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Dipartimento di Matematica “Tullio Levi Civita”

Master Degree in Mathematics

Structure of solutions and initial data identification for conservation laws with spatial discontinuous flux

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Contents

Introduction

In this thesis we deal with some control problems for a scalar conservation law with discontinuous flux

$$\begin{cases} u_t + f(u,x)_x = 0, & x \in \mathbb{R}, \quad t \geq 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{1}$$

where $f = \mathcal{H}(x)f_l(u) + (1 - \mathcal{H}(x))f_r(u)$, f_l, f_r are uniformly convex and $\mathcal{H}$ is the Heaviside function. In recent years PDEs like (1) have been a topic of intense research due to the great number of applications, such as traffic flow ([20]), two phase flow in porous media ([17]), sedimentation problems and Saint Venant model of blood flow, but also inverse problems for standard (with continuous flux) conservation laws, and reformulation of balance laws (i.e. conservation laws with source terms) in terms of conservation laws with discontinuous flux (see [8] and the references therein). The discontinuity of the flux represents some heterogeneity in the physical reality of the model, such as a road with changing surface conditions in the case of traffic flow, or an abrupt change in the properties of a porous medium. Despite the numerous applications, control problems for equations like (1) are still basically absent in the existing literature and here we study two of them, in particular we deal with controllability and initial data identification.

The main difficulty when studying well posedness for problems like (1) can be traced back to the fact that, in addition to the usual lack of regularity of solutions to a (classical) conservation law, imposing that the conservation of the quantity u through the discontinuity interface $x = 0$ holds (i.e. the Rankine-Hugoniot condition) is not enough to prove uniqueness. In fact, for a given initial datum, (1) has infinitely many solutions, depending on the entropy conditions that we require to hold at the interface $x = 0$. In this thesis we will be concerned with entropy solutions of type AB. More specifically, we work with infinitely many concepts of solution, each one associated to a *connection* (A, B), namely a specific pair of values such that the function $c^{AB}(x) = \mathcal{H}(x)A + (1 - \mathcal{H}(x))B$ is an *undercompressive* stationary solution of (1) (i.e. the characteristics emerge from the interface discontinuity $x = 0$). For a fixed connection AB, we say that a function u is a solution of (1) if it is an entropy solution at the left and at the right of the discontinuity interface, separately, with fluxes f_l, f_r. Moreover, we require that at the interface the only type of undercompressive wave that is allowed is only the one of the stationary solution c^{AB}, i.e. the wave with values precisely the values of the connection. All of this is encoded in a

Kružkov type inequality where, instead of the constant values "k", one has the function c^{AB}.

In this thesis two main results are achieved concerning control problems for (1). The first one concerns the full characterization of the attainable set at time $T > 0$

$$\mathcal{A}^{AB}(T) \doteq \{\mathcal{S}_T^{AB} u_0 : u_0 \in \mathbf{L}^\infty\} \tag{2}$$

in terms of some Oleinik-type inequalities and some further conditions motivated by the fact that there might be some profiles $\omega \in \mathcal{A}^{AB}(T)$ that are reachable only by solutions containing a shock in at least a semiplane $\{x > 0\}$, $\{x < 0\}$. Here $\mathcal{S}_t^{AB}$ is the semigroup operator associated to a connection AB. This result completes the characterization already obtained in [4], where only the profiles reachable with locally Lipschitz solutions outside the line $x = 0$ were characterized. The result is achieved using an adaptation of the method of generalized characteristics, originally developed for conservation laws with strictly convex flux by Dafermos in [11].

The second new main result obtained in this thesis is the full characterization of the set of initial data that lead to a fixed profile ω, i.e. the set

$$\mathcal{I}_T^{AB}\omega \doteq \{u_0 \in \mathbf{L}^\infty : \mathcal{S}_t^{AB} u_0 = \omega\} \tag{3}$$

in terms of two integral inequalities. Using this kind of characterization, we manage to prove that the set $\mathcal{I}_T^{AB}\omega$ is an infinite dimensional cone in $\mathbf{L}^\infty$. This is a generalization to the discontinuous flux setting of the recently obtained results in [10] for a convex conservation law; however we prove that the set $\mathcal{I}_T^{AB}\omega$ is not anymore (in general) convex when dealing with AB-entropy solutions. In order to achieve this results we will introduce a new kind of generalized characteristic that is allowed to travel along the interface discontinuity only when the flux is minimal (i.e. the flux is exactly the flux of the connection $f_l(A) = f_r(B)$).

The thesis is structured as follows. In Chapter 1 we recall some basics of the theory of conservation laws with Lipschitz continuous flux. In Chapter 2 we present the theory of entropy solution of type AB. In Chapter 3 we prove the characterization of the attainable set (2). In Chapter 4 we prove the results about the structure of (3).

Chapter 1

Preliminaries on conservation laws

Abstract. In the first part of this chapter, mainly following [7] (see also [12]), we present the basic theory of scalar conservation laws in one dimension. The last section will be dedicated to the method of generalized characteristics developed by Dafermos in [11] for conservation laws with strictly convex flux.

A scalar one dimensional conservation law is a non linear partial differential equation of the form

$$u_t + f(u)_x = 0 \tag{1.1}$$

where $u(x,t)$ is the state variable and f is the so called flux function, that is usually assumed to be Lipschitz. The name *conservation* laws follows from the fact that the solution u represents the evolution in time of the conserved quantity, while f is the flux. In particular,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\,\mathrm{d}x &= \int_a^b u_t(x,t)\,\mathrm{d}x \\ &= \int_a^b -f(u(x,t))_x\,\mathrm{d}x \\ &= f(u(a,t)) - f(u(b,t)) = [\text{inflow at } a] - [\text{outflow at } b]\end{aligned} \tag{1.2}$$

Accordingly, the conserved quantity is neither created nor destroyed, and the only way to increase or decrease the total amount of u contained in some interval $[a,b]$ is with the flux through the two endpoints a, b.

In this chapter we will be mainly concerned with the Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0, \qquad x \in \mathbb{R}, \quad t \geq 0 \\ u(\cdot, 0) = u_0(\cdot) \end{cases} \tag{1.3}$$

For the first part of this chapter we will mainly follow the lines of [7]. In the last section we present the theory of generalized characteristics, following the original paper by Dafermos [11].

1.1. Classical solutions

We say that a function u is classical solution of (1.3) if it is a continuously differentiable function whose partial derivatives pointwise satisfy the equation.

In case the flux f is a constant, i.e. $f(u) = \lambda u$ for every $u \in \mathbb{R}$, the conservation law becomes the transport equation $u_t + \lambda u_x = 0$. In this case, the solution is $u(x,t) = u_0(x - \lambda t)$ and u is constant along the lines $x = x_0 + \lambda t$.

This fact can be generalized to a classical solution of (1.3) with a general (smooth) flux function f. In fact, let $x(t,y)$ be the solution of the following differential equation

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}x(t,y) = f'(u(x(t,y),t)),\\ &x(0,y) = y \end{aligned} \tag{1.4}$$

Then the solution u satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t,y),t) = u_x(x(t,y),t)f'(u(x(t,y),t) + u_t(x(t,y),t) = 0 \tag{1.5}$$

This means that u is constant along the lines $t \mapsto x(t,y)$; accordingly

$$x(t,y) = y + tf'(u_0(y)) \tag{1.6}$$

Using the method of characteristics, we can define (at least for small times and compactly supported initial data u_0) a solution u in the following way: for a point (x,t), we go back along the (unique, at least for t small) line $x(s,y)$ such that $x(t,y) = x$ and define $u(x,t) = u_0(y)$. One can easily check that the function defined in this way is a classical solution. Of course this construction fails as soon as two different characteristics cross each other. In this case the concept of classical solution is not anymore sufficient to ensure the existence globally in time. The aim of the next sections is then to derive a well posedness theory for a conservation law (1.3).

1.2. Weak solutions

The main feature of conservation laws is that, regardless of how smooth the initial data are, they can develop shocks (discontinuities) in finite time so that, after this time, there is no hope of recovering a classical solution of (1.3). This happens as, seen in the previous section, as soon as two characteristics intersect.

For this reason, in order to have achieve well-posedness for the Cauchy problem, it arises the need to consider weak distributional solutions of (1.1).

Definition 1.1. We say that a function u is a *weak solution* of (1.3) if u is continuous as a function $[0,+\infty) \to \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$, $u_0 = u(\cdot,0)$ and for every test function $\varphi \in \mathcal{D}(\mathbb{R} \times (0,+\infty)$ holds

$$\int_0^{+\infty}\int_{\mathbb{R}} u\varphi_t + f(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{1.7}$$

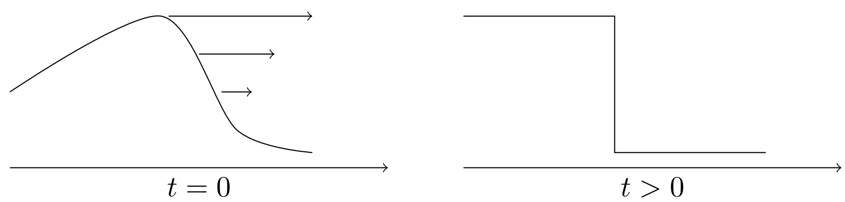

Figure 1.1: Formation of a shock in the case $f = \frac{u^2}{2}$ (Burger's equation). Higher points moving with higher speed make the profile steeper and steeper, eventually forming the shock.

It is easy to understand under which conditions a function like

$$u(x,t) = \begin{cases} u_l & x < \lambda t \\ u_r & x > \lambda t \end{cases} \tag{1.8}$$

is a distributional solution of (1.1). In fact, a simple application of the divergence theorem yields the following necessary and sufficient condition for (1.8) to be a distributional solution:

$$f(u_r) - f(u_l) = \lambda(u_r - u_l) \tag{1.9}$$

This is the so called Rankine-Hugoniot condition. The Rankine-Hugoniot condition uniquely determines at which speed a shock with states u_l, u_r must travel in order to be an admissible shock for a solution u. This condition can be used to determine weather or not u is a weak solution for a wide class of functions:

Definition 1.2. We say that $u(x,t)$ enjoys piecewise Lipschitz regularity if u is measurable, bounded, and there exists finitely many Lipschitz curves $\gamma_i :]a_i, b_i[\to \mathbb{R}$ and finitely many points P_i such that

1. every point P lying outside the curves γ_i's and different from all the P_i's has a neighborhood in which u is Lipschitz continuous.

2. Every point Q lying on a curve γ_i has a neighborhood V in which u is Lipschitz continuous in the domains $\{(x,t) \in V \mid x < \gamma_i(t)\}$ and $\{(x,t) \in V \mid x > \gamma_i(t)\}$.

One can prove the following proposition (see [7]):

Proposition 1.3. *Let u be piecewise Lipschitz. Then u is a weak solution of* (1.1) *if and only if*

1. *for a.e.* $x \in \mathbb{R} \times (0, +\infty) \setminus \cup_i \gamma_i$, *u satisfies*

$$u_t + f(u)_x = 0$$

2. *for every curve* γ_i *and for a.e.* $t \in]a_i, b_i[$, *the Rankine-Hugoniot condition holds:*

$$f(u(\gamma_i(t)+, t)) - f(u(\gamma_i(t)-, t)) = \dot{\gamma}_i(u(\gamma_i(t)+, t) - u(\gamma_i(t)-, t)$$

In other words, to decide if a piecewise Lipschitz function u (as in Definition 1.2) is a solution, it is sufficient to check separately that the equation holds pointwise in the regions where u is Lipschitz and that the Rankine-Hugoniot condition holds at the points of discontinuity of u.

1.3. Entropy admissibility conditions

The concept of weak solution of (1.3), is not enough to select a unique solutions to the initial value problem, as we can see in the following example.

Example 1.4. Consider Burgers' equation, namely the conservation law with flux $f = \frac{u^2}{2}$. Let the initial data be

$$u_0(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

One can set $u_l = 0$, $u_r = 1$ and use the Rankine-Hugoniot condition to find that for $\lambda = \frac{1}{2}$ (1.8) is a weak solution. But one can also solve the Cauchy problem using two discontinuities, for example, for each $\alpha \in [0, 1]$, the function

$$u_\alpha(x, t) = \begin{cases} 0 & x < \alpha t/2 \\ \alpha & \alpha t/2 < x < (\alpha + 1)t/2 \\ 1 & x > (\alpha + 1)t/2 \end{cases}$$

is a weak solution to the Cauchy problem. It is also possible to find a continuous solution using a *rarefaction wave*. In particular, the function

$$u(x, t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases}$$

is a Lipschitz continuous (classical) solution.

A method to select a unique solution is via the so called *vanishing viscosity*. The idea is to add a small viscosity term like εu_{xx}, that usually has a regularizing effect, to obtain

$$u^\varepsilon + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \tag{1.10}$$

and then let $\varepsilon \to 0$.

Definition 1.5 (Viscosity solution)**.** A weak solution u of (1.1) is *admissible* in the vanishing viscosity sense if there exists a sequence of solutions u^ε of (1.10) that converges to u in $\mathbf{L}^1_{\text{loc}}$ as $\varepsilon \to 0+$.

Finding such a sequence of u^ε is usually very difficult. For this reason, it's convenient to introduce further conditions that are easier to verify. Let $\eta(u)$ be a convex function. Let $q(u)$ be such that $q'(u) = \eta'(u)f'(u)$. We call the couple (η, q) an *entropy-entropy flux* pair. If u is a classical solution of (1.1), it's immediate to see that

$$\eta(u)_t + q(u)_x = 0$$

Now assume that u is a weak viscosity solution, i.e. there is a sequence u^ε as above such that $u^\varepsilon \to u$ in $\mathbf{L}^1_{\text{loc}}$. Multiplying (1.1) by $\eta'(u)$ and using the chain rule, we obtain

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x = \varepsilon(\eta(u^\varepsilon)_{xx} - \eta''(u^\varepsilon)(u^\varepsilon_x)^2) \le \varepsilon\eta(u^\varepsilon)_{xx}$$

Assuming that u^ε remains uniformly bounded, one gets, in distributional sense, for $\varepsilon \to 0+$, that

$$\eta(u)_t + q(u)_x \le 0 \tag{1.11}$$

Thus the idea is to use the above entropy inequality to single out the unique solution of the conservation law.

Definition 1.6 (Entropy inequality)**.** A weak solution is *entropy admissible* if for every entropy-entropy flux pair (η, q) it holds

$$\eta(u)_t + q(u)_x \le 0 \tag{1.12}$$

in distributional sense.

There is a family of entropies (the Kružkov entropies) that are really easy to work with. For every $k \in \mathbb{R}$, we define

$$\eta_k(u) = |u - k|, \quad q_k(u) = \text{sgn}(u - k)(f(u) - f(k)) \tag{1.13}$$

When studying entropy admissible solutions, it is possible to prove that requiring (1.11) for all Kružkov entropies is already sufficient to single out a unique solution. One can prove that entropy admissible solutions coincide with solutions that satisfy the following condition, the Liu condition:

Definition 1.7 (Liu admissibility condition)**.** A piecewise Lipschitz weak solution to (1.1) is *Liu admissibile* if for almost every discontinuity point (t, x)

$$\frac{f(u^*) - f(u_l)}{u^* - u_l} \geq \frac{f(u_r) - f(u^*)}{u_r - u^*} \tag{1.14}$$

for all $u^* = \alpha u_l + (1 - \alpha)u_r$ and $\alpha \in [0, 1]$, where $u_l = u(x-, t)$ and $u_r = u(x+, t)$.

The Liu admissibility condition has a clear meaning from both the geometrical and the stability point of view. In particular, if $u_l < u_r$ ($u_r < u_l$), (1.14) tells us that the graph of f must lie above (below) the secant line through u_l, u_r in the interval $[u_l, u_r]([u_r, u_l])$ (see Figure (1.2))

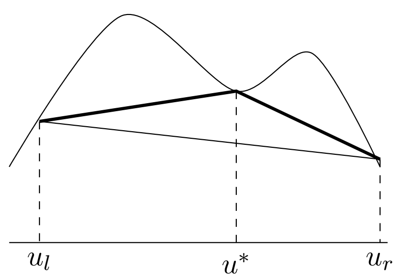

Figure 1.2: Geometrical interpretation of the Liu condition

The stability interpretation is the following. Let the function u in (1.8) be a weak solution of (1.1) and, to fix the ideas, let $u_l < u_r$. We can perturb the initial data adding a small intermediate step $u^* \in [u_l, u_r]$. In this way, the solution will be initially made of two shocks propagating with velocities

$$\frac{f(u^*) - f(u_l)}{u^* - u_l}, \quad \frac{f(u_r) - f(u^*)}{u_r - u^*}$$

If we want the solutions to be stable with respect to small perturbation on the initial data, we have to require that the speed of the shock behind (u_l, u^*) is greater than the speed of the shock ahead (u^*, u_r). This is equivalent to require the Liu condition.

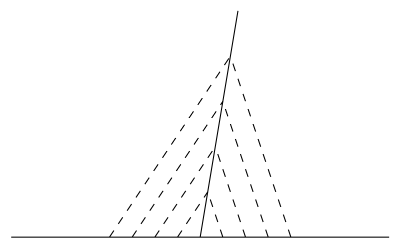

Figure 1.3: The graphical meaning of the Lax condition: for a shock to be admissible, characteristics must enter the shock.

Finally, let's present the Lax condition. It is not, in general, equivalent to the conditions above. Nevertheless, in the case of convex (or concave) flux, it can be proven to be equivalent to all the conditions stated until now.

Definition 1.8 (Lax condition)**.** A piecewise Lipschitz weak solution to (1.1) is Lax admissible if for almost every jump point (t, x) holds

$$f'(u_l) \geq f'(u_r) \tag{1.15}$$

where $u_l = u(x-, t)$, $u_r = u(x+, t)$.

One can easily check that only one of the solutions provided in Example 1.4 satisfies the Lax condition.

1.4. Front-tracking algorithm

The method of front-tracking is a classical method for proving existence of solutions for conservation laws. It can be used also for systems of conservation laws, but in that case one has to assume that the initial datum has small total variation and a general theory is not known. In the case of scalar conservation laws, the theory is a lot easier and can be presented in a couple of pages. The idea is to approximate both the flux f, that can be any Lipschitz function, and the initial datum u_0, with respectively piecewise affine functions f_ν and piecewise constant functions $u_{0,\nu}$. For each ν one obtains a piecewise constant front-tracking approximation u_ν. Letting $\nu \to +\infty$, it is possible to prove that u_ν converges to the solution of the problem (1.3).

First let us show how to build front tracking approximate solutions u_ν. Let $\nu \geq 1$ be an integer. Let f_ν coincide with f on the points $2^{-\nu} j$, with $j \in \mathbb{Z}$, and affine on the segments $[2^{-\nu}(j-1), 2^{-\nu} j]$. We are going to show that there is a piecewise constant entropy admissible weak solution u_ν, with discontinuities located on a finite number of segments, to the problem

$$(u_\nu)_t + f(u_\nu)_x = 0$$

with initial datum $u_{0,\nu}$, a piecewise constant function with compact support taking values in the set $2^{-\nu}\mathbb{Z}$. The solution u_ν will still have values in this set. First of all, we need to know how to solve a Riemann problem with flux f_ν, i.e. the Cauchy problem with initial datum

$$u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

1. $u_l < u_r$. Let $\text{conv}(f_\nu)$ be the largest convex function smaller than f_ν (it will still be piecewise affine). Let the jumps of the derivative of $\text{conv}(f_\nu)$ be located at $u_l = w_0 < w_1 < \ldots < w_q = u_r$. Let

$$\lambda_i = \frac{f(w_i) - f(w_{i-1})}{w_i - w_{i-1}}, \quad i \in \{1, \ldots, q\}$$

Then the solution of the Riemann problem with flux $\text{conv}(f_\nu)$ is

$$u_\nu(x,t) = \begin{cases} u_l & x < t\lambda_1 \\ w_i & t\lambda_{i-1} < x < t\lambda_i, \quad 2 \le i \le q \\ u_r & x > t\lambda_q \end{cases}$$

Indeed, along each jump both the Rankine-Hugoniot and the Liu admissibility condition are satisfied.

2. $u_l > u_r$. Let $\text{conc}(f_\nu)$ be the smallest concave function greater than f_ν. As above, it will still be piecewise affine. Let the jumps of the derivative of $\text{conc}(f_\nu)$ be located at $u_l = w_0 < w_1 < \ldots < w_q = u_r$. Let

$$\lambda_i = \frac{f(w_i) - f(w_{i-1})}{w_i - w_{i-1}}, \quad i \in \{1, \ldots, q\}$$

Then the solution of the Riemann problem is

$$u_\nu(x,t) = \begin{cases} u_l & x < t\lambda_1 \\ w_i & t\lambda_{i-1} < x < t\lambda_i, \quad 2 \le i \le q \\ u_r & x > t\lambda_q \end{cases}$$

Notice that in both cases the solution takes values in $2^{-\nu}\mathbb{Z}$.

Now we start with the more complex initial datum $u_{0,\nu}$. Let it's discontinuities be located at the finite number of points $x_1, \ldots, x_N$. In each point x_i, we solve the Riemann problem, obtaining a solution u_ν defined until the first time t_1 at which two shock collide for the first time. Since the function $x \mapsto u_\nu(x, t_1)$ is still piecewise constant with values in $2^{-\nu}\mathbb{Z}$ we can repeat the procedure and solve new Riemann problems at the discontinuity points of $u_\nu(\cdot, t_1)$, and so on. If we can prove that the number of interactions does not become infinite in finite time, we can prolong the solution u_ν globally in time.

Without loss of generality, with a small perturbation on the initial data, we can assume that at each time at most two shocks collide. Two cases can happen:

1. The fronts that interact at time t have the same sign. In this case $\mathrm{TotVar(u(\cdot, t-))} = \mathrm{TotVar(u(\cdot, t+))}$ and the number of fronts decreases by one.

2. The fronts that interact at time t have opposite sign. In this case , the number of fronts might increase, but the total variation decreases by at least $2^{-\nu}$ after the interaction time.

Case 2. can happen only finitely many times, since each time it happens, the total variation decreases by at least of a fixed positive quantity, and Case 1. leaves it unchanged. Then also Case 1. can happen only finitely many times. This proves that the total number of interactions is finite, and this concludes the construction of the front tracking approximation u_ν.

1.5. A contractive semigroup

The aim of this section is first to prove existence of an admissible solution for an initial datum in $\mathbf{L}^1$ with bounded variation, and then to extend (by density) the result to general integrable bounded initial data.

Let us take any initial datum $u_0 \in \mathbf{L}^1 \cap \mathbf{BV}$. Construct an approximating sequence of piecewise constant functions u_ν with the front-tracking algorithm, corresponding to a sequence $u_{0,\nu}$ of piecewise constant initial data with total variation smaller than the total variation of u_0, and that converge to u_0 in $\mathbf{L}^1$. We want to pass to the limit for $\nu \to +\infty$, and to do that we use Helly's theorem that we recall here.

Theorem 1.9. *Let $u_\nu : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ be a sequence of $\mathbf{L}^\infty$ functions such that there exists two constants $M, L > 0$ such that*

$$\mathrm{TotVar}(u(\cdot, t)) \leq M, \quad |u_\nu(x,t)| \leq M, \quad \forall x, t \tag{1.16}$$

$$\|u_\nu(\cdot, t) - u_\nu(\cdot, s)\|_{\mathbf{L}^1} \leq L\,|t - s|\,, \quad \forall\, t, s > 0 \tag{1.17}$$

Then there exists a subsequence u_μ converging to a function $u \in \mathbf{L}^1_{loc}(\mathbb{R} \times [0, +\infty)$, and u satisfies the same bounds.

Back to our problem, we want to apply Helly's theorem to the front tracking sequence. We can do that because we know

$$\mathrm{TotVar}(u(\cdot, t)) \leq \mathrm{TotVar}(u_{0,\nu}), \quad |u_\nu(x,t)| \leq M$$

Moreover, if L is a Lipschitz constant for f, the speed of the shocks will be smaller than L, so that we obtain the bound

$$\|u_\nu(\cdot, t) - u_\nu(\cdot, s)\| \leq L\,|t-s|\,\mathrm{TotVar}(u_{0,\nu})$$

This ingredients allow us to apply Helly's theorem and conclude that the sequence u_ν converges in $\mathbf{L}^1_{\mathrm{loc}}$ to a function u. In order to prove that u is an admissible solution, we only have to show that, for each $k \in \mathbb{R}$ and for every test function $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^+ \times \mathbb{R})$, it holds

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} |u - k|\,\varphi_t + \mathrm{sgn}(u-k)(f(u) - f(k))\,\mathrm{d}x\,\mathrm{d}t \geq 0 \tag{1.18}$$

Thanks to the dominated convergence theorem, we can write

$$\begin{aligned}\int_{\mathbb{R}^+} \int_{\mathbb{R}} &|u - k|\,\varphi_t + \mathrm{sgn}(u-k)(f(u) - f(k))\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\nu \to +\infty} \int_{\mathbb{R}} |u_\nu - k|\,\varphi_t + \mathrm{sgn}(u_\nu - k)(f(u_\nu) - f(k))\,\mathrm{d}x\,\mathrm{d}t \geq 0 \end{aligned} \tag{1.19}$$

Uniqueness for $\mathbf{L}^\infty$ solutions can be proved with the method of doubling variables by Kružkov, and for the proof we refer to [7] (a detailed description of an adapted version of this method is provided in the chapter on entropy solutions of (A, B) type). The following theorem is due to Kružkov.

Theorem 1.10. *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous. Let u, v be entropy admissible solutions of (1.1) defined for $t \geq 0$. Let M, L be constants such that*

$$\begin{aligned} &|u(t,x)| \leq M, \quad |v(t,x)| \leq M, \qquad \textit{for all}\, t, x, \\ &|f(w) - f(w')| \leq L\,|w - w'|\,, \qquad \textit{for all}\, w, w' \in [-M, M]. \end{aligned} \tag{1.20}$$

Then, for every $R > 0$ *and* $\tau \geq \tau_0 \geq 0$*, one has*

$$\int_{|x|\leq R} |u(x,\tau) - v(x,\tau)| \,\mathrm{d}x \leq \int_{|x|\leq R+L(\tau-\tau_0)} |u(x,\tau_0) - v(x,\tau_0)| \,\mathrm{d}x. \quad (1.21)$$

Corollary 1.11 (Uniqueness in $\mathbf{L}^\infty$)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz continuous. If* u, v *are bounded entropy solutions of* (1.1) *such that* $\|u(\cdot,0) - v(\cdot,0)\|_{\mathbf{L}^1} < \infty$*, for every* $t > 0$ *we have*

$$\|u(\cdot,t) - v(\cdot,t)\|_{\mathbf{L}^1} \leq \|u(\cdot,0) - v(\cdot,0)\|_{\mathbf{L}^1}. \quad (1.22)$$

For all initial data $u_0 \in \mathbf{L}^\infty$*, the problem* (1.3) *has at most one bounded entropy solution.*

By density of BV functions in $\mathbf{L}^1 \cap \mathbf{L}^\infty$, at the end one can prove the following theorem:

Theorem 1.12. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be Lipschitz continuous. There exists a continuous semigroup* $\mathcal{S} : [0,+\infty) \times \mathbf{L}^1 \to \mathbf{L}^1$ *such that the following properties hold*

(i) $\mathcal{S}_0 u_0 = u_0$ *and* $\mathcal{S}_{t+s} u_0 = \mathcal{S}_t(\mathcal{S}_s u_0)$ *for each* $t, s \geq 0$

(ii) $\|\mathcal{S}_t u_0 - \mathcal{S}_t v_0\| \leq \|u_0 - v_0\|$

(iii) For each $u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ *the trajectory* $t \mapsto \mathcal{S}_t u_0$ *yields the unique bounded entropy admissible solution of* (1.3)*.*

(iv) If $u_0(x) \leq v_0(x)$ *for every* $x \in \mathbb{R}$*, then* $\mathcal{S}_t u_0(x) \leq \mathcal{S}_t v_0(x)$ *for each* $x \in \mathbb{R},\ t > 0$*.*

Proof. Fix any initial datum $u_0 \in \mathbf{L}^1 \cap BV$. With the front tracking algorithm we can build an admissible solution with initial datum u_0 and by Corollary 1.11 we know that it is unique. Therefore the semigroup $\mathcal{S}$ is defined at least on $\mathbf{L}^1 \cap BV$. By density of BV functions in $\mathbf{L}^1$, we can extend the domain of $\mathcal{S}$ to the entire $\mathbf{L}^1$ by setting

$$\mathcal{S}_t u_0 = \lim_{\substack{w_0 \to u_0,\\ w_0 \in BV}} \mathcal{S}_t w_0 \quad (1.23)$$

Thanks to the contractive property of Corollary 1.11, the limit is well defined and satisfies (i) and (ii). Moreover, by dominated convergence theorem, (iii) holds. To prove (iv), we can assume that, by density of BV in $\mathbf{L}^1$, $u_0, v_0 \in BV \cap \mathbf{L}^1$. In this case, the be obtained as the limit of the front tracking approximations constructed above. Therefore we can reduces ourselves to prove that if u_0 and v_0 are piecewise constants initial data in $BV \cap L^1$ that satisfy $u_0 \leq v_0$, the corresponding approximate solutions u, v satisfy

$$u(x,t) \leq v(x,t), \quad \text{for all } t \geq 0, \ x \in \mathbb{R} \tag{1.24}$$

If (1.24) fails, there is a first time τ at which (1.24) ceases to hold (possibly $\tau = 0$). In particular, for all $t \leq \tau$ it holds $u(x,t) \leq v(x,t)$. Since u, v are piecewise constant, there is a small δ for which the solutions u, v, in $[\tau, \tau + \delta$, are obtained by solving the Riemann problems at the points of discontinuity for u, v. To obtain a contradiction, it will be sufficient to show that the (iv) holds for the solutions of the approximate Riemann problem with flux f_ν, that is, when u_0, v_0 are constant at the left and at the right of the origin and with values respectively u_l, u_r and v_l, v_r. Since if $\max\{u_l, u_r\} \leq \min\{v_l, v_r\}$, we only have to analyze two cases:

CASE 1. $u_l \leq v_l \leq u_r \leq v_r$. Observe that the solution of the Riemann problem with flux f_ν satisfies the following property: at a point (x,t) we have $u(x,t) = w$ if and only if the line with slope x/t supports the graph of f_ν at $(w, f_\nu(w))$ (restricted to the interval $[u_l, u_r]$); in mathematical terms

$$f_\nu(w) - \frac{x}{t} w = \min_{s \in [u_l, u_r]} \left\{ f_\nu(s) - \frac{x}{t} s \right\}$$

This implies that

$$u(x,t) = \text{argmin}_{[u_l,u_r]} \left\{ f_\nu(s) - \frac{x}{t} s \right\} \leq \text{argmin}_{[v_l,v_r]} \left\{ f_\nu(s) - \frac{x}{t} s \right\}$$

that proves the result in CASE 1.

