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# Convergence Analysis for Discrete-Time Quantum Semigroup

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**Abstract**

In the last years, models based on completely-positive trace-preserving maps have been proved to be effective for the description of a wide variety of quantum system, and in particular those of interest for quantum information and computation. This motivates a deeper analysis of this type of maps, aimed to develop more effective protocols for quantum information processing and their design. In this work their asymptotic behavior is analyzed. First, the probabilities of converging to invariant subspaces, in the limit of infinite iteration, are studied. Next, two different decompositions of the quantum system's Hilbert space are introduced, both aimed to analyze the convergence behavior and speed. Finally the possibilities that the dynamics converges to a subspace, after a finite amount of time, is investigated. The starting point for addressing all these issues is the Perron-Frobenius theory, and its specialization to completely-positive dynamics. The methods used are linear-algebraic, and follow the typical approach of linear system and Markov chain theories.

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# 1. Introduction

One of the major discoveries of the twentieth century is undoubtedly the theory of *quantum mechanics*. It owes its raise to a series of discrepancy between theoretical and experimental results, obtained at the turn of the century, that showed the limits of the established physical theories at that time, now dubbed *classical physics*. These arose applying classical laws to new fields of interest, like the description of the elements composing matter, the atoms, or their interaction with light. The solution arrived in the 1920s, with the introduction of a new mathematical framework to construct physical theories: quantum mechanics. Since then, an increasingly number of phenomenons have been modeled using this new set of rules; for example, the effort in describing the interaction between atoms and light has led to the theory called *quantum electrodynamics*.

In this framework, new effects, not possible in classical physics, can take place, some of which are quite far from everyday intuition on how nature works. Another element of novel is the description of composite systems, which is carried out using a tensor product, instead of a direct sum. This results in an exponential growth of the mathematical model, when the number of subsystems composing the system increases. As a consequence, simulations of quantum system on classical computers are inefficient, and practically impossible for large dimensional systems.

This problem was already noted by Richard Feynman, who suggested, in 1982, that using quantum effects to simulate quantum system could overcome these difficulties. These ideas, with different motivation, were used by David Deutsch in 1985 to define a new model of computation, the quantum analogues of a Turing machine, which is the main theoretical tool used in the analysis of algorithms (these concepts are not further developed here, the reader is referred to [1] for a deeper treatment of quantum information and computation theory, and related aspects).

What the model introduced by Deutsch suggested, was the possibility of executing simulation algorithms more efficient on devices based on quantum effects, compared to their classical counterparts. In the subsequent years, this has been confirmed by the design of algorithms that assure better performance than the known in the classical context. Among the most notably, there are Shor's algorithms for the factorization in prime number and the solution of the discrete logarithm, two problem still believed to have no efficient solution on classical computers.

But the applications of quantum effects are not limited to computation, one other possibility is the field of communications. This idea led to the creation of *quantum information theory*, in the 1990s. This, in analogy with its classical counterpart, starts form a fundamental unit of information: the *quantum bit* or *qubit*, defined by Ben Schumacher when he provided the analogue of Shannon's noiseless coding theorem, one of the cornerstones of classical information theory. The main difference between classical bit and qubit is, in a naive way, that the last can be in a superposition of values, instead of a well defined one, as for classical bits. This opens to the possibilities of more efficient communication, for example, Charles Bennett and Stephen Wiesner found a way to transmit two classical bit using one qubit, a result known as *superdense coding*.

It is also worth notice that with the increasingly interest for miniaturization in various fields, the quantum nature of systems at study is emerging. In this contexts, models based on quantum mechanics are attracting increasing interest, since quantum effects cannot be neglected anymore.

What all these fields share, is the necessity for models that include the system-environment iteration. For example, one of the major problem in the large scale implementations of quantum computer is *decoherence*, i.e. the irreversible decaying of quantum correlations, which, roughly speaking, converts quantum superposition in classical uncertainties. On the side of communications, every channel is affected by noise, so every model should take care of this aspect. Moreover, in any control loop, the system description has to include the effect of control laws on it.

This necessity has impacted the research in quantum mechanics, posing the attention to open system, rather than closed one, on which the theory was focused in the early years. With the works of Davies [2] and Kraus [3], the *Completely-Positive Trace-Preserving* (CPTP) maps have gained a central role in this context, being the mathematical tools for the description of Markovian Dynamics, which permits to model a variety of interesting phenomenons, some of which can be used to implements the results obtained in quantum information and computation [1, 4].

The passage from closed to open dynamics, i.e. from unitary evolutions to CPTP maps, has allowed for the maps describing the dynamics to exhibit contractive behavior, which in turn calls for the study of their ergodic properties. Part of the effort has been spent in the study of fixed points for these maps, whose structure is now well understood [5, 6]. Another topics of interest, especially for control of quantum system and quantum error correction, is the convergence to a state or a subspace of the dynamics induced by the maps [7, 8, 9, 10].

In this context, the identification of attractive subspaces, velocity of convergence as well as the possibility to obtain a state in finite time, are of great interest. Some of these problems have already been addressed, for continuous-time CPTP semigroups, notably in the works of Baumgartner and Narnhofer [11, 12], and Ticozzi and al. [8]. We here build on these ideas and develop the analysis directly in the discrete-time case. This is more general, as not all discrete-time semigroup can be seen as "sampled" continuous-time semigroups [13], and has its own peculiarities.

Our starting point is the Perron-Frobenious theory for linear maps preserving a cone, and its specialization in the case of CPTP maps. This is in analogy of what can be done with Markov chains, since CPTP maps can be thought as the quantum equivalent of them, and this parallel has already proved to be useful in many cases (see e.g. [14, 15], where it is exploited in different way). Taking this point of view, and borrowing some ideas from linear system theory, it is possible to develop a set of linear-algebraic tools for the analysis of the asymptotic behavior of a CPTP map, which can be used also to support the design of quantum devices that need state or subspace preparation protocols.

Specifically, the problem of finding closed formulas for some asymptotic probabilities, obtained iterating a CPTP map, is addressed directly following what can be done for Markov chains. Then two different decomposition of the Hilbert space, underlying a quantum system, are proposed. Both start from an invariant subspace and, based on the action of the map, define a sequence of subspaces, until the whole space is covered. The first is based on the support of the eigenvectors relative to the peripheral spectrum, while the second exploits the "dissipitave links" between subspaces. In both cases, the resulting decomposition permits to verify attractivity of the target subspace, and if it is not the case, to extend it to an attractive one; also an estimation of the convergence speed can be obtained. The approaches are built up on existing ergodicity results and representation of CPTP maps [7, 8]. Finally, the possibility of restricting the dynamics to a subspace after a finite number of iterations is explored.

The structure of the thesis is as follow.

- In Chapter 2 are collected the basic theorems of the Perron-Frobenius theory, for linear maps preserving a cone.
- The first part of Chapter 3 reviews its specialisation to CPTP maps, further developing some key concepts. The relevant notions regarding invariant structures are then recalled.
- Chapter 4 deals with the derivation of the closed formulas for asymptotic probabilities.
- In Chapter 5 are illustrated the two decompositions, the *nested-face decomposition*, based on spectral properties, and the *dissipation induced decomposition,* whose features are also explored in the dual picture.
- The possibility that the dynamics is restricted in a subspace after a finite amount of time is analyzed in Chapter 6.

# 2. Cones and Positive Map

Let $\mathcal{V}$ be a finite-dimensional, real vector space.

**Definition 1** (Cone)
*A set $C \subset \mathcal{V}$ is a cone if*

$$\forall x, y \in C \ a, b \geq 0 \Rightarrow ax + by \in C.$$

A cone $C$ is pointed if $C \cap -C = \{0\}$ and full if $\text{span}(C) = \mathcal{V}$, or equivalently if its interior is not void. A cone is proper if it is closed (in the topological sense), pointed and full. A proper cone $C$ induces a partial order over $\mathcal{V}$ defined by

$$x \geq 0 \quad \Leftrightarrow \quad x \in C, \tag{2.1}$$

$$x \geq y \quad \Leftrightarrow \quad x - y \geq 0; \tag{2.2}$$

if $x$ belongs also to the interior of $C$ it is customary to write $x > 0$. One example of proper cone is the set of positive semidefinite (PSD) hermitian matrices as a subset of the hermitian matrices. In this case the order induced by the cone is the well known order defined by $X \geq Y$, if $X - Y$ is positive semidefinite.

**Definition 2** (Face)
*A set $F \subset C$ is a face ($F \lhd C$) if it is a cone and $\forall x \in C$, $y \in F$ $y \geq x \Rightarrow x \in F$*

It is possible to see that if $v \in \text{int}(C)$ then for any $w \in C$ exists a $c > 0$ such that $cv > w$, since a face which contains an element contains also every positive multiple of it, we see that any nontrivial face of a cone is in its boundary.

For any set $S \subset C$, the application that associate to it the minimal face which contains it, is defined as:

$$\phi(S) := \bigcap \{F | F \lhd C, S \subset F\} \tag{2.3}$$

using the application (2.3) it is possible to define the lattice of faces, with *meet* ($\wedge$) and *join* ($\vee$) being:

$$F \wedge G := F \cap G, \quad F \vee G := \phi(F \cup G). \tag{2.4}$$

For a single element the face generated by it is

$$\phi(\{x\}) = \phi(x) = \{y \in C : \alpha x \geq y, \alpha > 0\}. \tag{2.5}$$

For the cone of PSD matrices, any non trivial face contains the operators with range on a fixed subspace, moreover this bijection between subspaces and faces is a lattice isomorphism [16].

If $T$ is a linear map, $\lambda(T)$ will denote its spectrum

$$\lambda(T) := \{\lambda_k \in \mathbb{C} \ : \ \exists \ v, Tv = \lambda_k v\}, \tag{2.6}$$

$\sigma(T)$ its spectral radius

$$\sigma(T) := \max\{|\lambda_k| \ : \ \lambda_k \in \lambda(T)\} \tag{2.7}$$

and $\nu_{\lambda_k}$ the index of $\lambda_k \in \lambda(T)$ i.e. its multiplicity as a root of the minimal polynomial of $T$. A linear function $T : \mathcal{V} \to \mathcal{V}$ which maps a proper cone into itself is called positive[1]. For these maps exist a generalization of Perron-Frobenius theory (see [17]).

**Theorem 1** (Positive linear operator)
*If $C \subset \mathcal{V}$ is a proper cone and $T : \mathcal{V} \to \mathcal{V}$ is positive then:*

*i.* $\sigma(T) \in \lambda(T)$,

*ii.* $\exists v \in C$ *such that* $T(v) = \sigma(T)v$,

*iii.* $\forall\, \lambda_k \in \lambda(T)$ *with* $|\lambda_k| = \sigma(T)$, $\nu_{\lambda_k} \leq \nu_{\sigma(T)}$.

Moreover, $T$ is said to be *irreducible* if it does not leave any nontrivial face invariant. If $v$ is an eigenvector by (2.5), $\phi(v)$ is easily seen to be invariant. Since any face is in the boundary of $C$ and $\phi(v) = C$ if and only if $v \in \mathrm{int}(C)$, $T$ is irreducible if and only if there are no eigenvectors in the boundary of $C$.

**Theorem 2** (Irreducible operator)
*If $T$ is irreducible then $\sigma(T)$ is a simple eigenvalue and the corresponding eigenvector $v$ can be chosen such that $v \in \mathrm{int}(C)$.*

A positive operator such that $T(C - \{0\}) \subset \mathrm{int}(C)$ is irreducible, moreover for these maps if $\lambda_k \in \lambda(T)$ is such that $|\lambda_k| = \sigma(T)$ then $\lambda_k = \sigma(T)$.

[1] Usually, in the theory of cone-preserving maps, these are called nonnegative maps while positive is reserved to them which verify $T(C - \{0\}) \subset \mathrm{int}(C)$.

# 3. Quantum Systems and Discrete-Time Dynamics

## 3.1 Quantum Systems of Finite Dimension

Given a complex Hilbert space $\mathcal{H}$, $\mathfrak{B}(\mathcal{H})$ will stand for the set of linear bounded operators on $\mathcal{H}$, $\mathfrak{H}(\mathcal{H})$ for the subset of hermitian ones and $\mathfrak{H}^+(\mathcal{H})$ for PSD operators. $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{H}^+(\mathcal{H})$ will denote the set of operators with trace one, the *density operators*.
The space $\mathfrak{B}(\mathcal{H})$ is equipped with the *Hilbert-Schmidt inner product*:

$$\langle \rho | \eta \rangle = \mathrm{Tr}(\rho^\dagger \eta) \;\; \rho, \eta \in \mathfrak{B}(\mathcal{H}). \tag{3.1}$$

A *quantum system* is associated to an Hilbert space, and a density operator $\rho$ describes our knowledge of the system and is usually referred as the *state*. Throughout this document only finite dimensional quantum systems are considered.

Physical observable quantities or simply *observables* are associated to elements of $\mathfrak{H}(\mathcal{H})$, and correspond to possible measurements on the system. If $X \in \mathfrak{H}(\mathcal{H})$ then there are orthogonal projections $\Pi_i$, such that:

$$X = \sum_i x_i \Pi_i, \qquad \sum_i \Pi_i = I, \tag{3.2}$$

where the $\Pi_i$ are unique if $x_i \neq x_j$ for $i \neq j$. The observables act on density operators via the inner product: the possible outcomes of the measurement are the $x_i$ and any of this has probability to be observed

$$\mathbb{P}[x_i] = \mathrm{Tr}(\Pi_i \rho). \tag{3.3}$$

It follows that the expected value of the measurement is

$$\mathbb{E}[X] = \mathrm{Tr}(X\rho). \tag{3.4}$$

The state after the measurement depends on the outcome; if it is $x_i$, then the new state $\rho'$ is

$$\rho' = \frac{\Pi_i \rho \Pi_i}{\mathrm{Tr}(\Pi_i \rho)}. \tag{3.5}$$

Since generally the outcome is not known, the average evolution has to be considered, which is given by

$$\mathbb{E}[\rho'] = \sum_i \Pi_i \rho \Pi_i. \tag{3.6}$$

A map on $\mathfrak{H}(\mathcal{H})$ is positive if it preserves the cone of PSD matrices.

## 3.2 Completely-Positive Trace-Preserving Maps

This work is concerned only with discrete-time evolution, characterized by Markovian dynamics. Then the state at a fixed time determines the whole evolution in the future, which is described by a fixed, time-invariant map $T : \mathfrak{H}(\mathcal{H}) \to \mathfrak{H}(\mathcal{H})$.

In the Schrödinger's picture the observables are time invariant, while the state evolves according to the iterations of the map:

$$\rho(n+1) = T(\rho(n)). \tag{3.7}$$

We are interested in dynamics of open systems, i.e. systems which interact with the environment. In this scenario a map, to be physically admissible in the context of quantum mechanic [1], has to be:

i. *Linear*:
$$\forall\ A, B \in \mathfrak{H}(\mathcal{H}): \quad T(cA + B) = cT(A) + T(B). \tag{3.8}$$

ii. *Trace preserving*:
$$\forall\ A \in \mathfrak{H}(\mathcal{H}): \quad \mathrm{Tr}(T(A)) = \mathrm{Tr}(A). \tag{3.9}$$

iii. *Completely positive* (CP):
$\forall n \in \mathbb{N}$, $T \otimes id_n$ is positive, where $id_n$ is the identity map on the operators of an Hilbert space of dimension $n$.

We shall refer to these maps as *Completely-Positive Trace-Preserving* (CPTP) maps. In this case the map $T$ is the generator of a *Quantum Dynamical Semigroup* (QDS), which gives the evolution of the state.

