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Supersymmetric observables of $\mathcal{N} = 1$ Quantum Field Theories on a twisted $\mathrm{S}^1 \times \mathrm{S}^3$

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CHAPTER 1

Introduction

Quantum field theory has repeatedly proven to be a solid framework to describe the physics of at least three of the fundamental interactions we know today. Its vast development over the last century led not only to a remarkable understanding of particle physics and to outstanding predictions in the same field, but also to the development of powerful tools that may be applied outside the field of high energy physics, such as in the study of statistical mechanics and critical phenomena. Nevertheless, QFT presents a fundamental issue in that it does not allow to quantise the gravitational interaction in a consistent way within its formalism. In particular, the quantised Einstein-Hilbert action results in a non-renormalisable theory. In order to overcome this problem, it has been a while now that physicists came out with a different approach: string theory. Though the precise relation between string theory and the phenomenology of particle physics and cosmology remains in part an open problem nowadays, string theory has proven to be a consistent quantum gravity formalism. On top of that, it is undeniable that it has driven large part of the recent developments in high energy theoretical physics, both conceptual and technical.

As it is well-known, supersymmetry should be introduced in string theory in order to remove some inconsistencies that arise in a purely bosonic string theory (the presence of tachyonic excitations). Remarkably, one can talk about supersymmetry without the need of string theory, implementing it in the simpler context of QFT and general relativity, and this is useful since it allows to study the effective physics, possibly arising from string theory, at energy regimes that are comprised between the scale at which supersymmetry is broken and the Planck scale where a modification of the point particle paradigm is needed. In such a context, supersymmetry is basically an additional spacetime symmetry that exchanges bosons with fermions and vice versa, and as such it requires each particle in a given theory to have its own superpartner. When supersymmetry is at work the Poincaré algebra is extended to include also the supersymmetry generators, called supercharges. This wider symmetry algebra has the noteworthy effect of making supersymmetric theories much more constrained compared to non-supersymmetric ones. The result is that supersymmetric theories are usually much easier to tackle from the technical point of view and quantum corrections are under better control due to cancellations between bosonic and fermionic contributions in the loops. Remarkably, this more constrained setup often allows us to compute one or more of the "observables" associated to the theory (e.g. generating functional, n-point correlation functions, etc.) in an exact way i.e. without relying on some kind of approximation as, for example, perturbation theory. This means that we can have insights on the physics at strong coupling, which is something extremely rare in ordinary QFT where it is usually hard to obtain significant results without relying on a perturbative analysis at weak coupling. In this view, supersymmetric quantum field theories constitute an extremely valuable playground for exploring the non-perturbative structure of QFTs and it is not surprising that the high energy physics community put a lot of effort into studying them over the the last couple of decades.

Within this general picture, in recent years there has been an increasing interest in the study of supersymmetric quantum field theories defined in curved spaces i.e. manifolds with a fixed non-flat metric, different from the usual Minkowski one. Such interest increased after that a systematic way to define such theories was developed [1]. Before that, defining a supersymmetric QFT on such a background was not a straightforward task and had to rely on a trial and error approach. Hence, once this obstacle was essentially removed, more and more results were derived and great developments in the understanding of this kind of setup were made.

There are three main reasons why studying supersymmetric quantum field theories on curved manifolds is worth of interest. In the first place, there are some technical motivations. A general approach for defining physical observables is to switch on some background sources and couple them to the suitable fields appearing in the theory of interest. Then one can obtain correlation functions by taking functional derivatives of the partition function with respect to these background sources, and sometimes supersymmetry allows to compute them exactly (for instance using the technique of supersymmetric localisation which essentially reduces the path integral to an ordinary integral [26, 31, 32]). If one wants to consider correlation functions involving the stress-energy tensor of the theory, then the background source it has to be coupled with is nothing but the metric tensor (and its supersymmetric completion of course); a background metric tensor amounts precisely to considering the theory on a curved space. Moreover, the presence of a finite length scale which is introduced on a compact manifold constitutes a natural IR regulator, hence in this context one should take care of the usual UV divergences only.

Secondly, the study of these theories may lead to a deeper understanding of the role of spacetime geometric properties in determining the physics of quantum fields. To be more concrete, the picture is similar to what happens with topological quantum field theories [26]. These are a class of QFTs which live on spaces whose topological properties constitute the only relevant information determining the physics of the theory. In particular, computing physical observables amounts to computing topological invariants associated to the space where the theory is living. This deep entanglement between physical and mathematical objects has led to progress in both sides: mathematical tools were used directly to compute physical quantities, and techniques arising from the physics world have taught us new ways for handling mathematical objects, in particular topological invariants indeed [27]. An analogous situation occurs in our context, though the interest is focused on the complex structure of the manifolds rather than on their topology. It can be shown that many observables arising from supersymmetric field theories placed on curved manifolds depend only on some of the geometric information at disposal [3, 4], in particular on those involving the complex structure. Thus, it would be interesting to explore thoroughly this relationship between complex geometry and physical observables.

Last but not least, results concerning observables of field theories on curved spaces can often be related to observables in gravity theories through the holographic principle. It has been a while now since the first time the AdS/CFT correspondence idea came out [28, 29, 30] and since then its understanding and usage has grown more and more. Essentially it is a strong/weak duality which states that in certain conditions there is a one to one mapping between parameters and physical quantities of a gravitational theory in a $d+1$ dimensional anti-de Sitter space and those of a conformal field theory living in its d dimensional boundary. Because of this relation between a bulk gravitational theory and a CFT living at the boundary, the duality is called holographic. Therefore, by computing exact observables in a conformal field theory living on a curved space, we can potentially learn a lot in regard of its dual gravity theory at both strong and weak coupling regimes.

The present thesis lays within this framework and in particular it aims to achieve some progress in the understanding of one particular spacetime geometry that is $S^1 \times S^3$. Such a background is particularly relevant for two reasons. On the one hand, it has a well-known complex structure, as $S^1 \times S^3$ is diffeomorphic to the complex manifolds belonging to the class of primary Hopf surfaces [25], and therefore there is quite a lot of literature that comes to help when needed. For example, the complex structure parameters can be encoded in the metric in a simple way by some deformations of the manifold with respect to the simple direct product of the circle times the round 3-sphere, in particular twisting and squashing. On the other hand, $S^1 \times S^3$ is the conformal boundary of an asymptotically AdS_5 space in global coordinates (after Wick rotation to Euclidean signature and compactification of the Euclidean time), hence it may be very useful from the holographic perspective. For instance, one specific application consists in the microscopic counting of black hole entropy. It is an established fact that black holes display a thermodynamical behaviour, yet in the context of non-quantum gravity (i.e. general relativity) we are able to tell only that their entropy is proportional to the horizon area, which is a macroscopic quantity. A viable theory of quantum gravity should provide a way for deriving this entropy through a statistical microstate counting. In specific contexts, string theory has proven to be able to do this [33] but there are still many cases where a proper microscopic description is not fully known. Supersymmetric asymptotically AdS_5 black holes constitute one of these cases and have the geometry of interest in this thesis as a conformal boundary. The AdS/CFT correspondence may be able to provide a tool for counting black hole microstates exploiting the dual description in terms of a CFT, whence the interest for CFTs in $S^1 \times S^3$ background.

Keeping in mind this context, the aim of this thesis is to tackle the computation of one particular physical observable arising from a simple $\mathcal{N} = 1$ SCFT living on the background $S^1 \times S^3$: the supersymmetric Casimir energy $\langle H_{susy} \rangle$. This quantity is defined to be the vacuum expectation value of the supersymmetric Hamiltonian governing the evolution of the theory, and it is related to the exponential prefactor that appears in the expression for the partition function. As we will see, it is not a priori obvious that $\langle H_{susy} \rangle$ is a well-defined physical quantity, and indeed in non-supersymmetric theories it is not. Luckily, in [6] it was shown that supersymmetry makes it non-ambiguous and physical, as well as $\langle H_{susy} \rangle$ was computed for the background corresponding to the simple direct product $S^1 \times S^3$ by performing a dimensional reduction over the 3-sphere. The supersymmetric Casimir energy has also been shown to be related to the microstate counting problem in an interesting though not yet entirely understood way [15]. In the present work we will exploit techniques similar to those in [6] in order to extend their result to the more general case of a twisted $S^1 \times S^3$ as background geometry. Below we sketch the outline of the thesis and illustrate our new results.

Chapter 2 is devoted to a general introduction to supersymmetric theories in curved spaces. Firstly we review the algorithmic method that allows to take a flat space supersymmetric field theory and put it onto a curved manifold without spoiling supersymmetry [1], with a particular focus on $\mathcal{N} = 1$ theories with two conserved supercharges of opposite R-charge [2]. Then we briefly speak about the dependence of the partition function of such theories on deformations of the geometry and sketch the main ideas leading to the proof that only the information concerning the complex structure of the manifold enters the partition function [3, 4].

In chapter 3 we start by introducing more specifically the background geometry corresponding to a non-deformed $S^1 \times S^3$, including its symmetries and the background field needed to define a supersymmetric theory. Then, we explain the reasoning carried out in [6] to show that the Casimir energy is unambiguous and well-defined. The crucial point is that when reducing to one dimension some Chern-Simons terms pop out and they cannot be obtained by integrating 4d counterterms. Once this will be established, we will consider a simple chiral multiplet on this background and go through the majority of the details of the actual computation leading to the Casimir energy expression. We will expand each field in spherical harmonics and integrate over S^3, and the result will be a one dimensional theory with infinite degrees of freedom whose ground state energy is fairly simple to obtain. The choice of the chiral multiplet is due to the fact that it is the simplest $\mathcal{N} = 1$ supersymmetric theory, and it can be regarded as a toy model for understanding this technique and subsequently apply it to some more relevant and complete theories. Though this chapter constitute essentially a review of [6], here we present explicitly most of the passages of the dimensional reduction that are not explained in that work.

The original work of this thesis is contained in chapter 4. We will introduce both a twisting in the background geometry, parametrised by two real numbers σ_1 and σ_2, and another more subtle deformation (not visible in the manifold) which is encoded in an integer parameter n_0. The effort we put in detailing the dimensional reduction and the subsequent derivation of the supersymmetric Casimir energy in chapter 3 will allow us to provide a natural extension of the procedure to this more general setting. The focus will be on highlighting the physical differences from the case studied in the previous chapter and discussing some subtleties that arise only when the parameters σ_1, σ_2, and n_0 are turned on. For the detailed computations one can still refer to chapter 3. The final result will be a generalised expression for the Casimir energy which includes the one obtained in [6] and extend it to the twisted $S^1 \times S^3$ background.

Finally, chapter 5 contains a summary of the obtained results as well as their interpretation in terms of the background complex structure. There we will define primary Hopf surfaces and see how the newly introduced twisting parameters σ_1 and σ_2 arrange into the complex structure parameters of an Hopf surface. The complex structure parameters which were purely real in [6] are completed to complex values thanks to the introduction of σ_1 and σ_2, hence our new findings results in an overcoming of some limitations of the previous work. We will also discuss how our results relate to others which can be found in literature as well as the possible directions for further developments.

CHAPTER 2

Supersymmetric field theories on curved spaces

2.1 From flat to curved space in a nutshell

Given that supersymmetry is a spacetime symmetry rather than an internal one, it is not a priori obvious how to couple a generic supersymmetric field theory defined on flat Minkowski space to a non-flat fixed metric without spoiling supersymmetry. The first approach one might think about is to substitute the curved metric everywhere in the flat-space lagrangian and then add suitable terms in powers of $\frac{1}{r}$ (where r is a relevand scale of the metric) in both the lagrangian and the supersymmetry variations so as to recover invariance a and meaningful supersymmetry algebra. Although this trial and error strategy lead to correct results, it presents considerable technical difficulties.

Luckily, it has been a while now that a more systematic approach has been developed [1, 2]. Roughly speaking, the main point of the procedure consists in coupling the supersymmetric theory of interest to off-shell supergravity, that is without integrating out the auxiliary fields lying inside the supergravity multiplet through their equations of motion; then one takes the rigid limit in order to keep the metric fixed and decouple the gravitino.

Let us go a little bit more through details.

1. The first step consists in writing a lagrangian that couples our supersymmetric model to supergravity. This is done by introducing a dynamical metric $g_{\mu\nu}$ together with its complements to a superfield i.e. the gravitino Ψ_μ and some auxiliary fields, and the coupling terms are constituted essentially by the minimal coupling of the supercurrent multiplet with the supergravity auxiliary fields.

2. Without solving the equations of motion for the auxiliary fields, we take the limit $M_P \to \infty$ while holding fixed the metric we are interested in (and therefore specifying the spacetime manifold $\mathcal{M}$). By doing this, the fields in the supergravity multiplet become a non-dynamical fixed background. Then, we also set $\Psi_\mu = 0$ so as to make the gravitino disappear.

3. We impose the supersymmetry variation of the gravitino to vanish so as it decouples completely from the theory:

$$\delta\Psi_\mu = \delta\bar{\Psi}_\mu = 0\,. \tag{2.1}$$

   These constraints are called Killing spinor equations and in general will be two linear differential equation for the supersymmetry parameters ζ and $\bar{\zeta}$. They admit solutions only for some specific values of the background auxiliary fields, and the number of independent solutions corresponds to the number of preserved supersymmetries. We stress the fact that the auxiliary fields has not to satisfy any equation of motion and they can take any arbitrary value.

4. Finally we substitute the expressions for the auxiliary fields obtained by requiring (2.1) to be integrable into the lagrangian. In general we will find something of the form:

$$\mathcal{L}_{\mathcal{M}} = \mathcal{L}'_{\mathbb{R}^4} + \sum_{n=1}^{+\infty} \frac{1}{r^n}\,\delta\mathcal{L}_n\,, \tag{2.2}$$

   where $\mathcal{L}'_{\mathbb{R}^4}$ is simply the flat space lagrangian with the flat metric replaced by $g_{\mu\nu}$ and r is some characteristic size of $\mathcal{M}$. Remarkably [1] proved that all terms with $n > 2$ vanish. Note that the flat space lagrangian is recovered in the limit $r \to \infty$.

If one manages to follow all this steps, the rigid supersymmetry algebra on $\mathcal{M}$ arises as a subalgebra of the local supersymmetry algebra of the supergravity theory we started with, and in general it is different from the supersymmetry algebra of the flat space theory.

2.2 $\mathcal{N} = 1$ theories with R-symmetry

2.2.1 Supergravity coupling and rigid limit

We now focus on four dimensional $\mathcal{N} = 1$ supersymmetric theories that admit a $U(1)_R$ symmetry, since this is the case which we are going to work with in the rest of the thesis. A broad discussion of this setting is contained in [2] which is also the source we will refer to in writing this section. However, we will use the conventions of [5] that are also summarised in appendix A. In particular we will consider a four dimensional Riemannian manifold $\mathcal{M}$ equipped with a real metric with euclidean signature.

On general ground, a flat-space field theory can be placed on $\mathcal{M}$ by coupling its stress-energy tensor to the metric on $\mathcal{M}$. However, when the theory is supersymmetric the stress-energy tensor $T_{\mu\nu}$ is part of the supercurrent multiplet, which includes also various other operators. Then, the correct generalisation is to couple the operators inside the supercurrent multiplet to the fields lying in the supergravity multiplet i.e. $g_{\mu\nu}$ to $T_{\mu\nu}$, the gravitino $\Psi_{\mu\alpha}$ to the supersymmetry current $S_{\mu\alpha}$, and so on. Now, there exist different formulations of $4d$ $\mathcal{N} = 1$ supergravity as well as there exist various different supercurrent multiplets. In our setting the most convenient choice is to work with the R-multiplet, whose existence is subjected to the presence of a $U(1)_R$ symmetry in the theory; this is precisely our case. The R-multiplet is described by the pair of superfields $(\mathcal{R}_\mu, \chi_\alpha)$, where $\mathcal{R}_\mu$ is a vector superfield (in lorentzian signature it is real) and χ_α is a chiral superfield (i.e. $\tilde{D}_{\dot\alpha}\chi_\alpha = 0$) such that they satisfy:

$$\tilde{D}^{\dot\alpha}\,(\sigma^\mu_{\alpha\dot\alpha}\mathcal{R}_\mu) = -\frac{1}{2}\,\chi_\alpha\,, \qquad\qquad D^\alpha\chi_\alpha = \tilde{D}_{\dot\alpha}\tilde{\chi}^{\dot\alpha}\,. \tag{2.3}$$

The expressions of the superfields $\mathcal{R}_\mu$ and χ_α in components is a bit cumbersome and not very useful for our purpose (see e.g. [2]); the important thing is that they contain the R-current J^μ_R, of course the stress-energy tensor $T_{\mu\nu}$ and the supersymmetry currents $S_{\mu\alpha}$ and $\tilde{S}^{\dot\alpha}_\mu$, and finally a closed two-form $\mathcal{F}_{\mu\nu}$. Note that $\mathcal{F}_{\mu\nu}$ can be seen as a field-strength for another vector field $\mathcal{A}_\mu$, that is:

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu\,, \tag{2.4}$$

modulo the usual ambiguity in that the transformation $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu\alpha$ leaves $\mathcal{F}_{\mu\nu}$ unchanged. All of these currents are determined only by the flat-space supersymmetric theory and are conserved.

The R-multiplet couples to the so called *"new minimal supergravity"* multiplet [8, 9] which contains the metric $g_{\mu\nu}$, the gravitino fields $\Psi_{\mu\alpha}$ and $\tilde{\Psi}^{\dot\alpha}_\mu$, and two auxiliary bosonic fields: the abelian vector field A_μ and a skew-symmetric two-index tensor field $B_{\mu\nu}$. However, it is more convenient to embed the latter inside another vector field V^μ defined by:

$$V^\mu \equiv \frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu B_{\rho\sigma}\,. \tag{2.5}$$

Such a vector field is covariantly conserved i.e. $\nabla_\mu V^\mu = 0$, ∇_μ being the Levi-Civita connection on $\mathcal{M}$. In euclidean signature the background fields A_μ and V_μ are complex rather than real as in lorentzian signature, and Ψ_μ and $\tilde{\Psi}_\mu$ are independent one of another as well as the supersymmetry currents S_μ and $\tilde{S}_\mu$. Yet, we will assume the metric $g_{\mu\nu}$ to be real. Each field inside the supercurrent multiplet acts as a source for a field in the new minimal supergravity multiplet. The minimal coupling lagrangian it is then given by:

$$\mathcal{L} = \frac{1}{2}\, g^{\mu\nu}\, T_{\mu\nu} - \Psi^{\mu} S_{\mu} - \tilde{\Psi}^{\mu} \tilde{S}_{\mu} - \left(A_{\mu} - \frac{3}{2}\, V_{\mu} \right) J_{R}^{\mu} - V^{\mu} \mathcal{A}_{\mu}\, . \tag{2.6}$$

At this point, a couple of comments have to be made. Firstly, note that the lagrangian should be invariant under an exact one-form shift of $\mathcal{A}_{\mu}$ for the theory to be consistent; this is indeed the case since V^{μ} is covariantly conserved as we stated above. Secondly, from (2.6) we can see that A_{μ} plays the role of gauge field for the local R-symmetry. The full supergravity theory is given by the non-linear completion of (2.6).

As explained in the previous section, now we have to take the rigid limit $M_P \to \infty$ without integrating out the supergravity auxiliary fields and holding the metric fixed. Since we want the gravitino to disappear from the theory, we set $\Psi_{\mu} = \tilde{\Psi}_{\mu} = 0$, which also make the supersymmetry variation of the supergravity bosonic fields to vanish [8]. This is precisely what we want because the supergravity fields should play the only role of a non-dynamical background for our field theory. Instead, the supersymmetry variations of the gravitino fields read:

$$\begin{aligned} \delta\Psi_{\mu} &= -2\, D_{\mu}\zeta - 2i\, V_{\mu}\zeta - 2i\, V^{\nu}\sigma_{\mu\nu}\zeta\, , \\ \delta\tilde{\Psi}_{\mu} &= -2\, D_{\mu}\tilde{\zeta} + 2i\, V_{\mu}\tilde{\zeta} + 2i\, V^{\nu}\tilde{\sigma}_{\mu\nu}\tilde{\zeta}\, , \end{aligned} \tag{2.7}$$

where the supersymmetry parameters ζ_{α} and $\tilde{\zeta}^{\dot{\alpha}}$ are two-components independent commuting spinors carrying respectively R-charge +1 and -1 [1] and $D_{\mu} \equiv \nabla_{\mu} - iqA_{\mu}$ is the covariant derivative, q being the R-charge of the field on which it is acting. Notice that the fact that in euclidean signature they are independent means that $\zeta^{\dagger} \neq \tilde{\zeta}$ and $\tilde{\zeta}^{\dagger} \neq \zeta$. Requiring (2.7) to vanish, we obtain the so called Killing spinor equations:

$$\begin{aligned} D_{\mu}\zeta + i\, V_{\mu}\zeta + i\, V^{\nu}\sigma_{\mu\nu}\zeta &= 0\, , \\ D_{\mu}\tilde{\zeta} - i\, V_{\mu}\tilde{\zeta} - i\, V^{\nu}\tilde{\sigma}_{\mu\nu}\tilde{\zeta} &= 0\, . \end{aligned} \tag{2.8}$$

A given configuration of the background i.e. of $g_{\mu\nu}$, A_{μ}, and V_{μ} preserves some degree of supersymmetry if the differential equations (2.8) admit at least one non-trivial solution for either ζ or $\tilde{\zeta}$, which in this context are referred to as Killing spinors. In particular, the number of independent solutions corresponds to the number of supercharges that are unbroken. Indeed, each independent Killing spinor is determined modulo a complex multiplicative constant which is nothing but a supersymmetry

[1] We can take ζ and $\tilde{\zeta}$ to be commuting since they are not dynamical fields, thus nothing forces them to be anti-commuting.

parameter of the field theory on $\mathcal{M}$. It is clear now why we did not eliminate the auxiliary fields through their equations of motion: we want them to take particular values such that the Killing spinor equations can be solved. Remarkably, (2.8) do not depend on the field theory involved but on the supergravity field configuration only, thus a given background that admits a solution of the Killing spinor equations will preserve supersymmetry in any theory that admits a coupling to the new minimal multiplet.

After applying this procedure with success, the supersymmetry algebra arise as a subalgebra of the local supersymmetry algebra of the supergravity theory. In particular if there is only one Killing spinor ζ, the superalgebra simply reads:

$$\delta_\zeta^2 = 0 \,, \tag{2.9}$$

where δ_ζ is the action of the operator $i\zeta\mathcal{Q}$, $\mathcal{Q}_\alpha$ being one of the supersymmetry generators (analogously $\delta_{\tilde{\zeta}}$ will be the action of the operator $i\tilde{\zeta}\tilde{\mathcal{Q}}$). If instead (2.8) admits more than one solution, then for each pair of ζ and $\tilde{\zeta}$ we can build the vector $K^\mu = \zeta\sigma^\mu\tilde{\zeta}$. Now, we define the twisted Lie derivative along $K = K^\mu\partial_\mu$ as the operator:

$$\hat{L}_K \equiv L_K - iq\, K^\mu A_\mu \,. \tag{2.10}$$

Then, the supersymmetry algebra reads:

$$\{\delta_\zeta,\, \delta_{\tilde{\zeta}}\} = 2i\hat{L}_K \,, \qquad\qquad \{\delta_\zeta,\, \delta_\zeta\} = \{\delta_{\tilde{\zeta}},\, \delta_{\tilde{\zeta}}\} = 0 \,. \tag{2.11}$$

Summarizing, by following the illustrated procedure the problem of defining an $\mathcal{N} = 1$ supersymmetric theory with an $U(1)_R$ symmetry on a given four dimensional curved manifold reduces to that of finding a suitable configuration of the new minimal supergravity auxiliary fields A_μ and V_μ and one or more Killing spinors such that the Killing spinor equations (2.8) are solved.

2.2.2 Solving the Killing spinor equations

As we said, equations (2.8) do not admit solutions for arbitrary values of $g_{\mu\nu}$, A_μ, and V_μ. A systematic analysis of the backgrounds admitting one or more solutions is carried over in details in [2]. Here we will only review their results concerning manifolds that admit two superchargers of opposite R-charge, that is one independent solution for each of the two equations (2.8). This is the case relevant for the follow of this work.

To begin with, it was shown that a manifold admits at least one supercharge if and only if it is Hermitian i.e. it is a complex manifold with a smooth Hermitian product on the tangent space. Provided that this condition is satisfied, let us assume that the existing solution of (2.8) is ζ. Then, we can build covariant objects as spinor bilinears, and in particular we define:

$$J_{\mu\nu} \equiv \frac{2i}{|\zeta|^2}\,\zeta^\dagger \sigma_{\mu\nu}\zeta\;, \tag{2.12}$$

where $|\zeta| = \zeta^\dagger\zeta$. One can show that this quantity is such that:

$$J^\mu{}_\nu\, J^\nu{}_\rho = -\delta^\mu{}_\rho\,. \tag{2.13}$$

This tells us that $J^\mu{}_\nu$ is an almost complex structure on the manifold $\mathcal{M}$. In particular, the Killing spinor equation ensure that $J^\mu{}_\nu$ is integrable and therefore a complex structure, which is equivalent to saying that $\mathcal{M}$ is an Hermitian manifold. The complexified tangent space is splitted in holomorphic and anti-holomorphic subspaces, according to the eigenvalue of their elements with respect to $J^\mu{}_\nu$. The values for the background fields A_μ and V_μ that allows for a solution ζ are of the form:

$$A_\mu = \frac{1}{4}\,J_\mu{}^\nu\,\partial_\nu\left(\log\sqrt{g}\right) - \frac{i}{2}\,\partial_\mu\left(\log s\right) - \frac{1}{4}\left(\delta^\nu{}_\mu - iJ_\mu{}^\nu\right)\nabla^\rho J_{\rho\nu} + \frac{3}{2}\,U_\mu\;, \tag{2.14}$$

$$V_\mu = -\frac{1}{2}\,\nabla^\rho J_{\rho\mu} + U_\mu\;, \tag{2.15}$$

where U^μ is a vector field we have the freedom to choose, provided that it is holomorphic i.e. $J^\mu{}_\nu U^\nu = iU^\mu$ and covariantly conserved i.e. $\nabla_\mu U^\mu = 0$, and s is an arbitrary nowhere vanishing complex function. In the complex frame adapted to the Hermitian metric, the solutions of the Killing spinor equation are the element of the one-dimensional vector space generated by:

$$\zeta_\alpha = \sqrt{\frac{s}{2}}\begin{pmatrix}0\\1\end{pmatrix}\,. \tag{2.16}$$

Equivalently, in case the Killing spinor is $\tilde\zeta$ rather than ζ, we can build the bilinear:

$$\tilde J_{\mu\nu} \equiv \frac{2i}{|\tilde\zeta|^2}\,\tilde\zeta^\dagger \tilde\sigma_{\mu\nu}\tilde\zeta\;, \tag{2.17}$$

where $|\zeta| = \tilde\zeta^\dagger\tilde\zeta$, and there are expressions for $\tilde\zeta$, A_μ, and V_μ analogous to (2.14), (2.15) and (2.16).

If now we admit for the presence of two supercharges of opposite R-charge, namely a solution ζ and another $\tilde\zeta$, we have two different complex structures $J^\mu{}_\nu$ and $\tilde J^\mu{}_\nu$ and it can be proved that they induce opposite orientations and that they commute i.e.:

$$J^\mu{}_\nu\tilde J^\nu{}_\rho - \tilde J^\mu{}_\nu J^\nu{}_\rho = 0\,. \tag{2.18}$$

We introduce also the vector K^μ as above:

$$K^\mu = \zeta\sigma^\mu\tilde\zeta\,. \tag{2.19}$$

Such a vector is holomorphic with respect to both complex structures and it satisfies $K^\mu K_\mu = 0$. It can be shown that the Killing spinor equations imply that K^μ is a Killing vector, that is:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \,. \tag{2.20}$$

We will also assume that the vector K commutes with its complex conjugate $\bar{K}$ [2] i.e.:

$$[K,\, \bar{K}] = \left(K^\mu\, \nabla_\mu \bar{K}^\nu - \bar{K}^\mu\, \nabla_\mu K^\nu\right) \partial_\nu = 0 \,. \tag{2.21}$$

The expressions (2.14) and (2.15) are still valid, but now the vector U^μ has to be proportional to K^μ i.e. $U^\mu = \kappa K^\mu$, where κ is a complex function. The fact that $\nabla_\mu V^\mu = 0$ constrains κ to be such that $K^\mu\, \partial_\mu \kappa = 0$. Thanks to the complex structure $J^\mu{}_\nu$, we can introduce holomorphic coordinates w, z such that $K = \partial_w$. In such coordinates the metric on $\mathcal{M}$ necessarily takes the form:

$$ds^2 = \Omega^2(z,\, \bar{z}) \left[(dw + h(z,\, \bar{z})\, dz) \left(d\bar{w} + \bar{h}(z,\, \bar{z})\, d\bar{z}\right) + c^2(z,\, \bar{z})\, dz\, d\bar{z}\right] \,, \tag{2.22}$$

where $h(z,\, \bar{z})$ is a complex function and $c(z,\, \bar{z})$ and $\Omega(z,\, \bar{z})$ are real positive functions. In particular the conformal factor is determined by the norm of K^μ:

$$\Omega^2 = 2\, \bar{K}_\mu K^\mu = 4\, |\zeta|^2\, |\tilde{\zeta}|^2 \,. \tag{2.23}$$

The complex frame where the expressions for the Killing spinors we will give in a second are valid is still the frame adapted to the Hermitian metric. In this case we can give its explicit expression:

$$\Theta^1 = \Omega\, c\, \mathrm{d}\bar{z} \,, \qquad\qquad \Theta^2 = \Omega\, (\mathrm{d}w + h\, \mathrm{d}z) \,. \tag{2.24}$$

The corresponding real frame is obtained as:

$$\begin{aligned} e^1 &= \frac{1}{2}\left(\Theta^1 + \bar{\Theta}^{\bar{1}}\right) \,, \qquad & e^2 &= \frac{1}{2i}\left(\Theta^1 - \bar{\Theta}^{\bar{1}}\right) \,, \\ e^3 &= \frac{1}{2}\left(\Theta^2 + \bar{\Theta}^{\bar{2}}\right) \,, \qquad & e^4 &= \frac{1}{2i}\left(\Theta^2 - \bar{\Theta}^{\bar{2}}\right) \,. \end{aligned} \tag{2.25}$$

In this frame the Killing spinors are elements of the vector space generated by:

$$\zeta_\alpha = \sqrt{\frac{s}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,, \qquad\qquad \tilde{\zeta}^{\dot{\alpha}} = \frac{\Omega}{\sqrt{2s}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,. \tag{2.26}$$

Lastly, note that K and $\bar{K}$ are killing vectors for the metric on $\mathcal{M}$ since it is real. However, the transformations they generate is not a symmetry of the background fields in general. If we require A_μ and V_μ to be invariant under the action of K and

[2] The case where $[K,\, \bar{K}] \neq 0$ is more complicated and furtherly restrict the geometry of $\mathcal{M}$, but it is not relevant for the following.

$\bar{K}$ (the former up to gauge transformations), we get additional constraints for the functions κ and s:

$$\bar{K}^\mu\,\partial_\mu\kappa = K^\mu\,\partial_\mu|s| = \bar{K}^\mu\,\partial_\mu|s| = 0\,. \tag{2.27}$$

We will assume these conditions so as to ensure that K and $\bar{K}$ generate symmetries of all the supergravity background fields, though one in principle is free to work with auxiliary fields that are not invariant under their action.

2.2.3 Chiral multiplet on a curved manifold

For our purposes we will need to define a free $\mathcal{N}=1$ chiral multiplet theory on a curved background. In euclidean signature the chiral multiplet contains a pair of complex scalar fields ϕ, $\tilde{\phi}$, a pair of Weyl spinors of opposite chirality ψ_α, $\tilde{\psi}^{\dot\alpha}$, and two complex auxiliary scalar fields F, $\tilde{F}$. Once again we stress that in euclidean signature each field is independent of its tilde version, some reality conditions relating them have to be specified only when passing to lorentzian signature. In flat space the free theory of a single chiral multiplet is described by the simple Kähler potential $K(\Phi,\,\tilde{\Phi}) = \tilde{\Phi}\Phi$ in terms of superfields, which upon integration in superspace gives rise to the Wess-Zumino lagrangian:

$$\mathcal{L}_{\mathbb{R}^4} = \int d^2\theta\, d^2\tilde{\theta}\,\tilde{\Phi}\Phi = \partial_\mu\tilde{\phi}\,\partial^\mu\phi - \tilde{F}F + i\tilde{\psi}\,\tilde{\sigma}^\mu\partial_\mu\psi\,, \tag{2.28}$$

where the indices are contracted with $\delta_{\mu\nu} = \mathrm{diag}\,(1,\,1,\,1,\,1)$. The lagrangian (2.28) is invariant under the supersymmetry transformations:

$$\begin{cases}\delta_s\phi = \sqrt{2}\,\zeta\psi\\ \delta_s\psi = \sqrt{2}\,F\zeta + i\sqrt{2}\,\sigma^\mu\tilde{\zeta}\,\partial_\mu\phi\\ \delta_s F = i\sqrt{2}\,\tilde{\zeta}\,\tilde{\sigma}^\mu\partial_\mu\psi\end{cases}, \qquad \begin{cases}\delta_s\tilde{\phi} = \sqrt{2}\,\tilde{\zeta}\tilde{\psi}\\ \delta_s\tilde{\psi} = \sqrt{2}\,\tilde{F}\tilde{\zeta} + i\sqrt{2}\,\tilde{\sigma}^\mu\zeta\,\partial_\mu\tilde{\phi}\\ \delta_s\tilde{F} = i\sqrt{2}\,\zeta\,\sigma^\mu\,\partial_\mu\tilde{\psi}\end{cases}. \tag{2.29}$$

Note that the spinorial supersymmetry parameters ζ and $\tilde{\zeta}$ are unrestricted by any equation up to now. This theory exhibits an $U(1)_R$ symmetry, thus we can apply the procedure described in section 2.2 to couple it to new minimal supergravity and then take the rigid limit and try to solve the Killing spinor equations. In particolar the R-charges of the fields inside the multiplet are:

$$R[\phi] = -R[\tilde{\phi}] = q_r\,, \qquad R[\psi] = -R[\tilde{\psi}] = q_r - 1\,, \qquad R[F] = -R[\tilde{F}] = q_r - 2\,, \tag{2.30}$$

where q_r is an arbitrary real number at this point.

Once we follow all the steps described so far, we end up with the following lagrangian:

$$\begin{aligned}\mathcal{L}_{\mathcal{M}} = D_\mu\tilde{\phi}\,D^\mu\phi + iV^\mu\Big(D_\mu\tilde{\phi}\,\phi - \tilde{\phi}\,D_\mu\phi\Big) + \frac{q_r}{4}\,(\mathcal{R} + 6V^\mu V_\mu)\,\tilde{\phi}\,\phi\,+\\ - \tilde{F}F + i\tilde{\psi}\,\tilde{\sigma}^\mu D_\mu\psi + \frac{1}{2}\,V_\mu\,\tilde{\psi}\,\tilde{\sigma}^\mu\psi\,,\end{aligned} \tag{2.31}$$

where the indices now are contracted with the metric $g_{\mu\nu}$ on $\mathcal{M}$ and we denoted with $\mathcal{R}$ its ricci scalar. Recall that the covariant derivative acting on a field of R-charge q is given by:

$$D_\mu = \nabla_\mu - iqA_\mu \,. \tag{2.32}$$

The supersymmetry transformations that leave the curved space theory (2.31) invariant are:

$$\begin{cases} \delta_s\phi = \sqrt{2}\,\zeta\psi \\ \delta_s\psi = \sqrt{2}\,F\zeta + i\sqrt{2}\,\sigma^\mu\tilde{\zeta}\,D_\mu\phi \\ \delta_s F = i\sqrt{2}\,D_\mu\big(\tilde{\zeta}\,\tilde{\sigma}^\mu\psi\big) \end{cases} , \qquad \begin{cases} \delta_s\tilde{\phi} = \sqrt{2}\,\tilde{\zeta}\tilde{\psi} \\ \delta_s\tilde{\psi} = \sqrt{2}\,\tilde{F}\tilde{\zeta} + i\sqrt{2}\,\tilde{\sigma}^\mu\zeta\,D_\mu\tilde{\phi} \\ \delta_s\tilde{F} = i\sqrt{2}\,D_\mu\big(\zeta\,\sigma^\mu\tilde{\psi}\big) \end{cases} , \tag{2.33}$$

where here instead ζ and $\tilde{\zeta}$ are constrained to satisfy the Killing spinor equations (2.8), thus in the local frame (2.24) they have to be complex multiples of (2.26).

