Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA
"TULLIO LEVI-CIVITA"

CORSO DI LAUREA MAGISTRALE IN MATEMATICA

Generic Vanishing in Geometria Analitica e Algebrica

Relatore:
Prof.
Ernesto Carlo Mistretta

Laureando:
Stefano Mammola
1132674

Anno Accademico 2016/2017
20 Aprile 2018

Introduction

The goal of this thesis is to illustrate the Generic Vanishing theorem (GVT), specifically the version that Green and Lazarsfeld proved in the late 80's in [GL1], focusing on its proof and some of its applications, in particular on the so called Ueno's conjecture K. To understand these problems we need some definitions and properties from the Hodge theory, in particular on a compact Kähler manifold. In fact, these are the main objects of our studying. A Kähler manifold is a complex manifold equipped with a Hermitian metric whose imaginary part, which is a 2-form of type $(1, 1)$ relative to the complex structure, is closed. We are interested in these objects mostly because the Kähler identities, which are identities between certain operators on differential forms, provide the Hodge decomposition of the de Rham cohomology. We also prove the Kodaira Vanishing theorem, an important theorem on vanishing of cohomology of positive line bundle. While Kodaira's theorem depends on the fact that the first Chern class $c_1(L)$ is positive, the GVT concerns line bundles with $c_1(L) = 0$. This kind of line bundles are topologically trivial, because their underlying smooth line bundle is the trivial bundle $X \times \mathbb{C}$, and they are parametrized by the points of $\mathrm{Pic}^0(X)$. In Chapter 1 we start with some definitions and properties about sheaves cohomology and singular cohomology. Then we compare them with the de Rham cohomology and we give a brief review of the Hodge theory. We start from compact and oriented Riemannian manifolds, here the classes in the de Rham cohomology are uniquely represented by harmonic forms. Then we move to compact Kähler manifolds, where the harmonic theory is compatible with the complex structure. We show the Hodge decomposition on Kähler manifolds. This chapter ends with a description of holomorphic line bundle with trivial first Chern class, extending the results about cohomology of a compact Kähler manifold to cohomology groups of the form $H^0(X, \Omega_X^p \otimes L)$ where $L \in \mathrm{Pic}^0(X)$, the identity component of the Picard group. In Chapter 2 we introduce the theory of complex tori and abelian varieties in order to study two complex tori that one can associate to a compact Kähler manifold, the Picard torus and the Albanese torus, which are dual of each other. Moreover, we describe the Albanese map $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$, with the property that $\mathrm{alb}^*\colon \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$ is an isomorphism. Finally we show that the Picard torus of a smooth projective variety is an abelian variety, therefore the Albanese torus is projective as well. In Chapter 3 we show the famous vanishing theorem of Kodaira, which states that if X is a compact Kähler manifold and L is an ample line bundle on X, then $H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$. In addition we explain the Kodaira Embedding theorem which gives a criterion for a Kähler manifold to be projective. The Chapter ends with a parallel between the Kodaira Vanishing theorem and the Generic Vanishing theorem of Green and Lazarsfeld.

In Chapter 4 we give the proof of the GVT, it says that the i^{th}-cohomology group of a generic topologically trivial line bundle L on a compact Kähler manifold X is zero for all $i < \dim \operatorname{alb}(X)$. We introduce the subsets $S^i_m(X) = \{\xi \in \operatorname{Pic}^0(X) \mid H^i(X, L_\xi) \geq m\} \subseteq \operatorname{Pic}^0(X)$, where L_ξ is the line bundle on X corresponding to $\xi \in \operatorname{Pic}^0(X)$. We study the infinitesimal properties of these loci, at first when X is a complex manifold, then we specify our studying in the case that X is a compact Kähler manifold. In order to prove the theorem an important role is played by the tangent cone theorem and some of its corollaries.

In Chapter 5 we see some applications of the GVT. We look at the Beauville theorem about the structure of $S^1(X)$, without giving the proof that can be found in [Be]. Then we state the Green-Lazarsfeld's theorem about the irreducible components of $S^i_m(X)$ which turn out to be translate of subtori of $\operatorname{Pic}^0(X)$, we follow the proof in [GL2]. The end of the thesis is focused on the varieties of Kodaira dimension zero and on Ueno's Conjecture K, due to Kenji Ueno, which is partially open. We prove the first point of the conjecture, following the proof given by Ein and Lazarsfeld in [EL] that makes use of the GVT.

Ringraziamenti

Ci tengo a ringraziare il mio relatore, il Prof. Ernesto Carlo Mistretta, per tutto il tempo e l'impegno che mi ha dedicato. La sua disponibilità, la sua pazienza e i suoi consigli sono stati di fondamentale importanza per il completamento di questa tesi.
Voglio ringraziare i miei genitori per tutto il sostegno, l'affetto e il coraggio che mi hanno donato in questo percorso e che continuano a darmi giorno dopo giorno. Vi sono grato per aver sempre potuto contare su di voi in qualsiasi situazione. Ringrazio mio fratello Marco, con il quale ho da sempre condiviso non solo tantissime risate e momenti di spensieratezza, ma anche molte discussioni di ogni genere, grazie alle quali sono diventato migliore sia come matematico, ma soprattutto, come persona.
Ringrazio infine una persona che mi sprona continuamente a dare il meglio e allo stesso tempo mi vuole bene per come sono, il mio Amore, Eleonora. Grazie per essermi vicina e riuscire sempre a farmi felice.

Contents

Chapter 1

Preliminaries

We will start with a review of basic facts about Hodge theory on a compact Kähler manifold recalling also the definition of the sheaf cohomology, the singular cohomology and the de Rham cohomology. The useful thing about Kähler manifold is that Kähler conditions makes the harmonic theory compatible with the complex structure.

1.1 Sheaf Cohomology

We want to introduce sheaf cohomology using acyclic resolutions.
Let M a complex manifold.

Definition 1.1.1. *A resolution of a sheaf $\mathscr{F}$ is a complex*

$$0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to \dots$$

together with a homomorphism $0 \to \mathscr{F} \to \mathscr{F}^0$ such that

$$0 \to \mathscr{F} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \dots$$

is an exact complex of sheaves.

Definition 1.1.2. *A sheaf $\mathscr{F}$ is called flasque if for any open subset $U \subseteq M$ the restriction map $r_{u,M} \colon \mathscr{F}(M) \to \mathscr{F}(U)$ is surjective.*

We have

Lemma 1.1.3. *If*

$$0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to 0$$

is a short exact sequence and $\mathscr{F}^0$ is flasque, then the induced sequence

$$0 \to \mathscr{F}^0(U) \to \mathscr{F}^1(U) \to \mathscr{F}^2(U) \to 0$$

is exact for any open $U \subseteq M$.

Now, in order to define sheaves cohomology we need to know if any sheaf can be resolved by flasque sheaves. In fact we have

Proposition 1.1.4. *Any sheaf $\mathscr{F}$ on M admits a resolution*

$$0 \to \mathscr{F} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \dots$$

such that F^i is flasque for all $i \geq 0$.

And now we can give the following

Definition 1.1.5. *The i-th cohomology group $H^i(M, \mathscr{F})$ of a sheaf $\mathscr{F}$ is the i-th cohomology of the complex*

$$\mathscr{F}^0(M) \xrightarrow{\varphi^0} \mathscr{F}^1(M) \xrightarrow{\varphi^1} \mathscr{F}^2(M) \xrightarrow{\varphi^2} \dots$$

induced by a flasque resolution $\mathscr{F} \to \mathscr{F}^\bullet$, i.e.

$$H^i(M, \mathscr{F}) = \frac{\ker(\varphi^i \colon \mathscr{F}^i(M) \to \mathscr{F}^{i+1}(M))}{\operatorname{im}(\varphi^{i-1} \colon \mathscr{F}^{i-1}(M) \to \mathscr{F}^i(M))}.$$

Note that, if $\mathscr{F}$ is flasque then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.
Moreover, from the definition it follows that $H^0(M, \mathscr{F}) = \Gamma(M, \mathscr{F}) = \mathscr{F}(M)$, for any sheaf $\mathscr{F}$.
We also have

Proposition 1.1.6. *If $\mathscr{F} \to \mathscr{F}^\bullet$ and $\mathscr{F} \to \mathscr{G}^\bullet$ are two flasque resolutions of the sheaf F then both define naturally isomorphic cohomology groups.*

This Proposition says that our definition of sheaf cohomology is independent of the chosen flasque resolution.
From this definition we also get a very good explanation of the non-exactness of short exact sequences at level of global section. Indeed we have

Proposition 1.1.7. *Let*

$$0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to 0$$

be a short exact sequence of sheaves on M. Then there exists a long exact cohomology sequence defined as follows

$$\begin{tikzcd}
0 \arrow[r] & H^0(M, \mathscr{F}^0) \arrow[r] & H^0(M, \mathscr{F}^1) \arrow[r] & H^0(M, \mathscr{F}^2) \arrow[dll] \\
& H^1(M, \mathscr{F}^0) \arrow[r] & H^1(M, \mathscr{F}^1) \arrow[r] & H^1(M, \mathscr{F}^2) \arrow[dll] \\
& H^2(M, \mathscr{F}^0) \arrow[r] & H^2(M, \mathscr{F}^1) \arrow[r] & H^2(M, \mathscr{F}^2) \dots
\end{tikzcd}$$

One more definition, by the fact that the space of sections $\Gamma(K, \mathscr{F})$ of $\mathscr{F}$ over the closed set $K \subset M$ can be defined as the direct limit of the spaces of sections over all open neighbourhoods of K, we have

Definition 1.1.8. *A sheaf $\mathscr{F}$ is called soft if the restriction $\Gamma(M, \mathscr{F}) \to \Gamma(K, \mathscr{F})$ is surjective for any closed subset $K \subseteq M$.*

Moreover, soft sheaves are acyclic, i.e. sheaves with trivial higher cohomology groups; and any sheaf of modules over a soft sheaf of commutative rings is soft and hence acyclic.

1.2 Singular (Co)homology

Let X be a topological space. To avoid technicalities, X will always be assumed to be a locally compact, Hausdorff space, and satisfying the second countability axiom.

The definition of singular homology and cohomology uses topological simplexes. The *topological n-simplex* Δ_n is defined as

$$\Delta_n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\}.$$

Singular (co)homology is defined by looking at all possible continuous maps from simplexes to X, where we denote by *singular n-th simplex* a continuous map $f\colon \Delta_n \to X$. And, if X is a differentiable manifold, we call a singular simplex f *differentiable* if it can be extended to a $\mathcal{C}^\infty$-map from a neighbourhood of $\Delta_n \subseteq \mathbb{R}^{n+1}$ to X.

Define the *group of singular n-chains* as the free abelian group

$$\mathcal{S}_n(X) = \mathbb{Z}\left[f\colon \Delta_n \to X \mid f \text{ singular } n\text{-simplex}\right].$$

Define similarly $\mathcal{S}_n^\infty(X)$ the free abelian group of differentiable singular n-chains by requiring that all f are differentiable. Define also the boundary map $\partial_n\colon \mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X)$ as

$$\partial_n f = \sum_{j=0}^{n} (-1)^j f|_{t_j}.$$

Now, we call *the group of singular n-cochains* the free abelian group

$$\mathcal{S}^n(X) = \mathrm{Hom}\left(\mathcal{S}_n(X), \mathbb{Z}\right),$$

and *the group of differentiable singular n-cochains* the free abelian group

$$\mathcal{S}^n_\infty(X) = \mathrm{Hom}\left(\mathcal{S}^\infty_n, \mathbb{Z}\right).$$

Then the adjoint of ∂_{n+1} define the boundary map

$$d_n \colon \mathcal{S}^n_\infty(X) \to \mathcal{S}^{n+1}_\infty(X).$$

So we have

Lemma 1.2.1.

$$\partial_{n-1}\partial_n = 0 \quad \textit{and} \quad d_{n+1}d_n = 0,$$

i.e. the groups $\mathcal{S}^\bullet(X)$ and $\mathcal{S}_\bullet(X)$ define complexes of abelian groups.

Then we can finally state the

Definition 1.2.2. *We define singular homology and cohomology with values in $\mathbb{Z}$ as follows*

$$H^i_{sing}(X, \mathbb{Z}) = H^i\left(\mathcal{S}^\bullet(X), d_\bullet\right),$$

$$H_i^{sing}(X, \mathbb{Z}) = H^i\left(\mathcal{S}_\bullet(X), \partial_\bullet\right).$$

If X is a manifold, we define the differentiable singular (co)homology as follows

$$H^i_{sing,\infty}(X, \mathbb{Z}) = H^i(\mathcal{S}^\bullet_\infty(X), d_\bullet) \ \text{ and } \ H_i^{sing,\infty}(X, \mathbb{Z}) = H^i(\mathcal{S}^\infty_\bullet(X), \partial\bullet).$$

We can extend easily these definitions to a general ring R. In the case $R = \mathbb{R}$ or $R = \mathbb{C}$ we have the following theorem, due to de Rham.

Theorem 1.2.3.

$$H^i_{sing}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} R = H^i(X, R).$$

The proof of this theorem can be done by using the natural map $A^k(X) \to \mathcal{S}^k_{sing}(X)$ given by integration:

$$\alpha \mapsto \left(f \mapsto \int_{\Delta_i} f^*\alpha\right).$$

Moreover, by Stokes' theorem, the map $\alpha \to \int \alpha$ gives a morphism of sheaves from the de Rham complex to the singular cochain complex, and this morphism of acyclic resolution leads to

$$H^k_{DR}(X) \simeq H^k_{sing}(X, \mathbb{R}).$$

To see a more specific description of these facts see [Vo] or [Fu].

1.3 Hodge Theory

1.3.1 On a compact and oriented Riemannian manifold

Let's start from the case of a compact smooth manifold M. Denote by $A^k(M,\mathbb{R})$ the space of smooth real-valued k-forms, and let d be the exterior derivative mapping $A^k(M,\mathbb{R}) \to A^{k+1}(M,\mathbb{R})$. Then define the de Rham cohomology groups of M as

$$H^k_{dR} = \frac{\ker\left(d\colon A^k(M,\mathbb{R}) \to A^{k+1}(M,\mathbb{R})\right)}{\operatorname{im}\ \left(d\colon A^{k-1}(M,\mathbb{R}) \to A^k(M,\mathbb{R})\right)}.$$

A class in $H^k_{dR}(M,\mathbb{R})$ is represented by a closed k-form ω, but it's not unique. Define the Laplace operator $\Delta\colon A^k(M,\mathbb{R}) \to A^k(M,\mathbb{R})$ by the formula $\Delta = d \circ d^* + d^* \circ d$, where d^* is the adjoint operator $d^*\colon A^k(M,\mathbb{R}) \to A^{k-1}(M,\mathbb{R})$ defined by the condition $(d^*\alpha, \beta) = (\alpha, d\beta)$ with $\alpha \in A^k(M,\mathbb{R})$ and $\beta \in A^{k-1}(M,\mathbb{R})$. Then we have that ω is d-closed and of minimal norm if and only if ω is harmonic, i.e. $\Delta\omega = 0$, and we denote by $\mathcal{H}^k(M,\mathbb{R}) = \ker\Delta$ the set of harmonic k-forms. So we have the

Theorem 1.3.1. *Let (M,g) be a compact and oriented Riemannian manifold of dimension n. Then the natural map $\mathcal{H}^k(M,\mathbb{R}) \to H^k_{dR}(M,\mathbb{R})$ is an isomorphism; i.e. every de Rham cohomology class contains a unique harmonic form.*

1.3.2 On a complex manifold

Here we want to extend Hodge theory to the case of a compact complex manifold X of dimension n. We denote by $A^k(X)$ the space of smooth complex-valued differential k-forms on X. Because of the complex structure, we get a decomposition

$$A^k(X) = \bigoplus_{p+q=k} A^{p,q}(X),$$

where $A^{p,q}(X)$ denote the space of differentials (p,q)-forms.

