

# UNIVERSITÀ DEGLI STUDI DI PADOVA

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Topological defects in conformal field theory: from  
vertex operator algebras to conformal nets

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# Contents

|                                                             |           |
|-------------------------------------------------------------|-----------|
| <b>Introduction</b>                                         | <b>5</b>  |
| <b>1 Conformal Field Theory</b>                             | <b>11</b> |
| 1.1 The conformal group . . . . .                           | 11        |
| 1.1.1 The conformal algebra . . . . .                       | 13        |
| 1.1.2 Correlation functions . . . . .                       | 14        |
| 1.1.3 Energy-momentum tensor and Ward identities . . . . .  | 15        |
| 1.2 CFTs in two dimensions . . . . .                        | 17        |
| 1.2.1 The infinite and central extensions . . . . .         | 17        |
| 1.2.2 Primary fields . . . . .                              | 18        |
| 1.2.3 The operator-state correspondence . . . . .           | 19        |
| 1.2.4 The conformal charge . . . . .                        | 21        |
| 1.2.5 The Hilbert space structure . . . . .                 | 23        |
| 1.2.6 Operator Product Expansion . . . . .                  | 25        |
| 1.2.7 Minimal models and Rational CFTs . . . . .            | 26        |
| 1.2.8 The partition function . . . . .                      | 29        |
| 1.3 The Ising Model . . . . .                               | 31        |
| <b>2 Topological defects in CFT</b>                         | <b>37</b> |
| 2.1 From global symmetries to topological defects . . . . . | 37        |
| 2.2 Topological defect lines . . . . .                      | 40        |
| 2.2.1 Fusion ring . . . . .                                 | 41        |
| 2.2.2 k-junctions . . . . .                                 | 42        |
| 2.2.3 Verlinde lines . . . . .                              | 46        |
| 2.3 Defects in partition functions . . . . .                | 47        |
| 2.4 Fusion category of TDLs . . . . .                       | 49        |
| 2.5 TDLs of the critical Ising model . . . . .              | 57        |
| <b>3 Algebraic quantum field theory</b>                     | <b>61</b> |
| 3.1 Algebraic formulation of quantum mechanics . . . . .    | 61        |
| 3.1.1 States and representations . . . . .                  | 63        |
| 3.2 Haag-Kastler axioms . . . . .                           | 65        |
| 3.3 Von Neumann Algebras . . . . .                          | 67        |
| 3.3.1 Application to quantum field theory . . . . .         | 70        |
| 3.4 Superselection sectors . . . . .                        | 73        |
| 3.4.1 Statistics . . . . .                                  | 78        |
| 3.5 The category of DHR endomorphisms . . . . .             | 82        |
| 3.6 The chiral Ising model . . . . .                        | 87        |
| <b>4 Conformal nets and extensions</b>                      | <b>93</b> |
| 4.1 Conformal nets . . . . .                                | 94        |
| 4.2 Q-systems and extensions . . . . .                      | 95        |
| 4.2.1 Application in quantum field theory . . . . .         | 98        |

|          |                                                                 |            |
|----------|-----------------------------------------------------------------|------------|
| 4.2.2    | Charged fields . . . . .                                        | 100        |
| 4.2.3    | The canonical extension . . . . .                               | 101        |
| 4.3      | Q-systems as algebra objects . . . . .                          | 102        |
| 4.4      | Constructing the two-dimensional Ising model CFT . . . . .      | 105        |
| <b>5</b> | <b>Topological defects in AQFT</b>                              | <b>109</b> |
| 5.1      | Defects and boundary conditions . . . . .                       | 110        |
| 5.2      | Operations with Q-systems . . . . .                             | 112        |
| 5.2.1    | Reduced Q-system . . . . .                                      | 112        |
| 5.2.2    | Maximal local subtheories of non-local extensions . . . . .     | 114        |
| 5.2.3    | Braided products . . . . .                                      | 115        |
| 5.3      | The universal construction . . . . .                            | 119        |
| 5.3.1    | Computation of the boundary conditions . . . . .                | 121        |
| 5.4      | Defects of the relativistic 2D Ising model CFT . . . . .        | 126        |
| <b>6</b> | <b>The bridge with Euclidean CFT</b>                            | <b>129</b> |
| 6.1      | Vertex operator algebras . . . . .                              | 130        |
| 6.2      | From Vertex Operator Algebras to conformal nets . . . . .       | 133        |
| 6.3      | The categories of modules and bimodules . . . . .               | 137        |
| 6.4      | Full center of Q-systems . . . . .                              | 143        |
| 6.5      | Bimodules-defects correspondence . . . . .                      | 144        |
| 6.6      | Modules and algebra objects of the chiral Ising model . . . . . | 150        |
|          | <b>Conclusion</b>                                               | <b>155</b> |
|          | <b>Appendix</b>                                                 | <b>157</b> |
| A -      | Covariant Form of a Conserved Charge . . . . .                  | 157        |
| B -      | Computation of the bubble diagrams . . . . .                    | 157        |
| C -      | The hypergroup of boundary conditions . . . . .                 | 159        |
| D -      | Calculations of the fermionic extension . . . . .               | 163        |
|          | <b>Bibliography</b>                                             | <b>165</b> |

# Introduction

The main objective of this thesis is to study the description of topological defects in 1+1-dimensional relativistic conformal field theories (CFTs), and to highlight their common features with the Euclidean formulation, with the ultimate goal of identifying a unified mathematical framework. We begin by briefly introducing the main concepts that will be discussed throughout the thesis.

Conformal field theories are a special class of quantum field theories characterized by an enhanced spacetime symmetry group, namely the conformal group. Conformal transformations are those coordinate transformations that preserve the metric up to a local scale factor; intuitively, they preserve angles but not necessarily distances.

CFTs appear in many different contexts, ranging from condensed matter physics to string theory. In many of these applications, the system is described by a two-dimensional Euclidean CFT. For instance, the worldsheet of a quantum string is described by such a theory, as well as two-dimensional critical models in statistical mechanics. In two Euclidean dimensions, CFTs exhibit an exceptionally rich mathematical structure, deeply connected to complex analysis: the local conformal transformations are enhanced to arbitrary holomorphic and antiholomorphic functions.

Conformal field theories are introduced in Chapter 1.

Besides spacetime symmetries, physical theories also exhibit global symmetries. The notion of symmetry can be generalized to include so-called non-invertible symmetries. A paradigmatic example is the Kramers–Wannier duality in the Ising model, which relates the high-temperature and low-temperature phases. At the critical point, these two different descriptions correspond to the same model, and the duality becomes an intrinsic symmetry of the theory. However, in contrast with ordinary global symmetries, this transformation is not invertible: it is not implemented by a unitary operator on the Hilbert space. More generally, non-invertible symmetries satisfy product laws which generalize the group multiplication law of ordinary symmetries.

These generalized symmetries can also be described in terms of defects of the theory. Roughly speaking, a defect is an operator supported on a submanifold of spacetime, rather than a point. We will focus on defects in two-dimensional CFTs with the property of being topological: when inserted in a correlation function, their continuous deformations do not change the correlator, unless the defect crosses the support of other operators.

Topological defects are studied in Chapter 2.

Since CFTs are mostly studied in the Euclidean setting, defects are also usually analyzed within this framework. In particular, a standard tool for their classification is modular invariance. However, in view of the Osterwalder–Schrader theorem, which relates Euclidean and Lorentzian quantum field theories, it is natural to investigate the connection between the relativistic and Euclidean descriptions of CFTs, including the corresponding notions of defects.

To this end, we introduce the axiomatic framework of Algebraic Quantum Field Theory (AQFT). This framework is based on the assignment of a net of local algebras, i.e. a map  $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$  that associates to each spacetime region  $\mathcal{O}$  an algebra of observables  $\mathcal{A}(\mathcal{O})$ . The central idea is that all relevant physical information is encoded in this net, together with its action under the Poincaré symmetry

group. In this sense, AQFT can be regarded as a natural extension of the Heisenberg picture of quantum mechanics to quantum field theory.

The net is required to satisfy a set of physically motivated conditions, known as the Haag–Kastler axioms. Roughly speaking, these axioms encode:

- *isotony*: larger spacetime regions contain more observables;
- *locality*: observables localized in spacelike separated regions commute;
- *covariance*: the net transforms consistently under spacetime symmetries;
- *vacuum positivity*: the theory admits a distinguished vacuum state with positive energy.

Once a quantum field theory has been reformulated in terms of a net of observable algebras, the next step is to understand its possible excitations and internal symmetries. Physical information is extracted through the representation theory of the pair consisting of the net and its symmetry action.

At this point, two equivalent descriptions arise. The first is formulated directly in terms of superselection sectors, namely inequivalent Hilbert-space representations of the observable algebra corresponding to different types of physical charges. This picture is conceptually close to the traditional description of quantum field theory in terms of states and particles.

The second description replaces representations by suitable localized endomorphisms of the observable algebra itself. Instead of enlarging the Hilbert space, one studies how the algebra of observables can be transformed while remaining within the vacuum representation. Remarkably, the full information about superselection sectors can be recovered from these endomorphisms. Moreover, they naturally carry additional structures such as tensor products, duals and statistics.

Although the representation-theoretic picture is often more intuitive from a physical perspective, the endomorphism framework is technically more convenient and will therefore be the main language adopted throughout this thesis.

Within this framework, one can also describe the statistics of particles in a model-independent way. In higher than three spacetime dimensions, locality implies that particle exchanges are governed by the permutation group, leading to the familiar Bose–Einstein and Fermi–Dirac statistics. In lower dimensions, however, the topology of spacetime allows for a richer structure: particle exchanges are described by the braid group rather than the permutation group. As a consequence, excitations may exhibit *braid statistics*, interpolating between bosonic and fermionic behaviour.

Algebraic Quantum Field Theory is discussed in Chapter 3.

After developing the general framework of AQFT, we return to conformal field theory. Starting from a one-dimensional conformal net, which carries conformal symmetry rather than Poincaré symmetry, we describe a general procedure to construct the associated two-dimensional CFT.

The starting point is a chiral net, namely a conformal net providing the AQFT description of a one-dimensional CFT living on one of the two light-rays of two-dimensional Minkowski spacetime, together with its superselection sectors. Although the formalism applies to arbitrary rational chiral symmetry algebras, throughout this thesis we will mainly focus on theories whose chiral symmetry is given by the Virasoro algebra. A two-dimensional theory is then obtained by combining two copies of the chiral net, corresponding to left- and right-moving degrees of freedom:  $\mathcal{A}_+ \otimes \mathcal{A}_-$ . However, this tensor product construction alone does not yet define a full local CFT. One must still specify which left- and right-moving sectors are allowed to combine and how they interact.

This additional information is encoded by a *Q-system*. A Q-system is a triple  $(\Theta, W, X)$  consisting of an object  $\Theta$  together with two morphisms  $W$  and  $X$  satisfying a set of algebraic relations. Roughly speaking,  $\Theta$  determines the spectrum of the full two-dimensional theory: it specifies which pairs of left- and right-moving sectors are present among all those available in the tensor product of the two

chiral theories. The morphism  $W$  embeds the vacuum sector into  $\Theta$ , ensuring that the resulting theory contains the vacuum representation, while the morphism  $X$  endows  $\Theta$  with a multiplication structure.

The information contained in a Q-system is then used to construct an extension of the observable net:  $\mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{B}$ . This larger theory contains charged fields, namely operators interpolating between the vacuum sector and non-trivial superselection sectors. Moreover, the multiplication  $X$  determines how such charged fields combine and therefore encodes their operator product expansion. In this way, a Q-system provides a complete algebraic description of both the spectrum of the full two-dimensional theory and the interactions among its charged fields.

Conformal nets and their extensions are the main focus of Chapter 4.

Once the construction of two-dimensional conformal field theories within AQFT has been established, we turn to the main subject of this thesis: the study of defects. In the Euclidean approach, topological defects are commonly represented as lines that can be continuously deformed without affecting correlation functions. In the algebraic framework, instead, one works directly in Minkowski spacetime and must formulate their defining properties in terms of localization and causality.

The basic idea is to represent a defect as a timelike curve separating spacetime into two connected regions. On each side of the defect, we place a copy, or a possibly different realization, of the same CFT; in other words, the theories on the two sides are assumed to be isomorphic. Observables localized entirely on one side should belong to the corresponding theory, while the defect encodes the way observables and charged fields can be transmitted across the separating line.

Algebraically, such a configuration is described by a net  $\mathcal{D}$  together with two distinguished subnets  $\mathcal{B}^L$  and  $\mathcal{B}^R$ , corresponding to the theories living on the left and right of the defect. The net  $\mathcal{D}$  should be thought of as the smallest theory simultaneously containing both sides and all the fields required to interpolate between them:  $\mathcal{D}$  may contain information not only about observables localized away from the defect, but also about defect fields intrinsically localized on the separating line. In our treatment, however, we focus on defects without such additional operators, since a general algebraic theory incorporating these defect degrees of freedom has not yet been developed. In parts of the AQFT literature, these simpler configurations are referred to as *boundary conditions*.

For a topological defect, the theories on the two sides preserve the same energy-momentum tensor and therefore share a common chiral symmetry. More precisely, both  $\mathcal{B}^L$  and  $\mathcal{B}^R$  are extensions of the same chiral subnet  $\mathcal{A}_+ \otimes \mathcal{A}_-$ . Since this subnet contains the generators of local conformal transformations, the local algebras of both theories can be defined for every spacetime region, even when the region lies on the opposite side of the defect. Such algebras, however, should not be interpreted as physical observables on that side: transporting a field across the defect may change its locality properties, and fields belonging to the two extensions need not commute in arbitrary spacelike configurations.

A crucial feature of this construction is therefore that, although the theories  $\mathcal{B}^L$  and  $\mathcal{B}^R$  are local, the global net  $\mathcal{D}$  is in general non-local. In particular, observables transported across the defect need not commute at spacelike separation. From the Euclidean perspective, this non-locality is the algebraic manifestation of the fact that local operators may undergo non-trivial transformations when transported through a defect line. The defect is therefore encoded precisely in the controlled failure of locality of the larger theory  $\mathcal{D}$ .

Causality is implemented through a weaker, oriented notion of locality. If a region  $\mathcal{O}_1$  lies on the spacelike left of a region  $\mathcal{O}_2$ , then the algebra  $\mathcal{B}^L(\mathcal{O}_1)$  is required to commute with  $\mathcal{B}^R(\mathcal{O}_2)$ :

$$[\mathcal{B}^L(\mathcal{O}_1), \mathcal{B}^R(\mathcal{O}_2)] = 0.$$

By conformal covariance, this requirement extends from regions lying strictly on opposite sides of the timelike defect to any pair of double cones with the same relative left–right ordering. This property is usually called *left-locality*, *right-locality*, or *half-locality*, depending on the orientation chosen.

The Q-systems introduced earlier provide the appropriate tool for constructing such a net. Let  $(\Theta^L, W^L, X^L)$  and  $(\Theta^R, W^R, X^R)$  be the Q-systems describing the extensions  $\mathcal{B}^L$  and  $\mathcal{B}^R$  of the

common chiral theory. Their *braided product* combines the two Q-systems into a single one, whose associated extension contains both  $\mathcal{B}^L$  and  $\mathcal{B}^R$ . The two extensions are not simply placed side by side: their charged generators are required to satisfy specific exchange relations. By choosing the appropriate braiding, one imposes precisely the half-locality condition described above, so that fields from the left theory commute with fields from the right theory whenever they occur in the required spacelike order.

The resulting braided-product extension  $\mathcal{D}$  is generally non-local, but its non-locality is controlled by the exchange relations between the two subtheories. Moreover, the construction is universal: every defect satisfying the required half-locality condition can be obtained as a representation of  $\mathcal{D}$ . Thus, rather than describing a single defect, the braided product initially contains all admissible ways of joining the two theories.

This universality is reflected in the fact that the extension

$$\mathcal{A} := \mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{D}$$

is generally reducible. In particular, its relative commutant is non-trivial:

$$\mathcal{A}' \cap \mathcal{D} \neq \mathbb{C}1.$$

A trivial relative commutant is instead required for  $\mathcal{D}$  to describe a single irreducible physical configuration. The braided-product extension should therefore be interpreted as a superposition of different defect configurations rather than as one individual defect.

Since the theories described by  $\mathcal{B}^L$  and  $\mathcal{B}^R$  are themselves local, the relative commutant  $\mathcal{A}' \cap \mathcal{D}$  turns out to coincide with the algebraic center

$$\mathcal{A}' \cap \mathcal{D} = \mathcal{D}' \cap \mathcal{D}.$$

The different physical configurations contained in  $\mathcal{D}$  can consequently be separated through its central decomposition. More precisely, each minimal central projection selects an irreducible representation of  $\mathcal{D}$  and therefore an elementary defect relating  $\mathcal{B}^L$  and  $\mathcal{B}^R$ . The classification problem is thus reduced to the finite-dimensional algebraic problem of determining the minimal projections of the center.

Defects in algebraic QFT are studied in Chapter 5.

In general, in the absence of a Wick rotation in the presence of defects, it is not a priori clear whether the Euclidean and relativistic approaches lead to the same classification. Nevertheless, the constructions developed in this thesis reproduce the expected number of defects obtained in the Euclidean theory, and, in the special case of diagonal CFTs, the two classifications are shown to coincide exactly. This suggests that their agreement is not accidental.

To understand this relation more deeply, we briefly introduce vertex operator algebras (VOAs), which provide an axiomatic framework closely related to the Euclidean formulation of two-dimensional CFTs. Their construction is based on the operator–state correspondence and emphasizes the holomorphic structure of the theory.

VOAs and conformal nets describe similar physical data from rather different perspectives. The VOA approach begins with fields and their operator product expansions and is therefore closely related to the Euclidean formulation of CFT. Conformal nets, by contrast, begin with algebras of bounded observables assigned to spacetime regions and emphasize locality, causality and the operator-algebraic structure of the theory. In this sense, VOAs play a role analogous to field-based axiomatic approaches such as the Wightman framework, whereas conformal nets implement the philosophy of algebraic quantum field theory.

Under suitable analytic assumptions, a unitary VOA can be used to construct a conformal net by smearing its vertex operators and considering the von Neumann algebras they generate. Conversely, one expects the chiral fields of a sufficiently regular conformal net to reconstruct an associated VOA.

Although a completely general equivalence between the two frameworks has not yet been established, important results are available for broad classes of unitary rational theories.

This comparison is especially relevant to the construction of full two-dimensional CFTs. In both approaches, one starts from chiral data and selects a consistent combination of left- and right-moving sectors. In the conformal-net setting, this information is encoded by a  $Q$ -system. It turns out that the VOA-based Euclidean framework likewise requires an algebraic structure playing the same role as a  $Q$ -system. In both cases, this structure specifies which chiral sectors appear in the full theory and how the corresponding fields combine through the operator product expansion. The two constructions therefore appear rather different at the analytic level, but rely on essentially the same underlying algebraic data.

Category theory provides the natural language for making this common structure explicit. A category consists of objects and morphisms between them, together with rules for composing morphisms. Further structures can then be introduced in order to encode the physical operations relevant to CFT. A tensor product describes the fusion of sectors, dual objects describe conjugate charges, and a braiding records the exchange statistics of excitations. When the number of irreducible sectors is finite and suitable non-degeneracy conditions hold, these data organize into a *modular tensor category*.

The representation theory of a rational VOA and the DHR representation theory of a rational conformal net are expected to give rise to equivalent modular tensor categories, at least under the unitarity and rationality assumptions relevant here. This categorical description therefore retains precisely the algebraic information shared by the two approaches, while abstracting away from their different analytic realizations.

Within this categorical framework, the algebraic structures appearing in the Euclidean approach are understood as algebra objects, while  $Q$ -systems provide their operator-algebraic counterpart. Their multiplication and unit generalize the familiar notions of multiplication and identity in an ordinary associative algebra, while their additional Frobenius and compatibility properties encode the consistency conditions required by the corresponding CFT.

Once a theory is represented by an algebra object, its boundaries, interfaces and defects admit a unified description in terms of modules and bimodules. A boundary condition is described by a module, i.e. an object of the underlying category carrying an action of the algebra associated with the bulk theory. More generally, an interface between theories described by algebra objects, or  $Q$ -systems,  $Q_1$  and  $Q_2$  is encoded by a  $Q_1$ - $Q_2$  bimodule, an object which carries compatible actions of the two algebras. When the same theory is present on both sides, these become  $Q$ - $Q$  bimodules and describe defects of that theory. Their relative tensor product reproduces the fusion of defects, while simple bimodules correspond to elementary, indecomposable defect lines.

The collection of such bimodules and their morphisms forms a categorical structure of its own. Thus, the same mathematical language simultaneously describes the chiral sectors, the full CFT and its topological defects. The classification obtained from the central decomposition of the braided-product extension in AQFT can then be compared directly with the classification by bimodules arising in the Euclidean categorical approach.

The development of this universal categorical description, and its relation to both VOAs and conformal nets, is the main subject of Chapter 6.

To illustrate these concepts, we study the critical Ising model in each chapter, analyzing it from different perspectives. Although the model is well known in the literature, it still provides a non-trivial and instructive example that allows for explicit computations.

Finally, although the Ising model has been extensively studied, many computations are often omitted in the literature. In this thesis, we reproduce these calculations in detail, together with several proofs that are usually only stated or sketched.



# Chapter 1

## Conformal Field Theory

In this chapter we introduce the basic structures of conformal field theory. We first discuss conformal transformations and the constraints they impose in generic dimension. We then specialize to two-dimensional CFTs, where the local conformal symmetry is enhanced to the infinite-dimensional Virasoro algebra. Finally, we review the minimal models and the critical Ising model, which will serve as the main example throughout the thesis. This chapter is based on [1–6].

### 1.1 The conformal group

Conformal field theories are theories that are conformally invariant. So the first step to study them is to introduce conformal transformations.

**Definition 1.1.1.** A conformal transformation is an invertible mapping  $x \mapsto x'$  which leaves the metric tensor invariant up to a scale:

$$g'_{\mu\nu}(x') = \Omega(x)^2 g_{\mu\nu}(x)$$

where  $\Omega$  is called the *Weyl factor*. The set of conformal transformations forms a group called the *conformal group*.

Intuitively, conformal transformations are transformations which preserve the angle of intersection between any two curves, as shown in figure 1.1. Note that by picking  $g_{\mu\nu} = \eta_{\mu\nu}$ , the Minkowski metric, we see that the Poincaré group is a subgroup of the conformal group with  $\Omega(x)^2 = 1$ .

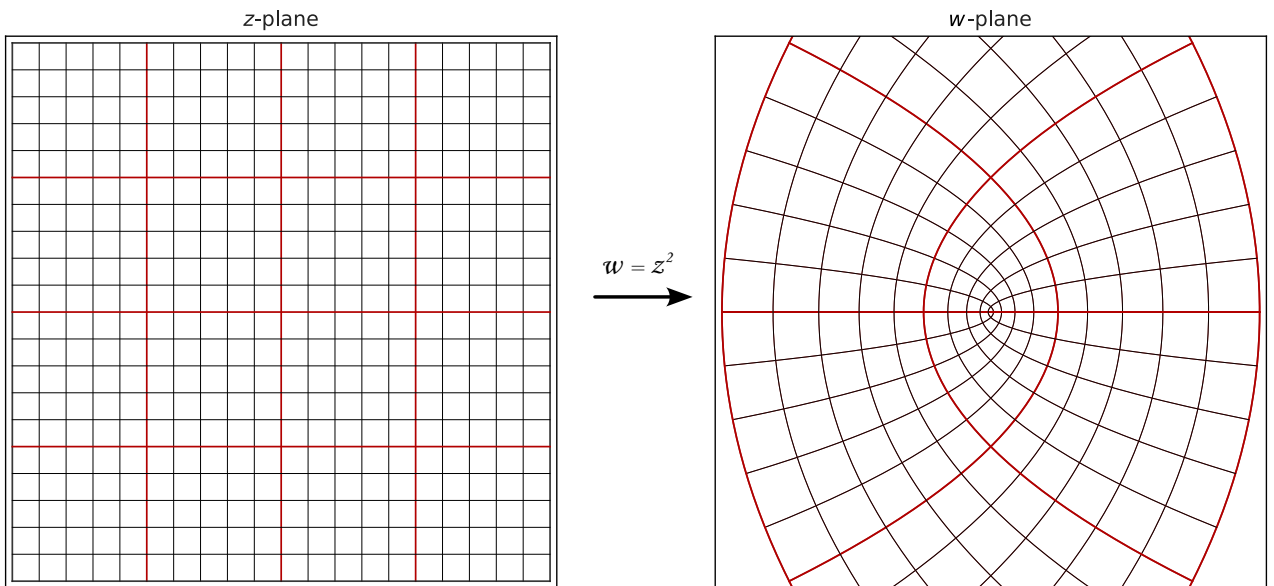


Figure 1.1: Representation of the conformal map  $z \rightarrow w = z^2$  on the complex plane.

To better characterize the conformal group, we can study an infinitesimal transformation  $x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu$ :

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) = (\delta_\mu^\alpha - \partial_\mu \epsilon^\alpha)(\delta_\nu^\beta - \partial_\nu \epsilon^\beta) g_{\alpha\beta} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + o(\epsilon^2)$$

where we are using the Einstein convention on repeating indices.

The requirement that the transformation be conformal implies:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu} \quad (1.1)$$

We now specialize for simplicity to a flat Euclidean metric  $g_{\mu\nu} = \delta_{\mu\nu}$ . The discussion wouldn't change much with the Minkowskian metric.

Taking the trace of (1.1) we find:

$$f(x) = \frac{2}{d} \partial_\rho \epsilon^\rho \quad (1.2)$$

Instead, if we further differentiate (1.1) and take a linear combination of the same equation but with permuted indices we get:

$$2\partial_\mu \partial_\nu \epsilon_\rho = \delta_{\mu\rho} \partial_\nu f + \delta_{\nu\rho} \partial_\mu f - \delta_{\mu\nu} \partial_\rho f \quad (1.3)$$

Taking the trace:

$$2\partial^2 \epsilon_\mu = (2-d) \partial_\mu f$$

We now further apply  $\partial_\nu$  to this equation, apply  $\partial^2$  to both sides of (1.1) and combine the two results:

$$(2-d) \partial_\mu \partial_\nu f = \delta_{\mu\nu} \partial^2 f \quad (1.4)$$

Finally, taking the trace we get:

$$(d-1) \partial^2 f = 0 \quad (1.5)$$

From the above equations we can derive the explicit form of the conformal transformations. We will do this derivation assuming  $d \geq 3$  and specialize to  $d = 2$  later.

Combining equations (1.4) and (1.5) we get  $\partial_\mu \partial_\nu f = 0$ , i.e.  $f = A + B_\mu x^\mu$  with  $A, B_\mu$  constant. Substituting this result in (1.3) we get that  $\partial_\mu \partial_\nu \epsilon_\rho$  is a constant, which implies

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \quad c_{\mu\nu\rho} = c_{\mu\rho\nu} \quad (1.6)$$

with  $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho}$  constant.

Since in all the previous equations  $\epsilon$  appears with a derivative, we deduce that  $a^\mu$  is free of constraints. This term describes an infinitesimal translation.

If we substitute (1.6) into (1.1) we get:

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b_\rho^\rho \delta_{\mu\nu}$$

meaning that  $b_{\mu\nu}$  can be decomposed into the sum of an antisymmetric part and a pure trace:

$$b_{\mu\nu} = \alpha \delta_{\mu\nu} + m_{\mu\nu}, \quad m_{\mu\nu} = -m_{\nu\mu}$$

The pure trace represents an infinitesimal scale transformation, whereas the antisymmetric part is an infinitesimal rigid rotation.

Finally, substituting (1.6) into (1.3) we get:

$$c_{\mu\nu\rho} = \delta_{\mu\rho} b_\nu + \delta_{\mu\nu} b_\rho - \delta_{\nu\rho} b_\mu, \quad b_\mu = \frac{1}{d} c_{\sigma\mu}^\sigma$$

and the corresponding infinitesimal transformation is:

$$x'^{\mu} = x^{\mu} + 2(x \cdot b)x^{\mu} - b^{\mu}x^2$$

called *special conformal transformation (SCT)*.

The finite transformations corresponding to the above infinitesimal transformations are:

$$\begin{aligned} \text{Translation: } x'^{\mu} &= x^{\mu} + a^{\mu} \\ \text{Dilation: } x'^{\mu} &= \alpha x^{\mu} \\ \text{Rotation: } x'^{\mu} &= M_{\nu}^{\mu} x^{\nu} \\ \text{SCT: } x'^{\mu} &= \frac{x^{\mu} - b^{\mu}x^2}{1 - 2b \cdot x + b^2x^2} \end{aligned}$$

Note that special conformal transformations are not globally defined if our space is  $\mathbb{R}^d$  or  $\mathbb{R}^{d-1,1}$ . In fact there always exists an  $x$  for which  $1 - 2b \cdot x + b^2x^2 = 0$ . The solution to this is provided by the implementation of the conformal compactification of our space, which basically means that we add the point  $\{\infty\}$  to our space.

### 1.1.1 The conformal algebra

The generators of the conformal algebra can be easily found by first considering the action of the conformal group on scalar fields satisfying  $\Phi'(x') = \Phi(x)$ .<sup>1</sup>

$$\begin{aligned} \text{Translation: } P_{\mu} &= -\partial_{\mu} \\ \text{Dilation: } D &= -x^{\mu}\partial_{\mu} \\ \text{Rotation: } L_{\mu\nu} &= x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} \\ \text{SCT: } K_{\mu} &= -2x_{\mu}x^{\nu}\partial_{\nu} + x^2\partial_{\mu} \end{aligned}$$

These generators satisfy the following algebra:

$$\begin{aligned} [D, P_{\mu}] &= P_{\mu} & [D, K_{\mu}] &= -K_{\mu} & [K_{\mu}, P_{\nu}] &= 2(\delta_{\mu\nu}D - L_{\mu\nu}) \\ [K_{\rho}, L_{\mu\nu}] &= \delta_{\rho\mu}K_{\nu} - \delta_{\rho\nu}K_{\mu} & [P_{\rho}, L_{\mu\nu}] &= \delta_{\rho\mu}P_{\nu} - \delta_{\rho\nu}P_{\mu} \\ [L_{\mu\nu}, L_{\rho\sigma}] &= \delta_{\nu\rho}L_{\mu\sigma} + \delta_{\mu\sigma}L_{\nu\rho} - \delta_{\mu\rho}L_{\nu\sigma} - \delta_{\nu\sigma}L_{\mu\rho} \end{aligned} \tag{1.7}$$

We can write the algebra in a compact form by defining:

$$J_{\mu\nu} = L_{\mu\nu}, \quad J_{-1,\mu} = \frac{1}{2}(P_{\mu} - K_{\mu}), \quad J_{-1,0} = D, \quad J_{0,\mu} = \frac{1}{2}(P_{\mu} + K_{\mu})$$

where  $J_{ab} = -J_{ba}$  and  $a, b \in \{-1, 0, 1, \dots, d\}$ . These new generators obey the commutation relations:

$$[J_{ab}, J_{cd}] = \eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}$$

where  $\eta = \text{diag}(-1, 1, 1, \dots, 1)$  if spacetime is Euclidean. Hence the conformal group in  $d$  dimensions is isomorphic to  $SO(d+1, 1)$ . In the relativistic setting, spacetime is  $\mathbb{R}^{d-1,1}$  and the conformal group is isomorphic to  $SO(d, 2)$ .

Note also that the Poincaré group together with the dilations forms a subgroup of the conformal group: this means that a theory invariant under translations, rotations and dilations is not necessarily invariant under SCT.

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<sup>1</sup>We are using the Euclidean CFT convention in which conformal generators are represented as differential operators without factors of  $i$ . The corresponding self-adjoint generators in the Lorentzian/unitary convention are obtained by multiplying these differential operators by  $i$ .

### 1.1.2 Correlation functions

We now briefly discuss how much conformal symmetry constrains correlation functions. We will focus on a massless scalar  $\Phi$  obeying the Klein-Gordon equation.

Consider a scalar operator  $O(x)$  with energy dimension  $\Delta_O$ . Extrapolating from the scale transformations it is tempting to postulate that under all conformal transformations:

$$O(x) \rightarrow \Omega(x)^{-\Delta_O} O(x) \quad (1.8)$$

But this is too strong since composite fields like  $(\partial\Phi)^2$  will pick up derivatives of the Weyl factor when  $\partial_\mu \Omega \neq 0$ .

**Definition 1.1.2.** Operators transforming homogeneously under the *whole* conformal group as in (1.8) are called *primary*.

The reason why we specified that the transformation property must be satisfied for the whole group will be clear shortly.

Since one can show that a SCT is just a combination of an inversion, i.e. the transformation  $x^\mu \mapsto \frac{x^\mu}{|x|^2}$ , a translation and another inversion, all conformal transformations are combinations of inversions and of transformations with constant  $\Omega$ , then to check whether an operator is primary we need only consider how it transforms under inversion.

Now consider the two-point function:

$$\langle O_1(x_1) O_2(x_2) \rangle := \frac{1}{\mathcal{Z}} \int D\Phi O_1(x_1) O_2(x_2) e^{-S[\Phi]}$$

where  $O_1$  and  $O_2$  are two scalar primaries,  $S$  is the action and  $\mathcal{Z}$  the generating functional of the theory.

Under a dilation  $x^\mu \mapsto \lambda x^\mu$  we see that:

$$\langle O_1(x'_1) O_2(x'_2) \rangle = \lambda^{-\Delta_1 - \Delta_2} \langle O_1(x_1) O_2(x_2) \rangle \quad (1.9)$$

Moreover, as a consequence of rotational and translational invariance the correlator must be a function of  $|x_1 - x_2|$ . Then equation (1.9) forces this function to be homogeneous of degree  $-\Delta_1 - \Delta_2$ , thus fixing:

$$\langle O_1(x_1) O_2(x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \quad (1.10)$$

where  $C_{12}$  is a constant. If we now perform an inversion, it is easy to show that:

$$x^\mu \mapsto \frac{x^\mu}{|x|^2} \Rightarrow |x_1 - x_2| \mapsto \frac{|x_1 - x_2|}{|x_1| |x_2|}, \quad \Omega(x) = \frac{1}{|x|^2}$$

which implies:

$$\langle O_1(x'_1) O_2(x'_2) \rangle = |x_1|^{2\Delta_1} |x_2|^{2\Delta_2} \langle O_1(x_1) O_2(x_2) \rangle$$

On the other hand:

$$\langle O_1(x'_1) O_2(x'_2) \rangle = \frac{C_{12}}{|x'_1 - x'_2|^{\Delta_1 + \Delta_2}} = (|x_1| |x_2|)^{\Delta_1 + \Delta_2} \langle O_1(x_1) O_2(x_2) \rangle$$

Hence, in order to satisfy conformal invariance,  $C_{12} \neq 0$  if and only if  $\Delta_1 = \Delta_2$ .

The same procedure fixes also three-point functions<sup>2</sup>:

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle = \frac{\lambda_{123}}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3} |x_2 - x_3|^{\Delta_2 + \Delta_3 - \Delta_1} |x_1 - x_3|^{\Delta_1 + \Delta_3 - \Delta_2}} \quad (1.11)$$

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<sup>2</sup>Clearly, correlation functions involving fields with spin are more complicated. Nevertheless, conformal invariance still strongly constrains their form, fixing their spacetime dependence up to a finite number of independent tensor structures.

where  $\lambda_{123}$  is a constant, called a *structure constant*.

Even though conformal invariance is a highly constraining symmetry, this trend stops here: starting from four (or more) point functions, one can construct conformally invariant coordinate ratios, known as *anharmonic ratios* (or cross-ratios). Any function of these quantities is allowed in the correlator.

As a consequence of the strong constraints imposed by conformal symmetry, CFTs possess the remarkable property of being definable without reference to a Lagrangian formulation. It is sufficient to specify the *CFT data*, namely the set  $\{O_i, \Delta_{O_i}, \lambda_{ijk}\}$ , consisting of the local operators of the theory, their conformal dimensions, and the three-point function structure constants. Once the CFT data are known, two- and three-point correlation functions are completely fixed by conformal symmetry, while four-point functions are further constrained by consistency conditions known as *crossing equations*. Correlation functions with a higher number of insertions can be recursively reduced to lower-point functions through the *Operator Product Expansion* (OPE), which will be introduced later. In this way, the full theory can, in principle, be reconstructed without the need for an explicit Lagrangian description.

### 1.1.3 Energy-momentum tensor and Ward identities

It is well known that the Noether currents associated with spacetime symmetries are expressed in terms of the energy-momentum tensor. Consequently, the latter plays a central role in conformal field theories.

For a theory described by a Lagrangian density  $\mathcal{L}$ , the canonical energy-momentum tensor is given by

$$T^\mu_\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta^\mu_\nu \mathcal{L}.$$

The canonical energy-momentum tensor is not necessarily symmetric. A symmetric tensor can be obtained through the Belinfante procedure, which modifies  $T^\mu_\nu$  by the addition of suitable total-derivative terms without affecting the associated conserved charges. Otherwise one can use the following trick: we can couple the system to a background metric  $g_{\mu\nu}(x)$  which under an infinitesimal coordinate transformation  $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$  transforms as  $\delta g_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ . Then we set:

$$\delta_\epsilon S = \frac{1}{2} \int \partial_\mu \epsilon_\nu \frac{\delta S}{\delta g_{\mu\nu}} d^d x \quad \Rightarrow \quad T_g^{\mu\nu} := \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \quad (1.12)$$

where  $g = \det(g_{\mu\nu})$ . For a flat spacetime, we take the limit  $g_{\mu\nu} \rightarrow \delta_{\mu\nu}$  and evaluate  $T_g^{\mu\nu}$  in this limit to get  $T^{\mu\nu}$  defined above. This produces automatically a symmetrized energy-momentum tensor.

We have used Lagrangian formalism to derive  $T^\mu_\nu$ , but a general CFT may not admit a Lagrangian. Therefore use the following definition.

**Definition 1.1.3.** The energy-momentum tensor  $T^{\mu\nu}$  is defined by the Noether current  $j^\mu = \epsilon_\nu T^{\mu\nu}$  corresponding to the coordinate transformation  $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$ .

One can then easily prove the following conserved currents:

$$\text{Translation: } j_P^\mu = \epsilon_\nu T^{\mu\nu}, \quad \epsilon_\nu = \text{const.}$$

$$\text{Rotation: } j_L^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$$

$$\text{Dilation: } j_D^\mu = T^\mu_\nu x^\nu$$

$$\text{SCT: } j_K^{\mu\nu} = 2x^\mu x_\rho T^{\rho\nu} - |x|^2 T^{\mu\nu}$$

Note that translation invariance implies that this tensor is conserved  $\partial_\mu T^{\mu\nu} = 0$ , whereas scale and SCT invariance both imply the tracelessness  $T^\mu_\mu = 0$ . This is not a surprise: if we look at equation (1.12) and combine it with (1.1) and (1.2):

$$\delta_\epsilon S = \int \partial_\mu \epsilon_\nu T^{\mu\nu} d^d x = \frac{1}{2} \int (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) T^{\mu\nu} d^d x = \frac{1}{d} \int \partial_\nu \epsilon^\nu T^\mu_\mu d^d x$$

Hence the tracelessness of the (symmetric and conserved) energy-momentum tensor implies the invariance of the action under conformal transformations.

One may be tempted to think that scale invariance, together with Poincaré invariance, automatically implies conformal invariance. In general, this is not the case, and a standard counterexample is provided by Maxwell theory in  $d \neq 4$ , which is scale invariant but not conformally invariant, since its energy-momentum tensor is not traceless.

One might argue that a non-vanishing trace  $T^\mu_\mu \neq 0$  would necessarily imply the breaking of scale invariance too. However, one can show that if

$$T^\mu_\mu = -\partial_\mu V^\mu,$$

for some vector operator  $V^\mu$ , known as the *virial current*, then it is possible to improve the dilation current in such a way that it remains conserved, and scale invariance is preserved.

Nevertheless, it is not always possible to extend this procedure to the special conformal currents, so that full conformal invariance is not guaranteed. An additional condition is required, namely that the virial current itself is a total derivative,

$$V^\mu = \partial_\nu S^{\nu\mu},$$

which allows one to improve the energy-momentum tensor to a traceless form, thereby restoring full conformal invariance.

For two-dimensional CFTs, a remarkable result by Zamolodchikov and Polchinski shows that, under suitable assumptions, namely unitarity (i.e. the absence of negative-norm states) and compactness (i.e. a discrete energy spectrum), scale invariance implies conformal invariance. In particular, for such theories one can always construct an improved energy-momentum tensor which is traceless.

At quantum level, these conservation laws are expressed through the Ward identities. Their general form is:

$$\partial_\mu \langle j^\mu_a(x) \Phi(x_1) \dots \Phi(x_n) \rangle = -i \sum_{i=1}^n \delta(x - x_i) \langle \Phi(x_1) \dots G_a \Phi(x_i) \dots \Phi(x_n) \rangle$$

where  $G_a$  is the representation of the hermitian quantum generator of the symmetry and  $j^\mu_a$  the associated conserved current.

In the case under consideration, one finds:

$$\begin{aligned} \text{Translation: } \partial_\mu \langle T^\mu_\nu(x) X \rangle &= - \sum_{i=1}^n \delta(x - x_i) \partial_{\nu_i} \langle X \rangle \\ \text{Rotation: } \epsilon_{\mu\nu} \langle T^{\mu\nu}(x) X \rangle &= -i \sum_{i=1}^n s_i \delta(x - x_i) \langle X \rangle \\ \text{Dilation: } \langle T^\mu_\mu(x) X \rangle &= -i \sum_{i=1}^n \Delta_i \delta(x - x_i) \langle X \rangle \end{aligned} \tag{1.13}$$

where  $X$  stands for a product of  $n$  primary operators,  $\epsilon_{\mu\nu}$  is an antisymmetric tensor coming from the definition of the spin generator,  $s_i$  is the spin of the operator  $O_i$  in  $X$  and  $\Delta_i$  its conformal dimension. The identity for SCT does not bring any additional information, so we have omitted it.

A crucial observation is the following: conformal invariance at the quantum level does not generically follow from conformal invariance at the classical level. A quantum field theory requires a regularization prescription, which typically introduces a scale into the theory. This scale breaks conformal symmetry, except at special values of the parameters corresponding to renormalization group fixed points. At such points, the classical tracelessness of the energy-momentum tensor is expected to persist at the quantum level (up to possible anomalies on curved backgrounds).

## 1.2 CFTs in two dimensions

Two-dimensional CFTs are interesting not only because of their numerous applications, but also because of their distinctive features. Indeed, in two dimensions the structure of these theories becomes significantly richer, as the finite-dimensional conformal group is promoted to an infinite-dimensional symmetry algebra.

### 1.2.1 The infinite and central extensions

Consider the Euclidean two-dimensional metric expressed in terms of the complex coordinate  $z = x^1 + ix^2$ :

$$ds^2[\mathbb{R}^2] = (dx^1)^2 + (dx^2)^2 = dzd\bar{z} \quad (1.14)$$

Now recall that the conformal group is isomorphic to  $SO(d+1, 1)$ . For  $d = 2$  one can show that the group  $SO^+(3, 1)$ , i.e. the connected component of  $SO(3, 1)$  containing the identity, is isomorphic to the projective special linear group  $PSL(2, \mathbb{C})$  or *Möbius group*:

$$z' = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1 \quad (1.15)$$

Remember that we had to compactify our spacetime adding a point at infinity. The Möbius group is indeed the group of complex automorphisms of the Riemann sphere as a complex manifold, i.e. the bijective conformal maps from the Riemann sphere to itself, where the Riemann sphere is the complex plane together with a point at infinity.

Although this is the global conformal group, note that any other holomorphic map  $z' = f(z)$  also multiplies the metric by a Weyl factor  $\Omega^2 = |\frac{\partial f}{\partial z}|^2$ . Expanding in a Laurent series,

$$f = \sum_n f_n z^{n+1}, \quad f_n \in \mathbb{C},$$

we identify the generators of these transformations as:

$$l_n = -z^{n+1} \partial \quad \bar{l}_n = -\bar{z}^{n+1} \bar{\partial}$$

These two infinite sets of generators form two commuting *Witt algebras*:

$$[l_n, l_m] = (n - m) l_{n+m} \quad [\bar{l}_n, \bar{l}_m] = (n - m) \bar{l}_{n+m} \quad [l_n, \bar{l}_m] = 0$$

Hence in two dimensions the local conformal group is infinitely extended! In particular, from the above commutation relations we see that the global conformal group is generated by  $\{l_{-1}, l_0, l_1\} \cup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$ .

If one naïvely attempts to quantize some of the most common classically conformally invariant theories, one does not obtain a representation of the Witt's algebra, but rather of one of its central extensions. Moreover, when the same theories are quantized on a curved manifold instead of a flat one, conformal symmetry becomes anomalous, and this anomaly is directly related to the central extension. A central extension of an algebra consists in the addition of a commuting (central) term to the usual commutation relations.

One can show that the only central extension compatible with the Witt's algebra reads:

$$[l_n, l_m] = (n - m) l_{n+m} + \frac{c}{12} (n^3 - n) \delta_{m+n, 0}$$

called the *Virasoro algebra* and  $c$  is called *central charge*.

In the case of Lorentzian signature, the conformal symmetry is enhanced to independent reparametrizations of the two lightlike coordinates,  $x^{\pm} = f^{\pm}(x^{\pm})$ , where

$$ds^2[\mathbb{R}^{1,1}] = -dt^2 + dx^2 = -dx^+ dx^-, \quad x^{\pm} = t \pm x$$

This is consistent with the fact that the Lie algebra  $\mathfrak{so}(2, 2)$  decomposes as

$$\mathfrak{so}(2, 2) \simeq \mathfrak{so}(2, 1)_+ \oplus \mathfrak{so}(2, 1)_-,$$

or equivalently that  $SO(2, 2)$  is locally, i.e. close to the identity, isomorphic to  $SO(2, 1)_+ \times SO(2, 1)_-$ . The two factors  $SO(2, 1)_\pm$  are themselves locally isomorphic to  $PSL(2, \mathbb{R})$ , whose action on the light rays is given by fractional linear transformations of the form (1.15), but with real coefficients. To make this action globally well-defined, each light ray is compactified by adding a point at infinity, yielding two copies of  $S^1$ . The infinite-dimensional extension of the global conformal group is then given by the group of orientation-preserving diffeomorphisms,

$$Diff^+(S^1) \times Diff^+(S^1),$$

whose Lie algebra is generated by two commuting copies of the Virasoro algebra.

Before ending this section let's note that for certain manipulations it can be useful to extend our variable space by making  $z$  and  $\bar{z}$  independent. Then one can always find the physical space in the two-dimensional plane  $z^* = \bar{z}$ , where the star indicates the complex conjugation.

### 1.2.2 Primary fields

We want to study more in detail primary fields. As we have said earlier, a primary operator transforms homogeneously as in (1.8) when a generic conformal transformation is performed. Now that the group is infinitely extended, there can be fields that satisfy such transformation rule only for the global conformal group.

**Definition 1.2.1.** We call an operator  $O(z, \bar{z})$  a *quasi-primary* if it transforms homogeneously under the global conformal group.

Clearly all primary operators are also quasi-primary, but the converse is not true.

Let's see how the transformation rule (1.8) is written in the complex coordinates. After performing holomorphic  $z \rightarrow w(z)$  and anti-holomorphic  $\bar{z} \rightarrow \bar{w}(\bar{z})$  transformations, the metric (1.14) picks up a Weyl factor given by the two corresponding complex jacobians. Then we have:

$$O(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} O(z, \bar{z}) \quad (1.16)$$

where we have defined the left/right *conformal weights*  $h, \bar{h} \in \mathbb{R}$ . To see how these two numbers are related to the dimension  $\Delta_O$  of the operator  $O$  let's perform a rotation plus a dilatation by multiplying the coordinates  $z \rightarrow \lambda z$  with  $\lambda = |\lambda|e^{i\theta} \in \mathbb{C}$ :

$$O(\lambda z, \lambda^* \bar{z}) = (|\lambda|e^{i\theta})^{-h} (|\lambda|e^{-i\theta})^{-\bar{h}} O(z, \bar{z}) = |\lambda|^{-(h+\bar{h})} e^{-i\theta(h-\bar{h})} O(z, \bar{z})$$

where we are working on the physical plane  $\lambda^* = \bar{\lambda}$ . Hence we have found that:

$$\Delta_O = h + \bar{h} \quad s_O := h - \bar{h}$$

where  $s_O$  is the spin of the operator.

We conclude this discussion with the following definition:

**Definition 1.2.2.** A holomorphic field, i.e. which depends only on  $z$ , is called *chiral*, whereas an anti-holomorphic field is called *anti-chiral*. In Lorentzian signature, the same terminology is used for fields depending respectively on one of the two light-cone coordinates  $x^+$  and  $x^-$ , and one also speaks of left-moving and right-moving fields.

Accordingly, one often refers to a theory depending only on one of these variables as a *chiral CFT*. In the Lorentzian setting this terminology has a direct geometric meaning: since  $x^+$  and  $x^-$  parametrize light rays, a chiral CFT may be regarded as a one-dimensional conformal theory living on a light ray,

or on its compactification  $S^1$ . In the Euclidean formulation, instead, a holomorphic theory depending only on  $z$  is still naturally defined on the complex plane, and should not be literally interpreted as a one-dimensional theory.

As will become clear when discussing the Hilbert space of a CFT, in both the Euclidean and Lorentzian settings a full two-dimensional theory is obtained by consistently combining two chiral components.

### 1.2.3 The operator-state correspondence

By inspecting the algebra (1.7), we observe that  $P_\mu$  and  $K_\mu$  act respectively as raising and lowering operators with respect to the grading defined by the dilatation operator  $D$ . This suggests that  $D$  may play a role analogous to that of a Hamiltonian.

To make this idea precise, let us consider the conformal map from the complex plane to the cylinder,

$$z \mapsto w = \log z = \log |z| + i\theta = \tau + i\theta.$$

Here  $\tau$  parametrizes the height along the cylinder, while  $\theta$  is the angular coordinate.

Writing the flat metric on the plane in polar coordinates,

$$ds^2[\mathbb{R}^2] = dr^2 + r^2 d\theta^2 = r^2 \left( \frac{dr^2}{r^2} + d\theta^2 \right),$$

and using  $r = e^\tau$ , we obtain

$$ds^2[\mathbb{R}^2] = e^{2\tau} (d\tau^2 + d\theta^2).$$

Up to the Weyl factor  $e^{2\tau}$ , this is precisely the metric of a cylinder.

Under this change of coordinates, the dilatation generator becomes

$$D = -x^\mu \partial_\mu = -r \partial_r = -\partial_\tau,$$

showing that radial evolution on the plane is mapped to time evolution on the cylinder. For this reason,  $D$  is often referred to as the *conformal Hamiltonian*.

The Hilbert space can then be constructed as the span of the eigenstates of the conformal Hamiltonian. Equivalently, to each circle of fixed radius  $r$  on the plane, or fixed time  $\tau$  on the cylinder, one associates a Hilbert space. This construction is known as *radial quantization* and it is shown in figure 1.2.

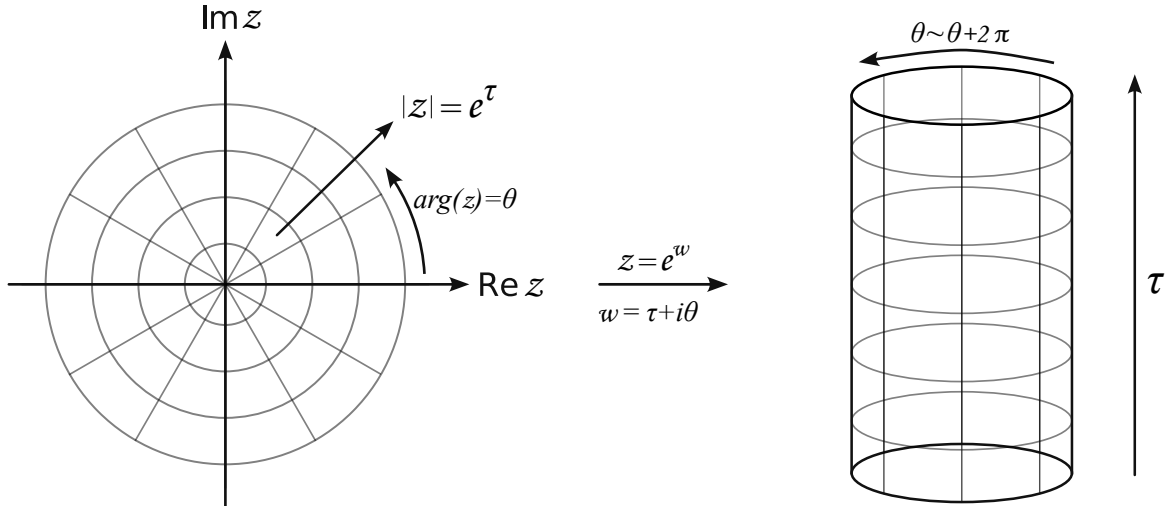


Figure 1.2: The plane-cylinder correspondence underlying radial quantization.

To further formalize this picture, let us assume the existence of a conformally invariant vacuum state  $|\Omega\rangle$ . In particular, a local operator  $\Phi(z, \bar{z})$  defines the state

$$|\Phi\rangle = \lim_{z, \bar{z} \rightarrow 0} \Phi(z, \bar{z}) |\Omega\rangle.$$

This establishes a one-to-one correspondence between local operators on the plane and states on the cylinder, known as the *operator-state correspondence*. In this picture, an insertion at the origin corresponds to a state defined at  $\tau \rightarrow -\infty$ , while insertions at larger radii correspond to later Euclidean times. The vacuum state itself can be regarded as the state associated with the insertion of the identity operator.

As we have already pointed out,  $P_\mu$  and  $K_\mu$  behave as raising/lowering operators with respect to the dilation generator  $D$ , which as said, plays the role of a Hamiltonian. Therefore, acting with  $P_\mu$  on a primary state generates its first excited states. However, in two-dimensional CFT the global conformal algebra is enhanced to the infinite-dimensional Virasoro algebra. As a consequence, there exists an infinite tower of additional raising operators, which generate a much richer set of excited states beyond those obtained from the global conformal symmetry. We will return to this later.

To endow the Hilbert space with an inner product, we must define Hermitian conjugation. In Minkowski spacetime, Hermitian conjugation does not act on the spacetime coordinates. In Euclidean signature, however, the Euclidean time coordinate must be reversed ( $\tau \rightarrow -\tau$ ) upon Hermitian conjugation. In radial quantization this corresponds to the mapping  $z \rightarrow 1/z^*$ . Then for a quasi-primary, we define:

$$\Phi(z, \bar{z})^\dagger := \bar{z}^{-2h} z^{-2\bar{h}} \Phi(1/\bar{z}, 1/z) \quad (1.17)$$

so that:

$$\langle \Phi | = \lim_{z, \bar{z} \rightarrow 0} \langle \Omega | \Phi^\dagger(z, \bar{z}) = \lim_{z, \bar{z} \rightarrow 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle \Omega | \Phi(1/\bar{z}, 1/z) \stackrel{\zeta := 1/\bar{z}}{=} \lim_{\zeta, \bar{\zeta} \rightarrow \infty} \zeta^{2h} \bar{\zeta}^{2\bar{h}} \langle \Omega | \Phi(\zeta, \bar{\zeta})$$

The prefactors in (1.17) guarantee that the scalar product  $\langle \Phi | \Phi \rangle$  has a well-defined limit when  $\zeta, \bar{\zeta} \rightarrow \infty$ .