2.3 Structure of the supersymmetric partition function

2.3.1 The importance of the partition function in QFT

Given a supersymmetric theory on a curved manifold with euclidean signature metric such as we described in the previous section, the most fundamental observable one can compute is the partition function, defined through the functional integral as follows. Let X be the collection of fields inside the theory and $S[X]$ its euclidean action. Then, the partition function is:

$$\mathcal{Z}_\mathcal{M} = \int \mathcal{D}X\, e^{-S[X]} \,. \tag{2.34}$$

The quantity $\mathcal{Z}_\mathcal{M}$ is so important because it generates all the other observables of the theory, namely the n-point functions, through functional derivation, therefore determining completely the quantum theory. Moreover, it is through the partition function that one is able to see the holographic principle at work and relate field theory results with gravity results. Indeed the AdS/CFT dictionary tells us that under certain conditions the logarithm of $\mathcal{Z}_\mathcal{M}$ coincides with the renormalised on-shell action of the dual supergravity theory living in the bulk whose conformal boundary is $\mathcal{M}$.

In the present work we will not address directly the problem of computing a partition function, nevertheless our results will be closely linked. In particular, we will see how one specific observable, the Casimir energy[3], fits into the context of the work in [3]. Remarkably, they found that under certain hypotheses well-suited

[3]We will speak more extensively of the Casimir energy and how it is defined in chapter 3.

for the case of our interest, the data entering the partition function is much less than that appearing in the lagrangian, with a particular focus on the dependence on the geometric properties of the underlying manifold $\mathcal{M}$. It is then useful to state precisely this result and to quickly review the reasoning behind its proof.

2.3.2 The role of $\mathcal{Q}$-cohomology in the quantum theory

One of the key passages of the aforementioned proof relies on a particular property that is present in supersymmetric theories. Limited to this section, let us denote with a simple δ the operator that implements supersymmetry. In general the operator δ squares to some bosonic operator B, and it is possible to define its cohomology. We call it $\mathcal{Q}$-cohomology, and we say that an object is $\mathcal{Q}$-closed if its supersymmetry variation vanishes and $\mathcal{Q}$-exact if it is itself the supersymmetry variation of something else. We say that two objects are in the same $\mathcal{Q}$-cohomology group if their difference is a $\mathcal{Q}$-exact term.

If a theory is supersymmetric, then $\delta S = 0$ which means that the action is $\mathcal{Q}$-closed. Now let us deform the action by an infinitesimal $\mathcal{Q}$-exact term δV. Then the deformed partition function is:

$$\begin{aligned}\mathcal{Z}'_{\mathcal{M}} = \int \mathcal{D}X\, e^{-S[X]-\delta V[X]} = \int \mathcal{D}X\, e^{-S[X]}\,(1-\delta V[X]) = \\ = \mathcal{Z}_{\mathcal{M}} - \int \mathcal{D}X\,\delta\left(V[X]\,e^{-S[X]}\right),\end{aligned} \tag{2.35}$$

where in the last step we exploited the fact that the action is $\mathcal{Q}$-closed. In general, for any transformation that keeps the path integral measure invariant, path integrals of the form of the last appearing in (2.35) vanish. To show that, consider the path integral of a generic functional $F[X]$ and replace the dummy integration fields X with their transformed version $X' = X + \delta X$. Then, if the measure is invariant we have:

$$\begin{aligned}\int \mathcal{D}X\, F[X] = \int \mathcal{D}X'\, F[X'] = \int \mathcal{D}X\, F[X+\delta X] = \int \mathcal{D}X\, \left(F[X] + \frac{\delta F[X]}{\delta X}\,\delta X\right) \\ \Longrightarrow \quad \int \mathcal{D}X\, \frac{\delta F[X]}{\delta X}\,\delta X = 0 \quad \Longrightarrow \quad \int \mathcal{D}X\,\delta F[X] = 0\,.\end{aligned}$$

Hence, we can conclude that $\mathcal{Z}'_{\mathcal{M}} = \mathcal{Z}_{\mathcal{M}}$ i.e. the partition function does not depend on the action itself but only on its $\mathcal{Q}$-cohomology group. In other words, we have just showed that $\mathcal{Q}$-exact term in the lagrangian do not contribute to the partition function of the theory.

Note that the same reasoning can be carried out for each and all of the possibly more than one supersymmetries at play. For instance, in our case where there are two supercharges one can define the $\tilde{\mathcal{Q}}$-cohomology starting from the operator $\tilde{\delta}$ and come to the conclusion that the partition function is not affected by $\tilde{\mathcal{Q}}$-exact terms in the lagrangian too.

2.3.3 Linear analysis of background geometry deformations

In a spirit somehow similar to that of topological field theories, where it does make sense to study the dependence of the QFT observables on the topological structure of the space where it lives, in our case it is particularly interesting to study how the $\mathcal{N} = 1$ SQFTs relate themselves with the geometric structure of $\mathcal{M}$. In the following we will assume that no background gauge fields are present other that the R-symmetry gauge field A_μ.

The geometric data that enters the lagrangian of the theory when two supercharges are present is constitued by the hermitian metric $g_{\mu\nu}$, the complex structure $J^\mu{}_\nu$, and the Killing vector K^μ, the last two through the background fields A_μ and V_μ. What we can do is to look at how the lagrangian changes when the geometry of $\mathcal{M}$ is a little deformed. To do that, we have to deform all the three objects:

$$\begin{aligned} g_{\mu\nu} &\longrightarrow g_{\mu\nu} + \Delta g_{\mu\nu} \\ J^\mu{}_\nu &\longrightarrow J^\mu{}_\nu + \Delta J^\mu{}_\nu \\ K_\mu &\longrightarrow K_\mu + \Delta K_\mu \end{aligned} \quad . \tag{2.36}$$

We must require them to keep satisfying the properties they had when not deformed, namely $J^\mu{}_\nu$ to be integrable, $g_{\mu\nu}$ to be compatible with the complex structure, and K^μ to be holomorphic with respect to the complex structure. These requirement will put constraints on the possible deformations, however their explicit expressions are not very relevant for our purposes. Furthermore, we should exclude from our analysis those transformations that are only changes of coordinates. It can be showed that the set of non-trivial complex structure deformations is parametrised by the moduli of the moduli space of complex structure deformations of $\mathcal{M}$. By properly going through the computations, one can work out the deformed lagrangian and it turns out that many of its terms are either $\mathcal{Q}$-exact or $\tilde{\mathcal{Q}}$-exact at linear order, therefore they do not contribute to the partition function, as we showed in the previous section.

The final outcome of this linearized analysis is that $\mathcal{Z}_{\mathcal{M}}$ depends only on the complex structure moduli corresponding to $\Delta J^w{}_{\bar{w}}$ and $\Delta J^z{}_{\bar{w}}$ and not on other geometric data such as the metric itself.

2.3.4 General parameter dependence of $\mathcal{Z}_{\mathcal{M}}$

The result above is remarkable, yet it is achieved by considering only infinitesimal geometry deformations. Such an assumption is quite strong and would restrict considerably its applications. Luckily, it has been shown that the statement holds for finite deformations too [4]. The proof relies on different tools than the ones we presented here and the linearised approach has the merit of being more intuitive from the physical point of view. We will not present the full non-linear analysis here. To summarise, we state precisely the general result:

Theorem 1. *The partition function $\mathcal{Z}_{\mathcal{M}}$ of an $\mathcal{N}=1$ SQFT with two supercharges of opposite R-charge defined on the background manifold $\mathcal{M}$ is an holomorphic function of the complex structure moduli corresponding to $\Delta J^{w}{}_{\bar{w}}$ and $\Delta J^{z}{}_{\bar{w}}$. Moreover, it does not depend on other geometric details of $\mathcal{M}$ such as its hermitian metric $g_{\mu\nu}$ or the other components of the complex structure deformation.*

Of course this theorem restricts also the parameter dependence of the n-point functions of the theory since they are obtained from $\mathcal{Z}_{\mathcal{M}}$ by taking derivatives.

CHAPTER 3

The Casimir energy of a simple $\mathcal{N} = 1$ SCFT on round $S^1 \times S^3$

3.1 Background geometry

The aim of this chapter is mainly to review the computation of the Casimir energy performed in [6], adding also some technical details that are missing in the paper. We will comment more about the Casimir energy itself and why it is interesting in section 3.2, but first of all it is necessary to define the background geometry we are going to consider and follow the procedure illustrated in chapter 2 to define a SQFT in such a curved space. We caution the reader that some of the conventions used here are slightly different from those in [6], hence intermediate results may differ.

3.1.1 A taste of holography: $S^1 \times S^3$ as conformal boundary of AdS_5

The manifold we will work with in this chapter is the direct product $S^1 \times S^3$. Aside from the fact that such a background has been and it is still being the target of a lot of high-energy physics research literature, the main motivation that pushed us to study observables on it is the fact that they can be related to gravitational observables in a 5-dimensional spacetime through the AdS/CFT correspondence. Hence, we open a brief parenthesis to show that $S^1 \times S^3$ is the conformal boundary of AdS_5 indeed.

The $d + 1$ dimensional euclidean Anti-de-Sitter space, AdS_{d+1} in short, is one of the three maximally symmetric solutions of the Einstein equations with constant curvature in $d+1$ dimensional spacetime. It corresponds to the solution with negative curvature which is obtained when there is a negative cosmological constant. The easiest way to define AdS_{d+1} consists in embedding it into $\mathbb{R}^{d+1,1}$ as an hyperboloid. Being $\{X^I\}$ with $I=0,\ldots,d+2$ the Cartesian coordinates on $\mathbb{R}^{d+1,1}$, AdS_{d+1} is the manifold defined by the constraint:

$$-(X^0)^2+(X^1)^2+\cdots+(X^{d+1})^2=-R^2\,,\qquad R\in\mathbb{R}\,,\ X^0>0\,. \tag{3.1}$$

We can reformulate the constraint 3.1 by introducing the coordinates $(\tau,\,\rho,\,\Omega^i)$, with $\tau,\rho\in(0,+\infty)$, $i=1,\ldots d$, defined by:

$$\begin{cases} X^0=R\cosh\tau\cosh\rho\\ X^i=R\,\Omega^i\sinh\rho\\ X^{d+1}=-R\sinh\tau\cosh\rho\end{cases}. \tag{3.2}$$

In this system of coordinates (3.1) translates into $\Omega^i\Omega^i=1$, which means that Ω^i define a $(d-1)$-sphere. The coordinates on this $(d-1)$-sphere together with $(\tau,\,\rho)$ are a suitable system of coordinates for AdS_{d+1}, and the metric induced by that of $\mathbb{R}^{d+1,1}$ is:

$$ds^2_{AdS}=R^2\left(\cosh^2\rho\,d\tau^2+d\rho^2+\sinh^2\rho\,d\Omega^2_{d-1}\right), \tag{3.3}$$

where $d\Omega_{d-1}$ is the surface element of the unitary $(d-1)$-sphere. These are called *global coordinates* for AdS_{d+1}. To understand what is the boundary of this space, we introduce yet another coordinate $\theta\in\left[0,\,\frac{\pi}{2}\right)$, which is related to ρ as $\tanh\rho=\sin\theta$. The metric (3.3) becomes:

$$ds^2_{AdS}=\frac{R^2}{\cos^2\theta}\left(d\tau^2+d\theta^2+\sin^2\theta\,d\Omega^2_{d-1}\right). \tag{3.4}$$

When we set θ to its extremum $\frac{\pi}{2}$ we obtain the metric of a cylinder $\mathbb{R}_+\times S^{d-1}$. We say that $\mathbb{R}_+\times S^{d-1}$ is the conformal boundary of the euclidean AdS_{d+1}. Notice that by extending the domain of τ to the complete real axis (namely considering the universal covering space of AdS_{d+1}), the conformal boundary is $\mathbb{R}\times S^{d-1}$, which is nothing but the background for the radial quantisation of a flat space CFT. By compactifying the coordinate τ, the bond with the space where we will work becomes clear: $\mathbb{R}$ gets compactified into S^1, hence we can consider the conformal boundary of AdS_{d+1} to be $S^1\times S^{d-1}$. In our specific case, $d=4$ and we have that the space $S^1\times S^3$, where we will setup our field theory, is the conformal boundary of AdS_5.

3.1.2 Manifold definition and background fields

As we already said, the manifold we will work with is the direct product $S^1\times S^3$. A suitable system of coordinates for this space is given by the set $(\tau,\,\theta,\,\varphi_1,\,\varphi_2)$ where $\tau\in[0,\,2\pi)$ is the coordinate on the circle and $\theta\in[0,\,\frac{\pi}{2}]$ and $\varphi_1,\,\varphi_2\in[0,\,2\pi)$ are the coordinates on the 3-sphere[4]. Of course we have the identifications $\tau \sim \tau + 2\pi$, $\varphi_1 \sim \varphi_1 + 2\pi$, and $\varphi_2 \sim \varphi_2 + 2\pi$. The metric reads:

$$ds^2 = \beta^2\, d\tau^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi_1^2 + r^2 \cos^2\theta\, d\varphi_2^2 \ , \tag{3.5}$$

where β and r are positive parameters determining the radius of respectively the circle and the 3-sphere. These coordinates describe S^3 as a Hopf fibration of a 2-torus (parametrised by φ_1, φ_2) over an interval parametrised by θ, and in fact they are called Hopf coordinates. Note that at first sight (3.5) may seem to be singular at $\theta = 0$ and $\theta = \frac{\pi}{2}$ since either the term in $d\varphi_1^2$ or that in $d\varphi_2^2$ disappear; however, one may check that these are just coordinate singularities and that the Riemann tensor is perfectly regular. For completeness we report also the determinant and the Ricci scalar of the metric (3.5):

$$g = \beta^2 r^6 \cos^2\theta \sin^2\theta \ , \tag{3.6}$$

$$\mathcal{R} = \frac{6}{r^2} \ . \tag{3.7}$$

The 3-sphere has $SO(4) \simeq SU(2)_L \times SU(2)_R$ symmetry and for each $SU(2)$ factor there exist an associated angular momentum operator: the left angular momentum $\vec{J}^L$ corresponds to $SU(2)_L$, and the right angular momentum $\vec{J}^R$ corresponds to $SU(2)_R$. J_3^L and J_3^R are the two Cartan operators and they generate symmetries along linear combinations of ∂_{φ_1} and ∂_{φ_2}. In particular, in the differential representation they are given by:

$$J_3^L = \frac{i}{2}\left(L_{\partial_{\varphi_1}} + L_{\partial_{\varphi_2}}\right) , \qquad J_3^R = \frac{i}{2}\left(L_{\partial_{\varphi_1}} - L_{\partial_{\varphi_2}}\right) , \tag{3.8}$$

where $L_{\partial_{\varphi_i}}$ is the Lie derivative along the vector ∂_{φ_i}. Such expressions are comprehensive of both the orbital angular momentum and the internal spin.

If the metric (3.5) can support an SQFT with two supercharges of opposite R-charge, then it must be possible to recast it in terms of holomorphic coordinates $(w,\, z)$ in such a way that it takes the form (2.22). This is indeed possible and a suitable change of coordinates is given by:

$$\begin{cases} w = \dfrac{i\beta}{r}\tau + \varphi_2 - i \log\cos\theta \\ z = e^{i(\varphi_1 - \varphi_2)} \tan\theta \end{cases} . \tag{3.9}$$

The real coordinates boundary conditions lead to the identifications $w \sim w + \frac{2\pi i \beta}{r}$ and $w \sim w + 2\pi$ while z is kept fixed. The functions of z and $\bar{z}$ appearing in (2.22) are given by:

$$\Omega = r \ , \qquad h = -\frac{i\bar{z}}{1+|z|^2} \ , \qquad c = \frac{1}{1+|z|^2} \ . \tag{3.10}$$

[4]We warn the reader that thes τ and θ have nothing to do with the ones introduced above in section 3.1.1 to parametrise the AdS space.

The vector $K = \partial_w$ is a Killing vector by construction and in real coordinates it reads:

$$K = -\frac{ir}{2\beta}\,\partial_\tau + \frac{1}{2}\,\partial_{\varphi_1} + \frac{1}{2}\,\partial_{\varphi_2}\,, \tag{3.11}$$

and lowering the index with the real metric, we get the associated 1-form:

$$\underline{K} = \frac{1}{2}\,r^2\left[-\frac{i\beta}{r}\,\mathrm{d}\tau + \sin^2\theta\,\mathrm{d}\varphi_1 + \cos^2\theta\,\mathrm{d}\varphi_2\right]. \tag{3.12}$$

Obviously it is the direction of (3.11) which is relevant, while the normalisation is arbitrary since the Killing spinors are defined up to a multiplicative complex constants. One may check that the Killing vector K^μ satisfies all the properties that we listed in 2.2.2 i.e. $K^\mu K_\mu = 0$ and $[K,\,\bar{K}] = 0$. The most general expressions for the two auxiliary background fields read:

$$V = \frac{i\beta\kappa r}{2}\,\mathrm{d}\tau + \left(1 - \frac{\kappa}{2}\,r^2\right)\left(\sin^2\theta\,\mathrm{d}\varphi_1 + \cos^2\theta\,\mathrm{d}\varphi_2\right), \tag{3.13}$$

$$\begin{aligned} A = \sin^2\theta\,\mathrm{d}\varphi_1 + \cos^2\theta\,\mathrm{d}\varphi_2 - \frac{i}{2}\,\mathrm{d}(\log s) + \\ + \frac{1}{2}\left(1 - \frac{3}{2}\,\kappa r^2\right)\left(-\frac{i\beta}{r}\,\mathrm{d}\tau + \sin^2\theta\,\mathrm{d}\varphi_1 + \cos^2\theta\,\mathrm{d}\varphi_2\right). \end{aligned} \tag{3.14}$$

A convenient choice of the function κ is the constant $\frac{2}{r^2}$, so that the components in $\mathrm{d}\varphi_1$ and $\mathrm{d}\varphi_2$ of A and V cancel out (except for those that may be contained in $\mathrm{d}(\log s)$):

$$V = \frac{i\beta}{r}\,\mathrm{d}\tau\,, \tag{3.15}$$

$$A = \frac{i\beta}{r}\,\mathrm{d}\tau - \frac{i}{2}\,\mathrm{d}(\log s)\,. \tag{3.16}$$

This choice has the advantage of making A and V regular everywhere. For now, we leave the function s unfixed.

In order to give an expression for the Killing spinors, we have to introduce a local frame. For purely technical purposes, it is more convenient to use a local frame which is rotated with respect to (2.25). Respectively in real and complex coordinates it is given by:

$$\begin{aligned} e^1 &= -r\sin\theta\cos\theta\,\sin(\varphi_1 + \varphi_2)\,(\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2) + r\cos(\varphi_1 + \varphi_2)\,\mathrm{d}\theta \\ e^2 &= -r\sin\theta\cos\theta\,\cos(\varphi_1 + \varphi_2)\,(\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2) - r\sin(\varphi_1 + \varphi_2)\,\mathrm{d}\theta \\ e^3 &= r\,\sin^2\theta\,\mathrm{d}\varphi_1 + r\,\cos^2\theta\,\mathrm{d}\varphi_2 \\ e^4 &= \beta\,\mathrm{d}\tau \end{aligned}\,, \tag{3.17}$$

$$
\begin{aligned}
\Theta^1 &= \Omega\, c\, e^{-2i\varphi_2}\, \mathrm{d}\bar{z} \\
\bar{\Theta}^{\bar{1}} &= \Omega\, c\, e^{2i\varphi_2}\, \mathrm{d}z \\
\Theta^2 &= \Omega\left(\mathrm{d}w + h\, \mathrm{d}z\right) \\
\bar{\Theta}^{\bar{2}} &= \Omega\left(\mathrm{d}\bar{w} + \bar{h}\, \mathrm{d}\bar{z}\right)
\end{aligned} \quad . \tag{3.18}
$$

In such a frame, the two Killing spinors are given by:

$$
\zeta_\alpha = \varsigma \sqrt{\frac{s}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \tilde{\zeta}^{\dot{\alpha}} = \tilde{\varsigma}\, \frac{r}{\sqrt{2s}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.19}
$$

where ς, $\tilde{\varsigma} \in \mathbb{C}$ are the complex constants that parametrise the supersymmetry transformations along the two directions selected by the Killing spinors. By imposing suitable boundary conditions on ζ and $\tilde{\zeta}$ we can fix also the function s. The two spinors should be anti-periodic when we go once around the circle parametrised by either φ_1 or φ_2; for what concerns the boundary condition under $\tau \to \tau + 2\pi$, it is consistent to take them either periodic or anti-periodic, but for now we will set them to be periodic, following [6], and we will analyse the most general case in chapter 4. In order to impose these boundary conditions, we have to work out how (3.19) transform under the rotations of τ, φ_1, and φ_2 by 2π. The operator that implements such transformations is the exponential of the Lie derivative[5]:

$$
\zeta \longrightarrow \exp\left(2\pi L_X\right)\zeta\,, \qquad \tilde{\zeta} \longrightarrow \exp\left(2\pi L_X\right)\tilde{\zeta}\,, \tag{3.20}
$$

where X is the derivative along one of τ, φ_1, φ_2. Luckily, both (3.19) are eigenfunctions of the spinorial Lie derivative:

$$
\begin{aligned}
&L_{\partial_\tau}\zeta = \left(\frac{1}{2s}\,\partial_\tau s\right)\zeta\,, \quad L_{\partial_{\varphi_1}}\zeta = \left(\frac{i}{2} + \frac{1}{2s}\,\partial_{\varphi_1} s\right)\zeta\,, \quad L_{\partial_{\varphi_2}}\zeta = \left(\frac{i}{2} + \frac{1}{2s}\,\partial_{\varphi_2} s\right)\zeta\,, \\
&L_{\partial_\tau}\tilde{\zeta} = -\left(\frac{1}{2s}\,\partial_\tau s\right)\tilde{\zeta}\,, \quad L_{\partial_{\varphi_1}}\tilde{\zeta} = -\left(\frac{i}{2} + \frac{1}{2s}\,\partial_{\varphi_1} s\right)\tilde{\zeta}\,, \quad L_{\partial_{\varphi_2}}\tilde{\zeta} = -\left(\frac{i}{2} + \frac{1}{2s}\,\partial_{\varphi_2} s\right)\tilde{\zeta}\,,
\end{aligned} \tag{3.21}
$$

hence the exponential of the spinorial Lie derivative simply returns the spinor itself multiplied by the exponential of its eigenvalue. Therefore, the simple choice of a constant s makes the Killing spinor satisfy the required boundary conditions, and the function s itself disappears from A_μ. In particular, choosing $s = r$ leads to two Killing spinors with the same normalisation.

Summarizing, our choices of the arbitrary functions are:

$$
\kappa = \frac{2}{r^2}\,, \qquad s = r \tag{3.22}
$$

[5]For the definition of the Lie derivative acting on spinors see appendix A.2.

and they lead to the following expressions for the background supergravity vector fields and the two Killing spinors:

$$A = V = \frac{i\beta}{r}\, \mathrm{d}\tau \,, \tag{3.23}$$

$$\zeta_\alpha = \varsigma \sqrt{\frac{r}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad\qquad \tilde{\zeta}^{\dot{\alpha}} = \tilde{\varsigma} \sqrt{\frac{r}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{3.24}$$

Note that the normalisation of the Killing vector used in (3.11) corresponds to $\varsigma = \tilde{\varsigma} = 1$.

3.2 Generalities on the Casimir energy in QFT

3.2.1 Definition and its ambiguity in ordinary QFT

Throughout our work we speak of Casimir energy referring to the ground state energy of a quantum field theory. More specifically, we will deal only with CFTs in the present thesis. Denoting the Casimir energy with E_0, it can be expressed as the integral of the vacuum expectation value of the $\tau\tau$-component of the stress-energy tensor:

$$E_0 = \int_{S^3} d^3x \, \sqrt{g_3}\, \langle T_{\tau\tau} \rangle \,, \tag{3.25}$$

where $g_3 = r^6 \cos^2\theta \sin^2\theta$ is the determinant of the metric (3.5) restricted to the 3-sphere. However, it is not guaranteed a priori that (3.25) is a meaningful definition in the sense that it is scheme independent. In fact, it is not in general: there exists a non-vanishing dimensionless counterterm that we can add to the action and that has the effect of shifting E_0:

$$S_{ct} \sim b \int_{S^1 \times S^3} d^4x \, \sqrt{g}\, \mathcal{R}^2 = 24\pi^3 b\, r\beta \,, \tag{3.26}$$

where b is an arbitrary dimensionless constant. Interpreting what multiplies β as a contribution to the energy, we can conclude that such a counterterm would shift E_0 by an amount proportional to b, therefore E_0 is ambiguous.

In principle, one could see this also from the vacuum expectation value of the trace of the stress-energy tensor $\langle T^\mu{}_\mu \rangle$. Recall that for a 4d CFTs in a curved space $\langle T^\mu{}_\mu \rangle$ is non-vanishing in general, and it takes the form:

$$\langle T^\mu{}_\mu \rangle \sim a E_{(4)} - c W^2 \,, \tag{3.27}$$

where $E_{(4)}$ is the Euler density, $W = W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$ is the square of the Weyl tensor[6], and a and c are the conformal anomaly coefficients i.e. two dimensionless constants

[6] See appendix A.1 for the definitions of $E_{(4)}$ and $W_{\mu\nu\rho\sigma}$.

depending on the details of the field theory at work. The counterterm (3.26) has the effect of adding a term at the RHS of (3.27):

$$\langle T^{\mu}{}_{\mu} \rangle \sim aE_{(4)} - cW^2 + b\Box\mathcal{R}\,. \tag{3.28}$$

However, for our particular background, the trace is untouched since $\mathcal{R}$ is constant and thus $\Box\mathcal{R} = 0$. therefore, this argument is not enough. In passing, note that luckily for us both $E_{(4)}$ and W^2 vanish for the background metric (3.5), hence we are guaranteed that, whatever CFT we define on the round $S^1 \times S^3$, the one-point function of the trace of the energy-momentum tensor will vanish.

3.2.2 Supersymmetry at work: removing the ambiguity

The presence of supersymmetry constrains our theory just enough to remove the ambiguity in the definition of the Casimir energy. The key idea is that (3.26) should be completed to a supersymmetric counterterm, but one can show that such a completion vanishes. However, we can show our claim in a more rigorous way.

In presence of supersymmetry the Casimir energy, we will denote it with E_{susy}, can equivalently be characterised as the vacuum expectation value of the supersymmetric Hamiltonian H_{susy}, which is the operator that generates translations along the lorentzian time direction $t = -i\tau$, i.e. $E_{susy} = \langle H_{susy}\rangle$. Note that this quantity can be linked to the partition function (2.34). In fact, the latter can be thought as a trace over the Hilbert space of states[7]:

$$Z_{S^1\times S^3} = \mathrm{Tr}\left[(-1)^F e^{-\beta H_{susy}}\right]. \tag{3.29}$$

In the limit where β is very big, this sum is clearly dominated by the state which has the smallest energy, that is nothing but the ground state whose energy is the supersymmetric Casimir energy. Hence E_{susy} can be worked out from the partition function as:

$$E_{susy} = -\lim_{\beta\to\infty}\frac{d}{d\beta}\log Z_{S^1\times S^3}\,. \tag{3.30}$$

This link between the Casimir energy and the partition function is not particularly relevant for now, but it is good to keep it in mind and also it will be useful for the discussion we will carry out in chapter 5.

In order to show our claim that the supersymmetric Casimir energy is unambiguous, we start from the superalgebra (2.11). By exploiting the linearity of the Lie derivative and plugging in the expressions (3.23) and (3.11) for A_μ and K_μ, we can rewrite the superalgebra in a more explicit form:

$$\left\{\delta_\zeta,\, \delta_{\tilde\zeta}\right\} = \varsigma\tilde\varsigma\left[\frac{r}{\beta}\, L_{\partial_\tau} + i\left(L_{\partial_{\varphi_1}} + L_{\partial_{\varphi_2}}\right) + R\right], \tag{3.31}$$

[7]The $(-1)^F$ is included since we are using periodic boundary condition of the dynamical fermionic fields under the transformation $\tau \to \tau + 2\pi$.

where R is the R-charge. In this expression we identify the third component of the left angular momentum (3.8) and the Hamiltonian $H_{susy} = iL_{\partial_t} = -L_{\partial_\tau}$ in the differential representation:

$$\{\delta_\zeta,\, \delta_{\tilde{\zeta}}\} = -\varsigma\tilde{\varsigma}\left[\frac{r}{\beta}\, H_{susy} - 2J_3^L - R\right]. \tag{3.32}$$

The strategy we adopt is to perform a dimensional reduction on S^3, whose outcome is a supersymmetric quantum mechanics with infinite degrees of freedom. The one-dimensional superalgebra is the same as (3.32), but from a purely 1d perspective the information on the angular momentum is lost, hence J_3^L and R are combined together to form the generator of a global symmetry Σ:

$$\{\delta_\zeta,\, \delta_{\tilde{\zeta}}\} = -\varsigma\tilde{\varsigma}\,\frac{r}{\beta}\,(H_{susy} - \Sigma)\ , \qquad\qquad \Sigma \equiv \frac{\beta}{r}\left(2J_3^L + R\right). \tag{3.33}$$

Moreover, both H_{susy} and Σ commute with the supercharges, as we will see when actually performing the dimensional reduction. Now, in order to show that E_{susy} is unambiguous we have to make the assumption that the vacuum of the theory is supersymmetric, namely $Q\,|0\rangle = \tilde{Q}\,|0\rangle = 0$. If this is the case, then by computing the vev of both sides of the superalgebra (3.33) we discover that $\langle H_{susy}\rangle = \langle\Sigma\rangle$. Hence, if the vacuum does not break supersymmetry, each consideration we do for $\langle\Sigma\rangle$ will automatically be valid also for $\langle H_{susy}\rangle$. Note that up to this point, we have still the freedom of shifting the two vevs of the same amount and the superalgebra (3.33) would still be satisfied. What we have to do is to show that $\langle\Sigma\rangle$ is physically well-defined.

The crucial point is that when reducing to 1d, the generating functional that computes $\langle\Sigma\rangle$ is necessarily a one dimensional Chern-Simons term built starting from the background gauge field A_τ^Σ associated to Σ:

$$\mathcal{W}[A_\tau^\Sigma] = \langle\Sigma\rangle \int d\tau\, A_\tau^\Sigma\ . \tag{3.34}$$

This is the only functional that returns $\langle\Sigma\rangle$ when taking a functional derivative with respect to A_τ^Σ. The fact that (3.34) is a Chern-Simons term has two major consequences.

- If we were considering a purely 1d model, we could have added again a counterterm in the form of a Chern-Simons term to shift $\langle\Sigma\rangle$, and $\langle H_{susy}\rangle$ consequently. Remarkably, since we are considering a 1d theory that arises from the dimensional reduction of a 4d theory, the allowed counterterms should come from 4d counterterms; but there are no 4d counterterms of mass-dimension four that look like a CS term upon integrating over the 3-sphere. It follows that $\langle\Sigma\rangle$ is a physical quantity and the supersymemtric Casimir energy is well-defined!

- When the symmetry group associated to the vector field that appears in the CS term is not simply connected, as it is in our case where the group is $U(1)$, the coupling constant of the CS term has to be quantised in order to ensure gauge invariance. It follows that the supersymmetric Casimir energy cannot depend on continuous coupling constants. In particular, this means that in order to compute E_{susy} we can consider the theory at any point of the RG flow, and if we assume the existence of a weakly coupled point, then we can use a simple free field theory.

Summarizing, we showed that the supersymmetric Casimir energy is actually a physical unambiguous quantity and that it can be computed by just considering a free field theory. Now we can get into the actual computations and find the expression for E_{susy} for a particular theory in the given background geometry. From now on we will omit the subscript *susy* and denote the supersymmetric Hamiltonian with a simple H and the Casimir energy with E.

3.3 Setting up the theory

3.3.1 Lagrangian and supersymmetry transformations

Though our results of the previous section concerning the Casimir energy are rather general and can be extended to other backgrounds, the geometry we consider here is the one detailed in section 3.1 which relies on the hypothesis of $\mathcal{N} = 1$ theories with a $U(1)_R$-symmetry. Hence the theories we are allowed to couple to this background are only the chiral multiplet and the vector multiplet (and their combinations). In the present work we will consider only the chiral multiplet, which is the simplest supersymmetric field theory.

We already introduced the chiral multiplet and its lagrangian in curved spacetime in section 2.2.3. We recall that the chiral multiplet is composed by a complex scalar ϕ whose R-charge is q_r, a left Weyl spinor ψ_α whose R-charge is $q_r - 1$, a complex auxiliary scalar F with R-charge $q_r - 2$ and their tilde versions. The lagrangian on our curved background is:

$$\mathcal{L}_{S^1\times S^3} = D_\mu\tilde{\phi}\, D^\mu\phi + iV^\mu\Big(D_\mu\tilde{\phi}\,\phi - \tilde{\phi}\, D_\mu\phi\Big) - \tilde{F}F + i\tilde{\psi}\,\tilde{\sigma}^\mu D_\mu\psi + \frac{1}{2}\, V_\mu\,\tilde{\psi}\,\tilde{\sigma}^\mu\psi\,, \quad (3.35)$$

where obviously $g_{\mu\nu}$ is given by (3.5) and A_μ and V_μ are given by (3.23). Note that compared to (2.31) this lagrangian has a missing term, but this is only because for our particular background it happens that $\mathcal{R} + 6V^\mu V_\mu = 0$, thus the term proportional to this quantity drops. We recall also the supersymmetry transformations:

$$\begin{cases}\delta_s\phi=\sqrt{2}\,\zeta\psi\\ \delta_s\psi=\sqrt{2}\,F\zeta+i\sqrt{2}\,\sigma^\mu\tilde\zeta\,D_\mu\phi\\ \delta_s F=i\sqrt{2}\,D_\mu\big(\tilde\zeta\,\tilde\sigma^\mu\psi\big)\end{cases}, \qquad \begin{cases}\delta_s\tilde\phi=\sqrt{2}\,\tilde\zeta\tilde\psi\\ \delta_s\tilde\psi=\sqrt{2}\,\tilde F\tilde\zeta+i\sqrt{2}\,\tilde\sigma^\mu\zeta\,D_\mu\tilde\phi\\ \delta_s\tilde F=i\sqrt{2}\,D_\mu\big(\zeta\,\sigma^\mu\tilde\psi\big)\end{cases}, \tag{3.36}$$

where ζ and $\tilde\zeta$ are constrained to be of the form (3.24). For the following, it is useful to rewrite the lagrangian (3.35) in a more explicit form and splitting the scalar part $\mathcal{L}_s$ and the fermionic part $\mathcal{L}_f$:

$$\mathcal{L}_s=\partial_\mu\tilde\phi\,\partial^\mu\phi+i\,(V^\mu-q_rA^\mu)\left(\partial_\mu\tilde\phi\,\phi-\tilde\phi\,\partial_\mu\phi\right)+q_r\big(q_rA^\mu A_\mu-2V^\mu A_\mu\big)\tilde\phi\,\phi-\tilde FF\,, \tag{3.37}$$

$$\mathcal{L}_f=i\tilde\psi\,\tilde\sigma^\mu\partial_\mu\psi-\frac{i}{2}\,\tilde\psi\,\tilde\sigma^\mu\,\omega_{\mu ab}\,\sigma^{ab}\psi+\frac{1}{2}\,(V_\mu+2(q_r-1)A_\mu)\,\tilde\psi\,\tilde\sigma^\mu\psi\,. \tag{3.38}$$

3.3.2 (Non-)Conserved charges

For the further computations, we will need the expressions for the conserved charges associated to the symmetries of our theory. The simplest one is the R-symmetry, whose conserved current is obtained by simply varying the action with respect to the background field A_μ [8]:

$$J^\mu_R=-\frac{1}{\sqrt{g}}\frac{\delta S}{\delta A_\mu}=iq_r\left(D^\mu\tilde\phi\,\phi-\tilde\phi\,D^\mu\phi\right)+2q_rV^\mu\,\tilde\phi\phi-(q_r-1)\,\tilde\psi\,\tilde\sigma^\mu\psi\,. \tag{3.39}$$

Such a current is covariantly conserved i.e. $\nabla_\mu J^\mu_R=0$ and integrating over S^3 we get the associated conserved charge, namely the R-charge:

$$\underline{R}=\int_{S^3}d^3x\,\sqrt{g_3}\,J^\tau_R=\int_{S^3}d^3x\,\sqrt{g_3}\Big[iq_r\left(D^\tau\tilde\phi\,\phi-\tilde\phi\,D^\tau\phi\right)+2q_rV^\tau\,\tilde\phi\phi+ \\ -(q_r-1)\,\tilde\psi\,\tilde\sigma^\tau\psi\Big], \tag{3.40}$$

where we used the bar under the letter to distinguish this operator from the one appearing in (3.32); both are the R-charge, but R is the abstract operator while $\underline{R}$ is one of its particular representations (the one acting through the (anti-)commutator), so they must not be confused one with the other. We will do the same for all the other operators.

