_1\frac{\partial Z}{\partial\eta_1} = b_1\delta\eta_1 + c\delta\eta_1\eta_2 \quad \text{but also} \quad \delta Z = \frac{\partial Z}{\partial\eta_1}\delta\eta_1 = b_1\delta\eta_1 - c\eta_2\delta\eta_1$$

Thus it defines the *left derivative* and the *right derivative*: $\dfrac{\partial_L Z}{\partial\eta_1} = b_1 + c\eta_2$ and $\dfrac{\partial_R Z}{\partial\eta_1} = b_1 - c\eta_2$.
Going back to N parameters, it is crucial to make an order choice for the integral measure:

$$\int d^N\eta = \int d\eta_N \ldots d\eta_1 \quad \text{ such that } \int d^N\eta\,(\eta_1\ldots\eta_N) = 1$$

treating each differentials as anticommuting $\{d\eta_i, d\eta_j\} = 0$ and $\{d\eta_i, \eta_j\} = 0$ for all $i, j = 1, \ldots, N$, let us suppose that we are changing the variables as $\eta_i = \sum_j U_{ij}\eta'_j$ thus each integral will look like $\int d\eta_i = \frac{\partial}{\partial\eta_i} = \sum_j \frac{\partial\eta'_j}{\partial\eta_i}\frac{\partial}{\partial\eta_j} = \sum_j \left(U^{-1}\right)_{ij}\frac{\partial}{\partial\eta'_j}$ i.e. opposite to bosonic integration:

$$\begin{aligned}\int d\eta^N &= \int d\eta_N \cdots \int d\eta_1 = \sum_{j_N\cdots j_1}\left(U^{-1}\right)_{Nj_N}\cdots\left(U^{-1}\right)_{1j_1}\int d\eta'_{j_N}\cdots\int d\eta'_{j_1} = \\ &= \sum_{j_N\cdots j_1}\left(U^{-1}\right)_{Nj_N}\cdots\left(U^{-1}\right)_{1j_1}\epsilon_{j_1\ldots j_N}\int d\eta'_N\cdots\int d\eta'_1 = \det\!\left(U^{-1}\right)\int d^N\eta' = \frac{1}{\det(U)}\int d^N\eta'\end{aligned} \tag{4.2}$$

Moreover, we can also treat an even number of Grassmann variables as complexes

Definition 4.1.4 (Complex Grassmann Algebra)**:**
*The real Grassmann algebra Λ_{2N} can also be written as a N dimensional **complex Grassmann algebra** with the following definitions $\forall\, i = 1, \ldots, N-1$*

$$\chi_i = \frac{1}{\sqrt{2}}\left(\eta_{2i} + i\eta_{2i+1}\right) \quad \overline{\chi_i} = \frac{1}{\sqrt{2}}\left(\eta_{2i} - i\eta_{2i+1}\right)$$

It is possible to show the following anticommutation relations:

$$\{\chi_i, \chi_j\} = \{\chi_i, \overline{\chi}_j\} = \{\overline{\chi}_i, \overline{\chi}_j\} = 0.$$

4.2 Fermionic Path Integral

Starting from the coherent states for the harmonic oscillators, i.e. states such that $\hat{a}\,|\lambda\rangle = \lambda\,|\lambda\rangle$ with $|\lambda\rangle = e^{\frac{-|\lambda|^2}{2}}e^{\lambda\hat{a}^\dagger}\,|0\rangle$ then we have that the fermionic coherent states would be $|\eta\rangle = e^{\eta\hat{\overline{\psi}}}\,|0\rangle$ and $\langle\overline{\eta}| = \langle 0|\,e^{\overline{\eta}\hat{\psi}}$ with the unusual normalization condition $\langle\overline{\eta}|\eta\rangle = e^{\overline{\eta}\eta}$, thus we are given of the resolution of the unit:

$$1_{\mathcal{H}} = \int d^2\eta e^{-\overline{\eta}\eta}\,|\eta\rangle\,\langle\overline{\eta}|$$

To give a brief explanation to the above lines we can derive them from the fermionic harmonic oscillator with the ladder operators $\hat{c}, \hat{c}^\dagger$ such that $\{\hat{c}, \hat{c}^\dagger\} = 1$ and $\{\hat{c}, \hat{c}\} = \{\hat{c}^\dagger, \hat{c}^\dagger\} = 0$ and the Hamiltonian $\hat{H} = \frac{\omega}{2}(\hat{c}^\dagger\hat{c} - \hat{c}\hat{c}^\dagger) = \omega\left(\hat{F} - \frac{1}{2}\right)$ with $\hat{F}$ the number operator with eigenvalues $\{0, 1\}$, therefore by considering the Hilbert space spanned by its eigenstates $\{|0\rangle, |1\rangle\}$ and taking two Grassmann numbers $\eta, \overline{\eta}$ we have $|\eta\rangle = |0\rangle + \eta|1\rangle$ and $\langle\overline{\eta}| = \langle 0| + \overline{\eta}\langle 1|$ thus $\langle\overline{\eta}|\eta\rangle = 1 + \overline{\eta}\eta = e^{\overline{\eta}\eta}$ and therefore

$$\int d\overline{\eta}d\eta\, |\eta\rangle\langle\overline{\eta}|\, e^{-\overline{\eta}\eta} = \int d\overline{\eta}d\eta\, (|0\rangle + \eta|1\rangle)(\langle 0| + \overline{\eta}\langle 1|)(1 - \overline{\eta}\eta) =$$
$$= \int d\overline{\eta}d\eta\, |0\rangle\langle 0| + \eta|1\rangle\langle 0| + \overline{\eta}|0\rangle\langle 1| + \eta\overline{\eta}|1\rangle\langle 1| - \overline{\eta}\eta|0\rangle\langle 0| = |1\rangle\langle 1| + |0\rangle\langle 0| = 1_{\mathcal{H}}$$

