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Functional Methods and Effective Field Theories

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Contents

Introduction

Canonical Quantization is so far the most successful attempt to combine the principles of Quantum Mechanics and Special Relativity into a unique theory. From an experimental point of view its success is undeniable, while from a mathematical point of view there is still a lot of work to do.

A possible way out is the axiomatic approach, in which the reconstruction theorem allows to build a fully consistent Quantum Field Theory from the sole knowledge of the Wightman distributions, the vacuum expectation values of products of fields satisfying the Wightman axioms. Therefore an approach one may think to use is: compute the Wightman distributions of a theory in any possible way, mathematically justified or not, and then use the Reconstruction theorem to give a rigorous structure to the theory at hand. Canonical Quantization offers a possibility to compute the desired distributions, but it suffers many problems.

In this thesis we consider alternative frameworks that solve or may solve some of the problems of Canonical Quantization. Two examples are Resurgence and the formulation on a lattice. The former is a research field in development in the very last years, the latter provides an attempt to build a sound definition of functional integral and is a tool for numerical, non perturbative simulations.

Despite the importance of Resurgence and the lattice, in this thesis we focus mainly on two other approaches. The first one is a versatile use of functional methods. The hope for the future is to find new functional tecniques in order to improve our knowledge on Quantum Field Theory. The second one is the Wilsonian approach. At first sight this approach may look an approximative method, but this is not true. On one hand, a way to systematically implement the Wilsonian ideas is by the Wetterich equation, a functional differential equation that is very useful both for perturbative and non perturbative results. On the other hand, in some cases it is possible to compute exactly the Wilsonian effective action, and the knowledge of the theory at low energy gives access to exact information on the full microscopic theory. This is what happens for instance in the Seiberg-Witten model.

Therefore, the content of this thesis is in short the following:

- in the first chapter we give a brief summary of the main troubles that affect Canonical Quantization and we introduce some formulations that should solve in part such problems, namely the axiomatic approach, Resurgence and the formulation of a Quantum Field Theory on a lattice. Since we want to analyze the problems from a mathematical point of view, in the first chapter we keep a rather rigorous language. In the other chapters we drop our attention to technical and mathematical details and we adopt an operative way of thinking.

- The second chapter is dedicated mainly to functional techniques, in particular to the study of different representations of the generating functional. For the sake of completeness, and for a better understanding of the next chapter, we give some space to more standard arguments, such as the effective action and the generating functional of connected Green's functions.

- In the third chapter we first introduce the Wilsonian approach and the Polchinski equation. Next we derive the Wetterich equation and show how we can start a perturbative expansion from it.

- In the last chapter we show the power of the Wilsonian approach by discussing the Seiberg-Witten model. We derive the exact solution for the low energy effective action and then we find some non perturbative results, such as the exact determination of all the instantonic contributions for the $N = 2$ super Yang-Mills theory in the low energy limit.

Chapter 1

Problems and outlooks in Quantum Field Theory

This is an introductory chapter where we summarize the problems of Canonical Quantization. We start from showing why the right way to define a field is through the so-called operator valued distributions. After that we see that taking a free field as starting point to build (perturbatively) an interacting one suffers many technical illnesses culminating in ultraviolet divergencies and divergencies in the perturbative expansion itself. After listing these problems, in the last three sections we pass to present some possible solutions, namely the axiomatic approach, Resurgence and Lattice Quantum Field Theory, other approaches being discussed in detail in the next chapters.

1.1 Short distance singularities

In the first three sections we follow essentially the first part of the work of Lutz Klaczynski [1]. The reader can find more details there. A problem of technical nature in Canonical Quantization already arises in the basic example of a free Hermitian scalar field $\phi(x)$. Following the steps of Canonical Quantization we can write it in the form

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[e^{-ip\cdot x} a(\mathbf{p}) + e^{ip\cdot x} a^\dagger(\mathbf{p}) \right]_{p_0 = E_p} , \tag{1.1}$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$ and $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ are the annihilation and creator operators satisfying

$$[a(\mathbf{p}), a(\mathbf{q})] = 0 = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})] \qquad [a(\mathbf{p}), a^\dagger(\mathbf{q})] = \delta^{(3)}(\mathbf{p} - \mathbf{q}) .$$

Expression (1.1) suggests that there exists an operator $\phi(x)$ at the sharp spacetime point x, but this is not true as it can be immediately seen by applying $\phi(x)$ to the vaccum Ψ_0 and trying to compute the norm of $\Psi = \phi(x)\Psi_0$: the result is infinite. The operator $\phi(x)$ in (1.1) is ill defined and there is no way to avoid this problem without falling into a triviality problem. Here triviality is to be understood in the sense that, if $\hat{\phi}(x)$ is assumed to be a well defined operator for each point x (here the hat is used just to distinguish the well defined operator $\hat{\phi}(x)$ from the ill defined $\phi(x)$), then it must be constant and just a multiple of the identity, as stated in the following theorem by Wightman:

Theorem 1 (Short distance singularities)**.** *Let $\hat{\phi}(x)$ be a Poincaré covariant Hermitian scalar field, that is*

$$U(a,\Lambda)\hat{\phi}(x)U^{\dagger}(a,\Lambda) = \hat{\phi}(\Lambda x + a)\,, \tag{1.2}$$

and suppose it is a well defined operator with the vacuum Ψ_0 in its domain. Then the function

$$F(x,y) = \langle \Psi_0 | \hat{\phi}(x)\hat{\phi}(y)\Psi_0 \rangle$$

is constant, call it c. Furthermore $\hat{\phi}(x)\Psi_0 = \sqrt{c}\Psi_0$ and thus

$$\langle \Psi_0 | \hat{\phi}(x_1)\dots\hat{\phi}(x_n)\Psi_0 \rangle = c^{n/2}\,.$$

Nonetheless, we can ignore all the technical problems that we do not like and formally compute the two point function from equation (1.1) to get

$$\langle \Psi_0 | \phi(x)\phi(y)\Psi_0 \rangle = \int \frac{d^3p}{(2\pi)^6}\frac{e^{-ip\cdot(x-y)}}{2E_p} = \Delta_+(x-y)\,, \tag{1.3}$$

which is a well defined function for $x \neq y$ and has singularities for $x = y$, where it gives the "infinite squared norm" of $\phi(x)\Psi_0$. A way to remedy the situation is to smooth out the field with respect to its spatial coordinates

$$\phi(t,f) = \int d^3x\, f(\mathbf{x})\phi(t,\mathbf{x})\,,$$

where $f \in \mathcal{S}(\mathbb{R}^3)$ is a Schwartz function on $\mathbb{R}^3$. In this way, the state $\Psi_f(t) = \phi(t,f)\Psi_0$ has the finite norm

$$\|\Psi_f(t)\| = \int \frac{d^3p}{(2\pi)^3}\frac{|\tilde{f}(\mathbf{p})|^2}{2E_p}\,,$$

where $\tilde{f}$ is the Fourier transform of f. In the case of a free scalar field, smearing in space is enough to avoid problems, but in the generel case of interacting theories the answer is unknown. This is why in the axiomatic approach by Wightman fields are introduced as completely smoothed-out operators

$$\phi(f) = \int d^4x\, f(x)\phi(x)$$

and the two point function (1.3) is recovered as (the kernel of) a tempered distribution

$$(f,g) \mapsto W(f,g) = \int d^4x d^4y\, f^*(x)W(x-y)g(y)\,, \qquad W(x-y) = \Delta_+(x-y)\,,$$

where f and g are Schwartz functions on $\mathbb{R}^4$ and $W(f,g) = \langle \Psi_0 | \phi(f)\phi(g)\Psi_0 \rangle$.

1.2 Triviality problems of interacting fields

A deeper problem of Canonical Quantization is given by Haag's theorem. For an interacting theory Canonical Quantization leads to the introduction of the so-called interaction picture in order to exploit perturbation theory. Let us consider an interacting scalar field $\phi(x)$ with Hamiltonian H. In order to relate the interaction picture with the Heisenberg picture, we split the full Hamiltonian H into the free Hamiltonian

H_0 and the interacting Hamiltonian H_I, $H = H_0 + H_\mathrm{I}$. If $\phi(\mathbf{x})$ denotes the scalar field in the Schroedinger representation, the other two pictures are obtained as

$$\begin{aligned} \phi(t,\mathbf{x}) &= e^{iHt}\phi(\mathbf{x})e^{-iHt} && \text{(Heisenberg picture)} \\ \phi_0(t,\mathbf{x}) &= e^{iH_0t}\phi(\mathbf{x})e^{-iH_0t} && \text{(interaction picture)} \end{aligned} \tag{1.4}$$

From equations (1.4) we see that $\phi_0(t,\mathbf{x})$ behaves like a free scalar field and that the two pictures are related by a unitary transformation

$$\phi(t,\mathbf{x}) = e^{iHt}e^{-iH_0t}\phi_0(t,\mathbf{x})e^{iH_0t}e^{-iHt} = V^\dagger(t)\phi_0(t,\mathbf{x})V(t)\,. \tag{1.5}$$

Here is where the problem arises. In words, Haag's theorem can be stated as follows:

Theorem 2 (Haag)**.** *If a scalar quantum field is unitarily equivalent to a free scalar quantum field, then it is also a free scalar field.*

Thus, either $\phi(t,\mathbf{x})$ is a free field or the unitary transformation $V(t)$ does not exist, that is the interaction picture does not exist. This theorem, which has been rigorously proven in the framework of the axiomatic approach, is quite general because it relies essentially only on the unitarity of $V(t)$. It does not state that interacting fields do not exist, it just says that interacting and free fields are different objects and are unitarily inequivalent. Perturbation theory relies on the splitting of the Hamiltonian into the free part and the interaction potential, an arbitrary splitting decided by our incapability to perform calculations in the non quadratic case. But it has been proven in the basic example of a scalar field that the free Hamiltonian H_0 and the full Hamiltonian H cannot exist as self-adjoint operators in the same Fock space, a result suggesting that the splitting $H = H_0 + H_\mathrm{I}$ is at the origin of the problems in perturbation theory.

Another interesting issue, which may be related to Haag's theorem, concerns the canonical commutation relations (CCR). Canonical quantization in $d = n + 1$ space-time dimensions relies on the fact that a scalar quantum field $\phi(x)$ and its conjugate momentum $\pi(x)$ satisfy the equal-time commutation relations

$$[\phi(t,\mathbf{x}),\pi(t,\mathbf{y})] = i\delta^{(n)}(\mathbf{x}-\mathbf{y}) \qquad [\phi(t,\mathbf{x}),\phi(t,\mathbf{y})] = 0 = [\pi(t,\mathbf{x}),\pi(t,\mathbf{y})]\,. \tag{1.6}$$

If we smear the fields with respect to their spatial coordinates

$$\phi(t,f) = \int d^nx\, f(\mathbf{x})\phi(t,\mathbf{x}) \qquad \pi(t,f) = \int d^nx\, g(\mathbf{x})\pi(t,\mathbf{x})\,,$$

relations (1.6) translate into

$$[\phi(t,f),\pi(t,g)] = i(f,g) \qquad [\phi(t,f),\phi(t,g)] = 0 = [\pi(t,f),\pi(t,g)]\,, \tag{1.7}$$

where (f,g) denotes the scalar product

$$(f,g) = \int d^nx\, f(\mathbf{x})g(\mathbf{x})\,.$$

A theorem by Baumann essentially states that in spatial dimension $n \geq 4$ a scalar field fulfilling (1.7) is free. Actually the theorem assumes a number of technicalities, among which a vanishing vacuum expectation value for $\phi(t,\cdot)$, but the important thing is that it poses the question whether the CCR (1.6) are so fundamental as it may be expected. They are certainly true for free theories, but nothing assures that they should hold for interacting fields as well, as suggested by Baumann's theorem. A similar result concerning fermions has been obtained by Powers for $n \geq 2$. The connection with Haag's theorem is still in the unitary transformation $V(t)$ defined in (1.5). If the respective conjugate momenta $\pi(x)$ and $\pi_0(x)$ of $\phi(x)$ and $\phi_0(x)$ are related by the same transformation $V(t)$, then of course $\phi(x)$ and $\pi(x)$ will satisfy the same commutation relations as $\phi_0(x)$ and $\pi_0(x)$.

1.3 Gell-Mann-Low formula and ultraviolet divergencies

In view of these facts, it should not be surprising that going on with perturbation theory divergencies arise all around. Let us see now how they arise. The interaction picture allows for the demonstration of the Gell-Mann-Low formula

$$\langle\Omega|\,\mathrm{T}\,\phi(x_1)\ldots\phi(x_n)|\Omega\rangle = \frac{\langle 0|\,\mathrm{T}\,\phi_0(x_1)\ldots\phi_0(x_n)e^{i\int d^4x\,V(\phi_0(x))}|0\rangle}{\langle 0|S|0\rangle}\,, \tag{1.8}$$

where T denotes the time ordered product, $|\Omega\rangle$ is the vacuum of the interacting theory and $|0\rangle$ is the free vacuum. S is the S-operator given by

$$S = \mathrm{T}\,e^{-i\int d^4x\,V(\phi_0(x))}$$

and this is the starting point for the perturbative expansion, since S can be expanded in terms of the Dyson's series

$$S = 1 + \sum_{n=1}^{+\infty}\frac{(-i)^n}{n!}\int d^4x_1\ldots d^4x_n\;\mathrm{T}\,V(\phi_0(x_1))\ldots V(\phi_0(x_n))\,.$$

Plugging this expansion into the Gell-Mann-Low formula (1.8), we get two kinds of divergencies. The first kind can be easily eliminated. We have seen above that the two point function (1.3), $\Delta_+(x-y)$, is well defined for $x \neq y$ and has poles at $x = y$. Thus, the vacuum expectation value of the product of two fields at the same spacetime point is still ill defined, but it can be cured by introducing the Wick powers, recursively defined by

$$\begin{aligned}
&:\phi_0(x) := \phi_0(x)\\
&:\phi_0^2(x) := \lim_{y\to x}\left[\phi_0(x)\phi_0(y) - \langle 0|\phi_0(x)\phi_0(y)|0\rangle\right]\\
&:\phi_0^n(x) := \lim_{y\to x}\left[\,:\phi_0^{n-1}(x):\phi_0(y) - (n-1)\langle 0|\phi_0(x)\phi_0(y)|0\rangle :\phi_0^{n-2}(x):\,\right],
\end{aligned}$$

and correspondig to normal ordered monomials. Substituting $\mathcal{H}_\mathrm{I}(x)$ with its normal ordered counterpart $:\mathcal{H}_\mathrm{I}(x):$ we schematically get something like

$$\langle 0|:\phi_0^{n_1}(x_1):\cdots:\phi_0^{n_k}(x_k):|0\rangle = \sum\prod\Delta_+(x_i - x_j)\,, \tag{1.9}$$

where the summation runs over Feynman diagrams without tadpoles and the products over the edges of a certain graph. Since products of $\Delta_+(x-y)$ are well defined, equation (1.9) is well defined. The second kind of divergencies arises when considering the time ordering. In this case we need to substitute the two point functions in equation (1.9) with their time ordered versions, i.e. the Feynman propagators

$$i\Delta(x-y) := \langle 0|\,\mathrm{T}\,\phi_0(x)\phi_0(y)|0\rangle = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\,.$$

Products of these objects are in general ill defined and are the origin of the UV divergencies. Renormalization is the procedure to cure these divergencies. In phycisist language it consists in introducing a regulator to make them finite, manipulating in some sense the so-obtained regulated objects and then removing the regulator in the right way to get meaningful results. More technically, renormalization can be given a mathematically sound meaning in terms of distributions.

1.4 Borel resummation

Still, even once renormalization produces finite coefficients for the perturbative expansion of equation (1.8), it turns out that the series do not converge (see [2] for more details). A simple way to see this fact is Dyson's argument. Consider the theory $\lambda\phi^4$ and assume that Feynman diagrams and renormalization produce the series $\sum_k \lambda^k c_k$ around the point $\lambda = 0$. If the radius of convergenge is larger than zero, then the series makes sense for negative λ as well. But if $\lambda < 0$, then the potential $V(\phi) = \lambda\phi^4$ is unbounded from below, which is physically unacceptable and should produce divergent vacuum expectation values. Thus the series $\sum_k \lambda^k c_k$ must have zero radius of convergence.

Yet, it is still possible to give a meaning to a divergent series with the notion of asymptotic series and Borel summation.

Definition 1 (Asymptotic series)**.** Let $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$. The series $\sum_{n=0}^{\infty} a_n z^n$ is called asymptotic to f as $z \to 0^+$ if

$$\forall N \in \mathbb{N} : \lim_{z\to 0^+} \frac{f(z) - \sum_{n=0}^{N} a_n z^n}{z^n} = 0\,.$$

The meaning of a (divergent) asymptotic series is that it approaches the function f to some extent, depending on z, before eventually diverging. Unfortunately a given series $\sum_{n=0}^{\infty} a_n z^n$ is not asymptotic to a unique function f because $g(z) = e^{-1/z}$ has null asymptotic series and thus the asymptotic series of f and $f+g$ coincide. We need a stronger notion of asymptotic series, given by

Definition 2 (Strong asymptotic series)**.** Let f be an analytic function in the interior of $S_\epsilon = \{z \in \mathbb{C} | |z| \leq R, |\arg z| \leq \frac{\pi}{2} + \epsilon\} \to \mathbb{R}$. The series $\sum_{n=0}^{\infty} a_n z^n$ is a strong asymptotic series if there exist C and σ such that $\forall N \in \mathbb{N}$, $z \in S_\epsilon$ the strong asymptotic condition

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| \leq C\sigma^{N+1}(N+1)!|z|^{N+1}$$

is fulfilled.

The function f is unique by virtue of Carleman's theorem and can be recovered thanks to the following theorem.

Theorem 3 (Watson)**.** *If $f : S_\epsilon \to \mathbb{R}$ has a strong asymptotic series $\sum_{n=0}^{\infty} a_n z^n$, we define the Borel transform*

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \,.$$

The Borel transform converges for $|z| < \frac{1}{|\sigma|}$. We obtained a convergent power series with finite radius of convergence, which, as it turns out, can be analytically continued to all complex z with $|\arg z| < \epsilon$. Then the function f is given by the Laplace transform

$$f(z) = \int_0^{+\infty} db\, g(bz) e^{-b} \,.$$

This Laplace transform is called inverse Borel transform and the method outlined here is known as Borel summability method. Unfortunately important theories like QED are not Borel summable. An attempt of solution to this problem is provided by Resurgence (see below).

1.5 The axiomatic approach

A major achievement towards a mathematically consistent formulation of a relativistic quantum theory is given by the axiomatic approach. It relies on a bunch of axioms that codify deep and simple physical requirements into rigorous mathematical language. An interesting review is given by Franco Strocchi in [3].

The quantum mechanical properties of a physical system are encoded in the following statements

QM1. (Hilbert space structure)**.** The states are described by vectors of a (separable) Hilbert space $\mathcal{H}$.

QM2. (Energy-momentum spectral condition)**.** The spacetime translations are a symmetry of the theory and are therefore described by strongly continuous unitary operators $U(a)$, $a \in \mathbb{R}^4$, in $\mathcal{H}$.

The spectrum of the generators P_μ is contained in the closed forward cone $\bar{V}_+ = \{p_\mu : p^2 \geq 0, p_0 \geq 0\}$. There is a vacuum state Ψ_0, with the property of being the unique translationally invariant state (uniqueness of the vacuum).

QM3. (Field operators)**.** The theory is formulated in terms of fields $\varphi_k(x)$, $k = 1, \ldots, N$, which are operator valued tempered distributions in $\mathcal{H}$, with Ψ_0 a cyclic vector for the fields, i.e. by applying polynomials of the smeared fields to the vacuum one gets a dense set $\mathcal{D}_0$.

We recall that strong continuity means that the function

$$a \mapsto \langle \Psi | U(a) \Phi \rangle$$

is continuous for all states Ψ, $\Phi \in \mathcal{H}$.

The relativistic invariance of the system translates into

R1. (Relativistic covariance)**.** The Lorentz transformations Λ are described by strongly continuous unitary operators $U(\Lambda(A))$, $A \in SL(2, \mathbb{C})$, and the fields transform covariantly under the Poincaré transformations $U(a, \Lambda) = U(a)U(\Lambda)$:

$$U(a, \Lambda(A))\varphi_i(x)U(a, \Lambda(A))^{-1} = S_{ij}(A^{-1})\varphi_j(\Lambda x + a) \,,$$

with S a finite dimensional representation of $SL(2, \mathbb{C})$.

R2. (Microcausality or locality)**.** The fields either commute or anticommute at space-like separated points

$$[\varphi_i(x), \varphi_j(y)]_\mp = 0 \qquad \text{for } (x-y)^2 < 0\,.$$

In the case for example of a scalar field, properties QM1-QM3, R1 and R2 imply the following properties of the vacuum expectation values

W1. $\mathcal{W}(x_1, \ldots, x_n) = \langle \Psi_0 | \phi(x_1) \cdots \phi(x_n) \Psi_0 \rangle$ are tempered distributions.