We can obtain other two currents by directly varying the action, namely the Ferrara-Zumino current and the stress-energy tensor:

$$J^\mu_{FZ}=\frac{2}{3\sqrt{g}}\frac{\delta S}{\delta V_\mu}=\frac{2i}{3}\left(D^\mu\tilde\phi\,\phi-\tilde\phi\,D^\mu\phi\right)+2q_r\,V^\mu\tilde\phi\phi+\frac{1}{3}\,\tilde\psi\,\tilde\sigma^\mu\psi\,, \tag{3.41}$$

[8] A discussion about the correct way to define the currents in euclidean signature is carried out in appendix A.4, where it is explained why we have to use the minus sign.

$$
\begin{aligned}
T_{\mu\nu} = \text{``}\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\text{''} &= -g_{\mu\nu}\Big[D_\rho\tilde{\phi}\, D^\rho\phi - iV^\rho\Big(D_\rho\tilde{\phi}\,\phi - \tilde{\phi}\, D_\rho\phi\Big)\Big] + 3q_r V_\mu V_\nu\,\tilde{\phi}\phi + \\
&+ \frac{q_r}{2}\,\mathcal{R}_{\mu\nu}\,\tilde{\phi}\phi + \Big[D_\mu\tilde{\phi}\, D_\nu\phi + iV_\mu\Big(D_\nu\tilde{\phi}\,\phi - \tilde{\phi}\, D_\nu\phi\Big) + (\mu\leftrightarrow\nu)\Big] + \\
&+ \frac{q_r}{2}\Big[g_{\mu\nu}\,\nabla_\rho\nabla^\rho(\tilde{\phi}\phi) - \nabla_\mu\nabla_\nu(\tilde{\phi}\phi)\Big] + \\
&- \frac{1}{4}\Big[iD_\mu\tilde{\psi}\,\tilde{\sigma}_\nu\psi - i\tilde{\psi}\,\tilde{\sigma}_\mu\, D_\nu\psi - V_\mu\,\tilde{\psi}\,\tilde{\sigma}_\nu\psi + (\mu\leftrightarrow\nu)\Big]\,.
\end{aligned}
\tag{3.42}
$$

The quotation marks surrounding the definition of the stress-energy tensor are due to the fact that the spinorial part is computed by varying the action with respect to the vielbeins rather than the metric; in appendix B we described the full derivation of $T_{\mu\nu}$ including all technical details. Neither J^μ_{FZ} nor $T_{\mu\nu}$ are covariantly conserved due to the presence of non-dynamical fields. Hence, the Hamiltonian is not defined through $T_{\tau\tau}$ since it would not be a conserved charge. In presence of background non-dynamical fields, the correct approach is to consider a suitable linear combination of (3.39), (3.41), and (3.42) and to project it onto the direction of a Killing vector (see e.g. [10]). In particular, consider a Killing vector ξ that is also a symmetry of the background auxiliary fields i.e.:

$$
L_\xi g_{\mu\nu} = L_\xi A_\mu = L_\xi V_\mu = 0\,. \tag{3.43}
$$

Then, we can define the following quantity:

$$
Y^\mu_\xi \equiv \xi^\nu\left(T^\mu{}_\nu + J^\mu_R\, A_\nu - \frac{3}{2}\, J^\mu_{FZ}\, V_\nu\right). \tag{3.44}
$$

One may check that the quantity Y^μ_ξ is indeed conserved i.e. $\nabla_\mu Y^\mu_\xi = 0$, and that it is the canonical Noether current associated to the spacetime symmetry generated by ξ. Thus, if we take $\xi = -\partial_t = -i\partial_\tau$ and we integrate the τ component of Y^μ_ξ over S^3 we get the Hamiltonian[9]:

$$
\begin{aligned}
\underline{H} &= -i\int_{S^3} d^3x\,\sqrt{g_3}\,Y^\tau_{\partial_\tau} = i\int_{S^3} d^3x\,\sqrt{g_3}\left(T^\tau{}_\tau + J^\tau_R\, A_\tau - \frac{3}{2}\, J^\tau_{FZ}\, V_\tau\right) \\
&= i\int_{S^3} d^3x\,\sqrt{g_3}\Big\{D^\mu\tilde{\phi}\, D_\mu\phi + iV^\mu\left(D_\mu\tilde{\phi}\,\phi - \tilde{\phi}\, D_\mu\phi\right) - D^\tau\tilde{\phi}\,\partial_\tau\phi - \partial_\tau\tilde{\phi}\, D^\tau\phi + \\
&\quad - iV^\tau\left(\partial_\tau\tilde{\phi}\,\phi - \tilde{\phi}\,\partial_\tau\phi\right) - \frac{q_r}{2}\Big[g^{\mu\nu}\,\nabla_\mu\partial_\nu(\tilde{\phi}\phi) - \nabla^\tau\partial_\tau(\tilde{\phi}\phi)\Big] + (q_r-1)\,A_\tau\,\tilde{\psi}\,\tilde{\sigma}^\tau\psi + \\
&\quad + \frac{i}{4}\left(D^\tau\tilde{\psi}\,\tilde{\sigma}_\tau\psi + D_\tau\tilde{\psi}\,\tilde{\sigma}^\tau\psi - \tilde{\psi}\,\tilde{\sigma}^\tau D_\tau\psi - \tilde{\psi}\,\tilde{\sigma}_\tau D^\tau\psi\right) + \frac{1}{4}\,\tilde{\psi}\,(\tilde{\sigma}^\tau\, V_\tau - \tilde{\sigma}_\tau\, V^\tau)\,\psi\Big\}\,.
\end{aligned}
\tag{3.45}
$$

[9] Recall that if X is a Killing vector, the differential operator that implements this symmetry on the space of fields is $-iX$; since the Hamiltonian acts as $i\partial_t$ on the fields, the corresponding Killing vector is $-\partial_t$.

Instead, if we take $\xi = -\frac{1}{2}(\partial_{\varphi_1} + \partial_{\varphi_2})$ or $\xi = -\frac{1}{2}(\partial_{\varphi_1} - \partial_{\varphi_2})$, we get respectively the left and the right angular momenta:

$$\begin{aligned}
\underline{J}_3^L &= \int_{S^3} d^3x \sqrt{g_3}\, Y^\tau_{-\frac{1}{2}(\partial_{\varphi_1}+\partial_{\varphi_2})} = -\frac{1}{2}\int_{S^3} d^3x \sqrt{g_3}\left(T^\tau{}_{\varphi_1} + T^\tau{}_{\varphi_2}\right) \\
&= -\frac{1}{2}\sum_{i=1}^{2}\int_{S^3} d^3x \sqrt{g_3}\Big[D^\tau\tilde{\phi}\,\partial_{\varphi_i}\phi + \partial_{\varphi_i}\tilde{\phi}\,D^\tau\phi + iV^\tau\left(\partial_{\varphi_i}\tilde{\phi}\,\phi - \tilde{\phi}\,\partial_{\varphi_i}\phi\right) + \\
&\qquad - \frac{q_r}{2}\nabla^\tau\partial_{\varphi_i}(\tilde{\phi}\phi) - \frac{i}{4}\left(D^\tau\tilde{\psi}\,\tilde{\sigma}_{\varphi_i}\psi + \nabla_{\varphi_i}\tilde{\psi}\,\tilde{\sigma}^\tau\psi - \tilde{\psi}\,\tilde{\sigma}^\tau\nabla_{\varphi_i}\psi - \tilde{\psi}\,\tilde{\sigma}_{\varphi_i}D^\tau\psi\right) + \\
&\qquad + \frac{1}{4}V^\tau\,\tilde{\psi}\,\tilde{\sigma}_{\varphi_i}\psi\Big],
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
\underline{J}_3^R &= \int_{S^3} d^3x \sqrt{g_3}\, Y^\tau_{-\frac{1}{2}(\partial_{\varphi_1}-\partial_{\varphi_2})} = -\frac{1}{2}\int_{S^3} d^3x \sqrt{g_3}\left(T^\tau{}_{\varphi_1} - T^\tau{}_{\varphi_2}\right) \\
&= \frac{1}{2}\sum_{i=1}^{2}\int_{S^3} d^3x \sqrt{g_3}\,(-1)^i\Big[D^\tau\tilde{\phi}\,\partial_{\varphi_i}\phi + \partial_{\varphi_i}\tilde{\phi}\,D^\tau\phi + iV^\tau\left(\partial_{\varphi_i}\tilde{\phi}\,\phi - \tilde{\phi}\,\partial_{\varphi_i}\phi\right) + \\
&\qquad - \frac{q_r}{2}\nabla^\tau\partial_{\varphi_i}(\tilde{\phi}\phi) - \frac{i}{4}\left(D^\tau\tilde{\psi}\,\tilde{\sigma}_{\varphi_i}\psi + \nabla_{\varphi_i}\tilde{\psi}\,\tilde{\sigma}^\tau\psi - \tilde{\psi}\,\tilde{\sigma}^\tau\nabla_{\varphi_i}\psi - \tilde{\psi}\,\tilde{\sigma}_{\varphi_i}D^\tau\psi\right) + \\
&\qquad + \frac{1}{4}V^\tau\,\tilde{\psi}\,\tilde{\sigma}_{\varphi_i}\psi\Big],
\end{aligned} \tag{3.47}$$

where we used the fact that all the components of A_μ and V_μ but the τ component vanish. Similarly to what we did above, here the bars under the names denote the fact that the operators (3.45), (3.46), and (3.47) are in the representation acting through the (anti-)commutators, while H, J_3^L and J_3^R are the same operators in the differential representation. Although we will not need the expression for the right angular momentum, it will be relevant for the discussion in chapter 4.

3.4 Dimensional reduction

3.4.1 Expansion in spherical/spinorial harmonics and 1d dofs

As already mentioned before, our strategy to compute the Casimir energy arising from the theory (3.35) is to perform a dimensional reduction over the 3-sphere. In order to do so, we expand all the fields in a suitable basis for the space of functions on S^3 with coefficients depending only on the coordinate on S^1; such a basis is given by the scalar harmonics $\{Y_l{}^{mn}\}$ and the spinorial harmonics $\{S^\lambda_{lmn}\}$, whose properties we summarised in appendix C:

$$\phi(\tau,\,\theta,\,\varphi_1,\,\varphi_2) = \sum_{l=0}^{+\infty}\sum_{m,n=-\frac{l}{2}}^{\frac{l}{2}} \phi_{lmn}(\tau)\,Y_l^{\,mn}(\theta,\,\varphi_1,\,\varphi_2)\,, \tag{3.48}$$

$$\tilde\phi(\tau,\,\theta,\,\varphi_1,\,\varphi_2) = \sum_{l=0}^{+\infty}\sum_{m,n=-\frac{l}{2}}^{\frac{l}{2}} \tilde\phi_{lmn}(\tau)\,Y_l^{\,mn}(\theta,\,\varphi_1,\,\varphi_2)^*\,, \tag{3.49}$$

$$\psi_\alpha(\tau,\theta,\,\varphi_1,\,\varphi_2) = \sum_{l,m,n}\sum_{\lambda=+,-} \psi^\lambda_{lmn}(\tau)\,S^\lambda_{lmn}(\theta,\,\varphi_1,\,\varphi_2)_\alpha\,, \tag{3.50}$$

$$\tilde\psi_{\dot\alpha}(\tau,\theta,\,\varphi_1,\,\varphi_2) = \sum_{l,m,n}\sum_{\lambda=+,-} \tilde\psi^\lambda_{lmn}(\tau)\,S^\lambda_{lmn}(\theta,\,\varphi_1,\,\varphi_2)^\dagger_{\dot\alpha}\,, \tag{3.51}$$

$$F(\tau,\,\theta,\,\varphi_1,\,\varphi_2) = \sum_{l=0}^{+\infty}\sum_{m,n=-\frac{l}{2}}^{\frac{l}{2}} f_{lmn}(\tau)\,Y_l^{\,mn}(\theta,\,\varphi_1,\,\varphi_2)\,, \tag{3.52}$$

$$\tilde F(\tau,\,\theta,\,\varphi_1,\,\varphi_2) = \sum_{l=0}^{+\infty}\sum_{m,n=-\frac{l}{2}}^{\frac{l}{2}} \tilde f_{lmn}(\tau)\,Y_l^{\,mn}(\theta,\,\varphi_1,\,\varphi_2)^*\,. \tag{3.53}$$

In the two spinorial decompositions (3.50) and (3.51) the sum over n ranges always from $-\frac{l}{2}$ to $\frac{l}{2}$, while $l \geq 1$ and $-\frac{l}{2} \leq m \leq \frac{l}{2} - 1$ when $\lambda = +$ and $l \geq 0$ and $-\frac{l}{2} - 1 \leq m \leq \frac{l}{2}$ when $\lambda = -$. From now on we will not write the sum extrema anymore, they will be understood. The set $\big(\phi_{lmn},\,\psi^+_{lmn},\,\psi^-_{lmn},\,f_{lmn}\big) \bigoplus \big(tilde\big)$ constitutes the degrees of freedom of the 1d theory, ϕ_{lmn} and f_{lmn} and their tilde versions being complex scalars while ψ^+_{lmn} and ψ^-_{lmn} and their tilde versions being complex Grassmann numbers. However we will see that in order to better understand the physics of the 1d theory it is convenient to rotate the basis for 1d fermions in the following way:

$$\begin{pmatrix}\psi_{lmn}\\ \lambda_{lmn}\end{pmatrix} \equiv V\begin{pmatrix}\psi^+_{lmn}\\ \psi^-_{lmn}\end{pmatrix}\,, \qquad\qquad \text{with}\quad V \equiv \begin{pmatrix}\cos\nu^+_{lm} & -\sin\nu^+_{lm}\\ \sin\nu^+_{lm} & \cos\nu^+_{lm}\end{pmatrix}, \tag{3.54}$$

where the sines and the cosines are given in terms of the quantum numbers l, m, and n (the full expressions are reported in appendix C). The same redefinition can be used for the tilde fields, leading to $\tilde\psi_{lmn}$ and $\tilde\lambda_{lmn}$. Note that thanks to the fact that $\sin\nu^-_{lm} = \cos\nu^+_{lm}$ and $\cos\nu^-_{lm} = -\sin\nu^+_{lm}$, the following equalities hold:

$$\sum_{\lambda=+,-} \psi^\lambda_{lmn}\,S^\lambda_{lmn} = \begin{pmatrix}\psi_{lmn}\,Y_l^{\,mn}\\ \lambda_{lmn}\,Y_l^{\,m+1,n}\end{pmatrix}\,, \tag{3.55}$$

$$\sum_{\lambda=+,-} \tilde\psi^\lambda_{lmn}\,S^\lambda_{lmn} = \begin{pmatrix}\tilde\psi_{lmn}\,(Y_l^{\,mn})^*\\ \tilde\lambda_{lmn}\,(Y_l^{\,m+1,n})^*\end{pmatrix}\,. \tag{3.56}$$

Before going on with the actual dimensional reduction, let us figure out what is the charge σ under the operator Σ (3.33) of the 1d dofs. The R-charge is obviously inherited from the four dimensional fields, hence it is respectively q_r for ϕ_{lmn}, $q_r - 1$ for ψ_{lmn} and λ_{lmn}, and $q_r - 2$ for f_{lmn} (the tilde fields simply take a minus sign). In order to compute the charge under the left angular momentum, we simply act with the operator J_3^L on the 4d fields and exploit the properties of the scalar and spinorial harmonics:

$$J_3^L\,\phi = \sum_{l,m,n} \frac{i}{2}\,(\partial_{\varphi_1} + \partial_{\varphi_2})\,(\phi_{lmn} Y_l^{\;mn}) = \sum_{l,m,n} m\,\phi_{lmn} Y_l^{\;mn}\,, \tag{3.57}$$

$$J_3^L\,\psi = \sum_{l,m,n,\lambda} \frac{i}{2}\big(L_{\partial_{\varphi_1}} + L_{\partial_{\varphi_2}}\big)\,(\psi^\lambda_{lmn}\, S^\lambda_{lmn}) = \sum_{l,m,n} \left(m + \frac{1}{2}\right) \begin{pmatrix} \psi_{lmn}\, Y_l^{\;mn} \\ \lambda_{lmn}\, Y_l^{\;m+1,n} \end{pmatrix}, \tag{3.58}$$

$$J_3^L\,F = \sum_{l,m,n} \frac{i}{2}\,(\partial_{\varphi_1} + \partial_{\varphi_2})\,(f_{lmn} Y_l^{\;mn}) = \sum_{l,m,n} m\,\phi_{lmn} Y_l^{\;mn}\,. \tag{3.59}$$

From these expressions it is quite clear that the eigenvalues of the left angular momentum are m for ϕ_{lmn} and f_{lmn} and $m + \frac{1}{2}$ for ψ_{lmn} and λ_{lmn}. Analogously one can work out the eigenvalues for the tilde fields, exploiting the fact that $(Y_l^{\;mn})^* = (-1)^{m+n} Y_l^{\;-m,-n}$. It turns out that they are $-m$ for $\tilde{\phi}_{lmn}$ and $\tilde{f}_{lmn}$ and $-m - \frac{1}{2}$ for $\tilde{\psi}_{lmn}$ and $\tilde{\lambda}_{lmn}$. It follows that the charge σ of the 1d fields is:

$$\sigma(\phi_{lmn}) = \sigma(\psi_{lmn}) = \sigma(\lambda_{lmn}) = -\sigma(\tilde{\phi}_{lmn}) = -\sigma(\tilde{\psi}_{lmn}) = -\sigma(\tilde{\lambda}_{lmn}) = \frac{\beta}{r}\,(q_r + 2m)\,, \tag{3.60}$$

$$\sigma(f_{lmn}) = -\sigma(\tilde{f}_{lmn}) = \frac{\beta}{r}\,(q_r + 2m - 2)\,. \tag{3.61}$$

At this point it may seem a little strange that f_{lmn} has a charge σ different from all the other fields, especially given that the operator Σ commutes with the supercharges; however this is actually correct since the supersymmetry transformations involve $f_{l,m+1,n}$, as we will see in a while, which has precisely the same charge σ as the other one dimensional fields.

Finally, let us compute also the charge σ of the Killing spinors. By construction, ζ and $\tilde{\zeta}$ have R-charge respectively 1 and -1. For what concerns the left angular momentum, it is computed again through the spinorial Lie derivative:

$$J_3^L\,\zeta = \frac{i}{2}\big(L_{\partial_{\varphi_1}} + L_{\partial_{\varphi_2}}\big)\zeta = -\frac{1}{2}\,\zeta\,. \tag{3.62}$$

Then, it follows that the charge σ is vanishing for ζ. Analogously one can show that $J_3^L\,\tilde{\zeta} = \frac{1}{2}\,\tilde{\zeta}$ and hence also $\tilde{\zeta}$ has vanishing charge σ.

3.4.2 Dimensional reduction: lagrangian

There is not much to tell about the core step of the dimensional reduction since it is mainly a matter of cumbersome computations. The bottom line consists in exploiting the orthogonality of the spherical harmonics (C.10) and (C.20) to perform the integral over S^3 appearing in the action. We will simply sketch how the integration works.

Let us begin from the scalar part of the lagrangian (3.37). Rearranging the different terms in the lagrangian, we have:

$$S_s = \int_{S^1} d\tau\, \beta \int_{S^3} d^3x\, \sqrt{g_3} \left\{ \frac{1}{\beta^2}\, \partial_\tau \tilde{\phi}\, \partial_\tau \phi + g^{ij}\, \partial_i \tilde{\phi}\, \partial_j \phi + i\big(V^\tau - q_r A^\tau\big) \Big(\partial_\tau \tilde{\phi}\, \phi - \tilde{\phi}\, \partial_\tau \phi \Big) + \right.$$
$$\left. + i\big(V^i - q_r A^i\big) \Big(\partial_i \tilde{\phi}\, \phi - \tilde{\phi}\, \partial_i \phi \Big) + q_r \big(q_r A^\mu A_\mu - 2 V^\mu A_\mu \big) \tilde{\phi}\, \phi - \tilde{F} F \right\}.$$

The first integral is nothing but the one dimensional measure, hence all what remains is identified with the 1d lagrangian. Doing a couple of integration by parts (keep in mind that the metric depends only on θ) and exploiting the fact that the τ component is the only non-vanishing of A_μ and V_μ, we arrive to[10]:

$$\mathcal{L}_s^{(1d)} = \int_{S^3} d^3x\, \sqrt{g_3} \left\{ \frac{1}{\beta^2}\, \partial_\tau \tilde{\phi}\, \partial_\tau \phi - \tilde{\phi}\, \nabla^2 \phi + 2i\, (V^\tau - q_r A^\tau)\, \partial_\tau \tilde{\phi}\, \phi + \right.$$
$$\left. + q_r \big(q_r A^\tau A_\tau - 2 V^\tau A_\tau \big) \tilde{\phi}\, \phi - \tilde{F} F \right\}. \tag{3.63}$$

Now it is quite straightforward to apply the properties of scalar harmonics (see appendix C.1) and perform the integral. The result can be recast in the following form:

$$\mathcal{L}_s^{(1d)} = \sum_{l,m,n} \left[\frac{1}{\beta^2}\, D_\tau \tilde{\phi}_{lmn}\, D_\tau \phi_{lmn} + \frac{\mu}{\beta^2} \Big(D_\tau \tilde{\phi}_{lmn}\, \phi_{lmn} - \tilde{\phi}_{lmn}\, D_\tau \phi_{lmn} \Big) + \right.$$
$$\left. + \frac{p^2}{\beta^2}\, \tilde{\phi}_{lmn}\, \phi_{lmn} - \tilde{f}_{lmn}\, f_{lmn} \right], \tag{3.64}$$

where the 1d covariant derivative is given by $D_\tau = \partial_\tau + \sigma$ and the two parameters μ and p are:

$$\mu = -\frac{\beta}{r}\, (2m+1)\,, \tag{3.65}$$

$$p = \frac{\beta}{r}\, \sqrt{(l-2m)\,(l+2+2m)}\,. \tag{3.66}$$

The covariant derivative D_τ can be interpreted as providing a minimal coupling between the fields it acts on and with a 1d background gauge field associated to the operator Σ.

[10] Recall that in curved space the Laplacian is defined as $\nabla^2 = \frac{1}{\sqrt{g_3}}\, \partial_i \left(\sqrt{g_3}\, g^{ij} \partial_j \right)$.

Now, let us consider the fermionic sector (3.38), where the situation is even simpler. One can figure out that the matrices involved are:

$$\tilde{\sigma}^{\tau} = -\frac{i}{\beta}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \omega_{\mu ab}\,\tilde{\sigma}^{\mu}\,\sigma^{ab} = -\frac{3i}{r}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{3.67}$$

From the first and property (C.21) it follows that:

$$\left(i\,\tilde{\sigma}^{\mu}\partial_{\mu}\right)\psi = \frac{1}{\beta}\,\partial_{\tau}\psi^{\lambda}_{lmn}\,S^{\lambda}_{lmn} + \alpha_{\lambda}\,\psi^{\lambda}_{lmn}\,S^{\lambda}_{lmn}\,, \tag{3.68}$$

where $\alpha_{+} = \frac{1}{r}(l+2)$ and $\alpha_{-} = -\frac{l}{r}$. So using the orthonormality it is easy to integrate over S^3, since all the matrices that appear are proportional to the identity. The result is:

$$\begin{aligned} S_f &= \int_{S^1} d\tau\,\beta \sum_{l,m,n,\lambda}\left[\frac{1}{\beta}\,\tilde{\psi}^{\lambda}_{lmn}\,D_{\tau}\psi^{\lambda}_{lmn} + \left(\alpha_{\lambda} + \frac{\mu}{\beta} - \frac{1}{r}\right)\tilde{\psi}^{\lambda}_{lmn}\psi^{\lambda}_{lmn}\right] \\ &= \int_{S^1} d\tau\,\beta \sum_{l,m,n}\left[\frac{1}{\beta}\begin{pmatrix}\tilde{\psi}^{+}_{lmn} & \tilde{\psi}^{-}_{lmn}\end{pmatrix}\mathbb{I}_2\,D_{\tau}\begin{pmatrix}\psi^{+}_{lmn}\\ \psi^{-}_{lmn}\end{pmatrix} + \begin{pmatrix}\tilde{\psi}^{+}_{lmn} & \tilde{\psi}^{-}_{lmn}\end{pmatrix} M \begin{pmatrix}\psi^{+}_{lmn}\\ \psi^{-}_{lmn}\end{pmatrix}\right], \end{aligned} \tag{3.69}$$

where we introduced the mass matrix M:

$$M = \frac{1}{r}\begin{pmatrix} l-2m & 0\\ 0 & -l-2m-2\end{pmatrix}. \tag{3.70}$$

Changing basis to (3.54) is straightforward for the kinetic term, since V is an orthogonal matrix i.e. $VV^T = V^TV = \mathbb{I}_2$. Instead the mass term in the new basis is given by:

$$VMV^T = \frac{1}{\beta}\begin{pmatrix}2\mu & -p\\ -p & 0\end{pmatrix}, \tag{3.71}$$

where μ and p are the same introduced above (3.65), (3.66). Thus, after the change of basis, the fermionic lagrangian becomes:

$$\begin{aligned} \mathcal{L}^{(1d)}_f = \sum_{l,m,n}\frac{1}{\beta}\Big[&\tilde{\psi}_{lmn}\,D_{\tau}\psi_{lmn} + \tilde{\lambda}_{lmn}\,D_{\tau}\lambda_{lmn} + 2\mu\,\tilde{\psi}_{lmn}\psi_{lmn} + \\ &- p\left(\tilde{\psi}_{lmn}\lambda_{lmn} + \tilde{\lambda}_{lmn}\psi_{lmn}\right)\Big]. \end{aligned} \tag{3.72}$$

3.4.3 Dimensional reduction: supersymmetry transformations

The next step we have to do is to find how the four dimensional supersymmetry transformations (3.36) translate in 1d. The modus operandi is pretty much the same: we expand every field in scalar/spinorial harmonics and exploit their properties.

However, the spinorial part is a little bit more tricky than that of the lagrangian, since some of the matrices that appear here are not diagonal. Thus, we will need to unpack the single scalar harmonics inside a spinorial harmonic. We also need to use the explicit expressions of the Killing spinors (3.24), which we recall here:

$$\zeta_\alpha = \varsigma\sqrt{\frac{r}{2}}\begin{pmatrix}0\\1\end{pmatrix} \qquad \Longrightarrow \qquad \zeta^\alpha = \epsilon^{\alpha\beta}\,\zeta_\alpha = \varsigma\sqrt{\frac{r}{2}}\begin{pmatrix}1\\0\end{pmatrix}, \tag{3.73}$$

$$\tilde{\zeta}^{\dot\alpha} = \tilde{\varsigma}\sqrt{\frac{r}{2}}\begin{pmatrix}1\\0\end{pmatrix} \qquad \Longrightarrow \qquad \tilde{\zeta}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\,\tilde{\zeta}^{\dot\alpha} = \tilde{\varsigma}\sqrt{\frac{r}{2}}\begin{pmatrix}0\\1\end{pmatrix}. \tag{3.74}$$

Substituting (3.73) and (3.74) into (3.36) and expanding in scalar/spinor harmonics, eventually one obtains:

$$\sum_{l,m,n}\delta_s\phi_{lmn}\,Y_l^{\;mn} = \varsigma\sqrt{r}\sum_{l,m,n}\psi_{lmn}\,Y_l^{\;mn}, \tag{3.75}$$

$$\sum_{l,m,n}\begin{pmatrix}\delta_s\psi_{lmn}\,Y_l^{\;mn}\\ \delta_s\lambda_{lmn}\,Y_l^{\;m+1,n}\end{pmatrix} = \sqrt{r}\sum_{l,m,n}\left\{\varsigma\,f_{lmn}\begin{pmatrix}0\\ Y_l^{\;mn}\end{pmatrix} + \tilde{\varsigma}\,\frac{1}{\beta}\begin{pmatrix}D_\tau\phi_{lmn}\,Y_l^{\;mn}\\ p\,\phi_{lmn}\,Y_l^{\;m+1,n}\end{pmatrix}\right\}, \tag{3.76}$$

$$\sum_{l,m,n}\delta_s f_{lmn}\,Y_l^{\;mn} = \tilde{\varsigma}\sqrt{r}\sum_{l,m,n,\lambda}\left\{\sin\nu_{lm}^{\lambda}\left(\frac{1}{\beta}\,D_\tau + \frac{\mu}{\beta} + \alpha_\lambda - \frac{1}{r}\right)\psi_{lmn}^{\lambda}\,Y_l^{\;m+1,n}\right\}, \tag{3.77}$$

$$\sum_{l,m,n}\delta_s\tilde{\phi}_{lmn}\,(Y_l^{\;mn})^* = -\tilde{\varsigma}\sqrt{r}\sum_{l,m,n}\tilde{\psi}_{lmn}\,(Y_l^{\;mn})^*, \tag{3.78}$$

$$\sum_{l,m,n}\begin{pmatrix}\delta_s\tilde{\lambda}_{lmn}\,(Y_l^{\;m+1,n})^*\\ -\delta_s\tilde{\psi}_{lmn}\,(Y_l^{\;mn})^*\end{pmatrix} = \sqrt{r}\sum_{l,m,n}\left\{\tilde{\varsigma}\,\tilde{f}_{lmn}\begin{pmatrix}(Y_l^{\;mn})^*\\ 0\end{pmatrix} + \varsigma\,\frac{1}{\beta}\begin{pmatrix}p\,\tilde{\phi}_{lmn}\,(Y_l^{\;m+1,n})^*\\ D_\tau\tilde{\phi}_{lmn}\,(Y_l^{\;mn})^*\end{pmatrix}\right\}, \tag{3.79}$$

$$\sum_{l,m,n}\delta_s\tilde{f}_{lmn}\,(Y_l^{\;mn})^* = \varsigma\sqrt{r}\sum_{l,m,n,\lambda}\left\{\sin\nu_{lm}^{\lambda}\left(\frac{1}{\beta}\,D_\tau - \frac{\mu}{\beta} - \alpha_\lambda + \frac{1}{r}\right)\tilde{\psi}_{lmn}^{\lambda}\,(Y_l^{\;m+1,n})^*\right\}. \tag{3.80}$$

At this point it is rather easy to read the 1d supersymmetry transformations, except maybe for (3.77) and (3.80), where introducing the rotated basis is not immediate.

By plugging in the expressions for $\sin\nu_{lm}^{\lambda}$, μ, and α_λ, we get:

$$\begin{aligned}\sin\nu_{lm}^{+}\left(\frac{\mu}{\beta}+\alpha_{+}-\frac{1}{r}\right) &= -\frac{1}{r}\sqrt{\frac{l+2m+2}{2\,(l+1)}}\,(l-2m) = \\ &= -\frac{p}{\beta}\sqrt{\frac{l-2m}{2\,(l+1)}} = -\frac{p}{\beta}\cos\nu_{lm}^{+}\,,\end{aligned} \tag{3.81}$$

$$\begin{aligned}\sin\nu_{lm}^{-}\left(\frac{\mu}{\beta}+\alpha_{-}-\frac{1}{r}\right) &= -\frac{1}{r}\sqrt{\frac{l-2m}{2\,(l+1)}}\,(l+2m+2) = \\ &= -\frac{p}{\beta}\sqrt{\frac{l+2m+2}{2\,(l+1)}} = -\frac{p}{\beta}\cos\nu_{lm}^{-}\,.\end{aligned} \tag{3.82}$$

Substituting these into (3.77) and (3.80) it is easy to identify also the 1d supersymmetry transformations of f_{lmn} and $\tilde{f}_{lmn}$. Removing all the sums by matching correctly the single addends we get:

$$\begin{cases}\delta_s\phi_{lmn} = \sqrt{r}\,\varsigma\,\psi_{lmn} \\ \delta_s\psi_{lmn} = \frac{\sqrt{r}}{\beta}\,\tilde{\varsigma}\,D_\tau\phi_{lmn} \\ \delta_s f_{l,m+1,n} = \frac{\sqrt{r}}{\beta}\,\tilde{\varsigma}\,D_\tau\lambda_{lmn} - \frac{\sqrt{r}}{\beta}\,p\,\tilde{\varsigma}\psi_{lmn} \\ \delta_s\lambda_{lmn} = \sqrt{r}\,\varsigma\,f_{l,m+1,n} + \frac{\sqrt{r}}{\beta}\,p\,\tilde{\varsigma}\phi_{lmn}\end{cases}, \tag{3.83}$$

$$\begin{cases}\delta_s\tilde{\phi}_{lmn} = -\sqrt{r}\,\tilde{\varsigma}\,\tilde{\psi}_{lmn} \\ \delta_s\tilde{\psi}_{lmn} = -\frac{\sqrt{r}}{\beta}\,\varsigma\,D_\tau\tilde{\phi}_{lmn} \\ \delta_s\tilde{f}_{l,m+1,n} = \frac{\sqrt{r}}{\beta}\,\varsigma\,D_\tau\tilde{\lambda}_{lmn} + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\tilde{\psi}_{lmn} \\ \delta_s\tilde{\lambda}_{lmn} = \sqrt{r}\,\tilde{\varsigma}\,\tilde{f}_{l,m+1,n} + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\tilde{\phi}_{lmn}\end{cases}. \tag{3.84}$$

Now that we found these transformations we can start to understand what is the physics of the 1d theory. But discussing this, let us complete the dimensional reduction by finding also the expressions for the conserved charges. We will return to the physics of the 1d theory in section 3.4.5.

3.4.4 Dimensional reduction: charges

Again, the dimensional reduction of the conserved charges is rather technical and the procedure is basically the same used in the previous two sections. One takes the expressions (3.40), (3.45), (3.46), and (3.47), expands in the suitable spherical harmonics and finally integrates over S^3 exploiting the orthonormality. Here we just quote the results of the integrals:

$$\begin{aligned}\underline{R} = \sum_{l,m,n} \Bigg[\frac{iq_r}{\beta^2} \Big(D_\tau \tilde{\phi}_{lmn}\, \phi_{lmn} - \tilde{\phi}_{lmn}\, D_\tau \phi_{lmn} - 2\mu\, \tilde{\phi}_{lmn} \phi_{lmn} \Big) + \\ + \frac{i}{\beta} (q_r - 1) \Big(\tilde{\psi}_{lmn} \psi_{lmn} + \tilde{\lambda}_{lmn} \lambda_{lmn} \Big) \Bigg],\end{aligned} \tag{3.85}$$

$$\begin{aligned}\underline{H} = \sum_{l,m,n} \Bigg[& - \frac{i}{\beta^2}\, \partial_\tau \tilde{\phi}_{lmn}\, \partial_\tau \phi_{lmn} + \frac{i}{\beta^2} \left(p^2 - \sigma^2 - 2\mu\sigma \right) \tilde{\phi}_{lmn}\, \phi_{lmn} + \\ & - \frac{i}{2\beta} \Big(\tilde{\psi}_{lmn}\, \partial_\tau \psi_{lmn} - \partial_\tau \tilde{\psi}_{lmn}\, \psi_{lmn} + \tilde{\lambda}_{lmn}\, \partial_\tau \lambda_{lmn} - \partial_\tau \tilde{\lambda}_{lmn}\, \lambda_{lmn} \Big) + \\ & - \frac{r\sigma_1}{4\beta^2} \Big(\tilde{\psi}_{lmn}\, D_\tau \psi_{lmn} - D_\tau \tilde{\psi}_{lmn}\, \psi_{lmn} - \tilde{\lambda}_{lmn}\, D_\tau \lambda_{lmn} + D_\tau \tilde{\lambda}_{lmn}\, \lambda_{lmn} + \\ & + 4\mu\, \tilde{\psi}_{lmn}\, \psi_{lmn} \Big) \Bigg],\end{aligned} \tag{3.86}$$

$$\begin{aligned}\underline{J}^L_3 = \sum_{l,m,n} \Bigg[& \frac{im}{\beta^2} \Big(D_\tau \tilde{\phi}_{lmn}\, \phi_{lmn} - \tilde{\phi}_{lmn}\, D_\tau \phi_{lmn} - 2\mu\, \tilde{\phi}_{lmn} \phi_{lmn} \Big) + \\ & + \frac{ir}{8\beta^2} \Big(\tilde{\psi}_{lmn}\, D_\tau \psi_{lmn} - D_\tau \tilde{\psi}_{lmn}\, \psi_{lmn} - \tilde{\lambda}_{lmn}\, D_\tau \lambda_{lmn} + D_\tau \tilde{\lambda}_{lmn}\, \lambda_{lmn} + \\ & - 4\mu\, \tilde{\lambda}_{lmn}\, \lambda_{lmn} \Big) \Bigg],\end{aligned} \tag{3.87}$$

$$\underline{J}^R_3 = \sum_{l,m,n} \Bigg[\frac{in}{\beta^2} \Big(D_\tau \tilde{\phi}_{lmn}\, \phi_{lmn} - \tilde{\phi}_{lmn}\, D_\tau \phi_{lmn} - 2\mu\, \tilde{\phi}_{lmn} \phi_{lmn} \Big) + (\underline{J}^R_3)_f \Bigg]. \tag{3.88}$$

Note that we did not report the fermionic part of the right angular momentum $(\underline{J}^R_3)_f$. This is due to the fact that we had some issues computing this integral since there appear terms with an angular dependence aside from the spinor harmonics. We will argue more on this topic in chapter 4, since for now we do not need this expression.

