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Infinite dimensional symplectic reduction and the dynamics of a rigid body moving in a perfect fluid

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Contents

Introduction

The purpose of this thesis is to give an introduction to the theory of infinite dimensional Hamiltonian systems and infinite dimensional symplectic reduction. These topics often arise in applications, when dealing with Hamiltonian PDEs. In the first part, we will see how most of the standard constructions of differential geometry and Hamiltonian systems extend naturally from the finite dimensional to the infinite dimensional Banach setting, while, in the last part, an example of infinite dimensional symplectic reduction is presented.

The first introductory chapter contains a brief review of the theory of calculus and of multilinear forms on Banach spaces. These notions will be crucial in chapter 2, where we give a description of smooth manifolds modeled on Banach spaces. Here, after introducing some basic notions, we will focus on equivalence relations and smooth tensor fields, that will be two equally important topics for symplectic reduction. The third chapter focuses on the notion of Banach Lie group and smooth actions of Lie groups on smooth manifolds. Again, up to some technicalities, the standard theory of finite dimensional Lie groups extends naturally to the Banach setting. This chapter is complemented by Appendix A, which contains a formulation of a slice theorem for smooth actions of infinite dimensional Banach Lie groups on Banach manifolds.

After these preliminaries, in chapter 4 we present the Hamiltonian formalism in the infinite dimensional Banach setting, focusing on symplectic geometry and Hamiltonian systems with symmetries. In the same context we present an infinite dimensional extension of the celebrated symplectic reduction theorem of Marsden, Weinstein [18]. Chapter 5 concludes the first part of the thesis by recalling (only in the finite dimensional setting) the symplectic reduction of cotangent bundles, namely the symplectic reduction procedure for finite dimensional mechanical systems with symmetries.

Finally, chapter 6 presents an infinite dimensional example of symplectic reduction. The aim is to understand the classical Kirchhoff equations of hydrodynamics, that regulate the dynamics of a rigid body moving in a perfect fluid, as the reduced equations obtained after a two-stage Hamiltonian reduction procedure. However, we will see how the infinite dimensional manifolds taken in consideration in this chapter are way more general topological spaces than a Banach manifold and so, in virtue of that, chapter 6 is complemented by Appendix B which gives a brief introduction and a review of some important properties of these topological spaces.

The main references about theory of smooth manifolds and Lie groups that are used in this thesis are the books [20] and [1], while the Hamiltonian formalism in the infinite dimensional case is is an adaptation of the one presented in the first chapters of [19] following the book [4]. We note that the core of chapter 4, namely the infinite dimensional extension of the symplectic reduction theorem of Marsden, Weinstein and its technical lemmas are taken from the unpublished notes [16]. Finally, the main references for chapter 6 are the papers [26] and [25], while Appendix B follows the book [13].

Chapter 1

Remarks of calculus on Banach spaces

We recall some basic notions of calculus on Banach spaces and of multilinear forms, stating the main definitions and some theorems like the inverse function theorem and the implicit function theorem that (implicitly) play an important part in the theory of manifolds modeled on Banach spaces which will be considered ahead. Our reference will be chapter 2 of [20].

Definition 1.1. *Let E be a real vector space. A* ***norm*** *on E is a function $\|\cdot\| : E \longrightarrow \mathbb{R}$, such that*

- *for each $v \in E$ we have $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$;*
- *for each $v \in E$ and $\lambda \in \mathbb{R}$ we have $\|\lambda v\| = |\lambda|\ \|v\|$;*
- *for each $v, w \in E$ we have $\|v + w\| \leq \|v\| + \|w\|$.*

The pair $(E, \|\cdot\|)$ is called a ***Banach space*** if it is a complete metric space with respect to the distance induced by the norm:

$$d(v, w) := \|v - w\|.$$

We proceed to recall some notions about spaces of linear and continuous transformations between Banach spaces. Let E and F be Banach spaces, we denote the set of all linear and continuous maps between E and F as $L(E, F)$. This set is a Banach space if equipped with the operator norm

$$\|f\| := \sup_{\|x\| \leq 1} \|f(x)\|.$$

In particular, if $F = \mathbb{R}$ the set $E' := L(E, \mathbb{R})$ is called the ***topological dual*** of E. This construction can be iterated in order to define the ***bidual space*** associated to E

$$E'' := L(E', \mathbb{R}),$$

that is the topological dual of E'. It is clear that there exists a natural inclusion of E into E'' given by the evaluation map:

$$\begin{aligned} \mathrm{ev} : E &\longrightarrow E'' \\ x &\longmapsto \mathrm{ev}_x : \ E' \longrightarrow \mathbb{R}, \end{aligned}$$

defined as:

$$\mathrm{ev}_x(f) := f(x).$$

Definition 1.2. *If the evaluation map is an isometry of Banach spaces (i.e. a vector space isomorphism which preserves the norm) then E is called a* ***reflexive space****.*

1.1 Differentiability and differentiable maps

The aim of this section is to recall the concept of differentiability for maps between Banach spaces.

Definition 1.3. *Let E, F be Banach spaces and $f : \ U \subseteq E \longrightarrow F$ where U is open in E. The map f is called (Fréchet)* ***differentiable*** *at $x \in E$, if there exists a linear and continuous operator $L_x \in L(E, F)$ such that the following limit exists:*

$$\lim_{y \to x} \frac{\|f(y) - f(x) - L_x(y - x)\|}{\|y - x\|} = 0. \tag{1.1}$$

The operator L_x is called the ***derivative*** *of f at x.*

Remark 1.4. The condition (1.1) is equivalent to require that for $y \to x$ in E:

$$f(y) = f(x) + L_x(y - x) + o(\|y - x\|).$$

In particular, if f is differentiable at x, then f is also continuous in x.

If a map $f : \ U \subseteq E \longrightarrow F$ is differentiable at any point of its domain U, then it is called ***differentiable***. Its derivative can be represented through the map between Banach spaces

$$\begin{aligned} Df : U \subseteq E &\longrightarrow L(E, F) \\ x &\longmapsto Df(x) := L_x. \end{aligned}$$

Proposition 1.5. *The following properties hold:*

- *let $f : \ U \subseteq E \longrightarrow V \subseteq F$ and $g : \ V \subseteq F \longrightarrow G$ be two differentiable maps, then the composition $g \circ f : \ U \subseteq E \longrightarrow G$ is differentiable and*

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x);$$

- *let $f_i : U \subseteq E \longrightarrow \mathbb{R}$ (i=1,2) be differentiable maps. Then $f_1 f_2 : U \subseteq E \longrightarrow \mathbb{R}$ is differentiable and*
$$D(f_1 f_2)(x) \cdot v = f_2(x) D f_1(x) \cdot v + f_1(x) D f_2(x) \cdot v,$$
where $g \cdot v$ denotes the evaluation of the linear continuous map $g \in L(E, F)$ on the vector $v \in E$.

In the case that Df is differentiable we are able to define the ***second derivative*** of f as the derivative of Df:
$$D^2 f := D(Df) : U \subseteq E \longrightarrow L(E, L(E, F)).$$
By Remark 1.4, if $D^2 f$ exists, then the derivative Df is a continuous function between U and $L(E, F)$ (equipped with the operator norm). Iterating this construction, fixed a positive integer $k \geq 1$, we say that a map $f : U \subseteq E \longrightarrow F$ is ***differentiable k-times*** if it is differentiable and, for each integer $1 \leq s \leq k-1$, the map $D^s f$ is differentiable. Again, thanks to Remark 1.4, if f is differentiable k-times, then, for each integer $1 \leq s \leq k-1$, the derivative $D^s f$ is a continuous map. In addition, if f is differentiable k-times and the k-derivative $D^k f$ is a continuous map between Banach spaces, then f is called a ***C^k map***.

Definition 1.6. *If $f : U \subseteq E \longrightarrow F$ is a C^k map for each $k \in \mathbb{N}$, then f is called a **smooth map** between Banach spaces. Moreoever if a smooth map f is bijective with a smooth inverse, then f is called a **smooth diffeomorphism**.*

Smooth maps between Banach spaces behave similarly to smooth maps in $\mathbb{R}^n$. In particular the classical formulations of the inverse function theorem and the implicit function theorem still hold.

Theorem 1.7 (Inverse function theorem)**.** *Let $f : U \subseteq E \longrightarrow F$ be a smooth map, $x_0 \in U$ and suppose that $Df(x_0) \in L(E, F)$ is a linear isomorphism of Banach spaces, then there exists an open neighborhood U_0 of x_0 in E and an open neighborhood V_0 of $y_0 := f(x_0)$ in F such that the map*
$$f : U_0 \longrightarrow V_0$$
is a smooth diffeomorphism and the derivative of f^{-1} satisfies
$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1},$$
for $y \in V_0$.

Theorem 1.8 (Implicit function theorem)**.** *Let E, F, G be Banach spaces, U be open in E and V be open in F. Let $f : U \times V \longrightarrow G$ be a smooth map. Assume that for some $x_0 \in U$ and $y_0 \in V$, $D_2 f(x_0, y_0) : F \longrightarrow G$ is an isomorphism of Banach spaces. Then there exist neighborhoods U_0 of x_0 and W_0 of $f(x_0, y_0)$ and a unique smooth map $g : U_0 \times W_0 \longrightarrow V$ such that, for all $(x, w) \in U_0 \times W_0$,*
$$f(x, g(x, w)) = w.$$

These theorems will guarantee, similarly to the theory of finite dimensional manifolds, the existence of an inverse function theorem and an implicit function theorem for smooth maps between Banach manifolds.

1.2 Multilinear k-forms

We conclude the chapter with a brief introduction to multilinear k-forms over a Banach space. This will be crucial at a later stage in order to introduce tensor fields over Banach manifolds.

Let E be a Banach space and k be a positive integer. Then the Cartesian product E^k is a Banach space (endowed with the product norm[1]). A ***multilinear k-form*** over E is a linear and continuous map $\tau : E^k \longrightarrow \mathbb{R}$. The space of all multilinear k-forms is the Banach space

$$\mathcal{T}_k(E) := L(E^k, \mathbb{R}).$$

We point out two particular classes of multilinear k-forms over a Banach space E.

Definition 1.9. *Let E be a Banach space, let $\tau \in \mathcal{T}_k(E)$ be a multilinear k-form over E and $\sigma : \{1, \dots, k\} \longrightarrow \{1, \dots, k\}$ be a permutation of indexes, then τ is called* ***skew-symmetric*** *if for each $v_1, \dots, v_k \in E$ it holds*

$$\tau(v_1, \dots, v_k) = sgn(\sigma)\tau(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Definition 1.10. *Let E be a Banach space, let $\tau : E \times E \longrightarrow \mathbb{R}$ be a bilinear form (i.e. a multilinear 2-form) over E, then τ is called an* ***inner product*** *over E if it is:*

- ***positive definite****: if for each $0 \neq v \in E$ we have that $\tau(v, v) > 0$;*
- ***symmetric****: if for every $v_1, v_2 \in E$ we have that $\tau(v_1, v_2) = \tau(v_2, v_1)$;*
- ***weakly non degenerate****: if, fixed $v \in E$ and supposed that for each $w \in E$, $\tau(v, w) = 0$, we have that $v = 0$.*

In general, given a bilinear form τ over a Banach space E, and a vector $v \in E$ there exists a unique linear and continuous map $\tau^v : E \longrightarrow \mathbb{R}$ such that $\tau^v(w) := \tau(v, w)$. This fact yields a linear map between E and E':

$$\begin{aligned} \tau^\flat : E &\longrightarrow E' \\ v &\longmapsto \tau^v. \end{aligned}$$

In addition, one can prove that if τ is weakly non degenerate, then the map B is injective. Moreover if τ is weakly non degenerate and the map B is a linear homeomorphism of Banach spaces, then τ is called ***(strongly) non degenerate***.

[1] That is the norm on E^k defined as $\|(v_1, \dots, v_k)\| := \max\{\|v_1\|, \dots, \|v_k\|\}$

Hilbert spaces. Let H be a Banach space. If there exists an inner product $\tau \in \mathcal{T}_2(H)$ such that, for each $v \in E$

$$\|v\| = \sqrt{\tau(v,v)},$$

then the pair (H, τ) is called a ***Hilbert space***. One can prove that for the inner product τ, the map $B : H \longrightarrow H'$ defined above is an isometry of Banach spaces. This yields that every Hilbert space H is reflexive (see Definition 1.2).

Chapter 2

Banach manifolds

In general, given a topological vector space V where some concept of differentiation exists, we are able to define what an abstract manifold modeled on V is. In this chapter we wish to look into the theory of smooth manifolds modeled on Banach spaces. Firstly we give the very basic definitions, after that several topics of classical differential geometry will be extended to this context.

The main reference is chapter 3 of [20], which also contains, in full details, all the omitted proofs of the claims given below.

Definition 2.1. *Let M be an Hausdorff topological space and E be a Banach space. A **smooth atlas** of M modeled on E is a collection $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ such that:*

- *for each $i \in I$, U_i is an open subset of M and $M = \bigcup_{i \in I} U_i$;*
- *for each $i \in I$, the map $\varphi_i : U_i \longrightarrow E$ is a homeomorphism onto its image $V_i := \varphi_i(U_i)$ which is open in E;*
- *for every $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$ the overlap map*

$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : \ \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j),$$

  is a smooth diffeomorphism between open sets of E.

*A pair $(M, \mathcal{A})$ is called a **Banach manifold** (modeled on E).*

If M is a Banach manifold and N is an open subset of M, it is clear that there exists a relative atlas modeled on E (induced by the atlas of M) that turns N into a Banach manifold. But, in general, there exist topological subspaces[1] of M that cannot be endowed with a smooth atlas modeled on some Banach space. The following definition states a condition that, if satisfied, allow us to endow a topological subspace $N \subseteq M$ with a smooth atlas modeled on some Banach space $F \subseteq E$.

[1]That is a subset N endowed with the relative topology from M. Namely a subset $A \subseteq N$ is open if and only if there exists an open set $U \subseteq M$ such that $A = U \cap N$

Definition 2.2. *Let M be a Banach manifold. A subset $N \subseteq M$ is called a* ***submanifold*** *of M if for each $n \in N$ there exists a chart $(U, \varphi : U \longrightarrow V)$ in M with $n \in U$ and a closed subspace $F \subseteq E$ such that*

$$\varphi(U \cap N) = V \cap F.$$

We also introduce the idea of smooth maps between Banach manifolds, that are continuous maps that are "locally" represented by smooth maps between Banach spaces.

Definition 2.3. *Let M and N be Banach manifolds modeled on the Banach spaces E and F. A continuous function $f : M \longrightarrow N$ is called a* ***smooth map between Banach manifolds*** *if, for each couple of charts $\varphi_M : U \longrightarrow V \subseteq E$ and $\psi_N : W \longrightarrow K \subseteq F$, the map*

$$\psi_N \circ f\big|_U \circ \varphi_M : V \longrightarrow K$$

is a smooth map between Banach spaces. If, in addition, the map $f : M \longrightarrow N$ is invertible with smooth inverse, then f is called a ***smooth diffeomorphism*** *between Banach manifolds.*

2.1 Tangent spaces

This section covers the idea of tangent vectors and tangent spaces to a Banach manifold at a point. Before starting we give a remark about derivatives of curves with values in Banach spaces.

Remark 2.4. Let $0 \in I \subseteq \mathbb{R}$ be an open interval and E be a Banach space, let $c : I \longrightarrow E$ be a smooth map. Then the derivative of c at $t \in I$ can be identified with an element of E through the limit:

$$c'(t) := \lim_{s \to 0} \frac{c(t+s) - c(t)}{s} \in E.$$

Now, let M be a Banach manifold and $m \in M$. A ***smooth curve at m*** is a smooth map $c : I \longrightarrow M$, where $0 \in I \subseteq \mathbb{R}$ is an open interval, such that $c(0) = m$. Any two smooth curves at m, c_1 and c_2, for which there exists a chart (U, φ) of M, with $m \in U$, such that

$$(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0),$$

are called ***tangent at m***. The following lemma assures that, for two smooth curves at m, the previous definition does not depend on the chart φ chosen.

Lemma 2.5. *Let M a Banach manifold, m be an element of M and c_1, c_2 be smooth curves that are tangent at m. Then, for each chart $\varphi : U \longrightarrow E$, with $m \in U$, we have that*

$$(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0).$$

Let $c: I \longrightarrow M$ be a smooth curve at m. The ***tangent vector (associated to c)*** to M at m is defined by

$$[c]_m := \{\tilde{c}: I \longrightarrow M: \tilde{c} \text{ and } c \text{ are smooth curves tangent at } m\}.$$

The set of all tangent vectors to M at m is denoted as

$$T_mM := \{[c]_m : c: I \longrightarrow M \text{ is a smooth curve at } m\},$$

and it is called the ***tangent space*** to M at m.

The following proposition shows that every tangent space T_mM is a real vector space isomorphic to the model space of M. In particular, it allows us to endow T_mM with the norm of E turning it into a Banach space.

Proposition 2.6. *Let M be a Banach manifold modeled on the Banach space E. Then, for each $m \in M$, the set T_mM is a real vector space and there exists a vector spaces isomorphism between T_mM and E.*

In the following, a tangent vector to M at m will be often denoted as $v_m \in T_mM$.

Remark 2.7. Following the classical theory, we wish to understand the tangent space to M at m, as a vector space in some sense "attached" to m. With this aim we define the ***tangent bundle*** of M

$$TM := \bigcup_{m \in M} \{m\} \times T_mM,$$

and its canonical projection (that is a surjective map of sets) onto M

$$\begin{aligned} p_M: \ TM &\longrightarrow M \\ (m, v_m) &\longmapsto m. \end{aligned}$$

Notice that, formally, a typical element of TM is a pair (m, v_m), but it is customary to write the only vector v_m.

Proposition 2.8. *Let TM be the tangent bundle of a Banach manifold M (modeled on the Banach space E). Then TM is a Banach manifold (modeled on $E \times E$) and the map p_M is smooth.*

2.1.1 The tangent map of a smooth map

For a smooth map between Banach manifolds, we use tangent spaces to give a definition of its derivative.

Definition 2.9. *Let M, N be Banach manifolds and $f: M \longrightarrow N$ be a smooth map. For each $m \in M$ the **tangent map** of f at m is a linear and continuous map between Banach spaces*

$$\begin{aligned} T_mf: \ T_mM &\longrightarrow T_{f(m)}N \\ [c]_m &\longmapsto [f \circ c]_{f(m)}. \end{aligned}$$

It is clear that the tangent map defined above can be naturally extended to a function between TM and TN

$$\begin{aligned} Tf\colon\ TM &\longrightarrow TN \\ (m,[c]_m) &\longmapsto (f(m), T_m f([c]_m)), \end{aligned}$$

that is called the ***tangent map*** of f.

The following proposition states some important properties related to the idea of tangent map.

Proposition 2.10. *Let $f : M \longrightarrow N$ and $g : N \longrightarrow K$ be smooth maps between Banach manifolds, then the following hold:*

- *$g \circ f\colon\ M \longrightarrow K$ is a smooth map between Banach manifolds and*

$$T(g \circ f) = Tg \circ Tf;$$

- *if $N = M$ and $f\colon\ M \longrightarrow M$ is the identity map, then $Tf\colon\ TM \longrightarrow TM$ is the identity map too;*

- *if $f\ :\ M \longrightarrow N$ is a smooth diffeomorphism, then Tf is a bijection and $(Tf)^{-1} = T(f^{-1})$.*

As anticipated in chapter 1, the classical formulations of the inverse function theorem and the implicit function theorem hold for smooth maps between Banach manifolds.