W2. (Covariance)**.** Setting $\xi_i := x_{i+1} - x_i$, we have

$$\mathcal{W}(x_1, \ldots, x_n) = W(\xi_1, \ldots, \xi_{n-1}) = W(\Lambda\xi_1, \ldots, \Lambda\xi_{n-1})\,.$$

W3. (Spectral condition)**.** The support of the Fourier transform $\tilde{W}$ of W is contained in the product of forward cones:

$$\tilde{W}(p_1, \ldots, p_n) = 0 \qquad \text{if } p_i \notin \bar{V}_+\,.$$

W4. (Locality)**.**

$$\mathcal{W}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \mathcal{W}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \qquad \text{if } (x_i - x_{i+1})^2 < 0\,.$$

W5. (Positivity)**.** For any terminating sequence $f = (f_0, f_1, \ldots, f_N)$, $f_i \in \mathcal{S}(\mathbb{R}^4)^i$ we have

$$\sum_{i,j} \int dxdy\, \bar{f}_i(x_i, \ldots, x_1) f_j(y_1, \ldots, y_j) \mathcal{W}(x_1, \ldots, x_i, y_1, \ldots, y_j) \geq 0\,.$$

W6. (Cluster property)**.** For any spacelike vector a and for $\lambda \to +\infty$

$$\mathcal{W}(x_1, \ldots, x_i, x_{i+1} + \lambda a, \ldots, x_n + \lambda a) \to \mathcal{W}(x_1, \ldots, x_i)\mathcal{W}(x_{i+1}, \ldots, x_n)\,.$$

From properties W1-W6 a number of successful results have been derived, among which we cite

- they provide a more general quantization rule than Canonical Quantization and they imply Canonical Quantization in the case of a free theory, whereas for a general class of interacting fields, as proved by Powers and Baumann, they exclude Canonical Quantization;
- existence of asymptotic fields and, under the assumption of asymptotic completeness, of a unitary S-matrix;
- proof of Pauli principle, i.e. the Spin Statistic Theorem;
- validity of the PCT symmetry;
- the Wightman functions have an analytic continuation to the so-called euclidean points.

The power of the axiomatic approach is due to the Reconstruction theorem, that allows one to recover the full theory, up to unitary equivalence, starting from a set of distributions satisfying W1-W6.

Theorem 4 (Reconstruction)**.** *Let $\{\mathcal{W}_n\}$ be a family of tempered distributions adhering to W1-W6. Then there is a scalar field theory fulfilling the Wightman axioms QM1-QM3, R1 and R2. Any other theory is unitarily equivalent.*

If we find a way to compute the set $\{\mathcal{W}_n\}$ of Wightman distributions, then we are able to build a mathematically sound Quantum Field Theory.

Unfortunately the constraints W1-W6 are highly non trivial to satisfy, even in the simple case of a scalar field, as indicated by non perturbative results on the triviality of the $\lambda\phi^4$ theory in four spacetime dimensions. It is believed that gauge theories do not suffer from triviality but other serious problems arise due to radical differences with respect to ordinary Quantum Field Theories. Let us see why. A gauge theory is invariant under an infinite dimensional Lie group $\mathcal{G}$ of local gauge transformations, which, by Noether theorem, imply the conservation of a current j^α_μ for each generator of the subgroup G of global gauge transformations. Such currents are the divergencies of antisymmetric tensors

$$j^\alpha_\mu = \partial^\nu G^\alpha_{\nu\mu}, \qquad G^\alpha_{\nu\mu} = -G^\alpha_{\mu\nu}\,. \tag{1.10}$$

This property has several important physical consequences, among which the non locality of charged fields. Take for simplicity the abelian case. A field φ has charge q if

$$[Q,\varphi] = q\varphi\,,$$

where

$$Q = \int d^3x\, j_0(\mathbf{x},t) = \int d^3x\, j_0(\mathbf{x},0)\,,$$

and by equation (1.10) we have

$$[Q,\varphi(y)] = \int d^3x\,[j_0(\mathbf{x},0),\varphi(y)] = [\Phi_\infty(\mathbf{E}),\varphi(y)]\,,$$

where $\Phi_\infty(\mathbf{E})$ denotes the flux at spacelike infinity of the electric field $E_i = G_{0i}$. But if $\varphi(y)$ and $E_i(x)$ have to satisfy the locality condition R2, then the right hand side vanishes, because the spacelike infinity is spacelike with respect to any spacetime point y, and φ must have zero charge. In conclusion, a modified version of the Wightmann axioms needs to be found in order to accomodate gauge theories.

1.6 Resurgence

We conclude this chapter giving two short outlooks on Resurgence and the formulation on a lattice of Quantum Field Theory. A nice reference for this section is [4], while for the formulation on a lattice is [5]. As we have mentioned before, the perturbative expansions in quantum field theories are not guaranteed to be Borel resummable. An attempt to cure this problem is provided by Resurgence and in this section we want to give the flavour of this theory with the help of a toy model.

Perturbation theory within a path integral formulation is an infinite dimensional generalization of the usual steepest descent method to evaluate ordinary integrals. The ideas of this method are more clear in the simple case of one dimensional ordinary integrals such as

$$Z(\lambda) = \frac{1}{\sqrt{\lambda}}\int_{-\infty}^{+\infty} g(x)e^{-f(x)/\lambda}\,dx\,. \tag{1.11}$$

Here g and f are functions of a proper class and λ plays the role of $\hbar$. We assume that the convergence properties of the integral are determined by f only. The perturbative expansion of $Z(\lambda)$ around $\lambda = 0$ corresponds to the saddle point approximation but in general this is not the only saddle point (i.e. stationary point). In order to apply the steepest descent method we need to continue the functions f and g to the complex plane $z = x + iy$ and view the integral $Z(\lambda)$ as an open contour integral in z:

$$Z(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{\mathcal{C}_x} g(z) e^{-f(z)/\lambda} \, dz \,,$$

where $\mathcal{C}_x$ is the real axis. We call z_σ the saddle points of $f(z)$. The contour of steepest descent passing through z_σ is defined as a flow line $z(u)$ satisfying the first order equations

$$\frac{dz}{du} = \eta \frac{\partial F^*}{\partial z^*} \,, \qquad \frac{dz^*}{du} = \eta \frac{\partial F}{\partial z} \,,$$

where $\eta = \pm 1$ and $F(z) = -f(z)/\lambda$. Notice that

$$\frac{dF}{du} = \eta \left| \frac{\partial F}{\partial z} \right|^2 ,$$

so that the cycles with $\eta = -1$ and $\eta = 1$ are denoted respectively downward and upward flows, since $\mathrm{Re}\, F$ is monotonically decreasing and increasing in the two cases and $\mathrm{Im}\, F$ is constant in both flows. We denote by $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$ the downward and upward flows passing through z_σ. The steepest descent path is $\mathcal{J}_\sigma$. We have two cases: either $\mathrm{Re}\, F(z(u))$ flows to $-\infty$ as $u \to -\infty$, in which case $\mathcal{J}_\sigma$ is called a Lefschetz thimble, or $\mathcal{J}_\sigma$ hits another saddle point and thus $F(z(u))$ approaches this second saddle point as $u \to -\infty$ and some care is required. In the second case the steepest descent path is called a Stokes line.

If the steepest descent paths of all the saddle points are thimbles, then, by means of the Picard-Lefschetz theory and in absence of singularities on the complex plane, the contour $\mathcal{C}_x$ can be deformed to match a combination $\mathcal{C}$ of thimbles $\mathcal{J}_\sigma$,

$$\mathcal{C} = \sum_\sigma \mathcal{J}_\sigma n_\sigma \,,$$

with integer coefficients n_σ. Thus, the original integral (1.11) is reduced to a sum of integrals over the thimbles $\mathcal{J}_\sigma$,

$$Z(\lambda) = \sum_\sigma n_\sigma Z_\sigma(\lambda) \,, \tag{1.12}$$

where

$$Z_\sigma(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{\mathcal{J}_\sigma} g(z) e^{-f(z)/\lambda} \, dz \,.$$

In the presence of a flow connecting two saddle points the decomposition into thimbles is no longer possible. This problem can be avoided by taking λ to be complex, modifying in this way the flow curves. The initial integral is then recovered in the limit $\mathrm{Im}\, \lambda \to 0$.

The great advantage of the decomposition (1.12) is that the power expansions of the integrals $Z_\sigma(\lambda)$ over thimbles are Borel resummable. This also show what is the condition under which the original integral $Z(\lambda)$ is Borel resummable: when the integration cycle $\mathcal{C}_x$ is already a thimble and the decomposition (1.12) is trivial, as in the case of a real $f(x)$ with a single real saddle (and possibly other complex saddles). The problem is that in higher dimensions, up to infinite dimensions (the path integral), the decomposition is in general an extremely difficult task unless it is trivial.

We can avoid this problem by means of the following trick. Consider the integral

$$\hat{Z}(\lambda, \lambda_0) = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} e^{-\hat{f}(x)/\lambda} \hat{g}(x, \lambda_0) \, dx \,,$$

where $\hat{f}(x) = f(x) + \delta f(x)$ and $\hat{g}(x, \lambda_0) = g(x) e^{\delta f(x)/\lambda_0}$ are functions that satisfy the same conditions as $f(x)$ and $g(x)$. The original integral is recovered for $\lambda_0 = \lambda$:

$$\hat{Z}(\lambda, \lambda) = Z(\lambda) \,.$$

At fixed λ_0 the thimble decomposition is determined by the saddles of $\hat{f}$ and not of f. By properly choosing the function δf, we can generally build a function $\hat{f}$ with only one real saddle x_0, which trivializes the thimble decomposition to $\mathcal{C} = \mathcal{C}_x$. In this way, the asymptotic expansion of $\hat{Z}(\lambda, \lambda_0)$ in λ is Borel resummable to the exact result $\hat{Z}(\lambda, \lambda_0)$. Setting then $\lambda = \lambda_0$ gives the original function $Z(\lambda_0)$. We call the series expansion of $\hat{Z}(\lambda, \lambda_0)$ in λ at fixed λ_0 exact perturbation theory.

These methods have been implemented to a large class of quantum mechanical systems and the generalization to quantum field theories is an open research field of the present days.

1.7 Quantum field theory on a lattice

In a Lattice Quantum Field Theory the continuum Minkowskian spacetime is substituted with a finite Euclidean volume on a discrete set of points, the lattice. The volume is usually taken to be a parallelepiped and all the points have the same distance a, the lattice spacing. Since the number of spacetime points is finite, there is a finite number of degrees of freedom. For this reason a lattice Quantum Field Theory is essentially an approximation of a Quantum Field Theory by a many body quantum mechanical system. Since Quantum Mechanics is under much better control than Quantum Field Theory, many statements in lattice field theory can be made on a rigorous level.

Moreover, the lattice provides natural regulators of the theory, both in the infrared due to the finite volume and in the ultraviolet because there is a minimum distance between points, the lattice spacing. Therefore it can be taken as a regularized version of the theory on the continuum and all loop integrals are finite.

The idea would be to take the continuum limit and the infinite volume limit in the end but this usually causes many problems of technical nature. Thus it is rarely known if the statements rigorously proven on the lattice keep on holding in the continuum and if the formulation on the lattice is a good approximation of the theory on the continuum. Fortunately, for many theories this seems to be the case, though there is usually only circumstantial evidence.

In any case there is an important operative reason for which the lattice formulation is a very useful and sometime indispensable tecnique: it is possible to approximate the path integral for any observable with, in principle, arbitrary precision by a numerical calculation and such calculations are often feasible on nowadays computers. This possibility in especially important for non perturbative computations like in QCD.

On a lattice the functional integration is reduced to a finite product of integrations, one for each spacetime point,

$$\int \mathcal{D}\varphi(x) \to \prod_x \int d\varphi(x)\,,$$

and an ordinary integral becomes a Riemann sum over the points,

$$\int d^4x \to a^4 \sum_x\,.$$

The only operation that is not defined in an obvious way is derivation. There are three possible definitions:

$$\partial_\mu\varphi(x) \to \partial^{\mathrm{f}}_\mu\varphi(x) = \frac{\varphi(x+ae_\mu)-\varphi(x)}{a} \qquad \text{forward derivative}$$

$$\partial_\mu\varphi(x) \to \partial^{\mathrm{b}}_\mu\varphi(x) = \frac{\varphi(x)-\varphi(x-ae_\mu)}{a} \qquad \text{backward derivative}$$

$$\partial_\mu\varphi(x) \to \partial^{\mathrm{m}}_\mu\varphi(x) = \frac{\varphi(x+ae_\mu)-\varphi(x-ae_\mu)}{2a} \qquad \text{midpoint derivative}$$

where e_μ is the unit vector in direction μ. In the limit $a \to 0$ all of them are equivalent, but at finite lattice spacing they may introduce discretization artifacts of order a. However, this problem does not persist for the Laplacian, whose defintion is univocal:

$$\partial^2\varphi(x) \to \partial^{\mathrm{f}}_\mu\partial^{\mathrm{b}}_\mu\varphi(x) = \sum_{\pm\mu} \frac{\varphi(x+ae_\mu)+\varphi(x-ae_\mu)-2\varphi(x)}{a^2}\,,$$

where the sum over $\pm\mu$ is over all positive and negative directions.

Chapter 2

Functional methods

Functional methods provide a powerfull tool of computation in Quantum Field Theory. Although most of the time it relies on formal expressions and many technical details are usually ignored, nonetheless the importance of the results achieved in this framework is compelling. While in the previous chapter we kept a rigorous language, from now on our attention to technical details will fall down.

In the first section we introduce functional methods deriving a few expressions for the generating functional of the Green's functions starting from Canonical Quantization. A good reference is the book by Vasiliev [6]. Although this approach is equivalent to Canonical Quantization, and hence suffers the same problems, we hope that a proper use of these techniques will shed light on new aspects. The path integral, which is just one of the functional tools we can use, has already proven itself to be of the uttermost importance for the understanding of certain notions.

After considering a generic theory with real bosonic fields only, and without gauge symmetries, we keep on developing the argument for spinorial fields and gauge theories, the latter in particular requiring special consideration.

In the last two sections we move our attention to the generating functional of connected Green's functions, the effective action and the Dyson-Schwinger equations. We do this not only for the sake of completeness, but also because some of the notions exposed in these sections are needed for the comprehension of the next chapter.

2.1 From Canonical Quantization to the path integral

We start with a few considerations about functionals. Since we deal with both operators and classical objects, in this section we will denote operators by $\hat{\cdot}$. Let $F[\hat{\varphi}]$ be an operator functional, i.e. an object of the form

$$F[\hat{\varphi}] = \sum_{k=0}^{+\infty} \int d^4x_1 \dots d^4x_k \, F_k(x_1, \dots, x_k)\hat{\varphi}(x_1) \dots \hat{\varphi}(x_k) \,, \tag{2.1}$$

where the F_k's are called coefficient functions. An operator functional is said to be symmetric if its coefficient functions are symmetric or antisymmetric according to the statistic of the field. A symmetric operator functional can be associated with a classical functional $F[\varphi]$ with classical argument φ. The operator $\hat{F} = F[\hat{\varphi}]$ is uniquely determined by the coefficient functions of the functional F, but if $\hat{\varphi}$ is on-shell, the converse is not true. For instance consider a theory with free action and full action given respectively by

$$S_0[\varphi] = \frac{1}{2}\left\langle \varphi\Delta^{-1}\varphi\right\rangle \qquad S[\varphi] = S_0[\varphi] + \langle V(\varphi)\rangle \, .$$

The brakets $\langle\cdot\rangle$ mean integration over all relevant variables and possibly summation over the components of φ:

$$\left\langle \varphi\Delta^{-1}\varphi\right\rangle = \int d^4x d^4y \, \varphi_i(x)\Delta^{-1}_{ij}(x-y)\varphi_j(y) \, .$$

The two functionals $F_1[\varphi]$ and

$$F_2[\varphi] = e^{\frac{i}{2}\left\langle \varphi\Delta^{-1}\varphi\right\rangle} F_1[\varphi]$$

are clearly different for a generic function φ. But if φ_0 satisfies the free equation of motion, then $\Delta^{-1}\varphi_0 = 0$ and we have $F_1 = F_2$ on-shell. In this section we assume that the operator Δ^{-1} is non degenerate and its inverse exists and is unique. With this choice we exclude gauge theories, which will be discussed below.

Wick theorem admits a functional form that reads

$$\begin{aligned} \mathrm{T}\, F[\hat{\varphi}_0] &=: e^{\chi\frac{i}{2}\left\langle \frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}\right\rangle} F[\varphi]\Big|_{\varphi=\hat{\varphi}_0} : \qquad \text{(real fields)} \\ \mathrm{T}\, F[\hat{\varphi}_0,\hat{\varphi}_0^\dagger] &=: e^{\chi i\left\langle \frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi^\dagger}\right\rangle} F[\varphi,\varphi^\dagger]\Big|_{\varphi=\hat{\varphi}_0} : \qquad \text{(complex fields)} \end{aligned} \tag{2.2}$$

where $\chi = 1$ for fields satisfying commutation relations and $\chi = -1$ for fields satisfying anticommutation relations. From now on we will focus for simplicity on the case of a real bosonic field. In the case of a fermionic field special care needs to be taken for signs.

The quantities

$$\mathcal{G}_n(x_1,\ldots,x_n) = \langle 0|\,\mathrm{T}\,\hat{\varphi}_0(x_1)\ldots\hat{\varphi}_0(x_n)e^{i\langle V(\hat{\varphi}_0)\rangle}|0\rangle \, , \tag{2.3}$$

appearing at the numerator of the right hand side of the Gell-Mann-Low formula (1.8), are called the full Green's functions, containing vacuum loops. On the left hand side appear the Green's functions (without vacuum loops)

$$G_n(x_1,\ldots,x_n) = \langle\Omega|\,\mathrm{T}\,\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)|\Omega\rangle \, ,$$

so that Gell-Mann-Low formula reads now

$$G_n(x_1,\ldots,x_n) = \frac{\mathcal{G}_n(x_1,\ldots,x_n)}{\mathcal{G}_0} \, ,$$

with $\mathcal{G}_0 = \langle 0|S|0\rangle$. The generating functional of the Green's functions is defined to be

$$Z[J] = \sum_{n=0}^{+\infty}\frac{i^n}{n!}\int d^4x_1\ldots d^4x_n\, G_n(x_1,\ldots,x_n)J(x_1)\cdots J(x_n)$$

and plugging in the definition (2.3) we get

$$Z[J] = \frac{1}{\mathcal{G}_0}\langle 0|\,\mathrm{T}\, e^{i\langle V(\hat{\varphi}_0)\rangle + i\langle\hat{\varphi}_0 J\rangle}|0\rangle \, .$$

We now use Wick theorem in the form (2.2) to reduce the T-product in terms of the N-product. Since $\langle 0|\,\mathrm{N}\,\hat{\varphi}_0(x_1)\dots\hat{\varphi}_0(x_n)|0\rangle = 0$, for an arbitrary functional we have $\langle 0|\,\mathrm{N}\,F[\hat{\varphi}_0]|0\rangle = F[0]$ and thus we get

$$Z[J] = \frac{1}{\mathcal{G}_0} e^{\frac{i}{2}\left\langle \frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}\right\rangle} e^{i\langle V(\varphi)\rangle + i\langle\varphi J\rangle}\Big|_{\varphi=0} . \tag{2.4}$$

Since

$$e^{i\langle V(\varphi)\rangle} e^{i\langle\varphi J\rangle} = e^{i\left\langle V(-i\frac{\delta}{\delta J})\right\rangle} e^{i\langle\varphi J\rangle} \qquad e^{\frac{i}{2}\left\langle \frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}\right\rangle} e^{i\langle\varphi J\rangle} = e^{-\frac{i}{2}\langle J\Delta J\rangle} e^{i\langle\varphi J\rangle} ,$$

we find

$$Z[J] = \frac{1}{\mathcal{G}_0} e^{i\left\langle V(-i\frac{\delta}{\delta J})\right\rangle} e^{-\frac{i}{2}\langle J\Delta J\rangle} . \tag{2.5}$$

This representation is usually derived from the path integral representation

$$N\int \mathcal{D}\varphi\, e^{iS[\varphi]+i\langle\varphi J\rangle} ,$$

leading to

$$\frac{N}{N_0} e^{i\left\langle V(-i\frac{\delta}{\delta J})\right\rangle} e^{-\frac{i}{2}\langle J\Delta J\rangle} , \tag{2.6}$$

where

$$\frac{1}{N_0} = \int \mathcal{D}\varphi\, e^{iS_0[\varphi]} \qquad \frac{1}{N} = \int \mathcal{D}\varphi\, e^{iS[\varphi]} .$$

By comparing equation (2.5) and (2.6) we read the meaning of the normalization constants,

$$\frac{N_0}{N} = \mathcal{G}_0 ,$$

and we find

$$Z[J] = N\int \mathcal{D}\varphi\, e^{iS[\varphi]+i\langle\varphi J\rangle} . \tag{2.7}$$

Taking derivatives with respect to J on both sides we find the Green's functions

$$\langle\Omega|\,\mathrm{T}\,\hat{\varphi}(x_1)\dots\hat{\varphi}(x_n)|\Omega\rangle = N\int \mathcal{D}\varphi\,\varphi(x_1)\dots\varphi(x_n)e^{iS[\varphi]} .$$

By linearity of the T-product we can extend this formula to a general operator functional of the form (2.1),

$$\langle\Omega|\,\mathrm{T}\,F[\hat{\varphi}]|\Omega\rangle = N\int \mathcal{D}\varphi\, F[\varphi]e^{iS[\varphi]} ,$$

or, by Gell-Mann-Low formula (1.8),

$$\langle 0|\,\mathrm{T}\,F[\hat{\varphi}_0]|0\rangle = N_0\int \mathcal{D}\varphi\, F[\varphi]e^{iS_0[\varphi]} . \tag{2.8}$$

The path integral formulation is usually seen as an alternative approach to Quantum Field Theory leading to the same Feynman rules as the ones provided by Canonical Quantization. Here we have seen how the two approaches are linked and the connection is given essentially by Wick theorem. Of course there would be a lot to say about the symbol $\int\mathcal{D}\varphi$, its definition, the integration space and so on, but in our approach this is quite irrelevant. We really need only a few properties of functional integration, like change of integration variables and commutativity between functional integration and functional derivation. This should shed light on why the path integral approach works so well and consistently with the operator formalism despite all the troubles in the definition of $\int\mathcal{D}\varphi$.

