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The n! Conjecture and the Isospectral Hilbert Scheme of Points

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Introduction

The goal of this thesis is to introduce and prove the $n!$ conjecture, this work is mainly based on the work of Mark Haiman from 1992 to 2001.
The $n!$ conjecture was for the first time approached to try to prove another conjecture, the positivity conjecture about the Kostka coefficients $K_{\lambda\mu}(q,t)$ which states that they belongs to the polynomial ring $\mathbb{N}[q,t]$.
It was known that the modules involved in the $n!$ conjecture are quotients of the ring R_n of coinvariants for the action of S_n on $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$, denoted as $\mathbb{C}[\mathbf{x},\mathbf{y}]$, and also that R_n was isomorphic to the space of diagonal harmonics. Unfortunately, despite the computations suggesting that the dimension of R_n should be $(n+1)^{n-1}$, proving it resulted very hard.
In the spring of 1992 Procesi and Haiman discussed the topic: Procesi suggested that the Hilbert scheme H_n and what we now call the isospectral Hilbert scheme X_n should be relevant to the determination of the dimension and character of R_n. Specifically, he observed that there is a natural map from R_n to the ring of global functions on the scheme-theoretic fiber in X_n over the origin in the symmetric power $S_n\mathbb{C}^2$, and with some luck this map could be an isomorphism! But let us make a step back and introducing the $n!$ conjecture properly.
Let $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ be a tuple of natural numbers such that $\sum_{i\in[m]} \mu_i = n$.
We define the Young Diagram associated to μ as the subset of $\mathbb{N}\times\mathbb{N}$ such that

$$d(\mu) = \{(p,q) \,|\, p < \mu_{q+1}\}.$$

The conjecture states that if we take the alternating polynomial Δ_μ defined as $\Delta_\mu = \det[x_i^{p_j} y_i^{q_j}]$ for $(p_j,q_j) \in \mu$ and we compute the space of all derivatives

$$D_\mu = \mathbb{C}[\partial\mathbf{x}, \partial\mathbf{y}]\Delta_\mu,$$

the dimension of D_μ is always $n!$.
Now it is important to see that there are three main topics to treat:

1. The $n!$ conjecture regarding the space D_μ
2. The positivity conjecture regarding the Kostka coefficients $K_{\lambda\mu}(q,t)$

3. The isospectral Hilbert scheme of points X_n and its natural map
$$\rho : X_n \to H_n$$
to the Hilbert scheme.

In this introduction my goal is to make clear the connections between the points 1 and 3 as they are the main focus of this thesis. In Chapter 1 the curious reader will also find a brief explaination of the connection between points 2 and 1.
Let us begin with some mathematics.
The first thing that we can notice is that, if we take the ideal generated by $x^p y^q$ for $(p, q) \notin \mu$ it is a monomial ideal, we will denote it by I_μ.
There is a very nice property of monomial ideals: they are the fixed points of the action of $T = (\mathbb{C}^*)^2$ on the Hilbert scheme! Let's see why.
It's clear that T acts on $\mathbb{C}^2$ sending (a, b) to (t_1a, t_2b), very similarly T acts on $H_n = \mathrm{Hilb}^n(\mathbb{C}^2)$ by

$$(t_1, t_2)I = (t_1, t_2)(f_1(x, y), \dots, f_m(x, y)) \to (f_1(t_1x, t_2y), \dots, f_m(t_1x, t_2y)),$$

so if I is monomial we can just factor the t_i out without modifying anything.
The other important class of points of H_n are the generic points denoted by $I = I(S)$, the ideals which vanishes on a specified finite set of distinct points $S \subseteq \mathbb{C}^2$ of cardinality n. In this very beautiful case I is radical and $\mathbb{C}[x, y]/I$ is reduced and isomorphic to $\mathbb{C}^n$. Intuitively we can think to I as a set of n points with multiplicity one and to I_μ as the origin with multiplicity n.
Notice that in H_n the order of the points does not matter whether in $\mathbb{C}^n$ it does, so it is natural to consider the map

$$\sigma : H_n \to \mathbb{C}^n/S_n$$

sending I to the unordered n-tuple $(P_1, \dots, P_n) = V(I)$ of points. Notice that each $P \in V(I)$ appears in the n-tuple a number of time equal to its multiplicity.
Now σ is called the *Hilbert Chow Morphism* and it is a morphism of algebraic varieties and note that for $S = (P_1, \dots, P_n)$ all distinct in $\mathbb{C}^n/S_n$ there is only one ideal $I = I(S) \in H_n$ such that $\sigma(I) = S$, thus, giving the fact that the generic locus is dense in H_n the map is *birational*.
Later we will see that H_n can also be described as a certain blowup of $\mathbb{C}^n/S_n$, so we can look at the Hilbert scheme of points as a resolution of the singularities of $\mathbb{C}^n/S_n$.
To recap let us look at the following diagram:

$$\begin{array}{ccc} & & \mathbb{C}^{2n} \\ & & \downarrow \\ H_n & \xrightarrow{\theta} & S_n\mathbb{C}^2 \end{array}$$

and notice that if we take a point $I(S) \in H_n$, we move it in $S_n\mathbb{C}^2$ and then we take the fiber in $\mathbb{C}^{2n}$ these fibers have lenght $n!$, in fact they are the sets of all possible orders of n distinct points.
Unfortunately this argument does not hold for the monomial ideals I_μ thus we have to find another way to prove the conjecture.
An important property of finite flat morphism of schemes is that each fiber has the same lenght.
Now suppose that we can find a scheme lying above H_n such that the map

$$\rho : Y \to H_n$$

is flat and the fibers of a generic ideal I have lenght $n!$, then we can use that property and conclude the proof! Sadly the trivial choice of completing the above diagram with the fiber product does not work, the map is not flat.
Haiman's is to complete the diagram above with the reduced fiber product of H_n and $\mathbb{C}^{2n}$ over $S_n\mathbb{C}^2$, we will call this space the *isospectral Hilbert scheme* and denote it with X_n.

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "\rho"] & \mathbb{C}^{2n} \arrow[d] \\ H_n \arrow[r, "\theta"] & S_n\mathbb{C}^2 \end{tikzcd}$$

Now because H_n is nonsingular and the projection $\rho : X_n \to H_n$ is finite, X_n being Cohen-Macaulay is equivalent to ρ being flat.
In particular the procedure is the following: we define the sheaf B over the Hilbert scheme of points H_n as the push-forward of $\mathcal{O}_F$ where F is the universal family of H_n. Then we prove that we can see X_n as $\operatorname{Spec}(B^{\otimes n}/\mathcal{J})$ for a certain sheaf of ideals $\mathcal{J}$ and we prove that the ring

$$B^{\otimes n}/\mathcal{J} \otimes_{\mathcal{O}_{H_n}} I_\mu$$

is Cohen-Macaulay and Gorenstein.
There exists a very strong result (see [5]) proving that, up to isomorphism, a local Artinian $\mathbb{C}$-algebra is Gorenstein if and only if it is of the form $\mathbb{C}[\mathbf{x}]/J$ where

$$J = \mathbb{C}[\partial\mathbf{x}]p,$$

in other words J is the vector space generated bya polynomial p its partial derivarives of all orders.
So, proving that our ring $B^{\otimes n}/\mathcal{J} \otimes_{\mathcal{O}_{H_n}} I_\mu$ is Gorenstein it is actually equivalent to proving that it is of the form $\mathbb{C}[\mathbf{x}]/J$. Subsequently, with some computations, we manage to identify this ideal J, and with it, the dimension and the structure of our ring.
The process of proving $B^{\otimes n}/\mathcal{J} \otimes_{\mathcal{O}_{H_n}} I_\mu$ Gorenstein is very insidious, approximately it goes like that:

- We prove that X_n is normal with a very ingenious argument using an algebraic structure called *Polygraphs*.
- We prove that the Gorenstein property is equivalent to the $n!$ conjecture, thus even the opposite implication works.
- We prove the $n!$ conjecture by hand for X_3, then we start with an induction argument.
- We use the equivalence: Cohen-Macaulay if and only if ρ flat for normal varieties to suppose
$$\rho : X_{n-1} \to H_{n-1}$$
flat.
- We use this ipothesis to prove that if X_{n-1} is Gorenstein then X_n is Gorenstein too.
- X_3 is Gorenstein because the $n!$ conjecture holds, thus X_n is Gorenstein and the $n!$ conjecture holds.

This thesis is organized into three chapters: the first one introduces the conjectures formally, gives an example of the $n!$ conjecture for small n and delves into some element of representation theory of finite groups. In the second chapter we dive into the algebraic geometry of the Hilbert scheme, the isospectral Hilbert scheme and we give a proof of the conjecture. During this proof we claim that the ideal

$$J = \mathbb{C}[\mathbf{x}, \mathbf{y}]A$$

where A is the space of alternating polynomials is a free $\mathbb{C}[\mathbf{y}]$-module, the proof of this fact will take the entire third chapter. Finally in the third chapter we introduce *polygraphs*, a particular union of linear subspaces in $E^n \times E^l$ where $E = \mathbb{A}^2(\mathbb{C})$.
The motivation behind the name is that their constituent subspaces are the graphs of linear maps from E^n to E^l.
The purpose of this section is to actually prove that the ring

$$\mathcal{O}(Z(n, l))$$

of the polygraph $Z(n, l)$ is a free $k[\mathbf{y}]$-module. Finally we find a map between this ring and J taht concludes the argument.

Chapter 1

The n! conjecture

1.1 The $n!$ conjecture

The aim of this chapter is to introduce the $n!$ conjecture and its connections to the Hilbert scheme.
Let us start with two fundamental definitions:

Definition 1.1. A partition of a positive integer n is a sequence of positive integers $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ satisfying $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l > 0$ and $n = \mu_1 + \mu_2 + \cdots + \mu_l$.

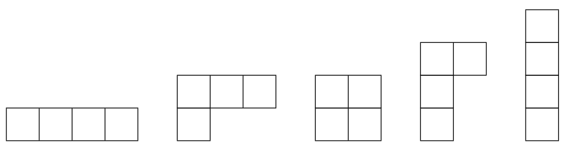
For instance, the number 4 has five partitions:

$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).$$

We can also represent partitions pictorially using Young diagrams as follows.

Definition 1.2. A Young diagram is a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing. The Young diagram associated to the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ is the one that has l rows, and μ_i boxes on the i-th row.

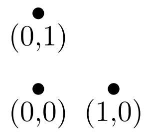
The Young diagrams corresponding to the partitions of 4 are:



Let $M \subseteq \mathbb{N} \times \mathbb{N}$ a finite subset of the first quadrant integer lattice with $|M| = n$.
In particular, M for us will be the Young diagram of a partition μ of n defined in the following way:

$$d(\mu) = \{(p, q) \mid p < \mu_{q+1}\}.$$

For example we could set $n = 3$ and $\mu = (2, 1)$, in this case our lattice will be:



Now, given our lattice M we can define a polynomial

$$\Delta_\mu(x_1, y_1, \dots, x_n, y_n) = \Delta_\mu(\mathbf{x}, \mathbf{y}) = \det[x_i^{p_j} y_i^{q_j}]_{1 \le i,j \le n}$$

where we are denoting by (p_j, q_j) the points in M.
If we consider again the previous example we find:

$$\Delta_\mu(\mathbf{x}, \mathbf{y}) = \det \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}.$$

In this setting the conjecture is the following:

Conjecture 1.3. Let us define the space D_μ as

$$D_\mu \coloneqq \mathbb{C}[\partial \mathbf{x}, \partial \mathbf{y}] \Delta_\mu$$

then

$$\dim D_\mu = n!.$$

Moreover S_n acts on it by the regular representation.

The next step is to explore the connections between this problem and the Hilbert scheme of points H_n.

1.2 The Positivity conjecture

Now let us try to give some definitions regarding the theory of symmetric functions and understand what is the connection between the Positivity conjecture and the $n!$ conjecture.

Definition 1.4. Schur functions are a family of symmetric functions denoted as $s_\mu(x_1, x_2, \dots, x_n)$, where μ is a partition of a non-negative integer d and $x_1, x_2, \dots, x_n$ are variables. The formal definition of a Schur function is given by:

$$s_\mu(x_1, x_2, \dots, x_n) = \frac{\det\left[x^{\mu+\delta}\right]}{\det\left[x^\mu\right]}$$

where $\delta = (n-1, n-2, \ldots, 0)$.
Notice that another formulation coming from the Vandermonde determinant formula of these polynomials is

$$s_\mu(x_1, \ldots, x_n) = \left(\sum_{w \in S_n} \epsilon(w) \cdot w(x^{\mu+\delta}) \right) \cdot \prod_{j<i} \frac{1}{x_i - x_j}.$$

One of the most important feature of this class of polynomials is that they form an orthogonal basis of the space of homogeneous degree d symmetric polynomials in n variables.

Definition 1.5. The Hall-Littlewood polynomials are symmetric polynomials defined for a partition μ with parts $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0$. They are given by the following expression:

$$P_\mu(x_1, \ldots, x_n; t) = \left(\prod_{i \geq 0} \prod_{j=1}^{m(i)} \frac{1-t}{1-t^j} \right) \sum_{w \in S_n} w \left(x_1^{\mu_1} \cdots x_n^{\mu_n} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right),$$

where μ is a partition of at most n with elements μ_i, and $m(i)$ elements equal to i, and S_n is the symmetric group of order $n!$.
Notice that when $q = 0$ we find that $P_\mu(x, 1)$ are Schur functions, in fact we find

$$P_\mu(x_1, \ldots, x_n, 0) = \left(\sum_{w \in S_n} \epsilon(w) \cdot w(x^{\mu+\delta}) \right) \cdot \prod_{j<i} \frac{1}{x_i - x_j}.$$

It is possible to expand the Schur polynomials in term of Hall-Littlewood polynomials using some particular coefficients $K_{\lambda\mu}(t)$ known as Kostka-MacDonald coefficients.

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(t) P_\mu(x, t)$$

In 1988 MacDonald introduced a new family of polynomials called *MacDonald Polynomials* to unify two families of polynomials: the Hall-Littlewood polynomials and the Jack polynomials (see [13]).
These polynomials, denoted as

$$P_\lambda(\mathbf{x}, q, t)$$

depend on a partition λ, a set of n variables $\mathbf{x} = x_1, \ldots, x_n$ and two real parameters q, t.
We stil need a few other definitions to be able to state the positivity conjecture:

Definition 1.6. We define the Macdonald Integral basis for the symmetric functions as the set of functins satisfying the following two conditions:

1.
$$J_\lambda = \prod_{s\in\lambda}\left(1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}\right) s_\lambda + \sum_{\mu<\lambda} s_\mu c_{\mu\lambda}(q,t)$$

2.
$$\langle J_\lambda, J_\mu \rangle_{q,t} = 0 \quad \text{for} \quad \lambda \neq \mu$$

where, given a partition λ we call $a_\lambda(s)$ the number of cells that lie to the east of s in λ and we call $l_\lambda(s)$ the number of cells that are strictly north to s.

It is a little bit technical to see how the scalar product in this space is defined, see [16] for the details.
Notice that the coefficients $c_{\mu\lambda}(q,t)$ are determined by conditions 1 and 2 and are rational functions in q and t.
The (q,t)-Kostka coefficients are then given by the expression

$$K_{\lambda\mu}(q,t) = \langle J_\mu(X;q,t), s_\lambda(X)\rangle.$$

As defined, they are rational functions of q and t, but conjecturally they are polynomials in q and t with nonnegative integer coefficients:

$$K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t].$$

The positivity conjecture has remained open since Macdonald formulated it at the time of his original discovery.
But what is the connection between this claim and the Hilbert scheme of points?
You can find an extensive explaination of the argument in [11]; as it is not the main focus of this work, here we will just give a brief taste of the topic.
Let us start defining the ring R_n, properties of which are described by our conjectures.
Let I be the ideal in $\mathbb{C}[\mathbf{x},\mathbf{y}]$ generated by all S_n invariant polynomials without constant term. We set

$$R_n = \mathbb{C}[\mathbf{x},\mathbf{y}]/I.$$

It is important to notice that I is an homogeneus ideal, since if $p(\mathbf{x},\mathbf{y})$ is an invariant polynomial without constant term, then so is each of its homogeneus components of various degrees.
Actually the same is true for bidegrees, where we say that $p(\mathbf{x},\mathbf{y})$ has bidegree (i,j) if it has degree i in $\mathbf{x}$ and j in $\mathbf{y}$. This means that I is a bihomogeneous ideal and consequently R_n is a doubly graded ring.
Clearly there is an action of S_n into R_n which preserves the bidegree, so if we write

$$R_n = \bigoplus_{i,j} (R_n)_{i,j} \tag{1.1}$$

each $(R_n)_{i,j}$ is a S_n submodule.
There is an alternative view of R_n, let us see a definition.

