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A comparison between standard slow-roll inflation and warm inflation

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Contents

Notation and conventions

We use natural units in which $\hbar = c = k_B = 1$. In these units, the reduced Planck mass is given by $M_{Pl} = (8\pi G)^{-1/2} \simeq 2.4 \cdot 10^{18}$GeV.

Greek indices μ, ν and so on go over the four spacetime coordinates $x^\mu = (x^0, x^1, x^2, x^3)$, where x^0 stands for the time coordinate.

Latin indices i, j, k and so on go over the three spatial coordinates.

Our metric signature is $(- + ++)$.

Spatial vectors are written in boldface.

Summation over repeated indices is assumed unless otherwise stated.

We use the symmetric Fourier convention

$$f(\mathbf{k}) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \qquad f(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \quad .$$

Introduction

The widely accepted Hot Big-Bang model of Cosmology states that the very early universe was dominated by matter distributions existing in the form of a hot and dense plasma of relativistic particles, which is simply referred to as radiation. This plasma was distributed in a highly, but not perfectly, homogeneous and isotropic manner, with the presence of very small anisotropies in the energy/matter density, known as *primordial cosmological perturbations*, that are supposed to constitute the seeds for the formation of the Large Scale Structures (LSS) we observe today in the universe. Such inhomogeneities have left a trace under the form of temperature anisotropies in an ubiquitous electromagnetic signal that we receive today from the early stage of the universe, the Cosmic Microwave Background signal (CMB). The CMB temperature anisotropies were first detected by the Cosmic Background Explorer (COBE) satellite [1], and subsequently they were analyzed in depth from the Wilkinson Microwave Anisotropy Probe (WMAP) mission and the Planck mission. The best CMB data are provided by the 2018 Planck measurements [2, 3], that provided a very precise characterization of the primordial cosmological perturbations, and that have allowed cosmological parameters to be constrained. In particular, the Planck data have established with extremely high precision that the primordial density perturbations show a small deviation from perfect scale invariance.

Inflation is a postulated period of accelerated expansion that took place well within the first second of the universe, before the radiation dominated era (necessarily before the epoch of primordial nucleosynthesis). It was initially introduced by Alan Guth in 1981 [4] in order to solve the horizon, flatness and monopoles problems that plague the standard Hot Big-Bang cosmological model, and, since its proposal, inflation has become the dominant paradigm able to provide a dynamical mechanism for the generation of the primordial energy density perturbations in the early universe. A period of accelerated expansion can be attained if the energy density of the universe is dominated by an unconventional cosmic fluid with a sufficiently negative equation of state (pressure divided by energy density). The most consolidated *slow-roll* inflationary models are based on the dynamics of a single spin-0 field, the inflaton ϕ, which dominates the energy budget with the vacuum energy associated to its potential. In these models the necessary conditions required to realize an inflationary expansion are obtained by the field ϕ moving very slowly in a very flat region of its potential. This kind of models predicts a nearly scale invariant power spectrum of Gaussian and adiabatic primordial scalar density perturbations, in excellent agreement with the experimental observations by Planck. Inflationary models also generally predict a stochastic background of primordial gravitational waves (SGWB), corresponding to tensor metric perturbations generated during the rapid expansion. This prediction is a unique and distinctive feature of inflationary cosmology, so that the detection of primordial gravitational waves (more precisely the detection of a contribution to the so called B-mode polarization in the CMB radiation) may be a ”smoking gun” probe of inflation, if alternative mechanism able to produce gravitational waves are ruled out. However, no such

primordial signal has yet been detected.
The literature contains a huge number of different models of inflation. Each model correspond to a given choice for the potential of the inflaton which fulfils the so called *slow-roll conditions* to attain an accelerated expansion. In this regard it is of fundamental importance to emphasize that the constraints on the physical cosmological observables provided by the Planck measurements also provide powerful constraints to the parameters of the inflationary models [5], which allow us to orient ourselves in the myriad of inflationary models proposed to date.

In canonical slow-roll models of inflation, the accelerated expansion occurs in a state which is practically empty of any other particle, since the couplings of the inflaton field with other possible degrees of freedom is assumed to be negligible, so that particle production is inhibited, and the traces of any matter distribution are quickly diluted away by the expansion. This scenario causes the universe to reduce to a supercooled state in which a thermalized radiation component is quasi totally absent, so that a subsequent phase must be added in order to recover the hot initial conditions of the standard cosmological model. Therefore, at the end of inflation, the inflaton field is supposed to decay into lighter relativistic fields. This process, known as *reheating* [6], generates the thermal bath of the hot Big-Bang era.
In some other models instead, the couplings of the inflaton can be relevant already during inflation, leading to an effective dissipative dynamics that is important also during the slow-roll phase. In this thesis we critically review the two distinct scenarios in which these dissipative effects are either negligible or relevant during inflation. The first type of models belong to the more standard class of theories of *cold inflation*. For the second case, we focus our attention on the so called models of *warm inflation*, in which the dissipation sustains a bath of relativistic particles at thermal equilibrium, so that, even though the inflaton vacuum energy remains still the dominant component of the universe energy budget for the accelerated expansion to take place, the inflationary universe is not more on the ground state, and the transition to a thermalized radiation dominated universe can happen smoothly, without the need of a separate reheating phase. The interaction between the inflaton field and the thermal bath entails an overdamped motion for the homogeneous inflaton configuration resulting from the backreaction of the produced particles, which allows the achievement of an inflationary expansion also through potentials that would otherwise be too steep to support the slow-roll conditions. Moreover, even in the case in which the dissipative effects are small compared to the Hubble expansion damping (weak dissipation regime), they can still significantly affect the generation of both scalar and tensor primordial perturbations, leading for example to a remarkable enhancement of the power spectrum of density perturbations for sufficiently high temperatures. This generally lowers the ratio between the amplitudes of tensor and scalar perturbations, providing a modified *consistency relation* for inflationary models.
The aim of these thesis is study the background evolution, the generation of both scalar and tensor primordial perturbations, and the phenomenology of these two distinct inflationary scenarios.
In chapter 1 we give an introduction to the mathematical and phenomenological description of our universe provided by the standard Big-Bang cosmological model, also focusing on the shortcomings that afflict the model and their possible resolution through the inflationary paradigm.
In chapter 2 we review the standard single-field model of slow-roll inflation. In particular we review the conditions that must be fullfilled in order to accomplish the inflationary expansion via the realization of a *slow-roll* dynamics for the inflaton field, and we study the amplification mechanism of quantum vacuum fluctuations of the inflaton and the metric tensor which yields the production of classical primordial perturbations. Then, we derive their power spectrum, the associated *spectral index* and the *tensor-to-scalar perturbation ratio*, which represent the physical observables constrained by the CMB measurements.

In chapter 3, initially, we briefly see how, in presence of a non-negligible coupling with a significant amount of thermalized radiation, it is more convenient the employment of a different mathematical formalism in order to derive the effective evolution equation for the inflaton field and its perturbations. In the context of a multi-component interacting system, a realistic study of the time evolution of its dynamical variables requires to take an average on some distribution function of states. Therefore, we are rather concerned with a thermal averaged background inflaton field and its small random thermal fluctuations. After that, we proceed with steps analogous to those of the previous chapter: we derive the slow-roll conditions for inflation, which are modified by the presence of dissipative effects, then we study the evolution of the primordial perturbations, that are of thermal origin in this case, and we finally compute the thermal contributions to the power spectrum for scalar and tensor metric perturbation. These thermal contributions are added to the standard ones arising from amplification of quantum fluctuations, and we recognize the regime in which the former dominate over the latter.
We will conclude by summing up the main results and comparing the relative advantages and disadvantages of the two scenarios.

Chapter 1

The Hot Big Bang cosmological model and the need of Inflation

1.1 A brief introduction

The standard cosmological model describing the evolution of our universe relies upon the validity of Einstein's theory of General Relativity (GR) and one fundamental assumption known as *cosmological principle*, which states that at sufficiently large scale (hundreds of Mpc) the observable universe appears to be homogeneous and isotropic. This assumption is born out of several observations, among which the almost perfect uniformity of the temperature of the cosmic microwave background radiation (CMB) coming from all direction in the sky, which gives us a picture of our universe when it was about 380000 years old. Indeed, the typical size of CMB temperature fluctuations, around the mean value $T_{CMB} = 2.725K$, is measured to be of order $\delta T/T_{CMB} \sim 10^{-5}$.
It is not so difficult to realize that the cosmological principle is violated today at relatively short scales, the ones at which we see the universe material clumped into celestial objects such as stars, galaxies and galaxy clusters. We think these structures originated by small primordial perturbations of matter density superimposed to a perfect homogeneous background, which have grown in time through the phenomenon of gravitational instability. The dynamics of the background corresponds to the large scale behaviour of the universe, and it is usually studied separately from the

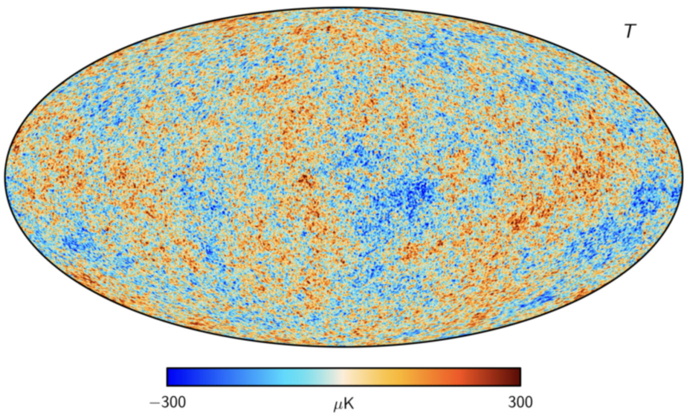

Figure 1.1: Temperature anisotropies in the CMB radiation around the background value measured by the Planck satellite

dynamics of the short scale inhomogeneities, whose evolution, as long as they remain small, can be faced via linear perturbation theory.
By astronomical observations performed by Edwin Hubble in the late 20's, we also have learned our universe is expanding on large scales, in the sense that the space itself between any pair of points separated by a large enough distance is increasing in time. In particular, far away galaxies are receding from us with a velocity proportional to their proper distance d, as stated by the experimental *Hubble law*:

$$v = H_0\, d \quad , \tag{1.1}$$

where the proportionality constant H_0 is known as the Hubble's constant.

According to GR, the geometrical structure of spacetime, mathematically represented through a 4-dimensional pseudo-Riemannian manifold, is encoded in the symmetric metric tensor $g \equiv g_{\mu\nu} dx^\mu \otimes dx^\nu$, whose evolution in presence of a distribution of matter/energy is determined by the *Einstein's field equations* (EFE), given by:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \quad . \tag{1.2}$$

Let us pause for a moment to give a short description of the quantities appearing on both sides of this equation.
On the left hand side (l.h.s.), we find the *Einstein's tensor* $G_{\mu\nu}$, constructed in terms of the *Ricci tensor* $R_{\mu\nu}$ and the *scalar curvature* R. Both this quantities are obtained by contraction of the *Riemann curvature tensor* $R^\mu_{\nu\rho\sigma}$, which describes the intrinsic geometrical properties of spacetime. It is defined as

$$R^\mu_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho} \quad , \tag{1.3}$$

with $\Gamma^\mu_{\nu\rho}$ the so called *Christoffel symbols*, whose explicit expression is

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}) \quad . \tag{1.4}$$

The Christoffel symbols are the coefficients of the particular *connection* defined on the spacetime manifold, known as the *Levi-Civita connection.* The introduction of a connection allows to specify the *covariant derivative* $\nabla_X T$ of a generic tensor field T along the direction of a tangent vector field $X = X^\mu \partial_\mu$.
In general, a connection is defined as a map which satisfies the following properties for all tensor fields T, W and vector fields X,Y on a manifold [7]:

- $\nabla_X (T + W) = \nabla_X T + \nabla_X W$;
- $\nabla_{(fX+gY)} T = f \nabla_X T + g \nabla_Y T$ for all functions f, g ;
- $\nabla_X (T\,W) = W\,\nabla_X T + T\,\nabla_X W$ (Leibiniz rule) .

The covariant derivative of a vector field $V = V^\mu \partial_\mu$ along a basis vector field ∂_μ reads [7]

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho \quad . \tag{1.5}$$

For a 1-form $\alpha = \alpha_\mu dx^\mu$ we have

$$\nabla_\mu \alpha_\nu = \partial_\mu \alpha_\nu - \Gamma^\rho_{\mu\nu} \alpha_\rho \quad , \tag{1.6}$$

and for a generic tensor $T = T^{\alpha\ldots\beta}_{\delta\ldots\gamma} \partial_\alpha \otimes \cdots \otimes \partial_\beta \otimes dx^\delta \otimes \cdots \otimes dx^\gamma$

$$\nabla_\mu T^{\alpha\ldots\beta}_{\delta\ldots\gamma} = \partial_\mu T^{\alpha\ldots\beta}_{\delta\ldots\gamma} + \Gamma^\alpha_{\mu\rho} T^{\rho\ldots\beta}_{\delta\ldots\gamma} + \cdots + \Gamma^\beta_{\mu\rho} T^{\alpha\ldots\rho}_{\delta\ldots\gamma} - \Gamma^\rho_{\mu\delta} T^{\alpha\ldots\beta}_{\rho\ldots\gamma} + \cdots - \Gamma^\rho_{\mu\gamma} T^{\alpha\ldots\beta}_{\delta\ldots\rho} \quad . \tag{1.7}$$

The peculiarity of the Levi-Civita connection relies in the fact that it is the only one among the infinite possible connections which satisfies the following properties [7]

$$\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu} \quad , \tag{1.8}$$

$$\nabla_\mu g_{\nu\rho} = 0 \quad , \quad \nabla_\mu g^{\nu\rho} = 0 \quad . \tag{1.9}$$

The first property says the Levi-Civita connection is symmetric in its lower indices. Such a connection is said to be "torsion-free", in the sense that the antisymmetric torsion tensor defined as $\mathcal{T}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$ is vanishing. The second property tells us the metric tensor $g_{\mu\nu}$ is covariantly constant, and we say that such a connection is "compatible with the metric". The metric compatibility, together with the Leibiniz rule of ∇, allows to freely move $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ inside and outside ∇_μ, i.e.

$$\nabla_\mu V^\nu = \nabla_\mu (g^{\nu\rho} V_\rho) = g^{\nu\rho} \nabla_\mu V_\rho \quad .$$

Working directly on the definition (1.3) one finds that the Riemann curvature tensor has the following symmetry properties [8, 9], usually expressed in terms of the covariant components $R_{\mu\nu\rho\sigma} = g_{\mu\alpha} R^{\alpha}_{\nu\rho\sigma}$:

$$\begin{aligned}
&R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \quad , \\
&R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \quad , \\
&R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad , \\
&R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \quad .
\end{aligned}$$

From these properties it is also possible to derive a differential identity for the curvature tensor, known as *Bianchi identity*, which reads

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\sigma R_{\mu\nu\lambda\rho} + \nabla_\rho R_{\mu\nu\sigma\lambda} = 0 \quad . \tag{1.10}$$

The Ricci tensor and the scalar curvature are then defined as

$$R^{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu} \quad , \quad R \equiv g_{\mu\nu} R^{\mu\nu} \quad . \tag{1.11}$$

The Ricci tensor results to be symmetric as a consequence of the various symmetries of the Riemman tensor, i.e. $R^{\mu\nu} = R^{\nu\mu}$.
Going back to the Einstein equations (1.2), on the right hand side (r.h.s.) we find the symmetric stress-energy tensor $T_{\mu\nu}$, which describes the matter content, and the cosmological constant Λ. The cosmological constant term can be seen as a contribution to the stress-energy tensor of the form $T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$, so that we can absorb it as a vacuum energy in the general definition of $T^{\mu\nu}$, which will be given in section 1.3.
The equations (1.2) are completed by other two important relations known as *(contracted) Bianchi identity* and *continuity equation*, respectively:

$$\nabla_\mu G^{\mu\nu} = 0 \quad , \tag{1.12}$$

$$\nabla_\mu T^{\mu\nu} = 0 \quad . \tag{1.13}$$

The first one is a differential identity for the Einstein tensor $G_{\mu\nu}$ which is obtained by contraction of the homonymous identity (1.10) for the curvature tensor. The second one, instead, is a physical statement about the local conservation of energy and momentum in the case we consider a flat Minkowski spacetime, whereas, in presence of a gravitational field, i.e. in the case of a curved spacetime, energy-momentum conservation does not hold in general and equation (1.12) rather provides the equations of motion of the matter distribution under the action of the gravitational field alone [8].

Focusing on the background dynamics of the universe, the cosmological principle and the Hubble's law provide strong constraints on the form of the metric tensor which should describe a spatially homogeneous and isotropic spacetime evolving in time because of the expansion. It can be shown [8, 10] that the most general metric holding these properties, solution of (1.2) and (1.13), is the *Friedmann-Lemaître-Robertson-Walker metric* (FLRW):

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] \quad . \tag{1.14}$$

Here, the coordinates (t, r, θ, ϕ) belong to a preferred reference frame, properly chosen to factor out the effect of the Hubble expansion. They are named *comoving coordinates*, and they are defined as the coordinates of an observer co-moving with the expansion, also referred to as the fundamental observer. The coordinate t is the proper time of the afore-mentioned observer, whose spatial coordinates (r, θ, ϕ) do not change in time during the evolution. The dimensionless function $a(t)$ is called *scale factor*, and its role is to account for the evolution of the universe by changing the spatial distances over time. For an expanding universe it is an increasing function of time, $\dot{a}(t) > 0$, and it yields the physical coordinates once it is multiplied by the constant comoving ones. As required by the cosmological principle, the spatial part of the metric, enclosed in the square brackets, is the squared line element of a 3-dimensional maximally symmetric space with constant scalar curvature. The quantity K is a constant related to the spatial curvature R of the space-like hypersurfaces at fixed time t by the relation $R = 6K$ [8], then it has the physical dimension of a length^{-2}. If $K > 0$ the spatial metric is the one of a 3-dimensional sphere $\mathbf{S}^3$, which corresponds to a spatially finite universe (closed) with positive spatial curvature. If $K < 0$, the spatial metric is the one of a 3-dimensional hyperboloid $\mathbf{H}^3$, then we have a spatially infinite universe (open) with negative spatial curvature. Finally, if $K = 0$, we obtain a spatial Euclidean metric, so the universe is spatially flat, hence infinitely extended.
It can be noticed the presence of a redundancy in the description of the metric, since the latter is left unchanged by a rescaling of the radial coordinate $r \to \tilde{r} = \lambda r$, which implies the following redefinitions of the scale factor and of the spatial curvature $a \to \tilde{a} = a/\lambda$, $K \to \tilde{K} = K/\lambda^2$, so that the physical lengths do not vary. This scaling freedom can be used either to normalize K to a constant $k = 0, \pm 1$ once and for all, or to normalize the scale factor in a convenient way, e.g. by setting its value at the present time t_0 to unity, $a(t_0) = a_0 = 1$. We will use the first convention, unless otherwise stated, which is simply obtained by choosing $\lambda = |K|^{1/2}$, if $K \neq 0$. In this case $a(t)$ inherits the dimension of a length, while the radial coordinate r becomes dimensionless.

On large scales, all the material filling the universe can be treated like a cosmic perfect fluid, defined as one such that there is no bulk motion of particles and no heat conduction in the rest local inertial frame (LIF) of any fluid element, hence any observer in this frame sees the fluid around him as isotropic. That is exactly the symmetry of the space seen by the comoving observer, indeed the comoving frame and the instantaneous rest frame of the fluid element coincide.
Let us consider the stress-energy tensor $T^{\mu\nu}$ of the cosmic perfect fluid at some point Q of the spacetime, and let us put ourselves in the rest LIF of a fluid element centered on Q. In this LIF, the mentioned properties of the perfect fluid constrain the components of $T^{\mu\nu}$ to be of the form:

$$T^{00} = \rho(t) \quad , \quad T^{0i} = T^{i0} = 0 \quad , \quad T^{ij} = P(t)\delta^{ij} \quad , \tag{1.15}$$

where ρ and P are the rest energy density and the rest isotropic pressure of the fluid, which do not depend on the spatial coordinates because isotropy and homogeneity.
Using the fact that, in the chosen frame, the metric tensor evaluated on Q is the Minkowskian one, i.e. $g_{\mu\nu}(Q) = \eta_{\mu\nu}$, and that the fluid element's four-velocity reads $u_\mu = (1,0,0,0)$, the stress-energy tensor of the perfect fluid can be written as

$$T^{\mu\nu} = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & P(t) & 0 & 0 \\ 0 & 0 & P(t) & 0 \\ 0 & 0 & 0 & P(t) \end{pmatrix} = (\rho + P)u^\mu u^\nu + P\eta^{\mu\nu} \,. \tag{1.16}$$

This is a tensor relation, so it holds in all coordinates system. Moreover the point P is arbitrary, hence this expression holds everywhere, and we can write

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}. \tag{1.17}$$

For each component of the cosmic fluid we have an equation of state relating ρ and P given by

$$P = w\rho \qquad w = constant \quad , \tag{1.18}$$

with w depending on the specific fluid component. The most commonly encountered cases are:

$$w = \begin{cases} 0 & \text{non relativistic matter} \\ \frac{1}{3} & \text{radiation} \\ -1 & \text{cosmological constant } \Lambda \end{cases} \quad . \tag{1.19}$$

Notice that the last case reproduces the contribution to $T_{\mu\nu}$ given by the cosmological constant in the case $\rho = \Lambda/8\pi G$.
By inserting the FLRW metric (1.14) and the perfect fluid stress-energy tensor (1.16) in equations (1.2) and (1.13), we find the *Friedmann equations* describing the dynamics of the expanding universe:

$$(00) \text{ component of (1.2)} \qquad \longrightarrow \qquad H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad , \tag{1.20}$$

$$(ij) \text{ component of (1.2)} \qquad \longrightarrow \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad , \tag{1.21}$$

$$\nu = 0 \text{ component of (1.13)} \qquad \longrightarrow \qquad \dot{\rho} + 3H(\rho + P) = 0 \quad , \tag{1.22}$$

with the dot indicating the derivative with respect to (w.r.t.) cosmic time. The function $H(t)$ is the Hubble parameter, defined as $H(t) \equiv \frac{\dot{a}}{a}$. It has the dimension of inverse time and can be viewed as the rate of the expansion of the universe. Moreover, the inverse of the parameter naturally sets a characteristic time scale for the expansion, namely the Hubble time[1] $t_H \equiv H^{-1}$ (see equation (1.32)).
This system contains three equations for three unknown variables $(\rho(t), P(t), a(t))$. However, only two of these equations are independent, due to the continuity equation (1.13). For instance, it possible to obtain the second equation (1.21) by differentiating in time (1.20), and by combining the resulting expression with third equation (1.22). The system is closed by the equation of state (1.18).

[1]Looking at its definition written as $t_H = dt\frac{a}{da}$, the Hubble time is roughly the time needed to the the scale factor to double.

According to the current experimental observations, today we live in a universe which can be considered spatially flat with good accuracy. In fact, as we will see in subsection 1.2.2, the degree of flatness can be expressed in terms of the quantity $\Omega_k \equiv -k/a^2H^2$, which results to be an increasing function of time. The Planck measurement [2] provide a current value of $\left|\Omega_k(t_0)\right| < 10^{-3}$. We therefore simplify our system of equations by assuming that the universe has always been flat, i.e. setting $k = 0$ in (1.20).
By solving (1.22) we find that the energy density of each component evolve independently according to:

$$\rho = \rho_* \left(\frac{a}{a_*}\right)^{-3(1+w)} \qquad \Longrightarrow \qquad \rho \propto \begin{cases} a^{-3} & \text{non relativistic matter} \\ a^{-4} & \text{radiation} \\ constant & \text{cosmological constant } \Lambda \end{cases} \quad , \tag{1.23}$$

where $*$ indicates a reference scale.
Then, considering separately each single component of the cosmic fluid[2], if we insert the solution for ρ in (1.20) and solve we find the explicit expression for the scale factor in a spatially flat universe:

$$a(t) = \begin{cases} a_* \left(\frac{t}{t_*}\right)^{\frac{2}{3(1+w)}} & , \quad w \neq -1 \\ e^{Ht} & , \quad w = -1 \end{cases} \qquad \Longrightarrow \qquad a \propto \begin{cases} t^{\frac{2}{3}} & \text{non relativistic matter} \\ t^{\frac{1}{2}} & \text{radiation} \end{cases} \\ \qquad\qquad \text{cosmological constant } \Lambda \quad . \tag{1.24}$$

In the last case of exponential expansion with a constant Hubble parameter, one says the universe goes through a *de Sitter stage*.
In deriving the solutions for $w \neq -1$ in (1.24), an integration constant is set to zero. This choice correspond to picking the time of what is usually called the "Big-Bang singularity", defined by

$$a(t_{BB}) = 0 \quad , \quad \rho(t_{BB}) = \infty \tag{1.25}$$

to be $t_{BB} = 0$.
This is a physical singularity that signals the breakdown of GR at arbitrary high energies.
From (1.23) we deduce the evolution of the universe goes through different epochs, during which a specific component of the cosmic fluid dominates the energy budget: if we go sufficiently backwards in time, which means for very small values of the scale factor, $a \ll 1$, the total energy density is radiation dominated (RD), i.e. most of the matter exists in the form of a very hot and dense primordial plasma of relativistic particles; as a increases with time, there is a moment t_{eq} in which the energy density of radiation and non-relativistic matter become comparable, followed by a matter dominated epoch (MD); likewise, if we still go further in time there is a transition, occurring at an instant t_Λ, from the MD epoch to the actual one, dominated by the constant vacuum energy density associated to the cosmological term, which is responsible for the accelerated Hubble expansion we observe today (we will clarify the nature of Λ in section 1.3). Actually, what we really know is that there are observational evidences [11] for the existence of an unknown form of energy (hence the name *dark energy*) responsible for the current accelerated expansion. This energy uniformly fills otherwise empty space without being diluted by the expansion, and it constitutes about the 68% of the total energy of the present-day observable universe [2]. Several models have been proposed over the years in order to explain the nature of dark energy [12], but if we assume that the source

[2]Strictly speaking, the variable ρ stands for the total energy density, which is the sum of the energy densities ρ_i of each single component of the cosmic fluid, which means that that the Friedmann equations are coupled differential equations in the variables ρ_i. However the different scaling laws in (1.23) suggest the universe was dominated by a single component for most of its history.

of this energy has an equation of state with a constant w parameter, experimental observations suggest $w < -0.95$ (95% C.L.) [2], hence the cosmological constant is the most direct explanation consistent with the data.

1.2 The shortcomings of the model and the inflationary solution

The Hot Big-Bang cosmological model allows us to follow the evolution on large scale of the universe from few instants before the Big-Bang singularity until today. Through this model we are able to understand various observed phenomena like the abundance of light elements, explained by the Big Bang Nucleosynthesis process (BBN), the origin of the CMB radiation and the Hubble's law. Nevertheless, it contains some issues in the form of fine-tuning problems.
A fine-tuning problem is present every time one has to require *unnatural* initial conditions on one or more parameters of a theory in order that the theoretical predictions fit the observed experimental data, otherwise inexplicable by the model. In this context, the term unnatural refers to the fact that these parameters must be tuned so precisely that such a solution appears very unlikely from a probabilistic point of view. Indeed, it usually comes out that the interval of acceptable values for the parameters has substantially null measure in the space of all the possible ones, meaning that this "ad hoc" solution must be very improbable. Usually, providing a mechanism which is able to yield the desired result in a dynamical way, without imposing any peculiar initial condition, is preferred. In most instances, this choice needs the introduction of new physics.
In the following we show how the three main shortcomings of the Hot Big-Bang model, namely the *horizon problem*, the *flatness problem* and the *unwanted relics problem*, can be economically solved by the inflationary mechanism, consisting in a long enough primordial period of accelerated expansion of the universe, i.e. $\ddot{a}(t) > 0$.

1.2.1 The horizon problem

In order to discuss properly the problem it is mandatory to introduce the notion of cosmological horizon. The existence in the model of a initial time t_{BB} sets a limit to the greatest comoving distance travelled by a light signal since the Big Bang singularity until today. Since no signal is faster than light, this quantity corresponds to the linear size of the *observable universe* of a fundamental observer, defined as the portion of the universe around a comoving observer which is causally connected with him. It can be directly computed from the FLRW metric (1.14), by setting $ds^2 = 0$, as light travels along null geodesics. Without loss of generality, we can exploit the homogeneity to set the radial coordinate r to the initial value $r_0 = 0$, and the isotropy to just move in a generic radial direction ($d\theta = d\phi = 0$) disregarding the initial angular variables θ_0, ϕ_0. Then, one obtains the following comoving distance:

$$r_{ph}(t) \equiv \int_0^t \frac{dt'}{a(t')} = \int_0^r \frac{dr'}{\sqrt{1-kr^2}} \quad , \tag{1.26}$$

while the physical distance travelled is

$$d_{ph}(t) \equiv a(t)\, r_{ph}(t) \quad . \tag{1.27}$$

If the integral converges (depending on the behaviour of $a(t)$ near $t = 0$), the physical distance (1.27) is finite and it is called *particle horizon.* At a given time t, particles separated by distances greater than $d_{ph}(t)$ *have never communicated* in the whole history of the universe.
For a spatially flat universe dominated by a perfect fluid with equation of state $P = w\rho$ with

$w \neq -1$, if we insert the explicit expression for the scale factor (1.24) in the definition (1.27) we obtain

$$d_{ph}(t) = \frac{3(1+w)}{1+3w} t \quad . \tag{1.28}$$

Then, in a FLRW universe $d_{ph}(t)$ is finite and positive for $w > -\frac{1}{3}$, which means, by looking at the acceleration equation (1.21), that $d_{ph}(t)$ is finite iff the acceleration of the expansion is negative, $\ddot{a}(t) < 0$. Indeed, using the equation of state, equation (1.21) becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1+3w) \quad \Longrightarrow \quad \ddot{a} < 0 \iff w > -\frac{1}{3} \quad . \tag{1.29}$$

The condition (1.29), known as *strong energy condition* (SEC), is satisfied by ordinary matter and radiation, for which, as we already mentioned, $w = 0$ and $w = 1/3$, respectively. This means that the MD and RD epochs consist into a period of negative accelerated expansion. A period of positive accelerated expansion of the universe can be only driven by a non conventional perfect fluid with $w < -\frac{1}{3}$ which must dominate the energy budget.

Another fundamental cosmological distance is represented by the *Hubble radius*, or its comoving counterpart, the *comoving Hubble radius*:

$$R_H(t) \equiv \frac{1}{H(t)} = \frac{a(t)}{\dot{a}(t)} \quad , \quad r_H(t) \equiv \frac{R_H(t)}{a(t)} = \frac{1}{a(t)H(t)} = \frac{1}{\dot{a}(t)} \quad . \tag{1.30}$$

Given a comoving observer, let us consider the points around him on a sphere of comoving radius r. At time t, the physical distance of these points from the observer is $d(t) = a(t)r$, and it increases with the expansion. The points recede from the observer with a velocity

$$v(t) \equiv \dot{d}(t) = \dot{a}(t)r = H(t)d(t) \quad . \tag{1.31}$$

This last expression allows to define the Hubble radius $R_H(t)$ as the physical radius of a sphere centered on a comoving observer whose points recede with a velocity equal to the speed of light. Reminding the definition of the characteristic Hubble time $t_H = H^{-1}$, the Hubble radius can also be regarded as the physical distance travelled by light within t_H. Then, at a given time t, particles separated by distances greater than $R_H(t)$ are not in causal contact *at that moment* but they could have communicated in an earlier stage. More precisely, they are not locally in causal contact, i.e. within the past Hubble time interval, but they may be globally, i.e. in the whole history of the universe.
The explicit computation of the Hubble radius for a universe dominated by a conventional perfect fluid yields

$$R_H(t) = H^{-1} = \frac{3(1+w)}{2} t \quad , \tag{1.32}$$

which combined with (1.28) leads to

$$d_{ph}(t) = \frac{2}{1+3w} R_H(t) \quad . \tag{1.33}$$

Then, in standard cosmology we have $d_{ph} \sim R_H$. This is the reason why one usually refers to both d_{ph} and R_H as the "horizon", even if their physical meaning is very different.
Usually, the evolution of the causal connection throughout the history of the universe is represented by plotting $R_H(t)$ / $r_H(t)$ over time, and comparing the physical/comoving length scales λ_{phys} /

λ with it. In a FLRW universe filled by an ordinary fluid the comoving Hubble radius is a strictly increasing monotonic function since

$$\dot{r}_H(t) = -\frac{\ddot{a}(t)}{\dot{a}^2(t)} \quad \Longrightarrow \quad \dot{r}_H(t) > 0 \quad \text{for} \quad w > -\frac{1}{3} \quad , \tag{1.34}$$

while the comoving lengths are constant. Equivalently, the Hubble radius always grows faster than the physical length scales. Indeed, from the Friedmann equation, one has $R_H(t) \sim a^2$ (RD) or $R_H(t) \sim a^{3/2}$ (MD), while $\lambda_{phys} \sim a$. In other words, the dimension of the causal connected region around a fundamental observer grows in time and in particular, for standard cosmology, all the length scales enter the causal horizon for the first time since $t = 0$.
The fact that the causal horizon is an increasing function of the cosmic time allows us to use the comoving distance $r_{ph}(t)$ as a different time coordinate known as conformal time, denoted with τ and such that $d\tau = \frac{dt}{a(t)}$. Then

$$\tau \equiv r_{ph}(t) = \int_0^t \frac{dt'}{a(t')} \quad . \tag{1.35}$$

We now have all the instruments to properly discuss the horizon issue. It arises by a careful observation of the CMB signal. This radiation was emitted about 380000 years after the Big Bang singularity, when the universe cooled enough to allow the formation of the first hydrogen atoms, and consequently the thermal decoupling of photons from the primordial plasma [13]. At that moment, the latter started to free stream along the geodesics of the FLRW universe, until they were revealed by our detectors.
Remarkably, as we said at the beginning of the chapter, the photons of this radiation share almost the same temperature, no matter the direction in the sky we look at. However, we now show that, according to standard cosmology, CMB photons coming from regions in the sky separated by an angular distance greater than $\Delta\theta \simeq 1.7^\circ$ were outside each other's particle horizon at the instant of decoupling, the last time microphysics can effectively act to smooth out temperature fluctuations. By $\Delta\theta$ we intend the angle which today is subtended by a region of the universe whose dimension at the instant t_{dec} of decoupling was of the order of the particle horizon $d_{ph}(t_{dec})$. At the present time t_0, this region has grown due to the expansion of the universe of a factor $a(t_0)/a(t_{dec}) \equiv a_0/a_{dec}$. Then, using the today particle horizon $d_{ph}(t_0)$ to estimate the linear dimension of our observable universe, and given that $d_{ph}(t_0) \gg d_{ph}(t_{dec})$, the angle $\Delta\theta$ is roughly obtained by

$$\Delta\theta \approx \frac{\frac{a_0}{a_{dec}} d_{ph}(t_{dec})}{d_{ph}(t_0)} \quad . \tag{1.36}$$

The emission of the CMB radiation took place during the MD epoch. If we make the approximation of considering the universe today as still matter dominated[3], from (1.24) and (1.28) we have:

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3} \quad , \quad H(t) = \frac{2}{3t} \quad , \quad d_{ph}(t) = 3t \quad , \tag{1.37}$$

where the above relations are valid for $t_{eq} \leq t \leq t_0$.
Using $H(t) = H_0 \left(\frac{a_0}{a(t)}\right)^{3/2}$, the particle horizon can be written as

$$d_{ph}(t) = 2H(t)^{-1} = 2H_0^{-1} \left(\frac{a(t)}{a_0}\right)^{3/2} \quad . \tag{1.38}$$

[3]The universe was matter dominated for most of the time since matter-radiation equality at $t_{\rm eq} \approx 70000$ years [2]. The age at which matter and the cosmological constant had equal energy density is relatively recent, at $t_\Lambda = 9.8 \pm 1.0\,\text{Gyr}$ [14].