Let $d\colon A^k(X) \to A^{k+1}(X)$ the exterior derivative. Then it decomposes into $d = \partial + \bar{\partial}$, where

$$\partial\colon A^{p,q}(X) \to A^{p+1,q}(X) \text{ and } \bar{\partial}\colon A^{p,q}(X) \to A^{p,q+1}(X).$$

We also have, from $d^2 = 0$, $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$.

Using these differentials we define to kinds of cohomology groups on X. The first is the *de Rham cohomology* defined as

$$H^k_{dR}(X,\mathbb{C}) = \frac{\ker\left(d\colon A^k(X) \to A^{k+1}(X)\right)}{\operatorname{im}\left(d\colon A^{k-1}(X)!A^k(X)\right)}.$$

Moreover, by Poincaré lemma, the complex of sheaves of smooth forms is a soft resolution of the constant sheaf $\mathbb{C}$; this makes $H^k_{dR}(X\mathbb{C})$ canonically isomorphic to $H^k(X,\mathbb{C})$.
The second is the *Dolbeault cohomology* defined as

$$H^{p,q}(X) = \frac{\ker\left(\bar\partial\colon A^{p,q}(X) \to A^{p,q+1}(X)\right)}{\operatorname{im}\left(\bar\partial\colon A^{p,q-1}(X) \to A^{p,q}(X)\right)}.$$

We have, by the holomorphic version of the Poincaré lemma that

$$H^{p,q}(X) \simeq H^q(X,\Omega^p_X).$$

Now, we want restate the results of the previous section in this setting. Then define the Laplace operator $\Delta\colon A^k(X) \to A^k(X)$ and the subspace of harmonic forms $\mathcal{H}^k(X) \subseteq A^k(X)$ as we did before. So, Theorem 1.3.1 becames

$$H^k(X,\mathbb{C}) \simeq H^k_{dR}(X,\mathbb{C}) \simeq \mathcal{H}^k(X).$$

1.3.3 On a compact Kähler manifold

Let X be a complex manifold of dimension n. Choose a Hermitian metric h on the holomorphic tangent bundle of X. Considered as a real vector space of dimension $2n$, the holomorphic tangent space is canonically isomorphic to the tangent space of X, viewed as a smooth manifold of dimension $2n$, then our Hermitian metric induces a Riemannian metric $g = \operatorname{Re} h$ on the tangent bundle of the smooth manifold X, and also a differential form $\omega = -\operatorname{Im} h \in A^2(X,\mathbb{R}) \cap A^{1,1}(X)$. We need the Hermitian metric h to be Kähler if we want the theory of Harmonic forms to interact well with the complex structure of X.

Definition 1.3.2. *A Kähler metric on a complex manifold is a Hermitian metric whose associated $(1,1)$-form is closed. A complex manifold that admits at least one Kähler metric is called a Kähler manifold.*

Proposition 1.3.3. *One has that on a Kähler manifold the Laplace operator satisfies the condition*

$$\frac{1}{2}\Delta = \partial\partial^* + \partial^*\partial = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

Moreover, we have $\Delta A^{p,q}(X) \subseteq A^{p,q}(X)$ and any holomorphic form is both ∂-closed and $\bar{\partial}$-closed.

The Laplace operator Δ also preserves the type of a form, in the following sense: if $\alpha \in A^k(X)$ is harmonic, then its components $\alpha^{p,q} \in A^{p,q}(X)$ are also harmonic, i.e.

$$0 = \Delta\alpha = \sum_{p+q=k} \Delta\alpha^{p,q}$$

since $\Delta\alpha^{p,q} \in A^{p,q}(X)$ then $\Delta\alpha^{p,q} = 0$.

Corollary 1.3.4. *On a compact Kähler manifold X, the space of harmonic forms decomposes by type as*

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$$

with $\mathcal{H}^{p,q}(X)$ denote the space of harmonic (p,q)-forms.

Hodge Decomposition

On a compact Kähler manifold we can use Theorem 1.3.1 to see that every cohomology class contains a unique harmonic representative, obtaining in this way the famous *Hodge decomposition* of the de Rham cohomology. We state it in a way that is independent of the choice of Kähler metric.

Theorem 1.3.5. *Let X be a compact Kähler manifold. Then the cohomology of X admits a direct sum decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

where $H^{p,q}$ is the set of those cohomology classes that are represented by a d-closed form of type (p,q). We have $H^{q,p} = \overline{H^{p,q}}$, where complex conjugation is with respect to the real structure $H^k(X, \mathbb{R})$.
Moreover $H^{p,q}$ is isomorphic to the Dolbeault cohomology group $H^{p,q}(X) \simeq H^q(X, \Omega^p_X)$.

Corollary 1.3.6. *Let X be a compact Kähler manifold , then every holomorphic form is harmonic, and so there is an embedding $H^0(X, \Omega^p_X) \hookrightarrow H^p(X, \mathbb{C})$ whose image is precisely the subspace $H^{p,0}$.*

1.3.4 Hodge theory for line bundles

Here we want to describe analytically all the holomorphic line bundles, on a compact Kähler manifold , with first Chern class equal to zero.

Lemma 1.3.7. *Let X a compact Kähler manifold and let L be a holomorphic line bundle with trivial first Chern class, endowed with the structure given by $\bar{\partial} + \tau$, for $\tau \in \mathcal{H}^{0,1}(X)$. Then the complex*

$$A^{p,0}(X) \to A^{p,1}(X) \to \ldots \to A^{p,n}(X)$$

with differential given by $\bar{\partial} + \tau$, computes the cohomology of $\Omega_X^p \otimes L$.

We can now extend the previous results about the cohomology of a compact Kähler manifold to cohomology groups of the form

$$H^q(\Omega_X^p \otimes L)$$

with $L \in \mathrm{Pic}^0(X)$ such that $c_1(L) = 0$.

We give the corresponding version of the Hodge theorem in our current situation

Theorem 1.3.8. *With the notation as above, every class in $H^q(\Omega_X^p \otimes L)$ is uniquely represented by a smooth form $\alpha \in A^{p,q}(X)$ satisfying*

$$(\bar{\partial} + \tau)\alpha = (\bar{\partial} + \tau)^*\alpha = (\partial - \overline{\tau})\alpha = (\partial - \overline{\tau})^*\alpha = 0$$

All global holomorphic sections of $\Omega_X^p \otimes L$ lie in the kernel of the connection $\nabla = d + \tau + \overline{\tau}$, which is integrable because $\nabla \circ \nabla = 0$.

This theory has one application which is very surprising. Let's suppose that $f \in A^0(X)$ is a smooth function such that $(\bar{\partial} + \tau)f = 0$. By the above Theorem, we automatically get $(\partial - \overline{\tau})f = 0$ as well. This means that the two differential equations are somehow coupled to each other, due to the fact that X is a compact Kähler manifold . This leads to the following

Corollary 1.3.9. *There is an isomorphism of vector spaces*

$$\overline{H^q(\Omega_X^p \otimes L)} \simeq H^p(\Omega_X^q \otimes L^{-1}).$$

Finally, another really interesting theorem

Theorem 1.3.10 (Cartan-Serre-Grothendieck). *Let X be a compact Kähler manifold and let L be a line bundle on X. The following are equivalent:*

1. *L is ample;*
2. *for every coherent sheaf $\mathscr{F}$ on X there exist a integer $m_0 = m_0(\mathscr{F}) \geq 0$ such that*
$$H^i(X, \mathscr{F} \otimes L^{\otimes m}) = 0 \text{ for all } i > 0, \text{ and } m \geq m_0;$$
3. *for every coherent sheaf $\mathscr{F}$ on X there exist a integer $m_1 = m_1(\mathscr{F}) \geq 0$ such that $\mathscr{F} \otimes L^{\otimes m}$ is generated by its global sections for all $m \geq m_1$;*
4. *there exist a integer $m_2 = m_2(\mathscr{F}) > 0$ such that $L^{\otimes m}$ is very ample for all $m \geq m_2$.*

Chapter 2

Albanese Variety

2.1 Complex Tori

A *lattice* in a complex vector space $\mathbb{C}^g$ is by definition a complex subgroup of maximal rank in $\mathbb{C}^g$. So, it's a free abelian group of rank $2g$. A *complex torus* is a quotient $X = \mathbb{C}^g/\Lambda$, where Λ is a lattice in $\mathbb{C}^g$. This complex torus X is a complex manifold of dimension g that inherits the structure of a complex Lie group from the vector space $\mathbb{C}^g$. Recall that a *Lie group* of dimension g is a compact connected complex manifold of dimension g with a group structure on the underlying set such that the maps defined by $(x, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are holomorphic. A meromorphic function on $\mathbb{C}^g$ periodic with respect to Λ can be considered as a function on X. An *abelian variety* is a complex torus admitting enough meromorphic functions.

More general, let V be a complex vector space of dimension g and let Λ be a lattice in V, then by definition Λ is a discrete subgroup of V of rank $2g$. Moreover the lattice Λ acts on V by addition. So, the quotient

$$X = V/\Lambda$$

is called *complex torus*. It's easy to see that it's a connected complex manifold, and X it's also compact; because Λ is of maximal rank as a subgroup of V. The addition of V induces a structure of complex Lie group on X. Moreover one can prove that any connected compact complex Lie group of dimension g is a complex torus. In fact, one can first show that any compact connected complex Lie group Y is commutative. Then we have that the exponential map $\exp\colon T_0Y \to Y$ is a surjective homomorphism of complex Lie groups whose kernel is a lattice in T_0Y, where T_0Y is the tangent space of Y at the point 0. Letting $\ker(\exp) = \Lambda$, exp induces an isomorphism of Lie groups

$$T_0Y/\Lambda \xrightarrow{\simeq} Y$$

Therefore Y is a complex torus.

For $n \in \mathbb{Z}$, $n \neq 0$, let X_n be the subgroup of elements annihilated by n, i.e. X_n is the kernel of the map $x \mapsto nx$, called *the group of n-division points* of X. Then

$$X_n \simeq (\mathbb{Z}/n\mathbb{Z})^{2g} .$$

For a complex torus $X = V/\Lambda$, the vector space V can be seen as the universal covering space. If we denote

$$\pi \colon V \to X$$

the universal covering map, then $\ker(\pi) = \Lambda$ can be identified with the fundamental group $\pi_1(X) = \pi_1(X, 0)$, that is isomorphic to the first homology group $H_1(X, \mathbb{Z})$, since Λ is abelian. So, the torus X is locally isomorphic to V, then we can consider V as the tangent space T_0X of X at the point 0. From the Lie theoretic point of view, $\pi \colon V = T_0X \to X$ is just the exponential map.

Now, we want to compute the singular cohomology groups of complex tori with values in $\mathbb{Z}$. As a real manifold X is isomorphic to $(\mathbb{R}/\mathbb{Z})^{2g} \simeq (S_1)^{2g}$, where S_1 is the circle group. From above, we have an identification $\pi_1(X) = H^1(X, \mathbb{Z}) = \Lambda$ that induces an isomorphism

$$H^1(X, \mathbb{Z}) \simeq \mathrm{Hom}(\pi_1(X), \mathbb{Z}).$$

Proposition 2.1.1. *We have a natural isomorphism induced by the cup product*

$$\bigwedge^n H^1(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$$

for every $n \geq 1$.

Then, we can identify $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$, and if we denote by $\mathrm{Alt}^n(\Lambda, \mathbb{Z}) = \bigwedge^n \mathrm{Hom}(\Lambda, \mathbb{Z})$ the group of $\mathbb{Z}$-valued alternating n-forms on Λ we get

Corollary 2.1.2. *There is, for every $n \geq 1$, a canonical isomorphism*

$$H^n(X, \mathbb{Z}) \simeq \mathrm{Alt}^n(\Lambda, \mathbb{Z}).$$

Moreover $H^n(X, \mathbb{Z})$ and $H_n(X, \mathbb{Z})$ are free $\mathbb{Z}$-modules of rank $\binom{2g}{n}$ for all $n \geq 1$.