CASE 2. $u_r \leq v_r \leq u_l \leq v_l$. In this case the solution of the Riemann problem with flux f_ν satisfies the following property: at a point (x,t) we have $u(x,t) = w$ if and only if the line with slope x/t supports from above the graph of f_ν at $(w, f_\nu(w))$ (restricted to the interval $[u_l, u_r]$); in mathematical terms

$$f_\nu(w) - \frac{x}{t} w = \max_{s \in [u_l, u_r]} \left\{ f_\nu(s) - \frac{x}{t} s \right\}$$

This implies that

$$u(x,t) = \text{argmax}_{[u_l,u_r]} \left\{ f_\nu(s) - \frac{x}{t} s \right\} \leq \text{argmax}_{[v_l,v_r]} \left\{ f_\nu(s) - \frac{x}{t} s \right\}$$

that proves the result also in CASE 2., and the proof is completed.

□

1.6. Lax-Oleinik formula

Lax-Oleinik formula is a classical formula that provides a representation for the unique entropy solution u of the problem (1.3) in case the flux f is convex and of superlinear growth. Let's briefly motivate idea behind the formula. One can prove that, for a given f convex and superlinear and $v_0 \in \mathrm{Lip}(\mathbb{R})$, the *value function*

$$v(x,t) = \min_{y \in \mathbb{R}} \left\{ v_0(y) + f^* \left(\frac{x-y}{t} \right) \right\} \tag{1.25}$$

is the unique *viscosity solution* (see [14]) of the initial value problem for the Hamilton-Jacobi equation in one space dimension

$$\begin{array}{ll} v_t(x,t) + f(v_x(x,t)) = 0 & (x,t) \in \mathbb{R} \times (0,+\infty), \\ v(x,0) = v_0(x) & x \in \mathbb{R} \end{array} \tag{1.26}$$

Let v be the solution of the problem above. We now proceed formally and differentiate (1.6) with respect to x to get

$$(v_x)_t + (f(v_x))_x = 0$$

Therefore, the function $u = v_x$ should be a good candidate to solve the conservation law (1.1) with initial data $(v_0)_x$. The idea is thus to differentiate the value function v with respect to x . In fact, one can prove that under our assumptions, v is Lipschitz (therefore differentiable almost everywhere by Rademacher theorem) and that the following theorem holds (see [14]):

Proposition 1.13 (Lax-Oleinik formula). *Assume $f : \mathbb{R} \to \mathbb{R}$ is smooth and uniformly convex and $u_0 \in L^\infty$.*

1. *For each $t > 0$ there exists for all but at most countably many points $x \in \mathbb{R}$ a unique point $y(x,t)$ such that*

$$v(x,t) = \int_0^{y(x,t)} u_0(\xi)\, \mathrm{d}\xi + t f^* \left(\frac{x - y(x,t)}{t} \right)$$

2. *The mapping $x \mapsto y(x,t)$ is nondecreasing*

3. *For each $t > 0$ one has*

$$v(x,t)_x = (f')^{-1} \left(\frac{x - y(x,t)}{t} \right)$$

for a.e. x.

Note that now it is immediate to check that $v(x,t)_x$ is the unique entropy solution of (1.3). Indeed, it is trivial to check that it is a weak distributional solution and that the initial condition is satisfied. We only need to check that at the points of jump there holds $v(x-,t)_x > v(x+,t)_x$. This is clear because the unique viscosity solution v is semiconcave (see [14]), so that the the points in which the derivative of $v(\cdot,t)$ is discontinuous can only be "$\bigwedge$"-shaped. Another, more immediate way, to see it, is to notice that $(f')^{-1}$ is increasing and $x \mapsto y(x,t)$ is increasing, so that at the jump points it must satisfy $y(x-,t) < y(x+,t)$. From this the conclusion follows immediately.

1.7. Generalized characteristics

In this section we discuss the method of generalized characteristics, introduced by Dafermos in the classical paper [11]. Here we briefly recall some results of [11] that will be widely used in the following chapters. Generalized characteristics really are a powerful tool in the context of conservation laws for studying the structure of solutions. The main drawback is that their use is essentially limited to conservation laws with stricly convex flux, and the theory does not apply in the general case, although recent developments show how to extend this approach for more general flux function (see [6]). Accordingly, through this chapter we assume $f \in \mathcal{C}^2$ and strictly (but not necessarily uniformly) convex, i.e. f'' is nonnegative and does not vanish identically on any non degenerate interval. We know that, under this hypothesis on the flux, the solution u of the conservation law (1.3) is in BV_{loc} (see the first section of the chapter on attainable profiles) and admits left and right limits $u(x-,t), u(x+,t)$ at every point. Then, with u being an admissible solution of (1.3), we can give the following definition

Definition 1.14 (Generalized characteristics)**.** A Lipschitz continuous curve $\zeta : [a,b] \to \mathbb{R}$ is called a *characteristic* if for almost all $t \in [a,b]$,

$$\dot{\zeta}(t) \in [f'(u(\zeta(t)+,t)), f'(u(\zeta(t)-,t))]. \tag{1.27}$$

By the theory of contingent equations [15], through any point (x,t) there is at least one *forward* characteristic defined on an interval $[t,t+\delta)$ and at least on *backward* characteristic, defined on an interval $(t-\delta,t]$. If there is more then one forward (backward) characteristic, the funnel spanned by a *minimal* and *maximal* characteristic is filled with forward (backward) characteristic through the point (x,t).

One of the main features of generalized characteristics is the following. Since the speed of ζ may belong to a nondegenerate interval, one might expect a characteristic to be able to travel with a lot of different speeds. Actually, the following proposition shows that, if ζ is characteristic, it can travel either with classical speed of with the Rankine-Hugoniot speed.

Proposition 1.15. *Let $\zeta : [a,b] \to \mathbb{R}$ be a characteristic for a solution u to (1.1). Then for a.e. $t \in [a,b]$,*

$$\dot{\zeta}(t) = \begin{cases} f'(u(\zeta(t)\pm, t)) & \text{if} \quad u(\zeta(t)-,t) = u(\zeta(t)+,t), \\ \\ \frac{f(u(\zeta(T)+,t)) - f(u(\zeta(t)-,t))}{u(\zeta(t)+,t) - u(\zeta(t)-,t)} & \text{if} \quad u(\zeta(t)-,t) > u(\zeta(t)+,t). \end{cases} \tag{1.28}$$

The content of this Proposition follows immediately from a more general result, that we state here, which will be of fundamental importance in our work on initial data identification (see the next chapters).

Lemma 1.16. *Let $\alpha : [a,b] \to \mathbb{R}$ and $\beta : [a,b] \to \mathbb{R}$ be Lipschitz curves, $0 \le a < b < \infty$. Then for almost all σ, τ with $a \le \sigma < \tau \le b$,*

$$\begin{aligned} \int_{\alpha(\tau)}^{\beta(\tau)} u(x,\tau)\,\mathrm{d}x - \int_{\alpha(\sigma)}^{\beta(\sigma)} u(x,\sigma)\,\mathrm{d}x \\ = \int_\sigma^\tau \{f(u(\alpha(t)-,t)) - \dot{\alpha}(t)u(\alpha(t)-,t)\}\,\mathrm{d}t \\ - \int_\sigma^\tau \left\{f(u(\beta(t)-,t)) - \dot{\beta}(t)u(\beta(t)-,t)\right\}\mathrm{d}t \end{aligned} \tag{1.29}$$

Proof. The result is a straightforward consequence of the fact that $u_t + f(u)_x = 0$ in the sense of distributions. □

Now we introduce the concept of genuine characteristic. Essentially, a genuine characteristic is a characteristic that travels with classical speed.

Definition 1.17. A characteristic $\zeta : [a,b] \to \mathbb{R}$ is called *genuine* if

$$u(\zeta(t)-,t) = u(\zeta(t)+,t) \quad \text{for almost all} \quad t \in [a,b]. \tag{1.30}$$

The next result establishes existence of genuine backward characteristics through any point $(\bar{x},\bar{t})$.

Theorem 1.18. *For any point $(\bar{x},\bar{t})$, $t > 0$, the minimal and maximal backward characteristics $\zeta_-(t;\bar{x},\bar{t})$ and $\zeta_+(t;\bar{x},\bar{t})$ through $(\bar{x},\bar{t})$ are genuine.*

Proof. We just try to give an idea of why this is true, delegating the details of the proof to [11] (see also [12] for a different approach).

Assume at some time $t \leq \bar{t}$ the characteristic $\zeta_-(\cdot, \bar{t}, \bar{x})$ is not genuine. Then by strict convexity and by the entropy admissibility conditions it must hold

$$f'(u(\zeta_-(t)-,t)) > f'(u(\zeta_-(t)-,t)), \quad f'(u(\zeta_-(t)+,t)) > f'(u(\zeta_-(t)+,t))$$

But then it should be true that from the point $(\zeta_-(t), t)$ there is a backward characteristic that starts with speed $f'(u(\zeta_-(t)-,t))$ that is strictly greater than the speed of the characteristic $\zeta_-(\cdot; \bar{x}, \bar{t})$, so that this is a contradiction to minimality. □

Theorem 1.19. *Let $\zeta(\cdot)$ be a genuine characteristic on $[a,b]$. Then there is a constant $\bar{u}$ such that $\zeta(\cdot)$ is a straight line with slope $f'(\bar{u})$ and in particular*

$$u(\zeta(t)-,t) = \bar{u} = u(\zeta(t)+,t), \quad a.e. \text{ on } (a,b) \tag{1.31}$$

Proof. Choose any $\sigma < \tau$ in the interval $[a,b]$. We apply Lemma 1.16 to the curves $\zeta(\cdot)|_{[\sigma,\tau]}$ and $\zeta(\cdot)|_{[\sigma,\tau]} - \varepsilon$, with $\varepsilon > 0$ and find

$$\begin{aligned}\int_{\zeta(\tau)-\varepsilon}^{\zeta(\tau)} u(x,\tau)\,\mathrm{d}x - \int_{\zeta(\sigma)-\varepsilon}^{\zeta(\sigma)} u(x,\sigma)\,\mathrm{d}x \\ = \int_\sigma^\tau f(u(\zeta(t)-\varepsilon-,t)) - f(u(\zeta(t)+,t) \\ - f'(u(\zeta(t)+,t))\,[u(\zeta(t)-\varepsilon-,t) - u(\zeta(t)+,t)]\,\mathrm{d}t \geq 0\end{aligned} \tag{1.32}$$

This implies that

$$u(\zeta(\tau)-,\tau) \geq u(\zeta(\sigma)-,\sigma), \qquad \sigma < \tau \tag{1.33}$$

Applying Lemma 1.16 to the curves $\zeta(\cdot)|_{[\sigma,\tau]}$ and $\zeta(\cdot)|_{[\sigma,\tau]} + \varepsilon$, with $\varepsilon > 0$, we find

$$\begin{aligned}\int_{\zeta(\tau)}^{\zeta(\tau)+\varepsilon} u(x,\tau)\,\mathrm{d}x - \int_{\zeta(\sigma)}^{\zeta(\sigma)+\varepsilon} u(x,\sigma)\,\mathrm{d}x \\ = \int_\sigma^\tau f(u(\zeta(t)-),t) - f(u(\zeta(t)+\varepsilon+,t) \\ - f'(u(\zeta(t)-,t))\,[u(\zeta(t)-,t) - u(\zeta(t)+\varepsilon+,t)]\,\mathrm{d}t \leq 0\end{aligned} \tag{1.34}$$

that yields

$$u(\zeta(\tau)+,\tau) \leq u(\zeta(\sigma)+,\sigma) \tag{1.35}$$

Using the fact that ζ is genuine and (1.33), (1.35), with $\sigma \in (a,b)$ and τ fixed, we obtain that

$$u(\zeta(\sigma)-,\sigma) = u(\zeta(\sigma)+,\sigma) = u(\zeta(\tau),\tau) \doteq \bar{u}, \qquad \text{for a.e.} \quad \sigma \in (a,b) \tag{1.36}$$

and the result is proved.

□

Remark 1.20. Notice that since the minimal and maximal backward characteristics through any point $(\bar{x},\bar{t})$ do not escape in finite time (they are straight lines), every other backward characteristic from $(\bar{x},\bar{t})$ must be defined for all times $(0,\bar{t})$.

Finally, we deduce a very important Corollary

Corollary 1.21. *Two genuine characteristics may intersect only at their end points.*

Remark 1.22. For every point $(\bar{x},\bar{t})$ there is only one forward characteristic. In fact, assume there are two of them, say $\alpha < \beta$ defined on an interval $(t, t+\delta)$. Take a point $s \in (t, t+\delta)$ and consider the maximal backward characteristic $\zeta_+(\cdot;\alpha(s),s)$ from $(\alpha(s),s)$ and the minimal backward characteristic $\zeta_-(\cdot;\beta(s),s)$ from $(\beta(s),s)$. Then we know that they are genuine, so that they intersect only at their end points. But on the other hand we have

$$\zeta_+(\bar{t};\alpha(s),s) \geq \bar{x} \geq \zeta_-(\bar{t};\beta(s),s)$$

and this is a contradiction.

Theorem 1.23. *At every point $(\bar{x},\bar{t})$, the minimal and maximal backward characteristics from $(\bar{x},\bar{t})$ have slopes $f'(u(\bar{x}-,t))$ and $f'(u(\bar{x}+,t))$, respectively.*

Proof. First assume that $(\bar{x},\bar{t})$ is a point of continuity for u, so that the minimal and maximal backward characteristics from $(\bar{x},\bar{t})$ coincide. In this case, with the same arguments of the proof of Lemma 1.19, with $\tau = t$, we find that the corresponding value $\bar{u}$ is $\bar{u} = u(\zeta(\bar{t}),\bar{t})$ and the result holds in this case.

Now let $(\bar{x},\bar{t})$ be a discontinuity point, i.e. it holds $u(\bar{x}-,\bar{t}) \geq u(\bar{x}+,\bar{t})$. Since $u(\cdot,\bar{t})$ is BV_{loc} it only has an at most countable number of discontinuity points. Therefore there exists a sequence of continuity points $x_n \uparrow \bar{x}$, $n \in \mathbb{N}$. Let $\zeta_n : [0,\bar{t}] \to \mathbb{R}$ be the unique backward (genuine) characteristic from $(\bar{x},\bar{t})$. Then, by Corollary 1.21, it must hold

$$\zeta_1(t) < \zeta_2(t) < \ldots < \zeta_n(t) < \ldots \qquad \forall t \in (0,\bar{t}) \tag{1.37}$$

and the sequence ζ_n converges uniformly to a line $\zeta : [0, \bar{t}] \to \mathbb{R}$ whose slope is the limit of the slopes of the lines ζ_n, i.e. $f'(u(\bar{x}-, \bar{t}))$. By the theory of contingent equations in [15], the line ζ satisfies the differential inclusion

$$\dot{\zeta}(t) \in [f'(u(\zeta(t)-, t), u(\zeta(t)+, t)], \qquad \text{for a.e.} \quad t \in (0, \bar{t}) \tag{1.38}$$

Therefore ζ is a backward characteristic from the point $(\bar{x}, \bar{t})$. Moreover, it must be the minimal backward characteristic, by Corollary 1.21. Then the result is proved for the minimal characteristic. The proof that the maximal characteristic has slope $f'(u(\bar{x}+, \bar{t}))$ is entirely analogous and is omitted. $\square$

CHAPTER 2

Entropy solutions of type (A,B)

ABSTRACT. In this chapter we present the theory of AB-entropy solutions for scalar conservation laws with discontinuous flux [1]. Following [9], we prove uniqueness with an adaptation of the Kružkov doubling of variables method, while existence of solutions can be proved with a modified version of the front-tracking algorithm (see [16]).

In this chapter we study the initial value problem for the scalar conservation law

$$u_t + f(x,u)_x = 0, \quad x \in (0,+\infty) \times \mathbb{R} \tag{2.1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R} \tag{2.2}$$

where the flux function f is such that

$$f(x,u) = \mathcal{H}(x) f_l(u) + (1 - \mathcal{H}(x)) f_r(u), \tag{2.3}$$

f_l, f_r are uniformly convex functions and $\mathcal{H}$ is the Heaviside function.

It's well known, as we saw in the previous chapter, that equations like (2.1) in general do not admit classical solutions, even in the continuous flux case, no matter how smooth the initial data are. Therefore it arises the need for weak distributional solutions. We recall also that even when the flux f is a Lipschitz continuous function, imposing that u satisfies the equation in distributional sense is not enough to select a unique solution. In order to achieve uniqueness one requires that the entropy conditions hold.

In the case when the flux f is the discontinuous function (2.3), one requires a solution u to (2.1) to be in particular an entropy admissible solution at the left and at the right, separately, of the interface. This, thanks to a result in [22], implies that the strong traces

$$u_l(t) = \lim_{x \to 0-} u(t,x), \quad u_r(t) = \lim_{x \to 0+} u(t,x) \tag{2.4}$$

exist. A straightforward consequence of is that a weak distributional solution of (2.1) has to satisfy the Rankine-Hugoniot condition along the discontinuity $x = 0$:

$$f_l(u_l(t)) = f_r(u_r(t)), \quad \text{for a.e. } t > 0 \tag{2.5}$$

However, this is provably not enough to achieve uniqueness, and additional constraints must be added along the interface $x = 0$. In fact, in [1], the authors pointed out that there are infinitely many $\mathbf{L}^1$-contractive semigroups, each one associated with a particular couple of values (A, B) (a *connection*) such that $f_l(A) = f_r(B)$. For this reason, there has been a great deal of research to develop a well posedness theory, and several different interface admissibility conditions were taken into consideration, each one leading to, possibly, a different solution (see e.g. [5, 9, 13, 18]). For us, u is a solution of (2.1), (2.2) if it satisfies an interface entropy condition associated to a interface connection (A, B), introduced for the first time in [1]. More specifically, once fixed a connection (A, B), u must satisfy, in the sense of distributions, with $c^{AB}(x) = \mathcal{H}(x)A + (1 - \mathcal{H}(x))B$,

$$\left|u - c^{AB}\right|_t + \left[\operatorname{sgn}(u - c^{AB})(f(x,u) - f(x, c^{AB}))\right]_x \leq 0, \quad \text{in } \mathcal{D}' \tag{2.6}$$

Using this kind of enforced entropy conditions, it's possible to prove uniqueness using an adapted version of the Kružkov doubling of variables argument (see [9, 16]). Existence can be obtained with a modified version of the front-tracking algorithm, described in [16], in which the authors also provide a new formulation of the problem in terms of Riemann solvers, that turns out to be equivalent to the concept of AB-entropy solutions described above.

In the following sections we provide a detailed introduction to the theory of AB-entropy solutions, exploring in more details what already mentioned in this introduction.

2.1. Basic definitions and general setting

Referring to the conservation law (2.1), we make the following assumptions on the fluxes f_l, f_r. As already said, we assume that they are $\mathcal{C}^2(\mathbb{R})$ uniformly convex functions, i.e. there exists $a > 0$ such that

$$f_l'', f_r'' \geq a > 0 \tag{2.7}$$

Moreover we assume that f_l and f_r coincide at two points of their domain. Up to a reparametrization of u, we assume that

$$f_l(0) = f_r(0), \quad f_l(1) = f_r(1) \tag{2.8}$$

We assume also that

$$\theta_l \geq 0, \quad \theta_r \leq 1 \tag{2.9}$$

Let's begin with the definition of connection.

Definition 2.1 (connection (A,B))**.** A pair of values (A, B) is called a connection if

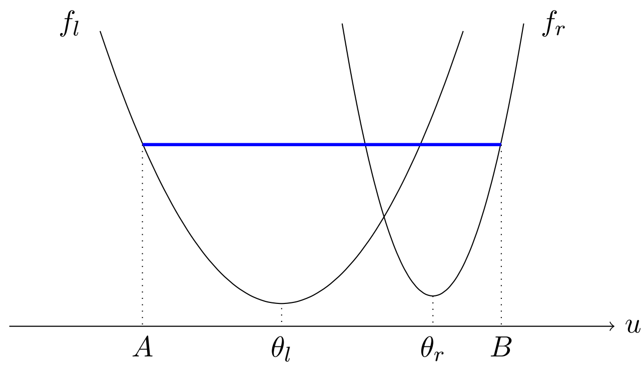

Figure 2.1: An example of connection (A, B) with f_l, f_r strictly convex fluxes

1. $f_l(A) = f_r(B)$
2. $A \leq \theta_l$ and $B \geq \theta_r$

Throughout the next chapters we will use the following notation: we denote by $f_{l,-}^{-1} \doteq (f_l|_{(-\infty,\theta_l]})^{-1}$, $f_{r,-}^{-1} \doteq (f_r|_{(-\infty,\theta_r]})^{-1}$ the inverse of f_l, f_r restricted to their decreasing part and by $f_{l,+}^{-1} \doteq (f_l|_{[\theta_l,+\infty)})^{-1}$, $f_{r,+}^{-1} \doteq (f_r|_{[\theta_r,+\infty)})^{-1}$ the inverse of f_l, f_r restricted to their increasing part. Moreover we set

$$\pi_{l,\pm}^{r} \doteq f_{l,\pm}^{-1} \circ f_r, \qquad \pi_{r,\pm}^{l} \doteq f_{r,\pm}^{-1} \circ f_l \tag{2.10}$$

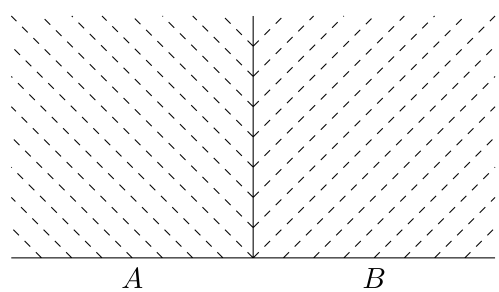

Figure 2.2: The stationary undercompressive solution c^{AB}.

For a connection (A, B) we let

$$c^{AB}(x) = \mathcal{H}(x)A + (1 - \mathcal{H}(x))B \tag{2.11}$$

In particular the second condition means that the function $c^{AB}(x)$ is a weak stationary undercompressive (or marginally undercompressive) solution of

(2.1), since the characteristics diverge, or are parallel to, the discontinuity interface (see Figure 2.2).

Definition 2.2 ((A, B)-entropy solution)**.** Let (A, B) be a connection. A function $u \in \mathbf{L}^\infty(\mathbb{R} \times [0, +\infty))$ is said to be and AB-entropy solution of the problem (2.1),(2.2) if the following holds

1. u is a distributional solution of (2.1), that is, for all test functions $\phi \in \mathcal{D}(\mathbb{R} \times [0, +\infty))$ holds

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} (u\phi_t + f(x, u)\phi_x) \,\mathrm{d}x \,\mathrm{d}t = 0 \tag{2.12}$$

2. u is a Kružkov entropy solution of (2.1),(2.2) on $(\mathbb{R} \setminus \{0\}) \times (0, +\infty)$, that is, $t \mapsto u(\cdot, t)$ is a continuous map from $[0, +\infty) \to \mathbf{L}^1_{\text{loc}}(\mathbb{R})$ and for any test function $0 \leq \phi \in \mathcal{D}((-\infty, 0) \times (0, +\infty))$ holds

$$\int_{-\infty}^{0} \int_{0}^{\infty} |u - k|\, \phi_t + \operatorname{sgn}(u - k)\, (f_l(u) - f_l(k))\, \phi_x \,\mathrm{d}x \,\mathrm{d}t \geq 0, \quad \forall k \in \mathbb{R} \tag{2.13}$$

and for any test function $0 \leq \phi \in \mathcal{D}((0, +\infty) \times (0, +\infty))$ holds

$$\int_{0}^{\infty} \int_{0}^{\infty} |u - k|\, \phi_t + \operatorname{sgn}(u - k)\, (f_l(u) - f_l(k))\, \phi_x \,\mathrm{d}x \,\mathrm{d}t \geq 0, \quad \forall k \in \mathbb{R} \tag{2.14}$$

3. u satisfies a Kružkov-type entropy inequality, depending on the connection (A, B), that is, for any test function $0 \leq \phi \in \mathcal{D}(\mathbb{R} \times (0, +\infty))$ holds

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \left|u - c^{AB}\right| \phi_t + \operatorname{sgn}(u - c^{AB}) \left(f(x, u) - f(x, c^{AB})\right) \phi_x \,\mathrm{d}x \,\mathrm{d}t \geq 0 \tag{2.15}$$

Remark 2.3. Notice that, since the fluxes f_l, f_r are uniformly convex, the solution u, that is in particular an entropy solution of (2.1) in $(-\infty, 0) \times (0, +\infty)$ and in $(0, +\infty) \times (0, +\infty)$, is such that $u(\cdot, t)$ is in $\mathbf{BV}_{\text{loc}}(\mathbb{R} \setminus \{0\})$. Moreover, thanks to a result in [22], since u is (in particular) a distributional solution of $u_t + f_l(u)_x = 0$ on $(-\infty, 0) \times (0, +\infty)$ and of $u_t + f_r(u)_x = 0$ on $(0, +\infty) \times (0, +\infty)$ and the fluxes f_l, f_r are strictly convex, it still admits left and right strong traces at $x = 0$, i.e. the limits

$$u(0-, t) =: u_l(t), \quad u(0+, t) =: u_r(t) \tag{2.16}$$

exist for a.e. $t > 0$.

The following Lemma is a direct consequence of the definition and of the existence of the left and right traces u_l, u_r.

Lemma 2.4. *A function $u \in \mathbf{L}^\infty(\mathbb{R} \times [0, +\infty))$ is an AB-entropy solution of the problem (2.1),(2.2) if and only if it is a Kružkov entropy solution of (2.1),(2.2) on $(\mathbb{R} \setminus \{0\}) \times (0, +\infty)$, and it is a function with traces (2.16) that satisfy a.e. $t > 0$ the Rankine-Hugoniot condition at the interface*

$$f_l(u_l(t)) = f_r(u_r(t)) \tag{2.17}$$

and

$$I^{AB}(u_l(t), u_r(t)) \geq 0 \tag{2.18}$$

where

$$\begin{aligned} I^{AB}(u_l(t), u_r(t)) \doteq \operatorname{sgn}(u_r(t) - B) \left(f_r(u_r(t)) - f_r(B)\right) \\ - \operatorname{sgn}(u_l(t) - A) \left(f_l(u_l(t)) - f_l(A)\right) \end{aligned} \tag{2.19}$$

Proof. Assume u is a solution in the sense of Definition 2.2. Fix any test function $\varphi \in \mathcal{C}^1_c(\mathbb{R}^+)$ and let

$$\theta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}(\varepsilon + x), & x \in [-\varepsilon, 0], \\ \frac{1}{\varepsilon}(\varepsilon - x), & x \in [0, \varepsilon], \\ 0, & |x| \geq \varepsilon, \end{cases} \tag{2.20}$$

With a density argument we find that the function $\phi(x,t) = \varphi(t)\theta_\varepsilon(x)$ can be used as an admissible test function in (2.12). Then we obtain

$$\begin{aligned} 0 = \int\int u\varphi'(t)\theta_\varepsilon(x) \,\mathrm{d}x \,\mathrm{d}t \\ + \frac{1}{\varepsilon}\int_{\mathbb{R}^+}\int_0^\varepsilon f_r(u)\varphi(t) \,\mathrm{d}x \,\mathrm{d}t - \int_{\mathbb{R}^+}\int_{-\varepsilon}^0 f_l(u)\varphi'(t) \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{2.21}$$

Clearly, as ε goes to zero, the first term goes to zero as well. Thanks to the existence of the strong traces u_l, u_r at the left and at the right of the interface, in the limit we obtain that

$$\int_{\mathbb{R}^+} (f_r(u_l(t)) - f_l(u_l(t)))\varphi(t) \,\mathrm{d}t = 0, \qquad \forall\, \varphi \in \mathcal{C}^1_c(\mathbb{R}^+) \tag{2.22}$$

and this implies the validity of (2.17).

Conversely, assume that (2.17) holds. Then for $\delta_\varepsilon = 1 - \theta_\varepsilon$, any test function $\phi \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R})$, and letting $\phi^\varepsilon(x,t) = \phi(x,t)\delta_\varepsilon(x)$, we have

$$\int\int u\phi_t^\varepsilon + f(u,x)\phi_x^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int\int (u\phi_t + f(u,x)\phi_x)\,\delta_\varepsilon(x)\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{1}{\varepsilon}\int_{\mathbb{R}^+}\int_0^\varepsilon f_r(u)\phi\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{-\varepsilon}^0 f_l(u)\phi\,\mathrm{d}x\,\mathrm{d}t \quad (2.23)$$

By approximation and by (2.13), (2.14) (that in particular imply that u is a distributional solution at the left and at the right of the interface, separately) the integral in the left hand side is equal to zero. Passing to the limit we get

$$\lim_{\varepsilon\to 0}\int\int (u\phi_t + f(u,x)\phi_x)\,\delta_\varepsilon(x)\,\mathrm{d}x\,\mathrm{d}t = \int\int (u\phi_t + f(u,x)\phi_x)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\mathbb{R}^+}\phi(0,t)(f_r(u_r(t)) - f_l(u_l(t))\,\mathrm{d}x\,\mathrm{d}t = 0 \quad (2.24)$$

and this concludes the proof. In an entirely similar way one can show also the equivalence between (2.15) and (2.18). ☐

Remark 2.5. Notice that under our hypothesis on the fluxes, it is not difficult to see that (2.18) is equivalent to

$$\begin{aligned} &f_l(u_l(t)) = f_r(u_r(t)) \geq f_l(A) = f_r(B) \\ &(u_l(t) \leq \theta_l, \quad u_r(t) \geq \theta_r) \Rightarrow u_l(t) = A, \quad u_r(t) = B \end{aligned} \quad (2.25)$$

for a.e. $t > 0$.

2.2. Uniqueness

We now prove uniqueness of the AB-entropy solutions. The proof of uniqueness exploits a modification of the Kružkov doubling method, adapted to (A,B) connections. The proof is taken from [9]. The key point of the proof, where the interface conditions come into play, is proving that the quantity E that appears in (2.35) is actually negative. That is the only point where we use the entropy conditions.

Theorem 2.6 ($\mathbf{L}^1$ stability and uniqueness)**.** *Let u and v be two (A,B)-entropy solutions of the problem (2.1) with initial data $u_0, v_0 \in \mathbf{L}^\infty(\mathbb{R})$. Let L be a Lipschitz constant for f_l, f_r in the interval $[-M,M]$, $M > 0$, and*

$$|u(x,t)| \leq M, \qquad |v(x,t)| \leq M, \qquad \forall\, x,t \quad (2.26)$$

Then for $t \in (0, +\infty)$

$$\int_{-r}^{r} |u(x,t) - v(x,t)| \, \mathrm{d}x \leq \int_{-r-Lt}^{r+Lt} |u_0(x) - v_0(x)| \, \mathrm{d}x, \quad \text{for all } r > 0 \tag{2.27}$$

In particular, there exists a unique entropy solution of type (A, B) *of* (2.1),(2.2).

Proof. The proof is based on Kružkov doubling method, with one extra difficulty due to the fact that we have to deal with the interface at $x = 0$.