From this result we can get the heat kernel from a fermionic initial state $|\chi\rangle$ to a final one $|\chi'\rangle$ in a rather closed way to the bosonic case[2]:

$$\langle\overline{\chi}'|\, e^{-i(t_f - t_i)H}|\chi\rangle = \langle\overline{\chi}'|\, e^{-i\Delta t H(\hat{\psi}, \hat{\overline{\psi}})} \dots e^{-i\Delta t H(\hat{\psi}, \hat{\overline{\psi}})}|\chi\rangle =$$
$$= \lim_{N\to\infty}\int \prod_{k=1}^{N-1} d^2\eta_k e^{-\overline{\eta}_k\eta_k}\langle\overline{\chi}'|\, e^{-i\Delta t H(\hat{\psi}, \hat{\overline{\psi}})}|\eta_{N-1}\rangle\langle\overline{\eta}_{N-1}|\dots|\eta_1\rangle\langle\overline{\eta}_1|\, e^{-i\Delta t H(\hat{\psi}, \hat{\overline{\psi}})}|\chi\rangle =$$
$$= \lim_{N\to\infty}\int \prod_{k=1}^{N-1} d^2\eta_k e^{-\overline{\eta}_k\eta_k}\prod_{p=1}^{N} e^{-i\Delta t H(\eta_{p-1}, \overline{\eta}_p)} e^{\overline{\eta}_p\eta_{p-1}} =$$
$$= \lim_{N\to\infty}\int \prod_{k=1}^{N-1} d^2\eta_k e^{\overline{\eta}_N\eta_N} e^{-i\Delta t\sum_{p=1}^{N}\frac{\overline{\eta}_k}{i\Delta t}(\eta_k - \eta_{k-1}) + H(\eta_{k-1}, \overline{\eta}_k)}$$

Whereas in the continuous approximation we tell the latter exponent to be the action itself, moreover, it is key here to switch into Euclidean time $\tau = it$, such that $S_E[\eta, \overline{\eta}] = \int_{\tau_i}^{\tau_f} d\tau\, (\overline{\eta}\dot{\eta} + H(\eta, \overline{\eta}))$. Therefore if $\psi(\tau_i) = \chi$ and $\psi(\tau_f) = \chi'$ then:

$$K(\chi', \tau_f; \chi, \tau_i) = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{\overline{\psi}(\tau_f)\psi(\tau_f)} e^{-S_E[\psi, \overline{\psi}]}$$

Furthermore, if $\beta = \tau_f - \tau_i$ and given anti-periodic boundary conditions, in short *APBC*, $\chi = \psi(0) = -\psi(\beta)$, we read the trace:

$$\mathrm{Tr}\left(e^{-\beta H}\right) = \sum_{n=0,1}\langle n|\, e^{-\beta H}|n\rangle = \sum_{n=0,1}\int d\overline{\eta}d\eta e^{-\overline{\eta}\eta}\langle n|\eta\rangle\langle\overline{\eta}|\, e^{-\beta H}|n\rangle =$$
$$= \sum_{n=0,1}\int d\overline{\eta}d\eta e^{-\overline{\eta}\eta}\langle n|\,(|0\rangle + \eta|1\rangle)(\langle 0| - \overline{\eta}\langle 1|)\, e^{-\beta H}|n\rangle =$$
$$= \sum_{n=0,1}\int d\overline{\eta}d\eta e^{-\overline{\eta}\eta}\Big(\langle n|0\rangle\langle 0|\, e^{-\beta H}|n\rangle +$$
$$+ \overline{\eta}\langle n|0\rangle\langle 1|\, e^{-\beta H}|n\rangle + \eta\langle n|1\rangle\langle 0|\, e^{-\beta H}|n\rangle + \eta\overline{\eta}\langle n|1\rangle\langle 1|\, e^{-\beta H}|n\rangle\Big)$$

We can invert the sign of $\overline{\eta} \to -\overline{\eta}$ in the second term, indeed it is not going to saturate the Grassmann integral anyway and if $\langle -\overline{\eta}| = \langle 0| - \overline{\eta}\langle 1|$ and therefore we read:

$$\mathrm{Tr}\left(e^{-\beta H}\right) = \int d^2\chi\,\langle -\overline{\chi}|\, e^{-\beta H}|\chi\rangle\, e^{-\overline{\chi}\chi} = \int_{APBC}\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S_E[\psi, \overline{\psi}]}$$

And finally

$$\mathrm{Tr}\left((-1)^F e^{-\beta H}\right) = \int d^2\chi\,\langle\overline{\chi}|\, e^{-\beta H}|\chi\rangle\, e^{-\overline{\chi}\chi} = \int_{PBC}\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S_E[\psi, \overline{\psi}]} \tag{4.3}$$

Where also the fermions are periodic such that $\psi(\tau + \beta) = \psi(\tau)$

[2] Check [8] for more details.

5

Computations

5.1 Euler number derivation

Let us consider a particle of unit mass moving along a Riemannian manifold (N, g) of dimension n, compact, oriented and with no boundaries, in which a set of local coordinates is given by $x^i = \{x^1, \dots, x^n\}$, let $t \in M$ denote time, with $M = \mathbb{R}$ or S^1 (for Euclidean time). The worldline field of our particle is the map $\phi : M \to N$. With a little abuse of notation, we call $\phi^i = x^i \circ \phi$ the pullback of the coordinates of the local chart in N to M. Similarly, $g_{ij}\dot{\phi}^i\dot{\phi}^j$ is the pullback of the metric g of N to M.
Next, we consider the free particle action:

$$S = \int_M dt \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j.$$

By varying it:

$$\delta S = \int_M dt \left(\frac{1}{2} g_{ij,k} \delta\phi^k \dot{\phi}^i \dot{\phi}^j + g_{ij} \frac{d}{dt}\left(\delta\phi^i\right)\dot{\phi}^j \right) = \quad \text{by parts}$$
$$\int_M dt \left[\frac{1}{2} g_{ij,k}\dot{\phi}^i\dot{\phi}^j - \frac{d}{dt}\left(g_{kj}\dot{\phi}^j\right)\right]\delta\phi^k + \int_M dt \frac{d}{dt}\left(g_{kj}\dot{\phi}^j\delta\phi^k\right)$$