Sometimes it will be useful to look at the Heisenberg's picture, in this case the density operator is fixed and observables are time varying. Since the action on density operators is determined by the inner product, the evolution is obtained by the dual map $T^*$, which means that the probabilities (3.3) are determined by:

$$\mathbb{P}[x_i] = \mathrm{Tr}(T^{*k}(\Pi_i)\rho), \tag{3.10}$$

while the expected value (3.4) evolves according to:

$$\mathbb{E}[X] = \mathrm{Tr}(T^{*k}(X)\rho). \tag{3.11}$$

In this context the map $T^*$ has to be linear and completely positive, while the trace preserving constrain implies *unitality* for the dual map:

$$T^*(I) = I. \tag{3.12}$$

A map which fulfills the above three conditions (in the Heisenberg's or Schrödinger's picture) is called a *quantum channel.*

The results in the following of the section can be found in [5].

A linear map $T : \mathfrak{H}(\mathcal{H}) \to \mathfrak{H}(\mathcal{H})$ is completely positive if and only if it admits an *Operator-Sum Representation* (OSR):

$$T(\rho) = \sum_{k=1}^{K} M_k \rho M_k^\dagger \tag{3.13}$$

where $M_k \in \mathfrak{B}(\mathcal{H})$ and $M^\dagger$ is the adjoint of $M$. For a fixed $T$ more than one OSR is possible: two sets of matrices $\{M_1, M_2, \ldots, M_m\}$ and $\{N_1, N_2, \ldots, N_m\}$ represent the same operator if and only if

$$M_k = \sum_h u_{kh} N_h, \tag{3.14}$$

where $u_{kh}$ are the entries of a unitary matrix. Since we can add any number of zero operators to a set of matrices, without changing the map represented, this property characterizes all the OSR of a map. In the OSR trace preserving is traduced in

$$\sum_{k=1}^{K} M_k^\dagger M_k = I. \tag{3.15}$$

For a linear map acting on a finite-dimensional vector space[1] a Jordan decomposition is possible:

$$T = \sum_{\lambda_k \in \lambda(T)} \lambda_k P_k + N_k \tag{3.16}$$

where $P_k$ stands for the projection onto the generalized eigenspace relative to $\lambda_k$ along the others generalized eigenspaces, and $N_k$ is nilpotent with index $\nu_{\lambda_k}$ ($N_k^{\nu_{\lambda_k}} = 0$). The next theorems characterize the spectrum and the projections of trace preserving maps. The following results, in a somewhat different formulation, can be found in [5].

**Theorem 3**
*Let $T$ be a (completely) positive, trace preserving map then:*

*i.* *$\sigma(T) = 1$ and any $\lambda_k \in \lambda(T)$ with $|\lambda_k| = 1$ has algebraic multiplicity equals to its geometric multiplicity, which means that $N_k = 0$ in the Jordan decomposition* (3.16).

*ii.* *$T_\infty := \sum_{k:\lambda_k=1} P_k = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} T^n$ is (completely) positive.*

*iii.* *$T_\phi := \sum_{k:|\lambda_k|=1} P_k$ and $T_\varphi := \sum_{k:|\lambda_k|=1} \lambda_k P_k$ are (completely) positive.*

Recall that the support of $X$ is $\mathrm{supp}(X) = \ker(X)^\perp$ where $\ker(X)$ is the kernel of $X$. If $X$ is hermitian the support coincides with the range.

**Theorem 4** (Fixed point)
*If $T$ is a trace preserving, positive map then*

*i.* *$\mathcal{X}_\mathcal{F} := T_\infty(\mathfrak{B}(\mathcal{H}))$ is the subspace containing all the fixed point of $T$.*

*ii.* *$\mathcal{X}_\mathcal{F}$ has a basis in $\mathfrak{H}^+(\mathcal{H})$.*

*iii.* *For any density operator $\rho \in \mathcal{X}_\mathcal{F}$, if $\eta \in \mathfrak{D}(\mathcal{H})$ and $\mathrm{supp}(\eta) \subset \mathrm{supp}(\rho)$ then $\mathrm{supp}(T(\eta)) \subset \mathrm{supp}(\rho)$.*

*iv.* *For any density operator $\rho \in \mathcal{X}_\mathcal{F}$, $\mathrm{supp}(\rho) \subset \mathrm{supp}(T_\infty(I))$.*

[1] Here and after we mostly deal with the space $\mathfrak{H}(\mathcal{H})$; since $\mathfrak{H}(\mathcal{H})$ is a real space, a Jordan decomposition in this form is not always possible, when this is needed its complex extension, $\mathfrak{B}(\mathcal{H})$, is considered. In this case a projection on a generalized eigenspace relative to a complex eigenvalue cannot be a real map. If projections relative to a pair of complex conjugate eigenvalues are summed, the sum is a real map.

## 3.3 Stability Properties of Subspaces

Consider a decomposition of the Hilbert space into two orthogonal subspaces:

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R. \tag{3.17}$$

This decomposition defines two (orthogonal) projections[2] $\Pi_S, \Pi_R \in \mathfrak{H}^+(\mathcal{H})$ and it is associated to a basis

$$\{|\varphi_l\rangle\} = \{|\phi_l^S\rangle\} \cup \{|\phi_l^R\rangle\}, \tag{3.18}$$

where $\{|\phi_l^S\rangle\}$ is a basis for $\mathcal{H}_S$ and $\{|\phi_l^R\rangle\}$ for $\mathcal{H}_R$. This basis induces a block structure on the matrices representing elements of $X \in \mathfrak{B}(\mathcal{H})$:

$$X = \begin{bmatrix} X_S & X_P \\ X_Q & X_R \end{bmatrix}. \tag{3.19}$$

Moreover on the space $\mathfrak{H}(\mathcal{H})$ a (orthogonal) decomposition is induced

$$\begin{aligned} \mathfrak{H}(\mathcal{H}) &= \mathfrak{H}_S \oplus \mathfrak{H}_{SR} \oplus \mathfrak{H}_R, \\ \mathfrak{H}_S &= \left\{ \rho \in \mathfrak{H}(\mathcal{H}) : \rho = \begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix} \right\}, \\ \mathfrak{H}_{SR} &= \left\{ \rho \in \mathfrak{H}(\mathcal{H}) : \rho = \begin{bmatrix} 0 & \rho_P \\ \rho_P^\dagger & 0 \end{bmatrix} \right\}, \\ \mathfrak{H}_R &= \left\{ \rho \in \mathfrak{H}(\mathcal{H}) : \rho = \begin{bmatrix} 0 & 0 \\ 0 & \rho_R \end{bmatrix} \right\}. \end{aligned} \tag{3.20}$$

In what follows we shall also use $\mathfrak{H}_S^+$ for the subset of PSD matrices in $\mathfrak{H}_S$ (which is a face of $\mathfrak{H}^+(\mathcal{H})$) and similarly $\mathfrak{H}_R^+$. Also $\mathfrak{D}_S$ will be used for the set of density operators acting on $\mathcal{H}_S$.

We are mostly interested in decomposition with useful properties for quantum engineering. The next results, taken from [7], are useful in this sense. The following characterizes the invariant property of the set $\mathfrak{D}_S$ in term of the matrices $M_k$, appearing in an OSR representation of $T$.

**Proposition 1**
*Let $T$ be a CPTP map described by an OSR as in* (3.13)*, then the set $\mathfrak{D}_S$ is invariant if and only if the matrices $M_k$, expressed in their block structure, have the following form:*

$$M_k = \begin{bmatrix} M_{k,S} & M_{k,P} \\ 0 & M_{k,R} \end{bmatrix}. \tag{3.21}$$

By this we see that invariance of $\mathfrak{D}_S$ corresponds to invariance of the corresponding subspace for the operators in a OSR, this motivates the following definition.

**Definition 3** (Invariant subspace)
*For a CPTP map $T$, $\mathcal{H}_S$ is an invariant subspace if $\mathfrak{D}_S$ is invariant under the action of $T$:*

$$\rho \in \mathfrak{D}_S \Rightarrow T(\rho) \in \mathfrak{D}_S$$

[2]These subspaces are also associated with faces of the cone $\mathfrak{H}^+(\mathcal{H})$, as noted in section 2. These are $\phi(\Pi_S)$ and $\phi(\Pi_R)$.

Another important concept about a subspace of the Hilbert space is attractivity:

**Definition 4** (Attractive subspace)
*A subspace $\mathcal{H}_S \subset \mathcal{H}$ is attractive for the CPTP map $T$ if*

$$\lim_{n\to\infty} \|T^n(\rho) - \Pi_S T^n(\rho)\Pi_S\| = 0,$$

$$\forall \rho \in \mathfrak{D}(\mathcal{H}).$$

A subspace is *Globally Asymptotically Stable* (GAS) if it is attractive and invariant. The next theorem gives a necessary and sufficient condition for an invariant subspace to be GAS.

**Theorem 5** (Characterization of GAS subspaces)
*Let $T$ be a CPTP map, $\mathcal{H}_S \oplus \mathcal{H}_R$ an orthogonal decomposition with $\mathcal{H}_S$ invariant. With regard to the block form* (3.21), *$\mathcal{H}_S$ is GAS if and only if there are no invariant density operators $\rho$ with* $\mathrm{supp}(\rho) \subset \bigcap_k \ker(M_{k,P})$.

# 4. Probabilities of Convergence to Invariant Subspace

In this sections some probabilities which could be of interest are derived. The purpose is to give some tools to analyze the asymptotic behavior of a quantum system which contains a GAS subspace. In what follows $T$ will stand for a CPTP map and the Hilbert space is decomposed as in (3.17), with $\mathcal{H}_S$ invariant.

## 4.1 Preliminaries

First of all we note that invariance of $\mathfrak{D}_S$ extends to the whole $\mathfrak{H}_S$ (and are in fact equivalent). Using a direct calculation and the property of Proposition 1, it is easily seen that $\mathfrak{H}_S \oplus \mathfrak{H}_{SR}$ is invariant too:

$$\begin{aligned}\sum_{k=1}^{K} M_k \rho M_k^\dagger = \sum_{k=1}^{K} \begin{bmatrix} M_{k,S} & M_{k,P} \\ 0 & M_{k,R} \end{bmatrix} \begin{bmatrix} \rho_S & \rho_P \\ \rho_P^\dagger & 0 \end{bmatrix} \begin{bmatrix} M_{k,S}^\dagger & 0 \\ M_{k,P}^\dagger & M_{k,R}^\dagger \end{bmatrix} = \\ = \sum_{k=1}^{K} \begin{bmatrix} M_{k,S}\rho_S + M_{k,P}\rho_P^\dagger & M_{k,S}\rho_P \\ M_{k,R}\rho_P^\dagger & 0 \end{bmatrix} \begin{bmatrix} M_{k,S}^\dagger & 0 \\ M_{k,P}^\dagger & M_{k,R}^\dagger \end{bmatrix} = \\ = \sum_{k=1}^{K} \begin{bmatrix} M_{k,S}\rho_S M_{k,S}^\dagger + M_{k,P}\rho_P^\dagger M_{k,S}^\dagger + M_{k,S}\rho_P M_{k,P}^\dagger & M_{k,S}\rho_P M_{k,R}^\dagger \\ M_{k,R}\rho_P^\dagger M_{k,S}^\dagger & 0 \end{bmatrix}.\end{aligned} \tag{4.1}$$

Using the decomposition (3.20) of $\mathfrak{H}(\mathcal{H})$, the action of $T$ could be split between the three subspaces:

$$\begin{aligned} T_S : \mathfrak{H}_S \to \mathfrak{H}_S, &\quad T_S(\rho_S) = \sum_{k=1}^{K} M_{k,S}\rho_S M_{k,S}^\dagger \\ T_R : \mathfrak{H}_R \to \mathfrak{H}_R, &\quad T_R(\rho_R) = \sum_{k=1}^{K} M_{k,R}\rho_R M_{k,R}^\dagger \\ T_{SR} : \mathfrak{H}_R \to \mathfrak{H}_S, &\quad T_{SR}(\rho_R) = \sum_{k=1}^{K} M_{k,P}\rho_R M_{k,P}^\dagger \\ T_P : \mathfrak{H}_{SR} \to \mathfrak{H}_{SR}, &\quad T_P(\rho_P) = \sum_{k=1}^{K} M_{k,S}\rho_P M_{k,R}^\dagger \\ T_{SP} : \mathfrak{H}_{SR} \to \mathfrak{H}_S, &\quad T_{SP}(\rho_P) = \sum_{k=1}^{K} M_{k,P}\rho_P^\dagger M_{k,S}^\dagger + M_{k,S}\rho_P M_{k,P}^\dagger \end{aligned} \tag{4.2}$$

$$T_{PR} : \mathfrak{H}_R \to \mathfrak{H}_{SR}, \qquad T_{PR}(\rho_R) = \sum_{k=1}^{K} M_{k,P}\rho_R M_{k,R}^\dagger$$

The maps from $\mathfrak{H}_{SR}$ are defined only in terms of the upper part of the elements: this simplified notation has no problem until we deal only with operators of $\mathfrak{H}(\mathcal{H})$. If we want to represent also elements of $\mathfrak{B}(\mathcal{H})$, this is still possible but $\rho_P$ has to be treated as an element of a real vector space, so the real and the imaginary (i.e. the hermitian and skewhermitian) parts have to be specified. However, this will be avoided and only hermitian operators are considered when using this notation. These maps are defined using a particular OSR of $T$, however it is easily seen that they do not depend on it; for example, for $T_R$ an alternative definition could be

$$T_R(\rho_R) = \Pi_R T(\rho_R)\Pi_R. \tag{4.3}$$

The action of the map can now be splitted

$$T(\rho) = \begin{bmatrix} T_S(\rho_S) + T_{SP}(\rho_P) + T_{SR}(\rho_R) & T_P(\rho_P) + T_{PR}(\rho_R) \\ (T_P(\rho_P) + T_{PR}(\rho_R))^\dagger & T_R(\rho_R) \end{bmatrix}, \tag{4.4}$$

and it is easily seen that:

$$\Pi_R T^n(\rho)\Pi_R = \begin{bmatrix} 0 & 0 \\ 0 & T_R^n(\rho_R) \end{bmatrix}. \tag{4.5}$$

The map $T_S$ is just the restriction to the invariant subspace $\mathfrak{H}_S$, and for this reason it is CPTP. The map $T_R$ is also CP, but generally just trace non increasing. The spectrum of the maps $T_R$, $T_S$ and $T_P$ is related to that one of $T$, to obtain this relation choose three bases $\{\rho_{Si}\}$ $\{\rho_{Pi}\}$ $\{\rho_{Ri}\}$ for the three subspaces. Their union is a basis for the whole space. Using coordinates relative to this basis the map $T$ can be represented by a matrix $\widehat{T}$, which, due to invariance of two subspaces, has the form:

$$\widehat{T} = \begin{bmatrix} \widehat{T}_S & \widehat{T}_{SP} & \widehat{T}_{SR} \\ 0 & \widehat{T}_P & \widehat{T}_{PR} \\ 0 & 0 & \widehat{T}_R \end{bmatrix}. \tag{4.6}$$

In the above representation the blocks are the matrix representation of the maps defined in (4.2). By (4.6) the product of the characteristic polynomials of $T_R$, $T_S$ and $T_P$ is the characteristic polynomial of $T$ (i.e. the spectrum of $T$ is the union of the spectrum of $T_R$, $T_S$ and $T_P$ counting multiplicities) and also their minimal polynomial divide the minimal polynomial of $T$.

The next Lemma will be needed later, and can be found in a more general form in [5].

**Lemma 1**
*If $A$ is a linear operator on a finite dimensional vector space and $\sigma(A) < 1$ then*

$$\sum_{n=0}^{+\infty} A^n = (I - A)^{-1}.$$

To be noted, if $A$ is positive, the limit is positive since any partial sum is of positive elements, and the set of positive maps is closed. To apply the lemma the next property will be needed.