Then, the angle $\Delta\theta$ becomes

$$\Delta\theta \approx \left(\frac{a_{\rm dec}}{a_0}\right)^{1/2} \quad . \tag{1.39}$$

The value of the scale factor in the past can be related to the present one via the cosmological redshift parameter z

$$\frac{a(t_0)}{a(t)} = 1 + z(t) \quad , \quad z(t) \equiv \frac{\lambda_0 - \lambda_t}{\lambda_t} \tag{1.40}$$

where $z(t)$ is the redshift at the time t, defined as the relative variation of the wavelength λ of a light signal emitted at time t and detected at time t_0. By definition, $z = 0$ today, and $1 + z \simeq z$ for $z \gg 1$. We can therefore write

$$\Delta\theta \approx (1 + z_{\rm dec})^{-1/2} \quad . \tag{1.41}$$

From the Planck data [2], $z_{\rm dec} = 1089.80 \pm 0.21\,(68\%\ \text{C.L.})$, then

$$\Delta\theta \approx z_{\rm dec}^{-1/2} \approx 0.03\text{rad} \implies \Delta\theta \approx 1.7^\circ \tag{1.42}$$

Now we can easily realize where the problem arises: the above considerations implies that patches of the universe separated by more than $\Delta\theta \simeq 1.7^\circ$ had never been in causal contact when the CMB radiation was emitted, but today they practically share the same thermodynamic properties! This is very counterintuitive, because we should expect to measure significant variations of temperature over the sky on such an angular scale, but instead we observe everywhere the same temperature T_{CMB}, up to very small fluctuations of order 10^{-5}.

As previously mentioned, there are mainly two ways to solve the problem. One is to assume an initial condition consisting of an almost perfectly homogeneous and isotropic universe, presenting the right small amount of anisotropies in order to get, by subsequent evolution, the large scale structure we observe today. This solution implies a fine tuning problem.
Alternatively, a more compelling solution is suggested by the following rewriting of the causal horizon $r_{ph}(t)$

$$r_{ph}(t) = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{a'}\frac{1}{a'H} = \int_{-\infty}^{\ln a} r_H \, d\ln a' \quad , \tag{1.43}$$

where we see how the the comoving Hubble radius r_H is the instantaneous contribution per logarithmic interval of the scale factor to the comoving particle horizon r_{ph}. The origin of the horizon problem lies in the fact that, since in standard cosmology r_H is always an increasing function, we expect the largest contribution to r_{ph} to come from recent times. Therefore, there is no hope that a super-Hubble sized region at early times could have been in causal contact at an earlier stage, because $r_{ph} \sim r_H$. Instead, this eventuality might occur in the case we have $r_{ph} \gg r_H$ today, which can be realized through the introduction of a new phase before the radiation dominated epoch, called inflation, characterized by a *decreasing* comoving Hubble radius, $\dot{r}_H(t) < 0$, so that r_{ph} receives most of its contribution from primordial epochs. A decreasing comoving Hubble radius implies that, during inflation, the physical length scales $\lambda_{phys} = a\lambda$ grow faster than the Hubble radius $R_H = H^{-1} \sim a^n\ (n < 1)$, as shown in figure 1.2. Thus, length scales which were outside the Hubble horizon at the time of CMB, and which are well inside the Hubble horizon today, could have already been inside R_H during the inflationary phase, which subsequently caused their horizon exit through the accelerated expansion. Hence, CMB photons emitted from causally disconnected regions at the moment of decoupling, which today share nearly the same temperature, had a chance to reach thermal equilibrium in an earlier epoch. This argument would explain the uniformity of

the CMB radiation.
We see from (1.34) that, contrary to the standard cosmological evolution, the requirement $\dot{r}_H(t) < 0$ leads to a period characterized by an accelerated expansion, $\ddot{a}(t) > 0$, hence dominated by an unconventional perfect fluid with $w < -\frac{1}{3}$. This condition defines what we call an inflationary phase.

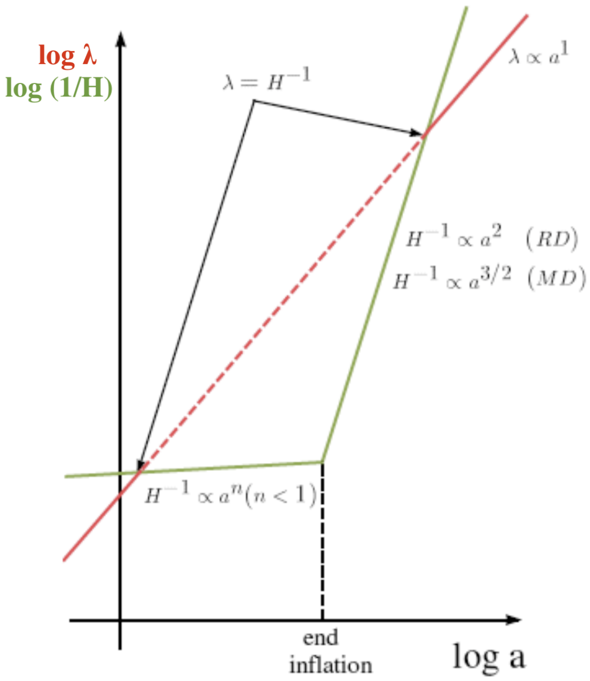

Figure 1.2: Graphical representation of the solution to the horizon problem [15]. Contrary to standard cosmology, where any physical length scale starts larger than R_H and then crosses the Hubble horizon only once, during inflation the latter remains almost constant, so that it is possible for physical length scales which were causally connected during inflation to leave the Hubble horizon because of the accelerated expansion and then re-enter the horizon at RD or MD epoch. The dotted line denotes the super-horizon stage of the length scale λ.

Let us now estimate the amount of inflation required to solve the horizon problem.
In order to quantify how much the universe has grown during the accelerated expansion we define the *number of e-folds* N as

$$e^{N(t)} \equiv \frac{a(t_f)}{a(t)} \quad , \quad t < t_f \tag{1.44}$$

where t_f stands for the instant of cosmic time at which inflation ends. Then, during inflation, N can be regarded as an inverse time variable, as $N(t)$ decreases as t increases, with $N = 0$ at $t = t_f$. Since everywhere we look in the sky we see a uniform CMB radiation with the same temperature, if we want the horizon problem to disappear we must at least require that the largest observable length scale we can probe today is inside the causal horizon at decoupling. However, in principle, this can also be true for larger scales and the problem would still be solved.
We can estimate the dimension of the actual observable universe with the comoving distance travelled by photons from the instant of decoupling until today (since before t_{dec} photons could not free stream because of the strong thermal coupling with the primordial plasma), given by

$$\int_{t_{dec}}^{t_0} \frac{dt}{a(t)} = \int_{t_{dec}}^{t_0} \frac{d\,\ln a}{aH} \quad . \tag{1.45}$$

With the introduction of the inflationary phase, the comoving particle horizon accumulated until the moment of decoupling is now

$$\int_{t_*}^{t_{dec}} \frac{dt}{a(t)} = \int_{t_*}^{t_{dec}} \frac{d\ln a}{aH} \quad , \tag{1.46}$$

with t_* a generic instant during inflation, $t_* < t_f$. Therefore, the afore-mentioned requirement reads

$$\int_{t_{dec}}^{t_0} \frac{d\ln a}{aH} \leq \int_{t_*}^{t_{dec}} \frac{d\ln a}{aH} \quad . \tag{1.47}$$

which sets an upper bound on t_*, or a lower bound on the number of e-folds $N(t_*) \equiv N_*$, represented respectively by the values t_{CMB} and $N(t_{CMB}) \equiv N_{CMB}$ at which the CMB scale enters the causal horizon, i.e. such that the inequality (1.47) is saturated.
The integrals on both sides of (1.47) are dominated by the moment when the comoving Hubble radius is the largest, i.e by $t = t_0$ for the l.h.s. and by $t = t_*$ for the r.h.s., so the previous inequality can be approximated as

$$\frac{1}{a_0 H_0} \leq \frac{1}{a_* H_*} \quad , \tag{1.48}$$

which, rearranged, yields

$$H_0^{-1} \leq \frac{a_0}{a_*} H_*^{-1} = \frac{a_f}{a_f}\frac{a_0}{a_*} H_*^{-1} \qquad \Longrightarrow \qquad \frac{a_f}{a_*} = e^{N_*} \geq \frac{a_f H_*}{a_0 H_0} \quad . \tag{1.49}$$

We want to rewrite the ratio of the scale factors on the r.h.s. in terms of the temperatures of the universe today and at the end of inflation, i.e. T_0 and T_f. After inflation, a thermalization process of the material filling the universe is necessary to obtain a transition to the initial RD epoch of the Hot Big Bang model. Hence, the universe is reheated up to a temperature T_{reh}, at which radiation becomes the dominant component. In this discussion we assume an instantaneous thermalization, i.e. the reheating temperature coincides with the temperature at the end of inflation, $T_f \equiv T_{reh}$. Assuming also an *adiabatic expansion*, we have that the entropy S in a comoving volume is conserved. The total entropy density of all particle species is defined as [16]

$$s \equiv \frac{2\pi^2}{45} g_{*S}(T) T^3 \quad , \tag{1.50}$$

where $g_{*S}(T)$ is the *effective number of relativistic degrees of freedom in entropy* at temperature T. Then, the conservation of S reads

$$S = sa^3 = \frac{2\pi^2}{45} g_{*S}(T) T^3 a^3 = const \qquad \Longrightarrow \qquad a \propto g_{*S}^{-1/3}(T) T^{-1} \quad . \tag{1.51}$$

The function $g_{*S}(T)$ undergoes significant variations everytime the temperature T of the primordial plasma drops below the mass threshold of a particle species that becomes non-relativistic and annihilates into relativistic species in the plasma.
To estimate the required number of e-folds, we approximate the behaviour of the scale factor as $a \propto T^{-1}$, neglecting the temperature dependence of $g_{*S}(T)$. Hence, the previous inequality becomes

$$e^{N_*} \gtrsim \frac{T_0}{H_0} \frac{H_*}{T_{reh}} \quad . \tag{1.52}$$

We now consider the case of a *quasi*-exponential accelerated expansion, instead of an exactly exponential one (the reason of the "quasi" will be clarified in section 1.3). This period coincides with a *quasi de Sitter* stage, during which the energy of the universe is dominated by a perfect fluid with $w \approx -1$, and the Hubble parameter $H_I(t)$, although a function of time, remains nearly constant. After these considerations, we can write

$$H_I \simeq constant \quad \Longrightarrow \quad H^2(t_*) \simeq H^2(t_f) \sim \frac{T_{reh}^4}{M_{Pl}^2} \quad , \tag{1.53}$$

where the dependence on the temperature in the second equality comes from the first Friedmann equation, and derives from the previous assumption of instantaneous reheating of the universe after inflation, which leads to a RD epoch with $\rho \propto T^4$. We also introduced the reduced Planck mass, $M_{Pl} = (8\pi G)^{-1/2} \sim 10^{18}$GeV.
From experimental measurements [2, 17] we know that

$$T_0 = (2.72548 \pm 0.00057)\text{K} \quad (95\% \text{ C.L.}) \quad , \tag{1.54}$$

$$H_0 = (67.66 \pm 0.42)\,\text{Km}\,\text{s}^{-1}\,\text{Mpc}^{-1} \quad (68\% \text{ C.L.}) \quad , \tag{1.55}$$

which converted in natural units yield $T_0 \simeq 2.4 \cdot 10^{-13}$GeV and $H_0 \simeq 1.54 \cdot 10^{-42}$GeV. Then, the inequality (1.52) becomes ($10 \simeq e^{2,3}$)

$$e^{N_*} \gtrsim 10^{29}\,\frac{T_{reh}}{M_{Pl}} \quad \Longrightarrow \quad N_* \gtrsim 67 + \ln\left(\frac{T_{reh}}{M_{Pl}}\right) \quad . \tag{1.56}$$

The Planck data [5] puts an upper bound on the energy scale of inflation, which implies an upper bound on the Hubble parameter H_I during the accelerated expansion given by

$$\frac{H_I}{M_{Pl}} < 2.5 \cdot 10^{-5} \quad (95\% \text{ C.L.}) \quad . \tag{1.57}$$

From (1.53) we have

$$\frac{T_{reh}}{M_{Pl}} \sim \left(\frac{H_I}{M_{Pl}}\right)^{1/2} < (2.5 \cdot 10^{-5})^{1/2} = 5 \cdot 10^{-3} \quad \longleftrightarrow \quad T_{reh} < 1.2 \cdot 10^{16}\text{GeV} \quad . \tag{1.58}$$

A lower bound is placed on the reheating temperature by primordial nucleosynthesis (BBN), $T_{reh} < T_{BBN} \sim 10^{-2}$GeV [18]. Then, from (1.56) we obtain

$$N_* > N_{CMB} \simeq [21, 62] \quad . \tag{1.59}$$

1.2.2 The flatness problem

Analogously to the horizon problem, also the flatness problem stems from an experimental observation, related, in this case, to the spatial curvature of the universe.
Let us start by the Friedmann equation (1.20) with the inclusion of the curvature term

$$H^2 = \frac{\rho}{3M_{Pl}^2} - \frac{k}{a^2} \quad , \tag{1.60}$$

and let us rewrite it so to make the problem more transparent. To this goal we define the critical energy density $\rho_c(t) \equiv 3H^2(t)M_{Pl}^2$, which corresponds to the energy density of flat universe ($k = 0$) with Hubble rate H. We also define the two parameters $\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)}$ and $\Omega_k(t) \equiv -\frac{k}{a(t)^2 H^2(t)}$. Dividing both side of (1.60) by $H^2(t)$, we obtain

$$\Omega(t) + \Omega_k(t) = 1 \quad . \tag{1.61}$$

Our universe is either flat, of very close to flat, to very high accuracy. Specifically, the parameter Ω_k, that measures the departure from flatness, is constrained by CMB observations [2] to be

$$\left|\Omega_k(t_0)\right| = \left|1 - \Omega(t_0)\right| = 0.0007 \pm 0.0019 \quad (68\% \text{ C.L.}) \quad . \tag{1.62}$$

In order to understand the reason why this experimental evidence leads to a fine tuning problem, let us study the time evolution of the parameter $\Omega_k(t)$ in standard cosmology. From equations (1.20) and (1.23) we have:

$$\left|\Omega_k(t)\right| = \frac{|k|}{a^2(t)H^2(t)} \propto \begin{cases} a^2(t) & \text{RD} \\ a(t) & \text{MD} \end{cases} \quad . \tag{1.63}$$

We can use this relation to relate the current value of the curvature parameter to the value it had at a time t_1 before the matter-radiation equality time $t_{\rm eq}$,

$$\frac{\Omega_k\left(t_1\right)}{\Omega_k\left(t_0\right)} = \frac{\Omega_k\left(t_1\right)}{\Omega_k\left(t_{\rm eq}\right)}\frac{\Omega_k\left(t_{\rm eq}\right)}{\Omega_k\left(t_0\right)} = \frac{a^2\left(t_1\right)}{a^2\left(t_{\rm eq}\right)}\frac{a\left(t_{\rm eq}\right)}{a\left(t_0\right)} = \frac{a^2\left(t_1\right)}{a^2\left(t_0\right)}\frac{a\left(t_0\right)}{a\left(t_{\rm eq}\right)} \tag{1.64}$$

We can therefore write

$$\frac{\Omega_k\left(t_1\right)}{\Omega_k\left(t_0\right)} \simeq \frac{z_{\rm eq}}{z^2(t_1)} \quad . \tag{1.65}$$

From the Planck data [2] $z_{\rm eq} = 3387 \pm 21\,(68\%$ C.L.). To estimate the redshift at early times, we use the fact that the temperature of the thermal bath scales as $T \propto a^{-1}$ (disregarding the change in the relativistic degrees of freedom). Given the current CMB temperature (1.54) and the fact that $\left|\Omega_k(t_0)\right| < 10^{-3}$, we therefore obtain

$$\left|\Omega_k\left(t_1\right)\right| \simeq \left|\Omega_k\left(t_0\right)\right| 3.387 \cdot 10^4 \left(\frac{10^{-13}\text{GeV}}{T\left(t_1\right)}\right)^2 \lesssim 10 \left(\frac{10^{-13}\text{GeV}}{T\left(t_1\right)}\right)^2 \quad . \tag{1.66}$$

If we consider a temperature close to the onset of BBN, which means $T(t_1 = t_{BBN}) \sim$ MeV, we find $\left|\Omega_k(t_{BBN})\right| \lesssim 10^{-19}$. Assuming a temperature at the Planck scale $T(t_1 = t_{Pl}) \sim 10^{19}$GeV results in $\left|\Omega_k(t_{Pl})\right| \lesssim 10^{-63}$. From this extreme fine-tuning, we conclude that a universe as close to flat as the observed one is very unnatural in standard cosmology.

One solution to this problem is to assume that the universe is exactly flat. Alternatively, since $|\Omega_k| \propto a^{-2} \simeq e^{-2N}$ in an inflationary universe ($H \simeq$ constant), one can imagine that the curvature parameter is of order one at the onset of inflation, and that it is then decreased by the inflationary expansion. The minimum duration of inflation required to produce the small values of the curvature just computed depends on what we assume as the reheating temperature at which the thermal bath of the standard RD epoch was formed (also in this discussion, for simplicity, we assume instantaneous reheating after inflation). Indeed, if we choose t_1 as the time at the end of inflation, $t_1 = t_f \equiv t_{reh}$, from (1.66) we obtain

$$N \gtrsim \ln\left(\frac{T_{\rm reh}}{T_0}\right) \simeq 53 + \ln\left(\frac{T_{\rm reh}}{10^{10}\,\text{GeV}}\right) \quad . \tag{1.67}$$

Given the previous bounds on T_{reh} used at the end of subsection 1.2.1, we have

$$N \gtrsim [25, 67] \quad . \tag{1.68}$$

1.2.3 The unwanted relics problem

There might be some additional challenges in embedding the standard cosmology in a particle physics theory. If the early universe had a temperature greater than $T \sim 10^{14} - 10^{16}$ GeV, and if Grand Unified Theories[4] (GUT) are realized in nature, then these symmetries were unbroken at those high temperatures. As the temperature drops below this value, a variety of stable, super-heavy particles called *topological defects* or *topological solitons*, are typically formed as a consequence of a second-order phase transition triggered by the spontaneous symmetry breaking (SSB) of the global subgroup of the GUT gauge group.
Before explaining the reason why these objects are "unwanted", let us give an insight into the mathematical framework.

In classical field theory, a topological soliton is defined to be a particular class of solutions of the field equation of motion corresponding to *finite energy* and *topologically stable* configurations of the field.
In general, given a theory with gauge group $\mathcal{G}$, which undergoes a spontaneous breaking into a subgroup $\mathcal{H}$, there will be soliton solutions associated to the symmetry breaking $\mathcal{G} \to \mathcal{H}$ if the degenerate vacuum manifold $\mathcal{G}/\mathcal{H}$ has a non trivial topological structure. More in detail, the homotopy group $\Pi_d(\mathcal{G}/\mathcal{H})$, defined as the set of homotopy classes of the maps $S^d \to \mathcal{G}/\mathcal{H}$, with d the dimension of the boundary of the physical space in which the theory lives, must be not trivial. In other words, not all the possible vanishing energy (vacuum) configuration are equivalent, in the sense that they cannot be all related to any other one by a gauge transformation. Then, the space of degenerate vacua splits up in different homotopy equivalence classes.
Moreover, the requirement of finite energy put a constraint on the asymptotic behaviour of the solitons: at the boundary of space they must behave as vanishing energy configurations, otherwise the integration of the Hamiltonian density over space would yield infinite energy. Hence, if not all vacua are homotopically equivalent, we can have as well the existence of homotopically distinct soliton solutions which interpolates between different vacua.
Then, topological stability refers to the fact that these solutions cannot be continuously mapped into each other and to the vacuum state, i.e. the corresponding quantum particles of the associated QFT do not decay. Instead, field configurations associated to elementary particles, are smooth fluctuations of the vacuum. From a physical point of view, the mapping from a soliton configuration to a vaccum configuration throughout the entire space is not allowed because it entails an amount of energy which becomes infinitely large in the thermodynamic limit [20].
The nature of a cosmological defect depends on the details of the symmetry breaking pattern. The most notable example are magnetic monopoles, i.e. point-like defects appearing in a $SO(3)$ gauge theory, in which $SO(3)$ is spontaneously broken to $U(1)$ by a Higgs triplet Φ^a transforming in the adjoint representation ('t Hooft Polyakov monopoles). We can also have one dimensional structures, as well as cosmic strings, or 2-dimensional ones like domain walls, and so on. The latter arise by the SSB of discrete symmetries. We refer to [13, 20] for more details about solitons and their role in the cosmological framework.

The cosmological production mechanism of topological defects is known as *Kibble mechanism* [21]. In short, it relies upon the fact that, during the GUT phase transition, the finite particle horizon

[4]A Grand Unified Theory is a high-energy completion of the Standard Model (SM) of particle physics which tries to combine the fundamental interactions of Nature, with the exception of gravity, into a unique gauge interaction specified by a gauge symmetry group containing the SM one, the latter given by $\mathrm{SO(3)} \times \mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$. We refer the reader to [19] for more details on the argument.

$d_{ph}(t)$ sets an upper bound on the correlation length ξ of the field inducing the spontaneous breaking of the symmetry group, call it Φ. The correlation length ξ corresponds with the distance over which the vacuum expectation value (VEV) $\langle \Phi \rangle$ acquired by the field after the SSB is correlated, implying that, if two generic points of the universe are separated by a distance $l > \xi$, then there is a possibility that the two points will be in two different vacua of the system. In that case, there could be a topological defect configuration which smoothly interpolates between these two regions of space being in different ground states, allowing for the existence of a transition region of false vacuum.
If we roughly estimate the characteristic size of a topological defect through the correlation length ξ, we expect that at least one defect per horizon volume should arise after the phase transition because of the causality bound on ξ, i.e.

$$n_{def}(t_{GUT}) \sim \xi^{-3} \gtrsim d_{ph}^{-3}(t_{GUT}) \quad . \tag{1.69}$$

This lower bound on n_{def} leads to a serious problem in standard cosmology, since, if these relics do not annihilate efficiently, we should expect an overabundance in their current number density [22–24], inconsistently with experimental observations. Furthermore, these massive relics behave as pressurless dust. Their energy density scales more slowly with the expansion than that of radiation, so they can become the dominant component of the universe at dangerously early times, e.g. before the BBN mechanism.
These unwanted particles, if present at the onset of inflation, were diluted away to a completely negligible (and completely unobservable) abundance by the inflationary expansion. It is important to notice that we have also to require that the reheating temperature, at which the standard RD era begins after inflation, must be smaller than the GUT energy scale, $T_{reh} < T_{GUT}$, otherwise the GUT symmetry would be restored and the cosmological defects would be formed again through the same mechanism. Another bound must also be imposed on T_{reh} in order to avoid the thermal production in the early RD stage of the universe of massive unwanted thermal relics predicted by supersymmetric theories, such as gravitinos [25, 26]. This last bound is very model dependent, since it depends on the specific particle content we are assuming.

1.3 The Inflaton field

From what we have seen so far, it is possible to summarize the definition of inflation and the condition necessary to achieve it as

$$\text{INFLATION} \equiv \ddot{a} > 0 \quad \Longleftrightarrow \quad P < -\frac{1}{3}\rho \quad . \tag{1.70}$$

We end the first chapter by showing how a period of accelerated expansion in the early universe can be implemented by assuming the existence of a quantum field, usually referred to as the *inflaton field*.

As said at the end of subsection 1.1, the current exponentially accelerated expansion is well explained by the cosmological term $\Lambda g_{\mu\nu}$ we find in the Einstein equations (1.2), which today results to be dominant w.r.t. the other sources of energy contained in the stress-energy tensor $T_{\mu\nu}$. We have already pointed out that this term is equivalent to an additional stress-energy tensor of the form $T^{\Lambda}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$, belonging to a perfect fluid filling the universe with constant energy density and isotropic pressure given by $P_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G}$ ($w = -1$).
In cosmology, within the context of GR, this form of energy is attributed to the vacuum, the latter simply understood as empty space. More precisely, in a Quantum Field Theory (QFT), the

vacuum is the lowest energy state of the theory. We call the energy of this state the "zero-point energy" of the system[5]. In fact, given a particle species with its stress-energy tensor, one can show [27] on very general grounds that the vacuum expectation value of the latter assumes the form $\langle 0| T_{\mu\nu} |0\rangle \equiv \langle T_{\mu\nu} \rangle = -\langle \rho \rangle g_{\mu\nu}$. Then, once inserted in the Einstein equations, $\langle T_{\mu\nu} \rangle$ mimics the effects of an effective cosmological constant $\Lambda_{eff} = 8\pi G \langle \rho \rangle$.
The true vacuum state of a QFT, corresponding to the absolute minimum configuration of the potential of the field theory, is stable, i.e. $\langle \rho \rangle$ is constant. This means that if inflation was driven by the true vacuum energy of a field theory, we would obtain a pure never ending de Sitter phase with $w = -1$ exactly. Instead, we want inflation to stop at some point, since we would like to recover the standard Big Bang cosmological model.
Historically, in the first model of inflation proposed (Alan Guth, 1981 [4]), the driving energy behind the exponential de Sitter expansion was the one associated to a false vacuum rather than a true vacuum, the former corresponding to a metastable local minumum of the potential. Once the universe cools enough due to the expansion, inflation can end because the system abruptly reaches the true stable vacuum through a first order phase transition[6]. The latter implies the phenomenon of *bubble nucleation*, i.e. the random formation of bubbles of true vacuum separated by spacetime regions of false vacuum, where the transition has not yet occured. These bubbles expand and if they met each other they merge together. Moreover, when the bubbles met, the collision of their walls heats up the universe by giving rise to the thermal plasma of standard cosmology. The problem with this idea is that the bubble coalescence should fill at least a portion of the universe equal to the size of the observable universe today, but the background spacetime of false vacuum around them is inflating, so these bubbles actually never met to form such a big region as we require, so we remain with separate empty bubbles.
This problem, known under the name of "Graceful exit problem", was solved by the works of Linde and Albrecht, Steinhardt [28, 29], who realized that a field seating in a local or global minimum of its potential is not necessary in order to have inflation. The accelerated expansion can also consists into a quasi de Sitter phase with $w \simeq -1$, which can be attained by a field moving very slowly in a region where its potential is very flat, so that, while in this region of the potential, the field mimics a vacuum energy.

Let's assume the inflaton field is a scalar field, function of the comoving coordinates, $\hat{\phi}(t, \mathbf{x})$. In this case, the total classical action in a curved spacetime is given by

$$S_{TOT} = S_{EH} + S_\phi + S_{\phi,\Psi,A,\Phi} \tag{1.71}$$

$$= \frac{1}{16\pi G} \int d^4x \sqrt{-g} \Big(R + \mathcal{L}\left[\phi, \phi_{;\mu}\right] + \mathcal{L}_{\phi,\Psi,A,\Phi} \Big) \quad , \tag{1.72}$$

where $g = det(g_{\mu\nu})$, R is the Ricci scalar, S_{EH} is the Einstein Hilbert action which gives rise, through the least action principle, to the l.h.s. of the Einstein equations (1.2), S_ϕ is the inflaton action and finally $S_{\phi,\Psi,A,\Phi}$ is the action describing the dynamics of all the other fundamental fields (matter fields, gauge fields and other scalar fields) and their coupling with the inflaton.
The generic form of a, minimally coupled [7], density lagrangian for the inflaton field is

$$\mathcal{L}\left[\phi, \partial_\mu \phi\right] = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \quad , \tag{1.73}$$

[5] It is fair to say that, although we can have a "pictorial" interpretation of this energy as due to the continuous formation and distruction of virtual pairs admitted by Heisenberg uncertainty principle, we currently do not have a way to reliably compute this vacuum energy.

[6] The first order phase transition entails the quantum tunnelling of the potential barrier separating the local minimum from the absolute one, which can only happen at sufficiently low temperatures, such that the barrier is low enough.

[7] We are not considering interaction terms with gravity as $\phi^2 R$.

where we replaced the covariant derivatives with ordinary partial derivatives, since the two coincide for the case of a scalar quantity.
The potential $V(\phi)$ coincides just with the quadratic mass term $\frac{1}{2}m_\phi^2\phi^2$ in the case of a free inflaton field, but in general it also accounts for self-interactions of ϕ and, possibly, also effective interactions with other fields, obtained by integrating them out[8].
The stress-energy tensor is defined as

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}}\frac{\delta(S_\phi + S_{\phi,\Psi,A,\Phi})}{\delta g^{\mu\nu}} \quad . \tag{1.74}$$

During the inflationary phase, the energy contribution of the other particle species, which are in the form of a radiation fluid, is subdominant w.r.t. the inflaton one. Then, the inflaton stress-energy tensor $T^\phi_{\mu\nu}$ is obtained from the definition (1.74), by neglecting the $S_{\phi,\Psi,A,\Phi}$ term of the total action. In the most general case of a non-minimally coupled scalar field, the infinitesimal variation δS of the action due to an infinitesimal variation $\delta g_{\mu\nu}$ of the metric tensor, up to linear order in $\delta g_{\mu\nu}$, is given by

$$\delta S\big(g^{\mu\nu}, \partial_\rho g^{\mu\nu}\big) = \int d^4x \left[\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta(\partial_\rho g^{\mu\nu})}\partial_\rho \delta g^{\mu\nu}\right] \tag{1.75}$$

$$= \int d^4x \left[\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} - \partial_\rho\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta(\partial_\rho g^{\mu\nu})}\right]\delta g^{\mu\nu} \quad , \tag{1.76}$$

where, passing from the first to the second line, we applied the Leibniz rule on the second term of (1.75), and then the divergence theorem, combined with the fact that the variations $\delta g_{\mu\nu}$ are taken to be vanishing at the boundaries of spacetime. Hence, we get

$$T^\phi_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}}\left[\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} - \partial_\rho\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta(\partial_\rho g^{\mu\nu})}\right] \tag{1.77}$$

For a minimally coupled scalar field, the second term of the last expression is null, while the first provides

$$T^\phi_{\mu\nu} = \frac{-2}{\sqrt{-g}}\left[-\frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}V(\phi)\right] \tag{1.78}$$

$$= \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left[-\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - V(\phi)\right] \tag{1.79}$$

$$= \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}_\phi \quad . \tag{1.80}$$

where in the first step we used

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu} \; . \tag{1.81}$$

As previously discussed, inflation cannot be driven by the energy of a stable vacuum configuration, $\langle\phi\rangle = constant$, otherwise it would last forever. For this reason, we require the expansion to be generated by a dynamical evolution of the ground state configuration.

$$\langle 0|\,\hat{\phi}(t,\mathbf{x})\,|0\rangle = \phi_0(t) \quad , \tag{1.82}$$

[8] In fact, a quadratic potential all throughout inflation is ruled out by CMB data [2].

which has no dependence on the spatial coordinates, due to the properties of homogeneity and isotropy of the background where the dynamics takes place. This condition allows the accelerated expansion to have an end when, because of some specific dynamical mechanism, the inflaton energy density is not more the dominant one.
In order to simplify the study of the evolution of $\hat{\phi}(t, \vec{x})$, we split the field in its classical ground state configuration, plus the inhomogeneous quantum fluctuations around this background value,

$$\hat{\phi}(t, \mathbf{x}) = \phi_0(t) + \hat{\delta\phi}(t, \mathbf{x}) \quad , \tag{1.83}$$

and, in addition, we assume the fluctuations to be small w.r.t. the classical background, i.e. :

$$\langle \delta\hat{\phi}^2(t, \mathbf{x}) \rangle \ll \phi_0^2(t) \quad . \tag{1.84}$$

In the next chapters, this last condition will enables us to tackle the equations of motion by means of perturbation theory.
In the reminder of this chapter we focus instead on the dominant classical dynamics. For the homogeneous and isotropic configuration $\phi_0(t)$ in a FLRW spacetime, the stress-energy tensor (1.80) is:

$$T^0_0 = -\left[\frac{1}{2}\dot{\phi}_0^2(t) + V(\phi_0)\right] = -\rho_\phi(t) \tag{1.85}$$

$$T^i_j = \left[\frac{1}{2}\dot{\phi}_0^2(t) - V(\phi_0)\right]\delta^i_j = \delta^i_j P_\phi(t) \quad . \tag{1.86}$$

The resulting form of $T^\phi_{\mu\nu}$ is the one of a perfect fluid at rest, with $\rho_\phi(t)$ and $P_\phi(t)$ respectively the energy density and the isotropic pressure associated to the inflaton field.
If, during the dynamical evolution, $V(\phi_0) \gg \dot{\phi}_0^2(t)$, then we get an equation of state $P_\phi \approx -\rho_\phi \approx -V(\phi_0)$, with $w \approx -1$, which yields a quasi de Sitter phase. Physically, this condition means that the inflaton is slowly rolling down its potential, which is the reason why the evolution in this period is said to follow a *slow-roll regime*. A slow-roll regime can be achieved by choosing a sufficiently flat potential, because the kinetic energy contribution is suppressed, while the potential $V(\phi) \simeq const$ comes to dominate the energy density.
We conclude that an inflationary expansion can be driven by the vacuum energy of a scalar field, which dominates the energy content of the universe with a suitably flat potential.