We get the bulk equation of motion:

$$0 = \frac{1}{2} g_{ij,k}\dot{\phi}^i\dot{\phi}^j - \frac{d}{dt}\left(g_{kj}\dot{\phi}^j\right) = \frac{1}{2} g_{ij,k}\dot{\phi}^i\dot{\phi}^j - \left(g_{kj,i}\dot{\phi}^i\dot{\phi}^j + g_{kj}\ddot{\phi}^j\right) =$$
$$= -\frac{1}{2}\left(g_{kj,i}\dot{\phi}^i\dot{\phi}^j + g_{ki,j}\dot{\phi}^j\dot{\phi}^i - g_{ij,k}\dot{\phi}^i\dot{\phi}^j\right) - g_{kj}\ddot{\phi}^j \iff$$
$$\ddot{\phi}^l + \frac{1}{2} g^{lk}\left(g_{ki,j} + g_{kj,i} - g_{ij,k}\right)\dot{\phi}^i\dot{\phi}^j = 0 \quad \iff \quad \ddot{\phi}^l + \Gamma^l_{ij}\dot{\phi}^i\dot{\phi}^j = 0$$

Hence, from a classical point of view, the particle freely moving is following the geodesics (2.3) of N.
Finally, we introduce n complex valued fermions $\psi^i = \begin{pmatrix}\psi^i_1 \\ \psi^i_2\end{pmatrix}$ where ψ^i_1, ψ^i_2 are real. These fermions should be thought as the local components of a fermionic field $\psi = \psi^i \frac{\partial}{\partial x^i}_{|\phi}$ [2].
Furthermore, taking the following action, the **supersymmetric non-linear sigma model** action [1]:

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + \frac{i}{2} g_{ij}(\phi)\overline{\psi}^i\gamma^0\frac{D}{dt}\psi^j + \frac{1}{12} R_{ijkl}\overline{\psi}^i\psi^k\overline{\psi}^j\psi^l \tag{5.1}$$

where

$$\frac{D}{dt}\psi^i = \frac{d}{dt}\psi^i + \Gamma^i_{jk}\dot{\phi}^j\psi^k \quad \text{and} \quad \overline{\psi}^i_\alpha = \left(\psi^i\right)_\beta\left(\gamma^0\right)_{\beta\alpha} \quad \forall\, \alpha, \beta = 1, 2$$

it is shown in the appendix A that is invariant under the supersymmetry transformations:

$$\begin{aligned} &\delta\phi^i = \bar{\epsilon}\psi^i \quad \text{and} \\ &\delta\psi^i = -i\gamma^0\dot{\phi}^i\epsilon - \Gamma^i_{jk}\bar{\epsilon}\psi^j\psi^k \end{aligned} \tag{5.2}$$

with $\epsilon = \begin{pmatrix}\epsilon_1 \\ \epsilon_2\end{pmatrix}$ and ϵ_1, ϵ_2 being the two real components of the anticommuting constant spinor and the matrix $\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, which is such that $\left(\gamma^0\right)^2 = \text{id}$.

Let us compute the supersymmetry charges through the use of Noether's theorem. Thus, on shell and since $\delta L = 0$ the following current is conserved

$$\begin{aligned}
j_0 &= \frac{\partial L}{\partial\left(\dot{\phi}^i\right)}\delta\phi^i + \frac{\partial_R L}{\partial\left(\dot{\psi}^i\right)}\delta\psi^i = \\
&= g_{ij}\dot{\phi}^i\delta\phi^j + \frac{i}{2}g_{ij}\overline{\psi}^i\gamma^0\Gamma^j_{kl}\psi^k\delta\phi^l + \frac{i}{2}g_{ij}\overline{\psi}^i\gamma^0\delta\psi^j = \\
&= g_{ij}\dot{\phi}^i\overline{\epsilon}\psi^j + \frac{i}{2}g_{ij}\overline{\psi}^i\gamma^0\Gamma^j_{kl}\psi^k\overline{\epsilon}\psi^l + \frac{1}{2}g_{ij}\overline{\psi}^i\gamma^0\gamma^0\dot{\phi}^j\epsilon = \\
&= \overline{\epsilon}g_{ij}\dot{\phi}^i\psi^j - \epsilon\frac{1}{2}g_{ij}\overline{\psi}^i\dot{\phi}^j = \\
&= \frac{i}{2}\epsilon Q + i\overline{\epsilon}\overline{Q} \quad \text{where } Q = ig_{ij}\dot{\phi}^i\overline{\psi}^j \text{ and } \overline{Q} = -ig_{ij}\dot{\phi}^i\psi^j
\end{aligned}$$

Next, if the conjugate momenta of ϕ^i, ψ^j are respectively $p_i = \frac{\partial L}{\partial\dot{\phi}^i} = g_{ij}\dot{\phi}^j + \frac{i}{2}g_{kj}\overline{\psi}^k\gamma^0\Gamma^j_{il}\psi^l$ and $\rho^j = \frac{i}{2}g_{ij}\overline{\psi}^i\gamma^0$, we quantize the theory imposing the commutation and anticommutation relations:[1]

$$[\phi^i, p_j] = i\delta^i_j \quad \{\psi^i, \overline{\psi}^j\} = g^{ij}$$

In particular, the Hilbert space of the non-supersymmetric system $\mathcal{H} = L^2(N)$ is given by square-integrable functions in the target space N with respect to the measure $\sqrt{g}d^nx$ where, as usual:

$$p_i \text{ acts as differentiation } \quad -i\frac{\partial}{\partial x^i}$$

The fermions generate a Fock space where the vacuum state is taken to be $\psi^i|0\rangle = 0$ for all $i = 1, \dots, n$ and all the other states are obtained acting by $\overline{\psi}^i$ on $|0\rangle$. Since $\overline{\psi}^i\overline{\psi}^j = -\overline{\psi}^j\overline{\psi}^i$ we can identify them as the basis of all the k-forms in N, i.e.