Now, we want to compute $H^q(X, \Omega^p_X)$, where Ω^p_X is the sheaf of the holomorphic p-forms on X. Let $T = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ the complex cotangent space to X at 0. By translation with respect to the group law on X, every complex p-covector $\alpha \in \bigwedge^p T$ extends to a translation invariant holomorphic p-form ω_α on X. In fact, let $x \in X$, the translation t_{-x} induces a vector space isomorphism $dt_{-x} \colon T_{X,x} \to T_{X,0}$; using the dual isomorphism $(dt_{-x})^* \colon \Omega^1_{X,0} \to \Omega^1_{X,x}$ we can define $(\omega_\alpha)_x = (\wedge^p (dt_{-x})^*)\,\alpha$. Moreover the map $\alpha \mapsto \omega_\alpha$ define a homomorphism of sheaves

$$\mathscr{O}_X \otimes_{\mathbb{C}} \bigwedge^p T \to \Omega^p_X \tag{2.1}$$

which is actually an isomorphism. Thus, Ω^p_X is a globally free sheaf of $\mathscr{O}_X$-modules. Since the only global sections of $\mathscr{O}_X$ are the constants, the global sections of Ω^p_X are the translation-invariant p-forms ω_α. In fact, for the isomorphism 2.1 we have

$$H^q(X, \Omega^p_X) \simeq H^q(X, \mathscr{O}_X \otimes_{\mathbb{C}} \bigwedge^p T) \simeq H^q(X, \mathscr{O}_X) \otimes_{\mathbb{C}} \bigwedge^p T.$$

We have the following

Theorem 2.1.3. *Let* $\overline{T} = \mathrm{Hom}_{\mathbb{C}-antilinear}(V, \mathbb{C})$. *Then there is a natural isomorphism*

$$H^q(X, \mathscr{O}_X) \simeq \bigwedge^q \overline{T}$$

for all $q \geq 0$. *In particular, for every pair* (p, q) *there is a natural isomorphism*

$$H^q(X, \Omega^p_X) \simeq \bigwedge^p T \otimes \bigwedge^q \overline{T}.$$

The proof depend on the Dolbeault resolution

$$0 \to \mathscr{O}_X \to \mathscr{C}^{0,0} \xrightarrow{\bar\partial} \mathscr{C}^{0,1} \xrightarrow{\bar\partial} \cdots$$

where $\mathscr{C}^{p,q}$ is the sheaf of $\mathscr{C}^\infty$ complex-valued differential forms of type (p, q) on X and $\bar\partial$ is the component of the exterior derivative d mapping $\mathscr{C}^{p,q}$ to $\mathscr{C}^{p,q+1}$. This resolution define an isomorphism

$$H^q(X, \mathscr{O}_X) = \frac{\left\{\bar\partial\text{-closed } (0, q)\text{-forms on } X\right\}}{\bar\partial\,\{\text{space of } (0, q-1)\text{-forms on } X\}}.$$

The same method can used to compute the cohomology of the de Rham complex. We know that $H^n(X, \mathbb{C}) = H^n(X, \mathbb{Z}) \otimes \mathbb{C}$, so denoting by $\mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C})$

the group of $\mathbb{R}$-linear alternating n-forms on V with values in $\mathbb{C}$ and applying the canonical isomorphism $\mathrm{Alt}^n(\Lambda, \mathbb{Z}) \otimes \mathbb{C} = \mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C})$ we get the canonical isomorphisms

$$H^n(X, \mathbb{C}) \simeq \mathrm{Alt}^n_{\mathbb{R}}(V, \mathbb{C}) \simeq \bigwedge^n \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \simeq \bigwedge^n H^1(X, \mathbb{C})$$

for all $n \geq 1$. Now, let $\mathscr{C}^n = \bigoplus_{p+q=n} \mathscr{C}^{p,q}$ the sheaf of $\mathscr{C}^\infty$ complex-valued n-forms. Then

$$0 \to \mathbb{C} \to \mathscr{C}^0 \xrightarrow{d} \mathscr{C}^1 \xrightarrow{d} \cdots$$

is a resolution of the constant sheaf $\mathbb{C}$, then

$$H^n(X, \mathbb{C}) \simeq H^n_{DR}(X) = \frac{\{d\text{-closed } n\text{-forms}\}}{d\,\{(n-1)\text{-forms}\}}.$$

Just as in the case of $(0, q)$-forms we obtain the result: in every class of n-forms in $H^n_{DR}(X)$ we can distinguish a uniquely determined representative. In fact, for all d-closed n-form ω, there is a unique translation invariant n-form ω_α, with $\alpha \in \bigwedge^n \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ such that

$$\omega - \omega_\alpha = d\mu$$

for some $(n-1)$-form μ. Then $H^n(X, \mathbb{C}) = \bigwedge^n \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$; taking cup product on the left side to the exterior product on the right side. Moreover, since we have $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = T \oplus \overline{T}$, this shows the following

Theorem 2.1.4. *The De Rham and Dolbeault isomorphism induces*

$$\begin{aligned} H^n(X, \mathbb{C}) &\simeq \bigwedge^n T \oplus \overline{T} \\ &\simeq \bigoplus_{p+q=n} \left(\bigwedge^p T \otimes \bigwedge^q \overline{T} \right) \\ &\simeq \bigoplus_{p+q=n} H^q(X, \Omega^p_X) \end{aligned}$$

for every $n \geq 0$, with $H^q(X, \Omega^p_X)$ the sheaf of holomorphic p-forms on X. and it's the Hodge Decomposition.

2.2 Abelian Varieties

We now want to study algebraic varieties over an algebraically closed field k.

Definition 2.2.1. *An abelian variety X is a complete algebraic variety over k, which implies in particular that it's irreducible, with a group law $m\colon X \times X \to X$ such that m and it's inverse map are both morphisms of varieties.*

Note that if $k = \mathbb{C}$, the underlying complex analytic space of an abelian variety is a compact complex analytic group, then it is a complex torus.

When $k \neq \mathbb{C}$ an abelian variety has the following properties which are similar to those of a complex torus.

Properties

1. X is a commutative and divisible group. Moreover, if n_X denotes the multiplication by a positive integer n, then its kernel X_n has the following structure
$$X_n \simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \text{ if } \operatorname{char} k \nmid n$$
$$X_{p^m} \simeq (\mathbb{Z}/p^m\mathbb{Z})^{i} \text{ if } p = \operatorname{char} k, m > 0$$
with $0 \leq i \leq g = \dim X$.

2. We have canonical isomorphism
$$H^q(X, \Omega_X^p) \simeq \bigwedge^p \left(H^0(X, \Omega_X^1)\right) \oplus_k \bigwedge^q \left(H^1(X, \mathscr{O}_X)\right)$$
and $\dim H^0(X, \Omega_X^1) = \dim H^1(X, \mathscr{O}_X) = g$.

3. There is an exact sequence
$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to \operatorname{NS}(X) \to 0$$
where $\operatorname{Pic}^0(X)$ has a natural structure of abelian variety, and $\operatorname{NS}(X)$ is a finitely generated abelian group.

4. An abelian variety is everywhere non-singular. In fact, if there exists a non-singular point $x_0 \in X$, then for $x \in X$ the translation morphism $T_{x,x_0^{-1}}\colon X \to X$ is an automorphism of X sending x_0 to x, so that x is again a non-singular point, which is a contradiction.

5. Let $T = T_0X$ be the tangent space of X at 0, and let $\Omega_{X,0}$ be its dual space $(T_0X)^*$ of differentials. Then there is a natural isomorphism
$$\Omega_{X,0} \otimes_k \mathscr{O}_X \to \Omega^1_X$$
where Ω^1_X is the sheaf of regular 1-forms on X. This map is defined as follows: for each $\theta \in \Omega_{X,0}$ consider the 1-form $\omega_\theta \in X$ defined by $(\omega_\theta)_x = T^*_{-x}(\theta)$, that is the unique translation invariant 1-form on X whose value at 0 is θ. It can easily be checked that is a regular 1-form on X. Moreover, since X is connected and complete and $H^0(X, \mathscr{O}_X) = k$, then, from that isomorphism, the everywhere regular forms on X are precisely the invariant forms.

The following theorem connects abelian varieties and complex tori.

Theorem 2.2.2. *Let $X = V/\Lambda$ a complex torus of dimension g. The following are equivalent*

(i) X is an abelian variety;

(ii) X is the complex space associated to a projective algebraic variety;

(iii) there are g algebraically independent meromorphic functions on X;

(iv) there exists a positive definite Hermitian form H on V such that its imaginary part is integral on $\Lambda \times \Lambda$. An hermitian form with these properties is called polarization *od the abelian variety X.*

2.3 Albanese Variety for Kähler manifolds

In this section we associate to any compact Kähler manifold X of dimension $n \geq 1$ the Albanese torus $\mathrm{Alb}(X)$ and the Picard torus $\mathrm{Pic}^0(X)$. (cf.[Ke] or [BH]).

Let X a compact Kähler manifold of dimension $n \geq 1$. From the Hodge decomposition
$$H^1(X, \mathbb{C}) = H^1(X, \mathscr{O}_X) \oplus H^0(X, \Omega^1_X), \tag{2.2}$$
recall that $\overline{H^1(X, \mathscr{O}_X)} \simeq H^0(X, \Omega^1_X)$, defined $H^1(X)_{\mathbb{Z}}$ as follows
$$H_1(X)_{\mathbb{Z}} = H_1(X, \mathbb{Z})/\text{torsion}$$
then it is a free abelian group of rank $2q$, where $q = h^0(X, \Omega^1_X)$. By Stoke's theorem every element $\gamma \in H_1(X)_{\mathbb{Z}}$ yields a linear form on the space $H^0(\omega_{\mathbb{C}})$ which we also denote by γ
$$\gamma \colon H^0(X, \Omega^1_X) \to \mathbb{C},\ \omega \mapsto \int_\gamma \omega.$$

We also have that the map $H_1(X)_{\mathbb{Z}} \to H^0(\Omega_X)^*$ is injective. It follows that $H_1(X)_{\mathbb{Z}}$ is a lattice in $H^0(X,\Omega^1_X)^*$, then the quotient defined by

$$\mathrm{Alb}(X) = H^0(X,\Omega^1_X)^*/H_1(X)_{\mathbb{Z}}$$

is a complex torus of dimension q, and it's called the *Albanese variety* of X.
Remark For any complex torus $Z = V/\Lambda$, we get $V = (H^0(Z,\Omega^1_Z))^*$ and $\Lambda = H_1(X,\mathbb{Z})$, then

$$\mathrm{Alb}(Z) = Z.$$

Definition 2.3.1. *Fix $x_0 \in X$, then for any point $x \in X$ we can choose a path γ from x_0 to x mod $H_1(X)_{\mathbb{Z}}$ and we have a map*

$$\begin{aligned} \mathrm{alb}_X \colon X &\to \mathrm{Alb}(X) \\ x &\mapsto \left(\omega \mapsto \int_{x_0}^{x} \omega\right) \quad \mathrm{mod}\ H_1(X)_{\mathbb{Z}} \end{aligned}$$

that it's called the Albanese map *of X with base point x_0, where $\int_{x_0}^{x} \omega = \int_\gamma \omega$. Note that by Stoke's theorem, the integral is independent from the choice of γ and so the map is well-defined.*

Corollary 2.3.2. *Let X be a smooth projective variety, and let $x_0 \in X$. Then there is an integer n such that the holomorphic map*

$$\begin{aligned} \mathrm{alb}^n_X \colon X^n &\to \mathrm{Alb}(X) \\ (x_1, \cdots, x_n) &\mapsto \sum_{j=1}^{n} \mathrm{alb}_X(x_j) \end{aligned}$$

is surjective. In particular $\mathrm{alb}_X(X)$ generates $\mathrm{Alb}(X)$ as a group.

Lemma 2.3.3. *The differntial $d\,\mathrm{alb}_X$ of the Albanese map alb_X in a point $x \in X$ is given by the linear map*

$$T_xX \to \mathrm{Hom}\big(H^0\left(X,\Omega^1_X\right), \mathbb{C}\big), v \mapsto (\omega \mapsto \omega(v)).$$

Then the codifferential $(d\,\mathrm{alb}_X)^* \colon H^0\left(X,\Omega^1_X\right) \otimes \mathscr{O}_X \to \Omega^1_X$ is just the evaluation map.

Theorem 2.3.4 (Universal Property of the Albanese Torus). *Let $\varphi \colon X \to M$ be a holomorphic map into a complex torus M. There exists a unique homomorphism $\tilde{\varphi} \colon \mathrm{Alb}(X) \to M$ of complex tori such that the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & M \\ \downarrow{\scriptstyle \mathrm{alb}_X} & & \downarrow{\scriptstyle t_{-\varphi(x_0)}} \\ \mathrm{Alb}(X) & \xrightarrow{\tilde{\varphi}} & M \end{array}$$

is commutative.

As a consequence we obtain that the Albanese map is functorial:

Corollary 2.3.5. *Let $f\colon X \to Y$ a morphism of compact Kähler manifolds. Then there is a homomorphism of complex tori $\tilde{f}$ such that the following*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{alb}_X"] & Y \arrow[d, "\mathrm{alb}_Y"] \\ \mathrm{Alb}(X) \arrow[r, "\tilde{f}"] & \mathrm{Alb}(Y). \end{tikzcd}$$

We want now to define the Picard torus of X. First, recall that we have that the following

$$H^1(X,\mathbb{R}) \to H^1(X,\mathbb{C}) = H^0(X,\Omega^1_X) \oplus \overline{H^0(X,\Omega^1_X)} \xrightarrow{pr} \overline{H^0(X,\Omega^1_X)}$$

is injective, since every real differential 1-form is of the form $\alpha + \overline{\alpha}$ for some $\alpha \in H^0(X,\Omega^1_X)$. So, define $H^1_{\mathbb{Z}}(X)$ as the image of $H^1(X,\mathbb{Z})$ in $\overline{H^0(X,\Omega^1_X)}$ and then we can consider the quotient

$$\mathrm{Pic}^0(X) = \overline{H^0(X,\Omega^1_X)}/H^1_{\mathbb{Z}}(X).$$

$\mathrm{Pic}^0(X)$ is the identity component of the Picard group of X. It's a complex torus, because $\mathrm{rk} H^1_{\mathbb{Z}}(X) = \dim_{\mathbb{C}} H^1(X,\mathbb{C}) = 2\dim_{\mathbb{C}} \overline{H^0(X,\Omega^1_X)}$. It is called the *Picard torus* of X. Note that the construction of $\mathrm{Pic}^0(X)$ if functorial, in the sense that if $f\colon X \to Y$ is a holomorphic map of compact Kähler manifolds, then the pull-back f^* of holomorphic 1-form induces a homomorphism of complex tori

$$f^*\colon \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X).$$

The Picard torus can be identified with the set of line bundles on X with trivial first Chern class.

Proposition 2.3.6. *For any compact Kähler manifold X there is a canonical isomorphism*

$$\mathrm{Pic}^0(X) \simeq \ker\left\{c_1\colon H^1(X,\mathscr{O}_X) \to H^2(X,\mathbb{Z})\right\}.$$

Proof. From the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_X \xrightarrow{\exp} (\mathscr{O}_X)^* \to 0$$

we get the long exact sequence in cohomology

$$H^1(X,\mathbb{Z}) \to H^1(X,\mathscr{O}_X) \to H^1(X,(\mathscr{O}_X)^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to \cdots.$$

Then, by Hodge duality we obtain

$$\ker c_1 = H^1(X,\mathscr{O}_X)/\left(\mathrm{im}\ H^1(X,\mathbb{Z})\right) = \overline{H^0(X,\Omega^1_X)}/H^1_{\mathbb{Z}}(X) = \mathrm{Pic}^0(X).$$

□

Our claim now is that if X is smooth and projective then $\mathrm{Pic}^0(X)$ is an abelian variety; then $\mathrm{Alb}(X)$ would be an abelian variety as well. To his end, let $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ be the first Chern class of the line bundle $\mathscr{O}_X(1)$.