Theorem 2.11 (Inverse function theorem)**.** *Let M, N be smooth manifolds, f : $M \longrightarrow N$ be a smooth map, and $m \in M$. Suppose that $T_m f\ :\ T_m M \longrightarrow T_{f(m)}N$ is an isomorphism of Banach spaces. Then there exist an open neighborhood U_m of $m \in M$ and an open neighborhood $W_{f(m)}$ of $f(m) \in N$ such that*

$$f\colon\ U_m \longrightarrow W_{f(m)}$$

is a smooth diffeomorphism between Banach manifolds.

Theorem 2.12 (Implicit function theorem)**.** *Let M_1, M_2, N be smooth manifolds (modeled on E_1, E_2, F), let $f\colon\ M_1 \times M_2 \longrightarrow N$ be a smooth map and $(p,q) \in M_1 \times M_2$ such that $T_2 f(p,q) : T_q M_2 \longrightarrow T_{f(p,q)}N$ is an isomorphism of Banach spaces, then there exist an open neighborhoods U of $p \in M_1$, W of $f(p,q) \in N$, and a unique smooth map $g\colon\ U \times W \longrightarrow M_2$ such that, for all $(x,w) \in U \times W$*

$$f(x, g(x,w)) = w.$$

2.1.2 Vector fields over a Banach manifold

We proceed to say something about smooth sections of the tangent bundle of a Banach manifold, namely the smooth vector fields, that will play a very important role at a later stage.

Let M be a Banach manifold and TM be its tangent bundle. A ***(smooth) vector field*** over M is a smooth function

$$X: M \longrightarrow TM$$

such that $p_M \circ X = \mathrm{id}_M$. The space of all vector fields over M is usually denoted as $\mathfrak{X}(M)$. Although it is clear that the set $\mathfrak{X}(M)$ is a real vector space, it is more difficult to see that it can be equipped with a Lie algebra structure. In order to do that we need to recall two crucial concepts covered by the following definitions.

Definition 2.13. *Let M, N be Banach manifolds, $\varphi: M \longrightarrow N$ be a smooth diffeomorphism and $Y \in \mathfrak{X}(N)$ be a vector field over N. The* ***pull-back*** *of Y by φ is the vector field $\varphi^* Y \in \mathfrak{X}(M)$ such that*

$$(\varphi^* Y)(m) := T_{\varphi(m)}\varphi^{-1}(Y(\varphi(m))).$$

It is verified in [20] that $\varphi^* Y: M \longrightarrow TM$ as defined above is smooth.

Definition 2.14. *Let M be a Banach manifold and $X \in \mathfrak{X}(M)$. An* ***integral curve of X at*** *$m \in M$ is a smooth curve at m, $c: I \longrightarrow M$, such that $c'(t) = X(c(t))$ for each $t \in I$ (where we denote $c'(t) := T_t c(1)$).*

The classical theory of ODE on finite dimensional manifolds can be adapted to the Banach setting, and one can prove that, given a smooth vector field $X \in \mathfrak{X}(M)$, for each $m \in M$ there exists an open neighborhood U_m of m and an open interval $I_m = (-\varepsilon_m, \varepsilon_m)$ for some $\varepsilon_m > 0$ such that, for each $m' \in U_m$ there exists an integral curve $c_{m'}: I_m \longrightarrow M$ of X at m'. This implies that, for each $t \in I$, there exists a smooth diffeomorphism onto its image:

$$\begin{aligned} F_t^X : U_m &\longrightarrow M \\ m' &\longmapsto c_{m'}(t) \end{aligned}$$

called the ***flow of X at*** $t \in I$ around $m \in M$.

We are finally ready to define a Lie algebra structure on $\mathfrak{X}(M)$, namely a bracket $[\cdot,\cdot]: \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ that is a bilinear, skew-symmetric map over $\mathfrak{X}(M)$ and satisfies Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Definition 2.15. *Let M be a Banach manifold and $X, Y \in \mathfrak{X}(M)$. Then the* ***Lie bracket*** *of X and Y is a vector field $[X,Y]$ over M defined by:*

$$[X,Y](m) := \frac{d}{dt}\Big|_{t=0} \left(\left(F_t^X\right)^* Y\right)(m).$$

2.2 Immersions, submersions and regular values

Now, we present a review about smooth maps whose images or level sets are Banach manifolds or submanifolds. We start with a definition that will be crucial in a short time.

Definition 2.16. *Let E be a Banach space and $A \leq E$ be a closed vector subspace. We say that A **splits** in E, if there exists a closed vector subspace $B \leq E$ (called a topological complement of A) such that*

$$E = A \oplus B.$$

2.2.1 Immersed and embedded submanifolds

Firstly, the idea of immersion is presented. These are the maps whose images can be endowed with a differential structure.

Definition 2.17. *Let M, N be Banach manifolds, $m \in M$ and $f : M \longrightarrow N$ be smooth map between Banach manifolds. f is called an **immersion at m** if the tangent map $T_m f$ is injective with closed split image in $T_{f(m)}N$. If f is an immersion at each m in M, f is called an **immersion**.*

Let M and N be Banach manifolds modeled on E and F, and $f : M \longrightarrow N$ be an injective immersion. The image $f(M)$ can be endowed with the final topology[2] induced by f, turning $f(M)$ into an Hausdorff topological space. In addition, there exists a smooth atlas (modeled on some closed subspace of F) over $f(M)$ turning it into a Banach manifold. However, since the quotient topology and the relative topology from N might be not compatible, $f(M)$ isn't always a submanifold of N (according to Definition 2.2). So, in general, $f(M)$ is called an ***immersed submanifold*** of N. But, if the two topologies are compatible, then the concepts of immersed and standard submanifold coincide.

Definition 2.18. *Let $f : M \longrightarrow N$ be smooth immersion between Banach manifolds. If f is an homeomorphism onto $f(M)$ (endowed with the relative topology from N), then f is called an **embedding**. In this case, the immersed submanifold $f(M)$ is also a submanifold (see Definition 2.2) of N, and it is called an **embedded submanifold**.*

The following proposition gives a sufficient condition for an injective immersion between Banach spaces to be an embedding.

Proposition 2.19. *Let $f : M \longrightarrow N$ be an injective smooth immersion between Banach manifolds. If f is either an open or closed map[3], then f is an embedding.*

[2] Namely, we define that $U \subseteq f(M)$ is open if and only if $f^{-1}(U)$ is open in M

[3] Namely, it has the property that images of open (respectively closed) subsets are open (respectively closed) subsets

2.2.2 Weakly regular values

Now, given a smooth map between Banach manifolds $f: M \longrightarrow N$, we wish to search values $n \in N$ such that:

- the level set $f^{-1}(n)$ is a submanifold of M;
- for each $m \in f^{-1}(n)$ the tangent spaces satisfy:

$$T_m f^{-1}(n) = \ker\,(T_m f).$$

An element $n \in N$ satisfying these two properties is called a ***weakly regular value*** for f. In order to easily identify these values, the concept of submersion was developed.

Definition 2.20. *Let M and N be Banach manifolds and $f: M \longrightarrow N$ be a smooth map between Banach manifolds. Let $n \in N$, then f is called a **submersion at n** if for each $m \in f^{-1}(n)$ the tangent map $T_m f$ is surjective with split kernel. If f is a submersion at each $n \in N$, then f is called a **submersion**.*

The following proposition states that if f is a submersion at $n \in N$ then n is a weakly regular value for f.

Proposition 2.21. *Let M, N be Banach manifolds and $f: M \longrightarrow N$ be a smooth map between Banach manifolds. Let $n \in N$, if f is a submersion at $n \in N$, then the level set $f^{-1}(n)$ is a submanifold of M and for each $m \in f^{-1}(n)$, we have $T_m f^{-1}(n) = ker(T_m f)$.*

2.3 Equivalence relations on manifolds

Let M be a set, which for our purposes should be thought of as a Banach manifold. We recall that a subset $R \subseteq M \times M$ is an ***equivalence relation*** over M if:

- for each $x \in M$, it holds $(x, x) \in R$;
- if $(x, y) \in R$, then $(y, x) \in R$;
- and if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

If $(x, y) \in R$ we say that x and y are related by R and write $x \sim_R y$. Let x be an element of M, the ***equivalence class*** of x with respect to the relation R is the subset

$$[x]_R := \{y \in M: \ x \sim_R y\} \subseteq M.$$

The collection of all equivalence classes is called the ***quotient space*** of M with respect to the relation R and it is denoted as

$$M/R := \{[x]_R : \ x \in M\};$$

there exists a natural projection (surjective map of sets) from M onto M/R:

$$\begin{aligned}\pi_R : M &\longrightarrow M/R\\ x &\longmapsto [x]_R.\end{aligned}$$

If the set M is actually a Banach manifold, it will be interesting for us to ask under which assumptions on the relation R the quotient space M/R is a Banach manifold.

Definition 2.22. *Let M be a Banach manifold. An equivalence relation R over M is called* ***regular*** *if the quotient space M/R is a Banach manifold and the natural projection $\pi_R : M \longrightarrow M/R$ is a submersion.*

In the case of a regular equivalence relation, M/R is called the ***quotient manifold*** associated to M with respect to R. The following proposition characterize smooth maps between Banach manifolds whose domain is a quotient manifold.

Proposition 2.23. *Let M be a Banach manifold and R be a regular equivalence relation over M. Then the following hold:*

- *a map $f : M/R \longrightarrow N$ is smooth, if and only if $f \circ \pi_R : M \longrightarrow N$ is smooth;*
- *any smooth map $g : M \longrightarrow N$ that satisfies $g(x) = g(y)$ if $x \sim_R y$, defines a unique smooth map $\hat{g} : M/R \longrightarrow N$ such that $\hat{g} \circ \pi_R = g$.*

We conclude the section by stating a theorem that gives necessary and sufficient conditions for a relation to be regular.

Theorem 2.24. *Let M be a Banach manifold and $R \subseteq M \times M$ be an equivalence relation over M. Then R is regular if and only if:*

- *R is a submanifold of $M \times M$*
- *and the projection onto the first factor $p_1 : R \longrightarrow M$, $p_1(x, y) := x$ is a submersion.*

2.4 Tensor fields over a Banach manifold

In this last section of the chapter we introduce the theory of smooth tensor fields over Banach manifolds, extending the theory about multilinear k-forms over a Banach space recalled in section 1.2 of chapter 1. Out of the class of tensor fields, we will focus on symplectic forms and Riemannian metrics.

Let M be a Banach manifold. We fix a positive integer k and we define a ***k-tensor*** over M as an element of the following disjoint union

$$\mathcal{T}_k(M) := \bigcup_{m \in M} \{m\} \times \mathcal{T}_k(T_m M).$$

We notice that there exists a natural projection from $\mathcal{T}_k(M)$ onto M,

$$\begin{aligned} \pi_{k,M} : \mathcal{T}_k(M) &\longrightarrow M \\ (m, \tau_m) &\longmapsto m. \end{aligned}$$

Using this projection one can endow $\mathcal{T}_k(M)$ with the initial topology[4] induced by $\pi_{k,M}$. But something more can be said:

Proposition 2.25. *Let M a Banach manifold modeled on E. Then $\mathcal{T}_k(M)$ endowed with the initial topology is a Banach manifold modeled on E^k and the projection $\pi_{k,M}$ is a submersion.*

Similar to the discussion in Section 2.1.2 for vector fields over a manifold, a smooth section of $\mathcal{T}_k(M)$, namely a smooth map $\tau : M \longrightarrow \mathcal{T}_k(M)$ such that $\pi_{k,M} \circ \tau = \mathrm{id}_M$, is called a ***(smooth) tensor field*** over M.

Cotangent bundle. We want to point out the very particular case when $k = 1$. Indeed the space of 1-tensors over a Banach manifold M takes the name of ***cotangent bundle*** of M,

$$T^*M := \mathcal{T}_1(M).$$

And the projection $\pi_{1,M} : \mathcal{T}_1(M) \longrightarrow M$ will be simply denoted as $\pi_M : T^*M \longrightarrow M$. Moreover, a 1-tensor field $\omega : M \longrightarrow T^*M$ is called a ***differential 1-form*** over M.

We continue by giving some definitions of classical differential geometry that will be very useful in later chapters. Let τ be a k-tensor field and X be a vector field over the Banach manifold M, we can define the ***interior product*** between X and τ as the $(k-1)$-tensor field over M such that, for every $m \in M$ and $v_1, \dots, v_k \in T_mM$:

$$(\mathbf{i}_X \tau)_m(v_1, \dots, v_{k-1}) := \tau_m(X(m), v_1, \dots, v_{k-1}).$$

Now, let M, N be Banach manifolds and $\varphi : M \longrightarrow N$ a smooth map between M and N. Let τ_1 be a k-tensor field over N, then the unique k-tensor field over M such that, for every $m \in M$ and $v_1, \dots, v_k \in T_mM$:

$$(\varphi^*\tau_1)_m(v_1, \dots, v_k) = (\tau_1)_{\varphi(m)}(T_m\varphi(v_1), \dots, T_m\varphi(v_k))$$

is called the ***pull-back*** of τ_1 by φ. Moreover, if τ_2 is a k-tensor field over M and the smooth map φ is a smooth diffeomorphism between M and N, then we define the ***push-forward*** of τ_2 by φ as the k-tensor field over N:

$$\varphi_*\tau_2 := (\varphi^{-1})^*\tau_2.$$

[4] $A \subseteq \mathcal{T}_k(M)$ is open if and only if there exists $U \subseteq M$ open such that $A = \pi_{k,M}^{-1}(U)$

We conclude with the idea of the Lie derivative. Let M be a Banach manifold and τ be a k-tensor field over M and $X \in \mathfrak{X}(M)$. The ***Lie derivative*** of τ along X is the k-tensor field over M defined by

$$(L_X\tau)(m) := \frac{d}{dt}\Big|_{t=0} \left(\left(F_t^X\right)^* \tau\right)(m),$$

where F_t^X is the flow of X around m (defined above).

We conclude the section giving some important examples of k-tensor fields over a Banach manifold.

Differential k-forms. Let M be a Banach manifold, then a ***differential k-form*** over M is a k-tensor field ω such that ω_m is an skew-symmetric multilinear k-form over the Banach space T_mM for each $m \in M$. The set of all differential k-forms over M is denoted by $\Omega^k(M)$.

As in the classical theory of differential geometry, one may introduce the idea of exterior derivative of a differential k-form. It is an operator $\mathbf{d} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ such that:

- $\mathbf{d}$ is a linear map;
- for every differential k-form ω we have that $\mathbf{d}(\mathbf{d}\omega) = 0$;
- for every smooth map between Banach manifolds $\varphi : M \longrightarrow N$ we have that $\varphi^*(\mathbf{d}\omega) = \mathbf{d}(\varphi^*\omega)$;
- for each vector field $X \in \mathfrak{X}(M)$ we have $L_X\omega = \mathbf{d}(\mathbf{i}_X\omega) + \mathbf{i}_X(\mathbf{d}\omega)$ and $\mathbf{d} \circ L_X = L_X \circ \mathbf{d}$.

We refer to [20] for a possible technical definition of $\mathbf{d}$. Interestingly, the standard definition of the exterior derivative cannot be adapted (without a lot of difficulties) to the infinite dimensional case.

Riemannian metrics. A ***weak Riemannian metric*** over a Banach manifold M is a 2-tensor field $\langle\!\langle\cdot,\cdot\rangle\!\rangle : M \longrightarrow \mathcal{T}_2(M)$ such that for each $m \in M$, $\langle\!\langle\cdot,\cdot\rangle\!\rangle_m$ is an inner product on T_mM (Definition 1.10). We also recall that for each $m \in M$ there exists a linear and continuous map between Banach spaces $B_m : T_mM \longrightarrow T_m^*M := L(T_mM, \mathbb{R})$ that is injective due to weak non degeneracy of the inner product. This map can be globalized to a smooth map between Banach manifolds:

$$\begin{aligned} \mathbb{F} : \; TM &\longrightarrow T^*M \\ (m, v_m) &\longmapsto (m, B_m(v_m)). \end{aligned}$$

Proposition 2.26. *Let M be a Banach manifold and $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ be a weak Riemannian metric over M. Then the map $\mathbb{F}$ defined above is a smooth diffeomorphism onto its image.*

Moreover, if $\langle\!\langle\cdot,\cdot\rangle\!\rangle_m$ is strongly non degenerate for each $m \in M$ (see discussion after Definition 1.10), then $(T_mM, \langle\!\langle\cdot,\cdot\rangle\!\rangle_m)$ is a Hilbert space and $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is called a ***(strong) Riemannian metric***.

Symplectic forms Let M be a Banach manifold. A differential 2-form $\omega \in \Omega^2(M)$ is called a ***weak symplectic form*** over M if $\mathbf{d}\,\omega = 0$ and for each $m \in M$, ω_m is a skew-symmetric, weakly non degenerate bilinear form over $T_m M$. The pair (M, ω) is called a (Banach) ***weak symplectic manifold***. Similarly to the case of Riemannian metrics, for every $m \in M$ we can build a linear and continuous map of Banach spaces

$$\Omega^\flat_m : \ T_m M \longrightarrow T^*_m M$$

that is also injective due to the weak non degeneracy of ω_m. A weak symplectic form ω is called a ***(strong) symplectic form*** if the map $\Omega^\flat_m$ is a linear homeomorphism of Banach spaces for every $m \in M$.

Let M be a Banach manifold. We already know that the cotangent bundle $T^*M = \mathcal{T}_1(M)$ is a Banach manifold. We proceed to indicate that it can be endowed with a weak symplectic structure. Firstly we build a differential 1-form over T^*M:

$$\theta_{can} : \ T^*M \longrightarrow T^*(T^*M)$$

defined by

$$\langle \theta_{can}(\alpha_m), V_{\alpha_m} \rangle := \langle \alpha_m, T_m \pi_M(V_{\alpha_m}) \rangle,$$

where $V_{\alpha_m} \in T_{\alpha_m}(T^*M)$ and $\pi_M : \ T^*M \longrightarrow M$ is the canonical projection.

Proposition 2.27 ([14], Theorem 2.4)**.** *The differential 2-form $\Omega_{can} := -\,\boldsymbol{d}\theta_{can}$ is a weak symplectic form over T^*M and it is a strong symplectic form if and only if the model space E is a reflexive Banach space.*

Chapter 3

Banach Lie groups

The aim of this chapter is to introduce the idea of Lie groups modeled on Banach spaces. Also, some theory about actions of Lie groups on Banach manifolds is treated. We refer to the books [20] and [1] for all the proofs omitted in this chapter.

We start with the basic definitions and some relevant observations.

Definition 3.1. *Let G be a Banach manifold (modeled on a Banach space E) which is a group with respect to some multiplication. If the group operation*

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (g,h) &\longmapsto gh^{-1} \end{aligned}$$

is a smooth map between Banach manifolds, then G is called a ***Banach Lie group*** *modeled on E.*

For any Banach Lie group G, given $g \in G$, the maps $L_g : G \longrightarrow G$, $h \longmapsto gh$ and $R_g : G \longrightarrow G$, $h \longmapsto hg$ are called the ***left and right translations by g***. In particular it is easy to see that both L_g and R_g are diffeomorphisms of G into itself. So is the composition $C_g := L_g \circ R_{g^{-1}}$, called the ***conjugation map by g***.

The Lie algebra of a Lie group

The tangent space at the identity T_eG is usually denoted by $\mathfrak{g}$ and it is called the ***Lie algebra*** of the Lie group G. It is equipped with a natural Lie algebra structure that is constructed through a procedure, explained below, that involves the following definition.