We can also expand a conformal field  $\Phi(z, \bar{z})$  of dimensions  $(h, \bar{h})$  in modes by considering its Laurent expansion:

$$\Phi(z, \bar{z}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \Phi_{m,n}, \quad \Phi_{m,n} = \frac{1}{2\pi i} \oint dz z^{m+h-1} \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+\bar{h}-1} \Phi(z, \bar{z}) \quad (1.18)$$

One can relate this expansion to the Fourier expansion of the field on the cylinder (given its periodic behaviour) by using the conformal map between the latter and the complex plane.

The usual Hermitian conjugation,

$$\Phi(z, \bar{z})^\dagger = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \bar{z}^{-m-h} z^{-n-\bar{h}} \Phi_{m,n}^\dagger,$$

compared to the definition of equation (1.17) gives:

$$\Phi_{m,n}^\dagger = \Phi_{-m, -n}$$

Note that to be well defined, the "in" and "out" states, respectively at  $z = 0$  and  $z = \infty$ , must obey:

$$\Phi_{m,n} |\Omega\rangle = 0, \quad m > -h, n > -\bar{h} \quad (1.19)$$

Within radial quantization, time ordering becomes a *radial ordering*, defined as:

$$\mathcal{R}[O_1(z)O_2(w)] = \begin{cases} O_1(z)O_2(w) & \text{if } |z| > |w| \\ O_2(w)O_1(z) & \text{if } |z| < |w| \end{cases}$$

If the two fields are fermions, a minus sign is added in front of the second expression. All operators in correlation functions are understood to be radially ordered.

### 1.2.4 The conformal charge

Let us count the free components of the energy momentum tensor in  $d = 2$ . Being a symmetric traceless rank-2 tensor, it has only two, out of four, free components. Considering the complex coordinates (1.14), these components are  $T_{zz}(z, \bar{z}) := T(z, \bar{z})$  and  $T_{\bar{z}\bar{z}}(z, \bar{z}) := \bar{T}(z, \bar{z})$ . Then the continuity equation  $\partial_\mu T^\mu_\nu = 0$  implies:

$$\partial_z \bar{T} = 0 \Rightarrow \bar{T} = \bar{T}(\bar{z}) \quad \partial_{\bar{z}} T = 0 \Rightarrow T = T(z)$$

It follows that there is an infinite number of conserved left-moving and right-moving currents:

$$(j_z^{(n)}, j_{\bar{z}}^{(n)}) = (z^{n+1} T(z), 0) \quad (\bar{j}_z^{(n)}, \bar{j}_{\bar{z}}^{(n)}) = (0, \bar{z}^{n+1} \bar{T}(\bar{z}))$$

The corresponding charges generate the infinite Virasoro extension of the two-dimensional conformal algebra discussed in the previous section. In radial quantization a fixed time slice corresponds to a contour of fixed radius. Therefore we define the conformal charge as:

$$Q_{\epsilon, \bar{\epsilon}} := \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) + \frac{1}{2\pi i} \oint d\bar{z} \bar{\epsilon}(\bar{z}) \bar{T}(\bar{z})$$

From now on we focus just on the holomorphic part. We can expand  $T(z)$  as in (1.18):

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \quad L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z)$$

whereas

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1}$$

Then it is easy to see that the conserved charge associated to the current  $j^{(n)} = z^{n+1} T(z)$  coincide with  $L_n$  and corresponds to the transformation given by  $\epsilon(z) = z^{n+1}$ . Moreover, the conformal charge can be expressed in modes as:

$$Q_\epsilon = \sum_{n \in \mathbb{Z}} \epsilon_n L_n$$

In conclusion, we have found that the expansion modes of the energy-momentum tensor are the generators of the conformal transformations. As already mentioned, the classical generators obey the Witt's algebra and the quantum generators obey the Virasoro algebra:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{m+n,0}$$

and similarly for the anti-holomorphic modes. The generators of the global conformal group  $PSL(2, \mathbb{C})$  are then identified with  $L_{-1}$ ,  $L_0$  and  $L_1$  (and their anti-holomorphic counterparts). In particular, the operator  $L_0 + \bar{L}_0$  generates dilations  $(z, \bar{z}) \rightarrow \lambda(z, \bar{z})$ , which are, as said, time translations in radial quantization. Thus,  $D = L_0 + \bar{L}_0$ .

Before moving on, let us note that the conformal Hamiltonian  $D$  must be self-adjoint in order to define a proper Hilbert-space generator of radial evolution. In radial quantization this follows from the adjointness relations  $L_n^\dagger = L_{-n}$  and  $\bar{L}_n^\dagger = \bar{L}_{-n}$ , which in particular implies  $L_0^\dagger = L_0$  and  $\bar{L}_0^\dagger = \bar{L}_0$ . Hence  $D^\dagger = D$ .

The same adjointness relations also expresses the well-known fact that, in radial quantization, translations and special conformal transformations are conjugate to each other. Indeed,  $L_{-1}, \bar{L}_{-1}$  generate translations on the plane, while  $L_1, \bar{L}_1$  generate special conformal transformations, and  $L_{-1}^\dagger = L_1$ , and  $\bar{L}_{-1}^\dagger = \bar{L}_1$ . This is the two-dimensional counterpart of the relation  $P_\mu^\dagger = K_\mu$ , which is required by compatibility of the conformal algebra (4.3.2) with  $D^\dagger = D$ .

From the quantum Noether theorem we know that this charge generates the variation of the fields as:

$$\delta_\epsilon O = [Q_\epsilon, O] \tag{1.20}$$

If we insert the definition of the conformal charge on the right-hand side we find:

$$\delta_\epsilon O = \frac{1}{2\pi i} \oint dz \epsilon(z) [T(z), O(w)]$$

How do we interpret this equation? All operator equations are to be viewed as living inside radial-ordered correlation functions. Therefore we have to choose the contour according to the order of the integrated operators:

$$\delta_\epsilon O = \frac{1}{2\pi i} \oint_{|z|>|w|} dz \epsilon(z) T(z) O(w) - \frac{1}{2\pi i} \oint_{|z|<|w|} dz \epsilon(z) T(z) O(w)$$

where the radial ordering of the integrand is, as always, understood. As shown in figure 1.3 this is equivalent to a single integration over a counterclockwise contour  $C$  encircling the insertion point  $w$  of the operator  $O$ . Therefore<sup>3</sup>:

$$\delta_\epsilon O = \frac{1}{2\pi i} \oint_C dz \epsilon(z) T(z) O(w) \quad (1.21)$$

Actually this expression could be derived by using the Ward identities (1.13) and, when generalized to an arbitrary product of operators with insertion points  $(w_i, \bar{w}_i)$ , is referred as the *Conformal Ward identity*.

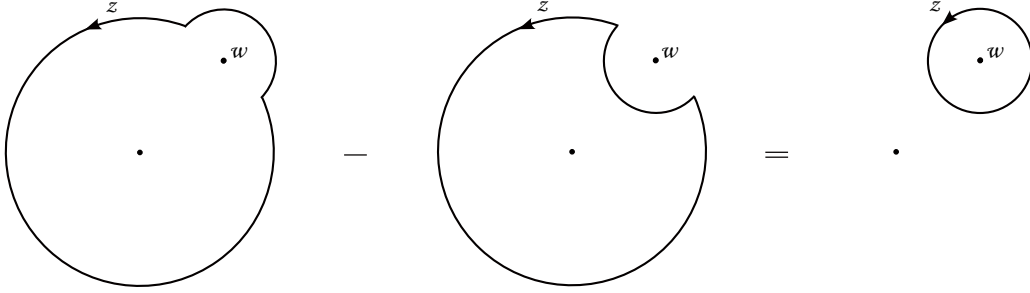


Figure 1.3: Evaluation of the radial commutator.

Now that we know how to compute the variation of any field under a generic conformal transformation, we can evaluate the transformation of the energy-momentum tensor itself. Using the mode expansion of  $T(w)$  and the Virasoro algebra, one can show that:

$$\delta_\epsilon T(w) = [Q, T(w)] = \sum_{n \in \mathbb{Z}} \epsilon_n [L_n, T(w)] = -\frac{c}{12} \partial_w^3 \epsilon(w) - 2T(w) \partial_w \epsilon(w) - \epsilon(w) \partial_w T(w)$$

Integrating this expression one finds:

$$T(w) = \left( \frac{dw}{dz} \right)^{-2} \left[ T(z) - \frac{c}{12} \{w, z\} \right], \quad \{w, z\} = \frac{d_z^3 w}{d_z w} - \frac{3}{2} \frac{d_z^2 w}{d_z w}$$

where  $\{w, z\}$  is called *Schwarzian derivative*. An explicit calculation gives:

$$\left\{ \frac{az+b}{cz+d}, z \right\} = 0, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1$$

Therefore the energy-momentum tensor is a quasi-primary (for  $c \neq 0$ ) field of conformal dimension 2.

Note that under the conformal transformation  $z = e^w$ , mapping the plane to the cylinder, the energy-momentum tensor transforms as:

$$T_{cyl}(w) = z^2 T_{pl}(z) - \frac{c}{24}$$

<sup>3</sup>Even though  $O(w)$  is evaluated at a fixed point, it cannot be regarded as a constant with respect to the contour integration, since the relevant object is the product  $T(z)O(w)$  inside correlation functions. By translational invariance, this product depends on the relative coordinate  $z - w$ , so the insertion at  $w$  influences the value of the contour integral.

On the plane, we will see shortly that  $\langle T_{pl} \rangle = 0$ , and therefore

$$\langle T_{cyl} \rangle = -\frac{c}{24}$$

This shows that the central charge can be interpreted as the Casimir energy of the theory on the cylinder. Including also the anti-holomorphic sector, the cylinder Hamiltonian is shifted to  $H_{cyl} = L_0 + \bar{L}_0 - \frac{c}{12}$ .

To conclude the section, since now we have all the ingredients, we formalize what we mean by a compact CFT:

**Definition 1.2.3.** A CFT is said to be *compact* if all conformal dimensions are real, the spectra of  $L_0$  and  $\bar{L}_0$  are discrete and the vacuum  $|\Omega\rangle$  is unique.

In the sequel we will always work with this assumption.

### 1.2.5 The Hilbert space structure

Consider now a primary operator  $O(z)$ , chiral for simplicity. We now consider an infinitesimal transformation of the coordinates given by  $w = z + \epsilon(z)$ . Then from (1.16) we get:

$$O(z) = (1 + \partial_z \epsilon(z))^h O(z + \epsilon(z)) = (1 + h\partial_z \epsilon + o(\epsilon^2))(O(z) + \partial_z O(z)\epsilon + o(\epsilon^2))$$

From which we find:

$$\delta_\epsilon O = (\epsilon \partial_z + h \partial_z \epsilon) O \quad (1.22)$$

If we now choose  $\epsilon(z) = z^{n+1}$ , according to (1.20) we find the commutation relation:

$$[L_n, O(z)] = (z^{n+1} \partial_z + h(n+1)z^n) O(z) \quad (1.23)$$

Now let's use this transformation to better characterize the Hilbert space of radial quantization. First of all note that equation (1.19) for the energy-momentum tensor reads:

$$L_n |\Omega\rangle = 0 \quad \bar{L}_n |\Omega\rangle = 0, \quad n \geq -1 \quad (1.24)$$

This condition implies the invariance of the vacuum with respect to the global conformal group and the vanishing of the expectation value of the energy-momentum tensor:

$$\langle \Omega | T(z) | \Omega \rangle = 0 \quad \langle \Omega | \bar{T}(\bar{z}) | \Omega \rangle = 0$$

Let's now consider a state created by a primary of conformal weights  $(h, \bar{h})$ :

$$|h, \bar{h}\rangle = O(0, 0) |\Omega\rangle$$

Using (1.24) and the commutation relations (1.23) we get:

$$L_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle \quad \bar{L}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle$$

i.e. primary states are eigenstates of the Hamiltonian. Likewise, we have:

$$L_n |h, \bar{h}\rangle = 0 \quad \bar{L}_n |h, \bar{h}\rangle = 0, \quad n > 0 \quad (1.25)$$

Using the Virasoro algebra we get:

$$[L_0, L_{-m}] = mL_{-m}$$

Meaning that the generators of the Virasoro algebra act as raising/lowering operators for the eigenstates of  $L_0$ . This was already pointed out for  $P_\mu$  and  $K_\mu$  in section 1.2.3; we now see that such behaviour adapts to the whole extended conformal group. This means that excited states may be obtained by successive applications of these operators on the state  $|h\rangle$ :

$$L_{-k_1} \dots L_{-k_n} |h\rangle, \quad 1 \leq k_1 \leq \dots \leq k_n \quad (1.26)$$

where the increasing order is a convention; a different ordering can always be brought into a linear combination of well ordered states by applying the Virasoro algebra. The above state is an eigenstate of  $L_0$  with eigenvalue  $h' = h + k_1 + \dots + k_n = h + l$ , where  $l \in \mathbb{N}$ ,  $0 \in \mathbb{N}$ , is the *level* of the state.

**Definition 1.2.4.** The states (1.26) are called *descendants*.

Descendants are clearly not primary states as they don't satisfy (1.25).

The subset of the full Hilbert space generated by the primary state  $|h\rangle$  and its descendants is closed under the action of the Virasoro generators and thus forms a *representation* (more correctly, a *module*) of the Virasoro algebra. This representation is called a *highest-weight representation* and the corresponding representation space is called a *Verma module*.

Thanks to (1.17), the inner product of two states in a Verma module is well defined:

$$L_{-k_1} \dots L_{-k_n} |h\rangle, \quad L_{-p_1} \dots L_{-p_n} |h\rangle \quad \Rightarrow \quad \langle h | L_{k_1} \dots L_{k_n} L_{-p_1} \dots L_{-p_n} |h\rangle$$

This product may be evaluated using the Virasoro algebra and vanishes unless the two states belong to the same level. Moreover, Hermiticity also forces  $h$  to be real:  $L_0^\dagger = L_0 \Rightarrow \langle h | h^* = (h | h)^\dagger = (L_0 | h)^\dagger = \langle h | L_0 = \langle h | h$ .

The same discussion can be carried over also for the anti-holomorphic generators. If we denote by  $\mathcal{V}_h^c$  and  $\bar{\mathcal{V}}_{\bar{h}}^c$  the Verma modules generated by the sets  $\{L_{-n}\}$  and  $\{\bar{L}_{-n}\}$ , respectively, for a given value  $c$  of the central charge and highest weights  $h$  and  $\bar{h}$ , the *whole* Hilbert space can be decomposed as

$$\mathcal{H} = \bigoplus_{h, \bar{h}} V_{h, \bar{h}} \otimes \mathcal{V}_h^c \otimes \bar{\mathcal{V}}_{\bar{h}}^c$$

Here  $V_{h, \bar{h}}$  is a finite-dimensional vector space accounting for the multiplicity  $Z_{h, \bar{h}} \in \mathbb{N}$  of the corresponding irreducible representation, with

$$\dim(V_{h, \bar{h}}) = Z_{h, \bar{h}}.$$

For instance, if the same representation appears more than once, one has multiple primary fields with identical conformal weights  $(h, \bar{h})$ . Any linear combination of such primary fields is still a primary with the same conformal weights. The space  $V_{h, \bar{h}}$  can thus be interpreted as the space spanned by these states, carrying the representation labeled by  $(h, \bar{h})$ . Equivalently, one can write

$$\mathcal{H} = \bigoplus_{h, \bar{h}} Z_{h, \bar{h}} \mathcal{V}_h^c \otimes \bar{\mathcal{V}}_{\bar{h}}^c, \quad Z_{h, \bar{h}} \in \mathbb{N}$$

Interpreting  $Z_{h, \bar{h}}$  as the number of copies in the direct sum of the subsequent Hilbert space, the equivalence follows from the fact that, for any vector space  $\mathcal{V}$ , the direct sum of  $Z$  copies of  $\mathcal{V}$  can be written as  $\mathcal{V} \oplus \dots \oplus \mathcal{V} \simeq V \otimes \mathcal{V}$ , where  $V \simeq \mathbb{C}^Z$  is a  $Z$ -dimensional vector space such that  $\dim(V) = Z$ .

However, not all states (1.26) in  $\mathcal{H}$  are physical: there can exist both null states and negative-norm states. For now, we will remove only the unphysical null states, while the removal of negative-norm states will be discussed later in the context of unitary CFTs.

**Definition 1.2.5.** A *null (or singular) state* is a non-vanishing descendant state  $|\chi\rangle \in \mathcal{V}_h^c$ , different from  $|h\rangle$ , which is itself a highest-weight state, i.e.

$$L_n |\chi\rangle = 0 \quad \text{for all } n > 0$$

Null states generate a submodule  $\mathcal{V}_\chi \subset \mathcal{V}_h^c$ , which is invariant under the action of the Virasoro algebra. Singular vectors are orthogonal to the entire Verma module:

$$\langle \chi | L_{-k_1} \dots L_{-k_n} |h\rangle = \langle h | L_{k_n} \dots L_{k_1} |\chi\rangle^* = 0$$

In particular this implies  $\langle \chi | \chi \rangle = 0$ . These properties also extend to all descendants of  $|\chi\rangle$ .

We have thus identified an invariant subspace under the action of the conformal algebra. By definition this means that the original representation is reducible. Therefore, from a Verma module containing

one or more singular vectors, one may construct an irreducible representation of the Virasoro algebra by quotienting out the null submodules:

$$M_h^c := \mathcal{V}_h^c / \mathcal{N}_h^c$$

where  $\mathcal{N}_h^c$  denotes the submodule generated by all null states.

These irreducible representations will be denoted as  $M_h^c$ , to distinguish them from the reducible Verma modules. The physical Hilbert space can then be written as:

$$\mathcal{H}_{phys} = \bigoplus_{h, \bar{h}} V_{h, \bar{h}} \otimes M_h^c \otimes \bar{M}_{\bar{h}}^c \simeq \bigoplus_{h, \bar{h}} Z_{h, \bar{h}} M_h^c \otimes \bar{M}_{\bar{h}}^c \quad (1.27)$$

In the sequel, by  $\mathcal{H}$  we will mean the physical Hilbert space.

### 1.2.6 Operator Product Expansion

In quantum physics, typical experiments probe composite operators made out of products of elementary fields and their derivatives. Mathematically, considering a scalar field  $\phi$ , for a distant observer two nearby insertions of  $\phi$  may effectively look like an insertion of  $\phi^2$ ,  $\phi \partial_\mu \phi$ , etc. However, the coincidence limit is in general a singular operation<sup>4</sup>. In a generic QFT, turning this intuition into a precise mathematical statement is difficult and usually leads only to asymptotic expansions. In a CFT, instead, where there are no scales other than the distance between the insertions, this procedure, called the *Operator Product Expansion* (OPE), is well-defined and has a finite radius of convergence.

Thanks to the operator-state correspondence, the insertion of two operators at the points  $(w, \bar{w})$  and  $(z, \bar{z})$  creates the state

$$|\Psi\rangle = O_1(z, \bar{z}) O_2(w, \bar{w}) |\Omega\rangle.$$

We can decompose this state in a complete basis consisting of all highest-weight states and their descendants:

$$|\Psi\rangle = \sum_k C_{ij}^k(z - w, \bar{z} - \bar{w}, \partial_w, \partial_{\bar{w}}) O_k(w, \bar{w}) |\Omega\rangle. \quad (1.28)$$

The above sum runs only over primary operators  $O_k$ , while the contribution of their descendants is encoded in the differential operators  $C_{ij}^k(z - w, \bar{z} - \bar{w}, \partial_w, \partial_{\bar{w}})$ . By translational invariance, these coefficients depend only on the separation between the two insertions. By studying three-point functions and exploiting conformal invariance, one can determine the coefficients  $C_{ij}^k$  up to an overall multiplicative constant. These constants are precisely the structure constants  $\lambda_{ijk}$  appearing in the three-point correlators. For this reason, together with the spectrum of conformal weights, they constitute the CFT data required to define a conformal field theory non-perturbatively, even in the absence of a Lagrangian description. Since the only singularities of CFT correlation functions are poles, we expect the expansion (1.28) to converge inside the disk  $|z - w| < |w' - w|$ , where  $w'$  denotes the nearest other operator insertion. This is illustrated in figure 1.4.

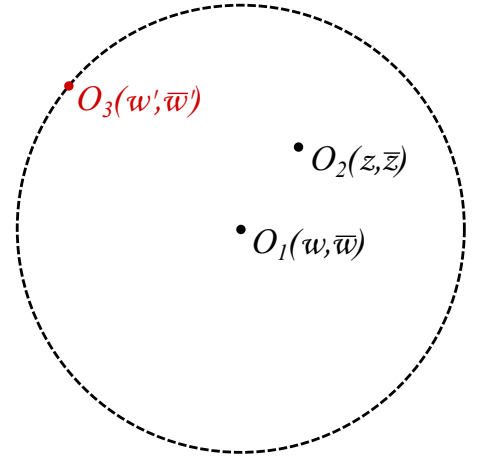


Figure 1.4: Schematics of the OPE in radial quantization.

In general, we write the OPE as

$$O_1(z, \bar{z}) O_2(w, \bar{w}) = \sum_k C_{ij}^k(z - w, \bar{z} - \bar{w}, \partial_w, \partial_{\bar{w}}) O_k(w, \bar{w}).$$

<sup>4</sup>For instance, consider the general form of the two-point function (1.10).

It is understood that this equation has a precise mathematical meaning only inside correlation functions. Moreover, radial ordering is also implicitly understood. Finally, other operator insertions may be present in the correlator, but according to the previous discussion the expansion remains valid provided that all such insertions lie outside the convergence disk, namely at distances greater than  $|z - w|$  from the expansion point.

The singular terms of the OPE as  $z \rightarrow w$  completely determine the action of conserved charges defined via contour integrals. Indeed, by Cauchy's residue theorem, any regular contributions vanish upon integration. Consequently, this singular behavior encodes all the relevant physics regarding the commutation relations of the symmetry generators (1.21).<sup>5</sup> For this reason, it is standard practice to omit the regular terms and display only the singular part of the OPE.

For example, from the conformal Ward identity one can derive that the OPE between  $T(z)$  and a generic operator  $O(w, \bar{w})$  with conformal weights  $(h, \bar{h})$  takes the form

$$T(z)O(w, \bar{w}) = \dots + \frac{h O(w, \bar{w})}{(z - w)^2} + \frac{\partial O(w, \bar{w})}{z - w} + \dots \quad (1.29)$$

and similarly for the expansion involving  $\bar{T}(\bar{z})$ . We see that the coefficient of the  $(z - w)^{-2}$  singularity contains the conformal weight of  $O(w, \bar{w})$ . Moreover, comparing this expansion with (1.21) and (1.22), one finds that for a primary operator the singular terms appearing in (1.29) are the only ones present.

Once the expansion (1.29) is known, one can derive the OPE of the energy-momentum tensor with itself:

$$T(z)T(w) = \frac{c/2}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w} + \dots$$

where  $c$  is the central charge.

Note that one may equivalently start from the Ward identities, derive the above OPE, which naturally introduces the constant  $c$ , and then obtain the commutation relations of the modes  $L_n$ , proving that they satisfy the Virasoro algebra. Here we have followed the opposite route, namely assuming the Virasoro algebra as the correct symmetry algebra of the quantum theory and motivating the appearance of the central extension through the Casimir energy. This allows for a more direct presentation while avoiding some technical computations.

### 1.2.7 Minimal models and Rational CFTs

When talking about the Hilbert space of a CFT, we have identified two issues: null vectors and negative-norm states. The first problem has already been addressed, and now we have to discuss the remaining one.

**Definition 1.2.6.** A CFT is said to be *unitary* if it contains no negative-norm states.

One can show, using the Virasoro algebra, that:

$$\langle h | L_n L_{-n} | h \rangle = \left( 2nh + \frac{c}{12} n(n^2 - 1) \right) \langle h | h \rangle. \quad (1.30)$$

Note that for  $n = 1$  we obtain, in a unitary theory:

$$\langle h | L_1 L_{-1} | h \rangle = 2h \langle h | h \rangle \geq 0 \quad \Rightarrow \quad h \geq 0.$$

In addition, if  $c < 0$ , the norm (1.30) becomes negative for sufficiently large  $n$ . Therefore, all representations with negative central charge are non-unitary.

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<sup>5</sup>Although the mode expansion of  $\epsilon(z)$  may introduce poles at  $z = 0$ , the integration contour in (1.21) is chosen to tightly encircle only the insertion point  $w$ . Since this contour is arbitrarily small and isolates  $w$  from any other poles or operator insertions, the regular terms of the  $TO$  expansion yield a vanishing integral, while ensuring that we are in the correct regime to perform the OPE.

The necessary and sufficient conditions for unitarity are obtained by considering the *Gram matrix* of inner products between all basis states. We denote by  $|i\rangle$  the basis states (1.26) of the Verma module and define:

$$M_{ij} = \langle i|j\rangle, \quad M^\dagger = M.$$

This matrix is block-diagonal, with blocks  $M^{(l)}$  corresponding to states at level  $l$ . A generic state is a linear combination  $|a\rangle = \sum_i a_i |i\rangle$ , and its norm is (in matrix notation)  $\langle a|a\rangle = a^\dagger M a$ . Since  $M$  is Hermitian, it can be diagonalized by a unitary matrix  $U$ :

$$M = U \Lambda U^\dagger.$$

Setting  $b = Ua$ , we obtain:

$$\langle a|a\rangle = \sum_i \lambda_i |b_i|^2,$$

where  $\lambda_i$  are the eigenvalues of  $\Lambda$ . Consequently, there exist negative-norm states if and only if  $M$  has at least one negative eigenvalue. Moreover, null vectors appear when one or more eigenvalues vanish, in which case the Verma module becomes reducible.

Hence, singular vectors occur whenever  $\det(M) = 0$ . There exists a general formula, due to Kac, for the determinant of the Gram matrix, the *Kac determinant*:

$$\det(M^{(l)}) = \alpha_l \prod_{\substack{r,s \geq 1 \\ rs \leq l}} (h - h_{r,s}(c))^{p(l-rs)}$$

Here  $p(l-rs)$  denotes the number of partitions of the integer  $l-rs$ , and  $\alpha_l$  is a positive constant independent of  $h$  and  $c$ :

$$\alpha_l = \prod_{\substack{r,s \geq 1 \\ rs \leq l}} ((2r)^s s!)^{m(r,s)}, \quad m(r,s) = p(l-rs) - p(l-r(s+1)).$$

The functions  $h_{r,s}(c)$  may be expressed in various equivalent forms. For example:

$$h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, \quad m = -\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{25-c}{1-c}}$$

with  $r$  and  $s$  positive integers. One can freely choose either the positive or negative branch of the square root in  $m$  thanks to a symmetry property of the functions  $h_{r,s}$ . In the  $(c, h)$  plane, the Kac determinant vanishes along the curves  $h = h_{r,s}(c)$ , called *vanishing curves*. Their intersections correspond to Virasoro representations containing more than one independent null vector. These curves are shown in figure 1.5.

Note that the Kac determinant does not provide the eigenvalues of the Gram matrix, but only their product. Therefore, a more detailed analysis is required to identify unitary theories. Since this analysis is rather involved, we only present the final results:

- For  $c > 1$  and  $h \geq 0$  there are no null states and no negative norm states;
- For  $c = 1$  one finds  $\det(M^{(l)}) = 0$  for  $h = \frac{n^2}{4}$ ,  $n \in \mathbb{Z}$ ;

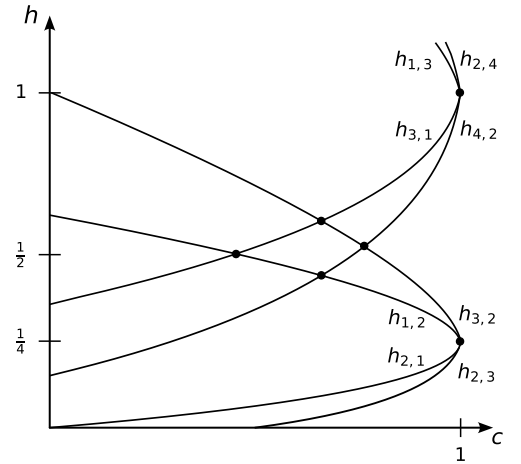


Figure 1.5: The vanishing curves  $h_{r,s}(c)$  in the  $(c, h)$ -plane.

- For  $c < 1$  and  $h \geq 0$  of course we have reducible representations along the vanishing curves. A more careful analysis shows that negative states are absent only on certain intersection points of such curves; these intersection points are parametrized by:

$$c(m) = 1 - \frac{6}{m(m+1)}, \quad h = h_{r,s}(m) \quad (1.31)$$

with  $m = 3, 4, \dots$ ,  $1 \leq r \leq m-1$  and  $1 \leq s \leq m$ .

The family of models with central charge (1.31) is called *minimal models*. Their importance relies on the fact that they are exactly solvable theories and, moreover, describe the continuum limit of discrete statistical models at criticality, such as the Ising model or the Potts model. The discovery of minimal models and their identification with known statistical systems at criticality constitute one of the greatest applications of conformal invariance.

Minimal models are characterized by only a finite number of irreducible representations of the Virasoro algebra. This property places them within a wider class of theories:

**Definition 1.2.7.** A *rational conformal field theory (RCFT)* is a CFT with a finite number of irreducible representations of its chiral algebra. Equivalently, it possesses only finitely many primary fields with respect to the action of the chiral algebra.

Note that in the above definition we refer to the *chiral algebra* rather than to the Virasoro algebra. This is because RCFTs include more general theories whose left- and right-moving symmetry algebras are enlarged to a larger chiral algebra containing the Virasoro algebra as a subalgebra.

Finally, we can introduce a useful concept that will appear repeatedly throughout this thesis: the *fusion algebra* of primary operators. Although fusion algebras will be defined more rigorously in the next chapter, one may think of them as a coarse-grained version of the operator algebra, retaining only the information about which conformal families may appear in the OPE of two primary fields while ignoring the corresponding coefficients and spacetime dependence.

The general form of a fusion algebra for primaries  $\{\Phi_i\}$  is:

$$\Phi_i \times \Phi_j = \sum_k N_{ij}^k \Phi_k, \quad N_{ij}^k \in \mathbb{N}, \quad (1.32)$$

where the symbol  $\times$  is sometimes replaced by  $\otimes$  or omitted altogether. The coefficients  $N_{ij}^k$  are called *fusion coefficients* and should not be confused with the OPE coefficients. Their role is to count the number of independent fusion channels through which the representations associated with  $\Phi_i$  and  $\Phi_j$  can combine into the representation associated with  $\Phi_k$ .

From the OPE point of view, the condition  $N_{ij}^k \neq 0$  means that the conformal family generated by  $\Phi_k$  may appear in the expansion of  $\Phi_i(z, \bar{z})\Phi_j(w, \bar{w})$ . More generally, when  $N_{ij}^k > 1$ , there exist several independent fusion channels associated with the same conformal family. Accordingly, the corresponding three-point functions are no longer characterized by a single coupling constant, but by several independent structure constants, one for each fusion channel.

Note that fusion rules can be defined in general for CFTs, although they take a particularly simple form in RCFTs, where the sum appearing on the right-hand side of (1.32) is finite.

In minimal models, one can show that  $N_{ij}^k \in \{0, 1\}$  and that the fusion rules of chiral fields have a precise form:

$$\phi_{(r_1, s_1)} \times \phi_{(r_2, s_2)} = \sum_{\substack{r=|r_1-r_2|+1 \\ r+r_1+r_2=1 \bmod 2}}^{\min(r_1+r_2-1, 2m-r_1-r_2-1)} \sum_{\substack{s=|s_1-s_2|+1 \\ s+s_1+s_2=1 \bmod 2}}^{\min(s_1+s_2-1, 2(m+1)-s_1-s_2-1)} \phi_{(r, s)} \quad (1.33)$$

### 1.2.8 The partition function

Let's briefly recall the heuristic argument that relates the thermal partition function to the generating functional of a CFT. Consider the matrix element of the evolution operator between position eigenstates:

$$\langle x_2 | e^{-iHt} | x_1 \rangle = \int_{x(0)=x_1}^{x(t)=x_2} Dx e^{-iS[x, \dot{x}]}$$

where  $H$  is the Hamiltonian and  $S$  is the particle action. Integrating over all configurations with  $x_1 = x_2$  and rotating to imaginary time, we obtain the thermal partition function as a path integral over periodic trajectories:

$$\mathcal{Z}(\beta) = \text{Tr}_{\mathcal{H}}(e^{-\beta H}) = \int_{x(0)=x(\beta)} Dx e^{-S_E[x, \dot{x}]}$$

where  $S_E$  is the Euclidean action and the length of the Euclidean time circle is the inverse temperature  $\beta = 1/T$ .

Extending this argument to two-dimensional CFTs, where spatial periodicity also emerges from the operator-state correspondence, we obtain:

$$\mathcal{Z}(\beta) = \text{Tr}_{\mathcal{H}}\left(e^{-\beta(L_0 + \bar{L}_0 - \frac{c}{12})}\right) = \int_{\mathbb{T}^2} D\Phi e^{-S_E}$$

where we have subtracted the vacuum energy contribution from the conformal Hamiltonian. Hence a two-dimensional CFT is naturally defined on a torus  $\mathbb{T}^2$ .

We now rewrite this quantity in order to separate holomorphic and anti-holomorphic contributions and to highlight the underlying torus structure. We first introduce  $q = e^{-\beta}$ , so that:

$$\mathcal{Z}(\beta) = \text{Tr}_{\mathcal{H}}\left(e^{-\beta(L_0 + \bar{L}_0 - \frac{c}{12})}\right) = \text{Tr}_{\mathcal{H}}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right) = \mathcal{Z}(q, \bar{q})$$

For the moment we have  $q = \bar{q}$ . We now introduce the modular parameters  $\tau$  and  $\bar{\tau}$  defined by

$$q = e^{2\pi i \tau} \quad \bar{q} = e^{-2\pi i \bar{\tau}}$$

Then:

$$\mathcal{Z}(q, \bar{q}) = \text{Tr}_{\mathcal{H}}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right) = \text{Tr}_{\mathcal{H}}\left(e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{\bar{c}}{24})}\right) \quad (1.34)$$

with  $\tau = \tau_1 + i\tau_2$ , where  $\tau_2 = \frac{\beta}{2\pi}$ , and  $\bar{\tau} = \tau^*$ ,  $\bar{q} = q^*$ .

While for a physical Euclidean torus one has  $\bar{\tau} = \tau^*$ , it is often convenient to treat the holomorphic and anti-holomorphic sectors formally as independent variables. In order to do so, one may also allow independent central charges  $c$  and  $\bar{c}$ . In this way one can regard the partition function as a function of two independent variables:

$$\mathcal{Z} = \mathcal{Z}(\tau, \bar{\tau})$$

The physical theory is recovered by imposing  $\bar{\tau} = \tau^*$ ,  $\bar{q} = q^*$  and  $c = \bar{c}$ .

The complex parameter  $\tau$  is called the *modular parameter* and encodes the geometry of the torus. Indeed, both spatial periodicity  $\theta \sim \theta + 2\pi$  on the cylinder and Euclidean time periodicity  $t_E \sim t_E + \beta$  are encoded in the identification

$$z \sim z + 2\pi\tau,$$

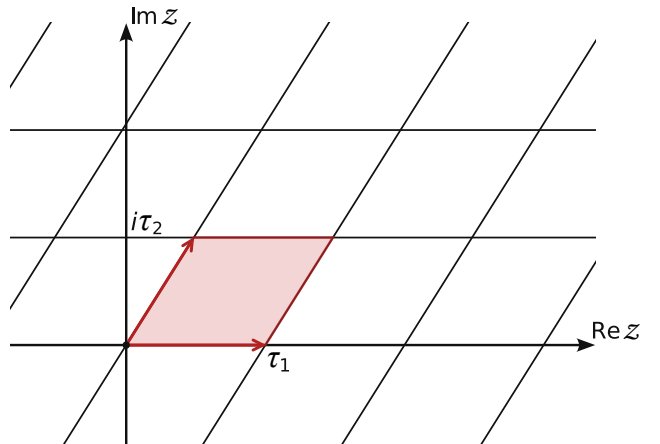


Figure 1.6: A complex lattice generated by  $\tau = \tau_1 + i\tau_2$ .

This equivalence generates a lattice in the complex plane, defining a torus whose shape is determined by  $\tau$  (see figure 1.6).

Expanding the expression (1.34), one finds that introducing the complex modular parameter  $\tau = \tau_1 + i\tau_2$  generalizes the thermal partition function to

$$\mathcal{Z}(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}} \left( e^{-2\pi\tau_2 H} e^{2\pi i\tau_1 (L_0 - \bar{L}_0)} \right)$$

The additional factor acts as a chemical potential for the momentum operator  $L_0 - \bar{L}_0$ , which in radial quantization generates rotations on the plane and is naturally related to the spin of the states. Its appearance has a simple geometric interpretation. For a rectangular torus ( $\tau_1 = 0$ ), a translation along the Euclidean time cycle returns to the same spatial point. For a tilted torus ( $\tau_1 \neq 0$ ), instead, a translation along the Euclidean time direction must be accompanied by a spatial translation in order to return to an identified point. The operator  $e^{2\pi i\tau_1 (L_0 - \bar{L}_0)}$  precisely implements this additional translation.

The symmetries of the torus become symmetries of the partition function. Different choices of cycles generating the same lattice are related by transformations in the modular group:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2.$$

This group is generated by two elements:

$$T : \tau \rightarrow \tau + 1, \quad S : \tau \rightarrow -\frac{1}{\tau}$$

These symmetries lift to the partition function, which becomes *modular invariant*:

$$\mathcal{Z}(\tau, \bar{\tau}) = \mathcal{Z}(\tau + 1, \bar{\tau} + 1), \quad \mathcal{Z}(\tau, \bar{\tau}) = \mathcal{Z}\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) \quad (1.35)$$

This is a consistency condition that every two-dimensional CFT must satisfy and has no higher-dimensional analogue.

Practically, the partition function is conveniently expressed in terms of characters:

$$\chi_h^{(c)} := \text{Tr}_{M_h^c} \left( q^{L_0 - \frac{c}{24}} \right), \quad \bar{\chi}_{\bar{h}}^{(\bar{c})} := \text{Tr}_{\bar{M}_{\bar{h}}^{\bar{c}}} \left( \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right)$$

Using the Hilbert space decomposition (1.27), we obtain:

$$\mathcal{Z}(\tau, \bar{\tau}) = \sum_{h, \bar{h}} Z_{h, \bar{h}} \chi_h^{(c)}(\tau) \bar{\chi}_{\bar{h}}^{(\bar{c})}(\bar{\tau})$$

The explicit form of the characters depends on the specific model.

In many theories, under the  $T$  transformation the characters acquire only a phase, making invariance under  $T$  straightforward to check. In particular, from (1.34):

$$\chi_h^{(c)}(\tau + 1) = e^{2\pi i(h - \frac{c}{24})} \chi_h^{(c)}(\tau)$$

Then modular invariance under  $T$  transformations implies that:

$$e^{2\pi i(h - \bar{h})} = 1$$

Meaning that in a modular invariant theory the spins must be integers. Note that this does not imply that the left/right conformal weights need to be integers.

Under the  $S$  transformation, instead, the characters transform linearly as:

$$\chi_h^{(c)}\left(-\frac{1}{\tau}\right) = \sum_{h'} S_{hh'} \chi_{h'}^{(c)}(\tau) \quad (1.36)$$

The matrix  $S$  is called the *modular S-matrix* and is symmetric and unitary. One then finds:

$$\mathcal{Z}\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \sum_{i,j,k,k'} \chi_i^{(c)}(\tau) S_{ik}^T Z_{k,k'} S_{k'j}^* \bar{\chi}_j^{(\bar{c})}(\bar{\tau}) = \mathcal{Z}(\tau, \bar{\tau}) \quad \Rightarrow \quad Z = SZS^\dagger \quad (1.37)$$

For simplicity we have used generic indices, although they run over the full set of conformal weights of the theory; we will adopt this convention throughout the thesis.

**Definition 1.2.8.** A *diagonal* CFT is a theory with  $Z_{i,j} = \delta_{i,j}$ .

Note that modular invariance under  $S$  follows immediately from (1.37) and the unitarity of  $S$  for diagonal theories.

Finally, we state a famous result due to Verlinde, which relates the fusion algebra (1.32) to the modular transformation properties of characters (1.36), namely the *Verlinde formula*:

$$N_{ij}^k = \sum_m \frac{S_{im} S_{jm} S_{km}^*}{S_{0m}} \quad (1.38)$$

This formula states that the unitary  $S$ -matrix diagonalizes the fusion matrices. More precisely, if we define  $(N_i)_j^k = N_{ij}^k$ , then the eigenvalues of  $N_i$  are given by  $\frac{S_{im}}{S_{0m}}$ . This formula will be extensively used in the sequel.

### 1.3 The Ising Model

Let's start from the discrete statistical model. Consider an  $N \times M$  lattice with spin variables  $\sigma_i \in \{-1, 1\}$  at each site, and suppose that they interact through a nearest-neighbour energy per link  $\langle ij \rangle$ :

$$E_{\langle ij \rangle} = -J \sigma_i \sigma_j$$

leading to the partition function:

$$\mathcal{Z} = \sum_{\{\sigma\}} e^{-\beta \sum_{\langle ij \rangle} E_{\langle ij \rangle}}$$

where  $\beta = 1/K_B T$ . Note that this system exhibits a  $\mathbb{Z}_2$  symmetry implemented by the transformation  $\sigma_i \rightarrow -\sigma_i$ .

This system undergoes a second-order phase transition at a critical value  $K = K_c$  of the coupling  $K := \beta J$ . The continuum limit at the critical point is described by a conformal field theory.

Before analyzing the associated CFT, let us briefly discuss the two phases separated by the critical point. There is a low-temperature ordered phase ( $K > K_c$ ) and a high-temperature disordered phase ( $K < K_c$ ). The partition function can be expanded as a power series in  $K$  and  $1/K$ , respectively, in these regimes.

In the high-temperature phase (small  $K$ ), using the identity:

$$e^{K \sigma_i \sigma_j} = \cosh(K) + \sigma_i \sigma_j \sinh(K) = \cosh(K) (1 + \sigma_i \sigma_j \tanh(K)), \quad \sigma_i, \sigma_j \in \{-1, 1\}$$

we find:

$$\mathcal{Z} = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \cosh(K) (1 + \sigma_i \sigma_j \tanh(K)) = (\cosh(K))^{2NM} \sum_{\{\sigma\}} \prod_{\langle ij \rangle} (1 + \sigma_i \sigma_j \tanh(K))$$

where  $2NM$  is the number of nearest-neighbour bonds.

The above product over nearest neighbours can be expanded as follows: pick a site; for each neighbour the product selects either 1 or  $\sigma_i \sigma_j \tanh(K)$ . Whenever the latter is chosen, one can imagine moving to the neighbouring site and repeating the procedure. In this way, the polynomial arising from the expansion of the product can be interpreted in terms of paths on the lattice.

We now perform the sum over spin configurations. Since  $\sigma_i \in \{-1, 1\}$ , only terms in which each spin appears an even number of times survive (a single  $\sigma_i$  would sum to  $1 - 1 = 0$ ). In the graphical representation, this implies that only configurations consisting of (possibly disconnected) closed loops contribute. Therefore, the partition function becomes:

$$\mathcal{Z}_{high} = [2 \cosh^2(K)]^{NM} \sum_{\text{loops}} [\tanh(K)]^{\text{length}} \quad (1.39)$$

where the factor  $2^{NM}$  arises from the sum over all spin configurations. This is the so-called *high-temperature expansion* of the Ising model. The procedure described above is illustrated in figure 1.7.

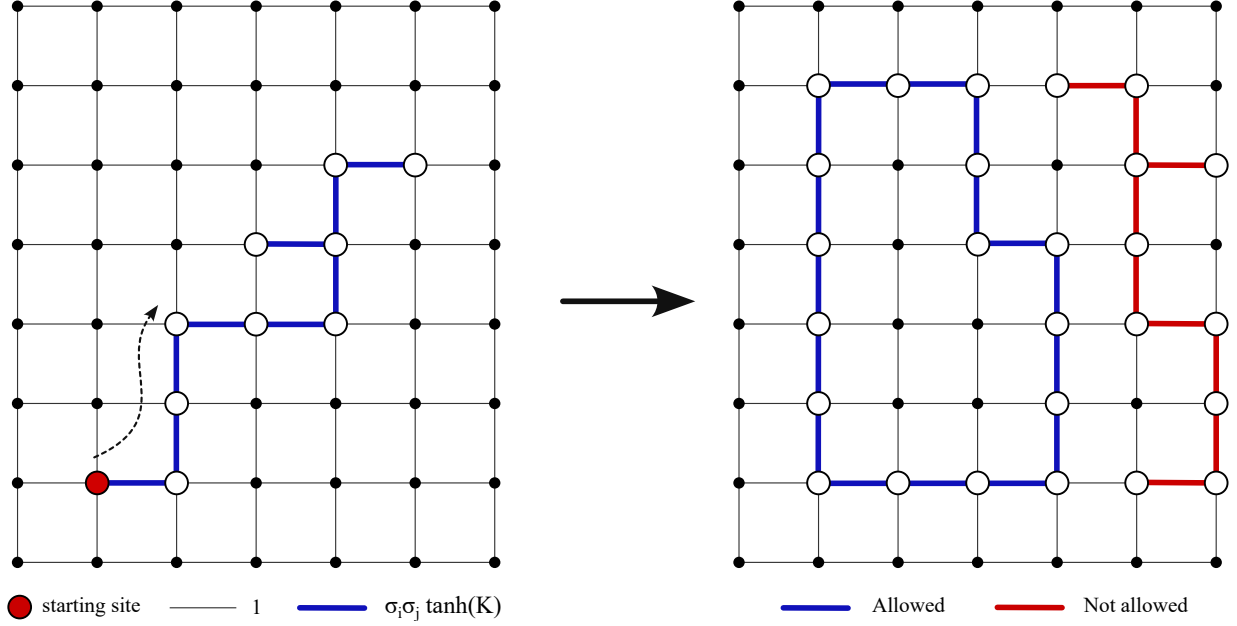


Figure 1.7: Graphical representation of the high-temperature expansion. In general, the path may branch and form more complicated closed configurations.

The spin chains arising in the high-temperature expansion can be interpreted as fluctuations of short-range correlations rather than as physical spin domains. Since the partition function contains no sources, only closed chains contribute, resulting in a gas of closed loops of correlated spins. In this respect, the high-temperature expansion is reminiscent of the perturbative expansion of a QFT partition function, which decomposes into a sum of closed vacuum bubble diagrams. Note that the contribution of long chains is suppressed in the low  $K$  regime.

In the low-temperature phase (large  $K$ ), the system is almost fully ordered, except for small clusters of flipped spins. A given configuration can thus be described in terms of the boundaries separating regions of opposite magnetization.

The lowest-energy configuration is the fully aligned one, with energy:

$$E_0 = -J \sum_{\langle ij \rangle} 1 = -J N_{bonds} = -2JNM$$

We factor this contribution out of the partition function:

$$\mathcal{Z} = e^{2KNM} \cdot \sum_{\{\sigma\}} e^{-\beta \sum_{\langle ij \rangle} E_{\langle ij \rangle} - E_0}$$

The remaining sum weights the energy cost of excitations above the ground state. For every anti-aligned nearest-neighbour pair we pay an energy cost:

$$\Delta E = E_{opposite} - E_{aligned} = +J - (-J) = 2J$$

Since the boundaries of flipped clusters form closed loops, we obtain:

$$\mathcal{Z}_{low} = 2e^{2NMK} \sum_{\text{loops}} e^{-2K(\text{length})} \quad (1.40)$$

where the factor 2 accounts for the degeneracy of the ground state. This situation is illustrated in figure 1.8.

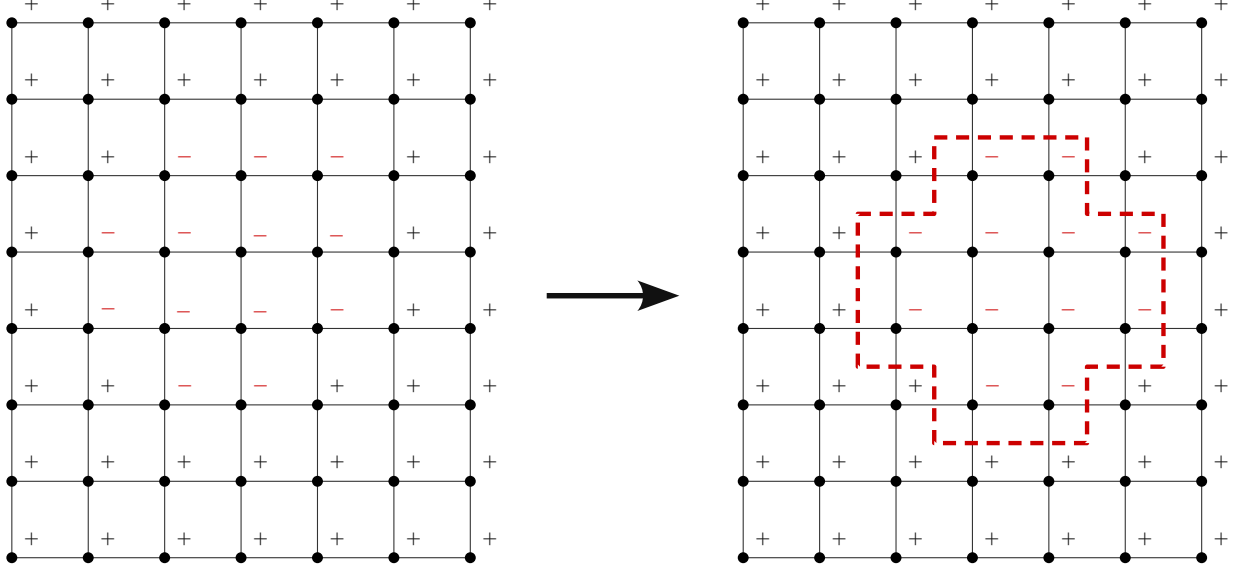


Figure 1.8: Graphical interpretation of the low-temperature expansion of the Ising model. Flipped spin clusters are encoded by closed domain walls on the dual lattice.

Note that in the high- $K$  regime, only small clusters give a significant contribution to the partition function.<sup>6</sup>

From expressions (1.39) and (1.40), we see that the two phases are mapped into each other through the identification:

$$e^{-2K'} = \tanh(K) \quad (1.41)$$

In particular:

$$\begin{aligned} \mathcal{Z}_{low}(K') &= 2e^{2NMK'} \sum_{\text{loops}} e^{-2K'(\text{length})} = 2 \tanh(K)^{-NM} \sum_{\text{loops}} \tanh(K)^{(\text{length})} = \\ &= 2 \sinh(K)^{-NM} \cosh(K)^{NM} \sum_{\text{loops}} \tanh(K)^{(\text{length})} = \\ &= 2 (2 \sinh(K) \cosh(K))^{-NM} 2^{NM} \cosh(K)^{2NM} \sum_{\text{loops}} \tanh(K)^{(\text{length})} = \\ &= 2 \sinh(2K)^{-NM} \mathcal{Z}_{high}(K) \end{aligned}$$

This transformation relating the high- and low-temperature phases of the Ising model is called the *Kramers–Wannier duality*. At the phase transition point we expect  $K = K'$ , and from equation (1.41):

$$K_c = -\frac{1}{2} \ln(\sqrt{2} - 1)$$

A crucial observation is the following. Equation (1.41) is a duality transformation relating two phases of the same physical system. However, at criticality the two phases coincide, meaning that (1.41)

<sup>6</sup>Although they are traditionally referred to as high- and low-temperature expansions, equations (1.39) and (1.40) are exact identities for the two-dimensional Ising model. The term "expansion" originates from the fact that these representations become naturally organized as power series in  $\tanh(K)$  and  $e^{-2K}$ , respectively, and analogous procedures in more general lattice models usually yield only perturbative expansions [7].

becomes an invariance of the critical theory. This is not a fundamental microscopic symmetry of the system, since it only emerges at the critical point. For this reason, we expect it to have special properties. Indeed, in the next chapter we will see that the transformation (1.41) corresponds to a *non-invertible symmetry* of the critical Ising model, which, although being a symmetry, exhibits features that go beyond those of ordinary symmetries.

The duality transformation (1.41) enables us to define an operator dual to the spin operator, called the *disorder operator*  $\mu$ , as follows. First consider the two-point function of two spin operators:

$$\langle \sigma(r_1) \sigma(r_2) \rangle = \frac{1}{\mathcal{Z}} \sum_{\{\sigma\}} \sigma(r_1) \sigma(r_2) e^{K \sum_{\langle ij \rangle} \sigma_i \sigma_j}$$

This is equivalent to picking an arbitrary path of  $n$ -steps from  $r_1$  to  $r_2$  along the lattice bonds and changing the coupling  $K \rightarrow K + i\pi/2$  for each bond on the path. Indeed, this introduces a factor  $e^{i\pi\sigma_i\sigma_j/2} = i\sigma_i\sigma_j$  per bond  $\langle ij \rangle$  of the path, leaving us with  $i^n \sigma(r_1) \sigma(r_2)$  as each intermediary spin  $\sigma_i$  appears twice and  $\sigma_i^2 = 1$ . One can show that the result is independent, up to a sign, of the path chosen.

Now we define the disorder operator correlation function  $\langle \mu(r_1) \mu(r_2) \rangle$ .<sup>7</sup> Choose any path connecting  $r_1$  and  $r_2$  and change the sign of the coupling,  $K \rightarrow -K$ , on all links intersected by the path. The resulting partition function, normalized by the original partition function  $\mathcal{Z}$ , defines the correlator. One can show that the result is independent of the particular choice of path.

Taking one endpoint to infinity defines a single disorder operator  $\mu(r)$ . Intuitively, it is called a disorder operator because it introduces a line along which neighbouring spins are more likely to be anti-aligned, thus favouring the presence of a domain wall. For this reason, its vacuum expectation value  $\langle \mu \rangle$  is expected to be non-vanishing only in the high-temperature disordered phase, in contrast with the spin operator, whose vacuum expectation value is non-vanishing only in the ordered phase.

This opposition is not accidental. Indeed, from the duality transformation (1.41) one finds

$$(-K)' = K' + i\frac{\pi}{2}, \quad (K + i\frac{\pi}{2})' = -(K').$$

These relations underlie the fact that Kramers–Wannier duality exchanges spin and disorder operators, mapping  $\langle \sigma(r_1) \sigma(r_2) \rangle$  to  $\langle \mu(r_1) \mu(r_2) \rangle$ . In the next chapter we will see how this correspondence persists in the continuum limit.

We can finally move on to the continuum limit of the Ising model. This limit can be analysed exactly through the Jordan–Wigner transformation, which maps the model to free fermionic degrees of freedom. In the continuum limit one obtains a theory of a real (Majorana) fermion. At criticality the fermion becomes massless and the action reads

$$S = \frac{1}{2\pi} \int d^2z (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) \quad (1.42)$$

The vicinity of the critical point is also described by a free-fermion theory, but with the addition of a mass term  $m\psi\bar{\psi} \propto (K - K_c)\psi\bar{\psi}$ , which explicitly breaks conformal invariance.