Lemma 2.3.7. *The hermitian form*

$$H\colon \overline{H^0(X, \Omega^1_X)} \times \overline{H^0(X, \Omega^1_X)} \to \mathbb{C}$$

given by

$$(\varphi, \psi) \mapsto -2i \int_X \omega^{n+1} \wedge \varphi \wedge \overline{\psi}$$

define a polarization on $\mathrm{Pic}^0(X)$*. We call it the canonical polarization of* $\mathrm{Pic}^0(X)$*.*

Proof. For $\varphi, \psi \in H^1_{\mathbb{Z}}(X) \subseteq \overline{H^0(X, \Omega^1_X)}$ the sum $\varphi + \overline{\varphi}$ and $\psi + \overline{\psi}$ are integral $(1,1)$-forms on $H^1_{\mathbb{Z}}(X)$. In fact we have

$$\begin{aligned}
\mathrm{Im}\, H(\varphi, \psi) &= \frac{1}{2i}\left(H(\varphi, \psi) - H(\psi, \varphi)\right) \\
&= -\int_X \omega^{n+1} \wedge \varphi \wedge \overline{\psi} + \int_X \omega^{n+1} \wedge \psi \wedge \overline{\varphi} \\
&= -\int_X \omega^{n+1} \wedge (\varphi + \overline{\varphi}) \wedge (\psi + \overline{\psi}) \\
&\in \mathbb{Z}.
\end{aligned}$$

So, we have an hermitian form $H\colon \Lambda \times \Lambda \to \mathbb{Z}$, we only have to show that it's positive definite. For this we are going to use the Hodge operator $*\colon H^{p,q}(X) \to H^{n-p,n-q}(X)$ (see [GH, section 4.4]). We get an inner hermitian product defined as follows

$$\langle\ ,\ \rangle\colon H^{p,q}(X) \times H^{p,q}(X) \to \mathbb{C},\ \langle \varphi, \psi \rangle = \int_X \varphi \wedge *\psi.$$

Now, it can be proved that for every 1-form $\varphi \in \overline{H^0(X, \Omega^1_X)}$ one has

$$*\varphi = \frac{-i}{(n-1)!}\omega^{n+1} \wedge \overline{\varphi}.$$

Using these results, from [Hu], we have now that for any nonzero $\varphi \in \overline{H^0(X, \Omega^1_X)}$

$$\begin{aligned}
H(\varphi, \varphi) &= -2i \int_X \omega^{n+1} \wedge \varphi \wedge \overline{\varphi} \\
&= 2(n-1)! \int_X \varphi \wedge *\varphi > 0.
\end{aligned}$$

□

Then, similarly we obtain the following

Corollary 2.3.8. *For any smooth projective variety X, the complex torus* $\mathrm{Alb}(X)$ *is an abelian variety.*

Proposition 2.3.9. *Let* $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ *be the Albanese map of a smooth projective variety X. Then*

$$\mathrm{alb}_X^* \colon \mathrm{Pic}^0\left(\mathrm{Alb}(X)\right) \to \mathrm{Pic}^0(X)$$

is an isomorphism.

Proof. We have that the pullback mapping

$$\mathrm{alb}_X^* \colon H^0\left(\mathrm{Alb}(X), \Omega_X^1\right) \to H^0(X, \Omega_X)$$

is an isomorphism, by Theorem 2.1.4 applied to $\mathrm{Alb}(X)$. This means that every holomorphic one-form on X is the pullback of a holomorphic one-form from $\mathrm{Alb}(X)$. And we know from (2.2) that every class in $H^1(X, \mathbb{C})$ can be uniquely written as the sum of holomorphic one-form and the conjugate of a holomorphic one-form

$$\mathrm{alb}_X^* \colon H^1(\mathrm{Alb}(X), \mathbb{C}) \to H^1(X, \mathbb{C})$$

is an isomorphism. □

Chapter 3

Vanishing Theorems

Recall some definitions. Let X a projective variety and let L a line bundle on X. L is said to be *very ample* if there exist a closed embedding $f\colon X \to \mathbb{P}^N$, where $N = \dim H^0(X, L) - 1$, such that

$$L = f^*\mathscr{O}_{\mathbb{P}^N}(1).$$

L is called *ample* if $L^{\otimes n}$ is very ample for some $n \in \mathbb{N}$.
Note that L is very ample if and only if there exist a closed embedding $f\colon X \to \mathbb{P}^n$ for some $n \in \mathbb{N}$ and there exists $m \in \mathbb{N}$ such that $L^{\otimes m} \simeq f^*\mathscr{O}_{\mathbb{P}^n}(1)$

3.1 Kodaira Vanishing Theorem

Here we want to give a proof of the Kodaira vanishing theorem using the results of Hodge theory.

Theorem 3.1.1 (Kodaira Vanishing). *Let L be an ample line bundle on a smooth projective variety X. Then*

$$H^i(X, \omega_X \otimes L) = 0$$

for every $i > 0$.

There is a very elegant proof based on Serre's Theorem 1.3.10 and the following theorem

Theorem 3.1.2. *Let X be a smooth projective variety, L a line bundle on X, and $s \in H^0(X; L^{\otimes N})$ a nontrivial section whose divisor is smooth. Then the mapping*

$$H^j(X, \omega_X \otimes L) \to H^j(X, \omega_X \otimes L^{\otimes N+1})$$

induced by multiplying by s is injective.

We can now prove the Kodaira Vanishing Theorem (cf. [EV]).

Proof. Suppose that L is an ample line bundle on a smooth projective variety X. Since $L^{\otimes N}$ is very ample for large N, it certainly has global sections whose divisors are smooth. Theorem 3.1.2 therefore gives us an injection

$$H^j(X, \omega_X \otimes L) \hookrightarrow H^j(X, \omega_X \otimes L^{\otimes N+1}).$$

But for sufficiently large values of N, the group on the right-hand side vanishes for $j > 0$ by Serre's theorem. Consequently, $H^j(X; \omega_X \otimes L) = 0$ for $j > 0$, as desired. □

3.2 Kodaira Embedding Theorem

Recall, first, the following definitions. A line bundle L is called *positive* if its first Chern class $c_1(L) \in H^2(X, \mathbb{R})$ can be represented by a closed positive real $(1,1)$-form.
Note that a compact complex manifold X that admits a positive line bundle L is automatically Kähler . Indeed, the closed positive real $(1,1)$-form representing $c_1(L)$ defines a Kähler structure on X.

Not every compact complex manifold is Kähler and not every compact Kähler manifold is projective. Of course, one would like to have a criterion that decides whether a Kähler manifold is projective. Such a criterion is provided by the Kodaira embedding theorem which will be proved now.

Let L be a holomorphic line bundle on a complex manifold X. A point $x \in X$ is a *base point* of L if $s(x) = 0$ for all $s \in H^0(X, L)$. The *base locus* $\mathrm{Bs}(L)$ is the set of all base points of L. Clearly, if $s_0, \dots, s_N \in H^0(X, L)$ is a basis of global sections then $\mathrm{Bs}(L) = Z(s_0) \cap \ldots \cap Z(s_N)$ is an analytic subvariety.
Moreover, if L is a holomorphic line bundle on a complex manifold X and suppose that $s_0, \dots, s_N \in H^0(X, L)$ is a basis. Then

$$\varphi_L \colon X \setminus \mathrm{Bs}(L) \to \mathbb{P}^N, x \mapsto [s_0(x), \dots, s_N(x)]$$

defines a holomorphic map such that $\varphi_L^* \left(\mathscr{O}_{\mathbb{P}^N}\right)(1) \simeq L|_{X \setminus \mathrm{Bs}(L)}$.

Remark The map φ_L is said to be associated to the *complete linear system* $H^0(X, L)$, whereas a subspace of $H^0(X, L)$ is simply called a *linear system* of L, and we usually denote it by $|L|$. We say that L is *globally generated* by the sections $s_0, \dots, s_N$ if $\mathrm{Bs}(L, s_0, \dots, s_N) = \emptyset$.

Equivalently, given $x \in X$, we can define $\varphi_L(x)$ to be the hyperplane H_x in $\mathbb{P}(H^0(X,L))$ consisting of those sections vanishing at x

$$\begin{aligned} \varphi_L \colon X &\to \mathbb{P}\left(H^0(X,L)\right) \\ x &\mapsto H_x = \left\{ s \in H^0(X,L) \mid s(x) = 0 \right\}. \end{aligned}$$

A question that rise spontaneously from this discussion is: Given a holomorphic line bundle L on X, when is that $\varphi_L \colon X \to \mathbb{P}^N$ an embedding? First, note that, in order fro φ_L to be well-defined, the linear system $|L|$ can't have any base points, it means that the restriction map

$$H^0(X,L) \to L(x)$$

must be surjective for every $x \in X$. Notice that this map sits in the long exact cohomology sequence associated to

$$0 \to L \otimes \mathscr{I}_x(L) \to L \to L(x) \to 0$$

where we denote $\mathscr{I}_x \subset \mathscr{O}$ the sheaf of holomorphic functions on X vanishing at x, and $\mathscr{I}_x(L)$ the sheaf of sections of L vanishing at x.

Now, we can say that φ_L will be an embedding if

1. φ_L is one-to-one. This happens if and only if for all x and y in X there is a section $s \in H^0(X,L)$ vanishing at x but not at y; i.e. if and only if the restriction map

$$H^0(X,L) \to L(x) \otimes L(y) \tag{3.1}$$

is surjective for all $x \neq y$ in X. We say that φ_L or L separates points. Note that if L satisfies this condition, then $|L|$ is base point free. Moreover, as before, this map is induced by a short exact sequence:

$$0 \to L \otimes \mathscr{I}_{x,y} \to L \to L(x) \otimes L(y) \to 0.$$

2. φ_L has non-zero differential everywhere. It means that for any $x \in X$ the differential $d\varphi_{L,x} \colon T_xX \to T_{\varphi_{L(x)}}\mathbb{P}^N$ is injective. Let ψ_a a local trivialization of L near x, this is the case if and only if for every $v^* \in \bigwedge_x^1 X$, the cotangent space at x, there exists $s \in H^0(X,L)$ such that $s_a(x) = 0$ and $ds_a(x) = v^*$, where $s_a = \psi^* s$. Let us reformulate this as follows. If s is any section in $\mathscr{I}_x(L)$, defined near x and ψ_α, ψ_β local trivializations of L in a neighbourhood U of x, so if $s_\alpha = \psi_\alpha^* s$, $s_\beta = \psi_\beta^* s$ and $s_\alpha = g_{\alpha,\beta} s_\beta$, then we have

$$\begin{aligned} ds_\alpha &= ds_\beta \cdot g_{\alpha,\beta} + dg_{\alpha,\beta} \cdot s_\beta \\ &= ds_\beta \cdot g_{\alpha,\beta} \end{aligned}$$

at x. So, consider the map $d_x\colon H^0(X, L\otimes \mathscr{I}_x) \to L\otimes \bigwedge_x^1 X$ defined by $s \mapsto d(\psi s)_x$, for any s vanishing at x. Then d_x is independent of the choice of ψ. Hence, $d\varphi_L$ is injective if and only if

$$d_x\colon H^0(X, L\otimes \mathscr{I}_x) \to L\otimes \bigwedge_x^1 X \tag{3.2}$$

is surjective. Also here the map d_x is induced by a short exact sequence which in this case takes the form

$$0 \to L\otimes \mathscr{I}_x^2 \to L\otimes \mathscr{I}_x \to L\otimes \bigwedge_x^1 X \to 0.$$

Summarizing, we find out that the complete linear system $|L|$ induces a closed embedding $\varphi_L\colon X\to \mathbb{P}^N$ if and only if the global sections of L separate points $x\neq y\in X$ and tangent directions $v\in T_xX$.

Now, recall that a line bundle L on a compact complex manifold X is called *ample* if and only if $L^{\otimes k}$, for some $k>0$, defines a closed embedding $\varphi_L\colon X\to\mathbb{P}^N$.

Theorem 3.2.1 (**Kodaira Embedding**). *Let X be a compact Kähler manifold, and L a line bundle on X. Then L is positive if and only if L is ample and so if an only if there is embedding*

$$\phi\colon X\hookrightarrow \mathbb{P}^n$$

of X into some projective space.

Proof. The proof is based on the observations above. To see more details see [GH, cap.4] or [Hu, section 5.3]. □

So far we know from Kodaira vanishing theorem that, if X a is compact Kähler manifold and L is an ample line bundle on X, then $H^i(X, L\otimes\omega_X)=0$ for $i>0$. We can go further, if we consider only a generic line bundle $P\in \mathrm{Pic}^0(X)$, what we are going to prove is that we still have a Kodaira-type vanishing theorem but not for all $i>0$. Indeed, for $q>n-\dim \mathrm{alb}(X)$ we have $H^q(X,\omega_X\otimes P^*)=0$, where $\mathrm{alb}\colon X\to \mathrm{Alb}(X)$ is the Albanese map

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}"] \arrow[dr, two heads] & \mathrm{Alb}(X) \\ & \mathrm{alb}(X) \arrow[u, hook] \end{tikzcd}$$

and $n-\dim\mathrm{alb}(X)$ is the dimension of the generic fiber.

Our purpose is to study the cohomological properties of topologically trivial holomorphic line bundles on a compact Kähler manifold, applying deformation theory to prove some conjectures of Beauville and Catanese concerning the vanishing of its cohomology groups. The essential idea for the proof of the Generic Vanishing Theorem is to study the deformation theory of the groups $H^i(X, L)$ as L varies over $\mathrm{Pic}^0(X)$. Roughly speaking, one wants to argue that if $i < \dim \mathrm{alb}(X)$ then one can "deform away" any non-zero cohomology class.

Let X a compact Kähler manifold and let $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ be the Albanese mapping of X for some choice of base point. Let $\mathrm{Pic}^0(X)$ denote the identity component of the Picard group of X, which parametrizes topologically trivial holomorphic line bundles on X. We wish to study the cohomology groups $H^i(X, L)$ for a general line bundle $L \in \mathrm{Pic}^0(X)$. To this end, for a given integer $i \geq 0$ let $S^i_m \subseteq \mathrm{Pic}^0(X)$ to be the analytic subvariety defined by

$$S^i_m(X) = \{L \in \mathrm{Pic}^0(X) \mid \dim H^i(X, L) \geq m\};$$

to simplify the notation we also set $S^i(X) = S^i_1(X)$.

Theorem 3.1. *Let X be a compact Kähler manifold. Then*

$$\mathrm{codim}_{\mathrm{Pic}^0(X)} S^i(X) \geq \dim \mathrm{alb}(X) - i.$$

In particular, if $L \in \mathrm{Pic}^0(X)$ is a generic line bundle, where general means on the complement of the proper analytic subset $S^i(X)$, then $H^i(X, L) = 0$ for every $i < \dim \mathrm{alb}(X)$.

Chapter 4

Generic Vanishing Theorem

Notation

(0.1) If V is a complex vector space, and if M is a complex manifold, we denote by V_M the trivial vector bundle on M with fibre V.

(0.2) Let $u\colon E \to F$ be a map of holomorphic vector bundles of ranks e and f respectively on a complex manifold M. For any integer $a > 0$, we denote by $\mathscr{J}_a(u)$ the ideal sheaf on M locally generated by the determinants of the $a \times a$ minors of u, with the conventions that $\mathscr{J}_0(U) = \mathscr{O}_M$ and $\mathscr{J}_a(u) = 0$ if $a > \min\{e, f\}$.

4.1 Poincaré Line Bundle

Recall that $\mathrm{Pic}^0(X)$ is defined to be the group of holomorphic line bundle with first Chern class equal to zero. Via the cohomology sequence of the exponential sequence we get

$$\mathrm{Pic}^0(X) \simeq \frac{H^1(X, \mathscr{O}_X)}{H^1(X, \mathbb{Z})}.$$

Denote with L_ξ the holomorphic line bundle on X corresponding to the point $\xi \in \mathrm{Pic}^0(X)$.