Definition 3.2. *Let $\xi \in \mathfrak{g}$ be a tangent vector to G at e. The* ***right invariant vector field associated to*** *ξ is the unique vector field $X_\xi \in \mathfrak{X}(G)$ defined by*

$$X_\xi(g) := T_eR_g(\xi) \in T_gG,$$

for each $g \in G$.

Remark 3.3. For each right invariant vector field X_ξ associated to $\xi \in \mathfrak{g}$, and for every $g \in G$, it holds $R_g^* X_\xi = X_\xi$, namely

$$(T_h R_g)(X_\xi(h)) = X_\xi(hg),$$

where $h \in G$.

One can prove that the set

$$\mathfrak{X}^R(G) := \{X_\xi \in \mathfrak{X}(G) : \xi \in \mathfrak{g}\} \subseteq \mathfrak{X}(G)$$

is a real vector space and, moreover, it is a closed Lie subalgebra of $\mathfrak{X}(G)$ if equipped with the Lie bracket of vector fields (see Definition 2.15). In addition, the natural vector space isomorphism

$$\begin{aligned} \alpha : \mathfrak{g} &\longrightarrow \mathfrak{X}^R(G) \\ \xi &\longmapsto X_\xi, \end{aligned}$$

induces a Lie bracket on $\mathfrak{g}$, defined by

$$[\![\xi, \eta]\!] := -[X_\xi, X_\eta](e),$$

for each ξ, $\eta \in \mathfrak{g}$.

Remark 3.4. There exists a formula useful to compute the Lie bracket of two elements $\xi, \eta \in \mathfrak{g}$. Indeed, one can prove that, for any $c_\xi : I_1 \longrightarrow G$ and $c_\eta : I_2 \longrightarrow G$ smooth curves at e such that $c'_\xi(0) = \xi$ and $c'_\eta(0) = \eta$, we have

$$[\![\xi, \eta]\!] = \frac{d}{dt}\frac{d}{ds}\Big|_{t=0,s=0} c_\xi(t) c_\eta(s) c_\xi(t)^{-1}$$

Exponential map of a Lie group

Let G be a Banach Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\xi \in \mathfrak{g}$ and X_ξ be the right invariant vector field associated to ξ. Then there exists a unique integral curve of X_ξ at $e \in G$ (see Definition 2.14), $\gamma_\xi : \mathbb{R} \longrightarrow G$, such that, for every $t, s \in \mathbb{R}$, we have

$$\gamma_\xi(s + t) = \gamma_\xi(s)\gamma_\xi(t).$$

The ***exponential map*** of the group G is defined by:

$$\begin{aligned} \exp_G : \mathfrak{g} &\longrightarrow G \\ \xi &\longmapsto \gamma_\xi(1), \end{aligned}$$

The following proposition states the most important properties that the exponential map of a Banach Lie group satisfies.

Proposition 3.5. *The following hold:*

1. $\exp_G : \mathfrak{g} \longrightarrow G$ *is a smooth map between Banach manifolds (where* $\mathfrak{g}$ *is understood as a Banach manifold modeled on itself with the identity chart).*

2. *For every* $\xi \in \mathfrak{g}$ *and* $t \in \mathbb{R}$*, we have* $\exp_G(t\xi) = \gamma_\xi(t)$.

3. *The tangent map of* $\exp_G$ *at* $0 \in \mathfrak{g}$ *is the identity of* $\mathfrak{g}$*, namely for every* $\xi \in \mathfrak{g}$*,*

$$T_0 \exp_G(\xi) = \xi$$

(Sometimes this result may be denoted as $\frac{d}{dt}\big|_{t=0} \exp_G(t\xi) = \xi$*).*

Thanks to Proposition 3.5, the exponential map satisfies the hypothesis of the inverse function theorem for smooth maps between Banach manifolds, and then it is a smooth diffeomorphism from an open neighborhood of $0 \in \mathfrak{g}$ onto an open neighborhood of $e \in G$.

Lie subgroups

Let G be a Banach Lie group and $H \subseteq G$ a subgroup, we say that H is a ***Lie subgroup*** of G if it is an injectively immersed submanifold of G which is itself a Lie group. If it is also an embedded submanifold of G, it is called a ***regular Lie subgroup***. And, moreover, the Lie algebra and the exponential map of a Lie subgroup are characterized by the following proposition.

Proposition 3.6. *Let H be a Lie subgroup of a Banach Lie group G. Then the Lie algebra $\mathfrak{h}$ associated to H is a closed subalgebra of $\mathfrak{g}$, and the following hold:*

- $\mathfrak{h} = \{\xi \in \mathfrak{g} : \exp_G(t\xi) \in H \text{ for all } t \in \mathbb{R}\}$;
- *the exponential maps satisfy*

$$\exp_H = \exp_G\big|_{\mathfrak{h}}.$$

We recall that in the finite dimensional case any closed subgroup $H \subseteq G$ is indeed a regular Lie subgroup. However, we stress that this result does not hold in the infinite dimensional case.

3.1 Actions of Banach Lie groups on Banach manifolds

This section presents an introduction to the theory of smooth actions of Banach Lie groups on Banach manifolds. Also, at the end of the section, we recall one of the most well known slice theorems for smooth actions of finite dimensional Lie groups on finite dimensional manifolds and we briefly discuss the difficulties in extending such a theorem to the infinite dimensional setting.

Definition 3.7. *Let M be a Banach manifold and G be a Banach Lie group. A* ***right action*** *of G on M is a smooth map $\Phi: M \times G \longrightarrow M$ such that:*

- $\Phi(m,e) = m$ *for all $m \in M$ (where $e \in G$ is the identity of the group);*
- $\Phi(\Phi(m,h),g) = \Phi(m,hg)$ *for all $g,h \in G$, $m \in M$.*

A ***left action*** *is a smooth map $\Psi: G \times M \longrightarrow M$ such that $\Psi(e,m) = m$ and $\Psi(g,\Psi(h,m)) = \Psi(gh,m)$ for every $g,h \in G$ and $m \in M$.*

We notice that, for each $g \in G$, the map $\Phi_g: M \longrightarrow M$, $m \longmapsto \Phi(m,g)$ is a smooth diffeomorphism of M, and its smooth inverse is

$$(\Phi_g)^{-1} = \Phi_{g^{-1}}.$$

Also, fixed $m \in M$, the smooth map $\Phi^m: G \longrightarrow M$, $g \longmapsto \Phi(m,g)$ is called the ***orbit map of Φ through m***. In particular, the image

$$\mathrm{Orb}_G(m) := \{\Phi^m(g): g \in G\} \subseteq M,$$

is called the ***orbit of Φ through m***, while the preimage $G_m := (\Phi^m)^{-1}(m)$ is called the ***isotropy group of Φ at m***. It is clear that $G_m = \{g \in G: \Phi_g(m) = m\}$ and that G_m is a subgroup of G that is closed with respect to its topology. Moreover, if G_m is a Lie subgroup of G, then its Lie algebra is denoted by $\mathfrak{g}_m$ and it is called the ***isotropy algebra at m***.

The following definition recalls some basic properties that an action can possess.

Definition 3.8. *Let $\Phi: M \times G \longrightarrow M$ be a right action of a Banach Lie group on a Banach manifold M. It is called:*

- ***transitive****: if there exists only one orbit, namely if for every $m_1, m_2 \in M$ there exists $g \in G$ such that $\Phi_g(m_1) = m_2$;*
- ***free****: if for each $m \in M$ the isotropy group at m is $G_m = \{e\}$;*
- ***proper****: if for any sequence $(m_n)_{n\in\mathbb{N}}$ that is convergent in M and for any sequence $(g_n)_{n\in\mathbb{N}}$ in G such that the sequence $(\Phi_{g_n}(m_n))_{n\in\mathbb{N}}$ converges in M, then g_n admits a convergent subsequence in G.*

3.1.1 Infinitesimal generators

Let Φ be an action of a Banach Lie group G on a Banach manifold M and $\xi \in \mathfrak{g}$. The ***infinitesimal generator*** of the action associated to ξ is the vector field $\xi_M \in \mathfrak{X}(M)$ defined as

$$\xi_M(m) := \frac{d}{dt}\Big|_{t=0} \Phi_{\exp_G(t\xi)}(m) = T_e\Phi^m(\xi).$$

We know that, for each $m \in M$, there exists a unique integral curve of ξ_M. One can prove that it is actually the smooth curve at m:

$$\mathbb{R} \ni t \longmapsto \Phi_{\exp_G(t\xi)}(m) \in M.$$

Also, one can prove that if the isotropy group G_m of Φ is a Lie subgroup of G, then its Lie algebra $\mathfrak{g}_m$ is characterized by the following:

$$\mathfrak{g}_m := T_e G_m = \{\xi \in \mathfrak{g} : \ \xi_M(m) = 0\}.$$

We proceed by recalling some important examples of actions that can be constructed for any Banach Lie group G.

Adjoint action. Fix $g \in G$. Recall the conjugation map by g,

$$C_g : \ G \longrightarrow G, \quad h \longmapsto ghg^{-1}.$$

We denote its tangent map at the identity $e \in G$ as $\mathrm{Ad}_g := T_e C_g : \ \mathfrak{g} \longrightarrow \mathfrak{g}$. One can prove that the map

$$\begin{aligned} \mathrm{Ad} : G \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (g, \xi) &\longmapsto \mathrm{Ad}_g(\xi), \end{aligned}$$

is a left action of G on $\mathfrak{g}$, called the ***adjoint action*** of G on $\mathfrak{g}$.

Coadjoint action. Let $\mathfrak{g}^*$ be the topological dual of $\mathfrak{g}$ and $\langle \mu, \xi \rangle$ denote the evaluation of $\mu \in \mathfrak{g}^*$ onto $\xi \in \mathfrak{g}$. Then, the ***coadjoint action*** of G on $\mathfrak{g}^*$ is the right action $\mathrm{Ad}^* : \ \mathfrak{g}^* \times G \longrightarrow \mathfrak{g}^*$, $(g, \mu) \longmapsto (\mathrm{Ad}^*)_g(\mu)$, where $(\mathrm{Ad}^*)_g : \ \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ is defined by

$$\langle (\mathrm{Ad}^*)_g(\mu), \xi \rangle := \langle \mu, \mathrm{Ad}_g(\xi) \rangle,$$

for each $\mu \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$.

We notice that, for each $\mu \in \mathfrak{g}^*$, the isotropy group at μ of the coadjoint action is

$$G_\mu = \{g \in G : \ \mathrm{Ad}^*_g(\mu) = \mu\},$$

and, in the case it is a Lie subgroup of G, its Lie algebra is $\mathfrak{g}_\mu = \{\xi \in \mathfrak{g} : \ \xi_{\mathfrak{g}^*}(\mu) = 0\}$.

3.1.2 The orbit space of an action, the finite dimensional case

For this last part, we consider a smooth right action $\Phi : \ M \times G \longrightarrow M$ of a finite dimensional Lie group G on a finite dimensional manifold M. We notice, that it determines an equivalence relation on M, defined by the following

$$m_1 \sim_G m_2 \ \text{ if and only if there exists } g \in G \text{ such that } \Phi_g(m_1) = m_2.$$

The quotient space, with respect to this equivalence relation, is called the ***orbit space*** of M with respect to the action of G and it is denoted M/G. The natural projection associated to this equivalence relation is the map

$$\begin{aligned} \pi_G : M &\longrightarrow M/G \\ m &\longmapsto [m] := \mathrm{Orb}_G(m). \end{aligned}$$

From [1] we report the statement of a classical theorem that states some sufficient conditions that turn this relation into a regular equivalence relation (according to Definition 2.22).

Theorem 3.9. *In the finite dimensional setting above, if the action Φ is free and proper, then M/G is a smooth (finite dimensional) manifold and the natural projection π_G is a smooth submersion.*

In addition, the following proposition gives a characterization of tangent spaces (and their duals) to M/G at some $[m] \in M/G$, that will play a role ahead.

Proposition 3.10. *Under the assumptions of Theorem 3.9, the following hold*

1. *for each $[m] \in M/G$ there exists a vector space isomorphism between the tangent space $T_{[m]}(M/G)$ and the quotient of vector spaces $T_m M/T_m \operatorname{Orb}_G(m)$;*
2. *every cotangent vector $\alpha_{[m]} \in T^*_{[m]}(M/G)$ is represented by a linear continuous functional $\alpha_m : T_m M \longrightarrow \mathbb{R}$, such that $\alpha_m\big|_{T_m \operatorname{Orb}_G(m)} = 0$.*

For further reference, we explicitly write the isomorphism in item 1 as the map

$$\beta : T_{[m]}(M/G) \longrightarrow T_m M/T_m \operatorname{Orb}_G(m), \quad v_{[m]} \longmapsto [v_m],$$

where $v_m \in T_m M$ such that $T_m \pi_G(v_m) = v_{[m]}$.

Infinite dimensional extensions of Theorem 3.9

Finally, we present a brief discussion about the extension of Theorem 3.9 to the Banach setting, namely, to the case of a infinite dimensional Banach Lie group acting on a infinite dimensional Banach manifold M.

We first notice that the theorem is false in this general Banach setting, a counterexample is presented in [6]. We now indicate some infinite dimensional generalizations that hold, adding some extra hypothesis, which however are rarely satisfied in applications:

- we first note Corollary 2.5 in [9] which states that for a free and proper smooth action of a Banach Lie group on a Hilbert manifold (namely a Banach manifold modeled on a Hilbert space), the orbit space M/G carries a unique Hilbert structure that turns the natural projection $\pi_G : M \longrightarrow M/G$ into a smooth submersion.
- Also, under appropriate hypothesis, [6] contains a sophisticated slice theorem that implies a generalized version of Theorem 3.9 that holds beyond the Banach setting.
- Finally, we note that Appendix A contains an more accessible set of hypotheses, but not the sharpest, under which an extension of the theorem can be stated.

Chapter 4

Infinite dimensional symplectic reduction

This chapter presents an extension of the celebrated symplectic reduction of Marsden and Weinstein [18] to the case of Banach manifolds. In general, we refer to the standard references [4] and [19], but the contents of sections 4.1 and 4.3 follow some unpublished notes which were kindly shared to us by Professor Ratiu [16]. We start with the following hypotheses:

(SR) 1. Let M be a Banach manifold modeled on a Banach space E, and Ω a weak symplectic form on M.

We recall from section 2.4 of chapter 2 that this implies the existence of a linear continuous injective map

$$\Omega_m^\flat : T_mM \longrightarrow T_m^*M$$

such that $\langle \Omega_m^\flat(v_m), w_m\rangle := \Omega_m(v_m, w_m)$ for every $v_m, w_m \in T_mM$, for all $m \in M$.

(SR) 2. Let G be a Banach Lie group that acts ***symplectically*** on M by a right action $\Phi : M \times G \longrightarrow M$, i.e., for each $g \in G$, it holds

$$\Phi_g^* \, \Omega = \Omega.$$

Assume also that the orbit space M/G is a Banach manifold and the natural projection $\pi_G : M \longrightarrow M/G$ is a smooth submersion.[1]

4.1 Symplectic linear algebra on Banach spaces

In this section we introduce some results of symplectic linear algebra over Banach spaces that will be crucial for symplectic reduction. Some preliminaries on locally convex topological vector spaces are required, for which we refer to Appendix B.1 and its references. The next theorem is an adaptation of some results of Kriegl [12],

[1] As explained in section 3.1.2 there is no simple set of hypothesis which guarantees this assumption which is useful in applications.

presented as Lemma B.2, Lemma B.3 and Theorem B.4 reported in Appendix B. The concepts appearing in its statement are also reviewed in Appendix B.

Theorem 4.1. *Let V be a locally convex real vector space, whose topology is generated by the family of seminorms $\mathcal{P}$.*

- *Let $\alpha : V \longrightarrow \mathbb{R}$ be a linear continuous map, then there exist $p_1, \dots, p_n \in \mathcal{P}$ and a positive constant $C > 0$ such that, for every $x \in V$,*
$$|\alpha(x)| \leq C \max\{p_1(x), \dots, p_n(x)\};$$
- *let $W \subseteq V$ be a closed subspace, and $v \notin W$, then there exists a linear continuous functional $\alpha : V \longrightarrow \mathbb{R}$, such that $\alpha\big|_W = 0$ and $\alpha(v) = 1$.*

Now, let E be a Banach space and Ω a skew-symmetric, weakly non degenerate bilinear form. By definition, there exists an injective, linear continuous map $\Omega^\flat : E \longrightarrow E'$ such that, for every $v, w \in E$,
$$\langle \Omega^\flat(v), w \rangle = \Omega(v, w).$$

In this context, one can prove that for each $y \in E$, the map $p_y : E \longrightarrow \mathbb{R}$, $x \longmapsto |\Omega(x, y)|$ is a seminorm over E. Due to the weakly non-degeneracy of Ω, the family $\{p_y\}_{y \in E}$ defines a locally convex Hausdorff topology on E, which, following [16], will be called the ***Ω-topology***.

Now, we recall the idea of symplectic orthogonal and we prove a very pleasant property concerning the symplectic orthogonal of any closed subspace of E.

Definition 4.2. *Let $F \subseteq E$ be a closed subspace, then the set*
$$F^\Omega := \{v \in E : \Omega(v, w) = 0 \quad \forall w \in F\},$$
is a closed subspace of E and it is called the ***symplectic orthogonal*** *to F with respect to Ω.*

The following lemma and the subsequent Proposition 4.4 are taken from [16]. The lemma will be used in section 4.3 and the proposition is needed for its proof.

Lemma 4.3. *Let E be a Banach space and Ω be a skew-symmetric, weakly non-degenerate bilinear form on E. Then for each closed subspace $F \subseteq E$, we have*
$$F = (F^\Omega)^\Omega.$$

Proposition 4.4. *Let $\alpha : E \longrightarrow \mathbb{R}$ be a linear continuous map with respect to the Ω-topology on E, then there exists a $y \in E$ such that $\alpha(x) = \Omega(x, y)$ for all $x \in E$.*

Proof. From Theorem 4.1 we know that there exist $y_1, \ldots, y_n \in E$ and a positive constant $C > 0$ such that

$$|\alpha(x)| \leq C \max_{1 \leq i \leq n} |\Omega(x, y_i)|.$$

Then, α vanishes on

$$F := \bigcap_{i=1}^{n} \ker \Omega^\flat(y_i) = (\operatorname{span}\{y_1, \ldots, y_n\})^\Omega;$$

F is clearly closed with respect to the Banach space topology on E and its codimension is less than n. Now, let $\hat{F}$ be an algebraic complement to F in E, since F is finite codimensional, every algebraic complement must be finite dimensional vector subspace, and hence closed in the norm topology, i.e. the closed subspace F splits in E.

In this setting, it is clear that the dual space $\hat{F}^*$ is finite dimensional vector space and it is spanned by $\{\Omega^\flat(y_1)\big|_{\hat{F}}, \ldots, \Omega^\flat(y_n)\big|_{\hat{F}}\}$ and thus we can write

$$\alpha\big|_{\hat{F}} = \sum_{k=1}^{n} \alpha_k \Omega^\flat(y_k)\big|_{\hat{F}} = \Omega^\flat\left(\sum_{k=1}^{n} \alpha_k y_k\right)\big|_{\hat{F}}.$$

Moreover, since both sides of this equality vanish on F, we get $\alpha = \Omega^\flat\left(\sum_{k=1}^{n} \alpha_k y_k\right)$ and so the proposition is proved.