Definition 1.7. The apolar form is the nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = f(\partial x_1, \ldots, \partial x_n, \partial y_1, \ldots, \partial y_n) g(x_1, \ldots, x_n, y_1, \ldots, y_n)_{|\mathbf{x}=\mathbf{y}=0}.$$

From this definition the following proposition follows fairly naturally:

Proposition 1.8. *If I is a homogeneous ideal, then its orthogonal complement $H = I^{\perp}$ is a homogeneous space of polynomials, closed under the taking of arbitrary partial derivatives. We also have $H = \{h \mid f(\partial\mathbf{x}, \partial\mathbf{y})h = 0 \,\forall f \in I\}$; in other words, regarding I as a system of polynomial partial differential equations, H is its space of solutions. If H is any homogeneous space of polynomials closed under partial derivatives, then $I = H^{\perp}$ is a homogeneous ideal with $H = I^{\perp}$.*

Definition 1.9. The space H_n of diagonal harmonics for S_n is I_1, where I is the ideal defining the ring R_n in Equation 1.1.

Now, given a bigraded space $A = \bigoplus_{i,j} A_{i,j}$ we can define its associated Hilbert series as

$$\mathcal{H}_A(t, q) = \sum_{i,j \in \mathbb{N}} t^i q^j \dim(A_{i,j}).$$

Conjecture 1.10. For the Hilbert series associated to R_n the following equation holds:

$$\mathcal{H}_n(1, 1) = (n + 1)^{n-1}.$$

In particular notice that $\mathcal{H}_n(1, 1)$ is precisely the dimension of the diagonal harmonics H_n associated to I_n which is the ideal generated by all S_n-invariant polynomials without constant term. This means that we can restate the rephrase the conjecture 1.10 in the following way:

Conjecture 1.11. It is conjectured that

$$\dim_{\mathbb{C}} R_n = (n + 1)^{n-1}.$$

Now we are ready to see how the conjecture relating the Kostka coefficients is related (in particular implies) the $n!$ conjecture.

Remark 1.12. Consider the ring $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \bigoplus_{r,s} \mathbb{C}[\mathbf{x}, \mathbf{y}]_{r,s}$ as a doubly graded ring with the S_n action respecting the grading. Clearly the polynomial Δ_μ is S_n alternating as it is defined as a determinant of a matrix, moreover it is doubly homogeneous. It follows that the space D_μ is S_n-invariant and has a double grading

$$D_\mu = \bigoplus_{r,s} (D_\mu)_{r,s}$$

by the S_n invariant subspaces $(D_\mu)_{r,s} = D_\mu \cap \mathbb{C}[\mathbf{x},\mathbf{y}]_{r,s}$.

Let us denote the irreducible S_n characters by χ^λ with the usual indexing by partitions λ of n.

Remark 1.13. In the next section we will define what a linear representation of a finite group is, for now it is enough to know that, given a representation of G we associate $g \in G$ to a matrix $[g] \in GL_n(\mathbb{C})$ and we define

$$\chi(g) = \mathrm{tr}([g]).$$

Conjecture 1.14. We have

$$\tilde{K}_{\lambda\mu}(q,t) = \sum_{r,s} t^r q^s \langle \chi^{\lambda, \mathrm{ch}(D_\mu)_{r,s}} \rangle.$$

where we set

$$\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q,t^{-1})$$

and $n(\mu) = \sum_i (i-1)\mu_i$.
Macdonald had shown that $K_{\lambda\mu}$ is equal to $\chi^\lambda(1)$ which is the degree of the irreducible S_n character χ^λ as it is the trace of the identity matrix or the dimension of the space.
Therefore this conjecture implies the $n!$ conjecture as it tells us the dimension of the space D_μ.

1.3 Representation theory

In this section we will introduce some useful concepts in Representation theory and the connection with our conjecture.
Let us start with some definitions:

Definition 1.15. Let G be a finite group, a representation of G on the vector space $\mathbb{C}^n$ is a morphism

$$G \to GL_n(\mathbb{C}).$$

Equivalently we can define it as a morphism

$$\Phi : G \times \mathbb{C}^n \to \mathbb{C}^n$$

such that:

- $\Phi(g, v)$ is linear over $\mathbb{C}$
- $\Phi(e, g) = g$ with e the identity of G

- $\Phi(g_1 g_2, v) = g_1 \Phi(g_2, v)$.

Definition 1.16. We define the *Regular representation* of a finite group G as a linear representation on the vector space

$$\mathbb{C}^{|G|} = \mathbb{C}g_1 \oplus \mathbb{C}g_2 \oplus \cdots \oplus \mathbb{C}g_n$$

in the following way:

$$\Phi(g, v) = \Phi(g, \sum_{i=1}^{n} a_i g_i) = \sum_{i=1}^{n} a_i g g_i.$$

Now our goal is not only to prove that the vector space D_μ has dimension $n!$ bus also that it carries a regular representation of the group S_n, which is a stronger claim!
Let us see what happens with small n's.

Example 1.17. If we set $n = 2$ we have only 2 cases which are clearly equivalent to each other: $\mu = (1, 1)$ and $\mu(2, 0)$, let us take the first.
The matrix is

$$M_\mu = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$$

with determinant $\Delta_\mu = x_2 - x_1$.
If we compute the space of derivatives we find

$$D_\mu = \mathbb{C} \cdot (1) \oplus \mathbb{C} \cdot (x_2 - x_1)$$

which has dimension equal to two.
Now we have to prove it carries the regular representation of S_2.
Basically we have to find a map Φ such that the following diagram commutes: calling $p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $-id$ the map sending x_1 to x_2 and viceversa we have

$$\begin{tikzcd} D_\mu \arrow[r, "\Phi"] \arrow[d, "-id"] & \mathbb{C}^2 \arrow[d, "p"] \\ D_\mu \arrow[r, "\Phi"] & \mathbb{C}^2 \end{tikzcd}$$

If we take $v_1 = x_2 - x_1$ and $v_2 = 1$ as a base of D_μ then we can send v_1 to $(1, -1)$ and v_2 to $(1, 1)$. It is easy to verify that it works.
If we set $n = 3$ then we have three different cases, two of which are equivalent:

1. $\mu = (1, 1, 1)$ with diagram

$$\underset{(0,0)}{\bullet} \quad \underset{(1,0)}{\bullet} \quad \underset{(2,0)}{\bullet}$$

2. $\mu = (2, 1)$ with diagram

$$\begin{array}{cc} \bullet & \\ (0,1) & \\ \bullet & \bullet \\ (0,0) & (1,0) \end{array}$$

3. $\mu = (3)$ with diagram

$$\begin{array}{c} \bullet \\ (0,2) \\ \bullet \\ (0,1) \\ \bullet \\ (0,0) \end{array}$$

Let us do some computations for the second case.
The basis of $D_{(2,1)} = D_\mu$ is

$$\{y_2 - y_3, y_3 - y_1, x_3 - x_2, x_1 - x_3, 1, \Delta_\mu\},$$

now we notice that if we permute two variables on Δ_μ the result is $-\Delta_\mu$ while if we act with an order three permutation the result is Δ_μ.
On the other hand if we permute two variables of one of the binomials the result is another binomial or its opposite, while if we act with an order three permutation the result is always another binomial.
We can now define the following map:

$$\Phi : D_\mu \to S_3$$

such that
$\Phi(1) = id$,
$\Phi(y_i - y_j) = id - (i, j) + (i, k) + (j, k) - (i, j, k) - (i, k, j)$,
$\Phi(\Delta_\mu) = id + (1, 2) + (2, 3) + (3, 1) + (1, 2, 3) + (1, 3, 2)$.
Notice that this map defines a group action

$$D_\mu \times S_3 \to D_\mu.$$

Now we want to verify that it is well defined. Let us try to do it for the only non trivial case: $y_i - y_j$ which we can set as $v = y_1 - y_2$ without losing generality.
If we apply $(1, 2)$ to v we find $-v$ thus we should have $(1, 2)\Phi(v) = -\Phi(v)$:

$$\begin{aligned} (1, 2)\Phi(v) = & \\ & = (1, 2)[id - (1, 2) + (2, 3) + (1, 3) - (1, 2, 3) - (1, 3, 2)] \\ & = (1, 2) - id + (1, 2, 3) + (1, 3, 2) - (2, 3) - (1, 3) = -\Phi(v). \end{aligned}$$

Chapter 2

The Hilbert scheme of points

In this chapter we will introduce the Hilbert scheme of points and we will present and prove some of its very useful properties.

2.1 The Hilbert scheme and its Universal Family

Let us start with the definition:

Definition 2.1. The Hilbert scheme of points H_n of the affine plane is the set of all ideals $I \subseteq \mathbb{C}[x, y]$ such that $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = n$. In other words H_n parametrizes subschemes $S \subseteq \operatorname{Spec} \mathbb{C}[x, y]$ for which $S = \operatorname{Spec} \mathbb{C}[x, y]/I$ is zero dimensional of lenght n.

Notice that this is the definition of the closed points of our scheme H_n but it says nothing about its scheme structure. We will talk about it when we will define the universal family F associated to H_n.

Theorem 2.2. *[6] The Hilbert scheme H_n is a non singular, irreducible variety over $\mathbb{C}$ of dimension $2n$.*

Now let us try to describe the scheme structure of H_n via explicit coordinates on open affine subsets, indexed by partitions μ.
Given μ let us define

$$\mathcal{B}_\mu = \{x^h y^k \mid (h, k) \in \mu\}$$

and

$$U_\mu = \{I \in H_n \mid \mathcal{B}_\mu \text{ spans } \mathbb{C}[x, y]/I\} \subseteq H_n.$$

This means that for every $I \in U_\mu$, $\mathcal{B}_\mu$ is a basis modulo I thus for each $x^r y^s$ there exists a unique expansion

$$x^r y^s \equiv \sum_{(h,k)\in\mu} c_{hk}^{rs}(I) x^h y^k \bmod I$$

where the coefficients $c_{hk}^{rs}(I)$ define a collections of functions on U_μ.

Proposition 2.3. *The sets U_μ are open affine subvarieties which cover H_n, moreover the affine coordinate ring $\mathcal{O}_{U_\mu}$ is generated by the functions c_{hk}^{rs} for $(h,k) \in \mu$ and all the (r,s),*

To prove that the sets U_μ cover H_n we rely on a theorem proved by Gordan:

Theorem 2.4. *[7] For every ideal I in a polynomial ring there is a basis $\mathcal{B}$ modulo I, consisting of monomials, such that every divisor of a monomial in $\mathcal{B}$ is also in $\mathcal{B}$.*

It is pretty clear that in our case such basis must be $\mathcal{B}_\mu$ for some partition μ. The next step is to notice that the scheme $S = \operatorname{Spec} \mathbb{C}[x,y]/I$ has always a finite number of points, in particular for each $p \in S$ we can assign a multiplicity m_p equal to the lenght of the local ring $(\mathbb{C}[x,y]/I)_p$. The sum of these multiplicities sum to n.
Now let us consider the scheme

$$\mathbb{C}^{2n} = \operatorname{Spec} \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] = \operatorname{Spec} \mathbb{C}[\mathbf{x}, \mathbf{y}]$$

the symmetric group S_n acts on it by permutating the factors. We will denote by

$$S_n\mathbb{C}^2 := \mathbb{C}^{2n}/S_n$$

The map

$$\sigma : H_n \to S_n\mathbb{C}^2$$

defined as

$$\sigma(I) = V(I)$$

can be shown to be a morphism and is called the Hilbert Chow morphism.

Proposition 2.5. *[9] The Hilbert Chow morphism $\sigma : H_n \to S_n\mathbb{C}^2$ is a projective morphism.*

The next step of our description of the Hilbert scheme is proving that we can look at it as a blow-up of $S_n\mathbb{C}^2$.
Let A be the space of alternating polynomials in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. For each subset $D = \{(p_1, q_1), \ldots, (p_n, q_n)\}$ of $\mathbb{N} \times \mathbb{N}$ the determinant

$$\Delta_D(\mathbf{x}, \mathbf{y}) = \det[x_i^{p_i} y_i^{q_i}]$$

is well defined and belongs to A. Moreover $\{\Delta_D\}$ for each D spans A.
When D is the diagram of a partition μ we have $\Delta_D = \Delta_\mu$.
Now let's denote by A^d the space of all products $f_1 \cdot f_2 \cdot \ldots, \cdot f_d$ with $f_i \in A$ and we set $A^0 = \mathbb{C}[\mathbf{x}, \mathbf{y}]/S_n$.

Theorem 2.6. *The space $R = A^0 \oplus A^1 \oplus \dots$ is a graded $\mathbb{C}[\mathbf{x},\mathbf{y}]/S_n$-algebra and*

$$H_n \cong \operatorname{Proj} R.$$

Moreover the natural morphism $\theta : \operatorname{Proj} R \to \operatorname{Spec} A^0 = \operatorname{Spec} \mathbb{C}[\mathbf{x},\mathbf{y}]/S_n$ is exactly the Hilbert Chow morphism.

Proof. Let Y be the open subset of $\operatorname{Spec} \mathbb{C}[\mathbf{x},\mathbf{y}]/S_n$ consisting of n-tuples of distinct points. For each set $S \in Y$ there exists a unique ideal $I \in H_n$ such that $V(I) = S$. This means that we have a bijection

$$\sigma : Y_H = \sigma^{-1}(Y) \subseteq H_n \to Y.$$

Now suppose $I \in U_\mu \cap Y_H$ with $S = V(I)$, the monomials $x^h y^k \in \mathcal{B}_\mu$ are a basis of $\mathcal{O}_{U_\mu}$ so must describe linearly independent functions on S, thus $\Delta_\mu(\mathbf{x},\mathbf{y}) \neq 0$.
Notice that $\Delta_\mu(\mathbf{x},\mathbf{y})$ depends on the ordering chosen for the points in S, but Δ_D/Δ_μ does not!
So we have that $\sigma^*(\Delta_D/\Delta_\mu)$ is a regular function on $U_\mu \cap Y_H$.
Now let's fix a partition $\mu = \{(h_1,k_1),\dots,(h_n,k_n)\}$ and remember our previous observation about $\mathcal{B}_\mu$ being a basis of our ring modulo I, then we find that the coefficients $c^{rs}_{hk}(I)$ satisfy:

$$[x_i^{h_j} y_i^{k_j}]_{i,j\in[n]} \cdot \begin{pmatrix} c^{rs}_{h_1k_1} \\ \cdot \\ \cdot \\ \cdot \\ c^{rs}_{h_nk_n} \end{pmatrix} = \begin{pmatrix} x_1^r y_1^s \\ \cdot \\ \cdot \\ \cdot \\ x_n^r y_n^s \end{pmatrix} \tag{2.1}$$

So given a diagram $D = \{(p_l, q_l)\}$ we find the matrix equation:

$$[x_i^{h_j} y_i^{k_j}]_{i,j\in[n]} \cdot [c^{p_l q_l}_{h_j k_j}]_{j.l\in[n]} = [x_i^{p_l} y_i^{q_l}]_{i.l\in[n]},$$

tanking the determinants we find:

$$\sigma^*(\Delta_D/\Delta_\mu) = \det[c^{p_l q_l}_{h_j k_j}]_{j.l\in[n]}$$

on $U_\mu \cap Y_H$, thus $\sigma^*(\Delta_D/\Delta_\mu)$ can be extent to a regular function f_D over the entire U_μ. For every two diagrams D_1 and D_2 we have

$$\sigma^*(\Delta_{D_1} \cdot \Delta_{D_2}) = \sigma^*(\Delta_{D_1}/\Delta_\mu \cdot \Delta_{D_2}/\Delta_\mu) \cdot \sigma^*(\Delta_\mu^2) = f_{D_1} \cdot f_{D_2} \cdot \sigma^*(\Delta_\mu^2).$$

Notice that this shows that $\sigma^*(A^2)$ is locally the principal ideal (Δ_μ^2) in $\mathcal{O}_{U_\mu}$.
Now it' clear that $\operatorname{Proj} R = \operatorname{Proj} R^2$ that is the same as the blow up of $\operatorname{Spec} \mathbb{C}[\mathbf{x},\mathbf{y}]/S_n$ along A^2, so from the universal property of the blow up there is a unique morphism $\alpha : H_n \to \operatorname{Proj} R$ such that $\theta \circ \alpha = \sigma$:

$$\begin{tikzcd} H_n \arrow[r, "\sigma"] \arrow[d, "\alpha"] & S_n\mathbb{C}^2 \\ \operatorname{Proj} R \arrow[ur, "\theta"] & \end{tikzcd}$$

Since $\operatorname{Proj} R$ is irreducible and birational to $S_n\mathbb{C}^2$, α is surjective.
To prove that its an isomorphism we have to show that α is an embedding, i.e. the map

$$\alpha^* : \mathcal{O}_{\operatorname{Proj} R} \to \mathcal{O}_{H_n}$$

is surjective.
If we try to solve the equations 2.1 using the Cramer's rule we find

$$c_{hk}^{rs}(I) = \Delta_D/\Delta_\mu(S)$$

with $D = \mu\backslash(h,k) \cup (r,s)$.
This means that on $U_\mu \cap Y_H$, $c_{hk}^{rs} = \sigma^*(\Delta_D/\Delta_\mu) = \alpha^*\theta^*(\Delta_D/\Delta_\mu)$, but since c_{hk}^{rs} generates $\mathcal{O}_{U_\mu}$ we have that α restricted to the closure of Y_H is an embedding.
Since H_n is irreducible Y_H is dense so we conclude the proof.