As seen, the conditions necessary to realize the inflationary mechanism merely require the inflaton field to dominate the energy budget of the universe, but they do not prohibit at all the presence of several other fields besides the inflaton. In presence of other particle species, the inflaton unavoidably interacts with them. This fact led to the development of two main dynamical realization of cosmological inflation, which substantially differ in the way the interactions are treated.
In the original picture, known as *cold inflation*, the inflaton interactions with the other species are totally negligible during inflation. The result is an adiabatic accelerated expansion during which our universe super-cools, reaching a final state with a temperature too low to allow a good thermalization of particles (recall $T \propto a^{-1}$). Hence, a primordial plasma must form after inflation, in order to recover the initial conditions of the Hot Big Bang cosmological model. In the simplest model, the interactions between the inflaton and other species cannot longer be neglected in the post-inflationary phase, giving rise to the decay of the inflaton into the lighter relativistic particles species which generate the plasma. Specifically, in this picture, the accomplishment of the slow-roll regime required to drive inflation constraints the inflaton potential $V(\phi)$ to be extremely flat.

When the potential starts to steepen, the slow-roll conditions are no longer met, inflation ends and interactions are no more considered negligible, so they are turned on. From a phenomenological point of view, the interactions imply the decay of the inflaton into relativistic particles, introducing a dissipative effect which converts the energy stored in the inflaton potential into radiation energy density, with a consequent production of entropy. The universe is then heated up, and the primordial plasma is achieved.

This first scenario is by far the most studied one in the literature. There is however an alternative scenario, known as *warm inflation*, in which the interactions of the inflaton with other species are relevant also during inflation, so that a thermal bath of non-negligible temperature T is also present during the inflationary expansion. The thermal bath present at some given time t during inflation is diluted by the expansion, so that the bath needs to receive continuous supply by the dissipation of the energy stored in the inflaton field due to its decay. In particular, the presence of the interactions leads to an effective form of friction on the motion of the inflaton, so that a slow-roll regime can be achieved even for a steep potential. Therefore, radiation production may occur concurrently with the inflationary expansion, and reheating is completed at the end of inflation by the same interactions that are effective also during the inflationary stage.

In the next two chapters we are going to outline the main aspects of these two different realizations of inflation, while also highlighting their differences.

Chapter 2

Cold Inflation

In this chapter we describe the general features of the first inflationary scenario, also known as *isoentropic inflation*. In particular, we discuss how the small seeds of primordial energy density perturbations are generated via the inflationary mechanism, starting by the coupled quantum fluctuations of the inflaton field and the metric tensor.
Before getting into the explicit calculations, let us first illustrate qualitatively how this physical process works. As mentioned, during inflation, the energy content of the universe is dominated by the vacuum energy of the inflaton field, so that the stress energy tensor $T^{\phi}_{\mu\nu}$ constitutes the main source of spacetime curvature present in the r.h.s. of the EFE (1.2). Intrinsic quantum fluctuations $\delta\phi$ of this field lead to perturbations of its stress energy tensor, $\delta T^{\phi}_{\mu\nu}$, which in turn gives rise, through the Einstein's equations, to the generation of ripples in the spacetime metric w.r.t. the homogeneous and isotropic FLRW background; at the same time, these fluctuations $\delta g_{\mu\nu}$ of the metric tensor enter the inflaton equations of motion, backreacting on the evolution of $\delta\phi$. Actually, as we will see, the perturbations $\delta\phi$ and $\delta g_{\mu\nu}$ are related by a *gauge choice* issue affecting the general definition of the cosmological perturbations, which can be traced back to the freedom of choice of the spacetime coordinate frame one uses to describe the system. It follows that it is sufficient to compute the evolution of a unique gauge invariant degree of freedom, called *curvature perturbation*, which accounts for the only independent dynamical degree of freedom from which the inflaton and the geometry can be obtained in any specific gauge.
The prolonged accelerated inflationary expansion stretches the Fourier modes of the perturbations from microphysical to cosmological scales by far larger than the causal Hubble horizon, which instead remains almost constant. As a consequence, quantum fluctuations are excited from the ground state, and on super-horizon scales they can be treated as classical perturbations. Once a mode is taken to such a large scale, it is unable to evolve because of causality reasons, and its amplitude becomes nearly frozen-in. After the end of accelerated expansion, the Hubble radius starts to grow faster than the physical length scales, and as soon as the different wavelengths of the fluctuations re-enter the horizon at matter or radiation dominated epoch, these perturbation start undergoing the gravitational collapse that results in the formation of the large scale structures in our universe (galaxies and clusters of galaxies). It is possible to select the time for the initial conditions of the primordial perturbations at $t \sim 1s$ after the Big Bang singularity, corresponding to temperatures around $T \sim 1MeV$, when the run-up to BBN begins. The reason for this choice, as observed in [16], comes from the fact that at this time all the length scales of cosmological interest, i.e. the ones which can potentially undergo the gravitational instability mechanism, are still well outside the Hubble horizon, so that the gravitational collapse cannot have been effective yet at those scales, and those modes are still frozen in the state set by inflation.

We proceed in steps. After deriving the equations of motion of the inflaton field, we first focus on the evolution of its classical background configuration on the FLRW spacetime, by specifying the conditions necessary to realize the slow-roll regime and their implications on the dynamical equations and on the potential $V(\phi)$. Then, we then move to the study of the evolution of the cosmological perturbations by means of linear perturbation theory. This perturbative approach is justified by the CMB experimental measurements, which yield temperature/density anisotropies of the order $\delta T/T \sim 10^{-5}$. It should be clear by now that, since inflaton field and spacetime metric fluctuations cannot be disentangled, an exhaustive treatment of the cosmological perturbations require the study of a system of coupled differential equations, given by the perturbed Einstein's and inflaton's equations of motion. For the sake of simplicity, we start by considering the evolution of just the quantum fluctuations of a test[1] scalar field in an unperturbed FLRW background metric, treating in particular the case of a massive real scalar field in a quasi-de Sitter stage. We then study the actual case of the inflaton perturbations coupled to the spacetime metric. Our final goal is the computation of the *power spectrum* of these perturbations, that, as we shall see, accounts for all the observed properties of these fluctuations as probed by the CMB anisotropies.
The arguments and the results reported in this chapter are mainly based on ref. [16, 30, 31].

2.1 Inflaton field evolution equation in an expanding universe

Let us start by deriving the equations of motion of a scalar field $\phi(t, \vec{x})$ on a FLRW background. We do so from a least action principle, i.e. by imposing

$$\frac{\delta S_{TOT}}{\delta \phi} = 0 \quad . \tag{2.1}$$

In the context of cold inflation, the contribution $S_{\phi,\Psi,A,\Phi}$ to S_{TOT}, containing the interactions of the inflaton with the radiation fluid, is neglected, and we have

$$\frac{\delta S_{TOT}}{\delta \phi} = 0 \quad \Longrightarrow \quad \frac{\delta S_{\phi}}{\delta \phi} = 0 \quad \Longleftrightarrow \quad \frac{\delta \mathcal{L}_{\phi}}{\delta \phi} - \left(\frac{\delta \mathcal{L}_{\phi}}{\delta \phi_{;\mu}} \right)_{;\mu} = 0 \quad , \tag{2.2}$$

from which, using the density lagrangian (1.73)

$$\frac{\partial V(\phi)}{\partial \phi} - (-\phi^{;\mu})_{;\mu} = 0 \quad , \tag{2.3}$$

where $;\mu$ is a short notation for the covariant derivative ∇_{μ}. The term $\phi^{;\mu}{}_{;\mu}$ is the D'Alambertian of ϕ in a curved spacetime, i.e. the Laplace-Beltrami operator for a pseudo-Riemannian manifold with signature $(3, 1)$, given by

$$\phi^{;\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}} \left(g^{\mu\nu} \sqrt{-g} \phi_{;\nu} \right)_{;\mu} \quad . \tag{2.4}$$

We now put ourselves in a fixed spatially flat FLRW spacetime, whose metric tensor is

$$g_{\mu\nu} = \mathrm{diag} \left(-1, \, a^2, \, a^2, \, a^2 \right) \tag{2.5}$$

[1]With this term we mean a field that contributes negligibly to the smacetime expansion.

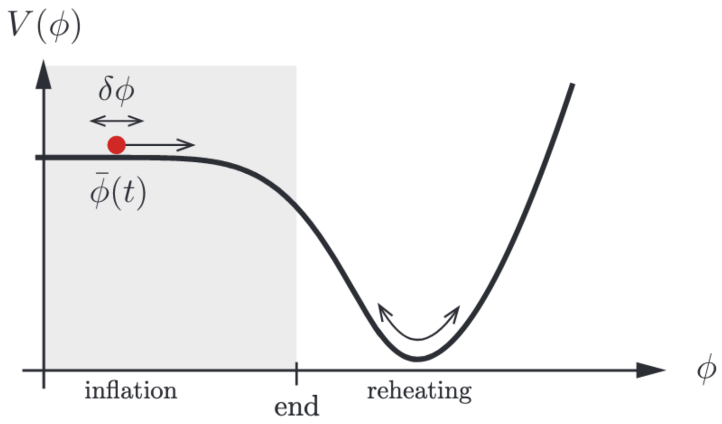

Figure 2.1

Hence, we have $\sqrt{-g} = a^3$, so from (2.4) follows the Klein-Gordon equation of motion for a scalar field in a flat FLRW spacetime:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} = -\frac{\partial V(\phi)}{\partial\phi} \quad . \tag{2.6}$$

Notice that, in the case the field does not evolve in an expanding background, i.e. $a(t) = constant$ and $H = 0$, we recover the Klein-Gordon equation of a scalar field in Minkowski spacetime.

2.2 Background dynamics

Let us focus on the dynamics of the background classical configuration $\phi_0(t) \equiv \phi(t)$. In this case, the equations determining the evolution of the inflaton and of the FLRW expanding universe are:

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \\ H^2 = \frac{\rho_\phi}{3M_{Pl}^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \quad , \end{cases} \tag{2.7}$$

where $' \equiv dV/d\phi$, and we used, in the Friedmann equation, the assumption that ϕ dominates the energy density.

Notice that, in the equation of motion, the potential gradient acts as a force $F = -V'(\phi)$, while the expansion of the universe is responsible for a friction term $F_f = -3H\dot{\phi}$ which opposes the evolution of phi down the potential.

The primordial density perturbations are nearly scale invariant. Namely, modes that leave the horizon at different times during inflation are produced with nearly the same amplitude. This requires a nearly time translational invariance of the system, that can be realized if the inflaton moves very slowly (so that modes that leave the horizon at different times probe nearly the same conditions on the inflaton). As we discuss below, this require the smallness of the so called slow-roll parameters (to be introduced shortly), which in turns requires

$$V(\phi) \gg \dot{\phi}^2 \quad , \quad |\ddot{\phi}| \ll 3H|\dot{\phi}| \quad . \tag{2.8}$$

With these approximations, the system of equations (2.7) simplifies into

$$\begin{cases} 3H\dot{\phi} \simeq -V'(\phi) \\ H^2 \simeq \frac{8\pi G}{3}V(\phi) \simeq const. \quad . \end{cases} \tag{2.9}$$

To check wether he conditions (2.8) are met, it is convenient to introduce the *slow-roll parameters*

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \qquad \eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \quad , \tag{2.10}$$

which are nothing but the relative variation of H and the $\dot{\phi}$ within a Hubble time interval. The significance of ϵ is more clear from:

$$\ddot{a} = \dot{(aH)} = aH^2(1-\epsilon) \quad , \tag{2.11}$$

from which one can see that $\epsilon < 1$ is sufficient to realize inflation. However, we now show that the conditions (2.8) are more restrictive, as they require $\epsilon \ll 1$. To see this, we differentiate the second of equation (2.7) to write

$$2H\dot{H} = \frac{8\pi G}{3}(\dot{\phi}\ddot{\phi} + V'(\phi)\,\dot{\phi}) \quad . \tag{2.12}$$

Using the first of (2.7) we rewrite this as

$$\dot{H} = -4\pi G\dot{\phi}^2 \quad . \tag{2.13}$$

From this, we obtain

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{4\pi G\dot{\phi}^2}{H^2} \simeq \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} \ll 1 \quad , \tag{2.14}$$

where the second of (2.9) has been used in the approximation and where the final condition is the first condition in (2.8). We thus proven that (2.8) require $\epsilon \ll 1$. It is then immediate to note that the second of (2.8) implies $|\eta| \ll 1$.
It is convenient to relate the slow roll quantities (2.10) to combinations of the potential and its derivative. From the first of (2.9) and from (2.14) we have

$$\epsilon \simeq \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} \simeq \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \equiv \epsilon_V \quad . \tag{2.15}$$

Moreover, differentiating the first of (2.9) with respect to time, and dividing the resulting expression by $3H^2\dot{\phi}$ we obtain

$$-\frac{\ddot{\phi}}{H\dot{\phi}} \simeq \frac{\dot{H}}{H^2} + \frac{V''}{3H^2} \quad . \tag{2.16}$$

Using (2.10) and the second of (2.9) we rewrite this as

$$\eta + \epsilon \simeq \frac{1}{8\pi G}\frac{V''}{V} \equiv \eta_V \quad , \tag{2.17}$$

which also needs to satisfy $|\eta_V| \ll 1$, as this is true for the l.h.s. of the expression. From this last equation, and from (2.15), we see that the slow-roll conditions can be cast as the smallness of the quantities

$$\epsilon_V = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \quad , \quad \eta_V = \frac{1}{8\pi G}\frac{V''}{V} \tag{2.18}$$

which are immediately related to the inflaton potential and its first two derivatives. Below, these combinations are related to properties of the observed fluctuation. These relations will enforce the smallness of the slow-roll parameters.

In slow roll inflationary models, the conditions (2.8), or, equivalently, the smallness of (2.18) typically hold throughout inflation. When the steepness of the potential starts to become significant, the slow-roll conditions, expressed by $\epsilon, |\eta| \ll 1$, begin to fail. From (2.11) we see that, by definition, inflation ends when ϵ reaches the value $\epsilon = 1$. From this moment, the inflaton starts to oscillate around the global minimum of its potential, as sketched in figure 2.1. While oscillating, the field decays into lighter relativistic particles, which heat up the universe through the production of entropy. This is the *reheating* phase, at the end of which, the universe becomes radiation dominated, so the initial conditions of the Hot Big Bang cosmological model are recovered.

2.3 Quantum fluctuations of a scalar field on an unperturbed background spacetime

Starting from the e.o.m (2.6), we use the splitting (1.83) in order to perform a linear expansion of the derivative of the potential in the fluctuation $\delta\phi$ around the background solution $\phi_0(t)$. We obtain

$$\ddot{\phi}_0 + \ddot{\delta\phi} + 3H(\dot{\phi}_0 + \dot{\delta\phi}) - \frac{\nabla^2\delta\phi}{a^2} = -\frac{\partial V}{\partial\phi}(\phi_0) - \frac{\partial^2 V}{\partial\phi^2}(\phi_0)\delta\phi + \mathcal{O}\Big(\delta\phi^2\Big) \quad , \tag{2.19}$$

which, considering the e.o.m. of $\phi_0(t)$ in (2.7), provides the linear e.o.m. for the fluctuations $\delta\phi$:

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{\nabla^2\delta\phi}{a^2} = -\frac{\partial^2 V}{\partial\phi^2}(\phi_0)\delta\phi \quad . \tag{2.20}$$

We convert this equation in conformal time τ, and rescale the field according to

$$\delta\chi(\tau, \mathbf{x}) = a(\tau)\delta\phi(\tau, \mathbf{x}) \quad . \tag{2.21}$$

This gives

$$\delta\chi'' - \nabla^2\delta\chi + \left(a^2 V_{,\phi\phi} - \frac{a''}{a}\right)\delta\chi = 0 \quad , \tag{2.22}$$

where prime denotes derivative w.r.t. conformal time $\frac{d}{d\tau} = a\frac{d}{dt}$ and where $,\phi$ denotes the derivative of the inflaton potential w.r.t. the inflaton field. The rescaling eliminated the term proportional to the first derivative out of this equation. The $V_{,\phi\phi}$ term in (2.22) corresponds to the square mass m_ϕ^2 of the perturbation $\delta\phi$. During slow-roll inflation, the second of (2.9) and the condition $|\eta_V| \ll 1$ implies

$$\frac{m_\phi^2}{H^2} = \frac{V_{,\phi\phi}}{H^2} \simeq \frac{3}{8\pi G}\frac{V_{,\phi\phi}}{V} = 3\eta_V \ll 1 \quad , \tag{2.23}$$

which puts a constraint on the inflaton mass.

In presence of a linear equation of motion invariant under spatial translations, as the one we have for the perturbation, it is convenient to perform a Fourier decomposition of the field. One can indeed show that, as a consequence of translation invariance, the different Fourier modes of $\delta\chi$ evolve independently [31]. Working in a spatially flat spacetime enables to expand the field using a complete set of plane waves, otherwise we should use the Helmholtz functions $Q_{\mathbf{k}}$, solutions of the generalized Helmholtz equation $\nabla_k^2 Q_{\mathbf{k}} + |\mathbf{k}|^2 Q_{\mathbf{k}} = 0$, with ∇_k^2 the Laplace-Beltrami operator for 3-dimensional curved Riemannian manifolds [32]. Then, the expansion reads

$$\delta\chi(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\delta\chi_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} \quad , \tag{2.24}$$

with the Fourier modes $\delta\chi_{\mathbf{k}}(\tau)$ satisfying equation (2.22) in momentum space,

$$\delta\chi''_{\mathbf{k}} + \left(k^2 + a^2 m_\phi^2 - \frac{a''}{a}\right)\delta\chi_{\mathbf{k}} = 0 \quad . \tag{2.25}$$

In addition, the reality condition $\delta\chi^* = \delta\chi$ implies $\delta\chi^*_{\mathbf{k}} = \delta\chi_{-\mathbf{k}}$.
For each mode $\mathbf{k}$, equation (2.25) coincides with the simple equation of motion of an harmonic oscillator with a time dependent frequency

$$\omega_k^2(\tau) = (k^2 + a^2 m_\phi^2 - a''/a) \quad , \tag{2.26}$$

due to the expansion of the universe.

2.3.1 Canonical quantization

To quantize the scalar perturbations $\delta\chi$ by using the canonical quantization technique: we promote the classical field $\delta\chi(\tau, \mathbf{x})$ to a quantum operator $\hat{\delta\chi}(\tau, \mathbf{x})$ satisfying the equal time Canonical Commutation Relations (CCR) together with its conjugate field. Equivalently, the Fourier modes $\delta\chi_{\mathbf{k}}$ are promoted to field operators $\hat{\delta\chi}_{\mathbf{k}}$, which, given the linearity of equation (2.25) and the reality condition, can be expressed as

$$\hat{\delta\chi}_{\mathbf{k}}(\tau) = u_k(\tau)\hat{a}_{\mathbf{k}} + u_k^*(\tau)\hat{a}^\dagger_{-\mathbf{k}} \quad . \tag{2.27}$$

Here, the mode function $u_k(\tau)$ and its complex conjugate $u_k^*(\tau)$ are two linearly independent solutions of (2.25), which are only functions of the modulus of the wavenumber $k = |\mathbf{k}|$ (and not also of the direction of the wavevector) as the same is true for $\omega_k(\tau)$. This is due to the isotropy of the FLRW geometry on which the perturbation is quantized. The constant operator $\hat{a}_{\mathbf{k}}$ and its hermitian conjugate $\hat{a}^\dagger_{\mathbf{k}}$ are, respectively, the annihilation and creation operators, corresponding to the quantized coefficients of the general solution, fixed by the initial conditions.
Then, the quantum field operator $\hat{\delta\chi}(\tau, \mathbf{x})$ reads:

$$\hat{\delta\chi}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[u_k(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\tau)\hat{a}^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \quad . \tag{2.28}$$

If the modes functions are normalized according to the Wronskian condition

$$u_k^* u_k' - u_k u_k'^* = -i \quad , \tag{2.29}$$

the ladder operators satisfy the CCR

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = 0 \quad , \quad \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = \delta^3(\mathbf{k} - \mathbf{k}') \quad . \tag{2.30}$$

The free[2] vacuum state of the theory is defined via the prescription

$$\hat{a}_{\mathbf{k}}\,|0\rangle = 0 \quad , \tag{2.31}$$

i.e. it is annihilated by $\hat{a}_{\mathbf{k}}$, for any $\mathbf{k}$, while multi-particle excited states are produced by repeated application of the creation operators,

$$\big|\{n_{\mathbf{k}}\}\big\rangle = \frac{1}{\sqrt{\prod_{\mathbf{k}} n_{\mathbf{k}}}}\prod_{\mathbf{k}}\left(\hat{a}^\dagger_{\mathbf{k}}\right)^{n_{\mathbf{k}}} \quad . \tag{2.32}$$

[2]By perturbing the equations of motion at linear order we are not considering contributions coming from self-interactions and effective interaction of ϕ, given by higher order derivatives of the potential.

Actually, the vacuum state is not uniquely determined. Indeed, we are totally free to write the general solution (2.27) using a different set of mode functions, e.g. $v_k(\tau)$, leading to

$$\hat{\delta\chi}_{\mathbf{k}}(\tau) = v_k(\tau)\hat{b}_{\mathbf{k}} + v_k^*(\tau)\hat{b}^\dagger_{-\mathbf{k}} \quad , \tag{2.33}$$

where $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ constitute a new set annihilation and creation operators, related to the old one by the Bogoliubov transformations[3]. Using the same prescription (2.31) used to define the a-vacuum state $|0\rangle_a$, we can then define a new b-vacuum state, $|0\rangle_b$, which in general contains particles created from the vacuum $|0\rangle_a$, and vice versa, i.e.

$${}_b\langle 0|\, \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\, |0\rangle_b \neq {}_a\langle 0|\, \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\, |0\rangle_a \neq 0 \quad . \tag{2.34}$$

Therefore, the non-uniqueness of the mode functions results in an ambiguity in the computation of the correlator $\langle 0|\hat{\delta\chi}_{\mathbf{k}}\hat{\delta\chi}_{\mathbf{k}'}|0\rangle$, which plays a crucial role in an inflationary model, as we will see.
In a static Minkowski spacetime, the normalization condition (2.29) in combination with the requirement that $|0\rangle$ must be the minimum energy state result to be sufficient to uniquely determine the mode function and, consequently, the vacuum state. In this case, one obtains [33]

$$u_k(\tau) = \frac{e^{-i\omega_k\tau}}{\sqrt{2\omega_k}} \quad , \tag{2.35}$$

with $\omega_k = (k^2 + m_\phi^2)^{1/2}$, since $a = 1$.
Instead, in the case of an expanding FLRW universe with a time dependent frequency, there is no time translation invariance and therefore no notion of constant energy, so the above conditions are not able to determine the vacuum. However, we have $\omega_k \longrightarrow k$ in the asymptotic past / deep UV regime, corresponding to $\tau \to -\infty$, when all comoving scales k were well inside the Hubble horizon, $k \gg aH$. In this regime the frequency ω_k is very slowly varying in time, i.e. $\omega_k'/\omega^2 \ll 1$. This adiabatic variation allows to choose the so called *adiabatic vacuum* in the asymptotic past / deep UV regime, which approximates the concept of Minkowski vacuum. Then we require

$$u_k(\tau) \xrightarrow{k\gg aH} \frac{e^{-ik\tau}}{\sqrt{2k}} \quad . \tag{2.36}$$

This procedure is called the *Bunch-Davis vacuum choice*. Once the general solution of equation (2.25) has been found, we will completely determine the mode functions by requiring the validity of (2.36).

2.3.2 Exact solution

As anticipated, we now assume a quasi de Sitter stage, so that we can write explicitly the temporal dependence of the frequency $\omega_k(\tau)$ in terms of the non vanishing slow-roll parameters. Exploiting the fact that $\epsilon, \eta \ll 1$ during inflation, we can work to first order in these slow-roll parameters. Within this approximation, ϵ and η can be treated as constants. Indeed, by deriving the definitions (2.10) in time, we have

$$\dot{\epsilon} = -\frac{\ddot{H}}{H^2} + 2\frac{\dot{H}^2}{H^3} = \epsilon\left(\frac{\ddot{H}}{\dot{H}} - 2\frac{\dot{H}}{H}\right) \tag{2.37}$$

$$= \epsilon\left(2\frac{\ddot{\phi}}{\dot{\phi}} + 2\epsilon H\right) = \epsilon(-2H\eta + 2\epsilon H) \sim \mathcal{O}\left(\epsilon^2, \eta^2\right) \quad , \tag{2.38}$$

[3]A Bogoliubov transformation is an isomorphism between two sets of annihilation and creation operators, providing two different representations of the algebra defined by the CCR.

where in the third equality we used equation (2.13) and its time derivative to write the ratio $\ddot{H}/\dot{H}$ in terms of time derivatives of the field ϕ.
In the the case of η, we start by rewriting the parameter in the following way

$$\eta = \epsilon - \frac{\dot{\epsilon}}{2\epsilon H} \quad , \tag{2.39}$$

where we used (2.38). The time derivative reads

$$\dot{\eta} = \frac{\dot{\epsilon}}{2} + \frac{1}{2H}\left(\frac{\dot{\epsilon}}{\epsilon}\right)^2 - \frac{\ddot{\epsilon}}{2\epsilon H} \sim \mathcal{O}\left(\epsilon^2, \eta^2\right) \quad . \tag{2.40}$$

Then, equations (2.38), (2.40) tell us that we can safely neglect $\dot{\epsilon}, \dot{\eta}$ if we work at first order in slow-roll.
By integrating the definition of the conformal time, one finds that, for small values of ϵ, the scale factor can be expressed as

$$a(\tau) \simeq -\frac{1}{\tau H(1-\epsilon)} \quad . \tag{2.41}$$

Starting by $H = \dot{a}/a = a'/a^2$, and using $\epsilon = -\dot{H}/H^2 = -H'/aH^2$, if we repeat the same steps used in (2.11) to obtain $\ddot{a}$ in terms of ϵ, we get an analogous relation for a'', which is

$$a'' = a^3 H^2 (2-\epsilon) \quad \Longrightarrow \quad \frac{a''}{a} = a^2 H^2 (2-\epsilon) \quad . \tag{2.42}$$

From the expression (2.41) for the scale factor we get

$$\frac{a''}{a} = a^2 H^2 (2-\epsilon) \simeq \frac{1}{\tau^2(1-\epsilon)^2}(2-\epsilon) \simeq \frac{1+2\epsilon}{\tau^2}(2-\epsilon) \simeq \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon\right) \quad , \tag{2.43}$$

and also

$$m_\phi^2 a^2 \simeq \frac{m_\phi^2}{H^2\tau^2(1-\epsilon)^2} \simeq \frac{m_\phi^2}{H^2\tau^2}(1+2\epsilon) \simeq \frac{3\eta_V}{\tau^2}(1+2\epsilon) \simeq \frac{3\eta_V}{\tau^2} \quad , \tag{2.44}$$

where we used (2.23).
Using the expressions (2.43) and (2.44), the equation of motion (2.25) for the mode functions can be rewritten as

$$u_k''(\tau) + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right) u_k(\tau) = 0 \quad , \tag{2.45}$$

where $\nu^2 = \frac{9}{4} + 3\epsilon - 3\eta_V$, which implies $\frac{3}{2} - \nu \simeq \eta_V - \epsilon$, at first order.
If we now make a change of variable, from τ to $z \equiv -k\tau > 0$, and trade the function $u_k(\tau)$ with $f_k(z) \equiv \frac{u_k(\tau)}{\sqrt{-\tau}} = \sqrt{\frac{k}{z}} u_k(z)$, the differential equation (2.45) can put in the form

$$\frac{\mathrm{d}^2 f_k}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}f_k}{\mathrm{d}z} + \left(1 - \frac{\nu^2}{z^2}\right) f_k = 0 \quad . \tag{2.46}$$

This is a homogeneous Bessel equation of order ν, and its general solution is given by

$$f_k(z) = c_1(k) H_\nu^{(1)}(z) + c_2(k) H_\nu^{(2)}(z) \quad , \tag{2.47}$$

which leads to

$$u_k(\tau) = \sqrt{-\tau}\left[c_1(k) H_\nu^{(1)}(-k\tau) + c_2(k) H_\nu^{(2)}(-k\tau)\right] \quad , \tag{2.48}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are two linear independent solutions of (2.47), known as Hankel functions. Let us study the behaviour of the solution in the two limiting case of $z = -k\tau \gg 1$, corresponding to sub-horizon scales $\lambda_{phys} \ll H^{-1} \longleftrightarrow k \gg aH$, and $z \ll 1$, corresponding to super-horizon scales $\lambda_{phys} \gg H^{-1} \longleftrightarrow k \ll aH$. Observe that $c_1(k)$ and $c_2(k)$ are free coefficients of the linear combination, which signal the non-uniqueness of the mode functions. They are fixed by imposing the asymptotic behaviour (2.36) on sub-horizon scales.

- **Sub-horizon regime ($k \gg aH$)**
  For large values of the argument z, the special functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ assume the asymptotic form [34]

$$H_\nu^{(1)}(z) \overset{z\gg 1}{\sim} \sqrt{\frac{2}{\pi z}} e^{i(z-\nu\frac{\pi}{2}-\frac{\pi}{4})} \quad , \quad H_\nu^{(2)}(z) \overset{z\gg 1}{\sim} \sqrt{\frac{2}{\pi z}} e^{i(-z-\nu\frac{\pi}{2}-\frac{\pi}{4})} \quad . \tag{2.49}$$

  Since we require the mode function u_k to approach the Minkowskian mode on small scales, $u_k(\tau) \overset{k\gg aH}{\sim} \frac{e^{-ik\tau}}{\sqrt{2k}}$, we must fix $c_1(k) = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}(\nu+\frac{1}{2})}$ and $c_2(k) = 0$, which provide the exact solution

$$u_k(\tau) = \sqrt{-\tau}\frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}(\nu+\frac{1}{2})} H_\nu^{(1)}(-k\tau) \quad . \tag{2.50}$$

- **Super-horizon regime ($k \ll aH$)**
  For small values of the argument z, the special function $H_\nu^{(1)}(z)$ has the asymptotic expansion [34]

$$H_\nu^{(1)}(z) \overset{z\ll 1}{\sim} \frac{e^{-i\frac{\pi}{2}}}{\pi} 2^\nu \Gamma(\nu) z^{-\nu} \quad , \tag{2.51}$$

  where Γ is the Euler Gamma function. Plugging this expression in (2.50), after some manipulations, gives

$$u_k(\tau) \overset{k\ll aH}{\sim} \frac{1}{\sqrt{2k}} e^{i\frac{\pi}{2}(\nu-\frac{1}{2})} 2^{\nu-\frac{3}{2}} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right) (-k\tau)^{\frac{1}{2}-\nu} \quad , \tag{2.52}$$

  with $\Gamma(3/2) = \sqrt{\pi}/2$.
  Hence, using the expression for the scale factor at zeroth order in slow roll, $a \approx -\frac{1}{\tau H}$, the behaviour of the mode functions $\delta\phi_k$ well outside the horizon results:

$$\delta\phi_k(\tau) \overset{k\ll aH}{\sim} e^{i\frac{\pi}{2}(\nu-\frac{1}{2})} 2^{\nu-\frac{3}{2}} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right) \frac{H}{\sqrt{2k^3}} (-k\tau)^{\frac{3}{2}-\nu} \quad . \tag{2.53}$$

  It's important to notice the very weak temporal dependence of the solution, since $\frac{3}{2} - \nu \simeq \eta_V - \epsilon \ll 1$. So, on super-horizon scales, the Fourier modes can be considered, with a very good approximation, as constant functions with amplitude

$$|\delta\phi_k| = 2^{\nu-\frac{3}{2}} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right) \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \tag{2.54}$$

$$\simeq \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \quad . \tag{2.55}$$

We conclude that, as long as the wavelength of the perturbations of the field ϕ is inside the Hubble horizon, they remain on the vacuum state. Instead, when the wavelength gets stretched to super-horizon scales by the accelerated expansion, the modes are frozen in. It can be argued, see for instance [35], that the large amplification leads to a *quantum to classical transition* of the modes, which mostly takes place at horizon crossing. After horizon crossing, the perturbations can be treated as classically evolving stochastic variables, whose statistics is encoded in their correlation functions. The details of this classicalization are not fully understood yet [35] and are beyond the scope of this thesis.

