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Tesi di Laurea

Non-equivalent primitive permutation representations of finite groups with the same set of derangements

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Chapter 1

Introduction

Let G be a finite group and $M \leqslant G$ a maximal subgroup of G. Define

$$\tilde{M} := \bigcup_{g \in G} M^g$$

and suppose that there exist two maximal subgroups M_1 and M_2 of G such that $\tilde{M_1} = \tilde{M_2}$. The question is to determine if this implies that M_1 and M_2 are conjugated in G.

In this work we investigate this problem in some types of groups. First, we prove that the answer is affermative if G is either the alternating or the symmetric group and the maximal subgroups considered are either intransitive or imprimitive. Secondly, we deal with soluble groups and we prove that the answer in this case is always affirmative. Then we prove that if G is the special linear group of degree 2 over a field of characteristic 2 then the answer is negative by showing a pair of non-conjugated maximal subgroups for which $\tilde{M_1} = \tilde{M_2}$ holds. Finally, we provide a complete answer for sporadic groups obtained computationally.

The first chapter contains the necessary definitions and some preliminary results which, although not directly connected to the problem, turn out to be useful in the proofs present in the following chapters.

Before moving on to our work, we state the problem in two equivalent ways. The first one justifies the title of this work.

Observation 1.0.1. *A group G has two non-conjugated maximal subgroups M_1, M_2 with $\tilde{M_1} = \tilde{M_2}$ if and only if it has two non-equivalent primitive actions with the same set of derangements.*

A proof of this equivalence will be given later.

The second one is a convenient way to see the problem. We have this in mind throughout this work, except for the case of soluble groups.

Observation 1.0.2. *Let M_1, M_2 be maximal subgroups of a finite group G. Let $\mathcal{K}_G$ be the set of conjugacy classes of G and for each $H \leqslant G$ define $\mathcal{K}_G^H := \{\mathfrak{K} \in \mathcal{K}_G \mid \mathfrak{K} \cap H \neq \emptyset\}$. With the above notation, $\tilde{M_1} = \tilde{M_2} \iff \mathcal{K}_G^{M_1} = \mathcal{K}_G^{M_2}$.*

Proof. $\Rightarrow$) We only prove the inclusion " $\subseteq$ ", the opposite can be done similarly. Let $\mathfrak{K} \in \mathcal{K}_G^{M_1}$ and let k be an element of $\mathfrak{K} \cap M_1$: since clearly $M_1 \subseteq \tilde{M_1}$, k is contained in $\tilde{M_1} = \tilde{M_2} = \{m^g \mid m \in M_2, g \in G\}$. Hence there exist m, g in M_2 and G respectively such that $m^g = k$. Conjugation on both sides by g^{-1} gives $m = k^{g^{-1}} \in \mathfrak{K}$: hence $\mathfrak{K} \cap M_2 \neq \emptyset$ and this implies that $\mathfrak{K} \in \mathcal{K}_G^{M_2}$.

$\Leftarrow$)Again we only prove the inclusion " $\subseteq$ ". Let $k \in \tilde{M_1} = \{m^g \mid m \in M_1, g \in G\}$ and let $\mathfrak{K}$ be the conjugacy class of k. There exist m, g in M_1 and G respectively such that $m^g = k \implies m = k^{g^{-1}} \in \mathfrak{K}$, hence $M_1 \cap \mathfrak{K} \neq \emptyset$ and this implies that $\mathfrak{K} \in \mathcal{K}_G^{M_1} = \mathcal{K}_G^{M_2}$. Now let n be in $\mathfrak{K} \cap M_2$: since k and n are in the same conjugacy class there exists $h \in G$ such that $k = n^h$. Hence $k \in M_2^h \subseteq \tilde{M_2}$ and we are done. □

Chapter 2

Preliminary results

2.1 Permutation groups

Definition 1 (Permutation group). *A permutation group is a subgroup of a symmetric group.*

Let G be a finite group and define an action of G on a set Ω; denote $\omega \cdot g$ the image of $\omega \in \Omega$ under the action of $g \in G$. We can associate in a natural way to each g of G a map $\varphi_g : \Omega \to \Omega$ according to the action: $\varphi_g(\omega) = \omega \cdot g$ for each $\omega \in \Omega$. φ_g can be seen as a element of the symmetric group on $|\Omega|$ elements, which we will denote as $S_{|\Omega|}$, and $g \mapsto \varphi_g$ is a homomorphism from G to $S_{|\Omega|}$. If the action of G on Ω is faithful, namely if 1_G is the unique element which stabilizes every element of Ω, then this homomorphism is actually an isomorphism and thus G can be seen as a permutation group.

Observation 2.1.1 (Cayley representation). *For each group G consider the set $\Omega = G$ and the operation of right multiplication. This is a faithful action, hence every group is isomorphic to a permutation group.*

Definition 2 (Regular group). *A transitive permutation group G acting on a set Ω is regular if only the action 1_G fixes any point.*

Definition 3 (Primitive group). *Let G be a permutation group acting transitively on a set Ω and denote ω^g the image of the action of g on the element $\omega \in \Omega$. For each nonempty subset Δ of Ω and for each $g \in G$ denote $\Delta^g = \{\delta^g \mid \delta \in \Delta\}$. We call Δ a block for G if for each $g \in G$ either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ and in particular if $\Delta = \{x\}$ for a certain $x \in \Omega$ or $\Delta = \Omega$ we call it a trivial block.*

We say that the group G has a primitive group action if it has no nontrivial blocks and imprimitive otherwise.

Definition 4 (Proper primitive group). *A primitive subgroup H of S_n, $n > 2$, is said to be improper if $A_n \subseteq H$ and proper otherwise.*

Definition 5 (Degree). *The degree of a permutation group G acting on a set Ω is $|\Omega|$.*

Definition 6 (Pointwise and setwise stabilizer)**.** *Let G be a group acting on a set Ω and let Δ be a subgroup of Ω. Then $G_{(\Delta)}$ and $G_{\{\Delta\}}$ denote respectively the pointwise and the setwise stabilizer of Δ, that is to say:*

$$G_{(\Delta)} = \{g \in G \mid \delta^g = \delta \quad \forall \delta \in \Delta\} \ , \ G_{\{\Delta\}} = \{g \in G \mid \Delta^g = \Delta\}$$

It is possible to connect this notion of primitivity to the action of G on the right cosets of a certain subgroup of G by right multiplication. This is the purpose of the following two lemmas.

Lemma 2.1.2. *Every transitive action of a group G is equivalent to an action of G on the right cosets of a subgroup H of G by right multiplication.*

Proof. Checking that for every $H \leqslant G$ G acts on the right cosets of H by right multiplication is immediate. Let Ω be the set on which G acts transitively, fix an element $\omega \in \Omega$ and consider the stabilizer H_ω, which is clearly a subgroup. Consider the map φ:

$$\begin{aligned} \varphi : \Omega &\to \{H_\omega g, g \in G\} \\ \vartheta = \omega^x &\mapsto H_\omega x \qquad \forall x \in G \end{aligned}$$

This map is well defined: indeed, for each $\vartheta \in \Omega$ there exists $x \in G$ such that $\omega^x = \vartheta$ because G acts transitively; moreover, if $x, y \in G$ are two distinct element for which $\omega^x = \omega^y = \vartheta$ then $\omega^{yx^{-1}} = \omega$, hence $H_\omega x = H_\omega y$.

We prove that this map is bijective. It is surjective because, for each right coset $H_\omega x$, $\omega^x \in \Omega$ and $\varphi(\omega^x) = H_\omega x$ and it is injective because if $\varphi(\vartheta) = \varphi(\eta) = H_\omega x$ for a certain $x \in G$ then $\vartheta = \omega^x = \eta$.

Finally we prove that φ is an equivalence of actions. For each $\vartheta = \omega^x \in \Omega$ and $g \in G$ we have

$$\varphi(\vartheta)g = H_\omega xg = \varphi(\omega^{xg}) = \varphi(\vartheta^g)$$

□

Before stating the second lemma, we recall the definition of core:

Definition 7 (Core)**.** *Let G be a group and $H \leqslant G$. The core $Core_G(H)$ of H in G is*

$$Core_G(H) := \bigcap_{g \in G} H^g$$

If $Core_G(H) = 1_G$ we say that H is corefree.

Lemma 2.1.3. *Consider the action of a finite group G on the right cosets of $M \leqslant G$ by right multiplication. Then G has a primitive action $\iff$ M is maximal and corefree.*

Proof. $\Rightarrow$) Suppose by contradiction that there exists a subgroup H with $M < H < G$ and let $\mathcal{H} = \{Mh \mid h \in H\}$. We prove that $\mathcal{H}$ is a block for G. For a fixed $g \in G$, if $\mathcal{H} \cap \mathcal{H}^g \neq \emptyset$ then $Mh = Mh'g$ for some $h, h' \in H$ which implies $g \in h'^{-1}Mh \subseteq H$. Hence $\forall h \in H$ we have $Mh = M(hg^{-1})g$ and this implies $\mathcal{H} = \mathcal{H}^g$. Since the action of G is primitive, either $\mathcal{H} = \{M\}$ or $\mathcal{H} = \{Mg \mid g \in G\}$: the first implies $H = M$, which is a contradiction, while the second implies $H = G$, which is again a contradiction. This proves the maximality of M. Moreover:

$$Core_G(M) = \{g \in G \mid g \in x^{-1}Mx \quad \forall x \in G\} = \{g \in G \mid (Mx)^g = Mx \quad \forall x \in G\} \tag{2.1}$$

and since G is a permutation group we can conclude that $Core_G(M) = 1_G$.

$\Leftarrow$)Let $\Delta \subseteq \{Mg \mid g \in G\}$ be the block containing M and consider $L = \{l \in G \mid \Delta^l = \Delta\}$, which is clearly a subgroup of G. Now for each $m \in M$ we have $\Delta^m = \Delta$ since $\Delta^m \cap \Delta \neq \emptyset$ as it contains M, thus $M \leqslant L$. Since the action must be transitive on the elements of Δ, if $\Delta \neq \{M\}$ then $M \neq L$.This implies $L = G$ because M is a maximal subgroup of G and thus the action has only one block.

The kernel of this action is the most right element in equation 2.1, hence it is equal to $Core_G(M) = 1_G$, thus this action is faithful. □

Now we are ready to prove the initial observation.

Proof of Observation 1.0.1. Let $M \leqslant G$ be a maximal subgroup and consider the action of G on the right cosets of M. We have:

$$\tilde{M} = \{g \in G \mid g = x^{-1}Mx \ \exists x \in G\} = \{g \in G \mid (Mx)g = Mx \ \exists x \in G\}$$

which means that $\tilde{M}$ is the subset of G which contains the elements whose action fixes at least one element of $\{Mx, x \in G\}$ and $G \setminus \tilde{M}$ is the set of derangements.

Moreover, let M_1, M_2 be maximal subgroups of G and suppose that the actions of G on their right cosets are equivalent with equivalence map $\varphi : \{M_1x, x \in G\} \to \{M_2y, y \in G\}$ such that $\varphi((M_1x)g) = \varphi(M_1x)g$ for all $g \in G$. If $\varphi(M_1) = M_2\bar{y}$ and $x \in M_1$ then:

$$M_2\bar{y} = \varphi(M_1) = \varphi(M_1x) = (M_2\bar{y})x \quad \implies x \in \bar{y}^{-1}M_2\bar{y} = M_2^{\bar{y}}$$

and since φ is a bijection $|M_1| = |M_2|$, thus $M_1 = M_2^{\bar{y}}$. □

2.2 Definitions for O'Nan-Scott Theorem

The following definitions are useful only in order to be able to give a precise statement of the O'Nan-Scott Theorem, which will be necessary in the case of symmetric groups.