□

Proof of Lemma 4.3. By definition it is clear that $F \subseteq (F^\Omega)^\Omega$. We prove the other inclusion by showing that $E \smallsetminus F \subseteq E \smallsetminus (F^\Omega)^\Omega$. Suppose $v \in E \smallsetminus F$. Theorem 4.1 implies the existence of a linear continuous functional $\alpha : E \longrightarrow \mathbb{R}$ (with respect to the Ω-topology on E), such that $\alpha(w) = 0$ for all $w \in F$ and $\alpha(v) = 1$; moreover, Proposition 4.4 implies that there exists a $y \in E$ such that $\alpha(x) = \Omega(x, y)$ for all $x \in E$. Thus, $\Omega(v, w) \neq 0$ and $\Omega(w, y) = 0$ for all $w \in F$. In other words $\Omega(v, y) \neq 0$ and $y \in F^\Omega$, i.e., $v \notin (F^\Omega)^\Omega$.

□

4.2 Hamiltonian systems and symmetries

Under the assumption **(SR) 1.**, a ***Hamiltonian system*** is a pair (M, X_H) where $X_H \in \mathfrak{X}(M)$ is the ***Hamiltonian vector field*** associated to a smooth function $H : M \longrightarrow \mathbb{R}$, i.e. a smooth vector field over M that satisfies

$$\mathbf{i}_{X_H} \Omega = -\mathbf{d} H,$$

where $\mathbf{d} H : M \longrightarrow T^*M$, $m \longmapsto T_m H$ is a differential 1-form[2].

[2] We remark that, as discussed in section 1 and 4 of [4], in the infinite dimensional setting, given a smooth function H on a weak symplectic manifold, an associated Hamiltonian vector field X_H need not exist. In virtue of that, whenever we deal with Hamiltonian functions on weak symplectic manifolds, we are also implicitly assuming that an associated Hamiltonian vector field actually exists.

A right action Φ of a Banach Lie group G defined as in **(SR) 2.** is called a ***symmetry*** of the Hamiltonian system (M, X_H) if it is Φ***-invariant***, namely, for each $g \in G$, we have

$$\Phi_g^* \, X_H = X_H.$$

One can check that this condition is equivalent to requiring that H is Φ***-invariant***, namely $H(m) = H(\Phi_g(m))$ for each $g \in G$ and $m \in M$.

Now, let (M, X_H) be a Hamiltonian system and $\Phi : M \times G \longrightarrow M$ be a symmetry. Assume that for each $\xi \in \mathfrak{g}$ the infinitesimal generator $\xi_M \in \mathfrak{X}(M)$ of the action Φ is a Hamiltonian vector field with Hamiltonian function $J_\xi : M \longrightarrow \mathbb{R}$. Assume, moreover, that there exists a smooth map $J : M \longrightarrow \mathfrak{g}^*$ such that

$$\langle J(m), \xi \rangle := J_\xi(m),$$

for each $m \in M$ and $\xi \in \mathfrak{g}$. If the map J is a smooth map between Banach manifolds, it is called a ***momentum map associated to*** Φ. We say that J is ***equivariant*** if for each $m \in M$ and $g \in G$ we have

$$J(\Phi_g(m)) = (\mathrm{Ad}^*)_g(J(m)), \tag{4.1}$$

where $\mathrm{Ad}^* : \mathfrak{g}^* \times G \longrightarrow \mathfrak{g}^*$ is the coadjoint action of G on $\mathfrak{g}^*$ defined in chapter 3.

Proposition 4.5. *For any equivariant momentum map associated to Φ, $J : M \longrightarrow \mathfrak{g}^*$, fixed $\mu \in \mathfrak{g}^*$, for each $\xi \in \mathfrak{g}$ and $m \in J^{-1}(\mu)$ we have*

$$T_m J(\xi_M(m)) = \xi_{\mathfrak{g}^*}(\mu),$$

where $\xi_{\mathfrak{g}^*}$ is the infinitesimal generator of Ad^*.

Proof. Fixed any $m \in M$, the claim is proved by differentiating formula (4.1) at $e \in G$. □

We conclude the section with a remark about the level sets of an equivariant momentum map J. Fixed $\mu \in \mathfrak{g}^*$ a weakly regular value (see section 2.2.2 of chapter 2) for the smooth map $J : M \longrightarrow \mathfrak{g}^*$, the following proposition will prove that the restriction of the Hamiltonian vector field $X_H\big|_{J^{-1}(\mu)}$ is actually a vector field on the manifold $J^{-1}(\mu)$.

Proposition 4.6. *In the setting given above, the following hold.*

1. *For each $\xi \in \mathfrak{g}$, $v_m \in T_m M$, we have*

$$\Omega_m(v_m, \xi_M(m)) = \langle T_m J(v_m), \xi \rangle.$$

2. *For every weakly regular value for an equivariant momentum map $J : M \longrightarrow \mathfrak{g}^*$, $\mu \in \mathfrak{g}^*$, $m \in J^{-1}(\mu)$ and Φ-invariant Hamiltonian vector field $X_H \in \mathfrak{X}(M)$, we have*

$$X_H(m) \in T_m(J^{-1}(\mu)).$$

Proof.

1. To conclude we just observe that:

$$\langle T_m J(v_m), \xi \rangle = T_m J_\xi(v_m) = \langle \mathbf{d}\, J_\xi(m), v_m \rangle = \langle -(\mathbf{i}_{\xi_M}\, \Omega)_m, v_m \rangle = \Omega_m(v_m, \xi_M(m)),$$

   because ξ_M is the Hamiltonian vector field associated to J_ξ by definition.

2. Since μ is a weakly regular value for J, we know that $X_H(m) \in T_m(J^{-1}(\mu))$ if and only if $T_m J\ (X_H(m)) = 0_{\mathfrak{g}^*}$. For each $\xi \in \mathfrak{g}$, by point 1., we have

$$\langle T_m J\ (X_H(m)), \xi \rangle = \Omega_m(X_H(m), \xi_M(m)) = -T_m H\ (\xi_M(m)).$$

   Now, by invariance of X_H, that is equivalent to Φ-invariance of the Hamiltonian H, we have $H(\Phi_{\exp_G(t\xi)}(m)) = H(m)$ for every $t \in \mathbb{R}$. Differentiation at $t = 0$ gives

$$T_m H(\xi_M(m)) = 0,$$

   which proves 2.

□

4.3 Symplectic reduction

Let (M, X_H) a Hamiltonian system with a symmetry $\Phi :\ M \times G \longrightarrow M$ that admits an equivariant momentum map $J :\ M \longrightarrow \mathfrak{g}^*$. We fix a weakly regular value $\mu \in \mathfrak{g}^*$ for the smooth map J and we recall, from section 3.1 of chapter 3, the definition of isotropy group G_μ with respect to the coadjoint action

$$G_\mu := \{ g \in G :\ \mathrm{Ad}_g^*\ \mu = \mu \} \le G,$$

which, for this section, is assumed to be a Lie subgroup of G with associated Lie algebra $\mathfrak{g}_\mu$. Moreover, we define the smooth map

$$\Phi_\mu := \Phi\big|_{J^{-1}(\mu)\times G_\mu} :\ J^{-1}(\mu) \times G_\mu \longrightarrow M,$$

which can be checked to be a smooth right action of G_μ on $J^{-1}(\mu)$, due to the equivariance of the momentum map. In addition, one can prove that the vector field $X_H\big|_{J^{-1}(\mu)} \in \mathfrak{X}(J^{-1}(\mu))$ is Φ_μ-invariant.

In the setting above, we assume that the orbit space (related to the action Φ_μ)

$$M_\mu := J^{-1}(\mu)/G_\mu,$$

is a Banach manifold and the natural projection $\pi_\mu :\ J^{-1}(\mu) \longrightarrow M_\mu$ is a smooth submersion. Under these hypotheses, the following theorem shows that this quotient manifold can be endowed with a unique weak symplectic form.

Theorem 4.7 (Symplectic reduction theorem)**.**

1. *There exists a unique weak symplectic form* Ω_μ *over the Banach manifold* M_μ*, that satisfies*
$$i_\mu^*\Omega = \pi_\mu^*\Omega_\mu, \tag{4.2}$$
where $i_\mu : J^{-1}(\mu) \longrightarrow M$ *is the inclusion.*

2. *Moreover if* Ω *is a strong symplectic form on* M*, then so is* Ω_μ*.*

Remark 4.8. Point 1. of this theorem and its proof may be found in standard references like [18] and [19]. On the other hand, point 2. and its proof is less known and is taken from the unpublished notes [16].

Lemma 4.9 (Reduction lemma)**.** *In the setting of Theorem 4.7, let* $\mu \in \mathfrak{g}^*$ *and* $m \in J^{-1}(\mu)$*, then:*

1. $\mathrm{Orb}_{G_\mu}(m) = \mathrm{Orb}_G(m) \cap J^{-1}(\mu)$*;*

2. *it holds*
$$T_m \,\mathrm{Orb}_{G_\mu}(m) = T_m \,\mathrm{Orb}_G(m) \cap T_m(J^{-1}(\mu));$$

3. $T_m(J^{-1}(\mu))$ *is the symplectic orthogonal to* $T_m \,\mathrm{Orb}_G(m)$ *with respect to* Ω_m*.*

Proof. 1. We just observe that $\Phi_g(m) \in J^{-1}(\mu)$ if and only if $\mu = J(\Phi_g(m)) = (\mathrm{Ad}^*)_g(J(m)) = (\mathrm{Ad}^*)_g(\mu)$ (by equivariance of J) if and only if $g \in G_\mu$.

2. Suppose that $v_m \in T_m \,\mathrm{Orb}_G(m) \cap T_m(J^{-1}(\mu))$. Then we know $v_m = \xi_M(m)$ for some $\xi \in \mathfrak{g}$ and $T_m J(v_m) = 0$. Thanks to Proposition 4.5 we deduce that

$$0 = T_m J(\xi_M(m)) = \xi_{\mathfrak{g}^*}(\mu),$$

i.e., $\xi \in \mathfrak{g}_\mu$. So, $v_m = \xi_M(m)$ with $\xi \in \mathfrak{g}_\mu$ and so $v_m \in T_m \,\mathrm{Orb}_{G_\mu}(m)$. The reverse inclusion is immediate since by point 1., $\mathrm{Orb}_{G_\mu}(m)$ is included in both $\mathrm{Orb}_G(m)$ and $J^{-1}(\mu)$.

3. Fix $\xi \in \mathfrak{g}$ and $v_m \in T_m M$. Point 1. of Proposition 4.6 implies

$$\langle T_m J(v_m), \xi\rangle = \Omega_m(v_m, \xi_M(m)).$$

Thus $v_m \in \ker(T_m J)$ if and only if $\Omega_m(v_m, \xi_M(m)) = 0$ for all $\xi \in \mathfrak{g}$, that is equivalent to ask $\Omega_m(v_m, w_m) = 0$ for all $w_m \in T_m \,\mathrm{Orb}_G(m)$.

□

In virtue of Lemma 4.3 and item 3 of the above lemma, we conclude that $T_m \,\mathrm{Orb}_G(m)$ and $T_m(J^{-1}(\mu))$ are one the symplectic orthogonal of the other with respect to Ω_m, namely

$$T_m \,\mathrm{Orb}_G(m)^{\Omega_m} = T_m(J^{-1}(\mu)) \quad \text{and} \quad T_m(J^{-1}(\mu))^{\Omega_m} = T_m \,\mathrm{Orb}_G(m) \tag{4.3}$$

This will be used in the proof below.

Proof of Theorem 4.7. Since π_μ is a surjective submersion, if Ω_μ exists, it is uniquely determined by the condition $\pi_\mu^*\Omega_\mu = i_\mu^*\Omega$. This relation also defines Ω_μ in the following way. According to Proposition 3.10 we denote by $[v] \in T_m(J^{-1}(\mu))/T_m \operatorname{Orb}_{G_\mu}(m)$ a tangent vector to M_μ at $[m] \in M_\mu$. Then $\pi_\mu^*\Omega_\mu = i_\mu^*\Omega$ is equivalent to saying that

$$(\Omega_\mu)_{[m]}([v],[w]) = \Omega_m(v,w)$$

for all $v, w \in T_m(J^{-1}(\mu))$. We proceed to prove that Ω_μ is a weak symplectic form over M_μ.

First of all, we fix $m \in J^{-1}(\mu)$, and we consider $v', w' \in T_m(J^{-1}(\mu))$ such that $[v'] = [v]$ and $[w'] = [w]$. This is equivalent to the existence of $\xi, \eta \in \mathfrak{g}_\mu$ such that $v' = v + \xi_M(m)$ and $w' = w + \eta_M(m)$. We observe that

$$\begin{aligned}\Omega_m(v', w') &= \Omega_m(v + \xi_M(m), w + \eta_M(m)) \\ &= \Omega_m(v,w) + \Omega_m(v, \eta_M(m)) + \Omega_m(\xi_M(m), w) + \Omega_m(\xi_M(m), \eta_M(m)) \\ &= \Omega_m(v,w).\end{aligned}$$

Indeed $\Omega_m(v, \eta_M(m)) = \Omega_m(\xi_M(m), w) = \Omega_m(\xi_M(m), \eta_M(m)) = 0$ by (4.3). Moreover, if $[m'] = [m]$, there exists $g \in G$ such that $\Phi_g(m) = m'$, and we have

$$\Omega_{m'}(T_m\Phi_g(v), T_m\Phi_g(w)) = (\Phi_g^*\Omega)_m(v,w) = \Omega_m(v,w),$$

where the last identity uses the fact that Φ is symplectic.

Thus Ω_μ is well-defined. Also, it is smooth since $\pi_\mu^*\Omega_\mu$ is smooth and, since $\mathbf{d}\,\Omega = 0$, it holds

$$\pi_\mu^*(\mathbf{d}\,\Omega_\mu) = \mathbf{d}(\pi_\mu^*\Omega_\mu) = \mathbf{d}(i_\mu^*\Omega) = i_\mu^*(\mathbf{d}\,\Omega) = 0.$$

Since π_μ is a surjective submersion, we conclude that $\mathbf{d}\,\Omega_\mu = 0$.

Finally, we prove that it is weakly non-degenerate. Fix $v \in T_m(J^{-1}(\mu))$ and assume that $(\Omega_\mu)_{[m]}([v],[w]) = 0$ for all $w \in T_m(J^{-1}(\mu))$. Then

$$\Omega_m(v,w) = \Omega_{[m]}([v],[w]) = 0,$$

and by weak non-degeneracy of Ω_m we obtain $v = 0$, that implies $[v] = 0$. Then, by definition, Ω_μ is a weak symplectic form over M_μ.

Now, suppose that Ω is a strong symplectic form and let $\overline{\alpha} \in T^*_{[m]}M_\mu$. According to Proposition 3.10, it can be represented by a linear continuous map $\alpha\colon T_m(J^{-1}(\mu)) \longrightarrow \mathbb{R}$ vanishing on the closed subspace $T_m \operatorname{Orb}_{G_\mu}(m)$, i.e., $\overline{\alpha}([w]) = \alpha(w)$ for all $w \in T_m(J^{-1}(\mu))$. Since Ω is strongly non degenerate, there exists a $v \in T_mM$ such that $\Omega_m(v,w) = \alpha(w)$ for all $w \in T_m(J^{-1}(\mu))$. Finally, from Lemma 4.9, it follows that $v \in (T_m \operatorname{Orb}_{G_\mu}(m))^\Omega \subseteq T_m(J^{-1}(\mu))$ and so

$$(\Omega_\mu)_{[m]}([v],[w]) = \overline{\alpha}([w]),$$

for all $w \in T_m(J^{-1}(\mu))$. Then Ω_μ is a strong symplectic form over M_μ.

□

4.3.1 Symplectic reduction of the Hamiltonian dynamics

In the setting of previous sections, from the restriction $X_H\big|_{J^{-1}(\mu)}$ of the Hamiltonian vector field X_H we can build the ***reduced vector field***, $\tilde{X}: M_\mu \longrightarrow T(M_\mu)$, defined by

$$\tilde{X}(\pi_G(m)) = T_m\pi_G(X(m)),$$

that is a smooth vector field over M_μ due to the fact that π_μ is a smooth submersion. In the following, we prove that the reduced vector field is a Hamiltonian vector field with respect to the weak symplectic form Ω_μ on M_μ.

Proposition 4.10. *The reduced vector field $\tilde{X} \in \mathfrak{X}(M_\mu)$ defined above is a Hamiltonian vector field over M_μ with respect to the weak symplectic form Ω_μ.*

Proof. We define the unique smooth function $\tilde{H}: M_\mu \longrightarrow \mathbb{R}$ that satisfies

$$\tilde{H} \circ \pi_\mu = H\big|_{J^{-1}(\mu)}.$$

We proceed to show that $\mathbf{i}_{\tilde{X}}\,\Omega_\mu = -\,\mathbf{d}\,\tilde{H}$. Firstly we observe that

$$\begin{aligned}
(\mathbf{i}_{\tilde{X}}\,\Omega_\mu)_{[m]}([v]) &= (\Omega_\mu)_{[m]}(\tilde{X}([m]),[v]) \\
&= (\Omega_\mu)_{[m]}(T_m\pi_\mu(X_H(m)),[v]) \\
&= \Omega_m(X_H(m),v) \\
&= (\mathbf{i}_{X_H}\,\Omega)_m(v) \\
&= -(\mathbf{d}\,H)_m(v).
\end{aligned}$$

Also it holds

$$\begin{aligned}
(\mathbf{d}\,\tilde{H})_{[m]}([v]) &= T_{[m]}\tilde{H}([v]) \\
&= T_{[m]}\tilde{H} \circ T_m\pi_\mu(v) \quad \text{(thanks to the chain rule)} \\
&= T_m(\tilde{H} \circ \pi_\mu)(v) \\
&= T_m H(v) = (\mathbf{d}\,H)_m(v).
\end{aligned}$$

So we obtain that $\tilde{X}$, by definition, is the Hamiltonian vector field associated to $\tilde{H}$ with respect to the weak symplectic form Ω_μ.

□

The Hamiltonian system $(M_\mu, \tilde{X})$ is called the ***reduced system*** of (M, X_H) with respect to the symmetry Φ and the momentum $\mu \in \mathfrak{g}^*$.

Chapter 5

Cotangent bundle reduction

The main topic of the chapter is one of the most known cases of symplectic reduction, namely the symplectic reduction of cotangent bundles. This procedure is largely used in applications, as we will exemplify in chapter 6.

In contrast to previous content of the thesis, here we will limit the discussion to the finite dimensional setting. Section 5.1 is based on chapter 4 of the standard reference [15], while the general ideas concerning symplectic reduction are taken from section 2.2 of [19].

5.1 Mechanical G-systems

A ***mechanical G-system*** consists of a configuration manifold Q with a kinetic energy that is invariant under the action of a Lie group G. The kinetic energy $T:\ TQ \longrightarrow \mathbb{R}$ is geometrically interpreted in terms of a Riemannian metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on Q. More precisely,

$$T(q, v_q) := \frac{1}{2} \langle\!\langle v_q, v_q \rangle\!\rangle_q^2,$$

for $v_q \in T_qQ$. The associated flat map (see section 2.4 of chapter 2)

$$\mathbb{F}:\ TQ \longrightarrow T^*Q$$

is a diffeomorphism of smooth manifolds and it is called the ***Legendre transformation*** of the mechanical system. Here, we refer to the image $T^*Q := \mathbb{F}(TQ)$ of the Legendre transformation as the ***phase space***[1] of the system, that, for our purposes, is always endowed with its canonical symplectic form Ω_{can}.

