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UNIVERSITÀ
DEGLI STUDI
DI PADOVA

Università degli studi di Padova
Dipartimento di Matematica "Tullio Levi-Civita"
Corso di Laurea Magistrale in Matematica

ALGANT Master Thesis

Galois categories and étale covers

presented by
Riccardo Carmine GILBLAS
1206393

advisor
Prof. **Dajano TOSSICI**

referent
Prof. **Marco GARUTI**

Academic year 2019-2020

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Introduction

The definition of the fundamental group related to a real manifold arose in the end of XIX century, in a paper of Henri Poincaré named *Analysis Situs*; in this paper, it was defined as the group of lateral classes of the loops. Nevertheless, already in the opus of Bernhard Riemann the idea of the fundamental group was present, throughout the study of the covers of Riemann surfaces, hence in a complex context. Later, Shreeram Shankar Abhyankar extended the notion of the fundamental group for varieties over a generic algebraically closed field: he defined the fundamental group as the inverse system of the Galois unramified extensions of the function field of the variety.

This construction motivated Alexander Grothendieck to define the fundamental group in an algebraic context: given an algebraic variety over a field, its étale fundamental group is given by the inverse limit of the Galois groups of the étale covers of the variety. Beyond the étale case, which remains nevertheless the central one, Grothendieck developed the concept of Galois category, with the aim of considering the most general construction of a category having associated a fundamental group.

Furthermore, Grothendieck linked the category of étale covers of an algebraic variety over $\mathbb{C}$ to the category of topological covers of the underlying topological space of the variety, with results in the so-called G.A.G.A. context (Algebraic geometry and Analytic geometry). In particular, the étale fundamental group of an algebraic variety over $\mathbb{C}$ turns out to be nothing but the profinite completion of the topological fundamental group.

In this writing, we study the properties of a Galois category and we show the construction of the fundamental group associated to it. Then, we prove that the étale covers of a connected scheme form in fact a Galois category and the fundamental group associated will be the étale fundamental group of the scheme. We present some results on étale fundamental groups in various settings and we end by recalling the link with the topological fundamental group that G.A.G.A. results give rise to.

Notation

Categories

Definition. A *category* $\mathcal{C}$ consists in the following data: a class of object of $\mathcal{C}$, denoted by $\mathrm{Obj}(\mathcal{C})$; for every pair of objects X, Y in $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphism between X and Y; for every X, Y, Z objects in $\mathcal{C}$, the composition law:

$$\begin{aligned} \circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

which is associative and such that for every object X there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X)$ called the identity such that for every Y, Z objects and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$ one has:

$$f \circ \mathrm{id}_X = f \quad \mathrm{id}_X \circ g = g.$$

We will usually write $X \in \mathcal{C}$ to mean $X \in \mathrm{Obj}(\mathcal{C})$.

Definition. Let $\mathcal{C}$ be a category. An object $C \in \mathcal{C}$ is said to be:

- *initial* if for every $X \in \mathcal{C}$ there exists a unique morphism $C \to X$;
- *terminal* if for every $X \in \mathcal{C}$ there exists a unique morphism $X \to C$;
- *zero object* if it is both initial and terminal.

Definition. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an *isomorphism* if there exists $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $f \circ g = \mathrm{id}_X$ and $g \circ f = \mathrm{id}_Y$. In this case we say that X is isomorphic to Y and we write $X \simeq Y$.

We will denote by $\mathrm{Isom}_{\mathcal{C}}(X, Y)$ the set of isomorphisms from X to Y and by $\mathrm{Aut}_{\mathcal{C}}(X)$ the set $\mathrm{Isom}_{\mathcal{C}}(X, X)$.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *covariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a law $F : \mathcal{C} \to \mathcal{D}$ that:

- to every $X \in \mathcal{C}$ assigns an object $F(X) \in \mathcal{D}$;
- to every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ assigns a morphism

$$F(f) \in \mathrm{Hom}_{\mathcal{C}}(F(X),F(Y))$$

  such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and if $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ one has $F(g \circ f) = F(g) \circ F(f)$.

We will say *functor* to mean covariant functor.

Definition. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* betweem F and G is a family $\eta = \{\, \eta_X \,\}_{X \in \mathcal{C}}$ of morphisms $\eta_X : F(X) \to G(X)$ such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

The natural transformation η is said *natural isomorphism* if for every $X \in \mathcal{C}$ the morphism η_X is an isomorphism; in this case we will write $F \approx G$.

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence* (of categories) if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \approx \mathrm{id}_{\mathcal{C}}$ and $F \circ G \approx \mathrm{id}_{\mathcal{D}}$.

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be:

- *essentially surjective* if for every $Y \in \mathcal{D}$ there exists $X \in \mathcal{C}$ such that $F(X) \simeq Y$;
- *fully faithful* if for every $X, Y \in \mathcal{C}$ the map

$$\begin{aligned} \Phi_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) &\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \\ f &\longmapsto F(f) \end{aligned}$$

  is bijective.

Proposition 0.0.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

We denote by SETS the category of sets (the morphisms are maps between sets), by FSETS the full sub-category of finite sets, by TOP the category of topological spaces (morphisms are continuous maps).

Algebraic geometry

Definition. Let X be a topological space and consider the category $\textsc{Top}(X)$ (the partially ordered set of open subsets of X). A *presheaf* of abelian groups (resp. of rings) $\mathcal{F}$ on X is a contravariant functor from $\textsc{Top}(X)$ to the category of abelian groups (resp. of rings). A morphism between two presheaves $\mathcal{F}, \mathcal{G}$ on X is a natural transformation between the two functors.

If $V \subset U \subset X$ and $s \in \mathcal{F}(U)$, we write $s_{|V}$ to mean the image of s under the map $\mathcal{F}(U) \to \mathcal{F}(V)$.

Definition. Let X be a topological space and $\mathcal{F}$ a presheaf on X. Given $x \in X$, the *stalk* of $\mathcal{F}$ at x is given by:

$$\mathcal{F}_x := \varinjlim_{x \in U \in \textsc{Top}(X)} \mathcal{F}(U).$$

Definition. Let X be a topological space; a *sheaf* of abelian groups (resp. of rings) $\mathcal{F}$ on X is a presheaf of abelian groups (resp. of rings) such that for any open covering $\{ U_i \}_{i \in I}$ of an open subset $U \subset X$ and for any family $\{ s_i \}_{i \in I}$ with $s_i \in \mathcal{F}(U_i)$ such that $s_{i|U_i \cap U_j} = s_{j|U_j \cap U_i}$ for any $i, j \in I$, then there exists a unique $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$ for any $i \in I$.

Definition. A *ringed topological space* X consists of a topological space $|X|$ (sometimes we will denote the topological space again by X if there is no possible confusion) together with a sheaf of rings $\mathcal{O}_X$ on X. We denote it by $(|X|, \mathcal{O}_X)$. A morphism $f : X \to Y$ between two ringed topological spaces $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ is given by a continuous map $|f| : |X| \to |Y|$ together with a morphism of sheaves $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$.
We say that a ringed topological space $(|X|, \mathcal{O}_X)$ is a *locally ringed topological space* if for every $x \in |X|$ the stalk $\mathcal{O}_{X,x}$ is a local ring.

Definition. An affine scheme is a locally ringed topological space $(|X|, \mathcal{O}_X)$ where $|X| = \mathrm{Spec} A$ with A a ring and the sheaf $\mathcal{O}_X$ is generated by the following definition on a basis: $\mathcal{O}_{\mathrm{Spec} A}(D(f)) := A_f$.

Definition. The category $\textsc{Sch}$ of schemes is the full subcategory of the category of locally ringed space whose objects are the locally ringed space $X = (|X|, \mathcal{O}_X)$ such that every point $x \in |X|$ has an open neighbourhood U such that $(U, \mathcal{O}_{X|U})$ is an affine scheme.

Definition. A morphism $f : X \to Y$ of schemes is said:

- *affine* if for every $V \subset Y$ open affine subset, $f^{-1}(V)$ is an affine open subset of X;

- *open immersion* if it induces an isomorphism with an open subscheme of Y;
- *closed immersion* if $|f|$ induces an homeomorphism between $|X|$ and a closed subset of $|Y|$ and $f^\sharp$ is surjective;
- *separated* if the diagonal morphism $\Delta_f : X \to X \times_{f,Y,f} X$ is a closed immersion;
- *universally closed* if for any morphism $Y' \to Y$ the base change morphism $X \times_Y Y' \to Y'$ is closed;
- *locally of finite type* if for every affine open subset $V \subset Y$ and for every affine open subset $U \subset f^{-1}V$ the morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ a finitely generated $\mathcal{O}_Y(V)$-algebra;
- *of finite type* if it is locally of finite type and quasi-compact;
- *proper* if it is of finite type, separated and universally closed.

Acknowledgements

Firstly, I would like to thank my advisor, professor Dajano Tossici, for his infinite willingness and precious kindness without whom I wound not have managed to work with the tranquillity and the serenity I had in the past months.

I thank my university friends in Bordeaux Bianca Gouthier, Francesco Stocco, Greta Narder, Rosa Marchesini and Simone Marini for the possibility of sharing together the mathematical problems we met and the soothing breaks from the study.

A special thank to Francesco Zerman, for being an inestimable study partner and friend.

Finally, I would like to thank my parents and my girlfriend Giulia for comforting me in bad moments and supporting me in every situation.

Chapter 1

Galois categories

In this chapter we develop one of the main tools of this production: the machinery of Galois categories. In the first section we give a quick recall on the fundamental topological group, an enlightening starting point to understand the construction of the Galois fundamental group; in the second section, we present the definition of a Galois category and some results on objects and morphisms of this category; the entire third section is dedicated to the proof of the Main Theorem, which tells that a Galois category is in fact equivalent to the category of finite sets endowed with the action of a suitable group, the Galois fundamental group.

1.1 The fundamental group of a topological space

In this section we briefly recall the construction of the fundamental group of a topological space; even though maybe it is already well known and this part could be tedious for someone, it will give a practical idea of the more general construction that we are developping for Galois categories. From our point of view, the reader may find usefull the short recovery we report here. The setting in this section is the category of topological spaces, Top.

Covers and group actions

Definition. Consider $X \in \text{Top}$. The category $\text{SpOv}(X)$ of spaces over X is a category whose objects are the couple (Y, p) where Y is a topological space and $p : Y \to X$ is a continuous map; a morphism between two objects $(Y_1, p_1), (Y_2, p_2) \in \text{SpOv}(X)$ is a continuous map $f : Y_1 \to Y_2$ such that $p_2 \circ f = p_1$.

Definition. The category $\text{Cov}(X)$ of covers of X is the full sub-category of $\text{SpOv}(X)$ whose objects (Y, p) satisfy the following condition: each point

$x \in X$ has an open neighbourhood V for which $p^{-1}(V)$ is a disjoint union of open subsets $U_i \subset Y$ such that $p_{|U_i} : U_i \to V$ is a homeomorphism.

It is possible to verify that if X is connected, the fibres of p are all homeomorphic to the same discrete space I. We define what is an automorphism of a cover:

Definition. Given $(Y, p) \in \textsc{Cov}(X)$, we define the group $\operatorname{Aut}(Y|X)$ of automorphism of (Y, p) as the group of automorphism of Y as topological space, i.e. homeomorphism $\phi : Y \to Y$, such that $p = p \circ \phi$.

We proceed by looking at actions of groups on topological spaces.

Definition. Let G a group acting continuously from the left on a topological space Y. We say that the action of G is *even* (sometimes one may find "properly discontinuous" instead) if each point $y \in Y$ has some open neighbourhood U such that the open sets gU are pairwise disjoint for all $g \in G$.

If G is a group acting on a topological space Y, one may consider the quotient space $G \setminus Y$ whose elements are the orbits under the action of G and the topology is the quotient one (the finest one making the projection $\pi : Y \to G \setminus Y$ continuous). We recall that the quotient space has the following universal property: if $f : Y \to Z$ is a continuous map which is G-invariant, there exists a unique map $\bar{f} : G \setminus Y \to Z$ such that $f = \bar{f} \circ \pi$.

Example. If $p : Y \to X$ is a cover, we may consider the action $\operatorname{Aut}(Y|X) \curvearrowright Y$ and look at the quotient space $\operatorname{Aut}(Y|X) \setminus Y$; notice that in this case, since $p \circ \phi = p$ for all $\phi \in \operatorname{Aut}(Y|X)$, for every $x \in X$ we may think that $\operatorname{Aut}(Y|X)$ acts on the fibre $p^{-1}(x)$.

Now we link the notion of an even action of a group on a topological space with the notion of a cover:

Lemma 1.1.1. *Given a group G acting evenly on a connected space Y, the projection $p : Y \to G \setminus Y$ makes (Y, p) a cover of $G \setminus Y$.*

Using the notions we introduced above, it is possible to prove the following statement:

Proposition 1.1.2. *If $(Y, p) \in \textsc{Cov}(X)$ is a connected cover of a locally connected topological space X, the action of $Aut(Y|X)$ on Y is even. Conversely, if G is a group acting evenly on a connected space Y, the automorphism group of the cover $p_G : Y \to G \setminus Y$ is precisely G.*

Notice now that if (Y, p) is a connected cover of X, we may consider the group $\mathrm{Aut}(Y|X)$ acting on Y and look at the quotient of this action $\mathrm{Aut}(Y|X)\backslash Y$. The cover map $p : Y \to X$ is $\mathrm{Aut}(Y|X)$-invariant by definition of $\mathrm{Aut}(Y|X)$, hence using the universal property of the quotient $\pi : Y \to \mathrm{Aut}(Y|X) \setminus Y$ we find a continuous map $\bar{p} : \mathrm{Aut}(Y|X) \setminus Y \to X$ making the following diagram commutative:

$$\begin{tikzcd}
Y \arrow[r, "\pi"] \arrow[dr, "p"] & \mathrm{Aut}(Y|X) \setminus Y \arrow[d, "\bar{p}"] \\
& X
\end{tikzcd}$$

Definition. $(Y, p) \in \textsc{Cov}(X)$ is *Galois* if Y is connected and the induced map $\bar{p}$ is an homeomorphism.

An important characterization for Galois covers is the following:

Proposition 1.1.3. *A connected cover $(Y, p) \in \textsc{Cov}(X)$ is Galois if and only if Aut$(Y|X)$ acts transitively on each fibre of p.*

The fundamental group

Definition. Let X a topological space;

- a *path* in X is a continuous map $f : [0, 1] \to X$;
- a *loop* in X is a path f such that $f(0) = f(1)$.;
- two path f, g in X are said *homotopic* if $f(0) = g(0)$, $f(1) = g(1)$ and there exists a continuous map $h : [0, 1] \times [0, 1] \to X$ with $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$. It is possible to verify that the relation of being homotopic is an equivalence, we write it $f \sim g$ and we indicate with $[f]$ the homotopy equivalence class of the path f.

If f, g are two paths in X with $f(1) = g(0)$, we may consider the path obtained by composing them:

$$f * g : [0, 1] \to X$$

defined by: $f * g(x) = g(2x)$ for $0 \leq x \leq 1/2$, $f * g(x) = f(2x - 1)$ for $1/2 \leq x \leq 1$. It is easy to see that this operation passes to a well defined operation on the quotient by homotopy equivalence: $[f]*[g] := [f*g]$. In fact, if we fix a point $x \in X$, the set of homotopy classes of loops starting (and ending) in x equipped with this operation is a group, which we will denote by $\pi_1(X, x)$: the identity is the class of the constant path $i(s) = x$ for all $s \in [0, 1]$ and if f is a generic loop, we have $[f]^{-1} := [\tilde{f}]$ where $\tilde{f}(s) = f(1-s)$ for all $s \in [0, 1]$. $\pi_1(X, x)$ is the fundamental group of X with base point x. The following technical lemma holds; it allows us to lift up a path through a cover map:

Lemma 1.1.4. *Let* $(Y, p) \in \textsc{Cov}(X)$*,* $y \in Y$ *with* $x = p(y)$*.*

1. *Given a path* $f : [0, 1] \to X$ *with* $f(0) = x$*, there is a unique path* $\tilde{f} : [0, 1] \to Y$ *with* $\tilde{f}(0) = y$ *and* $p \circ \tilde{f} = f$*.*
2. *Assume moreover given a second path* $g : [0, 1] \to X$ *with* $f \sim g$*. Then the unique lifting* $\tilde{g}$ *of the previous point is such that* $\tilde{f} \sim \tilde{g}$*.*

Now we are ready to state the theorem that we should keep in mind for the rest of the chapter, as it gives the idea of what we are going to construct with Galois category in a more general setting than covers of topological spaces. Let us fix a connected and locally simply connected topological space X and a point $x \in X$. We consider the category of sets with left $\pi_1(X, x)$-action, $\pi_1(X, x)$-$\textsc{Sets}$, and we define the functor

$$\begin{aligned} \mathrm{Fib}_x : \textsc{Cov}(X) &\longrightarrow \pi_1(X, x)\text{-}\textsc{Sets} \\ (Y, p) &\longmapsto p^{-1}(x) \\ (f : (Y, p) \to (Z, q)) &\longmapsto f_{|p^{-1}(x)} : p^{-1}(x) \to q^{-1}(x). \end{aligned}$$

It is a well defined functor since f respects the fibre by definition.

Theorem 1.1.5. *In the setting above, the functor* Fib_x *induces an equivalence of categories. In particular, connected covers correspond to* $\pi_1(X, x)$*-sets with transitive action and Galois covers to coset spaces of normal subgroups of* $\pi_1(X, x)$*.*

1.2 Definitions and first properties

Definition. Let $\mathcal{C}$ be a category; a morphism $u \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a *strict epimorphism* if there exists the fibre product in $\mathcal{C}$

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "u"] \\ X \arrow[r, "u"'] & Y \end{tikzcd}$$

and for any object $Z \in \mathcal{C}$ the map

$$\begin{aligned} \cdot \circ u : \mathrm{Hom}_{\mathcal{C}}(Y, Z) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z) \\ f &\longmapsto f \circ u \end{aligned}$$

is injective with image

$$\{\, h \in \mathrm{Hom}_{\mathcal{C}}(X, Z) \mid h \circ p_1 = h \circ p_2 \,\}\,.$$

Example. (Strict epimorphism in FSETS*)* As an easy example, we show that in the category FSETS being a strict epimorphism is equivalent to being surjective. Let us consider $X, Y \in$ FSETS and $u \in \mathrm{Hom}_{\mathrm{FSETS}}(X, Y)$;

Suppose u surjective, we prove that u is strict epimorphism. Firstly we notice that the fibre product always exists in FSETS:

$$X \times_Y X = \{\, (x, y) \in X \times X \mid u(x) = u(y) \,\}\,.$$

For every $Z \in$ FSETS the map $\cdot \circ u : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is clearly injective: if $f, g \in \mathrm{Hom}(Y, Z)$ with $f \circ u = g \circ u$, we have $f = g$ since u is surjective (i.e. an epimorphism in FSETS). We consider a map $h : X \to Z$ such that $h \circ p_1 = h \circ p_2$, where $p_i : X \times_Y X \to X$ are the projections given by the fibre product; we want to find $f : Y \to Z$ such that $f \circ u = h$. We define the map f in the natural way: $\forall y \in Y$

$$f(y) := h(u^{-1}(y)).$$

It remains just to verify that this is a proper definition: if $y = u(a) = u(b)$, then $(a, b) \in X \times_Y X$ and $h \circ p_1(a, b) = h(a) = h \circ p_2(a, b) = h(b)$, so $f(y) = h(a) = h(b)$ is well defined and $f \circ u = h$.

The converse is easy: if u strict epimorphism, then $\cdot \circ u$ is injective and this precisely means that u is epimorphism in FSETS so a surjective map.

Example. A more interesting example is the following: consider a topological space $X \in$ TOP and $U_1, U_2 \subset X$ open subsets such that $X = U_1 \cup U_2$. We have the map $u : U_1 \sqcup U_2 \to X$ induced by the inclusion; we prove that this map is a strict epimorphism in the category TOP. We have to show firstly that there exists the fibre product $(U_1 \sqcup U_2) \times_X (U_1 \sqcup U_2)$. As in the case above, we have:

$$(U_1 \sqcup U_2) \times_X (U_1 \sqcup U_2) = \{\, (x, y) \in (U_1 \sqcup U_2)^2 \mid u(x) = u(y) \,\}$$

and $u(x) = x$, $u(y) = y$. Hence we have

$$(U_1 \sqcup U_2) \times_X (U_1 \sqcup U_2) = \Delta_{(U_1 \sqcup U_2) \times (U_1 \sqcup U_2)}.$$

We have that for any topological space $Z \in \textsc{Top}$, $\cdot \circ u : \mathrm{Hom}(X,Z) \to \mathrm{Hom}(U_1 \sqcup U_2, Z)$ is injective: indeed u is surjective (hence an epimorphism in $\textsc{Top}$) as we are supposing $X = U_1 \cup U_2$.

Now consider $f \in \mathrm{Hom}(U_1 \sqcup U_2, Z)$ such that $f \circ p_1 = f \circ p_2$, where p_i is the projection on the i-th component of $(U_1 \sqcup U_2) \times_X (U_1 \sqcup U_2)$. f is defined on $U_1 \sqcup U_2$, hence it is given by $f_1 := f \circ e_1 :\in \mathrm{Hom}(U_1, Z)$ and $f_2 := f \circ e_2 \in \mathrm{Hom}(U_2, Z)$, where $e_i : U_i \to U_1 \sqcup U_2$ is the natural inclusion. Notice that f_1 and f_2 are compatible on $U_1 \cap U_2$: indeed if $x \in U_1 \cap U_2$ and we call $x_i = x$ the copy of x in U_i, we have:

$$f_1(x_1) = f \circ e_1(x_1) = f \circ p_1(x_1, x_2) = f \circ p_2(x_1, x_2) = f \circ e_2(x_2) = f_2(x_2).$$

But then f_1, f_2 are continuous maps and they coincide on the open subset $U_1 \cap U_2$, hence we can glue them and we find a continuous map $g : U_1 \cup U_2 = X \to Z$ such that $g_{|U_i} = f_i$; thus $f = g \circ u$.

Before being able to define what a Galois category is, we need the notion of categorical quotient by a group; maybe it is already well known but we recall it anyway.

Definition. Let $X \in \mathcal{C}$ and G a subgroup of the group $\mathrm{Aut}_{\mathcal{C}}(X)$. The *quotient* of X by G is an object $X/G \in C$ with a morphism $\pi : X \to X/G$ such that:

- $\pi \circ \sigma = \pi$ for every $\sigma \in G$;
- for every morphism $f : X \to Y$ such that $f \circ \sigma = f$ for all $\sigma \in G$, there exists a unique morphism $g : X/G \to Y$ such that $f = g \circ \pi$.

Now we can proceed in giving the definition of a Galois category:

Definition. A category $\mathcal{C}$ is a *Galois category* if there exists a covariant functor $F : \mathcal{C} \to \textsc{FSets}$, which we will call *fibre functor*, with the following properties:

(i) $\mathcal{C}$ has final object $e_{\mathcal{C}}$ and finite fibre products exist in $\mathcal{C}$;

(ii) finite coproducts exist in $\mathcal{C}$ and categorical quotients by finite groups of automorphisms exist in $\mathcal{C}$;

(iii) any morphism $u : X \to Y$ in $\mathcal{C}$ factors as $X \overset{u'}{\to} Y' \overset{u''}{\to} Y$ with u' a strict epimorphism and u'' a monomorphism that is an isomorphism on a direct summand of Y, i.e. we can write $Y \simeq Y' \sqcup Y''$ with u'' isomorphic to the immersion of the first component;

(iv) F sends final objects to final objects and commutes with fibre products;

(v) F commutes with coproducts and quotients by finite groups of automorphisms and sends strict epimorphisms to strict epimorphisms;

(vi) if $u \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, u is an isomorphism if and only if $F(u)$ is an isomorphism.

Remark. We will always suppose that a Galois category is essentially small. The definitions could be given without this assumption but it will become important later, when we will want to consider direct limits indexed in our category. However, this assumption is not so restrictive as every Galois category we are interested in is essentially small.

In a Galois category there exists automatically an initial object: indeed the category has coproducts and it is well known that the coproduct over the empty set is an initial object. We will denote it by $\emptyset_{\mathcal{C}}$.

We start to investigate the first properties of a Galois category: in the rest of this section, we describe the behaviour of monomorphism and epimorphism in a Galois category and we discover the usefulness of the notion of strict epimorphism in describing isomorphism. Indeed, as it is well known, in a general category a morphism that is both mono and epi is not necessarily an isomorphism: we find that this becomes true in a Galois category if we substitute *epimorphism* with *strict epimorphism.*

Lemma 1.2.1. *Let $\mathcal{C}$ a category admitting finite fibre products and let $u \in$ $Hom_{\mathcal{C}}(X, Y)$; the following are equivalent:*

(i) u is a monomorphism;

(ii) the first projection $p_1 : X \times_Y X \to X$ is an isomorphism;

(iii) the diagonal morphism $\Delta : X \to X \times_Y X$ is an isomorphism;

(iv) the two projections p_1 and p_2 coincide.

Proof. $(i) \Rightarrow (ii)$ We suppose u monomorphism and we prove that p_1 is an isomorphism. Consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "u"] \\
& X \arrow[r, "u"'] & Y
\end{tikzcd}$$

From $u \circ p_1 = u \circ p_2$ and using the fact that u is mono, we have $p_1 = p_2$. Now we prove that p_1 is an isomorphisms by showing that its inverse is Δ.

By the definition of Δ we have $p_1 \circ \Delta = id_X$; it remains to show $\Delta \circ p_1 = id_{X \times_Y X}$. We consider the following diagram:

$$\begin{tikzcd}
X \times_Y X \arrow[dr, "\Delta\circ p_1"] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "u"] \\
& X \arrow[r, "u"'] & Y
\end{tikzcd}$$

where we have:

- $p_1 \circ \Delta \circ p_1 = id_X \circ p_1 = p_1$;
- $p_2 \circ \Delta \circ p_1 = id_X \circ p_1 = p_1 = p_2$.

But there exists also the identity map $id_{X \times_Y X}$ satisfying the same condition, hence by the unicity in the universal property of the pull-back we conclude $\Delta \circ p_1 = id_{X \times_Y X}$ as wanted.