Definition 8 (Socle)**.** *Let G be a finite group. The socle of G is the subgroup generated by the set of all minimal normal subgroups of G, that is to say by the nontrivial normal subgroups that are minimal in the set of nontrivial normal subgroups of G with the order of the inclusion. It is denoted by $Soc(G)$.*

Definition 9 (Affine group)**.** *The affine group $AGL_m(p)$ is the group of affine matrices of order m on a finite field of order p.*

Definition 10 (Wreath product)**.** *Let K and H be groups and Γ be the set on which H acts. Let $Fun(\Gamma, K)$ be the group whose elements are the functions from Γ to K with the operation*

$$(fg)(\gamma) = f(\gamma)g(\gamma) \qquad \forall f, g \in Fun(\Gamma, K), \gamma \in \Gamma$$

The wreath product of K *by* H *with respect is* $K \wr_\Gamma H := Fun(\Gamma, K) \rtimes H$ *defined by the action:*

$$f^h(\gamma) = f(\gamma^{h^{-1}}) \qquad \forall f \in Fun(\Gamma, K), h \in H, \gamma \in \Gamma$$

If $\Gamma = H$ *and the action of* H *on itself is regular we can just write* $K \wr H$ *and call it standard wreath product.*

Observation 2.2.1. *If* $|\Gamma| = m < \infty$ *then it is immediate to see that* $\varphi : Fun(\Gamma, K) \to K^m$ *defined as* $\varphi(f) = (f(\gamma_1), \ldots, f(\gamma_m))$ *is a homomorhpism and* $\psi : K^m \to Fun(\Gamma, K)$ *such that* $\psi(k_1, \ldots, k_m)(\gamma_i) = k_i \ \forall i = 1, \ldots, m$ *is its inverse homomorphism. Hence* $Fun(\Gamma, K) \cong K^m$*.*

Definition 11 (Product action of the wreath product)**.** *Let* H *and* K *be groups acting on sets* Γ *and* Δ *respectively (to fix ideas, suppose that these objects are finite and in particular* $|\Gamma| = m < \infty$*) and define* $\Omega = Fun(\Gamma, \Delta)$*. Using these actions we define the product action of* $K \wr_\Gamma H$ *on* Ω *as the action*

$$\omega^{(f,h)}(\gamma) = (\omega(\gamma^{h^{-1}}))^{f(\gamma^{h^{-1}})} \qquad \forall (f,h) \in K \wr_\Gamma H, \omega \in \Omega, \gamma \in \Gamma$$

Proof. Recall that for the construction of the outer semidirect product we have $(f,h)(g,y) = (fg^{h^{-1}}, hy)$ and $g^{h^{-1}}(\gamma) = g(\gamma^h)$. For each $\omega \in \Omega, (f,h), (g,y) \in Fun(\Gamma, K) \rtimes H, \gamma \in \Gamma$ we have:

$$\begin{aligned}
(\omega^{(f,h)})^{(g,y)}(\gamma) &= (\omega^{(f,h)}(\gamma^{y^{-1}}))^{g(\gamma^{y^{-1}})} \\
&= ((\omega((\gamma^{y^{-1}})^{h^{-1}}))^{f((\gamma^{y^{-1}})^{h^{-1}})})^{g(\gamma^{y^{-1}})} \\
&= (\omega(\gamma^{(hy)^{-1}}))^{f(\gamma^{(hy)^{-1}})g(\gamma^{(hy)^{-1}h})} \\
&= (\omega^{(fg^{h^{-1}}, hy)})(\gamma)
\end{aligned}$$

Moreover the identity element of the semidirect product is $(1_{Fun(\Gamma,K)}, 1_H)$ and

$$\omega^{(1_{Fun(\Gamma,K)}, 1_H)}(\gamma) = \omega(\gamma)^{1_K} = \omega(\gamma) \qquad \forall \omega \in \Omega, \gamma \in \Gamma$$

Hence we have proved that the product action defined is indeed an action. □

Definition 12 (Diagonal action, diagonal type)**.** *Let* T *be a simple, non-abelian, regular subgroup of the symmetric group* $S_{|\Delta|}$ *acting on a certain set* Δ*,* $C_{S_{|\Delta|}}(T) \leqslant S_{|\Delta|}$ *its centralizer and* $\Gamma = \{1, \ldots, m\}$ *for a fixed* $m \in \mathbb{N}$*. By extension of the action of* $C_{S_{|\Delta|}}(T)$ *on* Δ *we can define an action of* $C_{S_{|\Delta|}}(T)$ *on* Δ^m *as*

$$(\delta_1 \ldots, \delta_m)^x = (\delta_1^x, \ldots, \delta_m^x) \qquad \forall (\delta_1 \ldots, \delta_m) \in \Delta^m, x \in C_{S_{|\Delta|}}(T)$$

It can be proved that the set Ω *of* $(C_{S_{|\Delta|}}(T))$*-orbits under this action constitutes a system of blocks for the product action of* $T \wr_\Gamma S_m$ *on* $\Delta^m \cong Fun(\Gamma, \Delta)$*. The action of the subgroup* $T^m = \{(t, 1_{S_{|\Delta|}}), t \in T\}$ *on* Ω *is called the diagonal action of* T^m*.*

In addition, let $N_{S_{|\Omega|}}(T^m)$ *be the normalizer of* T^m *(seen as a subgroup of* $S_{|\Omega|}$*). A group* G *such that* $T^m \leqslant G \leqslant N_{S_{|\Omega|}}(T^m)$ *is said to be of diagonal type.*

Definition 13 (Almost simple group)**.** *A group* G *is said to be almost simple if* $T \leqslant G \leqslant Aut(T)$ *where* T *is a simple and nonabelian group.*

2.3 Soluble groups

Definition 14 (Soluble group). *A group G is said to be soluble if it has a subnormal series $1 = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_n = G$ such that ${G_{i+1}}/{G_i}$ is abelian $\forall\ 0 \leq i \leq n-1$.*

Definition 15 (Characteristic subgroup). *A subgroup $H \leqslant G$ is characteristic in G if the image of H under every automorphism of G is H itself.*

Lemma 2.3.1. *Let $N \trianglelefteq G$ be a normal subgroup of G and $K \leqslant N$ be a characteristic subgroup of N. Then $K \trianglelefteq G$.*

Proof. For each $g \in G$, let $\varphi_g : G \to G$ the conjugacy map, namely $\varphi_g(x) = x^g$. Cleary φ_g is an automorphism of G and since $N \trianglelefteq G$ its restriction on N $\varphi_g|_N$ remains an automorphism. Thus $\varphi_g|_N(x) \in K$ for each $x \in K$ and it follows that $K \trianglelefteq G$. □

Definition 16 (Commutator). *Let G be a group and $x, y \in G$. We define the commutator of x and y as $[x, y] := x^{-1}y^{-1}xy$.*

Let G be a group and L, N be two subgroups of G. The commutator of L and N is the subgroup $[L, N]$ generated by the set $\{[l, n] \mid l \in L, n \in N\}$. If $L = N = G$ we denote $[G, G]$ as G' and call it the commutator subgroup of G.

The following observation lists some properties of the commutator that will be used in some proofs.

Observation 2.3.2.

- *$G' = 1_G \iff xy = yx \quad \forall\ x, y \in G \iff G$ is abelian;*
- *$[L, N] = 1_G \iff ln = nl \quad \forall\ l \in L, n \in N \iff L$ and N commute;*
- *G' is a normal subgroup of G. Indeed, for each $x \in G', g \in G$ we have that $gxg^{-1}x^{-1} = y \in G'$, hence $gxg^{-1} = yx \in G'$;*
- *G' is a characteristic subgroup of G. Indeed for each automorphism φ of G and for each $x, y \in G$ we have that $\varphi(x^{-1}y^{-1}xy) = \varphi(x^{-1})\varphi(y^{-1})\varphi(x)\varphi(y) = \varphi(x)^{-1}\varphi(y)^{-1}\varphi(x)\varphi(y)$.*

Definition 17. *(Commutator series) Given a group G, denote $G^{(n)}$ its nth commutator subgroup, namely $G^{(1)} = G'$ and $G^{(n)} = (G^{(n-1)})'$ and define commutator series the series*

$$G \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \ldots$$

We provide an alternative characterization of soluble groups, based on the commutator series.

Lemma 2.3.3. *A group G is soluble $\iff$ there exists $n \geq 0$ such that $G^{(n)} = 1_G$.*

Proof. ($\Leftarrow$) Let

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_n = G$$

be a subnormal series for G with abelian quotients. For each $0 \leq i \leq n-1$ and each $x, y \in G_{i+1}$, $xyG_i = yxG_i$, hence $[x, y] \in G_i$, consequently $(G_{i+1})' \subseteq G_i$.

It is easy to see that $H \subseteq G \implies H' \subseteq G'$, hence $G^{(2)} = (G'_n)' \subseteq G'_{n-1}$ and by induction $G^{(n)} \subseteq G^{(n)} \subseteq G_0 = 1_G$.

($\Rightarrow$) Let n be the first index for which $G^{(n)} = 1_G$. The series

$$1_G = G^{(n)} \trianglelefteq G^{(n-1)} \trianglelefteq \ldots \trianglelefteq G^{(1)} \trianglelefteq G^{(0)} = G$$

is a normal series ($G^{(i)}$ is normal in G for each $0 \leq i \leq n$) and with the same calculations as in the first part we see that the quotients are abelian, hence G is solvable. □

Definition 18 (Chief series)**.** *Given a group G, a normal series for G is a finite series of normal subgroups N_i of G, $i = 1, \ldots, k$, such that $N_0 = 1_G$, $N_k = G$ and $N_i \subseteq N_{i+1}$ for $i = 1, \ldots, k-1$. If ${}^{N_{i+1}}/_{N_i}$ is minimal normal in ${}^{G}/_{N_i}$ for each $i \in \{1, \ldots, k-1\}$, the series is called a chief series and ${}^{N_{i+1}}/_{N_i}$ is called a chief factor.*

Chapter 3

Symmetric and Alternating groups

As first strep in this chapter, we prove that the alternating group A_n and the symmetric group S_n are primitive for each n. Clearly this is true for $n = 2$ and $n = 3$ and for $n \geq 4$ both A_n and S_n are at least 2−transitive. The conclusion follows from the following lemma:

Lemma 3.0.1. *If a permutation group G is 2-transitive, then it is primitive.*

Proof. Let Ω be the set on which G acts and $\Delta \subset \Omega$ be a proper nonempty subset with at least two elements. Fix $\delta_1, \delta_2 \in \Delta$ and $\gamma \in \Omega \setminus \Delta$ and let $g \in G$ be an element of G such that $(\delta_1, \gamma)^g = (\delta_1, \delta_2)$. For this g, $\Delta \cap \Delta^g$ is neither empty nor Δ, hence it can not be a block. □

3.1 Symmetric groups

Now we want to classify all imprimitive maximal subgroups of S_n. The following two lemmas deal respectively with intransitive and transitive imprimitive subgroups. Both proofs use the well known fact that S_n is generated by the set $S = \{(1, k)|k = 2, \dots, n\}$.

Lemma 3.1.1. *The subgroup $S_{\{\Delta\}}$ of S_n stabilizer of a subset $\Delta \subset \Omega = \{1, \dots, n\}$, with $n > 2$ and $1 \leq |\Delta| < n/2$, is maximal in S_n.*

Proof. For $n = 3, 4$, $|\Delta|$ must be one and the stabilizers of a point are maximal in a primitive group. Now consider the case $n \geq 5$. Let $K \leqslant S_n$ be a subgroup of S_n properly containing $S_{\{\Delta\}}$ and take $h \in K \setminus S_{\{\Delta\}}$: there exists $x \in \Omega \setminus \Delta$ such that its image under h is $x' \in \Delta$ and since $|\Delta| < |\Omega \setminus \Delta|$ there exists $y \in \Omega \setminus \Delta$ such that its image under h is $y' \in \Omega \setminus \Delta$, Since $(x, y) \in S_{\{\Delta\}}$, K contains $(x, y)^h = (x', y')$. Now $S_{\{\Delta\}}$ contains (x', x'') $\forall x'' \in \Delta$ and (y', y'') $\forall y'' \in \Omega \setminus \Delta$, hence K contains $(x', z) \forall z \in \Omega$ and this implies $K = S_n$. □

Lemma 3.1.2. *The subgroup S_Π of S_n consisting of all permutations which preserve a partition $\Pi = \{\Delta_1, \dots \Delta_m\}$ of $\Omega = \{1, \dots, n\}$, where m is a proper divisor of n, is maximal in S_n.*

Proof. It is easy to see that S_Π is transitive: for each $x \in \Delta_i = \{x = \delta_{i,1}, \ldots, \delta_{i,n/m}\}$ and $y \in \Delta_j = \{y = \delta_{j,1}, \ldots, \delta_{j,n/m}\}$, the permutation $\sigma = (x, y) \prod_{l=2}^{n/m} (\delta_{i,l}, \delta_{j,l})$ is in S_Π.

Now let $K \leqslant S_n$ be a subgroup of S_n properly containing S_Π. First we prove that if K contains the transposition $(\delta_{i,1}, \delta_{j,1})$ with $\delta_{i,1} \in \Delta_i$, $\delta_{j,1} \in \Delta_j$ for some $i \neq j$, $1 \leq i, j \leq m$ and for an appropriate enumeration of the elements of Δ_i and Δ_j, then $K = S_n$. Fix an element $\delta_{1,a}$ of Δ_1 and choose an element $\delta_{t,b} \in \Delta_t$. If $t = 1$ clearly $(\delta_{1,a}, \delta_{1,b}) \in S_\Pi$. Otherwise, up to renumbering, take $a = b = 1$; then $h = (\delta_{1,1}, \delta_{i,1})(\delta_{t,1}, \delta_{j,1}) \prod_{l=2}^{n/m} (\delta_{1,l}, \delta_{i,l}) \prod_{s=2}^{n/m} (\delta_{t,s}, \delta_{j,s})$ is in S_Π and the images of $\delta_{1,1}$ and $\delta_{t,1}$ are $\delta_{i,1}$ and $\delta_{j,1}$ respectively. Hence $(\delta_{i,1}, \delta_{j,1}) = (\delta_{1,1}, \delta_{t,1})^h$, thus $(\delta_{1,1}, \delta_{t,1})$ is an element of K. The conclusion follows from the arbitrariness of $\delta_{t,b}$.