**Proposition 2** (Spectral characterization of GAS subspace)
*An invariant subspace $\mathcal{H}_S$ is GAS if and only if $\sigma(T_R) < 1$.*

*Proof.*
Since $T_R$ is a positive application, if $\sigma(T_R) = 1$ then there exists a density $\rho \in \mathfrak{H}_R$, such that $T_R(\rho) = \rho$. By this $\Pi_R T^n(\rho)\Pi_R = \rho$ for any $n$ and $\mathcal{H}_S$ is not GAS.
If $\sigma(T_R) < 1$ all the positive fixed points have support in $\mathcal{H}_S$ and by Theorem 5 $\mathcal{H}_S$ is GAS. □

From now on, until the end of the section, we assume that $\mathcal{H}_S$ is GAS, then the lemma can be applied with $A = T_R$.

## 4.2 Cumulative convergence error

Firstly we look at the sum of the probabilities to find the state in $\mathcal{H}_R$ during the evolution, which can be easily expressed in closed form:

$$\begin{aligned}\sum_{n=0}^{+\infty} \operatorname{Tr}(\Pi_R T^n(\rho)) = \sum_{n=0}^{+\infty} \operatorname{Tr}(\Pi_R T^n(\rho)\Pi_R) = \\ \operatorname{Tr}(\sum_{n=0}^{+\infty} T_R^n(\rho_R)) = \operatorname{Tr}((I - T_R)^{-1}(\rho_R)).\end{aligned} \tag{4.7}$$

Actually, the formula is also true in the case that $\Pi_R$ act as observable. First note that for the partial sums we have:

$$\sum_{n=0}^{N} \operatorname{Tr}(\Pi_R \rho(n)) = \operatorname{Tr}(\sum_{n=0}^{N} T_R^n(\rho_R)). \tag{4.8}$$

In fact for $N = 0$ the formula is true. Assume it for $N < k$, assume also that $\rho_R(n) = T_R^n(\rho_R)$ for $n < k$ and use (4.5) to obtain

$$\begin{aligned}&\rho_R(k) = T_R(\Pi_R\rho(k-1)\Pi_R) = T_R^k(\rho_R)\\ &\sum_{n=0}^{k} \operatorname{Tr}(\Pi_R T^n(\rho)) =\\ &= \operatorname{Tr}(\sum_{n=0}^{k-1} T_R^n(\rho_R)) + \operatorname{Tr}\Big(\Pi_R\, T\big(\Pi_S\rho(k-1)\Pi_S + \Pi_R\rho(k-1)\Pi_R\big)\,\Pi_R\Big) =\\ &= \operatorname{Tr}(\sum_{n=0}^{k} T_R^n(\rho_R)).\end{aligned}$$

So (4.8) is true, and letting $N$ go to infinity (4.7) is obtained.

If our purpose is to prepare a state in the subspace $\mathcal{H}_S$ the quantity $\operatorname{Tr}((I-T_R)^{-1}(\rho_R))$ could be thought as the error during the whole evolution.

## 4.3 Asymptotic Probabilities of Invariant Subspaces

Suppose that $\mathcal{H}_S$ is decomposed in $N$ orthogonal invariant subspaces $\mathcal{H}_{S_i}$. Choosing a suitable basis, the submatrices $M_{k,S}$ of an OSR are block diagonal:

$$M_{k,S} = \begin{bmatrix} M_{k,S_1} & 0 & \cdots & 0 \\ 0 & M_{k,S_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & M_{k,S_N} \end{bmatrix}. \tag{4.9}$$

In this case, if $\rho_S \in \mathfrak{D}_S$ and we look at the block structure induced by the decomposition, one block does not interact with the others, i.e. $\Pi_{S_i} T_S(\rho_S)\Pi_{S_j}$ depends only from $\Pi_{S_i}\rho_S\Pi_{S_j}$. Our assumptions imply that the support of the state tends to $\mathcal{H}_S$, for any initial condition; so it could be of interest to know what is the asymptotic probability to find the state in one of these subspaces, i.e. to evaluate

$$\lim_{n\to\infty} \mathrm{Tr}(\Pi_{S_i} T^n(\rho)). \tag{4.10}$$

This limit is easier to evaluate looking at the dual evolution. For $T^*$ the invariant subspaces are $\mathfrak{H}_{SR} \oplus \mathfrak{H}_R$ and $\mathfrak{H}_R$, since their orthogonal complements are invariant for $T$. Note also that $T^*$ can be partitioned by the dual maps of that ones in (4.4). Doing so, we can obtain the following preliminary result.

**Lemma 2**
*Let $\mathcal{H}_S = \bigoplus_{i=0}^N \mathcal{H}_{S_i}$ and any of the $\mathcal{H}_{S_i}$ invariant. If $\Pi_{S_i}$ is the orthogonal projections on $\mathcal{H}_{S_i}$, then for any $i$*

$$T^*(\Pi_{S_i}) = \Pi_{S_i} + T^*_{SR}(\Pi_{S_i}). \tag{4.11}$$

*Proof.*
We will firstly explicitly prove the case $N = 2$; due to invariance of $\mathcal{H}_{S_{1,2}}$ the matrices $M_k$ have the block-structure:

$$\begin{bmatrix} M_{k,S_1} & 0 & M_{k,P_1} \\ 0 & M_{k,S_2} & M_{k,P_2} \\ 0 & 0 & M_{k,R} \end{bmatrix}. \tag{4.12}$$

Rewriting the unitality condition, taking into account the structure (4.12), we obtain:

$$\sum_k \begin{bmatrix} M^\dagger_{k,S_1} & 0 & 0 \\ 0 & M^\dagger_{k,S_2} & 0 \\ M^\dagger_{k,P_1} & M^\dagger_{k,P_2} & M^\dagger_{k,R} \end{bmatrix} \begin{bmatrix} M_{k,S_1} & 0 & M_{k,P_1} \\ 0 & M_{k,S_2} & M_{k,P_2} \\ 0 & 0 & M_{k,R} \end{bmatrix} =$$
$$\sum_k \begin{bmatrix} M^\dagger_{k,S_1}M_{k,S_1} & 0 & M^\dagger_{k,S_1}M_{k,P_1} \\ 0 & M^\dagger_{k,S_2}M_{k,S_2} & M^\dagger_{k,S_2}M_{k,P_2} \\ M^\dagger_{k,P_1}M_{k,S_1} & M^\dagger_{k,P_2}M_{k,S_2} & M^\dagger_{k,P_1}M_{k,P_1} + M^\dagger_{k,P_2}M_{k,P_2} + M^\dagger_{k,R}M_{k,R} \end{bmatrix} = \tag{4.13}$$
$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Let us focus on $\mathcal{H}_{S_1}$, as the same reasoning applies to $\mathcal{H}_{S_2}$ up to a relabeling. In the same block-representation, the projection of interest is

$$\Pi_{S_1} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.14}$$

we thus have:

$$T^*(\Pi_1) = \sum_k \begin{bmatrix} M^\dagger_{k,S_1} M_{k,S_1} & 0 & M^\dagger_{k,S_1} M_{k,P_1} \\ 0 & 0 & 0 \\ M^\dagger_{k,P_1} M_{k,S_1} & 0 & M^\dagger_{k,P_1} M_{k,P_1} \end{bmatrix} = \Pi_{S_1} + T^*_{SR}(\Pi_{S_1}). \tag{4.15}$$

In the general case $\mathcal{H}_S = \bigoplus_i \mathcal{H}_{S_i}$, for any $j = 1, \dots, K$ we can consider the decomposition $\mathcal{H}_S = \mathcal{H}_{S_j} \oplus \bigoplus_{i \neq j} \mathcal{H}_{S_i}$. These two orthogonal subspaces in the sum are both invariant so by the reasoning above the evolution of $\Pi_{S_j}$ has the desired form. □

Using the above result, we can obtain a closed formula for (4.10), which depend on the initial state. To be noted, the limit is independent from the off diagonal block $\rho_P$.

**Proposition 3**
*Under the same hypothesis of the above proposition, if $\mathcal{H}_S$ is also GAS then*

$$\lim_{n\to\infty} \mathrm{Tr}(\Pi_{S_i} T^n(\rho)) = \mathrm{Tr}(\Pi_{S_i}\rho_S) + \mathrm{Tr}(\Pi_{S_i} T_{SR}((I - T_R)^{-1}(\rho_R))). \tag{4.16}$$

*Proof.*
The limit of $\Pi_{S_i}$ under the action of $T^*$ is easily computed:

$$\begin{aligned} T^*(\Pi_{S_i}) &= \Pi_{S_i} + T^*_{SR}(\Pi_{S_i}), \\ T^{*2}(\Pi_{S_i}) &= \Pi_{S_i} + (T^*_R(T^*_{SR}(\Pi_{S_i})) + T^*_{SR}(\Pi_{S_i})), \\ T^{*n}(\Pi_{S_i}) &= \Pi_{S_i} + (\sum_{k=0}^{n-1} T^{*k}_R(T^*_{SR}(\Pi_{S_i}))). \end{aligned} \tag{4.17}$$

Letting $n$ go to infinity

$$\begin{aligned} \lim_{n\to\infty} T^{*n}(\Pi_{S_i}) &= \lim_{n\to\infty} \Pi_{S_i} + (\sum_{k=0}^{n-1} T^{*k}_R(T^*_{SR}(\Pi_{S_i}))) \\ &= \Pi_{S_i} + ((I - T^*_R)^{-1}(T^*_{SR}(\Pi_{S_i}))). \end{aligned} \tag{4.18}$$

By the relation

$$\lim_{n\to\infty} \mathrm{Tr}(\Pi_{S_i} T^n(\rho)) = \lim_{n\to\infty} \mathrm{Tr}(T^{*n}(\Pi_{S_i})\rho), \tag{4.19}$$

the statement follows. □

The obtained formula highlights that the probability of convergence to an invariant subspace, inside a GAS subspace, is given by the sum of two terms: the initial probability of finding the state there (the term $\mathrm{Tr}(\Pi_{S_i})$), plus a term that can be computed explicitly knowing the map in (4.2).

## 4.4 Outcome Probabilities

In the same situation as above, we could be interested in the probability of a specific outcome of an observable, whose projections are the $\Pi_{S_i}$. Consider an observable $X_S = \sum_i x_i \Pi_i$ with $\sum_{i\neq 0} \Pi_i = \Pi_S$ and $x_i \neq 0$ if $i \neq 0$, $x_0 = 0$. Assuming the observable is measured at $n = 1, 2, 3 \dots$ (or at least until the first non zero outcome) it is possible to obtain the probability for the first non zero outcome to be $x_i$.

First note that measuring the observable at any step do not change the invariant property of $\mathcal{H}_S$, moreover during the computation that follows, we will see that the probability to have always zero as outcome is zero, so the GAS property is preserved as well. Also notice that we are interested in a subalgebra of the algebra of all observables, that one generated by the projections $\Pi_i$, since this is a commutative subalgebra the rule of classical probability may be applied. Thanks to this we can define the events

$$\begin{gathered}\mathrm{E}^i = \text{“the first non zero outcome is } x_i\text{”},\\ \mathrm{E}^i_k = \text{“the first non zero outcome is } x_i \text{ and happened at } n \leq k\text{”},\\ \mathrm{A}^i_k = \text{“the outcome at } n = k \text{ is } x_i\text{”},\\ \mathrm{B}_k = \text{“the outcomes are all zero for } n \leq k\text{”};\end{gathered}$$

and use with them rule of classical probability. We want to evaluate $\mathbb{P}(\mathrm{E}^i)$.

The relations between these events are:

$$\begin{aligned}\mathrm{E}^i_{k-1} &\subset \mathrm{E}^i_k,\\ \mathrm{E}^i &= \bigcup_{k=1}^{+\infty} \mathrm{E}^i_k,\\ \mathrm{E}^i_k &= \mathrm{E}^i_{k-1} \cup (\mathrm{A}^i_k \cap \mathrm{B}_{k-1}),\\ \mathrm{E}^i_k &\cap \mathrm{B}_k = \emptyset,\\ \mathrm{B}_k &= \mathrm{B}_{k-1} \cap \mathrm{A}^0_k.\end{aligned}$$

The latter relations thus imply:

$$\mathbb{P}(\mathrm{E}^i) = \lim_{k\to+\infty} \mathbb{P}(\mathrm{E}^i_k), \tag{4.20}$$

$$\mathbb{P}(\mathrm{E}^i_k) = \mathbb{P}(\mathrm{E}^i_{k-1}) + \mathbb{P}(\mathrm{A}^i_k|\mathrm{B}_{k-1})\mathbb{P}(\mathrm{B}_{k-1}), \tag{4.21}$$

$$\mathbb{P}(\mathrm{B}_k) = \mathbb{P}(\mathrm{B}_{k-1})\mathbb{P}(\mathrm{A}^0_k|\mathrm{B}_{k-1}). \tag{4.22}$$

**Lemma 3**
*For $i \neq 0$ and $k > 2$*

$$\mathbb{P}(A^i_k|B_{k-1})\mathbb{P}(B_{k-1}) = \mathrm{Tr}(\Pi_i T_{SR}(T_R^{k-1}(\rho_R))).$$

*Proof.*
Let $\rho(n|\mathrm{C})$ with C an event, be the state $\rho(n)$ assuming that C happened.
First of all

$$\rho(2|\mathrm{B}_1) = T\left(\frac{\Pi_R T(\rho)\Pi_R}{\mathrm{Tr}(\Pi_R T(\rho))}\right) = \frac{1}{\mathrm{Tr}(T_R(\rho_R))}\begin{bmatrix} T_{SR}(T_R(\rho_R)) & T_{PR}(T_R(\rho_R)) \\ (T_{PR}(T_R(\rho_R)))^\dagger & T_R^2(\rho_R)\end{bmatrix}.$$

Assume for $k \geq 2$

$$\rho(k|\mathrm{B}_{k-1}) = \frac{1}{\mathrm{Tr}(T_R^{k-1}(\rho_R))}\begin{bmatrix} T_{SR}(T_R^{k-1}(\rho_R)) & T_{PR}(T_R^{k-1}(\rho_R)) \\ (T_{PR}(T_R^{k-1}(\rho_R)))^\dagger & T_R^k(\rho_R)\end{bmatrix}, \tag{4.23}$$

then

$$\begin{aligned}\rho(k+1|\mathrm{B}_k) = T\left(\frac{\Pi_R\rho(k|\mathrm{B}_{k-1})\Pi_R}{\mathrm{Tr}(\Pi_R\rho(k|\mathrm{B}_{k-1})}\right) &=\\ = \frac{1}{\mathrm{Tr}(T_R^k(\rho_R))}\begin{bmatrix} T_{SR}(T_R^k(\rho_R)) & T_{PR}(T_R^k(\rho_R)) \\ (T_{PR}(T_R^k(\rho_R)))^\dagger & T_R^{k+1}(\rho_R)\end{bmatrix}.\end{aligned}$$

So (4.23) is true for any $k \geq 2$. By using it we have

$$\mathbb{P}(\mathrm{A}_k^i|\mathrm{B}_{k-1}) = \mathrm{Tr}(\Pi_i \rho(k|B_{k-1})).$$

If $i = 0$

$$\mathbb{P}(\mathrm{A}_k^0|\mathrm{B}_{k-1}) = \frac{\mathrm{Tr}(T_R^k(\rho_R))}{\mathrm{Tr}(T_R^{k-1}(\rho_R))}, \tag{4.24}$$

so by (4.22)

$$\begin{aligned}
\mathbb{P}(\mathrm{B}_1) &= \mathrm{Tr}(\Pi_R \rho(1)) = \mathrm{Tr}(T_R(\rho_R)),\\
\mathbb{P}(\mathrm{B}_2) &= \mathrm{Tr}(T_R(\rho_R))\frac{\mathrm{Tr}(T_R^2(\rho_R))}{\mathrm{Tr}(T_R(\rho_R))} = \mathrm{Tr}(T_R^2(\rho_R)),\\
\mathbb{P}(\mathrm{B}_k) &= \mathrm{Tr}(T_R^k(\rho_R)).
\end{aligned}$$

Now for $i \neq 0$ and $k \geq 2$

$$\begin{aligned}
\mathbb{P}(\mathrm{A}_k^i|\mathrm{B}_{k-1}) &= \mathrm{Tr}(\Pi_i \rho(k|B_{k-1}))) = \frac{\mathrm{Tr}(\Pi_i T_{SR}(T_R^{k-1}(\rho_R))}{\mathrm{Tr}(T_R^{k-1}(\rho_R))}\\
\mathbb{P}(\mathrm{A}_k^i|\mathrm{B}_{k-1})\mathbb{P}(\mathrm{B}_{k-1}) &= \mathrm{Tr}(\Pi_i T_{SR}(T_R^{k-1}(\rho_R)))
\end{aligned} \tag{4.25}$$

□

We are now ready to derive a closed form for the probabilities we are looking for.