Consider a specific test function ϕ, defined as follow. Let $0 \leq \delta \in \mathbb{R}$ and $\delta \in C^1(\mathbb{R})$ with compact support contained in $(-1, 1)$ such that its integral is 1, and let $\delta_h(t) := h\delta(ht)$. Let $\psi(X, T) \in \mathcal{D}((0, +\infty) \times (0, +\infty))$. Then define ϕ as

$$\phi(x, s, y, t) := \psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \delta_h\left(\frac{x-y}{2}\right) \delta_h\left(\frac{t-s}{2}\right)$$

Note that as h increases, the mass becomes more and more concentrated near the set $x = y$ and $t = s$. We rewrite (2.13), for two solutions u and v, using as test function for u the function $(x, s) \mapsto \phi(x, s, y, t)$ and for v the function $(y, t) \mapsto \phi(x, s, y, t)$. Hence we obtain

$$\int_0^{+\infty} \int_0^{+\infty} |u(x,s) - k| \, \phi_s(x, s, y, t) \\ + \operatorname{sgn}(u(x,s) - k) \left(f(x, u(x,s)) - f(x,k) \right) \phi_x(x, s, y, t) \, \mathrm{d}x \, \mathrm{d}s \geq 0 \tag{2.28}$$

$$\int_0^{+\infty} \int_0^{+\infty} |v(y,t) - k'| \, \phi_t(x, s, y, t) \\ + \operatorname{sgn}(v(y,t) - k') \left(f(y, v(y,t)) - f(y, k') \right) \phi_y(x, s, y, t) \, \mathrm{d}y \, \mathrm{d}t \geq 0 \tag{2.29}$$

Now let $k = v(y, t)$, $k' = u(x, s)$, integrate the first inequality w.r.t. y, t and the second inequality w.r.t. x, s, then take the sum. What we get is the following inequality:

$$\int_{(\mathbb{R}_+)^4} |u(x,s) - v(y,t)| \, (\phi_t + \phi_s) \\ + \operatorname{sgn}(u(x,s) - v(y,t))(f_r(u(x,s)) - f_r(v(y,t)))(\phi_x + \phi_y) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}y \, \mathrm{d}t \geq 0 \tag{2.30}$$

Notice that

$$(\phi_x + \phi_y)(x, s, y, t) = \psi_X\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \delta_h\left(\frac{x-y}{2}\right) \delta_h\left(\frac{t-s}{2}\right)$$

$$(\phi_s+\phi_t)(x,s,y,t)=\psi_T\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\delta_h\left(\frac{t-s}{2}\right)$$

Then one gets

$$\int_{(\mathbb{R}_+)^4}\Bigg[|u(x,s)-v(y,t)|\,\psi_T\left(\frac{x+y}{2},\frac{t+s}{2}\right) \\ +\operatorname{sgn}(u(x,s)-v(y,t))(f_r(u(x,s))-f_r(v(y,t))) \\ \psi_X\left(\frac{x+y}{2},\frac{t+s}{2}\right)\Bigg]\delta_h\left(\frac{x-y}{2}\right)\delta_h\left(\frac{t-s}{2}\right)\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}t\geq 0$$

With the change of variables $X=\frac{x+y}{2},\ Y=\frac{x-y}{2},;T=\frac{s+t}{2},\ S=\frac{s-t}{2}$ and sending $h\to+\infty$ we obtain

$$\int_0^{+\infty}\int_0^{+\infty}|u-v|\,\psi_t+\operatorname{sgn}(u-v)(f_r(u)-f_r(v))\psi_x\,\mathrm{d}x\,\mathrm{d}t\geq 0 \tag{2.31}$$

Now let $0\leq\alpha:\mathbb{R}\to\mathbb{R}$ be a continuous function with compact support in $[0,1]$. Then let

$$\alpha_h(x)=h\alpha\left(hx\right),\quad \beta_h(x)=\int_0^x\alpha_h$$

Let $\psi\in\mathcal{C}_c^1([0,+\infty)\times(0,+\infty)$. Notice

$$(\psi\beta_h)_x=\psi_x\beta_h+\psi\alpha_h$$
$$(\psi\beta_h)_t=\psi_t\beta_h$$

Then using (2.31) with test function $\psi\beta_h$, passing to the limit for $h\to\infty$ and the existence of traces at $x=0+$ yields

$$\int_0^{+\infty}\int_0^{+\infty}|u-v|\,\psi_t+\operatorname{sgn}(u-v)(f_r(u)-f_r(v))\psi_x\,\mathrm{d}x\,\mathrm{d}t \\ +\int_0^{+\infty}\operatorname{sgn}(u_r(t)-v_r(t))(f_r(u_r(t))-f_r(v_r(t)))\psi(0,t)\,\mathrm{d}t\geq 0 \tag{2.32}$$

We can do the same in the region $x<0$ and obtain a similar inequality, i.e.

$$\int_0^{+\infty}\int_{+\infty}^{0}|u-v|\,\psi_t+\operatorname{sgn}(u-v)(f_l(u)-f_l(v))\psi_x\,\mathrm{d}x\,\mathrm{d}t \\ -\int_0^{+\infty}\operatorname{sgn}(u_l(t)-v_l(t))(f_l(u_l(t))-f_l(v_l(t)))\psi(0,t)\,\mathrm{d}t\geq 0 \tag{2.33}$$

Then for every $\psi \in \mathcal{D}(\mathbb{R} \times (0, +\infty))$, adding (2.32), (2.33), one obtains:

$$-\int_0^{+\infty} \int_0^{+\infty} |u-v|\,\psi_t + \operatorname{sgn}(u-v)(f(u,x)-f(v,x))\psi_x \,\mathrm{d}x\,\mathrm{d}t \le E \quad (2.34)$$

where

$$E = \int_0^{+\infty} [\operatorname{sgn}(u(x,t)-v(x,t))(f(x,u(x,t))-f(x,v(x,t)))]_{x=0-}^{x=0+}\psi(0,t)\,\mathrm{d}t$$

and $[\cdot]_{x=0-}^{x=0+}$ denotes the limit from the right minus the limit from the left at $x = 0$. If we can prove that $E \le 0$, the proof is concluded in the exact same way of the classical continuous flux case.

Let's prove that for each t such that $f_l(u_l(t)) = f_r(u_r(t))$ and $f_l(v_l(t)) = f_r(v_r(t))$ it holds

$$\begin{aligned} &\operatorname{sgn}(u_r(t)-v_r(t))(f_r(u_r(t))-f_r(v_r(t))) \\ &\qquad - \operatorname{sgn}(u_l(t)-v_l(t))(f_l(u_l(t))-f_l(v_l(t))) \le 0 \end{aligned} \quad (2.35)$$

This is enough to ensure that $E \le 0$ since the Rankine-Hugoniot condition at the interface holds for almost every t. Without loss of generality, assume $u_r(t) \ge v_r(t)$. If $u_r(t) = v_r(t)$ or $u_l(t) = v_l(t)$, the left hand side of (2.35) is zero. Then assume $u_r(t) > v_r(t)$. If $u_l(t) > v_l(t)$, the left hand side of (2.35) is again zero, thanks to the Rankine-Hugoniot condition. Otherwise, if $u_l(t) < v_l(t)$, the left hand side of (2.35) is equal to

$$2(f_r(u_r(t)) - f_r(v_r(t)))$$

Assume by contradiction that $f_r(u_r(t)) > f_r(v_r(t))$. Since $u_r(t) > v_r(t)$, this implies $u_r(t) \ge B$. On the other hand, by Rankine-Hugoniot, also $f_l(u_l(t)) > f_l(v_l(t))$. Coupled with $u_l(t) < v_l(t)$, this implies $u_l(t) \le A$. Then either $u_r(t) = B$ and $u_l(t) = A$, and in this case the l.h.s. of (2.35) becames exactly (2.18) for v, and therefore (2.35) is satisfied, and this is a contradiction, or $u_r(t) > B$ and $u_l(t) < A$. This last case cannot happen, therefore the proof that $E \le 0$ is concluded.

Now the proof follows essentially in the same way of the continuous-flux case, since we know that for every test function ψ

$$\int_0^{+\infty} \int_0^{+\infty} |u-v|\,\psi_t + \operatorname{sgn}(u-v)(f(u,x)-f(v,x))\psi_x \,\mathrm{d}x\,\mathrm{d}t \ge 0 \quad (2.36)$$

The idea is to choose a test function that approximates the trapezoid

$$\Omega = \{(x,t) : \tau_0 \le t \le \tau, |x| \le R + L(\tau - t)\} \quad (2.37)$$

In order to do this we define

$$\psi(x,t) \doteq [\alpha_h(t-\tau_0) - \alpha_h(t-\tau)] \cdot [1 - \alpha_h(|x| - R - L(\tau - t))]$$

Now we use this ψ in inequality (2.36). With some easy calculations we find that

$$\begin{aligned}\int\int |u(x,t)-v(x,t)|\,[\delta_h(t-\tau_0)-\delta_h(t-\tau)]\cdot[1-\alpha_h(|x|-R-L(\tau-t))]\,\mathrm{d}x\,\mathrm{d}t \\ \geq \int\int \Big\{\mathrm{sgn}(x)[f(u(x,t),x)-f(v(x,t),x))]\mathrm{sgn}(u(x,t)-v(x,t)) \\ + L\,|u(x,t)-v(x,t)|\Big\} \\ \cdot[\alpha_h(t-\tau_0)-\alpha_h(t-\tau)]\delta_h(|x|-R-L(\tau-t))\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \quad (2.38)$$

Since L is a Lipschitz constant for both f_l, f_r on the interval $[-M, M]$ and by assumption $\|u\|_{\mathbf{L}^\infty}, \|v\|_{\mathbf{L}^\infty} \leq M$, we have $|f(u) - f(v)| \leq L\,|u - v|$. This implies that the right hand side of (2.38) is positive:

$$\int\int |u(x,t)-v(x,t)|\,[\delta_h(t-\tau_0)-\delta_h(t-\tau)]\cdot[1-\alpha_h(|x|-R-L(\tau-t))]\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad (2.39)$$

Letting $h \to \infty$ we obtain

$$\int_{|x|\leq R} |u(x,\tau)-v(x,\tau)| \leq \int_{|x|\leq R+L(\tau-\tau_0)} |u(x,\tau_0)-v(x,\tau_0)|\,\mathrm{d}x \quad (2.40)$$

and the statement of the theorem follows letting $\tau_0 \to 0$. □

2.3. Existence

In this section we prove existence for the problem (2.1), (2.2) with an adapted version of the front-tracking algorithm. In order to define the algorithm, we need to know how to solve a Riemann problem when a front interacts with the interface. In [16] the authors introduced a new formulation of the AB-entropy condition in terms of Riemann solvers, that they used to define the front-tracking approximations. In the following section we present that approach, starting with the definition of Riemann solver.

2.3.1. Riemann solvers As already said, we mainly follow the lines of [16]. A Riemann solver is a function that tells us how to solve a Riemann problem with piecewise constant initial data of the form

$$u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \tag{2.41}$$

This approach has the advantage of providing a complete description of how to solve a Riemann problem for (A, B) entropy solutions. For this reason, we report here the main lines of that paper. Also, for continuity with the rest of the thesis, we reformulate everything using convex fluxes, instead of concave as done in [16]. Let's give the definition of Riemann solver

Definition 2.7 (Riemann solver)**.** A Riemann solver is a function

$$R : [0,1] \times [0,1] \to [0,1] \times [0,1]$$

$$(u_l, u_r) \mapsto (R_1(u_l, u_r), R_2(u_l, u_r)) = (u^-, u^+)$$

such that

1. $f_l(u^-) = f_r(u^+)$
2. the waves (u_l, u^-) and (u^+, u_r) have respectively negative and positive speed
3. the function $(u_l, u_r) \mapsto (f_l(u^-), f_r(u^+))$ is continuous
4. $R(R(u_l, u_r)) = R(u_l, u_r)$ for every $u_l, u_r \in [0, 1]$
5. for every $(u_l, u_r) = R(u_l, u_r)$ and $\tilde{u}$ such that the wave $(\tilde{u}, R_1(u_l, u_r))$ has positive speed, it holds

$$f_l(R_1(\tilde{u}, u_r)) \in [\min\{f_l(u_l), f_l(\tilde{u})\}, \max\{f_l(u_l), f_l(\tilde{u})\}] \tag{2.42}$$

6. for every $(u_l, u_r) = R(u_l, u_r)$ and $\tilde{u}$ such that the wave $(R_2(u_l, u_r), \tilde{u})$ has negative speed, it holds

$$f_r(R_2(u_l, \tilde{u})) \in [\min\{f_r(u_r), f_r(\tilde{u})\}, \max\{f_r(u_r), f_r(\tilde{u})\}] \tag{2.43}$$

For every Riemann solver, one has an associated notion of solution to the problem (2.1),(2.2):

Definition 2.8 (Solution associated to a Riemann solver)**.** Fix a Riemann solver R. A function $u \in \mathbf{L}^\infty(\mathbb{R} \times [0, \infty))$ is an admissible solution to (2.1),(2.2) associated to the Riemann solver R if

1. u satisfies properties 1., 2. of Definition 2.1.

2. for almost every $t > 0$, it holds

$$R(u_l(t), u_r(t)) = (u_l(t), u_r(t))$$

that is, $(u_l(t), u_r(t))$ is an equilibrium for a.e. $t > 0$.

In [16], Riemann solvers are classified as follows. Notice that elements in the image of R are equilibria for the Riemann solver by property 4. of Definition 2.7. Consider the function

$$[0, \theta_l] \times [\theta_r, 1] \to \mathbb{R}$$

$$(u_l, u_r) \mapsto f_l(u^-) = f_r(u^+)$$

An application of property 3. of Definition 2.7 shows that this function is continuous. Let X be the image of this function. In particular it is a closed, non empty interval. Now assume that X is actually a singleton, say $X = \{\gamma\}$. Then we have the following

Proposition 2.9. *If R is a Riemann solver such that the set X is a singleton, the Riemann solver is completely determined by the value γ such that $X = \{\gamma\}$.*

Proof. We take into account four different cases.

1. $(u_l, u_r) \in [0, \theta_l] \times [\theta_r, 1]$. Since the wave (u_l, u^-) must have negative speed, $u^- \in [0, \theta_l]$. Similarly, since the wave (u^+, u_r) must have positive speed, $u^+ \in [\theta_r, 1]$. Then, by definition of X, we must have $f_l(u^-) = f_r(u^+) = \gamma$. Thus u^-, u^+ are uniquely determined.

2. $(u_l, u_r) \in [0, \theta_l] \times [0, \theta_r]$. As before, we must have $u^- \in [0, \theta_l]$. Moreover, either $u^+ = u_r$ or $u^+ \in [\theta_r, 1]$ with $f_r(u^+) > f_r(u_r)$. We now split the proof into two subcases

   (a) $f_r(u_r) < \gamma$. Consider the continuous function $[0, 1] \ni u_r \mapsto f_r(u^+)$, where $(u^-, u^+) = R(u_l, u_r)$. Notice that, by the previuous case, if $u_r \geq \theta_r$, one has $f_r(u^+) = \gamma$. Therefore, by continuity, it must be $f_r(u^+) = \gamma$ for all u_r such that $f_r(u_r) < \gamma$. Thus in this case the solution to the Riemann problem is $u^+ = f_{r,+}^{-1}(\gamma)$ and $u_l = f_{l,-}^{-1}(\gamma)$.

(b) $f_r(u_r) \geq \gamma$. In this case one must have $u^+ = u_r$ and $u^- = \pi^r_{l,-}(u_r)$. Otherwise should be $u^+ \in [\theta_r, 1]$ with $f_r(u^+) > f_r(u_r) \geq \gamma$, but since $(u^-, u^+) = R(u^-, u^+)$, this is in contradiction with the analysis of case 1.

3. The case $(u_l, u_r) \in [\theta_l, 1] \times [\theta_r, 1]$ is entirely similar to the previuous one. In particular

   (a) $f_l(u_l) \geq \gamma$ implies $u^- = u_l$ and $u^+ = \pi^l_{r,+}(u_l)$.

   (b) $f_l(u_l) < \gamma$ implies $u^- = f^{-1}_{l,-}(\gamma)$, $u^+ = f^{-1}_{r,+}(\gamma)$.

4. $(u_l, u_r) \in]\theta_l, 1] \times [0, \theta_r[$. By the second property of the Definition of Riemann solvers, we must have either $u^- \in [0, \theta_l]$ and $f_l(u^-) > f_l(u_l)$ or $u^- = u_l$. Analogously, either $u^+ \in [\theta_r, 1]$ and $f_r(u^+) > f_r(u_r)$ or $u^+ = u_r$. We have two cases

   (a) $\max\{f_r(u_r), f_l(u_l)\} \geq \gamma$. Assume that $f_r(u_r) \geq f_l(u_l)$.
   First let $f_r(u_r) > f_l(u_l)$.
   Assume by contradiction that $u^- = u_l$. It can't be $u^+ = u_r$, therefore it must be $u^+ \in [\theta_r, 1]$ with $f_r(u^+) > f_r(u_r)$ but this is not an equilibrium. Therefore $u^- \in [0, \theta_l]$ with $f_l(u^-) > f_l(u_l)$. This forces $u^+ = u_r$, and $u^- = \pi^r_{l,-}(u_r)$.
   Now let $f_r(u_r) = f_l(u_l)$. We have two possibilities: either $u^- \in [0, \theta_l]$ and $u^+ \in [\theta_r, 1]$ with $f_l(u^-) = f_r(u^+) \geq \gamma$ or $u^- = u_l$ and $u^+ = u_r$. The first one cannot happen by case 1, so that $u^- = u_l$ and $u^+ = u_r$.
   If instead we assume $f_r(u_r) < f_l(u_l)$, we obtain $u^- = u_l$, $u^+ = \pi^l_{r,+}(u_l)$.

   (b) Now assume $\max\{f_r(u_r), f_l(u_l)\} < \gamma$. Assume that $f_r(u_r) \geq f_l(u_l)$. First let $f_r(u_r) > f_l(u_l)$. Assume by contradiction that $u^+ = u_r$. Then it must be $u^- \in [0, \theta_l]$. This is impossible by case 2.a. Then $u^+ \in [\theta_r, 1]$ with $f_r(u^+) > f_r(u_r)$. This forces $u^- \in [0, \theta_l]$. By case 1., we finally find $u^- = f^{-1}_{l,-}(\gamma)$ and $u^+ = f^1_{r,+}(\gamma)$.
   Now let $f_r(u_r) = f_l(u_l)$. Either $u^- = u_l$ and $u^+ = u_r$ or $u^- = f^{-1}_{l,-}(\gamma)$ and $u^+ = f^1_{r,+}(\gamma)$. The first possibility can be proven to be impossible by passing to the limit in the previous case. Then $u^- = f^{-1}_{l,-}(\gamma)$ and $u^+ = f^1_{r,+}(\gamma)$.
   If we assume instead that $f_r(u_r) \leq f_l(u_l)$, we get the same result.

□

Let us summarize the content of the previous proposition. Given a pair of states (u_l, u_r) we can determine the left and right traces (u^-, u^+) of the solution to the Riemann problem with initial datum (2.41) as follows:

1. If $u_l \leq \theta_l$ and $u_r \geq \theta_r$, then $u^- = f_{l,-}^{-1}(\gamma)$ and $u^+ = f_{r,+}^{-1}(\gamma)$.
2. If $u_l \leq \theta_l$ and $u_r \leq \theta_r$ and $f_r(u_r) < \gamma$, then $u^- = f_{l,-}^{-1}(\gamma)$ and $u^+ = f_{r,+}^{-1}(\gamma)$.
3. If $u_l \leq \theta_l$ and $u_r \leq \theta_r$ and $f_r(u_r) \geq \gamma$, then $u^+ = u_r$ and $u^- = \pi_{l,-}^r(u_r)$.
4. If $u_l \geq \theta_l$ and $u_r \geq \theta_r$ and $f_l(u_l) \geq \gamma$, then $u^- = u_l$ and $u^+ = \pi_{r,+}^l(u_l)$.
5. If $u_l \geq \theta_l$ and $u_r \geq \theta_r$ and $f_l(u_l) < \gamma$, then $u^- = f_{l,-}^{-1}(\gamma)$, $u^+ = f_{r,+}^{-1}(\gamma)$.
6. If $u_l \geq \theta_l$ and $u_r \leq \theta_r$ and $f_r(u_r) \geq \gamma$ and $f_r(u_r) > f_l(u_l)$, then $u^- = \pi_{l,-}^r(u_r)$ and $u^+ = u_r$.
7. If $u_l \geq \theta_l$ and $u_r \leq \theta_r$ and $f_r(u_r) = f_l(u_l) \geq \gamma$, then $u^- = u_l$ and $u^+ = u_r$.
8. If $u_l \geq \theta_l$ and $u_r \leq \theta_r$ and $f_l(u_l) \geq \gamma$ and $f_r(u_r) < f_l(u_l)$, then $u^- = u_l$ and $u^+ = \pi_{r,+}^l(u_l)$.
9. If $\max\{f_r(u_r), f_l(u_l)\} < \gamma$, then $u^- = f_{l,-}^{-1}(\gamma)$, $u^+ = f_{r,+}^{-1}(\gamma)$.

Keeping in mind the previous description of the Riemann solver R_γ, it is immediate to reformulate Definition 2.8 in the following way

Lemma 2.10. *A function $u \in \mathbf{L}^\infty((0, +\infty) \times \mathbb{R})$ is a solution in the sense of Definition 2.8 if and only*

1. *u satisfies properties 1.,2. of Definition 2.1*
2. *for a.e. t, the couple $(u_l(t), u_r(t))$ satisfies the following conditions*
   (a) *$f_l(u_l(t)) = f_r(u_r(t)) \geq \gamma$.*
   (b) *if $u_l(t) \leq \theta_l$ and $u_r(t) \geq \theta_r$, then $f_l(u_l(t)) = \gamma = f_r(u_r(t))$.*

Remark 2.11. It is immediate to check that this definition is equivalent to Definition 2.1. Indeed, for a fixed connection (A, B), it is sufficient to take $\gamma = f_l(A) = f_r(B)$.

Now we want to use the above description in terms of Riemann solvers to obtain existence of solutions with the front tracking algorithm. As already said, this was done in [16] with a careful analysis of waves interactions at the junction.

2.4. Existence by front-tracking

In this section we face the question of existence of solutions using an adaptation of the classical front tracking algorithm. We mainly follow the lines of [16], here and there giving some further explanations to make the exposition more elementary. We remark that in [16] the fluxes f_l, f_r are assumed to be concave, while here they are convex, but the analysis remains essentially the same.

We briefly sketch how this adaptation of the front-tracking algorithm works. We fix an initial datum u_0 with bounded total variation and consider a sequence of piecewise constant approximations $u_{0,\nu}$ of u_0 such that TotVar $u_{0,\nu} \leq$ TotVar u_0. As in the continuous flux case, at every point of discontinuity $x \neq 0$ of the functions $u_{0,\nu}$ we solve the Riemann problem either with a single shock (in case $u_{0,\nu}(x-) > u_{0,\nu}(x+)$) or by rarefaction wave (in case $u_{0,\nu}(x-) < u_{0,\nu}(x+)$). Since we want the approximation u_ν to be piecewise constant, we split the rarefaction waves into rarefaction fans made of non entropic shocks, each shock violating the entropy condition of an amount that goes to zero as $\nu \to \infty$; we can do this for example by requiring that for a fixed ν the shocks into which we split the rarefaction wave do not have strength bigger than $1/\nu$. At $x = 0$ we solve the Riemann problem using the Riemann solver R_γ defined above. The approximate solution u_ν is defined until the first time in which there is a wave-wave interaction, or wave-junction interaction. When this happens, we solve a new Riemann problem. If we want to have the approximation u_ν defined globally in time, we need to control the growth of the number of fronts. In particular, we need that the number of interactions does not become infinite in finite time. Without loss of generality, we can assume that at every interaction time at most two waves interact (if not, it is always possible to perturb the initial data by a small quantity and avoid this situation). Assume there is a wave-wave interaction between two waves (u_l, u_m) and (u_m, u_r). Then, since the speed of the first wave must be bigger then the speed of the second wave, and since the fluxes f_l, f_r are convex, it must hold $u_l > u_r$. Then from the collision a single shock (u_l, u_r) emerges, so that the number of total fronts decreases by one. Assume that there is a wave-interface interaction from the left, so that there is wave $(\tilde{u}, u^-)$ that arrives at the junction from the left. If the shock $(\tilde{u}, u^-)$ is entropic, we will prove below that at most two shocks emerge from the interaction point, one at the left and one at the right of the interface. Moreover, they can interact again with the interface only after canceling one wave at left or at the right. If instead the shock $(\tilde{u}, u^-)$ is non entropic, we will see that it might be that from the point of interaction emerges a rarefaction wave. However the strength of the rarefaction wave will be $|u_r - u^+| = \mathcal{O}(1)\,|u^- - \tilde{u}| = \mathcal{O}(1)\nu^{-1}$. Therefore in this case we solve the problem with a non entropic shock with strenght at most $\mathcal{O}(1)\nu^{-1}$ where $\mathcal{O}(1)$ is constant that depends only on the choice of the fluxes f_l, f_r. This implies that there is a finite number of waves and the wave front approximation u_ν can be globally defined in time.

In order to construct the front-tracking approximation we need to decide how to solve the Riemann problem when a wave interacts with the junction. Of course we will use the Riemann solver R_γ. The problem is that when a non entropic shock reaches the junction: in that case we have to prove that a small rarefaction wave might emerge from the interaction. We study the case in which a wave $(\tilde{u}, u^-)$ reaches the junction from the left at some time $\bar{t}$ with the equilibrium before the interaction being (u^-, u^+), hence creating a new equilibrium (u_l, u_r) after time $\bar{t}$. Following [16] (although here the definition are not precisely the same since we are dealing with convex fluxes; moreover, for the fixed γ, we have (A, B) the corresponding connection) we say that an equilibrium (u^-, u^+) is of type

(i) G/G if $(u^-, u^+) \in [0, A] \times [B, 1]$ (and then $(u^-, u^+) = (A, B)$);

(ii) G/B if $(u^-, u^+) \in [0, A] \times [0, B]$;

(iii) B/G if $(u^-, u^+) \in [A, 1] \times [B, 1]$;

(iv) B/B if $(u^-, u^+) \in [0, A] \times [B, 1]$.

From the analysis of the Riemann solver R_γ above, we conclude that

(i) If (u^-, u^+) is G/G, since the wave $(\tilde{u}, u^-)$ has positive speed, $\tilde{u} > \bar{A}$. This implies that $(\tilde{u}, u^-)$ is a (entropy admissible) shock and that the Riemann problem at the point $(0, \bar{t})$ is solved with a single shock in quadrant I, and the new equilibrium is $(u_l, u_r) = (\tilde{u}, \pi^l_{r,+}(\tilde{u}))$. Hence the new equilibrium (u_l, u_r) will be B/G

(ii) If (u^-, u^+) is G/B, since the wave $(\tilde{u}, u^-)$ has positive speed, $\tilde{u} > \bar{A}$. This implies that $(\tilde{u}, u^-)$ is a (entropy admissible) shock and (exactly as in the previous case) that the Riemann problem at the point $(0, \bar{t})$ is solved with a single shock in quadrant I, and the new equilibrium is $(u_l, u_r) = (\tilde{u}, \pi^l_{r,+}(\tilde{u}))$. Hence the new equilibrium (u_l, u_r) will be B/G.

(iii) Assume (u^-, u^+) is B/G. Since the wave $(\tilde{u}, u^-)$ has positive speed, $\tilde{u} \in [\pi_{l,-}(u^-), 1]$.

(a) The wave $(\tilde{u}, u^-)$ is an entropic shock. Then the new equilibrium is $(u_l, u_r) = (\tilde{u}, \pi^l_{r,+}(\tilde{u}))$ and the Riemann problem is solved with a unique (entropic) shock in quadrant I. The new equilibrium is of B/G type.

(b) The wave $(\tilde{u}, u^-)$ is a rarefaction front, i.e. $\tilde{u} < u^-$. Then, the new equilibrium is

$$(u_l, u_r) = \begin{cases} (A, B), & \text{if} \quad \tilde{u} \in [\pi_{l,-}(\tilde{u}), A], \quad G/G \\ (A, B), & \text{if} \quad \tilde{u} \in [A, \bar{A}], \quad G/G \\ (\tilde{u}, \pi^l_{r,+}(\tilde{u})), & \text{if} \quad \tilde{u} \in [\bar{A}, u^-], \quad B/G \end{cases} \tag{2.44}$$

and the Riemann problem is solved, respectively, with two rarefaction fronts (one at the left, and one at the right of the interface); with a shock in quadrant I and a rarefaction front in quadrant II; with a single rarefaction front in quadrant II. Notice that each time a rarefaction front emerges, it has strength of order $\mathcal{O}(1)/\nu$, $\mathcal{O}(1)$ being a positive constant depending only on the fluxes f_l, f_r.

(iv) Assume (u^-, u^+) is B/B. Since the wave $(\tilde{u}, u^-)$ has positive speed, $\tilde{u} \in [\pi_{l,-}(u^-), 1]$.

(a) The wave $(\tilde{u}, u^-)$ is an entropic shock. Then the new equilibrium is $(u_l, u_r) = (\tilde{u}, \pi^l_{r,+}(\tilde{u}))$ and the Riemann problem is solved with a unique (entropic) shock in quadrant I. The new equilibrium is of B/G type.

(b) The wave $(\tilde{u}, u^-)$ is a rarefaction front, i.e. $\tilde{u} < u^-$. Then, the new equilibrium is $(u_l, u_r) = (\pi^r_{l,-}(u^+), u^+)$ and the Riemann problem is solved with a unique (entropic) shock in quadrant I. The new equilibrium is of G/B type.

It is important to notice that if until some time T at the interface do not arrive waves from the right, but only from the left, the state u^+ at the right of the junction can change only one time from B to G and then stays G until time T.

Now we prove a Lemma in which, for an approximate wave front tracking solution $\bar{u}$, we estimate the total variation of the flux $f(\bar{u}, x)$.

Lemma 2.12. *Let $\bar{u}$ be an approximate wave front tracking solution. For every $t \geq 0$ it holds*

$$\text{TotVar}(f(\bar{u}(\cdot, t), \cdot)) \leq \text{TotVar}(f(\bar{u}(\cdot, 0+), \cdot)) \tag{2.45}$$

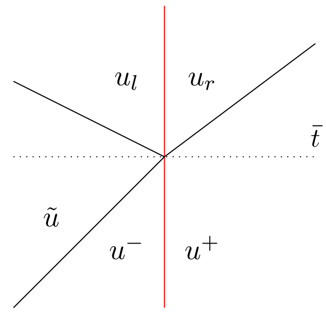

Figure 2.3: A wave reaches the junction from the left at time $\bar{t}$

Proof. If there is an interaction in $x < 0$ or $x > 0$ the total variation remains unchanged or decreases. Assume that a wave $(\tilde{u}, u_l)$ reaches the junction from the left and denote by t the time of interaction. Let (u^-, u^+) be the new equilibrium at time t. Then

$$\begin{aligned}&\mathrm{TotVar}(f(\bar{u}(\cdot, t+), \cdot)) - \mathrm{TotVar}(f(\bar{u}(\cdot, t-), \cdot))\\ &\qquad = |f_l(\tilde{u}) - f_l(u^-)| + \Big|f_r(u^+) - f_r(u_r)\Big| - |f_l(\tilde{u}) - f_l(u_l)| = 0 \qquad (2.46)\end{aligned}$$

thanks to point 5. of Definition 2.7 on the Riemann solver.