2.2 Other representations of the generating functional

There are a few functional relations we can exploit in order to write the functional generator in different, hopefully more efficient ways. One of this is the identity

$$e^{-\frac{1}{2}\langle IMI\rangle} F\Big[\frac{\delta}{\delta I}\Big] e^{\frac{1}{2}\langle IMI\rangle} = e^{\frac{1}{2}\left\langle \frac{\delta}{\delta I} M^{-1} \frac{\delta}{\delta I}\right\rangle} F[MI]\,, \tag{2.9}$$

holding for any functional F, function I and distribution M. As pointed out in [7] by Marco Matone, this is nothing else but the functional generalization of the relation between the Hermite polinomials and their Weierstrass representation

$$(-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} = e^{-\frac{1}{2}\frac{d^2}{dx^2}} x^n\,.$$

Equation (2.9) can be proved directly by Taylor expanding on both sides; otherwise it follows by an operatorial identity as shown in section 4 of [7]. Applying identity (2.9) to the Schwinger representation (2.5) we find

$$Z[J] = \frac{1}{\mathcal{G}_0} e^{-\frac{i}{2}\langle J\Delta J\rangle} e^{\frac{i}{2}\left\langle \frac{\delta}{\delta J}\Delta^{-1}\frac{\delta}{\delta J}\right\rangle} e^{i\langle V(-\Delta J)\rangle}\,. \tag{2.10}$$

This representation can be equivalently derived starting from the path integral representation. Consider the quantity

$$\varphi_c(x) = -\big(\Delta J\big)(x) = -\int d^4x\, \Delta(x-y) J(y)\,,$$

satisfying the classical equation of motion with external source $-J$,

$$\big(\Delta^{-1}\varphi_c\big)(x) = -J(x)\,.$$

The shift

$$\begin{cases} \varphi \to \varphi + \varphi_c \\ \mathcal{D}\varphi \to \mathcal{D}\varphi \end{cases}$$

in the generating functional of the free theory

$$Z_0[J] = N_0 \int \mathcal{D}\varphi\, e^{iS_0[\varphi] + i\langle\varphi J\rangle}$$

leads to

$$Z_0[J] = e^{-\frac{i}{2}\langle J\Delta J\rangle}\,.$$

The same shift in the interacting theory (2.7) leads to

$$Z[J] = N e^{-\frac{i}{2}\langle J\Delta J\rangle} \int \mathcal{D}\varphi\, e^{i\langle V(\varphi+\varphi_c)\rangle} e^{iS_0[\varphi]}\,.$$

Applying equation (2.8) we get

$$Z[J] = \frac{1}{\mathcal{G}_0} e^{-\frac{i}{2}\langle J\Delta J\rangle} \langle 0|\, \mathrm{T}\, e^{i\langle V(\hat{\varphi}_0+\varphi_c)\rangle} |0\rangle\,.$$

Using Wick theorem (2.2) as in the previous section we find again

$$Z[J] = \frac{1}{\mathcal{G}_0} e^{-\frac{i}{2}\langle J\Delta J\rangle} e^{\frac{i}{2}\left\langle \frac{\delta}{\delta\varphi_c}\Delta\frac{\delta}{\delta\varphi_c}\right\rangle} e^{i\langle V(\varphi_c)\rangle}\,.$$

If we express J in terms of φ_c, we get the dual representation

$$T[\varphi_c] := Z\big[J[\varphi_c]\big] = \frac{1}{\mathcal{G}_0} e^{-\frac{i}{2}\left\langle\varphi_c\Delta^{-1}\varphi_c\right\rangle} e^{\frac{i}{2}\left\langle \frac{\delta}{\delta\varphi_c}\Delta\frac{\delta}{\delta\varphi_c}\right\rangle} e^{i\langle V(\varphi_c)\rangle}\,. \tag{2.11}$$

Covariant derivatives

Applying Leibniz rule we find

$$\frac{\delta}{\delta J(x)} e^{-\frac{i}{2}\langle J\Delta J\rangle} = e^{-\frac{i}{2}\langle J\Delta J\rangle}\left[\frac{\delta}{\delta J(x)} - i\big(\Delta J\big)(x)\right]$$

and in general

$$F\Big[\frac{\delta}{\delta J}\Big] e^{-\frac{i}{2}\langle J\Delta J\rangle} = e^{-\frac{i}{2}\langle J\Delta J\rangle} F[\mathcal{D}_J]\,,$$

where $\mathcal{D}_J$ resembles a covariant derivative

$$\mathcal{D}_J(x) = \frac{\delta}{\delta J(x)} - i\big(\Delta J\big)(x)\,.$$

Turning back to the case of the generating functional, we can express the dual representation (2.11) in terms of covariant derivatives acting on 1 by applying the Leibniz rule to the Schwinger representation (2.5),

$$T[\varphi_c] = \frac{1}{\mathcal{G}_0} e^{-\frac{i}{2}\left\langle \varphi_c \Delta^{-1} \varphi_c\right\rangle} e^{i\langle V(\mathcal{D})\rangle} 1\,,$$

where now

$$\mathcal{D}(x) = i\Big(\Delta\frac{\delta}{\delta\varphi_c}\Big)(x) + \varphi_c(x)$$

Even the Green's functions can be written in terms of covariant derivatives

$$G_n(x_1,\ldots,x_n) = \frac{1}{\mathcal{G}_0}\mathcal{D}(x_1)\ldots\mathcal{D}(x_n) e^{i\langle V(i\mathcal{D})\rangle} 1\Big|_{\varphi_c=0}\,.$$

The advantage of this approach is that the covariant derivative has the following simple commutators

$$[\mathcal{D}(x),\mathcal{D}(y)] = 0\,, \qquad [\mathcal{D}(x),\varphi_c(y)] = i\Delta(x-y)\,.$$

2.3 Generating functional for the S-operator

The S-operator admits a functional representation by virtue of Wick theorem:

$$S = \mathrm{T}\, e^{i\langle V(\hat\varphi_0)\rangle} =: e^{\frac{i}{2}\left\langle \frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}\right\rangle} e^{i\langle V(\varphi)\rangle}\Big|_{\varphi=\hat\varphi_0} :\ .$$

It is natural to define the S-operator generating functional as

$$R[\varphi] = e^{\frac{i}{2}\left\langle \frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}\right\rangle} e^{i\langle V(\varphi)\rangle} = Z_0\Big[-i\frac{\delta}{\delta\varphi}\Big] e^{i\langle V(\varphi)\rangle} \tag{2.12}$$

in such a way that $S =: R[\hat\varphi_0] :$. We see that there is a simple relation with the dual representation (2.11) given by

$$T[\varphi] = \frac{1}{\mathcal{G}_0} e^{-\frac{i}{2}\left\langle \varphi\Delta^{-1}\varphi\right\rangle} R[\varphi]\,. \tag{2.13}$$

The first exponential drops if φ satisfies the free equation of motion, therefore we have

$$S = \mathcal{G}_0 : T[\hat\varphi_0] :\ .$$

At this point it is easy to write the expansion of S in terms of the Green's functions because

$$\frac{\delta}{\delta\varphi(x)}T[\varphi] = -\int d^4y\,\Delta^{-1}(x-y)\frac{\delta}{\delta J(y)}Z[J]$$

and thus

$$S = \mathcal{G}_0\sum_{n=0}^{+\infty}\frac{(-i)^n}{n!}\int d^4x_1d^4y_1\dots d^4x_nd^4y_n\times$$
$$\times\,\Delta^{-1}(x_1-y_1)\cdots\Delta^{-1}(x_n-y_n)G_n(y_1,\dots,y_n)\,:\hat{\varphi}_0(x_1)\dots\hat{\varphi}_0(x_n):\,.$$

2.4 Exponential interaction

The scalar field with exponential interaction is known to be trivial in four space-time dimensions. To be precise we cite [8], where it is shown that for the exponential interaction with space-time cutoff, $S_V[\phi] = \mu^d\int_\Lambda d^4x\,:e^{i\alpha\phi(x)}:$, where Λ is a bounded subset of $\mathbb{R}^d$, the Schwinger functions converge to the Schwinger functions for the free field for all α if $d>2$ or for all α such that $|\alpha|>\alpha_0$ if $d=2$ for a given $\alpha_0\in\mathbb{R}$. Nonetheless the exponential interaction is a good laboratory for the application of functional tecniques. To illustrate this example we follow [9]. The action is

$$S[\phi] = \int d^4x\Big[-\frac{1}{2}\phi(x)(\Box+m^2)\phi(x) - \mu^4e^{\alpha\phi(x)}\Big] = S_0[\phi] - \int d^4x\,V\big(\phi(x)\big)\,,$$

where μ and α are constants with mass dimensions respectively 1 and -1. The generating functional is

$$Z[J] = N\int\mathcal{D}\phi\,e^{iS_0[\phi]+i\langle V(\phi)\rangle+i\langle J\phi\rangle}\,,$$

which in the Schwinger representation becomes

$$Z[J] = \frac{N}{N_0}\exp\Big[i\mu^4\Big\langle e^{-i\alpha\frac{\delta}{\delta J}}\Big\rangle\Big]e^{-\frac{i}{2}\langle J\Delta J\rangle}\,. \tag{2.14}$$

What makes this theory so interesting is that the relation (see for instance [10])

$$e^{\left\langle B\frac{\delta}{\delta A}\right\rangle}F[A] = F[A+B]\,, \tag{2.15}$$

holding for any function A and B and functional F, can be used to evaluate the generating functional (2.14).

Equation (2.15) is the obvious generalization of

$$e^{y\frac{d}{dx}}f(x) = f(x+y)$$

and the proof is performed by means of the usual trick: we introduce the functional

$$F_0[A] = e^{\langle AI\rangle}\,,$$

with I an arbitrary function, in such a way that we can write

$$\begin{aligned}e^{\left\langle B\frac{\delta}{\delta A}\right\rangle}F[A] &= F\Big[\frac{\delta}{\delta I}\Big]e^{\left\langle B\frac{\delta}{\delta A}\right\rangle}e^{\langle AI\rangle}\Big|_{I=0}\\ &= F\Big[\frac{\delta}{\delta I}\Big]e^{\langle BI\rangle}e^{\langle AI\rangle}\Big|_{I=0}\\ &= F[A+B]\,.\end{aligned}$$

In equation (2.14) we expand the first exponential and we write

$$e^{-i\alpha \frac{\delta}{\delta J(x)}} = e^{-i\alpha \int d^4y\, \delta(x-y) \frac{\delta}{\delta J(y)}} = e^{-i\alpha \left\langle \delta_x \frac{\delta}{\delta J} \right\rangle}$$

in order to apply formula (2.15). Thus we get

$$\begin{aligned} Z[J] &= \frac{N}{N_0} \sum_{k=0}^{+\infty} (i\mu^4)^k \int d^{4k}x\, e^{-i\alpha \left\langle \delta_{x_1} \frac{\delta}{\delta J} + \cdots + \delta_{x_k} \frac{\delta}{\delta J} \right\rangle} e^{-\frac{i}{2}\langle J\Delta J\rangle} \\ &= \frac{N}{N_0} Z_0[J] \sum_{k=0}^{+\infty} (i\mu^4)^k \times \\ &\times \int d^{4k}x \exp\Big[-\alpha \sum_{j=1}^{k} (\Delta J)(x_j) - \alpha^2 \sum_{i<j}^{k} \Delta(x_i - x_j) + \frac{i\alpha^2}{2} \sum_{j=1}^{k} \Delta(x_j - x_j) \Big], \end{aligned} \tag{2.16}$$

where $d^{4k}x = d^4x_1 \dots d^4x_k$.

The exponential potential becomes more interesting when considering normal ordered potentials. With an abuse of notation we will write

$$: F[\phi] := e^{-\frac{i}{2} \left\langle \frac{\delta}{\delta\phi} \Delta \frac{\delta}{\delta\phi} \right\rangle} F[\phi] \tag{2.17}$$

to indicate normal ordered (or Wick ordered) functionals of ϕ even if ϕ is not an operator but a classical function. The exponential in equation (2.17) is the inverse of the exponential appearing in the functional formulation of Wick theorem and is called the Wick operator. The map

$$F[\phi] \mapsto e^{-\frac{i}{2} \left\langle \frac{\delta}{\delta\phi} \Delta \frac{\delta}{\delta\phi} \right\rangle} F[\phi]$$

is properly called the Wick transform of $F[\phi]$. In the case of the exponential interaction we have

$$: e^{\alpha\phi(x)} := e^{-\frac{i}{2} \left\langle \frac{\delta}{\delta\phi} \Delta \frac{\delta}{\delta\phi} \right\rangle} e^{\alpha\phi(x)} = e^{-\frac{i}{2}\alpha^2 \Delta(x-x)} e^{\alpha\phi(x)},$$

which has the effect to cancel the last summation in equation (2.16). Therefore, the generating functional for a scalar field with normal ordered exponential potential $\mu^4 : \exp(\alpha\phi) :$ is

$$\begin{aligned} Z'[J] &= \frac{N}{N_0} \sum_{k=0}^{+\infty} (i\mu^4)^k \int d^{4k}x\, e^{-i\alpha \left\langle \delta_{x_1} \frac{\delta}{\delta J} + \cdots + \delta_{x_k} \frac{\delta}{\delta J} \right\rangle} e^{-\frac{i}{2}\langle J\Delta J\rangle} \\ &= \frac{N}{N_0} Z_0[J] \sum_{k=0}^{+\infty} (i\mu^4)^k \int d^{4k}x \exp\Big[-\alpha \sum_{j=1}^{k} (\Delta J)(x_j) - \alpha^2 \sum_{i<j}^{k} \Delta(x_i - x_j) \Big]. \end{aligned} \tag{2.18}$$

The exponential interaction can be used to compute the generating functional associated with scalar field with polinomial interactions such as $\lambda\phi^n$. The key observation is that we can write

$$\phi^n = \frac{\partial^n}{\partial\alpha^n} e^{\alpha\phi}\big|_{\alpha=0},$$

so that we can rewrite the potential for the theory $\lambda\phi^n$ in the following way

$$V_n(\phi) = \frac{\lambda}{n!}\phi^n = \frac{\lambda}{n!} \frac{\partial^n}{\partial\alpha^n} e^{\alpha\phi}\big|_{\alpha=0}.$$

In this way the generating functional becomes

$$
\begin{aligned}
Z_n[J] &= \frac{N}{N_0}\exp\left[i\frac{\lambda}{n!}\frac{\partial^n}{\partial\alpha^n}\left\langle e^{-i\alpha\frac{\delta}{\delta J}}\right\rangle\Big|_{\alpha=0}\right]e^{-\frac{i}{2}\langle J\Delta J\rangle} \\
&= \frac{N}{N_0}\sum_{k=0}^{+\infty}\frac{(i\lambda)^k}{k!(n!)^k}\frac{\partial^n}{\partial\alpha_1^n}\cdots\frac{\partial^n}{\partial\alpha_k^n}\int d^{4k}x\, e^{-i\left\langle\alpha_1\delta_{x_1}\frac{\delta}{\delta J}+\cdots+\alpha_k\delta_{x_k}\frac{\delta}{\delta J}\right\rangle}e^{-\frac{i}{2}\langle J\Delta J\rangle}\Bigg|_{\alpha=0},
\end{aligned}
$$

where in the second line $\alpha = 0$ is shorthand for $\alpha_1 = 0, \ldots, \alpha_k = 0$. At this point it is natural to use the form (2.16) to express $Z_n[J]$ in a more explicit fashion. In the case of normal ordered interactions nothing changes apart from the fact that now we can use equation (2.18). In this way we get

$$
\begin{aligned}
&Z'_n[J] = \\
&= \frac{N}{N_0}\sum_{k=0}^{+\infty}\frac{(i\lambda)^k}{k!(n!)^k}\frac{\partial^n}{\partial\alpha_1^n}\cdots\frac{\partial^n}{\partial\alpha_k^n}\int d^{4k}x\,\exp\left[-\sum_{j=1}^{k}\alpha_j\big(\Delta J\big)(x_j)-\sum_{i<j}^{k}\alpha_i\alpha_j\Delta(x_i-x_j)\right].
\end{aligned}
$$

The derivatives with respect to the α's parameters commute with functional derivatives with respect to J and in evaluating the Green's functions we can set to zero the external sources before computing the derivatives with respect to the α's. For instance the expression for the N point Green's function is

$$
\begin{aligned}
&G_N(y_1,\ldots,y_N) = \frac{N}{N_0}\sum_{k=0}^{+\infty}\frac{(i\lambda)^k}{k!(n!)^k}\frac{\partial^n}{\partial\alpha_1^n}\cdots\frac{\partial^n}{\partial\alpha_k^n}\times \\
&\times\int d^{4k}x\, i^N\sum_{j_1,\ldots,j_N=1}^{k}\alpha_{j_1}\cdots\alpha_{j_N}\Delta(x_{j_1}-y_1)\cdots\Delta(x_{j_N}-y_N)\exp\left[-\sum_{i<j}^{k}\alpha_i\alpha_j\Delta(x_i-x_j)\right].
\end{aligned}
$$

The peculiarity of this method is that functional derivatives with respect to auxiliary functions have been substituted by ordinary derivatives with respect to auxiliary parameters. For a detailed discussion of the explicit calculation of these derivatives see [9].

2.5 Functional methods for anticommuting fields

If we deal with fields satisfying anticommutation relations, all the classical variables must be Grassmannian variables. In this case various minus signs appear here and there in the formulae we have seen. As spinorial fields are always complex fields, we will see here the case of a complex anticommuting field. The Green's functions are

$$
G_{mn}(x_1,\ldots,x_m,y_1,\ldots,y_n) = \langle\Omega|\,\mathrm{T}\,\hat\psi(x_1)\ldots\hat\psi(x_m)\hat\psi^\dagger(y_1)\ldots\hat\psi^\dagger(y_n)|\Omega\rangle
$$

and, as they have even grading, they are zero if $m+n$ is odd. The generating functional is

$$
\begin{aligned}
Z[\eta,\eta^*] = &\sum_{m,n=0}^{+\infty}\frac{i^{m+n}}{m!n!}\int d^4x_1\ldots d^4x_m d^4y_1\ldots d^4y_n\times \\
&\times G_{mn}(x_1,\ldots,x_m,y_1,\ldots,y_n)\eta(y_n)\cdots\eta(y_1)\eta^*(x_m)\cdots\eta^*(x_1)\,. \qquad (2.19)
\end{aligned}
$$

Proceding as for the bosonic real field and taking care for the signs we find

$$\begin{aligned} Z[\eta,\eta^*] &= \frac{1}{\mathcal{G}_0} e^{-i\left\langle \frac{\delta}{\delta\psi}\Delta\frac{\delta}{\delta\psi^*}\right\rangle} e^{i\langle V(\psi,\psi^*)\rangle} e^{i\langle \psi^*\eta-\eta^*\psi\rangle}\Big|_{\psi=0} \\ &= \frac{1}{\mathcal{G}_0} e^{i\left\langle V(i\frac{\delta}{\delta\eta^*},i\frac{\delta}{\delta\eta})\right\rangle} e^{i\langle\eta^*\Delta\eta\rangle} . \end{aligned} \tag{2.20}$$

Notice that here we use the convention $(AB)^* = A^*B^*$ for A and B Grassmannian variables and thus $i\,\langle\psi^*\eta-\eta^*\psi\rangle$ is imaginary. The path integral representation is

$$Z[\eta,\eta^*] = N\int\mathcal{D}\psi\mathcal{D}\psi^*\; e^{iS[\psi,\psi^*]+i\langle\psi^*\eta-\eta^*\psi\rangle}$$

and the Green's functions are

$$\begin{aligned} &G_{mn}(x_1,\ldots,x_m,y_1,\ldots,y_n) = \\ &\quad= (-1)^{m+n}N\int\mathcal{D}\psi\mathcal{D}\psi^*\;\psi(x_1)\cdots\psi(x_m)\psi^*(y_1)\cdots\psi^*(y_n)e^{iS[\psi,\psi^*]} . \end{aligned} \tag{2.21}$$

Since $G_{mn}=0$ if $m+n$ is odd, we can forget about the -1 in front of the integral.