**Proposition 4**
$\mathbb{P}(E^i) = \mathrm{Tr}(\Pi_i T_S(\rho_S)) + \mathrm{Tr}(\Pi_i T_{SP}(\rho_P)) + \mathrm{Tr}(\Pi_i T_{SR}((I - T_R)^{-1}(\rho_R)))$.

*Proof.*
By the lemma and (4.20)

$$\begin{aligned}
\mathbb{P}(\mathrm{E}_1^i) &= \mathrm{Tr}(\Pi_i \rho(1)) = \mathrm{Tr}(\Pi_i T_S(\rho_S)) + \mathrm{Tr}(\Pi_i T_{SP}(\rho_P)) + \mathrm{Tr}(\Pi_i T_{SR}(\rho_R))\\
\mathbb{P}(\mathrm{E}_2^i) &= \mathrm{Tr}(\Pi_i T_S(\rho_S)) + \mathrm{Tr}(\Pi_i T_{SP}(\rho_P)) + \mathrm{Tr}(\Pi_i T_{SR}(\rho_R)) + \mathrm{Tr}(\Pi_i T_{SR}(T_R(\rho_R))) =\\
&= \mathrm{Tr}(\Pi_i T_S(\rho_S)) + \mathrm{Tr}(\Pi_i T_{SP}(\rho_P)) + \mathrm{Tr}(\Pi_i T_{SR}(\rho_R + T_R(\rho_R)))\\
\mathbb{P}(\mathrm{E}_k^i) &= \mathrm{Tr}(\Pi_i T_S(\rho_S)) + \mathrm{Tr}(\Pi_i T_{SP}(\rho_P)) + \mathrm{Tr}(\Pi_i T_{SR}(\sum_{j=0}^{k-1} T_R^j(\rho_R)))
\end{aligned}$$

and letting $k$ go to infinity the proposition is proved. □

# 5. Convergence Tests and Subspace Decompositions

In this section, starting from an invariant subspace and a CPTP map, we derive two decompositions for the Hilbert space that will allow to decide if the subspace is also attractive for the dynamical semigroup generated by the given map.

The first algorithm considers this subspace as an invariant face of the cone of PSD matrices, and then constructs a sequence of other faces, which contain it and are invariant as well. These nested faces will have the property that the whole cone tends asymptotically to the smallest ones – and to the original subspace if it is attractive. The second is the *Dissipation-Induced Decomposition* (DID), which has first been proposed in [8], here adapted at the case of discrete time evolution. This decomposition is associated to a chain of subspaces which have to be "crossed" by the state trajectory in order to reach the invariant subspace.

## 5.1 Nested-Face Decomposition

### 5.1.1 Preliminary results

To obtain this decomposition we shall first need to generalize some known results on invariant subspaces from CPTP to CP maps. In the case of CPTP maps, for a subspace $\mathcal{H}_S$, we require that the set of density operators, with support on $\mathcal{H}_S$, is invariant under the action of the map. When dealing with CP maps the set of density operators is generally not preserved, so the natural extension is to require that the set of PSD matrices with support on $\mathcal{H}_S$, i.e. the face that corresponds to $\mathcal{H}_S$, is invariant. It is easily seen that if a map is CPTP, and the face associated to $\mathcal{H}_S$ is invariant, then $\mathcal{H}_S$ is an invariant subspace in the sense of definition 3. With a slight abuse of notations, in the following we shall call a subspace of $\mathcal{H}$ invariant for a CP map $A$, if the face which corresponds to it is invariant for $A$. We will need to adapt some results obtained for CPTP maps at the case of CP maps. These are mostly taken from [7] and the proofs use almost the same arguments. Some of them are stated in Section 3 and 4 in the case of CPTP maps.

For a generic set $W \subset \mathfrak{H}^+(\mathcal{H})$ its support can be defined as the minimal subspace of $\mathcal{H}$ which contains the supports of any element in $W$[1]

$$\text{supp}(W) := \bigvee_{\eta \in W} \text{supp}(\eta). \tag{5.1}$$

Correspondingly, if $F$ is a face, $\text{supp}(F)$ is the subspace which corresponds to $F$ in the lattice isomorphism of Section 2. Recall that $\phi(W)$ denotes the face generated by $W$.

[1] $\vee$ stands for the sum between subspaces.

**Lemma 4**
*For any set $W \subset \mathfrak{H}^+(\mathcal{H})$,* $\text{supp}(\phi(W)) = \text{supp}(W)$.

*Proof.*
Since the face generated by $W$ contains $W$, $\text{supp}(W) \subset \text{supp}(\phi(W))$. On the other hand the face which corresponds to $\text{supp}(W)$ contains any elements in $W$, so it must contains the face generated by $W$, by this $\text{supp}(\phi(W)) \subset \text{supp}(W)$ and then $\text{supp}(W) = \text{supp}(\phi(W))$. □

The next lemma relates invariance of a subset of the PSD matrices with the invariance of its support (or equivalently of the face generated by it).

**Lemma 5**
*Let $T$ be a positive map with $W \subset \mathfrak{H}^+(\mathcal{H})$ an invariant set. If $A \in \mathfrak{H}^+(H)$ is such that* $\text{supp}(X) \subset \text{supp}(W)$ *then* $\text{supp}(T(X)) \subset \text{supp}(W)$.

*Proof.*

Let $\widetilde{W}$ be the convex hull of $W$, this has the same support of $W$ and is also invariant. Moreover it contains an element $M$ of maximal rank such that

$$\text{supp}(M) = \text{supp}(\widetilde{W}) = \text{supp}(W). \tag{5.2}$$

If $A \in \mathfrak{H}^+(\mathcal{H})$ is such that $\text{supp}(A) \subset \text{supp}(W)$ then for some $c > 0$, $cM > A$. By this $T(cM) > T(A)$ and then $\text{supp}(T(A)) \subset \text{supp}(T(M)) \subset \text{supp}(W)$. □

The next proposition contains the generalizations announced above.

**Proposition 5**
*Suppose $T$ is a CP map on $\mathfrak{H}(\mathcal{H})$ and $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$ then*

*i. $\mathcal{H}_S$ is invariant if and only if in any OSR for $T$ the matrices $M_k$ have the block structure:*

$$M_k = \begin{bmatrix} M_{k,S} & M_{k,P} \\ 0 & M_{k,R} \end{bmatrix}. \tag{5.3}$$

*ii. $\mathcal{H}_S$ is invariant if and only if $\mathfrak{H}_S \oplus \mathfrak{H}_{SR}$ is invariant under the action of $T$.*

*Proof.*

i. Fix an OSR for $T$ and let $A \in \mathfrak{H}_S^+$

$$A = \begin{bmatrix} A_S & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.4}$$

Applying $T$ results in:

$$\begin{aligned} T(A) &= \sum_k M_k A M_k^\dagger \\ &= \sum_k \begin{bmatrix} M_{k,S} & M_{k,P} \\ M_{k,Q} & M_{k,R} \end{bmatrix} \begin{bmatrix} A_S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_{k,S}^\dagger & M_{k,Q}^\dagger \\ M_{k,P}^\dagger & M_{k,R}^\dagger \end{bmatrix} = \\ &= \sum_k \begin{bmatrix} M_{k,S} A_S M_{k,S}^\dagger & M_{k,S} A_S M_{k,Q}^\dagger \\ M_{k,Q} A_S M_{k,S}^\dagger & M_{k,Q} A_S M_{k,Q}^\dagger \end{bmatrix}. \end{aligned}$$

If $\mathcal{H}_S$ is invariant, we must have

$$\sum_k M_{k,Q} A_S M_{k,Q}^\dagger = 0,$$

for any choice of $A_S$. Since it is a sum of positive terms, for any $k$

$$M_{k,Q} A_S M_{k,Q}^\dagger = 0,$$

irrespective of $A_S$. This implies that $M_{k,Q} = 0$ for any $k$. Conversely, if $M_{k,Q} = 0$ for any $k$ then for $A \in \mathfrak{H}_S^+$, $T(A) \in \mathfrak{H}_S^+$ since all the others blocks are zero as well, and $\mathcal{H}_S$ is invariant.

ii. Suppose $\mathfrak{H}_S \oplus \mathfrak{H}_{SR}$ is invariant. Since $T$ is positive $\mathfrak{H}^+(\mathcal{H})$ is invariant, then their intersection is invariant too, and it is easily seen that

$$\mathfrak{H}_S \oplus \mathfrak{H}_{SR} \cap \mathfrak{H}^+(\mathcal{H}) = \mathfrak{H}_S^+.$$

The converse is obtained using the first point and computing $T(A)$ for $A \in \mathfrak{H}_S \oplus \mathfrak{H}_{SR}$ like in Section 4 for the case of CPTP maps.

□

From the proof follows also that if a CP map has one OSR with matrices of the form (5.4), then $\mathcal{H}_S$ is invariant (and then all the OSR have matrices of the form (5.4)). Using these results for a given CP map $T$ and an invariant subspace $\mathcal{H}_S$, it is possible to split the action of $T$ between the three subspaces of $\mathfrak{H}(H)$, as it is done in Section 4 for a CPTP map. In this case also similar properties apply: the characteristic polynomial of $T$ is the product of the characteristic polynomial of $T_S$, $T_P$ and $T_R$, their minimal polynomial divide the polynomial of $T$ and both $T_S$ and $T_R$ are CP maps.

Before turning to the construction, a remark is in order. If as above, $T$ is a CP map and $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$ with $\mathcal{H}_S$ invariant, we can consider $T_R$ which is a CP maps that acts on $\mathfrak{H}_R$. If we find in $\mathcal{H}_R$ an invariant subspace for $T_R$ i.e. $\mathcal{H}_R = \mathcal{H}_T \oplus \mathcal{H}_{R_1}$ and $\mathcal{H}_T$ is invariant, using the first point of the last proposition we see that $\mathcal{H}_S \oplus \mathcal{H}_T$ is invariant for $T$. In this situation we could think at two maps induced on $\mathfrak{H}_{R_1}$, one is obtained from $T$, the other from $T_R$, however using an OSR for $T$ follows immediately that these are actually the same, so we shall refer to it simply as $T_{R_1}$.

### 5.1.2 Construction of the Decomposition

Let $T$ be a CPTP map and $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$ with $\mathcal{H}_S$ an invariant subspace. The aim of this construction is to obtain a finite chain of $N$ invariant subspaces $\mathcal{H}_{S_i}$ such that

$$\mathcal{H}_{S_1} = \mathcal{H}_S, \tag{5.5}$$
$$\mathcal{H}_{S_i} \subset \mathcal{H}_{S_{i+1}}, \tag{5.6}$$
$$\mathcal{H}_{S_N} = \mathcal{H}. \tag{5.7}$$

To any of these subspaces will correspond a remainder (its orthogonal complement) $\mathcal{H}_{R_i}$ such that $\mathcal{H}_{S_i} \oplus \mathcal{H}_{R_i} = \mathcal{H}$, and the corresponding map $T_{R_i}$. We shall construct these maps in order for them to have strictly decreasing spectral radius

$$\sigma(T_{R_i}) > \sigma(T_{R_{i+1}}). \tag{5.8}$$

If we look at the subspaces $\mathcal{H}_{S_i}$ as faces of the PSD cone, under the action of $T$ the cone tends to restrict to smaller faces, until that one which correspond to the least GAS subspace containing $\mathcal{H}_S$ is reached. To any of the $\mathcal{H}_{S_i}$ we can associate the spectral radius of $T_{R_i}$, this permits to estimate the time needed to neglect $\mathcal{H}_{R_i}$.

The basic idea is to use the observation above and to look for $\mathcal{H}_{R_{i+1}}$ in $\mathcal{H}_{R_i}$, referring only at $T_{R_i}$. To start assume $\mathcal{H}_{S_1} = \mathcal{H}_S$ and $\mathcal{H}_{R_1} = \mathcal{H}_R$. We next describe the general construction step of $\mathcal{H}_{R_{i+1}}$ and $\mathcal{H}_{S_{i+1}}$ from $\mathcal{H}_{R_i}$ and $\mathcal{H}_{S_i}$. From now on we deal with $T_{R_i}$ which is a CP map, generally trace non increasing, so invariance of a subspace of $\mathcal{H}_{R_i}$ has to be intended as the invariance of the face associated to it.

i. Define

$$D_i := \ker((T_{R_i} - \sigma(T_{R_i})I)^{d_i^2}) \cap \mathfrak{H}^+_{R_i} \quad (5.9)$$

$$\mathcal{H}_{T_i} := \mathrm{supp}(D_i) \quad (5.10)$$

with $d_i = \dim(\mathcal{H}_{R_i})$. Note that, since $T_{R_i}$ is CP, $D_i$ contains elements different from zero. Furhtermore $D_i$ is the intersection of two invariant sets for $T_{R_i}$, so it is invariant. By Lemma 5 $\mathcal{H}_{T_i}$ is then invariant as well;

ii. put $\mathcal{H}_{S_{i+1}} = \mathcal{H}_{S_i} \oplus \mathcal{H}_{T_i}$, notice that this is an invariant subspace for $T$;

iii. if $\mathcal{H}_{T_i} = \mathcal{H}_{R_i}$ the construction is ended, otherwise put $\mathcal{H}_{R_{i+1}} = \mathcal{H}^{\perp}_{S_{i+1}}$, and iterate.

Since at any step $\mathcal{H}_{T_i}$ is not the zero space, $\mathcal{H}_{S_{i+1}}$ has dimension strictly larger than the dimension of $\mathcal{H}_{S_i}$, so in a finite number of steps its dimension must reach the dimension of $\mathcal{H}$ and the construction stops (i.e. $\mathcal{H}_{T_i} = \mathcal{H}_{R_i}$). When this happens we obtain the structure:

$$\mathcal{H}_{S_1} = \mathcal{H}_S, \quad (5.11)$$

$$\mathcal{H}_{S_i} \subset \mathcal{H}_{S_{i+1}}, \quad (5.12)$$

$$\mathcal{H}_{S_N} = \mathcal{H}. \quad (5.13)$$

Using the $\mathcal{H}_{T_i}$ the starting Hilbert space $\mathcal{H}$ can be written as a sum of orthogonal subspaces:

$$\mathcal{H} = \mathcal{H}_S \oplus \bigoplus_i \mathcal{H}_{T_i}. \quad (5.14)$$

It is then easy to see that, in a basis that reflect this structure, the matrices $M_k$ are in a block upper-triangular form, with the first diagonal block corresponding to $\mathcal{H}_S$ and each other diagonal block corresponding to one of the $\mathcal{H}_{T_i}$.

We next highlight some of the features of the above decomposition.

**Proposition 6**
*If the chain* (5.12) *is constructed as above, and we consider the maps $T_{R_i}$ then*

$$\sigma(T_{R_i}) > \sigma(T_{R_{i+1}}). \quad (5.15)$$

The proof is based on two lemmas. The first one shows that it is always possible to extend an eigenoperator of $T_R$ to one of $T$, both relative to the same eigenvalue.