Now we prove that the case above is actually the general case. Consider $g \in K \setminus S_\Pi$: there exist two different indices $i, j \in \{1, \ldots, m\}$ for which $(\Delta_i)^g \cap (\Delta_j)$ is neither empty nor Δ_j. Up to renumbering, we can suppose that $\delta_{i,1}, \delta_{i,2}$ are two elements of Δ_i whose images are $\delta_{j,1} \in \Delta_j$ and $\delta_{t,q}$ for a certain Δ_t, possibly $t = i$. Since $(\delta_{i,1}, \delta_{i,2}) \in S_\Pi$, K contains $(\delta_{i,1}, \delta_{i,2})^g = (\delta_{j,1}, \delta_{t,q})$ which is a transposition between two elements in different subsets of the partition, and thus we are done. □

Observation 3.1.3. *If n is even, the stabilizer $S_{\{\Delta\}}$ of $\Delta \subsetneq \Omega$ with $|\Delta| = n/2$ is contained in the imprimitive subgroup S_Π with $\Pi = \{\Delta, \Omega \setminus \Delta\}$, consequently it is not maximal in S_n. For this reason in the statement of Lemma 3.1.1 we have added the hypotesis that $|\Delta|$ is strictly less than half of $|\Omega|$.*

Now let $K < S_n$ be a proper intransitive subgroup and let $\Delta_1, \ldots \Delta_i, i > 2$, be its orbits. Then K is a proper subgroup of $S_{\{\Delta_1\}} \times S_{\{\Omega \setminus \Delta_1\}}$, hence K is not maximal.

Moreover, let $K < S_n$ be a proper transitive imprimitive subgroup and let Π be the partition of Ω on which it acts. Is it clear that $K \leqslant S_\Pi$ as defined in Lemma 3.1.2, and that Lemma proves that S_Π are maximal, hence we have described all transitive imprimitive maximal subgroups of S_n.

In conclusion we have proved the following:

Theorem 3.1.4. *Each maximal subgroup of the symmetric group S_n of finite degree n is either:*

- *a primitive subgroup;*
- *an intranstive $S_{\{\Delta\}}$ for $\Delta \subsetneq \Omega$, $1 \leq |\Delta| < n/2$;*
- *an imprimitive subgroup S_Π consisting of all permutations which preserve a partition Π of Ω into subsets of size m, $m|n$.*

Moreover each subgroups of the last two types is maximal in S_n.

At this point we are ready to provide a partial result to our problem.

Theorem 3.1.5. *If M_1, M_2 are maximal subgroups of S_n that are either intransitive or imprimitive then $\tilde{M}_1 = \tilde{M}_2 \iff M_1$ and M_2 are conjugated.*

Proof. Since $(\Leftarrow)$ is obvious, it is sufficient to prove $(\Rightarrow)$.

Define $\Omega = \{1, \ldots, n\}$ and, for each $K \leqslant S_n$, let $l(K)$ be the set of lengths of the cyclic permutations in K. If $\tilde{M_1} = \tilde{M_2}$, then M_1 and M_2 intersect the same conjugacy classes and in particular $l(M_1) = l(M_2)$. From Theorem 3.1.4, an intransitive maximal subgroup M consists of all permutations preserving a partition of Ω into Δ and $\Omega\backslash\Delta$ with $|\Delta| < |\Omega\backslash\Delta|$, hence $l(M) = \{1, \ldots, |\Omega\backslash\Delta|\}$. If otherwise M is an imprimitive maximal subgroup and $\Pi = \{\Delta_1, \ldots \Delta_m\}$ is the partition of Ω preserved by M, then M contains all the cycles in the stabilizer of Δ_i, which have lengths from 1 to n/m, and all the cycles of the type $(i_1, j_1, \ldots k_1, i_2, j_2 \ldots, k_{n/m})$ where $a_1, \ldots a_{n/m}$ are the elements of Δ_a, hence $l(M) = \{1, \ldots, n/m, 2n/m, \ldots, n\}$. Hence $l(M_1) = l(M_2)$ only if :

- M_1, M_2 are both intransitive and preserve respectively a partition in $\Delta_1, \Omega\setminus\Delta_1$ and $\Delta_2, \Omega \setminus \Delta_2$ with $|\Delta_1| = |\Delta_2|$. In this case $\exists g \in S_n$ such that $\Delta_1^g = \Delta_2$ which implies $M_1^g = M_2$;
- M_1, M_2 are both imprimitive and preserve a partition with the same number of subsets, respectively $\Pi^1 = \{\Delta_1^1, \ldots, \Delta_m^1\}$ and $\Pi^2 = \{\Delta_1^2, \ldots, \Delta_m^2\}$. In this case $\exists g \in S_n$ such that $(\Delta_i^1)^g = \Delta_i^2 \quad \forall i \in \{1, \ldots, m\}$ which again implies $M_1^g = M_2$.

□

Observation 3.1.6. *It is immediate to see that both imprimitive and intransitive maximal subgroups of S_n contains a single transposition, hence they can not intersect the same conjugacy classes as A_n.*

From the observation above we can conclude that, if there are no proper primitive groups of degree n, then two maximal subgroups of S_n intersect the same conjugacy classes of S_n if and only if they are conjugated.

In order to say something about the existence of proper primitive groups of a fixed degree n, consider the following, very important theorem, in the form presented in (6):

Theorem 3.1.7 (O'Nan-Scott)**.** *Let G be a finite primitive group of degree n, and let H be the socle of G. Then either:*

1. *H is a regular elementary abelian p-group for some prime $p, n = p^m = |H|$, and G is isomorphic to a subgroup of the affine group $AGL_m(p)$;*
2. *H is isomorphic to a direct power T^m of a nonabelian simple group T and one of the following holds:*
   - *(a) $m = 1$ and G is isomorphic to a subgroup of $Aut(T)$;*
   - *(b) $m \geq 2$ and G is a group of diagonal type with $n = |T|^{m-1}$;*
   - *(c) $m \geq 2$ and for some proper divisor d of m, $m = dt$, and some primitive group U with a socle isomorphic to T^d, G is isomorphic to a subgroup of the wreath product $U \wr_{\{1,\ldots,t\}} S_t$ with the product action, and $n = l^t$ where l is the degree of U;*
   - *(d) $m \geq 6$, H is regular and $n = |T|^m$.*

From the O'Nan-Scott theorem we see that every finite primitive group is either almost simple or its degree is of the form a^b, where a is the order of a finite simple group (cases 2b and 2d) or of a primitive group (cases 1 and 2c). The numbers of this form are "few" in the set of natural numbers. Define $E = \{n \in \mathbb{N} | \exists$ a proper primitive group of degree $n\}$: it can be proven (see (4)) that

$$|E \cap \{1, \dots, n\}| \sim \frac{2n}{logn}$$

Mathieu proved that $\{5, 6, \dots, 33\} \in E$. Using GAP, I have checked for $n = 5, \dots, 33$ whether there exist a primitive maximal subgroup of S_n which has the same exponent of another maximal subgroup. The answer was negative except for $n = 6, 10, 25$.

- for $n = 6$ the subgroup isomorphic to A_6 and two isomorphic to S_5, one intransitive and one primitive, have the same exponent. The two maximal subgroups isomorphic to S_5 have elements of order 6, while A_6 do not, but they intersect different conjugacy classes of elements of order 2.
- for $n = 10$ there is a primitive maximal subgroup isomorphic to $(A_6 \times C_2) \times C_2$ which has the same exponent of two other imprimitive maximal subgroups. However these two subgroups have elements of order 12, while the primitive maximal subgroups do not.
- for $n = 25$ there are two primitive maximal subgroups with the same exponent, but one has elements of order 24 and the other do not.

3.2 Alternating groups

Now we consider alternating groups.

Lemma 3.2.1. *Let M be a maximal subgroup of $A_n, n > 4$. Then*

- *if M is intransitive, then $M = (S_a \times S_{n-a}) \cap A_n$ where $1 \leq a < n/2$;*
- *if M is imprimitive, then $M = (S_a \wr_{\{1,\dots,b\}} S_b) \cap A_n$ where $ab = n, b > 1$.*

Proof. Let $K < A_n$ be a proper intransitive subgroup and let $\Delta_1, \dots \Delta_i, i \geq 2$ be its orbits. Considering A_n as a subgroup of S_n, K is a proper subgroup of $S_{\{\Delta_1\}} \times S_{\{\Omega \setminus \Delta_1\}}$ and $(S_{\{\Delta_1\}} \times S_{\{\Omega \setminus \Delta_1\}} \cap A_n) < A_n$. Indeed $\sigma = (\delta, x)(y, z)$, $\delta \in \Delta_1, x, y, z \in \Omega \setminus \Delta_1$ exists since $|\Omega \setminus \Delta_1| \geq 3$ for $n > 4$ and $\sigma \in A_n \setminus S_{\{\Omega \setminus \Delta_1\}}$. Now we have

$$K \leqslant (S_{\{\Delta_1\}} \times S_{\{\Omega \setminus \Delta_1\}}) \cap A_n < A_n$$

hence if K is maximal " $\leqslant$ " must be a equality.

On the other hand, we give an idea of proof of the maximality of $J = (S_{\Delta_1} \times S_{\Omega \setminus \Delta_1}) \cap A_n$ in the case in which all the elements that will be mentioned can be chosen one different from each other. Let $W \leqslant A_n$ properly containing J and consider h in $W \setminus J$: since h does not stabilize Δ_1, there exist $x \in \Omega \setminus \Delta_1$ whose image under h is $x' \in \Delta_1$ and since $|\Omega \setminus \Delta_1| > |\Delta_1|$, there exists another element $y \in \Omega \setminus \Delta_1$ whose image under h is $y' \in \Omega \setminus \Delta_1$. Moreover, let z be the image of y' under h. Since (x, y, y') is an element of J, $\bar{h} = (x, y, y')^h = (x', y', z) \in W$. Now we have two cases:

- $z \in \Delta_1$. Let u be an element of Ω: if $u \in \Delta_1, (u, x', z) \in J$, otherwise we can pick another element $a \in \Omega \setminus \Delta_1$ and obtain by conjugation $(\bar{h}^2)^{(u,a,y')} = (n, x', z)$;

- $z \in \Omega \setminus \Delta_1$. Let u be an element of Ω: if $u \in \Omega \setminus \Delta_1, (u, y', z) \in J$, otherwise we can pick another element $a \in \Delta_1$ and obtain by conjugation $(\bar{h})^{(u,a,x')} = (n, y', z)$.

Hence in both cases $A = \{(u, \cdot, z) | u \in \Omega \setminus \{\cdot, z\}\}$ where $\cdot$ is either x' or y', depending on the case; since A generates A_n this concludes the proof.

Let $K < A_n$ be a proper transitive imprimitive subgroup and let Π the partition of Ω on which it acts. Considering A_n as a subgroup of S_n, K is a subgroup of S_Π as defined in Lemma 3.1.2, and $S_\Pi \cap A_n < A_n$. Indeed:

- if there are at least three blocks $\Delta_1, \Delta_2, \Delta_3$, hence $\sigma = (x_1, x_2)(y_2, y_3)$, where $x_1 \in \Delta_1, \{x_2, y_2\} \subseteq \Delta_2$ and $x_2 \neq y_2$, $y_3 \in \Delta_3$, is in A_n but not in S_Π;
- if there are exactly two blocks Δ_1 and Δ_2, then they have at least 3 elements, hence $\sigma = (\delta, x)(y, z)$, where $\delta \in \Delta_1$ and x, y, z are distinct elements of Δ_2, is again in A_n but not in S_Π.

Now

$$K \leqslant (S_\Pi \cap A_n) < A_n$$

hence if K is maximal " $\leqslant$ " must be a equality. We omit the proof of the maximality of groups of these form. $\square$

We are ready to give a partial answer to our question in the case of alternating groups.

Theorem 3.2.2. *If M_1, M_2 are maximal subgroups of A_n, $n > 5$ and odd, that are either intransitive or imprimitive then $\tilde{M_1} = \tilde{M_2} \iff M_1$ and M_2 are conjugated.*

Proof. As in Theorem 3.1.5, we only need to prove ($\Rightarrow$).