We recall that the kinetic energy pulls-back via $\mathbb{F}^{-1}$ to a Hamiltonian function on the phase space $H:\ T^*Q \longrightarrow \mathbb{R}$, defined by

$$H(q, \alpha_q) = T(\mathbb{F}^{-1}(q, \alpha_q)).$$

[1]We stress that we labeled the image of the Legendre transformation as the phase space of the system, indeed, although in the finite dimensional case the map $\mathbb{F}:\ TQ \longrightarrow T^*Q$ is always a diffeomorphism and it seems to make sense to label in such a way the cotangent bundle T^*Q, we will see in chapter 6, that for infinite dimensional systems the Legendre transformation may be not surjective.

At this stage, we define what a ***symmetry*** of the mechanical system is. It is understood as a free, smooth right action of a finite dimensional Lie group G (with associated Lie algebra $\mathfrak{g}$) on the configuration space Q,

$$\Phi: \ Q \times G \longrightarrow Q,$$

such that the orbit space Q/G is a smooth manifold and the natural projection $\pi_G: \ Q \longrightarrow Q/G$ is a smooth submersion; also, we suppose that for every $g \in G$, the diffeomorphism $\Phi_g: \ Q \longrightarrow Q$ preserves the metric on Q, namely, for each $q \in Q$, $v_q, w_q \in T_qQ$, we have

$$\langle\!\langle T_q\Phi_g(v_q), T_q\Phi_g(w_q)\rangle\!\rangle_{\Phi_g(q)} = \langle\!\langle v_q, w_q\rangle\!\rangle_q.$$

Remark 5.1. Under these hypotheses, we know that there exists a well defined Riemannian metric on the quotient manifold Q/G, called the ***quotient metric*** (and also denoted $\langle\!\langle \cdot, \cdot \rangle\!\rangle$), that is defined by

$$\langle\!\langle [v], [w]\rangle\!\rangle_{[q]} = \langle\!\langle v, w\rangle\!\rangle_q$$

with associated Legendre transformation (namely the associated flat map):

$$\tilde{\mathbb{F}}: \ T(Q/G) \longrightarrow T^*(Q/G)$$

that will be useful at a later stage.

It is well known that the symmetry Φ lifts to a symplectic action of G on the phase space T^*Q through cotangent lift[2]. Namely, the map

$$\Phi^{T^*Q}: \ T^*Q \times G \longrightarrow T^*Q, \qquad (\alpha_q, g) \longmapsto T^*(\Phi_g)(\alpha_q),$$

is a free, smooth symplectic right action of G on the phase space T^*Q, called the ***cotangent lifted action*** of Φ. We recall from the standard reference [19] that the cotangent lifted action admits an equivariant momentum map $J: \ T^*Q \longrightarrow \mathfrak{g}^*$ defined by

$$\langle J(\alpha_q), \xi\rangle = \langle \alpha_q, \xi_Q(q)\rangle, \tag{5.1}$$

for each $\xi \in \mathfrak{g}$ and $\alpha_q \in T_q^*Q \subset T^*Q$.

5.2 Symplectic reduction of cotangent bundles

In this section, we perform symplectic reduction for the Hamiltonian system (T^*Q, X_H) associated to a mechanical system. The proofs of the statements reported below can be found in section 2.2 of [19].

[2]Given a diffeomorphism of finite dimensional manifolds $f: \ M \longrightarrow N$, it induces a symplectic diffeomorphism between the cotangent bundles with their canonical symplectic forms, $T^*f: \ T^*N \longrightarrow T^*M$, defined by $\langle T^*f(\alpha_n), v_m\rangle = \langle \alpha_n, T_mf(v_m)\rangle$ for each $\alpha_n \in T_n^*N$, $v_m \in T_mM$ and $n = f(m)$.

Let Q be the configuration space of a mechanical system with a symmetry $\Phi : Q \times G \longrightarrow Q$. We notice that, since we supposed Φ to be a free action, $0 \in \mathfrak{g}^*$ is a regular value for the canonical momentum map $J : T^*Q \longrightarrow \mathfrak{g}^*$ associated to the cotangent lifted action of Φ. Also, we recall that the isotropy group at $0 \in \mathfrak{g}^*$ of the coadjoint action $\mathrm{Ad}^* : \mathfrak{g}^* \times G \longrightarrow \mathfrak{g}^*$ is the whole group G and equivariance of the momentum map yields an induced action of G on the zero level set $J^{-1}(0)$,

$$\Phi_0 : J^{-1}(0) \times G \longrightarrow J^{-1}(0), \qquad (\alpha_q, g) \longmapsto \Phi_g^{T^*Q}(\alpha_q).$$

Here, Proposition 4.6 of chapter 4 implies that $X_H\big|_{J^{-1}(0)}$ is a Φ_0-invariant smooth vector field on $J^{-1}(0)$.

Provided that the orbit space $J^{-1}(0)/G$ is a smooth manifold and the natural projection $\pi_0 : J^{-1}(0) \longrightarrow J^{-1}(0)/G$ is a smooth submersion, Theorem 4.2 of chapter 4 implies the existence a reduced symplectic form Ω_0 on the quotient manifold $J^{-1}(0)/G$, namely a symplectic form Ω_0 such that

$$\pi_0^* \Omega_0 = i_0^* \Omega_{can},$$

where $i_0 : J^{-1}(0) \longrightarrow T^*Q$ is the inclusion. In addition, the following theorem implies that the reduced space $J^{-1}(0)/G$ is symplectomorphic to a cotangent bundle, that will be called the ***reduced phase space*** of the system.

Theorem 5.2 ([19], Theorem 2.2.2)**.** *There is a symplectic diffeomorphism between $J^{-1}(0)/G$ and $T^*(Q/G)$ with its canonical symplectic structure.*

5.2.1 Reduction of the Hamiltonian dynamics

Here, we focus on the reduction of the Hamiltonian vector field X_H. Proposition 4.10 of chapter 4 implies that the restricted vector field $X_H\big|_{J^{-1}(0)}$ is related through the projection $\pi_0 : J^{-1}(0) \longrightarrow J^{-1}(0)/G$ to a Hamiltonian vector field $X_{\tilde{H}} \in \mathfrak{X}(J^{-1}(0)/G)$ (with respect to the reduced symplectic form Ω_0) and the reduced Hamiltonian is the unique smooth map $\tilde{H} : J^{-1}(0)/G \longrightarrow \mathbb{R}$ such that

$$\tilde{H} \circ \pi_0 = H\big|_{J^{-1}(0)}.$$

Here, the symplectic diffeomorphism $\alpha_0 : J^{-1}(0)/G \longrightarrow T^*(Q/G)$, whose existence is guaranteed by Theorem 5.2, yields a Hamiltonian vector field

$$X_h := (\alpha_0)_* X_{\tilde{H}} \in \mathfrak{X}(T^*(Q/G)),$$

with associated Hamiltonian function $h : T^*(Q/G) \longrightarrow \mathbb{R}$, called the ***reduced Hamiltonian***, uniquely determined by the relation $h \circ \alpha_0 = \tilde{H}$.

From a Lagrangian point of view, the reduction of the Hamiltonian dynamics corresponds to a reduction of the kinetic energy $T : TQ \longrightarrow \mathbb{R}$. One can prove (see for example the standard reference [15]) that the kinetic energy $T_{red} : T(Q/G) \longrightarrow \mathbb{R}$

associated to the quotient metric on Q/G introduced in Remark 5.1, namely the smooth function on $T(Q/G)$ such that

$$T = T_{red} \circ T\pi_G,$$

where $\pi_G\colon\ Q \longrightarrow Q/G$ is the canonical projection, is related to the reduced Hamiltonian $h\colon\ T^*(Q/G) \longrightarrow \mathbb{R}$ by the formula

$$T_{red} = h \circ \tilde{\mathbb{F}},$$

where $\tilde{\mathbb{F}}\colon\ T(Q/G) \longrightarrow T^*(Q/G)$ is the Legendre transformation associated to the quotient metric on Q/G.

Chapter 6

The fluid-solid problem

In this chapter, we study the equations of motion for a planar rigid body moving in a potential two dimensional fluid in absence of external forces from the perspective of symplectic reduction. During the chapter we will refer to the setting described above as the ***fluid-solid system***.

The equations of motion for such a rigid body were first described by Kirchhoff [10] and are

$$\begin{cases} \dot{k} = V_y p_x - V_x p_y \\ \dot{p_x} = \Omega p_y \\ \dot{p_y} = -\Omega p_x \end{cases} \tag{6.1}$$

where $\mathbf{V} = (V_x, V_y)^T \in \mathbb{R}^2$ and $\Omega \in \mathbb{R}$ are the linear and angular velocity of the body, while $k \in \mathbb{R}$ and $\mathbf{p} = (p_x, p_y)^T \in \mathbb{R}^2$ in classical hydrodynamics are known as "impulsive pair" and "impulsive force" respectively. They are defined by $(k, \mathbf{p})^T = \mathbb{M}\ (\Omega, \mathbf{V})^T$ where

$$\mathbb{M} := \mathbb{M}_b + \mathbb{M}_a \tag{6.2}$$

is the ***total mass matrix*** of the rigid body, consisting of the ***mass matrix*** of the body

$$\mathbb{M}_b = \begin{pmatrix} I & 0 \\ 0 & m\mathbf{I}_2 \end{pmatrix},$$

where m is its mass, I its moment of inertia about its center of mass and $\mathbf{I}_2$ is the 2×2 identity matrix. The matrix $\mathbb{M}_a$ in (6.2) is called the ***matrix of added masses and inertia*** induced by the fluid, whose components only depends on the geometry of the rigid body.

We will see that, from a geometrical point of view, the fluid-solid system can be seen as an "infinite dimensional" mechanical system on which we can apply two commuting reduction procedures. As we will see, the assumption for the fluid to be potential fixes the motion of the system on the zero level set of an equivariant momentum map that is associated to an infinite dimensional symmetry. At this stage, performing the symplectic reduction yields a finite dimensional mechanical system which can be treated in an analogous way to the geometrical description of inertial motions of a free rigid body. Indeed, we will perform a Lie-Poisson reduction and we will notice that the reduced equations of motion are exactly the Kirchhoff equations (6.1).

6.1 The classical formulation of the fluid-solid problem

Consider a planar rigid body that occupies a simply connected smooth region of the plane $\mathcal{B} \subset \mathbb{R}^2$, moving inside a two dimensional perfect fluid (with constant density ρ) that fills the complement of the plane $\mathcal{F} := \mathbb{R}^2 \setminus \mathcal{B}$.

We introduce an orthonormal inertial frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\{\mathbf{e}_1, \mathbf{e}_2\}$ span the plane of motion of the body and $\mathbf{e}_3$ is perpendicular to it, and a moving frame $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ attached to the body, which is is determined by a rotation of angle $\theta \in [0, 2\pi[$ around the perpendicular axis $\mathbf{e}_3$ and whose origin is chosen to be a fixed point of the body, $\mathbf{x}_0 = x_0\mathbf{e}_1 + y_0\mathbf{e}_2 \in \mathcal{B}$, which we take to be the center of mass.

The ***scalar angular velocity*** of the body and ***translational velocity*** of its center of mass relative to the inertial frame are respectively

$$\omega = \dot{\theta} \qquad \text{and} \qquad \mathbf{v} = \dot{x}_0\mathbf{e}_1 + \dot{y}_0\mathbf{e}_2 = v_x\mathbf{e}_1 + v_y\mathbf{e}_2,$$

where dots denote derivatives with respect to time. Also, $\Omega := \omega \in \mathbb{R}$ and $\mathbf{V} := R_\theta^T\mathbf{v} = (V_x, V_y) \in \mathbb{R}^2$, with $R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, represents the scalar angular velocity of the body and the translational velocity of its center of mass with respect to the body frame. We recall that the kinetic energy of the body is given by the quadratic form

$$T_{body}(\Omega, \mathbf{V}) := \frac{I}{2}\Omega^2 + \frac{m}{2}(V_x^2 + V_y^2).$$

The Lie group SE(2). From a geometrical point of view, Euler's approach to the description of a free rigid body says that a configuration of the body $(\theta, \mathbf{x}_0) \in [0, 2\pi[\times\mathbb{R}^2$ determines an element of the Lie group of special Euclidean transformations of the plane, $SE(2)$, given by

$$g = \begin{pmatrix} R & \mathbf{x}_0 \\ 0 & 1 \end{pmatrix} \in SE(2), \qquad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In this context, a scalar angular velocity of the body ω and a translational velocity of its center of mass $\mathbf{v} = v_x\mathbf{e}_1 + v_y\mathbf{e}_2$ determines a tangent vector to $SE(2)$ at $g \in SE(2)$:

$$\dot{g} = \begin{pmatrix} -\sin\theta\,\omega & -\cos\theta\,\omega & v_x \\ \cos\theta\,\omega & -\sin\theta\,\omega & v_y \\ 0 & 0 & 0 \end{pmatrix} \in T_gSE(2),$$

while the same quantities expressed with respect the body frame Ω and $\mathbf{V} = V_x\mathbf{b}_1 + V_y\mathbf{b}_2$ are associated to the 3×3 matrix

$$\xi = \begin{pmatrix} 0 & -\Omega & V_x \\ \Omega & 0 & V_y \\ 0 & 0 & 0 \end{pmatrix} = g^{-1}\dot{g}.$$

Clearly, such a matrix ξ is an element of the Lie algebra $\mathfrak{se}(2)$, which we recall to be

$$\mathfrak{se}(2) = \left\{ \begin{pmatrix} 0 & -\Omega & V_x \\ \Omega & 0 & V_y \\ 0 & 0 & 0 \end{pmatrix} : \quad \Omega, V_x, V_y \in \mathbb{R} \right\},$$

and whose associated Lie bracket is the standard commutator of matrices. For future computations, it is convenient to identify the Lie algebra $\mathfrak{se}(2)$ with the space of angular and translational velocities of the body $(\Omega, \mathbf{V}) \in \mathbb{R}^3$ expressed with respect the body frame, endowed with the Lie bracket

$$[\![(\Omega, \mathbf{V}), (\tilde{\Omega}, \tilde{\mathbf{V}})]\!]_{\mathfrak{se}(2)} := (0, -\Omega \tilde{V}_y + \tilde{\Omega} V_y, \Omega \tilde{V}_x - \tilde{\Omega} V_x) \in \mathbb{R}^3.$$

Next, we consider the perfect fluid surrounding the body, whose formal geometrical description is given in the following section. We remark that the fluid's Eulerian velocity field, $\mathbf{u}$, is represented by a smooth divergence free vector field on $\mathcal{F}$ which we assume to decay at infinity. In order to avoid cavitation or penetration of the fluid into the body, we impose the condition that the normal component of $\mathbf{u}$ at a point of the boundary of $\mathcal{F}$ agrees with the normal component of the total velocity of the body at the same point. Namely, we require $\mathbf{u}$ to be a solution of the following problem:

$$\begin{cases} \operatorname{div} \mathbf{u} = 0 & \text{on } \mathcal{F}, \\ \mathbf{u}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = (\mathbf{v} + \omega \; \mathbb{J}(\mathbf{x}\text{-}\mathbf{x}_0)) \cdot \mathbf{n}(\mathbf{x}) & \text{for } \mathbf{x} \in \partial \mathcal{F}, \\ \|\mathbf{u}(\mathbf{x})\| \longrightarrow 0 & \text{as } \|\mathbf{x}\| \to \infty \end{cases} \tag{6.3}$$

where $\mathbb{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{x}_0$ represents the center of mass of the body and $\mathbf{n}$ is the outer normal to $\partial \mathcal{F}$ expressed with respect to the inertial frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

The kinetic energy of the fluid is hence given by

$$T_{fluid} := \frac{\rho}{2} \int_{\mathcal{F}} \|\mathbf{u}(\mathbf{x})\|^2 \; d\mathbf{x},$$

where $d\mathbf{x}$ is the the standard Euclidean area element of $\mathbb{R}^2$ restricted to $\mathcal{F}$.

The total kinetic energy of the system is given by the sum of the kinetic energies of the rigid body and the fluid:

$$T = T_{body} + T_{fluid} = \frac{I}{2}\Omega^2 + \frac{m}{2}(V_x^2 + V_y^2) + \frac{\rho}{2} \int_{\mathcal{F}} \|\mathbf{u}(\mathbf{x})\|^2 \; d\mathbf{x}.$$

The particle relabeling symmetry. It is well known that the kinetic energy of the fluid is invariant under the so called ***particle relabelling symmetry*** (see for example [17]) that is represented by a smooth right action of the Lie group of volume preserving diffeomorphisms, $\mathrm{Diff}_{vol}(\mathcal{F})$, on the configuration space of the system Q. We will show that this action makes the configuration space Q into the total space of a trivial principal fiber bundle over $SE(2)$. In other words, the orbit space of the action $Q/\mathrm{Diff}_{vol}(\mathcal{F})$ is actually a smooth manifold diffeomorphic to $SE(2)$.

Vorticity and circulation. In classical fluid dynamics, for a two dimensional fluid the ***scalar vorticity*** μ of the fluid is defined as the module of the curl of the Eulerian velocity field $\mathbf{u}$ and the ***circulation*** around the rigid body is the line integral of $\mathbf{u}$ along any closed curve $\mathcal{C}$ encircling it:

$$\nabla \times \mathbf{u} = \mu \ \mathbf{e}_3, \qquad \Gamma = \int_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{l}.$$

For our purposes, the crucial observation is that if we assume that the fluid is potential at the initial instant $t = 0$ of any motion of the system, then it must satisfy the same property for all subsequent times. This observation will be justified in the following sections where the Hamiltonian formalism is introduced. The assumption that the flow is potential means that we may write $\mathbf{u} = \nabla\phi$ for some potential ϕ. In particular the scalar vorticity μ and the circulation Γ must vanish. Equations (6.3) become:

$$\begin{cases} \Delta\phi = 0 & \text{on } \mathcal{F}, \\ \nabla\phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = (\mathbf{v} + \omega \ \mathbb{J}(\mathbf{x}\text{-}\mathbf{x}_0)) \cdot \mathbf{n}(\mathbf{x}) & \text{for } \mathbf{x} \in \partial\mathcal{F}, \\ \|\nabla\phi(\mathbf{x})\| \longrightarrow 0 & \text{as } \|\mathbf{x}\| \to \infty \end{cases} \tag{6.4}$$

which admits a unique (up to an additive constant) smooth solution $\phi : \mathcal{F} \longrightarrow \mathbb{R}$. Moreover, following Kirchhoff's original idea (see [10]), we note that a solution ϕ to the above problem clearly depends linearly by ω, $\mathbf{v}$ and hence it can be written as:

$$\phi = \omega \ \phi_\omega + v_x \ \phi_x + v_y \ \phi_y, \tag{6.5}$$

where ϕ_ω, ϕ_x, ϕ_y are smooth solutions of the Laplace equation that vanish at infinity and satisfy the Neumann type boundary conditions:

$$\nabla\phi_\omega(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = (\mathbf{x} \times \mathbf{n}(\mathbf{x})) \cdot \mathbf{e}_3 \qquad \nabla\phi_x(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \mathbf{e}_1$$

$$\nabla\phi_y(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \mathbf{e}_2$$

for any $\mathbf{x} \in \partial\mathcal{F}$. Finally, we observe that the smooth functions ϕ_ω, ϕ_x, ϕ_y depend only of the shape of the body, which does not change through any motion of the system.