$$\overline{\psi}^i \text{ is acting on } |0\rangle \text{ as } \quad dx^i\wedge$$

similarly, $\overline{\psi}^1 \dots \overline{\psi}^n|0\rangle$ is in correspondence with $dx^1 \wedge \dots \wedge dx^n$. The ψ's are acting as the contraction by a vector field:

$$\begin{aligned}
\psi^i\overline{\psi}^j\overline{\psi}^k \dots \overline{\psi}^m|0\rangle &= \{\psi^i, \overline{\psi}^j\}\overline{\psi}^k \dots \overline{\psi}^m|0\rangle - \overline{\psi}^j\psi^i\overline{\psi}^k \dots \overline{\psi}^m|0\rangle = \\
&= g^{ij}\overline{\psi}^k \dots \overline{\psi}^m|0\rangle - \overline{\psi}^k g^{il} \dots \overline{\psi}^m|0\rangle + \dots + \overline{\psi}^j\overline{\psi}^k \dots \overline{\psi}^m\psi^i|0\rangle = \\
&= g^{ij}\overline{\psi}^k \dots \overline{\psi}^m|0\rangle - \overline{\psi}^k g^{il} \dots \overline{\psi}^m|0\rangle + \dots
\end{aligned}$$

So that

$$\psi^i \text{ acts as the contraction of a } k\text{-forms by } \quad g^{ij}\frac{\partial}{\partial x^j}$$

The full supersymmetric Hilbert space could be regarded as the completion of[2]

$$\mathcal{H} = \Omega^*(N) \otimes \mathbb{C}$$

given by all the complex value forms on N square-integrable with respect to (2.5), $\mathbb{C}$-linearly extended to the following: $\forall\, \alpha, \beta \in \mathcal{H}$

$$(\alpha, \beta)_N = \int_N \alpha \wedge *\overline{\beta}$$

It is possible to decompose the Hilbert space as:

$$\mathcal{H} = \bigoplus_{p=0}^{n} \Omega^p(N) \otimes \mathbb{C} = \bigoplus_{p=0}^{n} \mathcal{H}^p$$

[1] All the others (anti)-commutation relations are trivial.

[2] By a slight abuse of notation we are denoting both $\Omega^*(N) \otimes \mathbb{C}$ and its completion by $\mathcal{H}$.

or by a $\mathbb{Z}_2$ grading given by the operator $(-1)^F$ such that it anticommutes with both $\psi^i, \overline{\psi}^i$ and $(-1)^F |0\rangle = |0\rangle$, as follow:

$$\mathcal{H} = \mathcal{H}^B \oplus \mathcal{H}^F$$

where in the former are laying the even forms $\mathcal{H}^B = \oplus_{\text{p even}} \mathcal{H}^p$ whilst in the latter the odd ones $\mathcal{H}^F = \oplus_{\text{p odd}} \mathcal{H}^p$. So that $\mathcal{H}^B$ and $\mathcal{H}^F$ are eigenspaces of $(-1)^F$ with eigenvalues $+1$ and -1 respectively.

Further, we might interpret Q and $\overline{Q}$ within this geometrical framework. Indeed by $-i\psi^i p_i = -ig_{ij}\psi^i\dot{\phi}^j + \frac{1}{2}g_{kj}\psi^i\overline{\psi}^k\gamma^0\Gamma^j_{il}\psi^l = -ig_{ij}\psi^i\dot{\phi}^j = \overline{Q}$ and analogously $Q = ig_{ij}\overline{\psi}^i\dot{\phi}^j = i\overline{\psi}^i p_i$, we associate Q as follow:

$$Q \mapsto idx^i \wedge \frac{-i\partial}{\partial x^i} = d$$

Thus, Q is acting as the exterior derivative and $\overline{Q}$ as d^*, i.e. its dual, which is given by the Hodge star operator as above (2.6).

Hence, from the algebra [1] we also have:

$$\{Q, \overline{Q}\} = 2H$$
$$\{Q, Q\} = \{\overline{Q}, \overline{Q}\} = 0$$

and therefore $H = \frac{1}{2}\{Q, \overline{Q}\}$ is identified with the Laplacian operator (2.2.7), properly with

$$H \mapsto \frac{1}{2}\Delta = \frac{1}{2}\{d, d^*\} = \frac{1}{2}(dd^* + d^*d) \tag{5.3}$$

5.1.1 $\mathrm{Tr}\,(-1)^F$

Let $|E\rangle$ be eigenstate of H. By

$$E = \langle E| H |E\rangle = \frac{1}{2}\langle E| Q\overline{Q} + \overline{Q}Q |E\rangle = \frac{1}{2}\left(\|Q|E\rangle\|^2 + \|\overline{Q}|E\rangle\|^2\right) \geq 0$$
$$\text{then } E = 0 \text{ if and only if } \quad Q|E\rangle = \overline{Q}|E\rangle = 0$$

the Hamiltonian is positive-definite. A zero energy state $H|0\rangle = 0$ is a ground state, and is a supersymmetric state, i.e. it is annihilated by both the supercharges Q and $\overline{Q}$.

Next, if $\mathcal{H}_{(n)}$ is the Hilbert space associated with the n-th energy level of value E_n such that the Hamiltonian $H_{|\mathcal{H}_{(n)}} = E_n$ then the operator $S = \frac{Q+\overline{Q}}{\sqrt{2}}$ satisfies $S^2 = 2H$ conserving each energy level so that for $E_n > 0$, the restriction of S to $\mathcal{H}_{(n)}$ is invertible:

$$S^2 = 2E_n > 0$$

S maps $\mathcal{H}^B_{(n)} \to \mathcal{H}^F_{(n)}$ and vice versa, so for $E_n > 0$ it realizes an isomorphism

$$\mathcal{H}^B_{(n)} \simeq \mathcal{H}^F_{(n)}$$

by which the number of fermions and bosons in each positive energy level coincides. For $E = 0$ a bijection might not exist and the number of bosonic and fermionic supersymmetric ground states does not need to equal anymore.