Formula (2.9) for Grassmannian variables turns to

$$e^{-\langle I^*MI\rangle}F\Big[\frac{\delta}{\delta I^*},-\frac{\delta}{\delta I}\Big]e^{\langle I^*MI\rangle} = e^{-\left\langle\frac{\delta}{\delta I}M^{-1}\frac{\delta}{\delta I^*}\right\rangle}F[MI,I^*M]$$

and if we apply it to equation (2.20) we get

$$Z[\eta,\eta^*] = \frac{N}{N_0}e^{i\langle\eta^*\Delta\eta\rangle}e^{i\left\langle\frac{\delta}{\delta\eta}\Delta^{-1}\frac{\delta}{\delta\eta^*}\right\rangle}e^{iV(-\Delta\eta,\eta^*\Delta)} .$$

The dual representation is

$$T[\psi_c,\psi_c^*] = \frac{N}{N_0}e^{-i\left\langle\psi_c^*\Delta^{-1}\psi_c\right\rangle}e^{-i\left\langle\frac{\delta}{\delta\psi_c}\Delta\frac{\delta}{\delta\psi_c^*}\right\rangle}e^{iV(\psi_c,\psi_c^*)} ,$$

where now

$$\psi_c = -\Delta\eta \qquad \psi_c^* = \eta^*\Delta .$$

Even for spinorial fields we can find a representation in terms of covariant derivatives. Applying Leibniz rule to equation (2.20) we find

$$\begin{aligned} Z[\eta,\eta^*] &= \frac{1}{\mathcal{G}_0}e^{i\left\langle V(i\frac{\delta}{\delta\eta^*},i\frac{\delta}{\delta\eta})\right\rangle}e^{i\langle\eta^*\Delta\eta\rangle} \\ &= e^{i\langle\eta^*\Delta\eta\rangle}e^{i\left\langle V(i\frac{\delta}{\delta\eta^*}-\Delta\eta,i\frac{\delta}{\delta\eta}+\eta^*\Delta)\right\rangle} . \end{aligned}$$

Looking at the definitions of ψ_c and ψ_c^* we define the covariant derivatives as

$$\mathcal{D}_{\psi_c^*}(x) = i\Big(\Delta\frac{\delta}{\delta\psi_c^*}\Big)(x)+\psi_c(x) \qquad \mathcal{D}_{\psi_c}(x) = -i\Big(\frac{\delta}{\delta\psi_c}\Delta\Big)(x)+\psi_c^*(x) ,$$

in such a way that

$$Z[\eta,\eta^*] = \frac{1}{\mathcal{G}_0}e^{-i\left\langle\psi_c^*\Delta^{-1}\psi_c\right\rangle}e^{i\left\langle V(\mathcal{D}_{\psi_c^*},\mathcal{D}_{\psi_c})\right\rangle} .$$

The covariant derivatives have the anticommutators

$$\begin{gathered} \{\mathcal{D}_{\psi_c^*}(x),\psi_c^*(y)\} = i\Delta(x-y) \qquad \{\psi_c(x),\mathcal{D}_{\psi_c}(y)\} = -i\Delta(x-y) \\ \{\mathcal{D}_{\psi_c^*}(x),\mathcal{D}_{\psi_c}(y)\} = 0 . \end{gathered}$$

2.6 Yang-Mills theories

The situation is more complicated for Yang-Mills theories, with free action and full action

$$S_0[A] = \frac{1}{2}\langle AKA\rangle \qquad S[A] = S_0[A] + S_V[A]\,.$$

Here we focus only on the vector fields $A^{a\mu}$ and the interaction term S_V has to be understood as some effective action including interactions with other fields. The brakets $\langle\cdot\rangle$ include summation over vector and color indices,

$$\langle AKA\rangle = \int d^4x d^4y\, A^{a\mu}(x) K^{ab}_{\mu\nu}(x-y) A^{b\nu}(y)\,.$$

One of the problems that we encounter in perturbative Yang-Mills theories is that K is degenerate and thus it is not obvious what the Feynman propagator $\Delta \sim K^{-1}$ should be. We have infinite possibilities to choose among. The degeneracy of K is due to its transversality, a property that in momentum space reads

$$p^\mu K^{ab}_{\mu\nu}(p) = 0\,, \tag{2.22}$$

where $K^{ab}_{\mu\nu}(p) = \delta^{ab}(-p^2\eta_{\mu\nu} + p_\mu p_\nu)$. It is clear from equation (2.22) that K has a null eigenvalue and thus is degenerate. These facts are strictly related to gauge invariance.

Following the Faddeev-Popov method we take as definition of the generating functional of the Green's functions

$$Z_f[J] = N \int \mathcal{D}A\, \delta[f(A)] \det M_f(A) e^{iS[A]+i\langle AJ\rangle}\,, \tag{2.23}$$

with gauge condition $f(A) = 0$ and $\det M_f(A)$ the Faddeev-Popov determinant. This is not a gauge invariant quantity because of the interaction term with the external source J. Before solving the integral (2.23) we shall treat the free case

$$Z^0_f[J] = N_0 \int \mathcal{D}A\, \delta[f(A)] e^{\frac{i}{2}\langle AKA\rangle + i\langle AJ\rangle}$$

in linear gauge $f(A) = n_\mu A^\mu(x) + c(x)$. Here $A^\mu(x) = A^{a\mu}(x)T^a$ and $c(x)$ is a given matrix function. The vector n_μ is either some c-number vector or a differential operator like ∂_μ. First of all we isolate the c dependence by means of the shift

$$\begin{cases} A^\mu \to A^\mu + l^\mu \\ \mathcal{D}A \to \mathcal{D}A\,. \end{cases}$$

The vector l^μ should be longitudinal in such a way that the kinetic term $\langle AKA\rangle$ does not change under the shift. Therefore we have $l^\mu = \partial^\mu\varphi$ with $\varphi = -(n\partial)^{-1}c$. As a consequence the term $\exp\langle ilJ\rangle$ is isolated from the integral. Next we need another shift,

$$\begin{cases} A \to A - \Delta J \\ \mathcal{D}A \to \mathcal{D}A\,, \end{cases}$$

in order to cancel the cross term $i\langle AJ\rangle$ as in ordinary theories. Due to the degeneracy of K, Δ is not uniquely determined by this only requirement. This is why the δ functional in the integral plays an important role. In fact, we need that the argument of $\delta[nA]$ is not changed by the shift in order to extract the depence on J from the integral. The additional requirement $n\Delta J = 0$ uniquely determines Δ in terms only of n. In momentum space we have

$$\Delta_n^{ab\mu\nu} = \delta^{ab}(p)\frac{1}{p^2}\left(\frac{p^\mu n^\nu + n^\mu p^\nu}{n\cdot p} - \eta^{\mu\nu} - n^2\frac{p^\mu p^\nu}{(n\cdot p)^2}\right).$$

Notice that this expression is transverse, which implies degeneracy for Δ_n. The resulting generating functional for free Yang-Mills theory is

$$Z_f^0[J] = e^{-\frac{i}{2}\langle J\Delta_n J\rangle + i\langle lJ\rangle}.$$

In the full theory (2.23) the determinant, as is well known, can be written in terms of auxiliary scalar anticommuting fields, the ghosts, and in practice its effect amounts to modify the effective action, $S_V[A] \to S_V^f[A]$. The evaluation of the integral is done by means of the following trick

$$\begin{aligned} Z_f[J] &= N\int \mathcal{D}A\,\delta[f(A)]e^{\frac{i}{2}\langle AKA\rangle}e^{iS_V^f[A]+i\langle AJ\rangle} \\ &= N\int \mathcal{D}A\,\delta[f(A)]e^{\frac{i}{2}\langle AKA\rangle}e^{iS_V^f[A+B]+i\langle (A+B)J\rangle}\Big|_{B=0} \\ &= N\int \mathcal{D}A\,\delta[f(A)]e^{\frac{i}{2}\langle AKA\rangle + \left\langle A\frac{\delta}{\delta B}\right\rangle}e^{iS_V^f[B]+i\langle BJ\rangle}\Big|_{B=0}. \end{aligned}$$

In the integral in the last line we recognize the free path integral $Z_f^0[-i\delta/\delta B]$ and we get

$$\begin{aligned} Z_f[J] &= \frac{N}{N_0}e^{\frac{i}{2}\left\langle \frac{\delta}{\delta B}\Delta_n\frac{\delta}{\delta B}\right\rangle}e^{\left\langle l\frac{\delta}{\delta B}\right\rangle}e^{iS_V^f[B]+i\langle BJ\rangle}\Big|_{B=0} \\ &= \frac{N}{N_0}e^{\frac{i}{2}\left\langle \frac{\delta}{\delta B}\Delta_n\frac{\delta}{\delta B}\right\rangle}e^{iS_V^f[B+l]+i\langle (B+l)J\rangle}\Big|_{B=0}. \end{aligned} \tag{2.24}$$

Apart from the shift of l this expression coincides with (2.4), leading to ordinary diagramatic interpretation.

2.7 S-operator generating functional for Yang-Mills theories

Looking at equation (2.12), the natural definition of the S-operator generating functional in Yang-Mills theory, in the linear gauge $nA + c = 0$, is

$$R_f[A] = e^{\frac{i}{2}\left\langle \frac{\delta}{\delta A}\Delta_n\frac{\delta}{\delta A}\right\rangle}e^{\left\langle l\frac{\delta}{\delta A}\right\rangle}e^{iS_V^f[A]} = Z_f^0\Big[-i\frac{\delta}{\delta A}\Big]e^{iS_V^f[A]}.$$

Applying Leibniz rule to the first line of equation (2.24) we find

$$\begin{aligned} Z_f[J] &= \frac{N}{N_0}e^{\frac{i}{2}\left\langle \frac{\delta}{\delta A}\Delta_n\frac{\delta}{\delta A}\right\rangle}e^{\left\langle l\frac{\delta}{\delta A}\right\rangle}e^{i\langle AJ\rangle}e^{iS_V^f[A]}\Big|_{A=0} \\ &= \frac{N}{N_0}e^{\frac{i}{2}\left\langle (\frac{\delta}{\delta A}+iJ)\Delta_n(\frac{\delta}{\delta A}+iJ)\right\rangle}e^{\left\langle l(\frac{\delta}{\delta A}+iJ)\right\rangle}e^{iS_V^f[A]}\Big|_{A=0} \\ &= \frac{N}{N_0}e^{-\frac{i}{2}\langle J\Delta_n J\rangle + i\langle lJ\rangle}e^{-\left\langle J\Delta_n\frac{\delta}{\delta A}\right\rangle}e^{\frac{i}{2}\left\langle \frac{\delta}{\delta A}\Delta_n\frac{\delta}{\delta A}\right\rangle}e^{\left\langle l\frac{\delta}{\delta A}\right\rangle}e^{iS_V^f[A]}\Big|_{A=0}. \end{aligned}$$

In the last line we recognize $R_f[A]$,

$$Z_f[J] = \frac{N}{N_0} e^{-\frac{i}{2}\langle J\Delta_n J\rangle + i\langle lJ\rangle} e^{-\left\langle J\Delta_n \frac{\delta}{\delta A}\right\rangle} R_f[A]\Big|_{A=0} \\ = \frac{N}{N_0} e^{-\frac{i}{2}\langle J\Delta_n J\rangle + i\langle lJ\rangle} R_f[-\Delta_n J] \,. \tag{2.25}$$

This is the analogous of relation (2.13) but now Δ_n is degenerate and we cannot write the dual relation $T[A_c] = Z[J]$, which would require $J = -\Delta_n^{-1} A_c$. From the relation (2.25) we see that the functional R_f determines Z_f uniquely, but the converse is not true because there are infinite possible choices for Δ_n^{-1}. The arbitrariness of R_f is made manifest by a few steps of functional algebra.

$$\begin{aligned} R_f[A] &= Z_f^0\Big[-i\frac{\delta}{\delta A}\Big] e^{iS_V^f[A]} \\ &= N_0 \int \mathcal{D}B\, \delta[f(B)] e^{\frac{i}{2}\langle BKB\rangle + \left\langle B\frac{\delta}{\delta A}\right\rangle} e^{iS_V^f[A]} \\ &= N_0 \int \mathcal{D}B\, \delta[f(B)] e^{\frac{i}{2}\langle BKB\rangle + iS_V^f[A+B]} \\ &= N_0 e^{\frac{i}{2}\langle AKA\rangle} \int \mathcal{D}B\, \delta[f(B-A)] e^{\frac{i}{2}\langle BKB\rangle - i\langle AKB\rangle + iS_V^f[B]} \,. \end{aligned}$$

Now recall that

$$e^{iS_V^f[B]} = \det M_f(B) e^{iS_V[B]} \,,$$

so that

$$R_f[A] = N_0 e^{\frac{i}{2}\langle AKA\rangle} \int \mathcal{D}B\, \delta[f(B-A)] \det M_f(B) e^{\frac{i}{2}\langle BKB\rangle - i\langle AKB\rangle + iS_V[B]} \,. \tag{2.26}$$

The Faddeev-Popov determinant was introduced together with $\delta[f(B)]$ and the values it takes off the surface $f(B) = 0$ were totally ininfluent and arbitrary. Now the argument of the δ functional is $f(B-A)$, with A arbitrary, and the values of $\det M_f(B)$ off the surface $f(B) = 0$ are still arbitrary but no longer ininfluent in the evaluation of $R_f[A]$. The arbitrariness we encountered in equation (2.25) in the choice of Δ_n^{-1} is the same arbitrariness we have in $\det M_f(B)$ off the surface $f(B) = 0$. We conclude this section by noticing that the S-operator generating functional we have introduce is not gauge invariant. This is not a problem because $R[A]$ is off-shell and does not have a direct physical interpretation. What is important is that the on-shell generating functional is gauge invariant and this turns out to be true.

2.8 Feynman gauge fixing

Since the on-shell generating functional will be gauge invariant, in particular it will be independent of the function parameter c and we can try to average expression (2.23) over c with weight $\exp\langle icdc\rangle$, where d is an arbitrary kernel. What we get is the following

$$Z_{nd}[J] = N \int \mathcal{D}A\mathcal{D}c\, \delta[nA + c] \det M_f(A) e^{iS[A] + i\langle cdc\rangle + i\langle AJ\rangle} \,. \tag{2.27}$$

The integration over c is easily performed by means of the delta functional and translates into a quadratic contribution to the action $S[A]$,

$$Z_{nd}[J] = N \int \mathcal{D}A \, \det M_f(A) e^{iS[A]+i\langle nAdnA\rangle + i\langle AJ\rangle} .$$

The quadratic contribution is added to the free action,

$$\frac{1}{2} \langle AKA\rangle + \langle nAdnA\rangle ,$$

with the advantage that now the kernel $K' = K + 2ndn$ is non transverse and invertible. In this way we get a generating functional of the standard form (2.7) and ordinary perturbation theory can start with propagator given by the inverse of K'. For instance we can take $n_\mu = \partial_\mu$, $d = -1/(2\rho)$ (where ρ is a number), to get

$$\frac{1}{2} \langle AK'A\rangle = \frac{1}{2} \left\langle A^{a\mu} \left[\eta_{\mu\nu}\Box + \frac{1-\rho}{\rho}\partial_\mu\partial_\nu\right] A^{a\nu} \right\rangle .$$

The propagator in momentum space reads

$$\delta^{ab} \frac{i}{p^2} \left[(1-\rho)\frac{p_\mu p_\nu}{p^2} - \eta_{\mu\nu}\right] ,$$

which, for $\rho = 1$, coincides with the usual expression for the propagator of the electromagnetic field in the Feynman gauge. Of course expression (2.27) is different from expression (2.23) but, again, it is only the on-shell generating functional that matters and the two choices turn out to be equivalent.

2.9 An example: QED

We now analyze the case of QED focusing in particular on the covariant derivatives formalism. The well-known QED action is

$$S[A, \psi, \bar{\psi}] = \int d^4x \left[\frac{1}{2} A^\mu(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu + \bar{\psi}(\not{\partial} - im)\psi + e\bar{\psi}\not{A}\psi\right] .$$

Since QED is an abelian Yang-Mills theory, the ghosts decouple from the rest of the Lagrangian and we can drop the Faddeev-Popov determinant. In order to make use of the functional methods discussed in sections 2.1 to 2.5 we integrate over the function parameter c as in section 2.8 with kernel $d = -1/2$ (in other words we are choosing the usual Feynman gauge). In this way we can take as generating functional of QED the following

$$Z[J_\mu, \eta, \bar{\eta}] = N \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi} \, e^{iS_F[A,\psi,\bar{\psi}] + i\langle A^\mu J_\mu\rangle + i\langle \bar{\psi}\eta - \bar{\eta}\psi\rangle} ,$$

where

$$S_F[A, \psi, \bar{\psi}] = \int d^4x \left[\frac{1}{2} A^\mu \eta_{\mu\nu}\Box A^\nu + \bar{\psi}(\not{\partial} - im)\psi + e\bar{\psi}\not{A}\psi\right] .$$

The factor N is the usual normalization constant

$$\frac{1}{N} = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi} \, e^{iS_F[A,\psi,\bar{\psi}]} .$$

We set

$$\Delta^{-1}_{\mu\nu}(x-y) = \eta_{\mu\nu}\Box\delta(x-y) \qquad \Delta^{-1}(x-y) = (\partial\!\!\!/ - im)\delta(x-y)\,,$$

their inverse being respectively the photonic and fermionic Feynman propagator, and

$$S_0[A,\psi,\bar\psi] = \int d^4x\,\Big[\frac{1}{2}A^\mu\eta_{\mu\nu}\Box A^\nu + \bar\psi(\partial\!\!\!/ - im)\psi\Big]\,.$$

The dual representation takes the form

$$\begin{aligned}&T[A_c,\psi_c,\bar\psi_c] =\\ &\quad= \frac{N}{N_0}e^{-i\left\langle\bar\psi_c\Delta^{-1}\psi_c\right\rangle}e^{-\frac{i}{2}\left\langle A_c\Delta^{-1}_{\mu\nu}A_c\right\rangle}e^{-i\left\langle\frac{\delta}{\delta\psi_c}\Delta\frac{\delta}{\delta\bar\psi_c}\right\rangle}e^{\frac{i}{2}\left\langle\frac{\delta}{\delta A_c}\Delta_{\mu\nu}\frac{\delta}{\delta A_c}\right\rangle}e^{ie\left\langle\bar\psi_c A\!\!\!/_c\psi_c\right\rangle}\,,\end{aligned}$$

where

$$\frac{1}{N_0} = \int\mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar\psi\;e^{iS_0[A,\psi,\bar\psi]}\,.$$

The representation in terms of covariant derivatives is

$$T[A_c,\psi_c,\bar\psi_c] = \frac{N}{N_0}e^{-i\left\langle\bar\psi_c\Delta^{-1}\psi_c\right\rangle}e^{-\frac{i}{2}\left\langle A_c\Delta^{-1}_{\mu\nu}A_c\right\rangle}e^{ie\left\langle\mathcal{D}_{\psi_c}\mathcal{D}\!\!\!\!/_A\mathcal{D}_{\bar\psi_c}\right\rangle}1$$

with

$$\mathcal{D}_{\bar\psi_c}(x) = i\Big(\Delta\frac{\delta}{\delta\bar\psi_c}\Big)(x) + \psi_c(x) \qquad \mathcal{D}_{\psi_c}(x) = -i\Big(\frac{\delta}{\delta\psi_c}\Delta\Big)(x) + \bar\psi_c(x)$$

$$\mathcal{D}^\mu_A(x) = i\left(\Delta^{\mu\nu}\frac{\delta}{\delta A^\nu_c}\right)(x) + A^\mu_c(x)$$

satisfying

$$\begin{gathered}[\mathcal{D}_{A\mu}(x), A_{c\nu}(y)] = i\Delta_{\mu\nu}(x-y)\\ \{\mathcal{D}_{\psi^*_c}(x),\psi^*_c(y)\} = i\Delta(x-y) \qquad \{\psi_c(x),\mathcal{D}_{\psi_c}(y)\} = -i\Delta(x-y)\,.\end{gathered}$$