The expressions (3.86) and (3.87) can be conveniently rewritten by using the fermionic equations of motion i.e.:

$$\partial_\tau \psi_{lmn} = -\,(\sigma + 2\mu)\, \psi_{lmn} + p\, \lambda_{lmn}\,, \tag{3.89}$$

$$\partial_\tau \tilde{\psi}_{lmn} = (\sigma + 2\mu)\, \tilde{\psi}_{lmn} - p\, \tilde{\lambda}_{lmn}\,, \tag{3.90}$$

$$\partial_\tau \lambda_{lmn} = -\sigma \lambda_{lmn} + p\, \psi_{lmn}\,, \tag{3.91}$$

$$\partial_\tau \tilde{\lambda}_{lmn} = \sigma \tilde{\lambda}_{lmn} - p\, \tilde{\psi}_{lmn}\,. \tag{3.92}$$

The resulting expressions are:

$$\underline{H}=\sum_{l,m,n}\Bigg[-\frac{i}{\beta^2}\,\partial_\tau\tilde{\phi}_{lmn}\,\partial_\tau\phi_{lmn}+\frac{i}{\beta^2}\left(p^2-\sigma^2-2\mu\sigma\right)\tilde{\phi}_{lmn}\,\phi_{lmn}+\\ \frac{i\sigma}{\beta}\left(\tilde{\psi}_{lmn}\,\psi_{lmn}+\tilde{\lambda}_{lmn}\,\lambda_{lmn}\right)+\frac{2i\mu}{\beta}\,\tilde{\psi}_{lmn}\,\psi_{lmn}-\frac{ip}{\beta}\left(\tilde{\psi}_{lmn}\,\lambda_{lmn}+\tilde{\lambda}_{lmn}\,\psi_{lmn}\right)\Bigg], \tag{3.93}$$

$$\underline{J^L_3}=\sum_{l,m,n}\Bigg[\frac{im}{\beta^2}\left(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn}-\tilde{\phi}_{lmn}\,D_\tau\phi_{lmn}-2\mu\,\tilde{\phi}_{lmn}\phi_{lmn}\right)+\\ +\frac{i}{\beta}\left(m+\frac{1}{2}\right)\left(\tilde{\psi}_{lmn}\psi_{lmn}+\tilde{\lambda}_{lmn}\lambda_{lmn}\right)\Bigg]. \tag{3.94}$$

Remarkably, in this form $\underline{J^L_3}$ resembles very much (3.85). Indeed, combining them to get $\underline{\Sigma}$ is now extremely easy:

$$\underline{\Sigma}=\sum_{l,m,n}\Bigg[\frac{i\sigma}{\beta^2}\left(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn}-\tilde{\phi}_{lmn}\,D_\tau\phi_{lmn}-2\mu\,\tilde{\phi}_{lmn}\phi_{lmn}\right)+\\ +\frac{i\sigma}{\beta}\left(\tilde{\psi}_{lmn}\psi_{lmn}+\tilde{\lambda}_{lmn}\lambda_{lmn}\right)\Bigg]. \tag{3.95}$$

3.4.5 Summary of the 1d theory

To conclude this section, we summarise what we have obtained through the dimensional reduction. Recall the definitions of the 1d covariant derivative and the parameters μ and p:

$$D_\tau=\partial_\tau+\sigma\,, \tag{3.96}$$

$$\mu=-\frac{\beta}{r}\,(2m+1)\,, \tag{3.97}$$

$$p=\frac{\beta}{r}\,\sqrt{(l-2m)\,(l+2+2m)}\,, \tag{3.98}$$

where σ is the charge of the field under the operator Σ and it reads $\frac{\beta}{r}\,(q_r+2m)$ for non-tilde fields and its opposite for tilde fields. The 1d lagrangian is given by the infinite sum $\mathcal{L}^{(1d)}=\sum_{l,m,n}\mathcal{L}_{lmn}$, where:

$$\mathcal{L}_{lmn}=\frac{1}{\beta^2}\,D_\tau\tilde{\phi}_{lmn}\,D_\tau\phi_{lmn}+\frac{\mu}{\beta^2}\left(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn}-\tilde{\phi}_{lmn}\,D_\tau\phi_{lmn}\right)+\frac{p^2}{\beta^2}\,\tilde{\phi}_{lmn}\,\phi_{lmn}+\\ +\frac{1}{\beta}\,\tilde{\psi}_{lmn}\,D_\tau\psi_{lmn}+\frac{2\mu}{\beta}\,\tilde{\psi}_{lmn}\psi_{lmn}+\\ +\frac{1}{\beta}\,\tilde{\lambda}_{lmn}\,D_\tau\lambda_{lmn}-\frac{p}{\beta}\,(\tilde{\psi}_{lmn}\lambda_{lmn}+\tilde{\lambda}_{lmn}\psi_{lmn})-\tilde{f}_{lmn}\,f_{lmn}\,, \tag{3.99}$$

and the supersymmetry transformations parametrised by ς and $\tilde{\varsigma}$ are:

$$\begin{cases} \delta_s \phi_{lmn} = \sqrt{r}\,\varsigma\,\psi_{lmn} \\ \delta_s \psi_{lmn} = \frac{\sqrt{r}}{\beta}\,\tilde{\varsigma}\,D_\tau \phi_{lmn} \\ \delta_s f_{l,m+1,n} = \frac{\sqrt{r}}{\beta}\,\tilde{\varsigma}\,D_\tau \lambda_{lmn} - \frac{\sqrt{r}}{\beta}\,p\,\tilde{\varsigma}\psi_{lmn} \\ \delta_s \lambda_{lmn} = \sqrt{r}\,\varsigma\,f_{l,m+1,n} + \frac{\sqrt{r}}{\beta}\,p\,\tilde{\varsigma}\phi_{lmn} \end{cases}, \tag{3.100}$$

$$\begin{cases} \delta_s \tilde{\phi}_{lmn} = -\sqrt{r}\,\tilde{\varsigma}\,\tilde{\psi}_{lmn} \\ \delta_s \tilde{\psi}_{lmn} = -\frac{\sqrt{r}}{\beta}\,\varsigma\,D_\tau \tilde{\phi}_{lmn} \\ \delta_s \tilde{f}_{l,m+1,n} = \frac{\sqrt{r}}{\beta}\,\varsigma\,D_\tau \tilde{\lambda}_{lmn} + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\tilde{\psi}_{lmn} \\ \delta_s \tilde{\lambda}_{lmn} = \sqrt{r}\,\tilde{\varsigma}\,\tilde{f}_{l,m+1,n} + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\tilde{\phi}_{lmn} \end{cases}. \tag{3.101}$$

We can now interpret the 1d theory obtained upon dimensional reduction. The free 4d chiral multiplet becomes a 1d theory with infinite degrees of freedom which can be divided into two kinds of supermultiplets:

- *Chiral multiplets*: (ϕ_{lmn}, ψ_{lmn}) and $(\tilde{\phi}_{lmn}, \tilde{\psi}_{lmn})$; these multiplets contain a complex scalar and a 1d spinor, both of them being dynamical.
- *Fermi multiplets*: $(f_{l,m+1,n}, \lambda_{lmn})$ and $(\tilde{f}_{l,m+1,n}, \tilde{\lambda}_{lmn})$; these multiplets contain a complex auxiliary scalar and a 1d dynamical spinor.

From (3.100) and (3.101) we learn that when $p = 0$ the two types of supermultiplets are completely decoupled, while when $p \neq 0$ they tie together to form a so called long multiplet. As we anticipated, f and $\tilde{f}$ have a shifted index with respect to the other fields, so that the charge σ is the same for every field in a given supermultiplet. This shows that the supercharges commute with the operator Σ as we claimed previously. In passing, notice that the superalgebra one finds from (3.100) and (3.101) is consistent with the general expression (3.33)[11].

For the discussion that follows, it is crucial to be careful with the ranges of the various quantum numbers, apart from the index n that ranges always from $-\frac{l}{2}$ to $\frac{l}{2}$. Let us consider separately the two supermultiplets.

- *Chiral multiplets*: we take the scalar fields as reference, for which $l \geq 0$ and $-\frac{l}{2} \leq m \leq \frac{l}{2}$. The question is whether the fermionic fields ψ_{lmn} and $\tilde{\psi}_{lmn}$ are well-defined in these ranges. For $l = 0$, we do not have the component proportional to $\cos\nu^+_{lm}$, but there is still the one proportional to $\cos\nu^-_{lm}$, so

[11] Keep in mind that $D_\tau\varsigma = D_\tau\tilde{\varsigma} = 0$ since the Killing spinors do not depend on τ and we showed above that their charge σ is zero.

they are indeed well-defined for $l \geq 0$. For what concerns m the potential problem is that fermions are defined also for the value $m = -\frac{l}{2} - 1$ through their component with $\lambda = -$, which is outside the range of definition of the scalars. Luckily, $\cos \nu^-_{l,-\frac{l}{2}-1} = 0$, so we can safely remove this m from the range of definition. Hence chiral multiplets are defined for $l \geq 0$ and $-\frac{l}{2} \leq m \leq \frac{l}{2}$.

- *Fermi multiplets*: again, we take the scalar fields as reference, for which $l \geq 0$ and $-\frac{l}{2} - 1 \leq m \leq \frac{l}{2} - 1$. The reasoning for what concerns l is the same as before, so $l \geq 0$ is good. For what concerns m, it is again similar. The fermion definition includes $m = \frac{l}{2}$ thanks to the $\lambda = -$ component, which is outside the range for the scalar fields; neverhteless $\sin \nu^-_{l,\frac{l}{2}} = 0$, hence we can remove it. We can conclude that Fermi multiplets are defined for $l \geq 0$ and $-\frac{l}{2} - 1 \leq m \leq \frac{l}{2} - 1$.

Notice that when the two types of multiplets are decoupled i.e. when $p = 0$, (3.99) includes only one of them: for $m = \frac{l}{2}$ the lagrangian $\mathcal{L}_{lmn}$ includes only the chiral multiplet, while for $m = -\frac{l}{2} - 1$ only the Fermi multiplet.

A further comment about the supersymmetry generators in 1d. If in 4d the operators that generates the susy transformations are $i\zeta\mathcal{Q}$ and $i\tilde{\zeta}\tilde{\mathcal{Q}}$, when performing the dimensional reduction they behaves exactly as (3.75) and (3.78), becoming respectively $i\sqrt{r}\,\varsigma\mathcal{Q}_{(1d)}$ and $-i\sqrt{r}\,\tilde{\varsigma}\tilde{\mathcal{Q}}_{(1d)}$.

To conclude this summary, we report again the expressions for the Hamiltonian and the operator $\underline{\Sigma}$. As the lagrangian, they can be written as infinite sums: $\underline{H} = \sum_{l,m,n} H_{lmn}$ and $\underline{\Sigma} = \sum_{l,m,n} \Sigma_{lmn}$, where the single addends are:

$$\begin{aligned} H_{lmn} = &-\frac{i}{\beta^2}\,\partial_\tau\tilde{\phi}_{lmn}\,\partial_\tau\phi_{lmn} + \frac{i}{\beta^2}\left(p^2 - \sigma^2 - 2\mu\sigma\right)\tilde{\phi}_{lmn}\,\phi_{lmn} + \\ &+ \frac{i\sigma}{\beta}\left(\tilde{\psi}_{lmn}\,\psi_{lmn} + \tilde{\lambda}_{lmn}\,\lambda_{lmn}\right) + \frac{2i\mu}{\beta}\,\tilde{\psi}_{lmn}\,\psi_{lmn} - \frac{ip}{\beta}\left(\tilde{\psi}_{lmn}\,\lambda_{lmn} + \tilde{\lambda}_{lmn}\,\psi_{lmn}\right), \end{aligned} \tag{3.102}$$

$$\begin{aligned} \Sigma_{lmn} = \frac{i\sigma}{\beta^2}&\left(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn} - \tilde{\phi}_{lmn}\,D_\tau\phi_{lmn} - 2\mu\,\tilde{\phi}_{lmn}\phi_{lmn}\right) + \\ &+ \frac{i\sigma}{\beta}\left(\tilde{\psi}_{lmn}\psi_{lmn} + \tilde{\lambda}_{lmn}\lambda_{lmn}\right). \end{aligned} \tag{3.103}$$

From now on we will suppress the indices l, m, and n of the 1d fields for ease of notation.

3.5 Spectrum of the Hamiltonian

3.5.1 Going to lorentzian signature

Once we obtained a one dimensional theory, we have to quantise it, and then finding the vev of the Hamiltonian will be quite straightforward. However, all this process is more easily carried out in lorentzian signature, so it is worth spending a few words on how the Wick rotation is performed. For a summary of our conventions in lorentzian signature and how they relate to those in euclidean signature see appendix A.3.

The Wick rotation amounts to decompactify the τ coordinate and make the identification $t = -i\tau$, so that the signature of the metric becomes $(-,+,+,+)$. As a consequence, every derivative with respect to τ picks a factor $-i$ too i.e. $\partial_\tau = -i\partial_t$. We do not have to transform other parameters appearing in the metric since $g_{\mu\nu}$ remains real also in lorentzian signature (unlike what will happen in chapter 4). However, notice that while in euclidean signature the parameter β is a well-defined quantity which specifies the length of the thermal circle, in lorentzian signature it is meaningless and it can be absorbed by rescaling t. In order not to confuse the reader, we will keep the β as a fixed parameter also in lorentzian signature, forbidding any t rescaling. The next step is to specify some reality conditions relating the fields, the supercharges, and the supersymmetry parameters in order to reduce the number of degrees of freedom, which in euclidean is doubled due to the fact that tilde objects are independent from the non-tilde ones. We will make the simplest choice, that is[12]:

$$\phi^\dagger = \tilde{\phi}\,, \qquad \psi^\dagger = \tilde{\psi}\,, \qquad f^\dagger = \tilde{f}\,, \qquad \lambda^\dagger = \tilde{\lambda}\,, \qquad \varsigma^\dagger = \tilde{\varsigma}\,, \qquad \mathcal{Q}^\dagger = \tilde{\mathcal{Q}}\,. \tag{3.104}$$

Recall that the dynamical spinors ψ and λ are grassmann scalars while the supersymmetry parameter ζ is a commuting scalar. With these identifications, the lorentzian lagrangian $\mathcal{L}^{(L)}_{lmn}$, which in our conventions coincides with minus the euclidean lagrangian (3.99), is given by:

$$\begin{aligned}\mathcal{L}^{(L)}_{lmn} =& \frac{1}{\beta^2}\, D_t\phi^\dagger D_t\phi + \frac{i\mu}{\beta^2}\left(D_t\phi^\dagger\phi - \phi^\dagger D_t\phi\right) + \frac{i}{\beta}\,\psi^\dagger D_t\psi - \frac{2\mu}{\beta}\,\psi^\dagger\psi + \\ &+ \frac{i}{\beta}\,\lambda^\dagger D_t\lambda + f^\dagger f - \frac{p^2}{\beta^2}\,\phi^\dagger\phi + \frac{p}{\beta}\,(\psi^\dagger\lambda + \lambda^\dagger\psi),\end{aligned} \tag{3.105}$$

where the lorentzian covariant derivative is $D_t = \partial_t + i\sigma$. The supersymmetry transformations become:

$$\begin{cases} \delta_s\phi = \sqrt{r}\,\varsigma\psi \\ \delta_s\psi = -\frac{i\sqrt{r}}{\beta}\,\varsigma^\dagger D_t\phi \\ \delta_s f = -\frac{i\sqrt{r}}{\beta}\,\varsigma^\dagger D_t\lambda - \frac{\sqrt{r}}{\beta}\,p\,\varsigma^\dagger\psi \\ \delta_s\lambda = \sqrt{r}\,\varsigma\, f + \frac{\sqrt{r}}{\beta}\,p\,\varsigma^\dagger\phi \end{cases}, \qquad \begin{cases} \delta_s\phi^\dagger = -\sqrt{r}\,\varsigma^\dagger\psi^\dagger \\ \delta_s\psi^\dagger = \frac{i\sqrt{r}}{\beta}\,\varsigma\, D_t\phi^\dagger \\ \delta_s f^\dagger = -\frac{i\sqrt{r}}{\beta}\,\varsigma\, D_t\lambda^\dagger + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\,\psi^\dagger \\ \delta_s\lambda^\dagger = \sqrt{r}\,\varsigma^\dagger f^\dagger + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\,\phi^\dagger \end{cases}. \tag{3.106}$$

At first it may seem that (3.106) are not consistent between each other due to the minus sign in some of the transformation laws of the hermitian conjugate fields.

[12] In 1d there is no distinction between left and right spinors, hence we can safely use the dagger as hermitian conjugation.

Yet it is not the case and we can realise it by looking more into the details of how the supersymmetry transformations are generated through the supercharges. Let us call Q and $Q^\dagger$ the supercharges in the representation acting on field space with the (anti-)commutator. Then we have:

- Scalars: $\delta_s\phi = [i\sqrt{r}\,\varsigma Q,\, \phi]\,, \quad \delta_s\phi^\dagger = [-i\sqrt{r}\,\varsigma^\dagger Q^\dagger,\, \phi^\dagger]$

$$\Longrightarrow \qquad (\delta_s\phi)^\dagger = [\phi^\dagger, -i\sqrt{r}\,\varsigma^\dagger Q^\dagger] = [i\sqrt{r}\,\varsigma^\dagger Q^\dagger,\, \phi^\dagger] = -\delta_s\phi^\dagger\,.$$

- Fermions: $\delta_s\psi = \{i\sqrt{r}\,\varsigma Q,\, \psi\}\,, \quad \delta_s\psi^\dagger = \{-i\sqrt{r}\,\varsigma^\dagger Q^\dagger,\, \psi^\dagger\}$

$$\Longrightarrow \qquad (\delta_s\psi)^\dagger = \{\psi^\dagger, -i\sqrt{r}\,\varsigma^\dagger Q^\dagger\} = \{-i\sqrt{r}\,\varsigma^\dagger Q^\dagger,\, \psi^\dagger\} = \delta_s\psi^\dagger\,.$$

So everything is consistent.

Lastly, we report here the hamiltonian $H^{(L)}_{lmn}$ and the charge $\Sigma^{(L)}_{lmn}$. These two are related to the corresponding operators in euclidean signature by a factor i, since the conserved charges in lorentzian signature are obtained by integrating the t component of a conserved current rather than the τ component:

$$\begin{aligned} H^{(L)}_{lmn} = -iH_{lmn} = \frac{1}{\beta^2}\,\partial_t\phi^\dagger\,\partial_t\phi + \frac{1}{\beta^2}\left(p^2-\sigma^2-2\mu\sigma\right)\phi^\dagger\phi + \frac{2\mu}{\beta}\,\psi^\dagger\psi + \\ + \frac{\sigma}{\beta}\left(\psi^\dagger\psi+\lambda^\dagger\lambda\right) - \frac{p}{\beta}\left(\psi^\dagger\lambda+\lambda^\dagger\psi\right)\Big]\,, \end{aligned} \tag{3.107}$$

$$\Sigma^{(L)}_{lmn} = -i\Sigma_{lmn} = \frac{\sigma}{\beta^2}\left(-iD_t\phi^\dagger\phi + i\phi^\dagger D_t\phi - 2\mu\,\phi^\dagger\phi\right) + \frac{\sigma}{\beta}\left(\psi^\dagger\psi+\lambda^\dagger\lambda\right). \tag{3.108}$$

From now on we will remove all the superscripts (L) and the lorentzian signature will be understood.

3.5.2 Quantising the 1D theory

In order to work out the energy spectrum of the quantum theory we have to quantise it. We will perform this in Lorentzian signature, yet adapting the procedure to Euclidean signature is straightforward. For our purpose, the canonical quantisation is very suitable. Firstly, starting from the lagrangian (3.105), we introduce the canonical conjugate fields:

$$\Pi_\phi = \frac{\partial\mathcal{L}^{(L)}}{\partial\partial_t\phi} = \frac{1}{\beta^2}\left(D_t\phi^\dagger - i\mu\,\phi^\dagger\right), \qquad \Pi^\dagger_\phi = \frac{\partial\mathcal{L}^{(L)}}{\partial\partial_t\phi^\dagger} = \frac{1}{\beta^2}\Big(D_t\phi + i\mu\,\phi\Big), \tag{3.109}$$

$$\Pi_\psi = \frac{\partial\mathcal{L}^{(L)}}{\partial\partial_t\psi} = \frac{i}{\beta}\,\psi^\dagger\,, \qquad \Pi_\lambda = \frac{\partial\mathcal{L}^{(L)}}{\partial\partial_t\lambda} = \frac{i}{\beta}\,\lambda^\dagger\,. \tag{3.110}$$

The non-dynamical fields f_{lmn} do not have a canonical conjugate field and they have been simply set to zero thanks to their equations of motion. We recall that each of these canonical conjugate fields should carry the indices l, m, and n, but we have suppressed them for the ease of notation. Now we impose the canonical (anti-)commutation relations:

$$\big[\phi,\, \Pi_\phi\big] = \big[\phi^\dagger,\, \Pi^\dagger_\phi\big] = i\,, \qquad\qquad \big\{\psi,\, \Pi_\psi\big\} = \big\{\lambda,\, \Pi_\lambda\big\} = i\,. \tag{3.111}$$

Every other commutator/anti-commutator (depending of what kind of fields are involved) is vanishing.

In terms of (3.109) canonical conjugate fields, the operators H_{lmn} and Σ_{lmn} are given by:

$$\begin{aligned} H_{lmn} = \beta^2\, \Pi^\dagger_\phi \Pi_\phi - i\,(\mu+\sigma)\left(\Pi_\phi\,\phi - \phi^\dagger \Pi^\dagger_\phi\right) + \frac{1}{\beta^2}\left(\mu^2+p^2\right)\phi^\dagger\phi + \\ + i\sigma\,(\psi\Pi_\psi + \lambda\Pi_\lambda) + 2i\mu\,\psi\Pi_\psi - ip\,(\lambda\Pi_\psi + \psi\Pi_\lambda) + \alpha_1\,, \end{aligned} \tag{3.112}$$

$$\Sigma_{lmn} = -i\sigma\left(\Pi_\phi\,\phi - \phi^\dagger\Pi^\dagger_\phi - \psi\Pi_\psi - \lambda\Pi_\lambda\right) + \alpha_2\,, \tag{3.113}$$

where we introduced two constants α_1 and α_2 that arise from the intrinsic ordering ambiguity of the operators in the quantum theory. The presence of these arbitrary constants may constitute a potential issue for our purpose, as it shifts the one-point functions of the respective operators, however there are some constraints that can fix them, as we will show in the next section. In passing, we point out that the Hamiltonian (3.112) is precisely equal to the canonical Hamiltonian that one can compute by considering the Legendre transform of the 1d lagrangian (3.105):

$$H_{lmn} = \Pi_\phi\,\partial_t\phi + \Pi^\dagger_\phi\,\partial_t\phi^\dagger + \Pi_\psi\,\partial_t\psi + \Pi_\lambda\,\partial_t\lambda - \mathcal{L}_{lmn}\,. \tag{3.114}$$

This provides a reassuring consistency check.

For the following, we will need the expression for the supercharge Q. It can be guessed rather easily by imposing that its action through the canonical (anti-)commutation relations generates the correct supersymmetry transformations of the fields (3.106). It turns out that Q reads:

$$Q = \sum_{l,m,n} Q_{lmn} = \sum_{l,m,n}\left[\psi\left(\Pi_\phi + \frac{i\mu}{\beta^2}\,\phi^\dagger\right) - \frac{ip}{\beta^2}\,\phi^\dagger\lambda\right], \tag{3.115}$$

and it has no ordering ambiguity since all the fields appearing commute with each other. We recall that the supersymmetry transformations (3.106) are obtained as:

$$\delta_s X = \big[i\sqrt{r}\,\varsigma Q,\, X\big\} + \big[-i\sqrt{r}\,\varsigma^\dagger Q^\dagger,\, X\big\}\,, \tag{3.116}$$

where X is a placeholder for a generic field and the usage of the commutator or the anti-commutator depends on whether it is a scalar or a spinor.

3.5.3 Fixing the ordering ambiguity

As anticipated, the two ordering constants appearing in (3.112) and (3.113) can be fixed by exploiting two consistency constraints. The first one comes from the superalgebra: given that the supercharge (3.115) has no ordering ambiguity, requiring the theory to satisfy the correct superalgebra (3.33) will put a constraint on α_1 and α_2. Figuring out what is this constraint, requires us to translate the superalgebra (3.33) in terms of the anticommutator $\{Q,\, Q^\dagger\}$. In order to do this, we have to understand how the actions of H and $\underline{H}$ and those of Σ and $\underline{\Sigma}$ are related one to each other (recall that H and Σ are operators in the differential representation while $\underline{H}$ and $\underline{\Sigma}$ are the same operators in the form of conserved Noether charges). One may check that upon using (3.111) it holds:

$$\big[\underline{H},\, X\big\} = -i\partial_t X = -H(X)\,, \qquad\qquad \big[\underline{\Sigma},\, X\big\} = -\sigma X = -\Sigma(X)\,. \tag{3.117}$$

So, the superalgebra (3.33) can be written as:

$$\{\delta_\varsigma,\, \delta_{\varsigma^\dagger}\}X = -\varsigma\varsigma^\dagger\,\frac{r}{\beta}\,(H-\Sigma)\,X = \varsigma\varsigma^\dagger\,\frac{r}{\beta}\,\big[\underline{H}-\underline{\Sigma},\, X\big\}\,. \tag{3.118}$$

On the other hand, assuming that X is a scalar, the same superalgebra can be obtained by the following computation:

$$\begin{aligned}\{\delta_\varsigma,\, \delta_{\varsigma^\dagger}\} &= \Big\{i\sqrt{r}\,\varsigma Q,\, \big[-i\sqrt{r}\,\varsigma^\dagger Q^\dagger,\, X\big]\Big\} + \Big\{-i\sqrt{r}\,\varsigma^\dagger Q^\dagger,\, \big[i\sqrt{r}\,\varsigma Q,\, X\big]\Big\}\\ &= r\,\varsigma\varsigma^\dagger\,\Big(\Big\{Q,\, \big[Q^\dagger,\, X\big]\Big\} - \Big\{Q^\dagger,\, \big[X,\, Q\big]\Big\}\Big) = -r\,\varsigma\varsigma^\dagger\,\Big[X, \{Q,\, Q^\dagger\}\Big]\,,\end{aligned} \tag{3.119}$$

where in the last passage we exploited the Jacobi identity for graded Lie algebras of grade 1. By comparing (3.118) with (3.119) we discover that:

$$\{Q,\, Q^\dagger\} = \frac{1}{\beta}\,(\underline{H}-\underline{\Sigma})\,. \tag{3.120}$$

One can follow the same steps assuming X to be a spinor instead and the result would be the same.

Now, if we compute directly the anticommutator of the supercharges (3.115) and its hermitian conjugate, we find:

$$\begin{aligned}\{Q_{lmn},\, Q^\dagger_{lmn}\} &= \frac{1}{\beta}\bigg[\beta^2\,\Pi^\dagger_\phi\Pi_\phi - i\mu\left(\Pi_\phi\,\phi - \phi^\dagger\Pi^\dagger_\phi\right) + \frac{1}{\beta^2}\left(\mu^2+p^2\right)\phi^\dagger\phi +\\ &\qquad\qquad - 2i\mu\,\Pi_\psi\psi - ip\,(\lambda\Pi_\psi + \psi\Pi_\lambda)\bigg]\\ &= \frac{1}{\beta}\,(H_{lmn} - \Sigma_{lmn} - \alpha_1 + \alpha_2 - 2i\mu\,\{\Pi_\psi,\, \psi\})\\ &= \frac{1}{\beta}\,(H_{lmn} - \Sigma_{lmn} - \alpha_1 + \alpha_2 + 2\mu)\,.\end{aligned} \tag{3.121}$$

Hence, to be consisted with (3.120), necessarily we have to require that $\alpha_1 = \alpha_2 + 2\mu$. One constant has thus been fixed.

In order to fix the second constant, we exploit a property of the renormalisation flow of Chern-Simons terms. Recall the low energy effective action (3.34) through which we compute the vev of Σ:

$$\mathcal{W}[A_\tau^\Sigma] = \langle \Sigma \rangle \int d\tau \, A_\tau^\Sigma \,. \tag{3.122}$$

Calling k the coefficient of a Chern-Simons term, it is a known fact [21] that when we integrate out a massive fermion of mass m, k gets shifted by an amount proportional to $\mathrm{sgn}(m)$, the constant of proportionality being linear in the charges at play.Thus, starting from a UV theory and integrating out all massive fermions, we get a relation between the value of k at high energy and at low energy:

$$\left(k_{UV} - k_{IR}\right) \propto \sum_i \mathrm{sgn}(m_i) \,, \tag{3.123}$$

where we assumed that all fermions have the same charge, which is the case relevant for us. In our case $k_{UV} = 0$ since we do not have any Chern-Simons term in the full theory (3.105). Moreover, from the diagonal mass matrix (3.70), we read that in each sector of the theory $\mathcal{L}_{lmn}$ that contains both the chiral and the Fermi multiplets, the two fermions appearing in our theory have masses of opposite sign; hence from (3.123) we can conclude that also k_{IR} should be zero. The conclusion is that $\langle \Sigma_{lmn} \rangle$ must vanish for long multiplets. This constraint imposes a further condition on the ordering constants, hence determining them completely. However to work out the explicit constraint we have to firstly find out what the ground state of the theory is. That is what we will do in the following section.

3.5.4 VEVs of the different multiplets

As anticipated, our aim is now to find out the ground state of each one of the Hamiltonians (3.112).

Let us begin by considering long multiplets i.e. those with $p \neq 0$. Starting with the scalar sector, we can introduce creation and annihilations operators a, $a^\dagger$ and b, $b^\dagger$ and express ϕ and its conjugate field in terms of them as:

$$\phi = \frac{(\mu^2 + p^2)^{-\frac{1}{4}}}{\sqrt{2}} \left(a + b^\dagger\right) \,, \qquad \Pi_\phi = \frac{i\,(\mu^2 + p^2)^{\frac{1}{4}}}{\sqrt{2}} \left(a^\dagger - b\right) . \tag{3.124}$$

The canonical commutation relations impose:

$$\left[a,\, a^\dagger\right] = \left[b,\, b^\dagger\right] = 1 \,, \tag{3.125}$$

$$\left[a,\, b\right] = \left[a^\dagger,\, b\right] = \left[a,\, b^\dagger\right] = \left[a^\dagger,\, b^\dagger\right] = 0 \,, \tag{3.126}$$

which are precisely the commutation relations we expect for creation and annihilation operators. The bosonic part of (3.112) can be written in terms of a and b as:

$$H^{(bos)}_{long} = \sqrt{\mu^2+p^2}\left(a^\dagger a + b^\dagger b + 1\right) + (\sigma+\mu)\left(a^\dagger a - b^\dagger b - 1\right). \tag{3.127}$$

Let us label with $|\Omega\rangle$ the state with no oscillators exited. All others states are obtained by acting on it with the creation operators and are labelled by the number of times they act:

$$|i,j\rangle = \left(a^\dagger\right)^i \left(b^\dagger\right)^j |\Omega\rangle \ . \tag{3.128}$$

The energy of the state $|i,j\rangle$ can be read by taking its braket with (3.127):

$$E(i,\,j) = \sqrt{\mu^2+p^2}-\mu-\sigma+j\left(\sqrt{\mu^2+p^2}-\mu-\sigma\right)+i\left(\sqrt{\mu^2+p^2}+\mu+\sigma\right). \tag{3.129}$$

Immediately we see that in order for the energy to be bounded from below we have to make the assumption:

$$\sqrt{\mu^2+p^2} > |\mu+\sigma| \,. \tag{3.130}$$

Otherwise, we could keep exciting the a-type oscillators and achieve an arbitrary small energy. Such a behaviour would not be possible to fix even when adding up the fermionic contribution since fermionic excitations are limited in number because of the anti-commuting algebra. Hence, in these conditions, the ground state is $|\Omega\rangle$ itself, which corresponds to $i=j=0$ and it has energy:

$$\langle H^{(bos)}_{long}\rangle = \sqrt{\mu^2+p^2}-\mu-\sigma \,. \tag{3.131}$$

Moving on to the fermionic sector, its Hamiltonian is given by:

$$H^{(fer)}_{long} = \frac{p}{\beta}\left(\lambda\psi^\dagger+\psi\lambda^\dagger\right)-\frac{1}{\beta}\left(\sigma+2\mu\right)\psi\psi^\dagger-\frac{\sigma}{\beta}\lambda\lambda^\dagger = \frac{1}{\beta}\begin{pmatrix}\psi & \lambda\end{pmatrix}\begin{pmatrix}-\sigma-2\mu & p\\ p & -\sigma\end{pmatrix}\begin{pmatrix}\psi^\dagger\\ \lambda^\dagger\end{pmatrix}. \tag{3.132}$$

From (3.111) it follows that $\left\{\psi,\,\psi^\dagger\right\} = \left\{\lambda,\,\lambda^\dagger\right\} = \beta$. In order to have the usual normalisation for the algebra of creation and annihilation operators, we rescale both fermions as $\psi \to \frac{1}{\sqrt{\beta}}\,\psi$ and $\lambda \to \frac{1}{\sqrt{\beta}}\,\lambda$. Since the matrix appearing in (3.132) is real and symmetric, we can diagonalise it while preserving the anti-commutation algebra by rotating the degrees of freedom. The matrix eigenvalues are:

$$x_\pm = -\mu-\sigma\pm\sqrt{\mu^2+p^2}\,. \tag{3.133}$$

Denoting with P the rotation matrix, the new degrees of freedom are:

$$\begin{pmatrix}u_+\\ u_-\end{pmatrix} = P\begin{pmatrix}\psi\\ \lambda\end{pmatrix}. \tag{3.134}$$

In terms of these new variables, the fermionic Hamiltonian reads simply:

$$H^{(fer)}_{long} = x_+\,u_+u_+^\dagger + x_-\,u_-u_-^\dagger\,, \tag{3.135}$$

with $\{u_+, u_+^\dagger\} = \{u_-, u_-^\dagger\} = 1$. Denoting again with $|\Omega\rangle$ the state with no fermionic oscillators excited, the possible fermionic states are just four:

State	$\lvert\Omega\rangle$	$u_+\lvert\Omega\rangle$	$u_-\lvert\Omega\rangle$	$u_+u_-\lvert\Omega\rangle$
Energy	0	x_+	x_-	$x_+ + x_-$

Given the assumption (3.130), the state with the least energy is $u_-|\Omega\rangle$, hence:

$$\langle H_{long}^{(fer)}\rangle = -\sqrt{\mu^2+p^2} - \mu - \sigma \,. \tag{3.136}$$

Summing up everything, we find that the ground state energy for long multiplets is given by:

$$\langle H_{long}\rangle = \langle H_{long}^{(bos)}\rangle + \langle H_{long}^{(fer)}\rangle + \alpha_2 + 2\mu = -2\sigma + \alpha_2 \,. \tag{3.137}$$

In terms of creation and annihilation operators, the charge Σ_{lmn} (3.113) of the long multiplets is given by:

$$\Sigma_{long} = \sigma\left(a^\dagger a - b^\dagger b - 1 - u_+u_+^\dagger - u_-u_-^\dagger\right) + \alpha_2. \tag{3.138}$$

Braketting this expression with the ground state we just found, we obtain that its vev is:

$$\langle\Sigma_{long}\rangle = -2\sigma + \alpha_2 \,. \tag{3.139}$$

Remarkably, $\langle H_{long}\rangle = \langle\Sigma_{long}\rangle$ as prescribed by the superalgebra. This is a good consistency check that corroborates what we are doing. As we explained in section 3.5.3, we must require $\langle\Sigma_{long}\rangle = 0$, and this fixes $\alpha_2 = 2\sigma$. The consequence of this choice is that long multiplets do not contribute to the Casimir energy of the theory, since the vev of their Hamiltonian is always vanishing. In passing, notice that this particular choice of the ordering constants coincide with requiring the operators H_{lmn} and Σ_{lmn} to be Weyl ordered[13].