The study of the conformally invariant action (1.42) leads to a theory with central charge

$$c_{\text{ising}} = c(3) = \frac{1}{2}$$

showing that the critical Ising model is described by the minimal model with  $m = 3$ . However, a modular invariant local CFT cannot be described solely in terms of the fermionic field. Indeed, modular invariance requires the left- and right-moving sectors to recombine into a bosonic theory

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<sup>7</sup>To be precise, as a consequence of the Osterwalder–Schrader reconstruction theorem, once all correlators involving such operators have been defined, the corresponding quantum field can be reconstructed, provided that the hypotheses of the theorem are satisfied.

whose spectrum consists of three primary fields: the identity<sup>8</sup>, the energy operator  $\epsilon(z, \bar{z})$  and the spin operator  $\sigma(z, \bar{z})$ . These are the continuum counterparts of the lattice operators  $E_{\langle ij \rangle}$  and  $\sigma(r)$  and have conformal weights  $(h, \bar{h}) = (0, 0), (1/2, 1/2), (1/16, 1/16)$  respectively. The allowed chiral conformal weights are also correctly reproduced by the Kac formula (1.31). Thus the partition function is

$$\mathcal{Z} = \chi_0^{(\frac{1}{2})} \bar{\chi}_0^{(\frac{1}{2})} + \chi_{\frac{1}{2}}^{(\frac{1}{2})} \bar{\chi}_{\frac{1}{2}}^{(\frac{1}{2})} + \chi_{\frac{1}{16}}^{(\frac{1}{2})} \bar{\chi}_{\frac{1}{16}}^{(\frac{1}{2})}$$

where an explicit expression for the characters can indeed be obtained from the fermionic description (1.42). Accordingly, the Hilbert space is

$$\mathcal{H} = (M_0^{(\frac{1}{2})} \otimes \bar{M}_0^{(\frac{1}{2})}) \oplus (M_{\frac{1}{2}}^{(\frac{1}{2})} \otimes \bar{M}_{\frac{1}{2}}^{(\frac{1}{2})}) \oplus (M_{\frac{1}{16}}^{(\frac{1}{2})} \otimes \bar{M}_{\frac{1}{16}}^{(\frac{1}{2})})$$

Therefore, the critical Ising model is a diagonal CFT.

For a diagonal partition function, the constraint (1.37) is satisfied once the characters transform among themselves through a unitary matrix  $S$ . However, modular invariance alone does not determine the explicit form of  $S$ . To compute it, one has to use the explicit expressions of the characters and study their transformation under  $\tau \mapsto -1/\tau$ . For the Ising model, one finds

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \quad (1.43)$$

In chapter 3 we will derive the same matrix in the framework of algebraic quantum field theory.

Since the model is diagonal, we can use the general fusion rules of minimal models (1.33) to determine the fusion algebra:

$$\epsilon \times \epsilon = 1, \quad \epsilon \times \sigma = \sigma = \sigma \times \epsilon, \quad \sigma \times \sigma = 1 + \epsilon.$$

For example, one finds

$$\epsilon(z, \bar{z})\epsilon(w, \bar{w}) = \frac{1}{|z - w|^2} + \dots \quad \sigma(z, \bar{z})\sigma(w, \bar{w}) = \frac{1}{|z - w|^{1/4}} + \frac{1}{2}|z - w|^{3/4}\epsilon(w, \bar{w}) + \dots$$

where, for the chosen normalization of the fields,  $\lambda_{\sigma\sigma 1} = 1$  and  $\lambda_{\sigma\sigma\epsilon} = 1/2$ .

One can also compute OPEs and fusion rules involving the fermionic fields and the disorder operator  $\mu(z, \bar{z})$ . The latter corresponds to the continuum limit of the discrete operator  $\mu(r)$  introduced above. In particular, one finds

$$\psi \times \sigma = \mu = \bar{\psi} \times \sigma, \quad \psi \times \mu = \sigma = \bar{\psi} \times \mu.$$

These fusion rules correspond to the OPEs:

$$\begin{aligned} \psi(z)\sigma(w, \bar{w}) &= \frac{e^{i\pi/4}}{\sqrt{2}(z - w)^{1/2}}\mu(w, \bar{w}) & \psi(z)\mu(w, \bar{w}) &= \frac{e^{-i\pi/4}}{\sqrt{2}(z - w)^{1/2}}\sigma(w, \bar{w}) \\ \bar{\psi}(\bar{z})\sigma(w, \bar{w}) &= \frac{e^{-i\pi/4}}{\sqrt{2}(\bar{z} - \bar{w})^{1/2}}\mu(w, \bar{w}) & \bar{\psi}(\bar{z})\mu(w, \bar{w}) &= \frac{e^{i\pi/4}}{\sqrt{2}(\bar{z} - \bar{w})^{1/2}}\sigma(w, \bar{w}) \end{aligned} \quad (1.44)$$

As a consequence of these relations, some authors [8] use the regularized product:

$$\mu(w, \bar{w}) \propto \lim_{z \rightarrow w} (z - w)^{1/2} \psi(z) \sigma(w, \bar{w}) \quad (1.45)$$

To define the disorder operator  $\mu$ . The proportionality constant in (1.45) may be different from the ones in (1.44) due to different field normalization.

<sup>8</sup>Note that when  $c \neq 0$  we have seen that  $T(z)$  and  $\bar{T}(\bar{z})$  are not primary operators. Indeed, they are descendants of the identity. For instance, placing the energy-momentum tensor  $T(z)$  at the origin gives  $|T\rangle = \lim_{z \rightarrow 0} T(z)|\Omega\rangle = \lim_{z \rightarrow 0} \sum_{n \in \mathbb{Z}} z^{-n-2} L_n |\Omega\rangle = \lim_{z \rightarrow 0} \sum_{n \leq -2, n \in \mathbb{Z}} z^{-n-2} L_n |\Omega\rangle = L_{-2} |\Omega\rangle$ .

Finally, note that at this stage we have no direct way to detect the presence of additional fields such as  $\psi$ ,  $\bar{\psi}$  and  $\mu$  starting directly from the continuous minimal model, since they are not encoded in the modular invariant partition function. In the next chapter we will see that these fields are naturally associated with defects of the theory. In particular, within the Euclidean framework, we will derive a generalization of the modular constraint (1.37), which allows one to detect and classify such additional sectors.

## Chapter 2

# Topological defects in CFT

In this chapter, we will introduce the concept of topological defects starting from the study of global symmetries, which provide the easiest example of a topological defect. After highlighting the main features of defects, we will specialize to 2d CFT where defects will be implemented by topological line operators, which will be characterized in the rest of the chapter. This chapter is based on [5, 6, 9–27] for the physical sections and on [28–32] for the notions of category theory.

### 2.1 From global symmetries to topological defects

In non-relativistic quantum mechanics, a global (dynamical) symmetry requires the existence of an operator  $U$  conserved under time evolution, i.e. commuting with the Hamiltonian,  $[U, H] = 0$ . For relativistic QFTs with Euclidean signature, time is on the same footing as any other spatial direction, hence the appropriate generalization of this condition is that the symmetry operator commutes with the energy-momentum tensor  $T_{\mu\nu}$ , thus the operator  $U$  should be topological.

To make this more concrete, consider a general QFT in  $d$  spacetime dimensions with a continuous  $U(1)$  global symmetry. It is associated with the conserved Noether current  $j_\mu(x)$  such that  $\partial^\mu j_\mu = 0$  on the solution of the equations of motion and the conserved charge operator is expressed as

$$Q = \oint d^{d-1}x j_0 \quad , \quad \partial_0 Q = 0$$

The conserved unitary operator  $U_\theta$  that implements a  $U(1)$  rotation with angle  $\theta$  is

$$U_\theta = e^{i\theta Q} = \exp[i\theta \oint d^{d-1}x j_0]$$

We can express this symmetry operator in a covariant way in Euclidean signature. Let  $M^{(d-1)}$  be a closed  $(d-1)$ -manifold with no boundary in the  $d$ -dimensional Euclidean spacetime. We define:

$$U_\theta(M^{(d-1)}) = \exp[i\theta \oint_{M^{(d-1)}} j_\mu n^\mu dM] = \exp[i\theta \oint_{M^{(d-1)}} \star j] \quad (2.1)$$

where  $n^\mu$  is a vector orthogonal to  $M^{(d-1)}$ ,  $dM$  is the infinitesimal volume element on  $M^{(d-1)}$ ,  $\star$  is the Hodge dual and  $\star j$  is a closed form, i.e.  $d\star j = 0$ , which is the Euclidean version of the Noether current (see appendix A). We can now see that if  $M_1^{(d-1)}$  and  $M_2^{(d-1)}$  are related by continuous deformation and  $M^d$  is the  $d$ -dimensional manifold with  $M_1^{(d-1)}$  and  $M_2^{(d-1)}$  as boundaries with opposite orientations as in figure 2.1, we get

$$0 = dQ = \oint_{M^{(d)}} d\star j = \oint_{M_1^{(d-1)}} \star j - \oint_{M_2^{(d-1)}} \star j \quad \Rightarrow \quad Q(M_1^{(d-1)}) = Q(M_2^{(d-1)})$$

And as a consequence

$$U_\theta(M_1^{(d-1)}) = U_\theta(M_2^{(d-1)})$$

Therefore, we see that a conserved current operator  $j_\mu(x)$  in a relativistic QFT leads to a topological operator  $U_\theta(M^{(d-1)})$  supported on a codimension-1 manifold in Euclidean spacetime. Note that  $U_\theta(M^{(d-1)})$  is best interpreted as a modification of the path integral rather than an operator acting on a Hilbert space. This is because if a particular time direction has been chosen, then the submanifold  $M^{(d-1)}$  does not need to be a spatial slice.

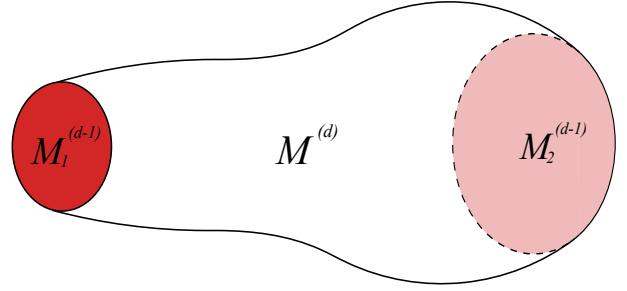


Figure 2.1: Deformation of  $M_1^{(d-1)}$  into  $M_2^{(d-1)}$ . The boundaries  $M_i^{(d-1)}$ ,  $i = 1, 2$ , of  $M^{(d)}$  are oriented in opposite ways and are assumed without boundary.

**Definition 2.1.1.** An  $n$ -dimensional defect is a modification of the path integral, supported on an oriented  $n$ -dimensional submanifold  $M^{(n)}$  of Euclidean spacetime,  $n < d$ , which changes the space of field configurations (for instance by imposing boundary conditions, singularities, or other constraints for the fields on  $M^{(n)}$ ):

$$\int_{\mathcal{C}} D\Phi (...) e^{-s[\Phi]} \rightarrow \int_{\mathcal{C}_{M^{(n)}}} D\Phi (...) e^{-s[\Phi]}$$

where (...) denotes possible field insertions,  $\mathcal{C}$  is the configuration space of the fields,  $\mathcal{C}_{M^{(n)}}$  is its modified version by the defect insertion. The modified partition function, denoted  $\mathcal{Z}_{M^{(n)}}$ , is called *twisted partition function*. The defect is called *topological*, if the path integral is invariant under deformations of  $M^{(n)}$  which do not cross the support of the other insertions.

Suppose for simplicity that  $n = d - 1$  as in the example above<sup>1</sup>. When the defect is at a fixed time and extends along the spatial direction, the modified correlators,

$$\langle \dots \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{C}} D\Phi (...) e^{-s[\Phi]} \rightarrow \frac{1}{\mathcal{Z}_{M^{(d-1)}}} \int_{\mathcal{C}_{M^{(d-1)}}} D\Phi (...) e^{-s[\Phi]} = \langle \dots \rangle_{M^{(d-1)}},$$

in the spirit of the Osterwalder-Schrader reconstruction theorem, describe a theory living on the Hilbert space  $\mathcal{H}$  determined by the theory without defect, but together with a new topological operator supported on  $M^{(d-1)}$ :

$$\langle \dots \rangle_{M^{(d-1)}} = \langle O(M^{(d-1)}) \dots \rangle = \langle \Omega | \mathcal{T}(\hat{O}(M^{(d-1)}) \dots) | \Omega \rangle$$

where  $|\Omega\rangle \in \mathcal{H}$  is the vacuum vector coming from the reconstruction and  $\mathcal{T}$  denotes time ordering. Considering the previous example, when  $M^{(d-1)} = M_t$  is a spatial slice at fixed time  $t$ , then  $U_\theta(M_t) = U(t)$  is the usual unitary operator acting on the Hilbert space  $\mathcal{H}$  implementing the  $U(1)$  symmetry at time  $t$ . Then the topological nature implies the conservation under time evolution.

Note that a precise orientation of the support of the defect is required. Indeed, reversing the orientation of  $M^{(d-1)}$  in equation (2.1) would give us a different operator, which in this example would coincide with the inverse operator  $U_\theta^{-1}(M^{(d-1)})$ , although this is not always the case.

When the defect is extended along the full time direction and localized in one of the spatial directions, then the associated Hilbert space  $\mathcal{H}_{M^{(d-1)}}$ , called *twisted*, differs from the one in the absence of the defect, since the insertion has modified the quantization by changing the configuration space of the fields along the whole time span.

<sup>1</sup>This is the case we are interested in, i.e. line defects in two spacetime dimensions, and also the most explored case in literature. The definition is still valid for generic supports and also the following considerations can be generalized.

Consider for example a free complex scalar field in  $d = 2$  and let the space be a circle  $S_x^1$  parametrized by  $x \sim x + 2\pi$ . The Hilbert space  $\mathcal{H}(S_x^1)$  on a circle is obtained by the canonical quantization of the free scalar field subject to the periodic boundary condition

$$\Phi(\tau, x) = \Phi(\tau, x + 2\pi)$$

The  $U(1)$  global symmetry gives rise to a unitary operator  $U_\theta(\mathcal{H}(S_x^1))$  acting on  $\mathcal{H}(S_x^1)$ . We can then insert a defect along the Euclidean (compact) time direction at  $x=0$ ,  $S_\tau^1$ , changing the boundary condition to

$$\Phi(\tau, x) = e^{i\theta} \Phi(\tau, x + 2\pi)$$

Canonical quantization subject to the above twisted boundary condition leads to a twisted Hilbert space  $\mathcal{H}_\theta(S_x^1)$  labeled by the  $U(1)$  group element  $\theta \in [0, 2\pi)$ .

Until now, ordinary global symmetries have been considered, which can only produce  $(d-1)$ -dimensional defects. We can relax the definition of global symmetry to also include defects supported on manifolds of general dimension.

**Definition 2.1.2** (p-form symmetry). A *p-form symmetry* is a codimension- $(p+1)$  operator  $U$ , i.e. supported on an oriented  $(d-p-1)$ -dimensional submanifold of the Euclidean spacetime, which is topological and invertible.

Therefore p-form symmetries are examples of  $(d-p-1)$ -defects and the usual global symmetries are just 0-form symmetries.

Consider now two  $(d-p-1)$ -dimensional topological defects with disjoint support, which can be freely deformed until they coincide without crossing any other operator insertion. Since they are topological, correlation functions do not depend on the distance between them. Hence, this configuration defines a *fusion product*  $\times$  between topological operators.

In general, p-form symmetries combine to form a p-form symmetry group  $G^{(p)}$ .<sup>2</sup> This means that  $\forall g \in G^{(p)}$  there are topological codimension- $(p+1)$  operators  $U_g$  obeying the *fusion rule*

$$U_g \times U_{g'} = U_{gg'} \quad (2.2)$$

Notice that in definition 2.1.1 there is no requirement for the defect operators to be invertible. As a consequence, general topological defects will not obey the group multiplication law (2.2), including also non-invertible operators associated with *non-invertible symmetries*.

In addition to the fusion product, we can also define a superposition of defects through

$$\langle \dots \rangle_{\alpha M_1^{(p)} + \beta M_2^{(q)}} = \alpha \langle \dots \rangle_{M_1^{(p)}} + \beta \langle \dots \rangle_{M_2^{(q)}} \quad \alpha, \beta \in \mathbb{N} \quad (2.3)$$

where  $p, q \leq d$  and  $d$  is the dimension of spacetime. Notice that the coefficients  $\alpha$  and  $\beta$  are restricted to be positive integers so that the defect  $\alpha M_1^{(p)} + \beta M_2^{(q)}$  is associated with the twisted Hilbert space  $\alpha \mathcal{H}_{M_1^{(p)}} \oplus \beta \mathcal{H}_{M_2^{(q)}}$ , where again the coefficients are interpreted as the number of copies in the direct sum of the subsequent Hilbert space. This would be impossible if the coefficients were real.

Finally, we can define an action of the defect on local operators. Notice that only  $(d-1)$ -dimensional defects can act on local operators, since otherwise one can simply deform the support of the topological operator past the local insertion. Consider now a defect supported on a sphere  $S^{d-1}$  and a local operator  $O(x)$  inside it. Then we can shrink the sphere to the insertion point. Due to the topological property of definition 2.1.1, the local operator should change accordingly in order to maintain the correlator untouched

$$\langle O(S^{d-1}) O(x) \rangle = \langle O'(x) \rangle$$

---

<sup>2</sup>One can show that a p-form symmetry group  $G^{(p)}$  is always abelian for  $p \geq 1$ . This is because we can use topological deformations to change the ordering of two topological operators of codimension greater than one. As a consequence the fusion product (2.2) is always commutative for  $p \geq 1$ .

if the defect is associated with a global symmetry

$$\langle U_g(S^{(d-1)})O(x) \rangle = \langle (gO)(x) \rangle \quad , \quad g \in G^{(0)}$$

Now that we have characterized topological defects in a generic quantum field theory, we can specify the case of interest: a 2-dimensional conformal field theory.

## 2.2 Topological defect lines

From now on, we restrict ourselves to the specific case of a 2d Euclidean CFT.

**Definition 2.2.1** (Topological defect lines). A *topological defect line* (TDL) in a 2d CFT  $\tau$  is a 1-dimensional topological defect  $\mathcal{L}$  in the sense of definition 2.1.1. The defect with opposite orientation, called dual defect, is denoted as  $\bar{\mathcal{L}}$ . The set of all topological defect lines is denoted as  $\text{TDL}(\tau)$ .

Following the previous discussion, we know that to the defect line  $\mathcal{L}$  we can associate an operator  $\hat{\mathcal{L}}$  acting on the same Hilbert space  $\mathcal{H}$  of the theory without defects and a twisted Hilbert space  $\mathcal{H}_{\mathcal{L}}$ . In particular, as a consequence of the invariance of correlators under deformations, all TDLs are "transparent" with respect to the holomorphic and anti-holomorphic parts of the energy-momentum tensor, i.e.

$$[\hat{\mathcal{L}}, T(z)] = 0 = [\hat{\mathcal{L}}, \bar{T}(\bar{z})] \quad [\hat{\mathcal{L}}, L_n] = 0 = [\hat{\mathcal{L}}, \bar{L}_n] \quad (2.4)$$

Hence, in an irreducible representation of the Virasoro algebra, all TDLs will be constant by Schur's lemma.

There are two main ways to insert a defect in the CFT in the cylinder. We can wrap the defect on the cylinder in the compact direction. By the operator-state map this corresponds to a line encircling the origin in the plane, as shown in figure 2.2. This defines an action of  $\hat{\mathcal{L}}$  on the Hilbert space  $\mathcal{H}$ : by inserting an operator  $O$  at the origin and shrinking the topological defect line around  $O$  produces a new local operator  $\hat{\mathcal{L}}(O)$  that, according to the operator-state correspondence, is a state  $\hat{\mathcal{L}}|O\rangle \in \mathcal{H}$  on the cylinder. Since  $\hat{\mathcal{L}}$  commutes with the Virasoro algebra, the new state has the same conformal weights as  $|O\rangle$ . In particular, since the vacuum is unique,  $\hat{\mathcal{L}}|0\rangle = \langle \hat{\mathcal{L}}|0\rangle$ , for some constant  $\langle \hat{\mathcal{L}} \rangle$ .

**Definition 2.2.2.**  $\langle \hat{\mathcal{L}} \rangle$  is called the *quantum dimension* of  $\hat{\mathcal{L}}$ .

The other possible way to insert a TDL is along the time direction. By the operator-state map, this corresponds to defect operators at the origin attached to  $\mathcal{L}$  as shown in figure 2.3. As already discussed in the previous section, this insertion corresponds to modifying the space of states on  $S^1$ , which becomes a " $\mathcal{L}$ -twisted" Hilbert space  $\mathcal{H}_{\mathcal{L}}$ . Recalling the fact that the defect is transparent with respect to the energy-momentum tensor, the new space will still be a representation of the holomorphic and antiholomorphic Virasoro algebras.

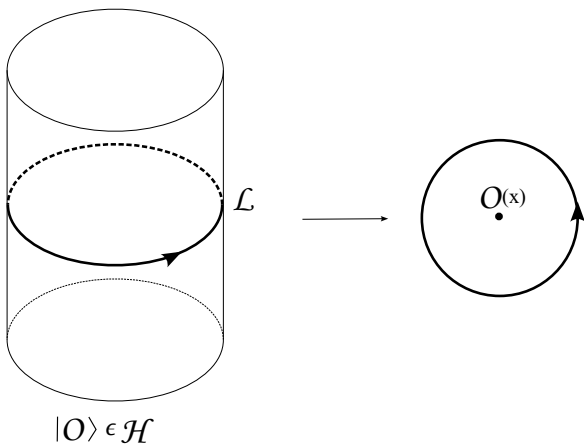


Figure 2.2: Insertion of the defect at fixed cylinder time.

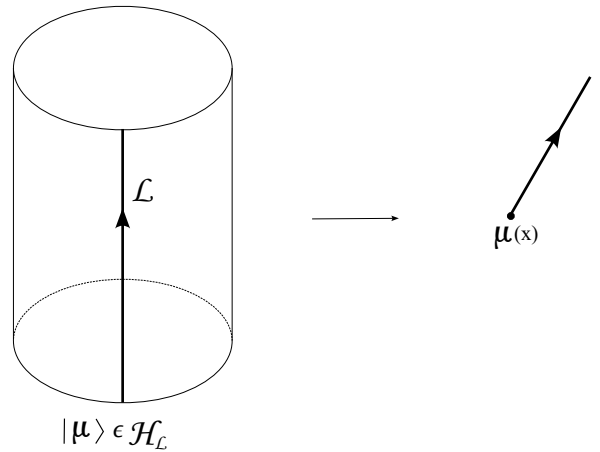


Figure 2.3: Insertion of the defect at fixed cylinder position.

### 2.2.1 Fusion ring

We now introduce the notions of sum and product of TDLs following the ideas of the previous section.

**Definition 2.2.3** (Direct sum). There exists a map,

$$\begin{aligned} + : TDL(\tau) \times TDL(\tau) &\rightarrow TDL(\tau) \\ (\mathcal{L}_i, \mathcal{L}_j) &\mapsto \mathcal{L}_i + \mathcal{L}_j \end{aligned}$$

called *direct sum* or *superposition*, such that correlation functions are additive with respect to it and the following properties hold:

- $(\mathcal{L}_i + \mathcal{L}_j) + \mathcal{L}_k = \mathcal{L}_i + (\mathcal{L}_j + \mathcal{L}_k)$ ;
- $\widehat{\mathcal{L}_i + \mathcal{L}_j} = \hat{\mathcal{L}}_i + \hat{\mathcal{L}}_j$ ;
- $\mathcal{H}_{\mathcal{L}_i + \mathcal{L}_j} = \mathcal{H}_{\mathcal{L}_i} \oplus \mathcal{H}_{\mathcal{L}_j}$ .

**Definition 2.2.4.** A defect  $\mathcal{L} \in TDL(\tau)$  is called *simple* if it cannot be written as a sum  $\mathcal{L}_1 + \mathcal{L}_2$  for any  $\mathcal{L}_1, \mathcal{L}_2 \in TDL(\tau)$ . If  $\mathcal{L}_i$  is simple, also its dual  $\bar{\mathcal{L}}_i = \mathcal{L}_i^\vee$  is simple.

From now on, we will assume that the number of simple defects is finite. Moreover, we will also assume that the set of TDLs is semi-simple, meaning that all non-simple defects can be written as a sum of simple defects. This is always verified if the CFT  $\tau$  is rational.

**Definition 2.2.5** (Fusion). There exists a map,

$$\begin{aligned} \otimes : TDL(\tau) \times TDL(\tau) &\rightarrow TDL(\tau) \\ (\mathcal{L}_i, \mathcal{L}_j) &\mapsto \mathcal{L}_i \otimes \mathcal{L}_j = \mathcal{L}_i \mathcal{L}_j \end{aligned}$$

called *fusion*, such that:

- $(\mathcal{L}_i \mathcal{L}_j) \mathcal{L}_k = \mathcal{L}_i (\mathcal{L}_j \mathcal{L}_k)$ ;
- $\widehat{\mathcal{L}_i \mathcal{L}_j} = \hat{\mathcal{L}}_i \hat{\mathcal{L}}_j$ ;
- $\mathcal{H}_{\mathcal{L}_i \mathcal{L}_j} = \mathcal{H}_{\mathcal{L}_i} \otimes \mathcal{H}_{\mathcal{L}_j}$

Here  $\otimes$  denotes also a notion of tensor product of Hilbert spaces induced by fusion of defects, and should not be confused with the usual tensor product of two independent Hilbert spaces [16].

Pictorially, the fusion of two defects describes what happens in the limit of coinciding defects, as shown in figure 2.4.

Among all defects, there is a particular one which is worth mentioning:

**Definition 2.2.6.** We denote as  $\mathcal{I} \in TDL(\tau)$  the *trivial* or *identity* defect, whose insertion does not change any correlation function. It is the neutral element with respect to the fusion operation.

Notice that commutativity is not required and in general does not hold. We can fuse two simple defects, but, in general, the result will fail to be simple. However, thanks to the semi-simplicity assumption,

$$\mathcal{L}_i \mathcal{L}_j = \bigoplus_k N_{ij}^k \mathcal{L}_k, \quad \mathcal{H}_{\mathcal{L}_i \mathcal{L}_j} = \mathcal{H}_{\mathcal{L}_i} \otimes \mathcal{H}_{\mathcal{L}_j} = \bigoplus_k N_{ij}^k \mathcal{H}_{\mathcal{L}_k}$$

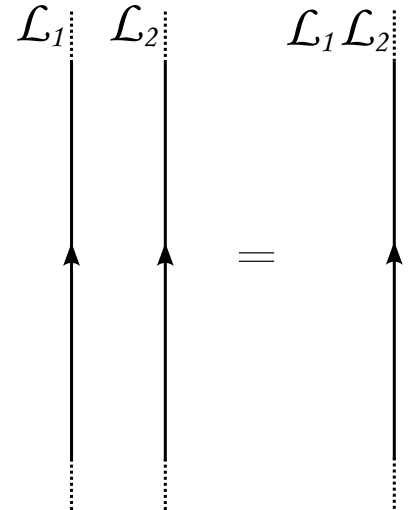


Figure 2.4: Fusion of two defects.

where the sums run over all simple defects and the numbers  $N_{ij}^k \in \mathbb{N}$  are called *fusion coefficients*. Note that we did not require the existence of a multiplicative inverse for each defect, which generally does not exist. For this reason, TDLs, equipped with fusion multiplication and direct sum, form a more complicated algebraic structure with respect to the usual group structure, called *fusion ring*.

### 2.2.2 k-junctions

Moving the parallel lines  $\mathcal{L}_1, \dots, \mathcal{L}_k$  on the cylinder very close to each other, we get the identification

$$\mathcal{H}_{\mathcal{L}_1, \dots, \mathcal{L}_k} \simeq \mathcal{H}_{\mathcal{L}_1 \dots \mathcal{L}_k} \simeq \mathcal{H}_{\mathcal{L}_1} \otimes \dots \otimes \mathcal{H}_{\mathcal{L}_k} \quad (2.5)$$

where  $\mathcal{H}_{\mathcal{L}_1, \dots, \mathcal{L}_k}$  is the *k-junction space*, i.e. the Hilbert space twisted by the presence of the  $k$  defects  $\mathcal{L}_1, \dots, \mathcal{L}_k$ . By the operator-state map, this corresponds to a *k-junction operator* at the origin attached to the  $k$  defect lines. We adopt the convention such that a junction  $v \in \mathcal{H}_{\mathcal{L}_1, \dots, \mathcal{L}_k}$  is represented with all TDLs pointing outwards.

In general, junctions are not topological. Moreover, note that the order of appearance of TDLs in  $\mathcal{H}_{\mathcal{L}_1, \dots, \mathcal{L}_k}$  matters. However, the subset of topological junctions is particularly interesting.<sup>3</sup>

**Definition 2.2.7.** The subspace  $\mathcal{V}_{\mathcal{L}_1, \dots, \mathcal{L}_k} \subset \mathcal{H}_{\mathcal{L}_1, \dots, \mathcal{L}_k}$  of states with conformal weights  $(0, 0)$  corresponds to *topological junction operators*, i.e. such that the junction point can be moved without changing any correlation function.<sup>4</sup>

By studying these topological junction spaces we can extract more properties about TDLs.

**Proposition 2.2.1.** *The 2-junction topological space  $\mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}}$  satisfies  $\dim(\mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}}) = N_{\mathcal{L}\bar{\mathcal{L}}}^0 \geq 1$*

*Proof.* From (2.5) we get

$$\mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}} \simeq \mathcal{V}_{\mathcal{L}\bar{\mathcal{L}}} = N_{\mathcal{L}\bar{\mathcal{L}}}^0 \mathcal{V}_{\mathcal{I}} \oplus \left( \bigoplus_{k \neq 0} N_{\mathcal{L}\bar{\mathcal{L}}}^k \mathcal{V}_{\mathcal{L}_k} \right) \quad (2.6)$$

where  $\mathcal{V}_{\mathcal{L}\bar{\mathcal{L}}}$  is the topological 1-junction space associated with the generally non-simple fused defect  $\mathcal{L}\bar{\mathcal{L}}$ . Note that the identification (2.5) naturally restricts to the topological-junction subspaces  $\mathcal{V}_{\mathcal{L}_1, \dots, \mathcal{L}_k} \subset \mathcal{H}_{\mathcal{L}_1, \dots, \mathcal{L}_k}$ . Indeed, it is obtained by bringing the defect lines together, an operation that preserves conformal weights because topological defects commute with the energy-momentum tensor.

But if  $|\psi\rangle \in \mathcal{V}_{\mathcal{L}}$ ,  $|\psi\rangle \neq 0$ , then one can open and join  $\mathcal{L}$  without changing the correlators<sup>5</sup>, making its action trivial on any operator, as shown in figure 2.5.

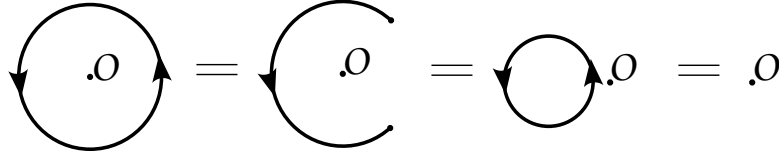


Figure 2.5: Trivial action of the topological 1-junction.

This implies that  $\mathcal{L} \propto \mathcal{I}$  and  $\dim(\mathcal{V}_{\mathcal{L}}) = \delta_{\mathcal{L}, \mathcal{I}}$ . Then in (2.6) just the first term of the direct sum survives and looking at the dimensions of both sides of the equation we get  $\dim(\mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}}) = N_{\mathcal{L}\bar{\mathcal{L}}}^0$ . Since there is always an obvious  $|\psi\rangle \in \mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}}$ ,  $|\psi\rangle \neq 0$ , given by the identity junction  $\mathbf{1}_{\mathcal{L}} : \mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}} \rightarrow \mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}}$  we get the statement.  $\square$

**Proposition 2.2.2.**  *$\mathcal{L} \in TDL(\tau)$  is simple if and only if  $\dim(\mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}}) = N_{\mathcal{L}\bar{\mathcal{L}}}^0 = 1$ .*

*Proof.* If  $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ , then  $\mathcal{H}_{\mathcal{L}} = \mathcal{H}_{\mathcal{L}_1} \oplus \mathcal{H}_{\mathcal{L}_2}$  and

$$\begin{pmatrix} \alpha \mathbf{1}_{\mathcal{H}_{\mathcal{L}_1}} & \mathbf{0} \\ \mathbf{0} & \beta \mathbf{1}_{\mathcal{H}_{\mathcal{L}_2}} \end{pmatrix} \in \mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}} \quad \forall \alpha, \beta \in \mathbb{C}$$

hence  $\dim(\mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}}) \geq 2$ . Instead, if  $\dim(\mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}}) = 1$ , the only element is the irreducible  $\mathbf{1}_{\mathcal{L}}$  and  $\mathcal{L}$  must be simple.  $\square$

<sup>3</sup>All the other junction operators can be obtained by suitable OPE with local operators [17].

<sup>4</sup>As long as the insertion point is not moved past the support of some other operator.

<sup>5</sup>Take a correlator with two of such junctions. Since they are topological, they can be moved freely. However, in the limit where they are brought close together, the two defect starting operators behave exactly as a single defect line  $\mathcal{L}$  without boundary. Therefore a correlator does not distinguish the two topological junctions with respect to the single joined defect.

Now we know that for a simple defect, the fusion with its dual contains the identity exactly once. This motivates the following definition.

**Definition 2.2.8** (Invertibility). A simple TDL  $\mathcal{L} \in TDL(\tau)$  is said to be invertible if

$$\mathcal{L}\bar{\mathcal{L}} = \mathcal{I}$$

In particular, this implies  $\hat{\mathcal{L}} = \hat{\mathcal{L}}^{-1}$

**Proposition 2.2.3.** Let  $\mathcal{L} \in TDL(\tau)$  be a simple defect for a unitary and compact theory  $\tau$  with unique vacuum. Then  $\mathcal{L}$  is invertible if and only if  $\langle \mathcal{L} \rangle = 1$ .

*Proof.* We give the idea of the proof starting from the following (the proof can be found in [6]) lemma

**Lemma 2.2.4.** For a unitary and compact theory  $\tau$  with unique vacuum  $\langle \mathcal{L} \rangle \geq 0$

Now consider a simple defect  $\mathcal{L}$  encircling the origin in the Euclidean plane; this represents the correlator  $\langle \mathcal{L} \rangle$ . Thanks to the conformal inversion  $w = z^{-1}$  we can reverse the orientation of  $\mathcal{L}$  without affecting the correlator. Hence  $\langle \mathcal{L} \rangle = \langle \bar{\mathcal{L}} \rangle$ . Thanks to proposition 2.2.2 we can express the fusion of a simple defect with its dual as<sup>6</sup>

$$\mathcal{L}\bar{\mathcal{L}} = \mathcal{I} + \sum_{k \neq 0} N_{\mathcal{L}\bar{\mathcal{L}}}^k \mathcal{L}_k \quad \Rightarrow \quad \langle \mathcal{L} \rangle^2 = 1 + \sum_{k \neq 0} N_{\mathcal{L}\bar{\mathcal{L}}}^k \langle \mathcal{L}_k \rangle \quad (2.7)$$

This combined with the previous lemma gives  $\langle \mathcal{L} \rangle \geq 1$  and in particular  $\langle \mathcal{L} \rangle = 1$  iff  $\mathcal{L}$  is invertible.  $\square$

In the last proof we have seen that  $\langle \mathcal{L} \rangle \geq 1$ . This result combined with the 1-dimensional representation of the fusion algebra provided by the quantum dimensions

$$\langle \mathcal{L}_i \rangle \langle \mathcal{L}_j \rangle = \sum_k N_{ij}^k \langle \mathcal{L}_k \rangle \quad (2.8)$$

implies that for every simple  $\mathcal{L}_i, \mathcal{L}_j$  of finite quantum dimension, there are only finitely many non-zero fusion coefficients  $N_{ij}^k$ . From now on we will always assume finite quantum dimensions.

We can rewrite equation (2.8) to see an interesting relation between the quantum dimension and the fusion coefficients. By assumption, the number of simple defects is finite and so it is the associated quantum dimension and we can define a finite vector of quantum dimensions  $\langle \underline{\mathcal{L}} \rangle$  and a finite-dimensional matrix  $N_i$ , called *fusion matrix*, whose entries are the fusion coefficients  $(N_i)_j^k = N_{ij}^k$ , and write

$$N_i \langle \underline{\mathcal{L}} \rangle = \langle \mathcal{L}_i \rangle \langle \underline{\mathcal{L}} \rangle$$

Hence,  $\langle \underline{\mathcal{L}} \rangle$  is a common eigenvector for all fusion matrices and the quantum dimension of the  $i$ -th simple defect is an eigenvalue of the corresponding fusion matrix. Moreover, since the entries of such matrix are all non-negative, if the matrix is also irreducible, by the Perron-Frobenius theorem<sup>7</sup>, we get that  $\langle \mathcal{L}_i \rangle$  is the maximal eigenvalue, also sometimes called *Perron-Frobenius dimension*. This relation between quantum dimension and fusion matrices will become interesting later on when dealing with algebraic QFT.

The study of topological 1- and 2-junctions has allowed us to gain more insight about how TDLs work. It remains to study 3-junctions.

First, we note that the fusion coefficients, which describe the fusion of defects as open lines (figure 2.4), also describe what type of topological junction can be created by joining three different TDLs.

<sup>6</sup>Note that from the definition of quantum dimension it follows that  $\langle \mathcal{L}_i \mathcal{L}_j \rangle = \langle \mathcal{L}_i \rangle \langle \mathcal{L}_j \rangle$

<sup>7</sup>The Perron-Frobenius theorem states that for a non-negative irreducible matrix the maximal eigenvalue is associated to the unique (up to multiplication by a positive constant) positive eigenvector. If  $N_i$  is irreducible, the theorem applies and since the entries of  $\langle \underline{\mathcal{L}} \rangle$  are all positive, we can conclude that  $\langle \mathcal{L}_i \rangle$  is the maximal eigenvalue.

**Proposition 2.2.5.** *The dimension of the Hilbert space of topological 3-junctions satisfies  $\dim(\mathcal{V}_{\mathcal{L}_i, \mathcal{L}_j, \bar{\mathcal{L}}_k}) = N_{ij}^k$*

*Proof.* We give the idea of the proof. Using the identification (2.5):

$$\mathcal{V}_{\mathcal{L}_i, \mathcal{L}_j, \bar{\mathcal{L}}_k} \simeq \mathcal{V}_{\mathcal{L}_i \mathcal{L}_j, \bar{\mathcal{L}}_k} = \bigoplus_p N_{ij}^p \mathcal{V}_{\mathcal{L}_p, \bar{\mathcal{L}}_k}$$

Now suppose that there exists a non-zero  $\psi \in \mathcal{V}_{\mathcal{L}_p, \bar{\mathcal{L}}_k}$ . Since this junction is topological, we deduce that  $\psi : \mathcal{H}_{\mathcal{L}_k} \rightarrow \mathcal{H}_{\mathcal{L}_p}$  must be an isomorphism<sup>8</sup>, i.e.  $\mathcal{L}_p \simeq \mathcal{L}_k$ . Therefore, looking at the dimension of both sides

$$\dim(\mathcal{V}_{\mathcal{L}_i \mathcal{L}_j, \bar{\mathcal{L}}_k}) = \sum_p N_{ij}^p \dim(\mathcal{V}_{\mathcal{L}_p, \bar{\mathcal{L}}_k}) = \sum_p N_{ij}^p \delta_{p,k} = N_{ij}^k$$

□

This result is not a coincidence: note that  $v \in \mathcal{H}_{\mathcal{L}_i, \mathcal{L}_j, \bar{\mathcal{L}}_k}$  can be seen as a map  $v : \mathcal{H}_{\mathcal{L}_k} \rightarrow \mathcal{H}_{\mathcal{L}_i, \mathcal{L}_j} \simeq \mathcal{H}_{\mathcal{L}_i \mathcal{L}_j}$ , that is, a homomorphism of twisted representations of the algebra of local operators. This interpretation is consistent with proposition 2.2.5, since the target space decomposes as  $\mathcal{H}_{\mathcal{L}_i \mathcal{L}_j} = \bigoplus_p N_{ij}^p \mathcal{H}_{\mathcal{L}_p}$ . Moreover, the same interpretation is valid for a generic k-junction due to identification (2.5).

This proposition has an important consequence.

**Corollary 2.2.6.** *Let  $\mathcal{L} \in TDL(\tau)$  be a simple defect. There is a distinguished junction  $\mathcal{I}_{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{I}} \in \mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{I}}$  such that*

*Proof.* From propositions 2.2.5 and 2.2.2 we get  $\dim(\mathcal{V}_{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{I}}) = N_{\mathcal{L} \bar{\mathcal{L}}}^0 = 1$

□

This formalizes the intuitive idea that the trivial defect can be attached to any other TDL in any place.

All k-junctions with  $k > 3$  can be obtained by fusion of 3-junctions. Consider, for example, four simple TDLs  $\mathcal{L}_1, \dots, \mathcal{L}_4$  such that  $\mathcal{L}_1 \mathcal{L}_2$  can fuse to  $\bar{\mathcal{L}}_5$  and  $\mathcal{L}_3 \mathcal{L}_4$  can fuse to  $\mathcal{L}_5$ , with  $\mathcal{L}_5$  simple. Therefore, by proposition 2.2.5 there exist two non-zero vectors  $u \in \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \bar{\mathcal{L}}_5}$  and  $\tilde{u} \in \mathcal{V}_{\bar{\mathcal{L}}_5, \mathcal{L}_3, \mathcal{L}_4}$ . The tensor product of these vectors represents the fusion of the common defect between the two junctions and describes a so-called *H-junction*, where the name comes from its particular topology, as can be seen from the left-hand side of figure 2.6. From this configuration, one can obtain a 4-junction by contracting the internal line, which amounts to a map

$$H_{\mathcal{L}_2 \mathcal{L}_3}^{\mathcal{L}_1 \mathcal{L}_4}(\mathcal{L}_5) : \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \bar{\mathcal{L}}_5} \otimes \mathcal{V}_{\bar{\mathcal{L}}_5, \mathcal{L}_3, \mathcal{L}_4} \rightarrow \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4}$$

This map is well defined due to the identification (2.5) and the properties of  $\mathcal{L}_5$ . In particular, it consists of the projection

$$\mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \bar{\mathcal{L}}_5} \otimes \mathcal{V}_{\bar{\mathcal{L}}_5, \mathcal{L}_3, \mathcal{L}_4} \simeq \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \bar{\mathcal{L}}_5 \mathcal{L}_5, \mathcal{L}_3, \mathcal{L}_4} = \bigoplus_k N_{\bar{\mathcal{L}}_5 \mathcal{L}_5}^{\mathcal{L}_k} \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_k, \mathcal{L}_3, \mathcal{L}_4} \rightarrow \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{I}, \mathcal{L}_3, \mathcal{L}_4} \simeq \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4}$$

<sup>8</sup>Consider a non-zero  $\psi^\dagger \in \mathcal{V}_{\mathcal{L}_k, \bar{\mathcal{L}}_p}$ . Then we can consider the map  $\psi \circ \psi^\dagger : \mathcal{H}_{\mathcal{L}_p} \rightarrow \mathcal{H}_{\mathcal{L}_p}$  is an element of  $\mathcal{V}_{\mathcal{L}_p}$  which contains just multiples of the identity.

We can then combine all such maps to find a full decomposition of a 4-junction w.r.t fusion of 3-junctions

$$H_{\mathcal{L}_2 \mathcal{L}_3}^{\mathcal{L}_1 \mathcal{L}_4} = \bigoplus_k H_{\mathcal{L}_2 \mathcal{L}_3}^{\mathcal{L}_1 \mathcal{L}_4}(\mathcal{L}_k) : \bigoplus_k \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \bar{\mathcal{L}}_k} \otimes \mathcal{V}_{\mathcal{L}_k, \mathcal{L}_3, \mathcal{L}_4} \rightarrow \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4} \quad (2.9)$$

where  $k$  runs over all simple defects.<sup>9</sup> This is shown in figure 2.6.

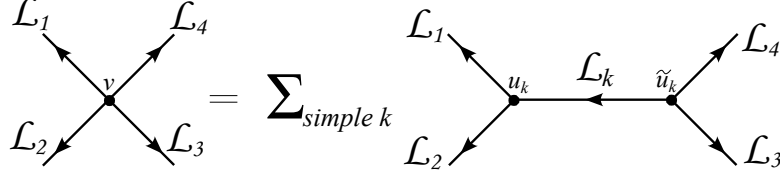


Figure 2.6: Decomposition of a 4-junction in terms of 3-junctions.

The topology of the H-junction comes from the fact that we have fused the pairs  $\mathcal{L}_1 \mathcal{L}_2$  and  $\mathcal{L}_3 \mathcal{L}_4$  and then fused the common channel. We can obtain a different topology by first combining  $\mathcal{L}_1 \mathcal{L}_4$  and  $\mathcal{L}_2 \mathcal{L}_3$ . Since both topologies come from the same 4-junction, they are equivalent. We call *fusion matrix* the map between the respective junction spaces, i.e. the linear transformation that maps the two decompositions of the 4-junction space  $\mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4}$ :

$$F_{\mathcal{L}_2 \mathcal{L}_3}^{\mathcal{L}_1 \mathcal{L}_4}(\mathcal{L}_5, \mathcal{L}_k) : \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \bar{\mathcal{L}}_5} \otimes \mathcal{V}_{\mathcal{L}_5, \mathcal{L}_3, \mathcal{L}_4} \rightarrow \mathcal{V}_{\mathcal{L}_1, \mathcal{L}_k, \mathcal{L}_4} \otimes \mathcal{V}_{\mathcal{L}_2, \mathcal{L}_3, \bar{\mathcal{L}}_k} \quad (2.10)$$

In figure 2.7 the equivalent configurations are shown. We will come back to the proper definition of the fusion matrix when dealing with category theory. The fusion matrix also admits a decomposition in terms of the H-junction map (2.9) and its inverse [11].

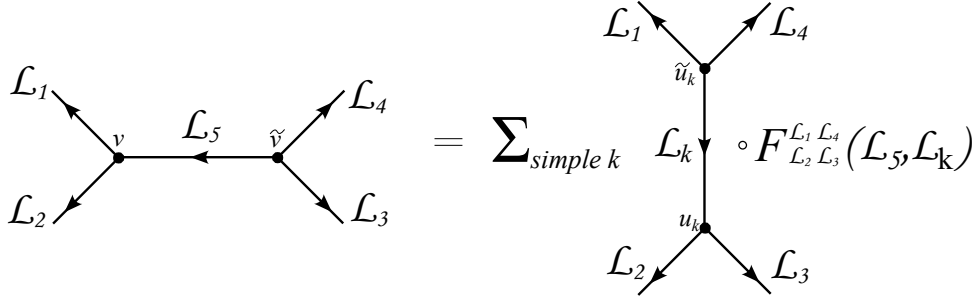


Figure 2.7: Equivalent configurations under the action of the fusion matrices.

A special configuration is called *partial fusion*, which describes what happens when two defects fuse partially, as shown in figure 2.8. Recalling from corollary 2.2.6 that we can always attach the identity defect to any TDL, partial fusion appears as an ordinary fusion, as in figure 2.7, with  $F_{\bar{\mathcal{L}}_1 \bar{\mathcal{L}}_2}^{\mathcal{L}_1 \mathcal{L}_2}(\mathcal{I}, \mathcal{L}_k)$ .

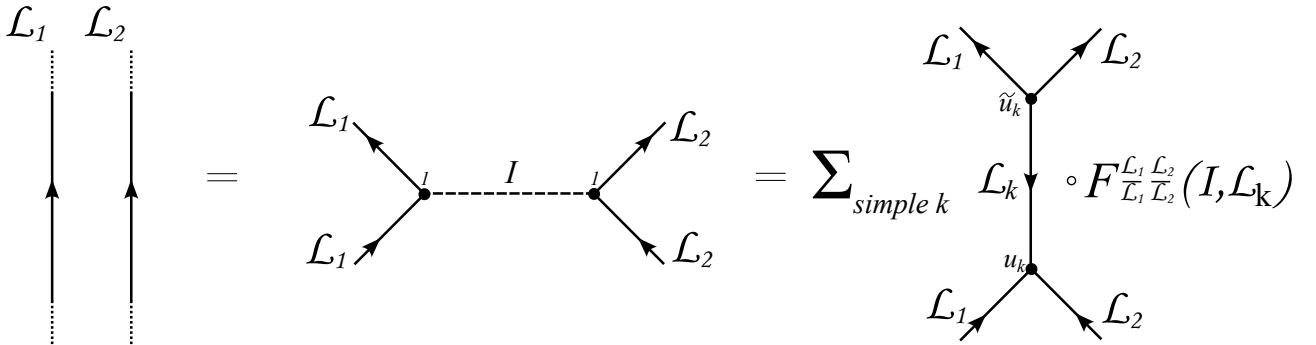


Figure 2.8: Partial fusion between two defects.

<sup>9</sup>If  $\mathcal{L}_k$  does not satisfy the required properties as  $\mathcal{L}_5$  above, then either  $\mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \bar{\mathcal{L}}_k}$  or  $\mathcal{V}_{\mathcal{L}_k, \mathcal{L}_3, \mathcal{L}_4}$ , or both, have vanishing dimension, so that such defect does not contribute.

This can be used to determine how the correlation function is modified when a simple defect  $\mathcal{L}$  is moved past a local operator  $\phi(z, \bar{z})$ , see figure 2.9. In particular, this process amounts to the mapping

$$\phi \mapsto \frac{\hat{\mathcal{L}}(\phi)}{\langle \mathcal{L} \rangle} + \hat{\mathcal{L}}^v(\phi) \quad (2.11)$$

where  $\hat{\mathcal{L}}^v(\phi) \in \mathcal{H}_{\mathcal{L}\bar{\mathcal{L}}}$  is a defect-starting operator with the same conformal weights as  $\phi$  attached to a superposition of simple TDLs, different from  $\mathcal{I}$ , appearing in the fusion of  $\mathcal{L}\bar{\mathcal{L}}$  and joined to  $\mathcal{L}$  by a junction  $v \in \mathcal{V}_{\mathcal{L}\bar{\mathcal{L}}, \bar{\mathcal{L}}, \mathcal{L}}$ .

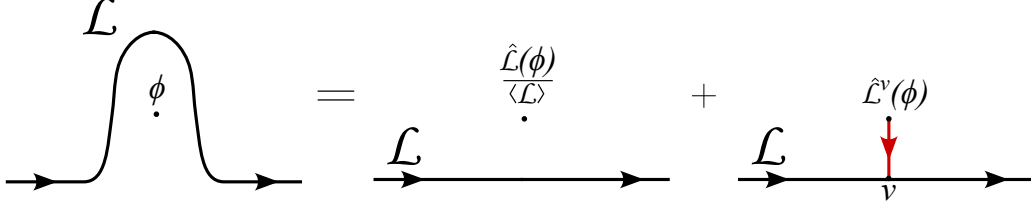


Figure 2.9: Moving of a simple defect line through the local operator  $\phi(z, \bar{z})$ .

We say that  $\mathcal{L}$  is transparent to  $\phi$  if and only if  $\hat{\mathcal{L}}(\phi) = \langle \mathcal{L} \rangle \phi$ , because in this case one can prove [17] that the term  $\hat{\mathcal{L}}^v(\phi)$  also vanishes.

The factor  $\frac{1}{\langle \mathcal{L} \rangle}$  is determined by observing that the defect must be transparent to the vacuum operator and to its descendants.

### 2.2.3 Verlinde lines

For a CFT with diagonal modular invariant, such as the Ising model, topological defect lines admit a particularly simple and explicit construction.

**Definition 2.2.9.** A *Verlinde line* is a topological defect line in a diagonal modular invariant CFT that commutes with the entire left- and right-moving chiral algebra.

The Verlinde lines are labelled by the same irreducible sectors as the primary fields, and their fusion algebra reproduces the fusion algebra of the theory:

$$\Phi_i \Phi_j = \sum_k N_{ij}^k \Phi_k \quad \implies \quad \mathcal{L}_i \mathcal{L}_j = \sum_k N_{ij}^k \mathcal{L}_k$$

Since a Verlinde line commutes with the full chiral algebra, it preserves each irreducible chiral sector<sup>10</sup>. Moreover, in a diagonal theory each sector appears with multiplicity one, so its action on the corresponding primary state must be proportional to the identity:

$$\hat{\mathcal{L}}_l |\phi_i\rangle = \lambda_i^{\mathcal{L}_l} |\phi_i\rangle \quad (2.12)$$

As we will shortly show, these eigenvalues are determined by the modular  $S$ -matrix:

$$\hat{\mathcal{L}}_l |\phi_i\rangle = \frac{S_{li}}{S_{0i}} |\phi_i\rangle \quad (2.13)$$

Finally, the quantum dimension of  $\mathcal{L}_l$  is obtained from its action on the vacuum sector. Setting  $i = 0$  in (2.13), one finds

$$\langle \mathcal{L}_l \rangle = \frac{S_{l0}}{S_{00}} \quad (2.14)$$

<sup>10</sup>In general CFTs, TDLs may preserve just a subalgebra of the whole chiral algebra.

## 2.3 Defects in partition functions

We still do not have a general procedure for finding all TDLs of a given CFT. Fortunately, for compact and unitary, local theories, we can use modular invariance to constrain the allowed TDLs as well as explicitly calculate their fusion rules.

The partition function is obtained by compactifying the cylinder time; therefore, there are two ways to insert a TDL, corresponding to the two insertions in the cylinder of figures 2.2 and 2.3. These are shown in figure 2.10.

Before studying the analytical realization of such defect insertions, we need to make a clarification. Recall that the Hilbert space decomposes as

$$\mathcal{H} = \bigoplus_{i,j} V_{ij} \otimes \mathcal{H}_{(i,j)}, \quad \dim V_{ij} = Z_{ij}$$

where  $\mathcal{H}_{(i,j)} = M_i^c \otimes \bar{M}_j^{\bar{c}}$  is an irreducible representation of the left- and right-moving Virasoro algebras, while  $V_{ij}$  is its multiplicity space. Since a topological defect operator commutes with both Virasoro actions, Schur's lemma implies that it acts trivially on each irreducible factor and arbitrarily on the corresponding multiplicity space:

$$\bigoplus_{i,j} L_{ij} \otimes \text{id}_{\mathcal{H}_{(i,j)}}, \quad L_{ij} \in \text{End}(V_{ij}).$$

More precisely, suppose that the multiplicity of the  $(i,j)$  representation is  $Z_{i,j} > 1$ . Then the defect operator must be a linear combination of operators intertwining the different copies of  $\mathcal{H}_{(i,j)}$ :

$$T^{(i,j,\alpha,\alpha')} : (\mathcal{H}_{(i,j)})^{\alpha'} \rightarrow (\mathcal{H}_{(i,j)})^{\alpha}, \quad \alpha, \alpha' = 1, \dots, Z_{i,j} \quad (2.15)$$

Thus, if  $|i, j, n, \bar{n}\rangle^{(\alpha)}$  is the  $(n, \bar{n})$ -th member of a basis of the vector space of the irreducible  $\alpha$ -th representation with highest weights corresponding to  $(i, j)$ :

$$(\hat{\mathcal{L}})_{i,j}^{\alpha,\alpha'} = L_{i,j}^{\alpha,\alpha'} \sum_{n,\bar{n}} |i, j, n, \bar{n}\rangle^{(\alpha)} \langle i, j, n, \bar{n}|^{(\alpha')}, \quad L_{i,j}^{\alpha,\alpha'} \in \mathbb{C}, \quad \alpha, \alpha' = 1, \dots, Z_{i,j}$$

For each highest weight then the action of the defect should be thought not as a constant value, but as a constant square matrix  $L_{i,j}^{\alpha,\alpha'}$ . Of course the matrix could be one-dimensional giving rise to the constant value interpretation.