The reduced dynamics of the system. From a dynamical point of view, we observe that for a potential velocity of the fluid $\mathbf{u} = \nabla\phi$, formula (6.5) allows us to rewrite the kinetic energy of the fluid in terms of the total velocity of the body:

$$T_{fluid} = \frac{\rho}{2} \int_{\mathcal{F}} \|\nabla\phi(\mathbf{x})\|^2 \ d\mathbf{x} = \frac{\rho}{2} \begin{pmatrix} \Omega & \mathbf{V} \end{pmatrix} \mathbb{M}_a \begin{pmatrix} \Omega \\ \mathbf{V} \end{pmatrix},$$

where the entries of the matrix $\mathbb{M}_a$ depends only on the shape of the body. This observation implies that the total energy of the system $T = T_{body} + T_{fluid}$ drops to a quadratic form on the Lie algebra $\mathfrak{se}(2)$,

$$T_{red}(\Omega, \mathbf{V}) = \frac{1}{2} \begin{pmatrix} \Omega & \mathbf{V} \end{pmatrix} \mathbb{M} \begin{pmatrix} \Omega \\ \mathbf{V} \end{pmatrix},$$

where $\mathbb{M} := \mathbb{M}_b + \mathbb{M}_a$ is the total mass matrix of the body. Clearly, we have that the quadratic form T_{red} is associated to kinetic energy Lagrangian $\mathcal{L} : TSE(2) \longrightarrow \mathbb{R}$ which is invariant under the lifted action of left multiplication on $SE(2)$. This invariance corresponds to the freedom of choice of origin and orientation of the inertial frame.

We note that the obtained mechanical system may be interpreted as describing the motions of a planar rigid body with added masses and inertia. In virtue of that, Lie-Poisson theory says that any motion of the body is associated to a unique solution of the so called ***Lie-Poisson*** equations on $\mathfrak{se}(2)^*$, which corresponds precisely to the classical Kirchhoff's equations of hydrodynamics (6.1) as we explicitly show below.

We recall that any element $\nu \in \mathfrak{se}(2)^*$ of the dual of the Lie algebra $\mathfrak{se}(2)$ can be identified with a vector of $\mathbb{R}^3$:

$$\nu = (k, p_x, p_y),$$

and the dual pairing with $\mathfrak{se}(2) \simeq \mathbb{R}^3$ is given by the Euclidean scalar product of $\mathbb{R}^3$ itself. We recall that the classical Lie-Poisson equations for a Lie group are $\dot{\nu} = \mathrm{ad}^*_\xi(\nu)$, where ad^* is the standard coadjoint representation of the Lie group $SE(2)$ and $\nu = \mathbb{M}\,\xi$, $\xi \in \mathfrak{se}(2)$. As anticipated, these equations become the classical Kirchhoff's equations of hydrodynamics:

$$\begin{cases} \dot{k} = p_x V_y - p_y V_x \\ \dot{p}_x = p_y \Omega \\ \dot{p}_y = -p_x \Omega \end{cases}$$

where we stress $\nu = (k, p_x, p_y)$, $\xi = (\Omega, V_x, V_y) = \mathbb{M}^{-1}\nu$ and

$$\mathrm{ad}^*_\xi(\nu) = (p_x V_y - p_y V_x, p_y \Omega, -p_x \Omega).$$

6.2 The geometry of the fluid-solid system

In this section we give some insights about the configuration space Q of the fluid-solid system. The geometrical description of a perfect fluid presented below is taken from the work [25], which presents an adaptation of the classical description given by Arnold in his celebrated paper [2], in order to consider the fact that the body moves inside the fluid.

The configuration and kinetic energy of a perfect fluid. Following section 2.1 of [25], we consider a configuration of the fluid as a smooth embedding $\varphi : \mathcal{F} \longrightarrow \mathbb{R}^2$ of the reference configuration $\mathcal{F}$ into $\mathbb{R}^2$. We require the embedding φ to be ***volume preserving*** to reflect the fact that the fluid is taken to be incompressible, namely if $d\mathbf{x}$ is the Euclidean area element on $\mathbb{R}^2$, then we require

$$\varphi^*\; d\mathbf{x} = i^*_{\mathcal{F}}\; d\mathbf{x},$$

where $i_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathbb{R}^2$ is the inclusion. In order to deal with a realistic fluid, we require that it is stationary at infinity. Mathematically, we assume that the embedding φ approaches the identity at infinity "suitably fast", namely such that the following integral converges:

$$\int_{\mathcal{F}} \|\varphi(\mathbf{x}) - \mathbf{x}\|^2 \, d\mathbf{x} < +\infty.$$

The collection of such volume-preserving embeddings, that is denoted by $\mathrm{Emb}_{vol}(\mathcal{F}, \mathbb{R}^2)$, is called the ***manifold of volume-preserving embeddings***[1].

Given a configuration of the fluid $\varphi \in \mathrm{Emb}_{vol}(\mathcal{F}, \mathbb{R}^2)$, a ***motion of the fluid*** through φ is a smooth curve $t \longmapsto \varphi_t \in \mathrm{Emb}_{vol}(\mathcal{F}, \mathbb{R}^2)$ such that $\varphi_0 = \varphi$. Moreover, associated to a motion φ_t through φ, the ***material velocity field*** of the fluid is defined as $\dot{\varphi}_t := \frac{d}{dt}\varphi_t$, that is the smooth map

$$\dot{\varphi}_t : \mathcal{F} \longrightarrow \mathbb{R}^2, \qquad \mathbf{X} \longmapsto \frac{d}{dt}\varphi_t(\mathbf{X}) \in T_{\varphi_t(\mathbf{X})}\mathbb{R}^2 = \mathbb{R}^2.$$

In contrast, the ***Eulerian velocity field*** of the fluid is given by

$$\mathbf{u}_t := \dot{\varphi}_t \circ \varphi_t^{-1},$$

which is a divergence free smooth vector field over the smooth manifold $\varphi_t(\mathcal{F})$[2]. Physically, $\mathbf{u}_t(\mathbf{x})$ is the velocity of the material of the fluid located at the current position $\mathbf{x} \in \varphi_t(\mathcal{F})$. In order to avoid confusions, in what follows we will omit the t.

Now, we define the ***kinetic energy*** of the fluid that is given by the function:

$$T_{fluid}(\varphi, \dot{\varphi}) = \frac{\rho}{2} \int_{\varphi(\mathcal{F})} \|\mathbf{u}(\mathbf{x})\|^2 \, d\mathbf{x},$$

where we recall $\mathbf{u} = \dot{\varphi} \circ \varphi^{-1}$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$. We remark that, under the assumption for the embedding φ to approach the identity at infinity suitably fast, the integral that defines the kinetic energy always converges for any admissible velocity field $\mathbf{u} = \dot{\varphi} \circ \varphi^{-1}$.

We notice that the kinetic energy T_{fluid} induces a weak Riemannian metric on the manifold $\mathrm{Emb}_{vol}(\mathcal{F}, \mathbb{R}^2)$, called the ***$L^2$ metric***:

$$\langle\langle (\varphi, \dot{\varphi}_1), (\varphi, \dot{\varphi}_2) \rangle\rangle_{Emb} := \rho \int_{\varphi(\mathcal{F})} \mathbf{u}_1 \cdot \mathbf{u}_2 \, d\mathbf{x}.$$

where $\mathbf{u}_i = \dot{\varphi}_i \circ \varphi^{-1}$, for $i = 1, 2$.

The configuration and kinetic energy of the fluid-solid system. We continue to follow [25] and define the configuration space for the fluid-solid system as the subset Q of $SE(2) \times \mathrm{Emb}_{vol}(\mathcal{F}, \mathbb{R}^2)$ given by

$$Q = \left\{ (g, \varphi) \in SE(2) \times \mathrm{Emb}_{vol}(\mathcal{F}, \mathbb{R}^2) : \ g(\partial\mathcal{B}) = \varphi(\partial\mathcal{F}) \right\}. \tag{6.6}$$

[1] such space is shown to be a Fréchet manifold for compact $\mathcal{F}$ in [21].

[2] which, as remarked in section 6.1 is represented by a smooth divergence free vector field on $\mathcal{F}$, $\varphi_t^* \mathbf{u}_t$.

Physically, since the region $\mathcal{B}$ occupied by the body is complementary to $\mathcal{F}$ which is filled with the fluid, we are asking that at any configuration of the system there is no cavitation or penetration of the fluid into the body.

We remark that the "no-penetration" condition imposed above is equivalent to require that the normal velocity of the fluid coincides with the normal velocity of the body when computed at a point of the boundary of it, while the tangential velocity can be arbitrary corresponding to the fact that there is no viscosity in the fluid. In other words, given a configuration of the system $(g, \varphi) \in Q$ and a tangent vector $\dot{g} \in T_g SE(2)$ associated to a pair of angular and translational velocities of the body expressed in the body frame

$$\begin{pmatrix} \Omega \\ \mathbf{V} \end{pmatrix} = g^{-1}\dot{g},$$

then any admissible velocity field of the fluid $\mathbf{u} = \dot{\varphi} \circ \varphi$ must satisfy the following problem:

$$\begin{cases} \operatorname{div} \mathbf{u} = 0 & \text{on } \varphi(\mathcal{F}), \\ \mathbf{u}(\mathbf{x}) \cdot \mathbf{n} = (\mathbf{V} + \Omega \ \mathbb{J}\mathbf{x}) \cdot \mathbf{n} & \text{for } \mathbf{x} \in \partial(\varphi(\mathcal{F})), \\ \int_{\varphi(\mathcal{F})} \|\mathbf{u}(\mathbf{x})\|^2 \ d\mathbf{x} < +\infty \end{cases} \tag{6.7}$$

as anticipated in section 6.1.

One can check that the configuration space Q is a submanifold of the product $SE(2) \times \mathrm{Emb}_{vol}(\mathcal{F}, \mathbb{R}^2)$ and, in this context, we describe the tangent bundle of Q:

$$TQ = \left\{ (g, \dot{g}, \varphi, \dot{\varphi}) : \ (g, \varphi) \in Q \text{ and } \mathbf{u} = \dot{\varphi} \circ \varphi^{-1} \text{ satisfies } (6.7) \right\}.$$

Finally, we define the kinetic energy of the fluid-solid system as the sum of the kinetic energies of the rigid body and the fluid, $T : \ TQ \longrightarrow \mathbb{R}$,

$$\begin{aligned} T(g, \dot{g}, \varphi, \dot{\varphi}) &= T_{body}(g^{-1}\dot{g}) + T_{fluid}(\varphi, \dot{\varphi}) \\ &= \frac{I}{2}\Omega^2 + \frac{m}{2}(V_x^2 + V_y^2) + \frac{1}{2}\int_{\varphi(\mathcal{F})} \|\mathbf{u}(\mathbf{x})\|^2 \ d\mathbf{x}, \end{aligned}$$

which induces a weak Riemannian metric on the configuration space Q, given by

$$\langle\!\langle (g, \dot{g}_1, \varphi, \dot{\varphi}_1), (g, \dot{g}_2, \varphi, \dot{\varphi}_2) \rangle\!\rangle_Q \coloneqq I \ \Omega_1\Omega_2 + m \ \mathbf{V}_1 \cdot \mathbf{V}_2 + \int_{\varphi(\mathcal{F})} \mathbf{u}_1 \cdot \mathbf{u}_2 \ d\mathbf{x},$$

where $(\Omega_i, \mathbf{V}_i) = g^{-1}\dot{g}_i$ and $\mathbf{u}_i = \dot{\varphi}_i \circ \varphi^{-1}$ for $i = 1, 2$.

6.3 The Lie group of volume preserving diffeomorphisms

In this section, we briefly recall the basic notions of the Lie group of volume preserving diffeomorphisms of $\mathcal{F}$ that will play a crucial role in the definition of a symmetry for the fluid at a later stage.

It is well known that the following set of diffeomorphisms of $\mathcal{F}$:

$$\mathrm{Diff}_{vol}(\mathcal{F}) = \left\{ \psi \in \mathrm{Diff}(\mathcal{F}) : \ \psi^* \ d\mathbf{x} = i_{\mathcal{F}}^* \ d\mathbf{x} \text{ and } \int_{\mathcal{F}} \|\psi(\mathbf{x}) - \mathbf{x}\|^2 \ d\mathbf{x} < +\infty \right\}$$

is a group with the composition of functions as multiplication. The unit element of the group is clearly the identity map $\mathrm{id}_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F}$.

The group $\mathrm{Diff}_{vol}(\mathcal{F})$ can be endowed with an infinite dimensional Lie group structure but its description is beyond the scope of this thesis. Indeed, the explicit construction of an atlas is very complicated and it deeply depends on the Euclidean metric induced on $\mathcal{F}$ from $\mathbb{R}^2$. Moreover, in our case, one has to face two more technical difficulties. The first one is the fact that $\mathcal{F}$ is chosen to be non-compact, in contrast to the compact case where, as shown in [24], a canonical differential structure is given. We refer to [21] and [8] for an abstract description of the most used differential structures concerning manifolds of smooth mappings over unbounded locally compact manifolds. The second difficulty is that $\mathcal{F}$ is understood as a manifold with boundary. We refer to [7] for some details about the diffeomorphism group of a Riemannian compact manifold with boundary.

The Lie algebra of divergence free vector fields and its dual. The Lie group $\mathrm{Diff}_{vol}(\mathcal{F})$ has an associated Lie algebra, denoted $\mathfrak{X}_{div}(\mathcal{F})$ and consisting of divergence free vector fields which are tangent to the boundary of $\mathcal{F}$ and such that the following integral converges:

$$\int_{\mathcal{F}} \|X(\mathbf{x})\|^2 \ d\mathbf{x} < +\infty.$$

The Lie bracket is the negative of the usual bracket for vector fields, namely for each $X, Y \in \mathfrak{X}_{div}(\mathcal{F})$ we have

$$[\![X, Y]\!] := -[X, Y].$$

As a topological vector space, the Lie algebra $\mathfrak{X}_{div}(\mathcal{F})$ is a Fréchet space (see Appendix B.2) whose topology cannot be induced by any norm. Hence, the topological dual of $\mathfrak{X}_{div}(\mathcal{F})$, denoted $\mathfrak{X}'_{div}(\mathcal{F})$ and consisting of all linear continuous functionals $T : \ \mathfrak{X}_{div}(\mathcal{F}) \longrightarrow \mathbb{R}$, cannot be a Fréchet space and it does not exists any suitable topology on $\mathfrak{X}'_{div}(\mathcal{F})$ such that the dual pairing

$$\langle \cdot, \cdot \rangle : \ \mathfrak{X}'_{div}(\mathcal{F}) \times \mathfrak{X}_{div}(\mathcal{F}) \longrightarrow \mathbb{R}, \quad (T, X) \longmapsto T(X)$$

is continuous.

In the following, we define the so called ***smooth part of the dual***, $\mathfrak{X}^*_{div}(\mathcal{F})$, which is actually a Fréchet vector subspace of $\mathfrak{X}'_{div}(\mathcal{F})$ and carries a continuous dual pairing with $\mathfrak{X}_{div}(\mathcal{F})$. We refer to section 3.B and section 8 of [3] for a rigorous geometrical description with proofs of the following construction.

Firstly, we observe that every bounded differential 1-form $\alpha \in \Omega^1(\mathcal{F})$[3] induces a linear continuous functional $T_\alpha : \ \mathfrak{X}_{div}(\mathcal{F}) \longrightarrow \mathbb{R}$ given by

$$T_\alpha(X) = \int_{\mathcal{F}} \alpha(X) \ d\mathbf{x},$$

[3]Namely a differential one form over $\mathcal{F}$ whose euclidean norm is bounded in $\mathcal{F}$.

and we notice that exact 1-forms give rise to the zero functional; one can check that the Stokes' theorem implies $T_{\mathbf{d}f}(X) = 0$ for all $X \in \mathfrak{X}_{div}(\mathcal{F})$ and $f \in \Omega^0(\mathcal{F}) := C^\infty(\mathcal{F})$. In virtue of that, any equivalence class $[\alpha]$ of the quotient of vector spaces $\Omega^1(\mathcal{F})/\,\mathbf{d}\,\Omega^0(\mathcal{F})$ induces a well defined linear continuous functional on the Lie algebra. So, this quotient of vector spaces, which can be proved to be a Fréchet space, is immersed in the topological dual through this identification. We denote it

$$\mathfrak{X}^*_{div}(\mathcal{F}) := \Omega^1(\mathcal{F})/\,\mathbf{d}\,\Omega^0(\mathcal{F}) \hookrightarrow \mathfrak{X}'_{div}(\mathcal{F}),$$

and it carries a well defined continuous dual pairing, namely:

$$\langle\cdot,\cdot\rangle_{\mathfrak{g}} : \ \mathfrak{X}^*_{div}(\mathcal{F}) \times \mathfrak{X}_{div}(\mathcal{F}) \longrightarrow \mathbb{R}, \qquad \langle[\alpha], X\rangle_{\mathfrak{g}} = \int_{\mathcal{F}} \alpha(X)\ d\mathbf{x}.$$

Also, since the domain $\mathcal{F}$ is just connected and not simply connected, there exists an isomorphism of vector spaces between $\mathfrak{X}^*_{div}(\mathcal{F})$ and $\mathbf{d}\,\Omega^1(\mathcal{F}) \times \mathbb{R}$, namely

$$[\alpha] \longmapsto \left(\mathbf{d}\,\alpha, \int_{\partial\mathcal{F}} \alpha\right). \tag{6.8}$$

which will be very useful at a later stage.

Adjoint and Coadjoint action. Even if we will not be use them during this thesis, for completeness we point out the standard adjoint and coadjoint actions of the Lie group $\mathrm{Diff}_{vol}(\mathcal{F})$. One can check that the adjoint action is represented by the smooth left action:

$$\mathrm{Ad} : \ \mathrm{Diff}_{vol}(\mathcal{F}) \times \mathfrak{X}_{div}(\mathcal{F}) \longrightarrow \mathfrak{X}_{div}(\mathcal{F}), \quad (\psi, X) \longmapsto \psi_* X,$$

where $\psi_* X$ denotes the push-forward of the vector field $X \in \mathfrak{X}_{div}(\mathcal{F})$ by the volume preserving diffeomorphism $\psi \in \mathrm{Diff}_{vol}(\mathcal{F})$. Due to technical reasons, the coadjoint action is not defined on the whole dual space $\mathfrak{X}'_{div}(\mathcal{F})$ but only on the smooth part of the dual, $\mathfrak{X}^*_{div}(\mathcal{F})$, and it is represented by the smooth right action:

$$\mathrm{Ad}^* : \ \mathfrak{X}^*_{div}(\mathcal{F}) \times \mathrm{Diff}_{vol}(\mathcal{F}) \longrightarrow \mathfrak{X}^*_{div}(\mathcal{F}), \ ([\alpha], \psi) \longmapsto [\psi^*\alpha].$$

6.4 The particle relabelling symmetry of the fluid-solid system

We observe that the Eulerian velocity field of the fluid is invariant if we replace φ by $\varphi \circ \psi$ and $\dot\varphi$ by $\dot\varphi \circ \psi$, where ψ is a volume preserving diffeomorphism of the reference configuration $\mathcal{F}$. This invariance represents the existence of a symmetry of the system, the so called ***particle relabeling symmetry***, that corresponds to the fact that we can label any particle of fluid in the reference configuration as we like to.

As anticipated in section 6.1, this symmetry is represented by a smooth right action of the Lie group $\mathrm{Diff}_{vol}(\mathcal{F})$ on the configuration space Q, defined by (6.6)

$$\Phi : \ Q \times \mathrm{Diff}_{vol}(\mathcal{F}) \longrightarrow Q, \ ((g,\varphi),\psi) \longmapsto (g, \varphi \circ \psi).$$

The algebraic and geometric structure of this action was studied, for instance, in [25] and [26] and, from these references, we report some facts that will be important in the following sections.