Let M be a maximal intransitive or imprimitive maximal subgroup of A_n. First, observe that $n \in l(M)$, where l is defined as above, if and only if M is imprimitive, hence in order to have $\tilde{M_1} = \tilde{M_2}$ it is necessary that are both imprimitive or intransitive.

If they are both imprimitive, $\exists\, 1 < a, b, c, d < n$ such that $a \leq c$, $ab = cd = n$, $M_1 = (S_a \wr_{\{1,\dots,b\}} S_b) \cap A_n$ and $M_2 = (S_c \wr_{\{1,\dots,d\}} S_d) \cap A_n$. Suppose that $a < c$. Since a and c must be odd by the oddness of n, $l(M_1) = \{1, \dots, a, 2a, \dots, ba\} \cap \{2m + 1, m \in \mathbb{N}\}$ contains in increasing order all odd numbers from 1 to a with a included and then $3a$, while $l(M_2)$ contains, again in increasing order, all odd numbers from 1 to c, hence it contains $a + 2 \leq c$ and this implies $a + 2 = 3a \implies a = 1$, which is a contradiction. Hence in this case we must have $a = c, b = d$.

Let $g \in S_n$ be a permutation whose action send a partition $\Pi_1 = \{\Delta_{1,1}, \dots, \Delta_{1,b}\}$ into $\Pi_2 = \{\Delta_{2,1} \dots, \Delta_{2,b}\}$. If g is even, then $\bar{g} = g \in A_n$; otherwise, let x, y be two elements in $\Delta_{2,1}$: then $\bar{g} = g(x, y) \in A_n$ send again Π_1 in Π_2. In both cases $M_1^{\bar{g}} = M_2$.

It they are both intransitive, there exist $b, d \in \mathbb{N}$, $n/2 < b \leq d < n$, such that $M_1 = (S_{n-b} \times S_b) \cap A_n$ and $M_2 = (S_{n-d} \times S_d) \cap A_n$. Suppose that $b < d$. Since $l(M_1) = \{1, \dots, b\} \cap \{2j + 1, j \in \mathbb{N}\}$ has b as largest value if b is odd and $b - 1$ otherwise and similarly M_2 has d as largest value if d is odd and $d - 1$ otherwise,

$l(M_1) = l(M_2)$ implies $b = d - 1$ and odd. Hence d is even and if $d < n-1$ then $\sigma = (d+1, d+2)(1, \dots, d)$ is a even permutation, consequently $\sigma \in \tilde{M_2} \notin \tilde{M_1}$ because $b < d$ and this contradicts our hypotesis. We conclude that $d = n - 1$ and $b = n - 2$. Now consider $\rho = (1, \dots, \frac{n-1}{2})(\frac{n+1}{2}, \dots, (n-1))$: $\rho \in \tilde{M_2}$ since it is a even permutation and it is contained in $S_{\{1,\dots,n-1\}}$. However ρ can not be contained in $\tilde{M_1}$. Indeed each permutation in $\tilde{M_1}$ permutes a subset of $n-2$ elements and the remaining 2 separately; hence the two disjoint cycles whose product is ρ can neither act on the same subset, since they move $n-1$ distinct points, nor one on a subset an the other on the other one, since $(n-1)/2 > 2$ by hypotesis. Hence $\rho \notin \tilde{M_1}$ and we obtain again a contradiction. We conclude that $b = d$. Moreover we see that we can turn a partition of $\{1, \dots, n\}$ into two subset of d and $n - d$ elements into another partition with the same structure with d or $n - d$ transpositions and at least one of this two numbers is even (otherwise n would be even), hence there is a permutation $g \in A_n$ such that $M_1^g = M_2$ and we are done. □

Observation 3.2.3. *For $n = 5$, A_5 has two maximal intransitive subgrups, $M_1 \cong S_3$ and $M_2 \cong A_4$, which intersect the same conjugacy classes but are not conjugated. In this case $M_1 = (S_2 \times S_3) \cap A_5$ and $M_2 = S_4 \cap A_5$ and, with the notation of Theorem 3.2.2 $\rho = (1,2)(3,4)$ has a conjugate element both in S_4 and in $S_2 \times S_3$.*

Theorem 3.2.4. *If M_1, M_2 are maximal subgroups of A_n with n even, that are either intransitive or imprimitive then $\tilde{M_1} = \tilde{M_2} \iff M_1$ and M_2 are conjugated.*

Proof. As in Theorem 3.1.5, we only need to prove ($\Rightarrow$).

Let M be a maximal intransitive or imprimitve maximal subgroup of A_n. First, observe that if $M = (S_a \wr_{\{1,\dots b\}} S_b) \cap A_n$ for some $1 < a, b < n, ab = n$ is imprimitive then $S_a \wr_{\{1,\dots b\}} S_b$ contains a n-cycle σ and M contains σ^2; this permutation has two orbits of size $n/2$, hence it can not be in the intransitive maximal subgroup $N = (S_c \times S_{n-c}) \cap A_n$ for any $1 < c < n/2$. Hence, in order to have $\tilde{M_1} = \tilde{M_2}$, it is necessary that are both imprimitive or intransitive.

If they are both imprimitive, $\exists\, 1 < a, b, c, d < n$ such that $a \le c$, $ab = cd = n$, $M_1 = (S_a \wr_{\{1,\dots,b\}} S_b) \cap A_n$ and $M_2 = (S_c \wr_{\{1,\dots,d\}} S_d) \cap A_n$. Suppose that $a < c$. If a, c are both odd we can proceed as in Theorem 3.2.2 and find a contradiction; if they are both even the largest value in $l(M_1)$ is $a - 1$ while the largest in $l(M_2)$ is $c - 1$, hence $a = c$. If a is even and c is odd, then the largest value in $l(M_1)$ is $a - 1$ while $l(M_2)$ contains $c > a - 1$ and we get a contradiction. Finally suppose that a is odd and c is even. In $l(M_1)$ we find in increasing order a followed by $3a$ if $3a < n$ and nothing otherwise, while in $l(M_2)$ we find a is followed by $a + 2$ if $a < c - 1$ and nothing otherwise; the case $a < c - 1$ gives $a + 2 = 3a \implies a = 1$ which is not a valid value for a, thus $a = c - 1$ and $n < 3a \implies n = 2a$. But this implies $c = a + 1 | 2a$, which has no solution for $a > 1$. Hence in this case we must have $a = c, b = d$. The proof that M_1, M_2 are conjugated is the same as in Theorem 3.2.2.

It they are both intransitive, $\exists\, n/2 < b \le d < n, b, d \in \mathbb{N}$ such that $M_1 = (S_{n-b} \times S_b) \cap A_n$ and $M_2 = (S_{n-d} \times S_d) \cap A_n$. If we suppose $b < d$, as in Theorem 3.2.2 we have $b = d-1$ and odd. Hence d is even and since n is even and stricly greater than d, then $d < n-1$ and consequently $\sigma = (d+1, d+2)(1, \dots, d)$

is a even permutation, $\sigma \in \tilde{M_2} \notin \tilde{M_1}$ because $b < d$ and this contradicts the hypotesis. We conclude that $b = d$.

Let $g \in S_n$ be a permutation for which the partition of $\{1, \ldots, n\}$ into two subset of d and $n - d$ elements has as image another partition with the same structure. If g is even then $g \in A_n$ and $M_1^g = M_2$, otherwise let x, y be two elements in the same subset of the second partition: then $\bar{g} = g(x, y) \in A_n$, $M_1^{\bar{g}} = M_2$ and we are done. □

We conclude this section with an observation about the case in which there exists a proper primitive group of degree n. We state a result due to Jordan (see (9), chapter 8):

Theorem 3.2.5. *Let G be a group with a primitive action on a set Ω, and let $\Lambda \subseteq \Omega$ with $|\Lambda| \geq |\Omega| - 2$. Suppose that $G_{(\Delta)}$ (the pointwise stabilizer of Δ) acts primitively on $\Omega \setminus \Delta$. Then the action of G on Ω is $(|\Delta| + 1)-$transitive.*

As a consequence of this theorem we can proof the following:

Lemma 3.2.6. *Let G be a primitive permutation group acting on a set Ω with cardinality n. If G contains a 3-cycle, then G is either S_n or A_n.*

In order to prove this lemma we need the following preliminary result:

Lemma 3.2.7. *Let G be a group acting on the set $\Omega = \{1, \ldots, n\}$. If G is $(n - 2)-$transitive, then G is either S_n or A_n*

Proof. Let $\mathcal{O}_{n-2}(\Omega)$ be the set of all $(n - 2)$-uples of distinct elements in Ω. Clearly $|\mathcal{O}_{n-2}(\Omega)| = n!/2$ and G is transitive on $\mathcal{O}_{n-2}(\Omega)$ in his componentwise action, hence $(n!/2)$ divides $|G|$. If $|G| = n!$ then $G = S_n$ and we are done, otherwise G has index 2 in S_n and in particular it contains the set of all 3-cycles, which is well known to be a set of generators for A_n. □

Now we are ready for the proof.

Proof of Lemma 3.2.6. Let $h = (x, y, z)$ be the transposition contained in G and consider $\Theta = \{x, y, z\}, \Delta = \Omega \setminus \Theta$. Then $G_{(\Delta)}$ contains h and it is easy to see $G_{(\Delta)}$ has a primitive action Θ. Hence by Theorem 3.2.5 G is $(n - 2)-$transitive on Ω and thanks to Lemma 3.2.7 we can conclude that G is either S_n or A_n. □

Moreover the analogous result can be proven for $n > 9$ and G containing a double transposition (see example 3.3.1 in (6)).

It is easy to see that for $n > 4$ every maximal subgroup of A_n which is either intransitive or imprimitive contains at least one double transposition or 3-cycle, thus, from the previous results, we can conclude the following:

Observation 3.2.8. *For $n > 9$, if M_1 and M_2 are two maximal subgroups of A_n for which $\tilde{M_1} = \tilde{M_2}$ holds and that are not conjugated, then they must be both primitive.*

Chapter 4

Soluble groups

Let G be a soluble group and M_1, M_2 two maximal subgroups of G. In this section we prove that $\tilde{M_1} = \tilde{M_2} \iff M_1$ and M_2 are conjugated.

As first step we prove the following:

Lemma 4.0.1. *Let G be a finite group, $M_1, M_2 \leqslant G$ maximal subgroups for which $\tilde{M_1} = \tilde{M_2}$ holds and $N \trianglelefteq G$. Then $N \leqslant M_1 \iff N \leqslant M_2$.*

In order to prove Lemma 4.0.1 we need a preparatory result:

Lemma 4.0.2. *Let H be a finite group and K a proper subgroup of H. Then the union of conjugates of K can not be the whole H.*

Proof. Suppose that there exists a proper subgroup K of H for which $\cup_{h\in H} K^h = H$ holds. If $|K| = k$ and $|H| = kn$, for a certain $n > 1$, let $\{x_1, \dots, x_n\}$ be a left transversal for K in H. For each $h \in H$, $h \in x_i K$ for a certain $i \in [1, \dots n]$, hence $hKh^{-1} = x_i K x_i^{-1}$ and so there are at most n distinct conjugates of K in H. Since each of them contains 1_H, we have:

$$|\bigcup_{h\in H} K^h| \leq (k-1)n + 1 < kn = |G|$$

which gives a contradiction. □

Now we are ready for the proof:

Proof of Lemma 4.0.1. We prove the result by contradiction. First, observe that the statement is symmetric, hence it is sufficient to prove " ⇒ ". If $N \nleqslant M_2$, then $G = M_2 N$ because of the maximality of M_2 in G. Since $N \leqslant M_1$ we have that $N \subseteq \tilde{M_1} = \tilde{M_2} = \cup_{g\in G} M_2^g = \cup_{n\in N} M_2^n$, thus

$$N = N \cap (\bigcup_{n\in N} M_2^n) = \bigcup_{n\in N} (N \cap M_2^n) \stackrel{1}{=} \bigcup_{n\in N} (N \cap M_2)^n$$

where 1 comes from the fact that $N \trianglelefteq G$. This equality contradicts Lemma 4.0.2, hence $N \leqslant M_2$. □

From Lemma 4.0.1 we can conclude in particular that $core_G(M_1) = core_G(M_2)$. Now define:

$$\bar{G} := {}^{G}\!/\!_{core_G(M_1)}, \qquad \bar{M_1} := {}^{M_1}\!/\!_{core_G(M_1)}, \qquad \bar{M_2} := {}^{M_2}\!/\!_{core_G(M_1)}$$

.