So, we have just learned that only decoupled chiral and Fermi multiplets contribute to the Casimir energy of our theory. We have to compute what are these contributions. Let us start with the Chiral multiplet. We set $p = 0$ and we discard λ so that the Hamiltonian and Σ_{lmn} are:

$$H_{chiral} = |\mu|\left(a^\dagger a + b^\dagger b + 1\right) + (\sigma+\mu)\left(a^\dagger a - b^\dagger b\right) - (\sigma + 2\mu)\,\psi\psi^\dagger + \mu + \frac{\sigma}{2}\,, \tag{3.141}$$

$$\Sigma_{chiral} = \sigma\left(a^\dagger a - b^\dagger b - \psi\psi^\dagger\right) + \frac{\sigma}{2}\,, \tag{3.142}$$

[13]We recall the definition of Weyl ordering. Given a generic field X with fermionic number F and its canonical conjugate field Π_X, the Weyl ordering of quadratic terms is:

$$W(X\,\Pi_X) \equiv \frac{1}{2}\left(X\,\Pi_X + (-1)^F\,\Pi_X\,X\right). \tag{3.140}$$

where we have already rescaled $\psi \to \frac{1}{\sqrt{\beta}}\psi$. In these expressions we removed a factor $\frac{\sigma}{2}$ which in the long multiplet comes from requiring the Weyl ordering of the terms containing λ. The condition (3.130) becomes:

$$|\mu| > |\mu + \sigma| \,, \tag{3.143}$$

hence in the vacuum all the bosonic oscillators are not excited. For what concerns the fermionic sector, there are only two possible states: $|\Omega\rangle$ with vanishing energy and $\psi\,|\Omega\rangle$ with energy $-\sigma - 2\mu$. From (3.143) we know that if $\mu > 0$, then $-2\mu < \sigma < 0$, therefore $\psi\,|\Omega\rangle$ is the state with smallest energy; instead if $\mu < 0$, then $0 < \sigma < -2\mu$ and $|\Omega\rangle$ is the state with the smallest energy. In both cases the vevs of H_{chiral} and Σ_{chiral} are given by:

$$\langle H_{chiral}\rangle = \langle \Sigma_{chiral}\rangle = \frac{|\sigma|}{2} \,. \tag{3.144}$$

We conclude by considering a Fermi multiplet. In this case the only field appearing is the fermion λ:

$$H_{Fermi} = \Sigma_{Fermi} = -\sigma\left(\lambda\lambda^{\dagger} - \frac{1}{2}\right) \,, \tag{3.145}$$

where again we have already rescaled $\lambda \to \frac{1}{\sqrt{\beta}}\lambda$ and discarded the contribution to the ordering constants coming from the Weyl ordering of the chiral multiplet. There are only two possible states, $|\Omega\rangle$ and $\psi\,|\Omega\rangle$; which one is the ground state depends on the sign of σ, however in both cases the vevs are:

$$\langle H_{Fermi}\rangle = \langle \Sigma_{Fermi}\rangle = -\frac{|\sigma|}{2} \,. \tag{3.146}$$

Now that we found the contributions to the vevs of H and Σ coming from all the different multiplets, all what is left is to sum them up to find the final result.

3.5.5 Infinte sum regularisation and final result

The vev of the Hamiltonian is obviously given by the infinite sum of the vevs of the Hamiltonians of each supermultiplet:

$$\langle H\rangle = \sum_{l,m,n} \langle H_{lmn}\rangle \,. \tag{3.147}$$

As we saw, every term of the sum (3.147) that corresponds to a long multiplet is vanishing and does not contribute. Hence we must restrict the domain of summation to indices such that $p = 0$. Looking back at (3.66), we realise that this condition holds either when $m = \frac{l}{2}$, which corresponds to a chiral multiplet, or when $m =$

$-\frac{l}{2}-1$, which corresponds to a Fermi multiplet (see the discussion about the ranges of the quantum numbers in section 3.4.5). Hence (3.147) reduces to:

$$\begin{aligned}\langle H\rangle &= \sum_{l,n}\langle H_{chiral}\rangle\Big|_{m=\frac{l}{2}} + \sum_{l,n}\langle H_{Fermi}\rangle\Big|_{m=-\frac{l}{2}-1} \\ &= \sum_{l=0}^{+\infty}\sum_{n=-\frac{l}{2}}^{n=\frac{l}{2}} \frac{\beta}{2r}\,|l+q_r| - \sum_{l=0}^{+\infty}\sum_{n=-\frac{l}{2}}^{n=\frac{l}{2}} \frac{\beta}{2r}\,|-l-2+q_r|\ .\end{aligned} \tag{3.148}$$

In order to understand what is the sign of the two terms inside the absolute values we need to make explicit the conditions imposed by the assumption (3.130):

$$\sqrt{\mu^2+p^2} = \frac{\beta}{r}\,(l+1)\,, \qquad\qquad \mu+\sigma = \frac{\beta}{r}\,(q_r-1)\,, \tag{3.149}$$

$$\Longrightarrow \qquad l+1 > |q_r-1|\,. \tag{3.150}$$

This inequality has to be satisfied for every $l \in \mathbb{N}$ in order for the spectrum of the Hamiltonian to be bounded from below. Hence, the range of possible values for the R-charge of the scalar field in the 4d chiral multiplet has to be restricted. The worst case scenario is achieved when $l=0$, hence we must have $0<q_r<2$. In this range, $l+q_r>0$ and $-l-2+q_r<0$, so the sum (3.148) becomes:

$$\langle H\rangle = \sum_{l=0}^{+\infty}\sum_{n=-\frac{l}{2}}^{n=\frac{l}{2}} \frac{\beta}{2r}\,(l+q_r) - \sum_{l=0}^{+\infty}\sum_{n=-\frac{l}{2}}^{n=\frac{l}{2}} \frac{\beta}{2r}\,(l+2-q_r)\,. \tag{3.151}$$

This sum is clearly divergent, therefore we should find a way to make sense of it. The usual approach when one has to face a divergent quantity consists in regularising it and then removing only the divergent terms. Different regularisation techniques can be used and they could potentially lead to different results. However, a physical quantity, such as the supersymmetric Casimir energy (that we showed to be unambiguous in section 3.2.2), cannot depend on some non-physical choice such as the regularisation method. The key point is that we have to regularise the sum in a supersymmetric fashion, since different supersymmetric regularisation methods should yield the same physical result. A viable approach consists in introducing a small parameter δ and multiplying each term of the sum by the exponential of a negative supersymmetric quantity multiplied by δ:

$$\langle H\rangle_\delta = \sum_{lmn}\langle H_{lmn}\rangle e^{-\delta\,|\langle H_{lmn}\rangle|}\,. \tag{3.152}$$

Then one computes the sum and finally takes the limit $\delta\to 0$, discarding the divergent terms in the expansion in powers of δ. In our case this method amounts to compute:

$$\langle H\rangle_\delta = \sum_{l=0}^{+\infty}\sum_{n=-\frac{l}{2}}^{n=\frac{l}{2}} \frac{\beta}{2r}\,(l+q_r)\,e^{-\frac{\delta\beta}{2r}(l+q_r)} - \sum_{l=0}^{+\infty}\sum_{n=-\frac{l}{2}}^{n=\frac{l}{2}} \frac{\beta}{2r}\,(l+2-q_r)\,e^{-\frac{\delta\beta}{2r}(l+2-q_r)}\,. \tag{3.153}$$

With the help of a computing software, one can verify that this sum reads:

$$\langle H\rangle_\delta = -\frac{4r}{\delta^2\beta}\,(q_r-1) + E + \mathcal{O}(\delta)\,, \tag{3.154}$$

where E is precisely what survives after discarding the diverging term and taking the limit $\delta \to 0$. Its expression gives the final result for the Casimir energy:

$$E = \frac{4\beta}{27r}\,(a+3c)\,, \tag{3.155}$$

where we introduced the two conformal anomaly coefficients [23]:

$$a = \frac{3}{32}\left[3(q_r-1)^3 - (q_r-1)\right]\,, \qquad c = \frac{1}{32}\left[9(q_r-1)^3 - 5(q_r-1)\right]\,. \tag{3.156}$$

We finally got to the end of the computation exposed in [6]. (3.155) is the expression for the Casimir energy of a chiral multiplet living on the direct product $S^1 \times S^3$. In addition to reviewing in detail the arguments presented there, we developed many steps of the computations which were not explicitly given, at least those more relevant for understanding the physics. In the next chapter we will present how to extend this procedure in the case where the metric is not a simple direct product. For many steps we will refer to what we did here since the general reasoning is pretty much the same, focusing the exposition on the differences we encounter.

CHAPTER 4

A further step: twisting the 3-sphere

4.1 Background geometry

This chapter will contain most of the original results of the present work. The aim is to follow the same reasoning based upon [6] we exposed in chapter 3 in order to compute the supersymmetric Casimir energy of a free chiral theory in a more general background. In particular we will consider a manifold that is still diffeomorphic to $S^1 \times S^3$ but now the 3-sphere will be twisted around the circle; such a deformation has a clear interpretation in terms of both the complex structure of the manifold itself and of the holographic duality. Here we will derive the results and we demand to chapter 5 most of the comments and their physical interpretation.

4.1.1 Manifold definition and background fields

First of all, we need to define our deformed manifold and the other background fields. A suitable system of coordinates for this space is the same as before i.e. the set $(\tau,\, \theta,\, \varphi_1,\, \varphi_2)$ where $\tau \in [0,\, 2\pi)$ is the coordinate on the circle and $\theta \in [0,\, \frac{\pi}{2}]$ and $\varphi_1,\, \varphi_2 \in [0,\, 2\pi)$ are the coordinates on the 3-sphere, with the identifications $\tau \sim \tau + 2\pi$, $\varphi_1 \sim \varphi_1 + 2\pi$, and $\varphi_2 \sim \varphi_2 + 2\pi$. Now we introduce also two real twisting parameters σ_1 and σ_2. The metric of the twisted $S^1 \times S^3$ can be written as:

$$ds^2 = \beta^2\, d\tau^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, (d\varphi_1 + \sigma_1\, d\tau)^2 + r^2 \cos^2\theta\, (d\varphi_2 + \sigma_2\, d\tau)^2 \;, \quad (4.1)$$

Compared to (3.5) we note that (4.1) has two new terms that mix the τ coordinate with the two coordinates of the Hopf fibration φ_1 and φ_2. We will give an intuitive explanation of this difference in the following section 4.1.2.

The determinant and the Ricci scalar of the metric (4.1) are the same as those of the direct product case:

$$g = \beta^2 r^6 \cos^2\theta \sin^2\theta \,, \tag{4.2}$$

$$\mathcal{R} = \frac{6}{r^2} \,. \tag{4.3}$$

The symmetries of the twisted 3-sphere are the same too, hence we have a left and a right angular momenta whose Cartan in the differential representation are still given by (3.8) and are comprehensive of both the orbital angular momentum and the internal spin.

The relations (3.9) between the real coordinates (τ, θ, φ_1, φ_2) and the complex holomorphic coordinates (w, z) get modified as:

$$\begin{cases} w = \left(\sigma_2 + \dfrac{i\beta}{r}\right)\tau + \varphi_2 - i\log\cos\theta \\ z = e^{i(\varphi_1+\sigma_1\tau)-i(\varphi_2+\sigma_2\tau)}\tan\theta \end{cases} . \tag{4.4}$$

The real coordinates boundary conditions now lead to the identification:

$$(w,\, z) \sim \left(w + 2\pi\sigma_2 + \frac{2\pi i\beta}{r}\,,\; z\, e^{2\pi i(\sigma_1-\sigma_2)}\right), \tag{4.5}$$

and also $w \sim w + 2\pi$ while z is kept fixed. These identifications will have a clear interpretation in terms of Hopf surfaces. The functions of z and $\bar{z}$ appearing in (2.22) in this case are still:

$$\Omega = r\,, \qquad h = -\frac{i\bar{z}}{1+|z|^2}\,, \qquad c = \frac{1}{1+|z|^2}\,. \tag{4.6}$$

The fact that it is possible to recast (4.1) in this form tells us that such a twisted $S^1 \times S^3$ background is indeed capable of supporting an $\mathcal{N} = 1$ SQFT with two supercharges of opposite R-charge. The vector $K = \partial_w$ is a Killing vector by construction and in real coordinates it reads:

$$K = -\frac{ir}{2\beta}\,\partial_\tau + \left(\frac{1}{2} + \frac{ir\sigma_1}{2\beta}\right)\partial_{\varphi_1} + \left(\frac{1}{2} + \frac{ir\sigma_2}{2\beta}\right)\partial_{\varphi_2}\,, \tag{4.7}$$

and lowering the index with the real metric, we get the associated 1-form:

$$\underline{K} = \frac{1}{2} r^2 \left[-\frac{i\beta}{r}\,\mathrm{d}\tau + \sin^2\theta\,(\mathrm{d}\varphi_1 + \sigma_1\,\mathrm{d}\tau) + \cos^2\theta\,(\mathrm{d}\varphi_2 + \sigma_2\,\mathrm{d}\tau)\right]. \tag{4.8}$$

In addition to the terms appearing in (3.11) and (3.12), here we have additional pieces that depend on the newly introduced parameters σ_1 and σ_2. One may check that, even after the twisting, K^μ satisfies all the properties that we listed in section

2.2.2 i.e. $K^\mu K_\mu = 0$ and $[K, \bar{K}] = 0$. The most general expressions for the two auxiliary background fields read:

$$V = \frac{i\beta\kappa r}{2}\,\mathrm{d}\tau + \left(1 - \frac{\kappa}{2}\,r^2\right)\left[\sin^2\theta\,(\mathrm{d}\varphi_1 + \sigma_1\,\mathrm{d}\tau) + \cos^2\theta\,(\mathrm{d}\varphi_2 + \sigma_2\,\mathrm{d}\tau)\right], \tag{4.9}$$

$$\begin{aligned} A = {}& \sin^2\theta\,(\mathrm{d}\varphi_1 + \sigma_1\,\mathrm{d}\tau) + \cos^2\theta\,(\mathrm{d}\varphi_2 + \sigma_2\,\mathrm{d}\tau) - \frac{i}{2}\,\mathrm{d}(\log s) - \frac{1}{2}\,(\sigma_1 + \sigma_2)\,\mathrm{d}\tau + \\ & + \frac{1}{2}\left(1 - \frac{3}{2}\,\kappa r^2\right)\left[-\frac{i\beta}{r}\,\mathrm{d}\tau + \sin^2\theta\,(\mathrm{d}\varphi_1 + \sigma_1\,\mathrm{d}\tau) + \cos^2\theta\,(\mathrm{d}\varphi_2 + \sigma_2\,\mathrm{d}\tau)\right]. \end{aligned} \tag{4.10}$$

By choosing the function κ to be the constant $\frac{2}{r^2}$, we can again make the components in $\mathrm{d}\varphi_1$ and $\mathrm{d}\varphi_2$ of A and V cancel out (except for those that may be contained in $\mathrm{d}(\log s)$):

$$V = \frac{i\beta}{r}\,\mathrm{d}\tau\,, \tag{4.11}$$

$$A = \left(\frac{i\beta}{r} - \frac{1}{2}\,(\sigma_1 + \sigma_2)\right)\mathrm{d}\tau - \frac{i}{2}\,\mathrm{d}(\log s)\,. \tag{4.12}$$

This choice has the advantage of making A and V regular everywhere. We will fix the other arbitrary function s momentarily.

Let us introduce the local frame we will use for the computations. Again, it is convenient to use a local frame which is rotated with respect to the general expression (2.25) we gave previously. Respectively in real and complex coordinates our choice is:

$$\begin{aligned} e^1 &= -r\sin\theta\cos\theta\,\sin(\varphi_1 + \varphi_2)\Big[(\mathrm{d}\varphi_1 + \sigma_1\,\mathrm{d}\tau) - (\mathrm{d}\varphi_2 + \sigma_2\,\mathrm{d}\tau)\Big] + r\cos(\varphi_1 + \varphi_2)\mathrm{d}\theta \\ e^2 &= -r\sin\theta\cos\theta\,\cos(\varphi_1 + \varphi_2)\Big[(\mathrm{d}\varphi_1 + \sigma_1\,\mathrm{d}\tau) - (\mathrm{d}\varphi_2 + \sigma_2\,\mathrm{d}\tau)\Big] - r\sin(\varphi_1 + \varphi_2)\mathrm{d}\theta \\ e^3 &= r\,\sin^2\theta\,(\mathrm{d}\varphi_1 + \sigma_1\,\mathrm{d}\tau) + r\,\cos^2\theta\,(\mathrm{d}\varphi_2 + \sigma_2\,\mathrm{d}\tau) \\ e^4 &= \beta\,\mathrm{d}\tau \end{aligned}, \tag{4.13}$$

$$\begin{aligned} \Theta^1 &= \Omega\,c\,e^{-2i\varphi_2 + i(\sigma_1 - \sigma_2)\tau}\,\mathrm{d}\bar{z} \\ \bar{\Theta}^{\bar{1}} &= \Omega\,c\,e^{2i\varphi_2 - i(\sigma_1 - \sigma_2)\tau}\,\mathrm{d}z \\ \Theta^2 &= \Omega\,(\mathrm{d}w + h\,\mathrm{d}z) \\ \bar{\Theta}^{\bar{2}} &= \Omega\,\big(\mathrm{d}\bar{w} + \bar{h}\,\mathrm{d}\bar{z}\big) \end{aligned}\,. \tag{4.14}$$

In such a frame, the two Killing spinors are given by:

$$\zeta_\alpha = \varsigma\,\sqrt{\frac{s}{2}}\begin{pmatrix}0\\1\end{pmatrix}, \qquad \tilde{\zeta}^{\dot{\alpha}} = \tilde{\varsigma}\,\frac{r}{\sqrt{2s}}\begin{pmatrix}1\\0\end{pmatrix}, \tag{4.15}$$

where ς, $\tilde{\varsigma} \in \mathbb{C}$ are the complex constants that parametrise the supersymmetry transformations along the two directions selected by the Killing spinors. The function s is fixed by imposing suitable boundary conditions on ζ and $\tilde{\zeta}$. The two spinors should be anti-periodic when we go once around the circle parametrised by either φ_1 or φ_2; however, as we already explained, it is consistent to take their boundary conditions under $\tau \to \tau + 2\pi$ to be either periodic or anti-periodic, so now we parametrise this freedom with an integer n_0, requiring the Killing spinors to be periodic when n_0 is even and anti-periodic when it is odd. Imposing the boundary conditions works exactly as in the round case: we compute the Lie derivative and then (4.15) transform with its exponential (3.20). Luckily, the Killing spinors are still eigenfunctions of the Lie derivatives along τ, φ_1, φ_2, and their eigenvalues are precisely the same as before. If we want the parameter $n_0 \in \mathbb{Z}$ to define the periodicity of the Killing spinors when we go once around the circle parametrised by τ, we cannot set $s = r$ as before, but rather we have to take $s = r\, e^{-in_0\tau}$. With this choice the background supergravity vector fields and the two Killing spinors become:

$$A = \left(\frac{i\beta}{r} - \frac{1}{2}\,(\sigma_1 + \sigma_2 + n_0) \right) \mathrm{d}\tau \,, \qquad V = \frac{i\beta}{r}\,\mathrm{d}\tau \,, \tag{4.16}$$

$$\zeta_\alpha = \varsigma\, e^{-\frac{i}{2} n_0 \tau} \sqrt{\frac{r}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad \tilde{\zeta}^{\dot{\alpha}} = \tilde{\varsigma}\, e^{\frac{i}{2} n_0 \tau} \sqrt{\frac{r}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{4.17}$$

so that the Lie derivatives are:

$$\begin{aligned}
&L_{\partial_\tau}\zeta = -\frac{in_0}{2}\,\zeta \,, && L_{\partial_{\varphi_1}}\zeta = \frac{i}{2}\,\zeta \,, && L_{\partial_{\varphi_2}}\zeta = \frac{i}{2}\,\zeta \,, \\
&L_{\partial_\tau}\tilde{\zeta} = \frac{in_0}{2}\,\tilde{\zeta} \,, && L_{\partial_{\varphi_1}}\tilde{\zeta} = -\frac{i}{2}\,\tilde{\zeta} \,, && L_{\partial_{\varphi_2}}\tilde{\zeta} = -\frac{i}{2}\,\tilde{\zeta} \,,
\end{aligned} \tag{4.18}$$

and indeed corresponds to the periodicity conditions highlighted above. As before, the normalisation of the Killing vector used in (4.7) corresponds to $\varsigma = \tilde{\varsigma} = 1$.

4.1.2 The effects of the twisting

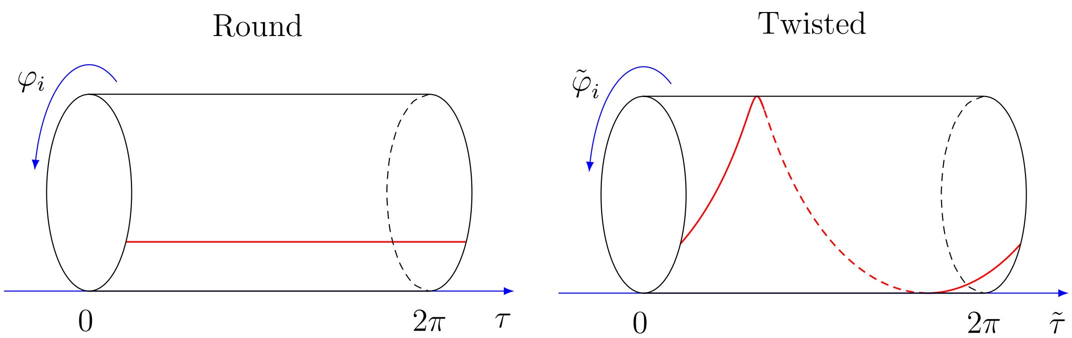
In the previous section we highlighted the differences in the background metric between the twisted and the round 3-sphere, yet the intuitive geometric picture may be a bit hidden behind the mathematical formulae. Therefore it is advisable to focus for a moment on understanding the core difference between the metrics (3.5) and (4.1). To do this, let us introduce a new set of coordinates for the twisted case:

$$\begin{cases} \tilde{\tau} = \tau \\ \tilde{\theta} = \theta \\ \tilde{\varphi}_1 = \varphi_1 + \sigma_1\,\tau \\ \tilde{\varphi}_2 = \varphi_2 + \sigma_2\,\tau \end{cases} . \tag{4.19}$$

In such coordinates the twisted metric (4.1) reads simply:

$$ds^2 = \beta^2\, d\tilde{\tau}^2 + r^2\, d\tilde{\theta}^2 + r^2 \sin^2\tilde{\theta}\, d\tilde{\varphi}_1^2 + r^2\cos^2\tilde{\theta}\, d\tilde{\varphi}_2^2\,, \tag{4.20}$$

which has the precise same form of the direct product metric (3.5). What is the difference between the two then? The difference resides in the range and the identifications of the coordinates $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ compared to φ_1 and φ_2. In particular, if we consider a line at constant φ_i in both cases (the red lines in the picture below), while in the round case moving along the coordinate τ does not affect the coordinate φ_i, from (4.19) we can read that in the twisted case $\tilde{\varphi}_i$ gets shifted too and thus a line at constant φ_i winds around the torus defined by the coordinates $\tilde{\tau}$ and $\tilde{\varphi}_i$. This is true simultaneously for $i = 1,\, 2$, and the rotation angle depends on the values of the twisting parameters σ_1 and σ_2.



Aside from the purely geometric interpretation, it would be also interesting to understand how this background deformation affects the gravity side of holographic duality. However, this is far beyond the aim of the present work and explaining it would require a consistent digression. What we can say is that if we consider the asymptotically AdS_5 gravity configuration having our twisted $S^1 \times S^3$ as conformal boundary, the twisting parameters σ_1 and σ_2 are related to the horizon angular velocities of a supersymmetric black hole (see e.g. [15]).

4.1.3 Supersymmetry algebra

The supersymmetry algebra preserved by the background we defined in section 4.1.1 is a bit more complicated than (3.32) and contains other operators. Unpacking the Lie derivative in (2.11), we get:

$$\begin{aligned}\{\delta_\varsigma,\, \delta_{\tilde{\varsigma}}\} = -\varsigma\tilde{\varsigma}\,\frac{r}{\beta}\Big[H - i(\sigma_1+\sigma_2)J_3^L - i(\sigma_1-\sigma_2)J_3^R - \frac{2\beta}{r}\,J_3^L + \\ - \frac{\beta}{r}\left(1+\frac{ir}{2\beta}\,(\sigma_1+\sigma_2+n_0)\right)R\Big].\end{aligned} \tag{4.21}$$

At first sight this may seem a little messy, however, we can rewrite this algebra in terms of three 1d operators that commute with the supercharges. We introduce:

$$\tilde{H} = H - \frac{i}{2} n_0 R \,, \tag{4.22}$$

$$\mathcal{J} = i(\sigma_1 - \sigma_2)\, J_3^R \,, \tag{4.23}$$

$$\Sigma = \left(\frac{\beta}{r} + \frac{i}{2}(\sigma_1 + \sigma_2)\right)\left(R + 2J_3^L\right) \tag{4.24}$$

$$\implies \quad \left\{\delta_\varsigma, \delta_{\tilde{\varsigma}}\right\} = -\varsigma\tilde{\varsigma}\,\frac{r}{\beta}\left(\tilde{H} - \Sigma - \mathcal{J}\right). \tag{4.25}$$

Compared to (3.33), here Σ includes also a piece which depends on the two twisting parameters and the algebra contains the new operator $\mathcal{J}$ where the right angular momentum appears. Moreover H does not commute any more with the supercharges, but rather it is the "twisted" hamiltonian $\tilde{H}$ that commutes[14]. Of course, if we switch off the twisting parameters by setting $\sigma_1 = \sigma_2 = 0$, we recover the algebra discussed in the previous chapter.

Though a little more complicated, the superalgebra (4.25) maintains basically the same structure we had when the background metric was a direct product, hence the considerations on the unambiguity of the Casimir energy we did in section 3.2.2 are still valid. In particular, the vacuum expectation values of the operators R, Σ, and $\mathcal{J}$ can all be interpreted as coefficients of suitable Chern-Simons terms, hence they are physical and do not depend on continuous coupling constants. Therefore, assuming the vacuum to be supersymmetric, we can conclude that:

$$\langle H \rangle = \frac{i}{2} n_0 \langle R \rangle + \langle \Sigma \rangle + \langle \mathcal{J} \rangle \,, \tag{4.26}$$

hence also the supersymmetric Casimir energy is physical and unambiguous, and it can be computed starting from a free theory in flat space.

4.2 Dimensional reduction of the 4d theory

4.2.1 1d degrees of freedom and their charges

As we did in chapter 3, we will consider the free chiral multiplet theory, but now on the background defined in section 4.1.1. The strategy is pretty much the same: we expand the 4d fields in scalar and spinor harmonics as in (3.48) and the following, and we find an infinite set of 1d degrees of freedom labelled by the quantum numbers l, m, and n.

The first thing we should do is work out the charges of the 1d fields under the relevant operators, in particular R, J_3^L and J_3^R. For what concerns the first two,

[14] One can understand this by noting that the killing spinors (4.17) are not independent of τ as in the direct product case.

the only thing thay might change from the direct product case is the left angular momentum of the spinorial degrees of freedom, given that the vielbeins enter the definition of the spinorial Lie derivative (A.19) and (A.20). However, this is not the case, and a direct computation as in (3.58) reveals that the left angular momentum eigenvalue is still $m + \frac{1}{2}$ for both ψ_{lmn} and λ_{lmn}. The right angular momentum can be computed analogously:

$$J_3^R\,\phi = \sum_{l,m,n} \frac{i}{2}\left(\partial_{\varphi_1} - \partial_{\varphi_2}\right)\left(\phi_{lmn} Y_l{}^{mn}\right) = \sum_{l,m,n} n\,\phi_{lmn} Y_l{}^{mn}\,, \tag{4.27}$$

$$J_3^R\,\psi = \sum_{l,m,n,\lambda} \frac{i}{2}\left(L_{\partial_{\varphi_1}} - L_{\partial_{\varphi_2}}\right)\left(\psi^{\lambda}_{lmn}\, S^{\lambda}_{lmn}\right) = \sum_{l,m,n} n \begin{pmatrix} \psi_{lmn}\, Y_l{}^{mn} \\ \lambda_{lmn}\, Y_l{}^{m+1,n} \end{pmatrix}, \tag{4.28}$$

$$J_3^R\,F = \sum_{l,m,n} \frac{i}{2}\left(\partial_{\varphi_1} - \partial_{\varphi_2}\right)\left(f_{lmn} Y_l{}^{mn}\right) = \sum_{l,m,n} n\,\phi_{lmn} Y_l{}^{mn}\,. \tag{4.29}$$

Hence we conclude that all the 1d fields corresponding to a given value of n have the same right angular momentum eigenvalue. Summarising, the charges of the 1d fields are the following:

	ϕ_{lmn}	ψ_{lmn}	λ_{lmn}	f_{lmn}
R	q_r	$q_r - 1$	$q_r - 1$	$q_r - 2$
J_3^L	m	$m + \frac{1}{2}$	$m + \frac{1}{2}$	m
J_3^R	n	n	n	n

Denoting with σ and ρ the charges respectively under the operators Σ and $\mathcal{J}$, we have the following situation:

$$\phi_{lmn},\ \psi_{lmn},\ \lambda_{lmn},\ f_{l,m+1,n} \quad \longrightarrow \quad \begin{cases} \sigma = \left(\dfrac{\beta}{r} + \dfrac{i}{2}\left(\sigma_1 + \sigma_2\right)\right)\left(q_r + 2m\right) \\ \rho = in\left(\sigma_1 - \sigma_2\right) \end{cases}, \tag{4.30}$$

$$\tilde{\phi}_{lmn},\ \tilde{\psi}_{lmn},\ \tilde{\lambda}_{lmn},\ \tilde{f}_{l,m+1,n} \quad \longrightarrow \quad \begin{cases} \sigma = -\left(\dfrac{\beta}{r} + \dfrac{i}{2}\left(\sigma_1 + \sigma_2\right)\right)\left(q_r + 2m\right) \\ \rho = -in\left(\sigma_1 - \sigma_2\right) \end{cases}. \tag{4.31}$$

We note that, as the operator Σ now has a new piece that comes from the twisting of the 3-sphere, its charge σ mimics exactly the same structure. Moreover, assuming that a given 1d supermultiplet is constituted by the set $\{\phi_{lmn},\ \psi_{lmn},\ \lambda_{lmn},\ f_{l,m+1,n}\}$, like in the direct product background (which is indeed the case as we will see later on), we can conclude that all the fields inside a supermultiplet have the same charges σ and ρ, which means that indeed the operators Σ and $\mathcal{J}$ commute with the supercharges as we claimed in section 4.1.3.

In the same way explained above we can compute the values of the charges σ and ρ for the two Killing spinors (4.17) and it turns out that they vanish for both ζ and $\tilde{\zeta}$.

4.2.2 Lagrangian and supersymmetry transformations

The 4d lagrangian and supersymmetry transformations of the free chiral multiplet theory have precisely the same form as (3.35) and (3.36), the difference being that the background fields $g_{\mu\nu}$, A_μ, and V_μ are now given by the expressions that include the twisting parameters and n_0, respectively (4.1) and (4.16). The dimensional reduction is carried out more or less as exposed in section 3.4. The differences in the computations are rather technical and are given mostly by the following two factors.

- When one unpacks the lagrangian (3.35) there appear new terms arising from the fact that the metric (4.1) contains mixed terms. For instance, in the scalar sector there are all the terms like $g^{\tau i}\,\partial_\tau\tilde{\phi}\,\partial_i\phi$ and those with i and τ exchanged; also terms containing A^i and V^i were vanishing on the direct product background while now they are not.

- Some properties of the spherical harmonics get slightly modified due to the presence of the twisting terms; in particular the action of the Laplacian is now (C.15) rather than (C.11) and that of the operator $i\tilde{\sigma}^\mu\partial_\mu$ is (C.23) rather than (C.21) because of the different vielbeins.

Eventually, after expanding all 4d fields in scalar/spinorial harmonics, applying their properties (see appendix C), and performing the integration over the coordinates on the twisted S^3, we get a one dimensional lagrangian that looks precisely like (3.99), with a 1d covariant derivative that is now generalised to:

$$D_\tau = \partial_\tau + \frac{in_0}{2}\,q + \sigma + \rho\,. \tag{4.32}$$

This covariant derivative is interpreted as providing a minimal coupling between the fields it acts on with some background gauge fields associated to the operators R, Σ, and $\mathcal{J}$.

The same situation repeats for the supersymmetry transformations. The differences from the direct product background are those highlighted above and the results one obtains are similar to (3.83) and (3.84) apart from the fact that there are some exponentials containing n_0 arising from the expressions (4.17) for ζ and $\tilde{\zeta}$:

$$\begin{cases} \delta_s\phi_{lmn} = \sqrt{r}\, e^{-\frac{i}{2}\,n_0\tau}\,\varsigma\,\psi_{lmn} \\ \delta_s\psi_{lmn} = \frac{\sqrt{r}}{\beta}\, e^{\frac{i}{2}\,n_0\tau}\,\tilde{\varsigma}\,D_\tau\phi_{lmn} \\ \delta_s f_{l,m+1,n} = \frac{\sqrt{r}}{\beta}\, e^{\frac{i}{2}\,n_0\tau}\,\tilde{\varsigma}\,D_\tau\lambda_{lmn} - \frac{\sqrt{r}}{\beta}\,p\,e^{\frac{i}{2}\,n_0\tau}\,\tilde{\varsigma}\psi_{lmn} \\ \delta_s\lambda_{lmn} = \sqrt{r}\, e^{-\frac{i}{2}\,n_0\tau}\,\varsigma\,f_{l,m+1,n} + \frac{\sqrt{r}}{\beta}\,p\,e^{\frac{i}{2}\,n_0\tau}\,\tilde{\varsigma}\phi_{lmn} \end{cases}, \tag{4.33}$$

$$\begin{cases} \delta_s\tilde{\phi}_{lmn} = -\sqrt{r}\, e^{\frac{i}{2}\,n_0\tau}\,\tilde{\varsigma}\,\tilde{\psi}_{lmn} \\ \delta_s\tilde{\psi}_{lmn} = -\frac{\sqrt{r}}{\beta}\, e^{-\frac{i}{2}\,n_0\tau}\,\varsigma\,D_\tau\tilde{\phi}_{lmn} \\ \delta_s\tilde{f}_{l,m+1,n} = \frac{\sqrt{r}}{\beta}\, e^{-\frac{i}{2}\,n_0\tau}\,\varsigma\,D_\tau\tilde{\lambda}_{lmn} + \frac{\sqrt{r}}{\beta}\,p\,e^{-\frac{i}{2}\,n_0\tau}\,\varsigma\tilde{\psi}_{lmn} \\ \delta_s\tilde{\lambda}_{lmn} = \sqrt{r}\, e^{\frac{i}{2}\,n_0\tau}\,\tilde{\varsigma}\,\tilde{f}_{l,m+1,n} + \frac{\sqrt{r}}{\beta}\,p\,e^{-\frac{i}{2}\,n_0\tau}\,\varsigma\tilde{\phi}_{lmn} \end{cases}. \tag{4.34}$$

The exponentials can be reabsorbed in the 1d supersymmetry parameters by rescaling them as:

$$\varsigma \to e^{-\frac{i}{2}\,n_0\tau}\,\varsigma\,, \qquad\qquad \tilde{\varsigma} \to e^{\frac{i}{2}\,n_0\tau}\,\tilde{\varsigma}\,. \tag{4.35}$$

Not only it is possible to perform this rescaling, but it is even convenient since the rescaled ς and $\tilde{\varsigma}$ have vanishing covariant derivative; in fact, given that the charges σ and ρ of the Killing spinors are vanishing, we have:

$$D_\tau\big(e^{-\frac{i}{2}\,n_0\tau}\,\varsigma\big) = \partial_\tau e^{-\frac{i}{2}\,n_0\tau}\,\varsigma + \frac{in_0}{2}\,e^{-\frac{i}{2}\,n_0\tau}\,\varsigma = 0\,, \tag{4.36}$$

and analogously for $\tilde{\varsigma}$. Summarising, in terms of the rescaled supersymmetry parameters (4.35), the 1d supersymmetry transformations in the twisted background are exactly the same as (3.100) and (3.101) with the covariant derivative given by (4.32), and the supersymmetry parameters satisfy $D_\tau\varsigma = D_\tau\tilde{\varsigma} = 0$.

4.2.3 Issues with the conserved current Y^μ_ξ

The next step is to reduce the four dimensional operators $\underline{R}$, $\underline{H}$, $\underline{J}^L_3$, and $\underline{J}^R_3$ to 1d ones. Note that their 4d expressions are still the same that we reported for the direct product case, namely (3.40), (3.45), (3.46), and (3.47). For what concerns the R-charge and the left angular momentum, everything works in the same way and we are able to find two expressions that are precisely equal to the direct product case but with the new covariant derivative (4.32). We will recall their expressions later on in section 4.2.4.