Proposition 4.1.1. *Fix a base point $x_0 \in X$. Then there exists a holomorphic line bundle, called the* Poincaré bundle, *$\mathcal{P}$ on $X \times \mathrm{Pic}^0(X)$ with the following properties:*

(a) For any $\xi \in \mathrm{Pic}^0(X)$ it satisfies

$$\mathcal{P}|_{X \times \{\xi\}} \simeq L_\xi;$$

(b) The restriction of $\mathcal{P}$ to $\{x_0\} \times \mathrm{Pic}^0(X)$ is the trivial line bundle, i.e.

$$\mathcal{P}|_{\{x_0\} \times \mathrm{Pic}^0(X)} \simeq \mathscr{O}_{\mathrm{Pic}^0(X)}.$$

Moreover, $\mathcal{P}$ is unique up to isomorphism.

Now we need a good model for computing the cohomology gropus $H^i(X, L_\xi)$ when $\xi \in \mathrm{Pic}^0(X)$ is allowed to vary. Fix $x_0 \in X$ and consider the Poincaré bundle $\mathcal{P}$. Let

$$p \colon X \times \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$$

the projection map. The cohomology groups $H^i(X, L_\xi)$ are related to the direct image sheaves $R^i p_* \mathcal{P}$, so recall some definitions first.

Definition 4.1.2. *Let $f \colon X \to Y$ be a morphism of complex manifold. Let $\mathscr{G}$ an $\mathscr{O}_Y$-sheaf on Y. Define the* pull-back sheaf

$$f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

where f^{-1} is the inverse functor. In this way $f^*\mathscr{G}$ is an $\mathscr{O}_X$-sheaf on X, indeed the map $f^\# \colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ induces a morphism of sheaves $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ and so we can consider $\mathscr{O}_X$ as an $f^{-1}\mathscr{O}_Y$-sheaf.

Definition 4.1.3. *Let $f \colon X \to Y$ a continuous map of topological space. Let $\mathscr{F}$ a sheaf of abelian group on X, define for all $i \geq 0$ the* higher direct image sheaf *$R^i f_*(\mathscr{F})$ on Y as the sheaf associated to the presheaf*

$$V \mapsto H^i\left(f^{-1}V, \mathscr{F}|_{f^{-1}V}\right)$$

on Y.

For more details see Hartshorne's book [Ha]. Now we can state a useful result from analysis.

Theorem 4.1.4. *Let $f\colon X \to Y$ be a proper morphism of complex manifolds, and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then for every $y \in Y$, there exists an open neighbourhood V and a bounded complex $E^\bullet$ of holomorphic vector bundles on V, with the following property: for every coherent sheaf $\mathscr{G}$ on Y, one has*

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{G}) \simeq \mathcal{H}^i(E^\bullet \otimes \mathscr{G})$$

and the isomorphism is functorial in $\mathscr{G}$.

For the proof, made in an algebraic setting, we refer to chapter III, Theorem 12.11 of [Ha] or [Mu, p.46].

This Theorem turns out to be very useful when applied to the Pioncaré bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^0(X)$. So, following the previous notation, $Y = \mathrm{Pic}^0(X)$ and $\mathscr{F} = \mathcal{P}$. So for any $\xi \in \mathrm{Pic}^0(X)$ we have an open neighbourhood V and a bounded complex of vector bundles $E^\bullet$. Let $\mathfrak{m}_\xi \subseteq \mathscr{O}_V$ the maximal ideal at ξ and let $\mathbb{C}(\xi) = \mathscr{O}_V/\mathfrak{m}_\xi$ the skyscraper sheaf at ξ. If we take $\mathscr{G} = \mathbb{C}(\xi)$, the isomorphism in the previous Theorem 4.1.4 becomes

$$H^i(X, E^\bullet \otimes \mathbb{C}(\xi)) \simeq R^i p_* \left(\mathcal{P} \otimes p^* \left(\mathbb{C}(\xi)\right)\right) \simeq H^i(X, L_\xi)$$

and so the complex $E^\bullet$ does compute cohomology of the line bundle corresponding to ξ.

4.2 Cohomology Support Loci

4.2.1 Infinitesimal Properties

Our purpose in this section is to prove useful results concerning the deformation theory associated to a complex of vector bundles on a complex manifold. Let M be a complex manifold, denote by $\mathscr{O}_M$ the sheaf of germs of holomorphic functions. For $y \in M$ denote by $\mathscr{O}_{M,y}$ the local ring at the point, by $\mathfrak{m}_y \subseteq \mathscr{O}_{M,y}$ its maximal ideal and by $\mathbb{C}(x) = \mathscr{O}_{M,x}/\mathfrak{m}_y$ the residue field. Let $E^\bullet$ a bounded complex of locally free sheaves on M, with $\mathrm{rk} E^i = e_i$

$$E^\bullet = \left[\cdots \xrightarrow{d^{i-1}} E^{i-1} \xrightarrow{d^i} E^i \xrightarrow{d^{i+1}} E^{i+1} \longrightarrow \cdots\right]$$

Given a point $y \in M$, we denote by $E^\bullet(y)$ the complex of vector spaces at y determined by the fibres of $E^\bullet$

$$E^\bullet(y) = E^\bullet/\mathfrak{m}_y E^\bullet = E^\bullet \otimes_{\mathscr{O}_M} \mathbb{C}(y)$$

where $\mathbb{C}(y) = \mathscr{O}_{M,y}/\mathfrak{m}_y$ is the residue field of M at y. We are interested in the study of how the cohomology groups of this complex of vector spaces depend on $y \in M$, in particular on the cohomology support loci

$$S^i_m(E^\bullet) = \{y \in M \mid \dim H^i(E^\bullet(y)) \geq m\},$$

and in particular, we want to understand these loci infinitesimally.

Lemma 4.2.1. *Each $S^i_m(E^\bullet)$ are closed analytic subvariety of M.*

Proof. Note first that

$$\begin{aligned}\dim H^i(E^\bullet(y)) \geq m &\iff \dim\ker d^i(y) - \dim\operatorname{im}\, d^{i-1}(y) \geq m \\ &\iff \mathrm{rk} d^{i-1}(y) + \mathrm{rk} d^i(y) \leq e_i - m,\end{aligned} \tag{4.1}$$

Then we have, as sets:

$$S^i_m(E^\bullet) = \bigcap_{\substack{a+b=e_i-m+1\\ a,b\geq 0}} \{y \in M \mid \mathrm{rk} d^{i-1}(y) \leq a-1 \text{ or } \mathrm{rk} d^i(y) \leq b-1\},$$

so we may take the ideal sheaf of $S^i_m(E^\bullet)$ to be

$$\mathscr{I}(S^i_m(E^\bullet)) = \sum_{\substack{a+b=e_i-m+1\\ a,b\geq 0}} \mathscr{I}_a(d^{i-1}) \cdot \mathscr{I}_b(d^i). \tag{4.2}$$

as in Notation (0.2). Note that we can do that because, after choosing local trivializations for the bundles E^i and E^{i+1}, the differential d^i is given by a matrix of holomorphic functions. □

We can now start to study the infinitesimal behaviour of $E^\bullet$. Fix a point $y \in M$, and denote by $T = T_yM$ the holomorphic tangent space to M at y, which is dual to $\mathfrak{m}_y/\mathfrak{m}_y^2$. Roughly, we want to keep only the first order terms in each differential d^i, and extract as much information s we can bout $S^i_m(E^\bullet)$. Algebraically, it means that we consider the complex $E^\bullet/\mathfrak{m}_y^2E^\bullet$. We have a short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}_yE^\bullet/\mathfrak{m}_y^2E^\bullet \arrow[r] \arrow[d, equal] & E^\bullet/\mathfrak{m}_y^2E^\bullet \arrow[r] & E^\bullet/\mathfrak{m}_yE^\bullet \arrow[r] \arrow[d, equal] & 0 \\ & E^\bullet(y) \otimes T^* & & E^\bullet(y) & \end{tikzcd}$$

It gives rise to the connecting homomorphism

$$D\left(d^i, y\right) : H^i\left(E^\bullet(y)\right) \to H^{i+1}(E^\bullet(y)) \otimes \left(T_yM\right)^*. \tag{4.3}$$

Then for each tangent vector $v \in T$, $D(d^i, y)$ determines a homomorphism

$$D_v\left(d^i, y\right) : H^i(E^\bullet(y)) \to H^{i+1}(E^\bullet(y))$$

and these vary linearly with v. Concretely, $D_v(d^i, y)$ may be described as follows: after choosing local trivializations of the E^i near y, d^i will be given by a matrix of holomorphic functions. Differentiating this matrix at y in the direction v gives a linear map $E^i(y) \to E^{i+1}(y)$, that we denote simply by δ_i. By hypothesis, we know that $d^i d^{i-1} = 0$, then we have

$$\delta_i d^{i-1}(y) + d^i(y)\delta_{i-1} = 0$$

which implies that δ_i passes to cohomology and it coincides with $D_v(d^i, y)$. Differentiating again we obtain

$$\delta_i^2 d^{i-1}(y) + 2\delta_i\delta_{i-1} + d^i(y)\delta_{i-1}^2 = 0$$

which, in cohomology, becomes $D_v\left(d^i, y\right) D_v\left(d^{i-1}, y\right) = 0$. It follows that we get a complex of vector spaces

$$D_v\left(E^\bullet, y\right) : \left[\cdots \to H^{i-1}(E^\bullet(y)) \to H^i(E^\bullet(y)) \to H^{i+1}(E^\bullet(y)) \to \cdots\right]$$

and we call it *derivative complex of $E^\bullet$ at y in the direction v*. Recall that the symmetric algebra

$$\mathrm{Sym} T^* = \bigoplus_{j=0}^{\infty} \mathrm{Sym}^j T^* \simeq \bigoplus_{j=0}^{\infty} \mathfrak{m}_y^j/\mathfrak{m}_y^{j+1}$$

where T^* is the space of linear functions on T. Note that, if M is smooth at y, then $T_yY = \left(\mathfrak{m}_y/\mathfrak{m}_y^2\right)^*$, and so $T^* = \mathfrak{m}_y/\mathfrak{m}_y^2$. So $\mathrm{Sym}\, T^* = \bigoplus \mathrm{Sym}^j\left(\mathfrak{m}_y/\mathfrak{m}_y^2\right)$ that is the algebra of all polynomial functions on T; and we have that $\mathrm{Spec}\left(\mathrm{Sym}\, T^*\right) \simeq T$. Now, we can consider a similar construction, $R = \bigoplus \left(\mathfrak{m}_y^j/\mathfrak{m}_y^{j+1}\right)$, then $\mathrm{Spec}\, R$ is the tangent cone of T.
Moreover, if we denote by $\mathscr{O}_T$ the sheaf of germs of holomorphic functions on T we have, as rings, $\mathrm{Sym}\, T^* \simeq \mathscr{O}_T$. Then, thinking of T itself as a complex manifold, (4.3) define a morphism

$$D(d^i, y)\colon H^i(E^\bullet(y)) \otimes \mathscr{O}_T \to H^{i+1}(E^\bullet(y)) \otimes \mathscr{O}_T$$

between trivial holomorphic vector bundles on T (cf.(Notation 0.2)).

Lemma 4.2.2. *We have $D(d^i, y) \circ D(d^{i-1}, y) = 0$.*

Proof. The short exact sequence of complexes

$$0 \to \mathfrak{m}_y E^\bullet / \mathfrak{m}_y^3 E^\bullet \to E^\bullet / \mathfrak{m}_y^3 \to E^\bullet / \mathfrak{m}_y \to 0$$

gives rise to the connection homomorphism

$$H^{i-1}(E^\bullet(x)) \to H^i \left(\mathfrak{m}_y E^\bullet / \mathfrak{m}_y^3 E^\bullet \right).$$

Now it's easy to see that if we project to $H^i \left(\mathfrak{m}_y E^\bullet / \mathfrak{m}_y^2 E^\bullet \right)$ we obtain $D\left(d^{i-1}, y\right)$. This gives a factorization as follows

$$\begin{tikzcd}
H^{i-1}(E^\bullet(x)) \arrow[d] \arrow[dr, bend left, "D(d^{i-1},y)"] & & \\
H^i \left(\mathfrak{m}_y E^\bullet / \mathfrak{m}_y^3 E^\bullet \right) \arrow[r] & H^i \left(\mathfrak{m}_y E^\bullet / \mathfrak{m}_y^2 E^\bullet \right) \arrow[r, "D(d^i,y)"] \arrow[d, equal] & H^{i+1} \left(\mathfrak{m}_y^2 E^\bullet / \mathfrak{m}_y^3 E^\bullet \right) \arrow[d, equal] \\
& H^i\left(E^\bullet(y)\right) \otimes T^* \arrow[r] & H^{i+1}\left(E^\bullet(y)\right) \otimes \mathrm{Sym}^2 T^*
\end{tikzcd}$$

Where the last inequality on the right follows from $\mathrm{Sym}^2 T^* \simeq \mathfrak{m}_y^2 / \mathfrak{m}_y^3$. In the middle row, we are using two consecutive morphisms in the long exact sequence coming from the short exact sequence of complexes

$$0 \to \mathfrak{m}_y^2 E^\bullet / \mathfrak{m}_y^3 E^\bullet \to \mathfrak{m}_y E^\bullet / \mathfrak{m}_y^3 E^\bullet \to \mathfrak{m}_y E^\bullet / \mathfrak{m}_y^2 E^\bullet \to 0.$$

Their composition is zero, and so the Lemma is proved. □

We can now make the

Definition 4.2.3. *The directional derivative complexes fit together into a complex of trivial vector bundles $D(E^\bullet, y)$*

$$\left[\ldots \to H^{i-1}(E^\bullet(y)) \otimes \mathscr{O}_T \to H^i(E^\bullet(y)) \otimes \mathscr{O}_T \to H^{i+1}(E^\bullet(y)) \otimes \mathscr{O}_T \to \ldots \right]$$

on T, with differential $D(d^i, y)$, that we call derivative complex of $E^\bullet$ at y*.*

In particular, the derivative complex is itself a complex of vector bundles on the tangent space T, so we can consider its cohomology support loci

$$S^i_m(D(E^\bullet, y)) = \left\{ v \in T \mid \dim H^i(D_v(E^\bullet, y)) \geq m \right\} \subseteq T.$$

Note that here

$$H^i\left(D_v(E^\bullet, y)\right) = \frac{\ker\left(H^i(E^\bullet(y)) \to H^{i+1}(E^\bullet(y))\right)}{\mathrm{im}\left(H^{i-1}(E^\bullet(y)) \to H^i(E^\bullet(y))\right)}.$$

Moreover since the differentials in $D(E^\bullet, y)$ are linear in the variable $v \in T$ the loci $S^i_m(D(E^\bullet, y))$ are cones in T, i.e. algebraic sets defined by homogeneous ideals in $\mathrm{Sym}(T^*)$. The deformation-theoretic significance of the derivative complex is illustrated by

Theorem 4.2.4. *Fix an integer $m \geq 0$. Suppose that $y \in S^i_m(E^\bullet)$. Then*

$$TC_y(S^i_m(E^\bullet)) \subseteq S^i_m(D(E^\bullet, y))$$

where $TC_y(S^i_m(E^\bullet))$ denote the tangent cone of $S^i_m(E^\bullet)$ at the point y.