Assume now that a wave $(u_r, \tilde{u})$ reaches the junction from the right at time $\bar{t}$. Then, this time using point 6. of Definition 2.7, we obtain

$$\begin{aligned}&\mathrm{TotVar}(f(\bar{u}(\cdot, t+), \cdot)) - \mathrm{TotVar}(f(\bar{u}(\cdot, t-), \cdot))\\ &\qquad = \Big|f_r(\tilde{u}) - f_l(u^+)\Big| + |f_r(u^-) - f_r(u_l)| - |f_l(\tilde{u}) - f_l(u_r)| = 0 \qquad (2.47)\end{aligned}$$

□

We notice that there exist $L, M > 0$ such that, for every $t, s > 0$,

$$f(u_\nu(\cdot, t), \cdot) \leq M, \qquad \|f(u_\nu(\cdot, t), \cdot) - f(u_\nu(\cdot, s), \cdot)\|_{\mathbf{L}^1} \leq L\,|t - s| \qquad (2.48)$$

Moreover, by Lemma 2.4, for every $t > 0$,

$$\mathrm{TotVar}(f(u_\nu(\cdot, t), \cdot)) \leq \mathrm{TotVar}(f(u_{\nu,0}(\cdot), \cdot)) \leq \mathrm{TotVar}(f(u_0(\cdot), \cdot)) \qquad (2.49)$$

Therefore we can apply Helly's theorem and conclude that, possibly passing to a subsequence, $f(u_\nu, \cdot)$ converges in $\mathbf{L}^1_{\mathrm{loc}}$ to a function $\bar{f}$.

Now we can finally prove that there is a subsequence of u_ν converging in $\mathbf{L}^1_{\text{loc}}$. To do this, the idea is to isolate the interface/junction by means of two curves $Y^\nu_-(t) \le 0 \le Y^\nu_+(t)$ defined by

1. $Y^\nu_-(0) = 0 = Y^\nu_+(0)$
2. $Y^\nu_\pm$ follow the generalized characteristics of the front tracking approximation u_ν and $Y^\nu_-(t) = 0$ (respectively $Y^\nu_+(t) = 0$) if $Y^\nu_-(t)$ (respectively $Y^\nu_+(t)$)) reaches the interface and $f'_l(u_\nu(t, 0-) \ge 0$ (respectively $f'_r(u_\nu(t, 0+) \le 0)$. Notice that, by the presence of rarefaction fronts, it might be that at time $\bar{t}$ a rarefaction front emerges from the interface $x = 0$ at the point $(0, \bar{t})$. In this case we choose to follow the maximal forward characteristic.

Now we define the sets

$$D^\nu_1 \doteq \{(x,t) : Y^\nu_-(t) \le x \le Y^\nu_+(t)\} \tag{2.50}$$

and $D^\nu_2 \doteq (\mathbb{R}^+ \times \mathbb{R}) \setminus D^\nu_1$. In the set D^ν_2 it is easy to control the total variation since it is by definition not influenced by the interface, and an a priori estimates holds. As of the set D^ν_1, we can observe that thanks to the previous analysis for every $t > 0$ in the intervals $(Y^\nu_-(t), 0]$ and $(0, Y^\nu_+(t)]$ there is at most a point $\tilde{x}$ such that

$$\text{sgn}(u^\nu(\tilde{x}-, t) - A)\text{sgn}(u^\nu(\tilde{x}+, t) - A) \le 0$$

$$\text{sgn}(u^\nu(\tilde{x}-, t) - B)\text{sgn}(u^\nu(\tilde{x}+, t) - B) \le 0$$

This means that we can always split the intervals $(Y^\nu_-(t), Y^\nu_+(t))$ in at most four sub-intervals in which we can invert f_l, f_r. This allows to conclude that there exists a subsequence of u^ν converging in $\mathbf{L}^1_{\text{loc}}$ to a function u.

We want to show that u is still an entropy solution at the left and the right of the interface. Take any compactly supported function $\varphi \in \mathcal{C}^1_c((-\infty, 0) \times (0, +\infty))$. We want to prove that for every $k \in \mathbb{R}$ it holds

$$\int_0^{+\infty} \int_0^{+\infty} |u - k|\, \varphi_t + \text{sgn}(u - k)(f_l(u) - f_l(k))\varphi_x \,\mathrm{d}x \,\mathrm{d}t \ge 0 \tag{2.51}$$

Notice that by dominated convergence theorem the above integral is equal to

$$\lim_{\nu \to +\infty} \int_0^{+\infty} \int_{-\infty}^{0} |u_\nu - k|\, \varphi_t + \text{sgn}(u_\nu - k)(f_l(u_\nu) - f_l(k))\varphi_x \,\mathrm{d}x \,\mathrm{d}t \tag{2.52}$$

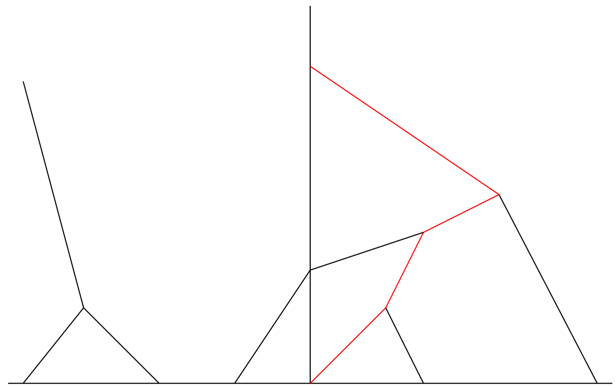

Figure 2.4: The front tracking algorithm. The red curve is Y^{ν}_{-}.

so that to prove (2.51) it is enough to prove that the integral in (2.52) is positive for every ν. Let $T > 0$ such that $\varphi(x,t) = 0$ for every (x,t) with $t > T$. For a fixed ν and $t \in (0,T)$ we let $x_1(t) < x_2(t) < \ldots < x_N(t) < 0$ be the points in which $u_\nu(\cdot,t)$ has a jump. In order to use the divergence theorem, we notice that the polygonal lines $x_\alpha(t)$ subdivide the domain $(-\infty,0)\times(0,T)$ in a finite number of regions Γ_j where u_ν is constant. Then, with

$$\Phi = (\varphi \cdot \mathrm{sgn}(u_\nu - k)(f_l(u_\nu) - f_l(k)), \varphi \cdot |u_\nu - k|)$$

we have

$$\begin{aligned}\int_0^{+\infty}\int_{-\infty}^{0} &|u_\nu - k|\,\varphi_t + \mathrm{sgn}(u_\nu - k)(f_l(u_\nu) - f_l(k))\varphi_x \,\mathrm{d}x\,\mathrm{d}t \\ &= \sum_j \int\int_{\Gamma_j} |u_\nu - k|\,\varphi_t + \mathrm{sgn}(u_\nu - k)(f_l(u_\nu) - f_l(k))\varphi_x \,\mathrm{d}x\,\mathrm{d}t \\ &= \sum_j \int\int_{\Gamma_j} \mathrm{div}\Phi \,\mathrm{d}x\,\mathrm{d}t = \sum_j \int_{\partial\Gamma_j} \Phi\cdot\mathbf{n}\,\mathrm{d}\sigma \qquad (2.53)\end{aligned}$$

where $\partial\Gamma_j$ is the oriented boundary of Γ_j and $\mathbf{n}$ is the outer normal. Observe that

$$\mathbf{n} = \pm(1, -\dot{x}_\alpha(t))$$

with the sign depending on which Γ_j we are considering. Let

$$\Delta\eta(x_\alpha(t),t) = |u_\nu(x_\alpha(t)+,t) - k| - |u_\nu(x_\alpha(t)-,t) - k|$$

$$\Delta q(x_\alpha(t),t) = \operatorname{sgn}(u_\nu - k)(f_l(u_\nu - f_l(k))(x_\alpha(t)+,t) \\ - \operatorname{sgn}(u_\nu - k)(f_l(u_\nu) - f_l(k))(x_\alpha(t)-,t)$$

Then, with $\mathcal{S}$ the indices for which x_α is an entropic shock and $\mathcal{R}$ the indices for which x_α is a non entropic shock, we find

$$\sum_j \int_{\partial\Gamma_j} \Phi \cdot \mathbf{n}\,\mathrm{d}\sigma = \sum_{\alpha\in\mathcal{S}} \int_{\partial\Gamma_j} \dot{x}_\alpha(t)\Delta\eta(x_\alpha(t),t) - \Delta q(x_\alpha(t),t)\,\mathrm{d}t \\ + \sum_{\alpha\in\mathcal{R}} \int_{\partial\Gamma_j} \dot{x}_\alpha(t)\Delta\eta(x_\alpha(t),t) - \Delta q(x_\alpha(t),t)\,\mathrm{d}t \quad (2.54)$$

The sum on $\alpha \in \mathcal{S}$ is bigger than zero by definition since $x_\alpha(t)$ is an entropy admissible shock. The sum on $\alpha \in \mathcal{R}$ might be negative, but very small and going to zero for $\nu \to +\infty$. In fact, following [7], we can establish

$$\dot{x}_\alpha(t)\Delta\eta(x_\alpha(t),t) - \Delta q(x_\alpha(t),t) = \mathcal{O}(1)\nu^{-1}\left|u_\nu(x_\alpha(t)+,t) - u_\nu(x_\alpha(t)-,t)\right| \quad (2.55)$$

Since the total strength of non entropic fronts remains uniformly bounded in ν, we can conclude.

There is only left to prove that the limit function u satisfies the interface conditions. The idea is to prove an estimate on the total variation of the flux of an approximate front tracking solution along the junction. In this way one can prove that the flux along the interface of the limit solution u is **BV**. For the details of the proof we refer to [16].

Lemma 2.13. *Let $\{u_\nu\}$ be the approximate front-tracking sequence constructed above and $u_{\nu,l}, u_{\nu,r}$ be the left/right traces at $x = 0$ of u_ν. Then for every ν the following estimate holds*

$$\mathrm{TotVar}(f_l(u_{\nu,l}),(0,T)) = \mathrm{TotVar}(f_l(u_{\nu,r}),(0,T)) \\ \leq 2\mathrm{TotVar}(f(u_0(\cdot),\cdot),\mathbb{R}) \quad (2.56)$$

Lemma 2.14. *The function u limit of the front-tracking approximations satisfies the interface conditions 2.a, 2.b of Lemma 2.10.*

Proof. Using Lemma 2.13 and invoking Helly's theorem, the sequence $f_l(u_{\nu,l}) = f_l(u_{\nu,r})$ converges in $\mathbf{L}^1$ to a BV function. Hence $f_l(u_l) = f_r(u_r)$ is BV. Of course every front-tracking approximation u_ν satisfies

$$f_l(u_{\nu,l}(t) = f_r(u_r(t)) \geq \gamma, \qquad \text{for a.e.} \quad t \geq 0$$

so that passing to the limit, u satisfies the same inequality and 2.a is proved. Now assume by contradiction that 2.b does not hold. This means that there is continuity point $\bar{t}$ for u_l, u_r such that

$$(u_l(\bar{t}), u_r(\bar{t})) \in ([0, A) \times (B, 1])$$

in a neighborhood $(\bar{t} - \delta, \bar{+}\delta)$ of $\bar{t}$. We know that u_ν converges in $\mathbf{L}^1_{\text{loc}}$ to u. Then if there is a subsequence such that the equilibria $(u_{\nu,l}, u_{\nu,r})$ are of type G/G on a subset of positive measure (uniformly on ν) we obtain a contradiction. Otherwise, there is a positive measure subset (uniformly in ν) such that at least one of the equilibria $u_{\nu,l}$ or $u_{\nu,r}$ are of bad type. However, this is clearly in contradiction with the fact that u_ν converges in $\mathbf{L}^1_{\text{loc}}$ to u. □

Remark 2.15. A more direct approach for proving that the limit function u is indeed an AB-entropy solution would be to directly pass to the limit in inequality (2.15) and conclude with the same arguments that we used above to prove that u is an entropy solution for f_l, f_r in quadrants I, II. In fact, it will be sufficient to treat the interface as a front, and use the fact that the (piecewise constant) traces of an approximate front-tracking solution satisfies (2.25) (with the connection (A, B) corresponding to the value γ).

Chapter 3

Attainable set

Abstract. In this chapter we characterize the attainable set at time $T > 0$

$$\mathcal{A}^{AB}(T) \doteq \{\mathcal{S}_T^{AB} u_0, u_0 \in \mathbf{L}^\infty\} \tag{3.1}$$

for AB-entropy solutions of the conservation law with discontinuous flux in terms of some Oleinik-type inequalities. We prove, adapting the method of generalized characteristics [11], that additional constraints to the ones of [4] must be added in order to characterize also those profiles that are reachable only with solutions that contain a shock in at least one of the semiplanes $\{x < 0\}$, $\{x > 0\}$. This is a major difference with respect to the continuous flux case where every attainable profile is reachable with a locally Lipschitz solution.

3.1. Continuous flux case

In this section we present the known results about the attainable set for a scalar one dimensional conservation law with uniformly convex flux

$$\begin{cases} u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+, \\ u(\cdot, 0) = u_0, \end{cases} \tag{3.2}$$

In this setting it is well known that the reachable set, that is

$$\mathcal{A}(T) := \{\mathcal{S}_T u_0 : u_0 \in \mathbf{L}^\infty\} \tag{3.3}$$

is fully characterized by the classical Oleinik inequalities. In particular, a function $\omega \in \mathbf{L}^\infty(\mathbb{R})$ is in $\mathcal{A}(T)$ if and only if

$$f'(\omega(y)) - f'(\omega(x)) \leq \frac{y - x}{T}, \quad \text{for a.e. } x \leq y \tag{3.4}$$

This is well known since the work [21]. We assume the flux f to be uniformly convex and of class $\mathcal{C}^2$. Here, inspired by an exercise in [7], we choose to base our proof on the front-tracking algorithm presented in the first chapter.

Proposition 3.1. *Let u be the solution of (3.2) and $\omega(x) = u(x, T)$. Then Oleinik estimates (3.4) hold.*

Proof. For a fixed $\nu \geq 1$, let u_ν be the piecewise constant front tracking approximation with values in $2^{-\nu}\mathbb{Z}$. For $t \geq 0$ we say that a line $x_\alpha(t)$ where u_ν has a jump is a *rarefaction front* if $u_\nu(x_\alpha(t)-,t) < u_\nu(x_\alpha(t)+,t)$, otherwise we say that it is a *shock front*.

We prove that all the rarefaction fronts have strength $u_\nu(x_\alpha(t)+,t) - u_\nu(x_\alpha(t)-,t) = 2^{-\nu}$. In fact, at time $t = 0$, a rarefaction front emerges from x if and only if $u_{\nu,0}(x-) < u_{\nu,0}(x+)$, and since f is convex, $\mathrm{conv}(f_\nu) = f_\nu$, so that from x emerge exactly $(u_{\nu,0}(x+) - u_{\nu,0}(x-))2^\nu$ rarefaction fronts with strength $2^{-\nu}$.

Assume that two fronts (u_l, u_m), from the left, and (u_m, u_r), from the right, collide. Notice that they cannot both be rarefaction fronts, otherwise it holds $u_l < u_m < u_r$ and the speed of the front from the left would be smaller than the speed of the right front, a contradiction. Therefore at least one of them is a shock. If they are both shocks, a single shock emerges. Since the strength of a front is always bigger then $2^{-\nu}$ and the strength of a rarefaction front is always $2^{-\nu}$, a single shock emerges also if one of them is a rarefaction front (unless there is a complete cancellation an no fronts emerge).

It is easy to see that if $x_\alpha(t) < x_{\alpha+1}(t)$ are two adjacent rarefaction fronts, it holds, with $0 < c = \min f''$,

$$\dot{x}_{\alpha+1} - \dot{x}_\alpha \geq (c - \mathcal{O}(1))\, 2^{-\nu} \tag{3.5}$$

where $\mathcal{O}(1)$ is a positive quantity approaching zero as $\nu \to \infty$. Now consider any two points $x < y$ and $t > 0$. Assume that (x,t) and $y(x,t)$ are separated by k rarefaction fronts x_α, $\alpha \in \{1, 2, \ldots, k\}$, and m shock fronts. Since every rarefaction front has strength equal to $2^{-\nu}$ and each shock front has strength at most $2^{-\nu}$, we only consider rarefaction fronts that are not "surrounded by shocks", i.e. rarefaction fronts that are adjacent to other rarefaction fronts: assume there are n of them. Then it holds

$$u_\nu(y,t) - u_\nu(x,t) \leq n \cdot 2^{-\nu} \tag{3.6}$$

Moreover, by (3.5) and since rarefaction fronts can only originate at time $t = 0$, it holds

$$y - x \geq \sum_{\alpha=1}^{n-1} x_{\alpha+1}(t) - x_\alpha(t) \geq \sum_{\alpha=1}^{n-1} (\dot{x}_{\alpha+1}(t) - \dot{x}_\alpha(t))t \geq (n-1)\,(c - \mathcal{O}(1))\, 2^{-\nu} t \tag{3.7}$$

This implies

$$(n-1)2^{-\nu} \leq \frac{y - x}{(c - \mathcal{O}(1))\, t} \tag{3.8}$$

and substituting this into (3.6) one obtains

$$u_\nu(y,t) - u_\nu(x,t) \le 2^{-\nu} + \frac{y-x}{(c - \mathcal{O}(1))\,t} \tag{3.9}$$

Now it is sufficient to pass to the limit for $\nu \to +\infty$ to find that

$$u(y,t) - u(x,t) \le \frac{y-x}{ct} \tag{3.10}$$

If a function $\omega \in \mathbf{L}^\infty$ satisfies (3.10), it is automatically in $\mathbf{BV}_{\text{loc}}$. Then using the theory of generalized characteristics (see the first chapter), imposing that backward genuine characteristics do not intersect, one finds the slightly stronger estimate (3.4). □

Remark 3.2. This is a striking example in which the PDE has a regularizing effect on the initial data. Notice that is related to the convexity of the flux. In fact, if for example the flux is linear, say $f(u) = \lambda u$, the conservation law becomes

$$u_t + \lambda u_x = 0 \tag{3.11}$$

that is just the transport equation with constant coefficients. Then the solution to (3.11) to the Cauchy problem with initial datum $u_0 \in \mathbf{L}^\infty$ is just $u(x,t) = u_0(x - \lambda t)$.

The strategy to prove that Oleinik estimates are also a sufficient condition to ensure that ω is reachable is the following. Assume that Oleinik estimates hold for ω. Then ω is $\mathbf{B}V_{\text{loc}}$ and we can use the theory of generalized characteristics [11]. We solve the problem

$$\begin{cases} \tilde{u}_t + f(\tilde{u})_x = 0 \\ \tilde{u}(x,0) = \omega(-x) \end{cases} \tag{3.12}$$

Oleinik estimates (3.4) give us a one sided Lipschitz constant for $\tilde{u}$. The additional assumption that Oleinik estimates hold also for ω will give us the other side-one sided Lipschitz constant. Therefore $\tilde{u}$ is Lipschitz, and reversing space and time it's easy to check that $\tilde{u}(-x,-t)$ is an admissible solution to (3.2), with initial datum $\tilde{u}(-x,0)$ and $\omega(x) = \tilde{u}(-x,T)$, so that ω is attainable. We prove this in the following proposition.

Proposition 3.3. *Let $\omega \in \mathbf{L}^\infty(\mathbb{R})$ satisfy Oleinik estimates (3.4). Then $\omega \in \mathcal{A}(T)$.*

Proof. Let $\tilde{u}$ be as above. By Oleinik estimates, for every $x_1 \leq x_2$ and $t \leq T$ we have

$$f'(\tilde{u}(x_2-,t)) - f'(\tilde{u}(x_1-,t)) \leq \frac{x_2 - x_1}{T}$$

On the other hand, consider the minimal backward characteristics starting from (x_1, t) and (x_2, t). Let ξ_1, ξ_2 be the values at zero of those characteristic, i.e.

$$\xi_1 = x_1 - tf'(\tilde{u}(x_1-,t)),\ \xi_2 = x_2 - tf'(\tilde{u}(x_2-,t))$$

Then we have

$$\begin{aligned} f'(\tilde{u}(x_2-,t)) - f'(\tilde{u}(x_1-,t)) &\geq f'(\tilde{u}(\xi_2-,0)) - f'(\tilde{u}(\xi_1+,0)) \\ &= f'(\omega(-\xi_2+)) - f'(\omega(-\xi_1-)) \geq -\frac{\xi_2 - \xi_1}{T} \\ &= \frac{x_2 - x_1 - t\left(f'(\tilde{u}(x_2-,t)) - f'(\tilde{u}(x_1-,t))\right)}{T} \end{aligned} \tag{3.13}$$

With some calculations, this implies

$$f'(\tilde{u}(x_2-,t)) - f'(\tilde{u}(x_1-,t)) \geq -\frac{x_2 - x_1}{T - \tau} \tag{3.14}$$

This implies that $x \mapsto f'(\tilde{u}(x,t))$ is Lipschitz for $t \in]0, T[$. Then $\tilde{u}$ is actually Lipschitz in compact sets of $\mathbb{R} \times (0, T)$, and by standard arguments one verifies that $\tilde{u}$ is an admissible solution for (3.2) and at time T produces the profile ω. □

Remark 3.4. There is another possible approach in order to prove that Oleinik estimates select the profiles ω are attainable for the conservation law (3.4). This method provides explicitly the solution and the idea is that, if Oleinik estimates hold for ω, the lines

$$\theta_{x,\pm}(t) := x - (T - t)f'(\omega(x\pm))$$

do not intersect each other in the interior of the domain. Then, roughly speaking, if a point (ξ, τ) lies on one of these lines, say $\theta_{x\pm}$, one defines the solution to be $u(\xi, \tau) = \omega(x\pm)$. In the regions not covered by these lines, the solution is defined to be a compression wave, generating a shock at time T (see Figure 3.1). This approach was used in [3], for a boundary problem, and we will use it again here for entropy AB-solutions. One can prove that this construction actually provides an admissible solution u, and that it is Lipschitz. Actually, the solution constructed in this way must coincide, by uniqueness of entropy solutions, with the solution constructed in Proposition

3.3. Indeed, being Lipschitz, u can be reversed in space time and still be an entropy solution of (3.4). The reverse u will solve the Cauchy problem with initial datum equal to $\omega(-x)$, therefore $u(-x,-t)$ must coincide with $\tilde{u}$.

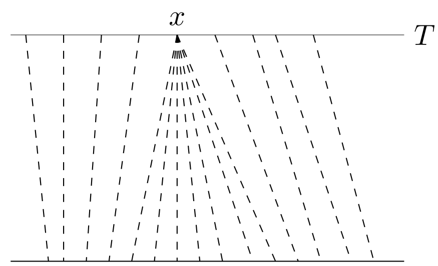

Figure 3.1: Example of characteristics of the unique Lipschitz solution that produces w at time T. In this case, a compression wave produces a shock at the point x.

3.2. Preliminaries and main theorem

Here we prove the result concerning exact controllability at time $T > 0$ for the discontinuous problem

$$\begin{cases} u_t + f(u,x)_x = 0, & x \in \mathbb{R}, \quad t \geq 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{3.15}$$

with flux function satisfying the hypothesis already stated in the previous chapter. In particular we are going to characterize the set $\mathcal{A}^{AB}(T)$ (3.1). Throughout the following

$$D^-\omega(x) = \liminf_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h}, \quad D^+\omega(x) = \limsup_{h\to 0} \frac{\omega(x+h)-\omega(x)}{h} \tag{3.16}$$

will denote, respectively, the lower and the upper Dini derivative of a function ω at x. Moreover, we introduce the following sets that characterize the left and right traces of an AB-entropy solution at the flux-discontinuity interface:

$$\begin{aligned}
\mathcal{T}_1 &= \Big\{(u_l, u_r) \in (\theta_l, +\infty) \times (\theta_r, +\infty);\ u_l \geq \pi_{l,+}(A), B \leq u_r \leq \pi^l_{r,+}(u_l)\Big\} \\
\mathcal{T}_2 &= \Big\{(u_l, u_r) \in (-\infty, \theta_l) \times (-\infty, \theta_r);\ \pi^r_{l,-}(u_r) \leq u_l \leq A,\ u_r \leq \bar{B}\Big\} \\
\mathcal{T}_{3,-} &= \Big\{(u_l, u_r) \in [\theta_l, +\infty) \times (-\infty, \theta_r);\ \bar{A} \leq u_l \leq \pi^r_{l,+}(u_r),\ u_r \leq \bar{B}\Big\} \\
\mathcal{T}_{3,+} &= \Big\{(u_l, u_r) \in (\theta_l, +\infty) \times (-\infty, \theta_r];\ u_l \geq \bar{A},\ \pi^l_{r,-}(u_l) \leq u_r \leq \bar{B}\Big\}
\end{aligned} \tag{3.17}$$

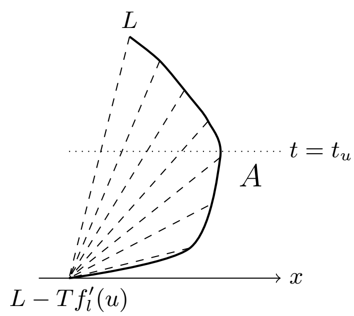

Figure 3.2: A curve $x_u(t)$.

Now fix $L \leq 0$ and a state u. Consider the differential equation

$$\begin{cases} \dot{y_u}(s) = -\lambda_l\left((f_l')^{-1}\left(\frac{y(s)-L+Tf_l'(u)}{T-t}\right), A\right), & s > 0 \\ y_u(0) = L \end{cases} \tag{3.18}$$

where $\lambda_l(u_1, u_2)$ is the Rankine-Hugoniot speed for the flux f_l of the shock with states u_1, u_2. By local uniqueness theorems and geometrical considerations, the solution y_u cannot escape in finite time; then (3.18) has a unique smooth solution $y_u : [0, T) \to \mathbb{R}$. We set $x_u(t) = y_u(T - t)$. The function $x_u(t)$ has a unique maximum in a point $t_u \in (0, T]$. Consider the function

$$[L/T, +\infty) \ni u \mapsto x_u(t_u) \tag{3.19}$$

One can easily see that $u \mapsto x_u(t_u)$ is strictly decreasing, continuous, and that for $u \geq \bar{A}$ one has $x_u(t_u) = x_u(T) = L < 0$. Define

$$\bar{u}_L \doteq \min\{u \geq L/T \mid x_u(t_u) \leq 0\} \tag{3.20}$$

Analogously, for $R \geq 0$, one defines $x_u(t) = y_u(T-t)$ where

$$\begin{cases} \dot{y}_u(s) = -\lambda_r \left(B, (f_r')^{-1}\left(\frac{y(s)-R+Tf_r'(u)}{T-t}\right)\right), & s > 0, \\ y_u(0) = R. \end{cases} \tag{3.21}$$

and $\lambda_r(u_1, u_2)$ is the Rankine-Hugoniot speed for the flux f_r of the shock with states u_1, u_2. Consider the map

$$(-\infty, R/T] \ni u \mapsto x_u(t_u). \tag{3.22}$$

The map $u \mapsto x_u(t_u)$ as defined here is decreasing, continuous, and if $\bar{u} \leq \bar{B}$ it holds $x_u(t_u) = x_u(T) = R > 0$. Then we define

$$\bar{u}_R \doteq \max\{u \leq R/T : x_u(t_u) \geq 0\} \tag{3.23}$$

Remark 3.5. If $A = \theta_l$, it's easy to see that $\bar{u}_L = (f_l')^{-1}(L/T)$. Analogously, if $B = \theta_r$, $\bar{u}_R = (f_r')^{-1}(R/T)$.

Let

$$\begin{aligned} t_L &= T - \tfrac{f_l'(\omega(L+))}{L}, & t_R &= T - \tfrac{f_r'(\omega(R-))}{R}, \\ L_A &= (T - t_{\bar{u}_R}) f_l'(A), & R_B &= (T - t_{\bar{u}_L}) f_r'(B), \end{aligned} \tag{3.24}$$

and

$$\varphi_1(x) \doteq \begin{cases} x - f_l'(\omega(x))T & x < 0 \\ -f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x)) \left(T - \frac{x}{f_r'(\omega(x))}\right) & 0 < x < R \\ x - f_r'(\omega(x))T & x > R \end{cases} \tag{3.25}$$

$$\varphi_2(x) \doteq \begin{cases} x - f_l'(\omega(x))T & x < L \\ -f_r' \circ f_{r,-}^{-1} \circ f_l(\omega(x)) \left(T - \frac{x}{f_l'(\omega(x))}\right) & L < x < 0 \\ x - f_r'(\omega(x))T & x > 0 \end{cases} \tag{3.26}$$

$$\varphi_{3,A}(x) \doteq \begin{cases} x - f_l'(\omega(x))T & x < L \\ -f_r' \circ f_{r,-}^{-1} \circ f_l(\omega(x)) \left(T - \frac{x}{f_l'(\omega(x))}\right) & L < x < L_A \\ x - f_r'(\omega(x))T & x > R \end{cases} \tag{3.27}$$

$$\varphi_{3,B}(x) \doteq \begin{cases} x - f_l'(\omega(x))T & x < L \\ -f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x)) \left(T - \frac{x}{f_r'(\omega(x))}\right) & R_B < x < R \\ x - f_r'(\omega(x))T & x > R \end{cases} \tag{3.28}$$

$$\varphi_3(x) \doteq \begin{cases} x - f_l'(\omega(x))T & x < L \\ x - f_r'(\omega(x))T & x > R \end{cases} \tag{3.29}$$

where we agree that if $L_A \leq L$ or $R \leq R_A$, we delete the second line of (3.27) or (3.28), respectively.

Moreover, for a point x such that

$$|f_u(\omega(x\pm), x)| \leq |x/T| \tag{3.30}$$

where $f(u, x)$ is the discontinuous flux and f_u is the derivative with respect to the first entry, we define the line $\theta_{x,\pm} : [0, T] \to \mathbb{R}$ to be respectively the minimal/maximal backward characteristic from (x, T). If instead

$$|f_u(\omega(x\pm), x)| > x/T, \tag{3.31}$$

this means that the minimal/maximal characteristic impact the interface in positive time and the time of impact is

$$t_{x,\pm} \doteq T - \frac{f_l'(w(x\pm))}{x} \tag{3.32}$$

In this last case, if $x < 0$ we let

$$\theta_{x,\pm}(t) = \begin{cases} x - (T - t) f_l'(\omega(x\pm)) & t \in (t_{x,\pm}, T)\,, \\ -(t_{x,\pm} - t)\left(f_r' \circ \pi_{r,-}^{l}(\omega(x\pm))\right) & t \in (0, t_{x,\pm})\,, \end{cases} \tag{3.33}$$

while if $x > 0$ we let

$$\theta_{x,\pm}(t) = \begin{cases} x - (T-t) f_r'(\omega(x\pm)) & t \in (t_{x,\pm}, T)\,, \\ -(t_{x,\pm} - t)\left(f_l' \circ \pi_{l,+}^r(\omega(x\pm))\right) & t \in (0, t_{x,\pm})\,, \end{cases} \tag{3.34}$$

The next theorem is a reformulation of the one in [4]. The theorem in [4] characterizes only the profiles reachable with a Lipschitz solution in the semiplanes $\{x < 0\}$ and $\{x > 0\}$. But, as firstly noticed in [2], there are profiles that are reachable only with solutions that contain at least a shock in one of the quadrants I, II. Here we extend the result obtained in [4] and characterize also the profiles that are reachable with a solution that has a shock in at least one of the semiplanes $\{x < 0\}$, $\{x > 0\}$. In order to do this, further conditions must be added (see in particular (3.37), (3.39), (3.41)). Since, as mentioned above, some profiles might be reachable only with solutions that contain a shock, the role of these new conditions is to ensure that such shocks can indeed be constructed. The difficult part of the proof is to prove that (3.37), (3.39), (3.41) are actually also necessary conditions: we prove this in Section 3.4. Let us now state the main theorem of this chapter, that completely characterizes the attainable set $\mathcal{A}^{AB}(T)$.