(ii) $\Rightarrow$ *(iii)* If p_1 is an isomorphism, it is immediate that Δ is its inverse: by definition of diagonal morphism, $p_1 \circ \Delta = \mathrm{id}_X$ and so if h is the inverse of p_1 we get:

$$h = h \circ \mathrm{id}_X = h \circ p_1 \circ \Delta = \mathrm{id}_{X \times_Y X} \circ \Delta = \Delta.$$

(iii) $\Rightarrow$ *(iv)* If Δ is an isomorphism, as in the previous points p_1 is its inverse. But then:

$$p_2 = p_2 \circ \mathrm{id}_{X \times_Y X} = p_2 \circ \Delta \circ p_1 = \mathrm{id}_X \circ p_1 = p_1.$$

(iv) $\Rightarrow$ *(i)* Assume $p_1 = p_2$ and consider $f, g \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$ with $u \circ f = u \circ g$; we want to prove that $f = g$. We have then the following commutative diagram:

$$\begin{tikzcd}
W \arrow[dr, "{(f,g)}"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "u"] \\
& X \arrow[r, "u"'] & Y
\end{tikzcd}$$

where (f, g) is the map given by the universal property of the pull-back applied to f and g. But since $p_1 = p_2$ one obtain:

$$f = p_1 \circ (f, g) = p_2 \circ (f, g) = g.$$

□

Lemma 1.2.2. *Let $\mathcal{C}$ be a Galois category and F be a fibre functor; consider $u \in Hom_{\mathcal{C}}(X, Y)$. Then u is a monomorphism if and only if $F(u)$ is a monomorphism.*

Proof. By Lemma 1.2.1, u is a monomorphism if and only if $p_1 : X \times_Y X \to X$ is an isomorphism; furthermore, from axiom (vi) p_1 is an isomorphism if and only if $F(p_1)$ is an isomorphism and from axiom (iv) F commutes with finite fibre product. Hence we have $F(X \times_Y X) = F(X) \times_{F(Y)} F(X)$ with the following pull-back diagram:

$$\begin{tikzcd}
F(X) \times_{F(Y)} F(X) \arrow[r, "F(p_2)"] \arrow[d, "F(p_1)"] & F(X) \arrow[d, "F(u)"] \\
F(X) \arrow[r, "F(u)"'] & F(Y)
\end{tikzcd}$$

and now again by Lemma 1.2.1, $F(p_1)$ is an isomorphism if and only if $F(u)$ is a monomorphism. □

Lemma 1.2.3. *Let $\mathcal{C}$ a category admitting finite fibre products and let $u \in Hom_{\mathcal{C}}(X, Y)$. If u is a monomorphism and a strict epimorphism then it is an isomorphism.*

Proof. Since u is a strict epimorphism, the map

$$\cdot \circ u : \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(X, X)$$

is injective with image the set of the morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that $f \circ p_1 = f \circ p_2$ where $p_i : X \times_Y X \to X$ is the projection on the i-th component. Since u is mono, one gets $p_2 = p_1$ by Lemma 1.2.1 and thus the image of the map $\cdot \circ u$ is in fact $\mathrm{Hom}_{\mathcal{C}}(X, X)$; so we find $h \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $h \circ u = id_X$. Finally, we have:

$$u \circ h \circ u = u \circ id_X = id_Y \circ u$$

and since u is an epimorphism we have $u \circ h = id_Y$. □

Lemma 1.2.4. *Let $\mathcal{C}$ be a Galois category and F a fibre functor; consider $u \in Hom_{\mathcal{C}}(X,Y)$ a morphism. Then $F(u)$ is an epimorphism if and only if u is a strict epimorphism.*

Proof. ($\Longleftarrow$) If u is a strict epimorphism, then $F(u)$ is a strict epimorphism by axiom (v), hence in particular $F(u)$ is an epimorphism.

($\Longrightarrow$) From axiom (iii), the morphism u factors as

$$X \xrightarrow{u'} Y' \xrightarrow{u''} Y = Y' \sqcup Y''$$

with u' a strict epimorphism and u'' a monomorphism. If $F(u)$ is an epimorphism, also $F(u'')$ is an epimorphism and from Lemma 1.2.2 $F(u'')$ is also a monomorphism since u'' is a monomorphism. But then $F(u'')$ is an isomorphism (remember that we are in the category FSETS) and by axiom (vi) we have that u'' is an isomorphism. Then u is a strict epimorphism. ☐

To finish this section, we show how a fibre functor behaves with respect to initial and final objects and we prove that a Galois category is artinian.

Lemma 1.2.5. *Let $\mathcal{C}$ be a Galois category and F a fibre functor. For any $X_0 \in \mathcal{C}$, we have that $F(X_0) = \emptyset$ if and only if $X_0 = \emptyset_{\mathcal{C}}$ and $F(X_0) = *$ (i.e. final in FSETS) if and only if $X_0 = e_{\mathcal{C}}$.*

Proof. For the first part:

($\Longrightarrow$) By definition of initial object, we have a unique morphism:

$$u : \emptyset_{\mathcal{C}} \to X_0$$

and so we have a morphism $F(u) : F(\emptyset_{\mathcal{C}}) \to F(X_0) = \emptyset$. But this implies that $F(\emptyset_{\mathcal{C}}) = \emptyset$ and then $F(u) = \mathrm{id}_{\emptyset}$, so $F(u)$ is an isomorphism. From axiom vi we conclude that u is an isomorphism and $X_0 \simeq \emptyset_{\mathcal{C}}$.

($\Longleftarrow$) We want to prove that $F(\emptyset_{\mathcal{C}}) = \emptyset$. For any $X \in \mathcal{C}$, there is the isomorphism:

$$(u_X, \mathrm{id}_X) : \emptyset_{\mathcal{C}} \sqcup X \to X$$

thus $F((u_X, \mathrm{id}_X)) : F(\emptyset_{\mathcal{C}} \sqcup X) \to F(X)$ is an isomorphism by axiom (vi). Now from axiom (v) we have $F(\emptyset_{\mathcal{C}} \sqcup X) = F(\emptyset_{\mathcal{C}}) \sqcup F(X)$ and so $F(X) \simeq F(\emptyset_{\mathcal{C}}) \sqcup F(X)$, which implies $F(\emptyset_{\mathcal{C}}) = \emptyset$.

For the second part:

($\Longleftarrow$) It follows from axiom (iv).

($\Longrightarrow$) Consider the unique morphism

$$v : X_0 \to e_{\mathcal{C}};$$

then $F(v) : * \to *$ is the identity and by axiom (vi) we get that v is an isomorphism, hence $X_0 \simeq e_{\mathcal{C}}$. ☐

Lemma 1.2.6. *A Galois category $\mathcal{C}$ is artinian.*

Proof. Let F be a fibre functor. Consider a decreasing sequence of monomorphisms:

$$\dots \xrightarrow{t_{n+1}} T_n \xrightarrow{t_n} T_{n-1} \xrightarrow{t_{n-1}} \dots$$

We show that for n big enough the sequence stabilizes, i.e. there exists $N \in \mathbb{N}$ such that T_n is an isomorphism for every $n \geq N$. Using Lemma 1.2.2, we get the decreasing sequence of monomorphism in FSETS:

$$\dots \xrightarrow{F(t_{n+1})} F(T_n) \xrightarrow{F(t_n)} F(T_{n-1}) \xrightarrow{F(t_{n-1})} \dots$$

which stabilizes since $F(T_i)$ is a finite set for every i. Hence there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $F(t_n)$ is an isomorphism and then t_n is an isomorphism by axiom (vi). ☐

1.3 Main Theorem of Galois categories

In this section we present the fundamental result about Galois category, which we will call *Main Theorem*; we give immediately the statement in order to help the reader to understand the importance of this theorem, but before seeing the proof we need to introduce some notions, as connected objects and Galois objects, and to analize the properties they have.

In analogy with the topological case, we start by defining what is going to play the role of the fundamental group and what the set of paths is going to be.

Definition. If $\mathcal{C}$ is a Galois category and $F_1, F_2 : \mathcal{C} \to$ FSETS two fibre functor; we define:

- $\pi_1(\mathcal{C}; F_i)$, the *fundamental group* of $\mathcal{C}$ with base point F_i to be the group of automorphism $\mathrm{Aut}_{\mathrm{Fct}}(F_i)$.
- $\pi_1(\mathcal{C}; F_1, F_2)$, the *set of paths* from F_1 to F_2 in $\mathcal{C}$ to be the group of isomorphism $\mathrm{Isom}_{\mathrm{Fct}}(F_1, F_2)$.

Theorem 1.3.1. ***(Main Theorem)*** *Let $\mathcal{C}$ be a Galois category. Then:*

1. *Any fibre functor $F : \mathcal{C} \to$ FSETS induces an equivalence of categories $\mathcal{C} \to \mathcal{C}(\pi_1(\mathcal{C}; F))$;*

2. *For any two fibre functors $F_1, F_2 : \mathcal{C} \to$ FSets, the set of paths $\pi_1(\mathcal{C}; F_1, F_2)$ is non-empty. The profinite group $\pi_1(\mathcal{C}; F_1)$ is non-canonically isomorphic to $\pi_1(\mathcal{C}; F_2)$ with an isomorphism that is canonical up to inner automorphism. In particular, the abelianization $\pi_1(\mathcal{C}; F)^{ab}$ does not depend on F up to canonical isomorphism.*

In the rest of this section, $\mathcal{C}$ will be a Galois category and $F : \mathcal{C} \to$ FSets a fibre functor.

Definition. We define the *pointed category* associated with $\mathcal{C}$ and F to be the category $\mathcal{C}^{\mathrm{pt}}$:

- the objects are the pairs (X, ζ) where $X \in \mathcal{C}$ and $\zeta \in F(X)$;
- a morphism $(X_1, \zeta_1) \to (X_2, \zeta_2)$ in $\mathcal{C}^{\mathrm{pt}}$ is a morphism $u \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$ such that $F(u)(\zeta_1) = \zeta_2$.

We have a forgetful functor:

$$\begin{aligned} For : \mathcal{C}^{\mathrm{pt}} &\longrightarrow \mathcal{C} \\ (X, \zeta) &\longmapsto X; \end{aligned}$$

notice that its sections are in 1-to-1 correspondence with:

$$\underline{\zeta} = (\zeta_X)_{X \in \mathcal{C}} \in \prod_{X \in \mathcal{C}} F(X).$$

Connected objects and their properties

We define now the connected objects in a category: this is the natural extension of connectedness in the topological case and it will play an important role in the Main Theorem, exactly as the connected components of a topological space are central in the construction of the topological fundamental group.

Definition. An object $X \in \mathcal{C}$ is *connected* if whenever we write it as a coproduct

$$X = X_1 \sqcup X_2,$$

either $X_1 = \emptyset_{\mathcal{C}}$ or $X_2 = \emptyset_{\mathcal{C}}$.

Proposition 1.3.2. *$X_0 \in \mathcal{C}$ is connected if and only if for any non-initial object $X \in \mathcal{C}$, a monomorphism $X \to X_0$ is automatically an isomorphism.*

Furthermore, any non-initial object $X \in \mathcal{C}$ can be written as finite coproduct of connected object:

$$X = \bigsqcup_{i=1}^{n} X_i$$

with X_i non-initial and connected; this decomposition is unique up to permutation.

Proof. For the first part of the statement:

($\Longleftarrow$) Suppose $X_0 = X_1 \sqcup X_2$ with X_1 non-initial; the morphism $i_{X_1} : X_1 \to X_0$ is a monomorphism since we can use Lemma 1.2.2 on $F(i_{X_1}) : F(X_1) \to F(X_0) = F(X_1) \sqcup F(X_2)$, where the last equality holds as F commutes with finite coproducts by axiom (v). But then i_{X_1} is an isomorphism, so $X_0 \simeq X_1$ and $X_2 = \emptyset_{\mathcal{C}}$.

($\Longrightarrow$) We take a monomorphism $i : X \to X_0$ where X is a non-initial object. Using axiom (iii), we find a decomposition for i:

$$X \xrightarrow{u} X_0' \xrightarrow{\varepsilon} X_0 = X_0' \sqcup X_0''$$

where u is a strict epimorphism and ε is a monomorphism. Since X_0 is connected, either $X_0' = \emptyset_{\mathcal{C}}$ or $X_0'' = \emptyset_{\mathcal{C}}$; but if $X_0' = \emptyset_{\mathcal{C}}$, we have $F(X) = \emptyset$ hence $X = \emptyset_{\mathcal{C}}$ by Lemma 1.2.5, against X non initial. Hence must be $X_0'' = \emptyset_{\mathcal{C}}$, but then ε is an isomorphism and so i is a strict epimorphism, which implies i isomorphism by Lemma 1.2.3.

For the second part: take $X \in \mathcal{C}$: if it is connected, we are done. Otherwise, we have

$$X = X_1 \sqcup X_1'$$

and we have a monomorphism $X_1 \to X$. Now if X_1 and X_1' are connected, we are done. If, without loss of generality, X_1 is not connected, we have:

$$X_1 = X_2 \sqcup X_2'$$

and a monomorphism $X_2 \to X_1$. We obtain a chain:

$$\cdots \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow X$$

and since $\mathcal{C}$ is artinian by Lemma 1.2.6 the chain stabilizes, so we arrive to a finite decomposition

$$X = \bigsqcup_{i=1}^{n} X_i$$

into connected objects.

Now, for the uniqueness of the decomposition: let exist another decomposition

$$X = \bigsqcup_{j=1}^{m} Y_j$$

and consider for every i an index $\sigma(i) \leq m$ such that $F(X_i) \cap F(Y_{\sigma(i)}) \neq \emptyset$. We look at the following diagram:

$$\begin{tikzcd}
X_i \times_X Y_{\sigma(i)} \arrow[r, "q"] \arrow[d, "p"'] & Y_{\sigma(i)} \arrow[d, "\iota_{Y_{\sigma(i)}}"] \\
X_i \arrow[r, "\iota_{X_i}"'] & X
\end{tikzcd}$$

Since ι_{X_i} is a monomorphism, also q is a monomorphism. Furthermore, since F commutes with fibre product by axiom (iv), we have

$$F(X_i \times_X Y_{\sigma(i)}) = F(X_i) \cap F(Y_{\sigma(i)}) \neq \emptyset$$

and by Lemma 1.2.5 we obtain $X_i \times_X Y_{\sigma(i)} \neq \emptyset_{\mathcal{C}}$, so the morphism

$$q : X_i \times_X Y_{\sigma(i)} \longrightarrow Y_{\sigma(i)}$$

is a monomorphism from a non-initial object to a connected object: by the previous part q is an isomorphism. In the same way we prove that also p is an isomorphism and we get

$$X_i \simeq Y_{\sigma(i)}.$$

□

The objects X_i of this decomposition are said *connected components* of X.

Lemma 1.3.3. *The following are true:*

1. *For any non-initial connected $X_0 \in \mathcal{C}$, for any non-initial $X \in \mathcal{C}$ and any $\zeta_0 \in F(X_0)$, $\zeta \in F(X)$, there is at most one morphism from (X_0, ζ_0) to (X, ζ) in $\mathcal{C}^{pt}$.*

2. *For any $(X_i, \zeta_i) \in \mathcal{C}^{pt}$ where $i = 1, 2, \ldots, n$ there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with X_0 connected such that (X_0, ζ_0) dominates (X_i, ζ_i) for every i. Furthermore, given $X \in \mathcal{C}$, there exists $X_0 \in \mathcal{C}$ connected such that the evaluation map*

$$\begin{aligned} ev_{\zeta_0} : Hom_{\mathcal{C}}(X_0, X) &\longrightarrow F(X) \\ u &\longmapsto F(u)(\zeta_0) \end{aligned}$$

   is a bijection.

Proof. 1. Suppose that there exist two morphisms $u_1, u_2 : X_0 \to X$ such that $F(u_1)(\zeta_0) = F(u_2)(\zeta_0) = \zeta$. We consider the equalizer $K = \ker(u_1, u_2)$, which exists in $\mathcal{C}$ by axiom (i). We want to use the Proposition 1.3.2 on the monomorphism

$$i : K \to X_0$$

to show that it is an isomorphism and then $u_1 = u_2$. The only thing we have to check before conclude is that $K \neq \emptyset_{\mathcal{C}}$. To do this, we use the fibre functor F: since by axiom (iv) F commutes with fibre product (and with kernels in particular), we have that $F(K)$ is the equalizer of $F(u_i)$. But $\zeta_0 \in F(K) \neq \emptyset$ and by Lemma 1.2.5 this implies $K \neq \emptyset_{\mathcal{C}}$.

2. We take $X := X_1 \times X_2 \times \cdots \times X_n$ and

$$\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in F(X_1) \times F(X_2) \times \cdots \times F(X_n) = F(X_1 \times X_2 \times \cdots \times X_n)$$

where the last equality holds by axiom (iv); notice that we have the canonical projections

$$\pi_i : X \to X_i \quad F(\pi_i(\zeta)) = \zeta_i.$$

Hence to conclude it is enough to find a connected object X_0 and an element $\zeta_0 \in F(X_0)$ such that $(X_0, \zeta_0) \geq (X, \zeta)$.

Now, if X is connected, we have nothing to prove. Otherwise, take the decomposition of X into connected components:

$$X = \bigsqcup_{i=1}^{s} C_i;$$

since by axiom (ii) F commutes with coproducts, $F(X) = \bigsqcup F(C_i)$ and if $\zeta \in F(X)$ there exists i such that $\zeta \in F(C_i)$. But then (C_i, ζ) dominates (X, ζ) as we wanted.

To prove that ev_{ζ_0} is a bijection, it is enough to consider $(X_i, \zeta_i)_{i \leq n}$ where $X_i = X$ and $\bigcup \{ \zeta_i \} = F(X)$ (they are finitely many since $F(X)$ is finite). Now we use the previous part and we find X_0 connected and $\zeta_0 \in F(X_0)$ such that $(X_0, \zeta_0) \geq (X, \zeta_i)$ for every i; but now the evaluation map ev_{ζ_0} is surjective and it is injective by the point 1.

□

Lemma 1.3.4. *Let $X_0 \in \mathcal{C}$ be connected and $X \in \mathcal{C}$. Then:*

1. *if X is non initial, any morphism $u \in Hom_{\mathcal{C}}(X, X_0)$ is a strict epimorphism;*

2. *if $u \in Hom_{\mathcal{C}}(X_0, X)$ is a strict epimorphism, X is connected;*
3. *any endomorphism $u \in Hom_{\mathcal{C}}(X_0, X_0)$ is automatically an automorphism.*

Proof. 1. From axiom (iii), the morphism u has a factorization:

$$X \xrightarrow{u'} X_0' \xrightarrow{u''} X_0 \simeq X_0' \sqcup X_0''$$

with u' a strict epimorphism and u'' a monomorphism. From Proposition 1.3.2 u'' is automatically an isomorphism, hence u is a strict epimorphism.

2. If $X_0 = \emptyset_{\mathcal{C}}$, there is nothing to prove. Otherwise, $X \neq \emptyset_{\mathcal{C}}$ and we suppose $X = X' \sqcup X''$; without loss of generality we may take $X' \neq \emptyset_{\mathcal{C}}$; we want to show that $X'' = \emptyset_{\mathcal{C}}$. Now we fix $\zeta' \in F(X')$ and $\zeta_0 \in F(X_0)$ such that $F(u)(\zeta_0) = \zeta'$; from Lemma 1.3.3 we have that there exists a connected object X_0' and $\zeta_0' \in F(X_0')$ such that in $\mathcal{C}^{\mathrm{pt}}$ there are morphisms $p : (X_0', \zeta_0') \to (X_0, \zeta_0)$ and $q : (X_0', \zeta_0') \to (X', \zeta')$. Notice that by point 1. the morphism $p : X_0' \to X_0$ is a strict epimorphism, so $u \circ p : X_0' \to X$ is a strict epimorphism. Now if $i_{X'} : X' \to X$ is the canonical monomorphism, we have the following diagram

$$\begin{tikzcd}
X_0' \arrow[r, "p"] \arrow[d, "q"'] & X_0 \arrow[d, "u"] \\
X' \arrow[r, "i_{X'}"'] & X = X' \sqcup X''
\end{tikzcd}$$

where we have

$$F(i_{X'} \circ q)(\zeta_0') = \zeta' = F(u \circ p)(\zeta_0');$$

thus, as X_0' is connected, from Lemma 1.3.3 the two morphism coincide: $i_{X'} \circ q = u \circ p$, i.e. the latter diagram is commutative. Hence $i_{X'} \circ q$ is a strict epimorphism and then $i_{X'}$ is both a monomorphism and a strict epimorphism. From Lemma 1.2.3 we have that $i_{X'}$ is an isomorphism and we conclude $X'' = \emptyset_{\mathcal{C}}$.

3. Using axiom (vi), it is enough to prove that $F(u)$ is an isomorphism. Since $F(X_0)$ is a finite set, in fact it suffices to prove that $F(u)$ is surjective. But by point 1., u is automatically a strict epimorphism hence $F(u)$ is surjective by Lemma 1.2.4.

□

Galois objects and their properties

Given a connected object $X_0 \in \mathcal{C}$ and $\zeta_0 \in F(X_0)$, the Lemma 1.3.3 says that the evaluation map $\mathrm{ev}_{\zeta_0} : \mathrm{Aut}_{\mathcal{C}}(X_0) \to F(X_0)$ is injective. For Galois object, we ask this map be also surjective for every $\zeta_0 \in F(X_0)$:

Definition. A connected object $X_0 \in \mathcal{C}$ is *Galois* if for any $\zeta_0 \in F(X_0)$ the evaluation map

$$\begin{aligned} \mathrm{ev}_{\zeta_0} : \mathrm{Aut}_{\mathcal{C}}(X_0) &\longrightarrow F(X_0) \\ u &\longmapsto F(u)(\zeta_0) \end{aligned}$$

is bijective.

We have the following equivalent properties for a connected object to be Galois:

Lemma 1.3.5. *Let $X_0 \in \mathcal{C}$ be a connected object; the following are equivalent:*

(i) X_0 is a Galois object;

(ii) $Aut_{\mathcal{C}}(X_0)$ acts transitively on $F(X_0)$;

(iii) $Aut_{\mathcal{C}}(X_0)$ acts simply transitively on $F(X_0)$;

(iv) $|Aut_{\mathcal{C}}(X_0)| = |F(X_0)|$;

(v) $X_0/Aut_{\mathcal{C}}(X_0) = e_{\mathcal{C}}$.

Observation. Notice that from (ii) in particular it is explicit the analogy with Galois covers in the topological situation (see Proposition 1.1.3). Furthermore, (v) shows that the notion of Galois object does not depend on the fibre functor F.

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ are quite immediate using the definition of Galois objects and the Lemma 1.3.3. (v) is equivalent to $F(X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)) = *$ by Lemma 1.2.5 and by axiom (v) the latter is equivalent to $F(X_0)/\mathrm{Aut}_{\mathcal{C}}(X_0) = *$, which is in fact equivalent to (i). ☐

Consider now a connected object $X \in \mathcal{C}$; we know from Lemma 1.3.3 that for any $\zeta \in F(\zeta)$ the evaluation map ev_{ζ} is injective. The idea now is to "enlarge" a little bit the object$X$ in order to make the evaluation map a bijection, i.e. we want to find a Galois object which, in a certain way, covers the connected object X. Furthermore, we would like to do this in a minimal way, so we would like to enlarge sharply our connected X in order to reach a

Galois object; the following lemma assures us that we can do it in a unique way, up to isomorphism. We will call this Galois object the *Galois closure* of X; the usefulness of the Galois objects and hence the importance of the existence of the Galois closure for connected objects will be clear within the following results.

Lemma 1.3.6. *For any connected object $X \in \mathcal{C}$ there exists a Galois object $\hat{X} \in \mathcal{C}$ which dominates X and which is minimal with respect to this property, i.e. if Y is another Galois object dominating X, we find a map $Y \to \hat{X}$ such that the following diagram is commutative:*

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X \\
\hat{X} \arrow[ur] &
\end{tikzcd}$$

Proof. First we prove the existence of this object. From the Lemma 1.3.3 we know that there exists $(X_0, \zeta_0) \in \mathcal{C}^{\mathrm{pt}}$ with X_0 connected such that the evaluation map $\mathrm{ev}_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \to F(X)$ is bijective. In particular $\mathrm{Hom}_{\mathcal{C}}(X_0, X)$ is a finite set so we enumerate all its elements:

$$\mathrm{Hom}_{\mathcal{C}}(X_0, X) = \{ u_1, \dots, u_n \} ;$$

we denote by $\zeta_i := F(u_i)(\zeta_0)$. Now we consider $X^n = \prod_{i=1}^{n} X$ and we call $p_i : X^n \to X$ the i-th projection. By the universal property of the product, we find a map $\pi : X_0 \to X^n$ such that $u_i = p_i \circ \pi$ for every i; now we decompose π using the axiom (iii) and we find the following commutative diagram :

$$\begin{tikzcd}
X_0 \arrow[r, "u_i"] \arrow[d, "\pi'"'] & X \\
\hat{X} \arrow[r, "\pi''"'] & X^n \arrow[u, "p_i"']
\end{tikzcd}$$

with π' a strict epimorphism and π'' a monomorphism. We want now to prove that $\hat{X}$ is the Galois object we are looking for.

From the Lemma 1.3.4 it is a connected object since X_0 is connected and π' is a strict epimorphism. We have to prove that for every $\hat{\zeta} \in F(\hat{X})$ we have that $\mathrm{ev}_{\hat{\zeta}} : \mathrm{Aut}_{\mathcal{C}}(\hat{X}) \to F(\hat{X})$ is bijective; notice that in fact it is enough to check this on $\hat{\zeta} = \hat{\zeta}_0 := F(\pi')(\zeta_0) = (\zeta_1, \dots, \zeta_n)$. As $\hat{X}$ is connected, we already know that $\mathrm{ev}_{\hat{\zeta}_0}$ is injective, so it remains to prove that it is surjective, i.e. for every $\zeta \in F(\hat{X})$ there exists $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$.

Let us fix an element $\zeta \in F(\hat{X})$; firstly we can suppose that there exists a morphism

$$\rho_\zeta : (X_0, \zeta_0) \longrightarrow (\hat{X}, \zeta).$$

Indeed by Lemma 1.3.3 there exists $(\tilde{X_0}, \tilde{\zeta_0}) \in \mathcal{C}^{\mathrm{pt}}$ with $\tilde{X_0}$ connected such that $(\tilde{X_0}, \tilde{\zeta_0}) \geq (X_0, \zeta_0)$ and $(\tilde{X_0}, \tilde{\zeta_0}) \geq (\hat{X}, \zeta)$ for every $\zeta \in F(\hat{X})$; so up to replace (X_0, ζ_0) with $(\tilde{X_0}, \tilde{\zeta_0})$, we can suppose the existence of ρ_ζ. Now, since from Lemma 1.3.3 there is at most one morphism $(X_0, \zeta_0) \to (\hat{X}, \zeta)$, we have that there exists $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta_0}) = \zeta$ if and only if there exists $\omega \in \mathrm{Aut}(\hat{X})$ such that $\omega \circ \pi' = \rho_\zeta$. Hence to conclude we can prove the latter.