Since $\dim_y TC_y(S^i_m(E^\bullet)) = \dim_y S^i_m(E^\bullet)$, one obtains the following corollaries.

Corollary 4.2.5. *Set $m = \dim H^i(E^\bullet(y))$. Then*

$$\dim_y(S^i_m(E^\bullet) \leq \dim \left\{v \in T \mid D_v(d^i, y) = D_v(d^{i-l}, y) = 0\right\}.$$

In particular, if either $D_v(d^i, y) \neq 0$ or $D_v(d^{i-l}, y) \neq 0$ for some tangent vector $v \in T$, then $S^i_m(E^\bullet)$ is a proper subvariety of M.

Proof. From (4.2.4) we get that

$$\begin{aligned}\dim_y S^i_m(E^\bullet) &\leq \dim_y S^i_m(D(E^\bullet, y)) \\ &= \dim \left\{v \in T \mid \dim H^i(D_v(E^\bullet, y)) \geq m\right\}.\end{aligned}$$

But $m = \dim H^i(E^\bullet(y))$ and $H^i(D_v(E^\bullet, y)) = \ker D_v(d^i, y)/\mathrm{im} D_v(d^{i-1}, v)$, so a vector $v \in T_y M$ can belong to the set $S^i_m(D(E^\bullet, y))$ only if both differentials $D_v(d^i, y)$ and $D_v(d^{i-l}, y)$ vanish at v, which are the conditions $D_v(d^i, y) = D_v(d^{i-l}, y) = 0$. $\square$

The following is a condition for $S^i_m(E^\bullet)$ to be a proper subset of M.

Corollary 4.2.6. *If $H^i(D_v(E^\bullet, y)) = 0$ for some point $y \in M$ and some tangent vector $v \in T_y M$, then $S^i_m(E^\bullet)$ is a proper subvariety of M, for some $m \geq 1$.*

Proof. Using again Theorem 4.2.4

$$\begin{aligned}\dim_y S^i_m(E^\bullet) &\leq \dim_y S^i_m(D(E^\bullet, y)) \\ &= \dim \left\{v \in T \mid \dim H^i(D_v(E^\bullet, y)) \geq m\right\}.\end{aligned}$$

Since there exist $v \in T$ such that $H^i(D_v(E^\bullet, y)) = 0$, then $S^i_m(E^\bullet) \subsetneq M$. $\square$

We can also use the derivative complex to detect isolated points of $S^i_m(E^\bullet)$

Corollary 4.2.7. *If $y \in S^i_m(E^\bullet)$, and if $H^i(D_v(E^\bullet, y)) = 0$ for every non-zero tangent vector $v \in T_yM$, then y is an isolated point of $S^i_m(E^\bullet)$.*

Proof. If $H^i(D_v(E^\bullet, y)) = 0$ then we there isn't $v \in T$ such that both $D_v(d^i, y) = D_v(d^{i-l}, y) = 0$, so either $D_v(d^i, y)$ or $D_v(d^{i-l}, y)$ must be nonzero. Using (4.2.5) we conclude that $\dim_y S^i_m(E^\bullet) = 0$. □

Now prove the tangent cone theorem:

Proof of Theorem 4.2.4. First, to simplify notation set $\mathfrak{m} = \mathfrak{m}_y$. Given vector spaces V and W, a linear map $\delta\colon V \to W \otimes T^*$ can be viewed as a matrix with entries in T^* after choosing some basis. We denote by $J_a(\delta)$ the homogeneous ideal of $\operatorname{Sym} T^*$ generated by the determinants of the $a \times a$ minors of δ, that are elements in $\operatorname{Sym}^a T^* = \mathfrak{m}^a/\mathfrak{m}^{a+1}$, and by $J_a(\delta)_k \subseteq \mathfrak{m}^k/\mathfrak{m}^{k+1}$ the degree k piece of this ideal. And second, if $\mathscr{J}$ is an ideal sheaf of $\mathscr{O}_M$, we set

$$Gr_k(\mathscr{J}) = \mathscr{J} \cap \mathfrak{m}^k / \mathscr{J} \cap \mathfrak{m}^{k+1}$$

and

$$Gr(\mathscr{J}) = \bigoplus Gr_k(\mathscr{J}).$$

We think about this last one as an homogeneous ideal of $Gr(\mathscr{O}_M) = \oplus \mathfrak{m}^k/\mathfrak{m}^{k+1} = \operatorname{Sym} T^*$.

Now let $\delta\colon E \to F$ be a morphism of holomorphic vector bundles on M, similar to (4.3), using the snake lemma applied to the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
E(y) \otimes T^* \arrow[r, "d(y)\otimes id"] \arrow[d] & F(y) \otimes T^* \arrow[d] \\
E/\mathfrak{m}^2 E \arrow[r] \arrow[d] & F/\mathfrak{m}^2 F \arrow[d] \\
E(y) \arrow[r, "d(y)"] \arrow[d] & F(y) \arrow[d] \\
0 & 0
\end{tikzcd}$$

gives rise to a "derivative" homomorphism $\delta\colon \ker d(y) \to \operatorname{coker} d(y) \otimes T^*$.

Lemma 4.2.8. *Let $r = \operatorname{rk} d(y)$. Then for $k \geq r$ we have the following*

1. $\mathscr{J}_k(d) \subseteq \mathfrak{m}^{k-r}$
2. $Gr_{k-r}\left(\mathscr{J}_k(d)\right) = J_{k-r}(\delta)_{k-r}$.

The proof will be done at the end of this section, for now suppose it's true.
The the differential $d^i \colon e^i \to E^{i+1}$ gives rise, as above, to a homomorphism $\delta^i \colon kerd^i(y) \to \operatorname{coker} d^i(y) \otimes T^*$. We have a commutative diagram

$$\begin{array}{ccc} \ker d^i(y) & \xrightarrow{\delta^i} & (\operatorname{coker} d^i(y)) \otimes T^* \\ \downarrow & & \uparrow \\ H^i(E^\bullet(y)) & \longrightarrow & H^{i+1}(E^\bullet(y)) \otimes T^* \end{array}$$

where the second horizontal map is $D(d^i, y)$ defined in (4.3). We have that $J_k(D(d^i, y) = J_k(\delta^i)$, then the previous Lemma implies

$$Gr_{k-r_i}(\mathscr{J}_k(d^i)) = J_{k-r_i}(D(d^i, y))_{k-r_i} \tag{4.4}$$

for $k \geq r_i$ with $r_i = \operatorname{rk} d^i(y)$.

Now set $h = \dim H^i(E^\bullet(y))$, then by (4.2), $S^i_m(E^\bullet, y)$ is defined in $T = T_yM$ by the homogeneous ideal

$$J = \sum_{\substack{a+b=h-m+1 \\ a,b \geq 0}} J_a(\delta^{i-1}) \cdot J_b(\delta^i) \subseteq \operatorname{Sym} T^*.$$

On the other hand we have

$$\mathscr{J} = \sum_{\substack{a+b=e_i-m+1 \\ a,b \geq 0}} \mathscr{J}_a(\delta^{i-1}) \cdot \mathscr{J}_b(\delta^i)$$

that denotes the ideal sheaf defining $S^i_m(E^\bullet)$ in M, as in (4.2). Then the tangent cone $TC_y(S^i_m(E^\bullet))$ is defined in T by the homogeneous ideal $Gr(\mathscr{J}) \subseteq \operatorname{Sym} T^*$. Since J is generated by polynomials of degree $h-m+1$, to complete the proof it enough to prove that

$$J_{h-m+1} \subseteq Gr_{h-m+1}(\mathscr{J}). \tag{4.5}$$

To this end, set $r_k = \operatorname{rk} d^k(y)$, and fix integers $a, b \geq 0$ such that $a + b = h - m + 1$. Then (4.4) gives

$$J_a\left(D(d^{i-1},y)\right)_a = Gr_a\left(\mathscr{J}_{a+r_{i-1}}(d^{i-1})\right)$$

and

$$J_b(D(d^i,y))_b = Gr_b\left(\mathscr{J}^{b+r_i}(d^i)\right).$$

Hence

$$\left(J_a(D(d^{i-1},y)) \cdot J_b(D(d^i,y))\right)_{a+b} \subseteq Gr_{a+b}\left(\mathscr{J}_{a+r_{i-1}}(d^{i-1}) \cdot \mathscr{J}_{b+r_i}(d^i)\right).$$

But $r_i + r_{i-1} = e_i - h$ thanks to (4.1); hence $(a+r_{i-1}) + (b+r_i) = e_i + m - 1$. Therefore the product of ideals on the right side of the inclusion above is contained in $\mathscr{J}$, and the theorem is proved. □

The last thing we need to do is the

Proof of Lemma 4.2.8. We can assume E and F to be trivial, because we are looking locally at y. We may choose bases $v_1, \ldots, v_n$ for E and $w_1, \ldots, w_m$ for F such that

$$d(v_i) \equiv w_i \,(\mod \mathfrak{m})$$

for $i \in \{1, \ldots, r\}$, and

$$d(v_i) \equiv \delta(v_i) \in \mathfrak{m}F \mod (w_1, \ldots, w_r) + \mathfrak{m}^2 F$$

for $i > r$. For $i_1 < i_2 < \ldots < i_k$ and $\ell_1 < \ell_2 < \ldots < \ell_k$ we denote by $\Delta^{\ell_1<\ell_2<\ldots<\ell_k}_{i_1<i_2<\ldots<i_k}(d)$ the determinant of the $k \times k$ minor corresponding to those indexes. Then we have

$$\Delta^{\ell_1<\ell_2<\ldots<\ell_k}_{i_1<i_2<\ldots<i_k}(d) \equiv \Delta^{\ell_1<\ell_2<\ldots<\ell_k}_{i_1<i_2<\ldots<i_k}(\delta) \mod \mathfrak{m}^{k-r+1}$$

if $i_1 = \ell_1, \ldots, i_r = \ell_r = r$, and

$$\Delta^{\ell_1<\ell_2<\ldots<\ell_k}_{i_1<i_2<\ldots<i_k}(d) \equiv 0 \mod \mathfrak{m}^{k-r+1}$$

otherwise. Then it follows the Lemma. □

4.2.2 On a Compact Kähler Manifold

Let X be a compact Kähler manifold of dimension n, we wish to apply the results of the previous section to study the following loci contained in $\mathrm{Pic}^0(X)$

$$S^i_m(X) = \{\xi \in \mathrm{Pic}^0(X) \mid \dim H^i(X, L_\xi) \geq m\}$$

and

$$S^i(X) = \{\xi \in \mathrm{Pic}^0(X) \mid \dim H^i(X, L_\xi) \geq 0\}$$

for $i, m > 0$. To this end, the basic fact is that, according to Theorem 4.1.4, locally on $\mathrm{Pic}^0(X)$ the groups $H^i(X, L_\xi)$ are computed as the pointwise cohomology of a complex of vector bundles. Therefore, fix once and for all a neighbourhood U of $\xi \in \mathrm{Pic}^0(X)$ and a complex $E^\bullet$ then one has

$$S^i_m(X) \cap U = S^i_m(E^\bullet).$$

Furthermore, from our construction of $\mathrm{Pic}^0(X) = H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z})$ the derivative complex $D_v(E^\bullet, \xi)$ takes the simple form:

Lemma 4.2.9. *A tangent vector $v \in T_\xi \mathrm{Pic}^0(X)$ corresponds to a harmonic $(0, 1)$-form $v \in \mathcal{H}^{0,1}(X)$ under the canonical identification*

$$T_\xi \mathrm{Pic}^0(X) = H^1(X, \mathscr{O}_X).$$

The derivative complex $D_v(E^\bullet; \xi)$ is isomorphic to

$$H^0(X, L_\xi) \to H^1(X, L_\xi) \to \cdots \to H^n(X, L_\xi)$$

with differential given by wedge product with $v \in \mathcal{H}^{0,1}(X)$.

Proof. Recall that the differentials in the derivative complex are constructed from the short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathscr{O}/\mathfrak{m}^2 \to \mathscr{O}/\mathfrak{m} \to 0$$

where $\mathfrak{m}$ is the ideal sheaf of the point $\xi \in \mathrm{Pic}^0(X)$. Tensor it by $E^\bullet$ and then we consider the connecting homomorphism of the corresponding cohomological long exact sequence. Recall that the isomorphism in Theorem 4.1.4 is functorial in $\mathscr{G}$, we can compute the connecting homomorphism on $X \times \mathrm{Pic}^0(X)$. In fact, we denote by P the restriction of the Pioncaré bundle $\mathcal{P}$ to the first infinitesimal neighbourhood, corresponding to $\mathfrak{m}^2$, of $X \times \{\xi\}$. Then locally on $\mathrm{Pic}^0(X)$ there is a finite complex $E^\bullet$ of vector bundles which computes the cohomology of $\mathcal{P}$ as in Theorem 4.1.4. On the other hand, a tangent vector $v \in \mathcal{H}^{0,1}(X)$ determines a class $e_v \in \mathrm{Ext}^1(P, P)$, which gives the homomorphism

$$H^i(X, L_\xi) \to H^{i+1}(X, L_\xi), [\alpha] \mapsto [v \wedge \alpha].$$

This concludes the proof. (See [GH, p.706].) $\square$

This description of the derivative complex is still not very tractable, since it involves cohomology groups. We can use Hodge theory to reinterpret the. Recall from (1.3.9) that we have an isomorphism of complex vector spaces

$$\overline{H^i(X, L_\xi)} \simeq H^0(X, \Omega^i_X \otimes L_\xi^{-1});$$

where the bar denotes the conjugate vector space. Concretely, take the harmonic $(i,0)$-form representing a given cohomology class for L_ξ, and conjugate to obtain a harmonic $(0,i)$-form that represents a cohomology class for $\Omega^i_X \otimes L_\xi^{-1}$. Furthermore we have a commutative diagram

$$\begin{array}{ccc} \overline{H^i(X,L_\xi)} & \xrightarrow{\bar{v}\wedge} & \overline{H^{i+1}(X,L_\xi)} \\ \downarrow & & \downarrow \\ H^0(X,\Omega^i_X \otimes L_\xi^{-1}) & \xrightarrow{\bar{v}\wedge} & H^0(X,\Omega^{i+1}_X \otimes L_\xi^{-1}) \end{array}$$

for every $v \in \mathcal{H}^{0,1}(X)$. Therefore if we conjugate the derivative complex

$$H^0(X,L_\xi) \to H^1(X,L_\xi) \to \cdots \to H^n(X,L_\xi),$$

in Lemma 4.2.9, using the isomorphism above, we obtain the complex

$$H^0(X,L_\xi^{-1}) \to H^0(X,\Omega^1_X \otimes L_\xi^{-1}) \to \cdots \to H^0(X,\Omega^n_X \otimes L_\xi^{-1})$$

where the differentials are given by wedge product with the holomorphic 1-form $\bar{v} \in \mathcal{H}^{1,0}(X)$. The advantage of this complex is that it involves only global sections of vector bundles. We can now apply our results about tangent cones to cohomology support loci:

$$TC_\xi S^i_m(E^\bullet) \subseteq S^i_m(D(E^\bullet,\xi)). \tag{4.6}$$

In our situation Corollary 4.2.5 becomes

Theorem 4.2.10. *Let X be a compact Kähler manifold . Fix a point $\xi \in \mathrm{Pic}^0(X)$, and set $m = \dim H^i(X,L_\xi)$. Then*

$$\dim_\xi S^i_m(X) \le \dim \left\{ \omega \in H^0(X,\Omega^1_X) \;\middle|\; \begin{array}{l} \omega\wedge\alpha = 0 \text{ for } \alpha \in H^0\left(X,\Omega^{i-1}_X \otimes L_\xi^{-1}\right) \\ \omega\wedge\beta = 0 \text{ for } \beta \in H^0\left(X,\Omega^{i}_X \otimes L_\xi^{-1}\right) \end{array} \right\}$$

Proof. This is the equivalent of (4.2.5). From (4.6) we get

$$\dim_\xi S^i_m(X) \le \dim \left\{ v \in T_\xi \mathrm{Pic}^0(X) \mid \dim H^i\left(D_v\left(E^\bullet,L_\xi\right)\right) \ge m \right\}.$$

We have that the i-th vector space in the derivative complex is $H^i(X,L_\xi)$, which is by assumption m-dimensional. So the only possibility for $\dim H^i(D_v(E^\bullet,L_\xi))$ to be $\ge m$ is that the two differential next to $H^i(X,L_\xi)$ are zero. After conjugation we obtain the requested condition on ω in $H^0(X,\Omega^1_X)$. □

As in (4.2.6) we have a criterion to say if $S^i(X)$ is a proper subset of $\mathrm{Pic}^0(X)$.