As a side result, from (2.15), we see that there are

$$\sum_{i,j} Z_{ij}^2 = \text{Tr}(Z^T Z) \quad (2.16)$$

independent operators commuting with the two Virasoro algebras, i.e. linearly independent solutions to equation (2.4).

Now we are ready to see how to implement analytically defects in the partition function. Starting from the insertion at fixed cylinder time, we get:

$$\mathcal{Z}^{\mathcal{L}}(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}}[\hat{\mathcal{L}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}] = \sum_{i,j} \text{Tr}_{V_{i,j}}[L_{i,j}] \text{Tr}_{\mathcal{H}_{i,j}}[q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}] = \sum_{i,j} \tilde{L}_{i,j} \chi_i^{(c)} \bar{\chi}_j^{(\bar{c})} \quad (2.17)$$

where  $\tilde{L}_{i,j} = \text{Tr}_{V_{i,j}}[L_{i,j}] = \sum_{\alpha} L_{i,j}^{\alpha,\alpha}$ .

We can make explicit the multiplicity matrix  $Z_{i,j}$  by setting  $\lambda_{i,j}^{\mathcal{L}} := \tilde{L}_{i,j}/Z_{i,j}$ . Clearly, since a defect

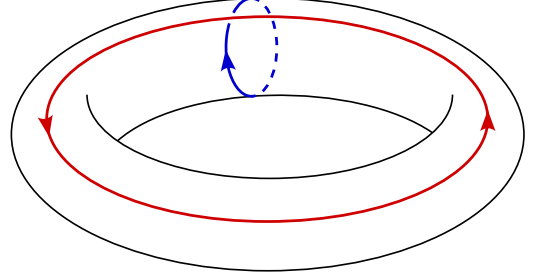


Figure 2.10: Possible ways of inserting a TDL on the torus.

along the spatial cycle does not modify the Hilbert space structure, if  $Z_{i,j} = 0$  for some  $(i,j)$ , such term would not be present in the partition function in the first place. Thus, we obtain:

$$\mathcal{Z}^{\mathcal{L}}(\tau, \bar{\tau}) = \sum_{i,j} Z_{i,j} \lambda_{ij}^{\mathcal{L}} \chi_i(\tau) \bar{\chi}_j(\bar{\tau}) \quad (2.18)$$

Looking at the insertion at constant cylinder position, we get the twisted partition function:

$$\mathcal{Z}_{\mathcal{L}}(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}_{\mathcal{L}}} [q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}] = \sum_{i,j} Z_{i,j}^{\mathcal{L}} \chi_i(\tau) \bar{\chi}_j(\bar{\tau}) \quad (2.19)$$

where we used the transparency of TDLs with respect to the energy-momentum tensor to express the trace over the twisted Hilbert space as a different combination of the characters.

By applying an S-transformation we exchange space and time directions, thus exchanging the two configurations of figure 2.10. According to the orientations in figures 2.2 and 2.3, this implies

$$\mathcal{Z}_{\mathcal{L}}(\tau, \bar{\tau}) = \mathcal{Z}^{\mathcal{L}}(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) \quad (2.20)$$

which reduces to the usual modular invariance for the trivial defect  $\mathcal{I}$ . Combining (2.20) with (2.18) and (2.19) and exploiting the transformation of the characters (1.36), we get

$$\sum_{i,j} Z_{i,j}^{\mathcal{L}} \chi_i(\tau) \bar{\chi}_j(\bar{\tau}) = \sum_{i,j,i',j'} Z_{i,j} \lambda_{ij}^{\mathcal{L}} S_{i,i'} S_{j,j'}^{\dagger} \chi_{i'}(\tau) \bar{\chi}_{j'}(\bar{\tau})$$

Renaming the indices and using the fact that  $S = S^T$ , we conclude with the constraint

$$Z_{i,j}^{\mathcal{L}} = \sum_{i',j'} S_{i,i'} Z_{i',j'} \lambda_{i',j'}^{\mathcal{L}} S_{j',j}^{\dagger} \quad (2.21)$$

The S-matrix can be calculated explicitly for any minimal model, while the requirement that  $Z_{i',j'}^{\mathcal{L}} \in \mathbb{N}$ , constrains the way a TDL can act, that is to say the possible values of  $\lambda_{ij}^{\mathcal{L}}$ . In particular, there are cases in which this fixes all the defects of the theory; for example, for diagonal theories, where (2.21) reduces to:

$$Z_{i,j}^{\mathcal{L}} = \sum_{i',j'} S_{i,i'} \lambda_{i',j'}^{\mathcal{L}} \delta_{i',j'} S_{j',j}^{\dagger} \quad (2.22)$$

In particular, we can invert this matrix equation and solve it for  $\lambda_{ii}^{\mathcal{L}_l} := \lambda_{li}$  by making the ansatz  $Z_{i',j'}^{\mathcal{L}_l} = N_{li'}^{j'}$ , i.e.  $Z^{\mathcal{L}_l} = N_l$ , and using Verlinde's formula (1.38), finding the eigenvalues (2.13):

$$\lambda_i^{\mathcal{L}_l} = \lambda_{ii}^{\mathcal{L}_l} = \sum_{i',j'} S_{i,i'}^{\dagger} N_{li'}^{j'} S_{j',i} = \sum_{i',j',k} S_{i,i'}^* \frac{S_{lk} S_{i'k} S_{j'k}^*}{S_{0k}} S_{i,j'} = \frac{S_{li}}{S_{0i}} \quad (2.23)$$

Here we have exploited the unitarity of the S-matrix. Note that defects are labeled with an index that belongs to the S-matrix. Therefore, we have as many simple defects as primary operators, which, in turn, amount to  $\text{Tr}[Z] = \sum_i \delta_{i,i}$ . However, this is also equal to  $\text{Tr}[Z^T]$ , so this exhausts all possible independent operators that satisfy (2.4). Therefore, in diagonal CFTs all TDLs are Verlinde lines.

As a consistency check, consider the integrality and positivity condition  $Z_{ij}^{\mathcal{L}} \in \mathbb{N}$ . Equation (2.22) implies that the allowed vectors  $\bar{\lambda}^{\mathcal{L}} := (\lambda_1^{\mathcal{L}}, \dots, \lambda_n^{\mathcal{L}})$  form the image of the lattice

$$\left( Z^{\mathcal{L}} \in \text{Mat}_n(\mathbb{N}) \mid S^{\dagger} Z S \text{ is diagonal} \right)$$

under the injective map  $Z^{\mathcal{L}} \mapsto (\lambda_1^{\mathcal{L}}, \dots, \lambda_n^{\mathcal{L}})$ . Hence, the allowed solution vectors  $\bar{\lambda}^{\mathcal{L}}$  form a lattice of rank at most  $n$ . The  $n$  Verlinde lines provide  $n$  linearly independent elements of this lattice and therefore show that it has maximal rank<sup>11</sup>. However, integrality alone does not imply that these are the only solutions, since arbitrary non-negative integral combinations correspond, in general, to non-simple defects.

The original derivation of Verlinde lines can be found in [27].

<sup>11</sup>Note that this argument alone does not show that the Verlinde lines form a basis of the full lattice; it only shows that they generate a sublattice of the same rank.

## 2.4 Fusion category of TDLs

Formally, the properties of TDLs described in section 2.2 can be formulated in terms of the language of category theory, in particular in terms of a *fusion category*. Since category theory will reveal to be the best language to describe defects, we shall give the first definitions. However, we will give just the most important ones and rely on the ideas for the others, in order to make the reading less heavy, since we do not need to develop the full machinery of category theory, which is quite involved. For precise definitions see [32].

**Definition 2.4.1** (Category). A *category*  $\mathcal{C}$  consists of

- a collection  $Obj(\mathcal{C})$  of objects;
- for each pair  $U, V \in Obj(\mathcal{C})$ , a set  $\text{Hom}(U, V)$  of morphisms  $f : U \rightarrow V$ ;
- for each  $U, V, Z \in Obj(\mathcal{C})$  a function

$$\begin{aligned} \circ : \text{Hom}(U, V) \times \text{Hom}(V, Z) &\rightarrow \text{Hom}(U, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

called *composition*;

- for each  $U \in Obj(\mathcal{C})$ , an element  $1_U$  of  $\text{Hom}(U, U)$ , called the *identity on  $U$* ;

such that the following axioms hold:

- associativity: for each  $f \in \text{Hom}(U, V)$ ,  $g \in \text{Hom}(V, Z)$  and  $h \in \text{Hom}(Z, H)$  we have  $(h \circ g) \circ f = h \circ (g \circ f)$ ;
- identity laws: for each  $f \in \text{Hom}(U, V)$ , we have  $f \circ 1_U = f = 1_V \circ f$ .

We will often write  $U \in \mathcal{C}$ , meaning  $U \in Obj(\mathcal{C})$ .

In our case the objects of the category will be TDLs and the additional structure of fusion ring will endow the category with analogous operations.

**Definition 2.4.2** (Fusion category). A *fusion category* is a rigid semisimple abelian  $\mathbb{C}$ -linear monoidal category with only finitely many isomorphism classes of simple objects and such that the unit object is simple.

We now decompose this definition to understand the meaning of each piece:

- a *monoidal* or *tensor* category is a category  $\mathcal{C}$  equipped with

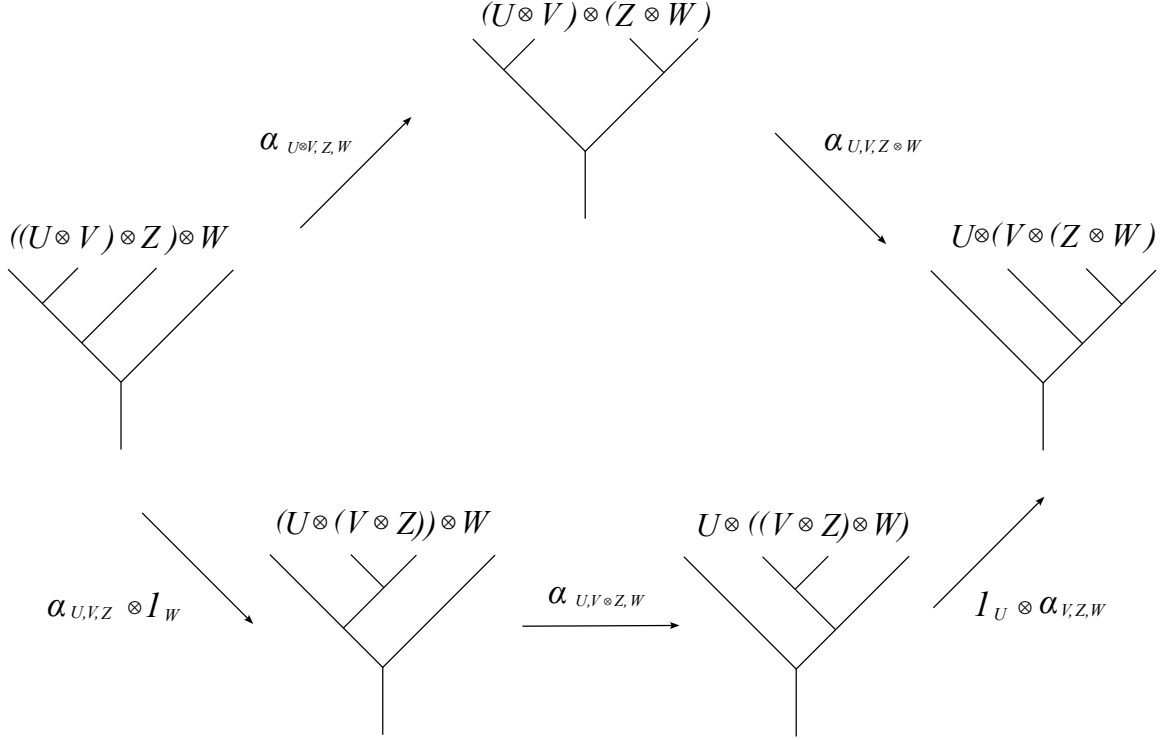
1. a tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ ;
2. a unit object  $1 \in \mathcal{C}$ ;
3. *associators*  $(U \otimes V) \otimes Z \xrightarrow{\alpha_{U,V,Z}} U \otimes (V \otimes Z)$
4. *unitors*  $\tilde{\rho}_U : 1 \otimes U \rightarrow U$  and  $\rho_U : U \otimes 1 \rightarrow U$

such that the following diagrams commute (*coherence relations*):<sup>12</sup>

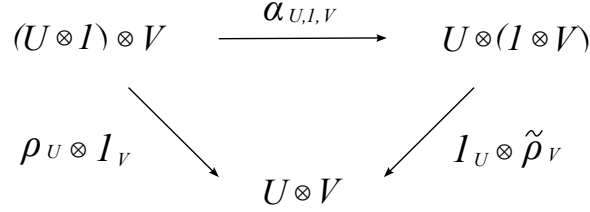
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<sup>12</sup>A tensor category is called *strict* if all these isomorphisms are identities, so that the tensor product of objects is associative and  $1 \otimes U = U = U \otimes 1$ . By the coherence theorem [33], there is no loss of generality in imposing this strictness property. This is why the associative fusion product we defined in 2.2.5 is always compatible with this definition.

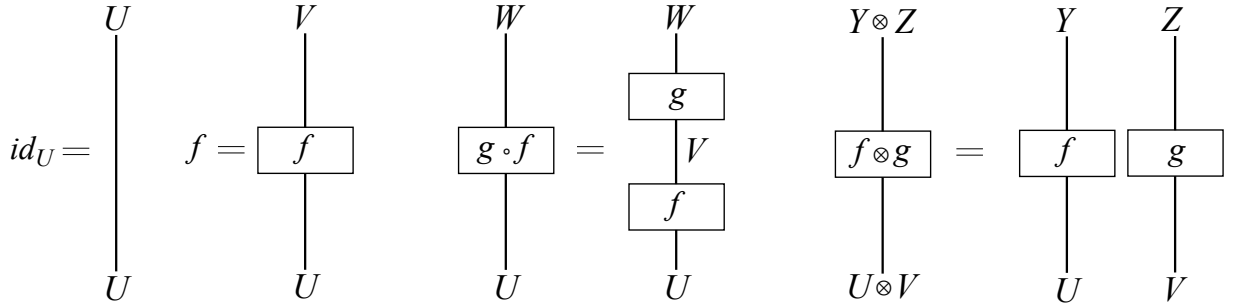
1. the pentagon identity:



2. the triangle identity:



We use string diagrams to represent objects and morphisms of monoidal categories. We follow the convention that such figures are read from bottom to top. Basic notations are the following:



Where  $f \in \text{Hom}(U, V)$ , composition of morphisms amounts to concatenation of lines, while the tensor product corresponds to juxtaposition.

- *abelian  $\mathbb{C}$ -linear* means that all  $\text{Hom}$ -sets carry the structure of a  $\mathbb{C}$  vector space and composition is  $\mathbb{C}$ -linear;  $\mathcal{C}$  behaves like a category of modules over a commutative ring. In particular, we have direct sums.
- *semisimple*: this means that every object of  $\mathcal{C}$  can be written as a finite direct sum of simple objects. An object  $U \in \text{Obj}(\mathcal{C})$  is called simple if  $\text{Hom}(U, U) := \text{End}(U) = \mathbb{C}1_U$ . Also in the categorical case we have a version of Schur's lemma:

**Lemma 2.4.1.** *If  $U, V$  are simple objects that are not isomorphic then  $\text{Hom}(U, V) \simeq 0$ .*

- *rigid*: this means that all objects of  $\mathcal{C}$  have left and right duals. Formally, a (right) duality on a tensor category  $\mathcal{C}$  associates to every  $U \in \text{Obj}(\mathcal{C})$  another object  $U^\vee \in \text{Obj}(\mathcal{C})$  and morphisms

$$b_U \in \text{Hom}(1, U \otimes U^\vee) \quad d_U \in \text{Hom}(U^\vee \otimes U, 1)$$

called *co-evaluation* and *evaluation* respectively and represented in (2.24) and respecting the *snake identities* (2.25), and to every morphism  $f \in \text{Hom}(U, V)$  the morphism

$$f^\vee := (d_V \otimes 1_{U^\vee}) \circ (1_{V^\vee} \otimes f \otimes 1_{U^\vee}) \circ (1_{V^\vee} \otimes b_U) \in \text{Hom}(V^\vee, U^\vee)$$

represented in (2.24). In a similar way left-duality is defined.

$$\begin{array}{ccc} \begin{array}{c} U \quad U^\vee \\ | \quad | \\ \boxed{b_U} \end{array} = \begin{array}{c} U \quad U^\vee \\ | \quad | \\ \curvearrowright \end{array} & \begin{array}{c} \boxed{d_U} \\ | \quad | \\ U^\vee \quad U \end{array} = \begin{array}{c} \curvearrowleft \\ | \quad | \\ U^\vee \quad U \end{array} & \begin{array}{c} U^\vee \\ | \\ \boxed{f^\vee} \\ | \\ V^\vee \end{array} = \begin{array}{c} U^\vee \\ | \\ \boxed{f} \\ | \\ V^\vee \end{array} \end{array} \quad (2.24)$$

$$\begin{array}{ccc} \begin{array}{c} U^\vee \\ | \\ U^\vee \end{array} = \begin{array}{c} \curvearrowright \\ | \\ U^\vee \end{array} & \begin{array}{c} U^\vee \\ | \\ U \end{array} = \begin{array}{c} \curvearrowleft \\ | \\ U \end{array} \end{array} \quad (2.25)$$

Once left/right dualities are defined, hence also the left duality morphisms  $\tilde{b}_U \in \text{Hom}(1, {}^\vee U \otimes U)$  and  $\tilde{d}_U \in \text{Hom}(U \otimes {}^\vee U, 1)$ , we can also define left and right traces of morphisms:

$$\text{Tr}_r(f) = \begin{array}{c} \curvearrowright \\ | \\ \boxed{f} \\ | \\ \curvearrowleft \end{array} \quad \text{Tr}_l(f) = \begin{array}{c} \curvearrowleft \\ | \\ \boxed{f} \\ | \\ \curvearrowright \end{array}$$

which are cyclic and factorize with respect to tensor products:

$$\text{Tr}_Y(g \circ f) = \text{Tr}_Y(f \circ g) \quad \text{Tr}_Y(f \otimes g) = \text{Tr}_Y(f) \text{Tr}_Y(g), \quad Y = l, r \quad (2.26)$$

In the case where left duality coincide the right duality, also the two notions of trace are equal; thus the category  $\mathcal{C}$  is called *spherical*. The trace of the identity morphism is known as the *quantum dimension* of an object:  $\dim(U) = d(U) := \text{Tr}(\text{id}_U)$ .

In the case of our interest, the fusion category of TDLs is characterized by TDLs as objects  $\mathcal{L} \in \text{Obj}(\mathcal{C})$  and the morphisms spaces are given by  $\text{Hom}(\mathcal{L}_1, \mathcal{L}_2) = \mathcal{V}_{\mathcal{L}_2, \bar{\mathcal{L}}_1}$ , which are  $\mathbb{C}$ -vector spaces. Moreover, the direct sum and the fusion of defects are compatible with the abelian  $\mathbb{C}$ -linear and the monoidal properties of a fusion category.

For each object  $\mathcal{L} \in \text{Obj}(\mathcal{C})$  there is also a dual  $\bar{\mathcal{L}} \in \text{Obj}(\mathcal{C})$ . To be precise, a general fusion category does not satisfy  ${}^\vee \mathcal{L} = \mathcal{L}^\vee$  and neither  $(\mathcal{L}^\vee)^\vee = \mathcal{L}$ . However, in the case of the fusion category of

defects, left and right dual coincides  ${}^\vee \mathcal{L} = \mathcal{L}^\vee =: \bar{\mathcal{L}}$ . Formally, this means that we are dealing with a *sovereign* or *pivotal* fusion category. Moreover, left and right traces coincide<sup>13</sup>, hence the category is also spherical and the categorical quantum dimension coincide with the physical one coming from the resolution of bubble diagrams:  $\text{Tr}(\text{id}_{\mathcal{L}}) = \langle \mathcal{L} \rangle = \langle \bar{\mathcal{L}} \rangle$ .

Since the morphism sets are vector spaces, their dimension is well defined. In particular, According to proposition 2.2.5, the fusion coefficients satisfy:

$$\dim(\text{Hom}(\mathcal{L}_k, \mathcal{L}_i \otimes \mathcal{L}_j)) = N_{ij}^k$$

for  $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$  simple objects.

Semisimplicity of a tensor category  $\mathcal{C}$  implies in particular *dominance* of  $\mathcal{C}$ , which means that there exists a family  $\{\mathcal{L}_i\}_{i \in \mathcal{J}}$  of simple objects with the following property: for any  $U, V \in \text{Obj}(\mathcal{C})$  every morphism  $f \in \text{Hom}(U, V)$  can be decomposed into a finite sum

$$f = \sum_r g_r \circ h_r, \quad g_r \in \text{Hom}(\mathcal{L}_i, V) \quad h_r \in \text{Hom}(U, \mathcal{L}_i) \quad (2.27)$$

for suitable members  $\mathcal{L}_i = \mathcal{L}_i(r)$  (possibly with repetitions) of this family. Physically, this reflects the property of substituting a junction between the non simple defects U and V, with a superposition of common simple defects. This property was used when we talked about how 3-junction could be fused to form higher rank junctions.

Let  $\{\mathcal{L}_i\}_{i \in \mathcal{J}}$  be the finite set of simple defects of the fusion category<sup>14</sup>. In particular, let  $\mathcal{L}_0 = \mathcal{I}$  be the trivial defect, corresponding to the tensor unit. We fix a base  $\{v_{(i,j)k}^\alpha\}$  in the junction spaces  $\text{Hom}(\mathcal{L}_i \otimes \mathcal{L}_j, \mathcal{L}_k)$ , as well as dual bases  $\{\tilde{v}_{\bar{\alpha}}^{(i,j)k}\}$  in  $\text{Hom}(\mathcal{L}_k, \mathcal{L}_i \otimes \mathcal{L}_j)$ , where  $\alpha, \bar{\alpha} = 1, \dots, N_{ij}^k$ . Pictorially:

$$\begin{array}{c} \mathcal{L}_k \\ | \\ \boxed{v_{(i,j)k}^\alpha} \\ | \quad | \\ \mathcal{L}_i \quad \mathcal{L}_j \end{array} = \begin{array}{c} k \\ | \\ \bullet \alpha \\ / \quad \backslash \\ i \quad j \end{array} \quad \begin{array}{c} \mathcal{L}_i \quad \mathcal{L}_j \\ | \quad | \\ \boxed{\tilde{v}_{\bar{\alpha}}^{(i,j)k}} \\ | \\ \mathcal{L}_k \end{array} = \begin{array}{c} i \quad j \\ \backslash \quad / \\ \bullet \bar{\alpha} \\ | \\ k \end{array} \quad (2.28)$$

Since the junction spaces are  $\mathbb{C}$ -Hilbert spaces the category carries a *dagger structure*

$$\dagger : \text{Hom}(\mathcal{L}_1, \mathcal{L}_2) \rightarrow \text{Hom}(\mathcal{L}_2, \mathcal{L}_1)$$

such that for  $v \in \text{Hom}(\mathcal{L}_1, \mathcal{L}_2)$  and  $u \in \text{Hom}(\mathcal{L}_2, \mathcal{L}_1)$  we have

$$(v \circ u)^\dagger = u^\dagger \circ v^\dagger \quad (v^\dagger)^\dagger = v$$

In particular, due to Schur's lemma, if  $\mathcal{L}_1$  is simple, then  $v^\dagger \circ v \in \text{End}(\mathcal{L}_1)$  must be proportional to the identity. The only option is  $v^\dagger \circ v = \langle v, v \rangle \text{id}_{\mathcal{L}_1}$ , where  $\langle \cdot, \cdot \rangle$  is the inner product in the topological junction Hilbert space. Therefore, the choice of the basis and its dual determine the proportionality constant between bubble diagrams, such as the one in (2.29), and the identity.

<sup>13</sup>This can be seen as a consequence of the fact that closed defect diagrams are evaluated on the Riemann sphere and are invariant under topological deformations

<sup>14</sup>In the following the Latin indices belong to  $\mathcal{J}$ , which contains all the indices labeling simple objects.

$$\begin{array}{c} i \\ | \\ \bullet^\alpha \\ \text{---} \bigcirc \text{---} \\ \bullet^{\bar{\beta}} \\ | \\ i \end{array} \begin{array}{c} j \quad k \end{array} = \langle \tilde{\mathbf{v}}_{\bar{\beta}}^{(j,k)i} | \mathbf{v}_{(j,k)i}^\alpha \rangle = \delta_\beta^\alpha \sqrt{\frac{\langle j \rangle \langle k \rangle}{\langle i \rangle}} \begin{array}{c} i \\ | \\ | \\ | \\ i \end{array} \quad (2.29)$$

In this case, we have chosen the normalization imposing the following conditions:

$$\langle \tilde{\mathbf{v}}_{\bar{\beta}}^{(j,k)0} | v_{(j,k)0}^\alpha \rangle = \delta_\beta^\alpha \delta^{j,k} \langle k \rangle, \quad \langle \tilde{\mathbf{v}}_{\bar{\beta}}^{(j,0)i} | v_{(j,0)i}^\alpha \rangle = \langle \tilde{\mathbf{v}}_{\bar{\beta}}^{(0,j)i} | v_{(0,j)i}^\alpha \rangle = \delta_\beta^\alpha \delta^{j,i}$$

where the first one comes from the fact that we want isolated bubbles to return the quantum dimension of the involved TDL and the second one ensures isotopy invariance<sup>15</sup>.

This allows us to see another special feature of this basis: for simple objects, the fusion spaces  $\text{Hom}(1, U \otimes \bar{U})$  and  $\text{Hom}(\bar{U} \otimes U, 1)$  are one-dimensional; as a result, the left/right evaluation and co-evaluation maps are uniquely determined up to normalization, and with the chosen normalization they coincide with the canonical basis morphisms describing fusion into the identity<sup>16</sup>:

$$\begin{array}{c} U_i \quad \bar{U}_i \\ | \quad | \\ \text{---} \bigcup \text{---} \\ | \quad | \\ \bar{U}_i \quad U_i \end{array} = \begin{array}{c} i \quad \bar{i} \\ | \quad | \\ \text{---} \bigcup \text{---} \\ \bullet^{\bar{\alpha}} \\ | \\ 0 \end{array} \quad \begin{array}{c} \text{---} \bigcup \text{---} \\ | \quad | \\ \bar{U}_i \quad U_i \end{array} = \begin{array}{c} 0 \\ | \\ \text{---} \bigcup \text{---} \\ \bullet^\alpha \\ | \quad | \\ \bar{i} \quad i \end{array}$$

So the traces used to define the quantum dimension coincide with the bubble diagrams produced by the basis morphisms.

We have also the following completeness relation:

**Proposition 2.4.2.** *With the normalization choice of equation (2.29) we satisfy the following completeness relation.*

<sup>15</sup>Note that the second condition corresponds to a diagram such as the one of figure (2.29), but with a side of the bubble replaced by the identity defect.

<sup>16</sup>The map  $U_i \mapsto \bar{U}_i$  induces an involution  $i \mapsto \bar{i}$  on the finite set of labels  $\mathcal{J}$  according to  $U_{\bar{i}} = \bar{U}_i$ , where we recall that  $\bar{U} := U^\vee = {}^\vee U$  with the last equality holding for the particular category of TDLs.

$$id_{i \otimes j} = \sum_k \sqrt{\frac{\langle k \rangle}{\langle i \rangle \langle j \rangle}} \sum_\alpha$$

*Proof.* By the dominance property (2.27) of  $\mathcal{C}$  we can decompose the identity morphism  $\text{id}_{i \otimes j}$  as:

$$id_{i \otimes j} = \begin{array}{c} i \\ | \\ j \end{array} = \sum_{k,\alpha} A_{ij}^{k\alpha} \begin{array}{c} i \quad j \\ \searrow \quad \swarrow \\ \bullet \quad \bar{\alpha} \\ | \\ k \\ | \\ \bullet \quad \alpha \\ \swarrow \quad \searrow \\ i \quad j \end{array}$$

Where  $A_{ij}^{k\alpha} \in \mathbb{R}$ , since each term of the sum is proportional to the morphism  $\tilde{v}_{\tilde{\alpha}}^{(i,j)k} \circ v_{(i,j)k}^{\alpha}$ . Now we compose two of these diagrams:

$$\begin{array}{c} i \\ | \\ | \\ j \end{array} = \sum_{\substack{k,\alpha \\ k',\beta}} A_{ij}^{k\alpha} A_{ij}^{k'\beta} \begin{array}{c} i \qquad j \\ \diagdown \quad \diagup \\ \bullet \quad \bar{\beta} \\ | \quad k' \\ \bullet \quad \beta \\ \bigcirc \quad i \quad j \\ \bullet \quad \bar{\alpha} \\ | \quad k \\ \bullet \quad \alpha \\ \diagup \quad \diagdown \\ i \qquad j \end{array} = \sum_{\substack{k,\alpha \\ k',\beta}} A_{ij}^{k\alpha} A_{ij}^{k'\beta} \delta_k^{k'} \delta_\beta^\alpha \sqrt{\frac{\langle i \rangle \langle j \rangle}{\langle k \rangle}} \begin{array}{c} i \qquad j \\ \diagdown \quad \diagup \\ \bullet \quad \bar{\beta} \\ | \quad k \\ \bullet \quad \alpha \\ \diagup \quad \diagdown \\ i \qquad j \end{array}$$

By consistency, we must have:

$$A_{ij}^{k\alpha} \sqrt{\frac{\langle i \rangle \langle j \rangle}{\langle k \rangle}} = 1$$

☐

Note that the completeness relation of proposition 2.4.2 corresponds to the partial fusion of TDLs.

Also, we satisfy the following relations:

<sup>16</sup>Usually, in the mathematical literature junctions are normalized to 1.

$$(2.30)$$

Note that the relations in (2.30) are the categorical analogue of the result of proposition 2.2.6.

Once we have chosen bases as above, there are two distinct distinguished bases for the morphism space  $\text{Hom}(\mathcal{L}_i \otimes \mathcal{L}_j \otimes \mathcal{L}_k, \mathcal{L}_l)$ :

$$\bigoplus_p \text{Hom}(\mathcal{L}_i \otimes \mathcal{L}_j, \mathcal{L}_p) \otimes \text{Hom}(\mathcal{L}_p \otimes \mathcal{L}_k, \mathcal{L}_l) \quad (2.31)$$

and

$$\bigoplus_q \text{Hom}(\mathcal{L}_j \otimes \mathcal{L}_k, \mathcal{L}_q) \otimes \text{Hom}(\mathcal{L}_i \otimes \mathcal{L}_q, \mathcal{L}_l) \quad (2.32)$$

Recalling that the Hom-spaces that appear on the right-hand side of both (2.31) and (2.32) are the topological three junction spaces, the coefficients of the basis transformation between the two, called *F-symbols* or *Fusion matrices*, are equivalent to the F-matrices introduced in equation (2.10):<sup>17</sup>

$$(2.33)$$

Notice that the F-matrices arise independently from the triviality of the associator, since they are just a change of basis. Nevertheless, the pentagon identity manifests itself in coordinates as [28]:

$$\sum_{\delta} F_{\alpha p \beta, \delta r \epsilon}^{(ijq)m} F_{\delta q \gamma, \nu s \mu}^{(rkl)m} = \sum_{\xi, t, \eta, \sigma} F_{\alpha q \gamma, \xi t \eta}^{(jkl)p} F_{\beta p \xi, \nu s \sigma}^{(itl)m} F_{\sigma t \eta, \mu r \epsilon}^{(ijk)t}$$

for all  $\nu, s, \mu, r, \epsilon$  and  $\alpha, p, \beta, q, \gamma$ . We finally state the following propositions:

**Proposition 2.4.3.** [28] *A fusion category is completely determined by the fusion coefficients and the F-symbols.*

**Theorem 2.4.4** (Ocneanu's rigidity). [34] *There are finitely many fusion categories with any given fusion ring*

<sup>17</sup>By comparison, the F-matrix defined in (2.10),  $F_{\mathcal{L}_2 \mathcal{L}_3}^{\mathcal{L}_1 \mathcal{L}_4}(\mathcal{L}_5, \mathcal{L}_k)$ , corresponds in this new notation to  $F_{\delta \mathcal{L}_5 \gamma, \beta \mathcal{L}_k \alpha}^{(\mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3) \mathcal{L}_4} = F_{\delta 5 \gamma, \beta k \alpha}^{(123)4}$  where  $\gamma$  and  $\delta$  are indices labeling respectively the junctions in  $\mathcal{V}_{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_5}$  and  $\mathcal{V}_{\mathcal{L}_5, \mathcal{L}_3, \mathcal{L}_4}$  and  $\alpha$  and  $\beta$  label respectively  $\mathcal{V}_{\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_k}$  and  $\mathcal{V}_{\mathcal{L}_1, \mathcal{L}_k, \mathcal{L}_4}$ .

Therefore, given the fusion coefficients of TDLs, there are finitely many F-matrices solving the pentagon identity. Moreover, combining the left-hand side with the dual of the diagram appearing on right-hand side of (2.33) we get a bubble diagram with the following property:

**Proposition 2.4.5.** *The following relation holds:*

$$= \sqrt{\frac{\langle i \rangle \langle j \rangle \langle k \rangle}{\langle l \rangle}} F_{ap\beta, \gamma q \delta}^{(ijk)l}$$

*Proof.* We use a F-move to bring the  $j$  line to the other side of the bubble. Then we use (2.29) to resolve the bubbles.

$$= \sum_{\theta, r, \rho} F_{ap\beta, \theta r \rho}^{(ijk)l} \sqrt{\frac{\langle j \rangle \langle k \rangle}{\langle r \rangle}} \sqrt{\frac{\langle i \rangle \langle r \rangle}{\langle l \rangle}} \delta_q^r \delta_\delta^\rho \delta_\gamma^\theta = F_{ap\beta, \gamma q \delta}^{(ijk)l} \sqrt{\frac{\langle i \rangle \langle j \rangle \langle k \rangle}{\langle l \rangle}}$$

By resolving also the last bubble we get the coefficient:

$$\sum_{\theta, r, \rho} F_{ap\beta, \theta r \rho}^{(ijk)l} \sqrt{\frac{\langle j \rangle \langle k \rangle}{\langle r \rangle}} \sqrt{\frac{\langle i \rangle \langle r \rangle}{\langle l \rangle}} \delta_q^r \delta_\delta^\rho \delta_\gamma^\theta = F_{ap\beta, \gamma q \delta}^{(ijk)l} \sqrt{\frac{\langle i \rangle \langle j \rangle \langle k \rangle}{\langle l \rangle}}$$

□

By resolving the bubble diagram on the left-hand side by topological manipulations, one can compute the F-symbols. Finally, note that when any of the labels  $i, j, k$  equals 0, labeling the trivial defect, then the left-hand side of proposition 2.4.5 degenerates (if non-zero) to the left-hand side of (2.29), leading to

$$F_{j\beta, l\delta}^{(0jk)l} = \sqrt{\frac{\langle j \rangle \langle k \rangle}{\langle l \rangle}} \delta_{\beta, \delta} \quad F_{i\beta, \gamma k}^{(i0k)l} = \sqrt{\frac{\langle i \rangle \langle k \rangle}{\langle l \rangle}} \delta_{\beta, \gamma} \quad F_{\alpha l, \gamma j}^{(ij0)l} = \sqrt{\frac{\langle i \rangle \langle j \rangle}{\langle l \rangle}} \delta_{\alpha, \gamma}$$

## 2.5 TDLs of the critical Ising model

Consider the Ising model CFT in  $d = 2$  spacetime dimensions. Being a diagonal theory, the treatment for the Ising model simplifies a lot. In particular, we know that TDLs are completely classified by Verlinde lines. However, we apply equation (2.22) anyway, to see how it works. To simplify notation, in (2.22) we call  $\lambda_{ii}^{\mathcal{L}} = \lambda_i$ ; then using the S-matrix (1.43), we get the matrix equation

$$\begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} =$$

$$= \frac{1}{4} \begin{pmatrix} \lambda_1 + \lambda_2 + 2\lambda_3 & \lambda_1 + \lambda_2 - 2\lambda_3 & \sqrt{2}(\lambda_1 - \lambda_2) \\ \lambda_1 + \lambda_2 + 2\lambda_3 & \lambda_1 + \lambda_2 - 2\lambda_3 & \sqrt{2}(\lambda_1 - \lambda_2) \\ \sqrt{2}(\lambda_1 - \lambda_2) & \sqrt{2}(\lambda_1 - \lambda_2) & 2(\lambda_1 + \lambda_2) \end{pmatrix}$$

where  $z_{i,j} \in \mathbb{N} \forall i, j$ . Comparing the two sides of the equation we get:

$$\begin{cases} \lambda_1 + \lambda_2 + 2\lambda_3 = z_{11} = z_{22} := n_1 \\ \lambda_1 + \lambda_2 - 2\lambda_3 = z_{12} = z_{21} := n_2 \\ \frac{1}{2\sqrt{2}}(\lambda_1 - \lambda_2) = z_{31} = z_{32} = z_{13} = z_{23} := n_3 \\ 2(\lambda_1 + \lambda_2) = z_{33} := n_4 \end{cases} \Rightarrow \begin{cases} \lambda_1 = n_1 + n_2 + \sqrt{2}n_3 \\ \lambda_2 = n_1 + n_2 - \sqrt{2}n_3 \\ \lambda_3 = n_1 - n_2 \\ n_1 + n_2 = n_4 \end{cases}$$

Then we see that the lattice of solutions is given by

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} n_1 + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} n_2 + \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{pmatrix} n_3$$

Each basis vector on the lattice represents the action of a different simple defect. Notice that for each one we have just one non-zero  $n_i$ , from which we can reconstruct the matrix of twisted Hilbert space multiplicities  $Z_{ij}^{\mathcal{L}}$ . The TDLs for the Ising model are:

- the trivial defect ( $n_1 = 1$ ):

$$\hat{\mathcal{I}}|1\rangle = |1\rangle \quad \hat{\mathcal{I}}|\epsilon\rangle = |\epsilon\rangle \quad \hat{\mathcal{I}}|\sigma\rangle = |\sigma\rangle \quad (2.34)$$

The corresponding partition function is the usual one for the Ising model.

- the  $\mathbb{Z}_2$ -defect ( $n_2 = 1$ ):

$$\hat{\eta}|1\rangle = |1\rangle \quad \hat{\eta}|\epsilon\rangle = |\epsilon\rangle \quad \hat{\eta}|\sigma\rangle = -|\sigma\rangle \quad (2.35)$$

This defect comes from the  $\mathbb{Z}_2$  global symmetry of the Ising model. The relative twisted partition function is<sup>18</sup>:

$$Z^\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathcal{Z}_\eta = \chi_{\frac{1}{2}} \bar{\chi}_0 + \chi_0 \bar{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{16}} \bar{\chi}_{\frac{1}{16}}$$

The three primaries in this twisted representation of the Virasoro algebra are denoted  $\psi, \bar{\psi}$  and  $\mu$  respectively, consisting of two Majorana fermions<sup>19</sup> and the disorder field of the Ising model. It is worth reminding here that all operators in twisted Hilbert spaces are not local operators, but are attached to a defect line as in figure 2.3.

<sup>18</sup>The superscript ( $c=1/2$ ) in the characters is understood.

<sup>19</sup>It is common to say that the critical Ising CFT is equivalent to a free Majorana fermion. Strictly speaking, however, one should distinguish the fermionic Majorana CFT from the bosonic Ising CFT. The former is a spin CFT, in which the Majorana fermions are genuine local fields, while in the bosonic Ising CFT they are not local bosonic operators; rather, they can be understood as defect fields attached to the  $\eta$ -line. The two theories are related by bosonization, or equivalently by gauging the fermion parity symmetry  $\mathbb{Z}_2^f$ , see [10]. This distinction is also visible at the level of symmetries. The free Majorana theory has a *global* symmetry group  $\mathbb{Z}_2^f \times \mathbb{Z}_2$ , generated by the fermion parity operator  $(-1)^F$  and chiral fermion parity operator  $(-1)^{F_L}$ . The latter has a mixed anomaly with fermion parity, and after bosonization this anomalous fermionic symmetry is transformed into the Kramers–Wannier symmetry of the bosonic Ising CFT [10].

- The Kramers-Wannier duality defect ( $n_3 = 1$ ):

$$\hat{N}|1\rangle = \sqrt{2}|1\rangle \quad \hat{N}|\epsilon\rangle = -\sqrt{2}|\epsilon\rangle \quad \hat{N}|\sigma\rangle = 0 \quad (2.36)$$

The fact that  $\lambda_3 = 0$  signals the non-invertibility of this defect. Indeed, we see that its quantum dimension is  $\langle N \rangle = \sqrt{2} \neq 1$ . As will be clear shortly, this TDL is associated to the Kramers-Wannier duality of the Ising model. The twisted partition function reads:

$$Z^N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \mathcal{Z}_N = \chi_{\frac{1}{16}} \bar{\chi}_0 + \chi_0 \bar{\chi}_{\frac{1}{16}} + \chi_{\frac{1}{16}} \bar{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{2}} \bar{\chi}_{\frac{1}{16}}$$

These primaries are usually denoted respectively as  $s, \tilde{s}, \Lambda$  and  $\tilde{\Lambda}$ . We see that for twisted Hilbert spaces, such as the one considered here, the usual spin selection rule does not apply, leading to primaries with general fractional spin<sup>20</sup>, that are called for this reason *anyons*. These fields obey more general statistic with respect to Bose-Einstein or Fermi-Dirac statistics and can be seen only in two or three spacetime dimensions, as in higher dimensions the spin-statistics theorem constraints the spin to the set of half integers<sup>21</sup>.

Before continuing, we make some observations:

1. the eigenvalues are coherent with the expected ones for Verlinde lines, i.e. equation 2.13.
2. The TDLs of the Ising model are all self dual:  $\bar{N} = N$  and  $\bar{\eta} = \eta$ ; we can represent them with unoriented lines.
3. from equations 2.34, 2.35 and 2.36 we can derive the fusion rules for the TDLs:

$$\eta N = N \eta = N \quad N^2 = \mathcal{I} + \eta \quad \eta^2 = \mathcal{I}$$

Again, this confirms the fact that Verlinde lines have the same fusion rules as primary operators.

4. Also the ansatz  $Z_{ij}^{\mathcal{L}^k} = N_{ki}^j$  that we made in section 2.3 is verified here.
5. having  $N_{ij}^k \in \{0, 1\}$  implies that we have from 2.7 that  $\langle \mathcal{L} \rangle^2$  counts the number of channels in the fusion of  $\mathcal{L}$  with its dual. Moreover, this combined with proposition 2.2.5, implies that the junction spaces are at most one-dimensional. For this reason, the F-symbols can be written without the indices labeling the junctions.

We now compute the F-symbols. The decomposition of the identity reads:

$$id_{N \otimes N} = \begin{array}{c} N \quad N \\ | \quad | \\ N \quad N \end{array} = \frac{1}{\sqrt{2}} \begin{array}{c} N \quad N \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ N \quad N \end{array} + \frac{1}{\sqrt{2}} \begin{array}{c} N \quad N \\ \diagdown \quad \diagup \\ \bullet \\ \eta \\ \bullet \\ \diagup \quad \diagdown \\ N \quad N \end{array} \quad (2.37)$$

The non-trivial F-symbols involve  $\eta$  and  $N$ . The 4-junction  $\mathcal{V}_{N,N,N,\eta}$  cannot be resolved using basis elements of 3-way junctions so it has to be zero.  $\mathcal{V}_{\eta,\eta,\mathcal{N},\mathcal{N}}$  is uniquely resolved so its F-symbol is also

<sup>20</sup>One can still find different spin selection rules which depends on the underlying fusion category of TDLs. This is done for example in [11].

<sup>21</sup>In quantum mechanics, Bose-Einstein and Fermi-Dirac statistics arises from the fact that in 3 or more spatial dimensions, the worldlines created by the permutations of  $n$  identical particles are homotopic, leading to a representation of the permutation group  $S_n$  on the Hilbert space of the  $n$  identical particles. Assuming non intersecting paths, for two spatial dimensions there is a well-defined orientation to exchanges, leading to representations of the *braid group*  $B_n$ . With one spatial dimension we cannot apply the same procedure, since we cannot exchange particles without overlaps. However, as we are going to see in chapter 3, we will have a representation of the Braid group at algebraic level.

trivial. The junction  $\mathcal{V}_{N,\eta,N,\eta}$  and its cyclic permutation, can be resolved in two ways. The F-symbol is  $F_{N,N}^{(N\eta N)\eta} = -1$ . The last and most interesting 4-junction is the one involving four N lines. The completeness relation (2.37) describing the partial fusion of two N's already gives the matrix elements  $F_{\mathcal{I},\mathcal{I}}^{(NNN)N}$  and  $F_{\mathcal{I},\eta}^{(NNN)N}$ . The remaining two can be found resolving the corresponding bubble diagram. This is shown in appendix B. The result is<sup>22</sup>:

$$F^{(NNN)N} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.38)$$

This also allows us to express the crossing relation:

We now have sufficient information to study what happens to the local primaries when we move a defect past them. The only interesting cases are shown in figures 2.11 and 2.12.

Figure 2.11: Moving  $\sigma$  past the  $\eta$  defect.

Figure 2.12: Moving  $\sigma$  past the N defect

We see that moving the spin field past the defect  $\eta$  changes its sign, as expected from a defect implementing the  $\mathbb{Z}_2$  symmetry.

On the other hand, moving the local order operator  $\sigma$  across the duality defect N transforms it into the non-local disorder operator  $\mu$ , attached to a  $\eta$ -line. In this sense, the defect N implements the Kramers–Wannier duality, exchanging order and disorder operators.

A particularly important property of the N defect is revealed by inserting an N-bubble into the partition function, expanding the bubble, and then fusing the defect segments that meet:<sup>23</sup>

<sup>22</sup>The matrix is expressed in the basis  $\{\mathcal{I}, \eta\}$ . We do not need to include N since it cannot appear as an internal line in the decomposition of  $\mathcal{V}_{N,N,N,N}$ .

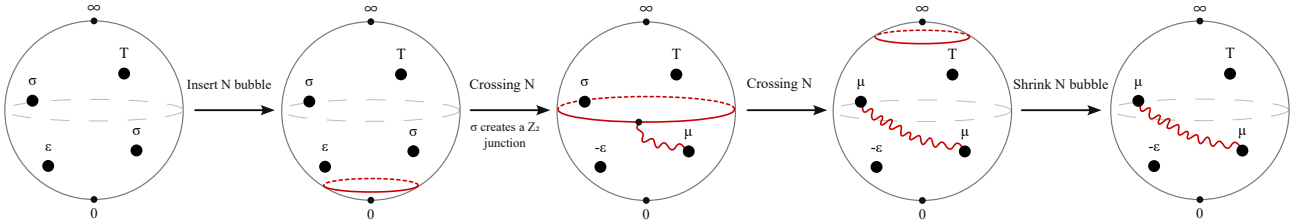
<sup>23</sup>See appendix B for the details of the calculation.

$$= \square + \square + \square + \square$$

We see that the resulting partition function becomes a sum over all possible configurations of  $\eta$ -defect lines. This is precisely the partition function of the theory obtained by gauging the  $\mathbb{Z}_2$  symmetry.

More generally, discrete gauging can be understood as the analogue of gauging a continuous symmetry. The topological defect  $\eta$  plays the role of a background  $\mathbb{Z}_2$  gauge field, and gauging amounts to summing over all possible  $\eta$ -defect networks on the underlying Riemann surface.

The resulting theory is denoted by  $Ising/\mathbb{Z}_2$  and is called the *orbifold theory*. For the critical Ising model, this orbifold theory is again equivalent to the original Ising CFT. One way to see this is that there is a unique  $c=1/2$  unitary CFT [35]. The duality defect  $N$  realizes this equivalence locally: correlators involving  $(T, \bar{T}, \epsilon, \sigma)$  are mapped to correlators involving  $(T, \bar{T}, -\epsilon, \mu)$ , consistently with the exchange between order and disorder sectors induced by Kramers–Wannier duality.<sup>24</sup> For instance, recalling that the complex plane is compactified to the Riemann sphere, by inflating an  $N$  bubble:<sup>25</sup>



Proving the equivalence of the correlators:

$$\langle T(z_4) \sigma(z_3, \bar{z}_3) \sigma(z_2, \bar{z}_2) \epsilon(z_1, \bar{z}_1) \rangle \leftrightarrow \langle T(z_4) \mu(z_3, \bar{z}_3) \mu(z_2, \bar{z}_2) (-\epsilon(z_1, \bar{z}_1)) \rangle$$

The same procedure also shows that the defect  $N$  imposes selection rules, such as the vanishing of correlators containing an odd number of  $\epsilon$  insertions.

In conclusion, non-invertible symmetries not only constrain correlation functions in a way analogous to ordinary global symmetries, but can also relate correlators of local operators to correlators involving non-local operators.

<sup>24</sup>The transformation  $\epsilon \mapsto -\epsilon$  can be understood from the fermionic description of the Ising model. The thermal perturbation of the critical theory is the Majorana mass term  $m \int i\psi\bar{\psi}$ , with  $m \propto K - K_c$ . Since the energy field is identified with the Majorana bilinear  $\epsilon \sim i\psi\bar{\psi}$  [10], the Kramers–Wannier exchange of the two massive phases, close to criticality given by  $m \mapsto -m$ , can equivalently be read as a sign flip of the energy operator. At the critical point, where the mass term is absent from the action, this duality is implemented by the topological defect  $N$ , hence  $N(\epsilon) = -\epsilon$ .

<sup>25</sup>The quantum dimension contribution arising after the first step simplifies with the one appearing in the last step.

## Chapter 3

# Algebraic quantum field theory

In this chapter, we introduce the axiomatic approach to quantum field theory known as algebraic quantum field theory. Algebraic QFT starts from a *net of local algebras*, i.e. the assignment  $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ , where  $\mathcal{O}$  is a spacetime region and  $\mathcal{A}(\mathcal{O})$  is the algebra of observables localized in  $\mathcal{O}$ . The idea at the foundation of this approach is that all relevant information about the system is contained in the net, together with its Poincaré automorphism. One may object that a "net of local algebras" is a structure too abstract for specifying a particular model; in particular, it does not involve an explicit specification of a Lagrangian. But recall that a model in non-relativistic Quantum Mechanics is fixed by specifying the set of observables on the Hilbert space and the Hamiltonian. The analogue is true in QFT.

We will start from a brief reformulation of quantum mechanics in a more algebraically focused setting. Then we will move to QFT by stating the Haag-Kastler axioms, which are at the base of AQFT. Then we will see what the physical consequences are of the axioms. This chapter is based on [33, 35–58].

### 3.1 Algebraic formulation of quantum mechanics

We recall that the key postulates of quantum mechanics say that:

- *Pure states* of a quantum system are described by rays in  $\mathcal{H}$ , i.e.  $[\psi] := \{\lambda\psi \mid \lambda \in \mathbb{C} \setminus \{0\}\}$ . *Mixed states* are described by density matrices, i.e. positive trace-class operators  $\rho : \mathcal{H} \rightarrow \mathcal{H}$  with unit trace;
- *Observables* are described by self-adjoint operators  $A$  on  $\mathcal{H}$ . However, the self-adjoint operators corresponding to observables may be a proper subset of all the self-adjoint operators on  $\mathcal{H}$ ; this occurs if the system is subject to superselection rules.

These principles can be formulated in algebraic terms as follows:

- A physical system is described by a *unital  $\ast$ -algebra*  $\mathcal{A}$ , whose self-adjoint elements are interpreted as the bounded observables. With abuse of language we call  $\mathcal{A}$  *algebra of observables*. In many situations we impose the stricter condition that  $\mathcal{A}$  be a unital  $C^\ast$ -algebra.
- states are identified with *positive, normalized linear functionals*  $\omega : \mathcal{A} \rightarrow \mathbb{C}$ ; i.e. we require  $\omega(A^\ast A) \in \mathbb{R}_{\geq 0} \quad \forall A \in \mathcal{A}$  and  $\omega(\mathbf{1}) = 1$  as well as  $\omega$  being linear. The state is mixed if it can be written as a convex combination of distinct states and pure otherwise.

We now give the precise definitions of the algebraic notions appearing above.

**Definition 3.1.1** ( $\ast$ -algebra). A  $\ast$ -algebra  $\mathcal{A}$  is an algebra over  $\mathbb{C}$  together with a map  $\ast : \mathcal{A} \rightarrow \mathcal{A}$ , called *involution*, such that:

- for all  $A, B \in \mathcal{A} : (A + B)^\ast = A^\ast + B^\ast$  and  $(AB)^\ast = B^\ast A^\ast$ ;
- for every  $\lambda \in \mathbb{C}$  and every  $A \in \mathcal{A} : (\lambda A)^\ast = \bar{\lambda} A^\ast$ ;

- for all  $A \in \mathcal{A} : (A^*)^* = A$

The  $*$ -algebra is *unital* if it has an element  $\mathbf{1}$  which is a unit for the algebraic product and therefore invariant under involution. Homomorphisms between  $*$ -algebras  $\alpha : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  are algebraic homomorphisms that respect involutions  $(\alpha(A))^* = \alpha(A^*)$  and preserves unit  $\alpha(\mathbf{1}_{\mathcal{A}_1}) = \mathbf{1}_{\mathcal{A}_2}$ .

The main example of a  $*$ -algebra is the algebra of bounded operators  $\mathcal{B}(\mathcal{H})$  on a Hilbert space  $\mathcal{H}$  with the adjoint as the involution. This algebra also carries additional structure.

**Definition 3.1.2** (normed algebra). A *normed algebra*  $\mathcal{A}$  is an algebra equipped with a norm  $\|\cdot\|$  satisfying

$$\|AB\| \leq \|A\| \|B\|$$

If  $\mathcal{A}$  is unital and  $\|\mathbf{1}\| = 1$ , then it is a *normed unital algebra*. If  $\mathcal{A}$  is complete in the topology induced by  $\|\cdot\|$ , then  $\mathcal{A}$  is a *Banach algebra*; if additionally,  $\mathcal{A}$  is a  $*$ -algebra and  $\|A^*\| = \|A\|$ , then  $\mathcal{A}$  is a *Banach  $*$ -algebra* or  *$B^*$ -algebra*.

A  $C^*$ -algebra is a particular  $B^*$ -algebra.