**Lemma 6**
*Let $T$ be a linear map on a vector space $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ of dimension $d$, with $\mathcal{V}_1$ invariant, so that in a suitable basis $T$ is represented by the matrix*

$$\widehat{T} = \begin{bmatrix} \widehat{T}_1 & \widehat{T}_2 \\ 0 & \widehat{T}_3 \end{bmatrix}. \tag{5.16}$$

*If $\eta \in \mathcal{V}_2$ has coordinates*

$$\widehat{\eta} = \begin{bmatrix} 0 \\ \widehat{\eta}_2 \end{bmatrix}, \tag{5.17}$$

*and $\widehat{T}_3\widehat{\eta}_2 = \sigma\widehat{\eta}_2$, then exists $\xi \in \mathcal{V}_1$ such that*

$$\xi + \eta \in \ker((T - \sigma I)^d). \tag{5.18}$$

*Proof.*
If $\xi \in \mathcal{V}_1$ then

$$\xi = \begin{bmatrix} \xi_1 \\ 0 \end{bmatrix},$$

and the lemma is proved if the equation in $\xi$

$$(T - \sigma I)^d(\xi + \eta) = \begin{bmatrix} (T_1 - \sigma I)^d \xi_1 + T_2\eta_2 \\ (T_3 - \sigma I)^d \eta_2 \end{bmatrix} = 0 \tag{5.19}$$

has solution.

Since $\eta_2$ is an eigenvector of $T_3$, $(T_3 - \sigma I)^d\eta_2 = 0$ and (5.19) reduces to

$$(T_1 - \sigma I)^d \xi_1 = -T_2\eta_2. \tag{5.20}$$

If $\sigma$ is not an eigenvalue of $T_1$, this system is clearly solvable in $\xi_1$. In the other case it is solvable only if $T_2\eta_2$ belongs to the image of $(T_1 - \sigma I)^d$ i.e. if $T_2\eta_2$ has no component with respect to the generalized eigenspace relative to $\sigma$. Since generalized eigenspaces are in direct sum, $T_2\eta_2$ admits a unique decomposition as sum of *generalized eigenvectors of* $T_1$:

$$T_2\eta_2 = \sum_k v_k, \tag{5.21}$$

where any $v_k$ is relative to a different eigenvalue $\lambda_k$, with $\lambda_j \neq \lambda_i$ if $j \neq i$. Also, due to the invariance of $\mathcal{V}_1$, any $v_k$ can be (trivially) extended to a generalized eigenvector $w_k$ for $T$, also relative to $\lambda_k$. We can thus write:

$$(T - \sigma I)^d \eta = \begin{bmatrix} T_2\eta_2 \\ 0 \end{bmatrix} = \sum_k \begin{bmatrix} v_k \\ 0 \end{bmatrix} = \sum_k w_k. \tag{5.22}$$

Notice that if $\sigma$ is an eigenvalue of $T_1$, as it is in the case we are discussing, then it is also an eigenvalue of $T$ since $\mathcal{V}_1$ is invariant. It is thus apparent that none of the $w_k$ can be a generalized eigenvalue corresponding to $\sigma$: in fact, due to the Jordan structure of $T$, $T - \sigma I$ restricted to the generalized $\sigma$-eigenspace is a nilpotent matrix of order at most $d$, and hence any generalized $\sigma$-eigenvector for $T$ is mapped to zero after $d$ applications of $T - \sigma I$. This allows us to conclude that none of the $v_k$ is relative to $\sigma$ and the system (5.19) is always solvable. □

**Lemma 7**
*Let $\mathcal{H}$ be a finite dimensional Hilbert space, and $\mathcal{H}_S \oplus \mathcal{H}_R$ an orthogonal decomposition. If $Z \in \mathfrak{H}(H)$ has block form*

$$Z = \begin{bmatrix} Z_S & Z_P \\ Z_P^\dagger & Z_R \end{bmatrix}, \tag{5.23}$$

*with $Z_R > 0$ and $X \in \mathfrak{H}(H)$*

$$X = \begin{bmatrix} X_S & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.24}$$

*with $X_S > 0$, then there exists a scalar $c > 0$ such that $Z + cX > 0$.*

*Proof.*
We wish to prove that for some $c$

$$\langle\varphi|Z + cX|\varphi\rangle > 0, \tag{5.25}$$

for any $|\varphi\rangle \neq 0$.
Decompose $|\varphi\rangle = |\varphi_S\rangle \oplus |\varphi_R\rangle$ according to the decomposition of $\mathcal{H}$ and rewrite (5.25) in its block form

$$\begin{gathered}\langle\varphi_S| \oplus \langle\varphi_R| \begin{bmatrix} Z_S + cX_S & Z_P \\ Z_P^\dagger & Z_R \end{bmatrix} |\varphi_S\rangle \oplus |\varphi_R\rangle = \\ \langle\varphi_S|cX_S + Z_S|\varphi_S\rangle + \langle\varphi_R|Z_P^\dagger|\varphi_S\rangle + \langle\varphi_S|Z_P|\varphi_R\rangle + \langle\varphi_R|Z_R|\varphi_R\rangle.\end{gathered} \tag{5.26}$$

Since $X_S > 0$ exists a $c_1$ such that $c_1 X_S + Z_S > 0$, so that redefining $c = c' + c_1$ we have:

$$\langle\varphi_S|cX_S + Z_S|\varphi_S\rangle \geq \langle\varphi_S|c' X_S|\varphi_S\rangle \tag{5.27}$$

for any $|\varphi_S\rangle$.
The set of vectors $|\varphi_S\rangle \oplus |\varphi_R\rangle$, under the condition $\langle\varphi_S|\varphi_S\rangle = \langle\varphi_R|\varphi_R\rangle = 1$, is compact and $\langle\varphi_R|Z_P^\dagger|\varphi_S\rangle + \langle\varphi_S|Z_P|\varphi_R\rangle$ is a real continuous function; then exists $m > 0$ such that

$$\langle\varphi_R|Z_P^\dagger|\varphi_S\rangle + \langle\varphi_S|Z_P|\varphi_R\rangle \geq -2m, \tag{5.28}$$

if [2] $|\varphi_S| = |\varphi_R| = 1$. So for $|\varphi_S\rangle$ and $|\varphi_R\rangle$ with $|\varphi_S| \neq 0$ and $|\varphi_R| \neq 0$

$$\begin{gathered}\langle\varphi_R|Z_P^\dagger|\varphi_S\rangle + \langle\varphi_S|Z_P|\varphi_R\rangle = \\ |\varphi_R||\varphi_S| \left(\langle \tfrac{1}{|\varphi_R|}\varphi_R|Z_P^\dagger|\tfrac{1}{|\varphi_R|}\varphi_S\rangle + \langle \tfrac{1}{|\varphi_S|}\varphi_S|Z_P|\tfrac{1}{|\varphi_R|}\varphi_R\rangle\right) \geq -2m|\varphi_R||\varphi_S|.\end{gathered} \tag{5.29}$$

By positiveness of $X_S$ and $Z_R$

$$\begin{aligned}\langle\varphi_S|X_S|\varphi_S\rangle &\geq a_1^2|\varphi_S|^2 \\ \langle\varphi_R|Z_R|\varphi_R\rangle &\geq a_2^2|\varphi_R|^2\end{aligned} \tag{5.30}$$

for some $a_1\,, a_2 > 0$ (for instance the roots of their minimum eigenvalues).
Suppose $|\varphi_S| \neq 0$ and $|\varphi_R| \neq 0$ and put (5.27), (5.29) and (5.30) in (5.26), to obtain

$$\langle\varphi_S| \oplus \langle\varphi_R|cX + Z|\varphi_1\rangle \oplus |\varphi_2\rangle \geq a_1^2|\varphi_S|^2 c' + a_2^2|\varphi_R|^2 - 2m|\varphi_S||\varphi_R|; \tag{5.31}$$

[2] $|\varphi| = \sqrt{\langle\varphi|\varphi\rangle}$ is the norm of $|\varphi\rangle$.

choosing $c' > \left(\frac{m}{a_1 a_2}\right)^2$

$$\begin{aligned} &\langle\varphi_S| \oplus \langle\varphi_R|cX + Z|\varphi_S\rangle \oplus |\varphi_R\rangle > \\ &\frac{m^2}{a_2^2}|\varphi_S|^2 + a_2^2|\varphi_R|^2 - 2m|\varphi_S||\varphi_R| = \\ &(\frac{m}{a_2}|\varphi_1| - a_2|\varphi_2|)^2 > 0, \end{aligned} \tag{5.32}$$

so $cX + Z > 0$. □

*Proof of proposition 6.*
Suppose by contradiction that $\sigma = \sigma(T_{R_i}) = \sigma(T_{R_{i+1}})$. It would be then possible to find $A \geq 0$ such that $T_{R_{i+1}}(A) = \sigma A$. $\mathcal{H}_{T'} = \mathrm{supp}(A)$ is invariant for $T_{R_{i+1}}$ so that $\mathcal{H}_{R'} = \mathcal{H}_{T_i} \oplus \mathcal{H}_{T'}$ is invariant for $T_{R_i}$. Consider $T_{R'}$ the restriction of $T_{R_i}$ to $\mathcal{H}_{R'}$. For this map $\mathfrak{H}(\mathcal{H}_{T_i}) \oplus \mathfrak{H}(\mathcal{H}_{T_iT'})$ is invariant and, being $A \in \mathfrak{H}(\mathcal{H}_{T'})$ an eigenoperator for $T_{T'}$, it is thus possible to apply Lemma 6 to extend $A$ to a generalized eigenoperator of $T_{R'}$, of the form

$$A' = \begin{bmatrix} A_1 & A_2 \\ A_2^\dagger & A \end{bmatrix}, \tag{5.33}$$

with $A > 0$. By the definition of $\mathcal{H}_{T_i}$ there exists, for $T_{R_i}$, a generalized eigenvector $X \geq 0$ relative to $\sigma$, such that $\mathrm{supp}(X) = \mathcal{H}_{T_i}$. Since $\mathcal{H}_{T_i} \subset \mathcal{H}_{R'}$, $X$ is an eigenvector of $T_{R'}$ as well. Now by Lemma 7 it is possible to find a constant $c > 0$ such that $B = cX + A > 0$. Since $B$ is a generalized eigenvector of $T_{R'}$ relative to $\sigma$, the same holds for $T_{R_i}$, however this is not possible since $\mathcal{H}_{T_i}$ is strictly contained in $\mathrm{supp}(B)$ in contradiction with its definition. □

This decomposition thus provides us with a nested sequence of faces to which the cone of PSD matrices asymptotically converge. In fact, if each of the $\mathcal{H}_{T_i}$ is characterized by a spectral radius strictly less than one, these subspaces tend to correspond to zero probability asymptotically. The next proposition shows how this is naturally related to attractivity.

**Proposition 7**
*The invariant subspace $\mathcal{H}_S$ is GAS if and only if $\sigma(T_{R_1}) < 1$. If this is not the case, $\mathcal{H}_{S_2} = \mathcal{H}_S \oplus \mathcal{H}_{T_1}$ is then the minimal GAS subspace containing $\mathcal{H}_S$.*

*Proof.*
The first part is just Proposition 2.
For the second part the same proposition assures that $\mathcal{H}_{S_2}$ is GAS, so we only need to check that any other GAS subspace which contains $\mathcal{H}_S$, contains $\mathcal{H}_{S_2}$ as well. By the definition of $D_1$ exists $M \in D_1$ with maximal rank (i.e. with support equal to $\mathcal{H}_{S_2}$), since the minimal polynomial of $T_{R_1}$ divides that one of $T$, $M$ must be a simple eigenvector so

$$T(M) = \begin{bmatrix} T_{SR}(M) & T_{PR}(M) \\ (T_{PR}(M))^\dagger & T_R(M) \end{bmatrix} = \begin{bmatrix} T_{SR}(M) & T_{PR}(M) \\ (T_{PR}(M))^\dagger & M \end{bmatrix} \tag{5.34}$$

and then any subspace which is GAS must contains $\mathrm{supp}(M)$, and the statement follows. □

We conclude the section with two remarks. First, all the construction is based on the maps $T_{R_i}$, which do not depend on a particular OSR, but only on the (whole) map $T$.

Secondarily, if $\mathcal{H}_S$ is the support of the fixed points $T_\infty(I)$ defined in Section 3, the above construction is similar in principle to a decomposition of the "transient subspaces" that has been outlined in [12] for continuous time evolutions. However, beside being developed for discrete-time semigroups, our results differs from the above in the following aspects: (i) we allow for the initial subspaces to be *any* invariant subspace, a generalization that is of interest in many control and quantum information protection tasks [9, 8, 18, 19, 20, 21]; (ii) we investigate in more detail the structure and the properties of the $\mathcal{H}_{T_i}$; (iii) we use the latter as tool for *deciding* asymptotic stability of the subspace.

## 5.2 Dissipation-Induced Decomposition

For this decomposition the framework is the same as before: $T$ is a CPTP map, and $\mathcal{H}$ is decomposed in two orthogonal subspaces

$$\mathcal{H}_S \oplus \mathcal{H}_R, \tag{5.35}$$

with $\mathcal{H}_S$ invariant. To obtain the desired decomposition we first work on an (arbitrary) OSR of the map. However, in the next section we derive a dual characterization, which will make clear that this decomposition does not depend on the particular OSR chosen. The aim of this construction is to divide $\mathcal{H}_R$ in $N$ subspaces:

$$\mathcal{H}_R = \bigoplus_{i=1}^{N} \mathcal{H}_{T_i}, \tag{5.36}$$

with the property that if $\mathrm{supp}(\rho) \subset \mathcal{H}_{T_i}$, the support of $T(\rho)$ cannot be in the $\mathcal{H}_{T_i}$ with $i < j - 1$ or $\mathcal{H}_S$. Assume $\mathcal{H}_S$ to be asymptotically stable: since the probability to find the state in a fixed subspace is greater than zero only if the support of the state has non zero intersection with that subspace, we see that for a state starting from a given $\mathcal{H}_{T_i}$ the dynamics has to generate a "probability flow" through the preceding $\mathcal{H}_{T_j}$ in order for it to reach $\mathcal{H}_S$. Our purpose is to check whether $\mathcal{H}_S$ is GAS, and if it is the case to estimate how fast the state reaches $\mathcal{H}_S$.

We first describe how this decomposition is carried out. Fix an OSR of $K$ matrices $M_k$ for $T$. To start let $\mathcal{H}_{S_1} = \mathcal{H}_S$ and $\mathcal{H}_{R_1} = \mathcal{H}_R$. We proceed iteratively: at each step we have a decomposition of the form

$$\mathcal{H} = \mathcal{H}_{S_i} \oplus \mathcal{H}_{R_i}, \tag{5.37}$$

where

$$\mathcal{H}_{S_i} = \mathcal{H}_S \oplus \bigoplus_{j=1}^{i-1} \mathcal{H}_{T_j}. \tag{5.38}$$

Let $M_{k,P_i'}$ and $M_{k,R_i}$ be the $P$ and $R$ blocks in (3.19), for the decomposition (5.37).