Corollary 4.2.11. *If the sequence*

$$H^0(X, \Omega_X^{i-1} \otimes L_\xi^{-1}) \xrightarrow{\omega\wedge} H^0(X, \Omega_X^i \otimes L_\xi^{-1}) \xrightarrow{\omega\wedge} H^0(X, \Omega_X^{i+1} \otimes L_\xi^{-1}) \tag{4.7}$$

is exact for some $\omega \in H^0(X, \Omega_X^1)$, then $S^i(X) \neq \mathrm{Pic}^0(X)$.

Finally, a version of (4.2.7), which gives a criterion for $\xi \in \mathrm{Pic}^0(X)$ to be an isolated point of $S^i(X)$.

Corollary 4.2.12. *If the sequence* (4.7) *is exact for every non-zero holomorphic* 1*-form $\omega \in H^0(X, \Omega_X^1)$, then ξ is an isolated point of $S^i(X)$.*

We used, in these cases, that we are working on global sections of coherent sheaves, that is much easier to use than general cohomology classes.

4.3 Proof of the GVT

The Generic Vanishing Theorem. *Let X be a compact Kähler manifold. Let* $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ *its Albanese mapping, having fixed $x_0 \in X$. Then*

$$\mathrm{codim}_{\mathrm{Pic}^0(X)} S_m^i(X) \geq \dim \mathrm{alb}(X) - i. \tag{4.8}$$

Proof. First, if we fix an irreducible component of $S^i(X)$, say Z, and we choose $\xi_0 \in Z$ such that $\dim H^i(X, L_\xi)$ is as small as possible, set $m = \dim H^i(X, L_{\xi_0})$; then $Z \subseteq S_m^i(X)$, so we need to show that

$$\dim_{\xi_0} S_m^i(X) \leq \dim \mathrm{Pic}^0(X) - \dim \mathrm{alb}(X) + i.$$

We are going to estimate $\dim_{\xi_0} S_m^i(X)$. Fix a non-zero section $\beta \in H^0(X, \Omega_X^i \otimes L_{\xi_0}^{-1})$, that is possible because $m = \dim H^0\left(X, \Omega_X^i \otimes L_{\xi_0}^{-1}\right) \geq 1$. Now, let $W = \{\omega \in H^0(X, \Omega_X^1) \mid \omega \wedge \beta = 0\} \subseteq H^0(X, \Omega_X^1)$, by Theorem 4.2.10 we have

$$\dim_{\xi_0} S_m^i(X) \leq \dim W,$$

so we only need to prove

$$\dim W \leq \dim \mathrm{Pic}^0(X) - \dim \mathrm{alb}(X) + i,$$

where $W(x) = \{\varphi \in (T_x X)^* \mid \varphi \wedge \beta(x) = 0\} \subseteq (T_x X)^*$. Consider the evaluation morphism $e(x)$ at a point $x \in X$

$$\begin{aligned} e(x)\colon H^0\left(X, \Omega_X^1\right) &\to \Omega_X^1 = \Omega_{X,x}^1 = (T_x X)^* \\ \omega &\mapsto \omega(x). \end{aligned}$$

If $\omega \in W$, then $\omega(x) \in W(x)$ and

$$\dim W - \dim \ker e(x) \leq \dim W(x).$$

On the other hand, recall that $H^0(X, \Omega^1_X)$ is the cotangent space to $\mathrm{Alb}(X)$ and $e(x)$ is the codifferential of the Albanese map at the point x, see Lemma 2.3.3. Consequently we can split $e(X)$ as follow

$$\Omega^1_{\mathrm{Alb}(X),\mathrm{alb}(x)} \twoheadrightarrow \Omega^1_{\mathrm{alb}(X),\mathrm{alb}(x)} \hookrightarrow \Omega^1_{X,x},$$

since $\dim \Omega^1_{\mathrm{Alb}(X),\mathrm{alb}(x)} = \dim \mathrm{Alb}(X) = \dim H^0(X, \Omega^1_X) = \dim \mathrm{Pic}^0(X)$ and $\dim \Omega^1_{\mathrm{alb}(X),\mathrm{alb}(x)} = \dim \mathrm{alb}(x)$ we get that at a *general* point $x \in X$

$$\dim \ker e(x) = \dim \mathrm{Pic}^0(X) - \dim \mathrm{alb}(X).$$

The inequality albove becomes

$$\dim_{\xi_0} S^i_m(X) \leq \dim \mathrm{Pic}^0(X) - \dim \mathrm{alb}(X) + \dim W(x).$$

So, for a general $x \in X$ with, $\beta(x) \neq 0$, we are reduced to proving that

$$\dim W(x) \leq i.$$

But this is a consequence of the following result of linear algebra:

Lemma 4.3.1. *Let V be a finite-dimensional vector space, and let $\beta \in \bigwedge^i V$ be a nonzero element. Then*

$$\dim \{v \in V \mid v \wedge \beta = 0\} \leq i.$$

Proof. Let $e_1, \ldots, e_m \in V$ be linearly independent vectors such that $e_j \wedge \beta = 0$ for $1 \leq j \leq m$. Let $n = \dim V$, complete $e_1, \ldots, e_m$ to a basis $e_1, \ldots, e_n$ of V. If $\beta \neq 0$, then exists $\alpha \in \bigwedge^{n-1} V$ such that $\alpha \wedge \beta \neq 0$. But since $n - i > n - m$, every term of α must involve one of the e_j, $1 \leq j \leq m$, then $\alpha \wedge \beta = 0$, contradiction. □

By this Lemma we get

$$\dim_{\xi_0} S^i_m(X) \leq \dim \mathrm{Pic}^0(X) - \dim \mathrm{alb}(X) + i$$

and so the theorem is proved. □

Chapter 5

Applications

5.1 Beauville's Theorem

We want to understand the structure of the set

$$S^1(X) = \{\xi \in \mathrm{Pic}^0(X) \mid H^1(X, L_\xi) \neq 0\}$$

when X is a compact Kähler manifold. This is the main subject of [Be]. Introduce some notation, denote by $\mathrm{Pic}^\tau(X)$ the set of holomorphic line bundle on X whose first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ is torsion. Now we have the exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^\tau(X) \to \ker\big(H^2(X, \mathbb{Z}) \to H^2(Z, \mathbb{C})\big) \to 0$$

Given a morphism $f \colon X \to C$ with connected fibres, we define

$$\mathrm{Pic}^\tau(X, f) = \{L \in \mathrm{Pic}^\tau(X) \mid L \text{ is trivial on every smooth fiber of } f\}$$

and set $\mathrm{Pic}^0(X, f) = \mathrm{Pic}^\tau(X, f) \cap \mathrm{Pic}^0(X)$. We'll see that this is the same as the image of $f^* \colon \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(X)$, the difference is in the singular fibres of f. Then we have the following theorem, which we are not going to prove.

Theorem 5.1.1 (**Beauville**). *Let X be a compact Kähler manifold. Let $\{f_i \colon X \to C_i\}_{i\in I}$ be the collection of all fibrations of X onto curves of genus ≥ 1. Then $S^1(X)$ is the union of the following subsets:*

1. $\mathrm{Pic}^0(X, f_i)$ *for every $i \in I$ with $g(C_i) \geq 2$;*

2. *Pic*$^0(X, f_i) \setminus \mathrm{Pic}^0(C_i)$ *for every $i \in I$ with $g(C_i) = 1$;*

3. *finitely many isolated points.*

5.2 The Structure of Cohomology Support Loci

We now specify the structure of cohomology support loci

$$S^i_m(X) = \left\{\xi \in \mathrm{Pic}^0(X) \mid \dim H^i(X, L_\xi) \geq m\right\}.$$

Suggested by Beauville, we have the following theorem proved by Green and Lazarsfeld

Theorem 5.2.1. *Let $Z \subseteq S^i_m(X)$ be an irreducible component. Then Z is a translate of a subtorus of* $\mathrm{Pic}^0(X)$.

During the proof of the generic vanishing theorem we use the fact that we can find locally on $\mathrm{Pic}^0(X)$ a bounded complex of vector bundles that computes the direct image sheaf of the Poincaré bundle. The proof of the theorem is based on the fact that, by Hodge theory once can write it explicitly. Is also used the construction

$$\mathrm{Pic}^0(X) = \frac{\mathcal{H}^{0,1}(X)}{\{\tau \in \mathcal{H}^{0,1}(X) \mid \bar{\tau} - \tau \text{ has period in } \mathbb{Z}(1)\}}.$$

Then for $\tau \in \mathcal{H}^{0,1}(X)$, the class $[\tau]$ in the quotient correspond to the smooth line bundle $X \times \mathbb{C}$, with complex structure given by $\bar{\partial} + \tau$. Details can be found in [GL2].

5.2.1 Consequences

We know that locally on $\mathrm{Pic}^0(X)$, the higher direct image sheaves $R^i p_2^* \mathcal{P}$ are computed by a linear complex. This is very interesting if we think that a linear complex is its own derivative complex, and so all the consequences of this result also become stronger: Theorem 4.2.4 now becomes

$$TC_x\left(S^i_m(E^\bullet)\right) = S^i_m\left(D(E^\bullet, x)\right). \tag{5.1}$$

And also Corollary 4.2.5 now gives a formula for the dimension of the cohomology support loci

Corollary 5.2.2. *Set $m = \dim H^i\left(E^\bullet(x)\right)$. Then*

$$\dim_x S^i_m\left(E^\bullet\right) = \dim\left\{v \in T_x X \mid D_v(d^i, x) = D_v(d^{i-1}, x) = 0\right\}.$$

Another useful improvement is a necessary and sufficient condition for isolated points, strengthening Corollary 4.2.7 we have

Corollary 5.2.3. *Set $m = \dim H^i\left(E^\bullet(x)\right)$. Then x is an isolated point of $S^i_m\left(E^\bullet\right)$ if and only if $H^i\left(D_v(E^\bullet, x)\right) = 0$ for every nonzero $v \in T_x X$.*

5.3 Ueno's Conjecture

The results of Green and Lazarsfeld have a lots of interesting application to algebraic geometry. One of these is a proof of theorem of Kawamata about varieties of Kodaira dimension equal to zero.

5.3.1 Kodaira Dimension

Let's introduce the subject first, starting with the definition of the Iitaka dimension. We will give a brief introduction to the problem, for more details see [La, section 2.1]

Let X be a smooth projective variety on an algebraically closed field k. The crucial invariant of X we will repeatedly refer to is its canonical bundle

$$\omega_X = \bigwedge^{\dim X} \Omega^1_X.$$

Definition 5.3.1. *Define the m-th plurigenus of X as*

$$P_m(X) = \dim H^0(X, \omega_X^{\otimes m})$$

for every $m \geq 0$.

Now, let L be a line bundle on X. If $L^{\otimes m}$ has nontrivial global sections, i.e. $\dim H^0(X, L^{\otimes m}) \neq 0$, then it defines a rational morphism

$$\phi_m \colon X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m})).$$

Let $Y_m = \overline{\phi_m(X)} \subseteq \mathbb{P}(H^0(X, L^{\otimes m}))$ denote the closure of its image, i.e. the image of the closure of the graph of ϕ_m.

Definition. *The* Iitaka dimension *of the line bundle L is defined to be*

$$\kappa(L) = \kappa(X, L) = \max_{m \in \mathbb{N}} \{\dim Y_m\}$$

if $H^0(X, L^{\otimes m}) \neq 0$ for all $m \geq 1$ we set $\kappa(X, L) = -\infty$. Thus either $\kappa(X, L) = -\infty$, or

$$0 \leq \kappa(X, L) \leq \dim X.$$

If $\kappa(X) = dim X$, then X is said to be of general type.

There is also an alternative characterization of the Iitaka dimension of a line bundle: let L a line bundle on an irreducible normal projective variety X, set $\kappa = \kappa(X, L)$ then there are constants $a, b > 0$ such that

$$a \cdot m^\kappa \leq \dim H^0(X, L^{\otimes m}) \leq b \cdot m^\kappa$$

for all sufficiently large and divisible m with $H^0(X, L^{\otimes m}) \neq 0$.

One can show that, for m sufficiently large, the rational mappings φ_m stabilize in the following sense: there is a morphism $\varphi_\infty \colon X_\infty \to Y_\infty$ between two smooth projective varieties, such that φ_m is birationally equivalent to φ_∞. This morphism is unique up to birational equivalence, and is called the *Iitaka fibration* of the line bundle L. By construction, $\dim Y_\infty = \kappa(X; L)$; moreover, φ_∞ is an algebraic fibre space, meaning that it has connected fibres. It is also known that the restriction of L to a very general fibre of φ_∞ has Iitaka dimension equal to zero.

Definition 5.3.2. *In the above setting, consider $L = \omega_X$, the canonical bundle of a smooth projective variety. We call $\kappa(X) = \kappa(X, \omega_X)$ the* Kodaira dimension *of X. We call φ_∞ the* Iitaka fibration *of X.*

Note that X_∞ is birational to X and a general fibre of φ_∞ has Kodaira dimension equal to zero.

In other words, the rough interpretation for the Kodaira dimension is that

$$P_m(X) \sim m^{\kappa(X)}$$

for sufficiently large and divisible m.

5.3.2 Varieties of Kodaira dimension zero

Interesting to study are the varieties whose Kodaira dimension is equal to zero.