**Definition 3.1.3** ( $C^*$ -algebra). A  $C^*$ -algebra  $\mathcal{A}$  is a  $B^*$ -algebra whose norm has the  $C^*$ -property

$$\|A^*A\| = \|A^*\| \|A\| = \|A\|^2, \quad \forall A \in \mathcal{A}$$

The algebra of bounded operators  $\mathcal{B}(\mathcal{H})$ , with the operator norm

$$\|A\| := \sup_{\|v\|=1} \|Av\|, \quad v \in \mathcal{H},$$

is a  $C^*$ -algebra.

In our second postulate, the role of the state in the algebraic approach is to assign expectation values: if  $A = A^*$ , we interpret  $\omega(A)$  as the expected value of  $A$  if measured in the state  $\omega$ .

Consider  $\mathcal{B}(\mathcal{H})$ , then every unit vector  $\psi \in \mathcal{H}$  induces a state on  $\mathcal{B}(\mathcal{H})$ , called *vector state*, by

$$\omega_\psi(A) = \langle \psi | A \psi \rangle, \quad A \in \mathcal{B}(\mathcal{H}) \quad (3.1)$$

since  $\langle \psi | \mathbf{1} \psi \rangle = 1$  and  $\langle \psi | A^* A \psi \rangle = \|A\psi\|^2 \geq 0$ . In addition, every density matrix (a positive operator  $\rho$  on  $\mathcal{H}$  with  $\text{Tr}(\rho) = 1$ ) induces a state on  $\mathcal{B}(\mathcal{H})$  according to

$$\omega_\rho(A) = \text{Tr}(\rho A), \quad A \in \mathcal{B}(\mathcal{H}) \quad (3.2)$$

indeed  $\text{Tr}(\rho \mathbf{1}) = 1$  and if  $\{|n\rangle\}_{n=1}^\infty$  is an orthonormal base for  $\mathcal{H}$  (assumed to be separable) of eigenvectors of  $\rho$ :

$$\text{Tr}(\rho A^* A) = \sum_n \langle n | \rho A^* A | n \rangle = \sum_n p_n \langle n | A^* A | n \rangle = \sum_n p_n \|A|n\rangle\|^2 \geq 0$$

It is important to realize that, in general, not all algebraic states on  $\mathcal{A}$  need arise from vectors or density matrices in a given Hilbert space representation; in this sense it is a more general notion of state. However, it is not guaranteed that all states correspond to physically realizable situations.

Before continuing, we prove for later purposes the following results.

**Proposition 3.1.1.** *Every state on a  $*$ -algebra  $\mathcal{A}$  satisfy*

$$\omega(A^*) = \bar{\omega}(A), \quad \forall A \in \mathcal{A}$$

where the bar represents complex conjugation.

*Proof.* Consider the combination  $\mathbf{1} + \lambda A$  for  $\lambda \in \mathbb{C}$ :

$$\omega((\mathbf{1} + \lambda A)(\mathbf{1} + \lambda A)^*) = 1 + \lambda \omega(A) + \bar{\lambda} \omega(A^*) + |\lambda|^2 \omega(AA^*) \in \mathbb{R}_{\geq 0}$$

Setting the imaginary part to zero leads to:

$$\text{Im}(\lambda) \text{Re}(\omega(A)) + \text{Re}(\lambda) \text{Im}(\omega(A)) - \text{Im}(\lambda) \text{Re}(\omega(A^*)) + \text{Re}(\lambda) \text{Im}(\omega(A^*)) = 0$$

Since this must hold for all  $\lambda \in \mathbb{C}$  we get  $\text{Re}(\omega(A)) = \text{Re}(\omega(A^*))$  and  $\text{Im}(\omega(A)) = -\text{Im}(\omega(A^*))$ .  $\square$

**Proposition 3.1.2.** *Every state on a  $*$ -algebra  $\mathcal{A}$  induces a Cauchy-Schwarz inequality*

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B), \quad \forall A, B \in \mathcal{A} \quad (3.3)$$

*Proof.* Let  $B \neq 0$  and consider the combination  $A - \lambda B$  for  $\lambda \in \mathbb{C}$ :

$$\omega((A - \lambda B)(A - \lambda B)^*) = \omega(AA^*) - \lambda\omega(BA^*) - \bar{\lambda}\omega(AB^*) + |\lambda|^2\omega(BB^*) \geq 0$$

Choosing  $\lambda = \omega(AB^*)/\omega(BB^*)$  and using proposition 3.3 to write  $\omega(BA^*) = \omega((AB^*)^*) = \bar{\omega}(AB^*)$  we get,

$$\omega(AA^*) - \frac{|\omega(AB^*)|^2}{\omega(BB^*)} \geq 0 \Rightarrow |\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$$

□

### 3.1.1 States and representations

The Hilbert space formulation of quantum mechanics is too useful to be abandoned entirely and the study of Hilbert space representations forms an important part of AQFT.

**Definition 3.1.4.** A *representation* of a unital  $*$ -algebra  $\mathcal{A}$  consists of a triple  $(\mathcal{H}, \mathcal{D}, \pi)$ , where  $\mathcal{H}$  is a Hilbert space,  $\mathcal{D}$  is a dense subspace of  $\mathcal{H}$ , and  $\pi$  is a map from  $\mathcal{A}$  to operators on  $\mathcal{H}$  with the following properties:

- each  $\pi(A)$  has domain  $D(\pi(A)) = \mathcal{D}$  and range contained in  $\mathcal{D}$ ;
- $\pi(1) = 1_{\mathcal{D}}$ ,
- $\pi$  respects linearity and products

$$\pi(A + \lambda B + CD) = \pi(A) + \lambda\pi(B) + \pi(C)\pi(D), \quad A, B, C, D \in \mathcal{A}, \lambda \in \mathbb{C}$$

- each  $\pi(A)$  has an adjoint with  $\mathcal{D} \subset D(\pi(A)^*)$ , whose restriction to  $\mathcal{D}$  obeys  $\pi(A)|_{\mathcal{D}}^* = \pi(A^*)$ .

The representation is called *faithful* if  $\ker(\pi) = \{0\}$  and is called *irreducible* if there are no subspaces of  $\mathcal{H}$  invariant under  $\pi(\mathcal{A})$  that are not either trivial or dense in  $\mathcal{H}$ .

We will use the shorthand notation  $(\mathcal{H}, \pi)$  for  $(\mathcal{H}, \mathcal{H}, \pi)$ .

**Definition 3.1.5.** Two representations  $(\mathcal{H}_1, \mathcal{D}_1, \pi_1)$  and  $(\mathcal{H}_2, \mathcal{D}_2, \pi_2)$  of a  $*$ -algebra  $\mathcal{A}$  are called *unitarily equivalent*, if there is a unitary map  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  which restricts to an isomorphism between  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , and satisfy the *intertwining relation*  $U\pi_1(A) = \pi_2(A)U$  for all  $A \in \mathcal{A}$ . They are *unitarily inequivalent* if they are not unitarily equivalent.

From equations (3.1) and (3.2), we have seen that from quantum mechanical states we can get algebraic states. Hence, the natural question is whether the knowledge of an algebraic state  $\omega$  is enough to reconstruct a Hilbert state such that  $\omega$  is the vector state  $\omega_\psi$  for some  $\psi \in \mathcal{H}$ . The answer of this question is given by the following famous theorem by Gel'fand, Naimark and Segal.

**Theorem 3.1.3** (GNS representation theorem). *Let  $\omega$  be a state on a unital  $*$ -algebra  $\mathcal{A}$ . Then there is a representation  $(\mathcal{H}_\omega, \mathcal{D}_\omega, \pi_\omega)$  of  $\mathcal{A}$  and a unit vector  $\Omega_\omega \in \mathcal{D}_\omega$  such that  $\mathcal{D}_\omega = \pi_\omega(\mathcal{A})\Omega_\omega$  and*

$$\omega(A) = \langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle, \quad \forall A \in \mathcal{A} \quad (3.4)$$

*Furthermore,  $(\mathcal{H}_\omega, \mathcal{D}_\omega, \pi_\omega, \Omega_\omega)$  are unique up to unitary equivalence. If  $\mathcal{A}$  is a  $C^*$ -algebra, then, additionally:*

1. *each  $\pi_\omega(A)$  extends to a bounded operator on  $\mathcal{H}_\omega$ ;*
2.  *$\omega$  is pure if and only if the representation is irreducible;*
3. *if  $\pi_\omega$  is faithful, then  $\|\pi_\omega(A)\| = \|A\|_{\mathcal{A}}$ .*

Due to the fact that  $\pi_\omega(\mathcal{A})\Omega_\omega$  is dense in  $\mathcal{H}_\omega$ , we say that  $\Omega_\omega$  is *cyclic* for the representation.

The link between purity and irreducibility is understood from the following example: if  $\psi, \varphi \in \mathcal{H}$  are linearly independent normalized vector states on a subalgebra  $\mathcal{A}$  of  $\mathcal{B}(\mathcal{H})$ , then the density matrix state

$$\rho = \lambda|\psi\rangle\langle\psi| + (1-\lambda)|\varphi\rangle\langle\varphi|, \quad 0 < \lambda < 1$$

can be realized as the vector state  $\omega_\rho = \sqrt{\lambda}\psi \oplus \sqrt{1-\lambda}\varphi$  in the reducible representation  $A \mapsto A \oplus A$  of  $\mathcal{A}$  on  $\mathcal{H} \oplus \mathcal{H}$ .

*Proof.* The proof is constructive and divided into several steps:

1. Define a subset  $\mathcal{I}_\omega \subset \mathcal{A}$  by

$$\mathcal{I}_\omega = \{A \in \mathcal{A} : \omega(A^*A) = 0\}$$

Using the Cauchy-Schwarz inequality (3.3), one may prove that  $\mathcal{I}_\omega$  is a left ideal<sup>1</sup>: if  $A \in \mathcal{I}_\omega$  then

$$|\omega((BA)^*(BA))|^2 = |\omega(C^*A)|^2 \leq \omega(C^*C)\omega(A^*A) = 0, \quad B \in \mathcal{A}$$

where  $C = B^*BA$ , so  $BA \in \mathcal{I}_\omega$ . A similar argument shows that  $\mathcal{I}_\omega$  is a subspace of  $\mathcal{A}$  and that the subspace  $\mathcal{I}_\omega^*$  is a right ideal.

2. Define a quotient vector space

$$\mathcal{D}_\omega := \mathcal{A}/\mathcal{I}_\omega = \{[A] \mid [A] := A + \mathcal{I}_\omega\}$$

and note that  $\mathcal{D}_\omega$  carries an inner product defined by

$$\langle [A] | [B] \rangle = \omega(A^*B) \quad A, B \in \mathcal{A} \quad (3.5)$$

which is well defined, i.e. has the properties of an inner product (linearity from the definition of  $\omega$ , conjugate symmetry from proposition 3.1.1 and positive-definiteness from the definition of  $\mathcal{D}_\omega$ ), and if  $A_1, A_2 \in [A]$  with  $A_1 = A_2 + I$  for some  $I \in \mathcal{I}_\omega$

$$\langle [A_1] | [B] \rangle = \omega(A_2^*B) + \omega(I^*B) = \omega(A_2^*B) = \langle [A_2] | [B] \rangle$$

where we used the right-ideal property of  $\mathcal{I}_\omega^*$ .

3. Define the Hilbert space  $\mathcal{H}_\omega$  as the completion of  $\mathcal{D}_\omega$  with respect to the inner product (3.5); by construction  $\mathcal{D}_\omega$  is dense in  $\mathcal{H}_\omega$ .
4. Define  $\Omega_\omega = [\mathbf{1}] \in \mathcal{D}_\omega \subset \mathcal{H}_\omega$ , noting that  $\|\Omega_\omega\|^2 = \omega(\mathbf{1}) = 1$ .
5. Define  $\pi_\omega$  as follows: for each  $A \in \mathcal{A}$ ,  $\pi_\omega(A)$  is the linear operator  $\mathcal{D}_\omega \rightarrow \mathcal{D}_\omega$  given by

$$\pi_\omega(A)[B] := [AB], \quad B \in \mathcal{A}$$

which is well-defined due to the left ideal property. The properties

$$\pi_\omega(\mathbf{1}) = \mathbf{1}_{\mathcal{H}_\omega} \quad \pi_\omega(A + \lambda B) = \pi_\omega(A) + \lambda\pi_\omega(B) \quad \pi_\omega(AB) = \pi_\omega(A)\pi_\omega(B)$$

are easily verified as identities of operators on  $\mathcal{D}_\omega$ , and it is clear that  $\pi_\omega(\mathcal{A})\Omega_\omega = \mathcal{D}_\omega$ , so  $\Omega_\omega$  is cyclic as  $\mathcal{D}_\omega$  is dense.

Now observe that

$$\langle \pi_\omega(A^*)[B] | [C] \rangle = \langle [A^*B] | [C] \rangle = \omega(B^*AC) = \langle [B] | \pi_\omega(A)[C] \rangle$$

for all  $[B], [C] \in \mathcal{D}_\omega$ . This shows that  $\pi_\omega(A)$  has an adjoint with dense domain including  $\mathcal{D}_\omega$  and indeed

$$\pi_\omega(A^*) = \pi_\omega(A)|_{\mathcal{D}_\omega}^*$$

---

<sup>1</sup>Given a ring  $\mathcal{R}$ , a left ideal is a subset  $\mathcal{I}$  of  $\mathcal{R}$  such that  $0 \in \mathcal{I}$  and for  $r \in \mathcal{R}$  and for every  $x, y \in \mathcal{I}$ , one has  $x+y \in \mathcal{I}$ ,  $-x \in \mathcal{I}$  and  $rx \in \mathcal{I}$ . If an ideal is two sided, the quotient  $\mathcal{R}/\mathcal{I}$  is still a ring.

Therefore  $(\mathcal{H}_\omega, \mathcal{D}_\omega, \pi_\omega)$  is a representation of  $\mathcal{A}$ , and the calculation

$$\langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle = \langle [\mathbf{1}] | \pi_\omega(A) [\mathbf{1}] \rangle = \langle [\mathbf{1}] | [A] \rangle = \omega(A)$$

verifies (3.4).

6. For uniqueness, suppose that another Hilbert space  $\mathcal{H}$ , dense domain  $\mathcal{D}$ , distinguished vector  $\Omega$  and representation  $\pi$  are given with the properties above. Now define a linear map  $U : \mathcal{D} \rightarrow \mathcal{D}_\omega$  by

$$U\pi(A)\Omega = \pi_\omega(A)\Omega_\omega$$

noting that it is well defined because

$$\pi(A)\Omega = 0 \Leftrightarrow 0 = \|\pi(A)\Omega\|_{\mathcal{H}}^2 = \omega(A^*A) = \|\pi_\omega(A)\Omega_\omega\|_{\mathcal{H}_\omega}^2 \Leftrightarrow \pi_\omega(A)\Omega_\omega = 0$$

A similar calculation shows that  $U$  is an isometry, and as it is clearly invertible,  $U$  therefore extends to a unitary  $U : \mathcal{H} \rightarrow \mathcal{H}_\omega$  so that  $U\mathcal{D} = \mathcal{D}_\omega$  and

$$U\pi(A)U^{-1} = \pi_\omega(A)$$

acting on any vector in  $\mathcal{D}_\omega$ .

7. The special features of  $C^*$ -algebras are addressed in [59].

□

## 3.2 Haag-Kastler axioms

In quantum mechanics there are two main descriptions: the Heisenberg picture, in which the algebraic relations between observables are the primary data, and the Schrödinger picture, based on spaces of wavefunctions on which the observables of the theory act in specific ways.<sup>2</sup>

For systems with finitely many degrees of freedom, the distinction is rather inessential, because the *Stone-von Neumann theorem* guarantees that any irreducible representation of the commutation relations is unitarily equivalent to the Schrödinger representation. However, in QFT, unitarily inequivalent representations have an important role, where they describe different physical situations [43] [60]. This motivates an approach based primarily on algebraic relations.

The starting point of AQFT is a series of assumptions called *Haag-Kastler axioms*, which reformulate the Wightman axioms without referring to individual fields as the generators of the local algebras:

1. *Local algebras*: there is a unital  $C^*$ -algebra<sup>3</sup>  $\mathcal{A}$  and, to each bounded open region of Minkowski spacetime  $\mathcal{O} \subset \mathbb{M}^d \simeq \mathbb{R}^{d-1,1}$ , a subalgebra  $\mathcal{A}(\mathcal{O})$  containing the unit of  $\mathcal{A}$ , so that the  $\mathcal{A}(\mathcal{O})$  collectively generate  $\mathcal{A}$ :

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O} \subset \mathbb{M}^d} \mathcal{A}(\mathcal{O})}$$

where the bar indicates the closure in norm. We call *observables* of a region  $\mathcal{O}$  the self-adjoint elements of  $\mathcal{A}(\mathcal{O})$ . With abuse of language we call  $\mathcal{A}(\mathcal{O})$  local observable algebra and we call  $\mathcal{A}$  the *quasi-local algebra* of the theory. The assignment

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

is called *net of local observables*.

---

<sup>2</sup>In standard quantum mechanics there is also a third possibility, the interaction or Dirac picture. However, due to Haag's theorem, this description is mathematically inconsistent in QFT.

<sup>3</sup>In principle it is sufficient the structure of a  $*$ -algebra. However, in most of the applications it is useful to require the additional  $C^*$  structure.

2. *Isotony*: whenever  $\mathcal{O}_1 \subset \mathcal{O}_2$ , the corresponding local algebras are nested:

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$$

3. *Einstein Causality*: if  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are causally disjoint, then:

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0 \quad \Rightarrow \quad [A_1, A_2] = 0 \quad \forall A_i \in \mathcal{A}(\mathcal{O}_i), i = 1, 2$$

4. *Poincaré covariance*:  $\mathcal{A}$  is the representation space of a representation of the restricted Poincaré group  $\mathcal{P}_+^\uparrow$  as a group of  $*$ -automorphisms  $\alpha_{a,\Lambda}$ ,  $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ , preserving the local structure of observables, i.e.

$$\alpha_{a,\Lambda}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O}_{a,\Lambda})$$

where  $\mathcal{O}_{a,\Lambda} := \{x \in \mathbb{M}^d | \Lambda^{-1}(x - a) \in \mathcal{O}\}$ .

5. *Vacuum*: there is a state  $\omega$  on  $\mathcal{A}$ , called the vacuum state, invariant under the previously mentioned representation of the Poincaré group. It is also assumed that this state is the unique invariant state. The GNS construction provides a Hilbert space  $\mathcal{H}_0$ , called *vacuum sector*, where  $\mathcal{A}$  can be represented as an algebra of bounded operators with a cyclic vector, the *vacuum vector*  $|\Omega\rangle$ . The representation  $\pi_0$  of  $\mathcal{A}$  in  $\mathcal{H}_0$  is called *vacuum representation*.

6. *Spectrum condition*: one can show [43] that the last condition implies that the  $*$ -automorphism  $\alpha_{a,\Lambda}$  extends to a unitary representation on  $\mathcal{H}_0$  of the covering group of the Poincaré group. Assuming that such representation is strongly continuous<sup>4</sup>, by Stone's theorem there are four commuting self-adjoint operators  $P^\mu$  such that

$$U(x) = e^{iP_\mu x^\mu}$$

represents spacetime translations. Then the spectrum condition requires that the joint spectrum of the operators  $P^\mu$  lies inside the closure of the forward light cone:

$$\sigma(P^\mu) \subset \overline{V^+} := \{p^\mu \in \mathbb{M}^d | p^\mu p_\mu \leq 0, p^0 \geq 0\}$$

Representations satisfying this condition are called *positive energy representations*.

7. *Weak additivity*: for any open region  $\mathcal{O}$ ,  $\pi_0(\mathcal{A})$  is contained in the weak closure<sup>5</sup> of the  $*$ -algebra generated by the algebras  $\pi_0(\mathcal{A}(\mathcal{O} + x))$  as  $x$  runs over  $\mathbb{M}^d$ .

The prototypical local region  $\mathcal{O}$  is a *double cone*, obtained intersecting a forward and a backward light cone. This is shown in figure 3.1.

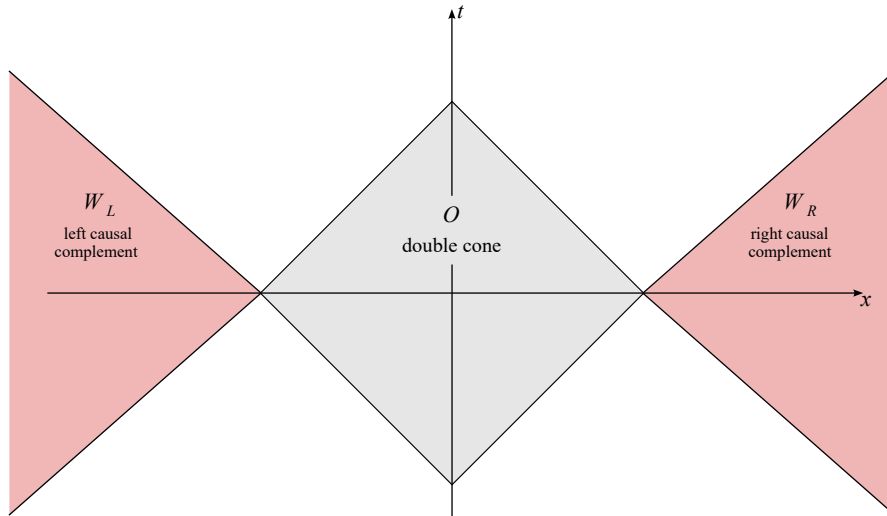


Figure 3.1: A double cone with the left and right wedges composing its spacelike complement  $\mathcal{O}' = \mathcal{W}_L \cup \mathcal{W}_R$ .

<sup>4</sup>I.e. the map  $x \mapsto U(x)\psi$  is continuous from  $\mathbb{M}^d$  to  $\mathcal{H}_0$  for each fixed  $\psi \in \mathcal{H}_0$ ,  $x \in \mathbb{M}^d$ .

The weak additivity axiom asserts that arbitrary observables can be built as limits of algebraic combinations of translates of observables in any given region  $\mathcal{O}$ . In many situations, this can be a consequence of the other axioms rather than an assumption; for example, this is the case for conformal nets [61], which we are going to introduce in the next chapter. The main important consequence of weak additivity is the following result:

**Theorem 3.2.1** (Reeh-Schlieder theorem). [42] *Let  $\mathcal{O}$  be any bounded open region. Then*

- *vectors of the form  $A|\Omega\rangle$  for  $A \in \pi_0(\mathcal{A}(\mathcal{O}))$  are dense in  $\mathcal{H}$ ;*
- *if  $A \in \pi_0(\mathcal{A}(\mathcal{O}))$  annihilates the vacuum,  $A|\Omega\rangle = 0$ , then  $A = 0$ .*

The first point says that  $|\Omega\rangle$  is cyclic for every local algebra; the other says that it is also *separating*. This means that the information contained in the local algebras  $\mathcal{A}(\mathcal{O})$  for any, arbitrarily small, open subset of  $\mathbb{M}^d$ , can be used to reconstruct the entire theory.

### 3.3 Von Neumann Algebras

So far we have expressed everything in terms of  $*$ - and  $C^*$ -algebras to be as general as possible. The theory is considerably enriched when expressed in terms of von Neumann algebras. This seems like an unmotivated extra complication, but as we shall see, one of the main success of AQFT has been to show how this is not the case.

First, we give the basic definitions regarding von Neumann Algebras.

**Definition 3.3.1** (von Neumann Algebra). Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{B}(\mathcal{H})$  the algebra of bounded linear operators on  $\mathcal{H}$ . Let  $\mathcal{M}$  be a  $*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$ . Then  $\mathcal{M}$  is a *von Neumann Algebra* or  *$W^*$ -algebra* if

- $\mathbf{1} \in \mathcal{M}$ ;
- $\mathcal{M}$  is closed in the weak operator topology.

An important result is the following:

**Theorem 3.3.1** (von Neumann double commutant). *For a  $*$ -algebra containing  $\mathcal{M}$  on  $\mathcal{B}(\mathcal{H})$  that contains  $\mathbf{1}$ , the following are equivalent:*

- $\mathcal{M}$  is weakly closed;
- $\mathcal{M}$  is strongly<sup>6</sup> closed;
- $\mathcal{M}'' = \mathcal{M}$  where the apex denotes the commutant:

$$\mathcal{M}' := \{A \in \mathcal{B}(\mathcal{H}) \mid [A, B] = 0, \quad \forall B \in \mathcal{M}\}$$

The most fundamental notion of a von Neumann algebra is its center.

**Definition 3.3.2.** For von Neumann algebras  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , we let  $\mathcal{M}_1 \wedge \mathcal{M}_2 = \mathcal{M}_1 \cap \mathcal{M}_2$ . We let  $\mathcal{M}_1 \vee \mathcal{M}_2$  denote the von Neumann algebra generated by  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , i.e. the intersection of all von Neumann algebras containing  $\mathcal{M}_1$  and  $\mathcal{M}_2$ .

**Definition 3.3.3.** We define the *center* of a von Neumann algebra as

$$\mathcal{Z}(\mathcal{M}) = \mathcal{M} \wedge \mathcal{M}'$$

A von Neumann algebra is called *factor* if

$$\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbf{1} \quad \text{equivalently}^7 \quad \mathcal{M} \vee \mathcal{M}' = \mathcal{B}(\mathcal{H})$$

<sup>5</sup>A sequence of operators converges in the weak topology,  $w - \lim A_n = A$ , if  $\langle \psi | A_n \varphi \rangle \rightarrow \langle \psi | A \varphi \rangle$  for all vectors  $\psi, \varphi \in \mathcal{H}$ .

<sup>6</sup>A sequence of operators converges in the strong topology if  $\|A_n \psi\| \rightarrow \|A \psi\|$  for all vectors  $\psi \in \mathcal{H}$ .

Factors are of great importance in the theory of von Neumann algebras. They are divided in three main categories, depending on the projections<sup>8</sup> belonging to them. The main types of projections are the following:

**Definition 3.3.4.** Let  $\mathcal{M}$  be a von Neumann algebra. A projection  $E \in \mathcal{M}$  is said to be:

- *equivalent* to a projection  $P \in \mathcal{M}$ , written  $E \sim P$ , if there exists  $V \in \mathcal{M}$  such that  $VV^* = E$  and  $V^*V = P$ ;
- A *subprojection* with respect to another projection  $P \in \mathcal{M}$ , denoted  $E \leq P$ , if  $PE = EP = E$ . If  $E \neq P$ , the subprojection is *proper* and we write  $E < P$ ;
- *central* if  $E \in \mathcal{Z}(\mathcal{M})$ ;
- *Minimal* if  $\mathcal{M}$  contains no proper subprojection of  $E$ ;
- *Abelian* if the von Neumann algebra over  $E\mathcal{H}$  defined as

$$EME := \{EAE \mid A \in \mathcal{M}\}$$

is abelian;

- *Infinite* if there exists a projection  $E_0 \in \mathcal{M}$  such that  $E_0 < E$ , i.e. it is a subprojection of  $E$ , and  $E_0 \sim E$ . The projection is said to be *finite* if it is not infinite;
- *Properly infinite* if  $E$  is infinite and for each central projection  $P \in \mathcal{Z}(\mathcal{M})$ , either  $PE = 0$  or  $PE$  is infinite.

Notice that every infinite projection  $E \in \mathcal{M}$  has infinite rank in the usual Hilbert-space sense. Indeed, if  $E$  is infinite in  $\mathcal{M}$ , then it is also infinite when regarded as a projection in  $\mathcal{B}(\mathcal{H})$ , and for  $\mathcal{B}(\mathcal{H})$  the above abstract notion of finiteness coincides with the ordinary distinction between finite- and infinite-rank projections. However, the converse is not true in general: for instance, the identity of  $\mathcal{M} = \mathbb{C}\mathbf{1}_{\mathcal{H}} \subset \mathcal{B}(\mathcal{H})$ , with  $\dim \mathcal{H} = \infty$ , has infinite rank but is finite in  $\mathcal{M}$ , because the only projections in  $\mathcal{M}$  are the null projection and  $\mathbf{1}$  and there is no proper subprojection  $\mathbf{1}$  equivalent to it.

Proper infinity is stricter, since it requires that each central decomposition of  $E$  is still infinite.

One can prove the following [54]:

**Proposition 3.3.2.** *There are the following relations for projections:*

$$\text{minimal} \Rightarrow \text{abelian} \Rightarrow \text{finite}$$

$$\text{properly infinite} \Rightarrow \text{infinite}$$

*For factors, the first arrows on both lines can be reversed.*

We can make a first classification based on the previous definition:

**Definition 3.3.5.** A von Neumann algebra  $\mathcal{M}$  is said to be:

- *finite*, if  $\mathbf{1}$  is finite in  $\mathcal{M}$ ;
- *infinite*, if  $\mathbf{1}$  is infinite in  $\mathcal{M}$ ;
- *properly infinite*, if  $\mathbf{1}$  is properly infinite in  $\mathcal{M}$ ;

In particular, we can classify factors as follows:

**Definition 3.3.6.** A von Neumann factor is said to be:

- *Type I* if it contains a minimal projection;
- *Type II* if it contains a finite projection, but no minimal projection;

<sup>7</sup>Indeed:  $(\mathcal{M} \cup \mathcal{M}')' = \mathbb{C}\mathbf{1}$ ; therefore  $(\mathcal{M} \cup \mathcal{M}')'' = \mathcal{B}(\mathcal{H})$ .

<sup>8</sup>A projection  $E \in \mathcal{M}$  is an idempotent and self-adjoint element:  $E = E^2 = E^*$ .

- *Type III* if it does not contain finite projections.

From this classification, we can state the fundamental result in the theory of von Neumann algebras:

**Proposition 3.3.3.** [62] *Every von Neumann algebra acting on a separable Hilbert space admits a central decomposition as a direct integral of factors.*

Thus, in this sense, every von Neumann algebra can be regarded as a generalized combination of factors. The precise construction is formulated in terms of direct integrals, but we will not formalize it here. The classification into type *I*, type *II* and type *III* is then applied to the factor components appearing in this decomposition.

The type *I* factors were already completely classified by Murray and von Neumann [54]: for each cardinal number  $k$  there is a unique (up to isomorphism) type  $I_k$  factor, namely  $\mathcal{B}(\mathcal{H})$  where  $\mathcal{H}$  is a Hilbert space of dimension  $k$ . The type *II* factors can be further subdivided according to whether or not the identity projection  $I$  is finite (type  $II_1$ ) or infinite (type  $II_\infty$ ). The type *III* factors can be subdivided into types  $III_\lambda$  with  $\lambda \in [0, 1]$ , although the basis for this subclassification depends on Tomita-Takesaki modular theory, which we very briefly discuss now. First consider the following definition:

**Definition 3.3.7.** Let  $\mathcal{M}$  be a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$ , and suppose that  $|\Omega\rangle \in \mathcal{H}$  is cyclic and separating for  $\mathcal{M}$ . Define an operator  $S_0$  on  $\mathcal{H}$  by setting

$$S_0 A |\Omega\rangle = A^* |\Omega\rangle$$

Then  $S_0$  extends to a closed<sup>9</sup> anti-linear operator  $S$  on  $\mathcal{H}$ . Let  $S = J\Delta^{1/2}$  be the polar decomposition of  $S$  (which is unique, also for densely defined unbounded operators), so that  $\Delta$  is positive (but generically unbounded) and  $J$  is anti-unitary. We call  $\Delta$  the *modular operator* and  $J$  the *modular conjugation* associated with the pair  $(\mathcal{M}, |\Omega\rangle)$ .

We will talk of the physical consequences of all these definitions in a bit. For now, just note that the hypotheses at the base of this definition are satisfied by a net of von Neumann algebras obeying the Haag-Kastler axioms, as a consequence of the Reeh-Schlieder theorem.

**Theorem 3.3.4** (Tomita's commutation theorem). *Let  $\mathcal{M}$  be a von Neumann algebra with cyclic and separating vector  $|\Omega\rangle$ . Then  $J|\Omega\rangle = |\Omega\rangle = \Delta|\Omega\rangle$  and*

$$\begin{aligned} \Delta^{it} \mathcal{M} \Delta^{-it} &= \mathcal{M}, \quad \forall t \in \mathbb{R} \\ J\mathcal{M}J &= \mathcal{M}' \end{aligned}$$

**Definition 3.3.8.** The *modular spectrum*  $S(\mathcal{M})$  of  $\mathcal{M}$  is given by

$$S(\mathcal{M}) = \bigcap_{\omega} sp(\Delta_{\omega})$$

where "sp" denotes the spectrum and  $\omega$  runs over the family of normal states of  $\mathcal{M}$ , and  $\Delta_{\omega}$  are the corresponding modular operators.<sup>10</sup>

The spectrum of the modular operator  $\Delta$  gives a rough measure of the periodicity of the modular automorphism group  $(\sigma_t^{\omega})_{t \in \mathbb{R}}$ , where  $\sigma_t^{\omega}(A) := \Delta^{it} A \Delta^{-it}$ ; i.e. the smaller the spectrum of  $\Delta$ , the closer the automorphism  $\sigma_t^{\omega}$  is to the identity. Conversely, as the spectrum of  $\Delta$  goes up to  $\mathbb{R}^+$ , the group  $(\sigma_t^{\omega})_{t \in \mathbb{R}}$  tends toward being ergodic (i.e. having no fixed points).

Finally, the classification reads:

<sup>9</sup>An operator  $A$  on  $\mathcal{H}$  with domain  $D(A) \subset \mathcal{H}$  is said to be *closed* if its *graph*,  $\mathcal{G}_A := \{(\psi, A\psi) : \psi \in D(A)\}$ , is a closed subset of  $\mathcal{H} \times \mathcal{H}$  with respect to the product topology and  $A$  is said to be *closable* if the closure of its graph is the graph of an operator  $\bar{A}$ , called the *closure* of  $A$ . One can show that a densely defined operator  $A$  on  $\mathcal{H}$  is closable iff its adjoint  $A^{\dagger}$  is densely defined and in this case  $\bar{A} = (A^{\dagger})^{\dagger}$  [44].

<sup>10</sup>A state is faithful if  $\omega(AA^*) = 0 \Rightarrow A = 0$ . A state  $\omega$  is said to be *normal* if for every increasing sequence  $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$  of semi-positive definite operators, namely  $A_n \geq 0$  and  $A_{n+1} - A_n \geq 0$ , with supremum  $A = \sup_n A_n$ , one has  $\omega(A) = \sup_n \omega(A_n)$ . Equivalently, a state is normal if it can be represented by a density operator, namely  $\omega(A) = \text{Tr}(\rho A)$ , for some positive operator  $\rho$  with  $\text{Tr}(\rho) = 1$  (see [52] for the details).

**Definition 3.3.9.** A factor of type *III* is said to be:

- Type  $III_0$  if  $S(\mathcal{M}) = \{0, 1\}$
- Type  $III_\lambda$  if  $S(\mathcal{M}) = \{\lambda^n \mid n \in \mathbb{Z}\}$
- Type  $III_1$  if  $S(\mathcal{M}) = \mathbb{R}^+$

Modular theory admits a remarkable physical interpretation in relativistic quantum field theory. In a celebrated result, Bisognano and Wichmann showed that the modular objects associated with the vacuum state and the algebra of observables localized in a wedge region of Minkowski spacetime have a direct geometric meaning. In particular, the modular flow is implemented by the Lorentz boosts preserving the wedge, while the modular conjugation corresponds to a reflection exchanging the wedge with its causal complement, combined with charge conjugation [63].

**Theorem 3.3.5** (Bisognano–Wichmann). [63] *Let  $\mathcal{A}(W)$  be the von Neumann algebra associated with a wedge region  $W$  and let  $\Omega$  be the vacuum vector. Then the modular objects  $(\Delta, J)$  associated with the pair  $(\mathcal{A}(W), \Omega)$  satisfy*

$$\Delta^{it}U(s)\Delta^{-it} = U(e^{-2\pi t}s), \quad JU(s)J = U(-s),$$

for all  $s, t \in \mathbb{R}$ .

To understand the geometric meaning of the first relation, consider a lightlike translation  $U(s)$ . Under a Lorentz boost of rapidity  $\lambda$ , the corresponding light-cone coordinate transforms as  $x^\pm \mapsto e^{\pm\lambda}x^\pm$ , so that

$$U(s) \mapsto U(e^\lambda s).$$

Comparing this with

$$\Delta^{it}U(s)\Delta^{-it} = U(e^{-2\pi t}s),$$

one sees that the modular group acts exactly as a Lorentz boost with rapidity  $\lambda = -2\pi t$ . Therefore the modular flow coincides with the one-parameter group of boosts preserving the wedge. The second relation shows instead that the modular conjugation reverses the direction of translations, corresponding geometrically to a spacetime reflection exchanging the wedge with its opposite.

### 3.3.1 Application to quantum field theory

Consider a net of local  $C^*$ -algebras obeying the Haag-Kastler axioms. Given a vacuum state  $\omega$  on  $\mathcal{A}$ , form its GNS representation. For any open bounded region  $\mathcal{O} \subset \mathbb{M}^d$  define the algebra:

$$\mathcal{M}_0(\mathcal{O}) := \pi_0(\mathcal{A}(\mathcal{O}))'' \tag{3.6}$$

Now consider the following proposition.

**Proposition 3.3.6.** *For any subalgebra  $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$  one has  $\mathcal{B} \subset \mathcal{B}''$  and  $\mathcal{B}''' = \mathcal{B}'$ .*

*Proof.* From the definition of commutant:

$$X \in \mathcal{B}'' \Leftrightarrow [X, Y] = 0 \quad \forall Y \in \mathcal{B}'$$

Since all elements of  $\mathcal{B}$  satisfy this request, we conclude that  $\mathcal{B} \subset \mathcal{B}''$ .

Now consider  $X \in \mathcal{B}'''$ . Similarly, this means that  $[X, Y] = 0 \quad \forall Y \in \mathcal{B}''$ . Then the same reasoning used before gives us  $\mathcal{B}' \subset \mathcal{B}'''$ .

Now we prove the other inclusion  $\mathcal{B}''' \subset \mathcal{B}'$ . Let  $X \in \mathcal{B}'''$ , hence  $X$  commutes with the whole  $\mathcal{B}''$ . But we have shown that  $\mathcal{B} \subset \mathcal{B}''$ . Therefore  $X$  commutes with  $\mathcal{B}$ , meaning that it is also an element of  $\mathcal{B}'$ .  $\square$

**Corollary 3.3.7.** *The algebra (3.6) is a von Neumann algebra*

*Proof.* By proposition 3.3.6:

$$\mathcal{M}_0(\mathcal{O})'' = (\pi_0(\mathcal{A}(\mathcal{O}))'' )'' = (\pi_0(\mathcal{A}(\mathcal{O}))')''' = \pi_0(\mathcal{A}(\mathcal{O}))'' = \mathcal{M}_0(\mathcal{O})$$

Since  $\mathcal{M}_0(\mathcal{O})$  also contains the unit operator, as a consequence of what is stated in the first axiom, by the von Neumann double commutant theorem we conclude that  $\mathcal{M}_0(\mathcal{O})$  is a von Neumann algebra.  $\square$

Einstein causality implies that the commutant  $\pi_0(\mathcal{A}(\mathcal{O}))'$  contains all local algebras  $\pi_0(\mathcal{A}(\tilde{\mathcal{O}}))$ , where  $\tilde{\mathcal{O}}$  is causally disjoint from  $\mathcal{O}$ . As  $\pi_0(\mathcal{A}(\mathcal{O}))' = \pi_0(\mathcal{A}(\mathcal{O}))''' = \mathcal{M}_0(\mathcal{O})'$ , we see that also  $\mathcal{M}_0(\tilde{\mathcal{O}}) \subset \mathcal{M}_0(\mathcal{O})'$ . It is natural to interpret  $\mathcal{M}_0(\mathcal{O})'$  in terms of (limits of) observables localized in the causal complement of  $\mathcal{O}$  and further natural to assume that there are no non-trivial observables localized in both  $\mathcal{O}$  and its causal complement. This motivates the assumption that:

$$\mathcal{M}_0(\mathcal{O}) \cap \mathcal{M}_0(\mathcal{O})' = \mathbb{C}\mathbf{1}$$

that is, that the *local algebras are factors*. This heuristic intuition can be formalized and further developed. Indeed, in 1985, Fredenhagen obtained one of the most striking results in the development of AQFT. In addition to the standard axioms he posed the following physically reasonable assumptions:

1. There is a dense domain  $\mathcal{D} \subset \mathcal{H}_0$  and a linear map  $\phi$  from real-valued test functions to symmetric operators  $f \mapsto \phi(f)$  defined on  $\mathcal{D}$  and obeying  $\phi(f)\mathcal{D} \subset \mathcal{D}$ , so that  $\phi(f)$  has a closure affiliated<sup>11</sup> to  $\mathcal{M}_0(\mathcal{O})$  for any  $\mathcal{O}$  containing  $\text{supp}(f)$ . Furthermore, the n-point functions of the fields  $\phi(f)$  define the distributions

$$W_n(f_1, \dots, f_n) = \langle \Omega_\omega | \phi(f_1) \cdots \phi(f_n) \Omega_\omega \rangle$$

2. Defining scalar maps on test functions by  $(\beta_{p,\lambda}f)(x) = f((x-p)/\lambda)$ , there should be a positive monotone function  $\nu$  so that the scaling limit n-point functions

$$W_n^{s.l.}(f_1, \dots, f_n) = \lim_{\lambda \rightarrow 0^+} \nu(\lambda)^n W_n(\beta_{p,\lambda}f_1, \dots, \beta_{p,\lambda}f_n)$$

exist and satisfy the (vacuum) Wightman axioms. One may think of this theory as living in the tangent space at  $p$ ; it is not usually the same theory we started with.

3. It is required that  $\bigcap_{\mathcal{O} \ni p} \mathcal{M}_0(\mathcal{O}) = \mathbb{C}\mathbf{1}$ , where the intersection is taken over all local regions containing the point  $p$ .

Assuming condition 1., we say that  $\omega$  has a regular scaling limit at  $p \in \mathbb{M}^d$  if conditions 2. and 3. hold. Then Fredenhagen proved the following [64]:

**Theorem 3.3.8.** *Let  $\mathcal{O}$  be a double-cone. If the state  $\omega$  has a regular scaling limit at some point of the spacelike boundary of  $\mathcal{O}$ , then  $\mathcal{M}_0(\mathcal{O})$  is a type  $III_1$  factor.*

Shortly afterwards, Haagerup proved that there was a unique *hyperfinite* type  $III_1$  factor [65].

**Definition 3.3.10.** Let  $\mathcal{M}$  be a von Neumann algebra. Then  $\mathcal{M}$  is said to be *hyperfinite* if there is a family  $(\mathcal{M}_i)_{i \in I}$  of finite-dimensional von Neumann algebras in  $\mathcal{M}$  such that

$$\mathcal{M} = \left( \bigcup_{i \in I} \mathcal{M}_i \right)'' = \bigvee_{i \in I} \mathcal{M}_i$$

Then a further paper of Buchholz, D'Antoni and Fredenhagen pointed out that the local algebras of QFT are hyperfinite [66], therefore fixing them uniquely up to isomorphism. Remarkably, the *local algebras themselves are completely independent of the theory*, and the distinction between different theories lies only in the "relative position" of the local algebras within  $\mathcal{B}(\mathcal{H})$ .

<sup>11</sup>Let  $\mathcal{M}$  be a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$ . A closed and densely defined operator  $A$  is said to be affiliated with  $\mathcal{M}$  if  $A$  commutes with every unitary operator in the commutant of  $\mathcal{M}$ . The notion of commutation for unbounded operators will be made precise in section 6.2.

Recall that every type I factor is isomorphic, as a von Neumann algebra, to  $\mathcal{B}(\mathcal{H}_1)$  for some Hilbert space  $\mathcal{H}_1$ . However, in general a factor is given as a subalgebra of  $\mathcal{B}(\mathcal{H})$  for a possibly different Hilbert space  $\mathcal{H}$ , and therefore it need not coincide with the whole algebra of bounded operators on  $\mathcal{H}$ . The following characterization describes how a type I factor can sit inside a larger algebra of operators: up to a unitary equivalence, it acts as the full algebra of bounded operators on one tensor factor and trivially on the other.

**Definition 3.3.11** (Alternative definition for type I factors). A von Neumann factor  $\mathcal{M}$  on a Hilbert space  $\mathcal{H}$  is of type *I* if there is a unitary  $U : \mathcal{H} \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$ , for some Hilbert spaces  $\mathcal{H}_i$ , so that  $\mathcal{M} = U^*(\mathcal{B}(\mathcal{H}_1) \otimes \mathbf{1}_{\mathcal{H}_2})U$ .

From this definition it is clear that type *I* factors arise in non-relativistic quantum mechanics: they are the observables of one party in a bipartite system. Many distinctions between quantum mechanics and quantum field theory arise due to this algebraic difference. We state some of them:

- *All eigenvalues have infinite degeneracy*: by definition of type *III* factors, the local algebra does not contain any finite projection. It follows that no self-adjoint element of a local algebra  $\mathcal{M}_0(\mathcal{O})$  can have a finite-dimensional eigenspace; moreover, the identity operator is an infinite projector, hence also the total Hilbert space is infinite-dimensional. In particular, if  $|\Omega\rangle$  is the vacuum vector, then the projection  $|\Omega\rangle\langle\Omega|$  belongs to no local algebra  $\mathcal{M}_0(\mathcal{O})$ , nor to any commutant  $\mathcal{M}_0(\mathcal{O})'$  (since the commutant of a type *III* is still a type *III*).
- *Local states are impure*: any state  $\omega$  on  $\mathcal{A}$  restricts in a natural way to a state  $\omega|_{\mathcal{A}(\mathcal{O})}$  on each local algebra  $\mathcal{A}(\mathcal{O})$ . We want to characterize it. To start, we note the following:<sup>12</sup>

**Proposition 3.3.9.** *Suppose  $\omega$  is a pure state on a  $C^*$ -algebra  $\mathcal{A}$ . Then the von Neumann algebra  $\mathcal{M}_\omega = \pi_\omega(\mathcal{A})''$  in the GNS representation of  $\omega$  is a type I factor.*

*Proof.* The GNS representation of a pure state is irreducible. So, for Schur's lemma, the commutant  $\pi_\omega(\mathcal{A})'$  consists only of multiples of the unit, and  $\mathcal{M}_\omega = \mathcal{B}(\mathcal{H}_\omega)$ , which is evidently type I.  $\square$

In the light of this result, the following theorem shows that local restrictions of states in QFT behave in a radically different way [43]:

**Theorem 3.3.10.** *Under the hypotheses of theorem 3.3.8, the algebra  $\pi_{\omega|_{\mathcal{A}(\mathcal{O})}}(\mathcal{A}(\mathcal{O}))''$  is of type  $III_1$ . Therefore  $\omega|_{\mathcal{A}(\mathcal{O})}$  is not pure, nor is it a normal state in the GNS representation of a pure state of  $\mathcal{A}(\mathcal{O})$ . Furthermore, when  $\mathcal{M}_\omega(\mathcal{O}) := \pi_\omega(\mathcal{A}(\mathcal{O}))''$  is a factor, the previous statements remain valid if the state  $\omega$  is replaced by any state belonging to the same Hilbert-space representation.*

Intuitively, the theorem shows that the restriction of a state to a local region in QFT is not a mixed state in the ordinary quantum-mechanical sense. In ordinary quantum mechanics, restricting a pure state to a subsystem produces a reduced density matrix by means of a partial trace over the complementary degrees of freedom. This construction presupposes a tensor product decomposition

$$\mathcal{H} = \mathcal{H}_{\mathcal{O}} \otimes \mathcal{H}_{\mathcal{O}^c}$$

such that the algebra of the subsystem is the type *I* factor

$$\mathcal{B}(\mathcal{H}_{\mathcal{O}}) \otimes \mathbf{1}.$$

If such an exact decomposition were available for a local region in QFT, the local von Neumann algebra would be of type *I*. This is incompatible with the type *III* nature of local algebras. The theorem gives an even stronger formulation of this obstruction: even if one represents the abstract local algebra in a type *I* way, through the GNS representation of a pure local state,

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<sup>12</sup>Note that here  $\omega$  is a generic state, not necessarily the vacuum. Hence  $(\mathcal{H}_\omega, \pi_\omega)$  denotes the corresponding GNS representation.

the physical restricted state  $\omega|_{\mathcal{A}(\mathcal{O})}$  is not normal in that representation. Hence it cannot be described as an ordinary density-matrix state, and the usual partial-trace picture of local mixed states does not apply.

- *Local experiments can be prepared locally:* suppose  $E$  is any projection in  $\mathcal{M}_0(\mathcal{O})$  and let  $\omega$  be any state on the  $C^*$ -algebra formed as the closure of  $\bigcup_{\mathcal{O}} \mathcal{M}_0(\mathcal{O})$  in  $\mathcal{B}(\mathcal{O})$ . For a type III factor one can show the following [52]:

**Proposition 3.3.11.** *If  $\mathcal{M}$  is a factor, then any two non-zero infinite projectors are equivalent.*

**Corollary 3.3.12.** *In a type III factor, any two non-zero projections are equivalent.*

Thanks to the above corollary,  $E \sim \mathbf{1}$  and we may write  $E = VV^*$  for some isometry  $V \in \mathcal{M}_0(\mathcal{O})$ . Then the modified state  $\omega_V(A) := \omega(V^*AV)$  satisfies

$$\omega_V(E) = \omega(V^*VV^*V) = 1, \quad \omega_V(A) = \omega(V^*AV) = \omega(V^*VA) = \omega(A), \quad \forall A \in \mathcal{M}_0(\mathcal{O})'$$

where we recall that  $\mathcal{M}_0(\tilde{\mathcal{O}}) \subset \mathcal{M}_0(\mathcal{O})'$  with  $\tilde{\mathcal{O}}$  causally disjoint from  $\mathcal{O}$ . Therefore, the property described by the projection  $E$  is guaranteed to be satisfied in the state  $\omega_V(E)$ , which can be prepared in our lab without changing the rest of the world.

### 3.4 Superselection sectors

In this section, we will see the general strategy for extracting the physics encoded in the net of observables. From now on, we will use the more commonly used dagger notation  $\dagger$  in physics, instead of the star  $*$ , to denote the algebraic involution.

Our base assumptions are the following: we will assume that our vacuum representation is faithful and irreducible, and that local algebras  $\pi_0(\mathcal{A}(\mathcal{O}))$  are von Neumann algebras<sup>13</sup>:  $\pi_0(\mathcal{A}(\mathcal{O})) = \pi_0(\mathcal{A}(\mathcal{O}))''$ . Furthermore, we assume that *Haag duality* is satisfied in the vacuum representation, meaning that

$$\pi_0(\mathcal{A}(\mathcal{O}')) = \pi_0(\mathcal{A}(\mathcal{O}))'$$

where  $\mathcal{O}'$  is the causal complement of  $\mathcal{O}$ .

The physics of a system characterized by a pair  $\{\mathcal{A}, \alpha\}$ , where  $\alpha$  is the Poincaré automorphism, can be derived from the representation theory of such pair. Of particular interest are the representations  $\pi_j$  that satisfy the following criterion.

**Definition 3.4.1** (Doplicher-Haag-Roberts criterion). A representation  $\pi_j$  of a net<sup>14</sup>  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  is a *DHR representation* if

$$\pi_j(\mathcal{A}(\mathcal{O}')) \simeq \pi_0(\mathcal{A}(\mathcal{O}'))$$

where " $\simeq$ " stands for unitary equivalence and  $\mathcal{O}$  are double cones. We say that such representations are *localizable in double cones*.

The physical motivation of this criterion is that a particular representation  $\pi_j$  should describe a particle with a Noether charge  $j$  localized in some region  $\mathcal{O}$ ; therefore its influence should be negligible far from that region. This translates on the unitarily equivalence of  $\pi_j$  with the vacuum representation on the causal complement  $\mathcal{O}'$ .<sup>15</sup>

Note that this criterion rules out long range interactions and, as a consequence of Strocchi's theorem [60], gauge theories in general. Fortunately, this criterion can be weakened: suppose that  $\mathcal{H}_j$  carries

<sup>13</sup>If this property does not hold, one can apply this discussion to the net  $\mathcal{O} \rightarrow \mathcal{M}_0(\mathcal{O}) := \pi_0(\mathcal{O})''$ .

<sup>14</sup>For "representation  $\pi$  of a net" we mean a representation of the quasi-local algebra  $\mathcal{A}$  together with the automorphism  $\alpha$  defined in the axioms, which from now on will be understood. Then  $\pi(\mathcal{A}(\mathcal{O}))$  is well defined for every double cone  $\mathcal{O} \subset \mathbb{M}^d$ .

<sup>15</sup>If two representations  $\pi_j$  and  $\pi_i$  are unitarily equivalent, i.e. exist an intertwining unitary  $U : \mathcal{H}_j \rightarrow \mathcal{H}_i$  such that  $U\pi_i(\cdot)U^\dagger = \pi_j(\cdot)$ , then the conserved charge operator  $Q$  in the two representations should be equivalent under the same mapping,  $\pi_j(Q) = j\mathbf{1}_{\mathcal{H}_j} = U\pi_i(Q)U^\dagger = iU\mathbf{1}_{\mathcal{H}_i}U^\dagger = i\mathbf{1}_{\mathcal{H}_j}$ , which is consistent if and only if  $j = i$ .

a strongly continuous unitary representation  $U_j$  of the covering group of the Poincaré group  $\tilde{\mathcal{P}}_+^\uparrow$  such that

$$\pi_j(\alpha_{a,\Lambda}(A)) = U_j(a, \tilde{\Lambda})\pi_j(A)U_j(a, \tilde{\Lambda})^\dagger$$

where  $(a, \tilde{\Lambda}) \in \tilde{\mathcal{P}}_+^\uparrow$  projects to  $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ ; we call such representation *Poincaré covariant*. Moreover, suppose that this representation is also a positive energy representation. Then Fredenhagen and Buchholz have shown in [45] that the DHR criterion is satisfied by such representations, in the absence of zero-mass particles, with spacelike cones  $\mathcal{C}$  instead of double cones  $\mathcal{O}$ ,

$$\pi_j(\mathcal{A}(\mathcal{C}')) \simeq \pi_0(\mathcal{A}(\mathcal{C}')),$$

which does not exclude a priori gauge theories. We will say that such representations are *localizable in spacelike cones*. Moreover, there is also evidence that this still holds in the presence of massless particles [39]. The space of physical states of the theory is defined as

$$\mathcal{H} = \bigoplus_{j \in \mathcal{J}} \mathcal{H}_j$$

where the sum extends over the list  $\mathcal{J}$  of all inequivalent and irreducible physical representations<sup>16</sup> of  $\mathcal{A}$ .

In the cases of interest here, namely two-dimensional conformal field theories, spacelike cone localization provides the natural framework for describing charged sectors and defect fields whose localization cannot be confined to bounded regions. The localization cone encodes the non-local structure attached to the field and provides a consistent notion of localization for the corresponding endomorphism.

However, as far as statistics is concerned, the two-dimensional setting offers a significant simplification. The statistics of charged sectors is not obtained by tracking the detailed geometry of the localization cone. Rather, it is extracted by comparing different spacelike separated localizations of the same sector through charge transporting operators. In two spacetime dimensions, spacelike separated regions admit a natural left-right ordering, which replaces the geometric notion of winding particles around one another familiar from higher dimensions. As a consequence, the relevant statistical information depends only on the relative localization of sectors and not on the particular choice of localization cone.