First $\mathcal{H}_{R_{i+1}}$ is defined as follows

$$\mathcal{H}_{R_{i+1}} = \bigcap_k \ker(M_{k,P_i'}). \tag{5.39}$$

Three cases are possible:

i. for all $k$ $M_{k,P_i'} = 0$ *i.e.* $\mathcal{H}_{R_{i+1}} = \mathcal{H}_{R_i}$. In this case we put $\mathcal{H}_{T_i} = \mathcal{H}_{R_i}$, obtaining $\mathcal{H}_{S_{i+1}} = \mathcal{H}$, and terminate the construction. We will see that we have just found an invariant subspace disjoint from $\mathcal{H}_S$.

ii. $\mathcal{H}_{R_{i+1}} = \{0\}$: in this case the construction is successfully concluded. We put again $\mathcal{H}_{T_i} = \mathcal{H}_{R_i}$, and later we will see that in this case $\mathcal{H}_S$ is GAS.

iii. If none of the above cases applies, we choose as $\mathcal{H}_{T_i}$ the orthogonal complement of $\mathcal{H}_{R_{i+1}}$ in $\mathcal{H}_{R_i}$

$$\mathcal{H}_{R_i} = \mathcal{H}_{T_i} \oplus \mathcal{H}_{R_{i+1}} \tag{5.40}$$

Note that since degenerate cases have been dealt separately, in this case both $\mathcal{H}_{T_i}$ and $\mathcal{H}_{R_{i+1}}$ have positive dimension, then $\mathcal{H}_{R_{i+1}}$ has dimension strictly less than $\mathcal{H}_{R_i}$. Now put $\mathcal{H}_{S_{i+1}} = \mathcal{H}_{S_i} \oplus \mathcal{H}_{T_i}$ and iterate.

First of all, notice that at any step the dimension of the $\mathcal{H}_{R_i}$ strictly decreases, so in a finite number of steps the procedure terminates.

If the DID runs to completion we obtain a decomposition of the form (5.36), if we choose a basis according to it, the matrices $M_k$ have the block structure:

$$M_k = \begin{bmatrix} M_{k,S} & M_{k,P_1} & 0 & \dots & \dots & 0 \\ 0 & M_{k,T_1} & M_{k,P_2} & 0 & \dots & 0 \\ 0 & M_{k,Q_{2,1}} & M_{k,T_2} & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \\ 0 & M_{k,Q_{N-1,1}} & M_{k,Q_{N-1,2}} & \dots & M_{k,T_{N-1}} & M_{k,P_N} \\ 0 & M_{k,Q_{N,1}} & M_{k,Q_{N,2}} & \dots & M_{k,Q_{N,N-1}} & M_{k,T_N} \end{bmatrix} \quad \forall k, \tag{5.41}$$

with $\cap_k \ker(M_{k,P_i}) = 0$ for $i = 1, \dots N$.
If instead the decomposition is terminated due to case one, and we choose a basis according to it, the matrices become

$$M_k = \begin{bmatrix} M_{k,S} & M_{k,P_1} & 0 & \dots & \dots & 0 \\ 0 & M_{k,T_1} & M_{k,P_2} & 0 & \dots & 0 \\ 0 & M_{k,Q_{2,1}} & M_{k,T_2} & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \\ 0 & M_{k,Q_{N-1,1}} & M_{k,Q_{N-1,2}} & \dots & M_{k,T_{N-1}} & 0 \\ 0 & M_{k,Q_{N,1}} & M_{k,Q_{N,2}} & \dots & M_{k,Q_{N,N-1}} & M_{k,T_N} \end{bmatrix} \quad \forall k. \tag{5.42}$$

The zero blocks in the last column over $M_{k,T_N}$, for *all* $k$, imply the invariance of the last subspace due to Proposition 1.

We focus now on the properties of $\mathcal{H}_S$ that can be inferred from the DID construction.

**Proposition 8**
*$\mathcal{H}_S$ is GAS if and only if the DID is terminated successfully.*

*Proof.*
If the DID is not terminated successfully the matrices have the structure (5.42), by this is easily seen that $\mathcal{H}_{T_N}$ is invariant, then $\mathcal{H}_S$ cannot be GAS (to see invariance we can permute the basis and use Proposition 1 or just verify it making a direct computation).

If instead the DID is terminated successfully then the statement is a consequence of Theorem 5. We can verify that no fixed point has support in $\mathcal{H}_R$, arguing by absurd. Suppose $\rho$ is a density operator with support in $\mathcal{H}_R$ and a fixed point for the map. Using Theorem 5 we see that actually

$$\text{supp}(\rho) \subset \bigoplus_{i=2}^{N} \mathcal{H}_{T_i}. \tag{5.43}$$

Then there exists a maximal $j$ between 2 and $N$ (note that the next subspace is minimal) such that

$$\mathrm{supp}(\rho) \subset \bigoplus_{i=j}^{N} \mathcal{H}_{T_i}. \tag{5.44}$$

Then $\rho_{T_j}$ (the component of $\rho$ corresponding to $\mathcal{H}_{T_j}$ in its block structure) is not zero, otherwise, due to positiveness of $\rho$, the corresponding columns and rows should be zero and (5.44) would hold with $j+1$. By the block structure (5.41) we see that

$$T(\rho)_{T_{j-1}} = \sum_k M_{k,P_j} \rho_{T_j} M_{k,P_j}^\dagger = 0, \tag{5.45}$$

since $\rho$ is a fixed point. This is a sum of PSD matrices so any terms must be zero.
If $\rho_{T_j} = CC^\dagger$ with $C \in \mathfrak{B}(\mathcal{H}_{T_j})$ we see that

$$M_{k,P_j} C = 0, \tag{5.46}$$

for any $k$, then the columns of $C$ are in $\cap_k \ker(M_{k,P_i}) = \{0\}$ which is impossible by construction. □

Consider now the case when the DID is completed successfully. We can compute the probability to find a state $\rho$, with support in a fixed $\mathcal{H}_{T_i}$, in $\mathcal{H}_{T_{i-1}}$ after applying $T$

$$\mathbb{P}_{T(\rho)}(\Pi_{T_{i-1}}) = \mathrm{Tr}(\Pi_{T_{i-1}} T(\rho)), \tag{5.47}$$

where $\Pi_{T_{i-1}}$ is the orthogonal projection onto $\mathcal{H}_{T_{i-1}}$. We can then obtain a bound on how this probability has grown (recall it was zero before the action of the map). We have:

$$\mathbb{P}_{T(\rho)}(\Pi_{T_{i-1}}) = \mathrm{Tr}(\Pi_{T_{i-1}} T(\rho)) = \mathrm{Tr}(\sum_k M_{k,P_i} \rho_{T_i} M_{k,P_i}^\dagger) = \mathrm{Tr}(\sum_k M_{k,P_i}^\dagger M_{k,P_i} \rho_{T_i}) \tag{5.48}$$

Calling $\gamma_i$ the minimum eigenvalue of $\sum_k M_{k,P_i}^\dagger M_{k,P_i}$ the increment is at least

$$\gamma_i \,\mathrm{Tr}(\rho_{T_i}), \tag{5.49}$$

as is seen putting $\sum_k M_{k,P_i}^\dagger M_{k,P_i}$ in its diagonal form. In the same way it is possible to give an upper bound for (5.47) by the maximum eigenvalue of $\sum_k M_{k,P_i}^\dagger M_{k,P_i}$. Note that these bounds are always meaningful: $\sum_k M_{k,P_i}^\dagger M_{k,P_i}$ is positive definite by construction, so its lowest eigenvalue cannot be zero. Moreover we can find a projection (e.g. the one on the subspace generated by an eigenvector corresponding to $\gamma_i$) for which the lower bound is reached, and in the same way we can reach the upper bound (this shows also that the maximum eigenvalue is less than or equal to one).

In the light of these observations, the $\gamma_i$ are indications of the (minimal) probabilities that a transition from $\mathcal{H}_{T_i}$ to $\mathcal{H}_{T_{i-1}}$ occurs. Knowing all these probabilities we can use them to find the convergence bottlenecks, and estimate the worst-case time needed to reach $\mathcal{H}_S$ starting from any state.

It should be noted that the respect of the bounds derived is assured only when $\mathrm{supp}(\rho) \in \mathcal{H}_{T_i}$, due to the possible presence of blocks that connect the subspaces $\mathcal{H}_{T_i}$ in the opposite way. However, since we assume $\mathcal{H}_S$ to be GAS, the transitions to it "dominate" the dynamics, so the $\gamma_i$ can be used as explained above.

### 5.2.1 Dual Characterization

The DID can be also studied, and in fact characterized, in the Heisenberg picture. As we shall see, in this dual framework some properties become more explicit, e.g. its independence from the chosen OSR. We start by characterizing invariance of subspaces by using the dual dynamics.

**Proposition 9**
*Let $T$ be a CPTP map, and $T^*$ its dual. A subspace $\mathcal{H}_S$ is invariant if and only if for any $n$*

$$T^{*n+1}(\Pi_S) \geq T^{*n}(\Pi_S). \tag{5.50}$$

*Proof.*
Suppose first $\mathcal{H}_\mathcal{S}$ is invariant, that implies

$$T^*(\Pi_S) = \Pi_S + T^*_{SR}(\Pi_S). \tag{5.51}$$

Iterating $T^*$ we obtain

$$T^{*n}(\Pi_S) = \Pi_S + \sum_{i=1}^{n-1} T_R^{*i}(T^*_{SR}(\Pi_S)) \tag{5.52}$$

since $T_R^{*n}(T^*_{SR}(\Pi_S)) \geq 0$ for any $n$, the sequence $T^{*n}(\Pi_S)$ is non decreasing.
Suppose now

$$T^*(\Pi_S) \geq \Pi_S. \tag{5.53}$$

Let $\mathcal{H}_R$ be the orthogonal complement of $\mathcal{H}_\mathcal{S}$ so that

$$\Pi_S + \Pi_R = I, \tag{5.54}$$

and then

$$T^*(\Pi_S) + T^*(\Pi_R) = I. \tag{5.55}$$

Rewriting these two terms in their block form we obtain:

$$T^*(\Pi_S) = \sum_k \begin{bmatrix} M^\dagger_{k,S} M_{k,S} & M^\dagger_{k,S} M_{k,P} \\ M^\dagger_{k,P} M_{k,S} & M^\dagger_{k,P} M_{k,P} \end{bmatrix}, \tag{5.56}$$

$$T^*(\Pi_R) = \sum_k \begin{bmatrix} M^\dagger_{k,Q} M_{k,Q} & M^\dagger_{k,Q} M_{k,R} \\ M^\dagger_{k,R} M_{k,Q} & M^\dagger_{k,R} M_{k,R} \end{bmatrix}, \tag{5.57}$$

Now by unitality

$$\sum_k M^\dagger_{k,S} M_{k,S} \leq I, \tag{5.58}$$

and from (5.53)

$$\sum_k M^\dagger_{k,S} M_{k,S} \geq I, \tag{5.59}$$

so

$$\sum_k M^\dagger_{k,S} M_{k,S} = I. \tag{5.60}$$

But now (5.55), together with (5.57), implies

$$M^\dagger_{k,Q} M_{k,Q} = 0, \tag{5.61}$$

for any $k$. Using Proposition 1, we have invariance of $\mathcal{H}_S$. □

From this result we immediately have the following:

**Corollary 1**
*Let $T$ be a CPTP map and $\mathcal{H}_S$ a subspace:*

*i. If $T^*(\Pi_S) \geq \Pi_S$ then $\mathcal{H}_S$ is invariant.*

*ii. If $\mathcal{H}_S$ is invariant then for any $n$*

$$\mathrm{supp}(T^{*n}(\Pi_S)) \subset \mathrm{supp}(T^{*n+1}(\Pi_S)). \tag{5.62}$$

Our main interest is for the second point of the corollary, since it helps in proving the following characterization.

**Proposition 10**
*Let $T$ be a CPTP map, $\mathcal{H}_S$ a GAS subspace and consider the decomposition induced by the DID. Then for $n \leq N$*

$$\mathrm{supp}(T^{*n}(\Pi_S)) = \mathcal{H}_S \oplus \bigoplus_{i=1}^{n} \mathcal{H}_{T_i} \tag{5.63}$$

*and for $n > N$ the support is the whole $\mathcal{H}$.*

*Proof.*
We shall prove that (5.63) holds by induction on $n$.

First the case $n = 1$. By using the matrix block-decomposition provided in (5.41), it is easy to show that:

$$T^*(\Pi_S) = \sum_k M_k^\dagger \Pi_S M_k = \sum_k \begin{bmatrix} M_{k,S}^\dagger M_{k,S} & M_{k,S}^\dagger M_{k,P_1} & 0 & \cdots & 0 \\ M_{k,P_1}^\dagger M_{k,S} & M_{k,P_1}^\dagger M_{k,P_1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & 0 \end{bmatrix}. \tag{5.64}$$

Recall that by (5.62)

$$\mathcal{H}_S \subset \mathrm{supp}(T^*(\Pi_S)), \tag{5.65}$$

so, by (5.64), it suffices to show that $\mathcal{H}_{T_1}$ is contained in $\mathrm{supp}(T^*(\Pi_S))$. Choosing a set of vector $|\varphi_{1,h}\rangle$ in $\mathcal{H}_{T_1}$, applying $T^*(\Pi_S)$ results in:

$$T^*(\Pi_S)|\varphi_{1,h}\rangle = \begin{bmatrix} M_{k,S}^\dagger M_{k,P_1}|\varphi_{1,h}\rangle \\ M_{k,P_1}^\dagger M_{k,P_1}|\varphi_{1,h}\rangle \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} |\psi_{1,h}\rangle \\ |\phi_{1,h}\rangle \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.66}$$

By a proper choice of $|\psi_{1,h}\rangle$, it is possible to obtain that the $|\phi_{1,h}\rangle$ are a basis for $\mathcal{H}_{T_1}$, since by construction the range of $\sum_k M_{k,P_1}^\dagger M_{k,P_1}$ is $\mathcal{H}_{T_1}$. Hence, the case $n = 1$ is completed.

Now assume (5.63) true for $n < l$, and call $\Pi_i$ the projection on $\mathcal{H}_{\mathcal{S}} \oplus \bigoplus_{j=1}^{i} \mathcal{H}_{T_j}$ so that

$$\mathrm{supp}(T^{*l-1}(\Pi_S)) = \mathcal{H}_{\mathcal{S}} \oplus \bigoplus_{j=1}^{l-1} \mathcal{H}_{T_j} = \mathrm{supp}(\Pi_{l-1}), \tag{5.67}$$

and for some real constant $c > 0$ and $C > 0$

$$c\Pi_{l-1} \leq T^{*l-1}(\Pi_S) \leq C\Pi_{l-1}. \tag{5.68}$$

This implies that the support of $T^*(\Pi_{l-1})$ and $T^{*l}(\Pi_S)$ is the same and the statement is proved if we show that $\mathrm{supp}(T^*(\Pi_{l-1}))$ coincides with $\mathcal{H}_{\mathcal{S}} \oplus \bigoplus_{j=1}^{l} \mathcal{H}_{T_j}$, which can be verified proceeding as above. By (5.62) in Corollary 1 we have:

$$\mathcal{H}_{\mathcal{S}} \oplus \bigoplus_{j=1}^{l-1} \mathcal{H}_{T_j} \subset \mathrm{supp}(T^{*l}(\Pi_S)) = \mathrm{supp}(T^*(\Pi_{l-1})). \tag{5.69}$$

Again if we apply $T^*(\Pi_{l-1})$ to a set of vector $|\varphi_{l,h}\rangle$ in $\mathcal{H}_{T_l}$ we obtain:

$$T^*(\Pi_{l-1})|\varphi_{l,h}\rangle = \sum_k \begin{bmatrix} 0 \\ M^\dagger_{k,Q_{l-1,1}} M_{k,P_l}|\varphi_{l,h}\rangle \\ M^\dagger_{k,Q_{l-1,1}} M_{k,P_l}|\varphi_{l,h}\rangle \\ \vdots \\ M^\dagger_{k,T_{l-1}} M_{k,P_l}|\varphi_{l,h}\rangle \\ M^\dagger_{k,P_l} M_{k,P_l}|\varphi_{l,h}\rangle \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} |\psi_{l,h}\rangle \\ |\phi_{l,h}\rangle \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.70}$$

where $|\phi_{l,h}\rangle = \sum_k M^\dagger_{k,P_l} M_{k,P_l}|\varphi_{l,h}\rangle$ and $|\psi_{l,h}\rangle$ accounts for the firsts blocks of elements. By (5.70) $|\psi_{l,h}\rangle \oplus |\phi_{l,h}\rangle$ are in $\mathrm{supp}(T^*(\Pi_{l-1}))$. Again choosing properly $|\varphi_{l,h}\rangle$, and noting that $\sum_k M^\dagger_{k,P_l} M_{k,P_l}$ is strictly positive definite, we prove that (5.63) holds for $l$ as well.