By varying the parameters of the theory, the states of non-zero energy jingle around in energy, moving in Bose-Fermi pairs, at a certain value they might fall down to $E = 0$ states,

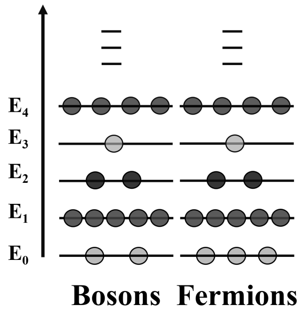

Figure 5.1: Isomorphism fermions-bosons for $E \neq 0$, [2].

such that $\dim \mathcal{H}^B_{(0)}$, $\dim \mathcal{H}^F_{(0)}$, which respectively indicate the number of bosonic and fermionic zero-energy states, would both increase by one. Vice versa it is not possible for a singlet of zero-energy to escape since it has to be paired with its supersymmetric partner, thus also in this case both would decrease by one.

Summing it up, we see that their difference does not change and therefore $\dim \mathcal{H}^B_{(0)} - \dim \mathcal{H}^F_{(0)}$ is invariant and independent of all parameters. A noteworthy property is that it may be regarded as the supertrace (4.3) of the grading operator, given $\beta > 0$:

$$\dim \mathcal{H}^B_{(0)} - \dim \mathcal{H}^F_{(0)} = \mathrm{Tr}\left((-1)^F e^{-\beta H}\right)$$

Indeed the non-zero energy states do not contribute to the trace since they come in opposite sign pair, thus the trace is evaluated just for states at $E = 0$.

Recalling the results anticipated in 2.2.2, we have that for each harmonic form there exists a unique representative of the De Rham cohomology by Hodge decomposition:

$$\mathrm{Harm}^p(N, g) \simeq H^p_{DR}(N)$$

By (5.3) the supersymmetric ground state can be regarded as the space of the harmonic forms of the Riemannian manifold:

$$\mathcal{H}_{(0)} = \mathrm{Harm}(N, g) = \bigoplus_{p=0}^{n} \mathrm{Harm}^p(N, g)$$

Finally we have the following identities:

$$\mathrm{Tr}\left((-1)^F e^{-\beta H}\right) = \sum_{p=0}^{n} (-1)^p \dim \mathrm{Harm}^p(N, g) = \sum_{p=0}^{n} (-1)^p \dim H^p_{DR}(N) = \chi(N) \tag{5.4}$$

where $\chi(N)$ is the Euler number (2.2) of N.

Proceeding as in [2], recalling (4.3), we have that the previous index, also known as Witten index, has a Path Integral representation:

$$\mathrm{Tr}\left((-1)^F e^{-\beta H}\right) = \int_{PBC} \mathcal{D}\phi \mathcal{D}\overline{\psi} \mathcal{D}\psi e^{-S_E[\phi, \psi, \overline{\psi}]} \tag{5.5}$$

5.2 Gauss-Bonnet formula derivation

Because the Witten index is independent of β [10], we can evaluate (5.5) for $\beta \to 0$.

Taking the action (5.1) in the basis in which γ^0 is diagonal is equivalent to the following:

$$\frac{1}{2} g_{ij}(\phi) \dot{\phi}^i \dot{\phi}^j + i \psi^{*i} g_{ij} \frac{D}{dt} \psi^j - \frac{1}{4} R_{ijkl}(\phi) \psi^{*i} \psi^{*j} \psi^k \psi^l \tag{5.6}$$

In order to evaluate the above Path Integral (5.5), one would Wick rotate the action into its Euclidean form, taking τ to be $\tau = it$, moreover, pursuing the stationary phase approximation (3.6), we could split the integration variables into zero-modes and non-constant modes as:

$$\phi^i(t) = \phi_0^i + \xi^i(t)$$
$$\psi^i(t) = \psi_0^i + \eta^i(t)$$

In order to take the limit $\beta \to 0$ we could substitute as follow:

$$L_E = \frac{L'_E}{\beta} \quad \tau = \tau'\beta \quad \xi^i = \xi'^i\sqrt{\beta} \quad \psi_0^i = \frac{\psi_0'^i}{\beta^{\frac{1}{4}}}$$

While $\eta^i = \eta'^i$. From now on, we will drop the prime notation from the variable names. In the limit $\beta \to 0$ at first order in β, we obtain:

$$\begin{aligned}
\frac{L_E}{\beta} &= \frac{1}{2}g_{ij}(\phi_0)\frac{\dot{\xi}^i}{\beta}\frac{\dot{\xi}^j}{\beta}\beta + \left(\frac{\psi_0^{*i}}{\beta^{\frac{1}{4}}} + \eta^{*i}\right) g_{ij}(\phi_0)\left[\frac{\dot{\eta}^j}{\beta} + \Gamma^j_{kl}(\phi_0)\frac{\dot{\xi}^k}{\beta^{\frac{1}{2}}}\left(\frac{\psi_0^l}{\beta^{\frac{1}{4}}} + \eta^l\right)\right] - \\
&- \frac{1}{4}R_{ijkl}(\phi_0)\left(\frac{\psi_0^{*i}}{\beta^{\frac{1}{4}}} + \eta^{*i}\right)\left(\frac{\psi_0^{*j}}{\beta^{\frac{1}{4}}} + \eta^{*j}\right)\left(\frac{\psi_0^{k}}{\beta^{\frac{1}{4}}} + \eta^{k}\right)\left(\frac{\psi_0^{l}}{\beta^{\frac{1}{4}}} + \eta^{l}\right) = \\
&= \frac{1}{\beta}\left[\frac{1}{2}g_{ij}(\phi_0)\dot{\xi}^i\dot{\xi}^j + \frac{1}{\beta^{\frac{1}{4}}}\psi_0^{*i}g_{ij}(\phi_0)\dot{\eta}^j + \eta^{*i}g_{ij}(\phi_0)\dot{\eta}^j - \frac{1}{4}R_{ijkl}(\phi_0)\psi_0^{*i}\psi_0^{*j}\psi_0^k\psi_0^l + \mathcal{O}(\beta)\right]
\end{aligned}$$