- The particle relabeling symmetry Φ is a free action.
- For any fixed $(g_0,\varphi_0) \in Q$, the orbit of Φ through $(g_0,\varphi_0) \in Q$ is

  $$\mathrm{Orb}(g_0,\varphi_0) = \{(g,\varphi) \in Q \mid g = g_0 \text{ and } \varphi : \ \mathcal{F} \longrightarrow \varphi_0(\mathcal{F}) \text{ is a diffeomorphism}\}.$$

- Given any $X \in \mathfrak{X}_{div}(\mathcal{F})$, the infinitesimal generator of the particle relabeling symmetry associated to X is the smooth vector field $X_Q \in \mathfrak{X}(Q)$ defined by

  $$X_Q(g,\varphi) = T_e\Phi^{(g,\varphi)}(X) = (0, T\varphi \circ X) \in T_{(g,\varphi)}Q,$$

  where $T\varphi : \ T\mathcal{F} \longrightarrow T\mathbb{R}^2$ is the tangent map of the embedding $\varphi : \ \mathcal{F} \longrightarrow \mathbb{R}^2$.

However, it is well known that the action Φ is not proper. Unfortunately, as in many applications, this prevents us to apply any general slice theorem in order to equip the orbit space with a differential structure. Despite that, in Proposition 6.2 below we will construct a finite dimensional smooth structure for the quotient $Q/\mathrm{Diff}_{vol}(\mathcal{F})$ such that the natural projection $\pi : \ Q \longrightarrow Q/\mathrm{Diff}_{vol}(\mathcal{F})$ will be a smooth submersion. But before of that, we shall give a technical lemma.

Lemma 6.1. *The projection onto the first factor $p : \ Q \longrightarrow SE(2)$, $(g,\varphi) \longmapsto g$ is a surjective smooth submersion, and, in particular, it is a principal bundle, with structure group $\mathit{Diff}_{vol}(\mathcal{F})$, that is isomorphic to the trivial bundle* $\mathrm{pr}_1 : \ SE(2) \times$ *$\mathit{Diff}_{vol}(\mathcal{F}) \longrightarrow SE(2)$.*

Proof. We start the proof by saying that there exists a smooth map

$$\sigma : \ SE(2) \longrightarrow Q,$$

such that $p \circ \sigma = \mathrm{id}_{SE(2)}$, namely a global section for $p : \ Q \longrightarrow SE(2)$. The proof of this claim is quite technical and we refer to section 4.5.3 of [22] for it.

The existence of a map $\sigma : \ SE(2) \longrightarrow Q$ such that $p \circ \sigma = \mathrm{id}_{SE(2)}$ implies that p is surjective. Also, p is smooth since it is the composition of the smooth maps $p = \mathrm{pr}_1 \circ i_Q$ where $i_Q : \ Q \longrightarrow SE(2) \times \mathrm{Emb}_{vol}(\mathcal{F},\mathbb{R}^2)$ is the inclusion and $\mathrm{pr}_1 : \ SE(2) \times \mathrm{Emb}_{vol}(\mathcal{F},\mathbb{R}^2) \longrightarrow SE(2)$, $(g,\varphi) \longmapsto g$, is the projection onto the first factor.

At this stage, we consider the smooth right action Φ of $\mathrm{Diff}_{vol}(\mathcal{F})$ on Q. The action is free and, since a typical fiber of p is a orbit of Φ (namely $p^{-1}(g) = \{g\} \times \mathrm{Diff}_{vol}(\mathcal{F}, g(\mathcal{F}))$), it is clear that $\mathrm{Diff}_{vol}(\mathcal{F})$ acts transitively on the fibers of p.

Finally, the existence of a global section $\sigma: SE(2) \longrightarrow Q$, implies that the bundle $p: Q \longrightarrow SE(2)$ is a globally trivial fibration and so it is a principal bundle with structure group $\mathrm{Diff}_{vol}(\mathcal{F})$. The global trivialization is the diffeomorphism

$$SE(2) \times \mathrm{Diff}_{vol}(\mathcal{F}) \longrightarrow Q, \ (g, \psi) \longmapsto \Phi_\psi(\sigma(g)).$$

Now, we are able to imply the submersion property for p, namely we will prove that, for each $(g, \varphi) \in Q$, the tangent map

$$T_{(g,\varphi)} \ p: \ T_{(g,\varphi)}Q \longrightarrow T_g SE(2), \ (\dot{g}, \dot{\varphi}) \longmapsto \dot{g},$$

is surjective[4]. Let $\dot{g} \in T_g SE(2)$ and $(g, \varphi) \in Q$, we consider the tangent vector

$$v = T_g \ (\Phi_\psi \circ \sigma)(\dot{g}) \in T_{(g,\varphi)}Q,$$

where $\psi \in \mathrm{Diff}_{vol}(\mathcal{F})$ is the unique volume preserving diffeomorphism such that $(g, \varphi) = \Phi_\psi(\sigma(g))$. By definition of σ, we know that $T_{(g,\varphi)}p \ (v) = \dot{g}$, that concludes the proof.

□

As mentioned before, the following proposition will give a finite dimensional smooth structure to the orbit space $Q/\mathrm{Diff}_{vol}(\mathcal{F})$ which turns the natural projection $\pi: \ Q \longrightarrow Q/\mathrm{Diff}_{vol}(\mathcal{F})$ into a smooth submersion.

Proposition 6.2. *The orbit space of the particle relabeling symmetry $Q/\mathit{Diff}_{vol}(\mathcal{F})$ can be endowed with a finite dimensional smooth manifold structure diffeomorphic to $SE(2)$ such that the natural projection $\pi: \ Q \longrightarrow Q/\mathit{Diff}_{vol}(\mathcal{F})$ is a smooth submersion.*

Proof. Let

$$\beta: \ Q/\mathrm{Diff}_{vol}(\mathcal{F}) \longrightarrow SE(2), \ [g, \varphi] \longmapsto g$$

and use the global section $\sigma: \ SE(2) \longrightarrow Q$, introduced in the proof of Lemma 6.1, to define

$$\gamma: \ SE(2) \longrightarrow Q/\mathrm{Diff}_{vol}(\mathcal{F}), \ g \longmapsto [\sigma(g)].$$

Clearly, it holds $\beta \circ \gamma = \mathrm{id}_{SE(2)}$ and $\gamma \circ \beta = \mathrm{id}_{Q/\mathrm{Diff}_{vol}(\mathcal{F})}$ and then γ is a bijection of sets, with $\beta = \gamma^{-1}$.

Now, we equip the quotient $Q/\mathrm{Diff}_{vol}(\mathcal{F})$ with the final smooth structure that turns the bijection γ into a smooth diffeomorphism of finite dimensional manifolds and we prove that the natural projection $\pi: \ Q \longrightarrow Q/\mathrm{Diff}_{vol}(\mathcal{F})$ is a smooth submersion using the following commuting diagram:

[4] Even if, for a general smooth map $f: \ M \longrightarrow N$ between infinite dimensional manifolds, the surjectivity of the tangent map may be not enough to prove the submersion property, paragraph 1.56 of [23] showes that this is actually enough if the codomain N is a finite dimensional manifold (as in our case).

$$\begin{tikzcd} & Q \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Q/\mathrm{Diff}_{vol}(\mathcal{F}) & & SE(2) \arrow[ll, "\gamma"'] \end{tikzcd}$$

Indeed $p: \; Q \longrightarrow SE(2)$ is a smooth submersion (as shown in Lemma 6.1) and $\gamma: \; SE(2) \longrightarrow Q/\mathrm{Diff}_{vol}(\mathcal{F})$ is a diffeomorphism.

□

6.5 The Hamiltonian formulation of the fluid-solid system and the symplectic reduction

In this section we give some technical insights about the Hamiltonian formulation associated to the fluid-solid system. The main goal here is to geometrically contextualize the fact that if the fluid is initially potential, it will remain potential throughout the motion. This will be a direct consequence of Noether theorem that will explicit the conservation of the circulation around the body and the advection of the scalar vorticity of the fluid.

Before starting let us fix the notation. Let Q be the configuration space of the fluid-solid system given in section 6.2, TQ its tangent bundle and

$$T'Q := \bigsqcup_{(g,\varphi)\in Q} (T_{(g,\varphi)}Q)'$$

its dual bundle. Since Q is endowed with a weak Riemannian metric $\langle\!\langle \cdot,\cdot \rangle\!\rangle_Q$, we define the ***Legendre transformation*** $\mathbb{F}: \; TQ \longrightarrow T'Q$, given by

$$\langle \mathbb{F}(g,\dot{g},\varphi,\dot{\varphi}), (g,\dot{g}_1,\varphi,\dot{\varphi}_1) \rangle_{T'Q} = \langle\!\langle (g,\dot{g},\varphi,\dot{\varphi}), (g,\dot{g}_1,\varphi,\dot{\varphi}_1) \rangle\!\rangle_Q,$$

which we stress not to be a surjective map.

6.5.1 The phase space of the fluid-solid system

We say that the ***phase space*** of the fluid-solid system is a cotangent bundle of the weak Riemannian manifold Q given by the image of the Legendre transformation

$$T^*Q := \{(g,p_g,\varphi,p_\varphi) := \mathbb{F}(g,\dot{g},\varphi,\dot{\varphi}): \text{ for some } (g,\dot{g},\varphi,\dot{\varphi}) \in TQ\},$$

which is a smooth vector bundle over Q (isomorphic to the tangent bundle TQ) with vector bundle projection

$$\tau: \; T^*Q \longrightarrow Q, \qquad (g,p_g,\varphi,p_\varphi) \longmapsto (g,\varphi).$$

Here, the dual pairing between T^*Q and TQ is determined by the Riemannian metric $\langle\!\langle \cdot,\cdot \rangle\!\rangle_Q$:

$$\langle (g,p_g,\varphi,p_\varphi), (g,\dot{g}_1,\varphi,\dot{\varphi}_1) \rangle_{T^*Q} = \langle\!\langle (g,\dot{g},\varphi,\dot{\varphi}), (g,\dot{g}_1,\varphi,\dot{\varphi}_1) \rangle\!\rangle_Q,$$

where $(g, p_g, \varphi, p_\varphi) = \mathbb{F}(g, \dot{g}, \varphi, \dot{\varphi})$.

Similar to the theory of finite dimensional manifolds, we know that there exists a canonical symplectic form Ω_{can} over the phase space T^*Q, that is the infinite dimensional analogous of the canonical symplectic form defined on a finite dimensional cotangent bundle. We refer to Appendix B.2 (and its references) for some details about cotangent bundles in infinite dimension and their symplectic structures.

6.5.2 The momentum map of the fluid solid system and symplectic reduction

We know that the particle relabelling symmetry $\Phi : \, Q \times \mathrm{Diff}_{vol}(\mathcal{F}) \longrightarrow Q$ lifts to an action of the Lie group $\mathrm{Diff}_{vol}(\mathcal{F})$ on the phase space T^*Q through cotangent lift. The cotangent lifted action will also be denoted $\Phi : \, T^*Q \times \mathrm{Diff}_{vol}(\mathcal{F}) \longrightarrow T^*Q$ and it is well known that it is a smooth free right action, which preserves the canonical symplectic form Ω_{can} of the phase space T^*Q.

In finite dimension, we know that any cotangent lifted action admits a canonical momentum map J, defined by formula (5.1) of chapter 5, which takes values on the dual of the Lie algebra. In our context formula (5.1) becomes

$$\langle J(g, p_g, \varphi, p_\varphi), X \rangle_{\mathfrak{g}} = \langle (g, p_g, \varphi, p_\varphi), X_Q(g, \varphi) \rangle_{T^*Q}. \tag{6.9}$$

In what follows we will show that formula 6.9 defines a canonical momentum map $J : \, T^*Q \longrightarrow \mathfrak{X}^*_{div}(\mathcal{F})$ for the cotangent lifted action Φ, which takes values in the smooth part of the dual of the Lie algebra $\mathfrak{X}_{div}(\mathcal{F})$, $\mathfrak{X}^*_{div}(\mathcal{F})$ (see section 6.3).

We observe that, given a momentum of the system $(g, p_g, \varphi, p_\varphi) \in T^*Q$ associated to some $(g, \dot{g}, \varphi, \dot{\varphi}) \in TQ$, we have

$$\langle (g, p_g, \varphi, p_\varphi), X_Q(g, \varphi) \rangle_{T^*Q} = \int_{\varphi(\mathcal{F})} \mathbf{u}^\flat(\varphi_* X) \; d\mathbf{x} = \int_{\mathcal{F}} (\varphi^* \mathbf{u}^\flat)(X) \; d\mathbf{x},$$

where $\mathbf{u}^\flat = \mathbf{u}_x \; dx + \mathbf{u}_y \; dy \in \Omega^1(\varphi(\mathcal{F}))$ is the differential one form associated to the smooth vector field $\mathbf{u} = \dot{\varphi} \circ \varphi^{-1} =: (\mathbf{u}_x, \mathbf{u}_y)$. In virtue of that, we define the map

$$T^*Q \ni (g, p_g, \varphi, p_\varphi) \longmapsto J(g, p_g, \varphi, p_\varphi) := [\varphi^* \mathbf{u}^\flat] \in \mathfrak{X}^*_{div}(\mathcal{F}),$$

which, by definition of the dual pairing $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ (see section 6.3), satisfies formula (6.9) above.

According to Noether theorem, the momentum map J defined above is conserved through any motion of the fluid-solid system. Moreover, isomorphism (6.8) implies that both

$$\mathbf{d}\,\varphi^* \mathbf{u}^\flat \qquad \text{and} \qquad \Gamma = \int_{\partial \mathcal{F}} \varphi^* \mathbf{u}^\flat$$

are conserved. From a physical point of view, we have that $\mathbf{d}\,\varphi^* \mathbf{u}^\flat = \mu \circ \varphi \; dx \wedge dy$, where $\mu = \mathbf{e}_3 \cdot (\nabla \times \mathbf{u}) \in C^\infty(\varphi(\mathcal{F}))$ is a scalar vorticity of the fluid, and $\Gamma = \int_{\partial \varphi(\mathcal{F})} \mathbf{u} \cdot d\mathbf{l}$. Clearly, the conservation of these quantities yields the conservation of the circulation Γ around the body of the fluid and the advection of its vorticity for any motion of the fluid-solid system.

In particular, the assumption to deal with a potential fluid is equivalent to require both to be 0. Geometrically speaking, we proved that any motion of the body in a potential perfect fluid lies on the zero level set of the momentum map $J^{-1}(0)$.

Symplectic reduction and reduced kinetic energy. The reduction of the kinetic energy made in section 6.1 is well justified by a symplectic reduction procedure. Formally, as recalled in chapter 5, the reduction of the kinetic energy corresponds to a Hamiltonian reduction of the dynamics by a symplectic reduction procedure, that identifies the quotient manifold $J^{-1}(0)/\mathrm{Diff}_{vol}(\mathcal{F})$ with the reduced phase space $T^*(Q/\mathrm{Diff}_{vol}(\mathcal{F})) \simeq T^*SE(2)$ endowed with its canonical symplectic form. We recall that the reduced Hamilton's equations, which determine the motions of the body in a potential fluid, are equivalent to the Euler-Lagrange equations of the reduced kinetic energy T_{red}.

Appendix A

An infinite dimensional version of Theorem 3.9

As anticipated in section 3.1.2 of chapter 3, in this appendix we provide a suitable set of hypotheses that extends Theorem 3.9 to the infinite dimensional Banach setting.

We briefly recall the notation. Let G be a Banach Lie group with associated Lie algebra $\mathfrak{g}$, M be a Banach manifold and $\Phi : M \times G \longrightarrow M$ be a smooth right action of G on M which is free and proper according to Definition 3.8 given in chapter 3. Moreover, we recall from section 3.1.2 that the action Φ induces an equivalence relation $\sim_G$ on the Banach manifold M with quotient space M/G and natural projection $\pi_G : M \longrightarrow M/G$.

Before starting, we prove a technical lemma.

Lemma A.1. *Let $\Phi : M \times G \longrightarrow M$ be a smooth, free and proper right action of a Banach Lie group G on a Banach manifold M. Then, for each $m \in M$, the tangent map*

$$T_e\Phi^m : \mathfrak{g} \longrightarrow T_mM$$

is injective and its image is a closed vector subspace of T_mM.

Proof. Firstly we prove the injectivity, it is enough to prove that $\ker(T_e\Phi^m) = \{0\}$. Let $\xi \in \mathfrak{g}$ such that $T_e\Phi^m(\xi) = 0$, namely we have

$$\xi_M(m) = T_e\Phi^m(\xi) = 0.$$

The discussion on infinitesimal generators made in section 3.1.1 of chapter 3 implies that the unique integral curve to ξ_M at m is the constant curve $\mathbb{R} \ni t \longmapsto \Phi_{\exp_G(t\xi)}(m) = m$, and, since the action is free, we have that $\exp_G(t\xi) \subseteq G_m = \{e\}$ for all $t \in \mathbb{R}$. It clearly implies $\exp_G(t\xi) = e$ for every $t \in \mathbb{R}$. Differentiating this identity at $t = 0$ we obtain

$$\xi = \frac{d}{dt}\Big|_{t=0} \exp_G(t\xi) = 0,$$

that concludes the proof.

Finally, we refer to Theorem 2.1 of [9] which guarantees that the image of the linear continuous map $T_e\Phi^m$ is a closed vector subspace of T_mM. □

In virtue of Lemma A.1, for every $m \in M$, the there exists a vector space isomorphism, $T_e\Phi^m : \mathfrak{g} \longrightarrow T_mM$, between the Lie algebra $\mathfrak{g}$ and the closed vector subspace of T_mM which is the image of the linear map $T_e\Phi^m$. Using this fact, we state the following definition.

Definition A.2. *Fix $m \in M$, we say that $\mathfrak{g}$ **splits** in T_mM if and only if the closed subspace $Im(T_e\Phi^m)$ splits in T_mM (see Definition 2.16).*

We now prove that under the additional hypothesis that $\mathfrak{g}$ splits in T_mM for each $m \in M$ an infinite dimensional version of Theorem 3.9, stated as Theorem (A.5), holds. The first steps of this generalization are coded in the following proposition and subsequent lemma. We stress that the proof of Proposition (A.3) is an adaptation of Corollary 4.1.22 of the standard reference [1].

Proposition A.3. *Let $\Phi : M \times G \longrightarrow M$ be a smooth, free and proper right action of a Banach Lie group G on a Banach manifold M, that satisfies assumption **(H)**. Then, for each $m \in M$, the orbit $Orb_G(m)$ is an embedded submanifold of M, which is diffeomorphic to G.*

Proof. It is enough to show that, for any $m \in M$, the orbit map through m, $\Phi^m : G \longrightarrow M$, is an embedding (see Definition 2.18 of chapter 2).

Lemma A.1 (toghether with assumption (**H**)) implies that the tangent map $T_e\Phi^m$ is injective and has a closed split image. In the following we prove the same property for every $T_g\Phi^m : T_gG \longrightarrow T_{\Phi_g(m)}M$, $g \in G$. Indeed, let $g \in G$, then, since Φ is a right action, the following identity holds:

$$\Phi^m \circ R_g = \Phi_g \circ \Phi^m. \tag{A.1}$$

We differentiate identity (A.1) at $e \in G$ and, due to the chain rule, we obtain

$$T_g\Phi^m \circ T_eR_g = T_m\Phi_g \circ T_e\Phi^m,$$

where both T_eR_g and $T_m\Phi_g$ are Banach space isomorphisms (because both R_g and Φ_g are diffeomorphisms), and $T_e\Phi^m$ is an injective linear map with closed split image in T_mM. Here, it is clear that also the map $T_g\Phi^m$ in injective and its image is closed and splits, for all $g \in G$. So, according to Definition 2.17 of chapter 2, the orbit map Φ^m is an immersion. In addition, it is injective due to the freeness of the action Φ.