Now $\bar{G}$ is a quotient of a solvable group and hence it is solvable itself, and it has a maximal core-free subgroup $\bar{M_1}$, hence $\bar{G}$ is primitive in its action on the right cosets of $\bar{M_1}$.

The following lemma proves that we can restrict our analysis to $\bar{G}$:

Lemma 4.0.3. *Let G be a group and let M, L and N be respectively two maximal subgroups and a normal subgroup of G contained both in M and L. If M/N and L/N are conjugated in G/N, then L and M are conjugated in G.*

Proof. Let $gN \in G/N$ be the element of G/N for which $M/N = (L/N)^{gN}$ holds. For each $m \in M$ there exists $l \in L$ such that $mN = (lN)^{gN} = g^{-1}lgN$ in G/N, hence $m^{-1}(g^{-1}lg) = n \in N$. This implies $m = g^{-1}(n'l)g$ for some $n' = gn^{-1}g^{-1}$ and since $N \subseteq L$ we are done. □

In addition, the following two lemmas can be applied to $\bar{G}$.

Lemma 4.0.4. *If G is a primitive permutation group on Ω, the action of a nontrivial normal subgroup $N \trianglelefteq G$ on Ω is transitive.*

Proof. Fix an element $\gamma \in \Omega$ and an element $g \in G$ and for each $x \in G$ denote γ^x the image of γ under the action of x. Consider $\Gamma = \{\gamma^n | n \in N\}$. Since $N \trianglelefteq G$, for each $n \in N$ there exists $m \in N$ such that $\gamma^{ng} = \gamma^{gm}$. Thus

$$\Gamma^g = \{(\gamma^n)^g | n \in N\} = \{(\gamma^g)^m | m \in N\}$$

is a N-orbit and this implies that $\Gamma \cap \Gamma^g$ is either $\emptyset$ or Γ. Since G is primitive, it follows that either $\Gamma = \Omega$ and N is transitive or $\Gamma = \{\gamma\}$ and $N = 1_G$. □

Lemma 4.0.5. *A finite soluble and primitive group G has a unique minimal normal subgroup.*

Proof. Let N be a minimal normal subgroup of G and N' its commutator subgroup. N is a normal subgroup of a soluble group and thus it is soluble itself, which implies $N' \lneq N$. Moreover N' is characteristic in N and thus $N' \triangleleft G$; since N is minimal normal in G, $N' = 1_G$ and it follows that N is abelian.

Let M be a maximal corefree subgroup of G: $M \cap N \trianglelefteq M$ since $N \trianglelefteq G$ and $M \cap N \trianglelefteq N$ since N is abelian, hence $M \cap N \trianglelefteq MN = G$ and this implies that $M \cap N = 1_G$ because M is corefree.

Now we prove that $C_G(N) = N$. Since N is abelian we have $C_G(N) = C_M(N)N$. Let $m \in C_M(N)$ and $n \in N$: with the action of the group G on the right cosets of M by right multiplication we have $Mn = M(mn) = M(nm) = (Mn)m$. Since N is normal, by Lemma 4.0.4 its action is transitive, hence m fixes every right coset. In addition, since the action is faithful, $m = 1_G$ and consequently $C_G(N) = N$.

Finally, suppose that there exists another minimal normal subgroup $L \neq N$ of G. Then $[L, N] \leqslant L \cap N = 1_G$ and hence L and N commute. This implies $L \subseteq C_G(N) = N$ and $N \subseteq C_G(L) = L$, consequently $L = N$, which is a contradiction. □

Given a group G and a normal subgroup N of G, we say that G is *represented* in N by the subgroup S if $G = SC_G(N)$. Clearly, for each $N \trianglelefteq G$, G is represented in N by G itself, hence there is at least one such subgroup.

We state a result from (1):

Theorem 4.0.6. *If N is a minimal normal subgroup of G such that ${}^{G}/_{C_G(N)}$ contains a normal subgroup different from 1_G, whose order is prime to the order of N, then:*

- *Every minimal subgroup S representing G in N satisfies $S \cap N = 1_G$;*
- *The two minimal subgroups H and K representing G in N satisfy $NH = NK$ if and only if there exists an element $x \in N$ such that $H = x^{-1}Kx$.*

Consider the group $\bar{G}$. It is a primitive soluble group, thus by Lemma 4.0.5 it has a unique minimal normal subgroup $\bar{N}$ and $C_{\bar{G}}(\bar{N}) = \bar{N}$. Now ${}^{\bar{G}}/_{\bar{N}}$ is a quotient of a soluble group and hence soluble itself, thus again in (1) is proved that it contains a normal subgroup different from $1_{\bar{G}}$, whose order is prime to the order of $\bar{N}$. Hence the hypothesis of Theorem 4.0.6 are satisfied.

Now we are ready to answer our question:

Theorem 4.0.7. $\tilde{M_1} = \tilde{M_2} \iff M_2 = M_1^g$.

Proof. Since ($\Leftarrow$) is obvious, it is sufficient to prove ($\Rightarrow$).

In the proof of Lemma 4.0.5 we have already proved that both $\bar{M_1}$ and $\bar{M_2}$ are complements of $\bar{N}$ since they are corefree and maximal. Hence from Theorem 4.0.6 $\bar{M_2} = \bar{M_1^g}$ for some $g \in \bar{N}$ and the conclusion follows from Lemma 4.0.3. □

Chapter 5

An example in $SL_2(2^f)$

In this chapter we find two non-conjugated maximal subgroups of the special linear group $SL_2(2^f), f \in \mathbb{N}^*$, which intersect the same conjugacy classes.

We begin by recalling the definition of special linar group:

Definition 19. *Let $\mathbb{F}_q$ be a finite field of order q. The special linear group $SL_n(q)$ is the group of all invertible $n \times n$ matrices over $\mathbb{F}_q$ with determinant 1.*

Consider the following two subgroups of $SL_2(2^f)$:

$$M_1 = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{F}_q,\ ab = 1\} \cup \{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{F}_q,\ ab = 1\}$$

$$M_2 = \{\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in \mathbb{F}_q,\ ab = 1\}$$

In order to prove the maximality of M_2, we need the following:

Lemma 5.0.1. *The action of $SL_2(q)$ on the projective line $\mathbb{P}^1(\mathbb{F}_q)$ is 2-transitive.*

Proof. Let $(A, B), (P, Q)$ be two pairs of points of the projective line with $A \neq B, C \neq D$. We can choose an appropriate coordinate system in which $A = [1 : 0]$, $B = [0 : 1]$, $P = [p_0 : p_1]$ and $Q = [q_0, q_1]$. Now the matrix

$$C = \begin{pmatrix} kp_0 & q_0 \\ kp_1 & q_1 \end{pmatrix}, k = (p_0 q_1 - q_0 p_1)^{-1}$$

send A and B to P and Q respectively and has determinant 1, hence $C \in SL_2(q)$. □

Now it is immediate to prove the maximality of M_2:

Lemma 5.0.2. *M_2 is a maximal subgroup of $SL_2(q)$.*

Proof. From the previous Lemma and Lemma 3.0.1 we obtain that $SL_2(q)$ with its action on the projective line is primitive. From Lemma 2.1.2 we see that this action is equivalent to the action on the right cosets of the stabilizer of a point, and from Lemma 2.1.3 we can conclude that this stabilizer is a maximal subgroup of $SL_2(q)$. Since M_2 is the stabilizer of $[1 : 0]$, the conclusion follows. □

We omit the proof of the maximality of M_1, see (8). The following lemma summarizes the necessary informations for M_1, M_2:

Lemma 5.0.3. *M_1 and M_2 are maximal subgroups of $SL_n(q)$ of order $2(q-1)$ and $q(q-1)$ respectively.*

Proof. The maximality has already been proved or stated, hence we focus only on the orders. For each of the two sets which consitute M_1 we can choose $a \in \mathbb{F}_q^*$ in $q-1$ different ways and each of them determines a unique $b \in \mathbb{F}_q$. Hence $|M_1| = 2(q-1)$. Regarding to M_2, each choice of $a \in \mathbb{F}_q^*$ determines a unique $b \in \mathbb{F}_q$ and for each pair a, b we are free to chose $c \in \mathbb{F}_q$ (c can also be 0) and all these matrices are distinct, thus $|M_2| = q(q-1)$. □

Our purpose now is to find all the conjugacy classes of $SL_2(q)$ in order to determine which are instersected by M_1 and M_2 respectively. Recall that $SL_2(q)$ is a group of size $q(q^2-1)$.

The elements of $SL_2(q)$ can be divided into four types:

1. Elements that are diagonalizable over $\mathbb{F}_q$ with 2 distinct eigenvalues;
2. Elements that are diagonalizable over $\mathbb{F}_q$ with only one eigenvalue;
3. Elements that are not diagonalizable over $\mathbb{F}_q$ but have eingenvalues in $\mathbb{F}_q$;
4. Elements that have eigenvalues in $\mathbb{F}_{q^2}$.

Elements of two different type cannot be in the same conjugacy class because eigenvalues are preserved by conjugation, hence we can deal separately with each type.

Elements of type 1 These elements can be written in an appropriate basis as $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), a \neq 0, 1$ and the centralizer is composed by the matrices which are diagonal in that basis, hence has cardinality $q-1$. This implies that each of these classes has order $q(q+1)$. Since we can choose $a \in \mathbb{F}_q \setminus \{0, 1\}$, which has $q-2$ elements, and that choosing either a or a^{-1} gives the same element, there are at most $\frac{q-2}{2}$ conjugacy classes of this type. Since the trace is invariant by change of basis, if $A = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} b & 0 \\ 0 & b^{-1} \end{smallmatrix}\right), a, b \neq 0$, are in the same conjugacy class, we have

$$a+a^{-1} = b+b^{-1} \implies a^2b+b = b^2a+a \implies (a-b)(ab-1) = 0 \implies a = b \text{ or } a = b^{-1}$$

hence there are exactly $\frac{q-2}{2}$ conjugacy classes of type 1. Finally, observe that there are no elements of order 2 of this type, since $a^2 = 1$ implies $a = a^{-1} = 1$ in $\mathbb{F}_{2^f}$.

Elements of type 2 Since the eigenvalue in this case bust be an element $a \in \mathbb{F}_{2^f}$ such that $a^2 = 1$, the only possibility is $a = 1$. Hence this class is constituted only by the identity matrix.

Elements of type 3 These elements have only one eigenvalue $a \in \mathbb{F}_{2^f}$ for which $a^2 = 1$, hence again the only possibility is $a = 1$ and they can be written in an appropriate basis as $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence there is only one conjugacy classes and by simple calculation it is possible to prove that an element of this form has as centralizer $\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), b \in \mathbb{F}_q\}$ which has q elements. Thus the class has size q^2-1.

Elements of type 4 The eigenvalues in this case are the zeros $\xi, \xi' \in \mathbb{F}_{q^2}$ of a irreducible polynomial $x^2 + ax + 1 = 0, a \in \mathbb{F}_q$ ($\xi\xi'$ must be 1 because it is the determinant of the diagonalized matrix, which is invariant under conjugation in $SL_2(q)$). Since there are q polynomials of this form over the field $\mathbb{F}_q$ and $1 + \frac{q-2}{2}$ are reducible, there are at most $q/2$ conjugacy classes of this type and, with analogous calculations done in the case of type 1, we see that there are actually $q/2$ conjugacy classes.

We are ready to determine which conjugacy classes are intersected by M_1 and M_2.

First consider the subgroup $M_1 = K \cup H$, where K, H are respectively the subset of diagonal and antidiagonal matrices. The elements of K are clearly of type 1 and 2 and each conjugacy class of elements of this type contains a diagonal matrix, hence K intersects all and only these conjugacy classes. The elements of H have characteristic polynomial $x^2 + 1 = 0$, hence they have 1 as unique eigenvalue. Moreover they have order 2, hence cannot be of type 1 or 2. Consequently, these elements are all of type 3. In conclusion M_1 intersect all and only the conjugacy classes of elements of type 1, 2 and 3.

Now consider the subgroup M_2. Since $K \subseteq M_2$, M_2 intersects all conjugacy classes of elements of type 1 and 2. Moreover M_2 contains $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, hence it intersects the conjugacy class of elements of type 3. Finally, clearly each matrix in M_2 has its eigenvalues in $\mathbb{F}_q$, hence we can conclude that also M_2 intersects all and only the conjugacy classes of elements of type 1, 2 and 3.

This proves that $\tilde{M}_1 = \tilde{M}_2$.

Chapter 6

Sporadic groups

This chapter contains a complete answer to the question for sporadic groups. The first table summarizes the results, while each sporadic group, except Fi'_{24}, is discussed in detail in one of the following tables.

The approach in this case was the one presented in the Observation 1.0.2, namely to check whether there are non-conjugated maximal subgroups which intersect the same conjugacy classes. All the calculations were made using **GAP** (7).