Recalling we have required periodic boundary condition, so the fluctuations are such that $\eta(0) = \eta(\beta) = 0$ and that we have switched the first and the third terms into a Vielbein basis[3], we get the Euclidean action:

$$S_E = \int_0^1 d\tau L_E = \int_0^1 d\tau \left(\frac{1}{2}g_{ij}(\phi_0)\dot{\xi}^i\dot{\xi}^j + \frac{1}{\beta^{\frac{1}{4}}}\psi_0^{*i}g_{ij}(\phi_0)\dot{\eta}^j + \eta^{*i}g_{ij}(\phi_0)\dot{\eta}^j - \frac{1}{4}R_{ijkl}(\phi_0)\psi_0^{*i}\psi_0^{*j}\psi_0^k\psi_0^l + \mathcal{O}(\beta)\right) \tag{5.7}$$

Where we used:

$$\int_0^1 d\tau\psi_0^{*i}g_{ij}(\phi_0)\dot{\eta}^j = \int_0^1 d\tau\frac{d}{d\tau}\left(\psi_0^{*i}g_{ij}(\phi_0)\eta^j\right) - \int_0^1 d\tau\frac{d}{d\tau}\left(\psi_0^{*i}g_{ij}(\phi_0)\right)\eta^j = 0$$

Hence, performing the limit for $\beta \to 0$, it reads:

$$S_E = \int_0^1 d\tau\left(\frac{1}{2}\delta_{ab}\dot{\xi}^a\dot{\xi}^b + \delta_{ab}\eta^{*a}\dot{\eta}^b - \frac{1}{4}R_{ijkl}(\phi_0)\psi_0^{*i}\psi_0^{*j}\psi_0^k\psi_0^l\right)$$

We start by carrying out the integral over the constant modes of the bosonic and fermionic fields separately

$$\int_{PBC}\mathcal{D}\phi e^{-S_E[\phi_0,\xi]} = \int_{\xi(0)=0}^{\xi(1)=0}\mathcal{D}\xi e^{-\frac{1}{2}\int_0^1 d\tau\delta_{ab}\dot{\xi}^a\dot{\xi}^b} = \mathcal{N}(2\pi)^{\frac{n}{2}}\det\left[-\frac{d^2}{d\tau^2}\right]^{-\frac{1}{2}}$$

Whereas, for the Grassmann fields, recalling (4.2) shifting $\eta_i = \sum_j\left[\frac{d}{d\tau}^{-1}\right]_{ij}\eta'_j$ and treating η'_i, η^*_j as independent:

$$\int_{PBC}\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S_E[\psi_0,\eta]} = \int_{\eta(0)=0}^{\eta(1)=0}\mathcal{D}\overline{\eta}\mathcal{D}\eta e^{-\int_0^1 d\tau\delta_{ab}\eta^{*a}\dot{\eta}^b} = \mathcal{N}'\frac{1}{\det\left[\frac{d}{d\tau}^{-1}\right]} = \mathcal{N}'\det\left[\frac{d}{d\tau}\right]$$

[3]Recall 2.7.

Hence, fixing the normalization constants and without any need of regularization (3.9), because the functional determinants identity elide each other, indeed by Binet's theorem:

$$\sqrt{\det\left[\frac{d^2}{d\tau^2}\right]}=\det\left[\frac{d}{d\tau}\right]$$

therefore, we read:

$$\mathrm{Tr}\left((-1)^F e^{-\beta H}\right)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int d\,(\mathrm{Vol})\int d\overline{\psi}_0^n d\psi_0^n\dots d\overline{\psi}_0^1 d\psi_0^1 e^{-\frac{1}{4}R_{ijkl}(\phi_0)\psi_0^{*i}\psi_0^{*j}\psi_0^k\psi_0^l}$$

Next, if we are in a odd dimensional space we see that we cannot saturate all the Grassmann measures and therefore the integral vanishes, whereas if we are in an even dimensional space, expanding out the exponent and re-casting it back to the right order we have, through the use of the Levi-Civita symbols:

$$\begin{aligned}
&\int d\overline{\psi}_0^n d\psi_0^n\dots d\overline{\psi}_0^1 d\psi_0^1 e^{-\frac{1}{4}R_{ijkl}(\phi_0)\psi_0^{*i}\psi_0^{*j}\psi_0^k\psi_0^l}=\\
&=\int d\overline{\psi}_0^n d\psi_0^n\dots d\overline{\psi}_0^1 d\psi_0^1\Bigg[1-\frac{1}{4}R_{ijkl}(\phi_0)\psi_0^{*i}\psi_0^{*j}\psi_0^k\psi_0^l+\frac{1}{2}\left(-\frac{1}{4}R_{ijkl}(\phi_0)\psi_0^{*i}\psi_0^{*j}\psi_0^k\psi_0^l\right)^2+\dots+\\
&+\frac{1}{\left(\frac{n}{2}\right)!}\left(-\frac{1}{4}R_{ijkl}(\phi_0)\psi_0^{*i}\psi_0^{*j}\psi_0^k\psi_0^l\right)^{\frac{n}{2}}\Bigg]=\\
&=\int d\overline{\psi}_0^n d\psi_0^n\dots d\overline{\psi}_0^1 d\psi_0^1\frac{(-1)^{\frac{n}{2}}}{4^{\frac{n}{2}}\left(\frac{n}{2}\right)!}\psi_0^1\psi_0^{*1}\dots\psi_0^n\psi_0^{*n}\epsilon^{i_1j_1,\dots,i_{\frac{n}{2}}j_{\frac{n}{2}}}\epsilon^{k_1l_1,\dots,k_{\frac{n}{2}}l_{\frac{n}{2}}}R_{i_1j_1k_1l_1}(\phi_0)\dots R_{i_{\frac{n}{2}}j_{\frac{n}{2}}k_{\frac{n}{2}}l_{\frac{n}{2}}}(\phi_0)=\\
&=\frac{(-1)^{\frac{n}{2}}}{(2)^n\left(\frac{n}{2}\right)!}\epsilon^{i_1j_1\dots i_{\frac{n}{2}}j_{\frac{n}{2}}}\epsilon^{k_1l_1\dots k_{\frac{n}{2}}l_{\frac{n}{2}}}R_{i_1j_1k_1l_1}(\phi_0)\dots R_{i_{\frac{n}{2}}j_{\frac{n}{2}}k_{\frac{n}{2}}l_{\frac{n}{2}}}(\phi_0)
\end{aligned}$$