□

Let us see and example for $n = 2$.

Example 2.7. In this case, as we have seen in the first chapter, we have only two possible Young diagrams:

- $\mu = (2,0)$ with $\Delta_\mu = x_1 - x_2$,
- $\mu = (1,1)$ with $\Delta_\mu = y_1 - y_2$.

This means that a basis for the alternating polynomials A is $J = \{x_1 - x_2, y_1 - y_2\}$.
The blow-up $\operatorname{Proj} S$ is covered by two affine open subsets

- $U_{x_1-x_2} = \operatorname{Spec} A^0[\frac{y_1-y_2}{x_1-x_2}]$,
- $U_{y_1-y_2} = \operatorname{Spec} A^0[\frac{x_1-x_2}{y_1-y_2}]$.

while the Hilbert scheme H_2 is covered by

- $W_x = \{I = (x^2 - e_1x + e_2, y - a_1x - a_0)\}$,
- $W_y = \{I = (y^2 - e'_1y + e'_2, x - a'_1y - a_0)\}$.

Intuitively we can think about this covering as a way to paramtrize sets of two points in the affine plane: consider the pair $(u, v), (u', v')$, it can be seen as the intersection between the three lines $x = u, x = u'$ and $(x - u)(v' - v) = (y - v)(u' - u)$ which are clearly elements in W_x.
In fact if we set $e_1 = -u - u'$, $e_2 = u \cdot u'$, $a_1 = \frac{v-v'}{u-u'}$ and $a_0 = \frac{u' \cdot v - u \cdot v'}{u-u'}$ then we find these lines.
This gives us explicit morphisms between W_x and $U_{x_1-x_2}$, and similarly between W_y and $U_{y_1-y_2}$.

To fix the ideas we do some other computations proving the $n!$ conjecture for $n = 3$. This will be very useful in the last section of this chapter as we will require the conjecture to be true for this particular case.

Example 2.8. Let us consider the Young diagram $\mu = (2, 1)$ and it's ideal

$$I_\mu = (x^2, xy, y^2).$$

Now let's take the open set

$$U_\mu = \{I \in H_3 \,|\, \{1, x, y\} \text{ is a basis of } R/I\}$$

that is the set of I of H_3 corresponding to three points non collinear.
We know that the tautological bundle $B = \pi_* \mathcal{O}_F$ has fibers $B(I) = R/I$, moreover we can caracterize U_μ as the non vanishing locus of the section $1 \wedge x \wedge y$ of the line bundle $\wedge^3 B$.
Now notice that this section is represented by the alternarnating polynomial $\Delta_\mu(\mathbf{x}, \mathbf{y})$ in the sense that it is the set in which this polynomial don't vanish.
We will prove the ring morphism

$$\mathbb{C}[\Delta_L/\Delta_\mu \;\forall\; L] \to \mathcal{O}(U_\mu)$$

is a homomorphism so $\mathbb{C}[\Delta_L/\Delta_\mu \;\forall\; L]$ represents the ring of regular functions on U_μ where L varies in the set of subsets of $\mathbb{N} \times \mathbb{N}$ of cardinality 3 different from μ.
We can try to describe this ring in a more explicit way:
every $I \in U_\mu$ is in fact generated by

$$\begin{aligned} &x^2 - ax - by - g, \\ &xy - cx - dy - h, \end{aligned}$$

or, by symmetry, by

$$\begin{aligned} &y^2 - ex - fy - j, \\ &xy - cx - dy - h, \end{aligned}$$

so in R/I we can reduce the monomial x^2y in many ways:

$$\begin{aligned} x^2y &= (ax+by+g)y = axy + by^2 + gy = \\ &= a(cx+dy+h) + b(ex+fy+j) + gy, \end{aligned}$$

or

$$\begin{aligned} x^2y &= x(cx+dy+h) = cx^2 + dxy + hx = \\ &= c(ax+by+g) + d(cx+dy+h) + hx. \end{aligned}$$

We can do the same thing for the monomial xy^2 finding:

$$\begin{aligned} xy^2 &= e(ax+by+g) + f(cx+dy+h) + jx \\ &= c(cx+dy+h) + d(ex+fy+j) + hy. \end{aligned}$$

These equations allow us to express some parameters with respect to others for example:

$$\begin{aligned} h &= be - cd, \\ g &= b(c-f) + d(d-a), \\ j &= e(d-a) + c(c-f). \end{aligned}$$

This means that we can describe U_μ with just 6 parameters thus

$$U_\mu = \operatorname{Spec} \mathbb{C}[a,b,c,d,e,f] \cong \mathbb{C}^6.$$

Now let's consider the scheme X_3 and let's try to describe the set $\rho^{-1}(U_\mu)$ where $\rho : X_n \to H_n$ is the natural projection.
We know that its coordinate ring is

$$\mathbb{C}[\mathbf{x}, \mathbf{y}, \Delta_L/\Delta_\mu] = \mathbb{C}[\mathbf{x}, \mathbf{y}, a, b, c, d, e, f]/\sim .$$

with $\sim$ some equivalence relations.
Our goal is to prove that this ring is Cohen-Macaulay and Gorenstein above

$$Q_\mu = (I_\mu, 0, 0, 0) = \rho^{-1}(I_\mu).$$

The previous properties are equivalent to $\mathbb{C}[\mathbf{x}, \mathbf{y}, \Delta_L/\Delta_\mu]$ being a free $\mathbb{C}[a, b, \ldots, f]$ module of rank 3! (because ρ has degree 6). We can express the coordinate ring of the scheme theoretic fiber of $\rho^{-1}(I_\mu)$ as $\mathbb{C}[\mathbf{x}, \mathbf{y}]/J$, let's try to get some element of J.
Take

$$\rho_* : \mathbb{C}[\Delta_L/\Delta_\mu] \xrightarrow{\sim} \mathbb{C}[\mathbf{x}, \mathbf{y}]/J \cong \mathbb{C}[\mathbf{x}, \mathbf{y}, \Delta_L/\Delta_\mu]/(a, b, \ldots, f)$$

and notice that, since S_n respect the fibers of ρ, if $q(\mathbf{x}, \mathbf{y}) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]/J$ is in $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$ then

$$q(\mathbf{x}, \mathbf{y}) = \sigma(q(\mathbf{x}, \mathbf{y})) = \rho_*(\sigma t(a, \ldots, f)) = -\rho_*(t(a, \ldots, f)) = -q(\mathbf{x}, \mathbf{y})$$

so $\mathbb{C}[\mathbf{x},\mathbf{y}]^{\epsilon} \subseteq J$.
Moreover we have x_i^2, x_iy_i and y_i^2 in J because each of them are equal to a linear equation with parameters $(a,b,\dots,f)$ so they are in $(a,b,\dots,f)$.
Now it is time to copute J_μ which is the annihilating ideal of Δ_μ:

$$\det\begin{pmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{pmatrix} = x_2y_3 + x_1y_2 + y_1x_3 - y_1x_2 - x_1y_3 - y_2x_3$$

thus the annihilating ideal is generated by:

$$\mathbb{C}[\mathbf{x},\mathbf{y}]^{\epsilon} \oplus < x_i^2, y_i^2, x_iy_i > .$$

Now it's clear that $J = J_\mu$ thus

$$\mathbb{C}[\mathbf{x},\mathbf{y}]/J = \mathbb{C}[\mathbf{x},\mathbf{y}]/J_\mu,$$

so $\mathbb{C}[\mathbf{x},\mathbf{y}]/J$ is Gorenstein because J_μ is the annihilating ideal of a Macaulay inverse system generated by one element.
Finally

$$\dim(D_\mu) = \dim(\mathbb{C}[\mathbf{x},\mathbf{y}]/J_\mu) = 3! = 6.$$

Now that we have proved this very important theorem we can define another useful feature of the Hilbert scheme of points, its universal family. Let us start with the set theoretic definition of it:

Definition 2.9. Let F be the subscheme of $H_n \times \mathbb{C}^2$ defined on the closed points as

$$F = \{(I,P) \mid P \in V(I)\} \tag{2.2}$$

We will call F the universal family of H_n.
Now our goal is to describe how to give a scheme structure to F.
Notice that F comes with a projection $\pi : F \to H_n$ and that the fibers of a point $I \in H_n$ will be the subscheme $V(I) \in \mathbb{C}^2$.

It is important to notice that we can define the universal family of the Hilbert scheme of points in a much more general way, it is, in fact, a direct consequence of the functorial definition of this scheme; let us see how.

Definition 2.10. Let $\mathcal{C}$ be a locally small category and *Set* be the category of sets. For each object A of $\mathcal{C}$ we define the functor

$$\mathrm{Hom}(-,A) : \mathcal{C} \to Set$$

mapping the elements B of $\mathcal{C}$ to the set $\mathrm{Hom}(B,A)$.

Definition 2.11. Given a functor $F : \mathcal{C} \to Set$ we say it is *representable* if it is naturally isomorphic to $\mathrm{Hom}(-,A)$ for some object A of $\mathcal{C}$.

Now according to Yoneda's Lemma, natural transformations from the functor $h_A : \mathcal{C} \to Set$ mapping to an object X to $\hom(X, A)$ are in one-to-one correspondence with the elements of $F(A)$.
Now we define the Hilbert scheme of points functorially:

Definition 2.12. Let *Sch* be the category of schemes over $\mathbb{C}$ and let *Set* be the category of sets.
For each $X \in Sch$ we define

$$\mathrm{Hilb}^n_{\mathbb{C}^2} : Sch \to Set$$

sending T to G defined as:
$\{Z \subseteq \mathbb{C}^2 \times T \mid Z \to T$ is flat, $Z \to T$ is proper, and $Z_t \subset \mathbb{C}^2$ is 0 dimensional of lenght $n \; \forall t \in T\}$, where Z_t is the fiber in Z over $t \in T$.

Grothendieck proved that this functor is representable [8], this means that it is equivalent to a functor

$$H_n : Sch \to Set$$

such that T goes to $\mathrm{Hom}(T, H_n)$.
We have defined the Hilbert scheme of points H_n.
From this very general definition, we notice that we can consider

$$H_n(H_n) = \mathrm{Hom}(H_n, H_n)$$

which contains the identity, thus, going back to definition 2.12 and taking the corresponding $Z \subseteq H_n \times \mathbb{C}^n$ is enough to define our space. This is the universal family of H_n.

Now we can define the sheaf B as $\pi_* \mathcal{O}_F$ and notice that we have a homomorphism between $\mathcal{O}_{H_n} \to B$ so that B has a structure of $\mathcal{O}_{H_n}$-algebras. Moreover local coordinates on F are generated by local coordinates on H_n pulled back by π plus the coordinates (x, y) of $\mathbb{C}^2$, that makes B a sheaf of $\mathcal{O}_{H_n}$-algebras generated by (x, y).

Proposition 2.13. *The following two statements are true:*

- *F is flat and finit of degree n over H_n,*
- *if $Y \subset T \times \mathbb{C}^2$ closed subscheme which is flat and finite of degree n over a scheme T there is a unique morphism $\phi : T \to H_n$ making the following diagram commutative.*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & T \times \mathbb{C}^2 \arrow[r] \arrow[d] & T \arrow[d, "\phi"] \\ F \arrow[r] & H_n \times \mathbb{C}^2 \arrow[r] & H_n \end{tikzcd}$$

Proof. A special case of the Grothendieck's construction claims the existence of the Hilbert scheme $\hat{H}_n = \text{Hilb}(\mathbb{P}^2)$[8] with a universal family $\hat{F}$ for which the universal property holds.
We identify $\mathbb{C}^2$ as the complement of the projective line Z at infinity in $\mathbb{P}^2$. Now consider the projection of $\hat{F} \cap (\hat{H}_n \times Z)$ onto $\hat{H}_n$: the image is a closed subset of $\hat{H}_n$, moreover its complement H_n is the largest subset such that the restriction F of $\hat{F}$ to H_n is contained in $H_n \times \mathbb{C}^2$.
Intuitively we can think about this construction in the follwing way:
$\hat{H}_n \times Z$ can be seen as the set of tuples $(p_1, \dots, p_n) \times p_{n+1}$ such that p_{n+1} is in Z, imagine $\hat{F}$ as (I, p_i) where $p_i \in V(I)$ and I as $V(I)$.
In this way we can see that the image of the projection is the set of tuples $(p_1, \dots, p_n)$ where at least one p_i is in Z thus the complementar is exactly the set of tuples with all the points in the complex plane which is H_n.
The required universal property of H_n and F now follows immediately from that of $\hat{H}_n$ and $\hat{F}$.

□

2.2 The Isospectral Hilbert scheme

Now we are ready to introduce the isospectral Hilbert scheme of points and prove that, like the simple Hilbert scheme, it can be seen as a blow-up.

Definition 2.14. The isospectral Hilbert scheme X_n is the reduced fiber product

$$\begin{array}{ccc} X_n & \xrightarrow{f} & \mathbb{C}^{2n} \\ \downarrow{\scriptstyle\rho} & & \downarrow \\ H_n & \xrightarrow{\sigma} & S^n\mathbb{C}^2 \end{array}$$

Intuitively we can think about it as the closed points

$$\{(I, P_1, \dots, P_n) \mid \sigma(I) = (P_1, \dots, P_n)\} \subseteq H_n \times \mathbb{C}^{2n}.$$

Theorem 2.15. *Defining $J^d = \mathbb{C}[\mathbf{x}, \mathbf{y}]A^d$ we have*

$$X_n \cong \text{Proj}\, \mathbb{C}[\mathbf{x}, \mathbf{y}][tJ] = \text{Proj}\, T$$

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} \text{Proj}\, T & \longrightarrow & \mathbb{C}^{2n} \\ \downarrow & & \downarrow \\ H_n = \text{Proj}\, S & \longrightarrow & S^n\mathbb{C}^2 \end{array}$$

Remember that X_n is the reduced subscheme of $H_n \times \mathbb{C}^{2n}$ so since $\operatorname{Proj} T$ is reduced we have $\operatorname{Proj} T \subset X_n$.(01J3 [15])
Now it's possible to prove that, thanks to the irreducibility of H_n, also X_n is irreducible. We found a closed proper subset of X_n, so we must have $\operatorname{Proj} T = X_n$.

□

Now we need to state and prove a few easy properties of this structure that we will use during the proof of the main theorem.

Lemma 2.16. *Let k and l be positive integers such that $k + l = n$. Suppose $U \subseteq \mathbb{C}^{2n}$ is an open set consisting of points $(P_1, \dots P_k, Q_1, \dots, Q_l)$ where $P_i \neq Q_j$ for all i, j.*
Now identify $\mathbb{C}^{2n}$ with $\mathbb{C}^{2k} \times \mathbb{C}^{2l}$, then the preimage $f^{-1}(U)$ in X_n is isomorphic as a scheme over $\mathbb{C}^{2n}$ to the preimage $f_k^{-1}(U) \times f_l^{-1}(U)$ in $X_k \times X_l$.

Proof. Consider the diagram of Proposition 2.13 with X_n instead of T.
Let $Y = (\rho \times 1)^{-1}(F) \subseteq X_n \times \mathbb{C}^2$ be the universal family of X_n. The fiber $V(I)$ of Y over a point $(I, P_1, \dots, P_k, Q_1, \dots, Q_l) \in f^{-1}(U)$ is the disjoint union of closed subschemes $V(I_k)$ and $V(I_l)$ in $\mathbb{C}^2$ of lengths k and l, respectively, with $\sigma(I_k) = (P_1, \dots, P_k)$ and $\sigma(I_l) = (Q_1, \dots, Q_l)$. Thus over $f^{-1}(U)$, Y is the disjoint union of flat families Y_k, Y_l of degrees k and l.
By the universal property, we get induced morphisms $\Phi_k : f^{-1}(U) \to H_k$, $\Phi_l : f^{-1}(U) \to H_l$ and $\Phi_k \times \Phi_l : f^{-1}(U) \to H_k \times H_l$. The equations $\sigma(I_k) = (P_1, \dots, P_k)$, $\sigma(I_l) = (Q_1, \dots, Q_l)$ imply that $\Phi_k \times \Phi_l$ factors through a morphism $\alpha : f^{-1}(U) \to X_k \times X_l$ of schemes over $\mathbb{C}^{2n}$.
Conversely, on $(f_k \times f_l)^{-1}(U) \subseteq X_k \times X_l$, the pullbacks of the universal families from X_k and X_l are disjoint and their union is a flat family of degree n. By the universal property there is an induced morphism $\Psi : (f_k \times f_l)^{-1}(U) \to H_n$, which factors through a morphism $\beta : (f_k \times f_l)^{-1}(U) \to X_n$ of schemes over $\mathbb{C}^{2n}$. By construction, the universal families on $f{-}1(U)$ and $(f_k \times f_l)^{-1}(U)$ pull back to themselves via $\beta \circ \alpha$ and $\alpha \circ \beta$, respectively. This implies that $\beta \circ \alpha$ is a morphism of schemes over H_n and $\alpha \circ \beta$ is a morphism of schemes over $H_k \times H_l$. Since they are also morphisms of schemes over $\mathbb{C}^{2n}$, we have $\beta \circ \alpha = 1_{f^{-1}(U)}$ and $\alpha \circ \beta = 1_{(f_k \times f_l)^{-1}(U)}$. Hence α and β induce mutually inverse isomorphisms $f^{-1}(U) \cong (f_k \times f_l)^{-1}(U)$.