For each group, I have taken the list of its maximal subgroups from the ATLAS (5) and divided them on the basis of the prime factors of their orders. The fist cell of each row indicates the position in the list in the ATLAS of the maximal subgroup. When a maximal subgroup shares its set of prime factors with no other, the corresponding row in the table is gray, while when the orders of two maximal subgroups have the same prime factors, then the corresponding rows of the table are adjacent and have the same color, blue or green. When the calculation was not excessively expensive in terms of memory, I have made a further division according to the exponent of the maximal subgroups; this division is shown in the table by shades of the same color. It was then sufficient to consider maximal subgroups kept together by these divisions. The most frequent way to prove that they intersect different conjugacy classes was to find an order such that there are elements of this order in a subgroup and not in another; this was checked either with **GAP** or looking at the character tables. For the Monster group also Table 14 in (2) turned out to be useful. When two or more maximal subgroups intersect the same conjugacy classes their rows are adjacent and colored in yellow or, when there are two subsets of maximal subgroups which intersect the same conjugacy classes, in orange.

For the group $Fi24'$ the approach was different: since fusion maps between the character tables of the maximal subgroups of $Fi24'$ and the character table of the group itself is available in **GAP**, it was sufficient to compare them. The result is that the pairs of maximal subgroups which intersect the same conjugacy classes are the non-conjugated subgroups $U_3(3).2$ and the two non-conjugated subgroups $L_2(13).2$, while the two non-conjugated subgroups $He : 2$ intersect different conjugacy classes of elements of order 12.

Tables

Table 6.1: Summary table

Group	Maximal subgroups intersecting the same conjugacy classes
M_{11}	$M_9 : 2$ and $2S_4$
M_{12}	None
M_{22}	The two copies of A_7; $2^4 : A_6$ and $2^4 : S_5$
M_{23}	$L_3(4) : 2_2$ and $2^4 : A_7$
M_{24}	None
J_2	None
Suz	The two copies of $L_3(3) : 2$
HS	The two copies of $U_3(5) : 2$
McL	The two copies of M_{22}; the two copies of $2^4 : A_7$ and $L_3(4) : 2$
Co_3	None
Co_2	None
Co_1	None
He	The two copies of $2^6 : 3.S_6$
Fi_{22}	The two copies of S_{10}; the two copies of $O_7(3)$
Fi_{23}	None
Fi'_{24}	The two copies of $U_3(3).2$; the two copies of $L_2(13).2$.
HN	The two copies of $M_{12} : 2$
Th	None
B	None
M	None
J_1	$2^3 : 7 : 3$ and $7 : 6$
$O'N$	The two copies of A_7
J_3	The two copies of $L_2(19)$
Ly	None
Ru	None
J_4	None

Table 6.2: M_{11}

M_{11}	Subgroup	Factors	Exponent	Other informations
3	$M_9 : 2$	[2 , 3]	24	They have elements of orders [1 , 2 , 3 , 4 , 6 , 8] and for each of these except 8 there is only one conjugacy class in M_{11}. There are two classes of elements of order 8 and both are intersected by both subgroups.
5	$2S_4$	[2 , 3]	24	
1	M_{10}	[2 , 3 , 5]	120	
4	S_5	[2 , 3 , 5]	60	
2	$L_2(11)$	[2 , 3 , 5 , 11]		

Table 6.3: M_{12}

M_{12}	Subgroup	Factors	Exponent	Other informations
6	$3^2 : 2S_4$	[2 , 3]	24	There are two conjugacy classes of elements of order 8 in M_{12}: the 6th and the 7th maximal subgroups intersect just one of this and not the same one, the 9th and 10th maximal subgroups intersect both but they intersect different classes of elements of order 6.
7	$3^2 : 2S_4$	[2 , 3]	24	
9	$2^{1+4} : S_3$	[2 , 3]	24	
10	$4^2 : D_{12}$	[2 , 3]	24	
11	$A_4 \times S_3$	[2 , 3]	6	
3	$A_6.2^2$	[2 , 3 , 5]	120	Intersect different classes of elements of order 8
4	$A_6.2^2$	[2 , 3 , 5]	120	
8	$2 \times S_5$	[2 , 3 , 5]	60	
1	M_{11}	[2 , 3 , 5 , 11]	1320	Intersect different classes of elements of order 8
2	M_{11}	[2 , 3 , 5 , 11]	1320	
5	$L_2(11)$	[2 , 3 , 5 , 11]	330	

Table 6.4: M_{22}

M_{22}	Subgroup	Factors	Exponent	Other informations
6	$2^3 : L_3(2)$	[2 , 3 , 7]		
2	$2^4 : A_6$	[2 , 3 , 5]	120	It has elements of order 6
5	$2^4 : S_5$	[2 , 3 , 5]	120	It has elements of order 6
7	M_{10}	[2 , 3 , 5]	120	It has no elements of order 6
1	$L_3(4)$	[2 , 3 , 5 , 7]	420	It has no elements of order 6
3	A_7	[2 , 3 , 5 , 7]	420	It has elements of order 6
4	A_7	[2 , 3 , 5 , 7]		It has elements of order 6
8	$L_2(11)$	[2 , 3 , 5 , 11]		

Table 6.5: M_{23}

M_{23}	Subgroup	Factors	Exponent	Other informations
6	$2^4 : (3 \times A_5) : 2$	[2 , 3 , 5]		
2	$L_3(4) : 2_2$	[2 , 3 , 5 , 7]	840	For orders 7 and 14 they intersect the same conjugacy classes, for orders 2, 3, 4, 5, 6 and 8 there is only one conjugacy class.
3	$2^4 : A_7$	[2 , 3 , 5 , 7]	840	
4	A_8	[2 , 3 , 5 , 7]	420	
1	M_{22}	[2 , 3 , 5 , 7 , 11]		
5	M_{11}	[2 , 3 , 5 , 11]		
7	$23 : 11$	[11 , 23]		

Table 6.6: M_{24}

M_{24}	Subgroup	Factors	Exponent	Other informations
7	$2^6 : (L_3(2) \times S_3)$	[2 , 3 , 7]	168	
9	$L_2(7)$	[2 , 3 , 7]	84	
5	$2^6 : 3.S_6$	[2 , 3 , 5]		
3	$2^4 : A_8$	[2 , 3 , 5 , 7]	840	It has no elements of order 21
6	$L_3(4) : S_3$	[2 , 3 , 5 , 7]	840	It has elements of order 21
2	$M_{22} : 2$	[2 , 3 , 5 , 7 , 11]		
1	M_{23}	[2 , 3 , 5 , 7 , 11 , 23]		
4	$M_{12} : 2$	[2 , 3 , 5 , 11]		
8	$L_2(23)$	[2 , 3 , 11 , 23]		

Table 6.7: J_2

J_2	Subgroup	Factors	Exponent	Other informations
4	$2^{2+4} : (3 \times S_3)$	[2 , 3]		
1	$U_3(3)$	[2 , 3 , 7]	168	It has elements of order 12
7	$L_3(2) : 2$	[2 , 3 , 7]	168	It has no elements of order 12
2	$3.A_6.2$	[2 , 3 , 5]	120	It has elements of order 15
3	$2^{1+4} : A_5$	[2 , 3, 5]	120	It has no elements of order 15
5	$A_4 \times A_5$	[2 , 3 , 5]	30	It has elements of order 15
6	$A_5 \times D_{10}$	[2 , 3 , 5]	30	It has elements of order 15 and one element of order 5 of the 6th maximal subgroup is in a different conjugacy class with respect to the two conjugacy classes containing the elements of order 5 of the 5th maximal subgroup.
8	$5^2 : D_{12}$	[2 , 3 , 5]	30	It has elements of order 10 and it has no elements of order 15
9	A_5	[2 , 3 , 5]	30	It has no elements of order 10 and it has no elements of order 15

Table 6.8: Suz

Suz	Subgroup	Factors	Exponent	Other informa-tions
11	$3^{2+4}:2.(A_4 2^2)2$	[2 , 3]	72	
4	$2^{1+6}.U_4(2)$	,[2 , 3 , 5]	360	
7	$2^{4+6}:3A_6$	[2 , 3 , 5]	120	It has no elements of order 20
9	$2^{2+8}:(A_5\times S_3)$	[2 , 3 , 5]	120	It has elements of order 20 and intersect different classes of elements of order 15
12	$(A_6\times A_5).2$	[2 , 3 , 5]	120	It has elements of order 20
13	$(A_6\times 3^2:4).2$	[2 , 3 , 5]	60	
2	$3_2U_4(3).2_{3'}$	[2 , 3 , 5 , 7]	2520	
6	$J_2:2$	[2 , 3 , 5 , 7]	840	It has no elements of order 21
8	$(A_4\times L_3(4)):2$	[2 , 3 , 5 , 7]	840	It has elements of order 21
17	A_7	[2 , 3 , 5 , 7]	420	
1	$G_2(4)$	[2 , 3 , 5 , 7 , 13]	10920	
16	$L_2(25)$	[2 , 3 , 5 , 13]	780	
3	$U_5(2)$	[2 , 3 , 5 , 11]	3960	Intersect different classes of elements of order 12
5	$3^5:M_{11}$	[2 , 3 , 5 , 11]	3960	
10	$M_{12}:2$	[2 , 3 , 5 , 11]	1320	
14	$L_3(3):2$	[2 , 3 , 13]	312	
15	$L_3(3):2$	[2 , 3 , 13]	312	

Table 6.9: HS

HS	Subgroup	Factors	Exponent	Other informations
7	$4^3 : L_3(2)$	[2 , 3 , 7]	168	
6	$2^4.S_6$	[2 , 3 , 5]	120	It has no elements of order 20 and it has elements of order 12
10	$4.2^4.S_5$	[2 , 3 , 5]	120	It has elements of order 20
11	$2 \times A_6.2^2$	[2 , 3 , 5]	120	It has elements neither of order 20 nor of order 12
12	$5 : 4 \times A_5$	[2 , 3 , 5]	60	
2	$U_3(5) : 2$	[2 , 3 , 5 , 7]	840	It has no elements of order 15 and it has elements of order 20
3	$U_3(5) : 2$	[2 , 3 , 5 , 7]	840	
4	$L_3(4) : 2_1$	[2 , 3 , 5 , 7]	840	It has elements neither of order 15 nor of order 15
5	S_8	[2 , 3 , 5 , 7]	840	It has elements of order 15
1	M_{22}	[2 , 3 , 5 , 7 , 11]		
8	M_{11}	[2 , 3 , 5 , 11]	1320	They intersect different conjugacy classes of elements of order 8
9	M_{11}	[2 , 3 , 5 , 11]	1320	

Table 6.10: McL

McL	Subgroup	Factors	Exponent	Other informations
5	$3^{1+4}:2.S_5$	[2 , 3 , 5]	360	It has elements of order 15
6	$3^4:M_{10}$	[2 , 3 , 5]	360	It has no elements of order 15
12	$5^{1+2}:3:8$	[2 , 3 , 5]	120	
1	$U_4(3)$	[2 , 3 , 5 , 7]	2520	
4	$U_3(5)$	[2 , 3 , 5 , 7]	840	It has no elements of order 14
8	$2.A_8$	[2 , 3 , 5 , 7]	840	It has elements of order 14 and of order 15
7	$L_3(4):2$	[2 , 3 , 5 , 7]	840	It has elements of order 14 and it has no elements of order 15
9	$2^4:A_7$	[2 , 3 , 5 , 7]	840	It has elements of order 14 and it has no elements of order 15
10	$2^4:A_7$	[2 , 3 , 5 , 7]	840	
2	M_{22}	[2 , 3 , 5 , 7 , 11]		
3	M_{22}	[2 , 3 , 5 , 7 , 11]		
11	M_{11}	[2 , 3 , 5 , 11]		

Table 6.11: Co_3

Co_3	Subgroup	Factors	Exponent	Other informations
12	$[2^{10}.3^3]$	[2 , 3]		
13	$S_3 \times L_2(8) : 3$	[2 , 3 , 7]		
8	$3^{1+4} : 4S_6$	[2 , 3 , 5]	360	
14	$A_4 \times S_5$	[2 , 3 , 5]	60	
3	$U_4(3).(2^2)_{133}$	[2 , 3 , 5 , 7]	2520	It has no elements of order 30
6	$2.S_6(2)$	[2 , 3 , 5 , 7]	2520	It has elements of order 30
7	$U_3(5) : S_3$	[2 , 3 , 5 , 7]	840	It has elements of order 30
9	$2^4.A_8$	[2 , 3 , 5 , 7]	840	It has elements neither of order 30 nor of order 21
10	$L_3(4) : D_{12}$	[2 , 3 , 5 , 7]	840	It has no elements of order 30 and it has elements of order 21
1	$McL : 2$	[2 , 3 , 5 , 7 , 11]	27720	
2	HS	[2 , 3 , 5 , 7 , 11]	9240	
4	M_{23}	[2 , 3 , 5 , 7 , 11 , 23]		
5	$3^5 : (2 \times M_{11})$	[2 , 3 , 5 , 11]	3960	
11	$2 \times M_{12}$	[2 , 3 , 5 , 11]	1320	