In the present work, however, we shall not make explicit use of defect starting operators. Their treatment would require a more careful analysis and lies beyond the scope of the present discussion. For local modular invariant two-dimensional CFTs, all fields of the full theory have integral spin and are therefore naturally described within the framework of double-cone localization. Nevertheless, the algebraic framework still retains the information about more exotic forms of statistics encoded in the chiral components of the theory, which will play an important role later on. Since our analysis will be entirely formulated in terms of the observable net and its DHR sectors, localization in double cones will be sufficient throughout.

As a consequence of the DHR criterion, for the physical representations considered (also for the non-irreducible ones) for an arbitrary double cone  $\mathcal{O}$  there is a unitary  $V_j^\mathcal{O} : \mathcal{H}_0 \rightarrow \mathcal{H}_j$  such that

$$\pi_j(A) = V_j^\mathcal{O} \pi_0(A) (V_j^\mathcal{O})^\dagger, \quad \forall A \in \mathcal{A}(\mathcal{O}')$$

Using  $V_j^\mathcal{O}$  we can construct a representation  $\tilde{\rho}_j^\mathcal{O}$  of  $\mathcal{A}$  in  $\mathcal{H}_0$  equivalent to  $\pi_j$  by setting

$$\tilde{\rho}_j^\mathcal{O}(A) := (V_j^\mathcal{O})^\dagger \pi_j(A) V_j^\mathcal{O}, \quad \forall A \in \mathcal{A}$$

The fundamental result is the following:

**Proposition 3.4.1.** *The representation  $\tilde{\rho}_j^\mathcal{O}$  extends to a  $*$ -endomorphism  $\rho_j^\mathcal{O}$  of  $\mathcal{A}$ .*

---

<sup>16</sup>i.e. covariant, positive energy and DHR.

*Proof.* Consider a bigger double cone  $\tilde{\mathcal{O}}$ ,  $\mathcal{O} \subset \tilde{\mathcal{O}}$ , and  $A \in \mathcal{A}(\tilde{\mathcal{O}})$ ,  $B \in \mathcal{A}(\tilde{\mathcal{O}}')$ . Since  $\mathcal{A}(\tilde{\mathcal{O}})' \subset \mathcal{A}(\mathcal{O})'$  and, due to the DHR criterion,  $\tilde{\rho}_j^\mathcal{O} = \pi_0$  on  $\mathcal{A}(\mathcal{O})'$ , we have

$$[\tilde{\rho}_j^\mathcal{O}(A), \pi_0(B)] = [\tilde{\rho}_j^\mathcal{O}(A), \tilde{\rho}_j^\mathcal{O}(B)] = \tilde{\rho}_j^\mathcal{O}([A, B]) = 0$$

so  $\tilde{\rho}_j^\mathcal{O}(A) \in \pi_0(\mathcal{A}(\tilde{\mathcal{O}}'))' = \pi_0(\mathcal{A}(\tilde{\mathcal{O}}))'' = \pi_0(\mathcal{A}(\tilde{\mathcal{O}}))$ , where we used Haag duality and the fact that local algebras are von Neumann algebras. We conclude that  $\tilde{\rho}_j^\mathcal{O}(\mathcal{A}(\tilde{\mathcal{O}})) \subset \pi_0(\mathcal{A}(\tilde{\mathcal{O}}))$ , so  $\tilde{\rho}_j^\mathcal{O}(\mathcal{A}) \subset \pi_0(\mathcal{A})$ , as local operators are dense in  $\mathcal{A}$ . Since  $\pi_0$  is faithful, there exists a linear map  $\rho_j^\mathcal{O}$  of the abstract  $C^*$ -algebra  $\mathcal{A}$  into itself with  $\rho_j^\mathcal{O} = \pi_0^{-1} \circ \tilde{\rho}_j^\mathcal{O}$ . From the properties of representations, shown in definition 3.1.4, it follows that

$$\rho_j^\mathcal{O}(AB) = \rho_j^\mathcal{O}(A)\rho_j^\mathcal{O}(B) \quad \rho_j^\mathcal{O}(A^*) = \rho_j^\mathcal{O}(A)^*$$

Hence this linear map is a  $*$ -endomorphism of  $\mathcal{A}$ .  $\square$

**Proposition 3.4.2.** [43] *Endomorphisms  $\rho_j^\mathcal{O}$  obtained from representations satisfying the DHR criterion for a given  $\mathcal{O}$  have the following properties:*

- **Localized in  $\mathcal{O}$ :**  $\rho_j^\mathcal{O}(A) = A$ ,  $\forall A \in \mathcal{A}(\mathcal{O}')$ ;
- **Transportable:**  $\forall \mathcal{O}_1, \mathcal{O}_2$  such that  $\mathcal{O} \cup \mathcal{O}_1 \subset \mathcal{O}_2$ , there is a unitary, sometimes referred to as "charge transport operator",  $U \in \mathcal{A}(\mathcal{O}_2)$  with  $Ad_U \circ \rho_j^\mathcal{O}(A) = U\rho_j^\mathcal{O}(A)U^\dagger = A$ ,  $\forall A \in \mathcal{A}(\mathcal{O}_1')$ , i.e. for every region  $\mathcal{O}_1$  there exists an endomorphism unitarily equivalent to  $\rho_j^\mathcal{O}(A)$  that is localized in  $\mathcal{O}_1$ . For this reason, we will sometimes omit the superscript  $\mathcal{O}$ , except when necessary to avoid confusion.
- **Subfactor:**  $\rho_j^\mathcal{O}(\mathcal{A}(\mathcal{O}_1)) \subset \mathcal{A}(\mathcal{O}_1)$  for every  $\mathcal{O}_1$  such that  $\mathcal{O} \subset \mathcal{O}_1$ . This means that  $\rho_j^\mathcal{O}$  maps any local algebra defined on a double cone containing the localization region into itself. Actually, this property can be derived from the previous two.

We can use such properties as defining properties for endomorphisms.

**Definition 3.4.2.** Given a double cone  $\mathcal{O}$ , let  $\Delta(\mathcal{O})$  be the set of all transportable endomorphisms of  $\mathcal{A}$  localized in  $\mathcal{O}$ , i.e. those satisfying the properties of proposition 3.4.2. We define  $\Delta := \bigcup_{\mathcal{O} \subset \mathbb{M}^d} \Delta(\mathcal{O})$ .

Then one can show that the equivalence classes of

$$\Delta/\sim := \{[\rho_j], j \in \mathcal{J} | \rho_j \sim \tilde{\rho}_j, \rho_j, \tilde{\rho}_j \in \Delta \Leftrightarrow \rho_j = \iota \circ \tilde{\rho}_j \text{ for some inner automorphism}^{17} \iota \text{ of } \mathcal{A}\}$$

are in one to one correspondence with unitary equivalence classes of representations satisfying the DHR criterion [43]. This provides us with another alternative language to describe the superselection sectors of a theory.

We can carry structure from a language to the other. In particular,  $\Delta/\sim$  is equipped with a natural product (given by the composition of endomorphisms of  $\mathcal{A}$ ), which induces a product on the space of sectors, namely:

$$[\pi_i \times \pi_j] := [\pi_0 \circ \rho_i \rho_j]$$

where  $\pi_i \simeq \pi_0 \circ \rho_i$  and  $\rho_i \rho_j = \rho_i \circ \rho_j$ . The corresponding representation space,  $\mathcal{H}_{i \times j}$  describes states in which a charged particle with superselection quantum number  $i$  and a particle with superselection quantum number  $j$  are simultaneously present.

We can say more about this product.

**Proposition 3.4.3.** *Composition of endomorphisms is locally commutative, i.e. for  $\mathcal{O}_1 \subset \mathcal{O}_2'$  we have*

$$\rho_i^{\mathcal{O}_1} \rho_j^{\mathcal{O}_2} = \rho_j^{\mathcal{O}_2} \rho_i^{\mathcal{O}_1}$$

<sup>17</sup>I.e. one that can be written as  $\iota(A) = UAU^\dagger$  for some unitary operator  $U \in \mathcal{A}$ .

*Proof.* Consider the configuration of figure 3.2.

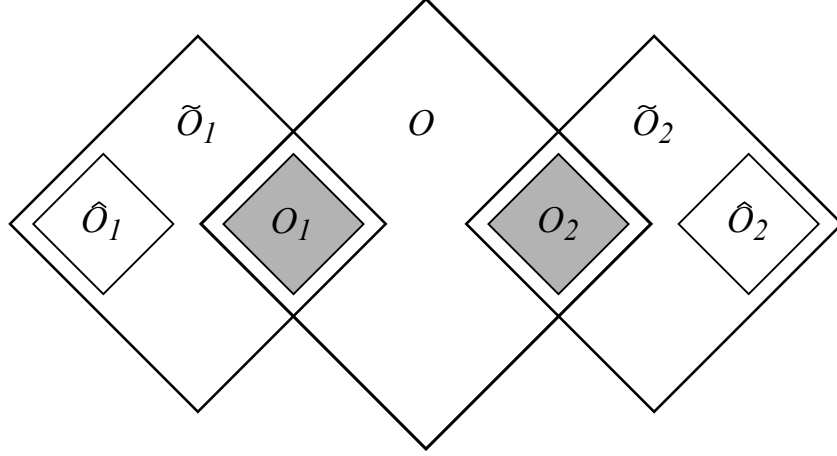


Figure 3.2: Chosen configuration for the proof.

We can transport the endomorphisms to  $\hat{\mathcal{O}}_k$ ,  $k = 1, 2$ , with two unitaries  $U_k \in \mathcal{A}(\tilde{\mathcal{O}}_k)$ :  $\rho_i^{\hat{\mathcal{O}}_1} = \text{Ad}_{U_1} \circ \rho_i^{\mathcal{O}_1}$  and  $\rho_j^{\hat{\mathcal{O}}_2} = \text{Ad}_{U_2} \circ \rho_j^{\mathcal{O}_2}$ . Hence, for any  $A \in \mathcal{A}(\mathcal{O})$  we have

$$\rho_i^{\mathcal{O}_1}(A) = \text{Ad}_{U_1^\dagger} \circ \rho_i^{\hat{\mathcal{O}}_1}(A) = \text{Ad}_{U_1^\dagger}(A) \quad \rho_j^{\mathcal{O}_2}(A) = \text{Ad}_{U_2^\dagger} \circ \rho_j^{\hat{\mathcal{O}}_2}(A) = \text{Ad}_{U_2^\dagger}(A)$$

since  $\mathcal{O}$  is causally disjoint from  $\hat{\mathcal{O}}_k$ ,  $k = 1, 2$ . Hence

$$\begin{aligned} \rho_i^{\mathcal{O}_1} \rho_j^{\mathcal{O}_2}(A) &= \rho_i^{\mathcal{O}_1} \circ \text{Ad}_{U_2^\dagger}(A) = \rho_i^{\mathcal{O}_1}(U_2^\dagger A U_2) = \rho_i^{\mathcal{O}_1}(U_2^\dagger) \rho_i^{\mathcal{O}_1}(A) \rho_i^{\mathcal{O}_1}(U_2) = U_2^\dagger \text{Ad}_{U_1^\dagger}(A) U_2 = \\ &= \text{Ad}_{U_2^\dagger U_1^\dagger}(A) = \text{Ad}_{U_1^\dagger U_2^\dagger}(A) = \rho_j^{\mathcal{O}_2} \rho_i^{\mathcal{O}_1}(A) \end{aligned}$$

where we used  $\rho_i^{\mathcal{O}_1}(U_2) = U_2$  since  $U_2 \in \mathcal{A}(\tilde{\mathcal{O}}_2)$ , while  $U_1^\dagger U_2^\dagger = U_2^\dagger U_1^\dagger$  follows from the locality axiom. We can repeat the same reasoning for any double cone  $\mathcal{O}$ , so  $\rho_i^{\mathcal{O}_1} \rho_j^{\mathcal{O}_2} = \rho_j^{\mathcal{O}_2} \rho_i^{\mathcal{O}_1}$ .  $\square$

**Corollary 3.4.4.** *Consider two endomorphisms  $\rho_i, \rho_j \in \Delta(\mathcal{O})$ . Then  $\rho_i \rho_j$  is unitarily equivalent to  $\rho_j \rho_i$ .*

*Proof.* Choose two spacelike separated double cones  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . There exist unitaries  $U_k$ ,  $k = 1, 2$ , such that  $\tilde{\rho}_i = \text{Ad}_{U_1} \circ \rho_i$  and  $\tilde{\rho}_j = \text{Ad}_{U_2} \circ \rho_j$  are, respectively, localized in  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . From the last proposition we get:

$$\rho_j \rho_i = \text{Ad}_{\rho_j(U_1^\dagger)U_2^\dagger} \circ \tilde{\rho}_j \tilde{\rho}_i = \text{Ad}_{\rho_j(U_1^\dagger)U_2^\dagger} \circ \tilde{\rho}_i \tilde{\rho}_j = \text{Ad}_{\epsilon(\rho_i, \rho_j)} \rho_i \rho_j$$

where

$$\epsilon(\rho_i, \rho_j) := \rho_j(U_1^\dagger)U_2^\dagger U_1 \rho_i(U_2)$$

is called *statistics operator* or *braiding*.  $\square$

We will say more on the statistics operator in a bit. For now, let us focus on representations; the result we wanted is the following:

**Corollary 3.4.5.** *The product of representations is commutative, i.e.*

$$[\pi_i \times \pi_j] = [\pi_j \times \pi_i]$$

*Proof.* It follows from the unitary equivalence of endomorphisms proven in the last corollary.  $\square$

From now on, we will refer to generic representatives of the equivalence classes of representations, keeping the square brackets understood.

Many interesting properties of the representations are valid in the *finite index case*. We briefly explain what this means: consider an inclusion  $\mathcal{N} \subset \mathcal{M}$ ; in particular, if  $\mathcal{N}$  is a factor, we call it *subfactor*. The index of inclusion, denoted  $[\mathcal{M} : \mathcal{N}]$  is a number that measures "how big the inclusion  $\mathcal{N} \subset \mathcal{M}$  is". More concretely, it corresponds to the dimension of a GNS constructed Hilbert space starting from a state induced by a *trace* over  $\mathcal{M}$ , which is a particular linear functional from the semi-positive defined elements of  $\mathcal{M}$  to the positive real numbers. This notion was initially introduced by Jones for Type  $II_1$  factors, since for them the existence of an appropriate trace is ensured, and later generalized to arbitrary factors. It has been shown that:

$$[\mathcal{M} : \mathcal{N}] \in \{4\cos^2(\frac{\pi}{n}), n \in \mathbb{N} \text{ and } n \geq 3\} \cup [4, +\infty]$$

The index allows for a notion of *dimension* of a unit preserving and injective endomorphism  $\varphi \in \text{End}(\mathcal{M})$  [38]:

$$\dim(\varphi) = d(\varphi) := [\mathcal{M} : \varphi(\mathcal{M})]^{1/2} \quad (3.7)$$

One can show that the dimension is stable under direct sums and composition:

$$\dim(\varphi_1 \oplus \varphi_2) = \dim(\varphi_1) + \dim(\varphi_2) \quad \dim(\varphi_1 \circ \varphi_2) = \dim(\varphi_1) \cdot \dim(\varphi_2)$$

In our case we are interested in the subfactors arising through the DHR endomorphisms, as a consequence of the third property in proposition 3.4.2.<sup>21</sup> In particular we define the *statistical dimension*, or just *dimension*, of  $\rho_i \in \Delta(\mathcal{O})$  as:

$$\dim(\rho_i) = d(\rho_i) := [\mathcal{A}(\mathcal{O}) : \rho_i(\mathcal{A}(\mathcal{O}))]^{1/2} \quad (3.8)$$

In particular, this index can be shown to be independent of the localization region  $\mathcal{O}$  [39]. For rational conformal field theories such indices are all finite. [49]

**Proposition 3.4.6.** [39] *Suppose that the representations labeled by  $\mathcal{J}$  have finite index. The following properties hold:*

- *Every covariant, positive energy representation of  $\mathcal{A}$  of finite index is completely reducible into a finite sum of irreducible covariant positive energy representations belonging to  $\mathcal{J}$ :*

$$\pi_i \times \pi_j \simeq \bigoplus_k N_{ij}^k \pi_k, \quad N_{ij}^k \in \mathbb{N} \quad (3.9)$$

- *There exists a unique involution ("charge conjugation") on  $\mathcal{J}$ :  $j \in \mathcal{J} \rightarrow \bar{j} \in \mathcal{J}$ , such that  $\pi_j \times \pi_{\bar{j}}$  contains the vacuum representation exactly once as a subrepresentation.*
- *The set of covariant, positive energy representation of  $\mathcal{A}$  which are localizable in double cones is stable under taking direct sums, conjugations, subrepresentations and composition.*

For rational CFTs, the Hilbert space decomposes into finitely many irreducible sectors of the observable algebra. These sectors are represented, in the field-theoretic description, by primary fields of the theory; in a full two-dimensional CFT the corresponding labels may be pairs of left- and right-chiral labels. Therefore, the fusion rules (3.9) coincide with the fusion rules arising from the OPE of the corresponding primary fields, and the set  $\mathcal{J}$  labels these primary sectors<sup>22</sup>. We can combine this observation with the following proposition [39].

**Proposition 3.4.7.** *Let  $\rho_j \in \Delta$ . Then the statistical dimension  $d(\rho_j)$  is the largest eigenvalue of the fusion matrix  $(N_j)_i^k := N_{ji}^k$ .*

<sup>21</sup>It is easy to verify that  $\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$  is a subfactor. As we are going to prove in section 3.6, DHR endomorphisms are faithful, i.e.  $\rho(A) = 0 \Rightarrow A = 0$ . Then note that if  $\rho(A) \in \rho(\mathcal{A}(\mathcal{O}))' \cap \rho(\mathcal{A}(\mathcal{O}))$ , then  $\rho(A)\rho(B) = \rho(B)\rho(A) \forall B \in \mathcal{A}(\mathcal{O})$  implying by faithfulness  $\rho([A, B]) = 0 \Rightarrow [A, B] = 0$ , i.e.  $A \in \mathcal{A}(\mathcal{O})' \cap \mathcal{A}(\mathcal{O}) = \mathbb{C}\mathbf{1}$ .

<sup>22</sup>Recall that for general RCFTs the chiral algebra may be larger than the Virasoro algebra.

This result relates several structures appearing in rational CFT. In a diagonal RCFT, all topological defect lines are Verlinde lines and are labeled by the same data as the chiral primary fields. Moreover, their fusion rules coincide with the fusion rules of the corresponding primary sectors. Since the DHR superselection sectors of the observable net are governed by the same fusion algebra, proposition 3.4.7, together with the discussion of section 2.2.2, shows that the statistical dimension of a sector coincides with the quantum dimension of the corresponding Verlinde defect.

### 3.4.1 Statistics

Now we see how the language of endomorphisms just introduced can be used to describe the statistical properties of the theory. In particular, we remark the fact, briefly mentioned before in section 2.5, that in low dimensional theories there exists more general statistics with respect to the usual Bose-Einstein and Fermi-Dirac statistics. The particles that obey this general statistics are called *anyons*. The reason why this phenomenon appears in low-dimensional theories is rooted in the topology of spacetime. In particular:

- In  $d = 3$ , there are two *distinguished* possibilities to exchange identical particles in a fixed time slice, which are counter clockwise and clockwise exchanges;
- In  $d = 2$ , the spacelike complement of a double cone consists of two *disconnected* wedges;
- In chiral CFTs, the theory is effectively defined on a one-dimensional light-ray, parametrized by  $x^+ = t + x$  or  $x^- = t - x$ . Since intervals on a line are naturally ordered, the exchange of two localized sectors depends on their relative left-right localization.

Although these are different properties, the consequence of all three is that the statistics of the particles are more exotic. For example, the difference between  $d = 4$  and  $d = 3$  is the following: in  $d = 4$  the oriented exchanges clockwise and counterclockwise can be deformed into each other continuously without ever intersecting the paths. For example, if the paths lie on a plane, as shown in figure 3.3 [47], it is sufficient to rotate the plane by an angle  $\pi$  around an axis passing through the positions of the two particles. In  $d = 3$  this is impossible and therefore we can assign a well-defined orientation to the exchanges. Therefore, in  $d = 4$  exchanges between  $n$  identical particles generate the group of permutations,  $S_n$ , whereas in  $d = 3$ , exchanges with orientation generate a different group: the *group of braids*,  $B_n$  [47].

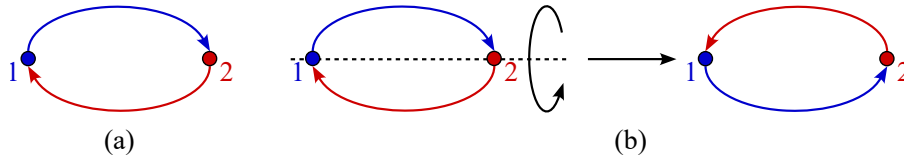


Figure 3.3: (a) An oriented exchange (b) Its deformation into an exchange with opposite orientation in  $d = 4$ .

Before we see how the braid group arises in  $d = 1, 2$ , we shall give its definition.

**Definition 3.4.3** (Braid group). The *Artin braid group* or *braid group on  $n$  strands*, denoted  $B_n$  is the finitely generated group with generators  $\zeta_1, \dots, \zeta_{n-1}$ , called *braids*, which satisfy the *braid relations*:

$$\begin{aligned} \zeta_i \zeta_j &= \zeta_j \zeta_i \quad \text{when} \quad |i - j| \geq 2 \quad \text{for} \quad i, j \in \{1, \dots, n - 1\} \\ \zeta_i \zeta_{i+1} \zeta_i &= \zeta_{i+1} \zeta_i \zeta_{i+1} \quad \text{for} \quad i, j \in \{1, \dots, n - 2\} \end{aligned} \quad (3.10)$$

We can geometrically interpret the braid group  $B_n$  as if its element acts on a space of  $n$ -strands by exchanging them. In particular,  $\zeta_i$  exchanges the  $i$ -th strand with the  $(i + 1)$ -th by passing over it. With this interpretation, we can draw the second braid relation, also called *Yang-Baxter equation*, as in figure 3.4.

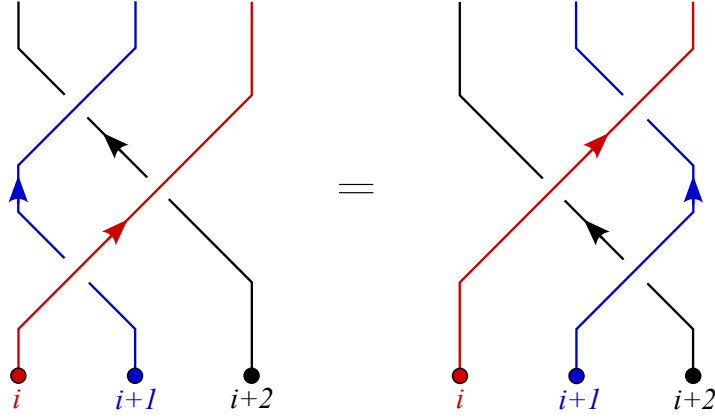


Figure 3.4: Graphical representation of the Yang-Baxter equation [47].

Notice that if we impose the additional relation  $\zeta_i^2 = 1$  we get the permutation group instead, consistent with the fact that now the exchanges are not oriented.

Now recall from corollary 3.4.4 that if  $\rho_i, \rho_j \in \Delta(\mathcal{O})$ , then

$$\epsilon(\rho_i, \rho_j) \rho_i \rho_j = \rho_j \rho_i \epsilon(\rho_i, \rho_j), \quad \epsilon(\rho_i, \rho_j) := \rho_j(U_1^\dagger) U_2^\dagger U_1 \rho_i(U_2) \quad (3.11)$$

where  $U_k$ ,  $k = 1, 2$ , are unitaries such that  $\tilde{\rho}_i = \text{Ad}_{U_1} \circ \rho_i$  and  $\tilde{\rho}_j = \text{Ad}_{U_2} \circ \rho_j$  are, respectively, localized in spacelike separated  $\mathcal{O}_1$  and  $\mathcal{O}_2$ .

Actually, there are two ways of separating the two endomorphisms. To be precise we define the following:

**Definition 3.4.4.** Let  $\mathcal{O}$  be a double cone in the Minkowski plane. Then its causal complement decomposes in two disconnected regions,  $\mathcal{O}' = \mathcal{W}_L \cup \mathcal{W}_R$ , called, respectively, left and right wedges. In the one-dimensional case we have intervals instead of double cones and the wedges are two half-lines, called future and past half-rays.

We can have either  $\mathcal{O}_1 \subset \mathcal{W}_{2,L} \subset \mathcal{O}'_2$  or  $\mathcal{O}_1 \subset \mathcal{W}_{2,R} \subset \mathcal{O}'_2$ . This would give the same result, but with two different braiding unitaries. Suppose that  $\epsilon(\rho_i, \rho_j)$  as defined in (3.11) realizes the second situation, i.e.  $\rho_i$  gets localized in the spacelike right (respectively future, in the chiral case) of  $\rho_j$ , then we have

$$\epsilon_{\rho_i, \rho_j}^+ := \epsilon(\rho_i, \rho_j) = \mathbf{1}, \quad \text{whenever } \rho_i \text{ is localized in the spacelike right of } \rho_j. \quad (3.12)$$

because one may just take  $U_1, U_2 = \mathbf{1}$ ; we denote this configuration with  $\rho_j < \rho_i$  or equivalently  $\mathcal{O}_2 < \mathcal{O}_1$ . The opposite choice of ordering would have resulted in an *opposite* braiding:  $\epsilon_{\rho_i, \rho_j}^- := (\epsilon_{\rho_j, \rho_i}^+)^{\dagger}$ . Note that the fact that one of the two braiding is not the identity when the two morphisms are spacelike separated (independently of how) does not mean that they do not commute, which is always the case as shown in proposition 3.4.3.

The braiding unitaries are particular examples of *intertwiners of morphisms*.

**Definition 3.4.5.** Intertwiners between localized endomorphisms  $\rho_i, \rho_j \in \Delta$  are operators  $T$  satisfying

$$T \rho_i(A) = \rho_j(A) T \quad \forall A \in \mathcal{A}$$

We write  $T : \rho_i \rightarrow \rho_j$  or  $T \in \text{Hom}(\rho_i, \rho_j)$ .<sup>23</sup>

Note that as a consequence of Haag duality, every intertwiner between two localized endomorphisms is an element of some local algebra. Indeed, if  $\rho_i \in \Delta(\mathcal{O}_1), \rho_j \in \Delta(\mathcal{O}_2)$  and  $T : \rho_i \rightarrow \rho_j$ :

$$T \rho_i(A) = \rho_j(A) T \quad \forall A \in \mathcal{A} \Rightarrow TA = AT \quad \forall A \in \mathcal{A}(\mathcal{O}'_1) \cap \mathcal{A}(\mathcal{O}'_2) \quad \Leftrightarrow \quad T \in (\mathcal{A}(\mathcal{O}'_1) \cap \mathcal{A}(\mathcal{O}'_2))'$$

<sup>23</sup>The reason of this notation will be clear in the next section.

In the specific case where  $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$  we get by Haag duality  $T \in \mathcal{A}(\mathcal{O}')' = \mathcal{A}(\mathcal{O})$ . Therefore  $T^*$  is a well-defined intertwiner  $\rho_j \rightarrow \rho_i$ .

The braiding is an example of intertwiner:  $\epsilon(\rho_i, \rho_j) \in \text{Hom}(\rho_i \rho_j, \rho_j \rho_i)$ . Furthermore:

**Proposition 3.4.8.** *The following fundamental relations hold<sup>24</sup>:*

1. For  $T_i \in \text{Hom}(\rho_i, \sigma_i)$  with  $\rho_i, \sigma_i \in \Delta$ ,  $i = 1, 2$ , we have:

$$\epsilon(\sigma_1, \sigma_2) \sigma_1(T_2) T_1 = T_2 \rho_2(T_1) \epsilon(\rho_1, \rho_2) \quad (3.13)$$

2. Let  $\rho_i \in \Delta$ ,  $i = 1, 2, 3$ , then:

$$\epsilon(\rho_1 \rho_2, \rho_3) = \epsilon(\rho_1, \rho_3) \rho_1(\epsilon(\rho_2, \rho_3)) \quad \epsilon(\rho_1, \rho_2 \rho_3) = \rho_2(\epsilon(\rho_1, \rho_3)) \epsilon(\rho_1, \rho_2) \quad (3.14)$$

*Proof.* 1. We start by showing that

$$\sigma_1(T_2) T_1 = T_2 \rho_2(T_1) \in \text{Hom}(\rho_1 \rho_2, \sigma_1 \sigma_2) \quad (3.15)$$

where  $\rho_i, \sigma_i \in \Delta(\mathcal{O}_i)$  and  $\mathcal{O}_i$  spacelike separated. For all  $A \in \mathcal{A}$  we have:

$$\begin{aligned} \sigma_1(T_2) T_1 \rho_1 \rho_2(A) &= \sigma_1(T_2) T_1 \rho_1(\rho_2(A)) = \sigma_1(T_2) \sigma_1(\rho_2(A)) T_1 = \sigma_1(T_2 \rho_2(A)) T_1 = \\ &= \sigma_1 \sigma_2(A) \sigma_1(T_2) T_1 \end{aligned}$$

Therefore  $\sigma_1(T_2) T_1 \in \text{Hom}(\rho_1 \rho_2, \sigma_1 \sigma_2)$ . Since  $T_1 \in \mathcal{A}(\mathcal{O}_1)$  and  $T_2 \in \mathcal{A}(\mathcal{O}_2)$  and the two double cones are spacelike separated we get

$$\sigma_1(T_2) T_1 = T_2 T_1 = T_2 \rho_2(T_1)$$

We are not going to do the explicit computation for the complete equation (3.13) since it would take a long time; note that with the morphisms spacelike separated with the correct ordering ( $\rho_2 < \rho_1$  and  $\sigma_2 < \sigma_1$ ) the braiding operators simplify (as a consequence of equation (3.12)) and we get back the just proven formula. If that is not the case, consider morphisms and intertwiners for which equation (3.15) holds and move their localization to arbitrary double cones by using charge transporting unitary operators<sup>25</sup>:  $\sigma_i = \text{Ad}_{U_i^\dagger} \tilde{\sigma}_i$ ,  $\rho_i = \text{Ad}_{U_i^\dagger} \tilde{\rho}_i$  and  $T_i = \text{Ad}_{U_i^\dagger} \tilde{T}_i$ ,  $i = 1, 2$ . Note that by multiplying both sides of (3.13) by  $\rho_1 \rho_2$  it is immediate that both members are intertwiners  $\rho_1 \rho_2 \rightarrow \sigma_2 \sigma_1$ .

2. We just show that  $\epsilon(\rho_1, \rho_3) \rho_1(\epsilon(\rho_2, \rho_3)) \in \text{Hom}(\rho_1 \rho_2 \rho_3, \rho_3 \rho_1 \rho_2)$ . For all  $A \in \mathcal{A}$ :

$$\begin{aligned} \epsilon(\rho_1, \rho_3) \rho_1(\epsilon(\rho_2, \rho_3)) \rho_1(\rho_2(\rho_3(A))) &= \epsilon(\rho_1, \rho_3) \rho_1(\epsilon(\rho_2, \rho_3) \rho_2(\rho_3(A))) = \\ &= \epsilon(\rho_1, \rho_3) \rho_1(\rho_3(\rho_2(A))) \rho_1(\epsilon(\rho_2, \rho_3)) = \\ &= \rho_3(\rho_1(\rho_2(A))) \epsilon(\rho_1, \rho_3) \rho_1(\epsilon(\rho_2, \rho_3)) \end{aligned}$$

Note that the equality with  $\epsilon(\rho_1 \rho_2, \rho_3)$  is trivial when  $\rho_3 < \rho_2 < \rho_1$  and simplifies to  $\epsilon(\rho_1 \rho_2, \rho_3) = \epsilon(\rho_1, \rho_3)$  when  $\rho_3 < \rho_2$ ; then the same strategy given before can be used to prove the equality for every localization.

□

Now we have all the ingredients to construct the desired representations. For simplicity, we restrict ourselves to the case of  $n = 3$  strands.

<sup>24</sup>With abuse of notation, the subscript of endomorphism will sometimes be used just to distinguish endomorphisms and not to label the corresponding representation. The distinction should be clear depending on the context.

<sup>25</sup>We have seen that  $\rho = \text{Ad}_{U^\dagger} \tilde{\rho}$  moves the localization of  $\rho$  to a different double cone depending on the charge transporter  $U$ . Then if we consider an intertwiner  $T \in \text{Hom}(\rho, \sigma)$ , where  $\rho$  and  $\sigma$  have different localizations, we get:  $T\rho = \sigma T \Rightarrow TU_\rho^\dagger \tilde{\rho} U_\rho = U_\sigma^\dagger \tilde{\sigma} U_\sigma T$ , from which we get  $\tilde{T} = U_\sigma T U_\rho^\dagger$ . In case the two morphisms share the same localization before and after the transport:  $\tilde{T} = \text{Ad}_U \circ T$  and  $T = \text{Ad}_{U^\dagger} \circ \tilde{T}$ .

**Proposition 3.4.9.** Consider  $\rho_i \in \Delta$ ,  $i = 1, 2, 3$ . Then the following equation holds:

$$\rho_3(\epsilon(\rho_1, \rho_2))\epsilon(\rho_1, \rho_3)\rho_1(\epsilon(\rho_2, \rho_3)) = \epsilon(\rho_2, \rho_3)\rho_2(\epsilon(\rho_1, \rho_3))\epsilon(\rho_1, \rho_2) \quad (3.16)$$

*Proof.* Starting from the left-hand side, we use equation (3.14):

$$\rho_3(\epsilon(\rho_1, \rho_2))\epsilon(\rho_1, \rho_3)\rho_1(\epsilon(\rho_2, \rho_3)) = \rho_3(\epsilon(\rho_1, \rho_2))\epsilon(\rho_1\rho_2, \rho_3) \quad (3.17)$$

Now we apply equation (3.13) with the following identification:

$$T_2 = \text{id}_{\rho_3}, \quad T_1 = \epsilon(\rho_1, \rho_2), \quad \rho_1 \rightarrow \rho_1\rho_2, \quad \rho_2 \rightarrow \rho_3, \quad \sigma_1 \rightarrow \rho_2\rho_1, \quad \sigma_2 \rightarrow \rho_3 \quad (3.18)$$

which becomes

$$\epsilon(\rho_2\rho_1, \rho_3)\rho_2\rho_1(\text{id}_{\rho_3})\epsilon(\rho_1, \rho_2) = \text{id}_{\rho_3}\rho_3(\epsilon(\rho_1, \rho_2))\epsilon(\rho_1\rho_2, \rho_3) \quad (3.19)$$

i.e. the right-hand side of (3.18) coincides with the right-hand side of (3.17). Thus, we have found:

$$\rho_3(\epsilon(\rho_1, \rho_2))\epsilon(\rho_1, \rho_3)\rho_1(\epsilon(\rho_2, \rho_3)) = \epsilon(\rho_2\rho_1, \rho_3)\epsilon(\rho_1, \rho_2)$$

Finally, we apply again (3.14) and conclude with:

$$\epsilon(\rho_2\rho_1, \rho_3)\epsilon(\rho_1, \rho_2) = \epsilon(\rho_2, \rho_3)\rho_2(\epsilon(\rho_1, \rho_3))\epsilon(\rho_1, \rho_2)$$

□

Equation (3.16) can be seen as a Yang-Baxter equation for three colored braids, and admits a graphical representation identical to the one of figure 3.4, but with the additional information that each strand corresponds to a particular endomorphism among  $\rho_1$ ,  $\rho_2$  and  $\rho_3$ . This equation is more general than the braid relation (3.10): each term in (3.16) does not represent a generator of  $B_n$  since the braids are different and each term belongs to a different space, thus making it impossible to create a representation on a single Hilbert space. However, if we now set  $\rho_i = \rho \ \forall i$ , we get

$$\rho(\epsilon_{\rho,\rho})\epsilon_{\rho,\rho}\rho(\epsilon_{\rho,\rho}) = \epsilon_{\rho,\rho}\rho(\epsilon_{\rho,\rho})\epsilon_{\rho,\rho}, \quad \epsilon(\rho, \rho) = \epsilon_{\rho,\rho} \quad (3.20)$$

which is easily mapped to the Yang-Baxter equation for  $B_3$ . In general, if we now define:

$$\zeta_{i+1} \mapsto \rho^i(\epsilon_{\rho,\rho}), \quad i = 0, \dots, n-2, \quad \rho^0(\epsilon_{\rho,\rho}) := \epsilon_{\rho,\rho} \quad (3.21)$$

we get a unitary representation of  $B_n$  into  $(\rho^n(\mathcal{A}))'$ .<sup>26</sup> Indeed, note that by applying  $\rho$  to both sides of equation (3.20) we find that the braid relation (3.10) holds for all generators. Moreover, note that

$$\epsilon_{\rho,\rho}\rho^i(\epsilon_{\rho,\rho}) = \rho^i(\epsilon_{\rho,\rho})\epsilon_{\rho,\rho} \quad \forall i \geq 2$$

which is the first braid relation.

The same procedure can be performed with  $\epsilon_{\rho,\rho}^- = (\epsilon_{\rho,\rho})^\dagger$  instead of  $\epsilon_{\rho,\rho}^+$ . In particular, note that if  $\epsilon_{\rho,\rho}^+ = \epsilon_{\rho,\rho}^-$ , then  $\epsilon_{\rho,\rho}^2 = \mathbf{1}$  and the representation collapses to a representation of the permutation group.

Consider now a theory with a charged particle with self conjugate sector (i.e.  $\pi_{\bar{j}} = \pi_j$ ) and such that  $N_{jj}^k \neq \delta_0^k$ . Then if we group  $n$  identical particles together, depending on the particular fusion rules, we end up with a superposition of different states, each related to a particular fusion tree.<sup>27</sup> We can now express the generators of the braid group representations in this base and the matrices we get describe how such states get mixed when we make an ordered exchange of a pair of identical particles. In this case the braid group is associated to a matrix representation, which is in general non abelian, and for this reason the corresponding particles are called *non-abelian anyons*. We make two observations on this:

<sup>26</sup>Of course  $\epsilon_{\rho,\rho} \in (\rho^2(\mathcal{A}))'$  from which it is immediate that  $\rho(\epsilon_{\rho,\rho}) \in (\rho^3(\mathcal{A}))'$  and so on and so forth. In general, since  $\rho$  is an endomorphism of  $\mathcal{A}$  we have  $\rho(\mathcal{A}) \subset \mathcal{A}$ , therefore  $\rho^{n+1}(\mathcal{A}) \subset \rho^n(\mathcal{A})$  and so  $(\rho^n(\mathcal{A}))' \subset (\rho^{n+1}(\mathcal{A}))'$ .

<sup>27</sup>For fusion tree we mean a diagram such as the one on the left-hand side of (2.33).

1. As a consequence of equation (3.16), braid statistics can appear also between non identical particles! However, in that case we do not have a representation of  $B_n$ , but rather a representation of the "groupoid of colored braids" [50];
2. The non-abelianity of anyons has nothing to do with their spin, but it is rather a property of the fusion algebra.

Consider again the same theory, but now with  $N_{jj}^k = \delta_0^k$ , i.e.  $\pi_j \times \pi_j = \pi_0$ . Then the fusion of identical particles among themselves generates just one single state, but we can still have one-dimensional representations of the braid group where the generators are the complex phases acquired by the fields According to a particular exchange. In this case we talk about *abelian anyons*. Even in this case, the particles need not to be identical, but if so, then one can show that the acquired phase is related to the spin of the representation [39]. Note that in case of half-integer spin this reduces to the common  $S_n$  statistics.<sup>28</sup>

Before ending this section, we just mention the fact that the base we heuristically introduced above, used to express the braid group representation in coordinates, is nothing more than a base of the intertwiner space  $\text{Hom}(\rho_j^n, \rho_j)$ . In the general case of  $n$  different particles, the coordinate form is still written with respect to a basis of intertwiners, but such treatment is highly non-trivial and far beyond the scope of this thesis; see, for example, [47] and [39].

Usually, the coordinate expressions of the representations of the (colored) braids are called statistical R matrices. However, we will use this notation for a slightly different (but still related) notion, which is directly connected to the categorical language we are going to introduce in the next section.

The physical implication of this statistical structure will be explored in a concrete way in the Ising model at the end of this chapter.

### 3.5 The category of DHR endomorphisms

We now focus on one spacetime dimension, i.e. to chiral rational CFTs. The discussion about two-dimensional CFTs will be resumed in the next chapter.

The structure introduced in the previous two sections, namely the set of DHR endomorphisms  $\Delta$ , the braiding unitary operator  $\epsilon$  and, more in general, intertwiners between endomorphisms, turn the DHR representation theory of  $\mathcal{A}$  into a *braided  $C^*$  tensor category*  $\mathcal{C} = \mathcal{C}^{DHR}(\mathcal{A})$ .

From now on we also assume *complete rationality*, i.e.  $\mathcal{A}$  possesses only finitely many inequivalent positive-energy representations with finite statistical dimension; all of them have conjugates and obey the properties proposition 3.4.6. This also implies *modularity* [35]; hence we deal with a *modular fusion category*.

**Definition 3.5.1.** A *modular fusion category* or *modular tensor category* is a ribbon fusion category with an invertible  $s$ -matrix.

We now unpack this definition. First note that the starting point is a fusion category, which was already well characterized in section 2.4. Then we have:

- *Ribbon*: this means that the category is endowed with a *braiding* and a *twist*. A braiding on a tensor category  $\mathcal{C}$  allows one to 'exchange' objects; it consists of a family of isomorphisms  $\epsilon_{U,V} \in \text{Hom}(U \otimes V, V \otimes U)$ , one for each pair  $U, V \in \text{Obj}(\mathcal{C})$ . Finally, a twist is a family of

---

<sup>28</sup>Indeed, in these cases we have  $\epsilon_{\rho,\rho} = \pm 1 \Leftrightarrow \epsilon_{\rho,\rho}^2 = 1$ .

isomorphisms  $\theta_U$ , one for each  $U \in \text{Obj}(\mathcal{C})$ . We can represent them graphically as follows

$$\begin{array}{cccc} \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{\mathcal{E}_{U,V}} \\ | \quad | \\ U \quad V \end{array} = \begin{array}{c} V \quad U \\ | \quad | \\ \text{crossing} \\ | \quad | \\ U \quad V \end{array} & \begin{array}{c} U \quad V \\ | \quad | \\ \boxed{\mathcal{E}_{V,U}^{-1}} \\ | \quad | \\ U \quad V \end{array} = \begin{array}{c} V \quad U \\ | \quad | \\ \text{inverse crossing} \\ | \quad | \\ U \quad V \end{array} & \begin{array}{c} U \\ | \\ \boxed{\theta_U} \\ | \\ U \end{array} = \begin{array}{c} U \\ | \\ \text{twist} \\ | \\ U \end{array} & \begin{array}{c} U \\ | \\ \boxed{\theta_U^{-1}} \\ | \\ U \end{array} = \begin{array}{c} U \\ | \\ \text{inverse twist} \\ | \\ U \end{array} \end{array}$$

The terminology "Ribbon" comes from the fact that one can represent pictorially the twist and the braiding using ribbons as in figure (3.22). However, we will stick to the string notation.

$$\theta_U = \text{twisted ribbon } U \quad \mathcal{E}_{U,V} = \text{crossing of ribbons } U \text{ and } V \quad (3.22)$$

Both the braiding and the twist must satisfy different consistency conditions with the already present structures in the category. In (3.23) is shown the compatibility with the rigid structure, as well as the functoriality of the twist and the braiding, i.e. compatibility with morphisms.

$$\begin{array}{ccccccc} \begin{array}{c} U \quad U^\vee \\ | \quad | \\ \text{twist} \\ | \quad | \\ U \quad U^\vee \end{array} = \begin{array}{c} U \quad U^\vee \\ | \quad | \\ \text{cap} \\ | \quad | \\ U \quad U^\vee \end{array} & \begin{array}{c} Z \quad W \\ | \quad | \\ \text{braiding} \\ | \quad | \\ \boxed{f} \quad \boxed{g} \\ | \quad | \\ U \quad V \end{array} = \begin{array}{c} Z \quad W \\ | \quad | \\ \boxed{g} \quad \boxed{f} \\ | \quad | \\ \text{braiding} \\ | \quad | \\ U \quad V \end{array} & \begin{array}{c} U \\ | \\ \text{twist} \\ | \\ \boxed{f} \\ | \\ U \end{array} = \begin{array}{c} U \\ | \\ \boxed{f} \\ | \\ \text{twist} \\ | \\ U \end{array} & \begin{array}{c} U \\ | \\ \text{braiding} \\ | \quad | \\ \boxed{f} \quad \boxed{f} \\ | \quad | \\ U \quad U \end{array} = \begin{array}{c} U \\ | \\ \boxed{f} \\ | \quad | \\ \text{braiding} \\ | \quad | \\ U \quad U \end{array} \end{array} \quad (3.23)$$

The compatibility of the braiding with the monoidal structure as well as the compatibility between the braiding and the twist are shown in (3.24).

$$\begin{array}{cccc} \begin{array}{c} Z \quad U \quad V \\ | \quad | \quad | \\ \text{braiding} \\ | \quad | \quad | \\ \boxed{\mathcal{E}_{U \otimes V, Z}} \\ | \quad | \quad | \\ U \quad V \quad Z \end{array} = \begin{array}{c} Z \quad U \quad V \\ | \quad | \quad | \\ \text{braiding} \\ | \quad | \quad | \\ \boxed{\mathcal{E}_{U,Z}} \quad \boxed{\mathcal{E}_{V,Z}} \\ | \quad | \quad | \\ U \quad V \quad Z \end{array} & \begin{array}{c} U \quad V \\ | \quad | \\ \text{twist} \\ | \quad | \\ \text{braiding} \\ | \quad | \\ U \quad V \end{array} = \begin{array}{c} U \quad V \\ | \quad | \\ \text{twist} \\ | \quad | \\ \text{braiding} \\ | \quad | \\ \theta_{U \otimes V} \quad \theta_V \quad \theta_U \\ | \quad | \\ U \quad V \end{array} \end{array} \quad (3.24)$$

For a ribbon category, left duality can be defined starting from the notion of right duality and a natural isomorphism between  $U$  and  $(U^\vee)^\vee$  can be obtained with the help of the twist, making the category automatically sovereign. [33]

- *Modular*: Suppose that  $\mathcal{J}$  is the set of labels of simple objects in the category. Then the modularity consists of a non-degenerate  $|\mathcal{J}| \times |\mathcal{J}|$ -matrix  $s = (s_{i,j})_{i,j \in \mathcal{J}}$  by

$$s_{i,j} = \text{Tr}(\epsilon_{U_i, U_j} \epsilon_{U_j, U_i})$$

This latter property provides in fact an explanation of the qualification "modular": when combined with the other axioms, it implies that the matrices  $s$  and  $t = \text{diag}(\theta_{U_i})$  (when multiplied with overall constants that are expressible through the  $\theta_{U_j}$  and the quantum dimensions  $\dim(U_j)$ ) generate a projective representation of the modular group  $SL(2, \mathbb{Z})$  [33]. Pictorially:

$$s_{i,j} = \text{diagram of braiding } i \text{ and } j, \quad t_{i,j} = \frac{\delta_{i,j}}{\langle i \rangle} \text{diagram of twist } i$$

Note that for simple objects  $s_{i,0} = \langle i \rangle$  with  $i \in \mathcal{J}$ .

In our case, the category  $\mathcal{C} = \mathcal{C}^{DHR}(\mathcal{A})$  is the category with DHR endomorphisms<sup>29</sup> as objects and intertwiners between endomorphisms as morphisms;  $\mathcal{C}^{DHR}(\mathcal{A})$  is a fusion category due to the underlying fusion ring induced by equation (3.9), with composition of endomorphisms as tensor product, and it is a ribbon category due to the braiding  $\epsilon_{\rho_i, \rho_j}$ ,  $\rho_i, \rho_j \in \text{Obj}(\mathcal{C}^{DHR}(\mathcal{A}))$  and the twist determined by the fractional part of the conformal weight of the associated representation:

$$\theta_\rho = e^{2\pi i h_\rho} \text{id}_\rho \quad (3.25)$$

In this category, the quantum dimension corresponds to the dimension defined in equation (3.8).

An important observation is the following: note that the axioms that the braiding has to satisfy, namely the functoriality of the braiding of (3.23) and the compatibility with tensor products of (3.24) correspond to the properties of proposition 3.4.8 by using the representation (3.21) to interpret each term as a move on braids. This is of course not an accident and shows indeed that the braiding unitary defined in the previous section is compatible with the ribbon category axioms. Furthermore, we can impose these rules as axioms also in the algebraic setting: indeed, these properties together with an initial condition such as the one of equation (3.12), uniquely fix the braiding. [58]

Modularity is ensured by the complete rationality of the underlying theory, which guarantees the non-degeneracy of the braiding and therefore the invertibility of the s-matrix [35]. The matrix  $s$  coincides, up to normalization, with the modular S-matrix of the CFT that implements the modular transformation  $\tau \rightarrow -\frac{1}{\tau}$  on the characters of primary fields [33]:<sup>30</sup>

$$s_{i,j} = \frac{S_{i,j}}{S_{0,0}} \quad (3.26)$$

Thus,  $\mathcal{C}^{DHR}(\mathcal{A})$  is a modular tensor category.

Introducing for this category a base and its dual as done previously in (2.28), we can express the braiding morphisms in coordinates, which provides us with the *braiding matrices*  $R$ :

<sup>29</sup>As we have discussed in the previous section the language of endomorphisms is equivalent to the one of representations. Hence one could take representations as objects instead.

<sup>30</sup>This normalization is consistent with the categorical definition of quantum dimension and our previous result of equation (2.14) for diagonal theories, in the context of the category of TDLs.

$$\begin{array}{c} k \\ | \\ \bullet \alpha \\ \text{loop} \\ | \\ i \quad j \end{array} = \sum_{\beta} R_{\alpha,\beta}^{(ij)k} \begin{array}{c} k \\ | \\ \bullet \beta \\ \text{loop} \\ | \\ i \quad j \end{array}$$

The number obtained when the braiding  $\epsilon_{\rho_i \rho_j}$  ("over-braiding") is replaced by  $\epsilon_{\rho_j \rho_i}^{-1}$  ("under-braiding") is denoted by  $R_{\alpha,\beta}^{-(ij)k}$ , and instead of  $R_{\alpha,\beta}^{+(ij)k}$  one often writes  $R_{\alpha,\beta}^{(ij)k}$ . One easily checks that

$$\sum_{\beta} R_{\alpha,\beta}^{+(ij)k} R_{\beta,\gamma}^{-(ji)k} = \delta_{\alpha,\gamma}$$

and, using the mutual compatibility of twist and braiding and functoriality of the twist of (3.23) and (3.24) we can prove the following:

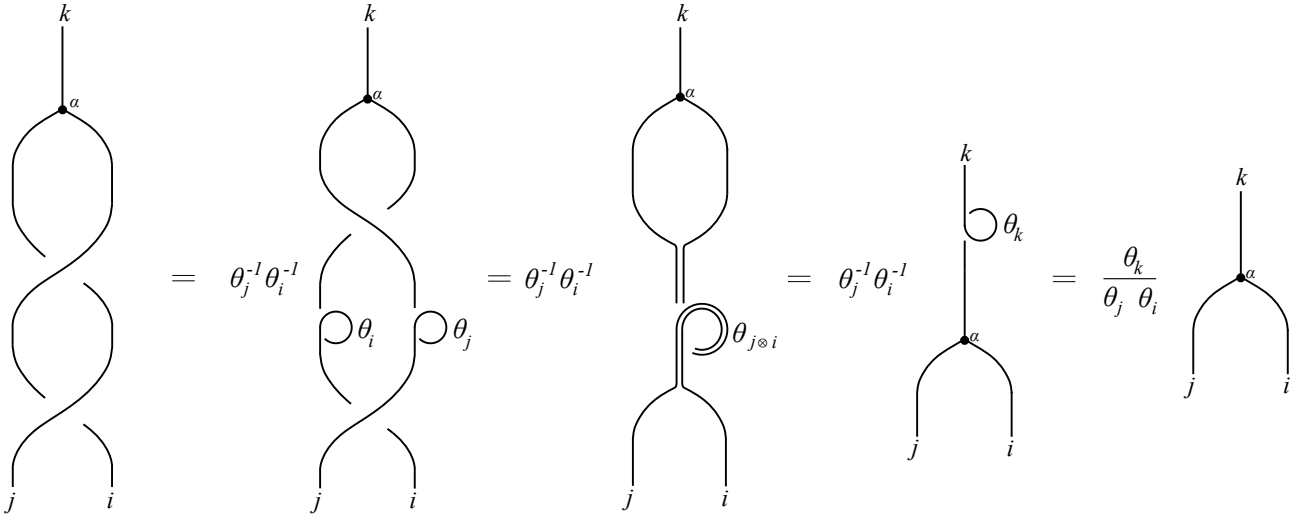
**Proposition 3.5.1.** *The following property holds:*

$$\sum_{\beta} R_{\alpha,\beta}^{(ij)k} R_{\beta,\gamma}^{(ji)k} = \frac{\theta_k}{\theta_i \theta_j} \delta_{\alpha,\gamma} \quad (3.27)$$

*Proof.* Consider two successive braidings. We start by resolving the knots by using the definition of the R matrix:

$$\begin{array}{c} k \\ | \\ \bullet \alpha \\ \text{loop} \\ | \\ i \quad j \end{array} = \sum_{\beta} R_{\alpha,\beta}^{(ij)k} \begin{array}{c} k \\ | \\ \bullet \beta \\ \text{loop} \\ | \\ i \quad j \end{array} = \sum_{\beta,\gamma} R_{\alpha,\beta}^{(ij)k} R_{\beta,\gamma}^{(ji)k} \begin{array}{c} k \\ | \\ \bullet \gamma \\ \text{loop} \\ | \\ i \quad j \end{array}$$

Now we now use the compatibility of twist and braiding and functoriality of the twist:



Comparing both sides we obtain:

$$\sum_{\beta} R_{\alpha, \beta}^{(ij)k} R_{\beta, \gamma}^{(ji)k} = \frac{\theta_k}{\theta_i \theta_j} \delta_{\alpha, \gamma}$$

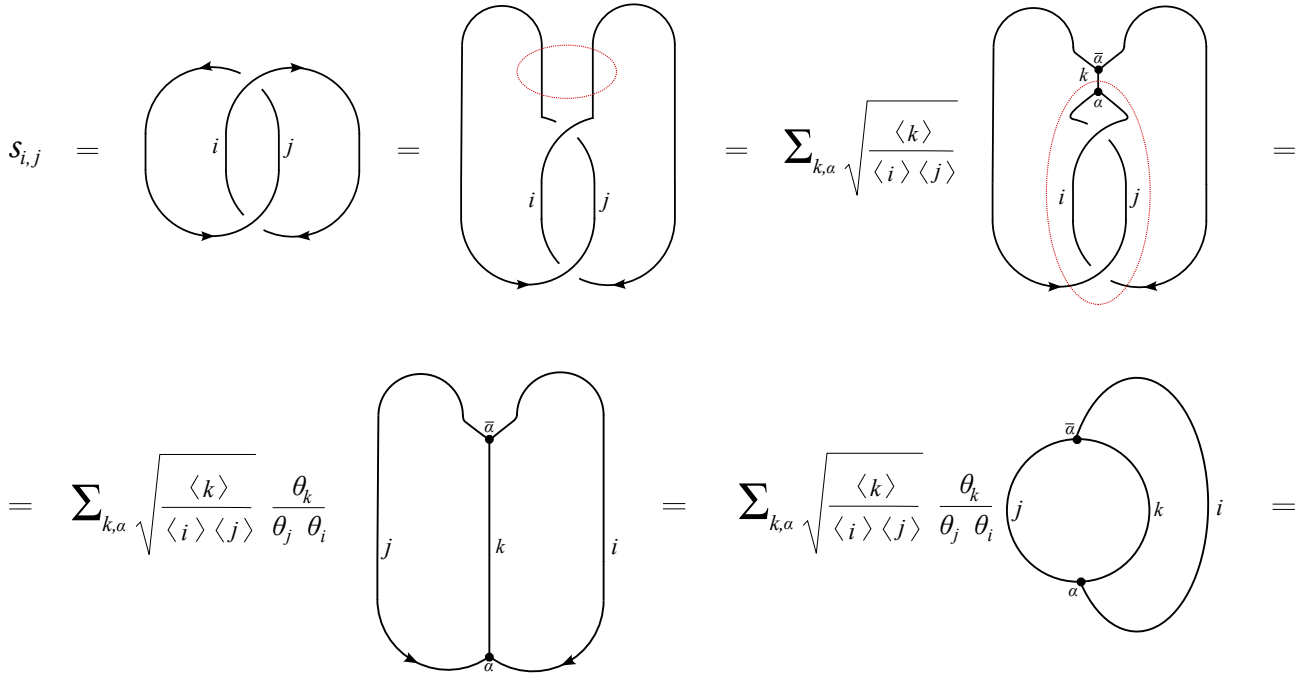
□

With this property we can prove a useful formula for computing the s-matrix.