Theorem 3.6. *Let (A, B) be a connection and $T > 0$. Then the set $\mathcal{A}^{AB}(T)$ is given by*

$$\mathcal{A}^{AB}(T) = \mathcal{A}_1(T) \cup \mathcal{A}_2(T) \cup \mathcal{A}_3^{AB}(T), \tag{3.35}$$

where $\mathcal{A}_1(T), \mathcal{A}_2(T), \mathcal{A}_3^{AB}(T)$ are sets of function $\omega \in \mathbf{L}^\infty(\mathbb{R})$ having essential left and right limits at $x = 0$, defined as follows.

$\mathcal{A}_1(T)$ is the set of all functions ω that satisfy $(\omega(0-), \omega(0+)) \in \mathcal{T}_1$, and for which there exists $R > 0$ such that the following conditions hold.

$$\begin{aligned} &\omega(x) \geq \max\left\{(f_r')^{-1}(x/T), B\right\}, \quad \forall x \in (0, R), \\ &\omega(R+) < (f_r')^{-1}(R/T), \end{aligned} \tag{3.36}$$

the map φ_1 is nondecreasing and

$$\omega(R+) \leq \bar{u}_R \tag{3.37}$$

$\mathcal{A}_2(T)$ is the set of all functions ω that satisfy $(\omega(0-), \omega(0+)) \in \mathcal{T}_2$, and for which there exists $L < 0$ such that the following conditions hold.

$$\omega(L-) > (f_l')^{-1}(L/T)\,,$$
$$\omega(x) \leq \min\left\{(f_l')^{-1}(x/T)\,, A\right\},\ \forall x \in (L, 0), \tag{3.38}$$

the map φ_2 is nondecreasing and

$$\omega(L-) \geq \bar{u}_L \tag{3.39}$$

$\mathcal{A}_3^{AB}(T)$ is the set of all functions ω for which there exists $L \leq 0 \leq R$, such that the following conditions hold.

$$(\omega(0-), \omega(0+)) \in \begin{cases} \mathcal{T}_{3,-} \cup \mathcal{T}_{3,+} & \text{if} \quad L = R = 0, \\ \{(A, B)\} & \text{otherwise,} \end{cases} \tag{3.40}$$

if $L = 0 = R$ the map φ_3 is nondecreasing,

otherwise:

$$\omega(L-) \geq \bar{u}_L, \qquad \omega(R+) \leq \bar{u}_R, \tag{3.41}$$

and, if (A, B) is not critical

- *if $t_L \geq t_R$, the map $\varphi_{3,B}$ is nondecreasing, and*

$$\begin{aligned} &\omega(x) = A \quad \forall\, x \in (L, 0), \qquad \omega(x) = B \quad \forall\, x \in (0, \min\{R, R_B\}), \\ &\omega(x) \geq B \quad \forall x \in (R_B, R), \qquad \omega(R+) \leq \tfrac{x}{T}, \\ &\textit{if } R_B < R: \quad \omega(R_B+) = B \end{aligned} \tag{3.42}$$

- *if $t_L \leq t_R$, the map $\varphi_{3,A}$ is nondecreasing, and*

$$\begin{aligned} &\omega(x) = A \quad \forall\, x \in (\max\{L, L_A\}, 0), \qquad \omega(x) = B \quad \forall\, x \in (0, R), \\ &\omega(x) \leq A \quad \forall x \in (L, L_A), \qquad \omega(L+) \leq \tfrac{x}{T}, \\ &\textit{if } L < L_A: \quad \omega(L_A-) = A \end{aligned} \tag{3.43}$$

if instead (A,B) is critical, the map φ_3 is nondecreasing, and

$$\omega(x) = A \quad \forall\, x \in (L,0), \quad \omega(x) = B \quad \forall\, x \in (0,R), \tag{3.44}$$

Remark 3.7. Consider the set $\mathcal{A}_1(T)$ and the condition on the monotonicity of the map φ_1. It is proved in [4] that this is equivalent to require that

$$D^+\omega(x) \leq \begin{cases} \frac{1}{f_l''(\omega(x))T} & \forall \quad x \in (-\infty,0), \\ \\ \frac{h_1(x)}{h_2(x)} & \forall \quad x \in (0,R), \\ \\ \frac{1}{f_r''(\omega(x))T} & \forall \quad x \in (R,+\infty), \end{cases} \tag{3.45}$$

with

$$h_1(x) = f_r'\left[f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]^2$$

and

$$h_2(x) = \left[f'' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right][f_r'(\omega(x))]^2(f_r'(\omega(x))T - x)$$

$$+ x\left[f_l' \circ f_{l,+}^{-1} \circ f_r(\omega(x))\right]^2 f_r''(\omega(x))$$

In light of this, since the functions f_l, f_r are supposed to be uniformly convex, we deduce that the right hand side (3.45) is always nonnegative and it's bounded on any set bounded away from zero. This implies that every element in $\mathcal{A}_1(T)$ has (a representative in it's equivalence class that has) finite total increasing variation (and hence finite total variation as well) on all bounded subsets of $\mathbb{R}$ bounded away from $x = 0$. This, together with the assumption that ω admits left and right limits at $x = 0$, implies that every element $\omega \in \mathcal{A}_1(T)$ has (essential) left and right limits at every point $x \in \mathbb{R}$.

3.3. Some technical lemmas

The aim of this section is to prove that for a given $\omega \in \mathcal{A}^{AB}(T)$, the left and right limits $(\omega(0-), \omega(0+))$ fall into one of the four classes (3.17) and that no rarefaction waves can be crated at time $t > 0$ from a point of the interface. Notice that, knowing that $\omega = u(\cdot, T)$ for some solution u, the existence the limits $(\omega(0-), \omega(0+))$ can be easily established using the property of non-intersection of genuine characteristics. Everything follows quite easily once we proved the following technical lemma.

Lemma 3.8. *For every $\bar{t} \geq 0$ the temporal unidirectional limits of the fluxes of the traces $f_l(u_l(\bar{t}\pm)) = f_r(u_r(\bar{t}\pm))$ exist.*

Proof. We prove that the limit $f_l(u_l(\bar{t}+))$ exists, the other cases being entirely similar. If for some $\delta > 0$ we have that $u_l(t) \geq \theta_l$ for every $t \in (\bar{t}, \bar{t}+\delta)$, we conclude that the limit $u_l(\bar{t}+)$ exists. In fact, if it didn't exist we could find two sequences $t_n, s_n \downarrow \bar{t}$ such that $u_l(t_n), u_l(s_n) \geq \theta_l$ and converge to different limits. But it is easy to see that this is not possible since backward genuine characteristics from the points $(0, s_n), (0, t_n)$ would intersect for a big enough n.

Assume that such a δ does not exist. Define two sets E_1, E_2 as

$$E_1 = \{t > \bar{t} : u_l(t) < \theta_l\}, \quad E_2 = \{t > \bar{t} : u_l(t) \geq \theta_l\} \tag{3.46}$$

It is easy to see that E_1 is open (actually, it is easy to see is that its complement, E_2, is closed). As of the set E_2, with the same arguments as above we actually get that the limit

$$\lim_{\substack{t \to \bar{t} \\ t \in E_2}} u_l(t) \tag{3.47}$$

exists. Analogously, define two sets F_1, F_2 as

$$F_1 = \{t > \bar{t} : u_l(t) > \theta_r\}, \quad F_2 = \{t > \bar{t} : u_l(t) \leq \theta_r\} \tag{3.48}$$

The set F_1 is open and the limit

$$\lim_{\substack{t \to \bar{t} \\ t \in F_2}} u_r(t) \tag{3.49}$$

exists.

We have to show that the limits

$$l_i \doteq \lim_{\substack{t \to \bar{t} \\ t \in E_i}} f_l(u_l(t)), \qquad i = 1, 2 \tag{3.50}$$

exist and they coincide.

The intersection $E_1 \cap F_1$ is still open and thanks to the interface conditions we find that $(u_l(t), u_r(t)) = (A, B)$ for a.e. $t \in E_1 \cap F_1$. Actually, it holds for *every* $t \in E_1 \cap F_1$. In fact since the values $u_l(t)$ are just the limits $u(0-, t)$, for a fixed $t \in E_1 \cap F_1$, tracing the backward characteristics (with negative slope) from a sequence of points (x_n, t), $x_n \uparrow 0$, since $E_1 \cap F_1$ is open, we find that the limit $u(0-,t)$ must actually be equal to A. Therefore the limit of u_l restricted to $E_1 \cap F_1$ exists and

$$\lim_{\substack{t \to \bar{t} \\ t \in E_1 \cap F_1}} u_l(t) = A \tag{3.51}$$

Now we have some cases:

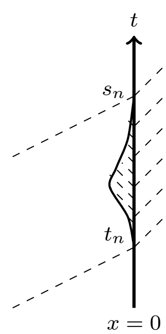

Figure 3.3: Case 1. There exists a $\delta > 0$ such that $E_1 \cap B(\bar{t}, \delta) \subset F_1$.

Case 1: The intersection $E_1 \cap F_2 \cap B(\bar{t}, \delta)$ is empty for some $\delta > 0$.

Since E_1 is open, it is an at most countable union of disjoint intervals, say $E_1 = \cup_n (t_n, s_n)$, that accumulate in a right neighborhood of $\bar{t}$. For each n and each such interval (t_n, s_n) there must be a shock $(t_n, s_n) \ni t \mapsto (y_n(t), t)$ such that $y_n(t) < 0$ for every $t \in (t_n, s_n)$, $y_n(t_n) = y_n(s_n) = 0$, with right state equal to A, because $E_1 \subset F_1$. By non-intersection of backward genuine characteristics, the limit of the left state of the shock $y_n(\cdot)$ exists and it is equal to the limit of u_l in the set E_2, in the sense that

$$\forall\, \varepsilon > 0\ \exists\, \bar{n} \in \mathbb{N}\ :\ \sup_{t \in (t_n, s_n)} \left| u(y_n(t)-, t) - \lim_{\substack{t \to \bar{t} \\ t \in E_2}} u_l(t) \right| < \varepsilon\ \forall\, n \geq \bar{n} \tag{3.52}$$

Therefore the velocities of the shocks y_n converge uniformly in n to the value

$$\lambda_l \left(\lim_{\substack{t \to \bar{t} \\ t \in E_2}} u_l(t), A \right) \tag{3.53}$$

But of course since for every n we have $y_n(s_n) = y_n(t_n) = 0$, the only choice for the limit speed is the zero speed. Therefore we obtain

$$\lim_{\substack{t \to \bar{t} \\ t \in E_2}} = \bar{A}$$

and this implies that the limit $f_l(u_l(\bar{t}+))$ exists.

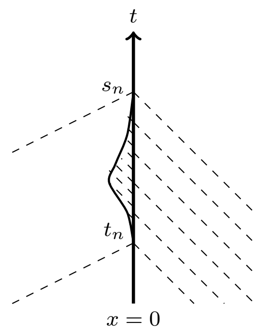

Figure 3.4: Case 2. There exists a $\delta > 0$ such that $E_1 \cap B(\bar{t}, \delta) \subset F_2$.

Case 2: The intersection $E_1 \cap F_1 \cap B(\bar{t}, \delta)$ is empty for some $\delta > 0$. Then the limit of u_r restricted to the set $E_1 = E_1 \cap F_2 \subset F_2$ exists (since the limit of u_r restricted to F_2 exists). Therefore the limit of u_l restricted to E_1 exists: in fact $u_l(t) = \pi^r_{l,-}(u_r(t-))$ for every $t \in E_1 \cap F_2$. As above the limit of the left state of the shocks $y_n(\cdot)$ exists in the sense (3.52), but since the limit of u_l restricted to E_1 exists also the right state converges. Therefore the velocities of the shocks converge uniformly in n to the value

$$\lambda_l \left(\lim_{\substack{t \to \bar{t} \\ t \in E_2}} u_l(t), \lim_{\substack{t \to \bar{t} \\ t \in E_1}} u_l(t) \right) \tag{3.54}$$

and since the only possible limit speed is the zero speed obtain

$$\lim_{\substack{t \to \bar{t} \\ t \in E_2}} u_l(t) = (f_l|_{[\theta_l, +\infty)})^{-1} \circ f_l \left(\lim_{\substack{t \to \bar{t} \\ t \in E_1}} u_l(t) \right) \tag{3.55}$$

so that the limit $f_l(u_l(\bar{t}+))$ exists.

Case 3: The intersections $E_1 \cap F_1 \cap B(\bar{t}, \delta)$ and $E_1 \cap F_2 \cap B(\bar{t}, \delta)$ are non-empty for every $\delta > 0$. It's clear that for every $t \in E_1 \cap F_1$ we have $u_l(t) = A$. The set F_2 is open and hence an at most countable union of intervals (T_n, S_n) (assume $T_n \geq S_{n+1}$) and as above there is, for each n, a shock $Y_n(\cdot)$ such that $Y_n(T_n) = Y_n(S_n) = 0$ and $Y_n(t) > 0$ for every $t \in (T_n, S_n)$. There is a sequence of points T_n or S_n such that T_n or S_n belongs to (t_k, s_k) for some k: without loss of generality assume there is a sequence of T_n such that for

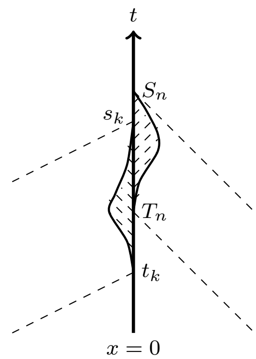

Figure 3.5: Case 3. $E_1 \cap F_1 \cap B(\bar{t}, \delta) \neq 0$, $E_1 \cap F_2 \cap B(\bar{t}, \delta) \neq 0$ for every $\delta > 0$.

each n exists k such that $T_n \in (t_k, s_k)$. From each point $(0, T_n)$ emerges a shock with positive slope with left state equal to B. Then by the Rankine-Hugoniot conditions it must hold $u_r(T_n) \geq \bar{B}$. Assume by contradiction that $u_r(T_n) > \bar{B}$. Then we have three cases. Firstly, it could happen that for a small $\varepsilon > 0$ we have $(T_n, T_n - \varepsilon) \subset F_2$: a contradiction because from the interface conditions, in this last set, it must hold $u_r(t) \leq \bar{B}$ a.e. and this leads to intersection of backward genuine characteristics. Secondly it could be that $T_n = S_{n+1}$. This is again a contradiction since at $(0, S_{n+1})$ a shock with negative slope and left state equal to B arrives, by the R-H conditions we must have $u_r(S_{n+1}) = u_r(T_n) \leq \bar{B}$. Finally, it could happen that there is a sequence $S_{n_k} \uparrow T_n$, at each S_{n_k}, as above, it holds $u_r(S_{n_k}) \leq \bar{B}$ and this leads to intersection of backward characteristics: a contradiction. Then at each T_n it must hold $u_r(T_n) = \bar{B}$. This means that the limit of u_r in the set F_2 must be equal to $\bar{B}$ since 1) it exists and 2) $\lim_{k\to\infty} u_r(T_{n_k}) = \bar{B}$. Then of course also the limit of u_r in the smaller set $E_1 \cap F_2$ exists and is equal to $\bar{B}$. Moreover, the limit of u_r in the set $E_1 \cap F_1$ is clearly equal to B, with the same arguments as above. We deduce that the limit of the flux

$$\lim_{\substack{t\to\bar{t}+ \\ t\in E_1}} f_r(u_r(t)) \tag{3.56}$$

exists. Since for a.e. $t \in E_1$ it holds $u_l(t) = (f_l|_{(-\infty,\theta_l]})^{-1} \circ f_r(u_r(t))$, also the essential limit

$$\operatorname{ess\,lim}_{\substack{t\to\bar{t}+ \\ t\in E_1}} u_l(t) \tag{3.57}$$

exists. Actually, the limit (3.57) exists in classical sense. In fact, tracing the backward characteristics (with negative slope) from a sequence (x_n, t),

$x_n \uparrow 0$, for times t belonging to sequences converging to $\bar{t}+$ with different limits, we would deduce that also the essential limit of the flux in E_1 does not exist, a contradiction.

We showed that the limits $u_l|_{E_i}(\bar{t}+)$, $i = 1, 2$ exist, and therefore we conclude the proof exactly as in the previous two cases. □

Lemma 3.9. *Let $\omega \in \mathcal{A}^{AB}(T)$. Then backward genuine characteristics cannot intersect at the interface. In particular, there are no rarefaction waves that start from a point of the interface.*

Proof. Assume that from a point $(0, \bar{t})$ a rarefaction wave opens in quadrant I and let $\zeta_1 < \zeta_2$ be two forward genuine characteristics from the point $(0, \bar{t})$ lying in quadrant I, and defined at least in $(\bar{t}, \bar{t} + \delta)$. We know that the unidirectional limits $f_l(u_l(\bar{t}\pm))$ and $f_r(u_r(\bar{t}\pm))$ exist. If we show that the limits $u_l(\bar{t}\pm)$ and $u_r(\bar{t}\pm)$ exist, then one can conclude as in [4].

First we show that $u_r(\bar{t}) \geq \theta_r$. Assume by contradiction that $u_r(\bar{t}) \leq \theta_r$. Then a shock emerges from $(0, \bar{t})$ in quadrant I lying at the right of the characteristic ζ_2. Then the minimal backward characteristics from the points of the shock all impact the interface at the point $(0, \bar{t})$, but this implies that the solution is unbounded, and this is a contradiction. Then since the limit $f_r(u_r(\bar{t}-))$ exists and it holds $u_r(t) \geq \theta_r$ for $t \in (\bar{t}, \bar{t} - \varepsilon)$, for some $\varepsilon > 0$, the limit $u_r(\bar{t}-)$ exists. Moreover from the interface conditions we obtain $u_r(\bar{t}-) \geq B$ and, by tracing the backward minimal characteristics (with positive slope) from a sequence of points $(x_n, \bar{t})$ with $x_n \downarrow 0$, we obtain that $u_r(\bar{t}-) = u(0+, \bar{t})$.

Moreover, since the limit $f_r(u_r(\bar{t}+))$ exists, and since by non-intersection of genuine characteristics it holds $u_r(t) \geq \theta_r$ for $t \in (\bar{t}, \bar{t}+\delta)$ (in fact, if it was $u_r(t) < \theta_r$, the maximal backward characteristic from $(0, t)$ would intersect ζ_1 and ζ_2), we conclude that the limit $u_r(\bar{t}+) \geq B$ exists.

Now we have some cases:

1. $u_r(\bar{t}+) = u_r(\bar{t}-) = B$ $(= u(0+, \bar{t}))$. Then from $(0, \bar{t})$ would emerge a single genuine characteristic: a contradiction.

2. $u_r(\bar{t}+) > B$ and $u_r(\bar{t}-) = B$ $(= u(0+, \bar{t}))$. In this case a single shock would emerge in quadrant I from $(0, \bar{t})$: again a contradiction.

3. $u_r(\bar{t}+) = B$ and $u_r(\bar{t}-) > B$ $(u_r(\bar{t}-) = u(0+, \bar{t}))$. By the interface conditions, this implies that the limit $u_l(\bar{t}-)$ exists and $u_l(\bar{t}-) = \pi^r_{l,+}(u_r(\bar{t}-))$. Then, if the limit $u_l(\bar{t}+)$ exists, we showed that all the four limits $u_l(\bar{t}\pm)$, $u_r(\bar{t}\pm)$ exist, and therefore we can use the analysis

of [4] and conclude. If $u_l(\bar{t}+)$ does not exist, in any case we know that the limit $f_l(u_l(\bar{t}+))$ exists and is equal to $f_l(A)$. Therefore there exists a sequence of points $t_n \downarrow \bar{t}$ such that the minimal backward characteristics from $(0, t_n)$ have slopes converging to $f_l'(\bar{A})$. Moreover, consider a sequence of points $s_n \uparrow \bar{t}$ and trace the backward minimal characteristics from $(0, s_n)$. Their slopes converge to $f_l'(u_l(\bar{t}-)) > \bar{A}$. Then there exists a $\bar{n}$ big enough such that the characteristics from $(0, t_n)$ and $(0, s_n)$ intersect each other in the interior of the domain: contradiction.

4. $u_r(\bar{t}+) > B$ and $u_r(\bar{t}-) > B$ ($u_r(\bar{t}-) = u(0+, \bar{t})$. In this case the limits $u_l(\pm)$ exist thanks to the interface conditions and they are equal to $\pi^r_{l,+}(u_r(\bar{t}\pm))$, so that we can use the analysis of [4].

□

Lemma 3.10. *Let $\omega \in \mathcal{A}^{AB}(T)$. Then*

$$(\omega(0-), \omega(0+)) \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{3,-} \cup \mathcal{T}_{3,+} \cup \{(A, B)\} \tag{3.58}$$

Proof. The lemma was proved in [4] in the case the limits $u_l(T\pm)$ and $u_r(T\pm)$ exist. The fact that the limits $(\omega(0-), \omega(0+))$ belong to one of the sets (3.17) also in the general case can be deduced with small adaptations to the analysis in [4], together with Lemma 3.8 and Lemma 3.9 . □

3.4. The sets $\mathcal{A}_1(T)$, $\mathcal{A}_2(T)$

Let u_0 be an initial data that produces a profile $\omega \doteq \mathcal{S}_T u_0$ at time T. By Lemma 3.10 we know that the pair $(\omega(0-), \omega(0+))$ belongs to one of the sets (3.17). We start by assuming that $(\omega(0-), \omega(0+)) \in \mathcal{T}_2$. We want to prove that then (3.38), (3.39) and the monotonicity properties of the map φ_2 hold. Everything was proved in [4] except (3.39) and the fact that φ_2 is nondecreasing in $(L, 0)$. First we prove this last property.

3.4.1. The map φ_2 is nondecreasing in $(L, 0)$. We prove that φ_2 is nondecreasing in $(L, 0)$. Notice that if $\omega(x+) < A$ for all $x \in (L, 0)$, the monotonicity of φ_2 in $(L, 0)$ is trivial, by non-intersection of backward characteristics. In fact, if this is the case, for every $x \in (L, 0)$ the maximal backward characteristic from (x, T) is the polygonal line $\theta_{x,+}$, that changes slope when crossing the interface. The problem arises when for some $x \in (0, L)$ it holds $\omega(x+) = A$. In fact, in this case, it might happen that at the point $t_{x,+}$ of impact of the maximal backward characteristic from (x,T) with the interface, it holds $u_r(t_{x,+}) = B$, so that the characteristic cannot be prolonged on the other side of the interface, and this in turn prevents us to prove Oleinik-type estimates in the usual way, and some extra considerations are needed. Let's prove that in any case

$$\varphi_2(x_1) \le \varphi_2(x_2), \quad \forall\, L < x_1 < x_2 < 0 \tag{3.59}$$

As already said, if $\omega(x_i+) \neq A$, $i = 1,2$, inequality (3.59) is clearly true, as well as if $\omega(x_i+) = A$, $i = 1,2$. Otherwise, only one of them is equal to A.

Case 1. $\omega(x_1+) = A$, $\omega(x_2+) \neq A$ (and therefore smaller than A, by the interface conditions). Assume $u_r(t_{x_1,+}) = B$ (otherwise the result is clear). Then since $\omega(x_2+) < A = \omega(x_1+)$, for fixed x, the map

$$u \mapsto -f_r' \circ f_{r,+-}^{-1} \circ f_l(u) \left(T - \frac{x}{f_l'(u)} \right)$$

is decreasing in u, we have

$$\begin{aligned} \varphi_2(x_1) &< -f_r' \circ f_{r,+-}^{-1} \circ f_l(\omega(x_2+)) \left(T - \frac{x_1}{f_l'(\omega(x_2+))} \right) \\ &< -f_r' \circ f_{r,+-}^{-1} \circ f_l(\omega(x_2+)) \left(T - \frac{x_2}{f_l'(\omega(x_2+))} \right) = \varphi_2(x_2) \end{aligned} \tag{3.60}$$

and the estimate (3.59) follows.

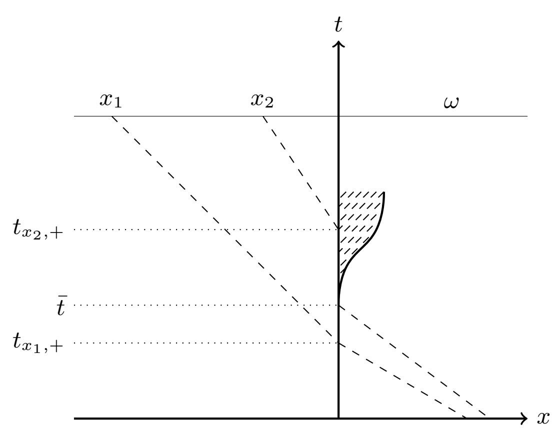

Figure 3.6: The situation described in Case 2.

Case 2. $\omega(x_2+) = A$, $\omega(x_1+) \neq A$ (and therefore smaller than A, by the interface conditions). Assume $u_r(t_{x_2,+}) = B$ (otherwise the result is clear). Let (see Figure 3.6)

$$\bar{t} \doteq \inf\{t : (u_l(s), u_r(s)) = (A, B),\ \forall s \in (t, t_{x_2,+})\}$$

Notice that $\bar{t} \in (t_{x_1,+}, t_{x_2,+})$. Then at the point $(0, \bar{t})$ emerges a shock with zero slope, lying in quadrant I, and $u_r(\bar{t}) = \bar{B}$. By non-intersection of genuine characteristics, it must hold

$$\varphi_2(x_1+) \leq -\bar{t} f_r'(\bar{B})$$

Moreover, since $\bar{t} \leq -t_{x_2,+}$,

$$-\bar{t} f_r'(\bar{B}) \leq -t_{x_2,+} f_r'(\bar{B}) = \varphi_2(x_2+)$$

and this concludes the proof.

3.4.2. Condition (3.39) holds. We let

$$\Omega = \Big\{(x,t) \mid \theta_{L,-}(t) < x < \theta_{L,+}(t)\Big\} \tag{3.61}$$

Assume by contradiction that (3.39) does not hold. Then

Claim: There is a solution u that yields ω at time T with a shock $y : (\bar{s}, T) \to \mathbb{R}$ such that for $t \in (\bar{s}, T)$

$$u(y(t)+, t) \leq A, \quad u(y(t)-, t) \geq y(t)/t \tag{3.62}$$

and $y(\bar{s}) = 0$.

Proof. (Claim) Step 1: The profile ω cannot be reached with a Lipschitz/compression wave-like solution in Ω. Indeed, assume by contradiction that ω is reachable with a Lipschitz solution in Ω. Then it must be a compression wave, partially reflected by the interface. But for this to be admissible, by the interface conditions, it must hold $L/T \leq f_l'(A)$. However in this case, the curve $(x_{\omega(L-)}(t), t)$, $t \in (0, T)$, is always strictly negative, so that (3.39) holds, and this is a contradiction.

Step 2: There is $\varepsilon > 0$ and a solution u that yields ω at time T with a shock $(y(t), t)$, $t \in (T - \varepsilon, T)$, such that (3.62) holds for $t \in (T - \varepsilon, T)$. To prove this, notice that by Step 1., every solution leading to ω must have a shock arriving at (L,T), since it cannot be Lipschitz. Moreover, assume there is a genuine backward characteristic ζ from (L,T) with slope L/T. Then, since $L/T > f_l'(A)$, from $(0,0)$ emerges a shock $y(t)$ with negative slope that has the desired properties (of the claim). Accordingly, in the following we assume that there is not a genuine backward characteristic from (L,T) with slope L/T. Let $\zeta_{L,\pm}(t)$ be the minimal and maximal backward characteristics from (L,T). Consider the set

$$\mathfrak{S} = \Big\{\zeta : [t_\zeta, T] \to \mathbb{R},\ \zeta(t_\zeta)t_\zeta = 0 \mid \zeta \text{ back. genuine char. from } (L,T)\Big\} \tag{3.63}$$

The set $\mathfrak{S}$ is totally ordered by "$<$". Let

$$\zeta_1 = \max\{\zeta \in \mathfrak{S} \mid t_\zeta = 0\}, \qquad \zeta_2 = \min\{\zeta \in \mathfrak{S} \mid \zeta(t_\zeta) = 0\}$$

The lines ζ_i, $i = 1,2$, are genuine characteristics, because they are uniform limits of genuine characteristics. Moreover, since there are no genuine characteristics from (L,T) with slope L/T, it must hold $t_{\zeta_1} = 0$, $\zeta(t_{\zeta_1}) < 0$ and $t_{\zeta_2} > 0$. This implies that there is a shock arriving at (L,T) with the desired properties.

Step 3: Assume condition (3.62) holds for $t \in (T-\varepsilon, T)$. Then we have

$$x_{\omega(L-)}(t) \le y(t), \quad t \in (T-\varepsilon, T)$$

Indeed, for every $t \in (T-\varepsilon, T)$

$$\dot{x}_{\omega(L-)}(t) = \lambda_l\left((f_l')^{-1}\left(\frac{x(t) - L + Tf_l'(\omega(L-))}{t}\right), A\right)$$

and

$$\dot{y}(t) = \lambda_l\Big(u(y(t)-,t), u(y(t)+,t)\Big)$$

Assume by contradiction at some point $t_1 > T-\varepsilon$ it holds $x_{\omega(L-)}(t_1) > y(t_1)$. Then there is a point $t_2 > t_1$ in which $x_{\omega(L-)}(t_2) > y(t_2)$, their derivatives exist and $\dot{y}(t_2) > \dot{x}_{\omega(L-)}(t_2)$. But this is a contradiction. In fact, by (3.62) and non-crossing of backward genuine characteristics,

$$u(y(t_2)+, t_2) \le A \tag{3.64}$$

$$\begin{aligned} u(y(t_2)-, t_2) &\le (f_l')^{-1}\left(\frac{y(t_2) - L + Tf_l'(\omega(L-))}{t_2}\right) \\ &\le (f_l')^{-1}\left(\frac{x_{\omega(L-)}(t_2) - L + Tf_l'(\omega(L-))}{t_2}\right) \end{aligned} \tag{3.65}$$

so that, since $u_1 \mapsto \lambda_l(u_1, u_2)$ and $u_2 \mapsto \lambda_l(u_1, u_2)$ are both increasing by convexity of f_l, one has

$$\dot{x}_{\omega(L-)}(t_2) = \lambda_l \left((f_l')^{-1} \left(\frac{x_{\omega(L-)}(t_2) - L + T f_l'(\omega(L-))}{t_2} \right), A \right)$$
$$\geq \lambda_l \left(u(y(t_2)-, t_2), u(y(t_2)+, t_2) \right) = \dot{y}(t_2) \quad (3.66)$$

and this is a contradiction.
Step 4: Prolong backwards the shock $y(t)$ until the point $\bar{s}$ in which it can't be prolonged anymore without violating the condition (3.62). By contradiction, assume that $\bar{s} > 0$ and $y(\bar{s}) < 0$ (notice that since $y(t) \geq x_{\omega(L-)}(t)$ and $x_{\omega(L-)}$ by assumption impacts the interface in positive time, it must hold $\bar{s} > 0$). Then, consider the same set $\mathfrak{S}$ and lines ζ_i, $i = 1, 2$, defined in Step 2., but replacing (L, T) with $(y(\bar{s}), \bar{s})$. Since by Step 3. $x_{w(L-)}(\bar{s}) \leq y(\bar{s})$, one has $y(\bar{s})/\bar{s} \geq L/T > f_l'(A)$. Therefore, with the same arguments of Step 2., it must hold $t_{\zeta_1} = 0$, $\zeta(t_{\zeta_1}) < 0$ and $t_{\zeta_2} > 0$ and, since $y(\bar{s})/\bar{s} > f_l'(A)$, the solution cannot be Lipschitz in the region

$$\left\{ (x, t) \mid \zeta_1(t) < x < \zeta_2(t) \chi_{(0, t_{\zeta_2})} \right\}$$

Therefore, with the same arguments of Step 2., we find an ε such that for $t \in (\bar{s} - \varepsilon, \bar{s})$ the shock $y(t)$ can be prolonged backwards and (3.62) holds, and this is a contradiction, so the claim is proved. □

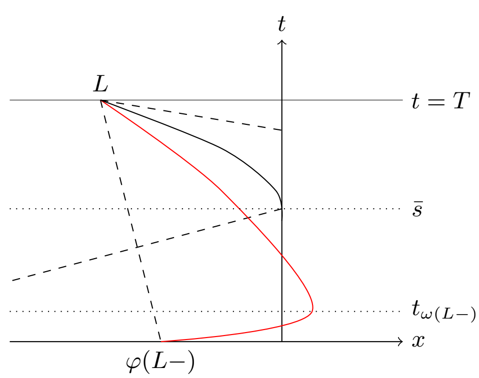

Figure 3.7: The situation in case (3.39) does not hold. In red the curve $(x_{\omega(L-)}(t), t)$ and in black the shock of the solution, $(y(t), t)$. However this cannot happen because two genuine characteristics would intersect.