Eventually, we read:

$$\mathrm{Tr}\left((-1)^F e^{-\beta H}\right)=\frac{(-1)^{\frac{n}{2}}}{(2)^n(2\pi)^{\frac{n}{2}}\left(\frac{n}{2}\right)!}\int d\,(\mathrm{Vol})\,\epsilon^{i_1j_1\dots i_{\frac{n}{2}}j_{\frac{n}{2}}}\epsilon^{k_1l_1\dots k_{\frac{n}{2}}l_{\frac{n}{2}}}R_{i_1j_1k_1l_1}(\phi_0)\dots R_{i_{\frac{n}{2}}j_{\frac{n}{2}}k_{\frac{n}{2}}l_{\frac{n}{2}}}(\phi_0) \tag{5.8}$$

Let us end by musing on this final result: if we recall that the L.H.S of the equation is the Euler Characteristic of the manifold, then we have obtained exactly the Gauss-Bonnet formula.

5.2.1 Examples

From the formula (5.8), from the example 2.2.2.1 and by recalling the volume form of the 2-sphere $d\,(\mathrm{Vol})=\hat{\theta}^1\wedge\hat{\theta}^2=\sin\theta d\theta\wedge d\phi$ we have:

$$\begin{aligned}
\chi(S^2)&=\frac{-1}{8\pi}\int_{S^2}\hat{\theta}^1\wedge\hat{\theta}^2\epsilon^{ij}\epsilon^{kl}R_{ijkl}=\frac{-1}{8\pi}\int_{S^2}\hat{\theta}^1\wedge\hat{\theta}^2\left(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk}\right)R_{ijkl}=\frac{-1}{8\pi}\int_{S^2}\hat{\theta}^1\wedge\hat{\theta}^2\sum_{ij}\left(R_{ijij}-R_{ijji}\right)=\\
&=-\frac{1}{8\pi}\int_{S^2}\hat{\theta}^1\wedge\hat{\theta}^2\left(R_{1212}+R_{2121}-R_{1221}-R_{2112}\right)=-\frac{1}{8\pi}\int_{S^2}\hat{\theta}^1\wedge\hat{\theta}^2 4R_{1212}=\frac{1}{2\pi}\int_{S^2}\hat{\theta}^2\wedge\hat{\theta}^1=\\
&=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta=2
\end{aligned}$$

And similarly the volume form of the torus is $d\,(\mathrm{Vol})=\hat{\theta}^1\wedge\hat{\theta}^2=r\,(R+r\cos\theta)\,d\theta\wedge d\phi$:

$$\chi(T^2)=-\frac{1}{8\pi}\int_{T^2}\hat{\theta}^1\wedge\hat{\theta}^2 4R_{1212}=\frac{1}{2\pi}\int_{T^2}\hat{\theta}^2\wedge\hat{\theta}^1\frac{\cos\theta}{r\,(R+r\cos\theta)}=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_0^{2\pi}d\theta\cos\theta=0$$

These examples clearly show, recalling (5.4), that the 2-sphere is endowed only with the constant harmonic 0-form and with the volume harmonic 2-form. On the other hand, the torus, in order for χ to vanish, must also be equipped with two harmonic (actually, constant) 1-forms.

A

Appendix

For simplicity sake let us consider the particle swinging around the same space where it is been rid of the curvature term and where the Riemannian manifold is bounded by the Levi-Civita connection 2.2.1, as a consequence of that, the ψ's transformation (5.2) read[1] $\delta\psi^i = -i\gamma^0\dot{\phi}^i\epsilon$ and given γ^0 to be hermitian $\delta\overline{\psi}^i = i\epsilon^\dagger\dot{\phi}^i = i\bar{\epsilon}\gamma^0\dot{\phi}^i$, so that:

$$\begin{aligned}\delta L = &\frac{1}{2}g_{ij,k}\delta\phi^k\dot{\phi}^i\dot{\phi}^j + g_{ij}\frac{d}{dt}(\delta\phi^i)\dot{\phi}^j + \frac{i}{2}g_{ij,k}\delta\phi^k\overline{\psi}^i\gamma^0\frac{D}{dt}\psi^j + \frac{i}{2}g_{ij}\delta\overline{\psi}^i\gamma^0\frac{D}{dt}\psi^j + \\ &+ \frac{i}{2}g_{ij}\overline{\psi}^i\gamma^0\left(\frac{d}{dt}\delta\psi^j + \Gamma^j_{kl,m}\delta\phi^m\dot{\phi}^k\psi^l + \Gamma^j_{kl}\frac{d}{dt}\left(\delta\phi^k\right)\psi^l + \Gamma^j_{kl}\dot{\phi}^k\delta\psi^l\right)\end{aligned}$$