2.3.3 Power spectrum of a stochastic field

All the cosmological perturbation fields, as well as the perturbations in the energy density and in the inflaton field, are treated as stochastic/random fields. A stochastic field is a function $\delta(t, \mathbf{x})$ which takes on each point a random configuration according to a probability distribution functional $Pr[\delta]$. We will just consider perturbation fields with zero mean, $\big\langle\delta(t,\mathbf{x})\big\rangle = 0$, where the brackets denote the ensemble average, i.e.

$$\big\langle\delta(t,\mathbf{x})\big\rangle = \int \mathcal{D}\delta Pr[\delta]\delta(t,\mathbf{x}) = 0 \quad . \tag{2.56}$$

In general, one defines the *n-point correlation function* as the expectation value of the product of n fields δ evaluated at different spatial points. Thus, for a given field, we have an infinite set of correlators.
In particular, the 2-point correlation function is

$$\xi(\mathbf{x},\mathbf{y}) \equiv \big\langle\delta(t,\mathbf{x})\delta(t,\mathbf{y})\big\rangle = \int \mathcal{D}\delta Pr[\delta]\delta(t,\mathbf{x})\delta(t,\mathbf{y}) \quad . \tag{2.57}$$

On the basis of the validity of the cosmological principle, we require statistical homogeneity and isotropy, meaning that the statistical properties of the translated field and the rotated field are the same of the original one, i.e.

$$Pr\big[\delta(t,\mathbf{x})\big] = Pr\big[\delta(t,\mathbf{x}-\mathbf{a})\big] = Pr[\delta(t,\hat{R}\mathbf{x})] \quad , \tag{2.58}$$

for any constant vector $\mathbf{a}$ and rotation $\hat{R}$. For the two point correlation function, these conditions separately imply the relations

$$\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{a},\mathbf{y}-\mathbf{a}) \quad \forall\mathbf{a} \tag{2.59}$$

$$\xi(\mathbf{x},\mathbf{y}) = \xi(\hat{R}^{-1}\mathbf{x},\hat{R}^{-1}\mathbf{y}) \quad \forall\hat{R} \quad , \tag{2.60}$$

which, combined, provide

$$\xi(\mathbf{x},\mathbf{y}) = \xi(\hat{R}^{-1}(\mathbf{x}-\mathbf{y})) \quad \forall\hat{R} \qquad \Longrightarrow \qquad \xi(\mathbf{x},\mathbf{y}) = \xi(|\mathbf{x}-\mathbf{y}|) \quad . \tag{2.61}$$

So, the two point correlator depends only on the distance between the two points.
Considering the Fourier transform of the stochastic field δ

$$\delta(t,\mathbf{k}) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}}\delta(t,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \quad , \tag{2.62}$$

one can construct the n-point correlators in Fourier space. Demanding the invariance of $\xi(\mathbf{x},\mathbf{y})$ under translations and rotations puts a constraint on the form of the two-point correlation function in Fourier space, which reads

$$\big\langle\delta(t,\mathbf{k})\delta(t,\mathbf{k}')\big\rangle = P(k)\delta^3(\mathbf{k}+\mathbf{k}') \quad , \tag{2.63}$$

where $P(k)$ is the *power spectrum.* The presence of the delta function encodes the requirement of homogeneity, and it means that different modes are statistically independent. The dependence of P on just the modulus $k \equiv |\mathbf{k}|$, instead, comes from the isotropy. These features are valid also for higher correlators.
Inverting (2.62), one can compute the variance of the field δ as

$$\langle \delta^2(t,\mathbf{x})\rangle = \xi(0) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} P(k) = \int_0^\infty \frac{dk}{2\pi^2} k^2 P(k) = \int_0^\infty \frac{dk}{k}\frac{k^3}{2\pi^2} P(k) = \int d(lnk)\Delta(k) \quad , \tag{2.64}$$

where we defined the *adimensional power spectrum* as $\Delta(k) = \frac{k^3}{2\pi^2}P(k)$. Thus, the power spectrum is a measure of the amplitude of the perturbations at a given scale k.
The slope of the adimensional power spectrum is described by the *spectral index* $n(k)$, given by

$$n(k) - 1 = \frac{d\ \ln \Delta(k)}{d\ \ln k} \quad . \tag{2.65}$$

For a constant value of $n(k) \equiv n$, the adimensional power spectrum has a simple power law dependence from k, which can be written w.r.t. a reference "pivot" scale k_0

$$\Delta(k) = \Delta(k_0)\left(\frac{k}{k_0}\right)^{n-1} \quad . \tag{2.66}$$

A particular value is represented by $n(k) \equiv 1$. In this case $\Delta(k)$ is said to be *scale invariant*, and we have a so called *Harrison-Zel'dovich power spectrum.* The scale invariance refers to the fact that, for such a power spectrum, the two point correlation function is invariant under a rescaling of the spatial coordinates, namely if $\mathbf{x} \longrightarrow \lambda\mathbf{x}$, where $\lambda > 0$ is some constant, one finds that

$$\big\langle \delta(t,\lambda\mathbf{x})\delta(t,\lambda\mathbf{y})\big\rangle = \big\langle \delta(t,\mathbf{x})\delta(t,\mathbf{y})\big\rangle \quad . \tag{2.67}$$

Indeed, writing (2.57) in terms of the two point correlator in Fourier space (2.63), and using $\delta^3(\lambda(\mathbf{k}+\mathbf{k}')) = \lambda^{-3}\delta^3(\mathbf{k}+\mathbf{k}')$, one can simply show that (2.67) is verified if $P(k) \propto \frac{1}{k^3}$, or equivalently $\Delta(k) = constant$.

The simplest type of random field is a *Gaussian random field*, i.e. characterized by a Gaussian probability distribution functional. CMB data indicate that the primordial perturbations are consistent with Gaussianity within the experimental bounds [3]. For a Gaussian distribution with zero mean all the statistical informations are contained in the two-point correlation function computed above, or, equivalently, in their power spectrum. Specifically, odd-n point correlators vanish, while even-n point correlators can be written as products of two point correlators, for instance

$$\big\langle \delta(t,\mathbf{x}_1)\delta(t,\mathbf{x}_2)\delta(t,\mathbf{x}_3)\delta(t,\mathbf{x}_4)\big\rangle = \big\langle \delta(t,\mathbf{x}_1)\delta(t,\mathbf{x}_2)\big\rangle \big\langle \delta(t,\mathbf{x}_3)\delta(t,\mathbf{x}_4)\big\rangle + \text{two permutations.} \tag{2.68}$$

The observed Gaussianity is explained within the inflationary paradigm described above. As we have showed, within the linear approach, the solutions of the coupled equations of motion for the cosmological perturbations are determined by the inflationary initial conditions, consisting of the quantum fluctuations of the inflaton field in its vacuum state evaluated at the time of horizon exit. We have shown that the Fourier modes $\delta\chi_{\mathbf{k}}$ of the (rescaled) quantum fluctuations have the dynamics of a quantum harmonic oscillator, and, as we learn from the study of this kind of system, the probability distribution of each mode $\delta\chi_{\mathbf{k}}$ in the vacuum state is Gaussian, with a variance given by the modulus square of the mode function [33, 36],

$$\big|\langle \delta\chi_{\mathbf{k}}|0\rangle\big|^2 \propto \exp[-|\delta\chi_{\mathbf{k}}|^2/|u_k|^2] \quad . \tag{2.69}$$

More generally, using the CCR (2.30) for the ladder operators, the two point correlation function for the inflaton field fluctuations $\delta\phi = \delta\chi/a$ reads

$$\langle 0|\delta\hat{\phi}_{\mathbf{k}}\delta\hat{\phi}_{\mathbf{k}'}|0\rangle = \frac{|u_k|^2}{a^2}\delta^3(\mathbf{k}'-\mathbf{k}) = P(k)\delta^3(\mathbf{k}'-\mathbf{k}) \quad , \tag{2.70}$$

with

$$P(k) = |\delta\phi_k|^2 = \frac{|u_k|^2}{a^2} \quad . \tag{2.71}$$

Since the linear evolution does not mix different Fourier modes, these statistical properties are inherited by the primordial cosmological perturbations at horizon re-entry, and in particular by the CMB anisotropies, which are then predicted to be Gaussian distributed at linear order.
Actually, during inflation, the initial Gaussian perturbations stay almost, but not quite, Gaussian. Indeed, the very nature of the gravitational dynamics and the unavoidable presence of interactions of the scalar field ϕ with itself and other fields, are sources of non-linearities in the inflationary dynamics, which produce deviations from a pure Gaussian statistics for the perturbations, namely non-vanishing odd-n correlators. However, the flatness condition on the scalar potential constrains the interaction terms to be very small, hence the non-linearities are suppressed, as well as the non-Gaussian features. The detection of a certain amount of non-Gaussianity [3], or the determination of bounds on it, is a powerful probe of the theory of inflation, because it could put precise constraints on the inflaton potential, helping to discriminate between the many available inflationary models.

2.4 Quantum fluctuations of a scalar field on a perturbed spacetime

Before facing the problem of the coupled evolution of the inflaton field fluctuations and the spacetime metric ones, let's go first through a general definition of the cosmological perturbations and the associated gauge issue, hence the distinction between metric and matter perturbations and the introduction of the fundamental curvature perturbation, together with a description of its main properties. In particular, we will see how, under a specific assumption, this kind of perturbation has the attractive feature of remaining constant throughout the period between the crossing of the Hubble horizon by a given Fourier mode and the horizon re-entry, which includes the end of inflation, the reheating phase and the transition to the radiation dominated epoch. It is thanks to this property that we are able to predict cosmological observables as the power spectrum of curvature perturbations at horizon re-entry, because the physics involved during this stage of the evolution of the universe is not very well understood. The way the power spectrum computed at horizon re-entry, during the RD era, is related to the observed power spectrum of CMB temperature fluctuations, in the MD era, is beyond the aim of this thesis.

2.4.1 The gauge issue of cosmological perturbations

A cosmological perturbation in a given quantity represented by a generic tensor field T, is nothing but the difference between the value $T(t, \mathbf{x})$ assumed on the physical perturbed spacetime, and the value $T_0(t)$ assumed on a background unperturbed spacetime, represented by the FLRW one:

$$\delta T(t, \mathbf{x}) = T(t, \mathbf{x}) - T_0(t) \quad . \tag{2.72}$$

The background value T_0 depends only on the cosmic time because homogeneity and isotropy, and the perturbations δT are taken to be very small compared to T_0, $|\delta T| \ll |T_0|$.

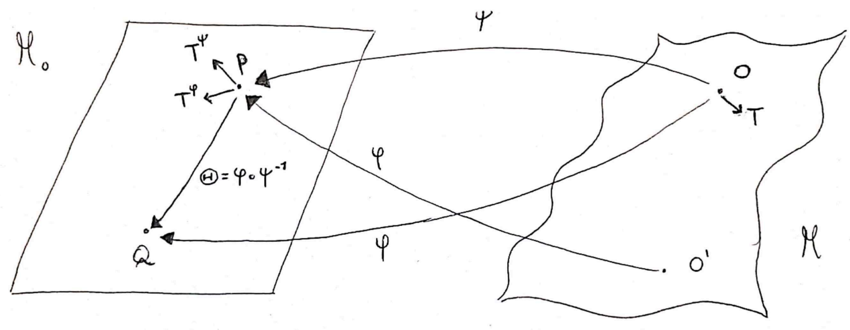

Figure 2.2: The gauge transformation can be seen as a change of the correspondence map between the perturbed and the background spacetime, or as an automorphism on the background spacetime.

In the context of GR, the difference (2.72) is meaningless, because we are comparing tensors evaluated on points of two different pseudo-Riemannian manifolds, i.e. elements belonging to two different vector spaces. Differential geometry teaches us that, in order to make the comparison meaningful, it's necessary to introduce a map, or more precisely a diffeomorphism, which establishes a one-to-one correspondence between the points of the manifolds $\mathcal{M}$ and $\mathcal{M}_0$, associated, respectively, to the physical spacetime and the background spacetime. Basically, the map enables to define a transportation law for tensors from a given point of $\mathcal{M}$ to another point of $\mathcal{M}_0$, or vice versa, which then allows to write (2.72) as a difference between tensors evaluated on the same point. Here, indeed, the generic coordinates $(t, \mathbf{x})$ are intended as those of the unperturbed spacetime.
A gauge choice, essentially, coincides with the choice of a specific correspondence map, while a change of this map is a gauge transformation. It's easy to understand how the freedom in choosing the map gives rise to an ambiguity in the definition of the perturbations: referring to figure 2.2, let's initially consider a map $\psi : \mathcal{M} \longrightarrow \mathcal{M}_0$, which identifies a generic point $O \in \mathcal{M}$ with a point $P = \psi(O) \in \mathcal{M}_0$; as said, the map ψ also endows the tensor field T, evaluated on O, with a representation on P, given by a tensor T^ψ, to be compared with the tensor T_0 evaluated on P. However, instead of ψ, we could use a new diffeomorphism φ, which will identify the same point P with a different point $O' \in \mathcal{M}$, such that $P = \varphi(O')$, and will provide another representation of T on P, namely T^φ. Then, we end up with two possible expressions for the perturbation δT,

$$\begin{aligned} \delta T^\psi &= T^\psi - T_0 \\ \delta T^\varphi &= T^\varphi - T_0 \quad , \end{aligned}$$

which, in general, are different, $\delta T^\psi \neq \delta T^\varphi$.

We would like to understand how a tensor changes under a gauge transformation, i.e. we want to establish a relation between different representations of a tensor on $\mathcal{M}_0$. In order to do this, it's necessary to vary our point of view. Instead of a change of the point on $\mathcal{M}$ with which P is identified, a gauge transformation can as well be seen as a one-to-one correspondence between different points in the background spacetime $\mathcal{M}_0$. Indeed, let x^μ be a coordinate system on $\mathcal{M}_0$, and $x^\mu(P)$ the coordinates of P. If $Q \in \mathcal{M}_0$ is the point, with coordinates $x^\mu(Q)$, such that $Q = \varphi(O)$, then the gauges ψ and φ uniquely determine a diffeomorphism $\Theta : \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ such that $\Theta(P) = Q$, given by $\Theta \equiv \varphi \circ \psi^{-1}$. Now the gauge transformation is represented by the map Θ. This is what is usually called an *active approach*, since the transformation Θ moves each point to another in $\mathcal{M}_0$ within a given coordinate system x^μ. If ξ^μ is the vector field on $\mathcal{M}_0$ whose local flux, defined by its integral curves, coincides with the map Θ, then we can write the transformation from P to Q as

$$\frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda} = \xi^\mu \quad \Longrightarrow \quad x^\mu(Q) = x^\mu(P) + \lambda \xi^\mu(x(P)) + \mathcal{O}\Big(\lambda^2, \xi^2\Big) \quad , \tag{2.73}$$

where λ is an arbitrary parameter for the integral curves. Notice we are considering an infinitesimal transformation with $|\xi^\mu| \ll 1$, so we can neglect higher order terms.
It's also possible to adopt a *passive approach* to gauge transformations, in which the point remains fixed: if y^μ is another coordinate system on $\mathcal{M}_0$ such that $y^\mu(Q) = x^\mu(P)$, we have at first order

$$y^\mu(Q) = x^\mu(P) \simeq x^\mu(Q) - \lambda\xi^\mu(x(P)) \tag{2.74}$$
$$\simeq x^\mu(Q) - \lambda\xi^\mu(x(Q)) \quad . \tag{2.75}$$

The above expression is nothing but a *coordinate transformation* of the point Q, and since the point Q is arbitrary, this argument extends to all points of $\mathcal{M}_0$. The passive approach results to be more convenient than the active one, because we know how tensors transform under a coordinate transformation. However, we have to point out that a gauge transformation is *not* exactly a coordinate transformation. A proof of this is the fact that, as we will see, scalar quantities do not transform trivially.
Using the passive approach, it comes out that, given a gauge transformation $\psi \longrightarrow \varphi$ defined by the vector field ξ^μ, the representation T^ψ of a generic tensor field T transforms, at linear level, according to

$$T^\varphi = T^\psi + \mathcal{L}_\xi T_0 \quad , \tag{2.76}$$

where $\mathcal{L}_\xi$ is the *Lie derivative* along the direction of the vector field ξ. This implies the following relation between the perturbations in the two different gauges:

$$\delta T^\varphi = \delta T^\psi + \mathcal{L}_\xi T_0 \quad . \tag{2.77}$$

We refer to [37] for a complete demonstration of (2.77).
The Lie derivative for a scalar field ϕ, for a covariant vector field V_μ and for a tensor field of rank (0,2) $T_{\mu\nu}$ are given by:

$$\mathcal{L}_\xi \phi = \xi^\mu \phi_{,\mu} \tag{2.78}$$
$$\mathcal{L}_\xi V_\mu = \xi^\rho V_{\mu,\rho} + V_\rho \xi^\rho_{,\mu} \tag{2.79}$$
$$\mathcal{L}_\xi T_{\mu\nu} = \xi^\rho T_{\mu\nu,\rho} + T_{\rho\nu}\xi^\rho_{,\mu} + T_{\mu\rho}\xi^\rho_{,\nu} \quad . \tag{2.80}$$

The important concept to keep in mind is that the gauge choice leads to the choice of a coordinate system on $\mathcal{M}_0$, which means to perform a threading of spacetime into time-like curves with fixed spatial coordinates x^i (corresponding to the worldlines of possible observers), and a slicing into space-like hypersurfaces with constant time coordinate x^0. Obviously, there is no preferred coordinate system/gauge from a physical point of view: the final result of a calculation cannot depend on this arbitrary choice.

2.4.2 Metric perturbations

The perturbation of the geometrical part of the Einstein's equations is entirely due to the perturbation of the spacetime metric. Analogously to the field decomposition (1.83), also the metric tensor can be written as

$$g_{\mu\nu}(\tau, \mathbf{x}) = g^{(0)}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\tau, \mathbf{x}) \quad , \tag{2.81}$$

with $g^{(0)}_{\mu\nu}$ the spatially flat FLRW metric tensor in comoving coordinates $(\tau, \mathbf{x})$, given by

$$ds^2 = a^2(\tau)(-d\tau^2 + \delta_{ij}dx^i dx^j) \quad . \tag{2.82}$$

The perturbation at linear order w.r.t. the homogeneous background solution can be put in the form [30]:

$$g_{00} = -a^2(\tau)(1 + 2\Phi(\tau, \mathbf{x})) \tag{2.83}$$
$$g_{0i} = a^2(\tau)\omega_i(\tau, \mathbf{x}) \tag{2.84}$$
$$g_{ij} = a^2(\tau)\big((1 - 2\Psi(\tau, \mathbf{x}))\delta_{ij} + \chi_{ij}(\tau, \mathbf{x})\big) \quad , \tag{2.85}$$

where Φ and Ψ are scalar functions, ω_i a vector, and χ_{ij} a traceless tensor, $\chi^i_i = 0$.
Following [31], one can decompose both matter and metric perturbations into scalar, vector and tensor components (SVT), according to how they transform under spatial rotations. The SVT decomposition is very useful because it can be shown that, at linear order, the different types of perturbations evolve independently, which means that can be studied separately assuming that the other types are absent, whether or not this is true in nature. Then, ω_i and χ_{ij} can be decomposed as:

$$\omega_i = \partial_i \omega_\parallel + \omega_i^\perp \tag{2.86}$$
$$\chi_{ij} = D_{ij}\chi_\parallel + 2\chi^\perp_{(i,j)} + \chi^T_{ij} \quad , \tag{2.87}$$

where $D_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2$ is a traceless derivative operator, $\omega_\parallel$ and $\chi_\parallel$ are the scalar contributions, $\omega_i^\perp$ and $\chi_i^\perp$ are the transverse (i.e. solenoidal) vector contributions[4] satisfying $\partial^i\omega_i^\perp = \partial^i\chi_i^\perp = 0$, and finally χ^T_{ij} is the tensor component, which is both traceless and transverse, ${\chi^T}^i_i = \partial^i\chi^T_{ij} = 0$. This last perturbation of the metric is the dynamical variable describing gravitational waves.
Since first order vector perturbations are not excited in presence of a scalar field driving inflation, and, if present, their amplitude would decay with the expansion of the universe [38], we can safely neglect them. Moreover, exploiting the fact that scalar and tensor components evolve independently at first order, we can just keep the scalar perturbations:

$$g_{00} = -a^2(\tau)(1 + 2\Phi(\tau, \mathbf{x})) \tag{2.88}$$
$$g_{0i} = a^2(\tau)\partial_i\omega_\parallel(\tau, \mathbf{x}) \tag{2.89}$$
$$g_{ij} = a^2(\tau)\Big((1 - 2\Psi(\tau, \mathbf{x}))\delta_{ij} + D_{ij}\chi_\parallel(\tau, \mathbf{x})\Big) \quad . \tag{2.90}$$

Let's now consider a gauge transformation. As seen, it implies an infinitesimal coordinate transformation defined by the vector field ξ^μ,

$$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu \quad . \tag{2.91}$$

The components of ξ^μ can also be decomposed into scalar and vector components,

$$\xi^0 = \alpha(\tau, \mathbf{x}) \tag{2.92}$$
$$\xi^i = \partial^i\beta(\tau, \mathbf{x}) + v^i(\tau, \mathbf{x}) \quad , \tag{2.93}$$

with $\partial^i v_i = 0$.
After a gauge transformation, the metric tensor transforms according to (2.77) as

$$\delta g_{\mu\nu} \longrightarrow \tilde{\delta g}_{\mu\nu} = \delta g_{\mu\nu} + \mathcal{L}_\xi g^{(0)}_{\mu\nu} \quad . \tag{2.94}$$

[4]Here, the terms parallel and transverse refer to the fact that, in Fourier space, these components of the perturbation are respectively parallel and perpendicular to the wavevector $\mathbf{k}$

Using the formula (2.80) for the Lie derivative of a tensor, which in the case of the metric tensor reduces to

$$\mathcal{L}_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} \quad , \tag{2.95}$$

we get the following transformations for the scalar perturbations

$$\tilde{\Phi} = \Phi + \alpha' + \mathcal{H}\alpha \tag{2.96}$$

$$\tilde{\Psi} = \Psi - \mathcal{H}\alpha - \frac{1}{3}\nabla^2\beta \tag{2.97}$$

$$\tilde{\chi}_\| = \chi_\| + 2\beta \tag{2.98}$$

$$\tilde{\omega}_\| = \omega_\| - \alpha + \beta' \quad . \tag{2.99}$$

The components v^i of ξ^μ contribute to the transformed vector perturbations $\tilde{\omega}_i^\perp$ and $\tilde{\chi}_i^\perp$, that we are neglecting, while, remarkably, the tensor perturbation χ_{ij}^T is gauge invariant at first order.

2.4.3 Matter perturbations

For matter perturbations we mean perturbations of the quantities appearing in the homogeneous stress energy tensor of a perfect fluid (1.16), namely perturbations of the energy density ρ, of the pressure P and of the four-velocity u_μ of the comoving observer with respect to their background FLRW values. It is also possible to account for imperfections of the cosmic fluid by adding to the definition (1.16) of the stress-energy tensor for a perfect fluid the so called anisotropic stress $\Sigma_{\mu\nu}$, constrained by the conditions $u_\nu\Sigma_{\mu\nu} = \Sigma^\mu_\mu = 0$ [31]. The anisotropic stress vanishes in the unperturbed FLRW universe, so in this case $\Sigma_{\mu\nu}$ represents a first order perturbation. However, for minimally coupled single-field models of inflation, the anisotropic stress vanishes at first order [39], then we disregard this kind of perturbations.
For our purposes, we are just interested in the scalar perturbation of the energy density and the pressure:

$$\rho(\tau, \mathbf{x}) = \rho_0(\tau) + \delta\rho(\tau, \mathbf{x}) \tag{2.100}$$

$$P(\tau, \mathbf{x}) = P_0(\tau) + \delta P(\tau, \mathbf{x}) \quad , \tag{2.101}$$

which undergo the gauge transformations

$$\delta\rho \quad \longrightarrow \quad \tilde{\delta\rho} = \delta\rho + \mathcal{L}_\xi\rho_0(\tau) = \delta\rho + \rho_0'\alpha \tag{2.102}$$

$$\delta P \quad \longrightarrow \quad \tilde{\delta P} = \delta P + \mathcal{L}_\xi P_0(\tau) = \delta P + P_0'\alpha \quad , \tag{2.103}$$

where we used (2.78).
Thus, although being scalar quantities, $\delta\rho$ and δP do not remain invariant as in the case of a simple coordinate transformation.
In a multi-component system, large scale matter perturbations[5] can be mainly distinguished in *adiabatic, or curvature, matter perturbations* and *isocurvature, or entropic, matter perturbations.* The first type are perturbations in all the cosmological species that affect the total energy density of the system while keeping constant the relative abundance of the different components. The second type, instead, are perturbations in the individual components of the cosmic fluid which leave unperturbed the total energy density. Therefore, while adiabatic perturbations induce a perturbation in the spatial curvature through the Einstein's equation, entropic perturbations does not. This explains why they are respectively called curvature and isocurvature.

[5]With large scale perturbations we mean perturbations which have been smoothed on a cosmological scale much greater than the Hubble horizon.

As we are going to argue, *single-field models of inflation predict that the primordial perturbations are purely adiabatic*, therefore we will not consider at all isocurvature perturbations.
The origin of the term adiabatic can be explained as follows: as shown in [40], for single-field inflation the large scale fluctuations of the inflaton field can be identified with a local shift along the trajectory of the homogeneous background solution $\phi_0(\tau)$, i.e. the inflaton field $\phi(\tau, \mathbf{x})$ at some spacetime point of the perturbed universe is the same as in the background universe but evaluated at a slight different time $\tau + \delta\tau(\mathbf{x})$, depending on the position,

$$\phi(\tau, \mathbf{x}) = \phi_0(\tau + \delta\tau(\mathbf{x})) \simeq \phi_0(\tau) + \phi_0' \delta\tau(\mathbf{x}) \quad \Longrightarrow \quad \delta\phi(\tau, \mathbf{x}) = \phi_0' \delta\tau(\mathbf{x}) \quad . \tag{2.104}$$

In other words, on scales well outside the horizon, each region evolves like a separate FLRW universe. The local time shift causes different regions of the universe to inflate by different amounts. From the first Friedmann equation (1.20) we can understand how these differences in the local expansion of the universe, described by a perturbation of the Hubble parameter $\delta H(\tau, \mathbf{x})$, induce a perturbation in the inflaton vacuum energy density, that is ultimately inherited by any scalar quantity X of the cosmic fluid after inflation. The time displacement causes the same relative change $\delta\tau = \delta X/\dot{X}$ for all quantities, meaning that the perturbation is democratically shared by all the species. However, it does not give rise to perturbations in the ratio between number density of different species, because, before horizon re-entry, there can be no effective particle flow (and hence no heat flow) between different regions on such cosmological scales. Then, in this case, all the perturbations of the cosmological fluid satisfy the *adiabaticity condition*

$$\delta\left(\frac{n_i}{n_j}\right) = 0 \quad \longleftrightarrow \quad S_{ij} \equiv \frac{\delta n_i}{n_i} - \frac{\delta n_j}{n_j} = 0 \quad \text{for all species } i \text{ and } j \quad , \tag{2.105}$$

where we defined the quantity S_{ij}, usually called *entropy perturbation*. If inflation is driven by more than one filed, isocurvature perturbations modes can arise [41], which means $S_{ij} \neq 0$.
In the most general case, a perturbation can be decomposed in its adiabatic and isocurvature contributions. For example, for the pressure perturbation we have

$$\delta P = \delta P_{ad} + \delta P_{iso} = P_0' \, \delta\tau(\mathbf{x}) + \delta P_{iso} = \frac{P_0'}{\rho_0'} \delta\rho + \delta P_{iso} \quad , \tag{2.106}$$

where the ratio P_0'/ρ_0' is the adiabatic sound-speed $c_s^2 = (\partial P/\partial\rho)\big|_S$, computed at constant entropy S. Notice that the isocurvature pressure perturbation

$$\delta P_{iso} = \delta P - \frac{P_0'}{\rho_0'} \delta\rho \quad , \tag{2.107}$$

is gauge invariant.

2.4.4 Common gauge choices and gauge invariant perturbations

There are two kind of approach we can adopt to tackle the gauge issue about the definition of the cosmological perturbations. One simply consists into a gauge fixing procedure, which implicitly defines a specific coordinate system on the perturbed spacetime. In this case, using certain coordinates rather than others can greatly simplify the computations, because we are free to exploit the gauge transformations to move to a particular gauge where some perturbations are vanishing. On the other hand, the removal of a matter perturbation could lead to the appearance of a metric perturbation, or vice versa, and this arbitrary trade can generate some confusion about the distinction between physical and fictitious perturbations. The alternative approach is indeed to work with quantities given by gauge invariant combinations of matter and metric perturbations. By definition, these perturbations are the real physical degrees of freedom, because they cannot be removed by a gauge transformation.

We now list some of the most common gauge choices, and then we will introduce the gauge invariant quantities which play a crucial role in the computation of the primordial energy density perturbations, by also specifying their physical interpretation.

- **Poisson gauge**
  It is defined by performing a coordinate transformation such that $\omega_\| = \chi_\| = 0$ in the new coordinate system.
  It is also known as the longitudinal or Newtonian gauge, because it can be shown, by unfolding the linearly perturbed Einstein's equations, that within this gauge the remaining two scalar perturbations of the metric Φ and Ψ satisfy a Poisson-like equation $\nabla^2 f(\mathbf{x}) = 4\pi G g(\mathbf{x})$, which reminds the behaviour of the Newtonian gravitational potential.

- **Spatially flat gauge**
  It is defined by selecting the constant-time hypersurfaces whose intrinsic spatial curvature, represented by the Ricci scalar R, is left unperturbed. At linear order, for a perturbed spatially flat FLRW spacetime, we have [38]

$$R = \frac{4}{a^2}\nabla^2\hat{\Psi} \quad , \tag{2.108}$$

  with $\hat{\Psi} = \Psi + \frac{1}{6}\nabla^2\chi_\|$. This quantity is called the *curvature perturbation*, since, as we can see from (2.108), in momentum space it is proportional to the perturbation of the spatial curvature w.r.t. the background value (the latter is zero in this case).
  The spatially flat gauge is then defined by performing a coordinate transformation such that $\Psi = \chi_\| = 0$.
- **Uniform expansion rate gauge**
  Let us introduce the expansion rate of the $t = const.$ hypersurfaces

$$\theta \equiv N^\mu_{;\mu} \quad , \tag{2.109}$$

  where N^μ is the unit time-like vector orthogonal to the hypersurfaces. In the unperturbed FLRW spacetime we have the uniform background value $\theta = 3H$. On a perturbed spacetime, at first order in the metric perturbations, the expansion rate is [39, 42]

$$\theta = \frac{3}{a}\left(\mathcal{H} - \mathcal{H}\Phi - \Psi' - \frac{1}{3}\nabla^2\omega_\|\right) \quad . \tag{2.110}$$

  Then the uniform expansion rate gauge is defined by imposing the condition

$$\delta\theta \equiv \mathcal{H}\Phi + \Psi' + \frac{1}{3}\nabla^2\omega_\| = 0 \quad . \tag{2.111}$$

- **Comoving gauge**
  This gauge is fixed by setting the coordinate system of an observer comoving with the cosmic

fluid, meaning that, in this frame, the fluid around the observer is isotropic and there is no flux of energy, i.e. the components T^0_i of the stress-energy tensor are vanishing. During inflation, this condition implies [30] $\delta\phi = 0$.

- **Uniform energy density gauge**
  It is found by selecting the constant time slicing where there is no perturbation in the energy density, namely $\delta\rho = 0$.

Let's now define the gauge invariant *curvature perturbation on uniform energy density hypersurfaces* as

$$-\zeta \equiv \hat{\Psi} + \mathcal{H}\frac{\delta\rho}{\rho_0'} \quad . \tag{2.112}$$

In a generic gauge, it is related to the curvature perturbation $\hat{\Psi}$ (i.e. the perturbed gravitational potential) and to the energy density perturbations $\delta\rho$ on that gauge.
Using the transformation laws (2.97), (2.98) and (2.102), it's straightforward to show that, despite $\hat{\Psi}$ and $\delta\rho$ are gauge dependent, this quantity assumes the same value on any gauge. Moreover, as anticipated at the beginning of this subsection, we clearly see how the physical perturbation ζ can be entirely put on a metric or a matter perturbation according to the chosen gauge, from which the physical interpretation also derives: as suggested by the name itself, if we move to the uniform energy density gauge, ζ represents the gravitational potential on the spatial slices where $\delta\rho = 0$, $-\zeta = \hat{\Psi}_{|\delta\rho=0}$; this can be achieved by just a time translation, since

$$\tilde{\delta\rho} = \delta\rho + \rho_0'\alpha = 0 \quad \Longrightarrow \quad \alpha = -\delta\rho/\rho_0' \quad . \tag{2.113}$$

Otherwise, if we put on the spatially flat gauge, $\hat{\Psi} = 0$, such a combination can be regarded as the perturbation of the total energy density.

Another useful gauge invariant variable is given by the *comoving curvature perturbation*, defined as

$$\mathcal{R} \equiv \hat{\Psi} + \mathcal{H}\frac{\delta\phi}{\phi_0'} \quad , \tag{2.114}$$

which is a combination of the curvature perturbation and the inflaton's fluctuation. Using the transformation law for the scalar field $\delta\phi \longrightarrow \tilde{\delta\phi} = \delta\phi + \phi_0'\alpha$, it's easy to verify that this quantity is gauge invariant.
In the comoving gauge, it coincides with the gravitational potential perturbation on the constant time slices with $\delta\phi = 0$, $\mathcal{R} = \hat{\Psi}_{|\delta\phi=0}$, reached by performing the time translation $\tau \longrightarrow \tau + \alpha$, with $\alpha = -\delta\phi/\phi_0'$.

Finally, a gauge invariant measure of the inflaton's fluctuations is given by

$$\mathcal{Q} \equiv \delta\phi + \hat{\Psi}\frac{\phi_0'}{\mathcal{H}} = \frac{\phi_0'}{\mathcal{H}}\mathcal{R} \quad , \tag{2.115}$$

known as the *inflaton perturbation on spatially flat slices*, or the *Sasaki-Mukhanov* variable, whose physical meaning can be easily deduced.
Notice that the definition of ζ is very general, in the sense that it applies to any epoch in the history of the evolution of the universe. During slow roll inflation, the total energy density coincides with the vacuum energy density of the inflaton, and we have

$$\dot{\rho}_0 = -3H(\rho_0 + P) = -3H\dot{\phi}_0^2 \quad , \tag{2.116}$$

$$\delta\rho \simeq \frac{\partial V}{\partial\phi_0}\delta\phi \simeq -3H\dot{\phi}_0\delta\phi \quad , \tag{2.117}$$

which imply

$$-\zeta \equiv \hat{\Psi} + \mathcal{H}\frac{\delta\rho}{\rho_0'} = \hat{\Psi} + H\frac{\delta\rho}{\dot{\rho}_0} \approx \hat{\Psi} + H\frac{\delta\phi}{\dot{\phi}_0} = \mathcal{R} \quad . \tag{2.118}$$

The linearly perturbed Einstein equations provide another relation between ζ and $\mathcal{R}$, given by the gauge transformation [43]

$$-\zeta_{\mathbf{k}} = \mathcal{R}_{\mathbf{k}} + \frac{k^2}{a^2H^2}\frac{2\rho_0}{3(\rho_0+P)}\Psi_{\mathbf{k}} \quad . \tag{2.119}$$

Hence, we see that the Fourier modes of ζ and $\mathcal{R}$ are equivalent on super-horizon scales,

$$-\zeta_{\mathbf{k}} \simeq \mathcal{R}_{\mathbf{k}} \quad \text{for} \quad k \ll aH. \tag{2.120}$$

Moreover, the perturbed continuity equation $\nabla_\mu T^{\mu\nu} = 0$ provides the following evolution equation for the perturbation ζ in the uniform energy density gauge [31]

$$\dot{\zeta}_{\mathbf{k}} = -H\frac{\delta P_{iso}}{\rho_0 + P_0} + \frac{k^2}{(aH)^2}(\dots) \quad , \tag{2.121}$$

where the dots stand for a finite quantity. As a consequence of (2.121), in absence of isocurvature pressure perturbations, $\delta P_{iso} = 0$, the curvature perturbation ζ is practically conserved on super-horizon scales, and the same is true for $\mathcal{R}$, due to (2.120). As argued in subsection 2.4.3, this is the case for single-field models of inflation. Basically, these two variables are not affected by the physics governing the super-horizon evolution of the cosmological perturbations between the moment when the Fourier modes wavelengths cross the horizon, during inflation, and the time they re-enter the horizon, during the radiation dominated epoch (see figure 2.3). This turns out to be a very nice feature, because we know very little about the details of what happens from after inflation up to the radiation dominated era, say the BBN. Then, given the physical meaning of the curvature perturbations ζ and $\mathcal{R}$, the importance of these two gauge invariant combinations lies in their ability to connect inflationary theoretical predictions made at horizon crossing, as the power spectrum for the inlaton field perturbations, with late-time observables computed at horizon re-entry, as the power spectrum for the primordial perturbations in the energy density/temperature. Indeed, this is exactly the next step: in the next subsection we will compute the comoving curvature perturbation $\mathcal{R}$ generated during inflation on super-horizon scales (as seen, the choice to compute $\mathcal{R}$ rather than ζ, is irrelevant), hence the associated power spectrum $\Delta_{\mathcal{R}}(k)$. We adopt a gauge fixing approach, by using the longitudinal gauge.