Theorem 2.17. *[10] The isospectral Hilbert scheme X_n is irreducible of dimension $2n$.*

Proposition 2.18. *[4] The closed subset $V(y_1, \dots, y_n)$ in X_n has dimension n.*

Lemma 2.19. *[2] Let G_r be the closed subset of H_n consisting of ideals I for which $\sigma(I)$ contains some points with moltiplicity at least r. Then G_r*

has codimension $r-1$ and has a unique irreducible component of maximal dimension.

□

2.3 The Nested Hilbert scheme

Definition 2.20. The nested Hilbert scheme $H_{n-1,n}$ is the reduced closed subscheme defined on the closed points as

$$H_{n-1,n} = \{(I_{n-1}, I_n) \mid I_n \subset I_{n-1}\} \subseteq H_{n-1} \times H_n.$$

There exists a result analog to the Fogarty's theorem but for the nested Hilbert scheme and it is the following.

Theorem 2.21. *[3] The nested Hilbert scheme is non-singular and irreducible of dimension $2n$.*

First let's notice that both H_n and $H_{n-1,n}$ are respectively open subsets of $\mathrm{Hilb}^n(\mathbb{P}^2)$ and $\mathrm{Hilb}^{n-1,n}(\mathbb{P}^2)$ and, given the projection between these two schemes, $H_{n-1,n}$ is exactly the preimage of H_n, thus the morphism

$$H_{n-1,n} \to H_n$$

is a projective morphism.
From the definition is clear that if $\sigma(I_{n-1}) = (P_1, \ldots, P_{n-1})$ then $\sigma(I_n) = (P_1, \ldots, P_{n-1}, P_n)$ for some P_n thus the coordinate of this last points are regular functions on $H_{n-1,n}$ resulting from the difference

$$x_n = (x_1 + \cdots + x_n) - (x_1 + \cdots + x_{n-1})$$

(same idea for y_n).
This means that we have a morphism

$$\sigma : H_{n-1,n} \to S_{n-1}\mathbb{C}^2 \times \mathbb{C}^2 = \mathbb{C}^{2n}/S_{n-1}$$

and

$$\alpha : H_{n-1,n} \to H_n \times \mathbb{C}^2$$

sending (I_{n-1}, I_n) to (I_n, P_n) that is precisely the universal family F over H_n.

Now we have to define the nested isospectral Hilbert scheme that will come necessary for the induction proof of our big theorem claiming the nice properties of X_n.

Definition 2.22. The nested isospectral Hilbert scheme $X_{n-1,n}$ is the reduced fiber product $H_{n-1,n} \times_{H_{n-1}} X_{n-1}$.

There is also an alternative formulation we will use during the proof: we can in fact identify $X_{n-1,n}$ as the reduced fiber product in the diagram

$$\begin{tikzcd} X_{n-1,n} \arrow[r] \arrow[d] & \mathbb{C}^{2n} \arrow[d] \\ H_{n-1,n} \arrow[r, "\sigma"] & S_{n-1}\mathbb{C}^2 \times \mathbb{C}^2 \end{tikzcd}$$

Intuitively we can think about it as the set of tuples

$$(I_{n-1}, I_n, P_1, \dots, P_n)$$

such that $\sigma(I_n) = (P_1, \dots, P_n)$ and $\sigma(I_{n-1}) = (P_1, \dots, P_{n-1})$.

Lemma 2.23. *Let $k+l = n$ and $U \subseteq \mathbb{C}^{2n}$ open such that $(P_1, \dots, P_k, Q_1, \dots, Q_l) \in U$ if and only if $P_i \neq Q_j$ for all i, j.*
Then the preimage of U in $X_{n,n-1}$ is isomorphic as a scheme over $\mathbb{C}^{2n}$ to the preimage of U in $X_k \times X_{l-1,l}$.

Proof. Thanks to Lemma 2.16 we know that $f^{-1}(U) \subseteq X_n$ is isomorphic to $(f_k \times f_l)^{-1}(U) \subseteq X_k \times X_l$ and same thing holds for X_{n-1}.
We can think about $X_{n,n-1}$ as a subset of $X_n \times X_{n-1}$ where $I_n \subseteq I_{n-1}$. Notice that this is just the closed subset of $(X_k \times X_{l-1}) \times (X_k \times X_l)$ where $I_k, P_1, \dots, P_K$ are the same and $I_l \subseteq I_{l-1}$. But that is exactly $X_k \times X_{l,l-1}$. $\square$

Proposition 2.24. *The closed subset $V(y_1, \dots, y_n) \subseteq X_{n-1,n}$ has dimension n.*

Lastly we need a technical lemma regarding the dimension of the fibers of the morphism

$$\alpha : H_{n-1,n} \to H_n \times \mathbb{C}^2$$

that we will use during the final proof.

Lemma 2.25. *Let d be the dimension of the fiber of the former morphism α over a point $(I, P) \in F$, and let r be the multiplicity of P in $\sigma(I)$. Then d and r satisfy the inequality*

$$r \geq \binom{d+2}{2}.$$

Proof. Remember that $H_{n-1,n}$ is defined as $\{(I_n, I_{n-1}) \mid I_n \subset I_{n-1}\}$ so we need to understand what are the possibile ideals I_{n-1} given $I_n = I$.
Notice that if we consider the local ring $(R/I)_P = (\mathbb{C}[\mathbf{x}, \mathbf{y}]/I)_P$ the I_{n-1} are the lenght one ideals in it. In fact suppose $I = \prod_{i=1}^{n}(x - a_i)$, $P = (x - a_1)$ and $I_{n-1} = \prod_{i \in [n] \setminus j}(x - a_i)$, then in $(R/I)_P$, I_{n-1} looks like $(x - a_1)$ which has lenght 1 in it.

We can also see them as the one dimensional subspaces of $\mathrm{soc}(R/I)_P$ where with this notation we indicate the socle of the localized ring $(R/I)_P$.
Therefore the fiber of α is the projective space $\mathbb{P}(\mathrm{soc}(R/I)_P)$ of dimension d thus $\dim(\mathrm{soc}(R/I)_P) = d + 1$.
Now notice that, under the action of $\mathbb{T}^2$, the closure of every orbit contains a monomial ideal I_μ, intuitively it is true because we can send $a_i \to x_i$ arbitrarly close to zero so morally zero, and since F is finite over H_n every point of F must have a pair $(I_\mu, 0) \in F$ in the closure of its orbit as well. This means that if we take a point such that the dimension of its fiber is maximized, there exists a I_μ in the closure of its orbit with the same dimension therefore the dimension of the fiber is maximized on a point I_μ. The socle of R/I_μ has dimension equal to the number of corners of the diagram μ, if this number is s we have

$$n \geq \binom{s+1}{2}.$$

This means that for every Artin local ring R/I generated over $\mathbb{C}$ by two elements, the socle dimension s and the lenght n of R/I will satisfy $n \geq \binom{s+1}{2}$.
Now the ring $(R/I)_P$ is an Artin local ring of lenght r generated by two elements with socle dimension $d + 1$ so we conclude the proof. $\square$

2.4 Calculation of canonical line bundles

This section is a key point for this thesis, and after two fundamental definitions we will able to understand why.

Definition 2.26. A Noetherian local ring R is called Cohen-Macaulay if there exists a R-regular sequence $x_1, \ldots, x_d$ of the maximal ideal such that $R/(x_1, \ldots, x_n)$ has dimension 0.

Definition 2.27. Noetherian local ring R of dimension zero (equivalently, with R of finite length as an R-module) is Gorenstein if and only if $\hom_R(k, R)$ has dimension 1 as a k-vector space, where k is the residue field of R.

If $\rho : X \to H$ is a finite morphism of equidimensional schemes of the same dimension, with H smooth, then X is Cohen-Macaulay if and only if ρ is flat (See Miracle Flatness 00R4 [15]). For X quasiprojective over $\mathbb{C}$ it follows from duality theory that X is Cohen-Macaulay if and only if the dualizing complex ω_X reduces to a sheaf on each connected component of X. (See 0AWT in [15] for the definitions and the proofs)
In particular, X is Gorenstein if and only if ω_X reduces to a line bundle (i.e., a locally free sheaf of rank 1) on each connected component of X.
Remember that we can see both H_n and X_n as blow-up constructions, which

means that we have a line bundle $\mathcal{O}(1)$ induced by this representation.
Moreover, given $H_n = \operatorname{Proj} T$ and $X_n = \operatorname{Proj} S[tJ] \cong \operatorname{Proj} \oplus_{d\geq 0} J^d$, we have

$$\mathcal{O}_{X_n}(k) = \rho^* \mathcal{O}_{H_n}(k).$$

There is a way to describe the tautological bundle $\mathcal{O}(1)$ in term of push forward of the bundle over F:
Let us take $\pi : F \to H_n$ and notice that since π is an affine morphism we have $F = \operatorname{Spec} B$ where B is $\pi_* F$.
Now B is flat and finite of degree n over H_n thus it is a locally free sheaf of $\mathcal{O}_{H_n}$-modules of rank n.
Now let us state the following very important proposition.

Proposition 2.28. *[9, Proposition 2.6] There is an isomorphism $\wedge^n B \cong \mathcal{O}(1)$ of line bundles over H_n.*

We still need to know the canonical sheaves on H_n and $H_{n,n-1}$ and to compute them we make use of the fact that invertible sheaves on a normal variety are isomorphic if they have isomorphic restrictions to an open set whose complement has codimension at least 2. (See [12, Proposition 1.11])

Definition 2.29. Let $z = ax + by$ a linear form in the variables x, y. We denote with U_z the open subset of H_n consisting of the ideals I for which z generates the $\mathbb{C}$ algebra R/I.
We will also denote U_z the preimmage of U_z under the projection $H_{n-1,n} \to H_n$.

Notice that z generates R/I if and only if $\{1, z, \ldots, z^{n-1}\}$ is a linearly independent set and this is an open condition.

Lemma 2.30. *The complement of $U_x \cup U_y$ has codimension 2 both in H_n and $H_{n-1,n}$.*

Proof. Let $Z = H_n \backslash (U_x \cup U_y)$ and let W be the generic locus of H_n. Remember that the Hilbert-Chow morphism induces an isomorphism between W and $S_n\mathbb{C}^2$, moreover the image of $Z \cap W$ is the set of points where some two of the P_i have the same x-coordinate and another two the same y-coordinate. Intuitively we can say that if we quotient our ring R by an ideal $\prod(x - a_i, y - b_i)$ and we want $V(I)$ not to be a subset of a smooth curve, we need some sort of fat point that comes out whenever we have $a_i = a_j$ or $b_i = b_j$ for some i, j.
This locus has codimension two.
The complement of W has one irriducible component of dimension $2n - 1$ and an open set of this component consists in ideals such that $\sigma(I)$ has a point of multiplicity two and the other all distinct. This open set is clearly

not contained in Z so Z intersected the complement of W has codimension at least 2.
This proves the statement for H_n, let's see what happens in $H_{n-1,n}$.
We know from proposition 2.25 that α has fibers of dimension d only over G_r for $r \geq \binom{d+2}{2}$ and it's possible to prove that the union of these fibers has codimension at leat $\binom{d+1}{2}$.
This solves the cases when $d > 1$, when $d = 1$ let's define the set $G_s \subset H_n$ as the set of ideals such that $V(I)$ contains a point of multiplicity at least S. We notice that the fibers of dimension 1 on G_3 occurs only over non curvilinear points, but the non curvilinear locus of G_3 has codimension at least 3 and this concludes the proof.

□

Lemma 2.31. *The canonical sheaf ω_{H_n} on the Hilbert scheme is trivial, i.e. $\omega_{H_n} \cong \mathcal{O}_{H_n}$.*

Proof. First it is clear that the $2n$-form $d\mathbf{x}d\mathbf{y} = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n$ is S_n invariant (if you apply an odd permutation to this form you get a minus sign from the $\mathbf{x}$ and a minus sign form the $\mathbf{y}$ coordinates thus they just cancel out).
This means that it defines a $2n$ form on the smooth locus in $S_n\mathbb{C}^2$ and therefore, thanks to the Hilbert Chow morphism, a rational $2n$ form on H_n.
Take $I \in H_n$, it is generated as an ideal in R by two polynomials

$$\begin{aligned} &x^n - e_1x^{n-1} + e_2x^{n-2} - \cdots + (-1)^n e_n \\ &y - (a_{n-1}x^{n-1} - a_{n-2}x^{n-2} + \cdots + a_0) \end{aligned} \tag{2.3}$$

where the first one indicates the x-coordinates of the n points and the second one determines the y coordinates.
Conversely for every choice of parameters $\mathbf{e}$ and $\mathbf{a}$ we have that these polynomials determine a point $I \in H_n$.
On the open set where each $\sigma(I)$ has different x coordinates the first of the two polynomials in 2.3 is of the form

$$\prod_{i \in [n]} (x - x_i)$$

and this implies that e_k is just the k elementary symetric function $e_k(\mathbf{x})$.
To determine a_k it is enough to find the interpolating polynomial $\Phi_{\mathbf{a}}(x)$ satisfying $\Phi_{\mathbf{a}}(x_i) = y_i$ for all i which means solving a system of linear equation that is just a matrix identity

$$(y_1, \ldots, y_n) = (a_0, \ldots, a_{n-1})M \tag{2.4}$$

where M is the Vandermonde matrix in the $\mathbf{x}$ variables.
This implies the following identity

$$d\mathbf{a} = \Delta(\mathbf{x})^{-1}d\mathbf{y}$$

where Δ is the determinant of M. Merging this equality with the well known equality

$$d\mathbf{e} = \Delta(\mathbf{x})d\mathbf{x}$$

given by the symetric functions we find

$$d\mathbf{a}\, d\mathbf{e} = d\mathbf{x}\, d\mathbf{y}$$

which proves that $d\mathbf{x}\, d\mathbf{y}$ is a nowhere vanishing regular section of ω on U_x. Clearly the same proof is valid for U_y so we find that $\omega \cong \mathcal{O}$ on $U_x \cup U_y$ thus everywhere thanks to Lemma 2.30. □

Proposition 2.32. *The canonical sheaf $\omega_{H_{n-1,n}}$ is isomorphic to $\mathcal{O}(1,1)$ where*

$$\mathcal{O}(l,k) = \mathcal{O}_{n-1}(k) \otimes \mathcal{O}_n(l)$$

and $\mathcal{O}_n$ is the pullback from H_n on to $H_{n-1,n}$.

Proof. We have tautological sheaves B_{n-1} and B_n pulled back from H_{n-1} and H_n. The kernel L of the canonical surjection $B_n \to B_{n-1}$ is the line bundle with fiber I_{n-1}/I_n at the point (I_{n-1}, I_n). From Proposition 2.28 we have $L = \mathcal{O}(-1,1)$. On the generic locus, the fiber I_{n-1}/I_n can be identified with the one-dimensional space of functions on $V(I_n)$ that vanish except at P_n. Thus the ratio of two sections of L is determined by evaluation at $x = x_n$, $y = y_n$.

Regarding the polynomials in 2.3 as regular functions on $U_x \times \mathbb{C}^2$, they are the defining equations of the universal family $F_x = \pi^{-1}(U_x)$ over $U_x \subseteq H_n$, as a closed subscheme of the affine scheme $U_x \times \mathbb{C}^2$. We can use these defining equations to eliminate e_n and y, showing that F_x is an affine cell with coordinates $x, e_1, \ldots, e_{n-1}, a_0, \ldots, a_{n-1}$.

Over the curvilinear locus, the morphism $\alpha : H_{n-1,n} \to F$ restricts to a bijective morphism of smooth schemes, hence an isomorphism. Under this isomorphism x corresponds to the x-coordinate x_n of the distinguished point, and modulo x_n we can replace the elementary symmetric functions $e_k(x)$ with $e'_k = e_k(x_1, \ldots, x_{n-1})$, for $k = 1, \ldots, n-1$. As in the proof of Proposition 3.6.3, we now calculate that a nowhere vanishing regular section of ω on $U_x \subseteq H_{n-1,n}$ is given by

$$t_x = \frac{1}{\prod_{i=1}^{n-1}(x_n - x_i)} \frac{dx_n \wedge d\alpha_n}{\prod_{i=1}^{n-1}(y_n - y_i)} dx \wedge dy.$$

By symmetry,

$$t_y = -\frac{1}{\prod_{i=1}^{n-1}(y_n - y_i)} \frac{dy_n \wedge d\alpha_n}{\prod_{i=1}^{n-1}(x_n - x_i)} dx \wedge dy$$

is a nowhere vanishing regular section of ω on U_y.
Now, at every point of U_x, the ideal I_{n-1} is generated modulo I_n by

$$x_n^{n-1} - e'_1 x_n^{n-2} + \cdots + (-1)^{n-1} e'_{n-1} = \prod_{i=1}^{n-1} (x - x_i),$$

so this expression represents a nowhere vanishing section s_x of L on U_x. Similarly, $\prod_{i=1}^{n-1}(y - y_i)$ represents a nowhere vanishing section s_y of L on U_y. By the observations in the first paragraph of the proof, the ratio s_x/s_y is the rational function $\frac{\prod_{i=1}^{n-1}(x_n - x_i)}{\prod_{i=1}^{n-1}(y_n - y_i)}$ on $H_{n-1,n}$. Since we have nowhere vanishing sections t_x, t_y of ω on U_x and U_y with $t_y/t_x = s_x/s_y$ it follows that we have $\omega \sim L^{-1} = \mathcal{O}(1, -1)$ on $U_x \cap U_y$ and hence everywhere, by Lemma 2.30.