Table 6.12: Co_2

Co_2	Subgroup	Factors	Esponenente
8	$2^{4+10}.(S_5S_3)$	[2 , 3 , 5]	240
10	$3^{1+4}.2^{1+4}.S_5$	[2 , 3 , 5]	360
11	$5^{1+2} : 4S_4$	[2 , 3 , 5]	120
4	$2^{1+8} : S_6(2)$	[2 , 3 , 5 , 7]	5040
6	$(2^4 \times 2^{1+6}).A_8$	[2 , 3 , 5 , 7]	840
7	$U_4(3) : D_8$	[2 , 3 , 5 , 7]	2520
1	$U_6(2) : 2$	[2 , 3 , 5 , 7 , 11]	55440
2	$2^{10} : M_{22} : 2$	[2 , 3 , 5 , 7 , 11]	18480
3	McL	[2 , 3 , 5 , 7 , 11]	27720
5	$HS : 2$	[2 , 3 , 5 , 7 , 11]	9240
9	M_{23}	[2 , 3 , 5 , 7 , 11 , 23]	

Table 6.13: Co_1

Co_1	Subgroup	Factors	Exponent	Other informations
15	$3^{3+4} : 2.(S_4 \times S_4)$	[2 , 3]		
21	$7^2 : (3 \times 2.S_4)$	[2 , 3 , 7]		
9	$24 + 12.(S_3 \times 3.S_6)$	[2 , 3 , 5]	240	
13	$3^{1+4} : 2.S_4(3).2$	[2 , 3 , 5]	360	
19	$5^{1+2} : GL_2(5)$	[2 , 3 , 5]	120	
18	$(D_{10} \times (A_5 \times A_5).2).2$	[2 , 3 , 5]	60	
20	$5^3 : (4 \times A_5).2$	[2 , 3 , 5]	60	It intersect only one conjugacy class of elements of order 30 and there is an element of order 30 in 18th maximal subgroup which is not in this class.
22	$5^2 : 2A_5$	[2 , 3 , 5]	60	It has no elements of order 30
5	$2^{1+8}.O8^+(2)$	[2 , 3 , 5 , 7]	5040	
8	$2^{2+12} : (A_8 \times S_3)$	[2 , 3 , 5 , 7]	1680	
10	$3^2.U_4(3).D_8$	[2 , 3 , 5 , 7]	2520	

12	$(A_5 \times J_2) : 2$	[2 , 3 , 5 , 7]	840	It has elements of order 42 and of order 60
14	$(A_6 \times U_3(3)).2$	[2 , 3 , 5 , 7]	840	It has no elements of order 42 and it has elements of order 60
17	$(A_7 \times L_2(7)) : 2$	[2 , 3 , 5 , 7]	840	It has elements of order 42 and it has no elements of order 60
16	$A_9 \times S_3$	[2 , 3 , 5 , 7]	1260	
7	$(A_4 \times G_2(4)) : 2$	[2 , 3 , 5 , 7 , 13]		
6	$U_6(2) : S_3$	[2 , 3 , 5 , 7 , 11]		
1	$Co2$	[2 , 3 , 5 , 7 , 11 , 23]	1275120	
3	$2^{11} : M_{24}$	[2 , 3 , 5 , 7 , 11 , 23]	425040	
4	$Co3$	[2 , 3 , 5 , 7 , 11 , 23]	637560	
2	$3.Suz : 2$	[2 , 3 , 5 , 7 , 11 , 13]		
11	$3^6 : 2.M_{12}$	[2 , 3 , 5 , 11]		

Table 6.14: He

He	Subgroup	Factors	Exponent	Other informations
5	$2^{1+6}.L_3(2)$	[2 , 3 , 7]	168	It has elements of order 12
6	$7^2 : 2.L_2(7)$	[2 , 3 , 7]	168	It has no elements of order 12
8	$7^{1+2} : (3 \times S_3)$	[2 , 3 , 7]	42	
9	$S_4 \times L_3(2)$	[2 , 3 , 7]	84	They intersect different classes of elements of order 4
10	$7 : 3 \times L_3(2)$	[2 , 3 , 7]	84	
3	$2^6 : 3.S_6$	[2 ,3 , 5]	120	
4	$2^6 : 3.S_6$	[2 , 3 , 5]	120	
11	$5^2 : 4A_4$	[2 , 3 , 5]	60	
2	$2^2.L_3(4).S_3$	[2 , 3 , 5 , 7]	840	
7	$3.S_7$	[2 , 3 , 5 , 7]	420	
1	$S_4(4) : 2$	[2 , 3 , 5 , 17]		

Table 6.15: Fi_{22}

Fi_{22}	Subgroup	Factors	Exponent	Other informations
11	$3^{1+6}:2^{3+4}:3^2:2$	[2 , 3]		
7	$(2\times 2^{1+8}):(U_4(2):2)$	[2 , 3 , 5]	720	
10	$2^{5+8}:(S_3\times A_6)$	[2 , 3 , 5]	240	
4	$O_8^+(2):S_3$	[2 , 3 , 5 , 7]	2520	It has elements of order 18 and of order 20
6	$2^6:S_6(2)$	[2 , 3 , 5 , 7]	2520	It has elements neither of order 18 nor of order 14
8	$U_4(3):2\times S_3$	[2 , 3 , 5 , 7]	2520	It has elements of order 18 and it has no elements of order 20
12	S_{10}	[2 , 3 , 5 , 7]	2520	It has no elements of order 18 and Iit has elements of order 14
13	S_{10}	[2 , 3 , 5 , 7]		
2	$O_7(3)$	[2 , 3 , 5 , 7 , 13]		
3	$O_7(3)$	[2 , 3 , 5 , 7 , 13]		
1	$2.U_6(2)$	[2 , 3 , 5 , 7 , 11]	27720	
5	$2^{10}:M_{22}$	[2 , 3 , 5 , 7 , 11]	18480	
9	${}^2F_4(2)'$	[2 , 3 , 5 , 13]		
14	M_{12}	[2 , 3 , 5 , 11]		

Table 6.16: Fi_{23}

Fi_{23}	Subgroup	Factors	Exponent
7	$3^{1+8}.2^{1+6}.3^{1+2}.2S_4$	[2 , 3]	
10	$(2^2 \times 2^{1+8}).(3 \times U_4(2)).2$	[2 , 3 , 5]	
11	$2^{6+8} : (A_7 \times S_3)$	[2 , 3 , 5 , 7]	1680
12	$S_6(2) \times S_4$	[2 , 3 , 5 , 7]	2520
4	$S_8(2)$	[2 , 3 , 5 , 7 , 17]	
2	$O_8^+(3) : S_3$	[2 , 3 , 5 , 7 , 13]	98280
5	$O_7(3) \times S_3$	[2 , 3 , 5 , 7 , 13]	32760
3	$2^2.U_6(2).2$	[2 , 3 , 5 , 7 , 11]	55440
9	S_{12}	[2 , 3 , 5 , 7 , 11]	27720
6	$2^{11}.M_{23}$	[2 , 3 , 5 , 7 , 11 , 23]	425040
1	$2.Fi_{22}$	[2 , 3 , 5 , 7 , 11 , 13]	720720
13	$S_4(4) : 4$	[2 , 3 , 5 , 17]	
8	$[3^{10}].(L_3(3) \times 2)$	[2 , 3 , 13]	
14	$L_2(23)$	[2 , 3 , 11 , 23]	

Table 6.17: HN

HN	Subgroup	Factors	Exponent	Other informations
6	$5^{1+4}.2^{1+4}.5.4$	[2 , 5]		
13	$3^4 : 2.(A_4 \times A_4).4$	[2 , 3]		
9	$2^{3+2+6}.(3 \times L_3(2))$	[2 , 3 , 7]		
3	$U_3(8) : 3$	[2 , 3 , 7 , 19]		
4	$2^{1+8}.(A_5 \times A_5).2$	[2 , 3, 5]	120	It has no elements of order 40
8	$(A_6 \times A_6).D_8$	[2 , 3 , 5]	120	It has elements of order 40
7	$2^6.U_4(2)$	[2 , 3 , 5]	360	
10	$5^{2+1+2}.4.A_5$	[2 , 3 , 5]	300	
14	$3^{1+4} : 4.A_5$	[2 , 3 , 5]	180	
5	$(D_{10} \times U_3(5)).2$	[2 , 3 , 5 , 7]		
1	A_{12}	[2 , 3 , 5 , 7 , 11]	27720	
2	$2.HS.2$	[2 , 3 , 5 , 7 , 11]	9240	
11	$M_{12} : 2$	[2 , 3 , 5 , 11]		
12	$M_{12} : 2$	[2 , 3 , 5 , 11]		

Table 6.18: Th

Th	Subgroup	Factors	Exponent	Other informations
15	$31 : 15$	[3 , 5 , 31]		
6	$3.3^2.3.(3 \times 3^2).3^2 : 2S_4$	[2 , 3]	216	It has elements of order 36
7	$3^2.3^3.3^2.3^2 : 2S_4$	[2 , 3]	216	It has no elements of order 36
11	$7^2 : (3 \times 2S_4)$	[2 , 3 , 7]		
4	$U_3(8) : 6$	[2 , 3 , 7 , 19]		
1	${}^3D_4(2) : 3$	[2 , 3 , 7 , 13]		It has elements of order 28
5	$(3 \times G_2(3)) : 2$	[2 , 3 , 7 , 13]		It has no elements of order 28
8	$3^5 : 2.S_6$	[2 , 3 , 5]	120	It has no elements of order 20 and it has elements of order 12
9	$5^{1+2} : 4S_4$	[2 , 3 , 5]	120	It has elements of order 20 and it has no elements of order 24
10	$5^2 : GL_2(5)$	[2 , 3 , 5]	120	It has elements of order 20 and of order 24
14	M_{10}	[2 , 3 , 5]	120	It has elements neither of order 20 nor of order 12
16	S_5	[2 , 3 , 5]	60	
3	$2^{1+8}.A_9$	[2 , 3 , 5 , 7]		
2	$2^5.L_5(2)$	[2 , 3 , 5 , 7 , 31]		
12	$L_2(19) : 2$	[2 , 3 , 5 , 19]		
13	$L_3(3)$	[2 , 3 , 13]		

Table 6.19: B

B	Subgroup	Factors	Other informations
30	$47 : 23$	[23 , 47]	
17	$[3^{11}].(S_4 \times 2S_4)$	[2 , 3]	
13	$3^{1+8}.2^{1+6}.U_4(2).2$	[2 , 3 , 5]	It has elements of order 9
21	$5^{1+4}.2^{1+4}.A_5.4$	[2 , 3 , 5]	It has no elements of order 9 and it has elements of order 60 and 25
22	$(S_6 \times S_6).4$	[2 , 3 , 5]	It has elements neither of order 9 nor of order 60
23	$5^2 : 4S_4 \times S_5$	[2 , 3 , 5]	It has elements neither of order 9 nor of order 25 and it has elements of order 60
10	$[2^{35}].(S_5 \times L_3(2))$	[2 , 3 , 5 , 7]	It has no elements of order 9 and it has elements of order 15 and 60
14	$(3^2 : D_8 \times U_4(3).2.2).2$	[2 , 3 , 5 , 7]	It has elements of order 9
19	$(S_6 \times L_3(4) : 2).2$	[2 , 3 , 5 , 7]	It has elements neither of order 9 nor of order 60 and it has elements of order 15
24	$L_2(49).2_3$	[2 , 3 , 5 , 7]	It has elements neither of order 9 nor of order 15
8	$[2^{30}].L_5(2)$	[2 , 3 , 5 , 7 , 31]	
4	$2^{9+16}.S_8(2)$	[2 , 3 , 5 , 7 , 17]	
12	$O_8^+(3) : S_4$	[2 , 3 , 5 , 7 , 13]	
5	Th	[2 , 3 , 5 , 7 , 13 , 19 , 31]	
6	$(2^2 \times F_4(2)) : 2$	[2 , 3 , 5 , 7 , 13 , 17]	
7	$2^{2+10+20}.(M_{22} : 2 \times S_3)$	[2 , 3 , 5 , 7 , 11]	It has elements of order 21 and It has no elements of order 35
15	$5 : 4 \times HS : 2$	[2 , 3 , 5 , 7 , 11]	It has no elements of order 21
18	$S_5 \times M_{22} : 2$	[2 , 3 , 5 , 7 , 11]	It has elements of order 21 and 35
2	$2^{1+22}.Co_2$	[2 , 3 , 5 , 7 , 11 , 23]	
11	$HN : 2$	[2 , 3 , 5 , 7 , 11 , 19]	
9	$S_3 \times Fi_{22} : 2$	[2 , 3 , 5 , 7 , 11 , 13]	
3	Fi_{23}	[2 , 3 , 5 , 7 , 11 , 13 , 17 , 23]	