To prove the existence of such ω, observe that the following equality holds:

$$\{ p_1 \circ \pi'' \circ \rho_\zeta, \dots, p_n \circ \pi'' \circ \rho_\zeta \} = \{ u_1, \dots, u_n \}.$$

Indeed the inclusion $(\subset)$ is immediate since $p_i \circ \pi'' \circ \rho_\zeta \in \mathrm{Hom}_{\mathcal{C}}(X_0, X) = \{ u_1, \dots, u_n \}$; for the inclusion $(\supset)$, it is enough to prove that $p_i \circ \pi'' \circ \rho_\zeta$ are all distinct. But from Lemma 1.3.4 the morphism ρ_ζ is a strict epimorphism as $\hat{X}$ is connected, hence $p_i \circ \pi'' \circ \rho_\zeta = p_j \circ \pi'' \circ \rho_\zeta$ implies $p_i \circ \pi'' = p_j \circ \pi''$, whence

$$p_i \circ \pi'' \circ \pi' = u_i \neq u_j = p_j \circ \pi'' \circ \pi'.$$

Thus we have the equality $\{ p_1 \circ \pi'' \circ \rho_\zeta, \dots, p_n \circ \pi'' \circ \rho_\zeta \} = \{ u_1, \dots, u_n \}$. Hence there exists a permutation $\sigma \in S_n$ such that:

$$p_{\sigma(i)} \circ \pi'' \circ \rho_\zeta = u_i = p_i \circ \pi'' \circ \pi'$$

for every i and by the universal property of the product X^n we find an automorphism $\sigma \in \mathrm{Aut}_{\mathcal{C}}(X^n)$ such that $p_{\sigma(i)} = p_i \circ \sigma$ for every i. Now if we rewrite the previous equality, we get:

$$p_i \circ \sigma \circ \pi'' \circ \rho_\zeta = p_i \circ \pi'' \circ \pi'$$

and using that p_i is an epimorphism we get:

$$\sigma \circ \pi'' \circ \rho_\zeta = \pi'' \circ \pi'.$$

But remember that $\pi'' \circ \pi'$ is the unique factorization of π given by the axiom (iii), so there exists an automorphism $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $\sigma \circ \pi'' = \pi'' \circ \omega$. Thus the previous equality becomes

$$\pi'' \circ \omega \circ \rho_\zeta = \pi'' \circ \pi'$$

and since π'' is a strict epimorphism we have $\omega \circ \rho_\zeta = \pi'$ as wanted.

For the minimality of $\hat{X}$. Let $Y \in \mathcal{C}$ another Galois object dominating X via $q : Y \to X$. Let $\eta_i \in F(Y)$ such that $F(q)(\eta_i) = \zeta_i$ for $i = 1, \dots, n$. As Y is Galois, for every i there exists $\omega_i \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $F(\omega_i)(\eta_1) = \eta_i$; we obtain a morphism

$$\kappa := (q \circ \omega_1, \dots, q \circ \omega_n) : Y \to X^n$$

i.e. a morphism $Y \to X^n$ such that $p_i \circ \kappa = q \circ \omega_i$. Using axiom (iii), we find for κ the factorization

$$Y \xrightarrow{\kappa'} Z' \xrightarrow{\kappa''} Z' \sqcup Z'' \simeq X^n$$

with κ' strict epimorphism and κ'' monomorphism; by Lemma 1.3.4 Z' is connected and $F(\kappa)(\eta_1) = (\zeta_1, \dots, \zeta_n) = \hat{\zeta}_0$, so Z' is the connected component of X^n that contains $\hat{\zeta}_0$ and by uniqueness of decomposition into connected components (Lemma 1.3.2) we get $Z' = \hat{X}$. Hence we found a morphism

$$\kappa' : Y \to \hat{X}$$

so $\hat{X}$ is minimal. $\square$

Galois correspondence

We arrive here to the core of the proof of the main theorem: the following proposition is indeed the proof of the main theorem in a "finite" version. But before continuing we would like to restrict our discussion from generic objects in $\mathcal{C}$ to connected objects, which have the rich variety of properties we illustrated previously. This is possible if we are studying the behaviour of morphisms from connected objects to generic objects: in this case we can restrict to morphisms between connected objects. Indeed the following lemma holds:

Lemma 1.3.7. *Let X be an object in $\mathcal{C}$ and consider its decomposition into connected components:*

$$X = \bigsqcup_{i=1}^{n} X_i;$$

given $X_0 \in \mathcal{C}$ connected, the map

$$\begin{aligned} \bigsqcup Hom_{\mathcal{C}}(X_0, X_i) &\longrightarrow Hom_{\mathcal{C}}(X_0, X) \\ (f : X_0 \to X_i) &\longmapsto e_i \circ f \end{aligned}$$

is bijective.

Proof. For injectivity: if X_0 is initial, the conclusion is trivial. Assume now X_0 non-initial, pick $u_i : X_0 \to X_i$ and $u_j : X_0 \to X_j$ two morphism and suppose that $e_i \circ u_i = e_j \circ u_j$; we prove that $u_i = u_j$. Notice that since e_k is a monomorphism, to conclude it is enough to show that $i = j$. Suppose by contradiction that $i \neq j$. From axiom (vi) one has $F(e_i \circ u_i) = F(e_j \circ u_j)$; furthermore, since F commutes with finite coproducts by axiom (v), we have $F(X) = \bigsqcup F(X_i)$ and so:

$$\operatorname{im}(F(e_i)) \cap \operatorname{im}(F(e_j)) \subset F(X_i) \cap F(X_j) = \emptyset.$$

But this implies $\operatorname{im}(F(e_i \circ u_i)) = \operatorname{im}(F(e_j \circ u_j)) = \emptyset$ and $F(X_0) \neq \emptyset$ since $X_0 \neq \emptyset_{\mathcal{C}}$ by Lemma 1.2.5; thus we find a contradiction and must be $i = j$.

For surjectivity: any morphism $u \in \operatorname{Hom}_{\mathcal{C}}(X_0, X)$ factors in the usual way, so:

$$X_0 \xrightarrow{u'} X' \xrightarrow{u''} X \simeq X' \sqcup X''$$

with u' strict epimorphism and u'' monomorphism. Since X_0 is connected, from Lemma 1.3.4 also X' is connected, hence it is one connected component X_i of X. Then $u = e_i \circ u'$ is the image of $u' \in \operatorname{Hom}(X_0, X_i)$ under the map above. □

Definition. Given a Galois object $X_0 \in \mathcal{C}$, we define $\mathcal{C}^{X_0} \subset \mathcal{C}$ as the full subcategory whose objects are $X \in \mathcal{C}$ such that X_0 dominates every connected component of X. We will write F^{X_0} the restriction of the fibre functor F to the subcategory $\mathcal{C}^{X_0}$.

From now on, we fix a Galois object $X_0 \in C$. The following lemma shows that the functor F^{X_0} is representable by $\operatorname{Hom}_{\mathcal{C}}(X_0, \cdot)_{|\mathcal{C}^{X_0}}$ and the natural trasformation between them is precisely the evaluation $\operatorname{ev}_{\zeta_0}$ for any $\zeta_0 \in F(X_0)$.

Lemma 1.3.8. *Any $\zeta_0 \in F(X_0)$ induces a natural isomorphism:*

$$ev_{\zeta_0} : Hom_{\mathcal{C}}(X_0, \cdot)_{|\mathcal{C}^{X_0}} \longrightarrow F^{X_0}.$$

In particular, this induces an isomorphism of groups:

$$u^{\zeta_0} : Aut_{Fct}(F^{X_0}) \to Aut_{Fct}(Hom_{\mathcal{C}}(X_0, \cdot)_{|\mathcal{C}^{X_0}}) = Aut_{\mathcal{C}}(X_0)^{op}.$$

Proof. Take $u \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$; we have the following diagram:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(X_0, X) \arrow[r, "(u\circ\cdot)"] \arrow[d, "\operatorname{ev}_{\zeta_0}(X)"] & \operatorname{Hom}_{\mathcal{C}}(X_0, Y) \arrow[d, "\operatorname{ev}_{\zeta_0}(Y)"] \\
F(X) \arrow[r, "F(u)"] & F(Y)
\end{tikzcd}$$

If $f \in \mathrm{Hom}_{\mathcal{C}}(X_0, Y)$, we have

$$(F(u) \circ \mathrm{ev}_{\zeta_0}(Y))(f) = (F(u) \circ F(f))(\zeta_0) = F(u \circ f)(\zeta_0)$$

where the last equality holds since F is a functor, while on the other side we have

$$\mathrm{ev}_{\zeta_0}(X)(u \circ f) = F(u \circ f)(\zeta_0).$$

Hence the diagram above is commutative and ev_{ζ_0} is a natural trasformation. It remains to prove that for every $X \in C$, $\mathrm{ev}_{\zeta_0}(X)$ is bijective; it is injective since X_0 is connected. For surjectivity, we can reduce to consider X connected: indeed we can take the decomposition

$$X = \bigsqcup X_i$$

in connected component as in Lemma 1.3.2; we can study the surjectivity of $\mathrm{ev}_{\zeta_0}(X_i)$ and using Lemma 1.3.7 and the fact that F commutes with coproducts (axiom (v)) we can conclude the surjectivity of $\mathrm{ev}_{\zeta_0}(X)$. Thus we can suppose X connected. Now let ζ be an element in $F(X)$; we want to prove that there exists $u \in \mathrm{Hom}_{\mathcal{C}}(X_0, X)$ such that $\mathrm{ev}_{\zeta_0}(u) = F(u)(\zeta_0) = \zeta$. Since we are supposing X connected and $X \in \mathcal{C}^{X_0}$, X_0 dominates X i.e. we find a morphism $u : X_0 \to X$ and an element $\hat{\zeta} \in F(X_0)$ such that $F(u)(\hat{\zeta}) = \zeta$. Since X_0 is Galois, we have the bijection:

$$\mathrm{ev}_{\zeta_0}(X_0)_{|\mathrm{Aut}_{\mathcal{C}}(X_0)} : \mathrm{Aut}_{\mathcal{C}}(X_0) \longrightarrow F(X_0)$$

so we are able to find $\omega \in \mathrm{Aut}_{\mathcal{C}}(X_0)$ such that $F(\omega)(\zeta_0) = \hat{\zeta}$. But now $u' = u \circ \omega \in \mathrm{Hom}_{\mathcal{C}}(X_0, X)$ is such that $F(u')(\zeta_0) = \zeta$, as wanted. Hence ev_{ζ_0} is bijective and a natural isomorphism. □

We have now all the requirements to proceed in the proof of the following proposition

Proposition 1.3.9. *The functor $F^{X_0} : \mathcal{C}^{X_0} \to$ FSETS factors as in the commutative diagram:*

$$\begin{tikzcd}
\mathcal{C}^{X_0} \arrow[r, "F^{X_0}"] \arrow[d, "F^{X_0}"'] & \mathrm{FSETS} \\
\mathcal{C}(Aut_{\mathcal{C}}(X_0)^{op}) \arrow[ru, "For"'] &
\end{tikzcd}$$

where $F^{X_0} : \mathcal{C}^{X_0} \to \mathcal{C}(Aut_{\mathcal{C}}(X_0)^{op})$ is an equivalence of categories.

Proof. For brevity we set $G := \mathrm{Aut}_{\mathcal{C}}(X_0)$; from Lemma 1.3.8 we know that the functor F^{X_0} is representable by $\mathrm{Hom}_{\mathcal{C}}(X_0, \cdot)$. Consider $X \in \mathcal{C}^{X_0}$ and notice that we have a right action of G on $F^{X_0}(X) \simeq \mathrm{Hom}_{\mathcal{C}}(X_0, X)$: if $g \in G$ and $u \in \mathrm{Hom}_{\mathcal{C}}(X_0, X)$, we have:

$$u * g := u \circ g;$$

it is easy to verify that it is actually a right action. Hence the functor F^{X_0} factors as:

$$\begin{tikzcd}
\mathcal{C}^{X_0} \arrow[r, "F^{X_0}"] \arrow[d, "F^{X_0}"'] & \mathrm{FS\small{ETS}} \\
\mathcal{C}(G^{\mathrm{op}}) \arrow[ur, "\mathrm{For}"'] &
\end{tikzcd}$$

It remains to verify that $F^{X_0} : \mathcal{C}^{X_0} \to \mathcal{C}(G^{\mathrm{op}})$ is an equivalence of categories. We prove that it is fully faithful and essentially surjective.

- *(essentially surjective)* Let us consider $E \in \mathcal{C}(G^{\mathrm{op}})$; similarly to what we did in the proof of Lemma 1.3.8, we can restrict to the case of E connected in $\mathcal{C}(G^{\mathrm{op}})$, i.e. E transitive G^{op}-set: indeed in $\mathcal{C}(G^{\mathrm{op}})$ we have a decomposition into connected objects just like we did in Lemma 1.3.2 and if we prove the essentially surjectivity for the connected components we will have automatically the essentially surjectivity on generic objects, using that F^{X_0} commutes with coproducts by axiom (v). Hence we can assume that G^{op} acts transitively on E; so fixed $e \in E$ we have an epimorphism:

$$\begin{aligned} p_e : G^{\mathrm{op}} &\longrightarrow E \\ g &\longmapsto g \cdot e \end{aligned}$$

  We set $f_e := p_e \circ \mathrm{ev}_{\zeta_0}^{-1} : F(X_0) \to E$; for any $s \in S_e := \mathrm{Stab}_{G^{\mathrm{op}}}(e)$ and any $g \in G^{\mathrm{op}}$, the following equalities hold:

$$\begin{aligned} f_e \circ F(s)(\mathrm{ev}_{\zeta_0}(g)) =& p_e \circ \mathrm{ev}_{\zeta_0}^{-1} \circ F(s) \circ F(g)(\zeta_0) \\ =& p_e \circ \mathrm{ev}_{\zeta_0}^{-1} \circ \mathrm{ev}_{\zeta_0}(s \circ g) \\ =& p_e(s \circ g) \\ =& (s \circ g) \cdot e \\ \overset{(*)}{=}& g \cdot (s \cdot e) \\ =& g \cdot e \\ =& f_e(\mathrm{ev}_{\zeta_0}(g)). \end{aligned}$$

where the equality $(*)$ holds since we are considering the action of G^{op}, thus the action of $s \circ g$ is given by the action of s and then the action of g. Hence f_e is S_e-invariant and it factors through the quotient $F(X_0)/S_e$:

$$\begin{tikzcd} F(X_0) \arrow[r, "f_e"] \arrow[d] & E \\ F(X_0)/S_e \arrow[ur, "\bar{f}_e"'] & \end{tikzcd}$$

We call againg S_e the subgroup considered in G instead of G^{op}; we remain sloppy on this in order to avoid further notation as it should not be too confusing for the reader. By axiom (ii) the quotient X_0/S_e exists in $\mathcal{C}$ (of course it is automatically in C^{X_0}) and by axiom (v) F commutes with quotients hence $F(X_0/S_e) \simeq F(X_0)/S_e$. Furthermore, since X_0 is a Galois object, by Lemma 1.3.5 G acts simply on X_0, so:

$$|F(X_0/S_e)| = |F(X_0)|/|S_e| = [G : S_e] = |E|.$$

This proves that the map $f_e : F(X_0)/S_e \to E$ is bijective and an isomorphism in G^{op}-sets.

- *(fully faithfull)* Let us consider $X, Y \in \mathcal{C}^{X_0}$. With the same argument used in Lemma 1.3.8, we can suppose that X and Y are connected. We show that F^{X_0} is faithful. Consider $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ two morphisms such that $F(f) = F(g)$; now let $u : (X_0, \zeta_0) \to (X, \zeta)$ be the unique map given by Lemma 1.3.3, where as usual $\zeta_0 \in X_0$ and $\zeta \in X$. Now $f \circ u$ and $g \circ u$ are morphisms from X_0 to Y and

  $$F(f \circ u)(\zeta_0) = F(f)(\zeta) = F(g)(\zeta) = F(g \circ u)(\zeta_0),$$

  so $f \circ u = g \circ u$ by uniqueness in Lemma 1.3.3. Now, since X is connected, u is a strict epimorphism by Lemma 1.3.4 and then we obtain $f = g$.

  We show that F^{X_0} is full. Let us consider a morphism

  $$u \in \mathrm{Hom}_{\mathcal{C}(G^{\mathrm{op}})}(X, Y)$$

  and fix an element $e \in F(X)$. Since u is G^{op}-invariant, we have $S_e \subset S_{u(e)}$ and thus the quotient morphism $p_{u(e)} : X_0 \to X_0/S_{u(e)}$ factors:

  $$\begin{tikzcd} X_0 \arrow[r, "p_e"] \arrow[d, "p_{u(e)}"'] & X_0/S_e \arrow[dl, "p"] \\ X_0/S_{u(e)} & \end{tikzcd}$$

Now if we consider $f_e : F(X_0) \to F(X)$ and $f_{u(e)} : F(X_0) \to F(Y)$ as in the previous part and we remember that both factors in the isomorphisms $\bar{f}_e : F(X_0)/S_e \to F(X)$ and $\bar{f}_{u(e)} : F(X_0)/S_e \to F(Y)$, the following diagram is commutative:

$$\begin{tikzcd}
F(X_0)/S_e \arrow[rr, "F(p)"] \arrow[dd, "\bar{f}_e"'] & & F(X_0)/S_{u(e)} \arrow[dd, "\bar{f}_{u(e)}"] \\
& F(X_0) \arrow[ul, "F(p_e)"] \arrow[ur, "F(p_{u(e)})"'] \arrow[dl, "f_e"'] \arrow[dr, "f_{u(e)}"] & \\
F(X) \arrow[rr, "u"] & & F(Y)
\end{tikzcd}$$

Now, since $\bar{f}_e$ and $\bar{f}_{u(e)}$ are isomorphisms, we may conclude that $u = F(p)$ thus F^{X_0} is full. $\square$

Conclusion of the proof

Pro-representability of the fibre functor. We briefly recall the definitions of pro-representability: given a category $\mathcal{C}$, one defines the category $\text{Pro}(\mathcal{C})$ to be the category whose objects are the projective systems

$$\underline{X} = (\phi_{ij} : X_i \to X_j)_{i,j \in I, i \geq j}$$

where $(I, \geq)$ is a partially ordered filtrant set and the morphisms are:

$$\text{Hom}_{\text{Pro}(\mathcal{C})}(\underline{X}, \underline{X}') := \varprojlim_{i \in I} \varinjlim_{j \in J} \text{Hom}_{\mathcal{C}}(X_j, X_i).$$

The category $\mathcal{C}$ can be seen as a full subcategory of $\text{Pro}(\mathcal{C})$. In the case of our interest, with $\mathcal{C}$ a Galois category and F a fibre functor for it, F can be extended canonically to a functor $\text{Pro}(F) : \text{Pro}(\mathcal{C}) \to \text{Pro}(\text{Sets})$.

Definition. The functor F is said *pro-representable* in $\mathcal{C}$ if there exists $\underline{X} \in \text{Pro}(\mathcal{C})$ such that the is a natural isomorphism:

$$\text{Hom}_{\text{Pro}(\mathcal{C})}(\underline{X}, \cdot) \xrightarrow{\sim} F;$$

F is said *stricly pro-representable* if it is pro-representable by an object $\underline{X} \in \text{Pro}(\mathcal{C})$ whose transition maps $\phi_{ij} : X_i \to X_j$ are epimorphism.

Now, we denote by $\mathcal{G}$ the set of Galois objects in $\mathcal{C}$ (considered up to isomorphism). Using Lemmas 1.3.3 and 1.3.6, we have that $\mathcal{G}$ is a directed;

now let us fix $\underline{\zeta} = (\zeta_X)_{X \in \mathcal{G}} \in \prod_{X \in \mathcal{G}} F(X)$. Again from Lemma 1.3.3, for any $X, Y \in \mathcal{G}$ such that $Y \geq X$ there exists a unique morphism $\phi^{\underline{\zeta}}_{X,Y} : Y \to X$ such that $F(\phi^{\underline{\zeta}}_{X,Y})(\zeta_Y) = \zeta_X$. Hence using this unicity we have that for all $X, Y, Z \in \mathcal{G}$ with $X \leq Y \leq Z$:

$$\phi^{\underline{\zeta}}_{X,Y} \circ \phi^{\underline{\zeta}}_{Y,Z} = \phi^{\underline{\zeta}}_{X,Z}.$$

Thus on $\mathcal{G}$ we have a structure of projective system whose transition maps are strict epimorphisms: indeed $\phi^{\underline{\zeta}}_{X,Y}$ is an epimorphism as X is Galois hence connected and Lemma 1.3.4 holds. We denote by $\mathcal{G}^{\underline{\zeta}}$ the partially ordered filtrant set

$$\{ \phi^{\underline{\zeta}}_{X,Y} \mid X, Y \in \mathcal{G}, X \leq Y \} .$$

One has the following lemma:

Proposition 1.3.10. *The fibre functor $F : \mathcal{C} \to$ FSETS is stricly pro-representable in $\mathcal{C}$ by $\mathcal{G}^{\underline{\zeta}}$.*

Proof. We have to prove that for any $Z \in \mathcal{C}$, there is an isomorphism

$$\varinjlim_{X \in \mathcal{G}} \mathrm{Hom}_{\mathcal{C}}(X, Z) \xrightarrow{\sim} F(Z);$$

as done before, we can restrict to the case when Z is connected. By Lemma 1.3.6 we know that there exists at least one $X \in \mathcal{G}$ dominating Z; now if we develop the right hand side we find:

$$\varinjlim_{X \in \mathcal{G}} \mathrm{Hom}_{\mathcal{C}}(X, Z) = \varinjlim_{X \in \mathcal{G}, X \geq Z} \mathrm{Hom}_{\mathcal{C}}(X, Z) \overset{(*)}{=} \varinjlim_{X \in \mathcal{G}, X \geq Z} F(Z) = F(Z)$$

where the equality $(*)$ comes from Lemma 1.3.8; this lemma assures that this isomorphism is natural as well and we are done. ☐

The fundamental group. Now that we have a clearer idea on how the functor F works, as we have the pro-representability via the object $\mathcal{G}^{\underline{\zeta}}$, we would like to improve our description of $\pi_1(\mathcal{C}, F) = \mathrm{Aut}_{\mathrm{Fct}}(F)$ as it is essential to conclude the proof of the main theorem. In order to achieve a better comprehension of this group, we have the next results. The first is a tecnical lemma:

Lemma 1.3.11. *For any $X, Y \in \mathcal{G}$ with $X \leq Y$, for any morphism $\phi, \psi : Y \to X$ in $\mathcal{C}$ and for any $\omega_Y \in Aut_{\mathcal{C}}(Y)$ there is a unique automorphism*

$\omega_X := r_{\phi,\psi}(\omega_Y) : X \to X$ *such that the following diagram commutes:*

$$\begin{tikzcd}
Y \arrow[r, "\omega_Y"] \arrow[d, "\psi"'] & Y \arrow[d, "\phi"] \\
X \arrow[r, "\omega_X"'] & X
\end{tikzcd}$$

Proof. Since X is connected, ψ is automatically a strict epimorphism and so the map $\cdot \circ \psi : \mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is injective. On the other hand from Lemma 1.3.8, $|\mathrm{Hom}_{\mathcal{C}}(Y, X)| = |F(X)|$ and since X is Galois also $|F(X)| = |\mathrm{Aut}_{\mathcal{C}}(X)|$, hence:

$$\psi : \mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

is bijective. In particular, there exists a unique morphism $\omega_X \in \mathrm{Aut}_{\mathcal{C}}(X)$ such that $\phi \circ \omega_Y = \omega_X \circ \psi$. ☐

From the previous lemma one gets that there exists a well defined map

$$r_{\phi,\psi} : \mathrm{Aut}_{\mathcal{C}}(Y) \longrightarrow \mathrm{Aut}_{\mathcal{C}}(X)$$

and in fact the following lemma holds:

Lemma 1.3.12. *The map $r_{\phi,\psi}$ is surjective and if $\phi = \psi$ it is a group morphism.*

Proof. We show that the map is surjective: fix $\omega_X \in \mathrm{Aut}_{\mathcal{C}}(X)$; we want to prove that there exists $\omega_Y \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $\phi \circ \omega_Y = \omega_X \circ \psi$. Firstly, pick $y \in F(Y)$ and denote by $x = \mathrm{ev}_y(\omega_X \circ \psi) = F(\omega_X \circ \psi)(y) \in F(X)$; since X is connected, ϕ is automatically a strict epimorphism by Lemma 1.3.4 and then $F(\phi)$ is surjective by Lemma 1.2.4. Hence we pick $y' \in F(\phi)^{-1}(x)$ and we take $\omega_Y = \mathrm{ev}_y^{-1}(y')$ where we remind that

$$\mathrm{ev}_y : \mathrm{Aut}_{\mathcal{C}}(Y) \xrightarrow{\sim} F(Y)$$

is a bijection by Lemma 1.3.8. But now also the evaluation

$$\mathrm{ev}_y : \mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{\sim} F(X)$$

is a bijection and we have $\mathrm{ev}_y(\phi \circ \omega_Y) = x = \mathrm{ev}_y(\omega_X \circ \psi)$ hence $\phi \circ \omega_Y = \omega_X \circ \psi$.