**Proposition 3.5.2.** *The s-matrix can be computed using the following formula:*

$$s_{i,j} = \sum_k \frac{\theta_k}{\theta_i \theta_j} N_{ij}^k \langle k \rangle$$

*Proof.* We start by deforming the two circles. Then we apply the completeness relation of proposition 2.4.2 to the first red dotted circle; successively, we apply the last proposition to the other red dotted circle, and finally we use the property of (2.29) to resolve bubble diagrams.



$$= \sum_{k,\alpha} \sqrt{\frac{\langle k \rangle}{\langle i \rangle \langle j \rangle}} \frac{\theta_k}{\theta_j \theta_i} \sqrt{\frac{\langle j \rangle \langle k \rangle}{\langle i \rangle}} \delta_a^\alpha \bigcirc_i = \sum_k \sqrt{\frac{\langle k \rangle}{\langle i \rangle \langle j \rangle}} \frac{\theta_k}{\theta_j \theta_i} \sqrt{\frac{\langle j \rangle \langle k \rangle}{\langle i \rangle}} N_{ij}^k \langle i \rangle$$

Note that we have used the fact that  $\alpha = 1, \dots, N_{ij}^k = \dim(U_i \times U_j, U_k)$  for  $U_i, U_j$  and  $U_k$  simple objects of the category. By simplifying the right-hand side, we find:

$$s_{ij} = \sum_k \frac{\theta_k}{\theta_i \theta_j} N_{ij}^k \langle k \rangle$$

□

Note that since  $s_{00} = 1$ , we deduce that  $s_{ij}$  and  $S_{ij}$  must be at least proportional by a factor  $S_{00}$ , which justifies (3.26). That factor is necessary to make the S-matrix unitary, since the s-matrix in general doesn't have this property [33]. In particular, from the unitarity of S:

$$SS^\dagger = \mathbf{1} \quad \Rightarrow \quad \sum_k S_{ik} S_{jk}^*$$

taking the first entry and using the fact that  $\langle k \rangle = s_{0k} = S_{0k}/S_{00}$ :

$$1 = \sum_k |S_{0k}|^2 = |S_{00}|^2 \sum_k |\langle k \rangle|^2 \quad \Rightarrow \quad |S_{00}| = \left( \sum_k |\langle k \rangle|^2 \right)^{-1/2} \quad (3.28)$$

where we have written  $|\langle k \rangle|^2$  since for a general MTC  $\langle k \rangle \in \mathbb{C}$ . In the specific case of  $\mathcal{C} = \mathcal{C}^{DHR}(\mathcal{A})$ , as we have seen, the quantum dimension is related to the Jones index, which is a positive real number.

### 3.6 The chiral Ising model

In this section, we are going to study the chiral Ising model with the just introduced notions. The idea is to start with the chiral model to build the two-dimensional theory in the next chapter and add defects subsequently. Since we will subsequently be dealing with equations containing elements from both chiral algebra and two-dimensional algebra, we shall denote the former by lowercase letters.

We are not going to explicitly build the net, which is rather difficult to do, but instead we study directly the superselection structure of the model. Indeed, the chiral Ising model is described by a *Virasoro net* with central charge  $c = 1/2$ ; its precise definition will be given later, but essentially it is just a net of observables generated by orientation preserving diffeomorphisms.<sup>31</sup> Therefore, the superselection structure is already known from the study of irreducible representations of the Virasoro algebra with  $c = 1/2$ . Nevertheless, one can actually build a net for the chiral Ising model using the CAR algebra; this is done in [36, 40].

We know that we have three primary operators, hence three superselection sectors,  $\pi_0$ ,  $\pi_\psi$  and  $\pi_\sigma$ . The associated DHR \*-endomorphisms of the net are denoted as the corresponding primary operators, i.e.  $1, \psi$  and  $\sigma$ . From the fusion rules (3.9) of superselection sectors, these endomorphisms are defined up to the following unitary equivalences:

$$\psi \times \psi \simeq 1 \quad \sigma \times \psi \simeq \psi \times \sigma \simeq \sigma \quad \sigma \times \sigma \simeq 1 + \psi$$

We choose the representative of the unitary equivalence class of  $\psi$  such that

$$\psi\sigma = \sigma,$$

<sup>31</sup>I.e. the infinite extension of the conformal group on the one-dimensional spacetime line.

where the product of endomorphisms is given by composition. Notice, however, that the analogous equality

$$\sigma\psi = \sigma$$

cannot be imposed for any non-trivial representative of the sector  $\psi$ .

To see this, let us first show that DHR endomorphisms are injective [67]. Indeed, let  $A \in \mathcal{A}(\mathcal{O})$  and suppose that  $\rho(A) = 0$ . By transportability, we can choose an equivalent endomorphism  $\tilde{\rho}$  localized in a region spacelike separated from  $\mathcal{O}$ . Therefore there exists a unitary charge transporter  $u$  such that

$$\tilde{\rho} = \text{Ad}_u \circ \rho.$$

Since  $\tilde{\rho}$  is localized spacelike to  $\mathcal{O}$ , it acts trivially on  $\mathcal{A}(\mathcal{O})$ . Hence

$$A = \tilde{\rho}(A) = u\rho(A)u^\dagger = 0.$$

Thus  $\rho(A) = 0$  implies  $A = 0$ , and  $\rho$  is injective<sup>32</sup>.

Now suppose, by contradiction, that there existed a non-trivial representative  $\psi'$  such that

$$\sigma\psi' = \sigma.$$

Then, for every local observable  $A$ ,

$$\sigma(\psi'(A)) = \sigma(A).$$

By the injectivity of  $\sigma$ , this implies  $\psi'(A) = A$  for all  $A$ , hence  $\psi' = 1$ . Therefore, while the equivalence  $\sigma\psi \simeq \sigma$  holds, it cannot be made into a strict equality.

It follows that:

- $\psi^2 = 1$ . Indeed from  $\psi \times \psi \simeq 1$  we know that there exists a unitary operator  $v$  such that:

$$v\psi \circ \psi(\cdot)v^\dagger = v\psi^2(\cdot)v^\dagger = 1(\cdot) \quad \Rightarrow \quad \psi^2(\cdot) = v^\dagger 1(\cdot)v = v^\dagger v 1(\cdot) = 1(\cdot)$$

which implies  $\psi^2 = 1$ . From now on the argument of endomorphisms  $(\cdot)$  will be usually omitted;

- Since  $\sigma\psi \simeq \sigma$  we have  $\text{Hom}(\sigma\psi, \sigma) = \mathbb{C} \simeq \text{Hom}(\sigma, \sigma\psi)$ , where the identification is given by the algebraic involution  $\dagger$ . By calling  $u \in \text{Hom}(\sigma\psi, \sigma)$  the unitary which realizes the equivalence  $\sigma\psi \simeq \sigma$ , we get  $\sigma\psi = \text{Ad}_u \sigma$  with  $u^2 = \mathbf{1} = (u^\dagger)^2$ . Indeed:

$$\sigma = \sigma\psi^2 = \sigma\psi(\psi(\cdot)) = \text{Ad}_u(\sigma(\psi(\cdot))) = \text{Ad}_u(\sigma\psi) = \text{Ad}_u(\text{Ad}_u(\sigma)) = \text{Ad}_u(u\sigma u^\dagger) = u^2\sigma(u^\dagger)^2$$

Hence  $u^2 \in \text{Hom}(\sigma, \sigma)$ ; since  $\sigma$  is a simple object, by Schur's lemma, we have  $u^2 = \lambda \mathbf{1}$ ,  $\lambda \in \mathbb{C}$ . From unitarity we get that this number must be a phase  $\lambda \bar{\lambda} \mathbf{1} = u^2(u^\dagger)^2 = \mathbf{1}$ . Since we can always redefine unitaries up to a phase we fix  $u^2 = \mathbf{1}$ . Note that we cannot have  $u \propto \mathbf{1}$ , since this would imply  $\sigma\psi = \sigma = \psi\sigma$  from which  $\psi = 1$ .

As a consequence of the fusion rules, we can choose a pair of intertwiners,  $r \in \text{Hom}(1, \sigma^2)$  and  $t \in \text{Hom}(\psi, \sigma^2)$ , satisfying the following relations:

$$r^\dagger r = \mathbf{1}, \quad t^\dagger t = \mathbf{1}, \quad r^\dagger t = 0 = t^\dagger r, \quad rr^\dagger + tt^\dagger = \mathbf{1} \quad (3.29)$$

The first two normalizations come from the fact that  $r^\dagger r$  and  $t^\dagger t$  must be proportional to  $\mathbf{1}$  due to Schur's lemma. The third relation follows from the fact that, being simple objects,  $\text{Hom}(\psi, 1) = \text{Hom}(1, \psi) = 0$ . The last equation is a normalization choice that comes from the fusion rule:  $\text{Hom}(\sigma^2, \sigma^2) = rr^\dagger \cdot \text{Hom}(1, 1) + tt^\dagger \cdot \text{Hom}(\psi, \psi) = rr^\dagger \cdot \mathbb{C} + tt^\dagger \cdot \mathbb{C}$ . In particular, this set of normalizations is called Cuntz algebra [35].

<sup>32</sup>This property is sometimes referred to as faithfulness. For a linear map, and in particular for a  $*$ -homomorphism of  $C^*$ -algebras, it is equivalent to injectivity: indeed, if  $\rho(A) = \rho(B)$ , then  $\rho(A - B) = \rho(A) - \rho(B) = 0$ , and therefore  $A - B = 0$ , i.e.  $A = B$ .

Since<sup>33</sup>  $u \in \text{Hom}(\sigma\psi, \sigma) \subset \text{Hom}(\sigma^2, \sigma^2) = rr^\dagger \cdot \mathbb{C} + tt^\dagger \cdot \mathbb{C}$  we can set  $u = \alpha rr^\dagger + \beta tt^\dagger$  with  $\alpha, \beta \in \mathbb{C}$ . From the last relation above, we can write:

$$u = \alpha rr^\dagger + \beta tt^\dagger = \alpha rr^\dagger + \beta(\mathbf{1} - rr^\dagger) = (\alpha - \beta)rr^\dagger + \beta\mathbf{1}$$

Since  $u$  cannot be proportional to the identity, we get  $\alpha \neq \beta$ . From the constraint  $u^2 = \mathbf{1}$ :

$$u^2 = (\alpha rr^\dagger + \beta tt^\dagger)(\alpha rr^\dagger + \beta tt^\dagger) = \alpha^2 rr^\dagger + \beta^2 tt^\dagger = \mathbf{1}$$

which implies  $\alpha^2 = 1 = \beta^2$ . Together with the other result we get  $u = rr^\dagger - tt^\dagger$ , upon fixing an overall free phase. Note that as a consequence of the properties of  $u$ , in particular from  $u^\dagger = u$ , we conclude that  $\text{Hom}(\sigma, \sigma\psi) = \text{Hom}(\sigma\psi, \sigma)$ .

Now we have to determine the action of morphisms on the intertwiners. We start from  $\psi(r)$ . Since  $r \in \text{Hom}(1, \sigma^2)$ ,  $\psi(r) \in \text{Hom}(\psi, \psi\sigma^2) = \text{Hom}(\psi, \sigma^2) = t \cdot \mathbb{C}$ . Let  $\psi(r) = \lambda t$ ,  $\lambda \in \mathbb{C}$ . Then

$$\psi(r^\dagger) = \psi(r)^\dagger = \bar{\lambda} t^\dagger \Rightarrow \psi(1) = 1 = \psi(r^\dagger r) = \lambda \bar{\lambda}$$

which implies that  $\lambda = e^{i\theta}$  with  $\theta \in S^1$ . Moreover,

$$\psi^2(r) = r = \lambda \psi(t) \Rightarrow \psi(t) = \lambda^{-1} r$$

We do not have any other constraint on the phase, hence we are free to choose  $\lambda = 1$ . Then we get:

$$\psi(u) = \psi(rr^\dagger - tt^\dagger) = tt^\dagger - rr^\dagger = -u$$

We conclude with:

$$\psi(r) = t \quad \psi(t) = r \quad \psi(u) = -u$$

Now we move on to  $\sigma$ . As before, we note that:

$$\sigma(r) \in \text{Hom}(\sigma, \sigma^3) = (rr^\dagger + tt^\dagger) \cdot \text{Hom}(\sigma, \sigma^3) = r \cdot \text{Hom}(\sigma, \sigma) + t \cdot \text{Hom}(\sigma, \psi\sigma) = r \cdot \mathbb{C} + t \cdot \mathbb{C}$$

Hence, we have  $\sigma(r) = \alpha r + \beta t$ ,  $\alpha, \beta \in \mathbb{C}$ . Now we impose  $\psi\sigma = \sigma$ :

$$\psi(\sigma(r)) = \sigma(r) \Leftrightarrow \alpha\psi(r) + \beta\psi(t) = \alpha r + \beta t \Leftrightarrow \alpha t + \beta r = \alpha r + \beta t$$

Therefore  $\alpha = \beta$ . Next:

$$1 = \sigma(1) = \sigma(r^\dagger r) = |\alpha|^2 (r + t)^\dagger (r + t) = |\alpha|^2 (r^\dagger r + t^\dagger t) = 2|\alpha|^2$$

Hence  $|\alpha| = 1/\sqrt{2}$ . Now we do the same for  $\sigma(t)$ :

$$\begin{aligned} \sigma(t) \in \text{Hom}(\sigma\psi, \sigma^3) &= (rr^\dagger + tt^\dagger) \cdot \text{Hom}(\sigma\psi, \sigma^3) = r \cdot \text{Hom}(\sigma\psi, \sigma) + t \cdot \text{Hom}(\sigma\psi, \psi\sigma) = \\ &= ru \cdot \text{Hom}(\sigma, \sigma) + tu \cdot \text{Hom}(\sigma, \psi\sigma) = ru \cdot \mathbb{C} + tu \cdot \mathbb{C} \end{aligned}$$

So we set  $\sigma(t) = \gamma ru + \delta tu$  and impose:

$$\psi(\sigma(t)) = \sigma(t) \Leftrightarrow -\gamma tu - \delta ru = \gamma ru + \delta tu$$

Therefore  $\gamma = -\delta$ . Next

$$1 = \sigma(1) = \sigma(t^\dagger t) = |\gamma|^2 (u^\dagger r^\dagger - u^\dagger t^\dagger)(ru - tu) = 2|\gamma|^2$$

Therefore  $|\gamma| = 1/\sqrt{2}$ . Finally, we impose the intertwining relation:

$$\sigma^2(r)t = t\psi(r) = tt \Leftrightarrow \alpha \sigma(r+t)t = tt \Leftrightarrow \alpha[\alpha(r+t) + \gamma(ru - tu)]t = tt$$

---

<sup>33</sup>Note that if  $u \in \text{Hom}(\sigma\psi, \sigma)$ , then  $\sigma^2(rt^\dagger)\sigma(u^\dagger) \in \text{Hom}(\sigma^2, \sigma^2)$ . Indeed:  $\sigma^2(rt^\dagger)\sigma(u^\dagger)\sigma^2(A) = \sigma^2(rt^\dagger)\sigma(u^\dagger\sigma(A)) = \sigma^2(rt^\dagger\psi(A))\sigma(u^\dagger) = \sigma^2(A)\sigma^2(rt^\dagger)\sigma(u^\dagger) \forall A \in \mathcal{A}$ .

Using the fact that  $u = rr^\dagger - tt^\dagger$  we find:

$$\alpha^2 rt + \gamma^2 tt - \gamma \alpha rt + \gamma \alpha tt = tt \Leftrightarrow \begin{cases} \alpha^2 + \alpha\gamma = 1 \\ \alpha^2 - \alpha\gamma = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha^2 = \gamma^2 = 1/2 \\ \alpha = \gamma \end{cases}$$

We fix an overall free sign by choosing  $\alpha = 1/\sqrt{2} = \gamma$ . Then we can compute  $\sigma(u)$ :

$$\sigma(u) = \sigma(rr^\dagger) - \sigma(tt^\dagger) = \frac{1}{2}(r+t)uu^\dagger(r+t)^\dagger - \frac{1}{2}(r-t)uu^\dagger(r-t)^\dagger = tr^\dagger + rt^\dagger$$

We have found:

$$\sigma(r) = \frac{1}{\sqrt{2}}(r+t) \quad \sigma(t) = \frac{1}{\sqrt{2}}(ru - tu) \quad \sigma(u) = tr^\dagger + rt^\dagger \quad (3.30)$$

The twists of the model, according to (3.25) and using the conformal weights  $h_{r,s}(3)$  of (1.31), are:

$$\theta_\psi = e^{i\pi} = -1 \quad \theta_\sigma = e^{\frac{\pi i}{8}}$$

Now we have to find the braidings. We start from  $\epsilon_{\psi,\psi}$ . We have  $\epsilon_{\psi,\psi} \in \text{Hom}(\psi^2, \psi^2) = \text{Hom}(1, 1) = \mathbb{C}$ , which means that  $\epsilon_{\psi,\psi} = \lambda$  for  $\lambda \in \mathbb{C}$ . Unitarity imposes  $\epsilon_{\psi,\psi} \epsilon_{\psi,\psi}^\dagger = 1 \Leftrightarrow |\lambda| = 1$ . Therefore  $\epsilon_{\psi,\psi} = e^{i\theta}$ ,  $\theta \in S^1$ . We also have  $\epsilon_{\psi,\psi} \epsilon_{\psi,\psi} = \epsilon_{\psi,\psi}^2 = 1$  from which  $\epsilon_{\psi,\psi} = \pm 1$ . Now we turn our attention to  $\epsilon_{\psi,\sigma} \in \text{Hom}(\psi\sigma, \sigma\psi) = \text{Hom}(\sigma, \sigma\psi) = \mathbb{C} u$ . Note that from what we have already seen, namely  $\text{Hom}(\sigma, \sigma\psi) = \text{Hom}(\sigma\psi, \sigma)$ ,  $\epsilon_{\psi,\sigma} = \epsilon_{\sigma,\psi}$ . We use the compatibility of braiding and twist (3.24):

$$\theta_{\psi\sigma} = \theta_\sigma = \epsilon_{\psi,\sigma} \theta_\psi \theta_\sigma \epsilon_{\sigma,\psi} \Rightarrow \epsilon_{\psi,\sigma}^2 = -1$$

Hence  $\epsilon_{\psi,\sigma} = \pm iu$ . We can now apply the other property (3.24) (i.e. equation (3.14)):

$$\epsilon_{\psi\sigma,\psi} = \epsilon_{\sigma,\psi} = \epsilon_{\psi,\psi} \psi(\epsilon_{\sigma,\psi}) = -\epsilon_{\psi,\psi} \epsilon_{\sigma,\psi}$$

which imposes  $\epsilon_{\psi,\psi} = -1$ , coherently with the  $1/2$  conformal dimension of  $\psi$ .

Finally, we look at  $\epsilon_{\sigma,\sigma}$ . We see that:

$$\epsilon_{\sigma,\sigma} \in \text{Hom}(\sigma^2, \sigma^2) = rr^\dagger \cdot \mathbb{C} + tt^\dagger \cdot \mathbb{C}$$

Therefore,  $\epsilon_{\sigma,\sigma} = \alpha rr^\dagger + \beta tt^\dagger$ ,  $\alpha, \beta \in \mathbb{C}$ . From unitarity:

$$\epsilon_{\sigma,\sigma} \epsilon_{\sigma,\sigma}^* = 1 = |\alpha|^2 rr^\dagger + |\beta|^2 tt^\dagger \Rightarrow |\alpha| = |\beta| = 1$$

So we can write:  $\epsilon_{\sigma,\sigma} = e^{i\theta}(rr^\dagger + e^{i\omega} tt^\dagger)$ , with  $\theta, \omega \in S^1$ . Then using compatibility with the twist:

$$\theta_{\sigma\sigma} = \epsilon_{\sigma,\sigma} \theta_\sigma \theta_\sigma \epsilon_{\sigma,\sigma} = \theta_\sigma^2 e^{2i\theta} (rr^\dagger + e^{i\omega} tt^\dagger)(rr^\dagger + e^{i\omega} tt^\dagger) = \theta_\sigma^2 e^{2i\theta} (rr^\dagger + e^{2i\omega} tt^\dagger)$$

Note that the left-hand side is not a scalar, since  $\theta_{\sigma\sigma} \in \text{Hom}(\sigma^2, \sigma^2)$ . We can use the compatibility of the twist with morphisms (3.23); in particular, intertwining  $\sigma^2$  with 1, which has twist 1, we get:

$$r^\dagger = r^\dagger \theta_{\sigma\sigma} \Rightarrow \theta_\sigma^2 e^{2i\theta} = 1 \Rightarrow e^{2\pi i/8 + 2i\theta} = 1$$

from which we find  $e^{i\theta} = \pm e^{-i\pi/8}$ . We can do the same intertwining  $\sigma^2$  with  $\psi$ :

$$\theta_\psi t^\dagger = t^\dagger \theta_{\sigma\sigma} \Rightarrow e^{2i\omega} = -1$$

from which  $e^{i\omega} = \pm i$ .

The relative sign inside  $\epsilon_{\sigma,\sigma}$  is related to the sign in front of  $\epsilon_{\psi,\sigma}$  by equation (3.14):

$$\epsilon_{\psi\sigma,\sigma} = \epsilon_{\sigma\sigma} = \epsilon_{\psi,\sigma} \psi(\epsilon_{\sigma,\sigma})$$

Suppose that we choose  $\epsilon_{\psi,\sigma} = -i$  and  $\epsilon_{\sigma,\sigma} = +e^{-i\pi/8}(rr^\dagger \pm itt^\dagger)$ . Then:

$$rr^\dagger \pm itt^\dagger = -i(rr^\dagger - tt^\dagger)(tt^\dagger \pm i rr^\dagger) = -i(-tt^\dagger \pm irr^\dagger) = (\pm rr^\dagger + itt^\dagger)$$

This is consistent if and only if we choose the plus sign. Choosing the other overall sign in  $\epsilon_{\psi,\sigma}$  would give the converse result; hence the two signs must be opposite.

Note that the remaining two free signs come from the two possible square roots of the statistical matrix<sup>34</sup> coming from equation (3.27):  $(R^{(ii)j})^2 = \frac{\theta_j}{\theta_i^2} \Rightarrow R^{(ii)j} = \nu_i \frac{\theta_j^{1/2}}{\theta_i}$ ,  $\nu_i = \pm 1$ . It can be shown that this sign, called *Frobenius-Schur indicator*, can be computed with  $\nu_i = \dim(U_i) F_{00}^{(iii)i}$  for a simple and self-dual object  $U_i$  of the category [33, 68]. In our case, these signs are plus for all our simple objects (which are all self-dual). In particular, we find  $R^{(\sigma\sigma)1} = +e^{-i\pi/8}$  and  $R^{(\sigma\sigma)\psi} = +e^{i\pi 3/8}$ , which also fixes  $R^{(\psi\sigma)\sigma} = R^{(\sigma\psi)\sigma} = +e^{-i\pi/2} = -i$  for the previous consideration.

In summary:

$$\epsilon_{\psi,\psi} = -1 \quad \epsilon_{\psi,\sigma} = \epsilon_{\sigma,\psi} = -i \quad \epsilon_{\sigma,\sigma} = e^{-i\pi/8}(rr^\dagger + itt^\dagger) = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi 3/8} \end{pmatrix} \quad (3.31)$$

Here we have also written the last braiding in matrix form with respect to a basis of  $\text{Hom}(\sigma, \sigma^3) \simeq r \cdot \mathbb{C} \oplus t \cdot \mathbb{C}$ . These basis elements can be intuitively interpreted as the states associated with the possible fusion trees obtained by fusing three spin fields into one, as discussed at the end of Section 3.4.1.

As anticipated, from the fusion rule  $\sigma \times \sigma \simeq 1 + \psi$ , the braiding of  $\sigma$  is non-trivial and provides an example of non-abelian braid statistics. In the chosen basis, this braiding is diagonal because it corresponds to exchanging the two innermost fields in the fusion tree, so that the intermediate fusion channel is preserved. As we shall see in a moment, braiding fields which are not adjacent in the chosen fusion tree requires a change of fusion basis and, in general, mixes the different fusion channels.

If we were in the Euclidean setting, all these calculations involving the braidings could have been avoided by using the OPE of two chiral fields  $\phi_i$  and  $\phi_j$ :

$$\lim_{z \rightarrow w} \phi_i(z) \phi_j(w) = \sum_k C_{i,j}^k (z - w)^{\Delta_k - \Delta_i - \Delta_j} \phi_k(w)$$

to determine the phase accumulated when two particles wrap around each other. In particular, note that an exchange of particles amounts to half of a full wrapping; meaning that if we set the conventional rotation direction as the counter-clockwise direction, the exchange of  $\psi$  with  $\sigma$  and the successive fusion to  $\sigma$  would accumulate an overall phase  $e^{\pi i (\Delta_\sigma - \Delta_\sigma - \Delta_\psi)} = -i$ ; the clockwise rotation would give  $+i$  instead; i.e. the opposite braiding.

The same procedure can be used to immediately compute all phases  $R^{(ij)k}$ , which agree with the components expressions of the braiding of equation (3.31):

$$R^{(\psi\psi)1} = -1 \quad R^{(\psi\sigma)\sigma} = R^{(\sigma\psi)\sigma} = -i \quad R^{(\sigma\sigma)1} = e^{-i\pi/8} \quad R^{(\sigma\sigma)\psi} = e^{i\pi 3/8}$$

Finally, we can compute the other element of the unitary representation of  $B_3$  associated to  $\sigma$ :

$$\begin{aligned} \sigma(\epsilon_{\sigma,\sigma}) &= \frac{e^{-i\pi/8}}{2} [(r+t)(r+t)^\dagger + i(r-t)uu^\dagger(r-t)^\dagger] = \\ &= \frac{e^{-i\pi/8}}{2} (rr^\dagger + rt^\dagger + tr^\dagger + tt^\dagger + irr^\dagger + itt^\dagger - itr^\dagger - irt^\dagger) = \\ &= \frac{e^{-i\pi/8}}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \end{aligned}$$

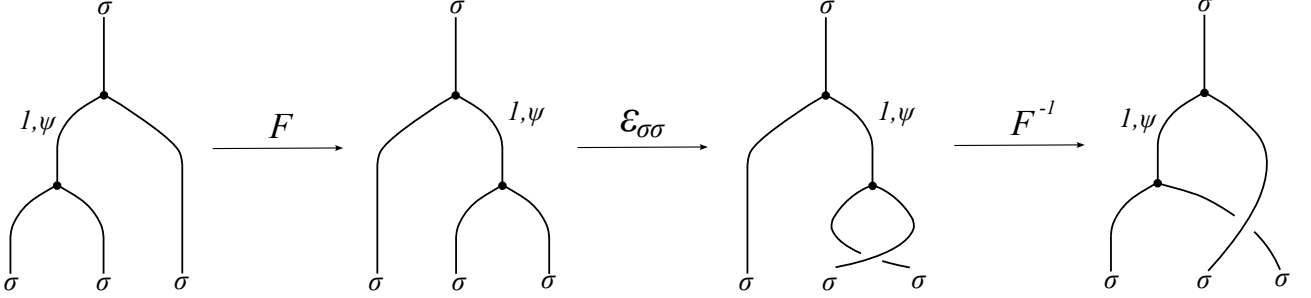
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<sup>34</sup>We suppressed the  $\alpha, \beta$  indices since  $N_{ij}^k \in \{0, 1\}$ .

The very same result can be achieved performing the operation  $F^{-1}\epsilon_{\sigma\sigma}F$ , i.e. we first make a basis change using  $F$  in order to determine how the two rightmost particles fuse in the fusion tree with three  $\sigma$ , then we rotate these two rightmost particles using the braiding, and finally undo the basis change<sup>35</sup>. Using the fusion matrix of equation (2.38), by noting that it is auto-inverse, and the matrix form of the braiding, we get:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{e^{-i\pi/8}}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{e^{-i\pi/8}}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

Coherent with the previous result. The procedure that we have followed is represented as:



We conclude the characterization of the chiral Ising model by computing the s-matrix. Using proposition 3.5.2:

$$s_{11} = 1 \quad s_{1\psi} = \frac{\theta_\psi}{\theta_1\theta_\psi} N_{1\psi}^\psi \langle \psi \rangle = 1 \quad s_{1\sigma} = \frac{\theta_\sigma}{\theta_1\theta_\sigma} N_{1\sigma}^\sigma \langle \sigma \rangle = \sqrt{2} \quad s_{\psi\psi} = \frac{\theta_1}{\theta_\psi\theta_\psi} N_{\psi\psi}^1 \langle 1 \rangle = 1$$

$$s_{\psi\sigma} = \frac{\theta_\sigma}{\theta_\psi\theta_\sigma} N_{\psi\sigma}^\sigma \langle \sigma \rangle = -\sqrt{2} \quad s_{\sigma\sigma} = \frac{\theta_1}{\theta_\sigma\theta_\sigma} N_{\sigma\sigma}^1 \langle 1 \rangle + \frac{\theta_\psi}{\theta_\sigma\theta_\sigma} N_{\sigma\sigma}^\psi \langle \psi \rangle = \sqrt{2} - \sqrt{2} = 0$$

While from (3.28) we have  $S_{00} = 1/2$ . This is consistent with the  $S$ -matrix (1.43) we expected from the Ising model.

<sup>35</sup>The first element of  $B_3$  is mapped to  $\epsilon_{\sigma,\sigma}$ . This element should make the exchange of the first two strands. However, in the language of ribbon categories, recall that the tensor product  $\otimes$  of the category corresponds, in this case, to composition of morphisms; then the algebraic action of  $\epsilon_{\sigma,\sigma}$  corresponds to the morphism  $\epsilon_{\sigma,\sigma} \otimes 1_\sigma$ . The fact that this morphism, after the change of basis, still correspond to the exchange of the innermost strands is understood by looking at the basis decomposition given by equations (2.31) and (2.32). With this in mind, it is clear why  $1_\sigma \otimes \epsilon_{\sigma,\sigma}$  corresponds algebraically to  $\sigma(\epsilon_{\sigma,\sigma})$ .

## Chapter 4

# Conformal nets and extensions

After introducing the general algebraic framework of quantum field theory and the DHR description of superselection sectors, we now specialize to the conformal case. The aim of this chapter is to explain how a full two-dimensional conformal field theory can be constructed starting from chiral data. Physically, the chiral theory describes one light-cone component of the system: its observables, its charged sectors, and the way these sectors can be fused. A two-dimensional CFT, however, contains both left- and right-moving degrees of freedom. Therefore, the first natural step is to consider two chiral nets and form their tensor product, which represents the independent propagation of the two chiral components.

This tensor product is not yet the most general full CFT. It only provides the kinematical arena in which left- and right-moving sectors are available. To obtain a genuine two-dimensional theory, one must specify which pairs of chiral sectors actually appear in the physical spectrum and how the corresponding fields multiply. This additional datum is encoded by a Q-system  $(\Theta, W, X)$ . From a physical point of view, the object  $\Theta$  specifies the charged content of the extension, while the morphisms  $W$  and  $X$  respectively implement the inclusion of the vacuum and the multiplication of charged fields. In this way, the Q-system plays the role of an algebraic substitute for the information usually contained in the operator product expansion.

The chapter then explains how a Q-system determines an extension of nets

$$\mathcal{A} \subset \mathcal{B},$$

where  $\mathcal{A}$  is the observable chiral theory and  $\mathcal{B}$  is the enlarged theory containing charged fields. These charged fields should be understood as operators which interpolate between the vacuum representation and non-trivial superselection sectors. Their algebraic relations, and in particular their locality properties, are dictated by the structure of the Q-system.

From a physical CFT point of view, they are the primary sector-generating field of the theory, and the Q-system completely determines their fusion rules.

We will first present this procedure in the simpler setting of a single von Neumann factor. Namely, given a factor  $\mathcal{N}$ , we will explain how a Q-system in  $\text{End}(\mathcal{N})$  determines an inclusion  $\mathcal{N} \subset \mathcal{M}$ , where  $\mathcal{M}$  is a larger von Neumann algebra containing  $\mathcal{N}$ . In general,  $\mathcal{M}$  need not be a factor; only under suitable irreducibility assumptions does this inclusion become an inclusion of factors. Then, recalling that the local algebras of the conformal nets considered here are type  $III_1$  factors, we will extend the same idea from a single algebra to a net of local observables, obtaining inclusions of nets  $\mathcal{A} \subset \mathcal{B}$ .

We also discuss the canonical extension, which provides the natural diagonal construction associated with a rational chiral theory. This construction is especially important because it gives the basic example of a local two-dimensional CFT obtained from chiral data. Finally, the abstract formalism is applied to the critical Ising model. This example makes the general construction explicit: the chiral Ising sectors are combined into a full two-dimensional theory, and the corresponding Q-system encodes both the spectrum and the algebraic structure of the resulting fields.

This chapter is based on [33, 35, 38, 69–71].

Note that, in order to simplify notation, we are not going to explicitly write the vacuum representation  $\pi_0$  from now on, identifying an observable algebra with its vacuum representation. Recall also that we are going to use lowercase letters for the elements of chiral algebras.

## 4.1 Conformal nets

We specify now the framework introduced in the previous chapter to chiral conformal field theories. A chiral CFT lives on a single light-ray of a two-dimensional Minkowski spacetime. We identify the light-ray with  $\mathbb{R}$  and consider its compactification  $S^1$ . Let  $\mathcal{I}$  be the set of non-empty non-dense open intervals on  $S^1$ . We also recall that the global conformal group is  $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$ , also called Möbius group as its complex version arising in Euclidean spacetime.

**Definition 4.1.1.** A *Möbius covariant net* on  $S^1$  is a net of von Neumann algebras  $\mathcal{I} \ni I \rightarrow \mathcal{A}(I)$  satisfying the Haag-Kastler axioms<sup>1</sup> where Poincaré covariance is substituted by Möbius covariance.

Note that Möbius covariance implies the existence of a strongly continuous unitary representation  $U$  of  $PSL(2, \mathbb{R})$  on the vacuum Hilbert space  $\mathcal{H}_0$ .

We know that the conformal group has an infinite extension in one dimension, given by the orientation preserving diffeomorphisms of the unit circle,  $\text{Diff}_+(S^1)$ .

**Definition 4.1.2.** A Möbius covariant net will be called a *conformal net* on  $S^1$  if in addition  $U$  extends to a strongly continuous projective unitary representation of  $\text{Diff}_+(S^1)$ , again denoted by  $U$ , such that for every  $I \in \mathcal{I}$

$$U(\gamma)\mathcal{A}(I)U(\gamma)^\dagger = \mathcal{A}(\gamma I), \quad \gamma \in \text{Diff}_+(S^1),$$

$$U(\gamma)xU(\gamma)^\dagger = x, \quad x \in \mathcal{A}(I), \quad \gamma \in \text{Diff}_+(I'),$$

where  $\text{Diff}_+(I)$  denotes the subgroup of orientation preserving diffeomorphisms of  $S^1$ ,  $\text{Diff}_+(S^1)$ , that are localized in  $I$  and  $I'$  is the complement of  $I$  in  $S^1$ .

We can have inclusion of nets describing subtheories.

**Definition 4.1.3.** A *conformal subnet* of a conformal net  $\mathcal{A}$  is a family  $\mathcal{B} = \{\mathcal{B}(I) : I \in \mathcal{I}\}$  of non-trivial von Neumann algebras acting on  $\mathcal{H}_0$  such that:

- $\mathcal{B}(I) \subset \mathcal{A}(I) \forall I \in \mathcal{I}$ ;
- $U(g)\mathcal{B}(I)U(g)^\dagger = \mathcal{B}(gI) \forall I \in \mathcal{I}, g \in PSL(2, \mathbb{R})$ ;
- $\mathcal{B}(I_1) \subset \mathcal{B}(I_2) \forall I_1, I_2 \in \mathcal{I}$  with  $I_1 \subset I_2$ .

One can show that a conformal subnet is a conformal net itself [71].

**Definition 4.1.4.** A net  $I \rightarrow \mathcal{A}(I)$  is said to be *irreducible* if  $(\bigvee_{I \in \mathcal{I}} \mathcal{A}(I))' = \mathbb{C}\mathbf{1}$  or, equivalently,  $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = \mathcal{B}(\mathcal{H}_0)$ . We call an inclusion  $\mathcal{B} \subset \mathcal{A}$  *irreducible* if  $\mathcal{B}(I)' \cap \mathcal{A}(I) = \mathbb{C}\mathbf{1}$  for some, hence for all,  $I \in \mathcal{I}$ , where the commutant is taken in  $\mathcal{B}(\mathcal{H}_0)$ .

Let  $U_{(c,0)}$  be the irreducible strongly continuous projective unitary and positive energy representation of  $\text{Diff}_+(S^1)$  with central charge  $c$  and lowest weight zero.

**Definition 4.1.5.** Let  $Vir_c$  be the *Virasoro net* with central charge  $c$  associated with  $U_{(c,0)}$ :

$$I \in \mathcal{I} \rightarrow Vir_c(I) := \{U_{(c,0)}(\gamma) : \gamma \in \text{Diff}_+(I)\}''$$

The vacuum vector  $|\Omega\rangle \in \mathcal{H}_0$  is the lowest weight vector of  $U_{(c,0)}$ .

---

<sup>1</sup>By definition of von Neumann algebra, this net comes already together with a Hilbert space, which is identified with the vacuum Hilbert space that one usually construct from the vacuum state defined in the axioms. This Hilbert space contains a unique Möbius invariant vacuum vector.

Any conformal net  $\mathcal{A}$  contains a copy of  $Vir_c$ , for some  $c$ , as a conformal subnet:

$$I \in \mathcal{I} \rightarrow Vir_{\mathcal{A}}(I) := \{U(\gamma) : \gamma \in \text{Diff}_+(I)\}''$$

the conformal inclusion  $Vir_{\mathcal{A}} \subset \mathcal{A}$  is automatically irreducible. Moreover, the Virasoro net is minimal, in the sense it does not contain any non-trivial conformal subnet [71].

The following result tells us that a subnet describes a subtheory if and only if the relative commutant of the subtheory in its extension is trivial; this ensures that the subtheory contains the total energy-momentum tensor of the full theory, since otherwise the relative commutant is a QFT of its own which commutes with the given subtheory [35].

**Proposition 4.1.1.** [71] *Let  $\mathcal{A}$  be a conformal net and  $\mathcal{B} \subset \mathcal{A}$  a conformal subnet. Then  $\mathcal{B}$  contains  $Vir_{\mathcal{A}}$ , namely it is intermediate  $Vir_{\mathcal{A}} \subset \mathcal{B} \subset \mathcal{A}$ , if and only if the inclusion  $\mathcal{B} \subset \mathcal{A}$  is irreducible. Moreover, the condition  $Vir_{\mathcal{A}} \subset \mathcal{B} \subset \mathcal{A}$  is equivalent to  $Vir_{\mathcal{B}} = Vir_{\mathcal{A}}$ .*

The chiral Ising model we have characterized in the previous chapter is the Virasoro net with  $c = 1/2$ , i.e.  $Vir_{1/2}$ .

Two conformal nets  $\mathcal{A}_+$  and  $\mathcal{A}_-$  can be combined to generate a two-dimensional net. In particular, we define

$$\mathcal{A}_{2D}(\mathcal{O}) := \mathcal{A}_+(I) \otimes \mathcal{A}_-(J) \quad I, J \in \mathcal{I}$$

The irreducible localized endomorphisms of  $\mathcal{A}_{2D}$  are the tensor products  $\rho = \rho^+ \otimes \rho^-$  of the chiral endomorphisms, and, in view of the convention (3.12), the two-dimensional braidings are given by the chiral ones by

$$\epsilon_{\rho, \sigma}^+ = \epsilon_{\rho^+, \sigma^+}^+ \otimes \epsilon_{\rho^-, \sigma^-}^- = \epsilon_{\rho^+, \sigma^+} \otimes \epsilon_{\sigma^-, \rho^-}^\dagger \quad \epsilon_{\rho, \sigma}^- = (\epsilon_{\sigma, \rho}^+)^\dagger = \epsilon_{\rho^+, \sigma^+}^- \otimes \epsilon_{\rho^-, \sigma^-}^+$$

The corresponding DHR category is the completion under direct sums<sup>2</sup> of the tensor product of categories  $\mathcal{C}^{DHR}(\mathcal{A}_+) \otimes \mathcal{C}^{DHR}(\mathcal{A}_-)^{opp}$ , denoted as

$$\mathcal{C}^{DHR}(\mathcal{A}_{2D}) := \mathcal{C}^{DHR}(\mathcal{A}_+) \boxtimes \mathcal{C}^{DHR}(\mathcal{A}_-)^{opp}$$

where  $\mathcal{C}^{opp}$  is the tensor category equipped with the opposite braiding  $\epsilon_{\rho^-, \sigma^-}^-$ .

This new two-dimensional net is, however, insufficient to describe an actual two-dimensional CFT, since we have just tensored two chiral theories without specifying any compatibility condition between them. It turns out that additional information is required to extend the net to a larger one that describes a specific theory [33]; such information is encoded in what is called Q-system in the algebraic literature, which corresponds to a Frobenius algebra over the DHR category. The extended net will contain new operators, called *charged fields*, that are the algebraic counterpart of the sector generating primary fields in CFT

## 4.2 Q-systems and extensions

We now describe a general mechanism to extend a net of observables in order to describe a larger theory. We will start by showing how this works for general von Neumann factors and then show the application in quantum field theory.

**Definition 4.2.1.** Let  $\mathcal{N}$  and  $\mathcal{M}$  be two type III factors. A pair of homomorphisms  $\varphi : \mathcal{N} \rightarrow \mathcal{M}$  and  $\bar{\varphi} : \mathcal{M} \rightarrow \mathcal{N}$  is called *conjugate* if there are intertwiners  $\mathcal{N} \ni W \in \text{Hom}(\text{id}_{\mathcal{N}}, \bar{\varphi}\varphi)$  and  $\mathcal{M} \ni V \in \text{Hom}(\text{id}_{\mathcal{M}}, \varphi\bar{\varphi})$  satisfying the conjugacy relations:

$$V^\dagger \varphi(W) = \mathbf{1}_{\mathcal{M}}, \quad W^\dagger \bar{\varphi}(V) = \mathbf{1}_{\mathcal{N}}$$

---

<sup>2</sup>This operation is called *Deligne product* [57].

One may normalize the pair such that  $W^\dagger W = d \mathbf{1}_{\mathcal{N}}$  and  $V^\dagger V = d \mathbf{1}_{\mathcal{M}}$ . Then the infimum of  $d$  among all solutions to the normalized conjugacy relations is denoted as  $\dim(\varphi) = \dim(\bar{\varphi})$ . If a solution of the conjugacy relations exists at all, then there exist solutions taking the infimum value. Such solutions are called *standard pairs*.

Note that for DHR endomorphisms, the definition of conjugate given in 3.4.6 coincide with this last definition [38].

Moreover, the above definition of dimension for homomorphisms generalizes the previous one, i.e. the one of equation (3.7). Indeed, one can show that if  $\varphi \in \text{End}(\mathcal{M})$ , then  $\dim(\varphi) = [\mathcal{M} : \varphi(\mathcal{M})]^{1/2}$ . In particular, if  $\mathcal{N} \subset \mathcal{M}$  is a subfactor and  $\iota : \mathcal{N} \rightarrow \mathcal{M}$  is the embedding homomorphism, then  $\dim(\iota) = [\mathcal{M} : \mathcal{N}]^{1/2}$  [35].

Standard pairs give rise to bijective maps [35],

$$\text{Hom}(\varphi_1, \varphi_2 \varphi_3) \rightarrow \text{Hom}(\bar{\varphi}_2 \varphi_1, \varphi_3) \quad \text{Hom}(\varphi_1, \varphi_2 \varphi_3) \rightarrow \text{Hom}(\varphi_1 \bar{\varphi}_3, \varphi_2),$$

called Frobenius reciprocity relations, which will be useful later on.

Finally, note that the conjugacy relations are the snake identities that also the evaluation/coevaluation must satisfy in a rigid category. For this reason, standard pairs, which are properly normalized, can be identified with the latter.

Given a subfactor  $\mathcal{N} \subset \mathcal{M}$  of a von Neumann algebra  $\mathcal{M}$ , there are two distinct homomorphisms.

**Definition 4.2.2.** Suppose that  $\mathcal{N} \subset \mathcal{M}$  is a subfactor and let  $\iota : \mathcal{N} \rightarrow \mathcal{M}$  be the embedding homomorphism. The following are called *canonical* and *dual canonical* endomorphisms respectively:

$$\Gamma = \iota \circ \bar{\iota} : \mathcal{M} \rightarrow \mathcal{M} \quad \Theta = \bar{\iota} \circ \iota : \mathcal{N} \rightarrow \mathcal{N}$$

**Proposition 4.2.1.** Let  $W \in \text{Hom}(\text{id}_{\mathcal{N}}, \Theta)$  and  $X := \bar{\iota}(V) \in \text{Hom}(\Theta, \Theta^2)$ , where  $W$  and  $V$  are the intertwiners defined in definition 4.2.1. Moreover they satisfy the following relations:

1. *Unit property:*  $W^\dagger X = \Theta(W^\dagger)X = \mathbf{1}_{\mathcal{N}}$ ;
2. *Associativity:*  $XX = \Theta(X)X$ ;
3. *Frobenius property:*  $\Theta(X^\dagger)X = XX^\dagger = X^\dagger\Theta(X)$ ;
4. *Specialness:*  $W^\dagger W = X^\dagger X = d \mathbf{1}_{\mathcal{N}}$

where  $d = \dim(\iota) = \sqrt{\dim(\Theta)}$ .

*Proof.* First we show that  $X := \bar{\iota}(V) \in \text{Hom}(\Theta, \Theta^2)$ . By definition of  $\Gamma$ ,  $\Gamma(A)V = VA$ ,  $\forall A \in \mathcal{M}$ . Therefore, if we apply  $\bar{\iota}$  to both sides we get:

$$\bar{\iota}(\Gamma(A))\bar{\iota}(V) = \bar{\iota}(V)\bar{\iota}(A) \tag{4.1}$$

However  $\bar{\iota}(\Gamma(A)) = \bar{\iota}(\iota\bar{\iota}(A)) = \Theta(\bar{\iota}(A))$ . Therefore (4.1) becomes:

$$\Theta(\bar{\iota}(A))\bar{\iota}(V) = \bar{\iota}(V)\bar{\iota}(A) \Rightarrow \Theta(\Theta(B))X = X\Theta(B)$$

where we have applied the relation to  $A = \iota(B)$ ,  $B \in \mathcal{N}$ .

Now we prove the remaining properties:

1.  $W^\dagger X = \text{id}_{\mathcal{N}}$  follows directly from the second conjugacy relation of standard pairs. Now, using the (adjoint) of the other conjugacy equation:  $\Theta(W^\dagger)X = \bar{\iota}(W^\dagger)\bar{\iota}(V) = \bar{\iota}(\iota(W^\dagger)V) = \bar{\iota}(\mathbf{1}_{\mathcal{M}}) = \mathbf{1}_{\mathcal{N}}$ ;
2. Using  $V \in \text{Hom}(\text{id}_{\mathcal{M}}, \bar{\iota})$ :  $XX = \bar{\iota}(V)\bar{\iota}(V) = \bar{\iota}(VV) = \bar{\iota}(\iota\bar{\iota}(V)) = \bar{\iota}\bar{\iota}(V)\bar{\iota}(V) = \Theta(X)X$ ;
3. Starting from  $XX^\dagger = \bar{\iota}(V)\bar{\iota}(V^\dagger) = \bar{\iota}(VV^\dagger)$ . Now by using the fact that  $V^\dagger \in \text{Hom}(\bar{\iota}, \text{id}_{\mathcal{M}})$ , i.e.  $V^\dagger\bar{\iota}(A) = AV^\dagger \forall A \in \mathcal{M}$ , we find  $\bar{\iota}(VV^\dagger) = \bar{\iota}(V^\dagger\bar{\iota}(V)) = \bar{\iota}(V^\dagger)\bar{\iota}\bar{\iota}(V) = X^\dagger\Theta(X)$ . On the other hand, using again  $V \in \text{Hom}(\text{id}_{\mathcal{M}}, \bar{\iota})$ :  $\bar{\iota}(VV^\dagger) = \bar{\iota}(\iota\bar{\iota}(V^\dagger)V) = \Theta(X^\dagger)X$ ;

4. Follows from the normalization of standard pairs.

□

We will give for these algebraic properties a more intuitive graphical representation in the next section, where we give the categorical description.

**Definition 4.2.3.** The triple

$$Q = (\Theta, W, X) \quad (\Theta \in \text{End}(N), W \in \text{Hom}(\text{id}_N, \Theta), X \in \text{Hom}(\Theta, \Theta^2))$$

satisfying the relations of proposition 4.2.1, is called *Q-system* associated to the subfactor  $\mathcal{N} \subset \mathcal{M}$ , and the number  $d = \dim(\iota) = \sqrt{\dim(\Theta)}$  is called the dimension  $d_Q$  of the Q-system. The Q-system is called *irreducible* if  $\dim \text{Hom}(\text{id}_N, \Theta) = 1$ .

Actually, the Q-system can be defined only referring to the subfactor  $\mathcal{N}$ . In this case we say that the triple  $Q = (\Theta, W, X)$ , defined as above, is a Q-system in the tensor category  $\text{End}(\mathcal{N})$  of endomorphisms of  $\mathcal{N}$ . We denote the subcategory of endomorphisms of finite dimension as  $\text{End}_0(\mathcal{N})$ .

The main result we are going to use tells us that a Q-system is exactly the information needed to reconstruct  $\mathcal{M}$  starting from the subfactor  $\mathcal{N}$ .

**Theorem 4.2.2.** [70] *Let  $\mathcal{N}$  be a type III factor, and  $Q = (\Theta, W, X)$  a Q-system in  $\text{End}_0(N)$  of dimension  $d_Q$ , such that  $\text{id}_N$  is a subendomorphism<sup>3</sup> of  $\Theta$ , written  $\text{id}_N \prec \Theta$ , with multiplicity 1. Then there is a (unique up to isomorphism) type III factor  $\mathcal{M}$  and an irreducible embedding  $\iota : \mathcal{N} \rightarrow \mathcal{M}$  such that  $Q$  is the Q-system associated with this subfactor. The dimension  $\dim(\iota)$  equals the dimension  $d_Q$ .*

We are not going to give the exact proof which would take too much time. Instead, we show the construction [35]:  $\mathcal{M}$  is the algebra generated by  $\mathcal{N}$  (regarded as a subalgebra written as  $\iota(\mathcal{N}) \subset \mathcal{M}$ ) and a single generator  $V \in \mathcal{M}$ , with the defining relations

$$V^\dagger = \iota(W^\dagger X^\dagger)V, \quad VV = \iota(X)V, \quad \iota(W^\dagger)V = \mathbf{1}_M, \quad V\iota(N) = \iota(\Theta(N))V \quad N \in \mathcal{N} \quad (4.2)$$

Note that the last relation means  $V \in \text{Hom}(\iota, \iota\Theta)$ . Every element  $M \in \mathcal{M}$  is of the form  $\iota(N)V$  with  $N \in \mathcal{N}$  and  $\iota(N)V = 0$  implies  $N = 0$ . The conjugate homomorphism  $\bar{\iota} : \mathcal{M} \rightarrow \mathcal{N}$  is given by:

$$\bar{\iota} : \iota(N)V \mapsto \Theta(N)X \quad (4.3)$$

and  $(W, V)$  is a standard solution of the conjugacy relations. Indeed,  $\bar{\iota}$  defined as above satisfies  $\bar{\iota}\iota = \Theta$ :

$$\iota(N) = \iota(N)\iota(W^\dagger)V = \iota(NW^\dagger)V \mapsto \Theta(NW^\dagger)X = \Theta(N)\Theta(W^\dagger)X = \Theta(N)$$

where we used the unit property of the Q-system and the third defining property of  $V$ .

Since every element of  $\mathcal{M}$  is of the form  $\iota(N)V$ , with  $N \in \mathcal{N}$ , the algebra  $\mathcal{M}$  is finitely generated, the weak topology on  $\mathcal{M}$  is inherited from the weak topology of  $\mathcal{N}$  and  $\mathcal{M}$  is weakly closed [35]. Hence  $\mathcal{M}$  is a von Neumann algebra.

The relative commutant  $\iota(\mathcal{N})' \cap \mathcal{M}$  is given by elements of the form  $\iota(H)V$  with  $H \in \text{Hom}(\Theta, \text{id}_N) = \mathbb{C}W^\dagger$ , by assumption, hence  $\iota(\mathcal{N})' \cap \mathcal{M} = \mathbb{C}\mathbf{1}_M$ . Thus, the inclusion is irreducible, and, since  $\iota(\mathcal{N}) \subset \mathcal{M} \Rightarrow \mathcal{M}' \subset \iota(\mathcal{N})'$ ,  $\mathcal{M}$  is a factor,  $\mathcal{M}' \cap \mathcal{M} = \mathbb{C}\mathbf{1}_M$ . One can show also the type III property.

Theorem 4.2.2 can be generalized by relaxing the condition that  $\text{Hom}(\text{id}_N, \Theta)$  is one-dimensional. In this case, the irreducibility  $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$  will fail, and  $\mathcal{M}$  may even fail to be a factor. On the other hand, if  $\dim \text{Hom}(\text{id}_N, \Theta) = 1$ , then the embedding  $\iota$  constructed from the Q-system is irreducible. This is the reason why Q-systems satisfying this condition are called irreducible.