Yet, the presence of the twisting terms inside our new vielbeins (4.13) introduces some issues with the dimensional reduction of the fermionic part of the Hamiltonian and the right angular momentum. In particular, the sigma matrices with a lower spacetime index turn out to present a dependence on the coordinates on S^3 that cannot be reabsorbed into the harmonics. For example, one may verify that $\tilde{\sigma}_\tau$ has the expression:

$$\tilde{\sigma}_\tau = e_{a\tau}\,\tilde{\sigma}^a = \begin{pmatrix} -i\beta - r\big(\sigma_1 \sin^2\theta + \sigma_2\cos^2\theta\big) & -ir(\sigma_1-\sigma_2)\sin\theta\,\cos\theta\, e^{i(\varphi_1+\varphi_2)} \\ ir(\sigma_1-\sigma_2)\sin\theta\,\cos\theta\, e^{-i(\varphi_1+\varphi_2)} & -i\beta + r\left(\sigma_1\sin^2\theta+\sigma_2\cos^2\theta\right) \end{pmatrix}. \tag{4.37}$$

A similar expression holds for $\tilde{\sigma}_{\varphi_1} - \tilde{\sigma}_{\varphi_2}$, which is the combination that enters $\underline{J}^R_3$, while instead $\tilde{\sigma}_{\varphi_1} + \tilde{\sigma}_{\varphi_2}$, which appears in $\underline{J}^L_3$ does not have the same problem since all the terms depending on θ, φ_1, and φ_2 cancel out. Unfortunately, inside (3.45) and (3.47) there are no other factors that summed up with these can remove the angular coordinates dependence, hence we cannot exploit the orthogonality of spherical harmonics to compute the integrals. One should try to compute them directly but this is a non-trivial challenge and, even if it is possible to do it by means of 3-j symbols (see e.g. [11]), the results are quite complicated and definitely different from what we expected to find (i.e. something quite similar to the direct product case). We suspect that these shortcomings are due to the fact that we are trying to compute quantities that are not left-invariant using a local a frame that is left-invariant, and that the expression (3.44) for the conserved current associated to the symmetry generated by ξ has to be adjusted in its fermionic part in order to take this into account. Though, at the moment we do not have a proof for this claim.

What to do then? A reasonable strategy is to guess what the expressions for $\underline{H}$ and $\underline{J}^R_3$ could be based on other considerations and clues. Starting from $\underline{H}$, we can try to use the canonical four dimensional Hamiltonian instead of (3.45):

$$\underline{H}^{(L)} = \frac{\partial \mathcal{L}^{(L)}_{S^1\times S^3}}{\partial\partial_t\phi}\,\partial_t\phi + \frac{\partial \mathcal{L}^{(L)}_{S^1\times S^3}}{\partial\partial_t\tilde{\phi}}\,\partial_t\tilde{\phi} + \frac{\partial \mathcal{L}^{(L)}_{S^1\times S^3}}{\partial\partial_t\psi}\,\partial_t\psi - \mathcal{L}^{(L)}_{S^1\times S^3}\,, \tag{4.38}$$

where $\mathcal{L}^{(L)}_{S^1\times S^3} = -\mathcal{L}_{S^1\times S^3}$, the RHS being (3.35). Expliciting the various terms, one obtains:

$$\begin{aligned}\underline{H}^{(L)} = \int_{S^3} d^3x\,\sqrt{g_3}\Big\{ & D^\mu\tilde{\phi}\,D_\mu\phi + iV^\mu\left(D_\mu\tilde{\phi}\,\phi - \tilde{\phi}\,D_\mu\phi\right) - D^\tau\tilde{\phi}\,\partial_\tau\phi + \\ & - \partial_\tau\tilde{\phi}\,D^\tau\phi - iV^\tau\left(\partial_\tau\tilde{\phi}\,\phi - \tilde{\phi}\,\partial_\tau\phi\right) - i\tilde{\psi}\,\tilde{\sigma}^\tau\partial_\tau\psi\Big\}.\end{aligned} \tag{4.39}$$

Upon expanding in harmonics, integrating over S^3, and using the equations of motion determined by the twisted 1d lagrangian we get:

$$
\begin{aligned}
\underline{H}^{(L)} = \sum_{l,m,n} \Bigg\{ & -\frac{1}{\beta^2}\,\partial_\tau\tilde{\phi}_{lmn}\,\partial_\tau\phi_{lmn} + \frac{1}{\beta^2}\Bigg[p^2 - \left(\frac{in_0}{2}\,q_r + \sigma + \rho\right)^2 + \\
& - 2\mu\left(\frac{in_0}{2}\,q_r + \sigma + \rho\right)\Bigg]\tilde{\phi}_{lmn}\,\phi_{lmn} + \frac{2\mu}{\beta}\,\tilde{\psi}_{lmn}\,\psi_{lmn} + \\
& + \frac{1}{\beta}\left(\frac{in_0}{2}\,(q_r - 1) + \sigma + \rho\right)\left(\tilde{\psi}_{lmn}\,\psi_{lmn} + \tilde{\lambda}_{lmn}\,\lambda_{lmn}\right) + \\
& - \frac{p}{\beta}\left(\tilde{\psi}_{lmn}\,\lambda_{lmn} + \tilde{\lambda}_{lmn}\,\psi_{lmn}\right)\Bigg\},
\end{aligned}
\tag{4.40}
$$

The fact that (4.40) has a structure similar to (3.93) is promising. Moreover, one can show that this expression coincides with the one obtained from integrating (3.45) when $\sigma_1 = \sigma_2$, therefore we can at least be sure that (4.40) is the correct Hamiltonian whenever the two twisting parameters are equal. Shifting our attention to the right angular momentum, the situation here is more tricky since there are no other simple ways one can follow to work out its expression. A reasonable guess is to take the expression (3.94) for the left angular momentum, of course with the covariant derivative suitable for the twisted background, and substitute the eigenvalues of J_3^L with those of J_3^R:

$$
\begin{aligned}
\underline{J}_3^R = \sum_{l,m,n} \Bigg[& \frac{in}{\beta^2}\left(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn} - \tilde{\phi}_{lmn}\,D_\tau\phi_{lmn} - 2\mu\,\tilde{\phi}_{lmn}\phi_{lmn}\right) + \\
& + \frac{in}{\beta}\left(\tilde{\psi}_{lmn}\psi_{lmn} + \tilde{\lambda}_{lmn}\lambda_{lmn}\right)\Bigg].
\end{aligned}
\tag{4.41}
$$

By construction this operator will return the expected value when acting through the (anti-)commutator on a given field, hence it is reasonable to assume that (4.41) is the correct expression. Notice that when $\sigma_1 = \sigma_2$ the operator J_3^R disappears completely from the supersymmetry algebra (4.25), therefore we know that at least in the case $\sigma_1 = \sigma_2$ our computations will be valid independently of the issue discussed in this subsection.

4.2.4 Summary of the 1d theory

We now summarise everything we have learned about the one dimensional theory that we obtain when performing the dimensional reduction over the 3-sphere of the 4d free chiral multiplet in the twisted $S^1 \times S^3$ background. Recall the definitions of the 1d covariant derivative and the parameters μ and p:

$$D_\tau = \partial_\tau + \frac{in_0}{2}\, q + \sigma + \rho \,, \tag{4.42}$$

$$\mu = -\frac{\beta}{r}\,(2m+1)\,, \tag{4.43}$$

$$p = \frac{\beta}{r}\,\sqrt{(l-2m)\,(l+2+2m)}\,, \tag{4.44}$$

where q is the R-charge, σ and ρ are the charges of the field under respectively the operators Σ and $\mathcal{J}$ and they read:

$$\begin{aligned}
&\phi_{lmn},\, \psi_{lmn},\, \lambda_{lmn},\, f_{l,m+1,n} \quad \longrightarrow \quad \begin{cases} \sigma = \left(\dfrac{\beta}{r} + \dfrac{i}{2}\,(\sigma_1+\sigma_2)\right)(q_r+2m) \\ \rho = in\,(\sigma_1-\sigma_2) \end{cases} , \\
&\tilde{\phi}_{lmn},\, \tilde{\psi}_{lmn},\, \tilde{\lambda}_{lmn},\, \tilde{f}_{l,m+1,n} \quad \longrightarrow \quad \begin{cases} \sigma = -\left(\dfrac{\beta}{r} + \dfrac{i}{2}\,(\sigma_1+\sigma_2)\right)(q_r+2m) \\ \rho = -in\,(\sigma_1-\sigma_2) \end{cases} .
\end{aligned} \tag{4.45}$$

It is good to keep in mind the above expression since we will make use of them many times in what follows. The 1d lagrangian is given by the infinite sum $\mathcal{L}^{(1d)} = \sum_{l,m,n} \mathcal{L}_{lmn}$, where $\mathcal{L}_{lmn}$ is the same as before apart from the covariant derivative:

$$\begin{aligned}
\mathcal{L}_{lmn} =& \frac{1}{\beta^2}\, D_\tau\tilde{\phi}_{lmn}\, D_\tau\phi_{lmn} + \frac{\mu}{\beta^2}\left(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn} - \tilde{\phi}_{lmn}\, D_\tau\phi_{lmn}\right) + \frac{p^2}{\beta^2}\,\tilde{\phi}_{lmn}\,\phi_{lmn} + \\
&+ \frac{1}{\beta}\,\tilde{\psi}_{lmn}\, D_\tau\psi_{lmn} + \frac{2\mu}{\beta}\,\tilde{\psi}_{lmn}\psi_{lmn} + \\
&+ \frac{1}{\beta}\,\tilde{\lambda}_{lmn}\, D_\tau\lambda_{lmn} - \frac{p}{\beta}\,(\tilde{\psi}_{lmn}\lambda_{lmn} + \tilde{\lambda}_{lmn}\psi_{lmn}) - \tilde{f}_{lmn}\, f_{lmn}\,,
\end{aligned} \tag{4.46}$$

The supersymmetry transformations parametrised by ς and $\tilde{\varsigma}$ are also the same as before:

$$\begin{cases}
\delta_s\phi_{lmn} = \sqrt{r}\,\varsigma\,\psi_{lmn} \\
\delta_s\psi_{lmn} = \frac{\sqrt{r}}{\beta}\,\tilde{\varsigma}\, D_\tau\phi_{lmn} \\
\delta_s f_{l,m+1,n} = \frac{\sqrt{r}}{\beta}\,\tilde{\varsigma}\, D_\tau\lambda_{lmn} - \frac{\sqrt{r}}{\beta}\, p\,\tilde{\varsigma}\psi_{lmn} \\
\delta_s\lambda_{lmn} = \sqrt{r}\,\varsigma\, f_{l,m+1,n} + \frac{\sqrt{r}}{\beta}\, p\,\tilde{\varsigma}\phi_{lmn}
\end{cases} , \tag{4.47}$$

$$\begin{cases} \delta_s\tilde{\phi}_{lmn} = -\sqrt{r}\,\tilde{\varsigma}\,\tilde{\psi}_{lmn} \\ \delta_s\tilde{\psi}_{lmn} = -\frac{\sqrt{r}}{\beta}\,\varsigma\, D_\tau\tilde{\phi}_{lmn} \\ \delta_s\tilde{f}_{l,m+1,n} = \frac{\sqrt{r}}{\beta}\,\varsigma\, D_\tau\tilde{\lambda}_{lmn} + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\tilde{\psi}_{lmn} \\ \delta_s\tilde{\lambda}_{lmn} = \sqrt{r}\,\tilde{\varsigma}\,\tilde{f}_{l,m+1,n} + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\tilde{\phi}_{lmn} \end{cases} . \tag{4.48}$$

As in the direct product case, the 4d theory reduces to a 1d theory of an infinite set of fields grouped in chiral supermultiplets (ϕ_{lmn}, ψ_{lmn}) and Fermi supermultiplets (λ_{lmn}, $f_{l,m+1,n}$). These supermultiplets are tied together to form a long multiplet when $p \neq 0$ while they are decoupled when $p = 0$.

The supersymmetry transformations are implemented by the two 1d operators $i\sqrt{r}\,\varsigma\mathcal{Q}_{(1d)}$ and $-i\sqrt{r}\,\tilde{\varsigma}\tilde{\mathcal{Q}}_{(1d)}$ and the two supersymmetry parameters have vanishing covariant derivative i.e. $D_\tau\varsigma = D_\tau\tilde{\varsigma} = 0$.

The operators we will need afterwards are the R-charge, the Hamiltonian, and the charges Σ and $\mathcal{J}$. Each of them is an infinite sum of operators acting only on the fields with a specific set of indices (l, m, n), like for example $\underline{R} = \sum_{l,m,n} R_{lmn}$ and other analogous expressions. These operators are:

$$\begin{aligned} R_{lmn} = \Bigg[\frac{iq_r}{\beta^2}\Big(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn} - \tilde{\phi}_{lmn}\,D_\tau\phi_{lmn} - 2\mu\,\tilde{\phi}_{lmn}\phi_{lmn}\Big) + \\ + \frac{i}{\beta}\,(q_r - 1)\,\Big(\tilde{\psi}_{lmn}\psi_{lmn} + \tilde{\lambda}_{lmn}\lambda_{lmn}\Big)\Bigg], \end{aligned} \tag{4.49}$$

$$\begin{aligned} H^{(L)}_{lmn} = -\frac{1}{\beta^2}\,\partial_\tau\tilde{\phi}_{lmn}\,\partial_\tau\phi_{lmn} + \frac{1}{\beta^2}\Bigg[p^2 - \bigg(\frac{in_0}{2}\,q_r + \sigma + \rho\bigg)^2 + \\ - 2\mu\bigg(\frac{in_0}{2}\,q_r + \sigma + \rho\bigg)\Bigg]\tilde{\phi}_{lmn}\,\phi_{lmn} + \frac{2\mu}{\beta}\,\tilde{\psi}_{lmn}\,\psi_{lmn} + \\ + \frac{1}{\beta}\bigg(\frac{in_0}{2}\,(q_r - 1) + \sigma + \rho\bigg)\Big(\tilde{\psi}_{lmn}\,\psi_{lmn} + \tilde{\lambda}_{lmn}\,\lambda_{lmn}\Big) + \\ - \frac{p}{\beta}\,\Big(\tilde{\psi}_{lmn}\,\lambda_{lmn} + \tilde{\lambda}_{lmn}\,\psi_{lmn}\Big)\,, \end{aligned} \tag{4.50}$$

$$\begin{aligned} \Sigma_{lmn} = \frac{i\sigma}{\beta^2}\,\Big(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn} - \tilde{\phi}_{lmn}\,D_\tau\phi_{lmn} - 2\mu\,\tilde{\phi}_{lmn}\phi_{lmn}\Big) + \\ + \frac{i\sigma}{\beta}\,\Big(\tilde{\psi}_{lmn}\psi_{lmn} + \tilde{\lambda}_{lmn}\lambda_{lmn}\Big)\,. \end{aligned} \tag{4.51}$$

$$\begin{aligned} \mathcal{J}_{lmn} = \frac{i\rho}{\beta^2}\,\Big(D_\tau\tilde{\phi}_{lmn}\,\phi_{lmn} - \tilde{\phi}_{lmn}\,D_\tau\phi_{lmn} - 2\mu\,\tilde{\phi}_{lmn}\phi_{lmn}\Big) + \\ + \frac{i\rho}{\beta}\,\Big(\tilde{\psi}_{lmn}\psi_{lmn} + \tilde{\lambda}_{lmn}\lambda_{lmn}\Big)\,. \end{aligned} \tag{4.52}$$

From now on we suppress the indices l, m, and n attached to the one dimensional fields.

4.3 Computing the Casimir energy

4.3.1 Lorentzian 1d theory

In order to carry out the computation of the Casimir energy more easily, we perform a Wick rotation and go to lorentzian signature. We trade τ for $t = -i\tau$, so that the metric signature becomes $(-, +, +, +)$. As before, we keep the parameter β also in lorentzian signature, though it has no physical meaning; it will be easier to return to euclidean signature when discussing the physical interpretation of the final results. In order to keep $g_{\mu\nu}$ real, now we have also to analytically extend the twisting parameters σ_1 and σ_2, otherwise we end up with imaginary terms of the form $2ir^2\sigma_i \sin^2\theta\, d\varphi_i\, dt$ inside the ds^2. Moreover, we should do the same thing with the parameter n_0 so as to ensure that the reality condition $\varsigma^\dagger = \tilde{\varsigma}$ holds. Hence, we define the lorentzian parameters as:

$$\sigma_1^{(L)} = i\sigma_1 \,, \qquad \sigma_2^{(L)} = i\sigma_2 \,, \qquad n_0^{(L)} = in_0 \,. \tag{4.53}$$

The one dimensional covariant derivative in lorentzian signature becomes:

$$D_t = \partial_t + \frac{in_0^{(L)}}{2}\, q + i\sigma^{(L)} + i\rho^{(L)} \,, \tag{4.54}$$

where $\sigma^{(L)}$ and $\rho^{(L)}$ are the charges under the operators:

$$\mathcal{J}^{(L)} = (\sigma_1^{(L)} - \sigma_2^{(L)})\, J_3^R \,, \tag{4.55}$$

$$\Sigma^{(L)} = \left(\frac{\beta}{r} + \frac{1}{2}\left(\sigma_1^{(L)} + \sigma_2^{(L)}\right)\right)\left(R + 2J_3^L\right) . \tag{4.56}$$

The reality conditions we require are the same we used in the direct product case:

$$\phi^\dagger = \tilde{\phi} \,, \qquad \psi^\dagger = \tilde{\psi} \,, \qquad f^\dagger = \tilde{f} \,, \qquad \lambda^\dagger = \tilde{\lambda} \,, \qquad \varsigma^\dagger = \tilde{\varsigma} \,, \qquad \mathcal{Q}^\dagger = \tilde{\mathcal{Q}} \,, \tag{4.57}$$

therefore also the 1d lagrangian and supersymmetry transformations look precisely like (3.105) and (3.106):

$$\begin{aligned} \mathcal{L}^{(L)}_{lmn} =& \frac{1}{\beta^2} D_t\phi^\dagger D_t\phi + \frac{i\mu}{\beta^2}\left(D_t\phi^\dagger\phi - \phi^\dagger D_t\phi\right) + \frac{i}{\beta}\,\psi^\dagger D_t\psi - \frac{2\mu}{\beta}\,\psi^\dagger\psi + \\ &+ \frac{i}{\beta}\,\lambda^\dagger D_t\lambda + f^\dagger f - \frac{p^2}{\beta^2}\,\phi^\dagger\phi + \frac{p}{\beta}\left(\psi^\dagger\lambda + \lambda^\dagger\psi\right) , \end{aligned} \tag{4.58}$$

$$\begin{cases} \delta_s\phi = \sqrt{r}\,\varsigma\psi \\ \delta_s\psi = -\frac{i\sqrt{r}}{\beta}\,\varsigma^\dagger D_t\phi \\ \delta_s f = -\frac{i\sqrt{r}}{\beta}\,\varsigma^\dagger D_t\lambda - \frac{\sqrt{r}}{\beta}\,p\,\varsigma^\dagger\psi \\ \delta_s\lambda = \sqrt{r}\,\varsigma\, f + \frac{\sqrt{r}}{\beta}\,p\,\varsigma^\dagger\phi \end{cases}, \qquad \begin{cases} \delta_s\phi^\dagger = -\sqrt{r}\,\varsigma^\dagger\psi^\dagger \\ \delta_s\psi^\dagger = \frac{i\sqrt{r}}{\beta}\,\varsigma\, D_t\phi^\dagger \\ \delta_s f^\dagger = -\frac{i\sqrt{r}}{\beta}\,\varsigma\, D_t\lambda^\dagger + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\,\psi^\dagger \\ \delta_s\lambda^\dagger = \sqrt{r}\,\varsigma^\dagger f^\dagger + \frac{\sqrt{r}}{\beta}\,p\,\varsigma\,\phi^\dagger \end{cases}. \tag{4.59}$$

From now on the lorentzian signature will be understood and we remove all the superscript (L).

Now, we can quantise the theory using the canonical quantisation. Let us introduce the canonical conjugate fields:

$$\Pi_\phi = \frac{\partial\mathcal{L}^{(L)}}{\partial\partial_t\phi} = \frac{1}{\beta^2}\left(D_t\phi^\dagger - i\mu\,\phi^\dagger\right), \qquad \Pi^\dagger_\phi = \frac{\partial\mathcal{L}^{(L)}}{\partial\partial_t\phi^\dagger} = \frac{1}{\beta^2}\Big(D_t\phi + i\mu\,\phi\Big), \tag{4.60}$$

$$\Pi_\psi = \frac{\partial\mathcal{L}^{(L)}}{\partial\partial_t\psi} = \frac{i}{\beta}\,\psi^\dagger\,, \qquad \Pi_\lambda = \frac{\partial\mathcal{L}^{(L)}}{\partial\partial_t\lambda} = \frac{i}{\beta}\,\lambda^\dagger\,. \tag{4.61}$$

Then we impose the canonical (anti-)commutation relations:

$$\left[\phi,\,\Pi_\phi\right] = \left[\phi^\dagger,\,\Pi^\dagger_\phi\right] = i\,, \qquad \left\{\psi,\,\Pi_\psi\right\} = \left\{\lambda,\,\Pi_\lambda\right\} = i\,. \tag{4.62}$$

At this point we can express the operators (4.49), (4.50), (4.51), and (4.52) in terms of the canonical conjugate fields:

$$R_{lmn} = -iq_r\left(\Pi_\phi\,\phi - \phi^\dagger\Pi^\dagger_\phi\right) + i(q_r-1)\left(\psi\Pi_\psi + \lambda\Pi_\lambda\right) + \alpha_r\,, \tag{4.63}$$

$$\begin{aligned} H_{lmn} = {} & \beta^2\,\Pi^\dagger_\phi\Pi_\phi - i\left(\mu + \frac{n_0}{2}\,q_r + \sigma + \rho\right)\left(\Pi_\phi\,\phi - \phi^\dagger\Pi^\dagger_\phi\right) + \frac{1}{\beta^2}\left(\mu^2+p^2\right)\phi^\dagger\phi + \\ & + i\left(\frac{n_0}{2}\left(q_r-1\right) + \sigma + \rho\right)\left(\psi\Pi_\psi + \lambda\Pi_\lambda\right) + 2i\mu\,\psi\Pi_\psi - ip\left(\lambda\Pi_\psi + \psi\Pi_\lambda\right) + \alpha_1\,, \end{aligned} \tag{4.64}$$

$$\Sigma_{lmn} = -i\sigma\left(\Pi_\phi\,\phi - \phi^\dagger\Pi^\dagger_\phi - \psi\Pi_\psi - \lambda\Pi_\lambda\right) + \alpha_2\,, \tag{4.65}$$

$$\mathcal{J}_{lmn} = -i\rho\left(\Pi_\phi\,\phi - \phi^\dagger\Pi^\dagger_\phi - \psi\Pi_\psi - \lambda\Pi_\lambda\right) + \alpha_3\,, \tag{4.66}$$

where we added the arbitrary constants α_r, α_1, α_2, and α_3 in order to keep track of the ordering ambiguity. We have checked that (4.64) coincides precisely with the canonical Hamiltonian obtained as a Legendre transform of the 1d lagrangian (4.58); this is a further evidence that our guess for the Hamiltonian is indeed correct. Notice that these expressions are in line with what one should have expected; in particular in the Hamiltonian, wherever there was a σ in the direct product case, now there is a factor $\frac{n_0}{2}\,q + \sigma + \rho$ due to the presence of the additional operators in the supersymmetry algebra (4.25).

Finally, the supercharge that acts on field space through the (anti-)commutator has precisely the same expression as (3.115):

$$Q = \sum_{l,m,n} Q_{lmn} = \sum_{l,m,n} \left[\psi \left(\Pi_\phi + \frac{i\mu}{\beta^2}\, \phi^\dagger \right) - \frac{ip}{\beta^2}\, \phi^\dagger \lambda \right], \tag{4.67}$$

where of course the canonical conjugate field is now (4.60). Once again, we stress that the supercharge Q has no ordering ambiguity and this is crucial for fixing the ordering constants appearing in the other operators.

Now that we have a clear picture of the one dimensional theory obtained by reducing the four dimensional one over S^3, we can carry out the actual computation of the supersymmetric Casimir energy. The modus operandi is once again analogous to that we used in chapter 3, however it is convenient to consider firstly the case in which the parameter n_0 is vanishing (so that the Hamiltonian commutes with the supercharges), and only afterwards the case $n_0 \neq 0$ which features a few more subtleties.

4.3.2 Case $n_0 = 0$

Let us begin by assuming that there is no n_0 i.e. that the Killing spinors have periodic boundary conditions under the transformation $\tau \to \tau + 2\pi$ in the original euclidean background. In this case the superalgebra is given by:

$$\left\{ Q,\, Q^\dagger \right\} = \frac{1}{\beta} \left(\underline{H} - \underline{\Sigma} - \underline{\mathcal{J}} \right). \tag{4.68}$$

An explicit computation tells us that the canonical (anti-)commutation relations imply that:

$$\left\{ Q_{lmn},\, Q^\dagger_{lmn} \right\} = \frac{1}{\beta} \left(H_{lmn} - \Sigma_{lmn} - \mathcal{J}_{lmn} + 2\mu - \alpha_1 + \alpha_2 + \alpha_3 \right). \tag{4.69}$$

It follows that (4.68) is valid if and only if the ordering constants satisfy:

$$\alpha_1 = \alpha_2 + \alpha_3 + 2\mu\,. \tag{4.70}$$

At this point we still have two constants to fix. We can repeat the reasoning we exposed in section 3.5.3 for both the operators $\underline{\Sigma}$ and $\underline{\mathcal{J}}$: since their vevs are the coefficients of a 1d Chern-Simons term, they should vanish on long multiplets because of the presence of fermions of opposite masses in the lagrangian. We anticipate that this conditions correspond again to taking the Weyl ordering of the operators. Thus the ordering constants are given by:

$$\alpha_1 = 2\left(\mu + \sigma + \rho \right), \tag{4.71}$$

$$\alpha_2 = 2\sigma\,, \tag{4.72}$$

$$\alpha_3 = 2\rho\,. \tag{4.73}$$

In passing, notice that these choices are consistent with the fact that $\underline{\Sigma}$ and $\underline{\mathcal{J}}$ are basically the same operator apart from a multiplicative factor, thus we should have expected their ordering constants to be equal up to the same factor i.e. $\alpha_2 = \frac{\sigma}{\rho}\alpha_3$.

Now we can start analysing the spectrum of the Hamiltonian. Instead of performing again the whole computation, we note that the Hamiltonian on the twisted background (4.64) is precisely equal to the Hamiltonian on the direct product background (3.112) provided that we make the substitution $\sigma \to \sigma + \rho$, where obviously the two σ are not the same but rather those corresponding to the respective backgrounds. As a consequence, much of the computations follow precisely what we did in section 3.5.4 and we do not repeat them here. Starting from the long multiplet, after introducing the creation and annihilation operators as in (3.124), we find that the Hamiltonian of the long multiplet is bounded from below if and only if:

$$\sqrt{\mu^2 + p^2} > |\mu + \sigma + \rho| \,. \tag{4.74}$$

Under this assumption, the vev of the scalar part is obtained when no oscillator is excited and it reads:

$$\langle H^{(bos)}_{long} \rangle = \sqrt{\mu^2 + p^2} - \mu - \sigma - \rho \,. \tag{4.75}$$

Eq. (4.74) also tells us that the vev of the fermionic part is $u_- \left|\Omega\right\rangle$ and its value is:

$$\langle H^{(fer)}_{long} \rangle = -\sqrt{\mu^2 + p^2} - \mu - \sigma - \rho \,. \tag{4.76}$$

Summing up $\langle H^{(bos)}_{long} \rangle$, $\langle H^{(fer)}_{long} \rangle$, and the ordering constant α_1, we get $\langle H_{long} \rangle = 0$ as expected. Following section 3.5.4, it is easy to check that $\langle \Sigma_{long} \rangle = \langle \mathcal{J}_{long} \rangle = 0$ too, hence we have the confirmation that Weyl ordering is indeed correct. Long multiplets still do not contribute to the Casimir energy.

In the same way, thanks to the considerations of section 3.5.4, we can directly tell that Chiral multiplets have a ground state with energy:

$$\langle H_{chiral} \rangle = \frac{|\sigma + \rho|}{2} \,, \tag{4.77}$$

and the ground state can be either $\left|\Omega\right\rangle$ or $\psi \left|\Omega\right\rangle$ depending on the sign of $\sigma + \rho$. Since chiral multiplets correspond to $m = \frac{l}{2}$, the expression (4.43) reads:

$$\mu = -\frac{\beta}{r}(l+1) < 0 \,, \tag{4.78}$$

where the inequality descends from the fact that the quantum number l is always positive. But then (4.74) for $p = 0$ implies that $0 < \sigma + \rho < -2\mu$, hence the ground state is $\left|\Omega\right\rangle$ and the absolute value can be safely removed from (4.77):

$$\langle H_{chiral} \rangle = \frac{\sigma + \rho}{2} \,. \tag{4.79}$$

For what concerns the vevs of the charges Σ and $\mathcal{J}$, they are easy to find starting from their expressions in terms of creation and annihilation operators. In particular Σ is given by (3.142) and $\mathcal{J}$ is a simple rescaling of the same operator by a factor $\frac{\rho}{\sigma}$. Eventually we obtain:

$$\langle \Sigma_{chiral} \rangle = \frac{\sigma}{2}\,, \qquad \langle \mathcal{J}_{chiral} \rangle = \frac{\rho}{2}\,. \tag{4.80}$$

Finally, for Fermi multiplets all the three operators H_{Fermi}, Σ_{Fermi}, and $\mathcal{J}_{Fermi}$ are proportional to $-\lambda\lambda^\dagger + \frac{1}{2}$, with constant of proportionality respectively $\sigma+\rho$, σ, and ρ. The only two states have energy $\pm\frac{1}{2}(\sigma+\rho)$ and we have to understand which one of the two is smaller. Knowing that Fermi multiplets are decoupled when $m = -\frac{l}{2} - 1$, from the expressions (4.45) we get:

$$\sigma + \rho = \frac{\beta}{r}(q_r - l - 2) + \frac{\sigma_1}{2}(q_r - l - 2 + 2n) + \frac{\sigma_2}{2}(q_r - l - 2 - 2n)\,. \tag{4.81}$$

The sign of this quantity depends on various factors and it is difficult to carry out a completely general discussion at this point. Hence, though what we did up to now is rather general, we will now make a further assumption in order to conclude the computations: we will assume that the two scales in our background i.e. β and r are very well separated. In particular, given that one of the main applications of this work is the microscopic counting of black holes entropy in AdS_5 through the holographic principle, the limit of big β is the one more relevant since it corresponds to low temperature black holes. Hence from now on we will assume $\beta \gg r$. As a consequence, if the twisting parameters are of order 1, which is reasonable, we have:

$$\sigma + \rho = \frac{\beta}{r}\left(q_r - l - 2 + \mathcal{O}\left(\frac{r}{\beta}\right)\right)\,. \tag{4.82}$$

It follows that the sign of $\sigma+\rho$ coincides with the sign of $q_r - l - 2$. But if $\beta \gg r$, the condition (4.74) is satisfied for $0 < q_r < 2$, precisely as in the direct product case; given that the quantum number l is always positive, this tells us that $\sigma + \rho < 0$, thus the ground state is $|\Omega\rangle$. Eventually, the vevs of the Fermi multiplets corresponds to:

$$\langle H_{Fermi} \rangle = \frac{\sigma+\rho}{2}\,, \qquad \langle \Sigma_{Fermi} \rangle = \frac{\sigma}{2}\,, \qquad \langle \mathcal{J}_{Fermi} \rangle = \frac{\rho}{2}\,. \tag{4.83}$$

Immediately we note that for all the three possible multiplets, we have:

$$\langle H_{lmn} \rangle - \langle \Sigma_{lmn} \rangle - \langle \mathcal{J}_{lmn} \rangle = 0\,, \tag{4.84}$$

which is perfectly consistent with the superalgebra (4.68) and provides a consistency check for what we are doing.

The final step consists in summing up all the non-vanishing contributions to the ground state energy of the theory, that are those arising from decoupled chiral

($m = \frac{l}{2}$) and Fermi ($m = -\frac{l}{2} - 1$) multiplets:

$$\begin{aligned}\langle H\rangle &= \sum_{chiral} \frac{1}{2}(\sigma+\rho) + \sum_{Fermi} \frac{1}{2}(\sigma+\rho) = \\ &= \sum_{l\geq 0}\sum_{n=-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{2}\left[\left(\frac{\beta}{r}+\frac{1}{2}(\sigma_1+\sigma_2)\right)(q_r+l)+n(\sigma_1-\sigma_2)\right]+ \\ &\quad +\sum_{l\geq 0}\sum_{n=-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{2}\left[\left(\frac{\beta}{r}+\frac{1}{2}(\sigma_1+\sigma_2)\right)(q_r-l-2)+n(\sigma_1-\sigma_2)\right].\end{aligned} \tag{4.85}$$

Before going on with the computations, it is convenient to introduce two new parameters:

$$\omega_1 \equiv \frac{\beta}{r}+\sigma_1\,, \qquad \omega_2 \equiv \frac{\beta}{r}+\sigma_2\,. \tag{4.86}$$

Aside from keeping the expressions more readable, in chapter 5 we will see that ω_1 and ω_2 are related to the complex structure parameters of the twisted $S^1 \times S^3$. In terms of (4.86), the sum that gives the Casimir energy reads:

$$\begin{aligned}\langle H\rangle = &\sum_{l\geq 0}\sum_{n=-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{2}\left[\frac{1}{2}(\omega_1+\omega_2)(q_r+l)+n(\omega_1-\omega_2)\right]+ \\ &+\sum_{l\geq 0}\sum_{n=-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{2}\left[\frac{1}{2}(\omega_1+\omega_2)(q_r-l-2)+n(\omega_1-\omega_2)\right].\end{aligned} \tag{4.87}$$

At first sight one may be tempted to perform immediately the sum over n, so that the term $n(\omega_1-\omega_2)$ drops out, however doing this would lead to a wrong result. In fact, the sums are again divergent and we have to regularise them in a supersymmetric fashion before performing any summation. Applying the regularisation method (3.152), we introduce a small parameter δ and we define:

$$\begin{aligned}\langle H\rangle_\delta = &\sum_{l\geq 0}\sum_{n=-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{2}\left[\frac{1}{2}(\omega_1+\omega_2)(l+q_r)+n(\omega_1-\omega_2)\right] e^{-\frac{\delta}{2}\left[\frac{1}{2}(\omega_1+\omega_2)(l+q_r)+n(\omega_1-\omega_2)\right]}+ \\ &-\sum_{l\geq 0}\sum_{n=-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{2}\left[\frac{1}{2}(\omega_1+\omega_2)(l+2-q_r)-n(\omega_1-\omega_2)\right] e^{-\frac{\delta}{2}\left[\frac{1}{2}(\omega_1+\omega_2)(l+2-q_r)-n(\omega_1-\omega_2)\right]}.\end{aligned} \tag{4.88}$$

We see that the terms linear in n do not cancel out anymore. Now one can compute explicitly the sums and expand everything in powers of δ:

$$\langle H\rangle_\delta = -\frac{2}{\delta^2}(q_r-1)\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)+E+\mathcal{O}(\delta)\,. \tag{4.89}$$

The finite part E is identified with the supersymmetric Casimir energy once we take the limit $\delta \to 0$ and we discard the diverging term, and it reads:

$$E = \frac{2}{3}(\omega_1 + \omega_2)(a - c) + \frac{2}{27}\frac{(\omega_1+\omega_2)^3}{\omega_1\omega_2}(3c - 2a)\,, \tag{4.90}$$

where obviously a and c are still the conformal anomaly coefficients (3.156). As a consistency check, we notice that the expression (4.90) reduces to (3.155) when the twisting parameters σ_1 and σ_2 vanish, so that we match the result of the untwisted case. Once again we stress that this result is valid for $\beta \gg r$ and $0 < q_r < 2$, and that the proof we gave is solid for $\sigma_1 = \sigma_2$; for $\sigma_1 \neq \sigma_2$ we believe that the result is the same but we have only given heuristic motivations, though rather convincing. We will comment further on this result in chapter 5; for now, let us move on and include in our scenario a non-vanishing n_0 too.