As an exercise we can compute the exact photonic propagator up to order e^2 using the (anti)commutators of covariant derivatives. The two point function is

$$\begin{aligned}G_{\mu\nu}(x-y) &= \langle\Omega|A_\mu(x)A_\nu(y)|\Omega\rangle\\ &= \frac{N}{N_0}\mathcal{D}_{A\mu}(x)\mathcal{D}_{A\nu}(y)e^{ie\left\langle\mathcal{D}_{\psi_c}\mathcal{D}\!\!\!\!/_A\mathcal{D}_{\bar\psi_c}\right\rangle}1\Big|_{\varphi_c=0}\,.\end{aligned}$$

In the expansion of the exponential only terms with an even number of covariant derivatives with respect to A_c survive after taking $\varphi_c = 0$ (here φ denotes any field of the theory). Simple functional algebra gives

$$\mathcal{D}_{A\mu}(x)\mathcal{D}_{A\nu}(y)1 = i\Delta_{\mu\nu}(x-y) + A_{c\mu}(x)A_{c\nu}(y)\,,$$

$$\begin{aligned}&\mathcal{D}_{A\mu}(x)\mathcal{D}_{A\nu}(y)\mathcal{D}_{A\rho}(z_1)\mathcal{D}_{A\sigma}(z_2)1\Big|_{\varphi_c=0} =\\ &= -\Delta_{\mu\nu}(x-y)\Delta_{\rho\sigma}(z_1-z_2) - \Delta_{\mu\rho}(x-z_1)\Delta_{\nu\sigma}(y-z_2) - \Delta_{\mu\sigma}(x-z_2)\Delta_{\rho\nu}(z_1-y)\end{aligned}$$

and

$$\mathcal{D}_{\psi_c}(z_1)\gamma_\rho\mathcal{D}_{\bar{\psi}_c}(z_1)\mathcal{D}_{\psi_c}(z_2)\gamma_\sigma\mathcal{D}_{\bar{\psi}_c}(z_2)\big|_{\varphi_c=0} = \\ = -\operatorname{tr}[\gamma_\rho\Delta(z_1-z_1)]\operatorname{tr}[\gamma_\sigma\Delta(z_2-z_2)] + \operatorname{tr}\left[\gamma_\rho\Delta(z_1-z_2)\gamma_\sigma\Delta(z_2-z_1)\right].$$

Expanding the exponential we get

$$\begin{aligned}
G_{\mu\nu}(x-y) &= i\frac{N}{N_0}\Delta_{\mu\nu}(x-y)+ \\
&+\frac{N}{N_0}\frac{e^2}{2}\Delta_{\mu\nu}(x-y)\int d^4z_1d^4z_2\,\Delta_{\rho\sigma}(z_1-z_2)\operatorname{tr}\left[\gamma_\rho\Delta(z_1-z_2)\gamma_\sigma\Delta(z_2-z_1)\right]- \\
&-\frac{N}{N_0}\frac{e^2}{2}\Delta_{\mu\nu}(x-y)\int d^4z_1d^4z_2\,\operatorname{tr}[\gamma_\rho\Delta(z_1-z_1)]\Delta_{\rho\sigma}(z_1-z_2)\operatorname{tr}[\gamma_\sigma\Delta(z_2-z_2)]+ \\
&+\frac{N}{N_0}e^2\int d^4z_1d^4z_2\,\Delta_{\mu\rho}(x-z_1)\operatorname{tr}\left[\gamma^\rho\Delta(z_1-z_2)\gamma^\sigma\Delta(z_2-z_1)\right]\Delta_{\sigma\nu}(z_2-y)- \\
&-\frac{N}{N_0}e^2\int d^4z_1\,\Delta_{\mu\rho}(x-z_1)\operatorname{tr}[\gamma^\rho\Delta(z_1-z_1)]\int d^4z_2\,\operatorname{tr}[\gamma^\sigma\Delta(z_2-z_2)]\Delta_{\sigma\nu}(z_2-y)\,,
\end{aligned}$$

which corresponds to what is given by Feynman graphs.

2.10 Connected Green's functions and proper vertex functions

The Green's functions we have met so far are the key objects to compute amplitudes and cross sections, but, from a theoretical point of view, they contain many redundancies. A general Green's function can be written in terms of simpler building blocks, the so-called connected Green's functions $G_n^{\rm c}$. More precisely an n point Green's function has the schematic structure

$$G_n = G_n^{\rm c} + \sum_{k<n}\prod_{m\leq k} G_m^{\rm c}$$

In a scattering involving n particles between initial and final states, the Green's function G_n includes the possibility that not all of the particles take part in the process, or that the process is actually composed of many subprocesses, each one independent of the other ones (cluster decomposition). From a diagrammatic point of view the connected Green's functions are the sum of all Feynman diagrams in which every vertex is connected to any other vertex through one or more lines.

The connected Green's functions have a generating functional

$$W[J] = \sum_{n=0}^{+\infty}\frac{i^n}{n!}\int d^4x_1\dots d^4x_n\,G_n^{\rm c}(x_1,\dots,x_n)J(x_1)\cdots J(x_n)$$

that is linked to the generating functional for the Green's functions $Z[J]$ in a very simple way, that is

$$Z[J] = e^{W[J]}\,. \tag{2.28}$$

Such a result is the cluster decomposition theorem and is a standard achievement of statistical mechanics. Taking derivatives with respect to J of equation (2.28) and evaluating at $J = 0$, we can see the precise structure of the Green's functions in terms of the connected ones. The first two examples are

$$\begin{aligned} G_1(x) &= G_1^c(x) \\ G_2(x,y) &= G_2^c(x,y) + G_1^c(x)G_1^c(y) \,. \end{aligned} \tag{2.29}$$

By turning our attention to the connected Green's functions we have not yet eliminated all the redundancies we can find. This is achieved by considering the Legendre transform of $W[J]$. Consider the quantity

$$\Gamma_J[\varphi_e] = -\int d^4x\, J(x)\varphi_e(x) - iW[J] \,, \tag{2.30}$$

where the subscript e stands for effective. The Legendre transform of $W[J]$ is defined to be the supremum with respect to J of $\Gamma_J[\varphi_e]$, that is

$$\Gamma[\varphi_e] = \sup_J\{\Gamma_J[\varphi_e]\} = -\int d^4x\, J_0(x)\varphi_e(x) - iW[J_0] \,. \tag{2.31}$$

This definition looks different from the one usually found in literature but we now see that it is not. Taking a J derivative of the definition (2.30) we find

$$\frac{\delta\Gamma_J}{\delta J(x)}[\varphi_e] = -\varphi_e(x) - i\frac{\delta W}{\delta J(x)}[J] \,.$$

Evaluating at J_0, the left hand side vanishes for stationarity and we are left with

$$\varphi_e(x) = \frac{1}{i}\frac{\delta W}{\delta J(x)}\bigg|_{J_0} \,.$$

Taking now a φ_e derivative of (2.31) we get

$$\frac{\delta\Gamma}{\delta\varphi_e(x)}[\varphi_e] = -\int d^4y\, \frac{\delta J_0(y)}{\delta\varphi_e(x)}\varphi_e(y) - J_0(x) + \frac{1}{i}\int d^4y\, \frac{\delta W}{\delta J(y)}\bigg|_{J_0}\frac{\delta J_0(y)}{\delta\varphi_e(x)} = -J_0(x) \,.$$

Thus we have found the same relations between J_0 and φ_e as are commonly known. From now on we will drop the subscript 0 from J_0, always understanding that J and φ_e are linked by the relations we have just found.

The importance of $\Gamma[\varphi_e]$ is that it generates the proper vertex functions Γ_n, in the sense that

$$\Gamma_n(x_1,\ldots,x_n) = \frac{\delta^n\Gamma[\varphi_e]}{\delta\varphi_e(x_1)\ldots\delta\varphi_e(x_n)} \,.$$

These functions are amputated and one particle irreducible connected Green's functions. Amputated here means that they lack the full propagators attached to their external legs. One particle irreducibles means that their Feynman graphs cannot become disconnected by cutting one single line. A graph that is one particle reducible can be factorized into the product of one particle irreducible graphs and Feynman propagators. For $n = 2$ the situation is different, Γ_2 is the inverse of the full propagator,

$$\int d^4z\, G_2^c(x,z)\Gamma_2(z,y) = i\delta(x-y) \,. \tag{2.32}$$

Non perturbatively the proper vertex functions can be used as building blocks to compute the connected Green's functions. They are the simplest building blocks we can extract from our Green's functions. Once the full set of proper vertex functions is known, a generic Green's function is computed by tree level diagrams of a theory with action $\Gamma[\varphi_e]$: the vertices are the proper vertex functions Γ_n with $n \geq 3$ and the lines are the inverse of Γ_2, the full propagator. What briefly explained in these lines is essentially the content of the Jona-Lasinio theorem. The functional $\Gamma[\varphi_e]$ is known as the effective action and reduces to the classical action in the limit $\hbar \to 0$.

In summary we have four functionals encoding all what we need to specify a theory. The first two ones are the classical action $S[\varphi]$ and the generating functional $Z[J]$. In a path integral representation the relation between them takes the form

$$Z[J] = N \int \mathcal{D}\varphi \, e^{iS[\varphi]} e^{i\langle J\varphi\rangle} \,.$$

This is the functional generalization of the Fourier transform and we may want to invert it:

$$e^{iS[\varphi]} = N' \int \mathcal{D}J \, Z[J] e^{-i\langle J\varphi\rangle} \,.$$

Next we met the generating functional for the connected Green's functions $W[J]$. By the cluster decomposition theorem it is linked to $Z[J]$ by the exponential map,

$$Z[J] = e^{W[J]} \qquad \Longleftrightarrow \qquad W[J] = \log Z[J] \,.$$

Finally the functional Legendre transform allows us to go from $W[J]$ to $\Gamma[\varphi_e]$ and vice verses:

$$\Gamma[\varphi_e] = -\int d^4x \, J(x)\varphi_e(x) - iW[J] \qquad \Longleftrightarrow \qquad W[J] = i \int d^4x \, J(x)\varphi_e(x) + i\Gamma[\varphi_e] \,.$$

2.11 Dyson-Schwinger equations

We want to derive the Dyson-Schwinger equations, the equations of motion for the effective action. Below we will see a similar equation for the Wilsonian effective action, which is the effective action with infrared cutoff. In the following it is convenient to consider Green's functions with the external current on. As at the end of section 2.1 for a generic functional F we have

$$\langle \Omega_J | \, \mathrm{T} \, F[\hat{\varphi}_J] | \Omega_J \rangle = N_J \int \mathcal{D}\varphi \, F[\varphi] e^{iS_J[\varphi]} \,, \tag{2.33}$$

where $S_J[\varphi] = S[\varphi] + \langle J\varphi\rangle$, $\hat{\varphi}_J$ satisfies the equations of motions of S_J, Ω_J is the vacuum of S_J and

$$\frac{1}{N_J} = \int \mathcal{D}\varphi \, e^{iS_J[\varphi]} = \frac{Z[J]}{N} \,.$$

Equation (2.33) becomes

$$\langle \Omega_J | \, \mathrm{T} \, F[\hat{\varphi}_J] | \Omega_J \rangle = \frac{N}{Z[J]} F\left[\frac{1}{i}\frac{\delta}{\delta J}\right] \int \mathcal{D}\varphi \, e^{iS[\varphi] + i\langle J\varphi\rangle} = \frac{1}{Z[J]} F\left[\frac{1}{i}\frac{\delta}{\delta J}\right] Z[J]$$

and, in particular,

$$G_n^J(x_1, \ldots, x_n) = \frac{(-i)^n}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \,. \tag{2.34}$$

For the connected Green's functions, notice that the generating functional for the Green's functions with external current is

$$Z_J[K] = N_J \int \mathcal{D}\varphi \, e^{iS[\varphi]+i\langle J\varphi\rangle+i\langle K\varphi\rangle} = \frac{Z[J+K]}{Z[J]} \,.$$

Therefore the connected Green's functions with external current are

$$\begin{aligned} G_n^{c,J}(x_1,\ldots,x_n) &= \frac{1}{i^n}\frac{\delta^n \log Z_J[K]}{\delta K(x_1)\ldots\delta K(x_n)}\bigg|_{K=0} \\ &= \frac{1}{i^n}\frac{\delta^n \log Z[J+K]}{\delta K(x_1)\ldots\delta K(x_n)}\bigg|_{K=0} \\ &= \frac{1}{i^n}\frac{\delta^n W[J+K]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{K=0} \,, \end{aligned}$$

that is

$$G_n^{c,J}(x_1,\ldots,x_n) = \frac{1}{i^n}\frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)} \,. \tag{2.35}$$

The Green's functions with external current are useful because we can write some nice relations such as

$$G_n^{J}(x_1,\ldots,x_n) = \Big[\frac{1}{i}\frac{\delta}{\delta J(x_1)} + G_1^J(x_1)\Big] G_{n-1}^J(x_2,\ldots,x_n) \tag{2.36}$$

$$G_n^{c,J}(x_1,\ldots,x_n) = \frac{1}{i}\frac{\delta}{\delta J(x_1)} G_{n-1}^{c,J}(x_2,\ldots,x_n) \,. \tag{2.37}$$

These relations follow by taking one J derivative of (2.34) and (2.35). Iterating (2.36) $n-1$ times we find

$$G_n^{J}(x_1,\ldots,x_n) = \prod_{i=1}^{n-1}\Big[\frac{1}{i}\frac{\delta}{\delta J(x_i)} + G_1^J(x_i)\Big] G_1^J(x_n) = \prod_{i=1}^{n}\Big[\frac{1}{i}\frac{\delta}{\delta J(x_i)} + G_1^J(x_i)\Big] 1 \,. \tag{2.38}$$

The same can be done with (2.37) with result

$$G_n^{c,J}(x_1,\ldots,x_n) = \prod_{i=1}^{n-1}\Big[\frac{1}{i}\frac{\delta}{\delta J(x_i)}\Big] G_1^{c,J}(x_n) \,. \tag{2.39}$$

Other useful relations involve the effective action $\Gamma[\varphi_e]$. First of all notice that

$$\varphi_e(x) = G_1^J(x) = G_1^{c,J}(x) \,,$$

obvious from the definition of φ_e, and

$$\frac{\delta}{\delta J(x)} = \int d^4y \, \frac{\delta\varphi_e(y)}{\delta J(x)}\frac{\delta}{\delta\varphi_e(y)} = i\int d^4y \, G_2^{c,J}(x,y)\frac{\delta}{\delta\varphi_e(y)} = i\Big(G_2^{c,J}\frac{\delta}{\delta\varphi_e}\Big)(x) \,.$$

We can now rewrite equations (2.38) and (2.39) in terms of φ_e:

$$G_n^{J}(x_1,\ldots,x_n) = \prod_{i=1}^{n}\Big[\Big(G_2^{c,J}\frac{\delta}{\delta\varphi_e}\Big)(x_i) + \varphi_e(x_i)\Big] 1 \tag{2.40}$$

$$G_n^{J,c}(x_1,\ldots,x_n) = \prod_{i=1}^{n-1}\Big[\Big(G_2^{c,J}\frac{\delta}{\delta\varphi_e}\Big)(x_i)\Big] \varphi_e(x_n) \,.$$

Performing the φ_e derivatives we can make explicit the structure of the Green's functions in terms of the proper vertex functions. Equation (2.40) can be extended to the case of an operator functional that admits an expansion in powers of $\hat{\varphi}_J$ like (2.1). This is simply

$$\langle\Omega_J|\,\mathrm{T}\,F[\hat{\varphi}_J]|\Omega_J\rangle = F\Big[G_2^{c,J}\frac{\delta}{\delta\varphi_e} + \varphi_e\Big]1\,. \tag{2.41}$$

With these notions it is easy to derive the Dyson-Schwinger equations. We start with

$$e^{i\langle J\varphi_e\rangle + i\Gamma[\varphi_e]} = N\int\mathcal{D}\varphi\, e^{iS[\varphi]+i\langle J\varphi\rangle}\,.$$

Performing the shift $\varphi\to\varphi+\varphi_e$, $\mathcal{D}\varphi\to\mathcal{D}\varphi$ in the integration variable we get

$$e^{i\Gamma[\varphi_e]} = N\int\mathcal{D}\varphi\, e^{iS[\varphi+\varphi_e]+i\langle J\varphi\rangle} \tag{2.42}$$

and deriving with respect to φ_e:

$$i\frac{\delta\Gamma}{\delta\varphi_e(x)}[\varphi_e]e^{i\Gamma[\varphi_e]} = N\int\mathcal{D}\varphi\, i\left[\frac{\delta S}{\delta\varphi(x)}[\varphi+\varphi_e] + \left\langle\frac{\delta J}{\delta\varphi_e(x)}\varphi\right\rangle\right]e^{iS[\varphi+\varphi_e]+i\langle J\varphi\rangle}\,.$$

The second term in the integral is zero as can be seen by performing back the shift $\varphi\to\varphi-\varphi_e$, $\mathcal{D}\varphi\to\mathcal{D}\varphi$. After that we divide by $i\exp(i\Gamma[\varphi_e])$ and use (2.42) to find

$$\frac{\delta\Gamma}{\delta\varphi_e(x)}[\varphi_e] = \frac{\int\mathcal{D}\varphi\,\frac{\delta S}{\delta\varphi(x)}[\varphi+\varphi_e]e^{iS[\varphi+\varphi_e]+i\langle J\varphi\rangle}}{\int\mathcal{D}\varphi\, e^{iS[\varphi+\varphi_e]+i\langle J\varphi\rangle}}$$

Shifting both the numerator and the denominator by $\varphi\to\varphi-\varphi_e$ we have

$$\frac{\delta\Gamma}{\delta\varphi_e(x)}[\varphi_e] = \langle\Omega_J|\,\mathrm{T}\,\frac{\delta S}{\delta\varphi(x)}[\hat{\varphi}_J]|\Omega_J\rangle$$

and using the representation (2.41) we get to the desired result

$$\frac{\delta\Gamma}{\delta\varphi_e(x)}[\varphi_e] = \frac{\delta S}{\delta\varphi(x)}\Big[\varphi = G_2^{c,J}\frac{\delta}{\delta\varphi_e} + \varphi_e\Big]1\,.$$

This is the master Dyson-Schwinger equation for the proper vertex functions. By deriving $n-1$ times with respect to φ_e we get an equation for Γ_n involving functions with a larger number of points. This feature makes it impossible to solve this infinite system of coupled equations analitically and some prescriptions or truncation schemes are needed in order to solve them numerically.

Just for an illustration we consider here the example of a $\lambda\phi^4$ scalar theory with action

$$S[\phi] = \int d^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right].$$

The first derivative is

$$\frac{\delta S}{\delta\phi(x)}[\phi] = -\Box\phi(x) - m^2\phi(x) - \frac{\lambda}{3!}\phi^3(x)\,.$$

To compute the right hand side of the Dyson-Schwinger equation we need

$$\frac{\delta}{\delta\phi_e(z)}G_2^{c,J}(x,y) = i\int d^4x'd^4y'\, G_2^{c,J}(x,x')\Gamma_3(z,x',y')G_2^{c,J}(y',y)$$

that can be obtained by taking a φ_e derivative of equation (2.32) where G_2^c is substituted with $G_2^{c,J}$. The Dyson-Schwinger equation becomes

$$\frac{\delta\Gamma}{\delta\phi_e(x)}[\phi_e] = -(\Box + m^2)\phi_e(x) - \frac{\lambda}{3!}\phi_e^3(x) + \\ - \frac{i\lambda}{3!}\int d^4x_1 d^4x_2 d^4x_3\, G_2^{c,J}(x,x_1)G_2^{c,J}(x,x_2)G_2^{c,J}(x,x_3)\Gamma_3(x_1,x_2,x_3) + \\ - \frac{\lambda}{6}G_2^{c,J}(x,x)\phi_e(x)$$

We remark that the first line is exactly the first derivative of the classical action evaluated on $\phi_e(x)$.

Chapter 3

The Wilsonian effective action

After a short reminder of the Euclidean formulation of a Quantum Field Theory, in this chapter we sketch the main ideas of the Wilsonian approach and introduce the Wilsonian effective action. The Wilsonian effective action is essentially the effective action with an infrared cutoff. In theories with massive particles only there is not a big difference between them, but in presence of massless fields this is no longer true.

Then we use simple functional techniques to derive two equations regulating the scaling on the infrared cutoff of the effective action and the generating functionals of a theory. The first one is the Polchinski equation and the other one is the Wetterich equation.