□

2.5 The ideal sheaf of X_n

Let us start recalling the notation and where we are with the proof of the conjecture:

- We define J_μ as the annihilating ideal of Δ_μ which means:

$$J_\mu = \{p \in \mathbb{C}[\mathbf{x}, \mathbf{y}] \,|\, p(\partial \mathbf{x}, \partial \mathbf{y}) \Delta_\mu = 0\}.$$

- We call R_μ the ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]/J_\mu$.
- R_μ is Gorenstein and it has the same dimension as a vector space as D_μ.(See [5, Proposition 4])

To give an idea of why the latter statement is true, we can notice that J_μ is the annihilating ideal of the Macaulay inverse system generatd by Δ_μ. In fact it is possible to prove that if $M \subseteq R$ is the annihilating ideal of a Macaulay inverse system generated by one element then R/M is Gorenstein. There is a nice way to describe the ideal J_μ by using the alternating operator

$$\Theta^\epsilon : \mathbb{C}[\mathbf{x}, \mathbf{y}] \to A$$

such that

$$\Theta^\epsilon(g) = \sum_{\omega \in S_n} \epsilon(\omega) \omega g$$

where $\epsilon(\omega)$ is the sign of the permutation.

Proposition 2.33. *The ideal J_μ is equal to the set of polunomials $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ such that the coefficient of Δ_μ in $\Theta^\epsilon(gp)$ is zero for all $g \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$.*

This statement makes sense because if we set $D = \{(p_1, q_1), \ldots, (p_n, q_n)\}$ the set of coordinates of a subset of $\mathbb{N} \times \mathbb{N}$ of cardinality n then the determinant of the matrix defined in the first chapter is exaclty

$$\Delta_D = \Theta^{\epsilon}(\mathbf{x}^p, \mathbf{y}^q).$$

This means that the set of all Δ_D is a basis of A.
Let us see the proof.

Proof. Observe that the constant term of $g(\partial\mathbf{x}, \partial\mathbf{y})p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu$ is, apart from a constant factor, the coefficient of Δ_μ in A. Hence if $p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu = 0$, the characterization certainly holds. Conversely, if the characterization holds, then $p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu$ has the property that it and all its partial derivatives of all orders have zero constant term.
By Taylor's theorem this implies that $p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu = 0$. $\square$

Now we notice that we can describe the closed points of X_n as a subset of the following set:

$$F^{\times^n}/_\sim = \{(I_1; P_1; \ldots; I_n; P_n) \mid P_i \in V(I_i)\ \forall\, i\}$$

where the equivalence relation $\sim$ means $I_1 = I_2 = \cdots = I_n$ and denote with ρ the projection $H_n \times \mathbb{C}^{2n} \to H_n$.
Thanks to this description we can define X_n as a scheme over H_n by

$$X_n = \operatorname{Spec} B^{\otimes n}/\mathcal{J}$$

for some sheaf of ideals $\mathcal{J}$. Our next goal is to describe it.

Proposition 2.34. *Let*

$$\Phi : B^{\otimes^n} \to (B^{\otimes^n})^* \otimes \wedge^n B$$

the homomorphism induced by

$$B^{\otimes^n} \otimes B^{\otimes^n} \to B^{\otimes^n} \to \wedge^n B$$

where the first map is the product and the second is the alternating operator. Then the ideal $\mathcal{J}$ is the kernel of Φ.

Proof. First let us specify where the map comes from:
Given a map

$$\alpha : B^{\otimes^n} \otimes B^{\otimes^n} \to \wedge^n B$$

it's clear it induces a map

$$\alpha' : B^{\otimes^n} \to \operatorname{Hom}(B^{\otimes^n}, \wedge^n B)$$

such that $\alpha'(s) = \alpha_s : B^{\otimes^n} \to \wedge^n B$. Now it's easy to see that $\mathrm{Hom}(B^{\otimes^n}, \wedge^n B) \cong (B^{\otimes^n})^* \otimes \wedge^n B$, using the map that associates the pair $f, w \in (B^{\otimes^n})^* \otimes \wedge^n B$ to the map that sends $s \in B^{\otimes^n}$ to $f(s) \cdot w$ belonging to $\mathrm{Hom}(B^{\otimes^n}, \wedge^n B)$.

Now let

$$\Theta^\epsilon f = \sum_{\omega \in S_n} \epsilon(\omega)\omega(f)$$

be the alternation operator.
Notice that $s \in \ker \Phi$ if and only if $\Theta(sg) = 0 \; \forall \, g \in B^{\otimes^n}$.
Now if $s \in \mathcal{J}$ then clearly $\Theta(sg)$ is still in $\mathcal{J}$ (as it is a combinations of elements in the ideal $\mathcal{J}$).
Let's call V the set of points in $\frac{F^n}{H_n}$ such that $\exists \; i \neq j$ such that $P_i = P_j$, notice that $X_n \sqcup V = \frac{F^n}{H_n}$.
Now consider $s \in \mathcal{J}$, then s must vanishes in X_n, moreover for all g, $\Theta(gs)$ is alternating so it vanishes on V as well as in X_n.
This implies that s belongs to $\ker \Theta$.
Then suppose $s \notin \mathcal{J}$, then it will not vanishes on X_n so there exists $x = (I; P_1; \ldots; P_n)$ with $P_i \neq P_j$ such that $s(x) \neq 0$, so we can find g such that $gs(\omega(x)) = 0$ for all $1 \neq \omega \in S_n$ but $gs(x) \neq 0$.
For example, we can take g as follows:

$$g = \sum_{i \in [n]} \prod_{j \in [n] \setminus i} (X_i - P_j)$$

$\square$

The inverse system of an S-submodule N of a module M is the annihilator I of N in S.
In other words it's the set of elements in S such that for all $n \in N$ we have $sn = 0$.
Now let's consider the set $D\mu = p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu$ and it's annihilating ideal $J_\mu = \{p \in \mathbb{C}[\mathbf{x}, \mathbf{y}] \mid p(\partial\mathbf{x}, \partial\mathbf{y})\Delta_\mu = 0\}$.

Remember that with I_μ we denote the ideal generated by all monomials with esponents not in μ:

$$I_\mu = (x^i y^j \mid (i, j) \notin \mu)$$

This implies that in the ring:

$$B^{\otimes^n}(I_\mu) = \mathbb{C}[\mathbf{x}, \mathbf{y}]/(\sum_{i \in [n]} I_\mu(x_i, y_i))$$

the image of Δ_L (that is a polynomial in $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$) vanishes for all $L \neq \mu$ and the image of Δ_μ spans the space $\wedge^n B(I_\mu)$.
The reason why the first claim is true is that if $L \neq \mu$ there exist a point (p_j, q_j) not in the Young diagram of μ so the monomial $x_i^{p_j} y_i^{q_j}$ will always be in $I_\mu(x_i, y_i)$. Now consider the following diagram:

$$\begin{tikzcd} \mathbb{C}[\mathbf{x},\mathbf{y}] \arrow[r, "\pi"] \arrow[rr, bend left, "\Theta_{|\Delta_\mu}"] & B^{\otimes^n}(I_\mu) \arrow[r, "\beta"] & \wedge^n B(I_\mu) \end{tikzcd}$$

and notice that it's commutative.
To verify this it's enough to check it on the generator Δ_μ, but $\Theta_{|\Delta_\mu}(\Delta_\mu) = \Delta_\mu = \beta \cdot \pi(\Delta_\mu)$. This observation yelds to the following proposition:

Proposition 2.35. *The ideal J_μ is the kernel of the composite map*

$$\mathbb{C}[\mathbf{x},\mathbf{y}] \to B^{\otimes^n}(I_\mu) \to B^{\otimes^n}(I_\mu)^* \otimes \wedge^n B(I_\mu)$$

Proof. We will prove this statement in the next very important theorem so let us try to understand clearly what we mean with the notation $B(I)$.
First consider the commutative diagram

$$\begin{tikzcd} \pi^{-1}(I) \arrow[r, hook] \arrow[d, "\pi"] & F \arrow[d, "\pi"] \\ \{I\} \arrow[r, hook] & H_n \end{tikzcd}$$

and remember that we defined the sheaf B as the push forward of the sheaf $\mathcal{O}_F$ through the projection map $\pi : F \to H_n$.
If we see F as $\mathrm{Spec}(B)$ then $\pi^{-1}(I)$ is exactly $\mathrm{Spec}(B \otimes_{\mathcal{O}_{H_n}} I)$ thus we denote

$$B(I) \coloneqq B \otimes_{\mathcal{O}_{H_n}} I.$$

In particular for the mononial ideal I_μ we have:

$$B(I_\mu) \coloneqq B \otimes_{\mathcal{O}_{H_n}} I_\mu.$$

□

Now let us state the key result of this work.

Theorem 2.36. *Let Q_μ the unique point of X_n lying over I_μ.*
The following are equivalent:
1) X_n is locally Cohen-Macaulay and Gorenstein at Q_μ
2) the $n!$-conjecture holds for the partition μ.

Proof. The sheaf homomorphism Φ in 2.35 can be identified with a linear homomorphism of vector bundles over H_n:

$$\Phi(I) : B^{\otimes^n}(I) \to (B^{\otimes^n}(I))^* \otimes \wedge^n B(I).$$

where I is a point in the generic locus of H_n.
Remember that for a linear map it's true that the map $\mathrm{rk}\,\Phi(I)$ is lower semicontiuous, that implies the set $\{I \in H_n \mid \mathrm{rk}\,\Phi(I) \geq r\}$ to be open for all r.

Now notice that if the rank of $\Phi(I)$ is constant on an open set U then the cokernel of Φ is locally free on U(and vicecersa).
The reason is that for each I you can see the cokernel as

$$\big((B^{\otimes^n})^*(I_\mu)\otimes\wedge^n B(I)\big)\,/\,\big(\Phi(I)(B^{\otimes^n}(I))\big)$$

and being both locally free sheaves, the cokernel will be locally free.
Clearly the same holds for the image of Φ.
Now notice that for a generic point I the fiber consists of $n!$ points on X_n (one for each possible permutation of the n points), so the generic rank of Φ is $n!$.
Now suppose that the $n!$ conjecture holds for μ:
remember the map:

$$\begin{tikzcd}
\mathbb{C}[\mathbf{x},\mathbf{y}] \arrow[rr, "\eta"] \arrow[drr, "\pi"'] & & (B^{\otimes^n})^*(I_\mu)\otimes\wedge^n B(I_\mu) \\
& & \mathbb{C}[\mathbf{x},\mathbf{y}]/I_\mu \arrow[u, "\Phi"]
\end{tikzcd}$$

since η and Φ have the same image we have that $\operatorname{rk}\Phi(I_\mu)=n!$.
Also $n!$ is the rank of $\Phi(I)$ for I in the generic locus which is dense, so $\operatorname{rk}\Phi$ is locally constant (equal to $n!$) around I_μ.
Using our first observation regarding the morphism 2.35 we find that the image of Φ

$$\Phi(\mathbb{C}[\mathbf{x},\mathbf{y}]/I_\mu)=\rho_*\mathcal{O}_{X_n}$$

is locally free, ρ is flat of degree $n!$ and X_n is locally Cohen-Macaulay at Q_μ.
Now we have to prove that $\mathcal{O}_{X_n,Q_\mu}$ is Gorenstein:
let M be the maximal ideal of the local ring $\mathcal{O}_{H_n,I_\mu}$, since X_n is finite over H_n the ideal $N=M\mathcal{O}_{X_n,Q_\mu}$ is a parameter ideal. A parameter ideal is an ideal generated by elements that are algebraically independent over the base ring. In this case if we take $p_1,\dots,p_m$ a set of independent generators of M and a ring homomorphism $\Phi:\mathcal{O}_{H_n,I_\mu}\to\mathcal{O}_{X_n,Q_\mu}$ it is clear that $\Phi(p_1),\dots,\Phi(p_m)$ must be independent over the base ring $\mathcal{O}_{H_n,I_\mu}$.
Since X_n is locally Cohen Macaulay at Q_μ, $\mathcal{O}_{X_n,Q_\mu}$ is Gorenstein if and only if $\mathcal{O}_{X_n,Q_\mu}/N$ is Gorenstein.
By definition we have $\mathcal{O}_{X_n,Q_\mu}/N\cong\Phi(I_\mu)\otimes_{\mathcal{O}_{H_n}}\mathcal{O}_{H_n,I_\mu}$ so the map $\Phi(I_\mu)$ factors as

$$\Phi(I_\mu):B^{\otimes^n}(I_\mu)\to\mathcal{O}_{X_n,Q_\mu}/N\to(B^{\otimes^n}(I_\mu))^*\otimes\wedge^n B(I_\mu). \tag{2.5}$$

Remember that the kernell of η is J_μ so its image is isomorphic to $\mathbb{C}[\mathbf{x},\mathbf{y}]/J_\mu$, moreover the second morphism above is surjective so we find

$$\mathcal{O}_{X_n,Q_\mu}/N\cong\mathbb{C}[\mathbf{x},\mathbf{y}]/J_\mu.$$

We already proved that J_μ is the annihilating ideal of a Macaulay inverse system generated by one element so the latter space is Gorenstein, this proves our initial claim.
Conversely suppose X_n locally Gorenstein at Q_μ, then $\mathcal{O}_{X_n,Q_\mu}/N$ is Gorenstein and so isomorphic to $\mathbb{C}[\mathbf{x},\mathbf{y}]/\mathcal{J}$ for some ideal $\mathcal{J}$.
We will prove something even stronger than the n! conjecture, in fact we will prove that $B(I)$ affords the regular S_n-representation for all I thus $B(I) \cong B(I_\mu)$.
We know that $p_*\mathcal{O}_{X_n}$ is locally free of rank $n!$ around I_μ. It's not difficult to see that $\mathcal{O}_{X_n}$ is S_n-equivariant, plus the morphism of sheaf p_* respects the action so $p_*\mathcal{O}_{X_n}$ is a S_n-module.
The isotypic components of such a bundle are direct summands of it and hence locally free themselves, so the character of S_n on the fibers is constant.
We know that we can write our bundle as a direct sum of irriducible components (the isotypic components of our representation), and since the starting bundle is locally free, its components must be locally free as well.
Therefore the character of S_n on the fibers must be constant because any automorphism of the vector bundle that permutes the isotypic components must preserve that character.
Finally if we take a general point I, the fibers are the coordinate rings of the S_n orbits of points $(P_1, \dots, P_n) \in \mathbb{C}^{2n}$ with all P_i distinct. Therefore every fiber affords the regular representation of S_n.
Now, since X_n is a scheme over H_n, the group scheme S_n acts on X_n via permutation of the n points. This action induces an S_n-action on the inverse image sheaf $p_*\mathcal{O}_{X_n}$ as follows: for any permutation $\sigma \in S_n$ and any open subset $U \subseteq H_n$, we define $\sigma(p_*\mathcal{O}_{X_n})(U)$ as $\mathcal{O}_{X_n}(\sigma(p^{-1}(U))$. Notice that if $(P_1, \dots, P_n)$ are all distinct, so I is in the generic locus, the fiber $p_*\mathcal{O}_{X_n}(I)$ is the coordinate ring of the orbit of those points, thus there exists an injective map

$$S_n \to GL(p_*\mathcal{O}_{X_n}(I)).$$

Since the action of S_n on X_n is compatible with the morphism $p : X_n \to H_n$ and also clearly with the projection $X_n \to X_n/S_n$ it follows that that $p_*\mathcal{O}_{X_n}$ is a sheaf of S_n-modules on H_n.
The socle of $\mathbb{C}[\mathbf{x},\mathbf{y}]/\mathcal{J}$ is a one dimensional S_n-invariant subspace.
It affords a regular representation, so such simple submodules are only $(\mathbb{C}[\mathbf{x},\mathbf{y}]/\mathcal{J})^{S_n}$ which are the constants and $(\mathbb{C}[\mathbf{x},\mathbf{y}]/\mathcal{J})^{\epsilon}$ the space of alternating polynomials modulo $\mathcal{J}$. The socle must therefore be the latter space. Now if we consider the factorization of the map $\Phi(I_\mu)$ in 2.5 we have $\mathcal{J} \subseteq J_\mu = \ker \eta$. Supose $J_\mu/\mathcal{J} \neq 0$, then we must have $\mathrm{soc}(\mathbb{C}[\mathbf{x},\mathbf{y}]/\mathcal{J}) \subseteq J_\mu/\mathcal{J}$ as the socle is contained in every non zero ideal. But this would imply $(\mathbb{C}[\mathbf{x},\mathbf{y}]/J_\mu)^{\epsilon} = 0$ (it means that J_μ contains all the alternating polynomials) hence $\Delta_\mu \in J_\mu$ which is absurd.