2.4.5 The comoving curvature perturbation in the longitudinal gauge

We are not going to explicitly perform all the calculations involved in the derivation of the curvature perturbation, which entails the linear perturbation of each term of the Einstein's equations (1.2) and of the Klein-Gordon equation (2.3) for the inflaton field. We will limit ourselves to report the main steps of this perturbative procedure, necessary to compute $\mathcal{R}$. The results which follows are summarized from [30].

The linearly perturbed Klein-Gordon equation provide the evolution equation for the Fourier mode $\delta\phi_{\mathbf{k}}$, which written in cosmic time t reads [30]

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} - \dot{\Phi}_{\mathbf{k}}\dot{\phi}_0 - 3\dot{\phi}_0\dot{\Psi}_{\mathbf{k}} + \frac{k^2}{a^2}\omega_{\|\mathbf{k}}\dot{\phi}_0 + \delta\phi_{\mathbf{k}}V_{,\phi\phi} + 2\Phi_{\mathbf{k}}V_{,\phi} = 0 \quad . \tag{2.122}$$

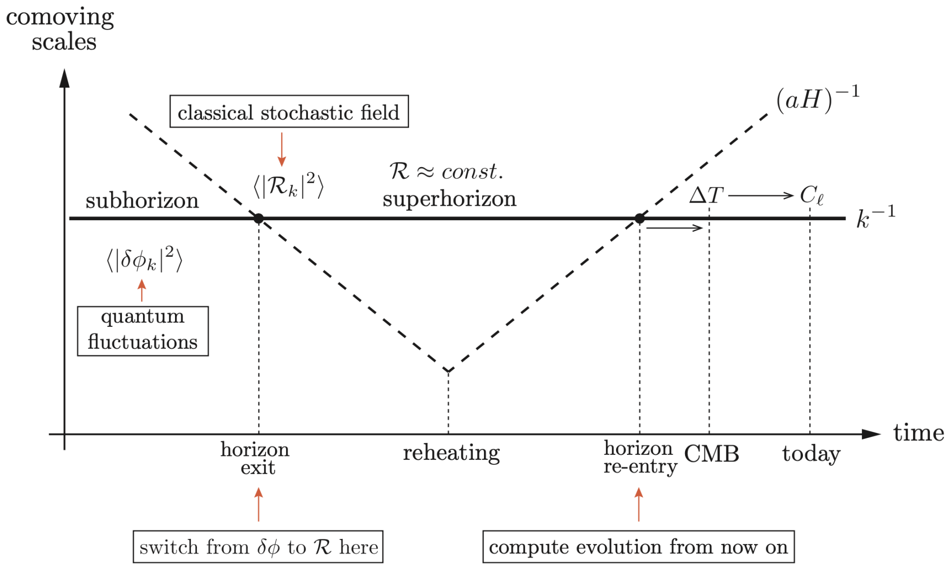

Figure 2.3

We now move in the longitudinal gauge, where the scalar perturbations $\omega_{\|}$ and $\chi_{\|}$ are set to zero. From the non-diagonal part $(i \neq j)$ of the (ij)-component of the linearly perturbed Einstein's equations one finds[6]

$$\delta G_{ij} = \delta T^{\phi}_{ij} \longrightarrow \partial_i \partial_j (\Psi - \Phi) = 0 \implies \Psi = \Phi \quad , \tag{2.123}$$

which means that we are left with a single independent scalar perturbation, say Ψ. The $(0i)$-component of the equation also provides

$$\dot{\Psi}_{\mathbf{k}} + H\Psi_{\mathbf{k}} = 4\pi G \dot{\phi}_0 \delta\phi_{\mathbf{k}} = \epsilon H^2 \frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}_0} \quad , \tag{2.124}$$

where we used (2.14).

We know that on super-horizon scale the fluctuations become practically frozen, and we can consider $\Psi_{\mathbf{k}}$ as nearly constant, in the sense that its relative variation within a time interval of the order of the characteristic expansion time H^{-1} is very small, i.e $|\dot{\Psi}_{\mathbf{k}}| \ll |H\Psi_{\mathbf{k}}|$. Then, the last equation implies a relation between $\Psi_{\mathbf{k}}$ and $\delta\phi_{\mathbf{k}}$ given by

$$\Psi_{\mathbf{k}} \simeq \epsilon H \frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}_0} \quad , \tag{2.125}$$

which allows to write the comoving curvature perturbation on super-horizon scale just in terms of the inflaton fluctuation $\delta\phi_{\mathbf{k}}$, as

$$\mathcal{R}_{\mathbf{k}} = \Psi_{\mathbf{k}} + H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}_0} = (1+\epsilon)H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}_0} \simeq H\frac{\delta\phi_{\mathbf{k}}}{\dot{\phi}_0} \quad . \tag{2.126}$$

It remains to solve equation (2.122), which in the longitudinal gauge becomes

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} - 4\dot{\phi}_0\dot{\Psi}_{\mathbf{k}} + 2\Psi_{\mathbf{k}}V_{,\phi} + \delta\phi_{\mathbf{k}}V_{,\phi\phi} = 0 \quad . \tag{2.127}$$

[6]The relation which follows is strictly true in absence of anisotropic stress.

Using again the fact that $|\dot{\Psi}_{\mathbf{k}}| \ll |H\Psi_{\mathbf{k}}|$ we have $|\dot{\Psi}_{\mathbf{k}}\dot{\phi}_0| \ll |\Psi_{\mathbf{k}}V_{,\phi}|$, and exploiting (2.125) and the slow roll relation $V_{,\phi} \simeq -3H\dot{\phi}_0$, equation (2.127) can be rewritten as

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \left(\frac{k^2}{a^2} + m_\phi^2 - 6\epsilon H^2\right)\delta\phi_{\mathbf{k}} = 0 \quad . \tag{2.128}$$

By trading the cosmic time t with the conformal time τ and the perturbation field $\delta\phi_{\mathbf{k}}$ with the usual rescaled field $\delta\chi_{\mathbf{k}} = a\delta\phi_{\mathbf{k}}$ we get

$$\delta\chi_{\mathbf{k}}'' + \left(k^2 + a^2 m_\phi^2 - \frac{a''}{a} - 6\epsilon a^2 H^2\right)\delta\chi_{\mathbf{k}} = 0 \quad , \tag{2.129}$$

which at first in order in the slow-roll parameters becomes

$$\delta\chi_{\mathbf{k}}'' + \left(k^2 + \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right)\delta\chi_{\mathbf{k}} = 0 \quad , \tag{2.130}$$

with $\nu^2 = 9/4 + 9\epsilon - 3\eta_V$, and $\nu \simeq 3/2 + 3\epsilon - \eta_V$ at first order.
We have already solved this kind of equation in subsection 2.3.2, in the case of an unperturbed background spacetime. On super-horizon scales, we have found the solution

$$|\delta\phi_k| \simeq \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \quad , \tag{2.131}$$

which yields the following adimensional power spectrum for the comoving curvature perturbation on super-horizon scales

$$\Delta_{\mathcal{R}}(k) = \frac{H^2}{\dot{\phi}_0^2}\Delta_{\delta\phi}(k) = \frac{H^2}{\dot{\phi}_0^2}\frac{k^3}{2\pi^2}|\delta\phi_k|^2 = \left(\frac{H^2}{2\pi\dot{\phi}_0}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \quad . \tag{2.132}$$

We have shown this quantity remains constant until horizon exit during the radiation dominated epoch, so we can evaluate it at the instant $t_H^*(k)$ of horizon crossing of some mode k, such that $k = aH$,

$$\Delta_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\dot{\phi}_0}\right)^2\Bigg|_{t_H^*(k)} \quad . \tag{2.133}$$

The scale dependence is now inside $t_H^*(k)$, because each mode cross the horizon at different time. The spectral index at first order in ϵ and η_V results

$$n_{\mathcal{R}}(k) - 1 = 3 - 2\nu = 2\eta_V - 6\epsilon \quad , \tag{2.134}$$

which means that single-field inflationary models predict a small, but not vanishing, deviation from a scale invariant Harrison-Zel'dovich power spectrum. From the Planck CMB data [2] we have a measured spectral index for the scalar perturbations given by

$$n_s = 0.9649 \pm 0.0042 \quad (68\%\ \text{C.L.}) \quad , \tag{2.135}$$

which is 8σ away from $n_s = 1$, consistently with theoretical predictions.
The nearly scale invariance is not so surprising if we look at equation (2.133): as we know, H and $\dot{\phi}_0$ change very slowly in time during inflation, resulting in a weak dependence on the instant of

horizon crossing, which in turn has a weak dependence on the scale, due to the extremely rapid expansion. Indeed, assuming an exponential expansion, $a = e^{Ht}$, we have

$$k = Ha = He^{Ht^*} \quad \Longrightarrow \quad t^*(k) = H^{-1} ln\left(\frac{k}{H}\right) \quad , \tag{2.136}$$

then a weak logarithmic scale dependence.
To summarize, we found that standard single-field models of slow-roll inflation predict that, starting by the vacuum fluctuations of a scalar field, the inflationary mechanism is able to generate nearly Gaussian and adiabatic primordial energy density perturbations with an almost scale invariant power spectrum.

2.5 Primordial tensor perturbations

Inflationary models also predict a stochastic background of primordial gravitational waves produced during inflation and amplified by the accelerated expansion. They consist of a signal coming from every direction in the sky that exhibit a spectrum in the whole frequency domain, whereas gravitational waves produced, for example, by the collision and merger of compact objects come from a specific direction and they have a spectrum peaked around a particular range of frequency. Gravitational waves are described by the traceless and transverse components χ^T_{ij} of the tensor perturbations of the spatial part of the metric, which satisfy the conditions

$$\chi^T_{ij} = \chi^T_{ji} \quad , \quad \chi^{T\,i}_{\ \ i} = 0 \quad , \quad \partial^i \chi^T_{ij} = 0 \quad . \tag{2.137}$$

The symmetric condition reduces the 9 initial degrees of freedom into 6, while the traceless and transverse conditions (defining the so called T-T gauge) provide other 4 constraints. Therefore, there remain 2 independent physical degrees of freedom, corresponding to two possible polarizations of the waves that are usually denoted with $\lambda = (+, \times)$. Then, the tensor perturbation can be decomposed in Fourier space as

$$\chi^T_{ij}(t, \mathbf{x}) = \sum_{\lambda=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \chi_\lambda(t, \mathbf{k}) \epsilon^\lambda_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \quad , \tag{2.138}$$

where ϵ^λ_{ij} are the polarization tensors, which have the following properties

$$\epsilon^\lambda_{ij} = \epsilon^\lambda_{ji} \quad , \quad \epsilon^{\lambda\,i}_{\ \ i} = 0 \quad , \quad k^i \epsilon^\lambda_{ij} = 0 \tag{2.139}$$

$$\left(\epsilon^\lambda_{ij}(\mathbf{k})\right)^* = \epsilon^\lambda_{ij}(-\mathbf{k}) \quad , \quad \epsilon^\lambda_{ij}(\mathbf{k}) \epsilon^{*ij}_{\lambda'}(\mathbf{k}) = \delta_{\lambda\lambda'} \quad . \tag{2.140}$$

At linear order in perturbation theory we can disregard scalar and vector perturbations of the metric, since they evolve independently, so that the perturbed Einstein equations provide the following equation of motion for the tensor mode[7] [31]

$$\ddot{\chi}^T_{ij} + 3H\dot{\chi}^T_{ij} - \frac{\nabla^2 \chi^T_{ij}}{a^2} = 0 \quad . \tag{2.141}$$

Treating the polarization states as scalar fields, i.e. setting $\chi_{+,\times} \equiv \sqrt{32\pi G} \phi_{+,\times}$[8], we can see that they satisfy an equation which has the same form of equation (2.20) for the fluctuations of

[7] This equation is valid during inflation in absence of anisotropic stress, otherwise the traceless and transverse component of the latter would constitute a source term on the r.h.s. of the equation.

[8] The normalization factor in front of the scalar field comes from the Einstein-Hilbert action, and from this writing it is clear that $\chi_{+,\times}$ is dimensionless.

a minimally coupled scalar field, but with a vanishing mass. Therefore, in order to solve (2.141), we can simply adapt the previous results found at the end of subsection 2.3.2 to the massless case $V_{,\phi\phi} = 0$.
On sub-horizon scales, $k \gg aH$ we find an oscillating solution with a decreasing amplitude, whereas on super-horizon scales, $k \ll aH$, the fluctuations become classical and they are nearly frozen in, with a quasi scale invariant amplitude given by

$$\left|\chi_{+,\times}\right| = \sqrt{32\pi G}\left|\phi_{+,\times}\right| = \sqrt{32\pi G}\frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \quad , \tag{2.142}$$

where in this case we have $\frac{3}{2} - \nu \simeq -\epsilon \ll 1$ at first order in slow-roll, since $\eta_V = 0$.
We define the power spectrum of tensor perturbations as the sum of the power spectra for the two polarizations

$$\Delta_T(k) \equiv 2\Delta_\chi(k) \quad , \quad \Delta_\chi(k) = \frac{k^3}{2\pi^2}P_\chi(k) = \frac{k^3}{2\pi^2}\left|\chi_{+,\times}\right|^2 \quad . \tag{2.143}$$

Therefore we obtain

$$\Delta_T(k) = \frac{2}{\pi^2}\left(\frac{H}{M_{Pl}}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon} \quad , \tag{2.144}$$

or

$$\Delta_T(k) = \frac{2}{\pi^2}\left(\frac{H}{M_{Pl}}\right)^2\Bigg|_{t_H^*(k)} \quad , \tag{2.145}$$

if evaluated at the moment of horizon crossing.
The spectral index for tensor perturbations is defined as

$$n_T(k) \equiv \frac{d\ln\Delta_T(k)}{d\ln k} \quad , \tag{2.146}$$

so, at first order in the slow-roll approximation, we have

$$n_T(k) = -2\epsilon \quad , \tag{2.147}$$

meaning that the tensor power spectrum is almost scale-invariant, as the scalar one.
The detection of the amplitude of the primordial gravitational waves is crucial to estimate the energy scale $E_{inf} \simeq V^{1/4}$ associated to inflation, since during inflation

$$\Delta_T \simeq \frac{2}{3\pi^2}\frac{V}{M_{Pl}^4} \quad . \tag{2.148}$$

We can ultimately define the *tensor-to-scalar perturbation ratio* as

$$r \equiv \frac{\Delta_T}{\Delta_\mathcal{R}} \quad . \tag{2.149}$$

The scalar power spectrum (2.133) can be rewritten in terms of the slow-roll parameter ϵ as

$$\Delta_\mathcal{R} = \frac{1}{8\pi^2\epsilon}\left(\frac{H}{M_{Pl}}\right)^2 \quad , \tag{2.150}$$

then the tensor-to-scalar ratio reads

$$r = 16\epsilon = -8n_T \quad . \tag{2.151}$$

This is a significant consistency relation since it holds for any single-field model of slow-roll inflation. The combined BICEP/Keck and Planck results [44] provide a constraint on the tensor-to-scalar ratio given by

$$r < 0.036 \quad (95\% \text{ C.L.}) \quad , \tag{2.152}$$

which translates into a constraint on the energy scale of inflation through the relation

$$V \simeq \frac{3\pi^2}{2} r \Delta_{\mathcal{R}} M_{Pl}^4 \quad . \tag{2.153}$$

Given the bound (2.152) on r and the value $\Delta_{\mathcal{R}} \simeq 2.1 \cdot 10^{-9}$ from the Plank measurements [2] we have

$$V^{1/4} < 1.4 \cdot 10^{16} \text{GeV} \quad . \tag{2.154}$$

Chapter 3

Warm Inflation

In this chapter we deal with an alternative realization of inflation, where the scalar inflaton field is no more assumed to be an isolated non-interacting quantum field[1], but it is regarded as an open sub-system which noticeably interacts with other quantum radiation fields.
The *warm inflationary phase* is defined as an accelerated expansion phase driven by the dominant potential energy of a scalar inflaton field whose motion is *overdamped* by the thermal contact with a radiation component, that arises from the thermalization of particles produced by the inflaton itself. We explore the warm inflationary scenario by proceeding in steps similar to those followed in the cold scenario. In section 3.1, we briefly discuss the microscopic basis of the inflaton field dynamics in a thermal environment. We will understand how the evolution of the field can be effectively described in the context of non-equilibrium *Thermal Quantum Field Theory* by an effective phenomenological equation of motion, where the backreaction of the radiation fields on the inflaton is partly incorporated by an extra friction term in addition to the Hubble damping.
In section 3.2 we look at the implications on the background evolution of the inflaton-bath system arising by the supplementary friction term, whose effect is to further slow down the inflaton field, allowing to trigger a slow-roll inflationary phase for a wider range of potentials. At the same time, the interaction at the basis of this friction induces a sufficiently strong dissipation of the inflaton vacuum energy which considerably feeds the radiation bath, so to prevent its otherwise total dilution caused by the Hubble expansion. In this way the universe does not super-cool as in standard cold inflation, because a substantial radiation component can survive throughout the accelerated expansion. It follows that the inflaton energy density ρ_ϕ falls more rapidly than the radiation energy density ρ_r because of decay, so that, in the warm scenario, the inflationary phase can directly end in a radiation dominated phase at the moment when a smooth crossover occurs between ρ_r and ρ_ϕ, hence a separate reheating phase must not be necessarily invoked.
In section 3.3 we investigate the impact of the thermal environment on the generation mechanism of curvature perturbations. In warm inflation, the main features to the scalar power spectrum arise from thermal fluctuations in the radiation component. The latter can be effectively encoded by a stochastic noise term in the evolution equation of the inflaton field which acts as a source for random inflaton thermal fluctuations. The fluctuations in the radiation and the inflaton field are generally coupled to each other due to the temperature dependence of the friction coefficient, and it is generally hard to obtain an exact solution for the coupled system of the fluctuation evolution equations. An approximated formula for the scalar curvature power spectrum can be analytically derived in the limit in which the friction coefficient is temperature-independent. In this regime, we

[1] Or better, the inflaton field interacts with nothing else besides gravity.

are able to obtain a result for the scalar spectral index. It turns out that, for temperatures $T > H$, the thermal contribution to the scalar curvature power spectrum dominates over the quantum contribution.
Finally, in section 3.4, we see how a thermal component in the spectrum of the primordial gravitational waves can be sourced by transverse and traceless modes of the anisotropic stress tensor of the radiation fluid arising by dissipative effects due to thermal fluctuations. For large enough temperatures, this contribution can overcome the one given by the quantum vacuum fluctuations.

3.1 Effective evolution equation for the inflaton field in the "in-in" CTP formalism

Intrinsically, warm inflation constitutes an out-of-thermal-equilibrium problem, due to the fact that the dynamical processes take place over an evolving background spacetime and the achievement of a possible thermalized state must pass through a non-equilibrium phase. This scenario essentially requires an overdamped relaxation of the inflaton field to an equilibrium point concurrently with radiation production due to dissipation of its vacuum energy. Since its appearance in literature [45, 46], the reliability of this picture as a description of the early universe has been hindered by the plausibility of its dynamic realization from first principles quantum field theory, mainly because of the lack of a theoretical understanding of non-equilibrium quantum field systems, but also because of the belief that inflation involves timescales that are too short for an effective particle production and thermalization to occur [47]. The description of this inflationary scenario was indeed an important motivation for delving into the study of non-equilibrium dynamics within Thermal Quantum Field Theory [48–50], a combination of QFT with the notions of statistical thermodynamics.
From a statistical mechanics perspective, the system as a whole would try to equally distribute the available energy, therefore the inflaton field is expected to dissipate its excess energy to the radiation fields, so that the thermal equilibrium of the universe is achieved as a result of an irreversible flow of energy. However, strictly speaking, this is a question that can be addressed only through a detailed calculation. Several studies [51–53] suggest that particle production through dissipative effects are naturally present in interacting fields systems. In particular these effects appear more manifest when a "system-environment" approach is applied, in which the effective evolution of a small portion of the whole system is analyzed by averaging out the remaining degrees of freedom of the rest of the world. This decomposition method constitutes a more economical way to study the non-equilibrium dynamics, since we only focus on the evolution of the modes of interest rather than keeping track of each mode, which could be a really demanding task. By the use of several techniques for finite temperature QFT, as the CTP functional integral formalism [48, 54, 55], it is shown [51, 56–62] that, when the small scale behaviour of the environment degrees of freedom is integrated out, a simple picture emerges where the evolution of a thermally averaged configuration of the inflaton field is determined by a stochastic *Langevine-like* evolution equation, typical of an open dissipative system feeling a random noise exerted by the external environment to which it is coupled. However, the non-equilibrium dissipative dynamics is well defined and understood only in a *close-to-local thermal equilibrium* (LTE) regime, realized when the macroscopic motion of the whole system is very slow compared to the time scales of the microscopic dynamics, so that the fields can quickly respond and adapt to changes in the thermodynamic variables. Still today we have a limited understanding of dissipative dynamics in strongly out-of-equilibrium conditions, therefore, up to now, a realization of the warm inflationary picture from first principle quantum field theory is achieved only under the restrictive assumption of quasi-equilibrium conditions [51, 58, 63].
We now provide some basics on Thermal QFT and its formulation within the Schwinger-Keldysh

CTP (or "in-in") formalism in Minkowski spacetime, also discussing the motivations for the introduction of this formalism [60, 64, 65]. Then, we will see how its application leads to the phenomenological evolution equation for the inflaton field.

Because of the non-negligible presence during inflation of a bath of thermally excited radiation fields, the whole quantum fields system does not remain in the vacuum state but it rather randomly samples all the possible microscopic thermal states belonging to a given statistical ensemble according to a statistical distribution operator $\hat{\rho}$. The latter represents the density matrix operator describing the mixed state of the entire quantum system.
The time dependent description of a non-equilibrium system is given by the time evolution of its density matrix $\hat{\rho}(t)$, determined by the Liouville-von Neumann equation (in the Schrodinger picture)

$$i\frac{\partial\hat{\rho}(t)}{\partial t} = [\hat{H}(t), \hat{\rho}(t)] \quad , \tag{3.1}$$

where the Hamiltonian of the system $\hat{H}(t)$ is time dependent. Once an initial statistical configuration of quantum fields $\hat{\rho}(t_0) \equiv \hat{\rho}_0$ is specified for some instant t_0 usually taken in the asymptotic past, $t_0 \longrightarrow -\infty$, equation (3.1) admits the formal solution

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}_0\hat{U}^\dagger(t) \quad , \quad \hat{U}(t) = \mathrm{T}\exp\left(\int_{t_0}^{t} dt'\hat{H}(t')\right) \quad , \tag{3.2}$$

where T denotes the time ordered product. Actually, the above evolution operator $\hat{U}$ holds only for closed systems.
In a quantum statistical framework, the physically interesting quantities of the system are represented by expectation values of the quantum operators, obtained by performing an ensemble average weighted by $\hat{\rho}$. Hence, in this context, the n-point correlation functions for a generic field theory of field $\hat{\phi}$ are given by

$$\langle \mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]\rangle \equiv \mathrm{Tr}\Big[\hat{\rho}_0\,\mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]\Big] \quad , \quad \mathrm{Tr}[\hat{\rho}] = \sum_i \langle\phi_i|\,\hat{\rho}\,|\phi_i\rangle = 1 \tag{3.3}$$

where $\{|\phi_i\rangle\}$ is a complete set of orthonormal states for the Hilbert space of the system. The dynamical information of the ensemble averages is contained in the time dependent fields $\hat{\phi}(x_i)$ in the Heisenberg picture.
The expectation values in (3.3) cannot be easily obtained within the conventional "in-out" formalism of QFT. In order to understand the reason, let us take a step back to the standard QFT approach. The latter is based on the "in-out" generating functional $Z[J]$, which is given by the vacuum persistence amplitude in presence of an external interaction source $J(x)$

$$Z[J] = e^{iW[J]} \equiv \langle 0_{out}|0_{in}\rangle_J = \langle 0_{out}|\mathrm{T}\exp\left(i\int_{-\infty}^{+\infty} d^4x J(x)\hat{\phi}(x)\right)|0_{in}\rangle \quad , \tag{3.4}$$

where $W[J]$ is the connected generating functional.
The above transition amplitude admits the following path integral representation

$$Z[J] = \int D\phi\, e^{i(S[\phi]+J\cdot\phi)} \quad , \tag{3.5}$$

with the dot "·" standing for a spacetime integration.
The states $|0_{in}\rangle$ and $|0_{out}\rangle$ in (3.4) denote the vacuum state in the interaction picture at $t = -\infty$

and $t = +\infty$ respectively, while $\hat{\phi}(x)$ is the Heisenberg field in the interaction picture evolving according to the "free" Hamiltonian $\hat{H}(t)$ of the system. Essentially, we let the in-vacuum evolve under the "interacting Hamiltonian" represented by the source term $\hat{H}^I_{int} = -\int d^3\mathbf{x} J(x)\hat{\phi}(x)$, and then we compare the resulting state at $t = +\infty$ with the out-vacuum.
In a non-equilibrium setting, in general, the states $|0_{in}\rangle$ and $|0_{out}\rangle$ are not equivalent, so that the correlation functions obtained by functional differentiation w.r.t. the source $J(x)$ are actually *matrix elements* rather than expectation values[2]. Therefore, whereas the "in-out" formulation is well suited to address problems of particle physics in absence of a thermal environment, where we are mostly interested in transition probability amplitudes (i.e. scattering S-matrix elements) between initial and final states constructed upon the vacuum state, the same formulation is not suitable to trace the time evolution of an expectation value in a non-equilibrium thermodynamic system.
Following the idea of Schwinger and Keldysh [54, 55], in order to overcome this problem we define the following *"in-in" generating functional* [64, 65]

$$Z[J_+, J_-] = e^{iW[J_+,J_-]} \equiv {}_{J_-}\langle 0_{in}|0_{in}\rangle_{J_+} \tag{3.6}$$

where instead we let the in-vacuum evolve under two distinct external sources J_- and J_+ and then we compare the resulting states in the far future. We can write the above expression by inserting a summation over a complete set of out states given by eigenvectors of the Heisenberg field at time $t = +\infty$, i.e. $\hat{\phi}(+\infty, \mathbf{x})\,|\psi_{out}\rangle = \psi(\mathbf{x})\,|\psi_{out}\rangle$. Then we have

$$Z[J_+, J_-] = \int D\psi\, \langle 0_{in}|\psi_{out}\rangle_{J_-}\, \langle \psi_{out}|0_{in}\rangle_{J_+} \quad . \tag{3.7}$$

In this case $Z[J_+, J_-]$ is seen to be a sum over the configurations ψ at $t = +\infty$ of the amplitude for the quantum state to evolve forward in time under the source J_+ from $|0_{in}\rangle$ to $|\psi_{out}\rangle$, times the amplitude for the state $|\psi_{out}\rangle$ to evolve backwards in time under the source J_- to the state $|0_{in}\rangle$. Explicitly, the expression (3.7) reads

$$\begin{aligned} Z[J_+, J_-] = \int D\psi\, \langle 0_{in}|\tilde{\mathrm{T}}\exp\left(-i\int_{-\infty}^{+\infty} d^4x J_-(x)\hat{\phi}(x)\right)|\psi_{out}\rangle \\ \times\ \langle \psi_{out}|\mathrm{T}\exp\left(i\int_{-\infty}^{+\infty} d^4x J_+(x)\hat{\phi}(x)\right)|0_{in}\rangle\ , \end{aligned} \tag{3.8}$$

where $\tilde{T}$ denotes the anti-temporal order operator.
The generalization of (3.8) to a quantum-statistical system described by a matrix density $\hat{\rho}$ is [60,

[2]In standard QFT of particle physics this problem is solved by assuming that the system adiabatically follows the non-degenerate ground state of the free theory upon slow switching off the interactions in the distant past and future. This assumption allows to directly relate the interacting vacuum state to the free vacuum state (Gell-Mann Low theorem [66]), and consequently to express the correlation functions written in terms of the interacting Heisenberg fields as a perturbative expansion of correlation functions written as expectation value, w.r.t. the free ground state, of free fields in the interaction picture. However, for an irreversible non-equilibrium dynamics the adiabatic assumption fails, since the system does not asymptotically come back to the same state.

67]

$$Z[J_+,J_-,\hat\rho] \equiv \mathrm{Tr}\left[\int D\psi\,\tilde{\mathrm{T}}\exp\left(-i\int_{t_0}^{+\infty} d^4x J_-(x)\hat\phi(x)\right)|\psi_{out}\rangle \right.$$
$$\left. \times\ \langle\psi_{out}|\,\mathrm{T}\,\exp\left(i\int_{t_0}^{+\infty} d^4x\, J_+(x)\hat\phi(x)\right)\hat\rho(t_0)\right] . \quad (3.9)$$

We can see that $Z[J_+,J_-,\hat\rho]$ is a normalized ($Z[0,0,\hat\rho]=1$) generating functional for the quantum expectation values (3.3), that are obtained by functional differentiation w.r.t. J_+ and J_- and then setting $J_+=J_-=0$. Notice that $Z[J_+,J_-,\hat\rho]$ also generates expectation values other than the time-ordered ones.
The so called CTP (Closed Time Path), or real-time, Schwinger-Keldysh formalism[3] provides a functional integral representation of the "in-in" generating functional (3.9) which yields real and causal effective actions, field equations and expectation values [48, 60, 64, 65].
To derive the path integral representation we evaluate the trace by considering two different complete sets of eigenvectors of the Heisenberg field at initial time $t=t_0$,

$$\hat\phi(t_0,\mathbf{x})\,|\phi_{in}\rangle = \phi(\mathbf{x})\,|\phi_{in}\rangle \quad , \quad (3.10)$$
$$\hat\phi(t_0,\mathbf{x})\,|\phi'_{in}\rangle = \phi'(\mathbf{x})\,|\phi'_{in}\rangle \quad , \quad (3.11)$$

so that equation (3.9) becomes

$$Z[J_+,J_-,\hat\rho] = \int D\phi D\phi' D\psi\ \langle\phi_{in}|\tilde{\mathrm{T}}\exp\left(-i\int_{t_0}^{+\infty} d^4x J_-(x)\hat\phi(x)\right)|\psi_{out}\rangle$$
$$\times\ \langle\psi_{out}|\mathrm{T}\exp\left(i\int_{t_0}^{+\infty} d^4x J_+(x)\hat\phi(x)\right)|\phi'_{in}\rangle\ \langle\phi'_{in}|\hat\rho_0|\phi_{in}\rangle\ , \quad (3.12)$$

which can be written as

$$Z[J_+,J_-,\hat\rho] = \int D\phi^+ D\phi^- e^{i\left[(S[\phi^+]+J_+\cdot\phi^+)-(S[\phi^-]+J_-\cdot\phi^-)\right]}\,\langle\phi_0^-|\hat\rho_0|\phi_0^+\rangle \quad , \quad (3.13)$$

where the functional integration is taken over field configurations $\phi^+(x)$ and $\phi^-(x)$ which coincide on the hypersurface at $t=+\infty$, $\phi^+(+\infty,\mathbf{x})=\phi^-(+\infty,\mathbf{x})$, and $|\phi_0^\pm\rangle$ is the quantum state corresponding to the field configuration $\phi^\pm(t_0,\mathbf{x})$.
Observe that the doubling of the sources implies a doubling of the degrees of freedom. Also, unlike the conventional "in-out" formalism, the time variable of the paths runs along the closed contour $C=C^+\cup C^-$ represented in figure 3.1, going from $t=t_0$ to $t=+\infty$ (forward branch C^+) and back (backward branch C^-). Hence the name CTP.
Analogously to the standard case, if the interactions are weak, the path integral representation allows a computation of the correlation functions (3.3) through a diagrammatic perturbative expansion written in terms of a specific set of Feynman rules obtained by the classical relativistic action of the theory.
As usual, the *"in-in" quantum effective action* for the averaged fields $\bar\phi^+$, $\bar\phi^-$ in presence of the

[3]This formalism was originally developed for non-relativistic quantum many-body field theory [54, 55], and then it was extended to the relativistic case, even on a curved spacetime background [64, 65].