Assume now $\psi = \phi$ and denote $r := r_{\phi,\phi}$; we take $\omega_Y, \omega'_Y \in \mathrm{Aut}_{\mathcal{C}}(Y)$ and we show that

$$r(\omega'_Y \circ \omega_Y) := r(\omega'_Y) \circ r(\omega_Y).$$

We have the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[r, "\omega_Y"] \arrow[d, "\phi"] & Y \arrow[r, "\omega'_Y"] \arrow[d, "\phi"] & Y \arrow[d, "\phi"] \\
X \arrow[r, "r(\omega_Y)"] \arrow[rr, bend right, "r(\omega'_Y \circ \omega_Y)"'] & X \arrow[r, "r(\omega'_Y)"] & X
\end{tikzcd}$$

where both $r(\omega'_Y) \circ r(\omega_Y)$ and $r(\omega'_Y \circ \omega_Y)$ close the big square, hence by Lemma 1.3.11 they coincide. $\square$

From what said above, we get a projective system of finite groups:

$$(r^{\underline{\zeta}}_{X,Y} := r_{\phi^{\underline{\zeta}}_{X,Y},\phi^{\underline{\zeta}}_{X,Y}} : \mathrm{Aut}_{\mathcal{C}}(Y) \to \mathrm{Aut}(X))_{X,Y\in\mathcal{G},X\leq Y}.$$

Set now

$$\Pi := \varprojlim_{X\in\mathcal{G}} \mathrm{Aut}_{\mathcal{C}}(X);$$

we want to show that in fact $\Pi^{\mathrm{op}} = \pi_1(\mathcal{C}, F)$. From the defition, Π^{op} acts on the right on

$$\varprojlim_{X\in\mathcal{G}} \mathrm{Hom}_{\mathcal{C}}(X, \cdot)_{|\mathcal{C}}$$

by precomposition; hence we have a group monomorphism:

$$\Pi^{\mathrm{op}} \longrightarrow \mathrm{Aut}_{\mathrm{Fct}}(\varprojlim \mathrm{Hom}_{\mathcal{C}}(X, \cdot)_{|\mathcal{C}}).$$

Furthermore, the natural isomorphism of Proposition 1.3.10

$$\mathrm{ev}_{\underline{\zeta}} : \varprojlim \mathrm{Hom}_{\mathcal{C}}(X, \cdot)_{|\mathcal{C}} \longrightarrow F$$

gives us the group monomorphism:

$$\begin{aligned} u^{\underline{\zeta}} : \pi_1(\mathcal{C}; F) \longrightarrow& \Pi^{\mathrm{op}} \\ \theta \longmapsto& (\mathrm{ev}^{-1}_{\zeta_X}(\theta(X)(\zeta_X)))_{X\in\mathcal{G}} \end{aligned}$$

which is in fact an isomorphism. Indeed:

Proposition 1.3.13. *$u^{\underline{\zeta}}$ is an isomorphism of profinite groups.*

Proof. We construct an inverse for $u^{\underline{\zeta}}$ as a group morphism: consider $(\omega_X)_{X\in\mathcal{G}} \in \Pi$; we want to construct an element$\theta_{\underline{\omega}}$ in $\pi_1(\mathcal{C}; F) = \mathrm{Aut}_{\mathrm{Fct}}(F)$ starting from (ω_X). As done before, it is enough to define θ on connected object, since we have Lemmas 1.3.2 and 1.3.7; if $Z \in \mathcal{C}$ is a connected object, we take its Galois closure $\hat{Z}$ as in Lemma 1.3.6 and construct the map $\theta_{\underline{\omega}}$ by the composition:

$$F(Z) \xrightarrow{\mathrm{ev}_{\zeta_{\hat{Z}}}^{-1}} \mathrm{Hom}_{\mathcal{C}}(\hat{Z}, Z) \xrightarrow{\cdot \circ \omega_{\hat{Z}}} \mathrm{Hom}_{\mathcal{C}}(\hat{Z}, Z) \xrightarrow{\mathrm{ev}_{\zeta_{\hat{Z}}}} F(Z).$$

We have to check that $\theta_{\underline{\omega}}$ is in fact a natural isomorphism and that $u^{\underline{\zeta}}(\theta_{\underline{\omega}}) = \underline{\omega}$. To show that it is a natural isomorphism, it suffices to show that $\cdot \circ \omega_{\hat{Z}}$ is a natural isomorphism, since we already have this result for the evaluation from Lemma 1.3.8. But bijectivity of $\cdot \circ \omega_{\hat{Z}}$ comes easily from $\omega_{\hat{Z}} \in \mathrm{Aut}_{\mathcal{C}}(\hat{Z})$, while naturality holds since for every $f \in \mathrm{Hom}_{\mathcal{C}}(Z_1, Z_2)$ with $\hat{Z} \geq Z_1, Z_2$ the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(\hat{Z}, Z_1) \arrow[r, "f\circ\cdot"] \arrow[d, "\cdot\circ\omega_{\hat{Z}}"'] & \mathrm{Hom}_{\mathcal{C}}(\hat{Z}, Z_2) \arrow[d, "\cdot\circ\omega_{\hat{Z}}"] \\
\mathrm{Hom}_{\mathcal{C}}(\hat{Z}, Z_1) \arrow[r, "f\circ\cdot"'] & \mathrm{Hom}_{\mathcal{C}}(\hat{Z}, Z_2)
\end{tikzcd}$$

We check that it is the inverse of $u^{\underline{\zeta}}$:

$$u^{\underline{\zeta}}(\theta_{\underline{\omega}}) = (\mathrm{ev}_{\zeta_X}^{-1} \circ \mathrm{ev}_{\zeta_X} \circ (\cdot \circ \omega_X) \circ \mathrm{ev}_{\zeta_X}^{-1}(\zeta_X))_{X \in \mathcal{G}}$$

Now $\mathrm{ev}_{(\zeta_X)}^{-1}(\zeta_X) = \mathrm{id}_X$ so we get:

$$u^{\underline{\zeta}}(\theta_{\underline{\omega}}) = (\mathrm{id}_X \circ \omega_X)_{X \in \mathcal{G}} = \underline{\omega}.$$

□

End of the proof. Lastly, we have all the tools to conclude the proof of the main theorem, which will be just a matter of putting together the results we collected until now.

Proof of the Main Theorem.

1. From the previous proposition we have

$$\pi_1(\mathcal{C}; F) = \Pi^{\mathrm{op}} = \varprojlim_{X \in \mathcal{G}} \mathrm{Aut}_{\mathcal{C}}(X)^{\mathrm{op}}$$

and from Proposition 1.3.10 we know that

$$F \simeq \mathrm{Hom}_{\mathrm{P{\scriptsize RO}}(\mathcal{C})}(\mathcal{G}^{\underline{\zeta}}, \cdot)_{|\mathcal{C}}.$$

So we have to show that the functor $\mathrm{Hom}_{\mathrm{PRO}(\mathcal{C})}(\mathcal{G}^{\underline{\zeta}}, \cdot)_{|\mathcal{C}} : \mathcal{C} \to \mathrm{FSETS}$ factors through an equivalence of categories $\mathcal{C} \to \mathcal{C}(\Pi^{\mathrm{op}})$. Since for every $X \in \mathcal{C}$ the profinite group Π acts on $\mathrm{Hom}_{\mathrm{PRO}(\mathcal{C})}(\mathcal{G}^{\underline{\zeta}}, X)$, we have the factorization

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathrm{FSETS} \\
\mathcal{C}(\Pi^{\mathrm{op}}) \arrow[ur] &
\end{tikzcd}$$

It remains to show that the induced functor

$$\mathrm{Hom}_{\mathrm{PRO}(\mathcal{C})}(\mathcal{G}^{\underline{\zeta}}, \cdot)_{|\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}(\Pi^{\mathrm{op}})$$

is an equivalence.

- *(essentially surjective)* If $E \in \mathcal{C}(\Pi^{\mathrm{op}})$, since E has the discrete topology, the action of Π^{op} on it factors through a finite quotient $\mathrm{Aut}_{\mathcal{C}}(X)$ for a suitable $X \in \mathcal{G}$, hence we reconduct to the case of Proposition 1.3.9 and so there exists $Z \in \mathcal{C}$ which is dominated by X and such that $F(Z) \simeq E$.
- *(fully faithful)* If $Z_1, Z_2 \in \mathcal{C}$, we are able to find $X \in \mathcal{G}$ such that $X \geq Z_1, Z_2$ and again we can use Proposition 1.3.9 with $\mathcal{C}^X$ to conclude the fully faithfulness.

2. Let us consider $F_1, F_2 : \mathcal{C} \to \mathrm{FSETS}$ two fibre functors; for $i = 1, 2$ fix $\underline{\zeta}^i \in \prod_{X \in \mathcal{G}} F_i(X)$. Then we have the natural isomorphism of Proposition 1.3.10:

$$\mathrm{ev}_{\underline{\zeta}^i} : \mathrm{Hom}_{\mathrm{PRO}(\mathcal{C})}(\mathcal{G}^{\underline{\zeta}^i}, \cdot) \longrightarrow F_i$$

hence to conclude it is enough to show that $\mathcal{G}^{\underline{\zeta}^1}$ and $\mathcal{G}^{\underline{\zeta}^2}$ are isomorphic in $\mathrm{PRO}(\mathcal{C})$. Remembering that

$$\mathcal{G}^{\underline{\zeta}^i} = (\phi^{\underline{\zeta}^i}_{X,Y})_{X,Y \in \mathcal{G}, X \leq Y},$$

it is enough to find for every $X, Y \in \mathcal{G}$ with $X \leq Y$ an isomorphism (compatible with the projective system)

$$\phi^{\underline{\zeta}^1}_{X,Y} \xrightarrow{\sim} \phi^{\underline{\zeta}^2}_{X,Y}.$$

The compatibility is assured by Lemma 1.3.11, since if we find $\omega_Y \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $F(\omega_Y)(\zeta^1_Y) = \zeta^2_Y$, then there exists automatically a unique $\omega_X \in \mathrm{Aut}_{\mathcal{C}}(X)$ such that $F(\omega_X)(\zeta_X^1) = \zeta_X^2$, i.e. making the following diagram commutative:

$$\begin{tikzcd}
Y \arrow[r, "\omega_Y"] \arrow[d, "\phi^{\zeta^1}_{X,Y}"'] & Y \arrow[d, "\phi^{\zeta^2}_{X,Y}"] \\
X \arrow[r, "\omega_X"'] & X
\end{tikzcd}$$

So it remains just to show that for every $Y \in \mathcal{G}$ and $\zeta_Y^1, \zeta_Y^2 \in F(Y)$ we actually find an automorphism $\omega_Y \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $F(\omega_Y)(\zeta_Y^1) = \zeta_Y^2$. But we are able to find such a morphism by Lemma 1.3.8 which is automatically an isomorphism by Lemma 1.3.4. □

Chapter 2

Étale covers

In this chapter we construct the category of the étale covers of a connected scheme. Given a connected scheme, one may want to consider an object analogous to a topological cover and possibly one would like to have a totally algebraic construction of it; furthermore, if these objects form a Galois category, it would be meaningful to consider the Galois fundamental group, in analogy to the topological fundamental group. The concept of étale morphism and in particular étale cover turns out to be the right one.

In the first section we give definitions and some basic properties that we are going to need further on; in the second section, we prove that étale covers form in fact a Galois category.

We briefly recall here some terminology for morphisms of schemes:

Definition. Consider $X, S \in \textsc{Sch}$ and $\phi \in \operatorname{Hom}(X, S)$; we that ϕ is:

- *locally of finite type* if for every affine open subset $V \subset S$ and for every affine open subset $U \subset X$ with $U \subset \phi^{-1}(V)$, the morphism induced by ϕ:
$$\mathcal{O}_S(V) \to \mathcal{O}_X(U)$$
makes $\mathcal{O}_X(U)$ a finitely generated $\mathcal{O}_S(V)$-algebra.
- *unramified at* $x \in X$ if $\mathfrak{m}_{\phi(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)$ is a finite separable extension of $k(\phi(x))$.
- *unramified* if it is unramified at every $x \in X$.
- *finite* if for every affine open subset $V \subset S$, $\phi^{-1}(V)$ is affine and $\mathcal{O}_S(V) \to \mathcal{O}_X(\phi^{-1}(V))$ makes $\mathcal{O}_X(\phi^{-1}(V))$ a finitely generated $\mathcal{O}_S(V)$-module.

- *flat* if for every $x \in X$ the map induced on the stalk
$$\phi_x^\sharp : \mathcal{O}_{S,\phi(x)} \to \mathcal{O}_{X,x}$$
is flat.

2.1 Definitions and first properties

In this section we give the definitions of étale morphism and étale covers. From now on, every scheme will be assumed to be locally noetherian; however, we will point out in the proofs where we use this assumption.

Definition. Let A be a finite-dimensional algebra over a field k. We say that A is an *étale algebra* over k is A is isomorphic to a finite product of finite separable field extension of k. It is possible to verify that finite étale algebras over k form a category, which we denote by $\mathrm{FEA{\small LG}}/k$.

Lemma 2.1.1. *Let A a finite-dimensional algebra over a field k. Then the following are equivalent:*

(i) A is an étale algebra over k;

(ii) $A \otimes_k \bar{k}$ is isomorphic to a finite product of copies of $\bar{k}$;

(iii) $A \otimes_k \bar{k}$ is reduced;

(iv) $\Omega_{A|k} = 0$.

Proof. We prove a preliminary result: a finite-dimensional algebra A over a field k is reduced if and only if it is isomorphic to a finite product of finite field extensions of k.

($\Longleftarrow$) is immediate: a finite product of finite field extensions of k is a reduced algebra.

($\Longrightarrow$) If we take the decomposition of A in connected components:
$$A = \prod_{i=1}^{n} A_i$$
where A_i is a connected finite-dimensional k-algebra, we notice that it is enough to prove the statement for A_i to conclude. Hence we reduce to the case when A is connected; we have now just to show that it is a field: being a finite-dimensional k-algebra would imply then that it is a finite field extension of k. To do so, we consider $a \in A \setminus \{ 0 \} = A^\times$ and we find an inverse for a in

$A^\times$. Since A is finite dimensional over k, it is artinian so there exists $n \in \mathbb{N}$ such that $Aa^n = Aa^{n+1}$. We are able to find $b \in A$ such that

$$a^n = ba^{n+1} = b^2 a^{n+2} = \cdots = b^n a^{2n};$$

so $(a^n b^n)^2 = a^{2n} b^{2n} = a^n b^n$ and since A has no non-trivial idempotents we get either $a^n b^n = 0$ or $a^n b^n = 1$. The case $a^n b^n = 0$ forces $a^n = a^n(a^n b^n) = 0$ which implies $a = 0$ since A is reduced, and we find a contradiction with $a \in A^\times$. Hence must be $a(a^{n-1b^n}) = a^n b^n = 1$ and we find an inverse for a.

Now we are ready to proceed in proving the equivalent properties:

$(ii) \iff (iii)$ it is immediate from the previous part.

$(ii) \implies (i)$ Set $\bar{A} = A/\sqrt{0}$; it is reduced so it is isomorphic to the product

$$\bar{A} = \prod_{i=1}^{n} K_i$$

with K_i a finite field extension of k, thanks to the previous part. Since every morphism $A \to \bar{k}$ factors through one of the K_i, we have:

$$N := |\mathrm{Hom}_{\mathrm{Alg}/k}(A, \bar{k})| = \sum_{i=1}^{n} |\mathrm{Hom}_{\mathrm{Alg}/k}(K_i, \bar{k})|.$$

Notice that $|\mathrm{Hom}_{\mathrm{Alg}/k}(K_i, \bar{k})| \leq [K_i : k]$ and the equality holds if and only if K_i is a finite separable field extension of k. On the other hand, we have:

$$\dim_k(\bar{A}) = \sum_{i=1}^{n} [K_i : k] \leq \dim_k(A)$$

so we obtain $N \leq \dim_k(A)$ and the equality holds if and only if $A = \bar{A}$ and K_i is a finite separable field extension of k for every i. Now we have:

$$\mathrm{Hom}_{\mathrm{Alg}/k}(A, \bar{k}) = \mathrm{Hom}_{\mathrm{Alg}/\bar{k}}(A \times_k \bar{k}, \bar{k})$$

hence $N = |\mathrm{Hom}_{\mathrm{Alg}/\bar{k}}(A \times_k \bar{k}, \bar{k})| = \dim_{\bar{k}}(A \times_k \bar{k}) = \dim_k(A)$. So we obtain $A = \bar{A}$ and it is a finite product of finite separable field extension of k, as wanted.

$(i) \implies (iv)$ We have $A = \prod_{i=1}^{n} K_i$ with K_i finite separable field extension of k for every i. The maximal ideals of A are of the type $m_i = \ker(A \to K_i)$ where $A \to K_i$ is the usual projection on the i-th component. Since $\Omega^1_{A|k} = 0$ if and only if $(\Omega^1_{A|k})_{m_i} = \Omega^1_{K_i|k} = 0$ and the latter holds since K_i is a separable extension of k (for details, see [Liu] Chapter 6 Lemma 1.13).

(iv) $\Longrightarrow$ *(iii)* We want to prove that $A \otimes_k \bar{k}$ is reduced; since $\Omega^1_{A|k} = 0$ implies that $\Omega^1_{A\otimes_k \bar{k}|\bar{k}} = \Omega^1_{A|k} \otimes_k \bar{k} = 0$, we can take A as $A \otimes_k \bar{k}$ before and prove that it is reduced in the case $k = \bar{k}$ algebraically closed. Now, since A is a finite-dimensional k-algebra, it is artinian so every prime ideal of A is maximal; furthermore, there are just finitely many of them. So let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ all the maximal ideals of A: as $\sqrt{0} = \bigcap \mathfrak{m}_i$, by Chinese remainder theorem we have the exact sequence:

$$0 \longrightarrow \sqrt{0} \longrightarrow A \xrightarrow{\phi} \prod_{i=1}^{n} A/\mathfrak{m}_i \to 1.$$

Now for every i one has $[A/\mathfrak{m}_i : k] < +\infty$ and since k is supposed algebraically closed we get $A/\mathfrak{m}_i = k$.
Claim. It is possible to find $e_i \in A$ for $i = 1, \dots, n$ satisfying the following properties:

(i) $\phi(e_i) = (\delta_{ij})_{1\leq j\leq n}$;

(ii) $e_i e_j \in \sqrt{0}^2$ for every $i \neq j$;

(iii) $e_i - e_i^2 \in \sqrt{0}^2$ for every i.

Proof of the claim. It is clear that we are able to find $\{ e_i \}$ satisfying the property (i) by the surjectivity of the map ϕ; it is clear as well that e_i^2 satisfy both (i) and (ii). Furthermore, since A is artinian, the descending chain of ideals

$$(e_i) \supset (e_i^2) \supset (e_i^3) \supset \dots$$

stabilizes, so we can find an index N_i such that $(e_i^{N_i}) = (e_i^{2N_i})$; taking $N = \max N_i$, we find for every i an element $a_i \in A$ such that:

$$a_i e_i^{2N} = e_i^N.$$

Set $\varepsilon_i := a_i e_i^N$; then $\{ \varepsilon_i \}$ still satisfy (i) and (ii) and one has:

$$\varepsilon_i^2 = (a_i e_i^N)^2 = a_i(a_i e_i^{2N}) = a_i e_i^N = \varepsilon_i$$

hence they satisfy also (iii). □

Set $\lambda_i = p_i \circ \phi : A \to A/\mathfrak{m}_i$ and define $\lambda : A \to A$ in the following way:

$$\lambda(a) = \sum_{i=1}^{n} \lambda_i(a) e_i.$$

For every $a \in A$, one has $a - \lambda(a) \in \sqrt{0}$ and it is possible to verify that the map:

$$\begin{aligned} d : A &\longrightarrow \sqrt{0}/\sqrt{0}^2 \\ a &\longmapsto a - \lambda(a) \mod \sqrt{0}^2 \end{aligned}$$

is a k-derivation, hence it is 0 by assumption. Thus we have $\sqrt{0} = \sqrt{0}^2$ and since A is artinian $\sqrt{0}$ is nilpotent, so $\sqrt{0} = 0$: we obtain A reduced as wanted. □

Étale covers. We introduce now the notion of étale cover:

Definition. We say that the morphism $\phi : X \to S$ is

- *étale at* $x \in X$ if it is both flat and unramified at x;
- *étale* if it is étale at every point of X;
- *étal cover* of S if it is finite, surjective and étale.

About the stability of properties of scheme morphisms, we remind the validity of the following lemma:

Lemma 2.1.2. *1. If P is a property of morphism of schemes which is stable under composition and arbitrary base-change, it is stable by fibre products.*

2. Furthermore, if closed immersions have P, then for any

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

where g is separated and $g \circ f$ has P, then f has P.

It is possible to verify that being surjective, flat, unramified, étale satisfy 1. and being separated, proper, finite satisfy 2. The reader may keep in mind these facts, since we are going to use them quite often.

With the following two lemmas, we present useful characterizations for a finite morphism to be flat and unramified.

Lemma 2.1.3. *If $\phi : X \to S$ is a finite morphism, then ϕ is flat if and only if $\phi_* \mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module.*

Proof. Since flatness is a local property, we can reduce to consider the case when $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine schemes and the map ϕ is induced by a finite ring morphism $\phi^\sharp : A \to B$. Furthermore, since we are supposing every scheme to be locally noetherian, we may consider A to be noetherian (and so it is B then).

($\Longleftarrow$) Suppose $\phi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module, then for every point $\mathfrak{p} \in S$ there is a neighbourhood $U \ni \mathfrak{p}$ such that $\mathcal{O}_X(\phi^{-1}(U))$ is a free $\mathcal{O}_S(U)$-module, thus it is flat. But then the localization $\mathcal{O}_{X,\phi^{-1}(\mathfrak{p})}$ is a flat $\mathcal{O}_{S,\mathfrak{p}}$-module, hence ϕ is flat.

($\Longrightarrow$) Suppose ϕ flat. For every $\mathfrak{p} \in S$, $B_{\phi^{-1}(\mathfrak{p})}$ is a flat $A_{\mathfrak{p}}$-module; now since $A_{\mathfrak{p}}$ is noetherian and $B_{\phi^{-1}(\mathfrak{p})}$ is a finitely generated $A_{\mathfrak{p}}$-module, $B_{\phi^{-1}(\mathfrak{p})}$ flat implies $B_{\phi^{-1}(\mathfrak{p})}$ free (see [Liu], Chapter 1, Theorem 2.16). Hence we have:

$$B_{\phi^{-1}(\mathfrak{p})} = \bigoplus_{i=1}^{n} A_{\mathfrak{p}} \frac{b_i}{d}$$

where $d, b_i \in B$ and $s = \phi^\sharp(d) \in A \setminus \mathfrak{p}$. Thus we have an exact sequence of A_s-modules:

$$0 \to A_s^n \to B_d \to Q \to 0.$$

Now as A_s is noetherian, K is a finitely generated A_s-module hence its support $\mathrm{supp}(K)$ is the closed subset $V(\mathrm{Ann}(K)) \subset \mathrm{Spec}(A_s)$. Using that B_d is a finitely generated A_s-module, similarly we get $\mathrm{supp}(Q) = V(\mathrm{Ann}(Q)) \subset \mathrm{Spec}(A_s)$. But $\mathrm{supp}(Q) \cap \mathrm{supp}(K) =: U_{\mathfrak{p}}$ is an open neighborhood of $\mathfrak{p}$ in S such that $\phi_*\mathcal{O}_{X|U_{\mathfrak{p}}} \simeq \mathcal{O}_{U_{\mathfrak{p}}}$ and we obtain what we wanted. □

Lemma 2.1.4. *If $\phi : X \to S$ is a finite morphism, the following are equivalent:*

- *(a) ϕ is unramified;*
- *(b) $\Omega^1_{X|S} = 0$;*
- *(c) $\Delta_{X|S} : X \to X \times_S X$ is an open immersion;*
- *(d) $(\phi_*\mathcal{O}_X)_s \otimes_{\mathcal{O}_{S,s}} \kappa(s) = \mathcal{O}_{X_s}(X_s)$ is a finite étale algebra over $\kappa(s)$ for every $s \in S$.*

Proof. $(a) \Rightarrow (b)$ To show that $\Omega^1_{X|S} = 0$ it suffices to prove that the localization $\Omega^1_{X|S,x}$ is 0 for every $x \in X$. As it is a local property, as in the previous lemma we may assume that $X = \mathrm{Spec}B, S = \mathrm{Spec}A$ are affine and ϕ is induced by $\phi^\sharp : A \to B$ making B a finitely generated A-module. In this situation, we know that $\Omega_{X|S}$ is a finitely generated B-module and to conclude we have to show that the $B_{\mathfrak{q}}$-module $\Omega^1_{B_{\mathfrak{q}}|A_{\mathfrak{p}}} =: M$ is zero for every $\mathfrak{q} \in \mathrm{Spec}B$.

In fact, it is enough to show that $M \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{q}) = 0$: indeed $B_\mathfrak{q}$ is a local ring with unique maximal ideal $\mathfrak{q}B_\mathfrak{q}$ and by Nakayama's lemma we have the implication:

$$M \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{q}) = M/\mathfrak{q}M = 0 \quad \Longrightarrow \quad M = 0.$$

Thus we reduced to prove $\Omega^1_{B_\mathfrak{q}|A_\mathfrak{p}} \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{q}) = 0$.

Now we use that ϕ is ramified: we have $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}$ for any $\mathfrak{q} \in X$ above $\mathfrak{p} \in S$, so:

$$B_\mathfrak{q} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}) = \kappa(\mathfrak{q}).$$

But then we compute:

$$\Omega^1_{B_\mathfrak{q}|A_\mathfrak{p}} \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{q}) = \Omega^1_{B_\mathfrak{q}|A_\mathfrak{p}} \otimes_{B_\mathfrak{q}} (B_\mathfrak{q} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p})) \overset{(\dagger)}{=} \Omega^1_{B_\mathfrak{q} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p})|\kappa(\mathfrak{p})} = \Omega^1_{\kappa(\mathfrak{q})|\kappa(\mathfrak{p})} = 0,$$

where the equality (†) is proved in ([Liu], Chapter 6, Proposition 1.8) and the last equality comes from the fact that $\kappa(\mathfrak{q})$ is a finite separable extension of $\kappa(\mathfrak{p})$ since ϕ is unramified.