□

2.6 Fundational results for the final proof

In this section we provide two other necessary results to prove the conjecture. To prove them we need a result claiming that the ring ideal generated by the alternating polynomials $\mathbb{C}[\mathbf{x},\mathbf{y}]^{\epsilon}$ is a $\mathbb{C}[\mathbf{y}]$-free module. To show this lemma we need polygraphs, a construction we will introduce in the third chapter.

Lemma 2.37. *Let $J = \mathbb{C}[\mathbf{x},\mathbf{y}]A$ the ideal generated by $A = \mathbb{C}[\mathbf{x},\mathbf{y}]^{\epsilon}$. Then J^d is a free $\mathbb{C}[\mathbf{y}]$-module for all d.*

The proof of this lemma is based on the fact that $R(n,l)$, that is the coordinate ring of the polygraph $Z(n,l)$, which we will define later, is a free $\mathbb{C}[\mathbf{y}]$-module.
In fact we will se that $R(n,l) = \mathbb{C}[\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}]/I(n,l)$ is a G-invariant ideal for G the cartesian product of d copies of S_n.
This means that G acts on $R(n,l)$ and we claim that J^d is isomorphic as a $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module to the space $R(n,l)^{\epsilon}$ of G-alternating elements of $R(n,l)$ which is a free $\mathbb{C}[\mathbf{y}]$-module.
Let us prove it formally.

Proof. Set $l = nd$, and let $Z(n,l)$ be the polygraph over $\mathbb{C}$, a subspace arrangement in $(\mathbb{C}^2)^n \times (\mathbb{C}^2)^l$. Let $G = S_{nd}$ be the Cartesian product of d copies of the symmetric group S_n, acting on $(\mathbb{C}^2)^n \times (\mathbb{C}^2)^l$ by permuting the factors in $(\mathbb{C}^2)^l$ in d consecutive blocks of length n. In other words, each $w \in G$ fixes the coordinates x, y on $(\mathbb{C}^2)^n$, and for each $k = 0, \dots, d-1$, it permutes the coordinate pairs a_{kn+1}, b_{kn+1} through a_{kn+n}, b_{kn+n} among themselves.

Let $R(n,l) = \mathbb{C}[\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}]/I(n,l)$ be the coordinate ring of $Z(n,l)$. By Theorem 3.5, $R(n,l)$ is a free $\mathbb{C}[\mathbf{y}]$-module. Due to the symmetry of its definition, $I(n,l)$ is a G-invariant ideal, so G acts on $R(n,l)$. We claim that J^d is isomorphic as a $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module to the space $R(n,l)^{\epsilon}$ of G-alternating elements of $R(n,l)$. Each $\mathbf{x}$-degree homogeneous component of $R(n,l)$ is a finitely generated $\mathbf{y}$-graded free $\mathbb{C}[\mathbf{y}]$-module. Since $R(n,l)^{\epsilon}$ is a graded direct summand of $R(n,l)$, it is a free $\mathbb{C}[\mathbf{y}]$-module, so the claim proves the Lemma.

Let $f_0 : [l] \to [n]$ be defined by $f_0(kn+i) = i$ for all $0 \le k < d$, $1 \le i \le n$. Restriction of regular functions from $Z(n,l)$ to its component subspace W_{f_0} is given by the $\mathbb{C}[\mathbf{x},\mathbf{y}]$-algebra homomorphism $\psi : R(n,l) \to \mathbb{C}[\mathbf{x},\mathbf{y}]$ mapping a_{kn+i}, b_{kn+i} to x_i, y_i. Observe that ψ maps $R(n,l)^{\epsilon}$ surjectively onto $\mathbb{C}[\mathbf{x},\mathbf{y}]A^d = J^d$.

Let p be an arbitrary element of $R(n,l)^{\epsilon}$. Since p is G-alternating, p vanishes on W_f if $f(kn+i) = f(kn+j)$ for some $0 \le k < d$ and some

$1 \leq i < j \leq n$. Thus, the regular function defined by p on $Z(n,l)$ is determined by its restriction to those components W_f such that for each k, the sequence $f(kn+1), \ldots, f(kn+n)$ is a permutation of $\{1, \ldots, n\}$. Moreover, for every such f, there is an element $w \in G$ carrying W_f onto W_{f_0}. Hence p is determined by its restriction to W_{f_0}. This shows that p vanishes on $Z(n,l)$ if $\psi(p) = 0$, that is, the kernel of the map $\psi : R(n,l)^\epsilon \to J^d$ is zero.

□

The consequence of this lemma is the following very important corollary:

Corollary 2.38. *The projection $X_n \to \mathbb{C}^n = \operatorname{Spec} \mathbb{C}[\mathbf{y}]$ of X_n of the $\mathbf{y}$ coordinates is flat.*

Proof. Notice that

$$X_n = \operatorname{Proj} \mathbb{C}[\mathbf{x}, \mathbf{y}][tJ]$$

and, being J^d a free $\mathbb{C}[\mathbf{y}]$-module, the projection induces a flat map between free $\mathbb{C}[\mathbf{y}]$ modules.
Remember that every projective module is flat and every free module is projective.(05CF [15]) □

Now observe that

$$J \subseteq \bigcap_{i<j} (x_i - x_j, y_i - y_j)$$

because every alternating polynomial vanishes whenever two or more coordinates coincides.
Moreover it is possible to prove that

$$J^d = \bigcap_{i<j} (x_i - x_j, y_i - y_j)^d \tag{2.6}$$

for all $d \geq 0$.(See [10])

Proposition 2.39. *The isospectral scheme X_n is arithmetically normal in its projective embedding over $\mathbb{C}^{2n}$ as the blow-up $X_n = \operatorname{Proj} S[tJ]$. In particular X_n is normal.*

Proof. Arithmetically normal means that $S[tJ]$ is a normal domain, that is an integral domain such that each localization is equal to its integral closure in its field of fraction.
$S[tJ]$ is already a normal domain: $S = \bigoplus_{d \geq 0} A^d$ is normal if A^d is normal, but each localization of it is an integral domain, that implies being an integrally closed domain.
The powers of an ideal generated by a regular sequence are integrally closed, as is an intersection of integrally closed ideals, so J^d is integrally closed. □

The next Lemma is a central result for our proof, but before stating it we try to give a general overview of what a derived functor is.
The derived direct image functor is a way of "sheafifying" a functor that assigns to each sheaf on a source scheme X a sheaf on a target scheme Y via a morphism of schemes $f : X \to Y$. More specifically, given a morphism of schemes $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, the direct image functor

$$f : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

assigns to $\mathcal{F}$ the sheaf $f(\mathcal{F})$ on Y defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

for any open subset $U \subseteq Y$. However, in general, the direct image functor is not exact, meaning that it may not preserve exact sequences of sheaves. To remedy this, one can define the derived direct image functor

$$Rf : \mathrm{D}(\mathrm{Sh}(X)) \to \mathrm{D}(\mathrm{Sh}(Y))$$

as a way of making f_* exact. Here, $\mathrm{D}(\mathrm{Sh}(X))$ and $\mathrm{D}(\mathrm{Sh}(Y))$ denote the derived categories of sheaves on X and Y, respectively, which are categories that encode cohomological information about sheaves.
Intuitively, the derived direct image functor Rf_* replaces the sheaf $f(\mathcal{F})$ with a complex of sheaves that encodes the cohomology of $f(\mathcal{F})$. This complex of sheaves is constructed by applying the "derived functor" of f_*, which involves taking a "resolution" of $\mathcal{F}$ (i.e., replacing $\mathcal{F}$ with a complex of sheaves that is quasi-isomorphic to $\mathcal{F}$) and then applying f_* to the complex.

Lemma 2.40. *[10, Lemma 3.8.5] Let $g : X \to Y$ be a proper morphism. Let $z_1, \ldots, z_m \in \mathcal{O}_X(X)$ be global regular functions, $Z = V(z_1, \ldots, z_n)$ and $U = X \backslash Z$.*
Suppose the following conditions hold:

1. *The z_i form a regular sequence in the local ring $\mathcal{O}_{X,P}$ for all $P \in Z$.*
2. *The z_i form a regular sequence in the local ring $\mathcal{O}_{Y,Q}$ for all $Q \in g^{-1}(Z)$.*
3. *Every fiber of g has dimension less then $m - 1$.*
4. *The canonical homomorphism $\mathcal{O}_X \to Rg_*\mathcal{O}_Y$ restricts to an isomorphism on U.*

Then $Rg_*\mathcal{O}_Y = \mathcal{O}_X$, i.e., the canonical homomorphism is an isomorphism.

2.7 Proof by induction: $X_{n-1,n}$ Cohen Macaulay

For technical reasons we want to start the induction for $n = 3$, but fortunately the cases $n = 1$ and $n = 2$ are very easy.
For $n = 1$ we have $S_1 = \{1\}$ so $\mathbb{C}[x, y]^{S_n} = \mathbb{C}[x, y]$ and $\mathbb{C}[x, y]^{\epsilon} = \mathbb{C}[x, y]$ as well because there are no odd permutations.
This means that we can describe H_2 as $\operatorname{Proj} \bigoplus \mathbb{C}[x, y]$ that is just $\operatorname{Spec} \mathbb{C}[x, y] = \mathbb{A}^2$.
For n=2 we have that every pair of points (P_1, P_2) are generated by complete intersection ideals, i.e. the sequence of generators is regular iff $P_1 \neq P_2$. The Hilbert scheme H_2 defined as $\operatorname{Proj} S$ is just the blow-up of $\mathbb{C}^4$ along the diagonal thus the fiber product over $S_2\mathbb{C}^2$ is itself. We know that H_2 is non singular and so is X_2.
Moreover for n=2 it's pretty easy to prove directly the $n!$-conjecture, and since we proved that it is equivalent to X_n being Cohen-Macaulay and Gorenstein we conclude.
Now assume by induction that X_{n-1} is Cohen-Macaulay and Gorenstein, then $p_{n-1} : X_{n-1} \to H_{n-1}$ is flat.
This implies that in the following diagram even $p^{'}$ is flat:

$$\begin{tikzcd} Y \arrow[r, "p^{'}"] \arrow[d] & H_{n-1,n} \arrow[d] \\ X_{n-1} \arrow[r, "p"] & H_{n-1} \end{tikzcd}$$

On the generic locus, where all points are distinct, the above diagram coincides locally with:

$$\begin{tikzcd} Y \arrow[r, "p^{'}"] \arrow[d] & S_{n-1}\mathbb{C}^2 \times \mathbb{C}^2 \arrow[d] \\ \mathbb{C}^{2(n-1)} \arrow[r, "p"] & S_{n-1}\mathbb{C}^2 \end{tikzcd}$$

which shows that Y is generically reduced (hence reduced) as well as irreducible and birational to $\mathbb{C}^{2n}$. This means that $Y = X_{n-1,n}$ and since $p^{'}$ is flat and finite and $H_{n-1,n}$ is non singular, $X_{n-1,n}$ is Cohen-Macaulay.

2.8 Proof by induction: X_n Gorenstein

Let us recap where we are:
we know that if X is a Noetherian scheme with dualizing complex ω_X, to prove that it is Gorenstein it is enough to prove that ω_X is a line bundle.(0AWT [15])

It's easy to prove that $X_{n-1,n}$ is Gorenstein, in fact we know from Proposition 2.32 that the canonical sheaf $\omega_{H_{n-1,n}}$ is $\mathcal{O}(1,-1)$.
Now by construction, supposing X_{n-1} Gorenstein, we have that

$$\omega_{X_{n-1,n}} = \mathcal{O}(-1,0) \otimes \mathcal{O}(1,-1) = \mathcal{O}(0,-1)$$

thus our scheme $X_{n-1,n}$ is Gorenstein.
Now we want to use this fact to prove that X_n is Gorenstein, in particular that $\omega_{X_n} = \mathcal{O}(-1)$.
The proof relies on the following claim, whose proof can be found in here [10]. Considering the projection

$$g : X_{n-1,n} \to X_n$$

we claim that

$$Rg_* X_{n-1,n} = \mathcal{O}_{X_n}.$$

By the projection formula, since $\mathcal{O}(0,-1) = g_* \mathcal{O} X_n(-1)$ is pulled back from X_n, this implies also $Rg_* \mathcal{O}(0,-1) = \mathcal{O} X_n(-1)$. Now

$$\mathcal{O}(0,-1)[2n] = \omega_{X_{n-1,n}}[2n]$$

is the dualizing complex on $X_{n-1,n}$, so by the duality theorem it follows that $\mathcal{O}(-1)[2n]$ is the dualizing complex on X_n. In other words, X_n is Gorenstein, with canonical sheaf $\omega_{X_n} = \mathcal{O}(-1)$, which is what we wanted to prove.
To prove the inductive base we know from Theorem 2.36 that it is not only true that if X_n is Gorenstein then the $n!$ conjecture holds, but even the other way around!
So with the previous induction process we reach the case $n = 3$ for which it is fairly easy to prove by hand our conjecture (see example 2.8; this means that X_3 is Gorenstein.
Finally remember that for Noetherian schemes, Gorenstein implies Cohen-Macaulay [1] thus we proved these two properties for our scheme X_n.
Let us remark the idea of the previous proof:

Remark 2.41. Let us remark the idea of the previous proof:

- We just proved that X_{n-1} Gorenstein implies X_n Gorenstein.
- Thanks to Theorem 2.36 X_n is Gorenstein if and only if the $n!$ conjecture holds.
- For X_3 we proved by hand the $n!$ conjecture, thus it is Gorenstein.
- X_n is Gorenstein thus the $n!$ conjecture holds.

Chapter 3

Polygraphs

In this chapter we will investigate polygraphs, particular arrangements of points in the space arising by graphs of linear functions from E^n to E^l where $E = \mathbb{A}^2$.
Let us start fixing some notation: let E be $\mathbb{A}^2(k)$ where k is a 0 characteristic field.
Given a function $f : [l] \to [n]$ there is a linear morphism $\pi_f : E^n \to E^l$ such that

$$\pi_f(P_1; \ldots; P_n) = (P_{f(1)}; \ldots; P_{f(n)}).$$

In the following lines we will denote the coordinates of $E^n \times E^l$ by

$$\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} = x_1, y_1, \ldots, x_n, y_n, a_1, b_1, \ldots, a_l, b_l.$$

Now let $W_f \subset E^n \times E^l$ be the graph of π_f, using the coordinates defined above we find that $W_f = V(I_f)$ where

$$I_f = \sum_{i \in [l]} (a_i - x_{f(i)}, b_i - y_{f(i)}).$$

(Remember that $V(I + J) = V(I) \cap V(J)$).
Notice that we have to describe a set of n points in dimension $n + l$, so we need $n(n + l)$ equations, but the first n coordinates of each point is already fixed so it is equivalent to describe n points in dimension l.
This requires only nl equations that are exactly:

$$\begin{cases} a_i - x_{f(i)} = 0 \\ b_i - y_{f(i)} = 0 \end{cases}$$

for each $i \in [l]$ and $P_{f(i)} = (x_{f(i)}, y_{f(i)}) \in (P_{f(1)}; \ldots; P_{f(n)})$.
Now we can define polygraphs:

Definition 3.1. The polygraph $Z(n, l) \subset E^n \times E^l$ is the subspace:

$$Z(n, l) = \bigcup_{f:[n]\to[l]} W_f$$

We can see $Z(n,l)$ as a subscheme of $E^n \times E^l$ so we can identify $P \in Z(n,l)$ as a prime ideal of $k[\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}]$.
In fact our space is simply an affine space and $Z(n,l)$ a set of points, thus we can think about it as a subspace of the Hilbert scheme of points of the starting affine space.
Our goal is to prove the following theorem:

Theorem 3.2. *The coordinate ring $R(n,l) = \mathcal{O}(Z(n,l))$ of the polygraph $Z(n,l)$ is a free $k[\mathbf{y}]$-module.*

To explain why we need this theorem let us give an example when $E = \mathbb{A}^1(k)$, so our coordinates are just $\mathbf{x},\mathbf{a}$.

Example 3.3. In this case the ideal of $Z(n,l)$ is

$$I = \sum_{i\in[l]} \left(\prod_{j\in[n]} (a_i - x_j) \right)$$

and notice that if $x_i \neq x_j$ for all $i,j \in [n]$ then the ideal is reduced.
Moreover it has l generators and its codimension is exactly l so it is a complete intersection ideal, hence $\mathcal{O}(Z(n,l))$ is Cohen-Maculay.
Moreover its finitely generated so $R(n,l)$ is a free $k[\mathbf{x}]$-module.
Now if we consider the finite flat morphism

$$Z(n,l) \to E^n$$

we can intuitively define the degree of the morphism as the number of times a point in E^n is covered by the map, so in our case the degree is equal to the number of graphs that is n^l.