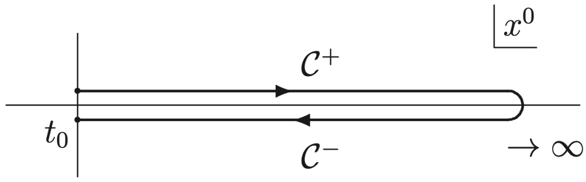

Figure 3.1

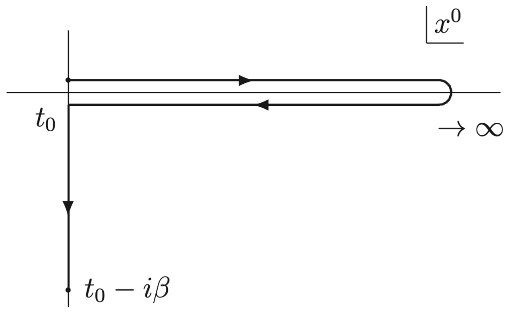

Figure 3.2

sources can be defined as the Legendre transform of the connected generating functional $W[J_+, J_-]$, i.e.

$$\Gamma[\bar{\phi}^+, \bar{\phi}^-] \equiv W[J_+, J_-] - J_+ \cdot \bar{\phi}^+ + J_- \cdot \bar{\phi}^- \quad , \tag{3.14}$$

where $\bar{\phi}^{\pm} = \pm\delta W[J_+, J_-]/\delta J_{\pm}$, and we assume that $\bar{\phi}^{\pm}(J_+, J_-)$ is invertible.
The field equations satisfied by $\bar{\phi}^{\pm}$ are

$$\frac{\delta\,\Gamma[\bar{\phi}^+, \bar{\phi}^-]}{\delta\bar{\phi}^{\pm}} = \mp J_{\pm} \quad . \tag{3.15}$$

When $J_+ = J_- = J$ the averaged fields coincide with the average $\langle\hat{\phi}\rangle_J$ of the Heisenberg field $\hat{\phi}$ in presence of the single source J. In particular, taking $J = 0$ we obtain the physical averaged field $\bar{\phi}^+ = \bar{\phi}^- = \bar{\phi} \equiv \langle\hat{\phi}\rangle$. Then, equation (3.15) becomes the dynamical equation for $\bar{\phi}$:

$$\left.\frac{\delta\,\Gamma[\bar{\phi}^+, \bar{\phi}^-]}{\delta\bar{\phi}^+}\right|_{\bar{\phi}^+ = \bar{\phi}^-} = 0 \quad . \tag{3.16}$$

The computations are generally performed in the Keldysh representation, defined by a linear transformation of the fields $\bar{\phi}^+$ and $\bar{\phi}^-$, namely

$$\phi_c = \frac{\bar{\phi}^+ + \bar{\phi}^-}{2} \quad , \quad \phi_\Delta = \bar{\phi}^+ - \bar{\phi}^- \quad . \tag{3.17}$$

In the limit $J_+ = J_- = 0$, ϕ_c coincide with physical field $\bar{\phi}$, while $\phi_\Delta = 0$. In this representation the effective equation of motion (3.16) for the averaged field becomes

$$\left.\frac{\delta\,\Gamma[\phi_c, \phi_\Delta]}{\delta\phi_\Delta}\right|_{\phi_\Delta = 0} = 0 \quad . \tag{3.18}$$

The CTP formalism is able to handle equilibrium as well as non-equilibrium dynamics.

Since in the warm iflation scenario the system is supposed to remain close to LTE throughout its evolution, and given that the thermalization process erases the memory of the initial condition, it is convenient to choose as initial condition a free theory with a thermal distribution of states[4] i.e. we take $\hat{\rho}_0 = \hat{\rho}_{eq} \propto e^{-\beta\hat{\mathcal{H}}}$, where $\hat{\mathcal{H}}$ depends on the ensemble chosen to represent the system. For a canonical ensemble, $\hat{\mathcal{H}}$ is equal to some time independent initial Hamiltonian $\hat{H}_0$. We therefore define the time dependent Hamiltonian as $\hat{H}(t) = \hat{H}_0$ for $t \leq t_0$, where t_0 denotes an initial time at which the CMB modes are well inside the horizon, and $\hat{H}(t) = \hat{H}_{dyn}(t)$ for $t > t_0$, where $\hat{H}_{dyn}(t)$ is the interacting Hamiltonian that determines the dynamics of the system. In our cosmological context, the time dependence of the Hamiltonian comes from the evolution of the background metric and of the inflaton field, and, due to this time dependence, the evolution at times $t > t_0$ will lead in general to a non-thermal density matrix. Moreover, we need to assume that the expansion is "slow" in order to achieve a near-equilibrium condition, where the precise meaning of "slow" will be specified later.

The initial thermal distribution can be regarded as a time evolution operator $e^{-i\hat{H}_0\Delta t}$ with imaginary-time interval $\Delta t = (t_0 - i\beta) - t_0 = -i\beta$, so that the CTP generating functional (3.13) becomes a path integral in which the time integration is taken along the three branched path represented in figure 3.2, namely from t_0 to $+\infty$, back to t_0 and finally from t_0 to $t_0 - i\beta$. Because of the trace in the definition of $Z[J_+, J_-, \hat{\rho}]$, the functional integration is now performed over time periodic field configurations with period $-i\beta$. Moreover, it can be proven that the choice of the time path is actually irrelevant as long as its extremities do not change and the imaginary part of t along the path is never increasing [69].

As argued in [60], the choice of the path depends on the problem at hand. If the system remains at thermal equilibrium throughout its evolution, that means $\hat{H}(t) = \hat{H}_0 \implies \hat{\rho}(t) = \hat{\rho}_{eq}\ \forall t$, then we are not interested on the time development of the ensemble averages, which are static, so the simplest choice for the time path is to go straight from t_0 to $t_0 - i\beta$ along the imaginary axis; in this case the "in-in" CTP formalism reduces to the imaginary-time Matsubara formulation of finite temperature QFT [70]. Instead, in the case of a non-equilibrium dynamics, we want to know the real time evolution of the averaged observables, therefore we choose to follow the three branched path which encompasses non-thermal states at times $t_0 < t < +\infty$.

The formalism described up to now is adopted, for example, in [59, 61], where the approach to equilibrium of a weakly self-coupled scalar field ϕ (the inflaton) is tackled by applying a somewhat blurred *"system-bath" separation*. Basically, the non-equilibrium effective equation of motion for the system of interest, given by the thermal averaged field $\langle\phi\rangle_\beta \equiv \bar{\phi}$, is obtained by integrating out the short wavelength modes of the field itself, which may act as a thermal bath driving the slowly varying background configuration to equilibrium. The procedure involves the perturbative computation of the CTP effective action and the use of the least action principle. In [58, 59] the same computation is also done in presence of other scalar fields which also serve as the bath, and it comes out that the form of the resulting effective equation of motion is the same.

Following [59], we can consider the classical action on flat Minkowski spacetime for a self-interacting scalar field ϕ

$$S[\phi, J] = \int_C d^4x \left(-\frac{1}{2}(\partial_\mu\phi)^2 - V_0(\phi) + J\phi \right) \quad , \tag{3.19}$$

[4]This is actually a limitation of the Schwinger-Keldysh formalism itself, which can only efficiently deal with thermal initial density matrices. Since there are problems where it is important to keep track of the initial conditions explicitly, extensions of the formalism have been developed to describe non-equilibrium dynamics of quantum systems starting from arbitrary initial density matrices [68].

where $V_0(\phi)$ is the zero temperature potential presenting a quartic self-interaction

$$V_0(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad . \tag{3.20}$$

The time integration is along the closed path C[5] going from $t_0 = -\infty$ to $+\infty$ and then back to $-\infty$. Moreover, the field satisfies periodic boundary conditions $\phi(t, \mathbf{x}) = \phi(t - i\beta, \mathbf{x})$, which will lead to the appearance of the finite temperature T in the computations.
Starting from the classical action (3.19), the CTP generating functional (3.13) is constructed by doubling the sources and the field variables along the path C, $\phi \to (\phi^+, \phi^-)$. Therefore, by decomposing the field as $\phi^\pm = \bar{\phi}^\pm + \eta^\pm$, with $\eta^\pm$ denoting the quantum fluctuations with zero mean, $\langle \eta^\pm \rangle_\beta = 0$, the CTP effective action $\Gamma[\phi_c, \phi_\Delta]$ for the thermal averaged fields in the Keldysh basis can be computed perturbatively via a diagrammatic loop expansion, in the limit of vanishing sources. Then, the application of the variational principle (3.18) provides an effective equation of motion for the physical background field $\phi_c = \bar{\phi}$ which exhibits space and time *non*-locality, and that can be expressed in the form [59, 61, 72]

$$\left[-\partial^2 + m^2 + \frac{\lambda}{3!}\bar{\phi}^2(x)\right]\bar{\phi}(x) + \int d^4x' \, \Sigma[\bar{\phi}](x, x')\theta(t - t')\bar{\phi}(x') = \xi_1(x)\bar{\phi}(x) + \xi_2(x) \quad , \tag{3.21}$$

where the function $\Sigma[\bar{\phi}](x, x')$ is related to self energy amplitude of the field, and ξ_1, ξ_2 are two random fields whose origin will be clarified shortly.
Heuristically, the appearance of non-local terms can be understood as follow [73]: the interactions with other fields imply quantum corrections to the classical action coming from loop diagrams where virtual particles are emitted off $\bar{\phi}$, propagate in space and time, and then are reabsorbed by $\bar{\phi}$. At thermal equilibrium, meaning for a space and time constant background field $\bar{\phi}$, these loops just yield quantum and finite temperature corrections to the classical potential $V_0(\phi)$, which are summarized by an effective potential $V_{eff}(\bar{\phi}, T)$ that can be identified with the thermodynamic *free-energy* density of the system (see Appendix A). Differently, in a non-equilibrium situation with an evolving background field $\bar{\phi}(t, \mathbf{x})$, the emission and the absorption points of virtual particles in the loop diagrams involve products of configurations of $\bar{\phi}$ on different spacetime points, thus the introduction of integral correction terms (non-local) that will give rise not only to effective potential corrections but also to dissipative effects.
The random terms on the r.h.s. of equation (3.21) arises from imaginary contributions to CTP effective action quadratic in ϕ_Δ that carry information about the fluctuations of the solution of the effective field equation. These quadratic terms in $\Gamma[\phi_c, \phi_\Delta]$ can be decoupled by means of a Hubbard–Stratonovich transformation [56, 59, 61], which allows to attribute the imaginary part $\mathrm{Im}\,\Gamma[\phi_c, \phi_\Delta]$ to the result of a Gaussian functional integration over some auxiliary random fields ξ_1 and ξ_2:

$$e^{i\Gamma[\phi_c,\phi_\Delta]} = \int D\xi_1 P[\xi_1] \int D\xi_2 P[\xi_2] \, e^{i\left(\mathrm{Re}\,\Gamma[\phi_c,\phi_\Delta] + \int d^4x [\phi_\Delta(x)\phi_c(x)\xi_1(x) + \phi_\Delta(x)\xi_2(x)]\right)} \, , \tag{3.22}$$

where $P[\xi_1]$ and $P[\xi_2]$ are Gaussian distribution functionals with zero mean, i.e.

$$\langle(\ldots)\rangle_{\xi_i} \equiv \int D\xi_i P[\xi_i](\ldots) = 0 \quad , \quad \langle \xi_i(x) \rangle_{\xi_i} = 0 \qquad (i = 1, 2) \quad . \tag{3.23}$$

[5]It can be shown [71] that, in the limit $t_0 \to -\infty$, this time path is actually equivalent to the one in figure 3.2, since correlation functions involving fields evaluated on the vertical imaginary segment vanishes due the Riemann-Lebesgue lemma.

This trick leads to the introduction of two source terms in the CTP effective action, i.e.

$$\Gamma[\phi_c,\phi_\Delta] \;\longrightarrow\; \Gamma'[\phi_c,\phi_\Delta,\xi_i] = \mathrm{Re}\,\Gamma[\phi_c,\phi_\Delta] + \int d^4x[\phi_\Delta(x)\phi_c(x)\xi_1(x) + \phi_\Delta(x)\xi_2(x)]\,. \tag{3.24}$$

When the variational principle (3.18) is applied to Γ', the real part gives rise to the l.h.s. of (3.21), while the source terms give rise to the stochastic terms $\phi\,\xi_1$ and ξ_2 on the r.h.s..
Stochastic non-linear and non-local differential equations such as (3.21) are impossible to solve analytically, and they are also notoriously difficult to tackle numerically. Consequently, one usually try to express the second term on the l.h.s. of (3.21) in an approximate local form, where the concept of localization depends on the particular length and time scales at play.
The spatial non-locality is handled by arguing that the inflaton field is presumably nearly homogeneous during inflation, i.e. its Fourier expansion is dominated by the large wavelength (or small wavenumber) oscillating modes, so that one can approximate by considering only the zero external momentum contribution to the loop diagrams. This removes the emergence of additional spatial gradient terms from the local description.
The treatment of the temporal non-locality is instead more involved. In ref. [74] appropriate conditions are identified that allow the approximation of the temporal non-local effects by local terms, and such that a local approximation for the equation of motion is in very good agreement with the full numerical solution of the non-local equation.
It emerges that the temporal localization requires the existence of a separation of timescales in the system, provided by the assumption of a near LTE evolution. In this case we can reasonably state that the background field changes *adiabatically*, namely its macroscopic motion is much slower than the microphysical dynamics. In other words, we can write the nth power of the inflaton field as

$$\bar\phi^n(t',\mathbf{x}') \simeq \bar\phi^n(t,\mathbf{x}') + n(t'-t)\bar\phi^{n-1}(t,\mathbf{x}')\,\dot{\bar\phi}(t,\mathbf{x}') \quad , \tag{3.25}$$

if we consider time intervals $(t'-t)$ of order of the timescale τ set by the microscopic degrees of freedom, since we are assuming $\tau\dot{\bar\phi} \ll \bar\phi$.
Therefore, inserting (3.25) in the integrand of equation (3.21), we obtain local terms that are corrections to the mass and the interaction vertex contributing to the effective potential, but also dissipative local terms proportional to $\dot{\bar\phi}$.
The result is a *Langevin-like stochastic differential equation* which, redefining for simplicity of notation $\bar\phi \equiv \phi$, reads [59]:

$$\ddot\phi(x) - \nabla^2\phi(x) + \gamma(\phi,T)\dot\phi(x) + \frac{\partial V_{eff}(\phi,T)}{\partial\phi} = \phi(x)\xi_1(x) + \xi_2(x) \quad . \tag{3.26}$$

Besides the correction V_{eff} to the potential V_0, the thermal environment effectively backreacts on the system through a local and deterministic *friction force* $\gamma\dot\phi$ and some stochastic *noise forces*, a multiplicative (field dependent) one $\phi\,\xi_1$, and an additive one ξ_2. These "forces" are physically interpreted as the action of the thermal noise due to the multiple interactions with the environment, which induce random inhomogeneities in the solution $\phi(t,\mathbf{x})$ of equation (3.26) even if ϕ had been initially prepared to be homogeneous. The source of stochastic evolution can be removed by taking the ensemble average $\langle\ \rangle_\xi$ of (3.26) over the noise fields. Therefore we split the complete solution $\phi(t,\mathbf{x})$ in the zero mode (homogeneous) configuration $\phi_0(t) \equiv \langle\phi(t,\mathbf{x})\rangle_\xi$, solution of the deterministic l.h.s. of (3.26), plus small noise-induced random thermal fluctuations $\delta\phi(t,\mathbf{x})$ around the deterministic trajectory. Then we write

$$\phi(t,\mathbf{x}) = \phi_0(t) + \delta\phi(t,\mathbf{x}) \quad , \quad \big\langle\delta\phi(t,\mathbf{x})\big\rangle_\xi = 0 \quad . \tag{3.27}$$

The quasi-equilibrium condition also allows to establish a *fluctuation-dissipation relation* between the amplitude of the fluctuations of the random noise fields ξ_1 and ξ_2 and the dissipation coefficient γ. In [59] it is shown that, in a weakly interacting model ($\lambda \ll 1$), the dominant contribution to γ comes from the multiplicative noise field ξ_1, since the contribution from ξ_2 results to be higher order in λ. In the high temperature regime the fluctuation-dissipation relation for the noise field ξ_1 reads [59]

$$\lim_{T\to\infty} \left\langle \xi_1(t,\mathbf{x})\xi_1(t',\mathbf{x}')\right\rangle_{\xi_1} = 2\,\gamma\, T\,\delta(t-t')\delta(\mathbf{x}-\mathbf{x}') \quad . \tag{3.28}$$

Actually, the noise field is generally colored, i.e. its correlation function is time dependent, with correlation time given by the relaxation time τ_{rad} of the heat bath fields generating the noise. However, as the interaction rate of radiation scales with T, in the limit of very high temperatures the noise becomes white, namely time uncorrelated, because the microscopic interactions with the radiation bath are virtually instantaneous ($\tau_{rad} \longrightarrow 0$) compared to the macroscopic motion of the classical field ϕ.

The *dissipation coefficient* γ quantifies the strength of the dissipative process, by describing the rate at which the scalar field ϕ transfers its energy to the thermal bath. Microscopically, it is generally related to the relaxation time (the inverse of the decay width) of the particles directly coupled with the inflaton field. Accordingly, we can distinguish *direct decay models* of warm inflation, in which the field ϕ directly decays into light radiation fields, from *indirect decay models*, in which the inflaton is coupled to a set of heavy bosonic and fermionic "catalyst" fields X whose mass m_X can even be larger than the temperature of the universe. These catalyst fields in turns couple with the bath fields, so that the decay of the inflaton into the radiation fields is mediated by a virtual particle channel. Therefore, γ can be related either to the relaxation time of radiation [62] or to the relaxation time of intermediate massive particles[6] [76].
The simpler direct decay models are particularly difficult to realize: in the high temperature regime, the direct coupling of the inflaton with thermally excited fields leads to harmful large thermal corrections to the effective potential which could make the model unstable, since the friction coefficient cannot counteract the increase in the steepness of the potential. These models require a highly fine tuned model building to work around this issue [63]. In this regard, indirect decay models proved to be a more consistent realization of warm inflation. Indeed, in this context, a possible solution to the problem is given by a two stage decay mechanism based on supersymmetry (SUSY) [51]: the key point is that the heavy X fields are basically in their ground state (they are not thermally excited), so the loop corrections to the inflaton potential are only of quantum origin, and they can be controlled by SUSY[7]. Nevertheless, also these kind of models presents a technical difficulty, since a large multiplicity of mediator fields is required to achieve an efficient energy transfer capable of sustaining the thermal bath for a sufficient number of e-folds [77, 78], and, although technically consistent, this would mean that warm inflation can be realized only in special scenarios [79].
The explicit temperature and field amplitude dependence $\gamma(\phi, T)$ is established by the model content. In particular, if we consider some specific indirect decay models treated in [80], the dissipation coefficient γ was found to have the following generic dependence on the inflaton field amplitude ϕ, temperature T and on the mass m_X of the X fields coupled to the inflaton [72, 80]

$$\gamma(\phi, T) = C_\phi \frac{T^c \phi^{2a}}{m_X^{2b}} \quad , \quad c + 2a - 2b = 1 \tag{3.29}$$

[6] In this case the decay width is related through the optical theorem [75] to the imaginary part of the self-energy diagram of the considered particles. The processes contributing to the decay width clearly involve out of shell particle states, otherwise the conservation of energy would be violated.

[7] Even if SUSY is explicitly broken, the corrections to the potential are proportional to the mass difference of the supersymmetric partners, and therefore they are smaller.

where C_ϕ is a dimensionless constant that carries the details of the microscopic model used to derive the dissipation coefficient, such as the different coupling constants and the multiplicity of the catalyst fields of the model.
For example, for what concerns the temperature dependence, at low temperature regimes (the latter understood as $T < m_X$) we have $c = 3$, while at high temperature regimes ($T > m_X$) we have $c = -1$.

The CTP formalism as well as the results quoted so far can be extended to curved spacetime [58, 65, 81]. It follows that, for a spatially flat expanding FLRW metric, the effective stochastic equation of motion for the thermal averaged field is given by

$$\ddot{\phi}(x) - a^{-2}(t)\nabla^2\phi(x) + [3H + \gamma(\phi, T)]\dot{\phi}(x) + \frac{\partial V_{eff}(\phi, T)}{\partial \phi} = \xi(x) \quad , \tag{3.30}$$

where the scale factor and the Hubble friction term arise due to the coupling of the field ϕ with the expanding background metric. For simplicity, we will restrict ourselves to an additive Gaussian white noise $\xi(x)$ approximation. The generalization of the high temperature fluctuation dissipation theorem (3.28) to an expanding universe is

$$\left\langle \xi(t, \mathbf{x})\xi(t', \mathbf{x}')\right\rangle_\xi = a^{-3}\left(2T\gamma_{eff}\right)\delta(t - t')\delta(\mathbf{x} - \mathbf{x}') \quad , \tag{3.31}$$

where now $(t, \mathbf{x})$ denote comoving cosmological coordinates, and we also account for effects of the expansion on the noise field through the coefficient $\gamma_{eff} \equiv \gamma + 3H$ [8].
Having reviewed the microscopic origin of the phenomenological equation (3.30), in the next section we proceed to its solution. However, we want to stress the fact that, even if the CTP formalism can be applied in principle in situations far from equilibrium, the local Langevin-like effective equation of motion (3.30) is only adequate to study the approach to equilibrium of the field ϕ in a *thermalized*, *adiabatic* and *perturbative* regime that can be realized in a close-to-thermal equilibrium situation.
Therefore, before moving on, we want to close this section with a summary of the *consistency conditions* [58, 83] which must be satisfied by any QFT based microscopic model of warm inflation in order to validate the assumptions leading to the derivation of equation (3.30), as well as of the other macroscopic equations we will encounter in the next sections, that describe the phenomenology of this inflationary scenario.
The **adiabatic-condition** requires that *all* the time scales of the microphysical dynamics determining the dissipative and noise effects must be faster than *all* the time scales associated to the macroscopic evolution of the system. In warm inflation there are two macroscopic time scales, provided by the Hubble expansion rate and the rate of change of the background inflaton field $\phi_0(t)$, so we can distinguish the $\boldsymbol{\phi_0}$ **adiabatic condition**

$$\Gamma_i^{-1} \ll \frac{\phi_0}{\dot{\phi}_0} \quad , \tag{3.32}$$

and the **thermal adiabatic condition**

$$\Gamma_i^{-1} \ll H^{-1} \quad , \tag{3.33}$$

where Γ_i denotes, for a given model, the decay rates of the fields responsible for the dissipative motion of ϕ, including also the bath fields in case of models in which radiation does not directly

[8] A more accurate estimate of γ_{eff} can be found in [82], obtained through a matching procedure in the sub-horizon limit. However, corrections on the given value of γ_{eff} have little effects on the final power spectrum generated by the thermal fluctuations.

interact with the inflaton field. In a cosmological setting the inequality (3.33) states that the microscopic dynamic timescales are much faster than the expansion time scale. It automatically implies the condition $\Gamma_{rad} \gg H$, which guarantees instantaneous thermalization of the relativistic particles produced via inflaton decay. This legitimizes the assumption, used in the entire discussion above, of a well defined temperature parameter T, that can be used to describe the state of radiation in an expanding universe. By further requiring that $\Gamma_{rad} \gg \dot{T}/T$, radiation can be manteined in a close to thermal equilibrium state. Moreover, since at high temperature Γ_{rad} is set by T [84] according to $\Gamma_{rad} \sim \alpha T$, for some model dependent coefficient α, the thermalization condition also implies (for $\alpha \sim \mathrm{O}(1)$)

$$T \gg H \quad . \tag{3.34}$$

We will see that this last condition actually represents the dividing point between cold and warm inflation, since it defines the regime in which the thermally induced fluctuations of ϕ will dominate over the vacuum quantum fluctuations amplified by the inflationary expansion.
Another very important bound on T is set by requiring that the temperature must be sufficiently high to make significant the thermal fluctuations of the fields, otherwise the fields excitations would be suppressed by Boltzmann exponential factors, as well as the dissipation coefficient [80]. Then we require

$$T \gg m_i(T) \quad , \tag{3.35}$$

with $m_i(T)$ the finite temperature effective masses of the particle fields.
Finally, the **infrared condition** states that the Compton wavelength of all the particle excitations of the model must be much smaller than the Hubble radius H^{-1} during inflation, i.e.

$$m_i^{-1}(T) \ll H^{-1} \quad . \tag{3.36}$$

which automatically implies (3.34) via the use of (3.35). Combined with the thermal-adiabatic condition, the infrared condition allows to perform approximate computation within a flat spacetime framework, ignoring the effects of the expansion on microphysics processes.

The Axion-like warm inflation model

An inflationary scenario driven by axion-like particles was initially suggested in 1990 [85], as *natural inflation*. An axion is a pseudo Nambu-Goldstone boson arising from the spontaneous symmetry breaking of a $U(1)$ global symmetry that is anomalous under a given gauge group. As a Nambu-Goldstone boson, the axion enjoys a shift symmetry which protects its potential from both large quantum and thermal corrections, since it requires that the interaction terms in the action involve only derivatives of the field. However, the shift symmetry is explicitly broken at the quantum level, and this anomaly leads to the appearance in the low-energy effective action for the axion of a non-renormalizable coupling with the gauge fields A^a_μ, that reproduces the variation of the action of the associated ultraviolet theory under an anomalous $U(1)$ transformation. The axion-gauge field coupling is given by

$$\mathcal{L}_{int} = \frac{\alpha}{4f}\phi\, \tilde{G}^{a\mu\nu} G_{a\mu\nu} \quad , \tag{3.37}$$

where $\tilde{G}^{a\mu\nu}$ is the dual gauge field strength, $\tilde{G}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^a_{\rho\sigma}$, $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, with g and f^{abc}, respectively, the coupling and the structure constant of the Yang-Mills gauge group. α is a dimensionless constant, while f is a constant with the dimension of a mass that is related to the SSB energy scale.
The explicit symmetry breaking generates a periodic effective potential for the axion of the form

$V(\phi) = V_0\Big[1-\cos\big(\phi/f\big)\Big]$ [86], which provides a mass term proportional to f^{-1} when expanded about the vacuum configuration.
These symmetry properties makes the axion the simplest spin-zero degree of freedom with a non-trivial radiatively stable potential, and hence a well motivated candidate for the inflaton field. Another interesting feature of natural inflation models that makes them very attractive in the context of slow-roll inflation, is that, if we consider an abelian gauge group $U(1)$, the axion-like coupling (3.37) generates an amplification of the gauge fields modes of one of the helicities by a factor $\propto \exp\{\pi\xi\}$, with $\xi = \frac{\alpha\dot{\phi}_0}{2fH}$, which leads to a strong particle production at the expense of the kinetic energy of the inflaton [87]. Due to the exponential dependence of the amplification, for large enough values of the parameter ξ, the gauge field production can strongly backreact on the evolution of the homogeneous inflaton field ϕ_0 through a term $\frac{\alpha}{4f}\tilde{G}^{a\mu\nu}G_{a\mu\nu}$ on the r.h.s. of its equation of motion that slows down the rolling of ϕ_0, so that the slow-roll regime can be realized in these models even for steep potentials [87, 88]. Moreover, the large amount of radiation quanta produced during inflation may largely enhance the gauge fields interaction rates, which could be able to overcome the exponential dilution, thus, naturally leading to thermalization and formation of a hot plasma [89]. Therefore, a warm inflation dynamics might emerge naturally in these types of models given appropriate parameters [89–92].

3.2 Background dynamics

In this section we see how the phenomenological picture of the warm inflationary scenario emerges from the study of the large scale behaviour of the inflaton-bath system, namely by the dynamical evolution equations of the homogeneous thermal averaged inflaton field $\phi_0(t)$ and the thermal radiation component on the expanding FLRW background spacetime. The arguments reported here are mainly based on refs. [45, 82, 93, 94].

From (3.30) we have that the equation of motion for the homogeneous component of the inflaton field is

$$\ddot{\phi}_0(t) + [3H + \gamma(\phi_0, T)]\dot{\phi}_0(x) + V_{,\phi}(\phi_0, T) = 0 \quad , \tag{3.38}$$

where we have redefined $V_{eff}(\phi_0, T) \equiv V(\phi_0, T)$. As mentioned, the effective finite temperature potential must be regarded as the Helmholtz free-energy density of the "inflaton + bath" system. For an effective number $g_*(T)$ of relativistic bath fields coupled to ϕ, the thermodynamic potential at temperature $T > H, m_\phi$ admits the following general expression[9] including both quantum and finite temperature corrections [95, 96]

$$V(\phi_0, T) = -\frac{\pi^2}{90}g_*(T)T^4 + \frac{1}{2}\delta m^2(\phi_0, T)\phi_0^2 + V_0(\phi_0) \quad . \tag{3.39}$$

The first term is minus the pressure exerted by the thermal radiation of relativistic fields. We can realize this by noticing that the factor $\pi^2T^4/90$ associated to each relativistic degree of freedom is exactly half the pressure of a black body radiation, since the electromagnetic radiation carries two independent polarization states. $V_0(\phi_0)$ refers to the inflaton effective potential at zero temperature, and the factor $\delta m^2(\phi_0, T)$ account for thermal corrections to the inflaton potential.
Therefore, in this new scenario, the cosmic fluid is a mixture of mutually interacting radiation and

[9] Actually, this expression for the effective potential is valid within thermal QFT on flat Minkowski spacetime. However, the set of consistency conditions listed at the end of the previous section allow to use it also in a FLRW background spacetime.

scalar field, whose total energy density and pressure are

$$\rho(\phi_0, T) = \frac{1}{2}\dot{\phi}_0^2 + U(\phi_0, T) \quad , \tag{3.40}$$

$$P(\phi_0, T) = \frac{1}{2}\dot{\phi}_0^2 - V(\phi_0, T) \quad , \tag{3.41}$$

with U the internal energy density of the whole system. The latter is related to the free-energy density (3.39) by the thermodynamic relation $U = V + Ts$, where s is the entropy density, quite dominated by the radiation component, $s = s_r$, and given by

$$s_r = -V_{,T} \quad . \tag{3.42}$$

where $_{,T}$ denotes the derivative w.r.t. the temperature.
The zero curvature Friedmann equation (1.20) relates the expansion rate H to the total energy density ρ

$$3H^2 = 8\pi G\rho \quad . \tag{3.43}$$

The last relevant equation is the one describing entropy density production due to the energy transfer from the inflaton field to the thermal bath. It is given by the $\nu = 0$ component of the continuity equation (1.13) for the total energy-momentum tensor,

$$\dot{\rho} + 3H(\rho + P) = 0 \quad , \tag{3.44}$$

which, making use of (3.40) and (3.41), becomes

$$\dot{\phi}_0\ddot{\phi}_0 + V_{,\phi}\,\dot{\phi}_0 + V_{,T}\,\dot{T} + \dot{T}s_r + T\dot{s}_r + 3H(\dot{\phi}_0^2 + Ts_r) = 0 \quad . \tag{3.45}$$

Then, using (3.38) and (3.42) we obtain

$$T(\dot{s}_r + 3Hs_r) = \gamma\dot{\phi}_0^2 \quad . \tag{3.46}$$

In general, the scalar field and radiation contributions to the total energy density ρ cannot be unambiguously separated. However, viable models of warm inflation need a mechanism, such as supersymmetry, that suppresses huge high temperature thermal corrections to the potential [97], which could prevent the slow-roll realization of the inflationary expansion. If we neglect the thermal corrections $\delta m^2(\phi_0, T)$ in the effective potential, we obtain a relationship between entropy and radiation energy density

$$s_r = -V_{,T} = \frac{2\pi^2}{45}g_*T^3 \quad \Longrightarrow \quad \rho_r = \frac{\pi^2}{30}g_*T^4 = \frac{3}{4}sT \quad , \tag{3.47}$$

where we have taken for granted that all the relativistic species are at thermal equilibrium at temperature T (i.e. that $g_* = g_{*s}$). As a consequence, the field and temperature dependence of the total energy density (3.40) can be disentangled

$$\rho(\phi_0, T) = \frac{1}{2}\dot{\phi}_0^2 + V_0(\phi_0) + \frac{\pi^2}{30}g_*T^4 = \rho_\phi + \rho_r \quad . \tag{3.48}$$

Only in this case, the time evolution is described by the following system of equations

$$\begin{cases} \ddot{\phi}_0 + 3H(1+Q)\dot{\phi}_0 + V_{,\phi} = 0 \quad , \\ 3H^2 = 8\pi G(\rho_\phi + \rho_r) \quad , \\ \dot{\rho}_r = -4H\rho_r + \gamma\dot{\phi}_0^2 \quad , \end{cases} \tag{3.49}$$

where we used the expressions in (3.47) to rewrite (3.46) in terms of ρ_r. We have also introduced a parameter Q describing the effectiveness of thermal dissipation relatively to the expansion damping, defined as

$$Q \equiv \frac{\gamma}{3H} \quad , \tag{3.50}$$

which allows to identify two different regimes of warm inflation, namely

- $Q \leq 1\,(\gamma \leq 3H)$, we are in the *weak dissipation regime*: dissipation is not strong enough to affect the background inflaton field evolution, but the thermal fluctuations of the radiation energy density are still able to significantly affect the inflaton field fluctuations and the spectrum of primordial perturbations;

- $Q \gg 1\,(\gamma \gg 3H)$, we are in the *strong dissipation regime*: dissipation dominates both the background dynamics and the fluctuations.

Given that also the ratio Q evolves during inflation, we may have models where the dissipation regimes alternate.
Notice how the above dynamical system reduces to the standard evolution equations (2.7) of the cold scenario in the limit $Q \ll 1$ in which the interactions of the inflaton field are ineffective.
It is now more evident from the last equation of (3.49) that the amount of radiation during inflation is determined by two competing effects: the first term on the r.h.s. is a sink term arising from the Hubble expansion which depletes the radiation away, while the second one acts like a source term feeding radiation via dissipation of the inflaton field energy.
Similarly, by rewriting the first of (3.49) in terms of the inflaton energy density ρ_ϕ we have

$$\dot{\rho}_\phi = -(3H + \gamma)\dot{\phi}_0^2 \quad , \tag{3.51}$$

which implies that ρ_ϕ is a monotonic decreasing function, because of the combined effect of Hubble redshift and decay.

We can assume that warm inflation is preceded by a RD era, and it starts when the approximately constant vacuum energy of the inflaton field equals the decreasing radiation energy density falling as $\rho_r \sim a^{-4}$. In a first moment the radiation component is redshifted away by the quasi-exponential expansion, but it is generally assumed that, when the inflaton decay becomes effective, the rate of radiation production through dissipation is sufficient to balance its depletion due to the expansion, so that the radiation energy density reaches a nearly steady state regime, meaning that

$$\frac{|\dot{\rho}_r|}{\rho_r} \ll H \quad , \tag{3.52}$$

which fulfills the mentioned consistency condition $\dot{T}/T \ll H \ll \Gamma_{rad}$.
The universe stops inflating and it smoothly enters a RD era at the moment when the quasi-stable radiation component exceeds the falling inflaton field energy.