3.1 Euclidean spacetime and Fourier conventions

Some of the notions we are going to discuss require the formulation on the Euclidean spacetime, but we will not go in the details of such a setup. We will just present the main and necessary features we will use in this work without justifications. For convenience we omit the subscript E in the quantities that would require it. This is the only chapter where we use the Euclidean framework. The path integral representation of the generating functional takes the form

$$Z[J] = N \int \mathcal{D}\varphi \, e^{-S[\varphi]+\langle J\varphi\rangle} \tag{3.1}$$

and the Green's functions are extracted according to

$$G_n(x_1, \ldots, x_n) = \left.\frac{\delta^n Z}{\delta J(x_1) \ldots \delta J(x_n)}\right|_{J=0} .$$

The action changes its own form according to the kind of particles it descibes. For example the action for a scalar field becomes

$$S[\phi] = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V(\phi)\right] .$$

The generating functional for the connected Green's functions is still given by

$$Z[J] = e^{W[J]}$$

but the effective action is defined in a different way:

$$\Gamma[\varphi_e] = \langle J\varphi_e\rangle - W[J] \tag{3.2}$$
$$\varphi_e(x) = \frac{\delta W}{\delta J(x)} \qquad\qquad J(x) = \frac{\delta \Gamma}{\delta \varphi_e(x)}\,.$$

This definition entails

$$\int d^4z\, G_2^c(x,z)\Gamma_2(z,y) = \delta(x-y)\,. \tag{3.3}$$

Given two vectors A_μ and B_μ on the spacetime, the scalar product is Euclidean, that is

$$A_\mu B_\mu = \sum_{\mu=1}^{4} A_\mu B_\mu,$$

so that, for instance $A^2 = A_\mu A_\mu \geq 0$.

This is what we need to know about an Euclidean Quantum Field Theory. For the Fourier transform we adopt the convention

$$\varphi(x) = \int \frac{d^4p}{(2\pi)^4}\tilde{\varphi}(p)e^{ip\cdot x} \qquad\qquad \tilde{\varphi}(p) = \int d^4x\,\varphi(x)e^{-ip\cdot x}\,.$$

In this way the quadratic part of the action takes the form

$$S_0[\varphi] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\tilde{\varphi}(p)\Delta^{-1}(p)\tilde{\varphi}(-p)\,.$$

3.2 Wilson renormalization group

In the Wilsonian approach to renormalization a UV cutoff Λ_0 is assumed in such a way that there are no divergencies. The cutoff is imagined very large (in the end we may want to take it to infinity) and its precise value is of no importance. We could take a smaller cutoff Λ and the observables of interest, at a energy scale far below Λ and Λ_0, would not change. This idea has a nice representation in terms of (Euclidean) path integral. Take a generic theory with path integral

$$Z[J] = N\int \mathcal{D}\varphi\, e^{-S_{\Lambda_0}[\varphi]+\langle J\varphi\rangle}$$

and split the field φ into low energy modes and high energy modes, $\varphi = \varphi_- + \varphi_+$, where φ_- has support for momenta $p < \Lambda$ and φ_+ for momenta $\Lambda < p < \Lambda_0$. Now we split the functional integration with the same logic in order to integrate out the high energy modes:

$$\begin{aligned} Z[J] &= N\int \mathcal{D}\varphi_-\, e^{\langle J\varphi_-\rangle}\int \mathcal{D}\varphi_+\, e^{-S_{\Lambda_0}[\varphi_-+\varphi_+]+\langle J\varphi_+\rangle} \\ &= N\int \mathcal{D}\varphi_-\, e^{-S_\Lambda[\varphi_-,J]+\langle J\varphi_-\rangle}\,, \end{aligned} \tag{3.4}$$

where we have defined

$$e^{-S_\Lambda[\varphi_-,J]} = \int \mathcal{D}\varphi_+\, e^{-S_{\Lambda_0}[\varphi_-+\varphi_+]+\langle J\varphi_+\rangle}\,. \tag{3.5}$$

The exponent S_Λ is called the Wilsonian effective action and describes the theory at energies below Λ. The independence of observables (at low energies) from Λ is codified by the following equation

$$\Lambda \frac{\partial}{\partial \Lambda} Z[J] = 0\,. \tag{3.6}$$

The Wilsonian effective action S_Λ can be identified with the (1PI) effective action with infrared cutoff Λ, as is shown by the following naive argument. Shifting the integration variable in (3.5) by $-\varphi_-$ we get

$$e^{-S_\Lambda[\varphi_-,J]} = e^{-\langle J\varphi_-\rangle} \int \mathcal{D}\varphi_+ \, e^{S_{\Lambda_0}[\varphi_+]+\langle J\varphi_+\rangle} = e^{W_+[J]-\langle J\varphi_-\rangle}\,, \tag{3.7}$$

that is

$$S_\Lambda[\varphi_-, J] = \langle J\varphi_-\rangle - W_+[J]\,. \tag{3.8}$$

The Legendre transform of $W_+[J]$ is

$$\Gamma_+[\varphi_e] = \langle J[\varphi_e]\,\varphi_e\rangle - W_+\big[J[\varphi_e]\big] \qquad \text{with} \qquad J[\varphi_e] = \frac{\delta\Gamma}{\delta\varphi_e}[\varphi_e]\,.$$

Therefore we have $\Gamma_+[\varphi_e] = S_\Lambda\big[\varphi_e, J[\varphi_e]\big]$. Besides, the shift in formula (3.7) leads to another remark. Plugging (3.7) in formula (3.4) gives

$$Z[J] = N \int \mathcal{D}\varphi_- \, e^{W_+[J]} = N' e^{W_+[J]}\,.$$

Even though this relation should not be taken too seriously, it shows that infinitely many interaction terms contained in the full theory $Z[J]$ are already generated by an integration over a momentum shell. Unfortunately integration over a momentum shell is a very difficult task.

Following the work of Polchinski [12] we want to translate the condition (3.6) into a flow equation for the Wilsonian effective action S_Λ. First of all we need to introduce a soft UV regulator to suppress the high energy modes. This is done by modifying the quadratic part of the action in the following way:

$$\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \tilde{\varphi}(p)\Delta^{-1}(p) K\left(\frac{p^2}{\Lambda_0^2}\right)\tilde{\varphi}(-p)\,,$$

where $K(p^2/\Lambda^2)$ is such that $K^{-1}(p^2/\Lambda^2) \sim 1$ for $p^2 \ll \Lambda^2$ and vanishes for $p^2 > \Lambda^2$. The generating functional is

$$Z[J] = \\ = N \int \mathcal{D}\varphi \, \exp\left[\int \frac{d^4p}{(2\pi)^4}\left(-\frac{1}{2}\tilde{\varphi}(p)\Delta^{-1}(p)K\left(\frac{p^2}{\Lambda_0^2}\right)\tilde{\varphi}(-p) + \tilde{J}(p)\tilde{\varphi}(-p)\right) - S_V[\varphi]\right].$$

We integrate out modes down to Λ and get an effective theory with action S_Λ. We lower the cutoff from Λ_0 to Λ and since the modes with momentum $p^2 > \Lambda^2$ are suppressed we take $\tilde{J}(p) = 0$ for $p^2 > \Lambda^2$. The path integral now reads

$$Z[J] = \\ = N \int \mathcal{D}\varphi \, \exp\left[\int \frac{d^4p}{(2\pi)^4}\left(-\frac{1}{2}\tilde{\varphi}(p)\Delta^{-1}(p)K\left(\frac{p^2}{\Lambda^2}\right)\tilde{\varphi}(-p) + \tilde{J}(p)\tilde{\varphi}(-p)\right) - S_\Lambda^V[\varphi]\right],$$

so that

$$\Lambda\frac{\partial}{\partial\Lambda}Z[J] =$$
$$= N\int\mathcal{D}\varphi\left[\int\frac{d^4p}{(2\pi)^4}\Big(-\frac{1}{2}\tilde{\varphi}(p)\Delta^{-1}(p)\Lambda\frac{\partial K}{\partial\Lambda}\tilde{\varphi}(-p)\Big)-\Lambda\frac{\partial}{\partial\Lambda}S^V_\Lambda[\varphi]\right]e^{-S_\Lambda[\varphi]+\langle J\varphi\rangle}\,. \tag{3.9}$$

On the other hand, since the functional integral of a total derivative is zero, we have

$$\int d^4p\,\Lambda\frac{\partial K^{-1}}{\partial\Lambda}\times$$
$$\times\int\mathcal{D}\varphi\,\frac{\delta}{\delta\tilde{\varphi}(p)}\left[\left(\tilde{\varphi}(p)K+\frac{1}{2}(2\pi)^4\Delta^{-1}(p)\frac{\delta}{\delta\tilde{\varphi}(-p)}\right)e^{-S_\Lambda[\varphi]+\langle J\varphi\rangle}\right]=0\,,$$

which becomes

$$\int\mathcal{D}\varphi\int\frac{d^4p}{(2\pi)^4}\Lambda\frac{\partial K^{-1}}{\partial\Lambda}\left[\frac{(2\pi)^4}{2}\delta(0)K-\frac{1}{2}\tilde{\varphi}(p)\Delta^{-1}(p)\tilde{\varphi}(-p)K^2+\right.$$
$$+\frac{1}{2}\tilde{J}(p)\Delta(p)\tilde{J}(-p)-(2\pi)^4\Delta(p)\tilde{J}(p)\frac{\delta S^V_\Lambda}{\delta\tilde{\varphi}(p)}+$$
$$\left.-\frac{(2\pi)^8}{2}\Delta(p)\frac{\delta^2S^V_\Lambda}{\delta\tilde{\varphi}(p)\delta\tilde{\varphi}(-p)}+\frac{(2\pi)^8}{2}\Delta(p)\frac{\delta S^V_\Lambda}{\delta\tilde{\varphi}(p)}\frac{\delta S^V_\Lambda}{\delta\tilde{\varphi}(-p)}\right]e^{-S_\Lambda[\varphi]+\langle J\varphi\rangle}=0\,. \tag{3.10}$$

The first term in the first line is field independent and can be neglected because it only changes $Z[J]$ by an overall factor. Since $\tilde{J}(p)=0$ for $p^2>\Lambda^2$, there is no overlap between $\tilde{J}(p)$ and $\partial_\Lambda K^{-1}$ and the second line drops. Therefore, if we choose

$$\Lambda\frac{\partial}{\partial\Lambda}S^V_\Lambda[\varphi,J]=\frac{1}{2}\int d^4p\,(2\pi)^4\Delta(p)\Lambda\frac{\partial K}{\partial\Lambda}\left(\frac{\delta^2S^V_\Lambda}{\delta\tilde{\varphi}(p)\delta\tilde{\varphi}(-p)}-\frac{\delta S^V_\Lambda}{\delta\tilde{\varphi}(p)}\frac{\delta S^V_\Lambda}{\delta\tilde{\varphi}(-p)}\right), \tag{3.11}$$

called the Polchinski equation, then the left hand side of equation (3.10) coincides with the right hand side of equation (3.9) and we recover the condition (3.6).

3.3 Functional renormalization group

In the previous section we have seen an introduction to the Wilson approach and given a flavour of the main ideas. Now we want to make use of the functional methods to illustrate an operative approach to these ideas. This functional renormalization group approach is based on a soft IR regulator similar to the one used by Polchinski, which allows to integrate out the modes above Λ. The main result is the Wetterich equation, a functional equation very close to the Polchinski equation. More details can be found in [13]

From an heuristic point of view the infrared regulator is obtained by adding to the quadratic part of the action a momentum dependent mass term of the form

$$\Delta S_\Lambda[\varphi]=\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\tilde{\varphi}(p)R_\Lambda(p^2)\tilde{\varphi}(-p)\,, \tag{3.12}$$

where $R_\Lambda(p^2)$ is called a soft IR regulator and is a function with the following properties:

(a) suppression of IR modes

$$\lim_{\frac{p^2}{\Lambda^2}\to 0} R_\Lambda(p^2) = \Lambda^2\,;$$

(b) physical limit

$$\lim_{\Lambda\to 0} R_\Lambda(p^2) = 0\,;$$

(c) UV limit

$$\lim_{\Lambda\to \Lambda_0} R_\Lambda(p^2) = +\infty\,.$$

The choice of R_Λ is by no means unique. Its specific functional form can be chosen according to the problems under investigation. A standard parametrization is

$$R_\Lambda(p^2) = p^2 r(y)\,,$$

where $y = p^2/\Lambda^2$ and r is a dimensionless shape function, in general only depending on y. For low momentum modes, $p^2 \to 0$, thanks to property (a) the mass function $R_\Lambda(p^2)$ becomes a constant and plays the role of a mass. In this way the momentum modes with $p^2 < \Lambda^2$ are suppressed just as they are in presence of a massive field. Property (b) is the natural requirement that once the cutoff is removed we get back the original theory. If we push the IR cutoff to infinity we expect that no mode is left to be integrated out. This is what is assured by (c).

The scale dependent generating functional is defined as

$$Z_\Lambda[J] = e^{-\Delta S_\Lambda[\frac{\delta}{\delta J}]} Z[J]\,. \tag{3.13}$$

In order to understand this definition we may use the path integral representation (3.1) for $Z[J]$ and get

$$Z_\Lambda[J] = N\int \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_\Lambda[\varphi]+\langle J\varphi\rangle}\,.$$

However we choose the form (3.13) as starting point because we want to show that our main result, the functional renormalization group equation, only relies on the existence of the generating functional without any reference to the path integral. Taking a Λ derivative of (3.13) and multiplying by Λ leads us to

$$\begin{aligned}\Lambda\partial_\Lambda Z_\Lambda[J] &= -\Lambda\partial_\Lambda\left(\Delta S_\Lambda\left[\frac{\delta}{\delta J}\right]\right) e^{-\Delta S_\Lambda[\frac{\delta}{\delta J}]} Z[J]\\ &= -\Lambda\partial_\Lambda\left(\Delta S_\Lambda\left[\frac{\delta}{\delta J}\right]\right) Z_\Lambda[J]\,.\end{aligned}$$

More explicitly, using the definition (3.12) of ΔS_Λ, we have

$$\begin{aligned}\Lambda\partial_\Lambda Z_\Lambda[J] &= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{\delta^2 Z_\Lambda[J]}{\delta\tilde{J}(p)\delta\tilde{J}(-p)}\Lambda\partial_\Lambda R_\Lambda(p^2)\\ &= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\tilde{G}_2^J(p;\Lambda)\Lambda\partial_\Lambda R_\Lambda(p^2)\,.\end{aligned}$$

This is the functional flow equation for the generating functional $Z[J]$. We can get its analogue for the connected Green's functions by decomposing the two point function G_2^J into its connected terms as in equation (2.29):

$$\Lambda\partial_\Lambda W_\Lambda[J] = -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\Big[\tilde{G}_2^{c,J}(p;\Lambda) + \tilde{G}_1^{c,J}(p;\Lambda)\tilde{G}_1^{c,J}(-p;\Lambda)\Big]\Lambda\partial_\Lambda R_\Lambda(p^2)\,. \tag{3.14}$$

Notice that this equation has the same structure as the Polchinski equation (3.11). This is obvious given the relation (3.8). Taking derivatives with respect to $\tilde{J}$ provides the scale dependence of connected Green's functions.

Finally we are interested in the flow equation for the effective action. In order to write it in the most essential form it is convenient to study the structure of equation (3.14) more closely. The term with the one point functions is nothing but the derivative of the regulator evaluated on φ_e, that is

$$\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\tilde{G}_1^{c,J}(p;\Lambda)\tilde{G}_1^{c,J}(-p;\Lambda)\Lambda\partial_\Lambda R_\Lambda(p^2) = \Lambda\partial_\Lambda \Delta S_\Lambda[\varphi_e]\,. \tag{3.15}$$

Therefore this term is a trivial one in terms of φ_e and we can subtract it from the effective action. In this way we are led to the following definition of the scale dependent effective action (corresponding to the Wilsonian effective action):

$$\Gamma_\Lambda[\varphi_e] = \langle J\varphi_e\rangle - W_\Lambda[J] - \Delta S_\Lambda[\varphi_e]\,. \tag{3.16}$$

Notice that $\Gamma_\Lambda[\varphi_e]$ reduces to the standard expression (3.2) in the limit $\Lambda \to 0$. Moreover the Legendre transform of $W_\Lambda[J]$ is $\Gamma_\Lambda[\varphi_e] + \Delta S_\Lambda[\varphi_e]$ and all the relations between $W[J]$ and $\Gamma[\varphi_e]$ apply to $\Gamma_\Lambda[\varphi_e] + \Delta S_\Lambda[\varphi_e]$. In particular we have

$$J(x) = \frac{\delta(\Gamma_\Lambda[\varphi_e] + \Delta S_\Lambda[\varphi_e])}{\delta\varphi_e(x)} \qquad \varphi_e(x) = \frac{\delta W_\Lambda[J]}{\delta J(x)}$$

and the analogue of (3.3) in momentum space

$$G_2^{c,J}(p;\Lambda) = \frac{1}{\Gamma_2(p;\Lambda) + R_\Lambda(p^2)}\,. \tag{3.17}$$

When deriving the definition (3.16) with respect to Λ we must take into account the hidden dependence of J on Λ (whereas φ_e is an independent variable here). What we get is

$$\Lambda\partial_\Lambda\Gamma_\Lambda[\varphi_e] = \langle\varphi_e\Lambda\partial_\Lambda J\rangle - \Lambda\partial_\Lambda W_\Lambda[J] - \left\langle \frac{\delta W_\Lambda}{\delta J}\Lambda\partial_\Lambda J\right\rangle - \Lambda\partial_\Lambda\Delta S_\Lambda[\varphi_e]\,.$$

and the first and the third term cancel out. Considering equations (3.14), (3.15) and (3.17) we find the desired result:

$$\Lambda\partial_\Lambda\Gamma_\Lambda[\varphi_e] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{\Lambda\partial_\Lambda R_\Lambda(p^2)}{\Gamma_2(p;\Lambda) + R_\Lambda(p^2)}\,. \tag{3.18}$$

This is called the Wetterich equation or functional renormalization group equation.

3.4 Perturbation theory by the Wetterich equation

The Wetterich equation is a very useful approach for many problems in Quantum Field Theory. For example it provides a simple proof that the effective action generates one particle irreducible diagrams. Here we want to sketch a perturbative approach based on the Wetterich equation.

In general the effective action admits many possible expansion schemes. Schematically we can write

$$\Gamma_\Lambda[\varphi_e] = \sum_{n=0}^{+\infty} g_n \mathcal{O}_n[\varphi_e]\,.$$

We might consider an expansion in monomials in the field, in which case we have

$$g_n \mathcal{O}_n[\varphi_e] = \int d^4x_1 \dots d^4x_n \, g_n(x_1, \dots, x_n) \varphi_e(x_1) \cdots \varphi_e(x_n) \, ,$$

where

$$g_n = \frac{1}{n!} \Gamma_n \Big|_{\varphi_e = 0} .$$

Another important expansion is the loop expansion:

$$g_n \mathcal{O}_n[\varphi_e] = \Delta\Gamma^n_\Lambda[\varphi_e] \, ,$$

where $\Delta\Gamma^n_\Lambda[\varphi_e]$ is the n loop contribution to the scale dependent effective action and $\Delta\Gamma^0_\Lambda[\varphi_e] = S[\varphi_e]$. This is the scheme leading to perturbation theory that we are going to discuss.

The Wetterich equation is very well suited for a perturbative approach because we can easily extract a recursive formula from it. The N loop approximation of the full scale dependent effective action is

$$\Gamma^{(N)}_\Lambda[\varphi_e] = S[\varphi_e] + \sum_{n=1}^{N} \Delta\Gamma^n_\Lambda[\varphi_e] \, .$$

What we want to show is that, plugging the N loop approximation of Γ_2 into the right hand side of the Wetterich equation, we get, on the left hand side, the approximation of the (scale derivative of the) full effective action up to $N+1$ loops. In formula this is

$$\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{\Gamma^{(N)}_2(p;\Lambda) + R_\Lambda(p^2)} = \Lambda \partial_\Lambda \Gamma^{(N+1)}_\Lambda[\varphi_e] + \dots \, , \tag{3.19}$$

where the dots stand for $N+2$ and higher loop contributions. The proof of this formula is very simple as it is enough to compute the difference between the left hand side of (3.19) and the right hand side of the full Wetterich equation (3.18). We have

$$\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \left[\frac{1}{\Gamma^{(N)}_2(p;\Lambda) + R_\Lambda(p^2)} - \frac{1}{\Gamma_2(p;\Lambda) + R_\Lambda(p^2)} \right] =$$

$$= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \left[\frac{1}{\Gamma_2(p;\Lambda) + R_\Lambda(p^2)} \big(\Gamma_2(p;\Lambda) - \Gamma^{(N)}_2(p;\Lambda) \big) \frac{1}{\Gamma^{(N)}_2(p;\Lambda) + R_\Lambda(p^2)} \right] =$$

$$= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \left[\frac{1}{\Gamma_2(p;\Lambda) + R_\Lambda(p^2)} \Big(\sum_{n=N+1}^{+\infty} \Delta\Gamma^n_2(p;\Lambda) \Big) \frac{1}{\Gamma^{(N)}_2(p;\Lambda) + R_\Lambda(p^2)} \right] .$$

The last line is clearly of $N+2$ loops at leading order, one is the loop in the flow (the integration over p) and the other $N+1$ are in the insertion in the round brackets. Therefore, starting with

$$\Gamma^{(0)}_2(p;\Lambda) = S_2(p;\Lambda) = \frac{\delta^2 S_\Lambda}{\delta\tilde{\varphi}(p)\delta\tilde{\varphi}(-p)} \Big|_{\varphi = \varphi_e}$$

to be inserted in (3.19), we can compute the one loop contribution to the effective action and repet the procedure iteratively to any loop order.