Finally the last statement follows directly from the first and the hypothesis of $\mathcal{H}_S$ being GAS, which in turn implies that the DID algorithm runs to completion and

$$\mathcal{H}_S \oplus \bigoplus_{i=1}^{N} \mathcal{H}_{T_i} = \mathcal{H}.$$

□

So the DID is determined by, and is in fact equivalent to, the sequence of supports (5.63). Since it depends only from the form of $T^*$ we readily obtain that the DID is independent from the chosen OSR. From the last result another useful property can be obtained.

**Corollary 2**

*Suppose $\mathcal{H}_S$ is an invariant subspace for the CPTP map $T$. Then it is GAS if and only if the sequence* $\mathrm{supp}(T^{*n}(\Pi_S))$ *is strictly increasing until it reaches the whole space.*

*Proof.*
One implication is just a restatement of the last proposition.
For the other implication it suffices to follow the last proof, taking into account that the failure of the DID returns the block structure of the form (5.42), from which we see that if a subspace is not GAS, the sequence $\mathrm{supp}(T^{*n}(\Pi_S))$ cannot cover the whole space. □

## 5.3 Example of Decomposition

In this last part of the chapter the two decomposition are illustrated on a simple example. Since this permits to apply the results presented in Chapter 4, some asymptotic probabilities are also evaluated.

### 5.3.1 Description of the dynamics

Consider a 7 level quantum system associated to the Hilbert space $\mathcal{H} = \mathrm{span}(\{|j\rangle\}_{j=1}^{7})$, on which, within each fixed time step, one of the following "noise actions" may occur:

i. with probability $\gamma_1 < 1$, level 1, 3 and 2, 4 are swapped,

ii. with probability $\gamma_2 < 1$, level 3 decays to 1 and 4 to 2,

iii. with probability $\gamma_3 \ll 1$, level 5 decays to level 4 and 3 in the same proportion,

iv. with probability $\gamma_4 < 1$, level 6 decays to 5,

v. with probability $\gamma_5 < 1$, level 7 decays to 5;

where $\sum_i \gamma_i = 1$, $\gamma_i > 0$ for any $i$ and $\gamma_3 < \gamma_4 < \gamma_5$. An OSR for the map $T$ jointly describing these processes can be obtained by the following matrices, associated to each of the processes in the ordered basis for $\mathcal{H}$ given above (see e.g. [1], Chapter 8 for details on phenomenological description of noise actions):

i. $N_1 = \begin{bmatrix} 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}$;

ii. $N_2 = \begin{bmatrix} 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{\sqrt{2}}&0&0\\ 0&0&0&0&0&\frac{1}{\sqrt{2}}&0\\ 0&0&0&0&0&0&\frac{1}{\sqrt{2}} \end{bmatrix}$, $N_3 = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{\sqrt{2}}&0&0\\ 0&0&0&0&0&\frac{1}{\sqrt{2}}&0\\ 0&0&0&0&0&0&\frac{1}{\sqrt{2}} \end{bmatrix}$;

iii. $N_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $N_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$;

iv. $N_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $N_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$;

v. $N_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $N_9 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Defining the probability-weighed matrices $M_1 = \sqrt{\gamma_1}N_1$, $M_2 = \sqrt{\gamma_2}N_2$, $M_3 = \sqrt{\gamma_2}N_3$, $M_4 = \sqrt{\gamma_3}N_4$, $M_5 = \sqrt{\gamma_3}N_5$, $M_6 = \sqrt{\gamma_4}N_6$, $M_7 = \sqrt{\gamma_4}N_7$, $M_8 = \sqrt{\gamma_5}N_8$ and $M_9 = \sqrt{\gamma_5}N_9$ we obtain a representation for the whole process $T$.

### 5.3.2 Checking GAS

By looking at the structure of the matrices, it is easy to note that the subspace $\mathcal{H}_{S_1} = \mathrm{span}(\{|1\rangle, |3\rangle\})$ is invariant. This allows us to employ the results of Section 5.2.1 in order to to check if it is also GAS. We must look at the sequence of supports $T^{*n}(|1\rangle\langle 1| + |3\rangle\langle 3|)$, obtaining:

$$\begin{aligned} \mathrm{supp}(T^*(|1\rangle\langle 1| + |3\rangle\langle 3|)) &= \mathrm{span}(|1\rangle, |3\rangle, |5\rangle), \\ \mathrm{supp}(T^{*2}(|1\rangle\langle 1| + |3\rangle\langle 3|)) &= \mathrm{span}(|1\rangle, |3\rangle, |5\rangle, |6\rangle, |7\rangle), \\ \mathrm{supp}(T^{*3}(|1\rangle\langle 1| + |3\rangle\langle 3|)) &= \mathrm{span}(|1\rangle, |3\rangle, |5\rangle, |6\rangle, |7\rangle). \end{aligned}$$

Since this sequence stops before covering the whole $\mathcal{H}$, by Corollary 2 $\mathcal{H}_{S_1}$ is not GAS.

### 5.3.3 Nested Faces

It is interesting find out what is the minimal subspace that contains $\mathcal{H}_{S_1}$ and is GAS. This can be done using the nested faces construction, thanks to the results in section 5.1.2. Decomposition (5.14) returns in this case the following subspaces, each characterized by the spectral radius of the corresponding $T_{R_i}$:

$$\begin{aligned} \mathcal{H}_{T_1} &= \mathrm{span}(\{|2\rangle, |4\rangle\}) & \sigma(T_{R_1}) &= 1, \\ \mathcal{H}_{T_2} &= \mathrm{span}(\{|5\rangle\}) & \sigma(T_{R_2}) &= 1 - \gamma_3, \\ \mathcal{H}_{T_3} &= \mathrm{span}(\{|6\rangle\}) & \sigma(T_{R_3}) &= 1 - \gamma_4, \end{aligned}$$

$$\mathcal{H}_{T_4} = \quad \text{span}(\{|7\rangle\}) \qquad \sigma(T_{R_4}) = 1 - \gamma_5.$$

As expected, given Proposition 7, $\sigma(T_{R_1}) = 1$; moreover, the same proposition permits to obtain the minimal GAS subspace, which is $\mathcal{H}_S = \mathcal{H}_{S_1} \oplus \mathcal{H}_{T_1}$. In our case where $\mathcal{H}_{T_1} = \mathcal{H}_{S_2} = \text{span}(\{|2\rangle, |4\rangle\})$, we obtain:

$$\mathcal{H}_S = \text{span}(\{|1\rangle, |3\rangle, |2\rangle, |4\rangle\}).$$

The subspace $\mathcal{H}_S$ can be used as the starting point for the DID; doing so, decomposition (5.36) is given by:

$$\begin{aligned} \mathcal{H}_{T_1'} &= \quad \text{span}(\{|5\rangle\}), \\ \mathcal{H}_{T_2'} &= \text{span}(\{|6\rangle, |7\rangle\}). \end{aligned}$$

For any of these subspaces there is a minimal and a maximal transition probability, as explained in Section 5.2, the least of which has value $\gamma_3$ (in this case it can be read out directly from the form of the dynamics, and in particular $M_4$). A comparison with the maximal spectral radius of the nested faces decomposition shows that both the constructions give the same estimation for the covergence speed towards $\mathcal{H}_S$.

### 5.3.4 Asymptotic probabilities

Knowing that $\mathcal{H}_S$ is GAS, it is possible to use the results in section 4.3, to evaluate the asymptotic probabilities of the two subspaces $\mathcal{H}_{S_1}$ and $\mathcal{H}_{S_2}$. In order to find the form of the fixed-point set, it is useful to note that representing the dynamics restricted to $\mathcal{H}_S$ in the basis $\{|1\rangle, |3\rangle, |2\rangle, |4\rangle\}$, one directly obtains a tensor structure. In fact, by relabeling these four states as $|1\rangle = |0_N\rangle \otimes |0_F\rangle$, $|2\rangle = |1_N\rangle \otimes |0_F\rangle$, $|3\rangle = |0_N\rangle \otimes |1_F\rangle$ and $|4\rangle = |1_N\rangle \otimes |1_F\rangle$, results in a decomposition of $\mathcal{H}_S$ in two "virtual" subsystem of dimension 2: $\mathcal{H}_S = \mathcal{H}_N \otimes \mathcal{H}_F$. With respect to this decomposition the matrices that generates the dynamics *inside* $\mathcal{H}_S$ can be written as:

$$\begin{aligned} B_1 &= \sqrt{\gamma_1} I_2 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; B_2 = \sqrt{\gamma_2} I_2 \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ B_3 &= \sqrt{\gamma_2} I_2 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B_4 = \sqrt{1 - \gamma_1 - \gamma_2} I_2 \otimes I_2. \end{aligned}$$

Any of the $B_i$ factorizes in an operator proportional to the identity on $\mathcal{H}_N$ times another on $\mathcal{H}_F$, this is a sufficient condition for $\mathcal{H}_N$ to be a *Noiseless Subsystem* [9, 20]. Moreover in this decomposition projecting onto the the subspaces $\mathcal{H}_{S_1}$ and $\mathcal{H}_{S_2}$, defined above, correspond to projecting onto the states $|0_N\rangle$ and $|1_N\rangle$. Thus evaluating the trace of the state projected onto one of them returns the probability of having prepared the corresponding state in $\mathcal{H}_N$. To do this in the asymptotic limit we can use the results of Section 4.3, since both subspaces are invariant.

Turning to the asymptotic probabilities, it is convenient to evaluate the limits of the projections, as is done in the proof of Proposition 3, and then apply them to the initial state:

$$\begin{aligned} \lim_{n\to\infty} \mathcal{E}^{*n}(\Pi_{S_1}) &= |1\rangle\langle 1| + |3\rangle\langle 3| + \frac{1}{2}(|5\rangle\langle 5| + |6\rangle\langle 6| + |7\rangle\langle 7|), \\ \lim_{n\to\infty} \mathcal{E}^{*n}(\Pi_{S_2}) &= |2\rangle\langle 2| + |4\rangle\langle 4| + \frac{1}{2}(|5\rangle\langle 5| + |6\rangle\langle 6| + |7\rangle\langle 7|). \end{aligned}$$

By these, if the initial state is $\rho_0 = \frac{1}{7} I_7$, we obtain:

$$\lim_{n\to\infty} \mathrm{Tr}(\Pi_{S_1}\mathcal{E}^n(\rho_0)) = \frac{1}{2},$$
$$\lim_{n\to\infty} \mathrm{Tr}(\Pi_{S_2}\mathcal{E}^n(\rho_0)) = \frac{1}{2}.$$

If instead the initial state is $\rho_0 = \frac{1}{2}(|1\rangle\langle 1| + |7\rangle\langle 7|)$, then we have

$$\lim_{n\to\infty} \mathrm{Tr}(\Pi_{S_1}\mathcal{E}^n(\rho_0)) = \frac{3}{4},$$
$$\lim_{n\to\infty} \mathrm{Tr}(\Pi_{S_2}\mathcal{E}^n(\rho_0)) = \frac{1}{4}.$$

# 6. Finite-Time Convergence to Subspaces

## 6.1 A Motivating Example

Consider the CPTP map $T$, described by the following matrices

$$M_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6.1}$$

in the basis $\{|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle\}$.

For this map the subspace generated by $|\varphi_1\rangle$ is GAS. The interesting characteristic is that this subspace is reached in a finite number of steps: whatever is the initial condition, after two steps the state is the only fixed point

$$\rho_f = |\varphi_1\rangle\langle\varphi_1|. \tag{6.2}$$

To convince ourselves of this, it suffices to consider how the identity operator is transformed under the action of $T$:

$$\begin{aligned} T(I) &= 2|\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|, \\ T^2(I) &= 3|\varphi_1\rangle\langle\varphi_1|. \end{aligned} \tag{6.3}$$

Since for any density operator we have $\rho \leq I$, applying $T$ leads to $T^2(\rho) \leq T^2(I)$, which implies that the support of $T^2(\rho)$ is contained in $\mathrm{span}(|\varphi_1\rangle)$ and then that it coincides with $\rho_f$. In analogy with classical linear system, this phenomenon is impossible in the case of continuous time evolution [12].

In this example the state $\rho_f$ is obtained in a finite number of steps from any initial condition. This is the most interesting case for applications, however in the following we look at a more general situation, this permits to obtain some results which are not immediate if only this special case is considered. We shall say that a subspace $\mathcal{H}_S$, is *dead-beat* in $k$ steps, if

$$T^n(\rho) \in \mathfrak{D}_S \quad \forall \rho \in \mathfrak{D}(\mathcal{H}), \;\; n \geq k, \tag{6.4}$$

and $k$ is the least integer for which (6.4) holds. In some situation we could know that (6.4) holds, but we do not know if $k$ is minimal. In this case clearly $\mathcal{H}_S$ is dead-beat, but in general not in $k$ steps, what we can easily check is that $\mathcal{H}_S$ is dead-beat in at least $k$ steps. Due to positiveness of density operators (6.4) is easily seen to be equivalent to

$$\mathrm{Tr}(\Pi_R T^n(\rho)) = 0 \quad \forall \rho \in \mathfrak{D}(\mathcal{H}), \;\; n \geq k, \tag{6.5}$$

where $\Pi_R$ is the orthogonal projection on $\mathcal{H}_R = \mathcal{H}_S^{\perp}$.

## 6.2 Characterization

To characterize the subspaces that fulfill condition (6.4), first we derive a condition which is easier to check, since it does not involve the whole evolution.

**Lemma 8**
*Let $T$ be a CPTP map, $\mathcal{H}_S$ a subspace of the underlying Hilbert space and $\Pi_R$ the orthogonal projection on $\mathcal{H}_S^\perp$. If for some $k$*

$$\mathrm{Tr}(\Pi_R T^k(\rho)) = 0 \quad \forall \rho \in \mathfrak{D}(\mathcal{H}), \tag{6.6}$$

*then $\mathcal{H}_S$ is dead-beat in at least $k$ steps.*

*Proof.*
This property is easily seen looking at the dual evolution. In this case the dead-beat condition (6.5) becomes

$$\mathrm{Tr}(T^{*n}(\Pi_R)\rho) = 0 \quad \forall \rho \in \mathfrak{D}(\mathcal{H}), n \geq k, \tag{6.7}$$

which is the same as

$$T^{*n}(\Pi_R) = 0 \quad \forall n \geq k. \tag{6.8}$$

Rewriting (6.6) we obtain

$$\mathrm{Tr}(T^{*k}(\Pi_R)\rho) = 0 \quad \forall \rho \in \mathfrak{D}(\mathcal{H}) \tag{6.9}$$

or equivalently

$$T^{*k}(\Pi_R) = 0. \tag{6.10}$$

Since $T^*$ is a linear map (6.10) implies (6.8). □

To obtain some results on dead-beat subspaces we shall mostly follow what is done in the example at the beginning, we restrict our attention on how the support of a density operator (or more generally of a PSD matrix) evolves during the evolution. To obtain general results looking only at how a particular support evolves, a direct consequence of positiveness for maps will be useful.

**Lemma 9**
*Let $T$ be a positive map. For any $X, Y \in \mathfrak{H}^+(\mathcal{H})$, if*

$$\mathrm{supp}(Y) \subset \mathrm{supp}(X) \tag{6.11}$$

*then*

$$\mathrm{supp}(T(Y)) \subset \mathrm{supp}(T(X)). \tag{6.12}$$

By a simple inductive argument this can be generalized at $\mathrm{supp}(T^n(Y)) \subset \mathrm{supp}(T^n(X))$, $\forall n$. Moreover if $\mathrm{supp}(Y) = \mathrm{supp}(X)$ then $\mathrm{supp}(T^n(Y)) = \mathrm{supp}(T^n(X))$.