By the arguments of Step 3., we find that $y(t) \geq x_{\omega(L-)}(t)$ for every $t \in (\bar{s}, T)$. Since $x_{\omega(L-)}(t_{\omega(L-)}) > 0$, we must have $\bar{s} \geq t_{\omega(L-)} > 0$ and

$y(\bar{s}) = 0$. This is a contradiction, because, since the shock $y(t)$ starts from $(0, \bar{s})$ with negative (or zero) slope, by the interface conditions we must have $u_l(\bar{s}) < \bar{A}$, but then (see Figure 3.7)

$$L - Tf_l'(\omega(L-)) = x_{\omega(L-)}(t_{\omega(L-)}) - t_{\omega(L-)}f_l'(\bar{A}) > -\bar{s}f_l'(u_l(\bar{s})) \quad (3.67)$$

and this contradicts the fact that genuine characteristics do not cross in the interior of the domain.

3.4.3. Construction of a solution that yields ω. Now we prove that the conditions of Theorem 3.6 on the set $\mathcal{A}_2(T)$ are also sufficient to guarantee that $\omega \in \mathcal{A}_2(T)$ is attainable. Let ω satisfy the conditions (3.38), (3.39), and the conditions on the monotonicity of the map φ_2. It is clear how to build the solution outside the region Ω (3.61), and this is done in [4]. In the region Ω, the compression-wave solution defined in [4] is AB-entropy admissible if and only if $L/T \leq f_l'(A)$. If instead $L/T > f_l'(A)$, there is no hope of having a Lipschitz solution in Ω. To prove attainability also in this case, we build a solution using the shock $x_{\bar{u}_L}(t)$. For the definition of $\bar{u}_L$ see (3.20). By definition, this is the "limit shock" that touches the interface with slope zero in positive time, so that $x_{\bar{u}_L}(t_{\bar{u}_L}) = 0$. Condition (3.39) tells us precisely that the final point of the minimal backward characteristic from (L, T) is smaller than the starting point of the rarefaction wave generating the shock $x_{\bar{u}_L}$, and therefore we can use it to build the solution (see Figure (3.8)). For a point $x < 0$ and a "state" $v \in \mathbb{R}$, we define the following polygonal lines. If $f_l'(v) \leq x/T$ let

$$\zeta_{x,v}(t) = \begin{cases} x - (T-t)f_l'(v) & t \in \left(T - \frac{f_l'(v)}{x}, T\right) \\ \left(\frac{f_l'(v)}{x} + t - T\right)\left(f_r' \circ \pi^l_{r,-}(v)\right) & t \in \left(0, T - \frac{f_l'(v)}{x}\right) \end{cases} \quad (3.68)$$

If instead $f_l'(v) > x/T$ let

$$\zeta_{x,v}(t) = x - f_l'(v), \quad t \in (0, T). \quad (3.69)$$

Let

$$\gamma(t) = x_{\bar{u}_L}(t)\chi_{(t_{\bar{u}_L}, T)} - (t_{\bar{u}_L} - t)f_l'(\bar{A})\chi_{(0, t_{\bar{u}_L})}$$

We define a partition of Ω as follows.

$$\Sigma \doteq \left\{(x,t) \in \Omega \mid x \leq L - (T-t)f_l'(\bar{u}_L)\right\} \quad (3.70)$$

$$\Pi \doteq \left\{(x,t) \in \Omega \mid x \leq \gamma(t)\right\} \setminus \Sigma \quad (3.71)$$

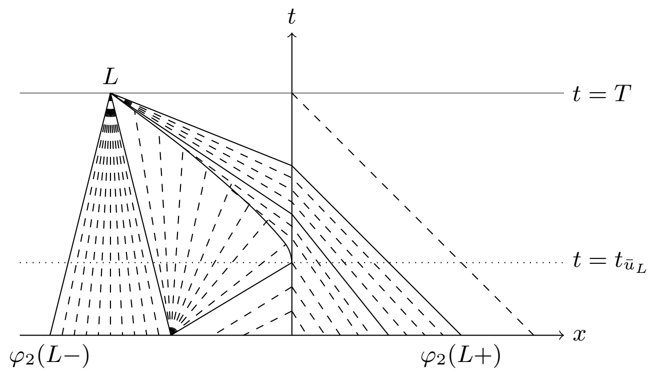

Figure 3.8: The characteristics of the solution defined in (3.74)

$$\Xi \doteq \Big\{(x,t) \in \Omega \mid -(t_{\bar{u}_L} - t) f_l'(\bar{A}) < x < 0, \quad 0 < t < t_{\bar{u}_L}\Big\} \tag{3.72}$$

$$\Lambda \doteq \Big\{(x,t) \in \Omega \mid x \le \zeta_{L,A}(t)\Big\} \setminus (\Sigma \cup \Pi \cup \Xi) \tag{3.73}$$

The solution in Ω is defined as

$$u^*(x,t) = \begin{cases} (f_l')^{-1}\left(\frac{L-x}{T-t}\right) & (x,t) \in \Sigma, \\ (f_l')^{-1}\left(\frac{x-\varphi(L-)}{t}\right) & (x,t) \in \Pi, \\ \bar{A} & (x,t) \in \Xi, \\ A & (x,t) \in \Lambda, \quad x < 0, \\ \bar{B} & (x,t) \in \Lambda, \quad x > 0, \\ v & x = \zeta_{L,v}(t), \quad \omega(L-) \le v \le A, \quad x < 0, \\ \pi_{r,-}^l(v) & x = \zeta_{L,v}(t), \quad \omega(L-) \le v \le A, \quad x > 0 \end{cases} \tag{3.74}$$

and the initial datum u_0 that generates the solution above is

$$u_0^*(x) = \begin{cases} (f_l')^{-1}\left(\frac{L-x}{T}\right) & x \in (\varphi_2(L-), L - T f_l'(\bar{u}_L)), \\ \bar{A} & x \in (L - T f_l'(\bar{u}_L), 0), \\ \bar{B} & x \in (0, \zeta_{L,A}(0)), \\ \pi_{r,-}^l(v) & x \in (\zeta_{L,A}(0), \varphi_2(L+)), \quad x = \zeta_{L,v}(0) \end{cases} \tag{3.75}$$

The analysis for the set $\mathcal{A}_1(T)$ is entirely symmetric and thus it's omitted.

3.5. The set $\mathcal{A}_3^{AB}(T)$

Let $\omega \in \mathbf{L}^\infty(\mathbb{R})$ such that $\omega((0-), \omega(0+)) = (A, B)$. Assume (A, B) is not a critical connection (if it is critical the proof follows similarly and it is easier). Let

$$L = \inf\{x \mid f_l'(\omega(x+)) \leq x/T\}, \qquad R = \sup\{x \mid f_r'(\omega(x-)) \geq x/T\}$$

We want to prove that all the conditions in the theorem for the set $\mathcal{A}_3^{AB}$ hold. The proof that condition (3.41) holds is entirely similar to what done in the previous section for condition (3.39).

3.5.1. Condition (3.43) holds. Assume that $t_L \leq t_R$. If $t_{\bar{u}_R} \leq t_L$ (and therefore $L_A \leq L$), we have to prove that

$$w(x) = A \quad \forall\, x \in (L, 0), \qquad w(x) = B \quad \forall\, x \in (0, R), \tag{3.76}$$

The fact that $u_l(t) = A,\ u_r(t) = B$ for $t \in (T - \delta, T)$ for some $\delta > 0$ is clear using the same arguments in [4]. Moreover, we know that in the first point in which the traces u_l, u_r change their value must emerge a shock with zero slope. Since $t_{\bar{u}_R} \leq t_L$, and since there cannot be shocks emerging from a point $(0, t)$ for $t > \max\{t_{\bar{u}_l}, t_{\bar{u}_R}\} \leq t_L$[1], we find that $(u_l(t), u_r(t)) = (A, B)$ for $t \in (t_L, T)$. Then condition (3.41) holds.

Assume now $t_L < t_{\bar{u}_R} < t_R$ (this is the situation represented in Figure (3.9)). As above, since there are no shocks that emerge from the interface from times bigger then $\max\{t_{\bar{u}_l}, t_{\bar{u}_R}\} = t_{\bar{u}_R}$, the traces have values $(u_l(t), u_r(t)) = (A, B)$ for every $t \in (t_{\bar{u}_R}, T)$, so that ω is forced to be A in $(L_A, 0)$ and B in $(0, R)$. The fact that $\varphi_{3,A}$ is increasing in (L, L_A) follows exactly in the same way in which we proved in the previous section that φ_2 is increasing, therefore we are done.

3.5.2. Construction of a solution that yields ω. To prove that they are also sufficient, for a function ω satisfying the conditions of Theorem 3.6, we explicitly construct a solution u and an initial data u_0 such that $\mathcal{S}_t^{AB} u_0(x) = u(x, t)$ and $\mathcal{S}_T^{AB} u_0 = \omega$. We do it in the case $t_L < t_{\bar{u}_R} < t_R$,

[1]Assume by contradiction a shock $(y(t), t)$ emerges from a point $\bar{t} > \max\{t_{\bar{u}_l}, t_{\bar{u}_R}\}$ in quadrant I with slope zero and with left state B. Then at every point $t \geq \bar{t}$ the speed of the shock $y(t)$ is strictly smaller then the speed of $x_{\bar{u}_R}(t)$, hence contradicting the fact that it must be $y(T) = x_{\bar{u}_R}(T) = R$.

(A, B) non critical connection, $L/T \leq f_l'(A)$, $R/T < f_r'(B)$, the other cases being entirely analogous. First define the set

$$\Omega \doteq \Big\{(x,t) \mid L - (T-t)f_l'(\omega(L-)) < x < R - (T-t)f_r'(\omega(R+))\Big\} \quad (3.77)$$

It will be sufficient to build the solution in Ω, since in the other regions the construction was done in [4] and it is entirely similar to the continuous-convex flux case. Since $t_L < t_{\bar{u}_R}$, it holds $L \leq L_A$.

The set of points $\theta_{x,\pm}(0)$, $x \in (L, L_A)$, covers the interval $(\theta_{L,+}(0), \theta_{L_A,+}(0))$ with the exception of at most a countable number of disjoint intervals (x_n^-, x_n^+), where the initial data will be defined so as to produce a compression wave generating a shock at the point $y_n \in (L, L_A)$ such that $\theta_{y_n,\pm}(0) = x_n^\pm$. Since we are assuming $L/T \leq f_l'(A)$, at the left of the interface the solution can be constructed using the compressione wave, partially refracted by the interface. At the right of the interface, since $R/T < f_r'(B)$, the solution cannot be Lipschitz and will be constructed using the shock $x_{\bar{u}_R}$, in similar way to what done for the profiles in $\mathcal{A}_2(T)$.

Define the following sets

$$\Xi \doteq \Big\{(x,t) \in \Omega \mid x \leq \theta_{L,+}(t)\Big\} \quad (3.78)$$

$$\Lambda \doteq \Big\{(x,t) \in \Omega \mid \theta_{L,+}(t) < x < \theta_{L_A,+}(t)\Big\} \quad (3.79)$$

$$\Delta \doteq \Big\{(x,t) \in \Omega \mid 0 < x < x_{\bar{u}_R}(t), t \in (t_{\bar{u}_R}, T)\Big\} \quad (3.80)$$

$$\Upsilon \doteq \Big\{(x,t) \in \Omega \mid x_{\bar{u}_R}(t)\chi_{(t_{\bar{u}_R},T)}(t) - (t_{\bar{u}_R} - t)f_r'(\bar{B})\chi_{(0,t_{\bar{u}_R})}(t) < x < \zeta_{R,\bar{u}_R}(t)\Big\} \quad (3.81)$$

For a set S we define $S^- = \{(x,t) \in S \mid x < 0\}$ and $S^+ = \{(x,t) \in S \mid x > 0\}$. Moreover let $\bar{R} = R - Tf_r'(\bar{u}_R)$. Then we define

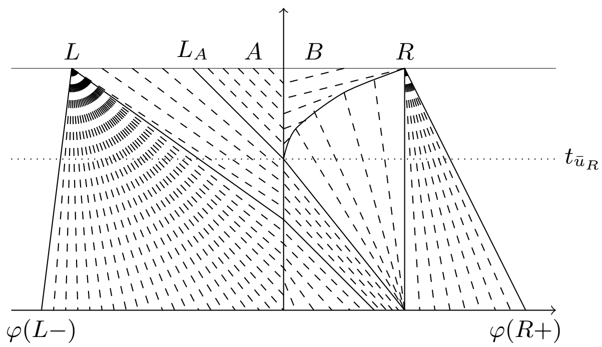

Figure 3.9: The characteristics of the solution defined in (3.82)

$$u^* = \begin{cases} v & (x,t) \in \Xi^-, \quad x = \zeta_{L,v}(t), \\ \pi^l_{r,-}(v) & (x,t) \in \Xi^+, \quad x = \zeta_{L,v}(t), \\ \omega(y\pm) & (x,t) \in \Lambda^-, \quad x = \theta_{y,\pm}(t), \quad y \in (L, L_A), \\ \pi^l_{r,-}(\omega(y\pm)) & (x,t) \in \Lambda^+, \quad x = \theta_{y,\pm}(t), \quad y \in (L, L_A), \\ v & (x,t) \in \Lambda^-, \quad x = \zeta_{y_n,v}(t), \quad \omega(y_n-) < v < \omega(y_n+), \\ \pi^l_{r,-}(v) & (x,t) \in \Lambda^+, \quad x = \zeta_{y_n,v}(t), \quad \omega(y_n-) < v < \omega(y_n+), \\ A & L_A - (T-t)f_l'(A) < x < 0, \quad t \in (t_{\bar{u}_R}, T), \\ B & (x,t) \in \Delta, \\ (f_r')^{-1}\left(\frac{x-\bar{R}}{t}\right) & (x,t) \in \Upsilon, \\ v & \zeta_{R,\bar{u}_R}(t) < x < \theta_{R,+}(t), \quad x = \zeta_{R,v}(t), \end{cases} \tag{3.82}$$

and the initial data that produces the solution u^* is, with $\bar{L} = \varphi_{3,A}(L+)$, $\bar{L}_A = \varphi_{3,A}(L_A+)$,

$$u_0^* = \begin{cases} v & \varphi_{3,A}(L-) < x < 0, \quad x = \zeta_{L,v}(0), \\ \pi^l_{r,-}(v) & 0 < x < \bar{L}, \quad x = \zeta_{L,v}(0)), \\ \pi^l_{r,-}(\omega(y\pm)) & \bar{L} < x < \bar{L}_A, \quad x = \theta_{y,\pm}(0), \quad y \in (L, L_A), \\ \pi^l_{r,-}(v) & \bar{L} < x < \bar{L}_A, \quad x = \zeta_{y_n,v}(0), \quad \omega(y_n-) < v < \omega(y_n+), \\ v & \zeta_{R,\bar{u}_R}(0) < x < \varphi_{3,A}(R+), \quad x = \zeta_{R,v}(t), \end{cases} \tag{3.83}$$

Remark 3.11. By the previous analysis, a profile $\omega \in \mathcal{A}_2(T)$ is attainable with a locally Lipschitz solution (separately in the I and II quadrants) if and only if $L/T \leq f_l'(A)$. Analogously, a profile $\omega \in \mathcal{A}_1(T)$ is attainable with a Lipschitz solution if and only if $R/T \geq f_r'(B)$. A profile $\omega \in \mathcal{A}_3^{AB}(T)$, is reachable with a Lipschitz solution if and only if $(\omega(0-), \omega(0+)) \neq (A, B)$ or, in the case $(\omega(0-), \omega(0+)) = (A, B)$, if and only if $L/T = f_l'(A)$ and $R/T = f_r'(B)$. In any case, every profile is attainable with a solution that has at most two shocks, one at the left and one at the right of the interface.

Example 3.12. Let's see what $\bar{u}$ is if $f_l = x^2/2$. Let's fix a state $u \in [L/T, +\infty)$ (or, equivalently, $\zeta_{L,u}(0)$). Solving the ode one finds that $x_u(t)$ is given by

$$x_u(t) = c\sqrt{t} + tA + \zeta_{L,u}(0)$$

with

$$c = \frac{L - \zeta_{L,u}(0) - TA}{\sqrt{T}}$$

With an explicit calculation one finds that

$$\bar{u}_L = -A - 2\sqrt{A\frac{L}{T}}$$

It holds $\bar{u} > L/T$ (unless $L/T = A$), so that the condition $\omega(L-) \geq L/T$ is not sufficient, in general, to guarantee that the profile ω is attainable.

Chapter 4

Initial data identification

Abstract. The aim of this chapter is to study the problem of initial data identification for the conservation law with discontinuous flux

$$\begin{cases} u_t + f(u,x)_x = 0, & x \in \mathbb{R}, \quad t \geq 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{4.1}$$

where $f = \mathcal{H}(x) f_l(u) + (1 - \mathcal{H}(x)) f_r(u)$, f_l, f_r are uniformly convex and $\mathcal{H}$ is the Heaviside function. The result will be achieved by an adaptation of the method of generalized characteristics [11] to the setting of AB-entropy solutions of (4.1), and represents a generalization of what done in [10], [19] for the case of a strictly convex flux. We prove that, as in the strictly convex flux case, the set of initial data

$$\mathcal{I}_T^{AB} \omega = \{u_0 \in \mathbf{L}^\infty : \mathcal{S}_T^{AB} u_0 = \omega\}$$

is an infinite dimensional cone that however it is not, in general, a convex cone. This represents a major difference with respect to the continuous flux case, where the set of initial data that yield a profile ω at time $T > 0$ is always convex.

4.1. Generalized characteristics

In this section we introduce a new object, that in some sense generalizes the concept of characteristic firstly introduced by Dafermos in [11] for conservation laws with strictly convex flux. Recall that a generalized characteristic for a conservation law with convex flux f in the sense of [11] is a Lipschitz continuous curve $\zeta : [0,T] \to \mathbb{R}$ such that for almost all $t \in [0,T]$,

$$\dot{\zeta}(t) \in [f'(u(\zeta(t)+,t)), f'(u(\zeta(t)-,t))]. \tag{4.2}$$

The following Lemma, that provides a straightforward generalization of Lemma 3.2 in [11], will be fundamental through the rest of the chapter.

Lemma 4.1. *Assume u is an admissible (A,B) entropy solution. Let $a, b \in [0,T]$ with $a < b$ and let $\alpha, \beta : [a,b] \to \mathbb{R}$ be two Lipschitz maps with $\alpha \leq \beta$.*

Then for $t_1 \leq t_2$ holds

$$\int_{\alpha(t_2)}^{\beta(t_2)} u(t_2,x)\,\mathrm{d}x - \int_{\alpha(t_1)}^{\beta(t_1)} u(t_1,x)\,\mathrm{d}x$$
$$= \int_{t_1}^{t_2} f(u(t,\alpha(t)-),\alpha(t)-) - \dot\alpha(t)u(t,\alpha(t)-)\,\mathrm{d}x -$$
$$\int_{t_1}^{t_2} f(u(t,\beta(t)+),\beta(t)+) - \dot\beta(t)u(t,\beta(t)+)\,\mathrm{d}x \quad (4.3)$$

Proof. Since u satisfies the conservation law with convex flux f_l, f_r at the left and at the right of the discontinuity interface $x = 0$, it will be sufficient to apply Lemma 3.2 twice (one time at the left and one time at the right of $x = 0$) and use the existence of the left and right traces at $x = 0$. □

Remark 4.2. An almost straightforward consequence of Lemma 4.1 is the following. Consider the set of all connections $\mathscr{C}_f \subset \mathbb{R}^2$ for the fluxes f_l, f_r. Fix an initial datum u_0 and consider the map

$$\mathscr{C}_f \ni (A,B) \mapsto \Phi_{u_0}(A,B) = (u_l,u_r) \in \mathbf{L}^\infty(\mathbb{R}^+) \times \mathbf{L}^\infty(\mathbb{R}^+) \quad (4.4)$$

where u_l, u_r are the left and right traces at $x = 0$ of the solution $u = \mathcal{S}^{AB}_{(\cdot)} u_0(\cdot)$. Then there exists $M > 0$ depending on $\|u_0\|_{\mathbf{L}^\infty}$ such that for every $(A,B) \in \mathscr{C}_f$ with $(A,B) \in B_{\mathbb{R}^2}(0,M)^c$ it holds

$$\Phi_{u_0}(A,B) = (A,B) \quad (4.5)$$

where A and B in the right hand side represent two constant functions $\mathbb{R}^+ \to \mathbb{R}$ with values respectively A, B. This might be useful if someone wants to minimize some cost depending only on the traces u_l, u_r. In fact in light of what said above it might be not (too) restrictive to work on a compact set of connections.

The proof follows in the following way: take a connection $(A,B) \in \mathscr{C}_f$ and assume at some point $t \in (0,T]$ it holds $u_l(t) > \theta_l$ or $u_r(t) < \theta_r$. Without loss of generality assume $u_l(t) > \theta_l$, the other case being entirely symmetric. Then the minimal backward genuine characteristic from the point $(0,t)$ has slope $f_l'(u_l(t)) > 0$ and we apply Lemma 4.1 with $t_2 = t$, $t_1 = 0$, $\alpha(s) = -(t-s)f_l'(u_l(t))$, $\beta(s) = 0$. Let $I = (-tf_l'(u_l(t)),0)$. Then by the interface conditions and by inequality (4.14) we find

$$\int_I u_0(x)\,\mathrm{d}x = \int_0^t f_l(u_l(s))\,\mathrm{d}s - \int_0^t f_l(u(\alpha(s)-,s)) - \dot\alpha(s)\cdot u(\alpha(s)-,s)\,\mathrm{d}s \geq \int_0^t f_l(\bar A) - \int_0^t f_l(\bar A) - \dot\alpha(s)\cdot \bar A\,\mathrm{d}s = \bar A t\dot\alpha \tag{4.6}$$

Since $t\dot\alpha = |I|$, we find

$$⨍_I u_0(x)\,\mathrm{d}x \geq \bar A \tag{4.7}$$

and this implies that $\|u_0\|_{\mathbf{L}^\infty} \geq \bar A$.

Definition 4.3. Let $v : \mathbb{R}\times[0,T]$ be an $\mathbf{L}^\infty$ function such that the limits $v(x-,t)$, $v(x+,t)$ exist for every $x\in\mathbb{R}$ and $t\in(0,T]$. Let $\alpha:[0,T]\to\mathbb{R}$ be a Lipschitz curve. We define

$$\mathcal{F}_t(v,\alpha\pm) := \int_t^T f(v(\alpha(t)\pm,t),\alpha(t)\pm) - \dot\alpha(t)v(\alpha(t)\pm,t)\,\mathrm{d}t \tag{4.8}$$

where $f(u,x)$ is the flux of the problem (4.1). We let also $\mathcal{F}(v,\alpha\pm) := \mathcal{F}_0(v,\alpha\pm)$.

Now we define the object that generalizes, in some sense that we will see below, the characteristics in the sense of [11] to the setting of AB-entropy solutions.

Definition 4.4. For a solution u of the problem (4.1) and $x\in\mathbb{R}$ we say that a Lipschitz (polygonal) curve $\zeta:[0,T]\to\mathbb{R}$ such that $\zeta(T)=x$, belongs to $\mathcal{C}(u,x)$ if for a.e. $t\in[0,T]$ one of the following holds:

1. $\zeta(t)\neq 0$ and
$$\dot\zeta(t) = f'(u(\zeta(t)-,t,),\zeta(t)) = f'(u(\zeta(t)+,t),\zeta(t)) \tag{4.9}$$

2. $\zeta(t)=0$ and
$$f_l(u(\zeta(t)-,t) = f_l(A) = f_r(B) = f_r(u(\zeta(t)+,t)). \tag{4.10}$$

We also define

$$\mathcal{C}_0(u,x) = \{\zeta(0) : \zeta\in\mathcal{C}(u,x)\} \tag{4.11}$$

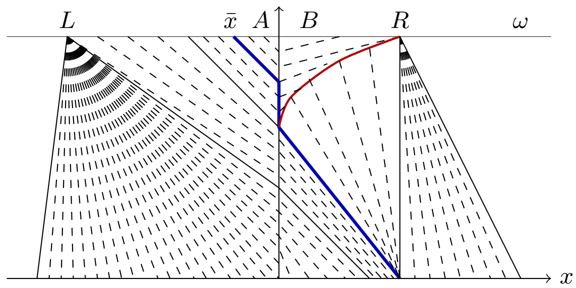

Figure 4.1: There are no backward generalized characteristics with time of existence $[0, T]$ from the point $\bar{x}$. If, instead, we consider elements in $\mathcal{C}_0(u^*, \bar{x})$ (the blue line), we see that, also if at the time at which the characteristic reaches the interface it cannot be prolonged on the other side in classical sense, there is at least an element in $\mathcal{C}_0(u^*, \bar{x})$ that is defined on the whole $[0, T]$.

In other words, $\zeta(t) \in \mathcal{C}(u, x)$ if and only if it is a classical generalized characteristic [11] for the fluxes f_l, f_r in the regions, respectively, $x < 0$ and $x > 0$, with the additional freedom that $\zeta(t)$ can "travel" along the discontinuity interface $x = 0$ in some intervals of time, but only if at those points the flux of the solution is the minimum possible, i.e. $f_l(u_l(\zeta(t)) = f_r(u_r(\zeta(t)) = f_l(A) = f_r(B)$, where we recall that u_l, u_r are the left and right traces of u at $x = 0$. Notice that in any case ζ is made of (up to) three segments. If $|x|$ is big enough, elements in $\mathcal{C}(u, x)$ coincide with backward genuine characteristics from the point x. Finally, the set $\mathcal{C}_0(u, x)$ is closed. A quick way to see this is that every sequence $\{\zeta_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(u, x)$ is uniformly bounded and uniformly Lipschitz (with Lipschitz constant equal to $\max\{L_l, L_r\}$ with L_l, L_r Lipschitz constants of f_l, f_r on some bounded set K), so that there exists a uniformly converging subsequence ζ_{n_k} whose limit will be an element of $\mathcal{C}(u, x)$ (uniform limit of genuine characteristics is still a genuine characteristic).

Remark 4.5. Let us briefly discuss the problem of initial data identification in the case of the conservation law (4.12) where f is a uniformly convex flux. As already mentioned, this problem was recently completely solved in [10], [19] using two different approaches. In [10], the proof is based on the Lax-Oleinik formula, while in [19] the authors used the method of generalized characteristics. When the flux f is continuous and uniformly convex the "special" solution u^* to the problem

$$u_t + f(u)_x = 0 \tag{4.12}$$

that yields ω at time T is always Lipschitz (see the previous chapter) and it is easy to prove that the initial datum u_0^* that produces u^* is completely identified among the elements of $\mathcal{I}_T\omega$ by property (i) below. This solution was used in [10], [19] to characterize all the other elements of of the set $\mathcal{I}_T\omega$ and it played a fundamental role because it was proved that $\mathcal{I}_T\omega$ is always a convex cone having u_0^* at its vertex.

If one looks at the proof of [19], it is possible to single out the three properties that make this kind of characterization possible.

(i) *For every* $y \in \mathbb{R}$ *there exists* $x \in \mathbb{R}$ *and a genuine characteristic* $\zeta : [0,T] \to \mathbb{R}$ *of* u^* *such that* $\zeta(0) = y$ *and* $\zeta(T) = x$.

(ii) *For every solution* u *of* (4.12) *such that* $u(\cdot, T) = \omega$, *for every* $x \in \mathbb{R}$, u^* *and* u *share at least one genuine characteristic* $\zeta : [0,T] \to \mathbb{R}$ *such that* $\zeta(T) = x$.[1]

(iii) Let u, v be solutions of (4.12). For every genuine characteristic $\zeta : [0,T] \to \mathbb{R}$ for u:

$$\mathcal{F}_t(v, \zeta\pm) \geq \mathcal{F}_t(u, \zeta\pm)$$

Here we would like to generalize $(i), (ii), (iii)$ to the setting of entropy solutions of type AB. This is not straightforward for two main reasons. The first one is that, while in the convex flux case it is clear that there is a privileged solution u^* (the Lipschitz one) that plays a key role in the initial data identification, in our case it is not so clear what solution should replace the role that u^* played in the convex case. Secondly, in the convex case a genuine characteristic can always be prolonged until time $t = 0$. In the discontinuous case this is cleary not anymore true, but we already gave a hint on how to generalize this property using the elements of $\mathcal{C}(u, \bar{x})$.

The following section will be dedicated to proving that the set $\mathcal{C}(u, x)$ allows to generalize the previous three properties to the discontinuous problem (4.1). Elements of $\mathcal{C}(u, x)$ will take the role of genuine characteristics. We will see below that there is a solution that generalizes, in some sense, the solution u^* to our discontinuous setting. More precisely, we will see that the following three properties hold:

$(i)^*$ *For every* $y \in \mathbb{R}$ *there exists* $x \in \mathbb{R}$ *and an element* $\zeta \in \mathcal{C}(u^*, x)$ *such that* $\zeta(0) = y$

$(ii)^*$ *For every solution* u *such that* $u(\cdot, T) = \omega$, *for every* $x \in \mathbb{R}$, $\mathcal{C}(u^*, x) \cap \mathcal{C}(u, x) \neq 0$.