Where by $,k$ we have intended the partial derivative with respect to ϕ^k, next carrying out all the substitutions lead to:

$$\begin{aligned}\delta L = &\frac{1}{2}g_{ij,k}\bar{\epsilon}\psi^k\dot{\phi}^i\dot{\phi}^j + g_{ij}\bar{\epsilon}\dot{\psi}^i\dot{\phi}^j + \frac{i}{2}g_{ij,m}\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\left(\dot{\psi}^j + \Gamma^j_{kl}\dot{\phi}^k\psi^l\right) - \frac{1}{2}g_{ij}\bar{\epsilon}\gamma^0\dot{\phi}^i\gamma^0\left(\dot{\psi}^j + \Gamma^j_{kl}\dot{\phi}^k\psi^l\right) + \\ &+ \frac{i}{2}g_{ij}\overline{\psi}^i\gamma^0\left(-i\gamma^0\ddot{\phi}^j\epsilon + \Gamma^j_{kl,m}\bar{\epsilon}\psi^m\dot{\phi}^k\psi^l + \Gamma^j_{kl}\bar{\epsilon}\dot{\psi}^k\psi^l - i\Gamma^j_{kl}\dot{\phi}^k\gamma^0\dot{\phi}^l\epsilon\right) = \\ &= \left[\frac{1}{2}g_{ij,k}\bar{\epsilon}\psi^k\dot{\phi}^i\dot{\phi}^j - \frac{1}{2}\bar{\epsilon}\frac{1}{2}\left(g_{ik,j} + g_{ij,k} - g_{kj,i}\right)\psi^k\dot{\phi}^i\dot{\phi}^j + \frac{1}{2}g_{ij}\overline{\psi}^i\Gamma^j_{kl}\dot{\phi}^k\dot{\phi}^l\epsilon\right] + \\ &+ \frac{1}{2}g_{ij}\bar{\epsilon}\dot{\phi}^i\dot{\psi}^j + \frac{1}{2}g_{ij}\overline{\psi}^i\ddot{\phi}^j\epsilon + \left[\frac{i}{2}g_{ij,m}\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\dot{\psi}^j - \frac{i}{4}\left(g_{ij,m} + g_{im,j} - g_{jm,i}\right)\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\dot{\psi}^j\right] + \\ &+ \left[\frac{i}{2}g_{ij,m}\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\Gamma^j_{kl}\dot{\phi}^k\psi^l + \frac{i}{2}g_{ij}\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\Gamma^j_{kl,m}\dot{\phi}^k\psi^l\right]\end{aligned}$$

Recall here, that the metric tensor is symmetric

$$\begin{aligned}\delta L = &\left[\frac{1}{4}\left(g_{kj,i} + g_{ij,k} - g_{ik,j}\right)\bar{\epsilon}\psi^k\dot{\phi}^i\dot{\phi}^j + \frac{1}{2}g_{ij}\overline{\psi}^i\Gamma^j_{kl}\dot{\phi}^k\dot{\phi}^l\epsilon\right] + \frac{1}{2}g_{ij}\bar{\epsilon}\dot{\phi}^i\psi^j + \frac{1}{2}g_{ij}\overline{\psi}^i\ddot{\phi}^j\epsilon + \\ &+ \left[\frac{i}{4}\left(g_{jm,i} + g_{ij,m} - g_{im,j}\right)\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\dot{\psi}^j\right] + \frac{i}{2}\left(g_{ij,m}\Gamma^j_{kl} + g_{ij}\Gamma^j_{kl,m}\right)\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\dot{\phi}^k\psi^l = \\ &= \frac{1}{2}g_{ij}\bar{\epsilon}\dot{\phi}^i\dot{\psi}^j - \frac{1}{2}g_{ij}\epsilon\ddot{\phi}^i\overline{\psi}^j + \frac{1}{2}g_{ki,j}\overline{\psi}^k\dot{\phi}^i\dot{\phi}^j\epsilon + \left[\frac{1}{4}g_{ij,k}\bar{\epsilon}\psi^k\dot{\phi}^i\dot{\phi}^j + \frac{1}{4}g_{ij,k}\epsilon\overline{\psi}^k\dot{\phi}^i\dot{\phi}^j\right] + \\ &+ \frac{i}{2}\partial_m\left(g_{ij}\Gamma^j_{kl}\right)\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\dot{\phi}^k\psi^l \qquad \text{by inserting} \qquad \epsilon\overline{\psi}^i = -\bar{\epsilon}\psi^i\end{aligned}$$

At this purpose, it is worth noticing by the antisymmetry of γ^0:

$$\sum_\alpha \epsilon_\alpha\overline{\psi}^i_\alpha = \sum_{\alpha,\beta}\epsilon_\alpha\psi^i_\beta\gamma^0_{\beta\alpha} = \sum_{\alpha,\beta}-\epsilon_\alpha\psi^i_\beta\gamma^0_{\alpha\beta} = \sum_\beta -\bar{\epsilon}_\beta\psi^i_\beta \tag{A.1}$$

$$\begin{aligned}\delta L = &\frac{1}{2}g_{ij}\bar{\epsilon}\dot{\phi}^i\dot{\psi}^j + \frac{1}{2}g_{ij}\bar{\epsilon}\ddot{\phi}^i\psi^j + \frac{1}{2}g_{ij,k}\bar{\epsilon}\dot{\phi}^k\dot{\phi}^i\psi^j + \frac{i}{2}\partial_m\left(g_{ij}\Gamma^j_{kl}\right)\bar{\epsilon}\psi^m\overline{\psi}^i\gamma^0\dot{\phi}^k\psi^l = \\ &= \frac{1}{2}\bar{\epsilon}\frac{d}{dt}\left(g_{ij}\dot{\phi}^i\psi^j\right) + \frac{1}{2}\bar{\epsilon}\partial_m\left(g_{ij}\Gamma^j_{kl}\psi^m\overline{\psi}^i\gamma^0\dot{\phi}^k\psi^l\right) \to 0\end{aligned}$$

In this way, up to total derivative terms and by choosing periodic boundary conditions to the ψ's, the action is invariant under the aforementioned transformation.

[1]Since the Christoffel symbol in the Riemannian connection is torsion-free and therefore symmetric.

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