$(b) \Rightarrow (c)$ We suppose that $\Omega^1_{X|S} = 0$ and we want to prove that $\Delta_{X|S}$ is an open immersion. As ϕ is finite, so affine and hence separated, the diagonal morphism $\Delta_{X|S}$ is a closed immersion; in particular we have:

$$\Delta_{X|S}(X) = \operatorname{supp}(\Delta_{X|S*}\mathcal{O}_X).$$

Now remind that we have the map $\Delta^\sharp_{X|S} : \mathcal{O}_{X\times_S X} \to \Delta_{X|S*}\mathcal{O}_X$ and consider

$$\mathcal{I} = \operatorname{Ker}(\Delta^\sharp_{X|S}),$$

the sheaf of ideals. As in ([Liu], Chapter 6, Remark 1.18), we have:

$$\Delta^*_{X|S}(\mathcal{I}/\mathcal{I}^2) = \Omega^1_{X|S} = 0;$$

hence for every $x \in X$:

$$(\Delta^*_{X|S}(\mathcal{I}/\mathcal{I}^2))_x = \mathcal{I}_{\Delta_{X|S}(x)}/\mathcal{I}^2_{\Delta_{X|S}(x)} = 0,$$

i.e. $\mathcal{I}_{\Delta_{X|S}(x)} = \mathcal{I}^2_{\Delta_{X|S}(x)}$ for every $x \in X$. But now, since we are in a local setting, we can suppose S noetherian and from ϕ finite we get X noetherian, thus $\mathcal{I}$ is coherent. Then we can apply the Nakayama's lemma and $\mathcal{I}_{\Delta_{X|S}(x)} = \mathcal{I}^2_{\Delta_{X|S}(x)}$ implies:

$$\mathcal{I}_{\Delta_{X|S}(x)} = 0 \quad \forall x \in X.$$

From this, we have that $\mathcal{I}(\Delta_{X|S}(X)) = 0$ so $\Delta_{X|S}(X)$ is contained in the open subset $U := X \times_S X \setminus \operatorname{supp}(\mathcal{I})$. We want to show that we have also the inclusion $U \subset \Delta_{X|S}(X)$. As $\mathcal{I}_{\Delta_{X|S}(x)} = 0$ for all $x \in X$, the induced morphism on the stalks:

$$\Delta^{\sharp}_{X|S,u} : \mathcal{O}_{X\times_S X,u} \to (\Delta_{X|S*}\mathcal{O}_X)_x$$

is an isomorphism, hence U is contained in $\Delta_{X|S}(X) = \mathrm{supp}(\Delta_{X|S*}\mathcal{O}_X)$ and we have $\Delta_{X|S}(X) = U$ is open in $X \times_S X$.

$(c) \Rightarrow (d)$ If $s \in S$, we have to prove that $\mathcal{O}_{X_s}(X_s)$ is a finite étale algebra over $\kappa(s) := k$. To do so, let us consider Ω an algebraic closure of κ; given $s \in S$, we may consider the geometric point $\bar{s} : \mathrm{Spec}(\Omega) \to S$, $\bar{s}(*) = s$. Now we have the fibre product $X_{\bar{s}} = X \times_S \mathrm{Spec}(\Omega)$, given by the following diagram:

$$\begin{tikzcd}
X_{\bar{s}} \arrow[r] \arrow[d] & \mathrm{Spec}(\Omega) \arrow[d, "\bar{s}"] \\
X \arrow[r, "\phi"'] & S
\end{tikzcd}$$

Notice that, if $X_s = X \times_S \kappa$ is the usual fibre of s, we have $X_{\bar{s}} = X_s \times_\kappa \mathrm{Spec}(\Omega)$: indeed

$$\begin{aligned}
X_s \times_\kappa \mathrm{Spec}(\Omega) =& (X \times_S \kappa) \times_\kappa \mathrm{Spec}(\Omega) \\
=& X \times (\kappa \times_\kappa \mathrm{Spec}(\Omega)) \\
=& X \times \mathrm{Spec}(\Omega) = X_{\bar{s}}.
\end{aligned}$$

and $\mathcal{O}_{X_s}(X_s) \otimes_k \Omega = \mathcal{O}_{X_{\bar{s}}}(X_{\bar{s}})$. Thus, using the equivalent property (ii) in Lemma 2.1.1, it is enough to show that $\mathcal{O}_{X_{\bar{s}}}(X_{\bar{s}})$ is a finite étale algebra over Ω. To do so, we call $\bar{x} : \mathrm{Spec}(\Omega) \to X_{\bar{s}}$ the geometric point corresponding to $\bar{s}$ and we look at the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spec}(\Omega) \arrow[r, "\bar{x}"] \arrow[d, "(\mathrm{id}\times\bar{x})"] & X_{\bar{s}} \arrow[r, "p_1"] \arrow[d, "\Delta_{X_{\bar{s}}|\Omega}"] & X \arrow[d, "\Delta_{X|S}"] \\
\mathrm{Spec}(\Omega) \times_\Omega X_{\bar{s}} \arrow[r, "(\bar{x}\times\mathrm{id})"'] & X_{\bar{s}} \times_\Omega X_{\bar{s}} \arrow[r, "(p_1\times p_1)"'] & X \times_S X
\end{tikzcd}$$

Since open immersion are stable under base changes and we are supposing that $\Delta_{X|S}$ is an open immersion, we get $\Delta_{X_{\bar{s}}|\Omega}$ open immersion and then $\bar{x}$ is open immersion. But hence $\bar{x}$ induces an isomorphism on a closed and open subscheme of $X_{\bar{s}}$, that is a connected component of $X_{\bar{s}}$ since $\mathrm{Spec}(\Omega)$ is connected and $X_{\bar{s}}$ is finite. Since we hit every point of $X_{\bar{s}}$ with a suitable geometric point $\bar{x}$, we obtain thus the following decomposition in connected components for $X_{\bar{s}}$:

$$X_{\bar{s}} = \bigsqcup_{\bar{x}} \mathrm{Spec}(\Omega).$$

But then $\mathcal{O}_{X_{\bar{s}}}(X_{\bar{s}})$ is a finite product of copies of Ω, hence it is a finite étale algebra over Ω as wanted.

$(d) \Rightarrow (a)$ We want to prove that ϕ is unramified, so that it is unramified at every $x \in X$. Since the question is local, we may assume that ϕ is induced by a finite A-algebra $\phi^\sharp : A \to B$ with A noetherian; hence an element $s \in S$ correspond to a prime ideal $\mathfrak{p} \subset A$. By assumption, we have:

$$B \otimes_A k(\mathfrak{p}) = \prod_{i=1}^{n} k_i$$

as a $k(\mathfrak{p})$-algebra, where k_i is a finite separable field extension of $k(\mathfrak{p})$ for every i. Hence, any prime ideal of $B \otimes_A k(\mathfrak{p})$ is maximal and it is of the type

$$\mathfrak{m}_j = \ker(B \otimes_A k(\mathfrak{p}) \to k_j) \quad j = 1, \dots, n.$$

Now, if $\mathfrak{q} \in X$ is a prime ideal above $\mathfrak{p} \in S$ and $\mathfrak{m}_j$ is its image under the map $\mathrm{Spec}(B) \to \mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$, using the commutation of tensor product and localization we get:

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) = (B \otimes_A k(\mathfrak{p}))_{\mathfrak{m}_j} = k_j$$

which is a separable field extension of $k(\mathfrak{p})$; thus ϕ is unramified. $\square$

Definition. If $\phi : X \to S$ is an étale cover, we define the *rank function*:

$$\begin{aligned} r(\phi) : S &\longrightarrow \mathbb{Z}_{\geq 0} \\ s &\longmapsto r_s(\phi) := \mathrm{rank}_{\kappa(s)}(\mathcal{O}_{X_s}(X_s)) \\ &\qquad = \dim_{\kappa(\bar{s})}(\mathcal{O}_s(X_s) \otimes_{\kappa(s)} \kappa(\bar{s})) = |X_{\bar{s}}| \end{aligned}$$

It is locally constant and hence it is constant if S is connected.

We have a usefull lemma which uses the rank notion:

Lemma 2.1.5. *Let S be a connected scheme. Then any finite étale morphism $\phi : X \to S$ si automatically an étale cover. Furthermore, ϕ is an isomorphism if and only if $r(\phi) = 1$.*

Proof. ϕ is finite flat morphism, hence it is both open and closed. But then $\phi(X)$ is both open and closed in S, which is connected. This implies $\phi(X) = S$ so ϕ is an étale cover. For the second part:

$(\Longrightarrow)$ If ϕ is an isomorphism, it is clear that $r(\phi) = 1$.

$(\Longleftarrow)$ If ϕ has rank 1, we have $|X_{\bar{s}}| = 1$ hence ϕ is a bijection. Furthermore, ϕ is open and continuous so it is an homeomorphism. It remains to check that the induced map on the sheaves $\phi^\sharp : \mathcal{O}_S \to \phi_*\mathcal{O}_X$ is an isomorphism; we prove equivalently that for every $s \in S$ the map induced on the stalk

$$(\phi^\sharp)_s : \mathcal{O}_{S,s} \to (\phi_*\mathcal{O}_X)_s$$

is an isomorphism. As the question is local, we can reduce to the affine case $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$ and ϕ induced by a finite faithfully flat A-algebra $A \to B$; the hypothesis $r(\phi) = 1$ implies $B = Ab$ for a suitable $b \in B$. To conclude, it is enough to prove that $A \to B$ is surjective, i.e. that $b \in A$. By assumption, there exists $a \in A$ such that $ab = 1$ and on the other hand, since B is finite over A, there exists a monic polynomial $P_b = T^d + \sum_{i=0}^{d-1} r_i T^i \in A[T]$ such that $P_b(b) = 0$. Now, multiplying this equality by a^{d-1}, we get $b = -\sum_{i=0}^{d-1} r_i a^{d-1-i} \in A$. □

2.2 Main theorem for étale covers

In this section we present the main theorem for étale covers: given a connected scheme S, the category of étale covers of S is a Galois category. As the reader will notice, the proof is quite linear and the unique aspect more tricky to treat is the one concerning the quotient. Before getting started with the proof, we provide some useful tools in the following results.

Lemma 2.2.1. *An affine surjective morphism $\phi : X \to S$ is an étale cover if and only if there exist a finite faithfully flat morphism $f : S' \to S$ such that the first projection $\phi' : X' \to S'$ in the following fibre product:*

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "f'"'] & S' \arrow[d, "f"] \\ X \arrow[r, "\phi"'] & S \end{tikzcd}$$

is a totally split étale cover of S'.

Proof. ($\Longleftarrow$) We want to prove that ϕ is an étale cover. For every $s \in S$ we consider an affine open subset $U = \operatorname{Spec}(A) \subset S$ containing s; since f is finite, $f^{-1}(U) = \operatorname{Spec}(A')$ is an affine open subset of X. Furthermore, by Lemma 2.1.3 $f_*\mathcal{O}_{S'}$ is a locally free $\mathcal{O}_S$-module. Hence

$$f_{|f^{-1}(U)} : f^{-1}(U) \longrightarrow U$$

is induced by a finite A-algebra $f^\sharp : A \to A'$ and $A' = A^r$. As ϕ is finite, also $\phi^{-1}(U) = \operatorname{Spec}(B)$ is affine and $\phi_{|\phi^{-1}}(U)$ is induced by an A-algebra

$\phi^\sharp : A \to B$. Now the fibre product X' coincides with $\mathrm{Spec}(B \otimes_A A')$; since ϕ' is supposed totally split, $B \otimes_A A' = A'^s$ as A'-algebra so $B \otimes_A A' = A^{rs}$ as A-module, while we have $B \otimes_A A' = B \otimes_A A^r = B^r$ as A-module. We obtain the equality

$$B^r = A^{rs}$$

as A-module, so B is a direct factor of A^{rs} and it is flat over A. Hence ϕ is a flat morphism. Furthermore, B is a submodule of the finitely generated module A^{rs} where we remember that A is noetherian: thus we get that B is finitely generated itself as A-module, so the morphism ϕ is finite. It remains to show that ϕ is unramified. To do this, we look at the commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "f'"'] & S' \arrow[d, "f"] \\ X \arrow[r, "\phi"'] & S \end{tikzcd}$$

Since ϕ' is an étale cover, from Lemma 2.1.4 we have $\Omega^1_{X'|S'} = 0$ and as the previous is a fibre product diagram, we obtain

$$f'^*(\Omega_{X|S}) = \Omega^1_{X'|S'} = 0.$$

In particular, for every $x' \in X'$:

$$(f'^*(\Omega^1_{X|S}))_{x'} = \Omega^1_{X|S,f'(x')} = 0.$$

Now, since f' is the base change of f and f is surjective, f' is surjective so the previous equation is equivalent to:

$$\Omega^1_{X|S,x} = 0 \quad \forall x \in X;$$

henceforth $\Omega_{X|S} = 0$ and ϕ is unramified again by Lemma 2.1.4.

($\Longrightarrow$) We proceed by induction on $r(\phi)$:

- if $r(\phi) = 1$, from Lemma 2.1.5 ϕ is an isomorphism; but then we can take $f = \mathrm{id}_S$ and the statement is trivially true.
- if $r(\phi) > 1$: as ϕ is a finite étale morphism, by Lemma 2.1.4 the morphism $\Delta : X \to X \times_S X$ is an open and closed immersion. So we have the decomposition:

  $$X \times_S X = X \sqcup X'$$

  with $i_{X'} : X' \to X \times_S X$ an open and closed immersion. In particular, since $i_{X'}$ is open immersion it is étale (see [Bos] Chapter 8 Lemma 7) and since it is closed immersion it is finite, hence $i_{X'}$ is finite étale morphism. Also, the projection on the first component $p_1 : X \times_S X \to X$ is finite étale itself, since these properties are stable under base change. Now we define $\phi' : X' \to X$ to be the composition:

$$X' \xrightarrow{i_{X'}} X \times_S X \xrightarrow{p_1} X;$$

ϕ' is a finite étale morphism as it is composition of finite étale morphisms. Δ is a section of p_1 so for every $x \in X$ we have $p^{-1}(x) \cap \Delta(X) = \Delta(x)$, thus the rank of ϕ' is :

$$r(\phi') = r(p_1) - 1 = r(\phi) - 1.$$

Then ϕ' is finite étale and it has rank stricly smaller than ϕ, hence by inductive hypothesis we are able to find $f : S' \to X$ which is a finite faithfully flat morphism such that the projection π_1

$$\begin{tikzcd}
S' \times_S X' \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X' \arrow[d, "\phi'"] \\
S' \arrow[r, "f"] & X
\end{tikzcd}$$

is a totally split étale cover of S'. But now the morphism $\tilde{f} = \phi \circ f : S' \to S$ is again a finite faithfully flat morphism and using the properties of fibre product we have:

$$\begin{aligned}
S' \times_{\tilde{f},\phi} &= S' \times_{f,p_1} (X \times_S X) = S' \times_{f,p_1} (X \sqcup X') \\
&= (S' \times_{f,p_1} X) \sqcup (S' \times_{f,p_1} X')
\end{aligned}$$

so the fibre product is totally split.

□

Lemma 2.2.2. *Consider the following commutative diagram of schemes:*

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "\psi"] & X \arrow[dl, "\phi"] \\
S &
\end{tikzcd}$$

if $\phi : X \to S$ and $\psi : Y \to S$ are finite étale morphism then $u : Y \to X$ is a finite étale morphism.

Proof. Let us write $u = p_2 \circ \Gamma_u$ where $\Gamma_u : Y \to Y \times_S X$ is the graph of u and $p_2 : Y \times_S X \to X$ is the projection on the second component. Consider the following fibre product diagrams:

$$\begin{tikzcd}
Y \arrow[r, "\psi"] \arrow[d, "\Gamma_u"] & X \arrow[d, "\Delta_{X|S}"] \\
Y \times_S X \arrow[r, "u \times \mathrm{id}_X"'] & X \times_S X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times_S X \arrow[r] \arrow[d, "p_2"] & Y \arrow[d, "\psi"] \\
X \arrow[r, "\phi"] & S
\end{tikzcd}$$

From Lemma 2.1.4 $\Delta_{X|S}$ is finite étale hence its base change Γ_u is finite and étale as well. In a similar way, p_2 is finite étale since it is the base change of ψ in the left hand side diagram, hence the composition $u = p_2 \circ \Gamma$ is finite and étale. □

Another result we will need is that in the category of S-schemes, a faithfully flat morphism of finite type is a strict epimorphism. The following technical lemma is necessary, but we omit the proof here; the interested reader might find it in [Mil] (Chapter 1 Proposition 2.18).

Lemma 2.2.3. *If a ring homomorphism $f : A \to B$ is faithfully flat, then we have the exact sequence:*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{d^0} B \otimes_A B \xrightarrow{d^1} \dots$$

where $d^{n-1} = \sum_{i=0}^{r-1} (-1)^i e_i$ *and*

$$e_i(b_0 \otimes \cdots \otimes b_{n-1}) = b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{n-1}.$$

Proposition 2.2.4. *In the category* SCH$/S$*, faithfully flat morphisms of finite type are strict epimorphisms.*

Proof. We consider a faithfully flat morphism of finite type $f : Y \to X$ in the category of SCH$/S$; we want to prove that it is a strict epimorphism. To do so, consider $h : Y \to Z$ a morphism of S-schemes such that $h \circ p_1 = h \circ p_2$, where $p_i : Y \times_X Y \to Y$ is the projection on the i-th component. We want to prove that there exists a morphism $g : X \to Z$ such that $g \circ f = h$. We divide the proof in three steps:

step 1. Suppose firstly that $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), Z = \mathrm{Spec}(C)$ are all affine schemes and f, h are induced respectively by $f^\sharp : A \to B$ faithfully flat and $h^\sharp : C \to B$. In this situation, the projections p_i are induced by the morphisms $e_{i-1} : B \to B \otimes_A B$. The condition

$h \circ p_1 = h \circ p_2$ becomes $e_0 \circ h^\sharp = e_1 \circ h^\sharp$ i.e. $(e_0 - e_1) \circ h^\sharp = 0$. We use the Lemma 2.2.3 on $f^\sharp$ and obtain the exact sequence:

$$0 \longrightarrow A \xrightarrow{f^\sharp} B \xrightarrow{e_0 - e_1} B \otimes_A B;$$

then $\mathrm{im}(h) \subset \ker(e_0 - e_1) = \mathrm{im}(f)$, hence find $g^\sharp : C \to A$ such that $f^\sharp \circ g^\sharp = h^\sharp$ and we are done.

step 2. Suppose now $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ and Z arbitrary. First we prove the uniqueness of g. Indeed suppose there are g_1, g_2 such that $g_1 \circ f = g_2 \circ f$. If we denote $|g_i| : X^{\mathrm{set}} \to Y^{\mathrm{set}}$ the topological component of g_i, we have that $|g_1|, |g_2|$ agree on $|X|$ since f is surjective. Now pick $x \in X$, let U be an open affine neighbourhood of $g_1(x) = g(_2x)$ in Z and consider $X_a = \mathrm{Spec}(A_a)$ an open neighbourhood of x in $\mathrm{Spec}(X)$ for a suitable $a \in A$ such that $g_1(X_a) = g_2(X_a) \subset U$. If $b \in B$ is the image of a under the morphism $A \to B$, we have that B_b is again faithfully flat over A_a and we reduced to the previous case with $Z = U$ affine; hence we can conclude that $g_{1|X_a} = g_{2|X_a}$. But know this is true on an open affine covering of Z and thus g_1 and g_2 agree on X.

Using the uniqueness we just proved, we can reduce to define g locally. Then fix $x \in X$, $y = f^{-1}(x)$ and consider U an open affine neighbourhood of $h(y)$ in Z. We have that $f(h^{-1}(U))$ is open in X, as any flat morphism locally of finite type is open. Clearly $x \in f(h^{-1}(U))$ and we find $a \in A$ such that $X_a = \mathrm{Spec}(A_a)$ is an open neighbourhood of x contained in $f(h^{-1}(U))$.

Now, notice that $f^{-1}(X_a) \subset h^{-1}(U)$: indeed if $y_1 \in f^{-1}(X_a)$, $f(y_1) \in f(h^{-1}(U))$ and then $f(y) = f(y')$ with $y' \in h^{-1}(U)$. If $y' \in Y \times_X Y$ is such that $p_1(y') = y_1$ and $p_2(y') = y_2$, we have:

$$h(y_1) = h(p_1(y')) = h(p_2(y')) = h(y_2) \in U$$

hence $y_1 \in h^{-1}(U)$.

But now, if $b \in B$ is the image of $a \in A$ and $Y_b = \mathrm{Spec}(B_b)$, we have $h(Y_b) = h(f^{-1}(X_a)) \subset h(h^{-1}(U)) = U$ and B_b is faithfully flat over A_a. Then we reduced the problem to $h_{|Y_b} : Y \to U$ and $f_{|Y_b} : Y_b \to X_a$ with U affine, hence we can conclude with the step 1.

step 3. Let X, Y, Z arbitrary. We can firstly reduce to the case X affine by taking an affine covering of X. Since f is faithfully flat and open, it is quasi-compact, hence we find a finite affine cover of $Y = Y_1 \cup \cdots \cup Y_n$.

Denote $Y' = Y_1 \sqcup \dots Y_n$; Y' is affine and the morphism $f' : Y' \to X$ is faithfully flat. Now we look at the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(X,Z) \arrow[r, "f^*"] \arrow[d, "\mathrm{id}"] & \mathrm{Hom}(Y,Z) \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] \arrow[d, "\alpha"] & \mathrm{Hom}(Y \times_X Y, Z) \arrow[d] \\
\mathrm{Hom}(X,Z) \arrow[r, "f'^*"'] & \mathrm{Hom}(Y',Z) \arrow[r, shift left, "p_1'^*"] \arrow[r, shift right, "p_2'^*"'] & \mathrm{Hom}(Y' \times_X Y', Z)
\end{tikzcd}$$

where $p_i^* = \mathrm{Hom}(p_i, Z)$, the map α is injective and the second raw is exact using step 2. Now $h \in \ker(p_1^*, p_2^*)$ hence $\alpha(h) \in \ker(p_1'^*, p_2'^*) = \mathrm{im}(f'*)$, then there exists $g \in \mathrm{Hom}(X,Z)$ such that $f'^*(g) = \alpha(h)$; but now, using the commutativity of the right hand side square, we obtain $f^*(g) = g \circ f = h$ as wanted.

□

Theorem 2.2.5. *Let S be a connected scheme and let $\mathcal{C}_S$ the full subcategory of* Sch/S *whose objects are étale covers of S. Then the category $\mathcal{C}_S$ is a Galois category and given a geometric point $\bar{s} : Spec(\Omega) \to S$ and denoted by $X_{\bar{s}}^{set}$ the underline set of $X_{\bar{s}}$, the functor:*

$$\begin{aligned} F_{\bar{s}} : \mathcal{C}_S &\longrightarrow \text{FSets} \\ (\phi : X \to S) &\longmapsto X_{\bar{s}}^{set} \end{aligned}$$

is a fibre functor. In analogy with the topological case, the profinite group:

$$\pi_1(S; \bar{s}) := \pi_1(\mathcal{C}_S; F_{\bar{s}})$$

is the étale fundamental group *of S with base point $\bar{s}$ and given $\bar{s}_1, \bar{s}_2$ two geometric point, the set*

$$\pi_1(S; \bar{s}_1, \bar{s}_2) := \pi_1(\mathcal{C}_S; F_{\bar{1}}, F_{\bar{s}_2})$$

is the set of étale paths *from $\bar{s}_1$ to $\bar{s}_2$.*

Proof. We verify the axioms of the definition of a Galois category.

(i) A final object is given by $\mathrm{id}_S : S \to S$; furthermore, the fibre product of two étale covers $\phi : X \to S$ and $\psi : Y \to S$ in the category Sch/S gives again an étale cover: from the cartesian diagram

$$\begin{tikzcd}
X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\phi"] \\
Y \arrow[r, "\psi"'] & S
\end{tikzcd}$$

we have that p_1 is finite, étale and surjective since it is a base-change of ψ and then the composition $\phi \circ p_1$ is an étale cover of S.

(ii) $\mathcal{C}_S$ has an initial object $\emptyset$; if we consider the coproduct of two étale covers in the category $\textsc{Sch}/S$, it is again an étale cover of S.

About the quotient, the matter is less immediate and we have to split the proof in several steps. Let us consider $\phi : X \to S$ an étale cover and $G \subset \mathrm{Aut}_{\textsc{Sch}/S}(\phi)$ a finite subgroup.

step 1. We assume firstly that $S = \mathrm{Spec}(A)$ is affine. Since ϕ is finite hence affine, we obtain that $\phi^{-1}(A) = X = \mathrm{Spec}(B)$ is affine and ϕ is induced by a finite A-algebra $\phi^\sharp : A \to B$. We remind the equivalence of category between the affine S-schemes $\textsc{Aff}/S$ and $(\textsc{Alg}/A)^{\mathrm{op}}$; thus G corresponds to the finite group $G^{\mathrm{op}} \subset \mathrm{Aut}_{(\textsc{Alg}/A)^{\mathrm{op}}}(B)$. Consider the subring

$$B^{G^{\mathrm{op}}} = \{\, b \in B \mid g(b) = b \quad \forall g \in G^{\mathrm{op}} \,\} \subset B;$$

we show that $X \xrightarrow{\pi} \mathrm{Spec}(B^{G^{\mathrm{op}}})$ is the quotient of X in $\textsc{Aff}/S$. It is clear by the construction that for every $g \in G$, $\pi \circ g = \pi$; furthermore, if $f : X \to Y = \mathrm{Spec}(C)$ is a morphism in $\mathcal{C}_S$ such that $f \circ g = f$ for every $g \in G$, using the equivalence of categories we have the correspondence of commutative diagrams:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[dl, "f"] & & B & B \arrow[l, "g^{\mathrm{op}}"'] \\
Y & \arrow[r, dash] & {} & C \arrow[u, "f^\sharp"] \arrow[ur, "f^\sharp"'] &
\end{tikzcd}$$

But then $f^\sharp$ factors through $B^{G^{\mathrm{op}}}$ and hence f factors through $\mathrm{Spec}(B^{G^{\mathrm{op}}})$. Hence $\mathrm{Spec}(B^{G^{\mathrm{op}}}) = X/G$.

step 2. We prove that $\phi_G : X/G \to S$ is an étale cover. We start from the étale cover ϕ: using Lemma 2.2.1 we find $f : S' = \mathrm{Spec}(A') \to S$ induced by a faithfully flat algebra $A \to A'$ such that $B \otimes_A A' = A'^n$ as A'-algebra. We consider now the morphism $\lambda : B \to \bigoplus_{g \in G^{\mathrm{op}}} B$ defined by:

$$\lambda(b) = (b - g \cdot b)_{g \in G^{\mathrm{op}}};$$

by the definition of $B^{G^{\mathrm{op}}}$, we have an exact sequence of A-algebras:

$$0 \longrightarrow B^{\mathrm{op}} \longrightarrow B \xrightarrow{\lambda} \bigoplus_{g \in G^{\mathrm{op}}} B$$

and if we tensor it by the flat A-algebra A' we get:

$$0 \longrightarrow B^{G^{\mathrm{op}}} \otimes_A A' \longrightarrow B \otimes_A A' \xrightarrow{\lambda \otimes \mathrm{id}'_A} \bigoplus_{g \in G^{\mathrm{op}}} B \otimes_A A'.$$

Now notice that

$$B^{G^{\mathrm{op}}} \otimes_A A' = (B \otimes_A A')^{G^{\mathrm{op}}} = (A'^n)^{G^{\mathrm{op}}}$$

and G^{op} acts on A'^n just like a permutation of the indices $1, \dots, n$, hence one has:

$$(A'^n)^{G^{\mathrm{op}}} = \bigoplus_{F \subset \{1,\dots,n\}} A'.$$

If we translate now this conclusion in terms of scheme and consider the following fibre product diagram:

$$\begin{tikzcd}
X' \arrow[r, "\phi'"] \arrow[d] & S' = \mathrm{Spec}(A') \arrow[d, "f"] \\
\mathrm{Spec}(B^{G^{\mathrm{op}}}) = X/G \arrow[r, "\phi_G"] & S = \mathrm{Spec}(A)
\end{tikzcd}$$

we have that $X' = \mathrm{Spec}((A'^n)^{G^{\mathrm{op}}}) = \bigsqcup_{i \in F} S'$ is a totally split étale cover, hence again by Lemma 2.2.1 we can conclude that ϕ_G is an étale cover.

step 3. We can reduce to the previous case in the following way: we cover S with affine open subsets S_i and we obtain the result using steps 1. and 2. Then, we can glue together the quotient objects to obtain an étale cover of S and using the uniqueness of quotient object we conclude that it is in fact the quotient.