1	$2.{}^2E_6(2):2$	[2 , 3 , 5 , 7 , 11 , 13 , 17 , 19]	
20	$5^3.L_3(5)$	[2 , 3 , 5 , 31]	It has elements of order 20
25	$L_2(31)$	[2 , 3 , 5 , 31]	It has no elements of order 20
16	$S_4 \times 2F_4(2)$	[2 , 3 , 5 , 13]	
26	M_{11}	[2 , 3 , 5 , 11]	It has no elements of order 10
29	$L_2(11):2$	[2 , 3 , 5 , 11]	It has elements of order 10
28	$L_2(17):2$	[2 , 3 , 17]	
27	$L_3(3)$	[2 , 3 , 13]	

Table 6.20: M

M	Subgroup	Factors	Other informations
43	$41:40$	[2 , 5 , 41]	
29	$7^{2+1+2}:GL_2(7)$	[2 , 3 , 7]	Ha elementi sia di ordine 21 sia di ordine 48
34	$(7^2:(3\times 2A_4)\times L_2(7)).2$	[2 , 3 , 7]	It has elements of order 21 and It has no elements of order 48
41	$7^2:SL_2(7)$	[2 , 3 , 7]	It has no elements of order 21
33	$13^2:2L_2(13).4$	[2 , 3 , 7 , 13]	
20	$(A_6\times A_6\times A_6).(2\times S_4)$	[2 , 3 , 5]	It has no elements of order 18 and It has elements of order 40
22	$5^{2+2+4}:(S_3\times GL_2(5))$	[2 , 3 , 5]	Non ha elementi nor of order 18 nor of order 40
31	$(S_5\times S_5\times S_5):S_3$	[2 , 3 , 5]	It has elements of order 18
10	$2^{3+6+12+18}.(L_3(2)\times 3S_6)$	[2 , 3 , 5 , 7]	It has elements of order 56 and 105
16	$5^{1+6}:2J_2:4$	[2 , 3 , 5 , 7]	It has elements of order 56 , it has elements neither of order 105 nor of order 42

24	$7^{1+4} : (3 \times 2S_7)$	[2 , 3 , 5 , 7]	It has elements of order 56 and 42, it has no elements of order 105
25	$(5^2 : [2^4] \times U_3(5)).S_3$	[2 , 3 , 5 , 7]	It has elements neither of order 56 nor of order 105
27	$(A_7 \times (A_5 \times A_5) : 2^2) : 2$	[2 , 3 , 5 , 7]	It has no elements of order 56 and it has elements of order 105
37	$L_2(71)$	[2 , 3 , 5 , 7 , 71]	
8	$2^{5+10+20}.(S_3 \times L_5(2))$	[2 , 3 , 5 , 7 , 31]	
40	$L_2(29) : 2$	[2 , 3 , 5 , 7 , 29]	
21	$(A_5 \times U_3(8) : 3_1) : 2$	[2 , 3 , 5 , 7 , 19]	
17	$(7 : 3 \times He) : 2$	[2 , 3 , 5 , 7 , 17]	It has no elements of order 34
23	$(L_3(2) \times S_4(4) : 2).2$	[2 , 3 , 5 , 7 , 17]	It has elements of order 34
5	$2^{10+16}.O_{10}^{+}(2)$	[2 , 3 , 5 , 7 , 17 , 31]	
13	$(3^2 : 2 \times O_8^{+}(3)).S_4$	[2 , 3 , 5 , 7 , 13]	
11	$3^8.O8_{-}(3).2^3$	[2 , 3 , 5 , 7 , 13 , 41]	
9	$S_3 \times Th$	[2 , 3 , 5 , 7 , 13 , 19 , 31]	
18	$(A_5 \times A_{12}) : 2$	[2 , 3 , 5 , 7 , 11]	
6	$2^{2+11+22}.(M_{24} \times S_3)$	[2 , 3 , 5 , 7 , 11 , 23]	
12	$(D_{10} \times HN).2$	[2 , 3 , 5 , 7 , 11 , 19]	
7	$3^{1+12}.2Suz.2$	[2 , 3 , 5 , 7 , 11 , 13]	
2	$2^{1+24}.Co_1$	[2 , 3 , 5 , 7 , 11 , 13 , 23]	
3	$3.Fi_{24}$	[2 , 3 , 5 , 7 , 11 , 13 , 17 , 23 , 29]	
4	$22.{}^2E_6 : S_3$	[2 , 3 , 5 , 7 , 11 , 13 , 17 , 19]	
1	$2.B$	[2 , 3 , 5 , 7 , 11 , 13 , 17 , 19 , 23 , 31 , 47]	
19	$5^{3+3}.(2 \times L_3(5))$	[2 , 3 , 5 , 31]	
38	$L_2(59)$	[2 , 3 , 5 , 29 , 59]	
42	$L_2(19) : 2$	[2 , 3 , 5 , 19]	
28	$5^4 : (3 \times 2L_2(25)) : 2_2$	[2 , 3 , 5 , 13]	
14	$3^{2+5+10}.(M_{11} \times 2S_4)$	[2 , 3 , 5 , 11]	It has elements of order 27

26	$(L_2(11) \times M_{12}) : 2$	[2 , 3 , 5 , 11]	It has no elements of order 27, it has elements of order 60 and of order 88
30	$M_{11} \times A_6.2^2$	[2 , 3 , 5 , 11]	It has elements neither of order 27 nor of order 60 and it has elements of order 40
32	$(L_2(11) \times L_2(11)) : 4$	[2 , 3 , 5 , 11]	It has elements neither of order 27 nor of order 88, it has elements of order 60
39	$11^2 : (5 \times 2A_5)4$	[2 , 3 , 5 , 11]	It has elements neither of order 27 nor of order 60 nor of order 40
15	$3^{3+2+6+6} : (L_3(3) \times SD_{16})$	[2 , 3 , 13]	The subgroup $3^6 : (L_3(3) \times SD_{16})$ has elements of order 9
35	$(13 : 6 \times L_3(3)).2$	[2 , 3 , 13]	It has no elements of order 9 and it has elements of order 104
36	$13^{1+2} : (3 \times 4S_4)$	[2 , 3 , 13]	It has elements neither of order 9 nor of order 104

Table 6.21: J_1

J_1	Subgroup	Factors	Other informations
5	11:10	[2 , 5 , 11]	
2	$2^3 : 7 : 3$	[2 , 3 , 7]	The 2nd and the 7th maximal subgroups have elements of orders [1 , 2 , 3 , 6 , 7] and for each of these there is only one conjugacy class of elements in J_1
7	7:6	[2 , 3 , 7]	
3	$2 \times A_5$	[2 , 3 , 5]	It has no elements of order 15
6	$D_6 \times D_{10}$	[2 , 3 , 5]	It has elements of order 15
1	$PSL_2(11)$	[2 , 3 , 5 , 11]	
4	19:6	[2 , 3 , 19]	

Table 6.22: $O'N$

$O'N$	Subgroup	Factors	Other informations
9	$4^3.L_3(2)$	[2 , 3 , 7]	
1	$L_3(7):2$	[2 , 3 , 7 , 19]	They intersect different conjugacy classes of elements of order 8
2	$L_3(7):2$	[2 , 3 , 7 , 19]	
5	$(3^2:4\times A_6).2$	[2 , 3 , 5]	It has elements of order 20
6	$3^4:2^{1+4}.D_{10}$	[2 , 3 , 5]	It has no elements of order 20
4	$4_2.L_3(4):2_1$	[2 , 3 , 5 , 7]	
12	A_7	[2 , 3 , 5 , 7]	They intersect the same conjugacy classes of elements of orders 4 and 7 and there is only one conjugacy class of elements of order 2,3,5 and 6
13	A_7	[2 , 3 , 5 , 7]	
3	J_1	[2 , 3 , 5 , 7 , 11 , 19]	
7	$L_2(31)$	[2 , 3 , 5 , 31]	They intersect different conjugacy classes of elements of order 8
8	$L_2(31)$	[2 , 3 , 5 , 31]	
10	M_{11}	[2 , 3 , 5 , 11]	They intersect different conjugacy classes of elements of order 8
11	M_{11}	[2 , 3 , 5 , 11]	

Table 6.23: J_3

J_3	Subgroup	Factors	Exponent	Other informations
7	$3^{2+1+2}:8$	[2 , 3]	72	
9	$2^{2+4}:(3\times S_3)$	[2 , 3]	24	
4	$2^4:(3A_5)$	[2 , 3 , 5]	60	
6	$(3A_6):2_2$	[2 , 3 , 5]	120	It has elements of order 15
8	$2^{1+4}:A_5$	[2 , 3 , 5]	120	It has no elements of order 15
2	$L_2(19)$	[2 , 3 , 5 , 19]		
3	$L_2(19)$	[2 , 3 , 5 , 19]		
1	$L_2(16):2$	[2 , 3 , 5 , 17]		
5	$L_2(17)$	[2 , 3 , 17]		

Table 6.24: Ly

Ly	Subgroup	Factors	Exponent	Other informations
5	$5^{1+4}:4.S_6$	[2 , 3 , 5]		It has an element of order 25
7	$3^{2+4}:2.A_5.D_8$	[2 , 3 , 5]	360	
1	$G_2(5)$	[2 , 3 , 5 , 7 , 31]		
2	$3.McL:2$	[2 , 3 , 5 , 7 , 11]		It has elements of order 33
4	$2.A_{11}$	[2 , 3 , 5 , 7 , 11]		It has no elements of order 33
3	$5^3.L_3(5)$	[2 , 3 , 5 , 31]		
6	$3^5:(2\times M_{11})$	[2 , 3 , 5 , 11]		
9	$37:18$	[2 , 3 , 37]		
8	$67:22$	[2 , 11 , 67]		

Table 6.25: Ru

Ru	Subgroup	Factors	Exponent	Other informations
12	$5^{1+2}:[2^5]$	[2 , 5]		
4	$2^{3+8}:L_3(2)$	[2 , 3 , 7]	336	
2	$2^6.U_3(3).2$	[2 , 3 , 7]	168	
13	$L_2(13):2$	[2 , 3 , 7 , 13]		
15	$5:4A_5$	[2 , 3 , 5]	60	
14	$A_6.2^2$	[2 , 3 , 5]	120	It has no elements of order 12
11	$3.A_6.2^2$	[2 , 3 , 5]	120	It has elements of order 15 and of order 12
10	$5^2:4.S_5$	[2 , 3 , 5]	120	It has elements of order 12 and it has no elements of order 15
6	$2^{1+4+6}.S_5$	[2 , 3 , 5]	240	
8	A_8	[2 , 3 , 5 , 7]	420	
5	$U_3(5):2$	[2 , 3 , 5 , 7]	840	
9	$L_2(29)$	[2 , 3 , 5 , 7 , 29]		
3	$(2^2Sz(8)):3$	[2 , 3 , 5 , 7 , 13]		
7	$L_2(25).2^2$	[2 , 3 , 5 , 13]	1560	
1	${}^2F4(2)$	[2, 3 , 5 , 13]	3120	

Table 6.26: J_4

J_4	Subgroup	Factors	Other informations
12	$43 : 14$	[2 , 7 , 43]	
11	$29 : 28$	[2 , 7 , 29]	
10	$U_3(3)$	[2 , 3 , 7]	
4	$2^{3+12}.(S5 \times L_3(2))$	[2 , 3 , 5 , 7]	
3	$2^{10} : L_5(2)$	[2 , 3 , 5 , 7 , 31]	
2	$2^{1+12}.3.M_{22} : 2$	[2 , 3 , 5 , 7 , 11]	It has elements of order 44
6	$M_{22} : 2$	[2 , 3 , 5 , 7 , 11]	It has no elements of order 44
1	$2^{11} : M_{24}$	[2 , 3 , 5 , 7 , 11, 23]	
7	$11^{1+2} : (5 \times 2S_4)$	[2 , 3 , 5 , 11]	
5	$U_3(11) : 2$	[2 , 3 , 5 , 11 , 37]	
8	$L_2(32) : 5$	[2 , 3 , 5 , 11 , 31]	
13	37:12 = F444	[2 , 3 , 37]	
9	$L_2(32) : 2$	[2 , 3 , 11 , 23]	

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