Chapter 4

The Wilsonian effective action in the Seiberg-Witten model

Explicitly computing the Wilsonian effective action is an extremely difficult task, often impossible. The best we can do in most cases, including the physically relevant ones, is to evaluate the first few terms in a perturbative expansion. This is not the case for the Seiberg-Witten model, where, thanks to some important features of supersymmetric theories, we are able to determine the exact Wilsonian effective action. This example shows the importance of the Wilsonian approach introduced in the last chapter because the knowledge of the Wilsonian action has been crucial for the derivation of exact results such as the instantonic contributions (see below) and the beta function of the theory (see [14]).

The model at hand is the $N = 2$ susy $SU(2)$ Yang-Mills theory. The discussion is in Minkowskian framework and we follow the pedagogical review by Adel Bilal [15].

4.1 From the microscopic theory to the low energy effective action

4.1.1 The action

The action of this theory is

$$S = \operatorname{Im}\operatorname{tr}\int d^4x\,\frac{\tau}{16\pi}\left[\int d^2\theta\, W^\alpha W_\alpha + \int d^2\theta d^2\bar\theta\,\Phi^\dagger e^{-2gV}\Phi\right]. \tag{4.1}$$

We recall the main ingredients of this action:

- Φ is a chiral superfield, made of a complex scalar field ϕ, a Weyl spinor ψ_α and an auxiliary scalar field F. In components it reads

$$\Phi = \phi(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar\theta^2\Box\phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar\theta + \theta^2 F(x)\,;$$

- V is a vector superfield with a vector field A_μ, its superpartner λ_α (the gaugino, another Weyl spinor) and another auxiliary scalar field D. In the Wess-Zumino gauge it is

$$V = -\theta\sigma^\mu\bar\theta A_\mu + i\theta^2(\bar\theta\bar\lambda) - i\bar\theta^2(\theta\lambda) + \frac{1}{2}\theta^2\bar\theta^2 D\,;$$

- all the fields are in the adjoint representation of $SU(2)$, that is $A_\mu = A_\mu^a T^a$, $\phi = \phi^a T^a$, etc...;
- the quantity

$$W_\alpha = \frac{1}{8g}\bar{D}^2\left(e^{2gV} D_\alpha e^{-2gV}\right) \tag{4.2}$$

  is a spinorial superfield defined in terms of $D_\alpha = \partial/\partial\theta^\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu$ and $\bar{D}_{\dot{\alpha}} = -\partial/\partial\bar{\theta}^{\dot{\alpha}} - i\sigma^\mu_{\alpha\dot{\alpha}}\theta^\alpha\partial_\mu$ in such a way that

$$-\frac{1}{4}\int d^4x d^2\theta \ \mathrm{tr}\, W^\alpha W_\alpha = \int d^4x \ \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\lambda\sigma^\mu\nabla_\mu\bar{\lambda} + \frac{1}{2}D^2\right).$$

- τ is the complex coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2},$$

  including the θ parameter (not to be confused with the anticommuting θ variables of superspace) and the coupling constant g.

The action (4.1) can be rewritten in terms of a $N = 2$ chiral superfield Ψ. We need another set of anticommuting variables $\tilde{\theta}_\alpha$ and $\bar{\tilde{\theta}}_{\dot{\alpha}}$. We take

$$\Psi = \Phi(\tilde{y},\theta) + \sqrt{2}\tilde{\theta}^\alpha W_\alpha(\tilde{y},\theta) + \tilde{\theta}^\alpha\tilde{\theta}_\alpha G(\tilde{y},\theta),$$

where $\tilde{y}^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}} = y^\mu + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}}$ and

$$G(\tilde{y},\theta) = -\frac{1}{2}\int d^2\bar{\theta}\left[\Phi(\tilde{y} - i\theta\sigma\bar{\theta},\theta,\bar{\theta})\right]^\dagger e^{-2gV(\tilde{y}-i\theta\sigma\bar{\theta},\theta,\bar{\theta})}.$$

The superfield $G(\tilde{y},\theta)$ is necessary to eliminate certain unphysical degrees of freedom. In terms of Ψ the action (4.1) reads

$$S = \mathrm{Im}\left[\frac{\tau}{16\pi}\int d^4x d^2\theta d^2\tilde{\theta}\,\frac{1}{2}\,\mathrm{tr}\,\Psi^2\right]. \tag{4.3}$$

An important remark is that the integrand only depends on Ψ and not on $\Psi^\dagger$ (holomorphicity condition). The quadratic dependence on Ψ is imposed by renormalizability. If we drop this condition (as in the case of effective field theories), then the most general $N = 2$ SUSY invariant action is forced by holomorphicity to take the form

$$\frac{1}{16\pi}\,\mathrm{Im}\int d^4x d^2\theta d^2\tilde{\theta}\,\mathcal{F}(\Psi), \tag{4.4}$$

where $\mathcal{F}$, called the $N = 2$ prepotential, depends only on Ψ and not on $\Psi^\dagger$. In $N = 1$ superspace language the general action (4.4) reads

$$\frac{1}{16\pi}\,\mathrm{Im}\int d^4x\left[\int d^2\theta\,\mathcal{F}_{ab}(\Phi)W^{a\alpha}W^b_\alpha + \int d^2\theta d^2\bar{\theta}\left(\Phi^\dagger e^{-2gV}\right)^a\mathcal{F}_a(\Phi)\right], \tag{4.5}$$

where

$$\mathcal{F}_a(\Phi) = \frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^a} \qquad \mathcal{F}_{ab}(\Phi) = \frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^a\partial\Phi^b}.$$

4.1.2 The moduli space

First of all we need to study the classical structure of the moduli space, which means that we need to study the vacua of the theory. The only non trivial part is the one concerning the scalar field ϕ. The part of the action (4.1) concerning the auxiliary fields is

$$S_{\text{aux}} = \frac{1}{g^2}\int d^4x \ \text{tr}\left(\frac{1}{2}D^2 - g\phi^\dagger[D,\phi] + F^\dagger F\right).$$

Solving the classical equations of motion for these fields and plugging the result back into the action gives

$$S_{\text{aux}} = -\frac{1}{2}\int d^4x \ \text{tr}\left([\phi^\dagger,\phi]\right)^2.$$

Thus, the scalar field ϕ has classical potential $V(\phi) = \frac{1}{2}\,\text{tr}\left([\phi^\dagger,\phi]\right)^2 \geq 0$.

We want to examine the case of unbroken susy, which is equivalent to requiring $V(\phi) = 0$ in the vacuum. This implies $[\phi^\dagger,\phi] = 0$. The scalar field ϕ is equivalent to six real scalar fields

$$\phi(x) = \frac{1}{2}\sum_{k=1}^{3}\big(a_k(x) + ib_k(x)\big)\sigma_k .$$

By a $SU(2)$ gauge transformation we can arrange $a_1(x) = a_2(x) = 0$. Then $[\phi^\dagger,\phi] = 0$ implies $b_1(x) = b_2(x) = 0$ and hence, with $a = a_3 + ib_3$ we have $\phi(x) = \frac{1}{2}a(x)\sigma_3$. Gauge transformations from the Weyl group (that is rotations by π around the first or second axis of $SU(2)$) can still change $a \to -a$, so a and $-a$ are gauge equivalent. We can take $u = \left\langle \text{tr}\,\phi^2 \right\rangle$ as the coordinate on the moduli space labeling gauge inequivalent vacua. The complex number a is defined by $\langle\phi\rangle = \frac{1}{2}a\sigma_3$ and labels the vacua of ϕ. Different values of a or u lead to physically different theories, as we will see. Classically the relation between them is

$$u = \frac{1}{2}a^2 ,$$

but, as we will see below, when quantum corrections are taken into account this relation does not hold anymore for any value of a and the quantum moduli space can be very different from the classical one.

4.1.3 The supersymmetric Wilsonian effective action

With the above conventions, as a consequence of the Higgs mechanism, the fields A_μ^b, λ^b and ψ^b, with $b = 1,2$, become massive with mass $m = \sqrt{2}a$. The fields with $b = 3$ stay massless. It can be shown that in a spontaneously broken gauge theory like this one there may arise solitons carrying magnetic charge and behaving like non singular magnetic monopoles. We can imagine that (in some circumstances) the massive fields form $N = 2$ susy bound states, like mesons and baryons in QCD, with magnetic charge.

The symmetry group $SU(2)$ is broken down to $U(1)$ while the $N = 2$ supersymmetry, as already said, remains unbroken. As a consequence the massless modes at low energy are described by a Wilsonian effective action with gauge group $U(1)$ and $N = 2$ supersymmetry. Low energy for the moment means far below the mass $m = \sqrt{2}a$ so that we can integrate out the fields with $b = 1,2$ as heavy fields (but this is not always correct, see below). Since the Wilsonian effective is not constrained by renormalizability, it takes the form (4.5) where all the colour indices drop. Moreover, V in (4.5) is in the adjoint representation and it is easy to see that from $e^{-2gV} = 1 - 2gV + \ldots$

only the 1 can contribute. Putting all together, the low energy Wilsonian effective action for the massless modes is

$$S_{\text{eff}} = \frac{1}{16\pi} \operatorname{Im} \int d^4x \left[d^2\theta\, \mathcal{F}''(\Phi) W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\, \Phi^\dagger \mathcal{F}'(\Phi) \right] . \tag{4.6}$$

with prepotential $\mathcal{F}$ to be determined. We remark that in this chapter we have followed none of the techniques of the previous one to find this effective action. The holomorphicity constraint imposed by supersymmetry has been enough to determine (4.6) without any direct passage from the microscopic theory.

4.1.4 Metric on the moduli space

Expanding in component fields the effective action (4.6), the kinetic part becomes

$$S^{\text{kin}}_{\text{eff}} = \frac{1}{4\pi} \operatorname{Im} \int d^4x\, \mathcal{F}''(\phi) \left[|\partial\phi|^2 - i\psi\sigma^\mu \partial_\mu \bar{\psi} - \frac{1}{4} F_{\mu\nu}(F^{\mu\nu} - i\tilde{F}^{\mu\nu}) - i\lambda\sigma^\mu \partial_\mu \bar{\lambda} \right] .$$

These kinetic terms can be seen as a four dimensional sigma-model with metric $\operatorname{Im}\mathcal{F}''(\phi)$. This is the metric on the moduli space as well, that is

$$ds^2 = \operatorname{Im}\mathcal{F}''(a) da d\bar{a} = \operatorname{Im}\tau(a) da d\bar{a}\,, \tag{4.7}$$

where $\tau(a)$ is the effective complexified coupling constant. If we can find the metric on the moduli space, we have the function $\mathcal{F}$ and of course the form of the effective action we are looking for.

The description of the effective action in terms of the fields Φ, W and the function $\mathcal{F}$ is not appropriate on all of the moduli space as can be easily seen by the following argument. The metric (4.7) should be positive definite, translating into the condition $\operatorname{Im}\tau(a) > 0$, but this cannot be the case: since $\mathcal{F}(a)$ is holomorphic, $\operatorname{Im}\tau(a) = \operatorname{Im}\mathcal{F}''(a)$ is a harmonic function and as such cannot have a minimum. This implies that it cannot be positive everywhere unless it is a constant as in the classical case. This fact means that the coordinates a and $\bar{a}$ are appropriate only in a certain region of the moduli space. When a singular point is approached, we need a new set of coordinates where the singularity does not appear. This is possible only provided the singularity is a coordinate singularity and not an intrinsic singularity.

4.2 Determination of the prepotential

In this section we see how, putting together concepts of Quantum Field Theory and complex analysis, we can build the prepotential $\mathcal{F}$.

4.2.1 The asymptotic region

We start the determination of the prepotential from regions with $a \to \infty$. For large a the dominant contribution when computing the effective action comes from regions of large momenta ($p \sim a$) where the microscopic theory (4.1) is asymptotically free. Thus, as $a \to \infty$, the effective coupling constant goes to zero and the perturbative expansion is reliable. The superfields Φ and V, or equivalently Ψ, are appropriate to describe the theory and we can safely say that the massive fields are integrated out as heavy fields.

The tree level and one loop contribution to $\mathcal{F}$ can be derived from the following symmetry argument as in [16]. In the classical theory (4.3) there is an important R symmetry $U(1)_R$ acting as

$$\theta \to e^{i\alpha}\theta \qquad \tilde{\theta} \to e^{i\alpha}\tilde{\theta} \qquad \Psi(x,\theta,\tilde{\theta}) \to \Psi'(x,\theta,\tilde{\theta}) = e^{2i\alpha}\Psi(x, e^{-i\alpha}\theta, e^{-i\alpha}\tilde{\theta})\,.$$

Without seeing the details, we say that this symmetry is anomalous and is broken by both perturbative and non perturbative effects down to a discrete symmetry $\mathbb{Z}_8$ with $\alpha = 2\pi n/8 = \alpha_n$, $n \in \mathbb{Z}$. This implies that the perturbative part of the Wilsonian effective Lagrangian,

$$L_{\text{pert}}[\Psi] = \frac{1}{16\pi}\,\text{Im}\int d^2\theta d^2\tilde{\theta}\,\mathcal{F}_{\text{pert}}\big[\Psi(x,\theta,\tilde{\theta})\big]\,,$$

is not invariant under $U(1)_R$ but has the variation

$$\delta_\alpha L_{\text{pert}} = -\frac{8\alpha}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = -\frac{\alpha}{8\pi^2}\,\text{Im}\int d^2\theta d^2\tilde{\theta}\,\Psi^2\,.$$

So, under an infinitesimal $U(1)_R$ transformation, on one hand we have

$$L_{\text{pert}} \to L_{\text{pert}} + \delta_\alpha L_{\text{pert}} = \frac{1}{16\pi}\,\text{Im}\int d^2\theta d^2\tilde{\theta}\left[\mathcal{F}_{\text{pert}}(\Psi) - \frac{2\alpha}{\pi}\Psi^2\right] \tag{4.8}$$

On the other hand, $L_{\text{pert}}[\Psi]$ transforms finitely to

$$\begin{aligned} L^\alpha_{\text{pert}}[\Psi'] &= \frac{1}{16\pi}\,\text{Im}\int d^2\theta d^2\tilde{\theta}\,\mathcal{F}_{\text{pert}}\big[e^{2i\alpha}\Psi(x,e^{-i\alpha}\theta,e^{-i\alpha}\tilde{\theta})\big] \\ &= \frac{1}{16\pi}\,\text{Im}\int d^2\theta d^2\tilde{\theta}\,e^{-4i\alpha}\mathcal{F}_{\text{pert}}\big[e^{2i\alpha}\Psi(x,\theta,\tilde{\theta})\big]\,. \end{aligned}$$

For infinitesimal α this becomes

$$\begin{aligned} L^\alpha_{\text{pert}}[\Psi'] &= \frac{1}{16\pi}\,\text{Im}\int d^2\theta d^2\tilde{\theta}\left[\mathcal{F}_{\text{pert}}(\Psi) + 2i\alpha\big(-2\mathcal{F}_{\text{pert}} + \Psi\mathcal{F}'_{\text{pert}}(\Psi)\big)\right] \\ &= \frac{1}{16\pi}\,\text{Im}\int d^2\theta d^2\tilde{\theta}\left[\mathcal{F}_{\text{pert}}(\Psi) + 4i\alpha\big(-\mathcal{F}_{\text{pert}} + \Psi^2\partial_{\Psi^2}\mathcal{F}_{\text{pert}}(\Psi)\big)\right], \end{aligned}$$

which, compared with (4.8), gives an equation for Ψ:

$$\Psi^2\partial_{\Psi^2}\mathcal{F}_{\text{pert}} - \mathcal{F}_{\text{pert}} = \frac{i}{2\pi}\Psi^2\,.$$

The solution to this equation is given by

$$\mathcal{F}_{\text{pert}}(\Psi) = \frac{i}{2\pi}\Psi^2\log\frac{\Psi^2}{\Lambda^2}\,, \tag{4.9}$$

where Ψ now is to be understood as the analogue of φ_- in section 3.2 and Λ is an energy reference scale that can be interpreted as the cutoff Λ of the previous chapter. Due to non renormalization theorems for $N = 2$ susy this is the full perturbative result.

There are however non perturbative corrections such as the instantonic ones that become more important for smaller a. We will not spend time here to introduce in the proper way such an important chapter as the one concerning instantons. We just say that their contribution is constrained by symmetry and dimensional arguments to take the form

$$\mathcal{F}_{\text{inst}}(a) = a^2 \sum_{k=0}^{+\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} .$$

An introduction to the instantonic calculus can be found in [17].

Neglecting for the moment instantonic contributions, in the limit $a \to \infty$, a and $\bar{a}$ provide local coordinates on the moduli space and we know exactly the form of $\mathcal{F}$ from perturbation theory:

$$\mathcal{F}(a) \sim \frac{i}{2\pi} a^2 \log \frac{a^2}{\Lambda^2}$$
$$\tau(a) \sim \frac{i}{\pi}\Big(\log \frac{a^2}{\Lambda^2} + 3\Big) .$$

Also, we have $u \sim a^2/2$ in this limit.

4.2.2 Duality transformation

A different set of coordinates is provided by a Legendre transformation in Φ and $\mathcal{F}(\Phi)$. A field dual to Φ is defined by

$$\Phi_D = \mathcal{F}'(\Phi)$$

and a function $\mathcal{F}_D(\Phi_D)$ dual to $\mathcal{F}(\Phi)$ by

$$\mathcal{F}_D(\Phi_D) = \mathcal{F}(\Phi) - \Phi\Phi_D ,$$

or equivalently

$$\mathcal{F}'_D(\Phi_D) = -\Phi . \tag{4.10}$$

Under these transformation the second term in the action (4.6) is invariant, that is

$$\text{Im} \int d^4x d^2\theta d^2\bar{\theta}\, \Phi^\dagger \mathcal{F}'(\Phi) = \text{Im} \int d^4x d^2\theta d^2\bar{\theta}\, \Phi_D^\dagger \mathcal{F}'_D(\Phi_D) .$$

The first term needs more manipulations. Recall that W_α is not arbitrary but has the constraint $\text{Im}(D_\alpha W^\alpha) = 0$, the analogue in superspace of the Bianchi identities $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$. In the path integral, apart from normalization factors, the integration over the superfield V can be translated into an integration over W imposing the constraint $\text{Im}(D_\alpha W^\alpha) = 0$ by a Lagrange multiplier superfield V_D in the following way

$$\int \mathcal{D}V \, \exp\left[\frac{i}{16\pi}\,\text{Im}\int d^4x d^2\theta\, \mathcal{F}''(\Phi) W^\alpha W_\alpha\right] \sim$$
$$\sim \int \mathcal{D}W\mathcal{D}V_D \, \exp\left[\frac{i}{16\pi}\,\text{Im}\int d^4x \left(\int d^2\theta\, \mathcal{F}''(\Phi) W^\alpha W_\alpha + \frac{1}{2}\int d^2\theta d^2\bar{\theta}\, V_D D_\alpha W^\alpha\right)\right] .$$

With some algebra the second term in the exponential becomes

$$\int d^2\theta d^2\bar{\theta}\, V_D D_\alpha W^\alpha = -4 \int d^2\theta\, (W_D)_\alpha W^\alpha ,$$

where $(W_D)_\alpha = -\bar{D}^2 D_\alpha V_D/4$ is the analogue of the abelian version of (4.2). Shifting the integration variable W to complete the square, the integral in W is factorized out and we are left with

$$\int \mathcal{D}V_D \, \exp\left[\frac{i}{16\pi}\,\mathrm{Im}\int d^4x d^2\theta \left(-\frac{1}{\mathcal{F}''(\Phi)}W_D^\alpha W_{D\alpha}\right)\right].$$

Taking a Φ derivative of (4.11) we see that $-1/\mathcal{F}''(\Phi) = \mathcal{F}_D''(\Phi_D)$ and we recover the first term in the effective action (4.6) in terms of the dual fields. The duality transformations provide a description of the theory in terms of a new set of fields which is equivalent to the original one. In this sense the action (4.6) is equivalent to

$$\frac{1}{16\pi}\,\mathrm{Im}\int d^4x \left[\int d^2\theta\, \mathcal{F}_D''(\Phi_D) W_D^\alpha W_{D\alpha} + \int d^2\theta d^2\bar{\theta}\, \Phi_D^\dagger \mathcal{F}_D'(\Phi_D)\right].$$

The transformation

$$\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \mapsto \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \tag{4.11}$$

is not the only duality transformation of the effective action leading to an equivalent description of the theory. Indeed, write the action (4.6) as

$$S_{\text{eff}} = \frac{1}{16\pi}\,\mathrm{Im}\int d^4x d^2\theta\, \frac{d\Phi_D}{d\Phi} W^\alpha W_\alpha + \frac{1}{32\pi i}\int d^4x d^2\theta d^2\bar{\theta}\left(\Phi^\dagger \Phi_D - \Phi_D^\dagger \Phi\right).$$

From this form it is easy to see that another duality transformation is

$$\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \mapsto \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Phi_D\\ \Phi\end{pmatrix} \qquad\qquad b \in \mathbb{Z}\,. \tag{4.12}$$

because the second term is clearly invariant and the first term gets shifted by

$$\frac{b}{16\pi}\,\mathrm{Im}\int d^4x d^2\theta\, W^\alpha W_\alpha = -\frac{b}{16\pi}\int d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu} = -2\pi b\nu\,,$$

where $\nu \in \mathbb{Z}$ is the instanton number. Since the action appears as e^{iS} in the path integral, the generating functional does not change under (4.12). The transformations (4.11) and (4.12) together generate the group $SL(2,\mathbb{Z})$ of duality transformations.