[1] Actually this property is true also if we replace u^* with any other element $v \in \mathcal{I}_T\omega$, but this is not important in the initial data identification

$(iii)^*$ Let v, v^* be solutions of (4.1). For every element $\zeta \in \mathcal{C}(x, v^*)$:

$$\mathcal{F}_t(v, \zeta\pm) \geq \mathcal{F}_t(v^*, \zeta\pm)$$

Hopefully, as in the convex flux case, there exists a unique solution u^* that satisfies $(i)^*$. In this case the initial data u_0^* that produces u^* will be a good candidate for being the vertex of the cone $\mathcal{I}_T^{AB}\omega$ (although at this point we still do not know if $\mathcal{I}_T^{AB}\omega$ is a cone, but it will become clear later). In the following section we will see that this is the case and prove these facts.

4.2. Properties of $\mathcal{C}(u, x)$

In the next lemmas we prove, respectively, $(iii)^*$, the existence (and uniqueness) of the special solution u^*, and $(ii)^*$. Property $(iii)^*$ is true essentially because of the same reasons that make property (iii) true, plus the fact that if $\zeta \in \mathcal{C}(u, x)$, in the interval in which ζ travels along the interface, the flux of the solution is the minimum possible in that interval (i.e. precisely the flux of the connection). The existence of u^* is proved in the previous chapter. Property $(ii)^*$ is deeper and has it's roots on the specific structural properties of an AB-entropy solution as well as the specific properties of the solution u^*. The following Lemma proves that $(iii)^*$ holds.

Lemma 4.6. *Let $v, v^* : \mathbb{R} \times [0,T]$ be solutions to the problem (4.1). Fix a point $x \in \mathbb{R}$ and let $\alpha \in \mathcal{C}(x, v^*)$. Then it holds*

$$\mathcal{F}_t(v, \alpha\pm) \geq \mathcal{F}_t(v^*, \alpha\pm) \tag{4.13}$$

Proof. We write the integral $\mathcal{F}$ as the sum of three parts depending on the sign of α. Then the proof follows from the following two observations:

1. In the segment (t_1, t_2) where $\alpha(t) = 0$, the integrand in $\mathcal{F}(v^*, \alpha\pm)$ is almost everywhere equal to $f_l(A)$, and by the interface conditions for v it holds $f_l(A) \leq f_l(v_l(t))$ for $t \in (t_1, t_2)$.

2. In a segment $(t_1, t_2) \ni t$ where $\alpha \neq 0$ (assume it's negative for fixing the ideas), using the convex estimate

$$f_l(u) - f_l'(w)u \geq f_l(w) - f_l'(w)w, \quad \forall\, u, w \in \mathbb{R} \tag{4.14}$$

one finds that (recall that since α is genuine for v^* one has $v^*(\alpha(t), t) = (f_l')^{-1}(\dot{\alpha}(t))$

$$\begin{aligned} f_l(v(\alpha(t), t)) &- \dot{\alpha}(t) v(\alpha(t), t) \\ &\geq f_l(v^*(\alpha(t), t)) - f_l'(v^*(\alpha(t), t)) v^*(\alpha(t), t) \end{aligned} \tag{4.15}$$

that is what we need to prove the Lemma.

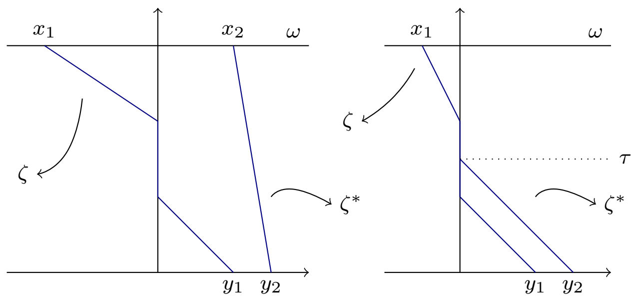

Figure 4.2: ζ and ζ^* do not intersect (left) and intersect (right).

□

Lemma 4.7. *Fix $\omega \in \mathcal{A}^{AB}(T)$. There exists a unique $u_0^* \in \mathcal{I}_T^{AB}\omega$ such that property $(i)^*$ holds.*

Proof. Existence is proved in the previous chapter, where u_0^* and the solution u^* produced by the initial datum u_0^* are explicitly defined and are used to prove the characterization of the reachable set $\omega \in \mathcal{A}^{AB}(T)$. Therefore we only have to prove uniqueness.

Assume $u_0 \in \mathcal{I}_T^{AB}\omega$ satisfies $(i)^*$. Let $y_1 < y_2$. Then there exist $\zeta \in \mathcal{C}(u, x_1)$ and $\zeta^* \in \mathcal{C}(u^*, x_2)$ such that $\zeta(0) = y_1$ and $\zeta^*(0) = y_2$ (see Figure 4.2). Assume that ζ and ζ^* do not intersect (if they intersect, the proof follows similarly integrating only up to the time τ of intersection, see the proof of Theorem 4.9 for a similar argument). Applying Lemma 4.1 to the curves ζ_1, ζ_2 and the solution u, and Lemma 4.6, we find that

$$\begin{aligned}\int_{y_1}^{y_2} u_0(x)\,\mathrm{d}x &= \int_{x_1}^{x_2} \omega(x)\,\mathrm{d}x + \mathcal{F}(u, \zeta^*+) - \mathcal{F}(u, \zeta-)\\ &\geq \int_{x_1}^{x_2} \omega(x)\,\mathrm{d}x + \mathcal{F}(u^*, \zeta^*+) - \mathcal{F}(u, \zeta-) \qquad (4.16)\end{aligned}$$

Analogously, applying Lemma 4.1, this time to the solution u^*, and Lemma 4.6, we find

$$\begin{aligned}\int_{y_1}^{y_2} u_0^*(x)\,\mathrm{d}x &= \int_{x_1}^{x_2} \omega(x)\,\mathrm{d}x + \mathcal{F}(u^*, \zeta^*+) - \mathcal{F}(u^*, \zeta-)\\ &\leq \int_{x_1}^{x_2} \omega(x)\,\mathrm{d}x + \mathcal{F}(u^*, \zeta^*+) - \mathcal{F}(u, \zeta-) \qquad (4.17)\end{aligned}$$

and subtracting (4.17) from (4.16) one obtains

$$\int_{y_1}^{y_2} u_0(x) - u_0^*(x)\,\mathrm{d}x \geq 0 \tag{4.18}$$

With the symmetric argument, with $y_1 < y_2$, $\zeta^* \in \mathcal{C}(u^*, x_1)$ and $\zeta \in \mathcal{C}(u, x_2)$, we find that also the opposite inequality holds, so that

$$\int_{y_1}^{y_2} u_0(x) - u_0^*(x)\,\mathrm{d}x = 0, \quad \forall\, y_1 < y_2 \tag{4.19}$$

that proves, thanks to Lebesgue differentiation theorem, that $u_0 = u_0^*$ as elements of $\mathbf{L}^\infty$. ☐

Now we pass to the proof of property $(ii)^*$. In the following we will often use the following fact:

Fact: Assume that for a solution u of (4.1) at some time it holds $u_l(t) = u(0-,t) = A$ and $u_r(t) = u(0+,t) = B$. Then there exists a $\delta > 0$ such that $(u_l(s), u_r(s)) = (A, B)$ for every $t \in (t-\delta, t)$. Moreover, if one of the traces u_l, u_r changes its value for the first time at some point $\bar{t}$ with $t > \bar{t} > 0$, it holds $u_l(\bar{t}) = \bar{A}$ and from the point $(0, \bar{t})$ emerges a shock with zero slope lying in quadrant II if u_l changed its value, while it holds $u_r(\bar{t}) = \bar{B}$ and emerges a shock with zero slope lying in quadrant I if u_r changed its value.

Lemma 4.8. *Fix $\omega \in \mathcal{A}^{AB}(T)$. Let $u^* = \mathcal{S}_{(\cdot)} u_0^*(\cdot)$, with $u_0^* \in \mathcal{I}_T^{AB}\omega$ the unique element that satisfies $(i)^*$, and let u any other solution such that $u(\cdot, T) = \omega$. Let $\bar{x} \in \mathbb{R}$. Then*

$$\mathcal{C}(u^*, \bar{x}) \cap \mathcal{C}(u, \bar{x}) \neq 0 \tag{4.20}$$

Proof. Without loss of generality assume $\bar{x} > 0$. Assume first that $f_r'(\omega(\bar{x}+)) \leq x/T$. Let $\theta_{\bar{x},+} : [0, T] \to \mathbb{R}$ be the maximal backward characteristic from $(\bar{x}, T)$. Then clearly

$$\theta_{\bar{x},+} \in \mathcal{C}(u^*, \bar{x}) \cap \mathcal{C}(u, \bar{x})$$

Now let $f_r'(\omega(\bar{x}+)) > x/T$. Let

$$t_{\bar{x},-} = T - \frac{f_r'(\omega(\bar{x}-))}{R}$$

the time at which the minimal backward characteristic from $(\bar{x}, T)$ impacts the interface. If $\omega(\bar{x}-) \neq B$, it must be $u_l(t_{\bar{x},-}) = \pi_{l,+}^r(\omega(\bar{x}-)) = u_l^*(t_{\bar{x},-})$ and the line

$$\theta_{\bar{x},-}(t) = \begin{cases} \bar{x} - f_r'(\omega(\bar{x}-)), & t_{\bar{x},-} < t < T, \\ -(t_{\bar{x},-} - t) f_l' \circ \pi_{l,+}^r(\omega(\bar{x}-)) & 0 < t < t_{\bar{x},-}, \end{cases}$$

is such that

$$\theta_{\bar{x},-} \in \mathcal{C}(u^*,\bar{x}) \cap \mathcal{C}(u,\bar{x})$$

It remains to prove the Lemma in the case $\omega(\bar{x}-) = B$. In this case, by the interface conditions, since the maximal backward characteristic from $(\bar{x},T)$ has slope smaller or equal than $f_r'(B)$, and by $f_r'(\omega(\bar{x}+)) > x/T$ it impacts the interface in positive time, we deduce that ω is continuous in $\bar{x}$ (and $\omega(\bar{x}\pm) = B$). Then we rename $t_{\bar{x},-} = t_{\bar{x}}$. We have two cases:

Case 1. $\omega \in \mathcal{A}_1(T)^{AB}$. By the observations above it must be $0 < \bar{x} < R$. By definition of u^* we have $u_l^*(t_{\bar{x}}) = \bar{A}$. Therefore, if $u_l(t_{\bar{x}}) = \bar{A}$, we are done because as before the line $\theta_{\bar{x},-}$ is in $\mathcal{C}(u^*,\bar{x}) \cap \mathcal{C}(u,\bar{x})$. Otherwise, $u_l(t_{\bar{x}}) = A$ (and of course $u_r(t_{\bar{x}}) = B$). Let

$$\bar{t} = \inf\{t \leq t_{\bar{x}} \mid (u_l(s), u_r(s)) = (A,B)\ \forall\, s \in (t, t_{\bar{x}})\}$$

We have two sub-cases:

1.a. $\bar{t} > t_{R,-}$. Then a shock for u lying in quadrant II starts from $(0,\bar{t})$ with zero slope and $u_l(\bar{t}) = \bar{A}$. We claim that $\omega(x) = B$ for $x \in (\bar{x}, (T-\bar{t})f_r'(B))$. In fact, assume that there is a point $x \in (\bar{x}, (T-\bar{t})f_r'(B))$ such that $f_r'(\omega(x-)) > B$. Then it must be $t_{x,-} \in (\bar{t}, t_{\bar{x}})$ and $u_r(t_{x,-}+) > B$: a contradiction with the definition of $\bar{t}$. Then the line

$$\alpha(t) = \begin{cases} \bar{x} - (T-t)f_r'(B), & t_{\bar{x}} < t < T, \\ 0, & \bar{t} < t < t_{\bar{x}}, \\ -(\bar{t}-t)f_l'(\bar{A}), & 0 < t < \bar{t}, \end{cases}$$

is in $\mathcal{C}(u^*,\bar{x}) \cap \mathcal{C}(u,\bar{x})$.

1.b. $\bar{t} \leq t_{R,-}$. This implies with same arguments above, that $\omega(x) = B$ for all $x \in (\bar{x}, R)$, and then $\omega(R-) = B$. If $\bar{t} = 0$, the line

$$\alpha(t) = \begin{cases} \bar{x} - (T-t)f_r'(B), & t_{\bar{x}} < t < T, \\ 0, & 0 < t < t_{\bar{x}}, \end{cases}$$

is in $\mathcal{C}(u^*,\bar{x}) \cap \mathcal{C}(u,\bar{x})$. If instead $\bar{t} > 0$, the solution u has a shock starting with zero slope from $(0,\bar{t})$, and either the shock lies in the I or in the II quadrant. If the shock lies in the second quadrant, the curve

$$\alpha(t)=\begin{cases}\bar{x}-(T-t)f_r'(B), & t_{\bar{x}}<t<T,\\ 0, & \bar{t}<t<t_{\bar{x}},\\ -(\bar{t}-t)f_l'(\bar{A}), & 0<t<\bar{t},\end{cases}$$

is in $\mathcal{C}(u^*,\bar{x})\cap\mathcal{C}(u,\bar{x})$. If the shock lies in the first quadrant, and $\bar{t}\le t_{\bar{u}_R}$, the curve

$$\alpha(t)=\begin{cases}\bar{x}-(T-t)f_r'(B), & t_{\bar{x}}<t<T,\\ 0, & \bar{t}<t<t_{\bar{x}},\\ -(\bar{t}-t)f_r'(\bar{B}), & 0<t<\bar{t},\end{cases}$$

is in $\mathcal{C}(u^*,\bar{x})\cap\mathcal{C}(u,\bar{x})$. We now prove that the last case cannot happen, namely, the case in which $\bar{t}\in(t_{\bar{u}_r},t_{R,-})$ and the shock starting from $(0,\bar{t})$ lies in the first quadrant. If this is the case, call $y(t)$ such shock. It must have left state equal to B and right state strictly bigger than the right state of the shock $x_{\bar{u}_R}(t)$, for each $t\ge\bar{t}$. Then $y(t)<x_{\bar{u}_R}(t)$ for $t\in(\bar{t},T)$ but this is a contradiction since it must happen that $y(T)=x_{\bar{u}_R}(T)=R$.

Case 2. $\omega\in\mathcal{A}_3^{AB}(T)$ and $(\omega(0-),\omega(0+))=(A,B)$. Again it must be $0<\bar{x}<R$, and let

$$\bar{t}=\inf\{t\le t_{\bar{x}}\mid(u_l(s),u_r(s))=(A,B)\ \forall\, s\in(t,t_{\bar{x}})\}$$

We have some different cases:

2.a . $t_{\bar{u}_R}\le t_L\le t_R$. Of course we also have $t_{\bar{u}_L}\le t_L\le t_R$. In this case $\omega(x)=A,\ x\in(L,0)$ and $\omega(x)=B,\ x\in(0,R)$. Therefore the traces of u^* satisfy

$$\begin{aligned} &u_l^*(t)=A, \quad t\in(t_{\bar{u}_L},T), \qquad u_l^*(t)=\bar{A}, \quad t\in(0,t_{\bar{u}_L}],\\ &u_r^*(t)=B, \quad t\in(t_{\bar{u}_R},T), \qquad u_r^*(t)=\bar{B}, \quad t\in(0,t_{\bar{u}_R}],\end{aligned} \tag{4.21}$$

We know that at $(0,\bar{t})$ a shock with zero slope emerges for the solution $\bar{u}$. If the shock emerges in the quadrant I, it must be $\bar{t}\le t_{\bar{u}_R}$, since, as above, $x_{\bar{u}_R}(t)$ is the minimal of all the possible shocks of solutions that yield ω with left state equal to B. Therefore in this case the curve

$$\alpha(t)=\begin{cases}\bar{x}-(T-t)f_r'(B), & t_{\bar{x}}<t<T,\\ 0, & \bar{t}<t<t_{\bar{x}},\\ -(\bar{t}-t)f_r'(\bar{B}), & 0<t<\bar{t},\end{cases}$$

is in $\mathcal{C}(u^*,\bar{x}) \cap \mathcal{C}(u,\bar{x})$. If instead the shock emerges in quadrant II, for the same, but specular, reason it must be $\bar{t} \le t_{\bar{u}_L}$ and the curve

$$\alpha(t) = \begin{cases} \bar{x} - (T-t)f_r'(B), & t_{\bar{x}} < t < T, \\ 0, & \bar{t} < t < t_{\bar{x}}, \\ -(\bar{t}-t)f_l'(\bar{A}), & 0 < t < \bar{t}, \end{cases}$$

is in $\mathcal{C}(u^*,\bar{x}) \cap \mathcal{C}(u,\bar{x})$.

2.b . $t_L \le t_{\bar{u}_R} \le t_R$. If $\bar{t} \le t_L$, we deduce that $\omega(x) = A,\ x \in (L,0)$ and $\omega(x) = B,\ x \in (0,R)$, and we conclude in the same way of Case 2.a. Otherwise, it must be $\bar{t} \in (t_L, t_{\bar{u}_R}]$. Then since the left trace is A for $t \in (\bar{t}, T)$ we deduce, as in Case 1, that $\omega(x) = A,\ x \in (T-\bar{t})f_l'(A)$. This means by definition of u^* that $u_r^*(\bar{t}) = \bar{B}$ and therefore the line

$$\alpha(t) = \begin{cases} \bar{x} - (T-t)f_r'(B), & t_{\bar{x}} < t < T, \\ 0, & \bar{t} < t < t_{\bar{x}}, \\ -(\bar{t}-t)f_r'(\bar{B}), & 0 < t < \bar{t}, \end{cases}$$

is in $\mathcal{C}(u^*,\bar{x}) \cap \mathcal{C}(u,\bar{x})$.

2.c,d . $t_{\bar{u}_L} \le t_R \le t_L$ (Case 2.c) or $t_R \le t_{\bar{u}_L} \le t_L$(Case 2.d). The result in this two cases follows in exactly the same way as in the previous two cases, therefore the proof is omitted.

□

4.3. Examples

Here we provide some examples and figures that hopefully will help to better understand how the sets $\mathcal{C}(u,x)$ behave and their structure. In order to do this, we use a relatively simple profile ω, but that in our opinion already captures the essence and the key points of Definition 4.4. In particular we let

$$\omega(x) = \begin{cases} v & x < L, \\ A & L < x < 0, \\ \bar{B} & 0 < x \end{cases} \tag{4.22}$$

with

$$f_l'(A) < L/T < f_l'(\bar{u}_l) < f_l'(v) \tag{4.23}$$

so that ω is attainable (see the previous chapter) but it is not attainable with a Lipschitz solution. The ”special” solution u^* is represented in Figure 4.3. In particular there is a compression wave that creates the shock at $(L, 0)$, a rarefaction wave that meets a shock with right state equal to A, and the left and right traces u_l^*, u_r^* are always equal to A and $\bar{B}$ respectively.

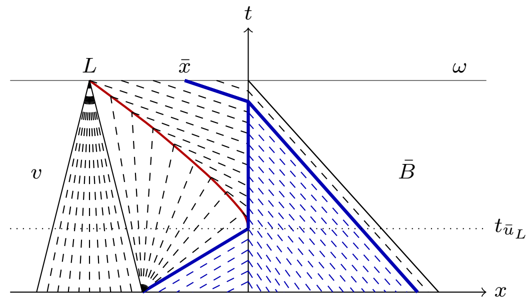

Figure 4.3: The solution u^*.

Now consider a point $\bar{x} \in (L, 0)$. There is a unique backward genuine characteristic with slope $f_l'(A)$ until it meets the interface. At that point we can prolong it on the other side with slope $f_r'(\bar{B})$, but another possible choice is to travel along the interface for some time and then go the right at any time, or go to the left at times $t \leq t_{\bar{u}_l}$ (the time at which the shock emerges from the interface). Therefore we will have distinct minimal and maximal lines in the set $\mathcal{C}(u^*, \bar{x})$, represented by the blue lines in Figure 4.3, while all the other blue dashed lines are the other elements in $\mathcal{C}(u^*, \bar{x})$.

If we consider other initial data $u_0 \in \mathcal{I}_T^{AB}\omega$, we produce different solutions u with different sets $\mathcal{C}(u, \bar{x})$ and $\mathcal{C}_0(u, \bar{x})$. Consider for example a solution like u^1 as in Figure 4.4. The red line at the left of the interface is a shock with left state v and right state A. Of course such a solution is admissible if and only if $L - T\lambda_l(v, A) \leq 0$. For simplicity assume this condition holds (it will be sufficient to take v big enough). In this case the set $C(u^1, \bar{x})$ is smaller since the left trace u_l is always equal to A.

In the case of a solution like u^2 (see Figure 4.5), it is easy to see that the set $\mathcal{C}_0(u^2, \bar{x})$ is not an interval. In fact, at time $t_{\bar{u}_L}$ the traces change their value due to two rarefaction waves arriving at the interface from both sides.

Finally, let's have a look at a solution u^3 for which $\max \mathcal{C}_0(u^*, \bar{x}) \neq \max \mathcal{C}_0(u^3, \bar{x})$ (see Figure 4.6). Here a shock emerges from a point of the interface and is reabsorbed by the interface itself after an interval of time.

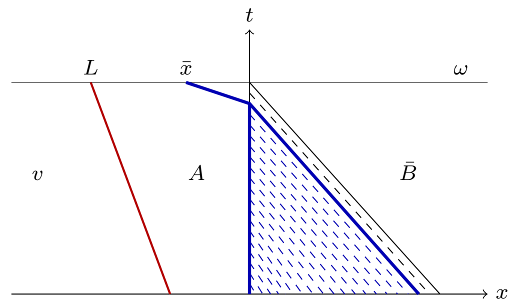

Figure 4.4: The solution u^1.

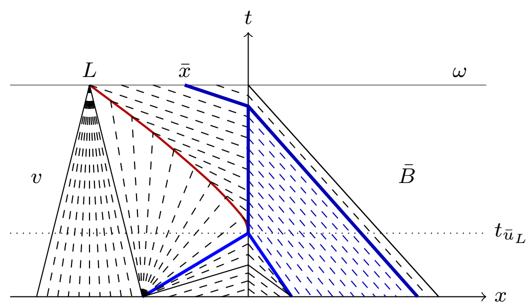

Figure 4.5: The solution u^2.

4.4. The set $\mathcal{I}_T^{AB}\omega$

In this section we prove a characterization of the set $\mathcal{I}_T^{AB}\omega$ in terms of some integral inequalities. The result is a generalization to the discontinuous flux setting of the corresponding results obtained in the convex case in [10], [19].

Let $\mathcal{X}(\omega)$ be the set of continuity points of ω, namely

$$\mathcal{X}(\omega) = \{x \in \mathbb{R} : \omega(x-) = \omega(x+)\} \tag{4.24}$$

The limits $\omega(x\pm)$ are well defined for every x if ω is assumed to be an element of the attainable set $\mathcal{A}^{AB}(T)$.

Theorem 4.9. *Let (A, B) be a connection. Then $\mathcal{S}_T^{AB} u_0 = \omega$ if and only if for every $\bar{x} \in \mathcal{X}(\omega)$ there exists $\bar{y} \in \mathcal{C}_0(u^*, \bar{x})$ such that*

$$\int_y^{\bar{y}} u_0(x)\,\mathrm{d}x \leq \int_y^{\bar{y}} u_0^*(x)\,\mathrm{d}x, \qquad \forall \quad y < \min \mathcal{C}_0(u^*, \bar{x}) \tag{4.25}$$

and

$$\int_{\bar y}^{y} u_0(x)\,\mathrm{d}x \geq \int_{\bar y}^{y} u_0^*(x)\,\mathrm{d}x, \qquad \forall \quad y > \max \mathcal{C}_0(u^*, \bar x) \tag{4.26}$$

Proof. We prove that the conditions (4.25), (4.26) are necessary: assume that for a solution u we have $u(\cdot, T) = \omega = u^*(\cdot, T)$ and $u = \mathcal{S}_{(\cdot)}^{AB} u_0(\cdot)$. Take $\bar x$ as in the statement of (4.25). Now, using Lemma 4.8, choose $\zeta_{\bar x} \in \mathcal{C}(u^*, \bar x) \cap \mathcal{C}(u, \bar x)$ and set $\bar y \doteq \zeta_{\bar x}(0)$. Choose any $y < \min \mathcal{C}_0(u^*, \bar x)$. From the structure of u^* (see property $(i)^*$ above), we know that there is some $x < \bar x$ such that $y = \zeta_x(0)$, $\zeta_x \in \mathcal{C}(u^*, x)$. From Lemma 4.6, we deduce that

$$\mathcal{F}(u, \zeta_x -) \geq \mathcal{F}(u^*, \zeta_x -) \tag{4.27}$$

Moreover, since $\zeta_{\bar x} \in \mathcal{C}(u^*, \bar x) \cap \mathcal{C}(u, \bar x)$, we have

$$\mathcal{F}(u^*, \zeta_{\bar x}) = \mathcal{F}(u, \zeta_{\bar x}) \tag{4.28}$$

Applying Lemma 4.1 one obtains

$$\int_x^{\bar x} \omega(x)\,\mathrm{d}x - \int_y^{\bar y} u_0^*(\xi)\,\mathrm{d}\xi = \mathcal{F}(u^*, \zeta_x -) - \mathcal{F}(u^*, \zeta_{\bar x}) \tag{4.29}$$

and, again, thanks to Lemma 4.1 and (4.27), (4.28),

$$\int_x^{\bar x} \omega(x)\,\mathrm{d}x - \int_y^{\bar y} u_0(\xi)\,\mathrm{d}\xi = \mathcal{F}(u, \zeta_x -) - \mathcal{F}(u, \zeta_{\bar x}) \geq \mathcal{F}(u^*, \zeta_x -) - \mathcal{F}(u^*, \zeta_{\bar x}) \tag{4.30}$$

Taking the difference of the two above equations one gets (4.25). The proof of (4.26) is entirely symmetric and therefore is omitted.

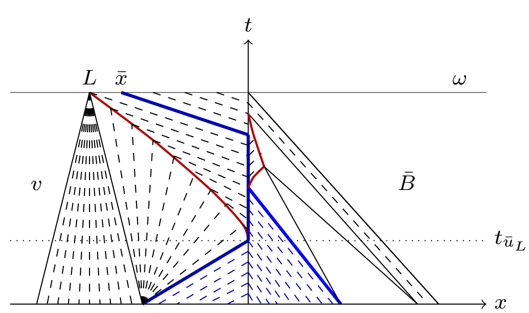

Figure 4.6: The solution u^3.

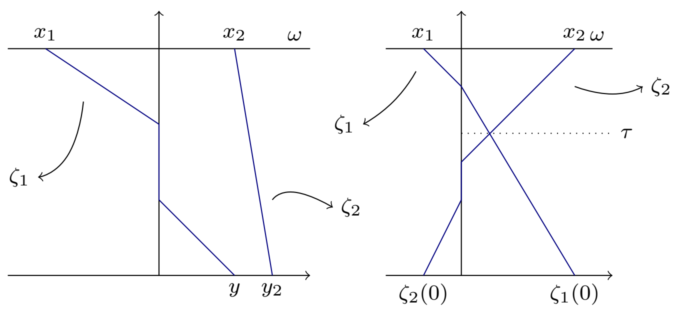

Figure 4.7: CASE 2: $\max \mathcal{C}_0(u, x_1) < \min \mathcal{C}_0(u^*, x_2)$ (left); CASE 1: $\max \mathcal{C}_0(u, x_1) \geq \min \mathcal{C}_0(u^*, x_2)$ (right)

Now we prove that if $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ satisfies (4.25), (4.26), then $\mathcal{S}_T^{AB} u_0 = \omega$. What we are going to prove is the following:

$$\int_{x_1}^{x_2} \omega(x) - \mathcal{S}_T u_0(x)\, \mathrm{d}x = 0, \quad \forall\, x_1 < x_2, \quad x_i \in \mathcal{X}(\omega),\ i = 1, 2 \tag{4.31}$$

By Lebesgue differentiation theorem this is enough to conclude that ω and $\mathcal{S}_T^{AB} u_0$ coincide as elements of $\mathbf{L}^\infty(\mathbb{R})$. We split the proof into two parts, in which we prove respectively the two opposite inequalities needed to obtain (4.31). We start with proving

$$\int_{x_1}^{x_2} \omega(x) - \mathcal{S}_T u_0(x)\, \mathrm{d}x \geq 0, \quad \forall\, x_1 < x_2, \quad x_i \in \mathcal{X}(\omega),\ i = 1, 2 \tag{4.32}$$

Take any two points $x_1 < x_2$. We have two cases

CASE 1. $\max \mathcal{C}_0(u, x_1) \geq \min \mathcal{C}_0(u^*, x_2)$. Then choose $\zeta_1 \in \mathcal{C}_0(u, x_1)$ and $\zeta_2 \in \mathcal{C}_0(u^*, x_2)$ such that $\zeta_2(0) \leq \zeta_1(0)$. By continuity there is a point $\tau \in [0, T)$ in which $\zeta_1(\tau) = \zeta_2(\tau)$. Apply Lemma 4.1 to the curves $\zeta_{i|[\tau,T]}$, $i = 1, 2$ and Lemma 4.6 to obtain

$$\int_{x_1}^{x_2} \omega(x)\, \mathrm{d}x = \mathcal{F}_\tau(u^*, \zeta_1-) - \mathcal{F}_\tau(u^*, \zeta_2+) \geq \mathcal{F}_\tau(u, \zeta_1-) - \mathcal{F}_\tau(u^*, \zeta_2+) \tag{4.33}$$

and

$$\int_{x_1}^{x_2} \mathcal{S}_T u_0(x)\, \mathrm{d}x = \mathcal{F}_\tau(u, \zeta_1-) - \mathcal{F}_\tau(u, \zeta_2+) \leq \mathcal{F}_\tau(u, \zeta_1-) - \mathcal{F}_\tau(u^*, \zeta_2+) \tag{4.34}$$

Taking the difference of the two above inequalities, one obtains (4.32).

CASE 2. $\max \mathcal{C}_0(u, x_1) < \min \mathcal{C}_0(u^*, x_2)$. Choose any $\zeta_1 \in \mathcal{C}_0(u, x_1)$, and set $y \doteq \zeta_1(0)$. Since $y < \min \mathcal{C}_0(u^*, x_2)$, the first condition of the Theorem gives us a $y_2 \in \mathcal{C}_0(u^*, x_2)$(and then a $\zeta_2 \in \mathcal{C}(u^*, x_2)$ such that $\zeta_2(0) = y_2$) such that

$$\int_y^{y_2} u_0(x)\,\mathrm{d}x \leq \int_y^{y_2} u_0^*(x)\,\mathrm{d}x \tag{4.35}$$

Apply Lemma 4.1 to the curves ζ_i, $i = 1, 2$ and Lemma 4.6 to obtain

$$\begin{aligned}\int_{x_1}^{x_2} \omega(x)\,\mathrm{d}x &= \mathcal{F}(u^*, \zeta_1-) - \mathcal{F}(u^*, \zeta_2+) + \int_y^{y_2} u_0^*(x)\,\mathrm{d}x \\ &\geq \mathcal{F}(u, \zeta_1-) - \mathcal{F}(u^*, \zeta_2+) + \int_y^{y_2} u_0^*(x)\,\mathrm{d}x \end{aligned}\tag{4.36}$$

and

$$\begin{aligned}\int_{x_1}^{x_2} \mathcal{S}_T u_0(x)\,\mathrm{d}x &= \mathcal{F}(u, \zeta_1-) - \mathcal{F}_\tau(u, \zeta_2+) + \int_y^{y_2} u_0(x)\,\mathrm{d}x \\ &\leq \mathcal{F}(u, \zeta_1-) - \mathcal{F}(u^*, \zeta_2+) + \int_y^{y_2} u_0(x)\,\mathrm{d}x \end{aligned}\tag{4.37}$$

Taking the difference of the two above equations and using (4.35), one obtains

$$\int_{x_1}^{x_2} \omega(x)\,\mathrm{d}x - \int_{x_1}^{x_2} \mathcal{S}_T u_0(x)\,\mathrm{d}x \geq \int_y^{y_2} u_0^*(x)\,\mathrm{d}x - \int_y^{y_2} u_0(x)\,\mathrm{d}x \geq 0 \tag{4.38}$$

and this proves (4.32) also in CASE 2.