4.3.3 Case $n_0 \neq 0$

When $n_0 \neq 0$ the R-charge appears explicitly in the superalgebra (4.25), therefore the ordering constant of (4.63) needs to be fixed too. In general, for all the operators R_{lmn}, Σ_{lmn}, and $\mathcal{J}_{lmn}$, it is possible to carry out the reasoning highlighted in section 3.5.3: their vevs are essentially Chern-Simons coefficients, hence when the theory contains pairs of fermions with masses of opposite signs, they should vanish. This tells us that all long multiplets do not contribute to any of $\langle R\rangle$, $\langle\Sigma\rangle$, $\langle\mathcal{J}\rangle$. As before, imposing this property will fix the ordering ambiguity.

First of all, from the superalgebra (4.25) we can read the first constraint as before. The result is:

$$\alpha_1 = \frac{n_0}{2}\,\alpha_r + \alpha_2 + \alpha_3 + 2\mu\,. \tag{4.91}$$

Let us consider only the long multiplets now. In terms of creation and annihilation operators, the operators (4.64), (4.63), (4.65), and (4.66) are given by:

$$\begin{aligned} H_{lmn} = \sqrt{\mu^2+p^2}\left(a^\dagger a + b^\dagger b + 1\right) + \left(\mu + \frac{n_0}{2}\,q_r + \sigma + \rho\right)\left(a^\dagger a - b^\dagger b - 1\right) +\\ + x_+\,u_+u_+^\dagger + x_-\,u_-u_-^\dagger + \alpha_1\,, \end{aligned} \tag{4.92}$$

$$R_{lmn} = q_r\left(a^\dagger a - b^\dagger b - 1\right) - (q_r - 1)\left(u_+u_+^\dagger + u_-u_-^\dagger\right) + \alpha_r\,, \tag{4.93}$$

$$\Sigma_{lmn} = \sigma\left(a^\dagger a - b^\dagger b - 1 - u_+u_+^\dagger - u_-u_-^\dagger\right) + \alpha_2\,, \tag{4.94}$$

$$\mathcal{J}_{lmn} = \rho\left(a^\dagger a - b^\dagger b - 1 - u_+u_+^\dagger - u_-u_-^\dagger\right) + \alpha_3\,, \tag{4.95}$$

where u_- and u_+ are the two fermionic degrees of freedom in the diagonal basis and:

$$x_\pm = -\left(\mu + \frac{n_0}{2}\,(q_r - 1) + \sigma + \rho\right) \pm \sqrt{\mu^2+p^2}\,. \tag{4.96}$$

In order to have an Hamiltonian bounded from below, we must impose the inequality:

$$\sqrt{\mu^2+p^2} > \left|\mu + \frac{n_0}{2}\, q_r + \sigma + \rho\right| , \tag{4.97}$$

where again we recall that μ, p, σ, ρ were defined in (4.43), (4.44), (4.45). Under this hypothesis, we have the minimum possible energy when no bosonic state is excited. Yet, this condition does not fix automatically the fermionic ground state as it did before when n_0 was vanishing. This complication arises from the fact that the R-charge does not commute with the supercharge. The fermionic oscillators give rise to four possible states:

$$|\Omega\rangle \quad\longrightarrow\quad (H_{lmn}-\alpha_1)\;|\Omega\rangle = 0 \tag{4.98}$$

$$u_+\,|\Omega\rangle \quad\longrightarrow\quad (H_{lmn}-\alpha_1)\,u_+\,|\Omega\rangle = x_+\,u_+\,|\Omega\rangle \tag{4.99}$$

$$u_-\,|\Omega\rangle \quad\longrightarrow\quad (H_{lmn}-\alpha_1)\,u_-\,|\Omega\rangle = x_-\,u_-\,|\Omega\rangle \tag{4.100}$$

$$u_+u_-\,|\Omega\rangle \quad\longrightarrow\quad (H_{lmn}-\alpha_1)\,u_+u_-\,|\Omega\rangle = (x_-+x_+)\,u_+u_-\,|\Omega\rangle \tag{4.101}$$

Given the values of $x_\pm$ in (4.96), telling which state is the one with lowest energy is not immediate. We can distinguish three different cases based on the relation between the two terms appearing in (4.96):

A) The parameters at work satisfy the following two inequalities:

$$\begin{cases} \sqrt{\mu^2+p^2} < \left|\mu + \frac{n_0}{2}\,(q_r-1)+\sigma+\rho\right| \\ \mu + \frac{n_0}{2}\,(q_r-1)+\sigma+\rho < 0 \end{cases} . \tag{4.102}$$

Under these conditions, the lowest energy state is $|\Omega\rangle$. Therefore the vevs of the operators (4.93), (4.94), and (4.95) are:

$$\langle R_{lmn}\rangle = -q_r+\alpha_r\,, \qquad \langle \Sigma_{lmn}\rangle = -\sigma+\alpha_1\,, \qquad \langle \mathcal{J}_{lmn}\rangle = -\rho+\alpha_2\,. \tag{4.103}$$

Requiring that $\langle R_{lmn}\rangle = \langle \Sigma_{lmn}\rangle = \langle \mathcal{J}_{lmn}\rangle = 0$ we fix the values of the ordering constants:

$$\begin{cases} \alpha_r = q_r \\ \alpha_2 = \sigma \\ \alpha_3 = \rho \end{cases} \qquad\Longrightarrow\qquad \alpha_1 = \frac{n_0}{2}\,q_r + \sigma+\rho+2\mu\,. \tag{4.104}$$

Eventually, the ground state energy in this case would be:

$$\langle H_{lmn}\rangle = \sqrt{\mu^2+p^2}+\mu \neq 0 = \frac{n_0}{2}\langle R_{lmn}\rangle + \langle \Sigma_{lmn}\rangle + \langle \mathcal{J}_{lmn}\rangle\,. \tag{4.105}$$

This is inconsistent with our assumptions, since we used the hypothesis that the vacuum does not break supersymmetry to arrive to this result, in which case (4.26) should have held. Necessarily there is something wrong in the computations we did when the parameters entering the theory satisfy the conditions (4.102), and we cannot say anything more.

B) The parameters at work satisfy the following two inequalities:

$$\begin{cases} \sqrt{\mu^2+p^2} < \left|\mu + \frac{n_0}{2}(q_r-1)+\sigma+\rho\right| \\ \mu + \frac{n_0}{2}(q_r-1)+\sigma+\rho > 0 \end{cases} . \tag{4.106}$$

Under these conditions, the lowest energy state is $u_+u_-|\Omega\rangle$. Therefore the vevs of the operators (4.93), (4.94), and (4.95) are:

$$\langle R_{lmn}\rangle = -3q_r+2+\alpha_r\,, \qquad \langle\Sigma_{lmn}\rangle = -3\sigma+\alpha_1\,, \qquad \langle\mathcal{J}_{lmn}\rangle = -3\rho+\alpha_2\,. \tag{4.107}$$

Requiring that $\langle R_{lmn}\rangle = \langle\Sigma_{lmn}\rangle = \langle\mathcal{J}_{lmn}\rangle = 0$ we fix the values of the ordering constants:

$$\begin{cases} \alpha_r = 3q_r-2 \\ \alpha_2 = 3\sigma \\ \alpha_3 = 3\rho \end{cases} \qquad \Longrightarrow \qquad \alpha_1 = \frac{n_0}{2}(3q_r-2)+3\sigma+3\rho+2\mu\,. \tag{4.108}$$

Eventually, the ground state energy in this case would be:

$$\langle H_{lmn}\rangle = \sqrt{\mu^2+p^2}-\mu \neq 0 = \frac{n_0}{2}\langle R_{lmn}\rangle + \langle\Sigma_{lmn}\rangle + \langle\mathcal{J}_{lmn}\rangle\,. \tag{4.109}$$

Also this case is inconsistent with our assumptions and we cannot say anything when the parameters satisfy (4.106).

C) The parameters at work satisfy the following inequality:

$$\sqrt{\mu^2+p^2} > \left|\mu + \frac{n_0}{2}(q_r-1)+\sigma+\rho\right|. \tag{4.110}$$

Under these conditions, the lowest energy state is $u_-|\Omega\rangle$. Therefore the vevs of the operators (4.93), (4.94), and (4.95) are:

$$\langle R_{lmn}\rangle = -2q_r+1+\alpha_r\,, \qquad \langle\Sigma_{lmn}\rangle = -2\sigma+\alpha_1\,, \qquad \langle\mathcal{J}_{lmn}\rangle = -2\rho+\alpha_2\,. \tag{4.111}$$

Requiring that $\langle R_{lmn}\rangle = \langle\Sigma_{lmn}\rangle = \langle\mathcal{J}_{lmn}\rangle = 0$ we fix the values of the ordering constants:

$$\begin{cases} \alpha_r = 2q_r-1 \\ \alpha_2 = 2\sigma \\ \alpha_3 = 2\rho \end{cases} \qquad \Longrightarrow \qquad \alpha_1 = \frac{n_0}{2}(2q_r-1)+2\sigma+2\rho+2\mu\,. \tag{4.112}$$

This case corresponds to Weyl ordering the operators. Eventually, the ground state energy reads:

$$\langle H_{lmn}\rangle = 0 = \frac{n_0}{2}\langle R_{lmn}\rangle + \langle \Sigma_{lmn}\rangle + \langle \mathcal{J}_{lmn}\rangle \,. \tag{4.113}$$

This case is the only one consistent with our assumptions.

So let us restrict to consider case C, which corresponds to parameters satisfying the condition:

$$\sqrt{\mu^2+p^2} > \max\left\{\left|\mu + \frac{n_0}{2}\left(q_r-1\right)+\sigma+\rho\right|, \left|\mu+\frac{n_0}{2}\,q_r+\sigma+\rho\right|\right\}. \tag{4.114}$$

This inequality is even more complicated than the one we had when $n_0 = 0$, thus we will again assume to have $\beta \gg r$; under this hypotesis (4.114) reduces simply to:

$$l+1 > \left|q_r - 1 + \mathcal{O}\left(\frac{r}{\beta}\right)\right|, \tag{4.115}$$

which is satisfied when $0 < q_r < 2$ as before, and we do not have any further condition on the twisting parameters and n_0 since their contribution is suppressed by a factor $\frac{r}{\beta}$.

As showed above, long multiplets do not contribute to the Casimir energy, thus we have to consider only decoupled chiral and Fermi multiplets. Everything goes much like for the case $n_0 = 0$, hence we skip the details of the computations for Chiral and Fermi multiplets. The results are:

$$\langle H_{chiral}\rangle = \langle H_{Fermi}\rangle = \frac{1}{2}\left(\sigma+\rho+\frac{n_0}{2}\left(q_r-1\right)\right)\,, \tag{4.116}$$

$$\langle R_{chiral}\rangle = \langle R_{Fermi}\rangle = \frac{1}{2}\left(q_r-1\right)\,, \tag{4.117}$$

$$\langle \Sigma_{chiral}\rangle = \langle \Sigma_{Fermi}\rangle = \frac{\sigma}{2}\,, \tag{4.118}$$

$$\langle \mathcal{J}_{chiral}\rangle = \langle \mathcal{J}_{Fermi}\rangle = \frac{\rho}{2}\,. \tag{4.119}$$

Remarkably, for every supermultiplets the vevs satisfy the condition:

$$\langle H_{lmn}\rangle = \frac{n_0}{2}\langle R_{lmn}\rangle + \langle \Sigma_{lmn}\rangle + \langle \mathcal{J}_{lmn}\rangle\,, \tag{4.120}$$

which is consistent with (4.26) if we assume the vacuum does not break supersymmetry. The Casimir energy is then given by summing up all the contributions from the decoupled chiral and Fermi multiplets. As before, the sum is divergent and we regularise it using the method (3.152). We do not repeat all the steps, we refer the reader to section 4.3.2 for more details. The final results we obtain is:

$$E = -\frac{\left(\omega_1+\omega_2+n_0\right)}{3\omega_1\omega_2}\left(\omega_1^2+\omega_2^2\right)\left(a-c\right) + \frac{1}{27}\frac{\left(\omega_1+\omega_2+n_0\right)^3}{\omega_1\omega_2}\left(5a-3c\right), \tag{4.121}$$

where the parameters ω_1 and ω_2 are given by (4.86) and a and c are the conformal anomaly coefficients (3.156). It is easy to check that (4.121) reduces to (4.90) when n_0 vanishes. Recall that this result is only valid under the hypoteses $\beta \gg r$ and $0 < q_r < 2$. In the next chapter we will discuss the physical implications of this result.

CHAPTER 5

Physical interpretation and final comments

In this last chapter we will comment the original results concerning the supersymmetric Casimir energy we obtained in chapter 4 and see how they relate with the general picture of SQFT in curved backgrounds as well as with other literature results. We will also highlight what are the directions which may be interesting for further studies. For convenience, we recall here the most general expression for the Casimir energy we obtained (4.121):

$$E = -\frac{(\omega_1 + \omega_2 + n_0)}{3\omega_1\omega_2}\left(\omega_1^2 + \omega_2^2\right)(a - c) + \frac{1}{27}\,\frac{(\omega_1 + \omega_2 + n_0)^3}{\omega_1\omega_2}\,(5a - 3c)\,, \tag{5.1}$$

where the parameters ω_1 and ω_2 are (in euclidean signature):

$$\omega_1 = \frac{\beta}{r} + i\sigma_1\,, \qquad \omega_2 = \frac{\beta}{r} + i\sigma_2\,. \tag{5.2}$$

When $n_0 = 0$ the expression (5.1) reduces to (4.90), that is:

$$E = \frac{2}{3}(\omega_1 + \omega_2)(a - c) + \frac{2}{27}\,\frac{(\omega_1 + \omega_2)^3}{\omega_1\omega_2}\,(3c - 2a)\,. \tag{5.3}$$

We recall also that to derive these expressions we assumed $\beta \gg r$ which corresponds to a 3-sphere radius much smaller than that of the circle.

5.1 Hopf surfaces and complex structure parameters

5.1.1 $S^1 \times S^3$ as a primary Hopf surface

In section 2.3 we enunciated a general theorem that applies to field theories such as the chiral multiplet on $S^1 \times S^3$ we considered in the present work, hence it would be interesting to study how our results fit in this context. In order to do this, we firstly need to better understand the complex structure of the manifold we are working on.

Consider the set $\mathbb{C}^2 \setminus (0,\, 0)$ with coordinates $(z_1,\, z_2)$, and consider the infinite cyclic group Γ generated by the following automorphism on $\mathbb{C}^2 \setminus (0,\, 0)$:

$$(z_1,\, z_2) \quad \longrightarrow \quad (z_1',\, z_2') = (p z_1 + \lambda z_2^m,\, q z_2)\,, \tag{5.4}$$

where $m \in \mathbb{N}$ and p, q, and λ are complex parameters such that $0 < |p| \leq |q| < 1$ and $(p-q)^m \lambda = 0$. A primary Hopf surface is defined as the manifold $\mathcal{M}_{p,q,\lambda}$ obtained by taking the quotient:

$$\mathcal{M}_{p,q,\lambda} = \frac{\mathbb{C}^2 \setminus (0,\, 0)}{\Gamma}\,. \tag{5.5}$$

Obviously $\mathcal{M}_{p,q,\lambda}$ has two complex dimensions, thus four real dimensions which is the same dimensionality of $S^1 \times S^3$. In fact it has been shown that all primary Hopf surfaces are diffeomorphic to $S^1 \times S^3$ [25].

We focus our interest to the case where $\lambda = 0$ so that we can label a primary Hopf surface purely through $p, q \in \mathbb{C}$. These two quantities correspond to the complex structure parameters of an Hopf surface. However, it is convenient to trade p and q for two other parameters $\omega_1,\, \omega_2 \in \mathbb{C}$:

$$p = e^{-2\pi\omega_2}\,, \qquad\qquad q = e^{-2\pi\omega_1}\,. \tag{5.6}$$

It will be clear why we named them ω_1 and ω_2 later on. Their real and imaginary parts encode the information about how $S^1 \times S^3$ is deformed, and with a slight abuse of terminology we will refer to them as complex structure parameters too. For now, we will assume that $|p| = |q|$, which translates into the condition:

$$\mathfrak{Re}(\omega_1) = \mathfrak{Re}(\omega_2)\,. \tag{5.7}$$

Now, let us introduce the new coordinates $(w,\, z)$ on $\mathcal{M}_{p,q}$, defined as:

$$\begin{cases} w = -i \log \bar{z}_1 \\ z = \frac{\bar{z}_2}{\bar{z}_1} \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} z_1 = e^{-i\bar{w}} \\ z_2 = \bar{z}\, e^{-i\bar{w}} \end{cases}\,. \tag{5.8}$$

These new coordinates are well-defined everywhere but on the locus $z_1 = 0$. However, we can introduce another atlas with coordinates $(w',\, z')$, where $w' = w + i \log z$ and $z' = z^{-1}$, to cover the full manifold. In the definition of $\mathcal{M}_{p,q}$ (5.5), quotienting by the cyclic group Γ amounts to introduce the identifications:

$$(z_1,\, z_2) \sim (z_1\, e^{-2\pi\omega_2},\, z_2\, e^{-2\pi\omega_1})\,. \tag{5.9}$$

In terms of the new coordinates (5.8), these identifications translates into:

$$(w,\, z) \sim \left(w + 2\pi i\, \bar{\omega}_2,\, z\, e^{2\pi(\bar{\omega}_2 - \bar{\omega}_1)}\right)\,, \tag{5.10}$$

and clearly from (5.8) it also follows that $w \sim w + 2\pi$ with z fixed. We can recognise that these identifications are of the same form as (4.5). In particular, if we take ω_1 and ω_2 to be (5.2), the identifications (4.5) and (5.10) coincide precisely. This shows that the coordinates $(w,\, z)$ on an Hopf surface introduced in (5.8) are a system of complex holomorphic coordinates on $S^1 \times S^3$, and turning on the imaginary part of the complex structure parameters of the Hopf surface ω_1 and ω_2 correspond to twisting the 3-sphere over the circle. For completion, we add that the specific metric (4.1) corresponds to the following hermitian metric on $\mathcal{M}_{p,q}$:

$$ds^2_{\mathcal{M}_{p,q}} = \frac{r}{|z_1|^2 + |z_1|^2}\left(dz_1\, d\bar{z}_1 + dz_1\, d\bar{z}_1\right). \tag{5.11}$$

As we showed in section 3.2, the supersymmetric Casimir energy is directly related to the partition function of the theory as:

$$E = -\lim_{\beta\to\infty} \frac{d}{d\beta} \log Z_{S^1\times S^3}\,. \tag{5.12}$$

Therefore, the general statements concerning the dependence of the partition function on the background geometry which we discussed in section 2.3 should be valid for the Casimir energy too. In view of what we just illustrated concerning primary Hopf surfaces, our result (5.3) and the more general (5.1) confirm this. Indeed, the expression we found for the supersymmetric Casimir energy of a chiral multiplet theory on $S^1 \times S^3$ is an holomorphic function of the parameters ω_1 and ω_2, which we have just shown to be the complex structure parameters of an Hopf surface. There is no dependence on other geometric details of the background manifold nor on the complex conjugates of ω_1 and ω_2. The only other parameter entering (5.1) is n_0.

5.1.2 Squashing the 3-sphere

The most general $S^1 \times S^3$ background should allow also for real parts of ω_1 and ω_2 that are different one from each other, thus relaxing the assumption (5.7). It is possible to show that this further deformation can be realised as a squashing of S^3 [5, 3] i.e. to a metric of the form:

$$\begin{aligned} ds^2 = \beta^2\, d\tau^2 + &\sqrt{\chi_1^2 \cos^2\theta + \chi_2^2 \sin^2\theta}\, d\theta^2 + \\ &+ \chi_1^2 \sin^2\theta\, (d\varphi_1 + \sigma_1\, d\tau)^2 + \chi_2^2 \cos^2\theta\, (d\varphi_2 + \sigma_2\, d\tau)^2\ , \end{aligned} \tag{5.13}$$

where $\chi_1,\, \chi_2 \in \mathbb{R}$ are the squashing parameters and are related precisely to the real parts of the complex structure parameters:

$$\omega_1 = \frac{\beta}{\chi_1} + i\sigma_1\,, \qquad \omega_2 = \frac{\beta}{\chi_2} + i\sigma_2\,. \tag{5.14}$$

Correspondingly, the Killing vector (4.7) gets generalised to:

$$K = \frac{1}{2}\big(-i\partial_\tau + \omega_1\, \partial_{\varphi_1} + \omega_2\, \partial_{\varphi_2}\big)\,. \tag{5.15}$$

In principle one can try to apply again the same reasoning we exposed in the previous chapter, however the new squashing parameters introduce increasing technical difficulties, hence we leave a thorough analysis of this case for future work. Let us just say that one could follow a more effective approach in this case, following section 3 of [6] where essentially it is considered a metric even more general than (5.13), though without twisting parameters. The main trick comes from [26] and consists in trading the fermionic degrees of freedom in the 4d chiral multiplet for some new scalars built as bilinears:

$$\begin{aligned} B &= \frac{1}{\sqrt{2}}\,\frac{\zeta^\dagger\psi}{|\zeta|^2}\,, &\qquad C &= \sqrt{2}\,\zeta\psi\,,\\ \tilde{B} &= \frac{1}{\sqrt{2}}\,\frac{\tilde{\zeta}^\dagger\tilde{\psi}}{|\tilde{\zeta}|^2}\,, &\qquad \tilde{C} &= \sqrt{2}\,\tilde{\zeta}\tilde{\psi}\,. \end{aligned} \tag{5.16}$$

These are called “twisted variables” (for examples of their usage in a context similar to ours see [4, 22]). Without entering into the details, one can show without performing the actual dimensional reduction that the 1d chiral multiplets arise from the couple (ϕ, C) and the Fermi multiplets from (B, F). Moreover, using this redefinition of the degrees of freedom, the shortening conditions can be read directly from the 4d supersymmetry transformations. Then, knowing that long multiplets do not contribute to the Casimir energy, one can focus on the shortened multiplets right from the beginning.

Eventually, given that (5.1) does not have a separate dependence on the real and imaginary parts of ω_1 and ω_2, the result we should expect to find is still the same but with the most general complex structure parameters (5.14). Yet it would be nice to have a concrete confirmation of this reasonable claim.

5.1.3 The full partition function

The Casimir energy we considered so far constitutes only a piece of the full partition function of the theory. Given the discussion we illustrated in section 2.3, of course the latter has to be an holomorphic function of the complex structure parameters p and q. The usual technique employed to compute $\mathcal{Z}_{S^1\times S^3}$ is the supersymmetric localisation, which once again relies on the constrained setup proper of supersymmetric theories. It turns out that the partition function has the form:

$$\mathcal{Z}_{S^1\times S^3} = e^{-\beta E}\,\mathcal{I}(p,q)\,, \tag{5.17}$$

where E is indeed the supersymmetric Casimir energy and $\mathcal{I}(p,q)$ is the supersymmetric index on $S^1\times S^3$ [18, 19]. The supersymmetric index is essentially a trace over the Hilbert space of states on S^3 and the complex structure parameters p and q take the role of fugacitites from the index perspective (see e.g. [24]).

The explicit computation of (5.17) for completely general values of the complex structure parameters p and q has not been done yet, however we have the results for a number of specific theories living in backgrounds with various degrees of deformation. It would be interesting to try to reproduce the entire partition function, including the supersymmetric index, through the 1d theory obtained by dimensional reduction on S^3, thus making contact with the localisation result.

5.2 Comparison with existing literature

The results we obtained partially fill in the general picture of supersymmetric theories on $S^1 \times S^3$, which received a remarkable attention from the high energy physics community in recent years.

The expression (5.3) fits in this context as an independent check of an already known result. Indeed our expression, when written in terms of generic ω_1 and ω_2, is in agreement with the one that has been found through localisation in [5]. Differently from the present work, they considered an $\mathcal{N} = 1$ theory including also the vector multiplet and with a background metric that accounts for the squashing of the 3-sphere but not for its twisting. Nevertheless, one can make the comparison between the two Casimir energy expressions by writing them in terms of the Hopf surface complex structure parameters and assuming that they are valid for every possible value of p and q.

Somehow more interesting is the expression (5.1) we obtained for $n_0 \neq 0$. We recall that n_0 is an integer parametrising the periodicity condition of the Killing spinors when we go once aroung the thermal circle i.e. $\tau \to \tau + 2\pi$. Let us start by noticing that (5.1) can also be restated as:

$$E = \frac{1}{3}\frac{(\omega_1 + \omega_2 + n_0)}{\omega_1\omega_2}\left[2\omega_1\omega_2 + 2n_0\left(\omega_1 + \omega_2\right) + n_0^2\right](a - c) + \\ + \frac{2}{27}\frac{(\omega_1 + \omega_2 + n_0)^3}{\omega_1\omega_2}(3c - 2a)\,. \tag{5.18}$$

A background completely analogous to the one we considered, including the presence of the parameter n_0, was considered in [15]. Once again, they used the localisation technique to compute the complete partition function of a theory including both a chiral and a vector multiplet. Their result for the Casimir energy is:

$$E_{[15]} = \frac{1}{3}\frac{(\omega_1 + \omega_2 + n_0)}{\omega_1\omega_2}\left[2\omega_1\omega_2 + 4n_0\left(\omega_1 + \omega_2\right) + 16n_0^2\right](a - c) + \\ + \frac{2}{27}\frac{(\omega_1 + \omega_2 + n_0)^3}{\omega_1\omega_2}(3c - 2a)\,. \tag{5.19}$$

As the reader may notice, (5.18) and (5.19) are similar but not identical. We believe that the difference is due to the regularisation procedure used in [15], where the authors themselves argue that it is not clear which regularisation scheme they should use and how. On the other hand, our regularisation procedure does not break supersymmetry and we are quite confident that it is correct.

Another result one can compare (5.1) with is contained in [16, 17]. There, they studied the asymptotic behaviour of $\mathcal{I}(p,q)$ in the limit $\beta \ll r$ (opposite to our assumption) and $\sigma_1, \sigma_2 \ll 1$ and they found a result that is closely related to the Casimir energy. Its expression is:

$$E_{[16,\,17]} = \frac{4\pi^2}{3}\frac{(\omega_1+\omega_2+n_0)}{\omega_1\omega_2}(a-c) + \frac{1}{27}\frac{(\omega_1+\omega_2+n_0)^3}{\omega_1\omega_2}(5a-3c)\,. \tag{5.20}$$

In this case, it is not surprising that (5.1) and (5.20) are different, given that they are valid under different assumptions, however it is remarkable that the term proportional to $5a-3c$ is precisely the same.

Noteworthy, whenever $a=c$ the three independently derived results (5.1), (5.19), and (5.20) coincide. Hence, we can be reasonably confident that at least the piece of the Casimir energy proportional to $5a-3c$ is valid in general for any background diffeomorphic to $S^1 \times S^3$. In passing, notice that the case $a=c$ is the one relevant for the application of the AdS/CFT correspondence (more about this in the next section). For what concerns the term proportional to $a-c$, the situation is still not clear since the results do not match precisely. It is tempting to speculate that the more general form of the Casimir energy has to interpolate between (5.1) and (5.20) given that they are valid in opposite regimes, yet this would contrast with the expression (5.19). Further studies are needed to clarify completely this question, which is intimately related to the question of what is a supersymmetry-preserving regularisation scheme.

5.3 Further developments

As we said, the context of supersymmetric quantum field theories on curved spaces is experiencing quite a lot of attention. The present work constitutes a little progress in the understanding of the implications of a background diffeomorphic to $S^1 \times S^3$. However, there are still several open questions left, some of them we already mentioned. In this final section we try to put some order among ideas.

The most direct improvements of our computations would be to extend the results outside the regime $\beta \gg r$ and to consider the most general Hopf surface as a background, including also the squashing of the 3-sphere. On the latter we already commented in section 5.1.2 and there is not much else to say; we have a clear expectation for which expression for the Casimir energy one should end up with, and it would probably constitute just a completion of what we already know.

The former development instead would be more interesting, especially in the case $n_0 \neq 0$, since we still do not have an expression for E we can trust completely when this parameter is non-vanishing. Moreover, a more thorough analysis of the inequalities arising from requiring the one dimensional hamiltonian to be bounded from below may put physical constraints on the values of the parameters σ_1 and σ_2 allowed for a consistent result.

The other main line for further developments is to consider more articulated field theories. Remaining in the context of $\mathcal{N} = 1$ SQFT, it would be interesting to add also the vector multiplet to the game, and therefore gauge interactions. In principle the procedure should not be that different from the one we exposed in this thesis; the additional vector field that appears in 4d should be expanded in vector harmonics on S^3 [11] and then their properties should come in help in order to reduce the theory to 1d. One should end up with a one dimensional supersymmetric gauge theory extending the one we found here. We expect the resulting Casimir energy to be the one obtained in this thesis, with a and c being the conformal anomaly coefficients of the full theory including the vector multiplet too.

Subsequently, one may wish to add even further structure and perhaps consider the $\mathcal{N} = 4$ SuperYang-Mills theory. The theory $\mathcal{N} = 4$ SYM on $S^1 \times S^3$ is particularly interesting from the holographic perspective since its holographic dual is a known supergravity theory in AdS_5. The AdS/CFT dictionary states that the logarithm of the CFT partition function on $S^1 \times S^3$ at large N coincides with the renormalised on-shell action of the dual 5d supegravity theory. The former quantity is precisely the Casimir energy, therefore extending our findings to this field theory would result in a way to obtain the on-shell supergravity action.

Here we can make a last comment about our result (5.1): the fact that we are pretty confident about the expression of the Casimir energy when the two Weyl anomaly coefficients a and c are equal is relevant in the holography context since at large N we have $a = c$ indeed [20]. Therefore, if we would attempt to interpret our result from the dual supergravity perspective, in principle we will not have troubles caused by the fact that we are not sure about that part of E which is proportional to $a - c$. For $\sigma_1 = \sigma_2 = n_0 = 0$ this has been done in [14]. It would be interesting to reproduce our results for non-vanishing σ_1, σ_2 and n_0 from such a holographic perspective.

APPENDIX A

Conventions and definitions

A.1 Spacetime and geometric objects

For the most part of the present work, we will consider a four dimensional spacetime $\mathcal{M}$ with a fixed Riemannian metric $g_{\mu\nu}$ with euclidean signature $(+, +, +, +)$. Euclidean spacetime indices ranges from 1 to 4. The determinant of the metric will be denoted simply by g. Starting from the metric, we can build the Christoffel symbols:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\, g^{\mu\sigma} \left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right) , \tag{A.1}$$

and use them to define the Levi-Civita connection ∇_μ on $\mathcal{M}$ by specifying its action on a generic tensor:

$$\nabla_\sigma T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} = \partial_\sigma T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} + \Gamma^{\mu_1}_{\sigma\rho}\, T^{\rho \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} + \ldots - \Gamma^{\rho}_{\sigma\nu_1}\, T^{\mu_1 \ldots \mu_p}{}_{\rho \ldots \nu_q} - \ldots \tag{A.2}$$

Recall that the covariant four-divergence of a vector V^μ satisfies the following identity:

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\, \partial_\mu \left(\sqrt{g}\, V^\mu\right) . \tag{A.3}$$

Of course starting from the Christoffel symbols we can build the Riemann tensor:

$$\mathcal{R}_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\tau}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\rho}_{\nu\tau}\Gamma^{\tau}_{\mu\sigma} , \tag{A.4}$$

and then the Ricci tensor $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\rho\mu}{}^{\rho}{}_{\nu}$ and the Ricci scalar curvature $\mathcal{R} = g^{\mu\nu}\, \mathcal{R}_{\mu\nu}$. Another relevant quantity that measures the curvature of the manifold is the Weyl tensor, which is defined as:

$$W_{\mu\nu\rho\sigma} = \mathcal{R}_{\mu\nu\rho\sigma} - \frac{1}{2}\left(\mathcal{R}_{\mu\rho}\, g_{\nu\sigma} + \mathcal{R}_{\nu\sigma}\, g_{\mu\rho} - \mathcal{R}_{\mu\sigma}\, g_{\nu\rho} - \mathcal{R}_{\nu\rho}\, g_{\mu\sigma}\right) + \frac{\mathcal{R}}{6}\left(g_{\mu\rho}\, g_{\nu\sigma} - g_{\nu\rho}\, g_{\mu\sigma}\right). \tag{A.5}$$

We introduce now the Levi-Civita symbol $\varepsilon_{\mu\nu\rho\sigma}$ which is defined to be totally anti-symmetric with $\varepsilon_{1234} = 1$. $\varepsilon_{\mu\nu\rho\sigma}$ does not transform as a tensor, but we can define the Levi-Civita tensor as:

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{g}\, \varepsilon_{\mu\nu\rho\sigma} \,, \tag{A.6}$$

and it transforms correctly as a tensor. With this new quantity, we can define another relevant scalar related to the metric, the Euler density:

$$E_{(4)} = \frac{1}{4}\, \epsilon^{\mu_1\nu_1\mu_2\nu_2}\, \epsilon^{\rho_1\sigma_1\rho_2\sigma_2}\, \mathcal{R}_{\mu_1\nu_1\rho_1\sigma_1}\, \mathcal{R}_{\mu_2\nu_2\rho_2\sigma_2} \,. \tag{A.7}$$

We introduce also a flat local frame through the vielbein 1-forms $\{e^1,\, e^2,\, e^3,\, e^4\}$, where $e^a = e^a{}_\mu\, \mathrm{d}x^\mu$. Of course they satisfy:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \delta_{ab}\, e^a\, e^b \,. \tag{A.8}$$

It directly follows that $\det\left(e^a{}_\mu\right) = \sqrt{g}$. The volume form on the manifold $\mathcal{M}$ is defined through the vielbeins as:

$$\mathrm{vol} = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \,. \tag{A.9}$$

Most of the times we will indicate the volume form with a naive $d^4x\,\sqrt{g}$ in the main text. Once we have specified a local frame, we can introduce the spin connection:

$$\begin{aligned}\omega_{\mu ab} = \frac{1}{2}\big(&e_a{}^\nu\, \partial_\mu e_{b\nu} - e_b{}^\nu\, \partial_\mu e_{a\nu} - e_a{}^\nu\, \partial_\nu e_{b\mu} + e_b{}^\nu\, \partial_\nu e_{a\mu} + \\ &- e_a{}^\nu\, e_b{}^\rho\, e_{c\mu}\, \partial_\nu e^c{}_\rho + e_b{}^\nu\, e_a{}^\rho\, e_{c\mu}\, \partial_\nu e^c{}_\rho\big).\end{aligned} \tag{A.10}$$

By means of the spin connection, the Levi-Civita connection is extended to objects with flat indices as follows:

$$\nabla_\mu T^{a_1\ldots a_p}{}_{b_1\ldots b_q} = \partial_\mu T^{a_1\ldots a_p}{}_{b_1\ldots b_q} + \omega_\mu{}^{a_1}{}_c\, T^{c\ldots a_p}{}_{b_1\ldots b_q} + \ldots - \omega_\mu{}^c{}_{b_1}\, T^{a_1\ldots a_p}{}_{c\ldots b_q} - \ldots \tag{A.11}$$

Such a connection is compatible with both the metric and the vielbeins i.e.:

$$\nabla_\mu g_{\rho\sigma} = \nabla_\mu e^a{}_\nu = 0 \,. \tag{A.12}$$

A.2 Spinors

In the local flat frame we can introduce spinors. The symmetry group of the local flat space is $SO(4)$ whose universal covering group is $Spin(4) \simeq SU(2)_+ \times SU(2)_-$.

- Left-handed Weyl spinors ψ_α are two components spinors that carry an un-dotted index and transform in the $(\frac{1}{2},\, 0)$ representation of $Spin(4)$.

- Right-handed Weyl spinors $\tilde{\psi}^{\dot{\alpha}}$ are two components spinors that carry a dotted index and transform in the $(0,\, \frac{1}{2})$ representation of $Spin(4)$.

The conventions we will adopt for spinors in euclidean signature are basically the same as [5]. We introduce the 2×2 sigma matrices:

$$\sigma^a_{\alpha\dot{\alpha}} = (\vec{\sigma},\, -i\mathbb{I}_2)\,, \qquad \tilde{\sigma}^{a\dot{\alpha}\alpha} = (-\vec{\sigma},\, -i\mathbb{I}_2)\,, \tag{A.13}$$

where $\vec{\sigma} = (\sigma^1,\, \sigma^2,\, \sigma^3)$ is the vector of the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.14}$$

The sigma matrices generate the Clifford algebra and indeed they have the following properties:

$$\sigma_a\, \tilde{\sigma}_b + \sigma_b\, \tilde{\sigma}_a = -2\delta_{ab}\,, \qquad \tilde{\sigma}_a\, \sigma_b + \tilde{\sigma}_b\, \sigma_a = -2\delta_{ab}\,. \tag{A.15}$$

Starting from the sigma matrices, we introduce also two-indices sigma matrices, which are the generators of $SU(2)_+$ and $SU(2)_-$ respectively:

$$\sigma_{ab} = \frac{1}{4}\left(\sigma_a\, \tilde{\sigma}_b - \sigma_b\, \tilde{\sigma}_a\right), \qquad \tilde{\sigma}_{ab} = \frac{1}{4}\left(\tilde{\sigma}_a\, \sigma_b - \tilde{\sigma}_b\, \sigma_a\right). \tag{A.16}$$

Spinorial indices are raised and lowered by acting from the left with the antisymmetric two-indices symbols $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}}$, fixed by requiring $\epsilon^{12} = 1$. The convention for contracting indices is "NorthWest to SouthEast" for undotted indices and "SouthWest to NorthEast" for dotted indices i.e.:

$$\psi\chi = \psi^\alpha \chi_\alpha\,, \qquad \tilde{\psi}\tilde{\chi} = \tilde{\psi}_{\dot{\alpha}} \tilde{\chi}^{\dot{\alpha}}\,. \tag{A.17}$$

With these conventions, if both spinors are anti-commuting then they satisfy $\psi\chi = \chi\psi$ and $\tilde{\psi}\tilde{\chi} = \tilde{\chi}\tilde{\psi}$, while if one of the two is a commuting spinor, the two relations pick a minus sign. For other identities involving spinor bilinears we refer to [5].