(iii) Consider two étale covers $\phi : X \to S$ and $\psi : Y \to S$ and let $u : Y \to X$ be a morphism in $\mathcal{C}_S$. We want to prove that it factors as a composition of a monomorphism and a strict epimorphism. By Lemma 2.2.1 u is finite and étale, hence it is both closed and open. Hence $u(Y) = X'$ is both open and closed, so $X''X \setminus X'$ is open and closed as well and we have $X = X' \sqcup X''$. Then $u' = u_{|X'} : Y \to X'$ is faithfully flat so a strict epimorphism by Proposition 2.2.4, the inclusion $i_{X'} : X' \to X$ is an open immersion hence a monomorphism and $u = u'' \circ u'$.

(iv) A final object in $\mathcal{C}_S$ is isomorphic to $\mathrm{id}_s : S \to S$ and $F_{\bar{s}}(\mathrm{id}_S)$ is a single point, i.e. a final object in FSETS. We show that $F_{\bar{s}}$ commutes with fibre product: if $\phi : X \to S$ and $\psi : Y \to S$ are étale covers and

$$F_{\bar{s}}(X \times_S Y) = (X \times_S Y) \times_S \mathrm{Spec}(\Omega);$$

we want to show that it coincides with

$$F_{\bar{s}}(X) \times_S F_{\bar{s}}(Y) = (X \times_S \operatorname{Spec}(\Omega)) \times_S (Y \times_S \operatorname{Spec}(\Omega)).$$

But this comes directly using the basic properties of fibre product:

$$\begin{aligned}(X \times_S \operatorname{Spec}(\Omega)) \times_S (Y \times_S \operatorname{Spec}(\Omega)) &= X \times_S (\operatorname{Spec}(\Omega) \times_S (Y \times_S \operatorname{Spec}(\Omega)))\\ &= X \times_S (\operatorname{Spec}(\Omega) \times_S Y) \times_S \operatorname{Spec}(\Omega)\\ &= X \times_S (Y \times_S \operatorname{Spec}(\Omega)) \times_S \operatorname{Spec}(\Omega)\\ &= X \times_S (Y \times_S (\operatorname{Spec}(\Omega) \times_S \operatorname{Spec}(\Omega))\\ &= (X \times_S Y) \times_S \operatorname{Spec}(\Omega).\end{aligned}$$

(v) It is straightforward that $F_{\bar{s}}$ commutes with finite coproducts and sends strict epimorphisms to strict epimorphisms. We have to prove that $F_{\bar{s}}$ commutes with categorical quotients by finite groups of automorphisms. Fix $\phi : X \to S$ an étale cover and $G \subset \operatorname{Aut}_{\mathcal{C}_S}((X, \phi))$ a finite subgroup. Since the assertion is local on S, using Lemma 2.2.2 we can assume that ϕ is totally split and that G acts on X as a permutation of copies of S (as in the proof of axiom (ii)). But then, the quotient X/G is isomorphic to a finite coproduct of copies of S and $F_{\bar{s}}$ commutes with finite coproducts.

(vi) Consider two étale covers $\phi : X \to S$ and $\psi : Y \to S$ and a morphism $u : X \to Y$ in $\mathcal{C}_S$ such that $F_{\bar{s}}(u) : F_{\bar{s}}(X) \to F_{\bar{s}}(Y)$ is a bijection. By Lemma 2.2.2 u is a finite étale morphism; furthermore, since $F_{\bar{s}}(X) \simeq F_{\bar{s}}(Y)$, we have that u is surjective and it has rank 1. Then it is an isomorphism by Lemma 2.1.5.

□

Chapter 3

Some applications

In this chapter we use the results we showed in the previous chapters to give some examples of Galois fundamental groups and present some interesting constructions. The first section is dedicated to varied extra tools we are going to need, about profinite groups and fundamental functors in particular; in the second section there are two initial examples of computation of the Galois fundamental group; in the third section we present some results on the first homotopy sequence, while in the last section we give the example of the fundamental group of an elliptic curve.

3.1 Some facts and observations

In this section we present a bunch of results in different topics that we did not need so far but that become useful from now on. Most of them will be presented without proof, as they are maybe well known facts; however, we will furnish a reference for each one in case the interested reader would like to look into further.

On schemes

Definition. Let X be a scheme over the field k and if K field extension of k denote $X_K = X \times_k \operatorname{Spec}(K)$.

- X is *geometrically connected* over k if the scheme X_K is connected for every field extension K of k.

- X is *geometrically reduced* over k if the scheme X_K is reduced for every field extension K of k.

It is possible to prove that X being geometrically connected (resp. geometrically reduced) is equivalent to $X_{\bar{k}}$ connected (resp. reduced) and also that if k is perfect, X geometrically reduced over k is equivalent to X reduced (see [Liu] Section 3.2.2). We consider a morphism $f : X \to S$; we say that f has geometrically reduced fibres if for every $s \in S$, X_s is geometrically reduced over $k(s)$. It is possible to show that if $f : X \to S$ is also flat, then also its base change has geometrically reduced fibres and if $X' \to X$ is étale then the composition with f, $X' \to S$ has finite fibres.

The following theorem is the so-called Stein factorization of a proper morphism; the interested reader can find the proof at Section 03GX.

Theorem 3.1.1. *Let $f : X \to S$ be a morphism such that $f_*\mathcal{O}_X$ is a quasicoherent $\mathcal{O}_S$-algebra. Then $f_*\mathcal{O}_X$ defines an S-scheme $p : S' := Spec(f_*\mathcal{O}_X) \to S$ and f factors as:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] & S \\ S' \arrow[ur, "p"'] & \end{tikzcd}$$

Furthermore:

1. *if f is proper, then p is finite and f' is proper with geometrically connected fibres; furthermore, if $f_*\mathcal{O}_X = \mathcal{O}_S$ then f has geometrically connected fibres;*

2. *if f is proper, flat and with geometrically reduced fibres, then p is an étale cover; in particular, $f_*\mathcal{O}_X = \mathcal{O}_S$ if and only if $f : X \to S$ has geometrically connected fibres.*

The Stein factorization theorem has one useful corollary:

Corollary 3.1.2. *Let $f : X \to S$ be a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$; if S is connected, then X is connected.*

On profinite groups

Until now we used just the definition of a profinite group as a projective limit of finite groups with the discrete topology; we present now some basic results on profinite groups. Their proofs can be found in [Wil] Chapter 1.

Finite groups endowed with the discrete topology are clearly compact and Hausdorff. It is possible to prove that these properties are preserved by projective limit and to show that a profinite group is compact, Hausdorff and totally disconnected. In fact, also the converse is true: if G is a compact, Hausdorff and totally disconnected topological group, it is a projective limit of finite groups, in particular:

$$G = \varprojlim_{N \lhd_O G} G/N$$

where we write $N \lhd_O G$ for open normal subgroups of G. More completely, one has the following proposition:

Proposition 3.1.3. *Let G a topological group. The following are equivalent:*

(i) G is profinite;

(ii) G is isomorphic to a closed subgroup of a cartesian product of finite groups;

(iii) G is compact and $\bigcap(N | N \lhd_O G) = 1$;

(iv) G is compact, Hausdorff and totally disconnected.

Lemma 3.1.4. *Let G be a profinite group and H a finite index subgroup. Then H contains a normal subgroup of G of finite index.*

Lemma 3.1.5. *Given $G = \varprojlim G_i$ a profinite group and denoted by $\phi_i : G \to G_i$ the canonical morphism, a basis for the topology of G is given by $\phi_i^{-1}(U)$ with $U \subset G_i$ open.*

Lemma 3.1.6. *Let G be a profinite group and H a subgroup. It is open if and only if it is closed and it has finite index in G.*

Lemma 3.1.7. *Let $f : G \to A$ a map from a profinite group to a discrete space. f is continuous if and only if there is an open normal subgroup $N \lhd_O G$ such that f factors through G/N.*

Given an arbitrary group , we find a profinite group associated to it, its profinite completion:

Definition. Given an arbitrary group G, its *profinite completion* $\widehat{G}$ is the profinite group

$$\widehat{G} := \varprojlim G/N$$

where N runs through the normal subgroups of G of finite index.

The profinite groups form a subcategory of the category of topological groups, where morphisms are continuous group homomorphisms. We have the following lemmas:

Lemma 3.1.8. *A morphism $u : H \to G$ between profinite groups is an epimorphism if and only if it is surjective.*

Proof. ($\Longleftarrow$) is immediate.

($\Longrightarrow$) We call $L = \operatorname{im}(u) \subset G$; we want to prove that $L = G$. We know that $G = \varprojlim G/N_i$ where $N_i \lhd_O G$; by brevity, we write $G_i = G/N_i$ and we denote $q_i : G \to G_i$ the natural projections; they are epimorphisms since they are surjective. Now, $u_i := q_i \circ u : H \to G_i$ is an epimorphism for every i, as it is a composition of epimorphisms; but since G_i is finite, u_i is surjective. We have a claim:

Claim. If $Y \subset G$ and $q_i(Y) = G_i$ for every i, then Y is dense in G.

Proof of the claim. Since $q_i(Y) = G_i$, then for every open $U \subset G_i$ one has $q_i^{-1}(U) \cap Y \neq \emptyset$. Using the fact that

$$\{\, q_i^{-1}(U) \mid i \in I, U \subset G_i \text{ open } \}$$

is a basis of the topology on G from Lemma 3.1.5, we find immediately that Y is dense in G.

We use the claim with $Y = L = \operatorname{im}(u)$: $q_i(Y) = q_i \circ u(H) = G_i$ since $q_i \circ u$ is surjective: we get L dense in G. But now since H is compact and u is continuous, L is also compact and L is closed as it is compact in G which is Hausdorff: thus it is dense and closed in G and $G = L$. □

Lemma 3.1.9. *A morphism $u : H \to G$ is a monomorphism if and only if* $\ker(u) = \{\, 1 \,\}$.

Proof. ($\Longrightarrow$) Suppose u monomorphism; if $\ker(u) \neq \{\, 1 \,\}$, pick a non trivial $x \in \ker(u)$ and set L the subgroup of H generated by x. Now we consider the morphisms: $\mathrm{id} = L \to H$ and $f : L \to H$ defined by $f(x) = 1$; then $f \neq \mathrm{id}$ but $u \circ f = u \circ \mathrm{id}$.

($\Longleftarrow$) Suppose $\ker(u)$ trivial, and $f, g : H' \to H$ two morphisms from a profinite group H' such that $u \circ f = u \circ g$. If there exists $x \in H'$ such that $f(x) \neq g(x)$, we have $u(f(x)) = u(g(x))$ and since u is a group morphism we get $u(f(x)g(x)^{-1}) = 1$ thus $f(x)g(x)^{-1} \in \ker(u) = \{\, 1 \,\}$ and $f(x) = g(x)$. □

Using the previous lemmas, it is easy to prove the following corollary:

Corollary 3.1.10. *A morphism $u : G \to H$ of profinite groups is an isomorphism if and only if it is both surjective and injective.*

Definition. Given a profinite group G and a prime p, its pro-p completion is

$$G^{(p)} := \varprojlim G/N$$

where N runs through the open normal subgroups of index a power of p.

Notice that $G^{(p)}$ is a maximal p-subgroup of G, i.e. a p-Sylow subgroup of G. If G is pronilpotent (in fact we will be interested in the case when G is abelian), we have the following useful structure result; the interested reader might find the proof in [Wil] Proposition 2.4.3.

Proposition 3.1.11. *Let G be a profinite group. If it is pronilpotent, then it is isomorphic to the cartesian product of its Sylow subgroups.*

Fundamental functors and their properties

Now that we have some tools on profinite groups, we can look more specifically on the Galois fundamental group we introduced in Chapter 1.
Let $\mathcal{C}$ a Galois category; from the Main Theorem of Chapter 1, we know that in fact $\mathcal{C} \simeq \mathcal{C}(\Pi)$ with Π the Galois fundamental group of $\mathcal{C}$. Consider now $X \in \mathcal{C}$ a connected object: it is a connected object in the category $\mathcal{C}(\Pi)$, i.e. a finite set with a transitive continuous Π-action. Now if $x \in X$ and $U = \mathrm{Stab}_\Pi(x)$, U is an open subgroup of Π since the action is continuous; furthermore, Π/U is isomorphic to the orbit of x under the action of Π, which is precisely X since X is connected. Thus we have a very useful characterization of connected objects in a Galois category $\mathcal{C}(\Pi)$: they are of the kind Π/U with U an open subgroup of Π.

Definition. Let $\mathcal{C}, C'$ be Galois categories; a functor $H : \mathcal{C} \to C'$ is a *fundamental* functor from $\mathcal{C}$ to $\mathcal{C}'$ if for any fibre functor $F' : \mathcal{C}' \to \text{FSets}$ such that $F' \circ H : \mathcal{C} \to \text{FSets}$ is a fibre functor for $\mathcal{C}$.

Notice that for the second part of the Main Theorem of Galois categories, it is enough that the latter condition is true for one fibre functor F', as all fibre functors are isomorphic.

In the category of our interest, the étale covers $\mathcal{C}_S$ of a connected scheme S, a fundamental functor may be obtain in the following way: consider a morphism $f : S' \to S$ of connected schemes and a geometric point $\bar{s}' : \mathrm{Spec}(\Omega) \to S'$ whose image in S through f gives a geometric point $\bar{s} : \mathrm{Spec}(\Omega) \to S$. Then we can consider the functor:

$$\begin{aligned} f^* : \mathcal{C}_S &\longrightarrow \mathcal{C}_{S'} \\ (\phi : X \to S) &\longmapsto (\phi' : X' \to S') \end{aligned}$$

where X' is the fibre product:

$$\begin{tikzcd} X' = X \times_S S' \arrow[r, "\phi'"] \arrow[d] & S' \arrow[d, "f"] \\ X \arrow[r, "\phi"] & S \end{tikzcd}$$

$f*$ is a fundamental functor, as $F_{\bar{s}'} \circ f^* = F_{\bar{s}}$: indeed

$$X'_{\bar{s}'} = X' \times_{S'} \operatorname{Spec}(\Omega) = (X \times_S S') \times_{S'} \operatorname{Spec}(\Omega) = X \times_S \operatorname{Spec}(\Omega) = X_{\bar{s}}.$$

The result is clear as well looking at the commutative diagram:

$$\begin{tikzcd}
X' \times_{S'} \operatorname{Spec}(\Omega) \arrow[r] \arrow[d] & \operatorname{Spec}(\Omega) \arrow[d, "\bar{s}'"] \\
X' \arrow[r, "\phi'"] \arrow[d] & S' \arrow[d, "f"] \\
X \arrow[r, "\phi"] & S
\end{tikzcd}$$

noticing that by definition $f \circ \bar{s}' = \bar{s}$.

If we have a fundamental functor $H : \mathcal{C} \to \mathcal{C}'$ between two Galois categories and $F' : \mathcal{C}' \to$ FSETS is a fibre functor, take $F = F' \circ H$ as fibre functor on $\mathcal{C}$. Then for every $\theta' \in \Pi' = \pi_1(\mathcal{C}', F')$, we can think the action of θ' as a natural trasformation $\theta' : F' \to F'$: for every $X' \in C'$, we have

$$\begin{aligned} \theta'_{X'} : F'(X) &\longrightarrow F'(X) \\ x' &\longmapsto \theta' \cdot x'. \end{aligned}$$

In this fashion, for every $X \in \mathcal{C}$ it defines:

$$\theta'_{H(X)} : F'(H(X)) = F(X) \longrightarrow F'(H(X)) = F(X);$$

thus $\theta'_{H(X)} \in \Pi$ and this gives rise to a morphism of profinite groups:

$$\begin{aligned} u_H : \Pi' &\longrightarrow \Pi = \pi_1(\mathcal{C}, F) \\ \theta' &\longmapsto \theta'_{H(\cdot)} \end{aligned}$$

We have also the converse construction: given a profinite groups morphism $u : \Pi' \to \Pi$, for every $X \in \mathcal{C}(\Pi)$ one can consider on it an induced action of Π' via u; if $\theta' \in \Pi'$, one defines:

$$\theta' \cdot X := u(\theta') \cdot X.$$

This gives rise to a fundamental functor $H_u : \mathcal{C}(\Pi) \to \mathcal{C}(\Pi')$. It is clear by this construction that $u_{H_u} = u$ and that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[d, "F"] & \mathcal{C}' \arrow[d, "F'"] \\
\mathcal{C}(\Pi) \arrow[r, "H_{u_H}"] & \mathcal{C}(\Pi')
\end{tikzcd}$$

commutes, so H and H_{u_H} are isomorphic functors.

Proposition 3.1.12. *Consider $\mathcal{C}, \mathcal{C}'$ Galois categories, $H : \mathcal{C} \to \mathcal{C}'$ a fundamental functor, F' fibre functor on $\mathcal{C}'$ and $F = F' \circ H$ fibre functor on $\mathcal{C}$ giving fundamental groups $\Pi' = \pi_1(\mathcal{C}'; F')$ and $\Pi = \pi_1(\mathcal{C}; F)$ respectively. Denote $u : \Pi' \to \Pi$ the morphism associated to H. Then:*

(i) The following are equivalent:

  a. u is an epimorphism;

  b. H sends connected objects to connected objects;

  c. H is fully faithful.

(ii) u is a monomorphism if and only if for any object $X' \in \mathcal{C}'$ there exists $X \in \mathcal{C}$ such that a connected component of $H(X)$ dominates X'.

(iii) u is an isomorphism if and only if H is an equivalence of categories.

(iv) Given

$$\mathcal{C} \xrightarrow{H} \mathcal{C}' \xrightarrow{H'} \mathcal{C}''$$

a sequence of fundamental functors between Galois categories, consider the induced sequence of fundamental groups:

$$\Pi'' \xrightarrow{u'} \Pi' \xrightarrow{u} \Pi.$$

Then $\ker(u) \supset \mathrm{im}(u')$ if and only if for every $X \in \mathcal{C}$, $H'(H(X))$ is totally split in $\mathcal{C}''$; $\ker(u) \subset \mathrm{im}(u')$ if and only if for any connected object $X' \in \mathcal{C}'$ such that $H'(X')$ has a section in $\mathcal{C}''$, there exists $X \in \mathcal{C}$ and a connected component of $H(X)$ dominating X' in $\mathcal{C}$.

Proof. Firstly notice that since $\mathcal{C} \simeq \mathcal{C}(\Pi)$, $\mathcal{C}' \simeq \mathcal{C}(\Pi')$ and H is a fundamental functor, we can suppose that $H : \mathcal{C}(\Pi) \to \mathcal{C}(\Pi')$ is given by the identity on objects and the action of Π' on the image of H is given by $u(\Pi')$. More in details, for every $x \in H(X) \simeq X$ and every $g' \in \Pi'$, one has

$$g' \cdot x = u(g') \cdot x.$$

Hence we will often write X instead of $H(X)$ if it is clear the action of which group we are considering; otherwise, we will stress the difference.

(i) *(b. $\Rightarrow$ a.)*
Suppose that H sends connected objects in connected objects; in particular for every open normal subgroup N of Π, $H(\Pi/N) \simeq \Pi/N$ is connected in $\mathcal{C}'$.

Take $N' = u^{-1}(N)$; then N' is an open normal subgroup of Π'. We have a morphism induced by u:

$$\bar{u} : \Pi'/N' \longrightarrow \Pi/N$$

and, if considered as a morphism in the category Π', it is automatically a strict epimorphism as Π/N is connected. Thus we have a surjective group morphism

$$\Pi' \longrightarrow \Pi/N$$

for every $N \lhd_O \Pi$ and we conclude that u is an epimorphism using the claim of Lemma 3.1.8.

(*a.* $\Rightarrow$ *c.*)
Suppose that u is surjective (again using Lemma 3.1.8), we prove that H is fully faithful: from what we said above, it amounts to prove that given $X, Y \in \mathcal{C}(\Pi)$ and $f : X \simeq H(X) \to Y \simeq H(Y)$ a Π'-invariant map, f is Π-invariant. But since u is surjective, for every $g \in \Pi$ we find $g' \in \Pi'$ such that $u(g') = g$, and for every $x \in X$ one has:

$$f(g \cdot x) = f(u(g') \cdot x) = f(g' \cdot x) = g' \cdot f(x) = u(g') \cdot f(x) = g \cdot f(x).$$

In this way we proved that H is full; but faithfulness is for free, since H acts as the identity on morphism and two Π'-invariant coinciding maps coincide also as Π-invariant maps.

(*c.* $\Rightarrow$ *b.*)
We take a connected object $\Pi/U \in \mathcal{C}$, we want to prove that $H(X)$ is connected in $\mathcal{C}'$, i.e. Π' acts transitively on Π/U via the morphism u. Now consider a connected component $Y \subset \Pi/U$ of $H(\Pi/U) \in \mathcal{C}'$; we have the canonical immersion $i' : Y \to \Pi/U$ (as a morphism in $\mathcal{C}'$, so a Π' invariant map). Since H is fully faithful one has $i' = H(i)$ with a suitable morphism $i : Y \to \Pi/U$ in $\mathcal{C}$; but now since Π/U is connected, i is automatically a strict epimorphism by Lemma 1.3.4 and as $Y \subset \Pi/U$ we obtain $Y = \Pi/U$, hence Π/U is connected in $\mathcal{C}'$.

(ii) ($\Longrightarrow$) Suppose that u is a monomorphism, i.e. $\ker(u) = \{1\}$ by Lemma 3.1.9 and take $X' \in \mathcal{C}'$. Notice that we can suppose that X' is connected: if we prove the statement with X' connected, we are done also in general just using the connected decomposition (see Proposition 1.3.2).

Now as X' is connected, it is isomorphic to Π'/U' for a suitable open subgroup U' of Π'; U' is closed so it is compact since Π' is compact. Then $u(U')$ is compact as u is continuous, hence closed since Π is

Hausdorff. Then $u(U')$ is closed and has finite index in $\mathrm{im}(u)$, so it is open in it; hence we are able to find an open subgroup $U \subset \Pi$ such that $U \cap \mathrm{im}(u) \subset u(U')$. We take then $X := \Pi/U$ and we consider the connected component of $H(X)$:

$$\mathrm{im}(u)U/U \simeq \mathrm{im}(u)/(U \cap \mathrm{im}(u));$$

it is connected since if we denote $W := U \cap \mathrm{im}(u)$ and we pick $xW, yW \in \mathrm{im}(U)/(U \cap \mathrm{im}(U))$, as now since $u^{-1}(U) \subset U'$ and u is injective, we have the well defined morphism

$$\mathrm{im}(u)U/U \longrightarrow \Pi'/U'$$

given by the section of u.

($\Longleftarrow$) Let $g' \in \ker(u)$ and suppose $g' \neq 1$; consider now an object $X' \in \mathcal{C}'$ such that g' acts non trivially on X'. By assumption, we are able to find an object (that we can suppose connected, as above) X such that there is a connected component $Y \subset H(X) \simeq X$ dominating X'. But then the action of g' on Y is given by the action of $u(g') = 1$, which is trivial, while the action of g' on X' is non trivial. Hence we contradict the Π'-invariance of the map $Y \to X'$.

(iii) From results on profinite groups, we have that u being an isomorphism is equivalent to being a morphism of profinite groups both injective and surjective.

($\Longrightarrow$) We already know from point (i) that u surjective implies H fully faithful. Hence it is enough to prove that H is essentially surjective: but this is immediate by looking at the proof of point (ii). Indeed given a connected object $\Pi'/U' \in \mathcal{C}'$ with U' open subgroup of Π', we take $U = u(U')$; since u is an isomorphism, we have the isomorphism:

$$\Pi/U \xrightarrow{\sim} \Pi'/U'$$

given by the inverse of u. So $H(\Pi/U) \simeq \Pi'/U'$ and H is essentially surjective.

($\Longleftarrow$) As we are assuming H essentially surjective, we have that for every connected object $\Pi'/U' \in \mathcal{C}'$ there exists an open subgroup U of Π such that $\Pi/U \simeq H(\Pi/U) \simeq \Pi'/U'$. This defines a morphism $\Pi \to \Pi'$ which is the inverse of u, hence u is an isomorphism.

(iv) For the first part:
($\Longrightarrow$) Suppose $\mathrm{im}(u') \subset \ker(u)$; this implies that $u \circ u'$ is trivial, i.e.

for every $g'' \in \Pi''$ one has $u \circ u'(g'') = 1$. Now consider an object $X \in \mathcal{C}'$. We want to show that $H'(H(X))$ is totally split in $\mathcal{C}''$; for every $g'' \in \Pi''$, the action of g'' on an element $x \in H'(H(X)) \simeq X$ is given by:

$$g'' \cdot x = u \circ u'(g'') \cdot x = 1 \cdot x.$$

Thus the action of Π'' is trivial on $H'(H(X))$ and then that $H'(H(X))$ is totally split in $\mathcal{C}''$.