*Proof.*
Let $\Pi_Y$ and $\Pi_X$ be the orthogonal projections onto $\mathrm{supp}(Y)$ and $\mathrm{supp}(X)$ respectively. For some real numbers $a, b > 0$ we have

$$a\Pi_Y \leq Y \leq b\Pi_Y, \tag{6.13}$$

then applying $T$

$$T(a\Pi_Y) \leq T(Y) \leq T(b\Pi_Y), \tag{6.14}$$

which implies

$$\text{supp}(T(\Pi_Y)) = \text{supp}(T(Y)). \tag{6.15}$$

In the same way

$$\text{supp}(T(\Pi_X)) = \text{supp}(T(X)). \tag{6.16}$$

Now by (6.11)

$$\Pi_Y \leq \Pi_X, \tag{6.17}$$

and then, applying $T$ and looking at the supports we obtain

$$\text{supp}(T(Y)) = \text{supp}(T(\Pi_Y)) \subset \text{supp}(T(\Pi_X)) = \text{supp}(T(X)). \tag{6.18}$$

□

The last proposition offers a physical interpretation: if we think at the dimension of the support as an indicator of the degree of classical uncertainty in the knowledge of the state, this property says that the order of states with respect to this indicator is preserved by positive maps. Also, and maybe more interesting, if we are able to stabilize a state in finite time, starting from a full rank state (i.e. a maximal uncertain state), then we can stabilize it (with the same means) starting from any state (to be noted the property immediately extends even if different positive maps are applied).

If we combine the last two properties we obtain another simple way to check if a subspace is dead-beat: $\mathcal{H}_S$ is dead-beat in at least $k$ steps if

$$\text{supp}(T^k(I)) \subset \mathcal{H}_S. \tag{6.19}$$

This was exactly the idea we used in the example. Also notice that here the use of the identity is only a matter of convenience: any other operator of full rank can be used instead (and in fact by the preceding their supports evolve in the same way). We can make the last observation more precise.

**Lemma 10**
*$\mathcal{H}_S$ is dead-beat in $k$ steps if and only if*

$$\text{supp}(T^n(I)) \not\subset \mathcal{H}_S, \tag{6.20}$$

*for $n < k$, and*

$$\text{supp}(T^k(I)) \subset \mathcal{H}_S. \tag{6.21}$$

*Proof.*
By the former discussion if (6.21) holds $\mathcal{H}_S$ is dead-beat in at least $k$ steps, moreover by (6.20) this is the minimal $k$ for which the dead-beat condition holds.
If $\mathcal{H}_S$ is dead-beat in $k$ steps then (6.21) must hold, moreover if (6.20) does not hold for some $\overline{k} < k$ by the first part $\mathcal{H}_S$ is dead-beat in $\overline{k}$ steps which is impossible since our definition implies that a subspace could be dead-beat only for one number of steps. □

This suggests to look at how the support of $I$ evolves under the action of $T$, since by the last proposition it identifies the dead-beat subspaces, giving also the number of steps needed to reach them.

**Proposition 11**
*Let $T$ be a positive map then the sequence of supports* $\mathrm{supp}(T^n(I))$ *is non increasing*

$$\mathrm{supp}(T^{n+1}(I)) \subset \mathrm{supp}(T^n(I)), \tag{6.22}$$

*for any $n$.*
*Moreover if the sequence of support is stationary for some $k$:*

$$\mathrm{supp}(T^{k+1}(I)) = \mathrm{supp}(T^k(I)), \tag{6.23}$$

*then it is stationary for all $n \geq k$*

$$\mathrm{supp}(T^n(I)) = \mathrm{supp}(T^k(I)). \tag{6.24}$$

*Proof.*
The first part is a straightforward application of what precedes.
If $\mathrm{supp}(T^n(I))$ is the whole Hilbert space (6.22) is obvious.
If $\mathrm{supp}(T^n(I)) = \mathcal{H}_S$ then this subspace is dead-beat at least in $n$ steps, so for any $k > n$ the support of $T^k(I)$ is contained in it and then (6.22) holds.
The second part is a consequence of Lemma 9, if

$$\mathrm{supp}(T^{k+1}(I)) = \mathrm{supp}(T^k(I)) \tag{6.25}$$

then

$$\mathrm{supp}(T^{k+2}(I)) = \mathrm{supp}(T^{k+1}(I)) = \mathrm{supp}(T^k(I)), \tag{6.26}$$

and iterating

$$\mathrm{supp}(T^{n+k}(I)) = \mathrm{supp}(T^k(I)). \tag{6.27}$$

□

**Corollary 3**
*Let $T$ be a CPTP map and $\mathcal{H}_S$ a dead-beat subspace for $T$. Then it is dead-beat in at most $d-1$ steps, where $d$ is the dimension of the whole Hilbert space.*

*Proof.*
By the last proposition we know that $\mathrm{supp}(T^n(I))$ is decreasing and if at some step stops, then it remains stationary. Since a decrease in the sense of inclusion implies a decrease in the dimension this can happen only for the first $d-1$ steps.

If $k$ is the number of steps in which $\mathcal{H}_S$ is dead-beat, looking at Lemma 10, equations (6.20) and (6.21) imply that the decrease is strict at least until $k$, then it must be $k \leq d-1$. □

If we put together what we know until now, we see that the sequence of supports $T^n(I)$ gives us a sequence of dead-beat subspaces until it stops. These subspaces are minimal, since by Lemma 10 any subspace which is dead-beat in $k$ steps must contains $\mathrm{supp}(T^k(I))$. As we will see later this property will assure also invariance of these subspaces.

Before turning to this a discussion about how invariance and dead-beat behavior relate seems appropriate. Dead-beat behavior requests that the subspace is the only possible after a fixed $k$, on the other hand invariance requests that starting on a subspace we remains in it for any $n$. As will be clear later, the problem of dead-beat subspaces is that we do not request the minimality property necessary: if $\mathcal{H}_S$ is dead-beat then any subspace which contains it is dead-beat too. To see why this could be a problem we refer to the example at the beginning, we know that the subspace generated by $|\varphi_1\rangle$ is dead-beat in 2 steps and this subspace is also invariant. On the other hand span$(|\varphi_1\rangle, |\varphi_3\rangle)$ is also dead-beat in two steps but not invariant. The problem with this last subspace is the presence of $|\varphi_3\rangle$.

**Proposition 12**
*If $\mathcal{H}_S$ is dead-beat in $k$ steps, and exists $\rho \in \mathfrak{D}(\mathcal{H})$ such that*

$$\operatorname{supp}(T^k(\rho)) = \mathcal{H}_S \tag{6.28}$$

*then $\mathcal{H}_S$ is also invariant.*

By Lemma 9 $\frac{1}{d}I$ must be one of these $\rho$.

*Proof.*
By dead-beat behavior

$$\operatorname{supp}(T^n(\rho)) \subset \mathcal{H}_S, \tag{6.29}$$

for any $n \geq k$. If $\rho_S \in \mathfrak{D}_S$ then

$$\operatorname{supp}(\rho_S) \subset \mathcal{H}_S = \operatorname{supp}(T^k(\rho)), \tag{6.30}$$

which implies

$$\operatorname{supp}(T^n(\rho_S)) \subset \operatorname{supp}(T^{k+n}(\rho)) \subset \mathcal{H}_S, \tag{6.31}$$

or equivalently

$$T^n(\rho_S) \in \mathfrak{D}_S. \tag{6.32}$$

□

If we call

$$\mathcal{H}_{S_i} = \operatorname{supp}(T^{N-i+1}(I)), \tag{6.33}$$

for $i = 1, \ldots, N$, where $N$ is the maximal $n$ until $\operatorname{supp}(T^n(I))$ stops, we obtain a nested sequence of subspaces where any $\mathcal{H}_{S_i}$ is dead beat in $N-i+1$ steps and invariant. By this and (6.33), if we use a suitable basis (a basis such that if we restrict to the first $\dim(\mathcal{H}_{S_i})$ elements we obtain a basis for $\mathcal{H}_{S_i}$), the matrices $M_k$ become of the form

$$M_k = \begin{bmatrix} M_{k,S_1} & M_{k,P_{1,2}} & M_{k,P_{1,3}} & \cdots & M_{k,P_{1,N+1}} \\ 0 & 0 & M_{k,P_{2,3}} & & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & & M_{k,P_{N,N+1}} \\ 0 & 0 & \cdots & & 0 \end{bmatrix}. \tag{6.34}$$

$\mathcal{H}_{S_1}$ is the minimal dead-beat subspace, and the dynamics restrict to it after $N$ steps; since it is invariant we can consider the map induced on its orthogonal complement. If we look at this map, it is defined by the matrices:

$$M_k = \begin{bmatrix} 0 & M_{k,S_2} & M_{k,P_{2,3}} & \cdots & M_{k,P_{2,N+1}} \\ 0 & 0 & M_{k,P_{3,4}} & & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & & M_{k,P_{N,N+1}} \\ 0 & 0 & \cdots & & 0 \end{bmatrix}. \tag{6.35}$$

It is readily seen that these matrices define a nilpotent map of index $N$ (and also that any of these matrices is nilpotent of index at most $N$). Moreover, if for an invariant subspace the map induced on its orthogonal complement is nilpotent, clearly that subspace is dead-beat.

## 6.3 Finite-Time Convergence and Nested Faces

The structure of subspaces, highlighted in the previous section, shares some similarities with that one obtained applying the first decomposition of Chapter 5; both are a sequence of nested faces, to which the cone of PSD matrices restricts under the action of $T$. It turns out that these two structures are related: the subspace $\mathcal{H}_{S_1}$ is the union of the faces identified during the construction except the last, which has to be associated to zero spectral radius, otherwise there is no possibility of convergence in finite time. To prove this we shall prove an equivalent fact: if in the decomposition (5.12) all the $T_{R_i}$ have positive spectral radius then $\mathrm{supp}(T^n(X)) = \mathcal{H}$ for any $n$ for an appropriate full rank $X$.

**Lemma 11**
*Let $T$ be a CP map, if $\sigma(T) > 0$,*

$$D = \ker((T - \sigma(T))^{d^2}) \cap \mathfrak{H}^+(\mathcal{H}) \tag{6.36}$$

*and*

$$\mathrm{supp}(D) = \mathcal{H}, \tag{6.37}$$

*then exists $X \in \mathfrak{H}^+(\mathcal{H})$ such that, for any $n$*

$$\mathrm{supp}(T^n(X)) = \mathcal{H} \tag{6.38}$$

*Proof.*
The following construction is a refinement of the construction considered in Section 5.1. Let

$$D_1 = \ker(T - \sigma(T)) \cap \mathfrak{H}^+(\mathcal{H}), \tag{6.39}$$

since $T$ is positive $D_1$ contains non zero vectors. Pick $X_1$ of maximal rank in $D_1$. If $\mathrm{supp}(D_1) = \mathcal{H}$ then $X_1$ is an eigenvector of maximal rank and $X = X_1$ satisfies (6.38). In the other case by the results of Chapter 5 $\mathrm{supp}(D_1)$ is an invariant subspace, so we can consider the map $T_{R_1}$ on its complement. Due to condition (6.37) $\sigma(T) = \sigma(T_{R_1})$, and we can consider a $D_2$ to obtain an $X_2$. Iterating this we can find $X_i$ such that the sum of their supports is $\mathcal{H}$. By the construction is readily seen that for any $i$

$$\mathrm{supp}(X_i) \subset \mathrm{supp}(T(X_i)), \tag{6.40}$$

this is a consequence of the fact that they are eigenvectors for some "part" of the application, so their supports under the action of $T$ can only grow (here the hypothesis $\sigma(T) > 0$ is needed). Then $X = \sum_i X_i$ is what we are looking for:

$$\mathrm{supp}(T(X)) = \mathrm{supp}(\sum_i T(X_i))), \tag{6.41}$$

since any terms in the sum is a PSD matrix the support of the sum is the sum of the supports of the elements, then by (6.40), $X$ fulfills (6.38). □

**Proposition 13**
*Let $T$ be a CPTP map. If in the decomposition* (5.12) *all the $T_{R_i}$ have positive spectral radius then exists $X \in \mathfrak{H}^+(\mathcal{H})$ such that*

$$\begin{aligned} \mathrm{supp}(X) &= \mathcal{H}, \\ \mathrm{supp}(T(X)) &= \mathcal{H}. \end{aligned} \tag{6.42}$$

*Proof.*
If we consider the block form induced by the decomposition, the matrices in an OSR of $T$ are upper triangular. The diagonal blocks define CP maps $T_i$ on $\mathcal{H}_{T_i}$ (these subspaces are that ones defined in Section 5.1). For these maps we can apply the preceding lemma and obtain $X_i$, one for $\mathcal{H}_{T_i}$, such that its support remains the whole $\mathcal{H}_{T_i}$ when $T_i$ is applied. If we look at how these $X_i$ are transformed by $T$, thanks to upper triangular matrices we see that the same argument used in the last proof works again:

$$X = \sum_i X_i. \tag{6.43}$$

□

A question which comes out naturally is if even the DID is related with dead-beat behavior, looking at the example, we could note that if we put $\mathcal{H}_S = T^d(I)$ and apply the DID (clearly this subspace is GAS for any $T$) the sequence of subspaces found is the same sequence of $\mathrm{supp}(T^n(I))$ (in reversed order), and then ask if this is a general property, or at least if these two chains have always the same length.
It turns out that both these assertions could fail, a simple example is the map defined by the matrices

$$M_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.44}$$

in the basis $\{|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle\, |\varphi_4\rangle\}$. In this case the basis is already that one of the DID, which gives only one subspace, while the decomposition in dead-beat subspaces is composed by two subspaces.

# 7. Conclusion

With the increasing interest for quantum information, more attention has been posed on CP maps, which are commonly used to describe the evolution of open Markovian quantum systems. The passage from conservative to dissipative system has led to an in depth study of the structure of the set of fixed points and peripheral eigenvectors, which is now well known [5, 6].

In this work we focused on their asymptotic behavior, characterizing the probabilities and the structure of the "vanishing" subspace. Some works have already addressed parts of this problems in the continuous-time case [8, 11, 12], and the ideas beyond them have been reused here. However, developing the analysis directly for the discrete-time dynamics, permits to obtain some characteristic results, since discrete-time evolutions have their own peculiarities and cannot always be obtained by sampling of continuous one [13]. We have been concerned with different aspects of the evolution:

- in the first part we derived analytic formulas for the asymptotic probabilities which results from the iteration of a CPTP map, given an initial state, in different scenarios. These include simple iteration of the map, as well as iteration of maps and measurement of appropriate observables.

- The central part of the work deals with two Hilbert-space decompositions. The first is the *nested-face decomposition,* which can be seen as an extension to discrete-time system of part of the results in [12], with the notable difference that we allow for any invariant subspace as starting point. The following DID decomposition has been adapted from the analogous, continuos-time decomposition presented in [8], to which a dual characterization has been added. Both provide methods to evaluate the attractivity properties of an invariant subspace, and to estimate the converge speed.

- Finally, a well known aspect of discrete-time system has been analyzed in depth: the possibility that the map has zero eigenvalues, which permits to reach a subspace in finite time. This is of course a topic of interest in many control application [10, 7, 22].

These results complements known results in the theory of discrete-time quantum semigruops, and stand between a pure theoretical works in quantum dynamical system theory and its application of quantum information and control.

Further developments could aim to find methodologies to obtain state preparation in finite time or asymptotically, given some specific control capabilities. Particularly interesting is the case of preparation of entangled state, which has many application in quantum information [23, 24, 25, 26, 27]. If perfect preparation cannot be achieved, the focus could be on partially prepare a state, i.e. with a (possibly high) probability. In this context the asymptotic formulas derived can be some of the tools used during the design.

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