The extension of the Levi-Civita connection on spinors is given by:

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{2}\, \omega_{\mu ab}\, \sigma^{ab} \psi\,, \qquad \nabla_\mu \tilde{\psi} = \partial_\mu \tilde{\psi} - \frac{1}{2}\, \omega_{\mu ab}\, \tilde{\sigma}^{ab} \tilde{\psi}\,. \tag{A.18}$$

Another differential operator that we will use is the spinorial Lie derivative along a vector, which is defined as follows:

$$L_X \psi = X^\mu\, \nabla_\mu \psi - \frac{1}{4}\, \nabla_\mu X_\nu\, \sigma^\mu \tilde{\sigma}^\nu \psi\,, \tag{A.19}$$

$$L_X \tilde{\psi} = X^\mu\, \nabla_\mu \tilde{\psi} - \frac{1}{4}\, \nabla_\mu X_\nu\, \tilde{\sigma}^\mu \sigma^\nu \tilde{\psi}\,, \tag{A.20}$$

where the sigma matrices with spacetime indices are simply $\sigma^\mu = e_a{}^\mu\, \sigma^a$.

Finally, though they will not be used much, we report the definitions of the supercovariant derivatives in superspace:

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}} \tilde{\theta}^{\dot{\alpha}}\, \partial_\mu\,, \qquad \tilde{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \tilde{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\, \partial_\mu\,, \tag{A.21}$$

where θ_α and $\tilde{\theta}^{\dot{\alpha}}$ are the superspace coordinates.

A.3 From euclidean to lorentzian

It will be useful to know how to pass from euclidean to lorentzian signature and viceversa, thus it is better to collect our conventions regarding this transformation. For the lorentzian metric, we use the signature mostly plus i.e. $(-, +, +, +)$. The Wick rotation amounts to analytically continue the lorentzian time t to the complex plane and identify the euclidean time τ as the imaginary axis. This leads to the identification $t = -i\tau$ and, more in general, to the transformation laws of covariant and contravariant indices. Denoting with a subscript L and E respectively the objects in lorentzian signature with those in euclidean signature, we have:

$$v_L^t = -i v_E^\tau \,, \qquad (v_L)_t = i(v_L)_\tau \,. \tag{A.22}$$

All the other components of vectors and tensors remain unchanged. In the lorentzian the local frame is identified by the vielbeins $\{e^0, e^1, e^2, e^3\}$, where $e^0 = ie^4$. We define the lorentzian volume form to be:

$$\mathrm{vol} = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \,. \tag{A.23}$$

Then, the relation between the lorentzian and euclidean volume forms is:

$$\mathrm{vol}_E = e^1 \wedge e^2 \wedge e^3 \wedge e^4 = i\, e^0 \wedge e^1 \wedge e^2 \wedge e^3 = i\, \mathrm{vol}_L \,, \tag{A.24}$$

which is consistent with the naive identification $d^4x_L = -i\, d^4x_E$. Moreover, given a lagrangian QFT, conventionally we set $iS_L = -S_E$. This, together with (A.24), tells us that the lagrangians are related as $\mathcal{L}_L = -\mathcal{L}_E$.

Our choice of sigma matrices in lorentzian signature is to take $\sigma^0 = i\sigma^4 = \mathbb{I}_2$ and also $\bar\sigma^0 = i\tilde\sigma_4 = \mathbb{I}_2$. Matrices σ^i are the same as in euclidean signature, while $\bar\sigma^i = \tilde\sigma^i$. This choice is consistent with the fact that we want σ^μ and $\tilde\sigma^\mu$ to be vectors. In fact, we have:

$$\sigma_L^t = e_a{}^t\, \sigma_L^a = e_0{}^t\, \sigma_L^0 + e_i{}^t\, \sigma_L^i = e_4{}^t\, \sigma_E^4 + e_i{}^t\, \sigma_E^i = -ie_a{}^\tau\, \sigma_E^a = -i\sigma_E^\tau \,, \tag{A.25}$$

which is precisely what we want according to (A.22). Similarly one can verify that $\bar\sigma^\mu$ transform in the same way and everything is consistent. Spinors transforming in the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are related by hermitian conjugation in lorentzian signature:

$$(\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha} \,, \qquad (\bar\psi^{\dot\alpha})^\dagger = \psi^\alpha \,. \tag{A.26}$$

An important property of sigma matrices in lorentzian signature is that they are hermitian i.e. $(\sigma^\mu)^\dagger = \sigma^\mu$ and $(\bar\sigma)^\dagger = \bar\sigma^\mu$. Finally, sigma matrices satisfy the identity:

$$\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\, \epsilon^{\alpha\beta}\, \sigma^\mu_{\beta\dot\beta} \,. \tag{A.27}$$

A.4 Currents in euclidean signature

In the following we provide the definition of a conserved current in euclidean signature such that it is consistent with the standard definition in Lorentzian signature given the chosen conventions on the Wick rotation.

Let us begin from a general current in lorentzian signature:

$$J^\mu = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta A_\mu} \,. \tag{A.28}$$

This means that under a variation corresponding to the associated symmetry, the lorentzian action varies as:

$$\delta S = \int d^4x \sqrt{-g}\, J^\mu \, \delta A_\mu \,. \tag{A.29}$$

Denoting euclidean quantities with a bar (limited to this section), with our conventions (A.29) translates into:

$$\begin{aligned} \delta \bar{S} = - \int d^4\bar{x} \sqrt{\bar{g}}\, J^\mu \, \delta A_\mu = - \int d^4\bar{x} \sqrt{\bar{g}} \left(J^t \delta A_t + J^i \delta A_i\right) = \\ = - \int d^4\bar{x} \sqrt{\bar{g}} \left(i J^t \delta \bar{A}_\tau + J^i \delta \bar{A}_i\right) . \end{aligned} \tag{A.30}$$

Now, if we require our current to transform as (A.22) under a Wick rotation i.e. $\bar{J}^\mu = \left(\bar{J}^\tau, \bar{J}^i\right) = (iJ^t, J^i)$, we obtain:

$$\delta \bar{S} = - \int d^4\bar{x} \sqrt{\bar{g}} \left(\bar{J}^\tau \, \delta \bar{A}_\tau + \bar{J}^i \, \delta \bar{A}_i\right) = - \int d^4\bar{x} \sqrt{\bar{g}}\, \bar{J}^\mu \, \delta \bar{A}_\mu \,. \tag{A.31}$$

It follows that if we want the euclidean current to be consistent with our conventions on the Wick rotation, we have to take the following as its definition:

$$\bar{J}^\mu = - \frac{1}{\sqrt{\bar{g}}} \frac{\delta \bar{S}}{\delta \bar{A}_\mu} \,, \tag{A.32}$$

which has a further minus sign in front compared to the lorentzian definition (A.28).

APPENDIX B

Derivation of the stress-energy tensor

In this appendix we sketch briefly the computations leading to the expression for the stress-energy tensor of the chiral theory in a curved background. The discussion will be mainly based on [10], yet we have followed some slightly different steps here and there.

Recall that the lagrangian of our theory is given by:

$$\begin{aligned}\mathcal{L} = D_\mu\tilde{\phi}\, D^\mu\phi + V^\mu \left(iD_\mu\tilde{\phi}\,\phi - i\tilde{\phi}\, D_\mu\phi\right) + \frac{q_r}{4}\left(\mathcal{R} + 6V^\mu V_\mu\right)\tilde{\phi}\,\phi - \tilde{F}F + \\ + i\tilde{\psi}\,\tilde{\sigma}^\mu D_\mu\psi + \frac{1}{2}\,V_\mu\,\tilde{\psi}\,\tilde{\sigma}^\mu\psi\ .\end{aligned} \tag{B.1}$$

Notice that here we retained the term proportional to $\mathcal{R}+6V^\mu V_\mu$ even if it vanishes for $S^1\times S^3$ because its variation under an infinitesimal deformation of the metric will not be zero in general, and indeed we would miss some terms otherwise. In ordinary general relativity the stress-energy tensor is defined to be (in euclidean signature):

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{g}}\,\frac{\delta S}{\delta g^{\mu\nu}}\ , \tag{B.2}$$

however it is necessary to generalize this definition when the theory includes chiral matter, because it is not always possible to write the variation of the action in terms of that of the (inverse) metric. Indeed, as we will see, there will be terms whose variation is proportional to the variation of the vielbeins. It is then natural to generalize (B.2) by taking the variation of the action with respect to them. Imposing the matching of the two definitions on something known, as the metric itself for instance, it turns out that:

$$T_{\mu\nu} \equiv \frac{e_{c\mu}}{2\sqrt{g}}\,\frac{\delta S}{\delta e_c{}^\nu} + (\mu\leftrightarrow\nu)\,. \tag{B.3}$$

One can use either one or the other definition according to which one is more suitable for a given term contributing to the variation of the action.

Let us start by computing how the action varies under an infinitesimal variation of the background metric:

$$\delta S = \int_{\mathcal{M}} d^4x\, (\delta\sqrt{g}\,\mathcal{L} + \sqrt{g}\,\delta\mathcal{L}) = \int_{\mathcal{M}} d^4x \left(-\frac{\sqrt{g}}{2}\, g_{\mu\nu}\,\mathcal{L}_\phi\,\delta g^{\mu\nu} + \sqrt{g}\,\delta\mathcal{L}\right) . \tag{B.4}$$

Here we used the fact that $\delta\sqrt{g} = -\frac{\sqrt{g}}{2}\, g_{\mu\nu}\,\delta g^{\mu\nu}$ and that on-shell $F = \tilde{F} = 0$ and the fermionic lagrangian vanishes (with $\mathcal{L}_\phi$ we denote the terms in the lagrangian where the field ϕ appears). Now we have to compute $\delta\mathcal{L}$. Let us begin considering the scalar sector:

$$\delta\mathcal{L}_{scalar} = \left[D_\mu\tilde{\phi}\, D_\nu\phi + iV_\mu\left(D_\nu\tilde{\phi}\,\phi - \tilde{\phi}\, D_\nu\phi\right) + \frac{3q_r}{2}\, V_\mu V_\nu\,\tilde{\phi}\phi\right]\delta g^{\mu\nu} + \frac{q_r}{4}\,\tilde{\phi}\phi\,\delta\mathcal{R} . \tag{B.5}$$

Most of the terms are straightforward, except for the one involving the Ricci tensor $\mathcal{R}$. Its variation reads:

$$\delta\mathcal{R} = \mathcal{R}_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\,\delta\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\nu}\,\delta g^{\mu\nu} + g_{\mu\nu}\,\nabla^\rho\nabla_\rho(\delta g^{\mu\nu}) - \nabla_\mu\nabla_\nu(\delta g^{\mu\nu}) . \tag{B.6}$$

Plugging (B.6) into (B.5) and integrating by parts[15], we end up with:

$$\begin{aligned}\delta\mathcal{L}_{scalar} = \Bigg[&D_\mu\tilde{\phi}\, D_\nu\phi + iV_\mu\left(D_\nu\tilde{\phi}\,\phi - \tilde{\phi}\, D_\nu\phi\right) + \frac{3q_r}{2}\, V_\mu V_\nu\,\tilde{\phi}\phi + \\ &+ \frac{q_r}{4}\left(\mathcal{R}_{\mu\nu}\,\tilde{\phi}\phi + g_{\mu\nu}\,\nabla^\rho\nabla_\rho(\tilde{\phi}\phi) - \nabla_\mu\nabla_\nu(\tilde{\phi}\phi)\right)\Bigg]\delta g^{\mu\nu} .\end{aligned} \tag{B.7}$$

Now we come to the fermionic sector. Here the variation is due to the variation of the vielbeins and of the spin connection:

$$\delta\mathcal{L}_{fermion} = \left(i\tilde{\psi}\,\tilde{\sigma}^a D_\mu\psi + \frac{1}{2}\, V_\mu\,\tilde{\psi}\,\tilde{\sigma}^a\psi\right)\delta e_a{}^\mu - \left(\frac{i}{2}\,\tilde{\psi}\,\tilde{\sigma}^\mu\sigma^{bc}\psi\right)\delta\omega_{\mu bc} . \tag{B.8}$$

The tricky step is to find the variation of the spin-connection. We start from the equation that expresses the torsionlessness of the connection:

$$0 = \nabla_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu - \Gamma^\rho_{\mu\nu}\, e^a{}_\rho + \omega_\mu{}^a{}_b\, e^b{}_\nu .$$

Taking the variation of both sides of this equation and then isolating $\delta\omega_{\mu bc}$, we get:

$$\delta\omega_{\mu bc} = e_{b\rho}\, e_c{}^\nu\,\delta\Gamma^\rho_{\mu\nu} - e_c{}^\nu\,\nabla_\mu(\delta e_{b\nu}) . \tag{B.9}$$

[15]Recall that $\nabla_\mu g_{\nu\rho} = \nabla_\mu g = 0$ due to the connection being compatible with the metric, and that boundary terms vanish because S^3 has no boundary.

For the Christoffel symbols we have:

$$2g_{\rho\sigma}\,\Gamma^{\rho}_{\mu\nu} = \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}$$

$$\begin{aligned}\Longrightarrow \qquad 2g_{\rho\sigma}\,\delta\Gamma^{\rho}_{\mu\nu} =& -2\delta g_{\rho\sigma}\,\Gamma^{\rho}_{\mu\nu} + \partial_\mu(\delta g_{\sigma\nu}) + \partial_\nu(\delta g_{\sigma\mu}) - \partial_\sigma(\delta g_{\mu\nu}) \\ =& \nabla_\mu(\delta g_{\sigma\nu}) + \nabla_\nu(\delta g_{\sigma\mu}) - \nabla_\sigma(\delta g_{\mu\nu})\end{aligned}$$

$$\begin{aligned}\Longrightarrow \qquad \delta\Gamma^{\rho}_{\mu\nu} =& \frac{1}{2}\,g^{\rho\sigma}\left[\nabla_\mu(\delta g_{\sigma\nu}) + \nabla_\nu(\delta g_{\sigma\mu}) - \nabla_\sigma(\delta g_{\mu\nu})\right] \\ =& \frac{1}{2}\,g_{\mu\sigma}\,g_{\nu\lambda}\,\nabla^\rho(\delta g^{\sigma\lambda}) - \frac{1}{2}\big(g_{\nu\sigma}\,\nabla_\mu(\delta g^{\rho\sigma}) + g_{\mu\sigma}\,\nabla_\nu(\delta g^{\rho\sigma})\big)\,.\end{aligned} \tag{B.10}$$

Instead, the variation of the vielbein is found as follows:

$$\begin{aligned} e_{b\nu} = g_{\mu\nu}\,e_b{}^\mu \qquad \Longrightarrow& \qquad \delta e_{b\nu} = g_{\mu\nu}\,\delta e_b{}^\mu + e_b{}^\mu\,\delta g_{\mu\nu} \\ \Longrightarrow& \qquad \delta e_{b\nu} = g_{\mu\nu}\,\delta e_b{}^\mu - e_{b\rho}\,g_{\nu\sigma}\,\delta g^{\rho\sigma}\,. \end{aligned} \tag{B.11}$$

Substituting (B.10) and (B.11) into (B.9), we get the following expression for the variation of the spin connection:

$$\delta\omega_{\mu bc} = \nabla_\nu\left(g_{\mu\lambda}\,e_{[b}{}^\nu\,e_{c]\rho}\,\delta g^{\lambda\rho} + \frac{1}{2}\,e_{b\rho}\,e_{c\lambda}\,\delta^\nu{}_\mu\,\delta g^{\lambda\rho} - e_{c\rho}\,\delta^\nu{}_\mu\,\delta e_b{}^\rho\right)\,. \tag{B.12}$$

Now let us focus on the second term of (B.8). Substituting the expression just obtained for $\delta\omega_{\mu bc}$, it is clear that the second term of (B.12) does not contribute since it is symmetric under the exchange of the indices b and c and it is contracted with σ^{bc}, which is antisymmetric. Integrating by parts we get:

$$\left(\frac{i}{2}\,\tilde\psi\,\tilde\sigma^\mu\sigma^{bc}\psi\right)\delta\omega_{\mu bc} = \frac{i}{2}\,D_\nu\left(\tilde\psi\,\tilde\sigma^\mu\sigma^{bc}\psi\right)\left(g_{\mu\lambda}\,e_{[b}{}^\nu\,e_{c]\rho}\,\delta g^{\lambda\rho} - e_{c\rho}\,\delta^\nu{}_\mu\,\delta e_b{}^\rho\right)\,,$$

where we traded ∇_μ for D_μ since the object on which it acts is uncharged under the R-symmetry. Now we exploit the identity:

$$\tilde\sigma^d\,\sigma^{bc} = \frac{1}{2}\left(-\delta^{db}\,\tilde\sigma^c + \delta^{dc}\,\tilde\sigma^b - \epsilon^{dbca}\,\tilde\sigma_a\right)\,,$$

in order to get rid of two of the sigma matrices. Carrying on a few extra steps it is easy to arrive to the following expression:

$$\begin{aligned}\left(\frac{i}{2}\,\tilde\psi\,\tilde\sigma^\mu\sigma^{bc}\psi\right)\delta\omega_{\mu bc} =& \frac{i}{4}\Big[D_\lambda\big(\tilde\psi\,\tilde\sigma_\rho\psi\big) - g_{\lambda\rho}\,D_\nu\big(\tilde\psi\,\tilde\sigma^\nu\psi\big) + \\ & \qquad - D^\nu\big(\tilde\psi\,\tilde\sigma^c\psi\big)e_{a\lambda}\,e_{d\nu}\,e_{b\rho}\,\epsilon^{dabc}\Big]\delta g^{\lambda\rho} + \\ & + \frac{i}{4}\Big[D_\rho\big(\tilde\psi\,\tilde\sigma_\nu\psi\big)\,e^{a\nu} - D_\nu\big(\tilde\psi\,\tilde\sigma_\rho\psi\big)\,e^{a\nu} + \\ & \qquad - D^\nu\big(\tilde\psi\,\tilde\sigma^\mu\psi\big)e_{c\mu}\,e_{d\nu}\,e_{b\rho}\,\epsilon^{dabc}\Big]\delta e_a{}^\rho\,.\end{aligned}$$

Now notice that the last term inside the first parenthesis of the RHS is antisymmetric under the exchange of λ and ρ, hence it vanishes when contracted with $\delta g^{\lambda\rho}$. In the same spirit we should also symmetrize the first term in these indices. Then, notice that the second term inside the first parenthesis vanishes on-shell; indeed using the equations of motion for ψ and $\tilde{\psi}$ we find:

$$D_\nu\big(\tilde{\psi}\,\tilde{\sigma}^\nu\psi\big) = D_\nu\tilde{\psi}\,\tilde{\sigma}^\nu\psi + \tilde{\psi}\,\tilde{\sigma}^\nu D_\nu\psi = \frac{1}{2}\,V_\nu\,\tilde{\psi}\,\tilde{\sigma}^\nu\psi - \frac{1}{2}\,V_\nu\,\tilde{\psi}\,\tilde{\sigma}^\nu\psi = 0\,.$$

Finally, we can exploit the properties of the Levi-Civita symbol in the last term of the second parenthesis:

$$e_{c\mu}\,e_{d\nu}\,e_{b\rho}\,\epsilon^{dabc} = \sqrt{g}\,\varepsilon_{\nu\sigma\rho\mu}\,e^{a\sigma} = \epsilon_{\nu\sigma\rho\mu}\,e^{a\sigma}\,.$$

So, putting everything together, we have:

$$\begin{aligned}\left(\frac{i}{2}\,\tilde{\psi}\,\tilde{\sigma}^\mu\sigma^{bc}\psi\right)\delta\omega_{\mu bc} =& \frac{i}{8}\Big[D_\lambda\big(\tilde{\psi}\,\tilde{\sigma}_\rho\psi\big) + D_\rho\big(\tilde{\psi}\,\tilde{\sigma}_\lambda\psi\big)\Big]\delta g^{\lambda\rho} + \frac{i}{4}\Big[D_\rho\big(\tilde{\psi}\,\tilde{\sigma}_\nu\psi\big)\,e^{a\nu} +\\ &- D_\nu\big(\tilde{\psi}\,\tilde{\sigma}_\rho\psi\big)\,e^{a\nu} - D^\nu\big(\tilde{\psi}\,\tilde{\sigma}^\mu\psi\big)\epsilon_{\nu\sigma\rho\mu}\,e^{a\sigma}\Big]\delta e_a{}^\rho\,.\end{aligned}$$

We are finally ready to write down the variation of the fermionic lagrangian in a suitable form:

$$\begin{aligned}\delta\mathcal{L}_{fermion} =& \Big[i\tilde{\psi}\,\tilde{\sigma}^a D_\mu\psi + \frac{1}{2}\,V_\mu\,\tilde{\psi}\,\tilde{\sigma}^a\psi - \frac{i}{4}\,D_\mu\big(\tilde{\psi}\,\tilde{\sigma}_\nu\psi\big)\,e^{a\nu} + \frac{i}{4}\,D_\nu\big(\tilde{\psi}\,\tilde{\sigma}_\mu\psi\big)\,e^{a\nu} +\\ &+ \frac{i}{4}D^\nu\big(\tilde{\psi}\,\tilde{\sigma}^\rho\psi\big)\epsilon_{\nu\sigma\mu\rho}\,e^{a\sigma}\Big]\delta e_a{}^\mu - \frac{i}{8}\Big[D_\lambda\big(\tilde{\psi}\,\tilde{\sigma}_\rho\psi\big) + D_\rho\big(\tilde{\psi}\,\tilde{\sigma}_\lambda\psi\big)\Big]\delta g^{\lambda\rho}.\end{aligned} \tag{B.13}$$

The complete variation of the action under an infinitesimal variation of the background metric is obtained by substituting (B.7) and (B.13) into (B.4):

$$\begin{aligned}\delta S = \int_{\mathcal{M}} d^4x\,\sqrt{g}\,\bigg\{\bigg[&-\frac{1}{2}\,g_{\mu\nu}\Big(D_\rho\tilde{\phi}\,D^\rho\phi + iV^\rho\big(D_\rho\tilde{\phi}\,\phi - \tilde{\phi}\,D_\rho\phi\big)\Big) + D_\mu\tilde{\phi}\,D_\nu\phi +\\ &+ \frac{3q_r}{2}\,V_\mu V_\nu\,\tilde{\phi}\phi + iV_\mu\Big(D_\nu\tilde{\phi}\,\phi - \tilde{\phi}\,D_\nu\phi\Big) + \frac{q_r}{4}\Big(\mathcal{R}_{\mu\nu}\tilde{\phi}\phi + g_{\mu\nu}\nabla^\rho\nabla_\rho(\tilde{\phi}\phi) +\\ &- \nabla_\mu\nabla_\nu(\tilde{\phi}\phi)\Big)\bigg]\delta g^{\mu\nu} + \Big[i\tilde{\psi}\,\tilde{\sigma}^a D_\mu\psi + \frac{1}{2}\,V_\mu\,\tilde{\psi}\,\tilde{\sigma}^a\psi - \frac{i}{4}\,D_\mu\big(\tilde{\psi}\,\tilde{\sigma}_\nu\psi\big)\,e^{a\nu} +\\ &+ \frac{i}{4}\,D_\nu\big(\tilde{\psi}\,\tilde{\sigma}_\mu\psi\big)\,e^{a\nu} + \frac{i}{4}\,D^\nu\big(\tilde{\psi}\,\tilde{\sigma}^\rho\psi\big)\epsilon_{\nu\sigma\mu\rho}\,e^{a\sigma}\Big]\delta e_a{}^\mu +\\ &- \frac{i}{8}\Big[D_\lambda\big(\tilde{\psi}\,\tilde{\sigma}_\rho\psi\big) + D_\rho\big(\tilde{\psi}\,\tilde{\sigma}_\lambda\psi\big)\Big]\delta g^{\lambda\rho}\bigg\}\,.\end{aligned} \tag{B.14}$$

Applying the definitions (B.2) and (B.3) according to which one is more convenient for a given term, after a bit of algebra on the fermionic sector one gets the expression for the stress-energy tensor:

$$
\begin{aligned}
T_{\mu\nu} = -g_{\mu\nu}\Big[D_\rho\tilde{\phi}\, D^\rho\phi - iV^\rho\Big(D_\rho\tilde{\phi}\,\phi - \tilde{\phi}\, D_\rho\phi\Big)\Big] + 3q_r V_\mu V_\nu\, \tilde{\phi}\phi + \frac{q_r}{2}\,\mathcal{R}_{\mu\nu}\,\tilde{\phi}\phi + \\
+ \Big[D_\mu\tilde{\phi}\, D_\nu\phi + iV_\mu\Big(D_\nu\tilde{\phi}\,\phi - \tilde{\phi}\, D_\nu\phi\Big) + (\mu \leftrightarrow \nu)\Big] + \frac{q_r}{2}\Big[g_{\mu\nu}\,\nabla_\rho\nabla^\rho(\tilde{\phi}\phi) + \\
- \nabla_\mu\nabla_\nu(\tilde{\phi}\phi)\Big] - \frac{1}{4}\Big[iD_\mu\tilde{\psi}\,\tilde{\sigma}_\nu\psi - i\tilde{\psi}\,\tilde{\sigma}_\mu\, D_\nu\psi - V_\mu\,\tilde{\psi}\,\tilde{\sigma}_\nu\psi + (\mu \leftrightarrow \nu)\Big]\,.
\end{aligned}
\tag{B.15}
$$

APPENDIX C

Spherical harmonics on S^3

Here we present a brief review of scalar and spinorial harmonics on S^3, based mainly on appendix A of [10] and on [11], where the topic is developed more broadly.

C.1 Scalar harmonics

To start with, we describe the 3-sphere of radius r as an embedding inside $\mathbb{C}^2 \simeq \mathbb{R}^4$:

$$ds^2 = du\, d\bar{u} + dv\, d\bar{v}\,, \qquad u\bar{u} + v\,\bar{v} = r^2\,. \tag{C.1}$$

This manifold has isometry group $SO(4) \simeq SU(2)_L \times SU(2)_R$ with generators L_i^L and L_i^R, $i = 1, 2, 3$. In differential representation, the two Cartan generators are given by:

$$L_3^L = \frac{1}{2}\left(u\partial_u + v\partial_v - \bar{u}\partial_{\bar{u}} - \bar{v}\partial_{\bar{v}}\right)\,, \qquad L_3^R = \frac{1}{2}\left(u\partial_u - v\partial_v - \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}\right)\,. \tag{C.2}$$

In order to make contact with the main text, we introduce the following real coordinates:

$$\begin{cases} u = ir\sin\theta\, e^{-i\varphi_1} \\ v = r\cos\theta\, e^{-i\varphi_2} \end{cases}. \tag{C.3}$$

In such coordinates, the defining relation for the 3-sphere is automatically satisfied. The metric induced on S^3 reads:

$$ds^2_{S^3} = r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi_1^2 + r^2\cos^2\theta\, d\varphi_2^2\,, \tag{C.4}$$

and the Cartan generators become:

$$L_3^L = \frac{i}{2}\left(\partial_{\varphi_1} + \partial_{\varphi_2}\right)\,, \qquad L_3^R = \frac{i}{2}\left(\partial_{\varphi_1} - \partial_{\varphi_2}\right)\,. \tag{C.5}$$

Now we introduce the scalar spherical harmonics ${Y_l}^{mn}(\theta,\, \varphi_1,\, \varphi_2)$ as a basis for the space of functions on S^3, where $l \in \mathbb{N}$ and $-\frac{l}{2} \leq m, n \leq \frac{l}{2}$, m and n taking integer or semi-integer values according to the parity of l. Their expressions is given by the sum:

$$
{Y_l}^{mn} = N_{lmn} \sum_k \frac{r^l\,(-i)^{2k+m+n}\,(\sin\theta)^{2k+m+n}\,(\cos\theta)^{l-m-n-2k}\;e^{-i\varphi_1(m+n)}\,e^{-i\varphi_2(m-n)}}{k!\,(k+m+n)!\,\left(\frac{l}{2}-m-k\right)!\,\left(\frac{l}{2}-n-k\right)!}\,, \tag{C.6}
$$

where the index k runs on integers between $\max\{0, -m-n\}$ and $\frac{l}{2} - \max\{m,n\}$ and the coefficient in front of the sum is:

$$
N_{lmn} = \sqrt{\frac{(l+1)\,\left(\frac{l}{2}-m\right)!\,\left(\frac{l}{2}-n\right)!\,\left(\frac{l}{2}+m\right)!\,\left(\frac{l}{2}+n\right)!}{2\pi^2}}\,. \tag{C.7}
$$

Each function $f\colon S^3 \to \mathbb{C}$ can be decomposed in this basis with coefficients $f_{lmn} \in \mathbb{C}$:

$$
f(\theta,\, \varphi_1,\, \varphi_2) = \sum_{l=0}^{+\infty} \sum_{m,n=-\frac{l}{2}}^{\frac{l}{2}} f_{lmn}\,{Y_l}^{mn}(\theta,\, \varphi_1,\, \varphi_2)\,. \tag{C.8}
$$

The scalar harmonics so defined possess two important properties that will be useful for our purposes:

$$
\left({Y_l}^{mn}\right)^* = (-1)^{m+n}\,{Y_l}^{-m,-n}\,, \tag{C.9}
$$

$$
\int_{S^3} d^3x\,\sqrt{g_3}\,{Y_l}^{mn}\left({Y_{l'}}^{m'n'}\right)^* = \delta_{l,l'}\,\delta_{m,m'}\,\delta_{n,n'}\,, \tag{C.10}
$$

where g_3 is the determinant of the metric on the 3-sphere. Noteworthy, ${Y_l}^{mn}$ are eigenfunctions of both the Cartan generators and of the Laplacian on the 3-sphere:

$$
\nabla^2 {Y_l}^{mn} = -\frac{1}{r^2}\,l\,(l+2)\,{Y_l}^{mn}\,, \qquad L_3^L\,{Y_l}^{mn} = m\,{Y_l}^{mn}\,, \qquad L_3^R\,{Y_l}^{mn} = n\,{Y_l}^{mn}\,. \tag{C.11}
$$

From these relations we can derive the expressions for the derivatives of ${Y_l}^{mn}$ with respect to φ_1 and φ_2:

$$
\partial_{\varphi_1} {Y_l}^{mn} = -i(L_3^L + L_3^R)\,{Y_l}^{mn} = -i(m+n)\,{Y_l}^{mn}\,, \tag{C.12}
$$

$$
\partial_{\varphi_2} {Y_l}^{mn} = -i(L_3^L - L_3^R)\,{Y_l}^{mn} = -i(m-n)\,{Y_l}^{mn}\,. \tag{C.13}
$$

We will need also the derivative with respect to θ and this has to be computed directly using the explicit expression (C.6). The result one obtains is:

$$
\begin{aligned}
\partial_\theta {Y_l}^{mn} = (m+n)\tan^{-1}\theta\,{Y_l}^{mn} - (m-n)\tan\theta\,{Y_l}^{mn} + \\
- i\,\sqrt{(l+2m+2)\,(l-2m)}\,e^{i(\varphi_1+\varphi_2)}\,{Y_l}^{m+1,n}\,.
\end{aligned} \tag{C.14}
$$

Note that almost every property we listed so far is independent of the metric on the 3-sphere. However, the metric enters the definition of the Laplacian operator, hence it is natural to expect that the action of the Laplactian on the scalar harmonics changes when considering a twisted metric as we do in chapter 4. This is indeed the case and in particular, once we introduce the twisting, we have:

$$\nabla^2_{(twisted)} Y_l^{\ mn} = -\frac{1}{r^2}\, l\,(l+2)\, Y_l^{\ mn} - \frac{1}{\beta^2}\left[\sigma_1\,(m+n) + \sigma_2\,(m-n)\right]^2 Y_l^{\ mn} \,. \quad \text{(C.15)}$$

C.2 Spinor harmonics

Analogously, we can introduce the spinorial spherical harmonics on S^3 as the spinorial functions:

$$\left(S^\lambda_{lmn}\right)_\alpha = \begin{pmatrix} \cos\nu^\lambda_{lm}\, Y_l^{\ mn} \\ \sin\nu^\lambda_{lm}\, Y_l^{\ m+1,n} \end{pmatrix} \,, \quad \text{(C.16)}$$

where $\lambda = +,-$. For $\lambda = +$ the indices take values $l \geq 1$ and $-\frac{l}{2} \leq m \leq \frac{l}{2} - 1$, while for $\lambda = -$ they take valus $l \geq 0$ and $-\frac{l}{2} - 1 \leq m \leq \frac{l}{2}$, and in both cases $-\frac{l}{2} \leq n \leq \frac{l}{2}$. The sines and cosines appearing are defined to be:

$$\sin\nu^\pm_{lm} = \mp\sqrt{\frac{l+1 \pm (2m+1)}{2\,(l+1)}} \,, \qquad \cos\nu^\pm_{lm} = \sqrt{\frac{l+1 \mp (2m+1)}{2\,(l+1)}} \,. \quad \text{(C.17)}$$

Note that $\sin\nu^-_{lm} = \cos\nu^+_{lm}$ and $\cos\nu^-_{lm} = -\sin\nu^+_{lm}$. The functions (C.16) are a basis for left-handed spinorial functions. Taking the hermitian conjugate, we find a basis for right-handed spinors:

$$\left(S^\lambda_{lmn}\right)^\dagger_{\dot\alpha} = \begin{pmatrix} \cos\nu^\lambda_{lm}\, \left(Y_l^{\ mn}\right)^* \\ \sin\nu^\lambda_{lm}\, \left(Y_l^{\ m+1,n}\right)^* \end{pmatrix} \,, \quad \text{(C.18)}$$

Much like in the scalar case, each spinorial function ψ defined on S^3 can be decomposed in the basis $\{S^\lambda_{lmn}\}$ with coefficients $\psi^\lambda_{lmn} \in \mathbb{C}$:

$$\psi(\theta,\, \varphi_1,\, \varphi_2) = \sum_{l,m,n}\sum_{\lambda=+,-} \psi^\lambda_{lmn}\, S^\lambda_{lmn} \,, \quad \text{(C.19)}$$

where the extrema of the sums over l, m, and n are specified by the ranges above.

Much like the scalar harmonics, the spinorial harmonics satisfy the property:

$$\int d^3x\, \sqrt{g_3}\, \left(S^\lambda_{lmn}\right)^\dagger_{\dot\alpha}\, \mathbb{I}^{\dot\alpha\alpha}\, \left(S^{\lambda'}_{l'm'n'}\right)_\alpha = \delta_{l,l'}\, \delta_{m,m'}\, \delta_{n,n'}\, \delta_{\lambda,\lambda'} \,, \quad \text{(C.20)}$$

Moreover, in the left-invariant frame (3.17) the following additional property holds:

$$\left(i\tilde\sigma^\mu\, \partial_\mu\right)\, S^\lambda_{lmn} = \alpha_\lambda\, S^\lambda_{lmn} \,, \quad \text{(C.21)}$$

where:

$$\alpha_+ = \frac{1}{r}\,(l+2) \,, \qquad \alpha_- = -\frac{l}{r} \,. \quad \text{(C.22)}$$

However, this relation obviously depends on the metric on S^3. If we introduce the twisting of the 3-sphere, in the twisted left-invariant frame (4.13), it is modified as:

$$(i\tilde{\sigma}^{\mu}\,\partial_{\mu})\;S^{\lambda}_{lmn} = \alpha_{\lambda}\,S^{\lambda}_{lmn} + \frac{i}{\beta}\left[(\sigma_1+\sigma_2)\,L^L_3 + (\sigma_1-\sigma_2)\,L^R_3\right]S^{\lambda}_{lmn}\,, \tag{C.23}$$

where L^L_3 and L^R_3 are the same Cartan generators in the differential representation that we introduced above (C.5).

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