Now in two set of variables the ideal

$$I = \sum_{i\in[l]} \left(\prod_{j\in[n]} (a_i - x_j, b_i - y_j) \right)$$

still defines $Z(n,l)$ as a set but this time is not reduced anymore and $R(n,l)$ is not Cohen-Macaulay.

Definition 3.4. Let $Z(n,l)$ be a polygraph and consider $r \in [n] \cup \{0\}$, $k \in [l] \cup \{0\}$ and $m \in \mathbb{Z}$. We define

$$Y(m,r,k) = \bigcup_{f,T} V(x_j \mid j \in T) \cap W_f$$

where T ranges over subsets of $[n]$ such that

$$|T \cap [r] \backslash f([k])| \geq m.$$

Intuitively $Y(m, r, k)$ is built taking the points of each graph that belong to certain ideals, in partiular ideals generated setting some coordinates to zero, and unify them.
We define $I(m, r, k)$ as the ideal of $Y(m, r, k)$ seen as a closed reduced subscheme of $Z(n, l)$.
Now we can state the precise version of Theorem 3.2 using the previous definition.

Theorem 3.5. *The coordinate ring $R(n, l)$ of the polygraph $Z(n, l)$ is a free $k[y]$-module with a basis B such that every ideal $I(m, r, k)$ is spanned as a $k[y]$-module by a subset of B.*

To prove this theorem we will use induction building the basis B out of the basis of $R(n-1, l)$ and $R(n, l-1)$.
To show that what we build is a basis we will use the following fact:
if we take a space on which the $\mathbf{y}$ coordinates are independent, its coordinate ring is a torsion-free $k[\mathbf{y}]$-module.
If the $\mathbf{y}$-coordinates of the subspaces are dependent, it means that there exists a polynomial in $k[\mathbf{y}]$ that vanishes on all the $\mathbf{y}$-coordinates of the subspaces. This polynomial would then annihilate the corresponding elements in the coordinate ring of each subspace, which would make those elements torsion elements in the coordinate ring. In general, a module is torsion-free if and only if it has no nonzero elements that are annihilated by a nonzero element of the ring. So if there exists a nonzero polynomial in $k[\mathbf{y}]$ that vanishes on all the $\mathbf{y}$-coordinates of the subspaces being considered, then the coordinate rings of these subspaces would not be torsion-free modules over $k[\mathbf{y}]$. On the other hand, if the $\mathbf{y}$-coordinates of the subspaces are independent, then it is not possible for a nonzero polynomial in $k[\mathbf{y}]$ to vanish on all the $\mathbf{y}$-coordinates of the subspaces simultaneously.

To prove that a subset of a torsion-free $k[\mathbf{y}]$-module is a free module basis, it suffices to verify it locally on an open locus $U_2 \subset \operatorname{Spec} k[\mathbf{y}]$ whose complement has codimension two. [14]
Intuitively we could say that removing codimension greater than 2 subscheme does't change the module of global regular function, they are 'to small'.

Lemma 3.6. *Theorem 3.5 holds for $Z(n, 0)$.*

Proof. Clearly $Z(n, 0) = E^n$ and so $R(n, 0) = k[\mathbf{x}, \mathbf{y}]$ that is a free $k[\mathbf{y}]$-module.
Now we have to check what happen to the ideals $Y(m, r, k)$:
k must be 0 so we have union of subspaces where at least the firsts m coordinates are 0.
It is ideal $I(m, r, 0)$ is generated by

$$I(m, r, 0) = \sum_{T} \prod_{j \in T} x_j$$

where T is each subset of $[r]$ of size $r-m+1$.
The reason why it is true is the goal is to set a size of T such that for each choice of $\binom{r}{|T|}$ j we have at least m different x_j.
So suppose we select j_1 from each T where it is contained, then we remain with

$$\binom{r}{|T|} - \binom{r-1}{|T|-1} = \binom{r-1}{|T|}$$

different sets, let us do it again for the second element j_2, then j_3 untill we reach j_{m-1}, at that point we must have at least one set:

$$\binom{r-m+1}{|T|} = 1 \implies |T| = r-m+1$$

The set B of all monomials in the x coordinates is a free $k[\mathbf{y}]$-module basis of $R(n,0)$, with subsets spanning every ideal generated by monomials in x. In particular, each ideal $I(m,r,0)$ is spanned by a subset of B.

□

Now let us define the open sets $\hat{U}_k$ where we will reduce our arguments.

Definition 3.7. The set $\hat{U}_k$ is the open locus in $\operatorname{Spec} k[\mathbf{y}]$ where the coordinates $y_1;\dots;y_n$ assume at least $n-k+1$ distinct values.
Moreover for any scheme $\pi : Z \to \operatorname{Spec} k[\mathbf{y}]$ we define U_k to be $\pi^{-1}(\hat{U}_k)$.

To treat $R(n,l)$ as a $k[\mathbf{y}]$-module, we will want to localize with respect to prime ideals in $k[\mathbf{y}]$, that is, at points $Q \in \hat{U}_k \subset \operatorname{Spec} k[\mathbf{y}]$. To extract local geometric information about $Z(n,l)$ as a subscheme of $E_n \times E_l$, by contrast, we want to localize at points $P \in U_k \subset E_n \times E_l$. A simple technical lemma relates these two types of localization, as follows.

Lemma 3.8. *Let R be a $k[\mathbf{y}]$-algebra, let*

$$\pi : \operatorname{Spec} R \to \operatorname{Spec} k[\mathbf{y}]$$

be the projection on the y. If I, $J \subset R$ are ideals such that $I_P = J_P$ locally for all $P \in U$ (localized as R-modules), then $I_Q = J_Q$ for all $Q \in \hat{U}$ (localized as $k[\mathbf{y}]$-modules).

Proof. It is enough to notice that

$$(I_Q)_{P_Q} = I_P$$

then by ipothesis we have

$$I_P = J_P$$

so we find

$$(I_Q)_{P_Q} = I_P = J_P = (J_Q)_{P_Q}$$

and since it is true for all $P \in U$ we proved the lemma. □

Now let us talk about the local geometry of $Z(n,l)$ on U_1.

Lemma 3.9. *For $f_1 \neq f_2$ we have:*

$$(W_{f_1} \cap U_1) \cup (W_{f_2} \cap U_1) = \emptyset$$

This means that for each $P \in Z(n,l) \cap U_1$ there exists a unique W_f containing P so $Z(n,l)$ coincides locally with W_f:

$$I(Z(n,l))_P = (I_f)_P.$$

Proof. U_1 is the set of points such that each y coordinate is different and since $P \in W_{f_1} \cap W_{f_2}$ iff $y_i = y_j$ for all i,j such that $f_1(i) = f_2(j)$ then we have that if $f_1 \neq f_2$ there is no intersection. □

Now let us state a result about the lattice of ideals in $R(n,l)$ defined with the sum and teh intersections as operations.

Lemma 3.10. *Let $\mathcal{L}$ be the sublattice of the lattice of ideals in R generated by ideals of all subspaces of the form:*

$$V(x_j \mid j \in T) \cap W_f. \tag{3.1}$$

Then for every $I \in \mathcal{L}$, $V(I) \cap U_1$ is reuced i.e. $I_P = \sqrt{I_P}$ for all $P \in U_1$.

Proof. Take a point P in U_1, then $P \in W_g$ for some function $g : [l] \to [n]$. Thanks to the previous observations we see that W_g do not intersect any other W_f around P so locally $W_g \cong E^n$ and $\mathcal{O}(W_g) \cong \mathcal{O}(E^n) = k[\mathbf{x}, \mathbf{y}]$. Now let us consider the ideal generated by:

$$\bigcup_{f,T} V(x_j \mid j \in T) \cap W_f$$

and when we intersect it with U_1 all the W_f became disjointed so:

$$\bigsqcup_f \left(\bigcup_T V(x_j \mid j \in T) \cap W_f \cap U_1 \right)$$

this implies that around P the ideal will looks like the ideal generated by:

$$\bigcup_T V(x_j \mid j \in T) \cap W_g \cap U_1$$

that is just $k[\mathbf{x}, \mathbf{y}]$ modulo square-free monomials in the variables x that is reduced.

□

Corollary 3.11. *If i belongs to the lattice generated by the ideals $I(m,r,k)$ in $R(n,l)$ then $V(I) \cap U_1$ is reduced.*

What happen in U_2? The situation is the following:
Let $Z_2 \subset Z(n,l)$ be the union of those graphs W_f for which $f(i) = f(j) = c$ for a pair $i, j \in [l]$. So we have:

$$\begin{cases} a_i = x_c \\ a_j = x_c \\ b_i = y_c \\ b_j = y_c \end{cases}$$

so we can use these equations to eliminate the coordinates i, j and the result will be that Z is isomorphic to a polygraph $Z(n, l-2) \subset E^n \times E^{l-2}$.
Notice that here a notation problem arises because the indices of $[l-2]$ are not $1, \dots, l-2$, so to fix it we will denote the set of indices as N and L with $|N| = n$ and $|L| = l$.

Lemma 3.12. *Let P a point in $U_2 \backslash U_1$ and let $\{p, q\}$ the unique pair of indices such that $P \in V(y_p - y_q)$ (we are in U_2 so at most one pair of indices can coincide).*
Let $\sim$ the equivalence relation on functions $f : [l] \to [n]$ defined by $f \sim g$ if and only if $\{f(i), g(i)\} = \{p, q\}$ for all $i \in [l]$.
Then:

1. *We have $P \in W_f$ only for f in a unique $\sim$-equivalence class F, so $Z(n,l)$ coincides locally at P with*

$$Z = \bigcup_{f \in F} W_f.$$

2. *Let f be a member of F, let $N = \{p, q\}$ and let $L = f^{-1}(N)$ (note that L depends only on F). The projection of Z on the coordinates $\mathbf{x}, \mathbf{y}, \mathbf{a}_L, \mathbf{b}_L$ is an isomorphism*

$$Z \cong E^{[n] \backslash N} \times Z(N, L)$$

where $Z(N, L)$ is the polygraph in indices N and L.

Proof. 1. Clearly it is impossible to have $P \in \bigcap_{f \in F} W_f$ and $P \in \bigcap_{f \in G} W_f$ for $G \neq F$.

2. On the coordinates in $[n] \backslash L$ each function in F coincide, so the coordinate ring of Z is generated by the remaining variables named $\mathbf{x}, \mathbf{y}, \mathbf{a}_L, \mathbf{b}_L$, so the projection on these coordinates is an isomorphism

of Z into it is image which is $E^{[n]\backslash N} \times Z(N, L)$:
In fact as earlier we have a set of equations:

$$\begin{cases} a_i = x_{f(i)} \\ b_i = y_{f(i)} \end{cases}$$

for all $i \in [n]\backslash L$ that allow us to project the space without losing informations.

□

Now we can prove a very important result, morally it says that the informations that we can extract by localizing a module M over $Q \in \cap U_2$ are enough to gather informations about M.

Lemma 3.13. *Let M be a torsion free $k[\mathbf{y}]$-module and B a subset of M. Suppose that for every $Q \in \hat{U}_2$, M_Q is a free $k[\mathbf{y}]_Q$-module with basis B. Then M is a free $k[\mathbf{y}]$-module.*

Proof. The goal is to prove that for all $x \in M$ we have:

$$x = \sum_{\alpha} p_\alpha b_\alpha \tag{3.2}$$

with $p_\alpha \in k[\mathbf{y}]$ and $b_\alpha \in B$.
We know it is true for each image of x in M_Q where $p_\alpha \in k[\mathbf{y}]_Q$.
Now notice that $k[\mathbf{y}]_Q$ is a subring of $k[\mathbf{y}]_0 = k(\mathbf{y})$, this means that we can see each image embedded into $k(\mathbf{y})$ thus the coefficients p_α do not depend on Q.
Now since the complement of $\hat{U}_2$ has codimension 2 every rational function regular on $\hat{U}_2$ is regular everywhere. This is true because of the following result:

$$\bigcap_{\mathrm{ht}(p)=1} A_p = A$$

for each ring A. Now let p be a prime of height 1 and $\hat{U}_2 = \cup_{f \in I} D(f)$, from $\mathrm{ht}(I) \geq 2$ we deduce that there exists $f \in I$ with $f \notin p$, which means $A_f \subset A_p$. Take $\frac{a}{b}$ regular on U, so in particular regular on $D(f) \subset U$. We deduce $\frac{a}{b} \in \mathcal{O}(D(f)) = A_f \subset A_p$. Since p was arbitrary of height 1, we have shown

$$\frac{a}{b} \in \bigcap_{\mathrm{ht}(p)=1} A_p = A.$$

This proves that p_α belong to $k[\mathbf{y}]$ and since 3.2 holds on a dense set of a torsion free module, it holds everywhere.

□

Corollary 3.14. *Let I, J be free submodules of a torsion free $k[\mathbf{y}]$-module M such that $I_Q = J_Q$ for all $Q \in \hat{U}_2$. Then $I = J$.*

Proof. I, J have locally the same basis so, thanks to the previous lemma they have globally the same basis so they are equal. □

Lemma 3.15. *Let B be a basis of a free $k[\mathbf{y}]$-module M. Let J be a M submodule and $B_1 = B \cap J$ spans $k(\mathbf{y}) \otimes J$. Then $J = k[\mathbf{y}]B_1$.*

Proof. Consider $x \in J$ written as

$$x = \sum_{\alpha} p_\alpha b_\alpha$$

with $p_\alpha \in k[\mathbf{y}]$ and $b_\alpha \in B$.
There exists also a unique representation for x in terms of B as basis of $M \otimes k(\mathbf{y})$ so $p_\alpha = 0$ for $b_\alpha \notin B_1$ so we have $x \in k[\mathbf{y}]B_1$.

□

3.1 Hilbert series

To keep studing the properties of our modules we need Hilbert series which intuitively are a way to encode information about the dimensions of the graded components of an algebra or module. It helps to understand how the dimensions of these components grow as the degree increases. The coefficients of the Hilbert series reveal the structure of the algebra or module at each degree. Let us notice a few fact of our set up:

- The coordinate ring $\mathcal{O}(E^n \times E^l) = k[\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}]$ is doubly graded by the degree of $\mathbf{x}, \mathbf{a}$ variables and the degree of $\mathbf{y}, \mathbf{b}$ variables.
- The ideals defining $Z(n, l)$ are intersections of ideals I_f which are clearly doubly homogeneous thus

$$R(n, l) = k[\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}]/I(Z(n, l))$$

  is doubly graded.
- By construction $Z(n, l)$ is finite over E^n so its ring of coordinates $R(n, l)$ is a finitely generated $k[\mathbf{x}, \mathbf{y}]$-module. This means that if we take the graded ring by the x degree

$$R(n, l) = \bigoplus_d R(n, l)_d$$

  each $R(n, l)_d$ is a finitely generated $k[\mathbf{y}]$-module graded by the y degree.

First we need a version of the Nakayama's lemma for graded modules, then we will define the Hilbert series precisely.

Lemma 3.16. *Let M be a finitely generated graded $k[\mathbf{y}]$-module and B a set of homogeneus elements of M that spans $M/\mathbf{y}M$ as a k-vector space, then B generates M.*
Moreover if

$$|B| = \dim_{k(\mathbf{y})}(k(\mathbf{y}) \otimes M)$$

then M is a free $k[\mathbf{y}]$-module with basis B.

First let me clarify that with the notation $M/\mathbf{y}M$ I indicate

$$M/\oplus_{d>0} k[\mathbf{y}]M.$$

I will give an idea of the proof:

Proof. If we consider the map

$$\pi : M \to M/\mathbf{y}M$$

so it is clear that $B\mathbf{y}$ generates M.
This means that we can write each element of M as

$$m = \sum_j c_j b_j + \sum_i d_i(b_i\mathbf{y})$$

but since B spans $M/\mathbf{y}M$ we know that

$$m + \mathbf{y}M = \sum_k a_k(b_k + \mathbf{y}M)$$

thus

$$m + \mathbf{y}M = \sum_j c_j(b_j + \mathbf{y}M) + \sum_i d_i(b_i\mathbf{y} + \mathbf{y}M) = \sum_k a_k(b_k + \mathbf{y}M)$$

that implies $b_i = 0$ and

$$m = \sum_j c_j b_j.$$

Definition 3.17. Let $R = \oplus_{d\geq 0} R_d$ be a graded ring over k and $M = \oplus_{d\geq 0} M_d$ a graded ring. We define the Hilbert series of M as:

$$H_M(t) = \sum_{d\geq 0} \dim_k(M_d) t^d.$$

Now for some modules it is easy to determine its Hilbert series, thanks to the following lemma.

Lemma 3.18. *[10] Let R the coordinate ring of*

$$\bigcup_C V(x_j \mid j \in T) \cap W_f$$

where C is a collection of pairs $(T; f)$.
Then R is a torsion free $k[\mathbf{y}]$-module and the dimension of the x-degree homogeneus component is equal to the numbers of pairs $e \in \mathbb{N}^n$, f such that

$$\sum_{i\in[n]} e_i = d$$

and there is some $(T, f) \in C$ such that $e_j = 0$ for all $j \in T$.