Conjecture 5.3.3 (Ueno's Conjecture K). *Let X be a smooth projective variety with $\kappa(X) = 0$, and let $\mathrm{alb} \colon X \to \mathrm{Alb}(X)$ denote its Albanese mapping. Then*

1. alb *is surjective with connected fibres;*
2. *if F is a general fibre of* alb*, then $\kappa(F) = 0$;*
3. *after passing to a finite étale cover, X becomes birational to $F \times \mathrm{Alb}(X)$.*

The first one was proved by Yujiro Kawamata, using difficult arguments from Hodge theory. Subsequently, Lawrence Ein and Robert Lazarsfeld found a very simple proof ([EL]) based on Theorem 5.2.1, see also [CH].

Theorem 5.3.4. *Let X be a smooth projective variety of Kodaira dimension zero. Then the Albanese mapping* $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ *is surjective.*

The proof is based on two surprisingly simple observations. But first, a word about the meaning of the condition $\kappa(X) = 0$. By definition, the sequence of plurigenera $P_m(X)$ is bounded; actually, we even have $P_m(X) \leq 1$ for all m. Indeed, if $P_m(X) \geq 2$ for some m, then we could find two linearly independent sections of $\omega_X^{\otimes m}$, and by multiplying these together, we would get $P_{km}(X) \geq k+1$, contradicting $\kappa(X) = 0$. Thus we can say that if X has Kodaira dimension zero, then $P_m(X) = 1$ for m sufficiently large. So, we shall assume that $P_1(X) = P_2(X) = 1$. The first observation is that this condition has an effect on the locus $S^n(X)$, where $n = \dim X$.

Proposition 5.3.5. *If $P_1(X) = P_2(X) = 1$, then $\mathscr{O}_X$ is an isolated point of $S^n(X)$.*

Proof. Since $P_1(X) \neq 0$, we have

$$H^n(X, \mathscr{O}_X) \simeq \mathrm{Hom}(H^0(X, \omega_X), \mathbb{C}) \neq 0$$

and so $\mathscr{O}_X \in S^n(X)$. Suppose that it is not an isolated point. Then by Theorem 5.2.1, $S^n(X)$ contains a subtorus T of positive dimension. In particular, T is a subgroup, and so if $L \in T$, then also $L^{-1} \in T$. This means that the image of the multiplication map

$$H^0(X, \omega_X \otimes L) \otimes H^0(X, \omega_X \otimes L^{-1}) \to H^0(X, \omega_X^{\otimes 2})$$

is nonzero for every $L \in T$. Now $\omega_X^{\otimes 2}$ only has one global section because $P_2(X) = 1$; let D be the corresponding effective divisor on X. By the above, the divisor of any global section of $\omega_X \otimes L$ has to be contained in D; but because D has only finitely many irreducible components, we can find two distinct points $L_1, L_2 \in T$, and nontrivial sections $s_1 \in H^0(X; \omega_X \otimes L_1)$ and $s_2 \in H^0(X; \omega_X \otimes L_2)$, such that $\mathrm{div} s_1 = \mathrm{div} s_2$. But then $\omega_X \otimes L_1 \simeq \omega_X \otimes L_2$, which contradicts the fact that L_1 and L_2 are distinct points of T. □

The second observation of Ein and Lazarsfeld is that $S^n(X)$ is closely related to the geometry of the Albanese mapping.

Proposition 5.3.6. *If the origin is an isolated point of $S^n(X)$, then the Albanese mapping* $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ *is surjective.*

Proof. Since $\mathscr{O}_X$ lies in $S^n(X)$, we have $H^0(X; \omega_X) \geq 1$. Let $s \in H^0(X; \omega_X)$ be any nontrivial section. Because $\mathscr{O}_X$ is an isolated point of $S^n(X)$, the criterion in Corollary 5.2.3 shows (after conjugating) that the mapping

$$H^0(X, \Omega_X^{n-1}) \xrightarrow{\omega\wedge} H^0(X, \Omega_X^n)$$

is surjective for every nonzero $\omega \in H^0(X; \Omega_X^1)$. In particular, we have $s(x) = 0$ at every point $x \in X$ where $\omega(x) = 0$. From this, we can deduce without much difficulty that alb must be surjective.
Indeed, suppose that alb was not surjective. Take an arbitrary point $x \in X$. The differential $T_xX \to T_{\mathrm{alb}(x)}\mathrm{Alb}(X)$ of the Albanese mapping is obviously not surjective; after dualizing and using Lemma 2.3.3, we find that the evaluation mapping

$$H^0(X, \Omega_X^1) \to (T_x(X))^*$$
$$\omega \mapsto \omega(x)$$

is not injective. Thus, there is at least one nonzero holomorphic one-form with $\omega(x) = 0$. By the above, we then have $s(x) = 0$; but because x was an arbitrary point of X, this contradicts the fact that $s \neq 0$. □

Together, the two propositions prove Theorem 5.3.4 in the case when $P_1(X) = 1$. The general case requires a little bit of extra work.

Proof. Theorem 5.3.4. If $P_1(X) = 1$, then $\kappa(X) = 0$ forces $P_2(X) = 1$, and so we are done by the above arguments. If this is not the case, it can be found a smooth projective variety Y with $\kappa(Y) = 0$, and a generically finite morphism $f: Y \to X$, such that $P_1(Y) = 1$ (This is the *Fujita's lemma*). Then alb_Y is surjective, and we can use this to show that alb_X is so,too. The first observation is that $f^*: \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ has finite kernel: if f^*L is trivial, then we get, from the projection formula

$$L \otimes f_*\mathscr{O}_Y \simeq f_*\mathscr{O}_Y;$$

so, setting $r = \deg f$, it follows that

$$L^{\otimes r} \det(f_*\mathscr{O}_Y) \simeq \det(f_*\mathscr{O}_Y),$$

which shows that the r-th power of L is trivial.
Dually, we get that $\mathrm{Alb}(Y) \to \mathrm{Alb}(X)$ is surjective. Then, if we look at the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\mathrm{alb}_Y"] & X \arrow[d, "\mathrm{alb}_X"] \\ \mathrm{Alb}(Y) \arrow[r] & \mathrm{Alb}(X) \end{tikzcd}$$

it's clear that the composition $Y \to \mathrm{Alb}_Y \to \mathrm{Alb}(X)$ is therefore surjective. Then we can conclude that alb_X must be surjective as well. □

5.3.3 Cyclic Coverings

We discuss here some covering constructions that allow one to extract "roots" of divisors or line bundles. We begin with a local description of the m-fold cyclic covering branched along a given divisor on a variety. Suppose then that X is an affine variety, and $s \in \mathbb{C}[X]$ is a non-zero regular function, where $\mathbb{C}[X]$ is the corresponding finitely generated extension of function fields. We wish to define a variety Y on which the mth root $\sqrt[m]{s}$ of s makes sense. Let's start with the product $X \times \mathbb{A}^1$ of X and the affine line. Let $Y \subseteq X \times \mathbb{A}^1$ the subvariety defined by the equation $t^m - s = 0$, where t is the coordinate on $\mathbb{A}^1$.

$$\begin{tikzcd}
\{t^m - s = 0\} = Y \arrow[rr, hook] \arrow[dr, "\pi"'] & & X \times \mathbb{A}^1 \arrow[dl, two heads, "p_1"] \\
& X &
\end{tikzcd}$$

Let $D = div(s)$, the mapping $\pi \colon Y \to X$ is a cyclic covering branched along D. Then if we set $s' = t \mid Y \in \mathbb{C}[Y]$ we get

$$(s')^m = \pi^* s$$

as functions on Y, so we have extracted the desired root of s.
This local construction can be globalized in the following way

Proposition 5.3.7 (**Cyclic Coverings**). *Let X be a variety, and L a line bundle on X. Suppose, given a positive integer $m \geq 1$ and a nonzero section $s \in H^0(X, L^{\otimes m})$, which defines a divisor $D \subseteq X$. Then there exists a finite flat covering*

$$\pi \colon Y \to X,$$

where Y is a scheme having the property that the pull-back $L' = \pi^* L$ of L carries a section $s' \in H^0(Y, L')$ such that

$$(s')^m = \pi^* s.$$

The divisor $D' = \operatorname{div}(s')$ maps isomorphically D.

As above, one should think of s' as being the m-th root $\sqrt[m]{s}$ of s.

Proof. Let $\mathbb{L}$ be the total space of the line bundle L with $p \colon L \to X$ the bundle projection. In other words $L = \operatorname{Spec}_{\mathscr{O}_X} \operatorname{Sym}(L^*)$. Then there is a "tautological" section

$$T \in \Gamma(\mathbb{L}, p^* L).$$

In fact, a section of p^*L is specified geometrically by giving for each point $a \in L$ a vector in the fibre of p over $x = p(a)$. But this vector is a, and we set $T(a) = a$. More formally, T is determined by a homomorphism of $\mathscr{O}_{\mathbb{L}}$-moduli

$$\mathscr{O}_{\mathbb{L}} \to \mathscr{O}_{\mathbb{L}} \otimes p^*L;$$

or equivalently a mapping

$$\mathrm{Sym}_{\mathscr{O}_X}(L^*) \to L \otimes \mathrm{Sym}_{\mathscr{O}_X}(L^*)$$

of quasi-coherent sheaves on X. Here, the term on the left in is a naturally a summand of that on the right, and the map in question is the canonical inclusion. One should view T as a "global fibre coordinate" in $\mathbb{L}$: for instance $\{T = 0\}$ defines the zero-section of $\mathbb{L}$.
Now, let $Y \subseteq \mathbb{L}$ be the divisor of the section

$$T^m - p^*s \in \Gamma(\mathbb{L}, p^*L^{\otimes m})$$

and $s' = T \mid Y$. Then the proposition follows from the local construction. □

Note that the proof could be done by taking an affine open covering $\{U_i\}$ of X which locally trivializes L, and carrying out the previous construction over each U_i: the fact that s is a section of the m-th power of a line bundle allows one to glue together the resulting local coverings.

Remark 5.3.8. *It follows from the construction that there is a canonical isomorphism*

$$\pi^*\mathscr{O}_Y = \mathscr{O}_X \oplus L^{\otimes(-1)} \oplus \cdots \oplus L^{\otimes(1-d)}.$$

Now we want to generalize this construction to the following case.
Let $Y = \{(x, v) \in L \mid v^{\otimes m} = s(x)\}$, if it is singular we can use the following method to find a space X' with a $(m:1)$-covering as desired. Let's consider

$$\begin{tikzcd} X' \arrow[r] \arrow[rrd, "\pi"'] & Y^\nu \arrow[r] & Y \subseteq L \arrow[d, "m:1"] \\ & & X \end{tikzcd}$$

where we denoted with Y^ν the normalization of Y and with X' the desingularization of Y^ν. Let L a line bundle on X. Then we can consider L both as a line bundle on X either a scheme over X, in this case it can be thought as $\{(x, v) \mid v \in L(x)\}$. So, define the tautological section s_0 on L

as $s_0(x,v) = v \in L(x)$; in particular $s_0|_Y(x,v) = v \in L(x)$. Since $s_0|_Y \in Y$, then it satisfies

$$(s_0|_Y(x,v))^{\otimes m} = v^{\otimes m} = s(x).$$

The map $f\colon Y \to X$ above is a cyclic covering. Now, consider $L = \omega_X$ the canonical bundle, and let s a non-trivial section $s \in H^0(X, \omega_X^{\otimes m})$. Let $\pi^* s \in H^0(Y, \pi^*\omega_X^{\otimes m})$ on Y, then there exist $s_0 \in H^0(Y, \pi_Y^*\omega_X)$ such that

$$s_0^{\otimes m} = \pi^* s.$$

Now we want to extend this discussion to X', which is smooth. We know that $\omega_{X'} = \pi^*\omega_X^{\otimes m} \otimes \mathscr{O}(nR)$, where R is the ramification divisor. Then

$$H^0(X', \omega_{X'}^{\otimes m}) = H^0(X, \omega_X^{\otimes m} \otimes \pi_*\mathscr{O}(nR)).$$

And here we can conclude using Fujita's Lemma 5.3.9. (see [Mo])

Lemma 5.3.9 (Fujita's Lemma)**.** *Let M be a manifold with $\kappa(M) \geq O$. Then there is a surjective morphism $f\colon N \to M$ from a manifold N with* $\dim N = \dim M$*, $\kappa(N) = \kappa(M)$ and $P_1(N) > 0$.*

Proof. The idea of the proof is the following. Let $D \in |kK_M|$, we construct in a natural way a subvariety W in K_M such that the projection $K \to M$ restricted to W makes W a cyclic k-sheeted branched covering of M with branch locus D. A smooth model N of W has the desired property. $\square$

Bibliography

[Be] A. Beauville, *Annulation du H^1 pour les fibrés en droites plats,* Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Math., vol. 1507, Springer, Berlin, 1992, pp. 1–15

[BH] C. Birkenhake, H. Lange, *Complex Abelian Varieties.* Springer, Berlin Heidelberg 2004.

[CH] J. A. Chen and C. D. Hacon, *Characterization of abelian varieties.* Invent. Math. 143 , 2001, no. 2, 435–447.

[EL] L. Ein, R. Lazarsfeld, *Singularities of theta divisors and the birational geometry of irregular varieties.* J. Amer. Math. Soc. 10, no. 1, 243–258, 1997.

[EV] H. Esnault and E. Viehweg, *Lectures on vanishing theorems,* Birkhäuser, Basel, 1992.

[Fu] W. Fulton, *Algebraic Topology - A First Course.* Springer-Verlag, 1995.

[GH] P. Griffiths; J. Harris, *Principles of algebraic geometry.* Wiley- Interscience, New York, 1978.

[GL1] M. Green and R. Lazarsfeld, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville.* Invent. Math. 90, 389-407, 1987.

[GL2] M. Green and R. Lazarsfeld, *Higher obstructions to deforming cohomology groups of line bundles.* J. Amer. Math. Soc. 4, no. 1, 87–103, 1991.

[Ha] R. Hartshorne, *Algebraic Geometry.* Springer-Verlag, New York-Heidelberg, 1977.

[Hu] D. Huybrechts, *Complex Geometry - An Introduction.* Springer, 2005.

[Ka] Y. Kawamata, *Characterization of Abelian Varieties.* Compositio math. 43, 253 - 276, 1981.

[Ke] G. R. Kempf, *Complex Abelian Varieties and Theta Functions.* Springer, 1991.

[La] R. Lazarsfeld, *Positivity in Algebraic Geometry.* Springer, 2004.

[Mo] S. Mori, *Classification of Higher-Dimensional Varieties*, Algebraic Geometry - Bowdoin 1985, Part 1. AMS, 1987.

[Mu] D. Mumford, *Abelian varieties.* Oxford Univ. Press 1970.

[Ue] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Lecture notes in Math,. vol 439, Springer-Verlag, 1975.

[Vo] C. Voisin, *Hodge Theory and Complex Algebraic Geometry. Volume 1.* Cambridge University Press, 2002.