The duality transformation (4.11) exchange electric and magnetic degrees of freedom. Thus, magnetic monopoles are exchanged with electrically charged states as the ones described by hypermultiplets in the $N = 2$ supersymmetric theory we are considering. In $N = 2$ susy theories there are two types of multiplets: short ones, with 4 helicity states, and long ones with 16 helicity states. Massless states must be in short multiplets. Massive states in short multiplets have mass $m^2 = 2|Z|^2$, Z being the central charge of the $N = 2$ susy algebra, while massive states in long multiplets have mass $m^2 > 2|Z|^2$. The states that become massive by the Higgs mechanism must be in short multiplets since the Higgs mechanism cannot generate the missing $16 - 4 = 12$ helicity states. For purely electrically charged states the central charge Z is an_e, where n_e is the (integer) electric charge. By duality we have that a purely magnetically charged state has $Z = a_D n_m$, where n_m is the (integer) magnetic charge. A state with both types of charge, called a dyon, has $Z = an_e + a_D n_m$ since the central charge is additive. In summary, for short multiplets we have the BPS mass formula

$$m^2 = 2|Z|^2 \qquad\qquad Z = (n_m, n_e)\begin{pmatrix}a_D\\ a\end{pmatrix}. \tag{4.13}$$

4.2.3 Singularities and monodromy

Now we have all the tools we need to extend our knowledge of the moduli space. We have already seen that in the limit $a \to \infty$ the relation between a and u is

$$u = \frac{1}{2}a^2 ,$$

implying that a is a multi-valued function of u. Moreover, from $a_D = \mathcal{F}'(a)$ we have

$$a_D = \frac{i}{\pi} a \left(\log \frac{a^2}{\Lambda^2} + 1 \right) \qquad\qquad a \to \infty .$$

Studying the monodromy properties of a and a_D as multi-valued functions of u we can extract a lot of information about a, a_D and hence $\mathcal{F}$.

We start with the monodromy at infinity. If we take u around a counterclockwise contour of very large radius in the complex u plane, schematically written as $u \to e^{2\pi i}u$, we have $a \to -a$ and

$$a_D \to \frac{i}{\pi}(-a) \left(\log \frac{e^{2\pi i} a^2}{\Lambda^2} + 1 \right) = -a_D + 2a .$$

The monodromy transformations can be compactly written by means of a monodromy matrix as in

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} , \qquad\qquad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \in SL(2, \mathbb{Z}) .$$

The monodromy matrices must be in $SL(2, \mathbb{Z})$ otherwise a monodromy transformation would change the underlying theory.

Since a branch cut has to start and end somewhere, there has to be at least one more singular point. Actually one is not enough and we need at least two more singularities. As a consequence of the $U(1)_R$ symmetry the moduli space is invariant under $u \to -u$. Therefore, if we have a singularity at $u = u_0$, we must have another singularity at $u = -u_0$. The only fixed points of $u \to -u$ are 0 and ∞. If there are only two singularities, they must be 0 and ∞. By contour deformation we see that the monodromy around 0 is the same as the monodromy around ∞. Then a^2 is not affected by any monodromy and hence is a good global coordinate. But we have already seen that this is not possible. Moreover, as we will see later, it seems that not even more than three singularities is a possible choice. In conclusion we need exactly three singular points, one at infinity and the other two at $u = \pm u_0$.

The singularities at $u = \pm u_0$ signal the presence of some massive degrees of freedom becoming massless. The description of the light fields in terms of the Wilsonian effective action (4.6), where the massive fields have been integrated out, cannot be valid anymore if a massive field is not really massive, inducing a singularity in the moduli space.

We can try to guess what the nature is of these degrees of freedom becoming massless. First we can think that the singularities are due to the gauge bosons becoming massless. But massless gauge bosons would imply an asymptotically conformally invariant theory in the infrared limit and conformal invariance implies $u = 0$. Thus we can exclude gauge bosons.

Next we can try with solitons, since there are no other elementary multiplets in the theory. Consider a magnetic monopole described by a $N = 2$ susy hypermultiplet

M. From the BPS mass formula (4.13) its mass is $m^2 = 2|a_D|^2$, vanishing at $a_D = 0$. Call u_0 the value of u at which a_D vanishes. The hypermultiplet M couples locally to the dual fields Φ_D and W_D, in the same way as electrically charged hypermultiplets would couple locally to Φ and W. Thus, near u_0 we have a local theory with massless fields Φ_D and W_D and light field M. This theory is exactly $N = 2$ susy QED with light electrons. We know the β function of this theory and is

$$\mu \frac{d}{d\mu} g_D = \frac{g_D^3}{8\pi} \,. \tag{4.14}$$

The scale μ is proportional to a_D and, since in QED the θ parameter is zero, we have

$$\tau_D = \frac{4\pi i}{g_D^2} \,.$$

Writing equation (4.14) in terms of a_D and τ_D we get

$$a_D \frac{d}{da_D} \tau_D = -\frac{i}{\pi} \quad \Longrightarrow \quad \tau_D = -\frac{i}{\pi} \log a_D \,.$$

Recalling $\tau_D = \mathcal{F}_D''(a_D)$ and $\mathcal{F}_D'(a_D) = -a$, we can integrate τ_D and get

$$a \approx a_0 + \frac{i}{\pi} a_D \log a_D \,,$$

where we dropped a subleading term $-ia_D/\pi$ since we are in proximity of $a_D = 0$. Near u_0, a_D should be a good coordinate with leading order $a_D \approx c_0(u - u_0)^k$. The only consistent choice turns out to be $k = 1$ and hence

$$\begin{aligned} a_D &\approx c_0(u - u_0) \\ a &\approx a_0 + \frac{i}{\pi} c_0(u - u_0) \log(u - u_0) \,. \end{aligned}$$

From these expressions the monodromy matrix is easily extracted:

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \,.$$

The monodromy of the last singularity, $-u_0$, is determined by observing that the contour around $u = \infty$ is equivalent to a contour encircling u_0 and a contour encircling $-u_0$, both counterclockwise. Then we have $M_\infty = M_{u_0} M_{-u_0}$ and hence

$$M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \,.$$

The state of vanishing mass responsible for a singularity should be invariant under the monodromy. In particular its mass and hence its charge should be invariant. In general a monodromy transformation M induces a transformation on the charges as in

$$Z = (n_m, n_e) \begin{pmatrix} a_D \\ a \end{pmatrix} \to (n_m, n_e) M \begin{pmatrix} a_D \\ a \end{pmatrix} = (n_m', n_e') \begin{pmatrix} a_D \\ a \end{pmatrix} \,.$$

Then invariance implies that the charge vector of a state with vanishing mass is a left eigenvector of M with unit eigenvalue. We can check that this is true for M_{u_0}: $(1, 0)$ is the charge vector of a magnetic monopole and is the left eigenvector of M_{u_0} with unit eigenvalue. On the other hand, the left eigenvector of M_{-u_0} with unit eigenvalue is $(1, -1)$, corresponding to a dyon. More generally, (n_m, n_e) is the left eigenvector with unit eigenvalue of

$$M_{n_m n_e} = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix} .$$

This implies that, apart from M_∞, a general monodromy matrix should be of the form M_{nm} with integers n and m.

Finally we can answer the question of how many singularities there are. Suppose there are p singularities at $u_1, \ldots, u_p$ in addition to the one at infinity. Then we have the factorization condition

$$M_\infty = M_{u_1} \cdots M_{u_p}$$

where M_{u_i} is of the form M_{nm}. For several low values of $p > 2$ it has been checked that there is no solution to this problem, and it seems likely that the same is true for all $p > 2$.

4.2.4 The prepotential from complex analysis

So far we have determined the behaviour of a and a_D as multi-valued functions of u in the limit $a \to \infty$ and we have found the monodromies of the singular points $u = \infty, \pm u_0$. The respective matrices are

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \qquad M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} .$$

The knowledge of these things is enough to reconstruct the whole functions a and a_D, and hence the prepotential $\mathcal{F}$. The precise location of u_0 depends on the renormalization conditions, which can be chosen such that $u_0 = 1$. If one wants to keep u_0, all one has to do is to replace $u \pm 1$ by $\frac{u}{u_0} \pm 1$.

Consider the differential equation in the complex plane

$$\left[-\frac{d^2}{dz^2} + V(z) \right] \psi(z) = 0 \tag{4.15}$$

with meromorphic potential $V(z)$, having poles at $z_1, \ldots, z_p$ and at ∞. There are two linearly independent solutions, $\psi_1(z)$ and $\psi_2(z)$. We require that $V(z)$ is single valued as z goes around any of the poles z_i. Since the differential equation does not change as z goes around a pole, the two solutions $\psi_1(z)$ and $\psi_2(z)$, when continued around a pole must be linear combinations of $\psi_1(z)$ and $\psi_2(z)$:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \big(z + e^{2\pi i}(z - z_i) \big) = M_i \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (z)$$

with a constant monodromy matrix M_i for each pole z_i. From the theory of differential equations it is known that this implies that the poles of V are at most of second order. In the case of three singularities at $z = \pm 1$ and $z = \infty$, the form of V is constrained to be

$$V(z) = -\frac{1}{4} \left(\frac{1 - \lambda_1^2}{(z+1)^2} + \frac{1 - \lambda_2^2}{(z-1)^2} - \frac{1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2}{(z+1)(z-1)} \right) .$$

The corresponding differential equation is well known in the mathematical literature since it can be transformed into the hypergeometric differential equation by

$$\psi(z) = (z+1)^{\frac{1}{2}(1-\lambda_1)}(z-1)^{\frac{1}{2}(1-\lambda_2)} f\left(\frac{z+1}{2}\right).$$

The hypergeometric differential equation is

$$x(1-x)f''(x) + \big[c - (a+b+1)x\big]f'(x) - abf(x) = 0\,,$$

with solutions

$$\begin{aligned} f_1(x) &= (-x)^{-a}F(a, a+1-c, a+1-b; 1/x) \\ f_2(x) &= (1-x)^{c-a-b}F(c-a, c-b, c+1-a-b; 1-x)\,, \end{aligned}$$

where F is the hypergeometric function.

From the asymptotic behaviours of a and a_D one can find

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 0 \implies V(z) = -\frac{1}{4}\frac{1}{(z+1)(z-1)}\,. \tag{4.16}$$

Transforming the differential equation (4.15) with this potential $V(z)$ into the hypergeometric equation gives the solutions in terms of the hypergeometric function. In conclusions the two solutions we are looking for are

$$\begin{aligned} a_D(u) &= i\frac{u-1}{2}F\left(\frac{1}{2}, \frac{1}{2}, 2; \frac{1-u}{2}\right) \\ a(u) &= \sqrt{2(u+1)}F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u+1}\right). \end{aligned}$$

We can use the integral representation of the hypergeometric function to write

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_1^u \frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,dx \tag{4.17}$$

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^1 \frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,dx\,. \tag{4.18}$$

Inverting the second equation, $u(a)$ can be obtained to be inserted in the first one in order to find $a_D(a) = \mathcal{F}'(a)$. Upon integration the desired result, $\mathcal{F}$, is found.

4.3 Instantonic contributions

Of course the analytical solution of $\mathcal{F}$ is not achievable and even using approximation methods in the scheme oulined in the end of last section, the work is quite long. However, with some tricks, as shown in [18] by Matone, it is possible to find a recursion relation for the instantonic contributions. What follows is clear in the framework of uniformization theory and we defer the reader to [18] for more details. First of all, since a and a_D satisfy the same equation, the Wronskian $\eta(u) = a(u)a_D'(u) - a_D(u)a'(u)$ is a constant, call it c. Then define

$$g(u) = \int_1^u \eta(z)\,dz = c(u-1) \tag{4.19}$$

and notice that

$$\partial_u \mathcal{F} = \frac{1}{2}\big[\partial_u(aa_D) - \eta(u)\big] = \frac{1}{2}\big[\partial_u(aa_D) - \partial_u g(u)\big]\,,$$

or, apart from an additive constant, $g(u) = aa_D - 2\mathcal{F}$. Define the function

$$\mathcal{G}(a) = i\pi\Big(\mathcal{F}(a) - \frac{1}{2}a\mathcal{F}'(a)\Big) = i\frac{\pi}{2}\Big(2\mathcal{F}(a) - aa_D\Big)\,. \tag{4.20}$$

By (4.19) we have $\mathcal{G} = Au + B$, with some constants A and B.

The asymptotic expression of $\mathcal{F}$ when including the instantonic contributions is

$$\mathcal{F}(a) = \frac{i}{2\pi}a^2 \log\frac{a^2}{\Lambda^2} + a^2\sum_{k=0}^{+\infty}\mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{4k}\,,$$

translating into

$$\mathcal{G}(a) = a^2\sum_{k=0}^{+\infty}\mathcal{G}_k\left(\frac{\Lambda}{a}\right)^{4k} \qquad \text{with} \qquad \mathcal{G}_0 = \frac{1}{2}, \quad \mathcal{G}_k = 2\pi k i\mathcal{F}_k\,. \tag{4.21}$$

In the limit $a \to \infty$, we have $\mathcal{G}(a) \sim a^2/2 \sim u$ and so we read $A = 1$.

Finally recall that a satisfies the differential equation (4.15) with V given in (4.16),

$$a'' + \frac{a}{4(u^2-1)} = 0\,.$$

We can change the integration variable from u to a by means of

$$\frac{\partial a}{\partial u} = \left(\frac{\partial u}{\partial a}\right)^{-1} \qquad \frac{\partial^2 a}{\partial u^2} = \left(\frac{\partial u}{\partial a}\right)^{-1}\frac{\partial}{\partial a}\left(\frac{\partial u}{\partial a}\right)^{-1}$$

to get an equation for u:

$$(1-u^2)u'' + \frac{1}{4}a(u')^3 = 0\,,$$

or for $\mathcal{G} = u + B$:

$$\big[1 - (\mathcal{G} - B)^2\big]\mathcal{G}'' + \frac{1}{4}a(\mathcal{G}')^3 = 0\,. \tag{4.22}$$

Plugging the expansion (4.21) into this equation we can see that $B = 0$. Moreover we find the recursion relation we were looking for:

$$\mathcal{G}_{n+1} = \frac{1}{8\mathcal{G}_0^2(n+1)^2}\left[(2n-1)(4n-1)\mathcal{G}_n + 2\mathcal{G}_0\sum_{k=0}^{n-1}\mathcal{G}_{n-k}\mathcal{G}_{k+1}C_{kn} - 2\sum_{j=0}^{n-1}\sum_{k=0}^{j+1}\mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k D_{jkn}\right] \tag{4.23}$$

where $n \geq 0$, and

$$C_{kn} = 2k(n-k-1) + n - 1$$
$$D_{jkn} = [2(n-j)-1][2n-3j-1+2k(j-k+1)]\,.$$

Following [19] we can find a considerably simplified version of the recursion relation (4.23). Start with the equation (4.22) for $\mathcal{G}$:

$$V^{-1}\mathcal{G}'' = a(\mathcal{G}')^3\,, \tag{4.24}$$

where $V^{-1} = 4(\mathcal{G}^2 - 1)$. We introduce a function $\mathcal{H}(a)$ such that $\mathcal{H}'(a) = \mathcal{G}(a)$ and define the auxiliary function f by

$$V^{-1} + f\mathcal{H}'' = 0\,. \tag{4.25}$$

In terms of f and $\mathcal{H}$, equation (4.24) becomes

$$f\mathcal{H}''' + a\mathcal{H}''^2 = 0\,. \tag{4.26}$$

Taking a derivative of (4.25) with respect to a and using (4.26) we get

$$f'\mathcal{H}'' - a\mathcal{H}''^2 + \partial_a V^{-1} = 0\,.$$

Next use the trick $\partial_a V^{-1} = \mathcal{G}'\partial_{\mathcal{G}} V^{-1} = 8\mathcal{H}''\mathcal{H}'$ to get

$$f' = a\mathcal{H}'' - 8\mathcal{H}' = \partial_a(a\mathcal{H}' - 9\mathcal{H})\,,$$

or, setting to zero the integration constant,

$$f = a\mathcal{H}' - 9\mathcal{H}\,.$$

Plugging this expression for f into (4.25) we find

$$1 - \mathcal{H}'^2 = \frac{1}{4}\mathcal{H}''(a\mathcal{H}' - 9\mathcal{H})\,. \tag{4.27}$$

By this equation and the expansion for $\mathcal{H}$,

$$\mathcal{H}(a) = \sum_{k=0}^{+\infty} \frac{\mathcal{G}_k}{3-4k} a^{3-4k}\,,$$

we find the recursion relation

$$\mathcal{G}_n = (4n-3)\sum_{k=1}^{n-1} \frac{g_{kn}}{(4k-3)[4(n-k)-3]}\mathcal{G}_k\mathcal{G}_{n-k}\,, \tag{4.28}$$

where $n \geq 2$ and

$$g_{kn} = \frac{k(n-k)(2n-15)}{n^2} + 3\,.$$

Equation (4.24) has the cube of the first derivative, while equation (4.27) has only the square at most. This is why equation (4.24) produces the trilinear recursion relation (4.23) while equation (4.27) produces the much simpler bilinear relation (4.28).

We conclude our work with a few remarks about the results in this last section. Since $\mathcal{G} = u$, by the defintion (4.20) of $\mathcal{G}$ we find the explicit relation between u and the prepotential $\mathcal{F}$:

$$u = i\pi\Big(\mathcal{F}(a) - \frac{1}{2}a\mathcal{F}'(a)\Big)\,. \tag{4.29}$$

This relation can be interpreted as the renormalization group equation because it can be rewritten as

$$\Lambda\partial_\Lambda\mathcal{F} = -8\pi i b_1 u\,,$$

where $b_1 = 1/(4\pi^2)$ is the one loop coefficient of the beta function. See [14] for more details.

In this context we essentially followed the work of Seiberg and Witten and we have seen that their results imply the recursion relation (4.23) and the relation (4.29). However, the results of Seiberg and Witten in the way we have seen them are not fully rigorously proved. In particular the existence of exactly three singularities in the moduli space has been assumed and we have only seen a justification for that.

As discovered by Matone, Bonelli and Tonin in their work [20], we can start from the other way around to prove this conjecture in the framework of uniformization theory. As stressed in [20] the relation (4.29) has been rigorously proved in the context of multi-instanton calculations up to two instanton contributions by Fucito and Travaglini [21], and to all orders by Dorey, Khoze and Mattis [22]. Furthermore, it has been derived in the framework of superconformal Ward identities by Howe and West [23]. Starting from this relation, together with the one loop formula (4.9) and CPT arguments, they derived two symmetries of the moduli space, namely

$$\begin{aligned} \bar{u}(\tau) &= u(-\bar{\tau}) \\ u(\tau - n) &= (-1)^n u(\tau)\,. \end{aligned}$$

These symmetries are the key points to determine the moduli space and its fundamental domain. It turns out that the assumptions of Seiberg and Witten are right and that the moduli space is the Riemann sphere with punctures at $u = \pm u_0$ and $u = \infty$.

Finally, it can be shown that behind these results there is a non unique geometrical structure. On one hand, in their original paper Seiberg and Witten found the explicit form (4.18) for $a(u)$ and $a_D(u)$ as the two independent integrals of a suitable differential form on a torus. In this approach

$$\tau(u) = \frac{da_D/du}{da/du}$$

can be seen as the τ parameter describing the complex structure of the torus, and as such is garanteed to satisfy $\operatorname{Im}\tau(u) > 0$. On the other hand, Matone and others used the uniformization of the sphere with three punctures to get their results. In this context $\tau(u)$ is the inverse uniformizing map from the sphere with three punctures to the upper half complex plane endowed with the Poincaré metric. The uniformization of negatively curved Riemann surfaces is a rich mathematical topic that seems to be of interest in other fields of physics as well, among which (super)string theory.

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