First we denote with $W_{T,f}$ the subspace $V(x_j \mid j \in T) \cap W_f$, with $I_{T,f}$ the its ideal $I_f + (x_j \mid j \in T)$ and with $R_{T,f}$ its coordinate ring.
By definition we have $R = k[\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}]/I$ where I is the intersection of $I_{T,f}$ for each pair in C.
Notice that the $\mathbf{y}$ coordinates are independent on each $W_{T,f}$ so the coordinate rings $R_{T,f}$ are free and torsion-free $k[\mathbf{y}]$-modules.
Clearly R is isomorphic to a subring of $\bigoplus_C R_{T,f}$ so R is a free and torsion-free $k[\mathbf{y}]$-module.
Now we define C_f to be the set of pairs $(T, f) \in C$ for a given f and R_f as you expect. We have an injective homomorphism

$$R \hookrightarrow \bigoplus_f R_f,$$

by Lemma 3.12 the unions $Z_f = \bigcup_f W_{C_f}$ have disjoint restrictions to U_1 so the above morphism localize to an insomorphism at each point of $\hat{U}_1$. (Remember that if $R = C/\prod_j I_j$, $R_j = C/I_j$ and $Z(I_j) \cap Z(I_i) = \emptyset$ then $R = \oplus R_j$.)
Now the projection of W_f into E^n is an isomorphism so Z_f projects isomorphically on $\operatorname{Spec} k[\mathbf{y}] \times V$ where $V = \cup_{C_f}(x_j \mid j \in T) \subseteq \operatorname{Spec} k[\mathbf{x}]$.
The coordinate ring of V, say $k[\mathbf{x}]/J$ is the face of a simplicial complex, in fact J is the ideal of polynomials vanishing on V:

$$J = \prod_T \left(\sum_{j\in T} x_j \right)$$

that is spanned by $\mathbf{x}^e$ where for each e exists a T such that for each $j \in T$, $e_j = 0$.
The ring R_f is turn is a free $k[\mathbf{y}]$-module with the same basis and since $k(\mathbf{y}) \otimes R \cong \bigoplus_f k(\mathbf{y}) \otimes R_f$ the result follows.

□

Corollary 3.19. *The Hilbert series of $k(\mathbf{y}) \otimes R(n,l)$ as a $k(\mathbf{y})$-algebra x-graded is*

$$\sum_d \dim_{k(\mathbf{y})}(k(\mathbf{y}) \otimes R(n,l)_d)t^d = \frac{n^l}{(1-t)^n}.$$

Proof. For each d the number of $e = (e_1, \dots, e_n)$ fulfilling the requirements of lemma 3.18 is $\binom{n+d-1}{d}$, moreover there are n^l possible functions so we find:

$$\sum_d \dim_{k(\mathbf{y})}(k(\mathbf{y}) \otimes R(n,l)_d)t^d = \sum_d t^d n^l \binom{n+d-1}{d} = n^l(1-t)^{-n}$$

□

Corollary 3.20. *Let B a set of doubly homogeneus polynomials whose image in $R(n,l)$ span $R(n,l)/\mathbf{y}$ as k-vector space. Denoting the x-degree of $p \in B$ by $d(p)$ if the degree enumerator of B satisfies:*

$$\sum_{p \in B} t^{d(p)} = \frac{n^l}{(1-t)^n} \tag{3.3}$$

then $R(n,l)$ is a free $k[\mathbf{y}]$-module with basis B.

Proof. We use the corollary 3.19 and the Nakayama Lemma:

$$\begin{aligned} \frac{n^l}{(1-t)^n} &= \sum_d \dim_{k(\boldsymbol{y})}(k(\boldsymbol{y}) \otimes R(n,l)_d)t^d = \sum_d |B_d| t^d \\ &= \sum_d \left(\sum_{p \in B_d} t^d \right) = \sum_{p \in B} t^{d(p)}. \end{aligned} \tag{3.4}$$

□

Corollary 3.21. *The Hilbert series*

$$\sum_d t^d \dim_{k(\mathbf{y})}(k(\mathbf{y}) \otimes \mathcal{O}(Y(m,r,k))_d)$$

of $k(\mathbf{y}) \otimes \mathcal{O}(Y(m,r,k))$ as a graded $k(\mathbf{y})$ algebra is equal to the enumerator

$$\sum_{e,f} t^{|e|}$$

for $e \in \mathbb{N}^n$ and f such that $|[r]\S_k(e,f)| \geq m$.

3.2 The case n=2

We start by writing down explicit polynomials that form the common ideal basis required by theorem 3.5.
To each pair (e,f) with $e \in \mathbb{N}^2$ and $f:[l] \to [2]$ we associate a basis element $p[e,f]$ that is homogeneus of $\mathbf{x}$-degree $|e|$:
for $e=(0,0)$ we set

$$p[(0,0),f] = \prod_{f(j)\neq f(1),\, j>1} (b_j - b_1) \cdot \begin{cases} (b_1 - y_2) & \text{if } f(1)=1 \\ 1 & \text{otherwise.} \end{cases}$$

For $e=(0,h)$ with $h>0$, let $f^{-1}(\{1\}) = S \cup T$, where S and T are disjoint and S is the smallest h elements of $f^{-1}(\{1\})$ (or the whole set if $h \geq |f^{-1}(\{1\})|$).
We set

$$p[(0.h),f] = x_2^{h-|S|} \prod_{i\in S}(a_i - x_1 - x_2) \prod_{j\in T}(b_j - y_2).$$

Now for $e=(h,0)$ still with $h>0$ we set

$$p[(h,0),f] = x_1 \theta p[(0,h-1),\theta f],$$

where θ is the permutation $(1,2)$ that act on the polynomial ring $k[\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}]$ exchanging x_1 with x_2 and y_1 with y_2 while fixing $\mathbf{a}$, $\mathbf{b}$.
Finally for $e=(h_1,h_2)$ let $h=\min(h_1,h_2)$ (still h_i are positive) we define

$$p[e,f] = (x_1x_2)^h p[e-(h,h),f].$$

Lemma 3.22. *For $n=2$ the coordinate ring $Z(2,l)$ is a free $k[\mathbf{y}]$-module with basis the set of elements $p[e,f]$ defined above. We will denote the base B.*

Proof. The goal is to prove that B spans $R(2,l)/(\mathbf{y})$ as a k-vector space, in fact since the enumerator of B is $2^l(1-t)^{-2}$ thanks to the corollary 3.19 it would conclude the proof. Notice that for each d we have $\binom{2+d-1}{d}$ polynomials and 2^l functions, using the computations present in the proof of corollary 3.19 it is easy to compute the enumerator of B.
Now let $B_0 = \{[p(0,h),f] \mid h \geq 0\}$, then we have

$$B = (B_0 \cup x_1\theta B_0) \cdot \{1, x_1x_2, (x_1x_2)^2, \dots\}.$$

It suffices to show that $B_0 \cup x_1\theta B_0$ spans $S = R(r,l)/((x_1x_2)+(\mathbf{y}))$ that is if and only if B_0 spans $S/(x_1)$.
Notice that if B_0 spans $S/(x_1)$ then we already know that $x_1\theta B_0$ spans x_1S: in S $x_1x_2=0$ we have a well defined surjective homomorphism:

$$S/(x_2) \to x_1S$$

sending $p + (x_2) \to x_1 p + x_1 x_2 = x_1 p$ and since θB_0 spans $S/(x_2)$ thanks to the morphism we can conclude.
Let us prove that B_0 spans $S/(x_1)$.
We start noticing that the ideal

$$\sum_{i\in[l]}(a_i - x_1, b_i - y_1)(a_i - x_2, b_i - y_2) + \sum_{i,j\in[l]}(\det\begin{pmatrix} a_i & b_i & 1 \\ a_j & b_j & 1 \\ x_1 & y_1 & 1 \end{pmatrix})$$

is contained in $I(Z(2,l))$, for each $i \in [l]$ we have just two possibilities: $f(i) = 1$ or $f(i) = 2$.
Clearly the first sum vanishes on $Z(2,l)$, for the second one notice that if $f(i) = f(j)$ then the first two rows are equal else either i or j is sent to 1 making one of the first two rows equal to the third one.
Now let us consider the ideal

$$I = I(Z(2,l)) + (x_1) + (\mathbf{y})$$

and the elements $a_i^2 - a_i x_2,\ a_i b_i,\ b_i^2$ and $x_2 b_i$.
The first is in I because we can multiply $a_i - x_1$ times $a_i - x_2$ and add (x_1), for the second we can take $a_i - x_1$ times $b_i - y_2$, for the third one we use the two components with b_i and for the last one we take $a_i - x_2$ and $b_i - x_1$; so each of these 4 elements belong to our ideal I for all i. Moreover the elements $a_i b_j - a_j b_i$ are in I for i, j, to see it it enough to compute the determinant of the matrix.
Notice that $S/(x_1) = k[\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}]/I$ that, for what we saw before, is spanned by monomials in $k[x_2, \mathbf{a}, \mathbf{b}]$ not divisible by $a_i^2,\ a_i b_j,\ b_i^2$ for $i \le j$ and $x_2 b_i$.
Therefore the ideal $S/(x_1)$ is spanned by

$$x_2^k \prod_{i\in S} a_i \prod_{j\in T} b_j$$

and each of these terms is exactly the leading term of an element of B_0 and the tail of the element is in I, this proves that B_0 spans $S/(x_1)$.

□

Corollary 3.23. *The non trivial ideals $I(m,r,k)$ are generated as ideals in $R(2,l)$ as follows:*

$$I(2,2,0) = (\mathbf{x}, \mathbf{a}) \tag{3.5}$$

$$I(1,2,k) = (x_1 x_2) + \sum_{i\in[k]}(a_i - x_1 - x_2, b_i - b_1) \tag{3.6}$$

$$I(1,1,k) = (x_1) + \sum_{i\in[k]}(a_i - x_2, b_i - y_2) \tag{3.7}$$

Proof. It is easy to see that the ideals on the right is contained in $I(m,r,k)$, suppose for example we consider the case (3.5), $I(2,2,0)$ is the coordinate ring of

$$Y(2,2,0) = \bigcup_{f,T} V(x_j | j \in T) \cap W_f$$

where T is a subset of [2] such that

$$|T \cap [2]| \geq 2.$$

So T must be [2] so $Y(2,2,0) = V(x_1,x_2) \cap W_f$ thus its coordinate ring is $(\mathbf{x},\mathbf{a})$. □

Now we finally can state our theorem of the case $n = 2$:

Theorem 3.24. *For $n = 2$, each ideal $I(m,r,k) \subseteq R(2,l)$ is spanned as a $k[\mathbf{y}]$-module by the set of elements $p[e,f] \in B$ satisfying*

$$|[r] \backslash S_k(e,f)| < m$$

where $S_k(e,f) = \{j \mid e_j > 0\} \cup \{f([k]).$

Proof. We observe that for each m,r,k the ideals I displayed in Corollary 3.23 are generated by polynomials which vanishes on $Y(m,r,k)$ thus $I \subseteq I(m,r,k)$.
To prove the other inclusion we just have to go case by case and perform some routine computations. The tricky cases are well explained here [10].
To conclude the proof we use Lemma 3.15, thus we need to prove that our polynomials $p[e,f]$ span $k(\mathbf{y}) \otimes I(m,r,k)$.
Now B is a homogeneus basis of $k(\mathbf{y}) \otimes R(r,l)$ thus it is enough to show that the pairs e,f not satisfying the condition of this theorem are enumerated by the Hilbert series of $k(\mathbf{y}) \otimes \mathcal{O}Y(m,r,k)$. This is true thanks to Corollary 3.20.

□

3.3 The induction idea

We managed to build a basis for the case $n = 2$, now the goal is to lift these constructions for the general case required by Theorem 3.5.
The induction involve the construction of a basis of $R(n,l)/I(1,1,l)$ from a basis of $R(n-1,l)$, the construction of a basis of $R(n,l)/I(1,1,t-1)$ from a basis of $R(n,l)/I(1,1,t)$ and finally the construction of a basis of $R(n,l)$ from a basis of $R(n,l)/I(1,1,0)$.

Lemma 3.25. *Suppose that $R(n-1,l)$ has an homogeneus common ideal basis, then so does $R(n,l)/I(1,1,l)$.*

Proof. Let $B' \subseteq k[\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}]$ a common ideal basis of $R(n-1, l)$ represented by a set of homogeneus polynomials without involving the variables x_n, y_n. We want to show that $B = \theta B'$ is a basis of $R(n, l)/I(1, 1, l)$.
Notice that

$$Y(1, 1, l) \cong Z(N, l) \times \operatorname{Spec} k[y_1]$$

because if $1 \in f[l]$ then we have $|T \cap \emptyset| \geq 1$ that's clearly impossible, so 1 can't be in the image of f, so basically we are killing a variable.
So x_1 must be zero and y_1 doesn't influence the graph, thus we have our isomorphism.
This implies that $R(n, l)/I(1, 1, l)$ is a free $k[\mathbf{y}]$-module with basis B.
Moreover if we consider the subset $Y(m, r, k) \cap Y(1, 1, l)$ of $Y(1, 1, l)$, it corresponds to the subset $Y_{N,l}(m-1, r-1, k) \times \operatorname{Spec} k[y_1]$ in $Z(N, l) \times \operatorname{Spec} k[y_1]$.
This implies

$$\sqrt{I(m, r, k) + J}/J = I_{N,l}(m-1, r-1, k) \otimes k[y_1]$$

that shows that B is a common ideal basis.

□

Lemma 3.26. *Given $n > 1$, $l > 0$ and $t \in [l]$, suppose that $R(n, l-1)$ and $R(n, l)/I(1, 1, l)$ each have a homogeneus ideal common basis, then so does $R(n, l)/I(1, 1, t-1)$.*

Proof. □

Lemma 3.27. *Suppose that $R(n, l)/I(1, 1, 0)$ has a homogeneus ideal basis, then so does $R(n, l)$.*

Proof. First we have that $Y(1, 1, 0) = V(x_1)$ and $I(1, 1, 0) = \sqrt{(x_1)}$.
Since x_1 doesn't vanish identically on any W_f, it is not a zero devisor in $R(n, l)$ so the multiplication by x_1 is an isomorphism between $R(n, l)$ and (x_1).
(Notice that $\sqrt{(x_1)} = (x_1)$)
Let B' a common ideal basis of $R(n, l)/I(1, 1, 0)$ and suppose that in a given x-degree we can find a free $k[\mathbf{y}]$-module basis B_d of $R(n, l)_d$ such that every $I(m, r, k)_d$ is spanned by a subset of B_d.
Then we claim that $x_1 \theta B_d$ is a basis of $I(1, 1, 0)_{d+1}$ with subsets spanning each $(I(m, r, k) \cap I(1, 1, 0))_{d+1}$.
In this case we would have a common ideal basis of $I(1, 1, 0)_{d+1}$ and the common ideal basis B' of $R(n, l)/I(1, 1, 0)$ restricted to the degree $d+1$, with these two sets we can build a free $k[\mathbf{y}]$-module basis B_{d+1} of $R(n, l)_{d+1}$ which is a common ideal basis.
In degree zero, we can take $B_0 = B_0''$, since $R(n, l)_0 = (R(n, l)/I(1, 1, 0))_0$.
Now we have to prove the previous claim.

If B_d is a basis then so is θB_d therefore $x_1\theta B_d$ is a basis of $I(1,1,0)_{d+1}$. Now observe that for any ideal $I \subseteq R(n,l)$ we have $I \cap I(1,1,0) = I \cap (x_1) = x_1(I : (x_1))$ and if I is radical $= (I : (x_1))$, so $V(I : (x_1))$ is the union of those components of $V(I)$ on which x_1 doesn't vanish identically.
We can apply this to $I(m,r,k)$ for $r > 0$ finding

$$I(m,r,k) : (x_1) = \theta I(m,r-1,k) \implies I(m,r,k) \cap I(1,1,0) = x_1\theta I(m,r-1,k).$$

For $r > 0$, this shows that if B_d has a subset spanning $I(m,r-1,k)_d$, then $x_1 B_d$ has a subset spanning $(I(m,r,k) \cap I(1,1,0))_{d+1}$. This suffices, since $I(m,r,k)$ is trivially equal to 0 or (1) for $r = 0$.

□

We are finally ready to prove Theorem 3.5!

Proof. As we anticipated the proof is by induction on n and l.
The base case for $l = 0$ is given by Lemma 3.6.
For $n = 1$ we notice that $Z(1,l) \cong Z(1,0 \cong E$ and that the only non trivial ideal $I(m,r,k)$ is $I(1,1,0)$ which has already been addressed by the case $Z(1,0)$, thus the case $n = 1$ is contained in the case $l = 0$.
Now let us consider the case $n > 0$ and $l > 0$ assuming that $R(n,l-1)$ and $R(n-1,l)$ already have a common ideal basis.
By Lemma 3.25 also $R(n,l)/I(1,1,l)$ does have it and applying repeatedly Lemma 3.26 discending t from l to 1. Using this technique we find that $R(n,l)/I(1,1,0)$ has a common ideal basis.
Finally thanks to Lemma 3.27 we prove that so does $R(n,l)$. □

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