($\Longleftarrow$) Suppose now that for every $X \in \mathcal{C}$ we have that $H'(H(X))$ is totally split in $\mathcal{C}''$; we want to prove that $\mathrm{im}(u') \subset \ker(u)$, i.e. that $u \circ u'$ is trivial. Take $g'' \in \mathcal{C}''$; for every $X \in \mathcal{C}$, $H'(H(X))$ is totally split in $\mathcal{C}''$, so the action of g'' on it is trivial. But this means that for every $X \in \mathcal{C}$, $u \circ u'(g'')$ acts trivially on X, thus $u \circ u'(g'') = 1$.

For the second part, we use the following fact:

Remark. $\ker(u) \subset U'$ if and only if there exists an open subgroup $U \subset \Pi$ such that the connected component of 1 of $H(\Pi/U)$ dominates $(\Pi'/U', 1)$ in $\mathcal{C}'^{\mathrm{pt}}$. The proof is exactly the same we gave in the point (ii).

($\Longrightarrow$) Suppose $\ker(u) \subset \mathrm{im}(u')$ and pick $X' \in \mathcal{C}'$ connected such that $H'(X')$ has a section. This means that X' considered in $\mathcal{C}''$, so with the action of Π'', is :

$$X' = \{\, x \,\} \sqcup \tilde{X}'$$

for a suitable element $x \in X'$, i.e. the point x is fixed by the action of Π'' on $H'(X') \simeq X'$. Now if $U' = \mathrm{Stab}_{\Pi'}(x)$, we have $X' \simeq \Pi'/U'$ as X' is connected and $\mathrm{im}(u') \subset U'$ by construction, since every point in $\mathrm{im}(u')$ keeps fixed x thus it is in U'. But then $\ker(u) \subset \mathrm{im}(u') \subset U'$, and by the remark above we have that there exists an open subgroup $U \subset \Pi$ such that $(H(\Pi/U), 1)_0 \geq (\Pi'/U', 1)$ in $\mathcal{C}'^{\mathrm{pt}}$.

($\Longleftarrow$) Since $\mathrm{im}(u')$ is closed, we know from [Rib] Proposition 2.4.1 that

$$\mathrm{im}(u') = \bigcap_{U' \leq_O \Pi',\ \mathrm{im}(u') \subset U'} U$$

Thus to prove that $\ker(u) \subset \mathrm{im}(u')$ it is enough to show that $\ker(u) \subset U'$ for every open subgroup U' of Π' containing $\mathrm{im}(u')$. But from the remark above $\ker(u) \subset U'$ if and only if there exists an open subgroup U of Π such that $(H(\Pi/U), 1)_0 \geq (\Pi'/U')$, and this is true since $\mathrm{im}(u') \subset U'$ implies that $H'(\Pi'/U')$ has a section in $\mathcal{C}''$, and then by assumpition there is an object $X \in \mathcal{C}$ such that $(H(X), 1)_0$ dominates $(\Pi'/U', 1)$.

□

The previous proposition gives a very interesting and useful dictionary between the properties of the morphism of fundamental groups and the ones of fundamental functors. This dictionary will be intensely used in the following sections to understand some kinds of fundamental groups.

With the next result, the last we present in this section, we return on the category of étale covers and in particular on the étale fundamental group; it says that in fact the fundamental group of a scheme is the same as the fundamental group of the associated reduced scheme. We omit the proof; the interested reader might find it in [Gro] Chapter IX.

Lemma 3.1.13. *Let X be a connected scheme and $i : X_{red} \to X$ be the underlying reduced closed subscheme and fix a generic point $x_\Omega : Spec(\Omega) \to X$; as $Spec(\Omega)$ is reduced, x_Ω factors through a generic point $x_\Omega : Spec(\Omega) \to X_{red}$. Then i induces an isomorphism of profinite groups:*

$$\pi_1(X_{red}; x_\Omega) \xrightarrow{\sim} \pi_1(X; x_\Omega).$$

3.2 Examples of Galois groups

In this section we present two examples of Galois fundamental groups: the first one is intended to help the reader to connect the "new" fundamental group with the topological fundamental group, in a setting where both of them have significance; the second one instead is in an algebraic setting: we prove that the étale fundamental group of a point coincides with the absolute Galois group of the base field.

The relationship with the fundamental topological group

We started the first chapter with a quick review of the construction of the topological fundamental group; throughout the rest of the chapter, we presented the construction of the Galois fundamental group for a Galois category, abandoning the topological point of view for a totally algebraic construction. The reader might now be interested in the connection between these two concepts: given an arc-wise connected topological space X, is the category $\textsc{Cov}(X)$ a Galois category? If yes, which is the relation between its Galois fundamental group and the topological fundamental group of X?

The answer to the first question is "partially yes": to obtain a Galois category, we must restrict to the subcategory of finite covers. More precisely, given a connected and locally arc-wise connected topological space X, the category $\mathrm{FC\small{OV}}(X)$ of finite topological covers of X is a Galois category and if $x \in X$ a fibre functor is given by:

$$\begin{aligned} F_x : \mathrm{FCov}(X) &\longrightarrow \mathrm{FSets} \\ (p : Y \to X) &\longmapsto F_x(p) := p^{-1}(x). \end{aligned}$$

It is clear here that we need to consider only finite covers, as we want the fibre of every point to be a finite set. Hence to every point $x \in X$ is associated the Galois fundamental group $\pi_1(\mathrm{FCov}(X), F_x)$ which we will denote by $\pi_1(X, x)$ to remark the analogy with the topological fundamental group; to avoid confusion, we will denote by $\pi_1^{\text{top}}(X, x)$ the topological fundamental group of X at the point x.

On the other hand, Theorem 1.1.5 induces another category equivalence on $\mathrm{FCov}(X)$:

Proposition 3.2.1. *The functor*

$$Fib_x : \mathrm{Cov}(X) \longrightarrow \pi_1^{top}(X, x)\text{-}\mathrm{Sets}$$

of Theorem 1.1.5 induces an equivalence of the categories

$$\mathrm{FCov}(X) \xrightarrow{\sim} \widehat{\pi_1^{top}(X, x)}\text{-}\mathrm{FSets}.$$

Proof. We already know that the functor Fib_x is an equivalence of categories. If $p : Y \to X$ is a finite connected cover of X, the set $Fib_x(p) = p^{-1}(x)$ is finite so the action of $\pi_1^{\text{top}}(X, x)$ on $p^{-1}(x)$ factors through a finite quotient $\pi_1^{\text{top}}(X, x)/N$ with N normal subgroup of $\pi_1^{\text{top}}(X, x)$ of finite index: indeed we have a morphism

$$\pi_1^{\text{top}}(X, x) \to G := \mathrm{Aut}_{\mathrm{FSets}}(p^{-1}(x))$$

where G is finite as $p^{-1}(x)$ is finite; then the kernel N of the morphism above has finite index, as $\pi_1^{\text{top}}(X, x)/N \hookrightarrow G$. Thus it is defined on $p^{-1}(x)$ a continuous action of $\widehat{\pi_1^{\text{top}}(X, x)}$.

Conversely, a continuous action of $\widehat{\pi_1^{\text{top}}(X, x)}$ on a finite set factors through a finite quotient, which is also a quotient of $\pi_1^{\text{top}}(X, x)$ by definition of profinite completion; hence it gives rise to a finite cover $Y \to X$. □

Thus we find an equivalence of categories

$$\widehat{\pi_1^{\text{top}}(X, x)}\text{-}\mathrm{FSets} \xrightarrow{\sim} \pi_1(X, x)\text{-}\mathrm{FSets}$$

and by Proposition 3.1.12 we get the isomorphism of profinite groups:

$$\pi_1(X,x) \simeq \widehat{\pi_1^{\mathrm{top}}(X,x)}.$$

In conclusion, the Galois fundamental group of the category of finite covers of a topological space X is isomorphic to the profinite completion of the topological fundamental group of X.

Spectrum of a field

We present now an example in the context of étale covers: the fundamental group of a point. Even if it is a trivial example in a certain way, as we consider the simplest algebraic variety, the result is nevertheless interesting; indeed one finds out that the fundamental group of Spec(k) is the absolute Galois group of k.

But we proceed by order: fix k a field and let us consider $S = \mathrm{Spec}(k)$ and a geometric point $\bar{s} : \mathrm{Spec}(\Omega) \to S$; we initially describe what is a Galois object in $\mathcal{C}_S$. To do so, consider a connected étale cover

$$\phi : X \longrightarrow S = \mathrm{Spec}(k);$$

the finiteness of ϕ forces X to be equal to an affine scheme Spec(K) with K a finite field extension of k. Indeed ϕ is finite hence affine, so $X = \mathrm{Spec}(R)$ and ϕ is induced by a finite morphism $k \to R$. This forces $R = K$ to be a finite separable field extension of k.

Now, by Lemma 1.3.5, $\phi : X \to S$ is a Galois object in $\mathcal{C}_S$ if and only if $|\mathrm{Aut}(X)| = |F_{\bar{s}}(X)|$; from the definition of the rank $r(\phi)$, we have:

$$|F_{\bar{s}}| = |X_{\bar{s}}| = \mathrm{rank}_k(K) = [K : k].$$

Thus we can conclude that $\phi : \mathrm{Spec}(K) = X \to S = \mathrm{Spec}(k)$ is a Galois object in $\mathcal{C}_S$ if and only if K is a finite Galois field extension of k. We denote by $\mathrm{Aut}(K|k)$ the Galois group of K.

Proposition 3.2.2. *Let k be a field, k^s its separable closure and $\bar{k}$ an algebraic closure of k containing k^s. Denote by $\Gamma_k := Aut(k^s|k)$ the absolute Galois group of k. Set $S = Spec(k)$ and the geometric point $\bar{s} : Spec(\bar{k}) \to S$. Then there is an isomorphism of profinite groups:*

$$\pi_1(S;\bar{s}) \xrightarrow{\sim} \Gamma_k.$$

Proof. From what said above, $\phi : X \to S$ is a Galois object in $\mathcal{C}_S$ if and only if $X = \mathrm{Spec}(K)$ with K a finite Galois field extension of k; notice that we can consider $K \subset \bar{k}$, up to taking a field in $\bar{k}$ isomorphic (as a k-algebra) to K.

For every such K, we have the isomorphism:

$$(\Gamma_k)_{|K} = \mathrm{Aut}(K|k) \simeq \mathrm{Aut}_{k\text{-alg}}(K);$$

furthermore, remember that there is the equivalence of categories induced by the usual equivalence of affine schemes:

$$\begin{aligned} \mathcal{C}_S &\xrightarrow{\sim} (\mathrm{FE}\textsc{Alg}/k)^{\mathrm{op}} \\ (\phi : X \to S) &\longmapsto (\phi^\sharp(X) : k \to \mathcal{O}_X(X)) \end{aligned}$$

Hence, if $X = \mathrm{Spec}(K)$ is Galois, we have the isomorphism:

$$\mathrm{Aut}_{\mathcal{C}_S}(X)^{\mathrm{op}} \simeq \mathrm{Aut}_{k\text{-alg}}(K);$$

finally, one has:

$$\pi_1(S, \bar{s}) \simeq \varprojlim_{X \text{ Galois}} \mathrm{Aut}_{\mathcal{C}_S}(X)^{\mathrm{op}} = \varprojlim_{K \text{ Galois}} \mathrm{Aut}_{k\text{-alg}}(K) \simeq \Gamma_k$$

where the first equality comes from the proof of the Main Theorem(see Section 1.3). □

3.3 The first homotopy sequence

We present here some results related to the first homotopy sequence. We give immediately the proof of the theorem and then we provide some examples showing its usefulness.

Theorem 3.3.1. *Let S be a connected scheme, $f : X \to S$ a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and $s \in S$. Fix Ω an algebraically closed field, x_Ω : Spec(Ω) $\to X_{\bar{s}}$ a geometric point with image $x_\Omega \in X$ and $s_\Omega \in S$. Consider the sequence:*

$$(X_{\bar{s}}, x_\Omega) \longrightarrow (X, x_\Omega) \longrightarrow (S, s_\Omega)$$

and the induced sequence of profinite groups

$$\pi_1(X_{\bar{s}}; x_\Omega) \xrightarrow{i} \pi_1(X; x_\Omega) \xrightarrow{p} \pi_1(S; s_\Omega).$$

Then p is an epimorphims and im(i) $\subset \ker(p)$. If f is flat and has geometrically reduced fibres, then im(i) $= \ker(p)$.

Proof. We firstly prove that p is an epimorphism; by Proposition 3.1.12 point (i), it is enough to show that every connected étale cover $\phi : S' \to S$ is sent by the fundamental functor associated to p in a connected étale cover $X' \to X$. We remember that this cover is obtained by the base change:

$$\begin{tikzcd}
X' \arrow[r, "\phi'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r, "\phi"] & S
\end{tikzcd}$$

One has:

$$f'_*(\mathcal{O}_{X'}) = f'_*(\phi^*\mathcal{O}_X) \overset{(\dagger)}{=} \phi^* f_*\mathcal{O}_X \overset{(\ddagger)}{=} \phi^*\mathcal{O}_S = \mathcal{O}_{S'}$$

where $(\dagger)$ comes from [Har] Chapter III Proposition 9.3 and $(\ddagger)$ comes from $f_*\mathcal{O}_X = \mathcal{O}_S$ by assumption. Now since f' is proper as it is the base change of f, from Theorem 3.1.1 we have X' connected.

We show $\mathrm{im}(i) \subset \ker(p)$. We use Proposition 3.1.12 point (iv), thus to conclude it is enough to prove that if $\phi : S' \to S$ is an étale cover of S, its image in $\mathcal{C}_{X_{\bar{s}}}$ is totally split, i.e. it is a finite coproduct of copies of $X_{\bar{s}}$. This étale cover of $X_{\bar{s}}$ is given by $X'_{\bar{s}} \to X_{\bar{s}}$ in the following commutative diagram, where the two columns on the left are constructed just by base changes:

$$\begin{tikzcd}
X'_{\bar{s}} \arrow[r] \arrow[d] & X_{\bar{s}} = X \times_S k(\bar{s}) \arrow[d] \arrow[rd] & \\
X' \arrow[r] \arrow[d] & X \arrow[d, "f"] & \mathrm{Spec}(k(\bar{s})) \arrow[ld, "\bar{s}"] \\
S' \arrow[r, "\phi"] & S &
\end{tikzcd}$$

But now :

$$\begin{aligned}
X'_{\bar{s}} = X_{\bar{s}} \times_S S' &= (X \times_S k(\bar{s})) \times_S S' \\
&= X \times_S (k(\bar{s}) \times_S S') \\
&= X \times_S \bigsqcup_{S'_{\bar{s}}} \mathrm{Spec}(k(\bar{s})) \\
&= \bigsqcup_{S'_{\bar{s}}} X_{\bar{s}}.
\end{aligned}$$

Suppose now f flat with geometrically reduced fibres; we prove now $\ker(p) \subset \mathrm{im}(i)$. We use again the characterization of Proposition 3.1.12:

we pick a connected étale cover $\phi : X' \to X$; suppose that the base change of ϕ,

$$\bar{\phi} : X'_{\bar{s}} \to X_{\bar{s}}$$

admits a section $\sigma : X_{\bar{s}} \to X'_{\bar{s}}$. We have to prove that ϕ comes by base change from a connected étale cover $S' \to S$. Since ϕ is finite (so proper) and étale and f is proper, flat with geometrically connected fibres, $g = f \circ \phi$ is also proper, flat with geometrically connected fibres. By Theorem 3.1.1, g factors as:

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "g'"'] & S \\
S' \arrow[ur, "p"'] &
\end{tikzcd}$$

with p étale cover. Furthermore, we have that g' is surjective: indeed pick $s' \in \S'$ and consider $S'_{p(s')}$: by 3.1.1, there is a bijection between the set of the connected components of $S'_{p(s')}$ and the connected components of $X'_{p(s')}$, so there is a connected component of the latter that corresponds to the point s'. In particular then there is a point $x' \in X'$ that g' sends in s' and g' is surjective; as X' is connected, we obtain hence S' connected. Now consider the following diagram:

$$\begin{tikzcd}
X' \arrow[r, dashed, "\alpha"'] \arrow[rr, bend left, "g"] \arrow[dr, "\phi"'] & X'' \arrow[r, "f'"'] \arrow[d, "p'"] & S' \arrow[d, "p"] \\
 & X \arrow[r, "f"'] & S
\end{tikzcd}$$

where the square is given by a base change and the map α comes from the universal property of fibre product. Since S' is connected and we already proved the exactness on the right of the sequence, we have that X'' is connected. Notice that if we prove that α is an isomorphism, we are done: in this case, ϕ is given by the base change of the étale cover p. So to conclude we have to show that α is an isomorphism.

Since p is an étale cover, its base change p' is an étale cover; thus from $\phi = p' \circ \alpha$ by Lemma 2.2.2 we have that α is finite étale. But then from Lemma 2.1.5 α is automatically an étale cover and it remains to show that it has rank 1. To do this, consider the base change of the previous diagram via $\bar{s} : \mathrm{Spec}(k(\bar{s})) \to S$, i.e.:

$$\begin{tikzcd}
X'_{\bar{s}} \arrow[r, dashed, "\alpha_{\bar{s}}"'] \arrow[rr, bend left, "g_{\bar{s}}"] \arrow[dr, "\phi_{\bar{s}}"'] & X''_{\bar{s}} \arrow[r, "f'_{\bar{s}}"'] \arrow[d, "p'_{\bar{s}}"] & S'_{\bar{s}} \arrow[d, "p_{\bar{s}}"] \\
 & X_{\bar{s}} \arrow[r, "f_{\bar{s}}"'] & \mathrm{Spec}(k(\bar{s}))
\end{tikzcd}$$

We have that $\alpha_{\bar{s}}$ is an étale cover (it is the base change of an étale cover), thus it induces a surjective map

$$\pi_0(X'_{\bar{s}}) \longrightarrow \pi_0(X''_{\bar{s}})$$

where $\pi_0(\cdot)$ denotes the set of connected components. Furthermore, since both g' and f' are geometrically connected by 3.1.1, the latter map is in fact bijective. Thus to conclude it is enough to prove that $\alpha_{\bar{s}}$ induces an isomorphism from a connected component $X'_{\bar{s}0} \in \pi_0(X'_{\bar{s}})$ to $X''_{\bar{s}0} := \alpha_{\bar{s}}(X'_{\bar{s}0})$. Consider then

$$X'_{\bar{s}0} := \sigma(X_{\bar{s}});$$

we have that σ induces an isomorphism from $X_{\bar{s}}$ to $X'_{\bar{s}}$. Furthermore, $p_{\bar{s}}$ is totally split: indeed it is an étale cover hence it is finite, thus if $\{*\} = \mathrm{Spec}(k(\bar{s}))$ we have that $p_{\bar{s}}^{-1}(*)$ are finitely many points, hence a finite product of copies of $\mathrm{Spec}(k(\bar{s}))$. Then the base change $p'_{\bar{s}}$ is totally split as well and so it induces an isomorphism from $X''_{\bar{s}0}$ to $X_{\bar{s}}$. But now one has:

$$\sigma|^{X'_{\bar{s}}} \circ p_{\bar{s}'}|_{X''_{\bar{s}0}} \circ \alpha_{\bar{s}}|^{X''_{\bar{s}0}}_{X'_{\bar{s}0}} = \mathrm{id}_{X''_{\bar{s}0}}.$$

□

The previous theorem allows us to prove the following corollary, which tells us that under certain condition the fundamental group of the fibre product of two schemes is precisely the product of the respective fundamental groups, as one could hope. Notice that this result is not automatic: using the Artin-Schreier method, it is possible to produce a counterexample and find X not proper and k of positive characteristic such that $\pi_1(X \times_k X, (x, x))$ is not isomorphic to $\pi_1(X, x) \times \pi_1(X, x)$.

Corollary 3.3.2. *Let k be an algebraically closed field, X a connected, proper scheme over k and Y a connected scheme over k. For any $x : Spec(k) \to X$ and $y : Spec(k) \to Y$, the morphism of profinite groups induced by the projections $p_X : X \times_k Y \to X$ and $p_Y : X \times_k Y \to Y$:*

$$\pi_1(X \times_k Y; (x, y)) \longrightarrow \pi_1(X; x) \times \pi_1(Y, y)$$

is an isomorphism.

Proof. From Lemma 3.1.13 we may suppose that X is reduced, so X geometrically reduced over k as k is algebraically closed. Hence X is geometrically connected, proper, geometrically reduced and surjective over k; all these properties are stable under base change, hence

$$p_Y : X \times_k Y \longrightarrow Y$$

has all of them as well. In particular, p_Y has geometrically connected fibres and from Theorem 3.1.1 one has that $p_{Y*}\mathcal{O}_{X\times_k Y} = \mathcal{O}_Y$ so p_Y satisfies the extra hypothesis of Theorem 3.3.1. Now consider the composition

$$(X \times_k Y)_y \overset{p_1}{\to} X \times_k Y \overset{p_X}{\to} X$$

where p_1 is the projection on the first component. Using the properties of fibre product we have:

$$(X \times_k Y)_y = (X \times_k Y) \times_Y \operatorname{Spec}(k) = X \times_k \operatorname{Spec}(k) = X;$$

by construction $p_X \circ p_1 = \mathrm{id}_X$ thus p_X gives a left inverse of p_1. Now if we consider the sequence

$$(X, x) = ((X \times_k Y)_y, x) \xrightarrow{p_1} (X \times_k Y, (x, y)) \xrightarrow{p_Y} (Y, y)$$

by Theorem 3.3.1 we obtain the exact sequence:

$$\pi_1(X; x) \overset{\pi_1(p_1)}{\longrightarrow} \pi_1(X \times_k Y; (x, y)) \overset{\pi_1(p_Y)}{\longrightarrow} \pi_1(Y; y) \longrightarrow 1$$

and $\pi_1(p_X)$ gives a left inverse of $\pi_1(p_1)$; hence the sequence is split, thus:

$$\pi_1(X \times_k Y; (x, y)) \simeq \pi_1(X; x) \times \pi_1(Y, y).$$

□

We present now another interesting result regarding a proper scheme over an algebraically closed field: its fundamental group is invariant under algebraically closed field extensions.

Before we can proceed with the proof of this proposition, we need a technical lemma:

Lemma 3.3.3. *Let X be a connected scheme of finite type over a field k and let Ω a field extension of k. For any étale cover $\phi : Y \to X_\Omega$, there exists a finitely generated k-algebra $R \subset \Omega$ and an affine morphism of finite type $\tilde{\phi} : \tilde{Y} \to X_R$ such that $\phi : Y \to X_\Omega$ is a base change of $\tilde{\phi}$. Furthermore, if η is the generic point of Spec(R), then $\tilde{\phi}_{k(\eta)} : \tilde{Y}_{k(\eta)} \to X_{k(\eta)}$ is an étale cover.*

Proof. As X is quasi-compact, we have a finite covering of open affine subsets $X_i = \mathrm{Spec}(A_i)$ with $i = 1, \dots, n$ where A_i is a finitely generated k-algebra for every i. Then $X_{i\Omega} = \mathrm{Spec}(A_i \otimes_k \Omega)$ and since ϕ is finite hence affine, we have $U_i = \phi^{-1}(X_{i\Omega}) = \mathrm{Spec}(B_i)$ where

$$B_i = (A_i \otimes_k \Omega)[T_1, \dots, T_{r_i}]/(P_{i,1}, \dots, P_{i,s_i})$$

where $P_{i,j}$ are polynomials in the variables $T_1, \dots, T_{r_i}$. Now, these polynomials have coefficients in $A_i \otimes_k \Omega$ and then each coefficient is a finite sum of pure tensors $a \otimes \lambda$ with $a \in A_i$ and $\lambda \in \Omega$. Now, if we denote by R_i the subalgebra of Ω generated by these λ (they are finitely many), we get:

$$B_i = (A_i \otimes_k R_i)[T_1, \dots, T_{r_i}]/(P_{i,1}, \dots, P_{i,s_i}) \otimes_{R_i} \Omega;$$

we denote by R the subalgebra of Ω generated by all R_i for $i = 1, \dots, n$; it is again finitely generated over k. Now, we can glue the affine schemes

$$\mathrm{Spec}((A_i \otimes_k R)[T_1, \dots, T_n]/(P_{i,1}, \dots, P_{i,s_i}))$$

to get our $\tilde{Y}$; it is possible up to enlarging the algebra R, if needed, in order to make the gluing open subsets $U_i \cap U_j$ descend again to R. Furthermore, by construction the morphism $\tilde{\phi} : \tilde{Y} \to X_R$ is affine.

Now, if η is the generic point of $\mathrm{Spec}(R)$, we have the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "\phi"'] & \tilde{Y} \arrow[r] \arrow[d, "\tilde{\phi}"'] & \tilde{Y}_{k(\eta)} \arrow[d, "\tilde{\phi}_{k(\eta)}"'] \\
X_\Omega \arrow[r] & X_R \arrow[r] & X_{k(\eta)}
\end{tikzcd}$$

and since $k(\eta) \to \Omega$ is faithfully flat and ϕ is an étale cover, then also $\tilde{\phi}_{k(\eta)}$ is an étale cover. □

Proposition 3.3.4. *Let k an algebraically closed field, X a connected and proper scheme over k and Ω an algebraically closed field extension of k. Fix a geometric point $x_\Omega : Spec(\Omega) \to X_\Omega$ and denote by x_Ω its image in X. Then $(X_\Omega; x_\Omega) \to (X; x_\Omega)$ induces an isomorphism of profinite groups:*

$$\pi_1(X_\Omega; x_\Omega) \xrightarrow{\sim} \pi_1(X; x_\Omega).$$

Proof. Remind that in our case the functor $H : \mathcal{C}_X \to \mathcal{C}_{X_\Omega}$ associated to the morphism of groups above is given by the base change under the morphism

$X_\Omega \to X$. More precisely, if $\phi : Y \to X$ is an étale cover of X, we have that $H(\phi) : Y' \to X_\Omega$ is the map given by the fibre product:

$$\begin{tikzcd}
Y' = Y \times_X X_\Omega \arrow[r, "H(\phi)"] \arrow[d] & X_\Omega \arrow[d] \\
Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

Using the characterizations (i) and (ii) of Proposition 3.1.12, to conclude it suffices to prove that H sends connected étale covers to connected étale covers and that for any $\phi' : Y' \to X_\Omega$ connected étale cover, there exists a connected étale cover $\phi : Y \to X$ such that $H(\phi)$ dominates ϕ'.