Finally, we use the properness of the action to deduce that Φ^m is a closed map, namely images of closed subsets of G are mapped in closed subsets of M, and hence, by Proposition 2.19, it is an embedding.

Let A be a closed subset of G and $\{g_n\}_{n\in\mathbb{N}} \subseteq A$ a sequence of elements of A such that $\{\Phi^m(g_n)\}_{n\in\mathbb{N}}$ converges to some $m \in M$. Then, the properness of Φ implies that there exists a subsequence $\{g_{n_k}\} \subseteq \{g_n\} \subseteq A$ that converges to some $g \in G$, but, since A is closed in G, we have $g \in A$ and, by continuity of the map Φ^m, we have $\Phi^m(g_n) \longrightarrow \Phi^m(g)$ for $n \to +\infty$, which implies that $\Phi^m(A)$ is closed in M.

□

Lemma A.4. *Let E be a Banach space and $F \subseteq E$ be a closed vector subspace of E that splits. Let also $\Delta_E \subseteq E \times E$ be the vector subspace*

$$\Delta_E := \{(v, v) \in E \times E : \quad v \in E\}.$$

Then, the linear subspace

$$X := \Delta_E + \{0\} \times F \subseteq E \times E$$

is closed in $E \times E$ and splits.

Proof. We first show that X is closed. For this we prove that, given a sequence $\{(v_n, v_n + f_n)\}_{n \in \mathbb{N}} \subseteq X$ that it is convergent to some (v, w) in $E \times E$, then the limit (v, w) is actually an element of X.

It is clear that $(v_n, v_n + f_n) \longrightarrow (v, w)$ in $E \times E$ implies $v_n \longrightarrow v$ in E for $n \to +\infty$, and we check also $f_n \longrightarrow (w - v)$ in E:

$$\|f_n - (w - v)\| = \|f_n - w + v + v_n - v_n\| \leq \|(v_n + f_n) - w\| + \|v_n - v\| \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Now, the closedness of F implies $(w - v) \in F$. In particular we notice that the limit can be written as $(v, w) = (v, v + (w - v))$ that is an element of X.

Finally, to prove that X splits in $E \times E$ we exhibit a topological complement of X in $E \times E$. Let $\hat{F}$ be a topological complement of F in E (namely a closed vector subspace of E such that $E = F + \hat{F}$ and $F \cap \hat{F} = \{0\}$) and we consider the linear subspace

$$\hat{X} := \{0\} \times \hat{F} \subseteq E \times E.$$

It is clearly a closed subspace of $E \times E$ and we notice that, for any pair $(v, w) \in E \times E$, since $E = F + \hat{F}$, we have $w - v = f + \hat{f}$ for some $f \in F$ and $\hat{f} \in \hat{F}$. Hence we have

$$E \times E \ni (v, w) = (v, v + (w - v)) = (v, v + f + \hat{f}) = (v, v + f) + (0, \hat{f}) \in X + \hat{X},$$

and so $E \times E = X + \hat{X}$. To show that the sum is direct, we need to prove $X \cap \hat{X} = \{(0, 0)\}$. Let $(v, v + f) \in X$, we notice $(v, v + f) \in \hat{X}$ if and only if $v = 0$ and $f \in \hat{F}$, but since $F \cap \hat{F} = \{0\}$, we deduce $f = 0$, namely

$$X \cap \hat{X} = \{(0, 0)\}.$$

□

Finally, we are ready to state and prove the main theorem of this appendix. The proof presented below is inspired by Theorem 4.1.20 of [1].

Theorem A.5. *Let $\Phi : M \times G \longrightarrow M$ be a smooth, free and proper right action of a Banach Lie group G on a Banach manifold M that satisfies **(H)**. Then the orbit space M/G is a Banach manifold and the natural projection $\pi_G : M \longrightarrow M/G$ is a smooth submersion.*

Proof. We notice that the claim is equivalent to prove that $\sim_G$ is a regular equivalence relation, i.e. it satisfies the hypotheses of Theorem 2.24 from chapter 2.

Firstly, we show that $\sim_G$ is an embedded submanifold of $M \times M$. We need to prove that the map

$$\begin{aligned}\tilde{\Phi} : G \times M &\longrightarrow M \times M\\ (g, m) &\longmapsto (m, \Phi^m(g))\end{aligned}$$

is an embedding.

According to Theorem 2.19 of chapter 2, it is enough to show that $\tilde{\Phi}$ is an injective immersion and a closed map. Due to the freeness of Φ it is easy to see that $\tilde{\Phi}$ is injective and due to the properness, we know from the proof of Proposition A.3 that, since Φ^m is a closed map for each $m \in M$, so is $\tilde{\Phi}$.

Finally, we show that $\tilde{\Phi}$ is an immersion. Through an explicit computation, the tangent map at any pair (e, m) (where e is the identity of G and $m \in M$) of $\tilde{\Phi}$ is

$$\begin{aligned}T_{(e,m)}\tilde{\Phi} : \ \mathfrak{g} \times T_m M &\longrightarrow T_m M \times T_m M\\ (\xi, v_m) &\longmapsto (v_m, v_m + T_e\Phi^m(\xi)).\end{aligned}$$

Moreover, Lemma A.1 (together with hypothesis (**H**)) implies that, for any $m \in M$, the tangent map $T_e\Phi^m$ is injective with closed split image. We get the same result for $T_{(e,m)}\tilde{\Phi}$ by observing:

- $\ker(T_{(e,m)}\tilde{\Phi}) = \{0\} \times \ker(T_e\Phi^m) = \{0\} \times \{0\}$, and
- $\mathrm{Im}(T_{(e,m)}\tilde{\Phi}) = \Delta_{T_m M} + \{0\} \times \mathrm{Im}(T_e\Phi^m)$.

Here, Lemma A.4 guarantees that, since $\mathrm{Im}(T_e\Phi^m)$ is a closed vector subspace of $T_m M$ and admits a topological complement, then $\mathrm{Im}(T_{(e,m)}\tilde{\Phi})$ is a closed vector subspace of $T_m M \times T_m M$ and admits a topological complement. So, $\tilde{\Phi}$ is an immersion at each (e, m).

In order to conclude, we need to prove that $\tilde{\Phi}$ is an immersion at any $(g, m) \in G \times M$. With this aim, we fix $g \in G$ and we define the following diffeomorphisms:

$$\begin{aligned}\Lambda_g :\ & M \times M \longrightarrow M \times M,\ (m_1, m_2) \longmapsto (m_1, \Phi_g(m_2)),\\ \Sigma_g :\ & G \times M \longrightarrow G \times M,\ (h, m) \longmapsto (hg, m).\end{aligned}$$

Since Φ is a right action, we observe that the following identity holds

$$\Lambda_g \circ \tilde{\Phi} = \tilde{\Phi} \circ \Sigma_g.$$

We differentiate the identity above at $(e, m) \in G \times M$ and, due to the chain rule, we get

$$T_{(m,m)}\Lambda_g \circ T_{(e,m)}\tilde{\Phi} = T_{(g,m)}\tilde{\Phi} \circ T_{(e,m)}\Sigma_g. \tag{A.2}$$

Similar to the proof of Proposition A.3, we observe that $T_{(m,m)}\Lambda_g$ and $T_{(e,m)}\Sigma_g$ are Banach space isomorphisms, and the linear map $T_{(e,m)}\tilde{\Phi}$ is injective with closed split image. Hence, also the linear map $T_{(g,m)}\tilde{\Phi}$ is injective with a closed split image. In addition, the identity (A.2) implies

$$\mathrm{Im}(T_{(g,m)}\tilde{\Phi}) = \{(v_m, T_m\Phi_g(v_m) + T_g\Phi^m(\xi_g)) : \ \ v_m \in T_mM,\ \xi_g \in T_gG\}\,.$$

The second, and last, assumption of Theorem 2.24 requires that the projection onto the first factor $\mathrm{pr}_1 : \ Im(\tilde{\Phi}) \longrightarrow M$ is a smooth submersion. We observe that the tangent map,

$$T_{(m,\Phi^m(g))}\ \mathrm{pr}_1\ (v_m, T_m\Phi_g(v_m) + T_g\Phi^m(\xi_g)) = v_m,$$

is clearly surjective, with kernel

$$\ker(T_{(m,\Phi^m(g))}\ \mathrm{pr}_1) = \{0\} \times \mathrm{Im}(T_g\Phi^m),$$

which admits a topological complement in $T_mM \times T_{\Phi^m(g)}M$ (e.g. the closed subspace $E \times F$, where F is a topological complement of $\mathrm{Im}(T_g\Phi^m)$, is a topological complement of $\{0\} \times \mathrm{Im}(T_g\Phi^m)$).

$\square$

Appendix B

Remarks on locally convex spaces and Fréchet manifolds

In this appendix we make a brief review of a particular class of topological vector spaces, namely the class of Fréchet spaces, with the aim of complementing section 4.1 of chapter 4 and to contextualize some functional analytical issues presented in chapter 6.

B.1 Locally convex and Fréchet spaces

First of all we recall some basic definitions. We refer to the lecture notes [12] and the standard reference [13] for a detailed description of these notions. Also, some results and examples given in this section are taken from [5].

Definition B.1. *Let E be a real vector space, a function $p: E \longrightarrow \mathbb{R}$ is called a* ***seminorm*** *on E if*

- *for each $v \in E$ and $\lambda \in \mathbb{R}$ we have $p(\lambda v) = |\lambda|\, p(v)$, and*
- *for each $v, w \in E$ we have $p(v+w) \leq p(v) + p(w)$.*

According to section 1.4.1 of [12], a family of seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ defines a vector space topology on E and we call this topological vector space a ***locally convex vector space***. In the following we report some well known results concerning locally convex vector spaces, that we recalled in section 4.1 of chapter 4.

Lemma B.2 ([12], Lemma 1.4.2, item 2)**.** *A seminorm $p: E \longrightarrow \mathbb{R}$ is continuous in the topology generated by $\mathcal{P}$ if and only if there exist $p_1, \dots, p_n \in \mathcal{P}$, and $C > 0$ such that, for each $v \in E$,*

$$p(v) \leq C \max\{p_1(v), \dots, p_n(v)\}.$$

Lemma B.3 ([12], Lemma 3.4.2)**.** *Let E be a real locally convex vector space and $f: E \longrightarrow \mathbb{R}$ a linear functional. Then f is continuous in the locally convex topology if and only if the map $x \longmapsto |f(x)|$ is a seminorm on E.*

Theorem B.4 ([12], Corollary 5.1.8). *Let E be a real locally convex space, $F \subseteq E$ be a closed vector subspace and $a \in E \setminus F$. Then there exists a linear continuous functional $f: E \longrightarrow \mathbb{R}$ such that $f\big|_F = 0$ and $f(a) = 1$.*

Now, let E be a locally convex real vector space whose topology is generated by a countable family of seminorms $\mathcal{P} = \{p_i\}_{i \in \mathbb{N}}$, then Theorem 2.1.5 of [5] implies that the locally convex topology is metrizable, with associated translation invariant metric

$$d(v, w) := \sum_{i=0}^{+\infty} 2^{-i} \frac{p_i(v-w)}{1 + p_i(v-w)}.$$

Definition B.5. *A locally convex real vector space E whose topology is generated by a countable family of seminorms is called a* ***Fréchet space*** *if its topology is complete (with respect to the metric d described above).*

Example B.6. Let $\mathcal{F} \subseteq \mathbb{R}^2$ be an open submanifold of $\mathbb{R}^2$ and $\{K_i\}_{i \in \mathbb{N}}$ a family of compact sets in $\mathbb{R}^2$ such that $K_i \subseteq K_{i+1}$ and $\mathcal{F} = \bigcup_{i \in \mathbb{N}} K_i$. We recall that the vector space of smooth vector fields over $\mathcal{F}$, $\mathfrak{X}(\mathcal{F})$, is isomorphic to the space of all smooth functions from $\mathcal{F}$ to $\mathbb{R}^2$. Let $X: \mathcal{F} \longrightarrow \mathbb{R}^2$ be a smooth vector field over $\mathcal{F}$ and consider the family of norms:

$$\|X\|_i = \sup_{x \in K_i,\ |\alpha| \leq i} |D^\alpha X(x)| + \int_{\mathcal{F}} |X(x)|^2 \, dx,$$

for $i \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ multiindex. One can prove that the vector subspace

$$\mathfrak{X}_b(\mathcal{F}) := \{X \in \mathfrak{X}(\mathcal{F}) : \ \|X\|_i < \infty \text{ for each } i \in \mathbb{N}\}$$

is a Fréchet space.

In the context of Fréchet spaces some good topological properties are preserved like the following fact.

Lemma B.7 ([11], Lemma 1.7). *Let E be a Fréchet space and $F \subseteq E$ be a closed vector subspace. Then both F and the quotient of vector spaces E/F are Fréchet spaces.*

Example B.8. In the setting of Example B.6 we consider the standard Euclidean area element of $\mathbb{R}^2$ and we observe that the vector subspace of divergence free vector fields

$$\mathfrak{X}_{div}(\mathcal{F}) := \{X \in \mathfrak{X}_b(\mathcal{F}) : \ \mathrm{div}(X) = 0\} \subset \mathfrak{X}_b(\mathcal{F})$$

is closed with respect to the Fréchet topology of $\mathfrak{X}_b(\mathcal{F})$ and so it is a Fréchet space.

However, it is well known that even the very basic notions of calculus on Fréchet spaces could be very difficult to develop. The main issue is represented by the following theorem, that states that the strong dual

$$E' = \{f : \ E \longrightarrow \mathbb{R} : \ \text{is linear and continuous}\},$$

if endowed with the topology of uniform convergence on bounded sets, is never a Fréchet space when E is a non-Banach Fréchet space.

Theorem B.9 ([5], Theorem 2.1.12)**.** *The strong dual E' of a Fréchet space is metrizable if and only if E is normable. In particular, duals of non-Banach Fréchet spaces are not Fréchet.*

Another critical issue is that the natural dual pairing $\langle\cdot,\cdot\rangle : E' \times E \longrightarrow \mathbb{R}$, $(f, v) \longmapsto f(v)$, is not continuous with respect to any suitable topology on E'. However, as indicated in chapter 6, in applications we often have to deal with some dual space and pairing in the Fréchet realm. A very natural solution to this situation is to choose, in place of E', a suitable vector subspace, often denoted $E^* \subseteq E'$, which can be endowed with a Fréchet vector space topology and carries a continuous dual pairing $\langle\cdot,\cdot\rangle_{E^*} : E^* \times E \longrightarrow \mathbb{R}$.

Remark B.10. A very classical example of this construction is given by fixing an inner product on E. Indeed, let $\tau : E \times E \longrightarrow \mathbb{R}$ be an inner product on E (see, for instance, Definition 1.10 of chapter 1) and let $\tau^\flat : E \longrightarrow E'$, $v \longmapsto \tau^v$, be its flat map. We recall that for any $v \in E$ we have

$$\tau^v : E \longrightarrow \mathbb{R},\ w \longmapsto \tau(v, w).$$

We know that, in general, the map $\tau^\flat$ is an injective (but not surjective) linear map of vector spaces. Here, we choose $E^* := \tau^\flat(E) \subset E'$ and one can easily check that, since $\tau^\flat : E \longrightarrow E^*$ is an isomorphism of vector spaces, E^* can be endowed with a Fréchet topology (induced by E itself), turning the map $\tau^\flat$ into an isomorphism of Fréchet spaces. Moreover, E^* carries a continuous dual pairing with E with respect to this topology, that is:

$$\langle\cdot,\cdot\rangle_\tau : E^* \times E \longrightarrow \mathbb{R},\ (f, v) \longmapsto \tau\left((\tau^\flat)^{-1}(f), v\right).$$

As we know, before dealing with smooth manifolds, one needs to develop a theory of calculus on the topological vector spaces one wish to model the manifolds on. Unfortunately, in our case, Theorem B.9 forbids the existence of a Fréchet derivative for a map between Fréchet spaces as it is defined for maps between Banach spaces, and, for this reason, in the literature there are several, non equivalent theories of calculus on Fréchet spaces. All of these are characterized by the loss of some very important theorems of classical calculus, like the classical formulation of the inverse function theorem and the implicit function theorem. In this appendix we refer to the theory of Bastiani calculus, based on the concept of directional derivatives (an introduction to the topic can be found in [23]).

B.2 Fréchet manifolds

Up to a choice of concept of calculus on Fréchet spaces, one can define in the very classical sense what an abstract manifold M modeled on a Fréchet space E is. We refer to [23] and [5] for a detailed discussion about the topic.

Here, we recall that a Hausdorff topological space M, which possesses a smooth atlas modeled on a Fréchet space E, namely a collection of homeomorphisms $\varphi_i : U_i \subseteq M \longrightarrow V_i \subseteq E$, such that any change of charts $\varphi_i \circ \varphi_j^{-1}$ is a smooth map between Fréchet spaces, is called a ***Fréchet manifold***. We know that some of the standard constructions of finite dimensional and Banach differential geometry still hold in the Fréchet realm, like the definition of the tangent bundle

$$TM = \bigsqcup_{m \in M} T_m M,$$

which is actually a Fréchet vector bundle over M.

The cotangent bundle of a Fréchet manifold. In constrast to the tangent bundle, Theorem B.9 deeply affects the construction of the dual bundle of a Fréchet manifold M,

$$T'M = \bigsqcup_{m \in M} (T_m M)',$$

which cannot be even endowed with a suitable differential structure.

In virtue of that, we say that a ***cotangent bundle*** of a Fréchet manifold M is a subset of the dual bundle $T'M$, denoted $T^*M \subset T'M$, which can be endowed with a Fréchet vector bundle structure and carries a continuous dual pairing with TM. An example of such a choice is given in section 6.5.1 of chapter 6.

Differential forms over a Fréchet manifold. At this stage it is clear that the standard definition of differential forms over a smooth manifold M as smooth sections of the dual bundle is meaningless in the infinite dimensional Fréchet case. We refer to Appendix E of [23] and its references for the details of the alternative definition presented below. Briefly, we define a differential k-form as a smooth map

$$\omega : \; TM \oplus \cdots \oplus TM \longrightarrow \mathbb{R},$$

which induces a multilinear skew symmetric k-form $\omega_m : \; (T_m M)^k \longrightarrow \mathbb{R}$ for each $m \in M$. In order to build a full theory of differential forms, one adapts the definitions of the classical operators of differential geometry (like the exterior derivative, the Lie derivative, the pull-back and push-forward of differential forms) in order to deal with the alternative definition above. One can check that the resulting theory works in a similar way to the theories of differential forms over finite dimensional and Banach manifolds.

Symplectic structures on Fréchet manifolds. Let M be a Fréchet manifold and T^*M be a cotangent bundle of M with vector bundle projection $\tau_M : \; T^*M \longrightarrow M$ and continuous dual pairing $\langle \cdot, \cdot \rangle_M$. In this case, descriptions developed in Appendix A of [6] and in section 48 of [13] construct a canonical weak symplectic form over the Fréchet manifold T^*M.

The procedure is an infinite dimensional adaptation of the standard finite dimensional one. Indeed, we define the differential one form $\theta_{can} : \ T(T^*M) \longrightarrow \mathbb{R}$, given by

$$\theta_{can}(p, V) = \langle p, T_p\tau_M(V)\rangle_M$$

and one can check that the negative of its exterior derivative $\Omega_{can} := -\,\mathbf{d}\,\theta_{can}$ is a weak symplectic form over T^*M (adapting the standard definition of weak symplectic form to this setting).

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