The realization of a warm inflationary phase yielding a nearly scale invariant power spectrum requires the slow-roll conditions

$$V(\phi_0) \gg \dot{\phi}_0^2/2 + \rho_r \qquad , \qquad |\ddot{\phi}_0| \ll (3H + \gamma)|\dot{\phi}_0| \tag{3.53}$$

in combination with the additional strong dissipation condition[10]

$$Q \gg 1 \quad . \tag{3.54}$$

Within the slow-roll and strong dissipation regime the system of equations (3.49) is approximated by

$$\begin{cases} 3HQ\dot{\phi}_0 + V_{,\phi} \simeq 0 \quad , \\ 3H^2 \simeq 8\pi G V \quad , \\ 4H\rho_r \simeq \gamma\dot{\phi}_0^2 \quad , \end{cases} \tag{3.55}$$

where we have dropped the highest derivative terms. The last equation yields the quasi-stationary value reached by radiation energy density

$$\rho_r \simeq \frac{\gamma\dot{\phi}_0^2}{4H} = \frac{3Q}{2}\left(\frac{1}{2}\dot{\phi}_0^2\right) \quad , \tag{3.56}$$

which shows that the parameter Q must be large as supposed in order to obtain a substantial radiation component, since the kinetic energy is very small during the slow-roll regime of inflation.

Also in this case the validity of the slow-roll approximation can be cast as conditions on the size of a set of slow-roll parameters. If the thermal corrections are subdominant, as assumed up to now, we can neglect the temperature dependence of the inflaton potential V and the dissipation coefficient γ, and the set of slow-roll parameters is given by

$$\epsilon_V \equiv \frac{1}{16\pi G}\left(\frac{V_{,\phi}}{V}\right)^2 \quad , \quad \eta_V \equiv \frac{1}{8\pi G}\frac{V_{,\phi\phi}}{V} \quad , \quad \beta \equiv \frac{1}{8\pi G}\frac{\gamma_{,\phi}\,V_{,\phi}}{\gamma V} \quad , \tag{3.57}$$

where β is a new slow-roll parameter which constrains the field dependence of V and γ.
The slow-roll regime requires $\epsilon_V \ll Q$, $|\eta_V| \ll Q$, $|\beta| \ll Q$. In order to check this conditions let us compute the ϵ and η parameters (2.10) in the warm inflationary regime.
Using the second of (3.55) and its time derivative we have

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{1}{Q}\frac{1}{16\pi G}\left(\frac{V_{,\phi}}{V}\right)^2 = \frac{\epsilon_V}{Q} \ll 1 \quad . \tag{3.58}$$

To compute the η parameter we differentiate the first of (3.55), obtaining

$$\ddot{\phi}_0 \simeq \frac{V_{,\phi\phi}\,\dot{\phi}_0}{3HQ} - \frac{\gamma_{,\phi}\,V_{,\phi}\,\dot{\phi}_0}{9H^2Q^2} \quad , \tag{3.59}$$

which plugged in the definition of η, using the second of (3.55), provides

$$\eta \equiv -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} \simeq -\frac{1}{Q}(\eta_V - \beta) \ll 1 \quad . \tag{3.60}$$

Similarly, computing the relative variation of radiation energy density within a Hubble time we have

$$\frac{1}{H}\frac{\dot{\rho}_r}{\rho_r} \simeq -\frac{1}{Q}(2\eta_V - \epsilon_V - \beta) \ll 1 \quad . \tag{3.61}$$

[10]Actually, a warm inflation scenario is achieved when the thermal fluctuations significantly alter the spectrum of primordial perturbations, hence also in the weak dissipation regime, but the most interesting features arise for $Q \gg 1$, so we restrict to this case.

From (3.58), (3.60) and (3.61), the aforementioned conditions on ϵ_V, η_V and β follow.
As anticipated, in a strong dissipation regime ($Q \gg 1$) the flatness conditions on the inflaton potential are relaxed, so that the slow-roll regime can be accomplished even for steeper potentials. In the case we do not neglect the temperature dependence of V and γ, we have to introduce two additional slow-roll parameters

$$b = \frac{TV_{,\phi T}}{V_{,\phi}} \quad , \quad c = \frac{T\,\gamma_{,T}}{\gamma} \quad , \tag{3.62}$$

which quantify thermal contributions to, respectively, the effective potential and the dissipation coefficient. A stability analysis [97] of the dynamical system (3.49) shows that a stable and sufficiently prolonged attractor solution can only exist if

$$0 < b \ll 1 \quad , \quad |c| < 4 \quad . \tag{3.63}$$

As previously argued, the condition on b is necessary to suppress dangerous thermal corrections to the potential. Instead, the physical meaning of the condition on c is evident from the last equation of (3.49): the radiation energy density will reach a stable equilibrium point if radiation is produced with a rate $\gamma \propto T^c$ equating or exceeding the one at which it is depleted by the expansion, given by $4H\rho_r \propto T^4$; then we cannot obtain a stationary radiation energy density if γ falls as T^4 or faster. As seen in the previous section, calculation of the dissipation coefficient for different models yield values of $c = -1$ and $c = 3$ which are consistent with the condition imposed on c.
From (3.56), using the first and the second equation of (3.49), we can find a relation between the radiation energy density and inflaton energy density in terms of the slow-roll parameter ϵ_V

$$\rho_r \simeq \frac{3Q}{2}\left(\frac{1}{2}\dot{\phi}_0^2\right) \simeq \frac{1}{12Q}\left(\frac{V_{,\phi}}{H}\right)^2 \simeq \frac{\epsilon_V}{2Q}\rho_\phi \quad , \tag{3.64}$$

which shows that $\rho_r \ll \rho_\phi$ in the slow-roll regime, since $\epsilon \ll 1$, thus guaranteeing a period of accelerated expansion. This relation may not hold at the start and the end of the warm inflationary phase, as during this periods the steady-state condition on the radiation energy density is violated. However, it was verified by numerical modelling that (3.64) holds very well during most of the inflationary phase [93]. The same relation also establishes a connection between temperature and inflaton potential

$$T^4 \simeq \frac{\epsilon_V V}{2\alpha Q} \quad , \quad \alpha \equiv \frac{\pi^2 g_*}{30} \quad . \tag{3.65}$$

Recalling the meaning of ϵ, from (3.58) we have that warm inflation ends when $\epsilon_V \simeq Q$, which, looking at (3.64), is equivalent to say that $\rho_r \simeq \rho_\phi$.

3.3 Thermal fluctuations evolution

In this section we explore the behaviour of thermal fluctuations during warm inflation, and we see how they constitute the primary source of primordial density perturbations. Our analysis will closely follow refs. [82, 94].

The evolution of the inflaton field thermal fluctuations modes goes through three different regimes, depending on the relative importance between thermal noise, expansion and curvature perturbation. The transition between these regimes occurs at two instants, identified with the *freeze-out instant* t_F and the horizon crossing instant.

In cold inflation, these two moments coincide, since we recall that, given a quantum fluctuation mode of comoving wavenumber k, its amplitude is essentially frozen at the instant t of horizon crossing when $k = a(t)H(t)$, or ,equivalently, at the instant $t = t_F$ when the decreasing physical wavenumber $k_{ph}(t) = k/a(t)$ falls below the freeze-out wavenumber scale $k_F = H$. Analogously, also the inflaton thermal fluctuations in the warm scenario present such a freeze-out scale, but it is quite different from the horizon crossing scale.
Initially, as assumed, the thermal noise effects dominates over the expansion, so that the inflaton field is able to reach the thermal equilibrium with the radiation component. As the universe expands, the interactions with the thermal environment become less efficient, until the moment t_F at which the thermal noise starts to have an irrelevant effect on the development of the inflaton fluctuations, meaning that for $t > t_F$ the inflaton field and the radiation component are no more considered to be in thermal equilibrium, and the evolution of the inflaton fluctuations becomes increasingly deterministic.
We will see shortly that the warm inflation freeze-out scale k_F is always larger than H, meaning that the freeze-out time of a mode always precedes its horizon crossing instant. Whilst the small scale inflaton fluctuations are freezing out, the metric fluctuations resulting by the perturbations in the total energy density remain relatively small. In fact, at this stage, they can be safely neglected in the Langevine equation (3.30) with a suitable gauge choice [98, 99]. Eventually, once the wavelength of the perturbations exceed the cosmological horizon, the metric perturbations become important and we end up with the generation of a large scale curvature perturbation. Furthermore, differently from the cold scenario, the large scale perturbations generated are already classical on creation, since they are induced by classical thermal fluctuations. Thus, the warm scenario has no quantum-to-classical transition problem [35].

3.3.1 The origin of the *freeze-out* length scale and the perturbed system

Following [83], we can estimate the freeze-out wavenumber k_F by focusing on the stage when the fluctuation modes evolve on a dynamic timescale sufficiently fast for the mode to thermalize, such that we can ignore the effects of the expansion. In other words, we consider physical wavenumbers $k_{ph}(t)$ sufficiently large that the evolution of $k_{ph}(t)$, $H(t)$, and $\phi_0(t)$ is adiabatic relative to the evolution of $\delta\phi_{\mathbf{k}}(t)$ within a Hubble time interval $\Delta t = H^{-1}$. In this regime the flat spacetime equation of motion for the inflaton field (3.26) is approximately valid and the evolution equation for the modes $\delta\phi_{\mathbf{k}}(t)$ is obtained by substitution of the inflaton field splitting (3.27) in equation (3.26) and taking the Fourier transform. Retaining only the terms linear in the fluctuations we have

$$\delta\ddot{\phi}_{\mathbf{k}} + \gamma\delta\dot{\phi}_{\mathbf{k}} + \left(V_{,\phi\phi} + k_{ph}^2\right)\delta\phi_{\mathbf{k}} = \xi_{\mathbf{k}} \quad , \tag{3.66}$$

with correlation function for the Fourier transform of the noise field $\xi_{\mathbf{k}}$ given by

$$\left\langle \xi_{\mathbf{k}}(t))\xi_{-\mathbf{k}'}(t')\right\rangle_{\xi} = 2\gamma T\,\delta(t-t')\delta(\mathbf{k}-\mathbf{k}') \quad . \tag{3.67}$$

On the considered time scale we can ignore the time variation of H, k_{ph}, γ and $V_{,\phi\phi}$, and we can fix the value of these quantities at an intermediate instant during the considered time interval. In this case, (3.66) coincides with the equation of motion of a damped harmonic oscillator immersed in a fluid and in presence of an external stochastic driving force.
We are interested in the overdamped regime in which we can neglect the second order time derivative of $\delta\phi$, realized when $\gamma > (V_{,\phi\phi} + k_{ph}^2)^{1/2}$. Equation (3.66) is then approximated by

$$\gamma\delta\dot{\phi}_{\mathbf{k}} + \left(V_{,\phi\phi} + k_{ph}^2\right)\delta\phi_{\mathbf{k}} = \xi_{\mathbf{k}} \quad . \tag{3.68}$$

The above equation has the same form of the velocity equation for a free Brownian particle in a fluid, whose solution, given the initial condition $\delta\phi_{\mathbf{k}}(t_*)$ at some initial time t_*, reads

$$\delta\phi_{\mathbf{k}}(t) \approx \frac{1}{\gamma} e^{-(t-t_*)/\tau} \int_{t_*}^{t} dt' e^{(t'-t_*)/\tau} \xi_{\mathbf{k}}(t') + \delta\phi_{\mathbf{k}}(t_*) e^{-(t-t_*)/\tau} \quad , \quad \tau \equiv \frac{\gamma}{V_{,\phi\phi} + k_{ph}^2} , \tag{3.69}$$

where the first term on the r.h.s. is the stochastic contribution from the noise which acts to thermalize the fluctuation, while the second term contains the memory of the initial condition at time $t = t_*$. The stochastic average of the solution is given by

$$\langle \delta\phi_{\mathbf{k}}(t) \rangle_{\xi} = \delta\phi_{\mathbf{k}}(t_*) e^{-(t-t_*)/\tau} \quad . \tag{3.70}$$

Using (3.67) and $\langle \xi_{\mathbf{k}} \rangle = 0$, the two point correlation function in Fourier space is

$$\langle \delta\phi_{\mathbf{k}}(t) \delta\phi_{-\mathbf{k}'}(t) \rangle_{\xi} \approx \delta^3(\mathbf{k}-\mathbf{k}') \frac{T}{V_{,\phi\phi} + k_{ph}^2} \left(1 - e^{-2(t-t_*)/\tau} \right) + \langle \delta\phi_{\mathbf{k}}(t_*) \delta\phi_{-\mathbf{k}'}(t_*) \rangle_{\xi} \, e^{-2(t-t_*)/\tau} \quad . \tag{3.71}$$

We see that the averaged fluctuation may relax to the equilibrium value $\langle \delta\phi_{\mathbf{k}}(t) \rangle_{\xi} = 0$ within a timescale fixed by the relaxation time interval τ, once the exponential decaying terms becomes negligible. However, due to the presence of thermal noise, the average of the squared fluctuations approaches a non-vanishing value related to the equilibrium temperature T.
Therefore, the fluctuation mode can thermalize within a Hubble time, as required, only if its physical wavenumber k_{ph} is sufficiently large such that

$$\tau < t_H \;\longleftrightarrow\; \frac{\gamma}{V_{,\phi\phi} + k_{ph}^2} < H^{-1} \;\Longrightarrow\; \frac{V_{,\phi\phi} + k_{ph}^2}{\gamma H} > 1 \quad . \tag{3.72}$$

In a strong dissipative regime, the slow-roll condition $\eta_V \ll Q$ implies $V_{\phi\phi} \ll \gamma H$, then the inequality (3.72) holds approximately for physical wavenumbers k_{ph} that satisfy

$$k_{ph} > k_F \simeq \sqrt{\gamma H} \quad . \tag{3.73}$$

Notice that

$$k_F = \sqrt{\gamma H} = \sqrt{3Q} H \gg H \quad , \tag{3.74}$$

meaning that, as anticipated, the freeze-out time well precedes the moment of horizon crossing at $k_{ph} = H$.
Once the physical wavenumber of a fluctuation mode drops below k_F, the relaxation time becomes bigger and bigger than the Hubble time, so the averaged fluctuation remains essentially frozen at the value $\langle \delta\phi_{\mathbf{k}}(t_F) \rangle$ assumed at the freeze-out time. Actually the freeze-out scale k_F acquires a weak time dependence from the Hubble parameter, but in practise we can treat it as constant, since H varies very little during inflation.

A more accurate calculation of the freeze-out scale k_F and of the fluctuations amplitude can be performed by explicitly solving the set of cosmological perturbation equations of the inflaton-radiation system on an expanding background.
In the context of warm inflation there are cosmological perturbations in the inflaton field, the radiation and the gravitational field, and in general they are intimately coupled.
From equation (3.30), by perturbing at first order, we obtain the equation of motion for the inflaton fluctuations $\delta\phi$

$$\ddot{\delta\phi} - a^{-2}\nabla^2 \delta\phi + (3H + \gamma)\, \dot{\delta\phi} + (\delta\gamma)\dot{\phi}_0 + \delta V_{,\phi} = \xi \quad , \tag{3.75}$$

where $\delta\gamma$ and $\delta V_{,\phi}$ are the perturbations of the dissipation coefficient and of the derivative of the effective potential.
The inhomogeneous energy-momentum flux from the inflaton field to the thermal bath gives rise to fluctuation $\delta\rho_r$ in the energy density of radiation. The evolution equation for $\delta\rho_r$ can be obtained by perturbing a covariant energy-momentum balance equation written in terms of a total energy-momentum tensor [82, 100, 101]. We have

$$\delta\ddot{\rho}_r + 9H\delta\dot{\rho}_r + \left(20H^2 + \frac{1}{3}k^2a^{-2}\right)\delta\rho_r = -k^2a^{-2}J + 5H\delta E + \delta\dot{E} \quad , \tag{3.76}$$

where δE and J are, respectively, the perturbed fluxes of energy and momentum, given by [82]

$$\delta E = \dot{\phi}^2\delta\phi \quad , \quad J = -\gamma\dot{\phi}\delta\phi \quad . \tag{3.77}$$

The inflaton fluctuations will also generate metric inhomogeneities which enter the evolution equations (3.75) and (3.76) [94, 99, 101]. However, in linear perturbation theory, if we put ourselves in the uniform expansion rate gauge ($\delta\theta = 0$) the metric perturbations entering the evolution equations can be discarded on sub-horizon scale, $k \gg aH$, when we work at the leading order in the slow-roll approximation [98, 99]. Therefore, equations (3.75) and (3.76) apply to intermediate length scales that lie between the microscopic scales of the thermal bath and the cosmological horizon scale.
In the most general case in which both γ and V are functions of the field amplitude and the temperature we have

$$\delta\gamma = \gamma_{,\phi}\,\delta\phi + \gamma_{,T}\,\delta T \quad , \tag{3.78}$$

$$\delta V_{,\phi} = V_{,\phi\phi}\,\delta\phi + V_{,\phi T}\,\delta T \quad . \tag{3.79}$$

The δT terms couple the perturbations of the inflaton and the energy density of radiation since

$$\frac{\delta\rho_r}{\rho_r} = 4\frac{\delta T}{T} \quad . \tag{3.80}$$

Using the first and the third of the slow-roll equations (3.55), these contributions can be written in terms of the parameters b and c as

$$V_{,\phi T}\,\delta T = bV_{,\phi}\,\frac{\delta T}{T} \simeq -b\frac{H}{\dot{\phi}_0}\delta\rho_r \quad , \tag{3.81}$$

$$\gamma_{,T}\,\delta T = c\frac{H\delta\rho_r}{\dot{\phi}_0}\frac{\dot{\phi}_0^2\gamma}{4H\rho_r} \simeq c\frac{H}{\dot{\phi}_0}\delta\rho_r \quad . \tag{3.82}$$

Since we require $b \ll 1$ the $V_{,\phi T}\,\delta T$ term can be neglected, while the $\gamma_{,T}\,\delta T$ term is leading order in the slow-roll approximation and it should be considered in the analysis of the coupled system.

Furthermore, unlike the cold scenario, entropic (isocurvature) perturbations are also important in warm inflation. They can arise, for example, due to thermal fluctuations in the radiation or through particle production induced by the interactions between the cosmic fluid components, that can lead to fluctuations in the relative number densities of different species at fixed total energy density. The generation of isocurvature perturbations is very model dependent, and it can affect the evolution of the curvature perturbation, giving an additional contribution to the power spectrum of the primordial scalar perturbations. Indeed, the possible presence of non-adiabatic pressure perturbations δP_{iso} may provide a non trivial evolution of the curvature perturbation ζ (and $\mathcal{R}$)

on super-horizon scales, since it is no longer frozen as in single field cold inflationary models. However, a numerical integration of the linearly coupled perturbation equations of the inflaton-radiation system has been performed by the authors of [94], including also metric and entropic perturbations. This analysis demonstrated that, under certain model conditions and within the slow-roll approximation, the system rapidly converges to a solution with a single free parameter as the length scales exceed the horizon; in particular, it emerges that only one perturbation mode, the adiabatic curvature perturbation, survives on super-horizon scales and it approaches a constant value, while the entropic perturbation vanishes.
Even if the isocurvature perturbations decay on large scales, they can still influence the evolution of the curvature perturbation on sub-horizon scales, with an even more significant impact if we consider a temperature dependent dissipation coefficient [94]. Nonetheless, we follow assume a regime (as studied in [94]) in which the isocurvature modes are negligible, relaying on the fact that, according to the constraints on inflation set by the Plank measurements [5], isocurvature perturbations, if they exist, must provide a subdominant contribution to the scalar power spectrum. Given the above considerations, we can entirely describe the scalar perturbations through the linear curvature perturbation on uniform energy density hypersurfaces

$$\zeta = -\hat{\Psi} - H\frac{\delta\rho}{\dot{\rho}} \simeq -\hat{\Psi} - \frac{H}{\dot{\phi}}\delta\phi = \mathcal{R} \quad , \tag{3.83}$$

where we used that $\rho \simeq V$ during inflation.
If we consider the spatially flat gauge ($\hat{\Psi} = 0$) we obtain

$$\zeta \simeq -\frac{H}{\dot{\phi}}\delta\phi \quad \Longrightarrow \quad P_{\zeta}(k) = \frac{H^2}{\dot{\phi}^2}P_{\phi}(k) \quad . \tag{3.84}$$

In the spatially flat gauge the inflaton field fluctuation coincides with the gauge invariant Mukhanov-Sasaki variable (2.115). Evaluating this quantity in the uniform expansion rate gauge we can find a relationship between the inflaton field perturbation $\delta\phi$ in the two gauges:

$$\mathcal{Q}_{\hat{\Psi}=0} = \mathcal{Q}_{\delta\theta=0} \quad \Longrightarrow \quad \delta\phi_{\hat{\Psi}=0} = \delta\phi_{\delta\theta=0} + \frac{\dot{\phi}}{H}\hat{\Psi}_{\delta\theta=0} \quad . \tag{3.85}$$

The linearly perturbed Einstein equations provide the relation [99]

$$-\frac{k^2}{a^2}\hat{\Psi} - \delta\theta H = 4\pi G\delta\rho \quad , \tag{3.86}$$

which evaluated in the uniform expansion rate gauge yields

$$\hat{\Psi}_{\delta\theta=0} = -\frac{4\pi Ga^2}{k^2}\delta\rho_{\delta\theta=0} \simeq -\frac{4\pi Ga^2}{k^2}V_{,\phi}\delta\phi_{\delta\theta=0} \quad . \tag{3.87}$$

Therefore, in the slow-roll approximation, from (3.85) we obtain

$$\delta\phi_{\hat{\Psi}=0} \simeq \delta\phi_{\delta\theta=0} - \frac{\dot{\phi}}{H}\frac{4\pi Ga^2}{k^2}V_{,\phi}\delta\phi_{\delta\theta=0} \simeq \delta\phi_{\delta\theta=0} + 3\frac{a^2H^2}{k^2}\frac{\epsilon_V}{Q}\delta\phi_{\delta\theta=0} \quad , \tag{3.88}$$

which means that, to leading order in slow-roll, the inflaton fluctuation in the uniform expansion rate gauge and in the spatially flat gauge coincide. Then, we can use the leading order solution of equation (3.75) to compute the power spectrum (3.84).

3.3.2 Uncoupled fluctuations

Here we restrict to the case in which the friction coefficient does not depend on temperature, i.e.

$$\gamma \equiv \gamma(\phi) \quad \longleftrightarrow \quad c = 0 \quad . \tag{3.89}$$

This implies that the perturbations of the inflaton and the radiation do not couple, since the leading order coupling term (3.82) does not appear in the inflaton perturbation equation. Then, we can only focus on the evolution of the inflaton field fluctuations, which will determine the primordial energy density perturbations.
From (3.75), by taking the Fourier transform we obtain

$$\delta\ddot{\phi}_{\mathbf{k}} + (3H+\gamma)\delta\dot{\phi}_{\mathbf{k}} + \left(\gamma_{,\phi}\,\dot{\phi}_0 + V_{,\phi\phi} + \frac{k^2}{a^2}\right)\delta\phi_{\mathbf{k}} = \xi_{\mathbf{k}} \quad . \tag{3.90}$$

The correlation function for the noise field in the k-space is

$$\left\langle \xi_{\mathbf{k}}(t))\xi_{-\mathbf{k}'}(t')\right\rangle_{\xi} = a^{-3}\,(2\gamma_{eff}T)\,\delta(t-t')\delta(\mathbf{k}-\mathbf{k}') \quad . \tag{3.91}$$

Introducing the time coordinate $z \equiv \frac{k}{a(t)H(t)}$ and using the slow-roll parameters ϵ, η_V and β, we are led to

$$(1-\epsilon)^2\delta\phi''_{\mathbf{k}} + \left[\epsilon'(\epsilon-1) - (3Q+2)\frac{(1-\epsilon)}{z}\right]\delta\phi'_{\mathbf{k}} + \left[1 + \frac{3Q}{z^2}\left(\frac{\eta_V}{Q} - \frac{\beta}{1+Q}\right)\right]\delta\phi_{\mathbf{k}} = (2\gamma_{eff}T)^{1/2}(1-\epsilon)^{1/2}\hat{\xi}_{\mathbf{k}}\,, \tag{3.92}$$

where prime denotes the derivative with respect to z. Moreover we have rescaled the noise field ξ as

$$\xi_{\mathbf{k}}(t) \quad \longrightarrow \quad \hat{\xi}_{\mathbf{k}}(z) \equiv \frac{a^2}{k^2}(2T\gamma_{eff})^{-1/2}\,(1-\epsilon)^{-1/2}\xi_{\mathbf{k}} \tag{3.93}$$

so that, using the following relation

$$\delta(z-z') = \frac{a}{k(1-\epsilon)}\delta(t-t') \quad , \tag{3.94}$$

the correlation function for the noise field $\hat{\xi}_{\mathbf{k}}(z)$ reads

$$\langle \hat{\xi}_{\mathbf{k}}(z))\hat{\xi}_{-\mathbf{k}'}(z')\rangle_{\xi} = k^{-3}\,\delta(z-z')\delta(\mathbf{k}-\mathbf{k}') \quad . \tag{3.95}$$

At zero order in the slow-roll approximation, the fluctuation equation of motion becomes

$$\delta\phi''_{\mathbf{k}} - (3Q+2)z^{-1}\delta\phi'_{\mathbf{k}} + \delta\phi_{\mathbf{k}} = (2\gamma_{eff}T)^{1/2}\hat{\xi}_{\mathbf{k}} \quad . \tag{3.96}$$

To recast the above differential equation to a more familiar form we define the fluctuation field variable $\delta\tilde{\phi}_{\mathbf{k}} \equiv z^{-\nu}\delta\phi_{\mathbf{k}}$, where $\nu \equiv 3/2(Q+1)$. Under this change of variable, the previous differential equation becomes

$$z^2\delta\tilde{\phi}''_{\mathbf{k}} + z\delta\tilde{\phi}'_{\mathbf{k}} + (z^2-\nu^2)\delta\tilde{\phi}_{\mathbf{k}} = z^{2-\nu}(2\gamma_{eff}T)^{1/2}\hat{\xi}_{\mathbf{k}} \quad , \tag{3.97}$$

which is a inhomogeneous Bessel equation of order ν.
This equation can be solved via Green function method, which provides the following solution [82]

$$\delta\phi_{\mathbf{k}}(z) = \int_z^{\infty} dz' G(z,z')(z')^{1-2\nu}(2\gamma_{eff}T)^{1/2}\hat{\xi}_{\mathbf{k}}(z') \quad , \tag{3.98}$$

where the retarded Green function $G(z,z')$ can be expressed in terms of the Bessel functions of the first and second kind $J_\nu(z)$ and $Y_\nu(z)$ of order ν as (see Appendix B.1)

$$G(z,z') = -\frac{\pi}{2} z^\nu z'^\nu (J_\nu(z)Y_\nu(z') - J_\nu(z')Y_\nu(z)) \quad \text{for} \quad z' > z \quad . \tag{3.99}$$

Corrections due to the time dependence of ν are higher order in the slow-roll approximation.
Then, using the expression (3.95) for the correlation function of the noise field, the two point correlation function for the inflaton field thermal fluctuations in Fourier space reads

$$\left\langle \delta\phi_{\mathbf{k}}(z)\delta\phi_{-\mathbf{k}'}(z)\right\rangle_\xi = k^{-3}\int_z^\infty dz' G^2(z,z')(z')^{2-4\nu}(2\gamma_{eff}T)\,\delta^3(\mathbf{k}-\mathbf{k}') \quad , \tag{3.100}$$

from which we can identify the inflaton field power spectrum with

$$P_\phi(k,z) = k^{-3}\int_z^\infty dz' G^2(z,z')(z')^{2-4\nu}(2\gamma_{eff}(z')T(z')) \quad . \tag{3.101}$$

When the order ν is large ($Q \gg 1$), the above integral presents a saddle point at $z = z_F \equiv z(t_F)$ given by [82]

$$z_F = (3\nu)^{1/2} = \left(\frac{9}{2}(Q+1)\right)^{\frac{1}{2}} \simeq \left(3\,\frac{\gamma(z_F)}{2H(z_F)}\right)^{\frac{1}{2}} \quad \Longrightarrow \quad k_F \equiv \frac{k}{a(z_f)} \simeq \sqrt{\frac{3}{2}}(H(z_F)\gamma(z_F))^{1/2} \quad . \tag{3.102}$$

We identify the saddle point z_F (or t_F) with the freeze-out time of the fluctuation mode with comoving wavenumber k.
The presence of the saddle point allows to evaluate the $\gamma_{eff}T$ term in (3.101) at $z = z_F$ and to take it out of the integral. Then we rewrite (3.101) as ($\gamma_{eff} \simeq \gamma$ for $Q \gg 1$)

$$P_\phi(k,z) = k^{-3}(2\gamma(z_F)T(z_F))F(z) \quad , \quad F(z) \equiv \int_z^\infty dz' G^2(z,z')(z')^{2-4\nu} \quad . \tag{3.103}$$

The integral function $F(z)$ is examined in Appendix (C). For large values of ν and fixed z, this integral can be analytically approximated by equation (C.10), which inserted in the above expression for the inflaton field power spectrum yields

$$P_\phi(k,z) \approx \frac{\sqrt{\pi}}{2} k^{-3}\sqrt{\gamma H}\,T\left(1 + z^2\frac{H}{\gamma}\right) \quad . \tag{3.104}$$

Notice that at early times ($z \gg 1$) the amplitude of the inflaton field thermal fluctuations is very large. As the value of z decreases with time, the fluctuations amplitude decreases as well, and $P_\phi(k,z)$ approaches a nearly constant value $P_\phi(k)$ once z drops below the freeze-out time $z_F \sim (H/\gamma)^{1/2}$.
A heuristic derivation of the inflaton fluctuations amplitude, that is compatible with the above (more accurate) result, is obtained in [83] and showed in Appendix D.
From (3.84), we have that the curvature perturbation adimensional power spectrum generated by the inflaton thermal fluctuations at the moment $z = 1$ of horizon crossing is given by

$$\Delta_\zeta^{th}(k) = \frac{k^3}{2\pi^2}P_\zeta(k,1) \approx \frac{1}{4\pi^{3/2}}\frac{H^{5/2}\gamma^{1/2}\,T}{\dot{\phi}^2} \quad , \tag{3.105}$$

that, using the first and the second of (3.55), can be rewritten as

$$\Delta_\zeta^{th}(k) \approx \frac{\sqrt{3}}{4\pi^{3/2}}\frac{4\pi G}{\epsilon_V}Q^{5/2}H\,T \quad . \tag{3.106}$$

During warm inflation, both quantum and thermal fluctuations of the inflaton field are responsible for the generation of a scalar adiabatic curvature perturbation. In order to establish the relative importance of these two contributions we compare the result (3.106) with the scalar power spectrum (2.150) generated in cold inflation by the quantum fluctuations, that can also be expressed as

$$\Delta_\zeta^{qu}(k) \approx \frac{1}{4\pi^2}\frac{4\pi G}{\epsilon_V}H^2 \quad . \tag{3.107}$$

Their ratio is

$$\frac{\Delta_\zeta^{th}(k)}{\Delta_\zeta^{qu}(k)} \sim Q^{5/2}\frac{T}{H} \quad , \tag{3.108}$$

hence, in order for the thermal spectrum to dominate over the quantum spectrum of fluctuations it is required that, even in the weak realization $Q \leq 1$ of warm inflation[11], the condition $T > H$ must be satisfied.
The spectral index of the adiabatic scalar perturbations is

$$n_\zeta - 1 = \frac{d\ln P_\zeta}{d\ln k} \quad . \tag{3.109}$$

It can be expressed in terms of the slow-roll parameters by writing the logarithmic interval in comoving wavenumber as

$$d\ln k = d\ln a(t) = \frac{\dot{a}}{a}dt = H dt = \frac{H}{\dot{\phi}}d\phi \simeq -\frac{8\pi GVQ}{V_{,\phi}}d\phi \quad , \tag{3.110}$$

where we used the slow-roll equations (3.55) in the last equality. Therefore, we have

$$n_\zeta - 1 = -\frac{V_{,\phi}}{8\pi GVQ}\frac{d\ln}{d\phi}\left(\frac{H^{5/2}\gamma^{1/2}\,T}{\dot{\phi}^2}\right) \quad . \tag{3.111}$$

Expressing the temperature via equation (3.65) and using the slow-roll equations (3.55) we obtain

$$n_\zeta - 1 = \frac{1}{Q}\left(-\frac{9}{4}\epsilon_V + \frac{3}{2}\eta_V - \frac{9}{4}\beta\right) \quad , \tag{3.112}$$

to be compared with the result (2.134) provided by the cold scenario. Analogously, (3.112) shows a small deviation from a scale invariant power spectrum if the slow-roll conditions are satisfied, consistently with the experimental observations. The presence of the new slow-roll parameter β shows the dependence of the spectrum on the derivative of the dissipation coefficient.

3.4 Thermal spectrum of tensor perturbations

Following [102], in this section we estimate the thermal component of the power spectrum of tensor perturbations in warm inflation. This contribution is sourced by the transverse-traceless modes of the anisotropic stress tensor that, in this scenario, are developed by dissipative effects arising from thermal fluctuations and the interactions between the inflaton field and the radiation fluid. The final result is given in terms of the length scale ℓ_{mfp} denoting the the mean free path of the particles in the thermal bath. This is a model dependent quantity which captures the microphysical

[11] We remind that by weak realization we mean a regime in which the thermal bath provides a negligible backreaction on the evolution of the background inflaton, but it still controls the generation of its perturbations.

details, defined as $\ell_{mfp} \equiv (\sigma n)^{-1}$, where σ is the interaction cross section of the warm sector and n is the number density of radiation. The consistency requirement of radiation being in thermal equilibrium imposes the condition $\ell_{mfp}^{-1} \gg H$.