For the first part: if $\phi : Y \to X$ is connected étale cover, since k is supposed algebraically closed, Y connected implies Y algebraically connected and then Y_Ω is connected. But then $Y_\Omega \to X_\Omega$ is connected étale covering and we are done.

For the second part: fix $\phi : Y \to X_\Omega$ a connected étale cover. We apply Lemma 3.3.3 and we find a finitely generated k-algebra such that there exists an affine morphism of finite type $\phi^0 : Y^0 \to X_R$ such that ϕ is a base change of ϕ^0. Moreover, up to replacing R by R_r for some $r \in R \setminus \{0\}$ and using the second part of the same lemma, we may assume that ϕ^0 is an étale cover. Notice that since $Y^0_\Omega = Y$ (see the proof of Lemma 3.3.3) and Y is supposed connected, we have Y^0 connected. We may call ϕ this morphism ϕ^0, i.e. we may suppose that the étale cover $\phi : Y \to X_\Omega$ is

Now, denote $S = \mathrm{Spec}(R)$ and fix $s : \mathrm{Spec}(k) \to S$, $x : \mathrm{Spec}(k) \to X$. The étale cover $\phi^0 : Y^0 \to X \times_k S$ correspond to the open subgroup

$$U := \mathrm{Stab}_{\pi_1(X \times_k S;(x,s))}(\phi) \subset \pi_1(X \times_k S;(x,s))$$

and from Corollary 3.3.2 we have the isomorphism:

$$\pi_1(X \times_k S;(x,s)) \simeq \pi_1(X;x) \times \pi_1(S;s).$$

Thus we are able to find open subgroups $U_S \subset \pi_1(S;s)$ and $U_X \subset \pi_1(X;x)$ such that $U_X \times U_S \subset U$. U_S and U_X correspond to certain connected étale covers $\psi_S : \tilde{S} \to S$ and $\psi_X : \tilde{X} \to X$ respectively, such that ϕ^0 is given by the quotient $(\psi_X \times \psi_S)/U$.

We denote $\tilde{Y}^0 = Y^0 \times_{X \times_k S} (X \times_k \tilde{s})$ and in proceeding we keep in mind the the following diagram:

$$\begin{tikzcd}
\tilde{X} \times_k \tilde{S} \arrow[rd, dashed] \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\
& \tilde{Y}^0 \arrow[r] \arrow[d] & Y^0 \arrow[d] \\
& X \times_k \tilde{S} \arrow[r] & X \times_k S
\end{tikzcd}$$

where the square is a fibre product; we took $(\psi_X \times \mathrm{id}) : \tilde{X} \times_k \tilde{S} \to X \times_k \tilde{S}$ and $\tilde{X} \times_k \tilde{S} \to Y^0$ the quotient. By the universal property of fibre product, we find the dashed morphism.

If η is the generic point of R, $k(\eta) \subset \Omega$ and Ω is algebraically closed; so we may assume that every point $\tilde{s} \in \tilde{S}$ above $s \in S$ has residue field contained in Ω and we can consider the geometric point $\tilde{s}_\Omega : \mathrm{Spec}(\Omega) \to \tilde{S}$.

Now, since the following fibre product diagram holds:

$$\begin{tikzcd}
Y_\Omega \arrow[r] \arrow[d] & \tilde{Y}^0 \arrow[d] \\
X_\Omega \arrow[r, "\mathrm{id} \times \tilde{s}_\Omega"'] & X \times_k \tilde{S}
\end{tikzcd}$$

we have $\tilde{Y}^0$ connected as Y_Ω is connected by hypothesis, hence the étale cover

$$\tilde{Y}^0 \to X \times_k \tilde{S}$$

corresponds to an open subgroup

$$V \subset \pi_1(X \times_k \tilde{S}) \simeq \pi_1(X) \times U_S$$

where the last equivalence is again provided by Corollary 3.3.2 . Furthermore, $V \supset \pi_1(\tilde{X} \times_k \tilde{S}) = U_X \times U_S$, hence $V = V' \times U_S$ with $U_X \subset V \subset \pi_1(X)$. Finally, if $\tilde{\phi} : \tilde{Y} \to X$ is the connected étale cover of X corresponding to V', we have that the étale cover $\tilde{Y}^0 \to X \times_k \tilde{S}$ is given precisely by the fibre product by $\tilde{S}$ of $\tilde{\phi}$, as we wanted.

$\square$

We present one last result; it provides an interesting decomposition for the fundamental group of a geometrically connected scheme of finite type over a field k.

Proposition 3.3.5. *Let k be a field, S a geometrically connected scheme of finite type over k; denote by k^s the separable closure of k. Fix a geometric point $\bar{s} : Spec(k(\bar{s})) \to S_{k^s}$ with image denoted by $\bar{s}$ in S and in $Spec(k)$. Then the sequence*

$$(S_{k^s}, \bar{s}) \longrightarrow (S, \bar{s}) \longrightarrow (Spec(k), \bar{s})$$

induces the short exact sequence of profinite groups:

$$1 \longrightarrow \pi_1(S_{k^s}, \bar{s}) \xrightarrow{i} \pi_1(S, \bar{s}) \xrightarrow{p} \pi_1(Spec(k), \bar{s}) \longrightarrow 1.$$

Proof. We will use the characterizations of Proposition 3.1.12.

- i is a monomorphism: we prove that if $\phi : X \to S_{k^s}$ is an étale cover, then there exists an étale cover $\tilde{\phi} : \tilde{X} \to S$ such that its base change via $S_{k^s} \to S$ dominates ϕ. So pick $\phi : X \to S_{k^s}$ an étale cover; by Lemma 3.3.3, we know that there exists a finite separable extension K of k and an étale cover $f : \tilde{X} \to S_K$ such that ϕ is a base change of f. Now consider the composition $\tilde{\phi}$ given by:

  $$\tilde{X} \xrightarrow{f} S_K \xrightarrow{e_K} S;$$

  we show that $\tilde{\phi}$ is the wanted étale cover. Consider the following commutative diagram:

  $$\begin{tikzcd}
\tilde{X} \times_{\tilde{\phi},k,e} S_{k^s} \arrow[drr, bend left] \arrow[dr, dashed, "\alpha"] \arrow[ddr, bend right] & & & \\
& X \arrow[r, "\phi"] \arrow[d, "h"] & S_{k^s} \arrow[d, "e"] \arrow[dr, "e_{k^s}"] & \\
& \tilde{X} \arrow[r, "f"] \arrow[rr, bend right, "\tilde{\phi}"'] & S_K \arrow[r, "e_K"] & S
\end{tikzcd}$$

  We have $e_{k^s} = e_K \circ e$; hence $e_{k^s} \circ \phi = \tilde{\phi} \circ h$ and by universal property of fibre product we find a morphism of S_{k^s}-schemes $\alpha : \tilde{X} \times_k S_{k^s} \to X$, so the base change of $\tilde{\phi}$ dominates ϕ.

- $\ker(p) = \mathrm{im}(i)$: the fact that $\mathrm{im}(i) \subset \ker(p)$ can be proved in the same way we did in Theorem 3.3.1. It remains to show $\ker(p) \subset \mathrm{im}(i)$; we prove that if $\phi : X \to S$ is an étale cover such that its base change $\phi_{k^s} : X_{k^s} \to S_{k^s}$ admits a section $\sigma : S_{k^s} \to X_{k^s}$, then there exists an étale cover $Y \to \mathrm{Spec}(k)$ such that its base change via $S \to \mathrm{Spec}(k)$ dominates ϕ.

Now, let K be a finite separable field extension of k such that σ is a base change of $\sigma_K : S_K \to X_K$. Now if we consider the composition

$$S_K \xrightarrow{\sigma_K} X_K \longrightarrow X$$

we find a morphism of S-schemes $S_K \to X$. Thus the base change via $S \to \mathrm{Spec}(k)$ of the étale cover $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ dominates X, as we wanted.

- p is an epimorphism: we show that if $\phi : Y \to \mathrm{Spec}(k)$ is a connected étale cover, then its base change $Y \times_k S \to S$ is again connected. But ϕ being étale implies that $Y = \mathrm{Spec}(K)$ with K a finite separable field extension of k and then $Y \times_k S = S_K$ is connected since S is geometrically connected.

□

Thus with the previous proposition we proved that the fundamental group $\pi_1(S)$ of a geometrically connected scheme of finite type over k can be decomposed in $\pi_1(S_{k^s})$ and $\pi_1(\mathrm{Spec}(k))$, where the latter coincide with the absolute Galois group of k as we proved in section 3.2.

3.4 Elliptic curves

We present in this section an interesting result on elliptic curves: the fundamental group of an elliptic curve over $\mathbb{C}$ at the identity point is isomorphic to the product of its Tate module. However, this results remains true for a generic abelian variety over an algebraic closed field and the proof in this more general case is quite similar to the one we present here. We briefly recall here the basic definitions for elliptic curves; the interested reader might deepen in [Sil].

Definition. An *elliptic curve* (E, O_E) over $\mathbb{C}$ is a couple where E is a smooth algebraic curve of genus one over $\mathbb{C}$ and $O_E \in E$ is a fixed point. The point O_E is intended to be the identity of a suitable commutative group law on E.

If there is no possible confusion, we will denote an elliptic curve just by E and its identity point just by O. We write " + " for the group law on E.

Definition. Given two elliptic curves $(E_1, O_1), (E_2, O_2)$, an *isogeny* is a morphism of algebraic curves

$$\phi : E_1 \longrightarrow E_2$$

such that $\phi(O_1) = \phi(O_2)$.

The isogeny ϕ is said *separable* if it is an étale morphism, *purely inseparable* if it is purely inseparable as a scheme morphism. Furthermore, since E_1, E_2 have the same dimension, ϕ is automatically surjective with finite fibres, and we say that ϕ has degree n if it has degree n as algebraic variety morphism.

For every $n \in \mathbb{N}$, it is defined the multiplication morphism

$$[n_E] : E \longrightarrow E$$

that takes a point $P \in E$ and sends it to the point $P + P + \cdots + P$ (n-times). We denote by $E[n]$ the kernel of this morphism and by $E[n](\mathbb{C})$ the set of its points, which are said *n-torsion* points of E. It is clear that $[n_E]$ is an isogeny and it is possible to prove that if the base field has characteristic 0 (as $\mathbb{C}$ in our case), $[p_E]$ is a also separable for every prime p. Furthermore, the multiplication morphism $[p_E]$ induces a projective system structure on $E[p^n](\mathbb{C})$ for $n \geq 0$ and one defines the Tate module to be its projective limit:

$$T_p(E) := \varprojlim E[p^n](\mathbb{C}).$$

One finds out that $E[p^n](\mathbb{C}) = (\mathbb{Z}/p\mathbb{Z})^2$ and thus

$$T_p(E) \simeq \mathbb{Z}_p^2.$$

Theorem 3.4.1. *Consider (E, O) an elliptic curve over $\mathbb{C}$. Then there is a canonical isomorphism of profinite groups*

$$\pi_1(E, O) \xrightarrow{\sim} \prod_{p \ prime} T_p(E)$$

Proof. By brevity we write π instead of $\pi_1(E, O)$. We give here a sketch of the proof; the reader interested in filling in the details might read [Cad] Theorem 6.11.

We divide the proof in several steps:

step 1. We prove that π is abelian and

$$\pi = \prod_{p \text{ prime}} \pi^{(p)}.$$

Indeed the map $\mu : E \times_{\mathbb{C}} E \to E$ defining the sum induces a morphism of profinite groups by Corollary 3.3.2:

$$\pi_1(\mu) : \pi \times \pi \longrightarrow \pi.$$

Also, the section $\sigma_1 : E \to E \times_{\mathbb{C}} E$ of the first projection $p_1 : E \times_{\mathbb{C}} E \to E$ gives the morphism:

$$\begin{aligned} \pi_1(\sigma_1) : \pi \longrightarrow & \pi \times \pi \\ g \longmapsto & (g, 1) \end{aligned}$$

and one has $\pi_1(\mu) \circ \pi_1(\sigma_1) = \mathrm{id}$. One can do the same for the second projection p_2 and since σ_1 and σ_2 commutes, for every $g_1, g_2 \in E$ we obtain the following chain of equalities:

$$\begin{aligned} \pi_1(\mu)(g_1, g_2) = & \pi_1(\mu)(\pi_1(\sigma_1)(g_1)\pi_1(\sigma_2)(g_2)) \\ = & \pi_1(\mu)(\pi_1(\sigma_1)(g_1))\pi_1(\mu)(\pi_1(\sigma_2)(g_2)) = g_1 g_2 \\ = & \pi_1(\mu)(\pi_1(\sigma_2)(g_2)\pi_1(\sigma_1)(g_1)) \\ = & \pi_1(\mu)(\pi_1(\sigma_2)(g_2))\pi_1(\mu)(\pi_1(\sigma_1)(g_1)) = g_2 g_1. \end{aligned}$$

Thus the group is commutative and we find the decomposition as above using Proposition 3.1.11 .

step 2. We show that if $\phi : X \to E$ is an étale cover, then X carries a unique structure of elliptic curve such that ϕ becomes a separable isogeny.

We construct the group structure on one fibre; then we can extend it thanks to the formalism of Galois categories. Thus we pick a geometric point $x : \mathrm{Spec}(\mathbb{C}) \to X$ with image in X again denoted by x, such that $\phi(x) = O_E$. This x is going to play the role of the identity point of the group structure on X. The pointed connected étale cover

$$\phi : (X; x) \longrightarrow (E; O_E)$$

corresponds to a set M with a transitive π action together with a distinguished point $m \in M$. As π is abelian, for every $g_1, g_2 \in \pi$ the map

$$\begin{aligned} \mu_M : M \times M \longrightarrow & M \\ (g_1 \cdot m, g_2 \cdot m) \longmapsto & g_1 g_2 \cdot m = g_2 g_1 \cdot m \end{aligned}$$

is well defined and maps (m, m) to m. Furthermore, if we endow M with the structure of $(\pi \times \pi)$-set given by $\pi_1(\mu) : \pi \times \pi \to \pi$ the map μ_M is $(\pi \times \pi)$-equivariant. Thus the map μ_M as a morphism in $(\pi \times \pi)$-FSETS corresponds to the morphism μ_X:

$$\begin{tikzcd}
X \times_{\mathbb{C}} X \arrow[r, "\mu_X"] \arrow[d] & X \arrow[d, "\phi"] \\
E \times_{\mathbb{C}} E \arrow[r, "\mu"] & E
\end{tikzcd}$$

By the universal property of fibre product, the morphism μ_X factors as in the following following commutative diagram:

$$\begin{tikzcd}
X \times_{\mathbb{C}} X \arrow[r, "\mu_X^0"] \arrow[d, "\phi \times_{\mathbb{C}} \phi"] \arrow[rr, bend left, "\mu_X"] & X \times_{\phi, E, \mu} (E \times_{\mathbb{C}} E) \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "\phi"] \\
E \times_{\mathbb{C}} E \arrow[r, "\mathrm{id}"] & E \times_{\mathbb{C}} E \arrow[r, "\mu"] & E
\end{tikzcd}$$

The morphism μ_X defines the multiplication on X, mapping (x, x) to x. In a similar way, we can construct the morphism $i_X : X \to X$ (the inverse morphism) above $[-1_E] : E \to E$ again mapping x to x. Finally, it is possible to check that this endows X with an algebraic group structure making it an elliptic curve such that $\phi : X \to E$ becomes a morphism of algebraic groups, thus a separable isogeny as ϕ is already assumed étale.

step 3. Let $\phi : X \to E$ an étale cover; by the previous step we can endow X with a group structure such that ϕ is a separable isogeny; let n be its degree. Then $\ker(\phi) \subset \ker([n_X])$ (see [Sil] Theorem 4.3) hence we have the commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[rr, bend left, "\phi"] \arrow[dr, "{[n_X]}"] & X/\ker(\phi) \arrow[r, "\sim"] \arrow[d, "\psi"] & E \\
& X \arrow[ur, "\phi"] &
\end{tikzcd}$$

Since ϕ is surjective, we have $\phi \circ \psi = [n_E]$. Thus we proved that if $\phi : X \to S$ is an étale cover of degree n, it is a factor of the morphism $[n_E] : E \to E$.

In particular, take p a prime: $[p_E^n]$ is a separable isogeny, so an étale cover of E; it is sent by the fibre functor F_{0_E} in $[p_E^n]^{-1}(0_E) = E[p^n](\mathbb{C})$ in the category of π-SETS and since $E[p^n](\mathbb{C})$ has also group structure, we have that the subgroup of π corresponding to $[p_E^n]$ is precisely $E[p^n](\mathbb{C})$.

Now the factorization we found above tells us that $([p_E^n] : E \to E)_{n\geq 0}$ is cofinal among the étale covers of E having degree a power of p, which in fact correspond precisely to the group $\pi_1(E; O_E)^{(p)}$. Henceforth we find the equality:

$$\pi_1(E, O_E)^{(p)} = \varprojlim E[p^n](\mathbb{C}) = T_p(E).$$

□

Chapter 4

G.A.G.A theorems

In this last chapter we show how the results so called G.A.G.A. (*"Géométrie Algégrique et Géométrie Analytique"*) connecting algebraic geometry and analytic geometry give us further informations on the étale fundamental group. In the first section we present the construction of the analytification of an algebraic variety over $\mathbb{C}$ and we recall some G.A.G.A theorems; in the second section, we present a bunch of examples of interesting étale fundamental group whose computation uses these theorems. We will not present proofs here: the interested reader my find them in [Gro] Chapter XII.

4.1 Analytification

We construct the category $\textsc{An}(\mathbb{C})$ of complex analytic spaces over $\mathbb{C}$. A complex analytic space is obtained by gluing together objects that corresponds to the affine schemes, as one constructs the category $\textsc{Sch}$; thus we firstly define what an *affine* complex analytic space is.

Definition. Let $U \subset \mathbb{C}^n$ denote the polydisc of all $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ such that $|z_i| < 1$ for every $1 \leq i \leq n$ and consider the usual metric topology on it, induced by $\mathbb{C}^n$. Given analytic functions

$$f_1, \dots, f_r : U \longrightarrow \mathbb{C}$$

let $\mathfrak{U}(f_1, \dots, f_r)$ denote the locally ringed space in $\mathbb{C}$-algebra whose underlying topological space is the closed subset

$$\bigcap_{i=1}^{r} f_i^{-1}(0) \subset U$$

endowed with the topology induced by U and whose structural sheaf is given by $\mathcal{O}_U/(f_1, \ldots, f_r)$ where $\mathcal{O}_U$ is the sheaf of germs of analytic functions on U. We call such $\mathfrak{U}(f_1, \ldots, f_r)$ an *affine complex analytic space.*

Now we define an analytic space as the gluing of such affine spaces:

Definition. The category $\textsc{An}(\mathbb{C})$ is the full subcategory of the category $\textsc{LocRing}(\mathbb{C})$ of locally ringed spaces in $\mathbb{C}$-algebras whose objects $(X, \mathcal{O}_X)$ are locally isomorphic to affine complex analytic spaces.

Proposition 4.1.1. *Let X be a scheme locally of finite type over $\mathbb{C}$; the functor*

$$Hom_{\textsc{LocRing}(\mathbb{C})}(\cdot, X) : \textsc{An}(\mathbb{C})^{op} \longrightarrow \textsc{Sets}$$

is representable: there exists $X^{an} \in \textsc{An}(\mathbb{C})$ and a morphism

$$\phi_X \in Hom_{\textsc{LocRing}(\mathbb{C})}(X^{an}, X)$$

such that there is a natural isomorphism

$$\Phi_X : Hom_{\textsc{An}(\mathbb{C})}(\cdot, X^{an}) \xrightarrow{\sim} Hom_{\textsc{LocRing}(\mathbb{C})}(\cdot, X)_{|\textsc{An}(\mathbb{C})^{op}}.$$

Furthermore, for any $x \in X^{an}$, the morphism induced on the completions of local rings

$$\hat{\mathcal{O}}_{X,\phi_X(x)} \xrightarrow{\sim} \hat{\mathcal{O}}_{X^{an},x}$$

is an isomorphism.

The morphism ϕ_X is in fact unique up to a unique X-isomorphism and we call X^{an} the *analytification* of X. Furthermore, if we have a morphism $f : X \to Y$ of schemes locally of finite type over $\mathbb{C}$, then there exists a unique morphism

$$f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$$

such that $\phi_Y \circ f^{\mathrm{an}} = f \circ \phi_X$. Thus it is possible to verify that we have a functor

$$(\cdot)^{\mathrm{an}} : \textsc{Sch}_{LFT}(\mathbb{C}) \longrightarrow \textsc{An}(\mathbb{C})$$

where $\textsc{Sch}_{LFT}(\mathbb{C})$ is the category of schemes locally of finite type over $\mathbb{C}$. In addition, one finds out that many properties of X are reflected on X^{an} via this functor (and viceversa). For example, X is connected (resp. irreducible, regular, normal, reduced, of dimension d) if and only if X^{an} is; similarly for morphisms: a morphism $f \in \mathrm{Hom}_{\textsc{Sch}_{LFT}(\mathbb{C})}(X, Y)$ is surjective (resp. dominant, closed immersion, finite, isomorphism, monomorphism, open immersion, flat, unramified, étale, smooth) if and only if f^{an} is.

One arrives to prove the following theorem:

Theorem 4.1.2. *Let X be a scheme locally of finite type over $\mathbb{C}$; then the functor $(\cdot)^{an}$ induces an equivalence from the category of étale covers of X to the category of étale covers of X^{an}.*

But it is possible to make one step further: take a scheme X locally of finite type over $\mathbb{C}$, let X^{an} be its analytification and X^{top} the underlying topological space. Consider a finite topological cover $p : Y \to X^{\text{top}}$; if $V \subset X^{\text{top}}$, we have

$$p^{-1}(V) = \sqcup_{i=1}^{n} U_i$$

such that $U_i \simeq V$ for every i. It is possible to show that one can endow U_i with a unique analytic structure making $p_{|U_i} : U_i \to V$ an étale analytic cover. This induces an analytic structure on Y making $p : Y \to X^{\text{an}}$ an étale analytic cover. Conversely, we can see an étale analytic cover of X^{an} as a topological cover of X^{top} using local inversion theorem. In this way, one finds an equivalence between the category of finite topological covers of X^{top} and the category of the étale covers of X^{an}.

But now we have proved in Section 3.2 that the Galois fundamental group of the finite topological covers is isomorphic to the profinite completion of the topological fundamental group. Hence for any algebraic variety X over $\mathcal{C}$ and any point $x \in X$, we find the following isomorphism:

$$\pi_1^{\text{étale}}(X, x) \overset{(1)}{\simeq} \pi_1^{\text{étale}}(X^{\text{an}}, x) \overset{(2)}{\simeq} \pi_1^{\text{fin.cov.}}(X^{\text{an}}, x) \overset{(3)}{\simeq} \pi_1^{\text{top}}\widehat{(X^{\text{top}}, x)}$$

where (1) comes from Theorem 4.1.2, (2) comes from what we said above and (3) comes from the result of Section 3.2. Henceforth we managed to show that the étale fundamental group of an algebraic variety over $\mathcal{C}$ is nothing but the profinite completion of the topological fundamental group of its underlying topological space. Thanks to Proposition 3.3.4, we can apply this result to connected proper scheme over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. More in general, it is possible to extend this group equality to algebraically closed field of characteristic 0, as $\overline{\mathbb{Q}}_p$ (the algebraic closure of $\mathbb{Q}_p$) and its completion $\mathbb{C}_p$.

4.2 Further examples

Example. From the isomorphism of profinite groups of the previous section, we obtain that if X is an algebraic variety on $\mathcal{C}$ such that the underlying topological space is simply connected, then for every $x \in X$ the étale fundamental group is trivial. Indeed if X^{top} is simply connected, then $\pi_1^{\text{top}}(X^{\text{top}}, x) = \{1\}$ for every $x \in X$; but then its profinite completion is again the trivial group. In this way, we obtain for example

$$\pi_1(\mathbb{P}^1_{\mathbb{C}}) = \pi_1(\mathbb{A}^1_{\mathbb{C}}) = \{1\}$$

and more in general $\pi_1(\mathbb{A}^n_{\mathbb{C}}) = \{ 1 \}$ for any n.

Example. Take k a field of positive characteristic, consider $\mathbb{P}^1_k$ and fix a geometric point $\bar{x} : \mathrm{Spec}(\bar{k}) \to \mathbb{P}^1_{\bar{k}}$ with image x in $\mathbb{P}^1_{\bar{k}}$ and in $\mathbb{P}^1_k$. By Theorem 3.3.1, the sequence

$$\mathbb{P}^1_{\bar{k}} \longrightarrow \mathbb{P}^1_k \longrightarrow \mathrm{Spec}(k)$$

induces the exact sequence of profinite groups:

$$\pi_1(\mathbb{P}^1_{\bar{k}}, x) \longrightarrow \pi_1(\mathbb{P}^1_k, x) \longrightarrow \pi_1(\mathrm{Spec}(k), *) \longrightarrow 1$$

It is possible to prove that $\pi_1(\mathbb{P}^1_{\bar{k}}, x) = 1$, by showing that any étale cover of it has rank 1 and thus is the trivial cover (for details see [Mil] p. 42). But this implies $\pi_1(\mathbb{P}^1_{\bar{k}}, x) = 1$ hence from the previous exact sequence we obtain an isomorphism

$$\pi_1(\mathbb{P}^1_k, x) \simeq \pi_1(\mathrm{Spec}(k), *)$$

and we proved in Section 3.2 that the latter is the absolute Galois group of k. For example, in the case $k = \mathbb{F}_p$ we obtain $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$.

Example. Let X a smooth compact connected curve over $\mathbb{C}$, fix $x \in X$ and let g be its genus. In particular X with the complex topology is a compact orientable surface, hence it is homeomorphic to the connected sum of g tori by the classification theorem for compact surfaces (see [Mas] Chapter 5). Thus its topological fundamental group of X in x is Γ_g, the group generated by

$$a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g$$

with the relation

$$[a_1, b_1] \cdots [a_g, b_g] = 1$$

where $[a_i, b_i] = a_i^{-1} b_i^{-1} ab$ is the commutator. Applying the result of the previous section, we obtain

$$\widehat{\Gamma}_g \xrightarrow{\sim} \pi_1(X, x).$$

Notice that for $\mathbb{P}^1_{\mathbb{C}}$ we find again that the étale fundamental group is trivial.

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