The proof of the opposite inequality is entirely symmetric and is accordingly omitted. □

Example 4.10. Let us apply the Theorem for a couple of simple profiles ω. Let's start with

$$\omega(x) = \begin{cases} v_1 & x < 0, \\ v_2 & x > 0, \end{cases} \tag{4.39}$$

with $v_1 < A$ and $v_2 = \pi^l_{r,-}(v_1)$. The solution u^* in this case is

$$u^*(x,t) = \begin{cases} v_1 & x < 0, \\ v_2 & x > 0, \end{cases} \tag{4.40}$$

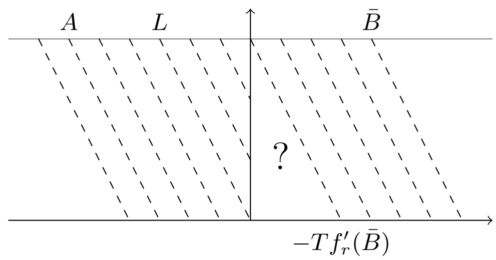

Figure 4.8: The second profile ω considered in Example (4.10)

Of course, for every $x > 0$, $\mathcal{C}(u^*, x)$ is a singleton and it's unique element is precisely the unique backward characteristic with slope $f_r' \circ \pi_{r,-}^l(v_1) = f_r'(v_2)$. The same holds for $x < L$. Thanks to the fact that $v_1 < A$, we can say that for every $L < x < 0$

$$\mathcal{C}(u^*, x) = \{\theta_{x,+} = \theta_{x,-}\}$$

so that the initial data in this case is uniquely determined.

Now we take $v_1 = A$, $v_2 = v_2 = \pi_{r,-}^l(v_1) = \bar{B}$ (see Figure 4.8). In this particular case, the initial datum is *not* uniquely determined. With the same arguments as above, we know that the initial datum is uniquely determined in the regions $x < 0$ and $x > -Tf_r'(\bar{B})$. This is true essentially because $\mathcal{C}(u^*, x)$ is a singleton and coincides with the unique genuine backward characteristic for every $x < L$ and $x > 0$. Now we want to find out what $\mathcal{C}_0(u^*, x)$ is for $x \in (L, 0)$. Fix such an x. It is clear that

$$\mathcal{C}_0(u^*, x) = [0, \theta_x(0)]$$

with θ_x being the polygonal line that changes slope at the interface, with slopes A and $\bar{B}$ at the left and at the right. Consider some $\bar{x} \in (L, 0)$ and any $y \notin \mathcal{C}(u^*, \bar{x})$. Then the statement of the theorem in this case is trivial, because one can choose always $\bar{y} = 0$, so that no additional conditions are added. This is true because, (if $y < 0$ is trivial, therefore assume $y > \theta_x(0)$) one can always take a sequence of points $x_n \uparrow L$ and apply condition (4.25) with $\bar{x}$ equal to those points (and y as y) in the limit obtaining the needed inequality. Then, what the Theorem says is that $u_0 \in L^\infty(\mathbb{R})$ is such that $\mathcal{S}_T u_0$ if and only if

$$u_0(x) = \begin{cases} A & x < 0, \\ v_0 & 0 < x < -Tf_r'(\bar{B}), \\ \bar{B} & -Tf_r'(\bar{B}) < x, \end{cases} \tag{4.41}$$

with $v_0 \in L^\infty(0, -Tf_r'(\bar{B}))$ such that

$$\int_y^{-Tf_r'(\bar{B})} v_0(x)\,\mathrm{d}x \leq \bar{B}(y + Tf_r'(\bar{B})), \quad y \in (0, Tf_r'(\bar{B})), \tag{4.42}$$

$$\int_0^y v_0(x)\,\mathrm{d}x \geq \bar{B}y, \quad y \in (0, Tf_r'(\bar{B})), \tag{4.43}$$

Notice that if $B = \bar{B} = \theta_r$, the initial data is uniquely identified also in this second case.

4.5. Geometrical and topological properties of $\mathcal{I}_T^{AB}(\omega)$

From Theorem 4.9 we can deduce some geometrical and topological properties of the set $\mathcal{I}_T^{AB}\omega$, that we collect in the following theorem.

Theorem 4.11. *Let $\omega \in \mathcal{A}^{AB}(T)$. Then, with respect to the $\mathbf{L}^1_{\mathrm{loc}}$ topology, the following holds:*

(i) For every $M > 0$, the set $\mathcal{I}_T^{AB}\omega \cap \{u_0 : \|u_0\|_{\mathbf{L}^\infty} \leq M\}$ is closed;

(ii) $\mathcal{I}_T^{AB}\omega$ is an F_σ set and has empty interior;

(iii) $\mathcal{I}_T^{AB}\omega$ reduces to a singleton if and only if $|\mathcal{C}_0(u^*, x)| = 1$ (see Definition 4.4) for every $x \in \mathbb{R}$. In particular only if (but not "if") ω is continuous outside the origin.

(iv) The set $\mathcal{I}_T^{AB}\omega$ is an affine cone having u_0^* as its vertex and unique extremal point.

(v) If $|\mathcal{C}_0(u^*, x)| = 1$ for every $x \in \mathcal{X}(\omega)$, the set $\mathcal{I}_T^{AB}\omega$ is convex; in particular, it is a convex affine cone having u_0^* as its vertex and unique extremal points. Otherwise, it could be non-convex.

Remark 4.12. It is noteworthy that in the discontinuous flux setting the convexity properties of $\mathcal{I}_T^{AB}\omega$ are, in general, lost. In fact, in the convex flux case, it was proved in [10], that the set of initial data is always convex. The property of being a cone, instead, remains true also in the discontinuous flux case.

Proof. The proof of (i) follows immediately from the semigroup properties of $\mathcal{S}_T^{AB}$. The fact that $\mathcal{I}_T^{AB}\omega$ is an F_σ set follows from (i).

We prove that $\mathcal{I}_T^{AB}\omega$ has empty interior. Take two points $0 < x_1 < x_2$ in $\mathcal{X}(\omega)$ such that the unique backward (genuine) characteristics, respectively

$\theta_{x_1}, \theta_{x_2}$, from x_1, x_2, do not interact with the interface. We can also choose them in such a way that $\theta_{x_1}(0) < \theta_{x_2}(0)$. Notice that in this case $\mathcal{C}_0(u^*, x_i) = \{\theta_{x_i}(0)\}$. Apply condition (4.25) with $\bar{x} = x_2, y = \theta_{x_1}(0)$ to find that any element in $\mathcal{I}_T^{AB}\omega$ satisfies

$$\int_{\theta_{x_1}(0)}^{\theta_{x_2}(0)} u_0(x)\,\mathrm{d}x \leq \int_{\theta_{x_1}(0)}^{\theta_{x_2}(0)} u_0^*(x)\,\mathrm{d}x \tag{4.44}$$

Now apply condition (4.26) with $\bar{x} = x_1, y = \theta_{x_2}(0)$ to find that any element in $\mathcal{I}_T^{AB}\omega$ satisfies

$$\int_{\theta_{x_1}(0)}^{\theta_{x_2}(0)} u_0(x)\,\mathrm{d}x \geq \int_{\theta_{x_1}(0)}^{\theta_{x_2}(0)} u_0^*(x)\,\mathrm{d}x \tag{4.45}$$

Inequalities (4.44), (4.5) together imply that every element $u_0 \in \mathcal{I}_T^{AB}\omega$ satisfies

$$\int_{\theta_{x_1}(0)}^{\theta_{x_2}(0)} u_0(x)\,\mathrm{d}x = \int_{\theta_{x_1}(0)}^{\theta_{x_2}(0)} u_0^*(x)\,\mathrm{d}x \tag{4.46}$$

From condition (4.46) one clearly sees that the set $\mathcal{I}_T^{AB}\omega$ has always empty interior.

Let's prove (iii). First assume that $|\mathcal{C}_0(u^*, x)| = 1$ for every $x \in \mathbb{R}$. Notice that this immediately implies that ω is continuous outside the origin. Take any two points $y_1 < y_2 \in \mathbb{R}$. By the property $(i)^*$ of the solution u^* there exist $\zeta_2 \in \mathcal{C}(u^*, x_2)$ such that $\zeta_2(0) = y_2$, and by our assumption x_2 is a continuity point of ω (or zero) and $\mathcal{C}(u^*, x_2)$ is a singleton. Without loss of generality we can assume that neither y_1 nor y_2 are in $\mathcal{C}_0(u^*, 0)$ (since it is a singleton by assumption). Then, by (4.25),

$$\int_{y_1}^{y_2} u_0(x)\,\mathrm{d}x \leq \int_{y_1}^{y_2} u_0^*(x)\,\mathrm{d}x \tag{4.47}$$

Now we exchange the role of y_1 and y_2. In particular, with the symmetrical argument (this time using (4.26)) we find the opposite inequality

$$\int_{y_1}^{y_2} u_0(x)\,\mathrm{d}x \geq \int_{y_1}^{y_2} u_0^*(x)\,\mathrm{d}x \tag{4.48}$$

We discovered that any initial data u_0 that leads to ω satisfies

$$\int_{y_1}^{y_2} u_0(x)\,\mathrm{d}x = \int_{y_1}^{y_2} u_0^*(x)\,\mathrm{d}x, \quad \text{a.e. } y_1 < y_2 \in \mathbb{R} \tag{4.49}$$

This uniquely identifies u_0 as an element of $\mathbf{L}^\infty(\mathbb{R})$, therefore $\mathcal{I}_T^{AB}\omega$ is a singleton.

Conversely, assume that $\mathcal{I}_T^{AB}\omega$ is a singleton. By contradiction assume there is some $x \in \mathbb{R}$ such that $|\mathcal{C}_0(u^*, x)| \neq 1$. If the elements in $\mathcal{C}(u^*, x)$ do not interact with the interface, this means that x is a discontinuity point of ω and it is clear how to build another element in $\mathcal{I}_T^{AB}\omega$ following the arguments used in the convex flux case in [10]. Then assume that the elements in $\mathcal{C}(u^*, x)$ interact with the interface. Without loss of generality assume $x < 0$. It might happen that x is a discontinuity point of ω with both the minimal and maximal characteristic that impact the interface. Then, instead of u^*, that creates the shock at (x, T) with a compression wave, we can use single shock with left state $\omega(x-)$ and right state $\omega(x+)$ at the left of the interface, and the corresponding values $\pi^l_{r,-}\omega(x\pm)$ at the right of the interface. Therefore $\mathcal{I}_T^{AB}\omega$ is not a singleton. If instead only the maximal characteristic impacts the interface in positive time, if u^* is made of a compression wave we can build another solution with a shock as above. If instead u^* has the shock that starts with slope zero from the interface, we can always modify the traces of the solution in a similar way to what done in point (v) below for the solution u_0^2 (see Figure 4.10). The last case to analyze is when ω is continuous at x. Since $\mathcal{C}(u^*, x)$ is not a singleton it must hold $\omega(x) = A$. There must be a positive δ such that $\omega(y) = A, y \in (x - \delta, x)$: this holds because, since $\mathcal{C}(u^*, x)$ is not a singleton, there must be a positive ε such that $u_l(t) = A$ for $t \in (t_x - \varepsilon, t_x)$. Then we can modify u^* to obtain a different admissible solution creating a shock with slope zero at the point $t_x - \varepsilon$, lying in quadrant I, that is re-absorbed by the interface at time t_x (as in Figure 4.6). If $x = 0$, with similar arguments one can prove that the same holds. The proof of (iii) is concluded.

To prove property (iv), notice that for every $u_0 \in \mathcal{I}_T^{AB}\omega$ and $\lambda \geq 0$ it holds $u_0^* + \lambda(u_0 - u_0^*) \in \mathcal{I}_T^{AB}\omega$. To see this, it's sufficient to prove that (4.25), (4.26) of Theorem 4.9 hold. Take $\bar{x} \in \mathcal{X}(\omega)$ and $y < \inf \mathcal{C}_0(u^*, \bar{x})$. Since $u_0 \in \mathcal{I}_T^{AB}\omega$ there is some $\bar{y}$ such that (4.25) holds. Then one has

$$\int_y^{\bar{y}} u_0^* + \lambda(u_0 - u_0^*)\,\mathrm{d}x \leq \int_y^{\bar{y}} u_0^* + \lambda(u_0^* - u_0^*)\,\mathrm{d}x = \int_y^{\bar{y}} u_0^*\,\mathrm{d}x \tag{4.50}$$

as wanted. The proof that also (4.26) holds is entirely symmetric.

Now we prove that u_0^* is an extremal point. Take any $\bar{x} \in \mathcal{X}(\omega)$ such that $\mathcal{C}_0(u^*, \bar{x})$ is a singleton and call its unique element $\bar{y}$. Assume by contradiction that there exists a $\lambda \in (0, 1)$ such that $u_0^* = \lambda u_0^1 + (1 - \lambda)u_0^2$ with

$u_0^* \neq u_0^i \in \mathcal{I}_T^{AB}\omega$, $i = 1, 2$. In particular it holds

$$\lambda \int_{\bar{y}}^{y} u_0^1 \,\mathrm{d}x + (1-\lambda) \int_{\bar{y}}^{y} u_0^2 \,\mathrm{d}x = \int_{\bar{y}}^{y} u_0^* \,\mathrm{d}x, \qquad \forall\, y \in \mathbb{R} \tag{4.51}$$

Then, since u_0^1, u_0^2 are different from u_0^*, there must be some $y \in \mathbb{R}$ such that

$$\int_{\bar{y}}^{y} u_0^1 \,\mathrm{d}x \neq \int_{\bar{y}}^{y} u_0^* \,\mathrm{d}x \neq \int_{\bar{y}}^{y} u_0^2 \,\mathrm{d}x \tag{4.52}$$

Assume that $y > \bar{y}$. Then by condition (4.26) of Theorem 4.9,

$$\int_{\bar{y}}^{y} u_0^1 \,\mathrm{d}x > \int_{\bar{y}}^{y} u_0^* \,\mathrm{d}x < \int_{\bar{y}}^{y} u_0^2 \,\mathrm{d}x \tag{4.53}$$

but this is in contradiction with (4.51). If instead $y < \bar{y}$, by condition (4.25) of Theorem 4.9,

$$\int_{\bar{y}}^{y} u_0^1 \,\mathrm{d}x < \int_{\bar{y}}^{y} u_0^* \,\mathrm{d}x > \int_{\bar{y}}^{y} u_0^2 \,\mathrm{d}x \tag{4.54}$$

that is, again, a contradiction with (4.51). This proves that u_0^* is an extremal point (and of course it is unique).

Finally, let's prove (v). Let $u_0^1, u_0^2 \in \mathcal{I}_T^{AB}\omega$ and $\theta \in (0,1)$. Let $\bar{x} \in \mathcal{X}(\omega)$ and $y < \inf \mathcal{C}_0(u^*, \bar{x})$. By hypothesis there exist $\bar{y}_1, \bar{y}_2 \in \mathcal{C}_0(u^*, \bar{x})$ such that

$$\int_{y}^{\bar{y}_1} u_0^1(x) \,\mathrm{d}x \leq \int_{y}^{\bar{y}_1} u_0^*(x) \,\mathrm{d}x, \qquad \int_{y}^{\bar{y}_2} u_0^2(x) \,\mathrm{d}x \leq \int_{y}^{\bar{y}_2} u_0^*(x) \,\mathrm{d}x \tag{4.55}$$

At this point (and only at this point) we use that $\mathcal{C}_0(u^*, \bar{x})$ is a singleton, deducing that $\bar{y}_1 = \bar{y}_2$. Then, for $\bar{y} = \bar{y}_1 = \bar{y}_2$, using (4.55),

$$\int_{y}^{\bar{y}} \theta u_0^1(x) + (1-\theta) u_0^2(x) \,\mathrm{d}x \leq \int_{y}^{\bar{y}} u^*(x) \,\mathrm{d}x \tag{4.56}$$

so that $\theta u_0^1 + (1-\theta) u_0^2$ satisfies (4.25). The proof that also (4.26) holds is entirely similar and is accordingly omitted.

Now we give an example in which the set $\mathcal{I}_T^{AB}\omega$ is not convex. We assume $f_l, f_r = u^2/2$. Let

$$\omega(x) = \begin{cases} \bar{A} & x \leq L, \\ A & x \in (L, 0), \\ v & x > 0, \end{cases} \tag{4.57}$$

with $v < \bar{B}$ negative enough. The strategy is to find two initial data $u_0^1, u_0^2 \in \mathcal{I}_T^{AB}\omega$ and show that for some $\theta \in (0,1)$, $\theta u_0^1 + (1-\theta) u_0^2 \notin \mathcal{I}_T^{AB}\omega$. We know, with some calculations, that $\bar{u}_L = -A - 2\sqrt{A\frac{L}{T}}$. In order to deal with easier calculations, we choose $T = 1$ and A, L such that $\bar{u}_L = 0$. One finds that in this case it must be $A = 4L$. With this choice, it also holds $\bar{A} = -4L = B$ and $\bar{B} = 4L = A$. We recall that for this profile ω, the "special" initial data u_0^*, that produces the solution u^*, has the form (see Figure 4.9)

$$u_0^*(x) = \begin{cases} \bar{A} & x < 5L, \\ L - x & 5L < x < L, \\ \bar{A} & L < x < 0, \\ \bar{B} & 0 < x < -4L, \\ -x & -4L < x < -v, \\ v & -v < x, \end{cases} \tag{4.58}$$

Now we define $u_0^1(x)$ as (see Figure 4.11)

$$u_0^1(x) = \begin{cases} \bar{A} & x < L, \\ A & L < x < 0, \\ B & 0 < x < -\lambda(B, v), \\ v & -\lambda(B, v) < x, \end{cases} \tag{4.59}$$

We choose v negative enough in order to have $-\lambda(B, v) > -4L$. Finally, let, with $\gamma > 1$ (see Figure 4.10),

$$u_0^2(x) = \begin{cases} \bar{A} & x < 5L, \\ L - x & 5L < x < L, \\ \gamma\bar{A} & L < x < 0, \\ \gamma\bar{B} & 0 < x < -L, \\ \bar{B} & -L < x < -4L, \\ -x & -4L < x < -v, \\ v & -v < x, \end{cases} \tag{4.60}$$

It's easy to see that $u_0^i \in \mathcal{I}_T^{AB}\omega$, $i = 1, 2$ (see Figures 4.9, 4.10, 4.11).

Fix any $\theta \in (0, 1)$ and let $u_0^\theta \doteq \theta u_0^1 + (1 - \theta)u_0^2$. We claim that if $u_0^\theta \in \mathcal{I}_T^{AB}\omega$ then there is some $\bar{y} \in [L, -3L]$ such that (4.25) holds with $y = 5L$. In fact, consider a sequence of points $\bar{x}_n \downarrow L$. Since they are points of continuity for ω, for each n there exists $\bar{y}_n \in \mathcal{C}_0(u^*, \bar{x}_n) = [L, \bar{x}_n - 4L]$ such that (4.25) holds with $y = 5L$ and $\bar{y} = \bar{y}_n$. Since the sequence y_n is bounded we can extract a converging subsequence, that we still call y_n.

Clearly the limit point (call it $\bar{y}$) of the sequence must be in $[L, -3L]$ and for this $\bar{y}$ it holds

$$\int_{5L}^{\bar{y}} u_0^{\theta}(x)\,\mathrm{d}x \leq \int_{5L}^{\bar{y}} u_0^{*}(x)\,\mathrm{d}x \tag{4.61}$$

Now we assume that $u_0^{\theta} \in \mathcal{I}_T^{AB}\omega$ and we see that this is in contradiction with what we have just proved. In particular we prove that

$$\int_{5L}^{\bar{y}} u_0^{\theta}(x) - u_0^{*}(x)\,\mathrm{d}x > 0, \quad \forall \bar{y} \in [L, -3L] \tag{4.62}$$

First assume that $\bar{y} \in [L, 0]$. With some easy calculations we find

$$\int_{5L}^{\bar{y}} u_0^{1}(x)\,\mathrm{d}x = 12L^2 + 4L\bar{y}, \quad \int_{5L}^{\bar{y}} u_0^{2}(x)\,\mathrm{d}x = 8L^2 - 4L\gamma(\bar{y} - L) \tag{4.63}$$

and

$$\int_{5L}^{\bar{y}} u_0^{*}(x)\,\mathrm{d}x = 8L^2 - 4L(\bar{y} - L) \tag{4.64}$$

so that for every $\bar{y} \in [L, 0]$,

$$\int_{5L}^{\bar{y}} u_0^{\theta} - u_0^{*}\,\mathrm{d}x = \theta 8L\bar{y} + (1-\theta)[4L(L-\bar{y})(\gamma - 1)] > 0. \tag{4.65}$$

Analogously, for every $\bar{y} \in [0, -L]$,

$$\int_{5L}^{\bar{y}} u_0^{\theta} - u_0^{*}\,\mathrm{d}x = \theta(-8L\bar{y}) + (1-\theta)[4L(L-\bar{y})(\gamma - 1)] > 0. \tag{4.66}$$

Finally, if $\bar{y} \in [-L, -3L]$,

$$\int_{5L}^{\bar{y}} u_0^{\theta} - u_0^{*}\,\mathrm{d}x = \theta[8L^2 - 8L(\bar{y} + L)] > 0. \tag{4.67}$$

and this is a contradiction with the fact that $u_0^{\theta} \in \mathcal{I}_T^{AB}\omega$, proving that $\mathcal{I}_T^{AB}\omega$ is not, in general, convex.

□

Remark 4.13. If for some $\bar{x}$ the set $\mathcal{C}_0(u^*, \bar{x})$ is not a singleton, nothing can be said about convexity of the set $\mathcal{I}_T^{AB}\omega$. In fact, it could also be convex. An easy example in which $\mathcal{I}_T^{AB}\omega$ is convex also if $\mathcal{C}(u^*, x)$ is not always a singleton, $x \in \mathcal{X}(\omega)$, is the second profile considered in Example 4.10. It is easy to see that in this case the set of initial data is convex because it is essentially characterized by conditions (4.42), (4.43), that clearly define a convex set by linearity of the integral.

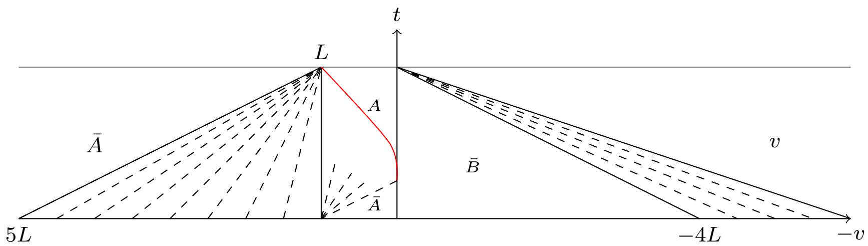

Figure 4.9: The solution produced by the initial datum u_0^*.

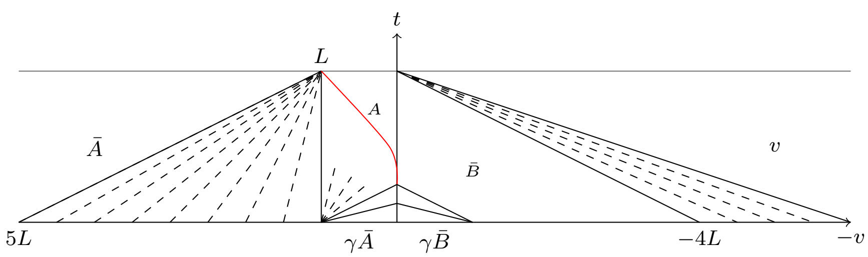

Figure 4.10: The solution produced by the initial datum u_0^2.

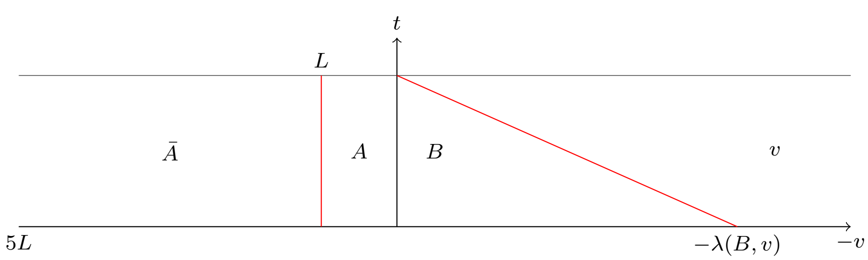

Figure 4.11: The solution produced by the initial datum u_0^1.

4.6. Final remarks

We want to remark another major structural difference between the set $\mathcal{I}_T^{AB}\omega$, of the discontinuous case, and the set $\mathcal{I}_T\omega$ of the classical convex case, that helps to understand from an intuitive point of view why the set $\mathcal{I}_T^{AB}\omega$ might be non-convex.

Let's consider, for a convex flux, a profile ω, whose set of discontinuity points is $\{x_i\}_{i\in\mathbb{N}}$. For each x_i consider the piece of initial data between the endpoints of the minimal/maximal backward characteristics from x_i, that is $(\theta_{x_i,-}(0), \theta_{x_i,+}(0)) = I_i$. The conditions that ensure that u_0 leads to ω at time T can be viewed as acting separately on each interval I_i, that is: what the initial data is in I_i does not influence what the initial data can be in I_j, $i \neq j$. In other words, with $I_T^i\omega$ being the set

$$\mathcal{I}_T^i\omega = \{u_0|_{I_i} : u_0 \in \mathcal{I}_T\omega\} \tag{4.68}$$

we can write

$$\mathcal{I}_T\omega \cong \prod_i \mathcal{I}_T^i\omega \tag{4.69}$$

Since each $\mathcal{I}_T^i\omega$ is clearly convex by the characterization of [10], and since products of convex sets is again convex, one deduces the convexity of $\mathcal{I}_T\omega$.

In the discontinuous case this is not anymore true. In fact, for some, say, $x < 0$ such that $\theta_{x,+}(0) > 0$ and $\theta_{x,-}(0) < 0$, there might be interaction between what the solution is in the region between $\theta_{x,-}(t)$ and $\theta_{x,+}(t)$, and the solution at the right of $\theta_{x,+}$, in the sense that they are not independent blocks that can be glued as one likes.

Of course it is the fact that the sets $\mathcal{C}_0(u^*, \bar{x}) \cap \mathcal{C}_0(u_0^1, \bar{x})$ and $\mathcal{C}_0(u^*, \bar{x}) \cap \mathcal{C}_0(u_0^2, \bar{x})$ are disjoint that allows us to prove that a convex combination of u_0^1, u_0^2 is not in the set of initial data that lead to ω. Indeed if for every fixed $\bar{x}$, we can always choose the same $\bar{y}$ for every $u_0 \in \mathcal{I}_T^{AB}\omega$, we could prove that the set $\mathcal{I}_T^{AB}\omega$ is convex, using the same proof of point (v) of Theorem 4.11. In our case (of the proof of point (v) of Theorem 4.11) it's easy to see that for $\bar{x} \in (L, 0)$ (see Figures 4.9, 4.10, 4.11)

$$C_0(v, \bar{x}) = \begin{cases} [L, \bar{x} - 4L] & \text{if} \quad v = u_0^*, \\ \{0\} & \text{if} \quad v = u_0^1, \\ \{L\} \cup [-L, \bar{x} - 4L] & \text{if} \quad v = u_0^2 \end{cases} \tag{4.70}$$

so that it's not possible to choose "a priori" a $\bar{y}$ that works for both u_0^1 and u_0^2.

The sufficient condition for the convexity of Theorem 4.9, that is, the cardinality of $\mathcal{C}_0(u^*, \bar{x})$ is 1 for every $\bar{x} \in \mathcal{X}(\omega)$, that is a pretty strong condition, can be refined to a weaker condition as follows. Let

$$\tilde{\mathcal{X}}(\omega) \doteq \mathcal{X}(\omega) \cap \{x : |\mathcal{C}_0(u^*, x)| = 1\} \tag{4.71}$$

Then the set $\mathcal{I}_T^{AB}\omega$ is convex if

$$\mathcal{C}_0(u^*, \bar{x}) \cap \mathrm{cl}(\tilde{\mathcal{X}}(\omega)) \neq 0, \qquad \forall \bar{x} \in \mathcal{X}(\omega) \tag{4.72}$$

In fact, notice the following. For each initial datum $u_0 \in \mathcal{I}_T^{AB}\omega$, and for each $\bar{x}$, Theorem 4.9 gives us a $\bar{y}$ that works when plugged into (4.25), (4.26). The possibility to prove that $\mathcal{I}_T^{AB}\omega$ is convex with same proof of point (v) of Theorem 4.11, depends on the ability to choose $\bar{y}$ depending only on $\bar{x}$ and independently from the particular initial datum u_0. Of course whenever $\mathcal{C}_0(u^*, \bar{x})$ is a singleton the choice of $\bar{y}$ is forced to be the only element of $\mathcal{C}_0(u^*, \bar{x})$, so that it depends only on $\bar{x}$. The problem is that when $\mathcal{C}_0(u^*, \bar{x})$ is not a singleton, different initial data $u_0 \in \mathcal{I}_T^{AB}\omega$ might require different $\bar{y} \in \mathcal{C}_0(u^*, \bar{x})$. However, for such $\bar{x}$, if (4.72) holds, we are always able to choose $\bar{y}(\bar{x}) \in \mathcal{C}_0(u^*, \bar{x}) \cap \mathrm{cl}(\tilde{\mathcal{X}}(\omega))$ independently from the initial datum u_0. In fact, in order to show that (4.25), (4.26) hold for $\bar{x}$ and $\bar{y}(\bar{x})$, it is sufficient to write (4.25), (4.26) for a sequence of points $\bar{x}_n$, $\{\bar{y}_n\} = \mathcal{C}_0(u^*, \bar{x}_n)$, with $\bar{x}_n \in \tilde{\mathcal{X}}(\omega)$ and $\bar{x}_n \to \bar{x}$ for $n \to \infty$. Since $\mathcal{C}_0(u^*, \bar{x})$ is closed, the limit point of the sequence $\{y_n\}_{n \in \mathbb{N}}$ will be in $\mathcal{C}_0(u^*, \bar{x})$, and the proof is concluded.

It's easy to check that the profile in Example 4.10 satisfies property (4.72), and in fact the set of initial data is convex. At this point one might suspect that this is also a necessary condition for convexity, but this is not so clear, and the proof seems complicated.

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