From the linearly perturbed Einstein equations, the tensor fluctuations equation in Fourier space reads ($\chi_{ij}^T \equiv \chi_{ij}$)

$$\ddot{\chi}_{ij} + 3H\dot{\chi}_{ij} + \frac{k^2}{a^2}\chi_{ij} = \frac{2}{a^2 M_{Pl}^2}\Pi_{ij}^{kl}\,\Sigma_{kl} \quad , \tag{3.113}$$

where $\Pi_{ij}^{kl}(\mathbf{k}) \equiv \Pi_i^k(\mathbf{k})\Pi_j^l(\mathbf{k}) - \frac{1}{2}\Pi_{ij}(\mathbf{k})\Pi^{kl}(\mathbf{k})$ is the projector on the transverse-traceless modes, with

$$\Pi_{ij} \equiv \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j \quad , \tag{3.114}$$

and Σ_{ij} is the anisotropic stress associated to the radiation bath.
Rewriting the above equation in conformal time we have

$$\chi_{ij}'' + 2\frac{a'}{a}\chi_{ij}' + k^2\chi_{ij} = \frac{2}{M_{Pl}^2}\Pi_{ij}^{kl}\,\Sigma_{kl} \quad . \tag{3.115}$$

By solving this equation through the Green function method we have

$$\chi_{ij}(\tau,\mathbf{k}) = \frac{2}{M_{Pl}^2}\int\frac{d\tau'}{a^2(\tau')}G_{\mathbf{k}}(\tau,\tau')\,\Pi_{ij}^{kl}(\mathbf{k})\int\frac{d^3\mathbf{x}}{(2\pi)^{3/2}}e^{-i\mathbf{k}\cdot\mathbf{x}}\Sigma_{kl}(\tau',\mathbf{x}) \quad , \tag{3.116}$$

where the $a^2(\tau)$ factor comes from a field redefinition in order to obtain a canonically normalized kinetic term in the lagrangian density, which brings $\Sigma_{\mu\nu} \longrightarrow \frac{1}{a^2}\Sigma_{\mu\nu}$.
The given solution is valid for any form of the scale factor. In the approximation of an exact de Sitter background metric (i.e. at leading order in slow-roll), the Green function is (see Appendix B.2)

$$G_{\mathbf{k}}(\tau,\tau') = \frac{1}{k^3\tau'^2}[(1+k^2\tau\tau')\sin\bigl(k(\tau-\tau')\bigr) - k(\tau-\tau')\cos\bigl(k(\tau-\tau')\bigr)]\theta(\tau-\tau') \quad . \tag{3.117}$$

Here we restrict the computation of the thermal power spectrum into the so called *hydrodynamic* regime, in which we consider comoving distances and conformal time intervals larger than the comoving mean free path of the thermal bath, since modes with short wavelength ($\lambda_{ph} < \ell_{mfp}$) provide a subdominant contribution [102].
In the mentioned regime the thermal bath can be treated as a classical and relativistic fluid, and statistical mechanical fluctuations in such a radiation fluid can be described using the Landau and Lifshitz hydrodynamics theory for near thermal equilibrium random fluids [100, 103], where stochastic source terms are added to the deterministic hydrodynamics equations. The microscopic physics become manifest in the form of dissipative terms like bulk and shear viscosities, that are related to the two-point correlation functions of the stochastic sources through fluctuation-dissipation relations. In particular we have [103, 104]

$$\langle\Sigma_{ij}(\tau,\mathbf{x})\Sigma_{kl}(\tau',\mathbf{x}')\rangle = 2T^{(c)}\left[\eta^{(c)}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) + \left(\zeta^{(c)} - \frac{2}{3}\eta^{(c)}\right)\delta_{ij}\delta_{kl}\right]\delta(\tau-\tau')\delta^3(\mathbf{x}-\mathbf{x}') \quad , \tag{3.118}$$

where $\langle\ldots\rangle$ denotes a stochastic ensemble average, while η and ζ are, respectively, the shear and bulk viscosity. The superscript index refers to comoving quantities, $T^{(c)} = aT$, $\eta^{(c)} = a^3\eta$, $\zeta^{(c)} = a^3\zeta$.

From (3.116), the two point correlation function reads

$$\langle \chi_{ij}(\tau,\mathbf{k})\chi_{ij}(\tau,\mathbf{k}')\rangle = \frac{4}{M_{Pl}^4}\int \frac{d\tau'}{a^2(\tau')}\int \frac{d\tau''}{a^2(\tau'')} G_{\mathbf{k}}(\tau,\tau')G_{\mathbf{k}'}(\tau,\tau'')\,\Pi_{ij}^{kl}(\mathbf{k})\Pi_{ij}^{mn}(\mathbf{k}') \times \int \frac{d^3\mathbf{x}d^3\mathbf{x}'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'}\,\langle \Sigma_{kl}(\tau',\mathbf{x})\Sigma_{mn}(\tau'',\mathbf{x}')\rangle \quad , \tag{3.119}$$

which, using (3.118), becomes

$$\langle \chi_{ij}(0,\mathbf{k})\chi_{ij}(0,\mathbf{k}')\rangle = \frac{48}{M_{Pl}^4}\int \frac{d\tau'}{a^4(\tau')} G_{\mathbf{k}}^2(0,\tau')T^{(c)}(\tau')\eta^{(c)}(\tau')\,\delta^3(\mathbf{k}+\mathbf{k}') \quad , \tag{3.120}$$

where we ignored the term in the square brackets of (3.118) proportional to $\delta_{ij}\delta_{kl}$, that is projected out by Π_{ij}^{kl}, and we used the relation $\Pi_{ik}^{kl}(\mathbf{k})\Pi_{ik}^{kl}(\mathbf{k}) = 3$. Furthermore, we have also set $\tau = 0$, since we consider the tensor spectrum evaluated at the end of inflation, at large scales $k \gg aH$.
Then, the thermal adimensional power spectrum of tensor perturbations reads

$$\Delta_T^{th}(k) = \frac{24k^3}{\pi^2 M_{pl}^4}\int \frac{d\tau'}{a^4(\tau')} G_{\mathbf{k}}^2(0,\tau')T^{(c)}(\tau')\eta^{(c)}(\tau') \quad . \tag{3.121}$$

The explicit expression of the shear viscosity η depends on the specific interactions within the thermal bath, but it is possible to find a lower and an upper bound on its value that induce corresponding bounds on the thermal spectrum:

- **Lower bound**
  A lower bound on the shear viscosity is given by [105]

  $$\eta^{(c)} \geq \frac{s^{(c)}}{4\pi} = \frac{\pi}{90} g_* T^3 a^3 \quad , \tag{3.122}$$

  with $s^{(c)}$ the comoving entropy density. Therefore we have

  $$\Delta_T^{th}(k) \geq \frac{4k^3}{15\pi M_{Pl}^4} g_* T^4 \int d\tau' G_k^2(0,\tau') \quad , \tag{3.123}$$

  where we assumed a constant temperature. Evaluation of the integral provides

  $$\int d\tau' G_k^2(0,\tau') = \frac{\pi}{6k^3} \qquad \Longrightarrow \qquad \Delta_T^{th}(k) \geq \frac{2}{45M_{Pl}^4} g_* T^4 = \frac{4}{32\pi^2}\frac{\rho_r}{M_{Pl}^4} \quad . \tag{3.124}$$

  The inflationary expansion requires that radiation must be a subdominant component in the energy density, $\rho_r \ll \rho_\phi \simeq 3H^2 M_{Pl}^2$, or $T \ll \sqrt{HM_{Pl}}$. Then, if the bound is saturated, we obtain

  $$\Delta_T^{th}(k) \ll \frac{4}{\pi^2}\frac{H^2}{M_{Pl}^2} = 2\Delta_T^q(k) \tag{3.125}$$

  i.e. the contribution arising from the amplification of quantum vacuum fluctuations dominates over the contribution arising from thermal fluctuations.

- **Upper bound**
  In [106] the shear viscosity is computed for a model where the thermal bath is represented by a weakly coupled real scalar field with negligible mass and a quartic self-interaction. The computation yields the approximate result

  $$\eta^{(c)} \approx \ell_{mfp}^{(c)}\, T^{(c)\,4}, \tag{3.126}$$

which inserted in (3.121), and neglecting the time dependence of ℓ_{mfp} and T, gives

$$\Delta_T^{th}(k) \approx \frac{24k^3 \ell_{mfp} T^5}{\pi^2 M_{Pl}^4} \int d\tau' G_k^2(0,\tau') = \frac{4}{\pi} \frac{\ell_{mfp} T^5}{M_{Pl}^4} \quad . \tag{3.127}$$

From the thermal equilibrium requirement, $\ell_{mfp} \ll H^{-1}$, we obtain the inequality .

$$\Delta_T^{th}(k) \ll \frac{4T^5}{\pi M_{Pl}^4 H} \quad . \tag{3.128}$$

Using $T \ll \sqrt{HM_{Pl}}$, the above inequality can be rewritten as

$$\Delta_T^{th}(k) \ll \frac{4}{\pi} \frac{H^2}{M_{Pl}^2} \left(\frac{M_{Pl}}{H}\right)^{1/2} = 2\pi\, \Delta_T^q(k) \left(\frac{M_{Pl}}{H}\right)^{1/2} \quad . \tag{3.129}$$

From the bound (1.57) on H provided by Planck measurements [5] we have

$$\Delta_T^{th}(k) \ll 4\pi \cdot 10^2\, \Delta_T^q(k) \sim 10^3\, \Delta_T^q(k) \quad , \tag{3.130}$$

which means that $\Delta_T^{th}(k)$ may exceed $\Delta_T^q(k)$ for relatively large value of the temperature of the thermal bath.

In the case in which the spectrum of tensor modes retains its dominant vacuum form $\Delta_T^q(k)$, the tensor spectral index in the strong dissipation regime is simply

$$n_T(k) = -2\frac{\epsilon_V}{Q} \quad . \tag{3.131}$$

Moreover, since the power spectrum of scalar perturbations is enhanced by thermal contribution, in warm inflation the tensor-to-scalar perturbation ratio r is suppressed compared to the value (2.151) obtained in standard cold inflation. Therefore we have the following modified consistency relation

$$r \simeq \frac{\Delta_T^q}{\Delta_T^{th}} \simeq \frac{\frac{2}{\pi^2}\frac{H^2}{M_{Pl}^2}}{\frac{\sqrt{3}}{8\pi^{3/2}}\frac{H\,T}{\epsilon_V M_{Pl}^2} Q^{5/2}} = -8n_T \frac{1}{\sqrt{3\pi}} \frac{H}{T} \frac{1}{Q^{3/2}} \quad , \tag{3.132}$$

that implies $r \ll 8|n_T|$ for $T \gg H$, even in the the weak dissipation regime $Q \leq 1$, and that translates in smaller inflationary energy scales in the warm scenario than in the cold one. As a consequence, this feature enlarges the class of inflationary potentials that are observationally consistent with the data. For example, within standard slow-roll inflation, monomial potential are ruled out by the constraints provided by the Plank data [5], since the predicted value for the tensor–to–scalar ratio is beyond the upper bound (2.152) established by the measurements. However, in virtue of the modified consistency relation (3.132), these kind of potentials can be rehabilitated in a warm infation scenario, as shown in [107, 108]. Furthermore, this modified relation may allow to distinguish between the two inflationary scenarios in a model independent way if primordial gravitational waves from inflation are found and their spectrum accurately measured.

Conclusions

In this thesis we reviewed the main aspects of what, up to now, are believed to be the two possible dynamical realizations of the inflationary paradigm: cold and warm inflation.
A significant difference lies in the background evolution and the phenomenology of the system and in the origin of the primordial cosmological perturbations.
In standard cold single-field inflation possible dissipative effects are neglected, therefore the accelerated expansion, triggered by the slow-rolling of the scalar inflaton field, takes place in a practically perfect vacuum state and the primordial perturbations are sourced by amplification of the coupled quantum vacuum fluctuations of the inflaton and the metric tensor. As the wavelength of the fluctuation modes is stretched to length scales exceeding the Hubble cosmological horizon, they are essentially frozen in. After horizon crossing it is commonly believed that the modes behave as classical stochastic variables, whose amplitude retains a constant value until horizon re-entry, when they set the initial conditions for the formation and the evolution of the large scale structures. The amplitude of scalar curvature and tensor metric perturbation at horizon crossing, $k = aH$, is predicted to be $\Delta_\zeta^q \sim \frac{H^2}{2\pi\dot{\phi}}$ and $\Delta_T^q \sim \frac{H^2}{M_{Pl}^2}$.
In warm inflation, instead, the accelerated expansion is still triggered by the dominant vacuum energy of a scalar inflaton field which slowly rolls down its potential, but the interactions with the ambient degrees of freedom are no longer ignored. This leads to a dissipative dynamics in which the vacuum energy of the inflaton is transferred to the environment by production of relativistic particles through a continuous decay of a fraction of the inflation field. This production must be strong enough to sustain a significant, but still subdominant, amount of radiation during the slow-roll regime, so that the whole system is now described by a density matrix corresponding to some excited quantum-statistical mixed state. If, as it is commonly assumed, the interaction rates of the decay products are much faster compared to the dynamics of the expanding background metric and the homogeneous inflaton field, a nearly thermal equilibrium condition can be achieved with the generation of a thermal heat bath. Therefore, considering an initial thermal distribution of states, the integration of the environment degrees of freedom through the use of the CTP Schwinger-Keldysh formalism yields an effective dynamics for the inflaton field which exhibits dissipative and fluctuating phenomena representing the backreaction of the produced radiation. These effects are described, respectively, by non-local terms and random noise fields appearing in the stochastic evolution equation for the thermal average of the inflaton field, and they are connected by a fluctuation-dissipation relation. Under the adiabaticity condition on the evolution of the inflaton, justified by the assumed regime of quasi-equilibrium, it is possible to perform a local approximation of the equation of motion which, aside from contributions to the thermal effective inflaton potential, provides an additional local friction term. The strength of the additional friction force is controlled by a dissipation coefficient γ, which generally is function of the inflaton field amplitude and the temperature. The presence of dissipation modifies the standard slow-roll conditions on the inflationary potential $\epsilon_V \ll 1, \eta_V \ll 1$, providing $\epsilon_V \ll Q, \eta_V \ll Q, \beta \ll Q$, where the parameter Q is the ratio between the dissipation coefficient and the Hubble parameter, $Q \equiv \gamma/3H$, while β is

an additional slow-roll parameter which controls the field dependence of the dissipation coefficient. Accounting also for the temperature dependence of the potential and the dissipation coefficient, necessary conditions to obtain a stable attractor solution for the background system of equations must be imposed, namely the suppression of thermal corrections to the potential and a sufficiently fast production rate of radiation. From a model building perspective, the most interesting realization of warm inflation is in the so called strong dissipation regime, when the dissipative coefficient dominates over the Hubble expansion damping ($Q \gg 1$). In fact, in this case, the background inflaton is strongly slowed down by the friction force originated by the particle production, so that slow-roll inflation is realized even for steep potentials.
Also the dynamics of the noise-induced inflaton thermal fluctuations is radically modified: the perturbation modes get frozen much before than the instant of horizon crossing, at the moment when their physical wavenumber drops below the freeze-out value $k_F \simeq \sqrt{\gamma H} \gg H$ such that the thermalization effects of the noise exerted by the thermal bath become negligible. Isocurvature perturbations are also generated in general in the warm scenario due to interactions between the components of the cosmic fluid. These may affect the evolution of the adiabatic curvature perturbation on super-horizon scale, but numerical simulations of the system of cosmological perturbations [94] show that this type of perturbations decay on large scales, so we can still consider the conservation of the curvature perturbation on length scales larger than the horizon valid. Moreover, it can be assumed that their influence on sub-horizon scales can also be neglected, since, according to the Planck data, the eventual contribution of entropic perturbations to the power spectrum is suppressed compared to the adiabatic one.
The power spectrum of adiabatic scalar curvature perturbations receives an additional contribution arising from the coupled inflaton and radiation thermal fluctuations. In the simplified case in which the dissipation coefficient does not depend on temperature, the system of thermal fluctuations decouples and it is possible to perform an analytical computation of the scalar power spectrum. We find a thermal contribution $\Delta_\zeta^{th} \sim \frac{H^{5/2}\gamma^{1/2}T}{\dot{\phi}^2}$, that dominates over the one coming from quantum vacuum fluctuations when the ambient temperature is greater than the Hubble parameter, $T > H$, independently on the considered dissipation regime, so that this last inequality can be seen as the defining condition of warm inflation.
Regarding the tensor metric perturbations, we have seen that the power spectrum of primordial gravitational waves generated during warm inflation includes a thermal component mainly sourced by the hydrodynamic thermal modes with wavelength larger than the mean free path l_{mfp} of the heat bath. The thermal contribution reads $\Delta_T^{th} \sim l_{mfp} \frac{T^5}{M_{Pl}^4}$, and it can dominate over the vacuum component for sufficiently high temperatures. If the tensor modes are unaffected by the coupling with the thermal bath, warm inflation predict a tensor-to-scalar perturbation ratio that is suppressed by at least a factor T/H compared to the one of the cold scenario.
The most important results are represented by equations (2.134) and (3.112) for the spectral index of scalar adiabatic perturbations, which tell us that both inflationary pictures predict a nearly scale invariant power spectrum for primordial scalar perturbations if conditions for the slow-roll regime are satisfied by a suitable inflaton effective potential, in excellent agreement with the measurement (2.135) provided by the Planck CMB data. Also the derived theoretical predictions (2.147) and (3.131) for the tensor spectral index indicate a scale invariant spectrum of stochastic primordial gravitational waves generated by both cold and warm inflation. However, no tensor modes resulting from an inflationary expansion have been detected yet.

Generally speaking, we can say that warm inflation appears to be a broader picture, since the extent of radiation production during inflation is variable, so that cold inflation emerges as the

limiting case of zero radiation production.
As in the context of warm inflation, gaps in our comprehension of dissipative quantum field theory in far out-of-equilibrium conditions also leave incompleteness in the post-inflationary reheating phase associated to the cold scenario. However, apart from modifying the inflaton field effective potential through quantum corrections, in the standard cold inflation picture the interactions and the resulting particle production are tacitly assumed to exert a negligible influence on the generation of the initial conditions for large scale structures formation, namely on the main observational predictions. Nevertheless, this assumption is not justified. On the other hand, even if the warm inflation picture makes no a priory assumption that particle production does not affect large scale structure formation, up to now, a realistic realization of this scenario requires the achievement of a close-to-thermal equilibrium regime that needs to be verified by explicit computation, but that generally is simply assumed in most of the works present in literature through the imposition of appropriate self-consistency conditions, whose validity may be very arduous or even impossible to achieve [47].
Ultimately we can summarize the advantages deriving from the warm scenario as follows,

- it appears to be a more general and natural realization of inflation (at least from a thermodynamic point of view), since no a priory assumption on the couplings of the inflaton is made, at the expense of the imposition of few consistency conditions deriving from gaps in our knowledge about strongly out-of-thermal equilibrium phenomena;
- model building is facilitated: the dissipative evolution widely enlarges the class of inflaton potentials which are able to sustain a slow-roll regime. Furthermore, compelling monomial potentials that in the cold scenario are ruled out by constraints imposed by the Planck data, can be rehabilitated thanks to the prediction (3.132) of a suppressed tensor-to-scalar ratio;
- warm inflation overcomes the quantum-classical transition problem [35], since the macroscopic dynamics of the background field and thermal fluctuations are classical from the onset;
- warm inflation smoothly terminates into a radiation dominated era, thus the additional reheating phase is not required.

Perhaps, an adequate understanding of particle production in quantum field theory may help us in the future to understand which or to what extent either of these two pictures is valid. Anyway, a broader perspective on the description of the early universe is required until clear experimental evidence strongly supports one inflationary scenario over the other, while also acknowledging the possibility that neither of the two pictures could be a faithful representation of the primordial stage of the universe before the well consolidated radiation dominated era of the Hot Big-Bang model.

Appendix A

Thermodynamic free energy

For a thermodynamic system at thermal equilibrium with Hamiltonian $\hat{\mathcal{H}}$, the free-energy F is defines as

$$F \equiv -\frac{1}{\beta} \ln Z_\beta \quad , \quad Z_\beta = \mathrm{Tr}[e^{-\beta\hat{\mathcal{H}}}]. \tag{A.1}$$

where Z_β is the thermal partition function.
In presence of an external source $J(x)$ the (non-normalized) thermal generating functional is

$$Z_\beta[J] = \mathrm{Tr}[e^{i\int d^4x J(x)\hat{\phi}(x)} e^{-\beta\hat{\mathcal{H}}}] = e^{iW_\beta[J]} = e^{i(\Gamma[\bar{\phi}_J] + \int d^4x J(x)\bar{\phi}_J(x))} \quad , \tag{A.2}$$

where the last equality follows from the definition of $\Gamma[\bar{\phi}_J]$, and the time integration is now taken, as discussed in section 3.1, on the compact interval $[0, -i\beta]$. In general $\Gamma[\bar{\phi}_J]$ admits the derivative expansion [50]

$$\Gamma[\bar{\phi}_J] = \int d^4x \left[\frac{1}{2} Z_\phi (\partial_\mu \bar{\phi}_J)^2 - V_{eff}(\bar{\phi}_J) + \text{terms containing } p \geq 4 \text{ derivatives}\right] \quad , \tag{A.3}$$

where Z_ϕ is the wavefunction renormalization factor.
For constant $\bar{\phi}_J$, which in turns implies constant $J(x) = J$, if we denote with Ω the volume of the system, the expression (A.2) becomes

$$Z_\beta[J] = e^{-\beta\Omega(V_{eff}(\bar{\phi}_J) - J\bar{\phi}_J)} \xrightarrow{J\to 0} Z_\beta = e^{-\beta\Omega V_{eff}(\bar{\phi})} \quad \Longrightarrow \quad F = \Omega V_{eff} \quad . \tag{A.4}$$

For a canonical statistical ensemble, the free-energy F is called *Helmholtz* free-energy, usually denoted with A. In terms of thermodynamic variables it is defined as

$$A \equiv U - TS \quad , \tag{A.5}$$

where U and S are the internal energy and the entropy of the system, respectively. These quantities, together with the temperature T, the pressure P and volume Ω of the system are also related through the Euler relation $U = TS - P\Omega$. The comparison of the two thermodynamic relations gives $A = -P\Omega$, which simply states that the pressure of the system is minus of the Helmholtz free energy density.

Appendix B

Green functions

B.1 Inflaton field fluctuations

We must solve the inhomogeneous Bessel function (3.97) subject to the initial condition $\delta\tilde{\phi}_{\mathbf{k}}(z_i) = \delta\tilde{\phi}'_{\mathbf{k}}(z_i) = 0$ at the initial time $z_i \longrightarrow \infty$.
Using the Green function method, the solution for equation (3.97) can be expressed as

$$\delta\tilde{\phi}_{\mathbf{k}}(z) = \int_0^\infty dz' \tilde{G}(z,z')(z')^{2-\nu}(2\gamma_{eff}T)^{1/2}\xi_{\mathbf{k}} \quad , \tag{B.1}$$

where $\tilde{G}(z,z')$ is the Green function for the problem, i.e. the so called *fundamental solution* of the differential operator

$$\mathcal{L}_z \equiv z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2-\nu^2) \quad , \tag{B.2}$$

that satisfies

$$\mathcal{L}_z\tilde{G}(z,z') = \delta(z-z') \quad . \tag{B.3}$$

Being G'' proportional to the Dirac delta, the first derivative of the fundamental solution presents a jump discontinuity in $z = z'$. Then, $G(z,z')$ is solution of the associated homogeneous Bessel equation for $z \neq z'$, which is continuous in $z = z'$ and such that

$$\tilde{G}'(z'_+,z') - \tilde{G}'(z'_-,z') = p(z') \quad . \tag{B.4}$$

The function $p(z')$ is found by integrating (B.3) over the interval $[z'-\epsilon, z'+\epsilon]$, and then taking the limit $\epsilon \longrightarrow 0$, that yields

$$p(z') = \frac{1}{z'^2} \quad . \tag{B.5}$$

The homogeneous equation has the two linearly independent solutions $\delta\tilde{\phi}^{(1)}_{\mathbf{k}}(z) = J_\nu(z)$ and $\delta\tilde{\phi}^{(2)}_{\mathbf{k}}(z) = Y_\nu(z)$. The generic solution can be written as

$$\tilde{G}(z,z') = \begin{cases} C_1(z')J_\nu(z) + C_2(z')Y_\nu(z) & 0 < z < z' \\ D_1(z')J_\nu(z) + D_2(z')Y_\nu(z) & z' < z < \infty \quad . \end{cases} \tag{B.6}$$

Imposing the boundary conditions at $z = z_i$ we have $D_1(z') = D_2(z') = 0$, thus $\tilde{G}(z,z') = 0$ for $z' < z$. Imposing the continuity of the Green function and the discontinuity of its first derivative at $z = z'$ we have

$$\begin{cases} C_1(z')J_\nu(z') + C_2(z')Y_\nu(z') = 0 \\ -C_1(z')J'_\nu(z') - C_2(z')Y'_\nu(z') = 1/z'^2 \quad . \end{cases} \tag{B.7}$$

Solving the first equation for C_2 and substituting in the second equation we have

$$C_2(z') = -C_1(z')\frac{J_\nu(z')}{Y_\nu(z')} \quad \Longrightarrow \quad -C_1(z')[Y_\nu(z')J'_\nu(z') - J_\nu(z')Y'_\nu(z')] = \frac{Y_\nu(z')}{z'^2} \quad . \tag{B.8}$$

Using the relation [34]

$$Y_\nu(z)J'_\nu(z) - J_\nu(z)Y'_\nu(z) = \frac{2}{z\pi} \quad , \tag{B.9}$$

we obtain

$$C_1(z') = -\frac{\pi}{2z'}Y_\nu(z') \quad , \quad C_2(z') = \frac{\pi}{2z'}J_\nu(z') \quad . \tag{B.10}$$

Therefore, the Green function reads

$$\tilde{G}(z,z') = -\frac{\pi}{2z'}[J_\nu(z)Y_\nu(z') - Y_\nu(z)J_\nu(z')]\theta(z'-z) \quad , \tag{B.11}$$

which plugged in (B.1) yields the solution

$$\delta\tilde{\phi}_{\mathbf{k}}(z) = -\frac{\pi}{2}\int_z^\infty dz'[J_\nu(z)Y_\nu(z') - Y_\nu(z)J_\nu(z')](z')^{1-\nu}(2\gamma_{eff}T)^{1/2}\xi_{\mathbf{k}} \quad . \tag{B.12}$$

B.2 Tensor metric perturbations

Introducing the rescaled variable $h_{ij} \equiv a\chi_{ij}$, the equation of motion (3.115) for the metric tensor perturbations assumes the form,

$$h''_{ij} + \left(k^2 - \frac{2}{\tau^2}\right)h_{ij} = \frac{2a}{M_{Pl}^2}\Pi_{ij}^{kl}\,T_{kl} \quad , \tag{B.13}$$

where we used $a''/a \simeq 2/\tau^2$ at leading order in the slow-roll approximation.
We look for a fundamental solution $G(\tau,\tau')$ of the differential operator

$$\mathcal{L}_\tau \equiv \frac{d^2}{d\tau^2} + k^2 - \frac{2}{\tau^2} \quad , \tag{B.14}$$

written in the form

$$G(\tau,\tau') = \tilde{G}(\tau,\tau')\theta(\tau-\tau') \quad , \tag{B.15}$$

such that

$$\mathcal{L}_\tau\tilde{G}(\tau,\tau') = 0 \quad , \quad \tilde{G}(\tau,\tau) = 0 \quad , \quad \left.\frac{d\tilde{G}(\tau,\tau')}{d\tau}\right|_{\tau=\tau'} = 1 \quad . \tag{B.16}$$

In fact, one can verify that, using the above conditions, we have $\mathcal{L}_\tau G(\tau,\tau') = \delta(\tau-\tau')$.
The function $\tilde{G}(\tau,\tau')$ is given by a linear combination of two linearly independent solutions $F_{1,2}(\tau)$ of the homogeneous equation associated to (B.13)

$$\left(\frac{d^2}{d\tau^2} + k^2 - \frac{2}{\tau^2}\right)F_i(\tau) = 0 \quad , \quad i = 1,2 \quad . \tag{B.17}$$

We choose the solution F_1 that approaches the positive frequency adiabatic mode (2.36) at early times

$$\lim_{\tau\longrightarrow-\infty} F_1(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{B.18}$$

and we can also set $F_2 = F_1^*$, since they are linearly independent.
Such solutions are given by

$$F_1(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) \quad , \quad F_2(\tau) = F_1^*(\tau) \quad , \tag{B.19}$$

and a linear combination that satisfies the conditions (B.16) is

$$\tilde{G}(\tau, \tau') = \frac{F_1(\tau)F_2(\tau') - F_2(\tau)F_1(\tau')}{F_1'(\tau')F_2(\tau') - F_2'(\tau')F_1(\tau')} \quad . \tag{B.20}$$

Therefore we have

$$G(\tau, \tau') = \frac{1}{k^3\tau\tau'}[(1 + k^2\tau\tau')\sin\bigl(k(\tau - \tau')\bigr) - k(\tau - \tau')\cos\bigl(k(\tau - \tau')\bigr)]\,\theta(\tau - \tau') \quad . \tag{B.21}$$

The Green function $\bar{G}(\tau - \tau')$ for the original tensor perturbation χ_{ij} is given by

$$\bar{G}(\tau - \tau') = \frac{a(\tau')}{a(\tau)} G(\tau, \tau') = \frac{\tau}{\tau'} G(\tau, \tau') \quad , \tag{B.22}$$

that can be showed to satisfy

$$\left(\frac{d^2}{d\tau^2} + 2\frac{a'}{a}\frac{d}{d\tau} + k^2\right) \bar{G}(\tau, \tau') = \delta(\tau - \tau') \quad . \tag{B.23}$$

So, finally we have

$$\bar{G}(\tau, \tau') = \frac{1}{k^3\tau'^2}[(1 + k^2\tau\tau')\sin\bigl(k(\tau - \tau')\bigr) - k(\tau - \tau')\cos\bigl(k(\tau - \tau')\bigr)]\,\theta(\tau - \tau') \quad . \tag{B.24}$$

Appendix C

Integrals

Substituting the expression (3.99) of the Green function in the definition of $F(z)$ we have

$$F(z) = \frac{\pi^2}{4}\int_z^\infty dz'(J_\nu(z)Y_\nu(z') - J_\nu(z')Y_\nu(z))^2 z^{2\nu}(z')^{2-2\nu} \quad . \tag{C.1}$$

For large ν and fixed z the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ have the asymptotic forms [34]

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ez}{2\nu}\right)^\nu \quad , \quad Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}}\left(\frac{ez}{2\nu}\right)^{-\nu} \quad , \tag{C.2}$$

through which we can write

$$J_\nu(z)Y_\nu(z') \sim \frac{1}{\nu}\left(\frac{z}{z'}\right)^\nu \quad , \quad J_\nu(z')Y_\nu(z) \sim \frac{1}{\nu}\left(\frac{z'}{z}\right)^\nu \quad . \tag{C.3}$$

Given that $z' > z$, for $\nu \gg 1$ the combination $J_\nu(z)Y_\nu(z')$ in (C.1) is subdominant, so the leading contribution is

$$F(z) \approx \frac{\pi^2}{4} z^{2\nu} Y_\nu^2(z) \int_0^\infty dz' J_\nu^2(z')(z')^{2-2\nu} \quad . \tag{C.4}$$

where we also extended the integration interval to $[0, \infty)$, since the small z contributions to the integral are negligible.
We can approximate the Bessel function $Y_\nu(z)$ with the following ascending series [34]

$$Y_\nu(z) \overset{\nu \gg 1}{\simeq} -\frac{1}{\pi}\left(\frac{2}{z}\right)^\nu \sum_{k=0}^{\nu-1} \frac{(\nu-k-1)!}{k!}\left(\frac{z^2}{4}\right)^k \approx -\frac{2^\nu}{\pi z^\nu}\Gamma(\nu)\left(1 + \frac{z^2}{4\nu} + \ldots\right) \quad , \tag{C.5}$$

where $\Gamma(\nu)$ is the Gamma function, and we used $\Gamma(\nu) = (\nu - 1)!$. It follows that

$$z^{2\nu} Y_\nu^2(z) \overset{\nu \gg 1}{\approx} \frac{2^{2\nu}}{\pi^2}\Gamma^2(\nu)\left(1 + \frac{z^2}{2\nu} + \ldots\right) \quad . \tag{C.6}$$

The integral in (C.4) belongs the following class of standard integrals [34]

$$\int_0^\infty dz \frac{J_\mu(az)\, J_\nu(az)}{z^\lambda} = \frac{(\frac{1}{2}a)^{\lambda-1}\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\lambda + \frac{1}{2}\right)\Gamma(\lambda)}{2\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)} . \tag{C.7}$$

In our case we have $a = 1$ and $\lambda = 2\nu - 2$, which implies

$$\int_0^\infty dz' J_\nu^2(z')(z')^{2-2\nu} = \frac{2^{1-2\nu}\sqrt{\pi}\,\Gamma\left(2\nu-2\right)}{\Gamma\left(\nu-\frac{1}{2}\right)^2\Gamma\left(2\nu-\frac{1}{2}\right)} \quad . \tag{C.8}$$

Moreover, using the asymptotic approximation $\Gamma(x+\alpha) \sim \Gamma(x)x^\alpha$ for $x \gg 1$, we obtain

$$\int_0^\infty dz' J_\nu^2(z')(z')^{2-2\nu} \overset{\nu\gg1}{\approx} \frac{2^{-2\nu}}{\Gamma(\nu)^2}\sqrt{\frac{\pi}{2\nu}} \quad . \tag{C.9}$$

Plugging the approximate estimates (C.6) and (C.9) in (C.4) we have

$$F(z) \approx \frac{\pi^2}{4} z^{2\nu} Y_\nu^2(z) \int_0^\infty dz' J_\nu^2(z')(z')^{2-2\nu} \approx \sqrt{\frac{\pi}{32\nu}}\left(1 + \frac{z^2}{2\nu} + \ldots\right) \quad . \tag{C.10}$$

Appendix D

Heuristic derivation of the freeze-out fluctuation amplitude

Assuming a non trivial distribution of inflaton particles n_k, which for sufficiently fast interactions should approach the Bose-Einstein distribution at the ambient temperature T, $n_k = (e^{\beta\omega_k} - 1)^{-1}$, the square of the fluctuation amplitude at the moment of freeze-out can be computed through the formula for the variance of a bosonic field within equilibrium thermal QFT on flat Minkowski spacetime [96, 109], i.e.

$$\langle\delta\phi^2\rangle_\beta = \frac{1}{(2\pi)^3}\int_{k_F-shell} d^3k\,\frac{n_k}{\omega_k} = \frac{1}{2\pi^2}\int_{k_F-shell} dk\frac{k^2}{\sqrt{k^2+m_\phi^2}\left(e^{\beta\sqrt{k^2+m_\phi^2}}-1\right)} \quad , \tag{D.1}$$

where we retained only the contribution from wavenumbers within the k_F-shell $k_F e^{-1/2} < k < k_F e^{1/2}$.
Assuming $T \gg k_F \gg m_\phi^2$, the above integral yields the approximate result

$$\langle\delta\phi^2\rangle_\beta \approx \frac{1}{2\pi^2}\int_{k_F-shell} dk\,k^2\frac{T}{k^2+m_\phi^2} \approx \frac{k_F T}{2\pi^2} \quad , \tag{D.2}$$

which, using (3.73), yields

$$\langle\delta\phi^2\rangle_\beta \approx \frac{\sqrt{\gamma H}\,T}{2\pi^2} \quad . \tag{D.3}$$

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