**Theorem 4.2.3.** [35] *Let  $\mathcal{N}$  be a type III factor, and  $Q = (\Theta, W, X)$  a Q-system in  $\text{End}_0(\mathcal{N})$  of dimension  $d_Q$ . Then there is a (unique up to isomorphism) type III von Neumann algebra  $\mathcal{M}$  and an embedding  $\iota : \mathcal{N} \rightarrow \mathcal{M}$  with conjugate  $\bar{\iota} : \mathcal{M} \rightarrow \mathcal{N}$  such that  $Q$  is the Q-system associated with this conjugate pair. The dimension  $\dim(\iota) = \sqrt{\dim(\Theta)}$  equals the dimension  $d_Q$ .*

<sup>3</sup>Here  $\text{id}_N$  is the identity automorphism of  $\mathcal{N}$  and with  $\text{id}_N \prec \Theta$  we mean that  $\Theta$  is associated to a reducible representation of  $\mathcal{N}$  containing the vacuum representation.

### 4.2.1 Application in quantum field theory

Let  $\mathcal{A}$  be a Haag dual net of local von Neumann algebras  $\mathcal{A}(\mathcal{O})$ , and  $\mathcal{C}^{DHR}(\mathcal{A})$  the category of its DHR endomorphisms. The net may be two-dimensional or chiral; in the latter case “ $\mathcal{O}$ ” is understood to stand for an interval  $I \in \mathcal{I}$  and elements belonging to the chiral algebra are denoted with lowercase letters instead of uppercase letters.

In the AQFT context, a Q-system  $(\Theta, W, X)$  is built from a DHR endomorphism  $\Theta$ . Then the first link with subfactor theory comes from the inclusion of type III subfactors  $\Theta(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$ . Then a result analogue to theorem 4.2.2 associates to the Q-system an extension  $\mathcal{A} \subset \mathcal{B}$ , so that the local inclusions  $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$  provide a second realization of the same subfactor-theoretic structure.

**Definition 4.2.4.** We say that the triple  $(\Theta, W, X)$  is a Q-system in  $\mathcal{C}_0^{DHR}(\mathcal{A})$ , the category of DHR endomorphisms with finite dimension, if it is a Q-system in  $\text{End}(\mathcal{A}(\mathcal{O}))$  for some  $\mathcal{O} \subset \mathbb{M}^2$ , i.e.  $\Theta$  is a DHR endomorphism of  $\mathcal{A}$  localized in  $\mathcal{O}$  and the elements  $W$  and  $X$  belong to  $\mathcal{A}(\mathcal{O})$ .

Note that with the assumptions of section 3.5,  $\mathcal{C}_0^{DHR}(\mathcal{A}) = \mathcal{C}^{DHR}(\mathcal{A})$ .

Before giving the QFT version of theorem 4.2.2 we introduce the following notion:

**Definition 4.2.5.** Let  $\mathcal{N} \subset \mathcal{M}$  be two von Neumann algebras with common unit. A *conditional expectation*  $\mu : \mathcal{M} \rightarrow \mathcal{N}$  of  $\mathcal{M}$  into  $\mathcal{N}$  is a completely positive<sup>4</sup> unit preserving (or unital) map with the bimodule property:

$$\mu(N_1 M N_2) = N_1 \mu(M) N_2 \quad \forall N_1, N_2 \in \mathcal{N}, M \in \mathcal{M}$$

Now we are ready to state the theorem:

**Theorem 4.2.4.** [38] Let  $\mathcal{A}$  be a (conformal) QFT, and  $Q = (\Theta, W, X)$  an irreducible Q-system in  $\mathcal{C}_0^{DHR}(\mathcal{A})$  of dimension  $d_Q$ . Then there is a (unique up to isomorphism) QFT  $\mathcal{B}$  and an irreducible embedding  $\iota : \mathcal{A} \rightarrow \mathcal{B}$  such that  $Q$  is the Q-system associated with this extension. The dimension  $\dim(\iota)$  equals the dimension  $d_Q$ .  $\mathcal{B}$  is relatively local w.r.t.  $\mathcal{A}$ , and  $\mathcal{B}$  is local iff

$$\epsilon_{\Theta, \Theta} X = X$$

holds; in this case the Q-system is said to be “commutative”. There is a conditional expectation  $\mu : \mathcal{B} \rightarrow \mathcal{A}$  that extends the vacuum state  $\omega_0$  on  $\mathcal{A}$  to a vacuum state  $\omega^0 := \omega_0 \circ \mu$  on  $\mathcal{B}$ , such that its GNS representation, when restricted to  $\mathcal{A}$ , is equivalent to  $\pi_\Theta$ .

*Proof.* We sketch the construction as we have done for the general case, but we will be a bit more precise this time.

The construction is the following [35]. We construct an algebra  $\mathcal{B}$  by adding one generator  $V$  satisfying the relations (4.2). Then, again, every element of  $\mathcal{B}$  is of the form  $\iota(A)V$  with  $A \in \mathcal{A}$  and  $\iota(A)V = 0 \Rightarrow A = 0$ .

The local subalgebras of  $\mathcal{B}$  are defined as

$$\mathcal{B}(\mathcal{O}) := \iota(\mathcal{A}(\mathcal{O})) \hat{V}$$

where  $\hat{V} = \iota(U)V$  with  $U \in \mathcal{A}$  is a unitary such that  $\hat{\Theta} = \text{Ad}_U \Theta$  is localized in  $\mathcal{O}$ .

In principle we know that  $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}$ . Now we show that indeed  $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$ . Because  $\hat{W} = UW \in \text{Hom}(\text{id}_{\mathcal{A}}, \hat{\Theta})$  is in  $\mathcal{A}(\mathcal{O})$  and  $\iota(\hat{W}^\dagger)\hat{V} = \iota(W^\dagger)V = \mathbf{1}_{\mathcal{B}}$  we have  $\forall A \in \mathcal{A}(\mathcal{O})$ :

$$\iota(A) = \iota(A)\iota(W^\dagger)V = \iota(A)\iota(\hat{W}^\dagger)\hat{V} = \iota(A\hat{W}^\dagger)\hat{V} \in \mathcal{B}(\mathcal{O})$$

Hence  $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$ . Then  $\mathcal{B}(\mathcal{O})$  are  $*$ -algebras. Indeed, using  $VV = \iota(X)V$ , we have

$$\hat{V}^2 = \iota(U)V\iota(U)V = \iota(U)\iota(\Theta(U))VV = \iota(U\Theta(U)X)V = \iota(U\Theta(U)XU^\dagger)\hat{V}$$

---

<sup>4</sup>See appendix C for the definition.

where we have used the last and the second properties of equation (4.2). From the definitions it is easy to see that  $U\Theta(U)XU^\dagger \in \text{Hom}(\hat{\Theta}, \hat{\Theta}^2) \subset \mathcal{A}(\mathcal{O})$  meaning that  $\hat{V}^2 \in \mathcal{B}(\mathcal{O})$ . This implies that products of elements are well defined, i.e.

$$\iota(A_1)\hat{V}\iota(A_2)\hat{V} = \iota(A_1U\Theta(A_2)U^\dagger)\hat{V}^2 \quad A_1, A_2 \in \mathcal{A}(\mathcal{O})$$

is of the form  $\iota(A)\hat{V}$ . Similarly, using the first relation of equation (4.2), one shows that  $\hat{V}^\dagger = \iota(W^\dagger X^\dagger \Theta(U^\dagger)U^\dagger)\hat{V}$ , with  $U\Theta(U)XW \in \text{Hom}(\text{id}, \hat{\Theta}^2) \subset \mathcal{A}(\mathcal{O})$ , which allows us to write  $(\iota(A)\hat{V})^\dagger$  as an element of  $\mathcal{B}(\mathcal{O})$ .

Then we can see that  $\mathcal{A}(\mathcal{O})$  is relatively local to  $\mathcal{B}(\mathcal{O})$ :

$$\iota(U)V\iota(A) = \iota(U\Theta(A))V = \iota(\hat{\Theta}(A))\hat{V} = \iota(A)\hat{V}$$

whenever  $A \in \mathcal{A}(\mathcal{O}')$ . Moreover,  $\mathcal{B}$  is itself local if  $\iota(U_i)V$  localized in spacelike separated  $\mathcal{O}_i$  commute, which turns out to be equivalent to the condition  $\epsilon_{\Theta, \Theta}X = X$ ; indeed on one side we have:

$$\iota(U_1)V\iota(U_2)V = \iota(U_1\Theta(U_2))VV = \iota(U_1\Theta(U_2)X)V$$

on the other side:

$$\iota(U_2)V\iota(U_1)V = \iota(U_2\Theta(U_1))VV = \iota(U_2\Theta(U_1)X)V$$

imposing the equality:

$$\iota(U_1\Theta(U_2)X)V = \iota(U_2\Theta(U_1)X)V \Rightarrow \iota((U_1\Theta(U_2) - U_2\Theta(U_1))X)V = 0 \Rightarrow \iota(U_1\Theta(U_2)(1 - \epsilon_{\Theta, \Theta})X)V = 0$$

This implies  $\epsilon_{\Theta, \Theta}X = X$  since  $U_1\Theta(U_2)$  is unitary. Finally we define

$$\mu(B) = d_Q^{-1} W^\dagger \bar{\iota}(B)W \quad B \in \mathcal{B} \quad (4.4)$$

where  $\bar{\iota}$  is defined as in (4.3). The fact that this is a completely positive map follows from *Stinespring theorem* (see appendix C); modularity is shown by direct computation,

$$\begin{aligned} \mu(\iota(A_1)B\iota(A_2)) &= d_Q^{-1} W^\dagger \bar{\iota}(\iota(A_1)B\iota(A_2))W = d_Q^{-1} W^\dagger \Theta(A_1)\bar{\iota}(B)\Theta(A_2)W = A_1(d_Q^{-1} W^\dagger \bar{\iota}(B)W)A_2 = \\ &= A_1\mu(B)A_2, \end{aligned}$$

where we used the fact that  $W \in \text{Hom}(\text{id}_A, \Theta)$ . We can also show that  $\mu$  respects the local structure, i.e.  $\mu(\mathcal{B}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$ :

$$\begin{aligned} \mu(\iota(A)\hat{V}) &= d_Q^{-1} W^\dagger \bar{\iota}(\iota(A)UV)W = d_Q^{-1} W^\dagger \Theta(AU)XW = d_Q^{-1} AUW^\dagger XW = d_Q^{-1} AUW = \\ &= d_Q^{-1} A\hat{W} \in \mathcal{A}(\mathcal{O}), \quad A \in \mathcal{A}(\mathcal{O}) \end{aligned}$$

With the help of  $\mu$  one can show that the vacuum state  $\omega_0$  of  $\mathcal{A}$  extends to a vacuum state  $\omega^0 := \omega_0 \circ \mu$  of  $\mathcal{B}$ . The GNS representation of  $\omega^0$ , when restricted to  $\mathcal{A}$ , is unitarily equivalent to the representation  $\pi_\Theta = \pi_0 \circ \Theta$  given by the DHR endomorphism  $\Theta$  [35, 38].  $\square$

This theorem was initially proven in [38] assuming that  $\dim \text{Hom}(\text{id}_A, \Theta) = 1$ . Again, the theorem can be generalized; this will be useful when dealing with defects.

**Lemma 4.2.5.** [35] *With the assumptions of theorem 4.2.4, admitting  $\text{id} \prec \Theta$  with multiplicity  $> 1$ , the relative commutant is given by  $\iota(\mathcal{A})' \cap \mathcal{B} = \iota(\text{Hom}(\Theta, \text{id}_A))V$ . The center of  $\mathcal{B}$  is given by  $\iota(H)V$ ,  $H \in \text{Hom}(\Theta, \text{id}_A)$  satisfying*

$$HX = \Theta(H)X \quad (4.5)$$

*Moreover  $\iota(H)V$  is self-adjoint if and only if  $H^\dagger = \Theta(H)XW$  and idempotent if and only if  $H\Theta(H)X = HHX = H$ .*

*Proof.* By direct computation, pick any  $A \in \mathcal{A}$ :

$$\begin{aligned}\iota(H)V\iota(A)V &= \iota(H\Theta(A))VV = \iota(AH)VV = \iota(AHX)V = \iota(A)\iota(\Theta(H)X)V = \iota(A)\iota(\Theta(H))VV \\ &= \iota(A)V\iota(H)V\end{aligned}$$

where we used the second and fourth relations of (4.2). Then:

$$(\iota(H)V)^\dagger = V^\dagger\iota(H^\dagger) = \iota(W^\dagger X^\dagger)V\iota(H^\dagger) = \iota(W^\dagger X^\dagger\Theta(H^\dagger))V = \iota(H)V \Leftrightarrow H = \Theta(H)XW$$

For the last relation:

$$\iota(H)V\iota(H)V = \iota(H\Theta(H)X)V = \iota(H)V \Leftrightarrow H\Theta(H)X = HHX = H$$

□

## 4.2.2 Charged fields

The abstract procedure of extending a net via a Q-system has a more intuitive interpretation. If  $\rho \prec \Theta$  is an irreducible subrepresentation localized in  $\mathcal{O}$ , then  $\iota(\text{Hom}(\rho, \Theta))^\dagger V$  is a Hilbert space in  $\mathcal{B}(\mathcal{O})$ , and its elements  $\Psi_\rho = \iota(W_\rho)^\dagger V \in \mathcal{B}(\mathcal{O})$  with  $W_\rho \in \text{Hom}(\rho, \Theta)$  satisfy the commutation relation:

$$\Psi_\rho A = \rho(A)\Psi_\rho \quad \forall A \in \mathcal{A}$$

intertwining the vacuum representation with the representation  $\rho$ . Because  $\rho$  is localized in  $\mathcal{O}$ , it follows that  $\Psi_\rho$  commutes with  $\mathcal{A}(\mathcal{O}')$ , i.e. the extension is relatively local with respect to  $\mathcal{A}$ .

Since  $\mu(\Psi_\rho\Psi_\rho^\dagger) \in \text{Hom}(\rho, \rho) = \mathbb{C}\mathbf{1}$ , one may normalize  $\Psi_\rho$  such that  $\mu(\Psi_\rho\Psi_\rho^\dagger) = \mathbf{1}$ . Then, as a consequence of theorem 4.2.4,

$$\omega_\rho(A) = \omega^0(\Psi_\rho A\Psi_\rho^\dagger) = \omega_0 \circ \mu(\Psi_\rho A\Psi_\rho^\dagger)$$

is a state on  $\mathcal{A}$ . Its GNS representation is equivalent to the representation  $\pi_\rho = \pi_0 \circ \rho$ , justifying the interpretation of  $\Psi_\rho$  as *charged field operators*.<sup>5</sup>

By choosing a complete irreducible decomposition of  $\Theta$  by isometries  $W_\rho$  (possibly with multiplicities) such that  $\sum_\rho W_\rho W_\rho^\dagger = \mathbf{1}$ , every element of  $\mathcal{B}$  has a unique representation in terms of the charged field:

$$B = \iota(A)V = \sum_\rho \iota(AW_\rho W_\rho^\dagger)V = \sum_\rho \iota(A_\rho) \Psi_\rho$$

where  $A_\rho = AW_\rho$ . Hence, we can see an extension  $\mathcal{B}$  as the theory generated by  $\mathcal{A}$  with the addition of charged fields.

These bounded operators may therefore be regarded as the analogue of the primary fields in AQFT, such that a general (“descendant”) field is obtained by left multiplication (“OPE”) of  $\Psi_\rho$  with some element of  $\mathcal{A}$ . The formulae given in (4.2) for the product  $V^2$  and the adjoint  $V^\dagger$  determine the multiplication and adjoint of the charged field operators  $\Psi_\rho$ . Therefore, the Q-system determines the algebraic product of charged field operators: it specifies which fusion channels appear and with which intertwiner coefficients. However, these data should not be confused with the full Euclidean OPE coefficients, which also depend on the normalization of local fields and on the coordinate dependence of the operator product.

In four dimensions, the Doplicher–Roberts reconstruction theorem [72] states that for every DHR endomorphism  $\rho$  there is an orthonormal basis of charged field operators satisfying  $\sum_i \Psi_\rho^i (\Psi_\rho^i)^\dagger = \mathbf{1}$ , which implement  $\rho$  by the formula  $\rho(A) = \sum_i \Psi_\rho^i A (\Psi_\rho^i)^\dagger$ .

<sup>5</sup>Usually, in QFT, the term *charged field operator* refers to a field defined in the whole Hilbert space and intertwining between several charged superselection sectors. In the present context, however, charged fields are used in a more restricted sense: they intertwine the vacuum sector with a given charged sector.

In low dimensions, the non-triviality of the braiding poses an obstruction, so that the range projections of  $\iota(\text{Hom}(\rho, \Theta))^\dagger V$  in general do not exhaust the unit operator [35, 38]; but one has still the implementation via the conditional expectation:

$$\rho(A) = \sum_i \mu(\Psi_\rho^i A (\Psi_\rho^i)^\dagger) \quad \text{where} \quad \sum_i \mu(\Psi_\rho^i (\Psi_\rho^i)^\dagger) = \mathbf{1}$$

This is the case every time  $\rho \prec \Theta$  has non integer dimension [38]; in particular this happens with  $\sigma$  in the Ising model.

There are two interesting ways to normalize charged fields.

**Proposition 4.2.6.** *The following relations hold:*

- $\mu(\Psi_\rho \Psi_\rho^\dagger) = \mathbf{1}$  iff  $W_\rho^\dagger W_\rho = d_Q$ ;
- Assuming  $\iota$  is irreducible, then  $\Psi_\rho^\dagger \Psi_\rho = \mathbf{1}$ , i.e. charged fields are isometries, iff  $W_\rho^\dagger W_\rho = d_Q / \dim(\rho)$

*Proof.* 1. by direct computation, using the definitions of  $\mu$  and  $\bar{\iota}$ :

$$\begin{aligned} \mu(\Psi_\rho \Psi_\rho^\dagger) &= d_Q^{-1} W^\dagger \bar{\iota}(\Psi_\rho \Psi_\rho^\dagger) W = d_Q^{-1} W^\dagger \Theta(W_\rho^\dagger) X X^\dagger \Theta(W_\rho) W = d_Q^{-1} W_\rho^\dagger W^\dagger X X^\dagger W W_\rho \\ &= d_Q^{-1} W_\rho^\dagger W_\rho \end{aligned}$$

where we have also used  $X^\dagger W = \mathbf{1}$  and the fact that  $W \in \text{Hom}(\text{id}, \Theta)$ .

2. Since  $\Psi_\rho^\dagger \Psi_\rho \in \text{Hom}(\text{id}_\mathcal{A}, \text{id}_\mathcal{A})$  we have  $\Psi_\rho^\dagger \Psi_\rho \in \mathcal{A}(\mathcal{O})' \cap \mathcal{B}(\mathcal{O})$ . Hence in the irreducible case  $\Psi_\rho^\dagger \Psi_\rho$  is automatically multiple of  $\mathbf{1}$ . In this case  $\Psi_\rho^\dagger \Psi_\rho$  equals its image under  $\mu$ . Now we compute the normalization similarly to the previous point:

$$\mu(\Psi_\rho^\dagger \Psi_\rho) = d_Q^{-1} W^\dagger X^\dagger \Theta(W_\rho W_\rho^\dagger) X W$$

one can show [35], by exploiting properties of standard pairs, that this equals  $d_Q^{-1} \cdot \dim(\rho) W_\rho^\dagger W_\rho$ .<sup>6</sup>

□

This proposition can also be generalized in the case charged fields have multiplicities, i.e. in the case of  $\dim \text{Hom}(\rho, \Theta) > 1$ .

To conclude, note that charged fields  $\Psi_\rho$  are localized in  $\mathcal{B}(\mathcal{O})$  if  $\rho$  is localized in  $\mathcal{O}$ . They can be transported to any other region  $\tilde{\mathcal{O}}$  with the help of unitary intertwiners  $U \in \text{Hom}(\rho, \tilde{\rho})$ , where  $\tilde{\rho}$  is localized in  $\tilde{\mathcal{O}}$ :

$$\Psi_{\tilde{\rho}} = \iota(U) \Psi_\rho \quad (4.6)$$

### 4.2.3 The canonical extension

Consider a two-dimensional net built from two (possibly different) chiral nets  $\mathcal{A}_{2D} = \mathcal{A}_+ \otimes \mathcal{A}_-$  with  $\mathcal{C}^{DHR}(\mathcal{A}_{2D}) := \mathcal{C}^{DHR}(\mathcal{A}_+) \boxtimes \mathcal{C}^{DHR}(\mathcal{A}_-)^{opp}$  as described previously. Since theorem 4.2.4 is very general, we can apply it also in this case.

A Q-system  $(\Theta, W, X)$  in  $\mathcal{C}^{DHR}(\mathcal{A}_{2D})$  is characterized by

$$\Theta \simeq \bigoplus_{\sigma, \rho} Z_{\sigma, \rho} \sigma \otimes \rho \quad (4.7)$$

The multiplicity matrix  $Z$ , coupling the left- and the right-moving representations, is also called the *coupling matrix*. From this, we can construct the extension  $\mathcal{A}_{2D} \subset \mathcal{B}_{2D} = \iota(\mathcal{A}_{2D})V$  generated by  $\mathcal{A}_{2D}$  and the charged fields  $\Psi_{\sigma \otimes \rho}^i = \iota(W_{\sigma \otimes \rho}^i)^\dagger V$ , with  $i = 1, \dots, Z_{\sigma, \rho}$ , satisfying:

$$\Psi_{\sigma \otimes \rho}^i A = (\sigma \otimes \rho)(A) \Psi_{\sigma \otimes \rho}^i \quad \forall A \in \mathcal{A}_{2D}$$

<sup>6</sup>The omitted computation is actually identical to another one done in chapter 6, namely equation (6.4), and involves the cyclic property of traces.

where  $\sigma \otimes \rho \prec \Theta$  are irreducible subrepresentations.

In the case  $\mathcal{A}_+ = \mathcal{A}_-$ , there is a distinguished extension of  $\mathcal{A}_{2D} = \mathcal{A} \otimes \mathcal{A}$  whose coupling matrix is the conjugation matrix.

**Theorem 4.2.7.** [38] *If  $\mathcal{C}_0^{DHR}(\mathcal{A})$  has finitely many inequivalent irreducible objects  $\rho$ , then there is a canonical Q-system  $Q_{can}$  in  $\mathcal{C}_0^{DHR}(\mathcal{A} \otimes \mathcal{A})$  with*

$$\Theta \simeq \bigoplus_{\rho} \rho \otimes \bar{\rho}$$

where the sum extends over the irreducible sectors of  $\mathcal{A}$ . More precisely, choosing some isometries  $T_{\rho} \in \text{Hom}(\rho \otimes \bar{\rho}, \Theta)$ , one has  $W = d_Q^{1/2} T_{id}$  and

$$X = d_Q^{-1/2} \sum_{\rho, \sigma, \tau} \left( \frac{d_{\rho} d_{\sigma}}{d_{\tau}} \right)^{1/2} \Theta(T_{\sigma}) T_{\rho} \cdot \left( \sum_a t_a \otimes t_{\bar{a}} \right) \cdot T_{\tau}^{\dagger}$$

where the second sum extends over an orthonormal basis of isometries  $t_a \in \text{Hom}(\tau, \rho\sigma)$  and  $t_{\bar{a}} \in \text{Hom}(\bar{\tau}, \bar{\rho}\bar{\sigma})$ . The dimension is  $d_Q = \sqrt{\dim(\Theta_{can})} = \sqrt{\sum_{\rho} \dim(\rho)^2}$ .

According to proposition 4.2.6, the charged operators  $\Psi_{\rho \otimes \bar{\rho}}$  of the canonical extension are isometries if  $W_{\rho \otimes \bar{\rho}} \in \text{Hom}(\rho \otimes \bar{\rho}, \Theta)$  are normalized as  $W_{\rho \otimes \bar{\rho}}^{\dagger} W_{\rho \otimes \bar{\rho}} = \sqrt{\dim(\Theta)} / \dim(\rho \otimes \bar{\rho}) = \sqrt{\dim(\Theta)} / \dim(\rho)^2$ .

Moreover, the canonical extension is always local:

**Proposition 4.2.8.** [38] *The canonical Q-system satisfies*

$$\epsilon_{\Theta, \Theta} X = X$$

i.e. it is commutative in  $\mathcal{C}_0^{DHR}(\mathcal{A} \otimes \mathcal{A})$ .

Thus, with  $\mathcal{C}_0^{DHR}(\mathcal{A})$  for a chiral QFT  $\mathcal{A}$ , the canonical Q-system describes a local two-dimensional extension  $\mathcal{A} \otimes \mathcal{A} \subset \mathcal{B}_{2D}$  with charged fields  $\Psi_{\rho \otimes \bar{\rho}}$ . In other words, the canonical Q-system gives a prescription to build diagonal CFTs out of the chiral data only.

### 4.3 Q-systems as algebra objects

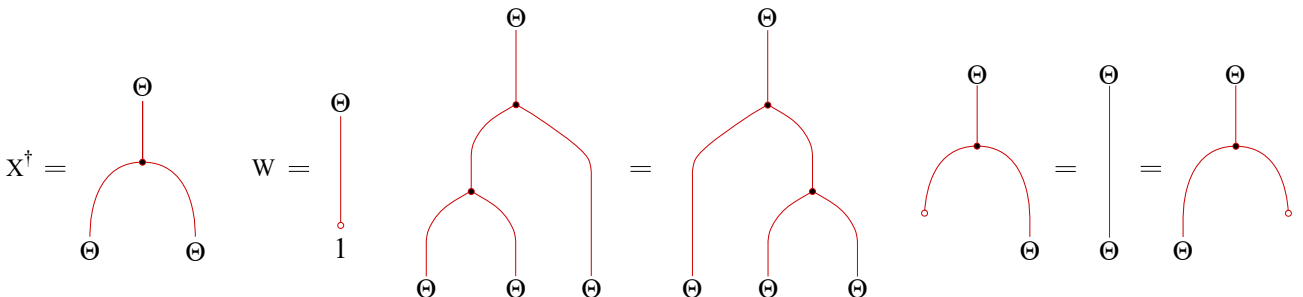
A Q-system is a particular case of a more general notion which can be defined for a tensor category. Nevertheless, we will keep the notation used for Q-systems in order to not create confusion.

**Definition 4.3.1.** An *algebra object*  $Q$  in a tensor category  $\mathcal{C}$  is a triple  $(\Theta, W, X^{\dagger})$ , where  $\Theta$  is an object of  $\mathcal{C}$ ,  $X^{\dagger} \in \text{Hom}(\Theta \otimes \Theta, \Theta)$  is called multiplication and  $W \in \text{Hom}(1, \Theta)$  is called unit, fulfilling:

$$X^{\dagger} \circ (X^{\dagger} \otimes \text{id}_{\Theta}) = X^{\dagger} \circ (\text{id}_{\Theta} \otimes X^{\dagger}) \quad X^{\dagger} \circ (W \otimes \text{id}_{\Theta}) = \text{id}_{\Theta} = X^{\dagger} \circ (\text{id}_{\Theta} \otimes W)$$

Note that with  $X^{\dagger}$  we are not referring to a dagger structure of the category and in principle different symbols should be used. However, since we are going to relate this definition to the one of Q-system, such notation permit an easier identification without introducing new letters.

Pictorially, we can represent the above relations as:

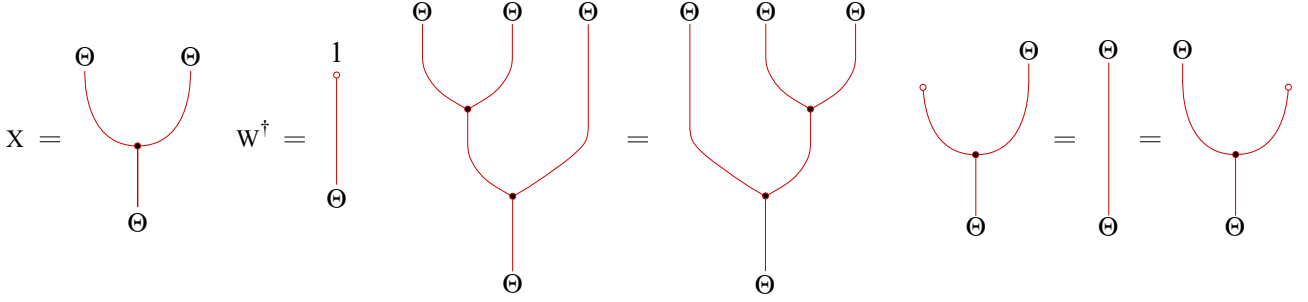


To call  $Q$  an algebra is appropriate because in the particular case that  $\mathcal{C}$  is the category of vector spaces over  $\mathbb{C}$  (or some other field), the prescription reduces to the conventional notion of an algebra [33]. Similarly also the following definitions are extensions to the category setting of the corresponding concepts in algebras over  $\mathbb{C}$ .

**Definition 4.3.2.** A *co-algebra*  $Q$  in a tensor category  $\mathcal{C}$  is a triple  $(\Theta, W^\dagger, X)$  where  $\Theta$  is an object of  $\mathcal{C}$ ,  $X \in \text{Hom}(\Theta, \Theta \otimes \Theta)$  is called comultiplication and  $W^\dagger \in \text{Hom}(\Theta, 1)$  is called counit, fulfilling:

$$(X \otimes \text{id}_\Theta) \circ X = (\text{id}_\Theta \otimes X) \circ X \quad (W^\dagger \otimes \text{id}_\Theta) \circ X = \text{id}_\Theta = (\text{id}_\Theta \otimes W^\dagger) \circ X \quad (4.8)$$

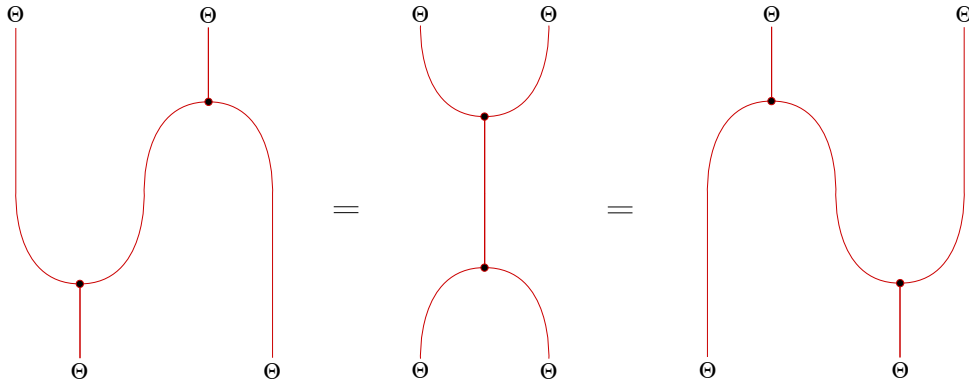
These relations are pictorially represented as follows:



**Definition 4.3.3.** A *Frobenius algebra* in a tensor category is an object in  $\mathcal{C}$  that is both an algebra and a co-algebra and for which the product and coproduct are related by:

$$(\text{id}_\Theta \otimes X^\dagger) \circ (X \otimes \text{id}_\Theta) = X \circ X^\dagger = (X^\dagger \otimes \text{id}_\Theta) \circ (\text{id}_\Theta \otimes X) \quad (4.9)$$

We can represent pictorially the Frobenius property as follows:



There are other additional properties that an algebra can satisfy:

**Definition 4.3.4.** An algebra in a tensor category is said to be:

- *special* if

$$W^\dagger \circ W = \beta_1 \text{id}_1 \quad X^\dagger \circ X = \beta_Q \text{id}_\Theta \quad (4.10)$$

for non-zero numbers  $\beta_1, \beta_Q \in \mathbb{C}$ . Pictorially:

- *symmetric* if the following equality between morphisms holds:

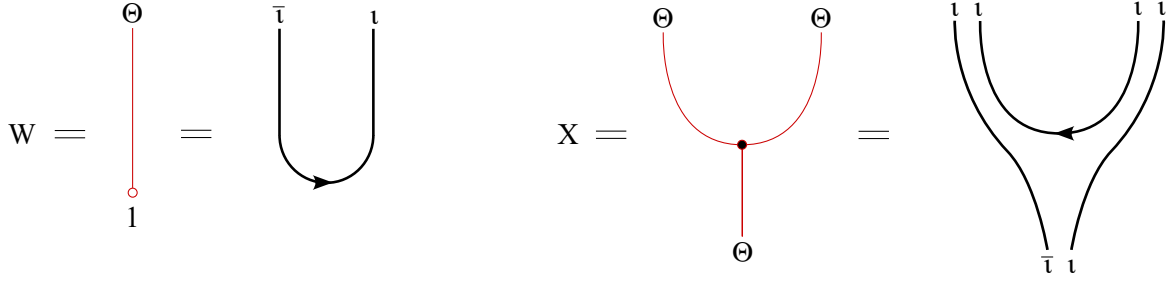
- *commutative* if the category is also braided and  $\epsilon_{\Theta, \Theta} \circ X = X$ . Note that multiplying both sides by  $\epsilon_{\Theta, \Theta}^-$  and exploiting the unitarity of the braiding we get:  $\epsilon_{\Theta, \Theta}^- \epsilon_{\Theta, \Theta}^+ X = \epsilon_{\Theta, \Theta}^- X \Rightarrow X = \epsilon_{\Theta, \Theta}^- X$ . The same relations hold also for  $X^\dagger$ ; hence pictorially:

- *haploid* if  $\dim \text{Hom}(1, Q) = 1$ . One can show that this condition also implies the symmetry property [33]. In the context of Q-systems, this is often referred to as *irreducibility*.

We now see that a Q-system consists of a symmetric special Frobenius algebra over  $\mathcal{C}_0^{DHR}(\mathcal{A})$  with  $\beta_1 = \beta_Q = d_Q$  and where  $\mathcal{A}$  is either a chiral or a two-dimensional net<sup>7</sup>. The symmetry property can be easily proved using the Q-system properties: since  $\Theta = \bar{u}$  is self-dual,  $\Theta = \Theta^\vee$ ; the unit property can be used to remove the "branching"  $W^\dagger$  factors, and the remaining snake diagrams are equal thanks to the Frobenius property. Finally, note that the defining properties of a Q-system in proposition 4.2.1 correspond to equations (4.8), (4.9) and (4.10).

<sup>7</sup>For Q-systems over  $\mathcal{C}_0^{DHR}(\mathcal{A})$  (or more generally  $\text{End}(\mathcal{N})$ ), morphisms are realized by intertwiners belonging to the underlying observable algebra (or to the von Neumann factor  $\mathcal{N}$ ). In this setting, the tensor unit is the identity automorphism of the algebra, and the identity morphisms  $\text{id}_1$  and  $\text{id}_\Theta$  are both implemented by the algebraic unit  $\mathbf{1}_\mathcal{A}$ .

We have introduced a slightly modified graphical notation for the algebra object in order to simplify diagrams. Nevertheless, in some situations it will be useful to revert to the original conventions. For this reason, we now show how  $W$  and  $X$  are represented in the previously defined notation for a Q-system:



The graphical representation makes also easier to see that  $W^\dagger W = X^\dagger X = \dim(\iota)\mathbf{1} = \sqrt{\dim(\Theta)}\mathbf{1}$ . In addition, note that we represent  $W$  as the co-evaluation for  $\iota$ ; this is not a coincidence:  $W$  is part of a standard pair, hence, as we have observed earlier in section 4.2, it is identified with the co-evaluation.

#### 4.4 Constructing the two-dimensional Ising model CFT

The two-dimensional Ising model is the extension of  $\mathcal{A}_{2D} = Vir_{1/2} \otimes Vir_{1/2}$  given by the canonical Q-system. Let  $\psi$  and  $\sigma$  localized in an interval  $I$  and let  $\mathcal{O} = I \times I$ . Then, choose any three isometries  $T_1, T_\psi, T_\sigma \in (Vir_{1/2} \otimes Vir_{1/2})(\mathcal{O})$  satisfying

$$T_i^\dagger T_j = \delta_{ij} \cdot \mathbf{1} \quad \sum_i T_i T_i^\dagger = \mathbf{1} \quad (4.11)$$

where  $i = 1, \psi, \sigma$ , and define:

$$\Theta(A) = T_1 A T_1^\dagger + T_\psi(\psi \otimes \psi)(A) T_\psi^\dagger + T_\sigma(\sigma \otimes \sigma)(A) T_\sigma^\dagger \quad A \in \mathcal{A}_{2D} \quad (4.12)$$

The intertwiners of the canonical Q-system,  $Q_{can} = (\Theta, W, X)$ , are given by proposition 4.2.7. In particular, note that:

$$d_Q = \sqrt{\dim(\Theta)} = \sqrt{\dim(1)^2 + \dim(\psi)^2 + \dim(\sigma)^2} = \sqrt{4} = 2$$

Hence we have  $W := \sqrt{2} \cdot T_1$  and

$$\begin{aligned} X = & \frac{1}{\sqrt{2}} (\Theta(T_1) T_1 T_1^\dagger + \Theta(T_1) T_\psi T_\psi^\dagger + \Theta(T_1) T_\sigma T_\sigma^\dagger + \Theta(T_\psi) T_1 T_1^\dagger + \Theta(T_\sigma) T_1 T_1^\dagger + \\ & + \Theta(T_\psi) T_\psi T_1^\dagger + \Theta(T_\sigma) T_\psi T_\sigma^\dagger + \Theta(T_\psi) T_\sigma (u \otimes u) T_\sigma^\dagger + \sqrt{2} \cdot \Theta(T_\sigma) T_\sigma (r \otimes r) T_1^\dagger + \\ & + \sqrt{2} \cdot \Theta(T_\sigma) T_\sigma (t \otimes t) T_\psi^\dagger) \end{aligned}$$

Properly normalizing as prescribed in proposition 4.2.6, i.e.  $W_{\rho \otimes \rho} = \sqrt{2}/\dim(\rho) \cdot T_\rho$ , one obtains the charged fields:

$$\Psi_{\psi \otimes \psi} = \iota(W_{\psi \otimes \psi}^\dagger) V \quad \Psi_{\sigma \otimes \sigma} = \iota(W_{\sigma \otimes \sigma}^\dagger) V$$

They are, respectively, the algebraic realization of the primary fields  $\epsilon$  and  $\sigma$  of the critical Ising model. By construction they satisfy the following intertwining relations:

$$\Psi_{\psi \otimes \psi} A = (\psi \otimes \psi)(A) \Psi_{\psi \otimes \psi} \quad \Psi_{\sigma \otimes \sigma} A = (\sigma \otimes \sigma)(A) \Psi_{\sigma \otimes \sigma} \quad \forall A \in \mathcal{A}_{2D}$$

Indeed, even though we have chosen the  $T_i$ 's as generic isometries, by writing  $\Theta$  as in equation (4.12), they become automatically intertwiners between  $\Theta$  and its subrepresentations. For example:

$$\begin{aligned} \Psi_{\psi \otimes \psi} A &= \Psi_{\psi \otimes \psi} \iota(A) = \iota(W_{\psi \otimes \psi}^\dagger) V \iota(A) = \iota(W_{\psi \otimes \psi}^\dagger \Theta(A)) V = \sqrt{2} \cdot \iota(T_\psi^\dagger \Theta(A)) V = \\ &= \iota((\psi \otimes \psi)(A)) \sqrt{2} \cdot \iota(T_\psi^\dagger) V = \iota((\psi \otimes \psi)(A)) \Psi_{\psi \otimes \psi} = (\psi \otimes \psi)(A) \Psi_{\psi \otimes \psi} \end{aligned}$$

where we used the properties of equation (4.11).

We can compute the adjoint of the charged fields. Starting from  $\Psi_{\psi \otimes \psi}$ :

$$\Psi_{\psi \otimes \psi}^\dagger = V^\dagger \iota(W_{\psi \otimes \psi}) = \iota(W^\dagger X^\dagger) V \iota(W_{\psi \otimes \psi}) = \iota(W^\dagger X^\dagger \Theta(W_{\psi \otimes \psi})) V = 2 \cdot \iota(T_1^\dagger X^\dagger \Theta(T_\psi))$$

Again, using the properties (4.11), we find:

$$\begin{aligned} \Psi_{\psi \otimes \psi}^\dagger &= 2 \cdot \iota(T_1^\dagger X^\dagger \Theta(T_\psi)) = \frac{2}{\sqrt{2}} \cdot \iota((T_1^\dagger \Theta(T_1^\dagger) + T_\psi^\dagger \Theta(T_\psi^\dagger) + \sqrt{2} \cdot (r \otimes r)^\dagger T_\sigma^\dagger \Theta(T_\sigma^\dagger)) \Theta(T_\psi)) = \\ &= \sqrt{2} \cdot \iota(T_\psi^\dagger) V = \Psi_{\psi \otimes \psi} \end{aligned}$$

where we have first resolved  $T_1^\dagger X^\dagger$ . We can do the same with  $\Psi_{\sigma \otimes \sigma}$ :

$$\Psi_{\sigma \otimes \sigma}^\dagger = \sqrt{2} \cdot \iota(T_1^\dagger X^\dagger \Theta(T_\sigma)) = \iota(\sqrt{2} \cdot (r \otimes r)^\dagger T_\sigma^\dagger) = \sqrt{2} \cdot (r^\dagger \otimes r^\dagger) \Psi_{\sigma \otimes \sigma}$$

Now we can compute the fusion rules:

- $\Psi_{\psi \otimes \psi}^2 = \iota(W_{\psi \otimes \psi}^\dagger) V \iota(W_{\psi \otimes \psi}^\dagger) V = 2 \cdot \iota(T_\psi^\dagger \Theta(T_\psi^\dagger) X) V = \frac{2}{\sqrt{2}} \iota(T_1^\dagger) V = \iota(W^\dagger) V = \mathbf{1}_{\mathcal{A}_{2D}}$ ;
- $\Psi_{\psi \otimes \psi} \Psi_{\sigma \otimes \sigma} = \sqrt{2} \cdot \iota(T_\psi^\dagger \Theta(T_\sigma^\dagger) X) V = \iota(T_\sigma^\dagger) V = \Psi_{\sigma \otimes \sigma}$ ;
- $\Psi_{\sigma \otimes \sigma} \Psi_{\psi \otimes \psi} = \sqrt{2} \cdot \iota(T_\sigma^\dagger \Theta(T_\psi^\dagger) X) V = \iota((u \otimes u) T_\sigma^\dagger) V = (u \otimes u) \Psi_{\sigma \otimes \sigma}$ ;
- $\Psi_{\sigma \otimes \sigma}^2 = \iota(T_\sigma^\dagger \Theta(T_\sigma^\dagger) X) V = \iota((r \otimes r) T_1^\dagger + (t \otimes t) T_\psi^\dagger) = \frac{1}{\sqrt{2}} \cdot ((r \otimes r) + (t \otimes t) \Psi_{\psi \otimes \psi})$

Hence, we recover the exact fusion rules we expect from the Ising model. Moreover, note that all charged fields are isometries; however, as claimed earlier, we do not satisfy  $\Psi_{\sigma \otimes \sigma} \Psi_{\sigma \otimes \sigma}^\dagger = \mathbf{1}$  due to the presence of the term  $(r^\dagger \otimes r^\dagger)$  arising from the non-trivial braiding of  $\sigma$ .

In summary:

$$\begin{aligned} \Psi_{\psi \otimes \psi} A &= (\psi \otimes \psi)(A) \Psi_{\psi \otimes \psi} & \Psi_{\sigma \otimes \sigma} A &= (\sigma \otimes \sigma)(A) \Psi_{\sigma \otimes \sigma} \\ \Psi_{\psi \otimes \psi}^\dagger &= \Psi_{\psi \otimes \psi} & \Psi_{\sigma \otimes \sigma}^\dagger &= \sqrt{2} \cdot (r^\dagger \otimes r^\dagger) \Psi_{\sigma \otimes \sigma} \\ \Psi_{\psi \otimes \psi} \Psi_{\sigma \otimes \sigma} &= \Psi_{\sigma \otimes \sigma} & \Psi_{\sigma \otimes \sigma} \Psi_{\psi \otimes \psi} &= (u \otimes u) \Psi_{\sigma \otimes \sigma} \\ \Psi_{\psi \otimes \psi}^2 &= \mathbf{1}_{\mathcal{A}_{2D}} & \Psi_{\sigma \otimes \sigma}^2 &= \frac{1}{\sqrt{2}} \cdot ((r \otimes r) + (t \otimes t) \Psi_{\psi \otimes \psi}) \end{aligned} \tag{4.13}$$

Finally, we show that these operators are local. This is expected on general grounds, since the canonical Q-system is commutative, namely  $\epsilon_{\Theta, \Theta} X = X$ .

Let  $\sigma \otimes \sigma$  be localized in a doublecone  $\mathcal{O}$ . We can shift the localization thanks to a charge transporter unitary  $U_x \in \text{Hom}((\sigma \otimes \sigma)^{\mathcal{O}_x}, \sigma \otimes \sigma)$  to a doublecone  $\mathcal{O}_x$  centered at the point  $x^\mu \in \mathbb{M}^2$ . According to (4.6), the associated charged field is:

$$\Psi_{\sigma \otimes \sigma}^x \in \mathcal{B}(\mathcal{O}_x)$$

Now, let's repeat the same procedure for a doublecone  $\mathcal{O}_y$  spacelike separated with respect to  $\mathcal{O}$ . Omitting for simplicity the embedding  $\iota$ , we have:

$$\Psi_{\sigma \otimes \sigma}^x \Psi_{\sigma \otimes \sigma}^y = U_x \Psi_{\sigma \otimes \sigma} U_y \Psi_{\sigma \otimes \sigma} = U_x \sigma \otimes \sigma (U_y) \Psi_{\sigma \otimes \sigma} \Psi_{\sigma \otimes \sigma}$$

Where we have used the intertwining relation of  $\Psi_{\sigma \otimes \sigma} \in \text{Hom}(\text{id}, \sigma \otimes \sigma)$ . On the other hand:

$$\Psi_{\sigma \otimes \sigma}^y \Psi_{\sigma \otimes \sigma}^x = U_y \Psi_{\sigma \otimes \sigma} U_x \Psi_{\sigma \otimes \sigma} = U_y \sigma \otimes \sigma (U_x) \Psi_{\sigma \otimes \sigma} \Psi_{\sigma \otimes \sigma}$$

If we impose the equality of both sides we find the condition:

$$\Psi_{\sigma \otimes \sigma}^x \Psi_{\sigma \otimes \sigma}^y = \Psi_{\sigma \otimes \sigma}^y \Psi_{\sigma \otimes \sigma}^x \quad \Leftrightarrow \quad U_x^\dagger \sigma \otimes \sigma (U_y^\dagger) U_y \sigma \otimes \sigma (U_x) \Psi_{\sigma \otimes \sigma} \Psi_{\sigma \otimes \sigma} = \Psi_{\sigma \otimes \sigma} \Psi_{\sigma \otimes \sigma}$$

The operator appearing on the left-hand side is precisely the two-dimensional DHR braiding associated with the exchange of the two localized morphisms. Therefore the condition becomes:

$$\epsilon_{\sigma \otimes \sigma, \sigma \otimes \sigma} \Psi_{\sigma \otimes \sigma}^2 = \Psi_{\sigma \otimes \sigma}^2,$$

i.e. the braiding must preserve the fusion. This relation can be seen as the general condition  $\epsilon_{\Theta, \Theta} X = X$  specialized to the  $\sigma \otimes \sigma$  endomorphism. Note that locality does not require the braiding itself to be trivial. What is required is only that the braiding acts trivially on the specific fusion channel selected by the multiplication  $X$ .

We now prove this condition for the braiding

$$\epsilon_{\sigma \otimes \sigma, \sigma \otimes \sigma}^+ = \epsilon_{\sigma \otimes \sigma, \sigma \otimes \sigma} := \epsilon_{\sigma, \sigma}^+ \otimes \epsilon_{\sigma, \sigma}^- = \epsilon_{\sigma, \sigma} \otimes \epsilon_{\sigma, \sigma}^\dagger,$$

where  $\epsilon_{\sigma, \sigma}^+ = \epsilon_{\sigma, \sigma}$  and  $\epsilon_{\sigma, \sigma}^- = \epsilon_{\sigma, \sigma}^\dagger$  are the chiral braidings and  $\epsilon_{\sigma \otimes \sigma, \sigma \otimes \sigma}^+ = \epsilon_{\sigma \otimes \sigma, \sigma \otimes \sigma}$  and  $\epsilon_{\sigma \otimes \sigma, \sigma \otimes \sigma}^- = \epsilon_{\sigma \otimes \sigma, \sigma \otimes \sigma}^\dagger$  are the corresponding two-dimensional versions. Using the braiding expressions (3.31), the Cuntz relations (3.29) and the products (4.13):

$$\begin{aligned} \epsilon_{\sigma \otimes \sigma, \sigma \otimes \sigma} \Psi_{\sigma \otimes \sigma}^2 &= ((rr^\dagger + itt^\dagger) \otimes (rr^\dagger - itt^\dagger)) \frac{1}{\sqrt{2}} \cdot ((r \otimes r) + (t \otimes t) \Psi_{\psi \otimes \psi}) = \\ &= \frac{1}{\sqrt{2}} [(rr^\dagger + itt^\dagger)r \otimes (rr^\dagger - itt^\dagger)r] + \frac{1}{\sqrt{2}} [(rr^\dagger + itt^\dagger)t \otimes (rr^\dagger - itt^\dagger)t] \Psi_{\psi \otimes \psi} = \\ &= \frac{1}{\sqrt{2}} \cdot ((r \otimes r) + (t \otimes t) \Psi_{\psi \otimes \psi}) = \Psi_{\sigma \otimes \sigma}^2 \end{aligned}$$

Therefore the spin field  $\Psi_{\sigma \otimes \sigma}$  is local. The locality of  $\Psi_{\psi \otimes \psi}$  can be easily proved as well.



## Chapter 5

# Topological defects in AQFT

Having constructed two-dimensional conformal field theories in the language of conformal nets and Q-systems, we now turn to the algebraic description of topological defects. In the Euclidean picture, a topological defect is represented by a line that can be freely deformed as long as it does not cross local field insertions. In the relativistic algebraic setting, this intuitive idea must be reformulated in terms of localization, causality and inclusions of nets. A defect is therefore described as an interface separating two regions of spacetime, with one copy of the theory living on one side and another, possibly equivalent, copy living on the other.

The physical meaning of the construction is the following. Away from the defect, each side behaves as an ordinary local CFT. However, once fields are transported across the defect, their locality properties may change. Thus the larger theory containing both sides is not expected to be fully local. Instead, it satisfies a weaker, oriented form of locality, usually called left-locality or half-locality. This controlled failure of locality is precisely what algebraically records the presence of the defect.

The technical tool used to implement this idea is the braided product of Q-systems. Given two extensions of the same chiral subnet, described by Q-systems  $Q^L$  and  $Q^R$ , their braided product produces a larger extension containing both theories. The choice of braiding fixes the exchange relations between charged generators belonging to the two sides and imposes the correct half-locality condition. In this sense, the braided product glues the two theories together while remembering on which side of the defect each field is supposed to live.

A crucial point is that this construction is universal. The braided product does not describe a single elementary defect, but rather contains all possible irreducible defect configurations compatible with the prescribed left- and right-theories. The problem of classifying defects is therefore reduced to decomposing this universal object. Algebraically, this is achieved by studying the center, or equivalently the relative commutant, of the braided-product extension. Minimal central projections select the irreducible representations of the universal extension, and hence the elementary defects.

Before applying the universal construction to concrete examples, we introduce two technical operations on Q-systems which clarify the algebraic meaning of this decomposition. The first is the reduction of a Q-system. Roughly speaking, this procedure allows one to extract from a given Q-system a smaller Q-system associated with a subextension, or subtheory, of the original one. In the present context this is precisely what is needed for the braided-product extension: since the latter describes a universal superposition of admissible defect configurations, its irreducible components are obtained by reducing the corresponding Q-system with respect to suitable central projections. In this way, the abstract problem of separating elementary defects becomes the concrete algebraic problem of decomposing and reducing a reducible Q-system.

The second operation concerns maximal local subtheories. Here the same idea of reduction is used in a different way: starting from a possibly non-local extension, one extracts the largest local subnet contained in it. This construction is useful for two reasons. First, it provides a more intrinsic formu-

lation of algebraic defects, since a defect can be described through a non-local extension whose left and right-local parts are recovered as maximal local subtheories. Second, it will play an important role in the following chapter, where the full center of a Q-system is introduced. The full center can be understood as a systematic way of producing local two-dimensional theories from chiral data, and its construction is naturally expressed in terms of maximal local subtheories.

After developing these tools, we apply the universal construction to the relativistic two-dimensional Ising model. In this example, the central decomposition of the braided product can be carried out explicitly, and the resulting irreducible components reproduce the expected elementary topological defects. This provides a concrete test of the algebraic formalism and prepares the comparison with the Euclidean categorical description discussed in the next chapter.

This chapter is based on [35, 38, 57, 70, 71, 73–75].

## 5.1 Defects and boundary conditions

We first recall the notion of a boundary between two possibly different CFTs, and then specialize it to the case of topological defects in the algebraic setting.

**Definition 5.1.1.** Consider two local CFTs in the two-dimensional Minkowski plane. A *boundary* is a timelike curve  $x(t)$  such that energy conservation is satisfied along it<sup>1</sup>, i.e.:

$$T_{01}^L(t, x(t)) = T_{01}^R(t, x(t)) \quad \Rightarrow \quad T_+^L(t) + T_-^R(t) = T_+^R(t) + T_-^L(t) \quad (5.1)$$

where  $T_{\mu\nu}^L$  and  $T_{\mu\nu}^R$  are the energy-momentum tensors describing the CFTs on the two sides of the boundary, and  $T_+^R(t+x), T_-^R(t-x), T_+^L(t+x), T_-^L(t-x)$  are the respective free components in the lightcone coordinates  $x^\pm = t \pm x$ .

Moreover, we call the boundary *transmissive* or *topological* if also momentum conservation is satisfied along the boundary:

$$T_{11}^L(t, x(t)) = T_{11}^R(t, x(t)) \quad \Rightarrow \quad T_+^L(t) - T_-^R(t) = T_+^R(t) - T_-^L(t) \quad (5.2)$$

Note that equations (5.1) and (5.2) can be combined into:

$$T_+^L(t) = T_+^R(t) \quad T_-^L(t) = T_-^R(t)$$

which clarifies why we call the boundary "transmissive". Note that both CFTs on the two sides of the boundary must have equal central charge. Since we are interested in the defects of a particular theory, we will always work with the transmissive assumption; otherwise the two theories on the two sides would be totally different CFTs.

According to [35], we make the following observations:

1. Because the two local theories at the boundary share the same energy-momentum tensor, they share the same unitary representation of the conformal group on the Hilbert space. So the field operators of the theory on one side of the phase boundary can be translated to the other side by the translation unitaries, where they will in general fail to coincide with the fields of the theory on the other side. It is important to notice that the Hilbert space necessarily supports both sets of fields everywhere in Minkowski spacetime, but on either side of the boundary only one of them represents physical observables.

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<sup>1</sup>Note that in this case we are interested in the conservation along a boundary and not in the local conservation. Consider for simplicity the line  $x = 0$ : heuristically, if we integrate the conservation equation  $\partial^\nu T_{\mu\nu} = 0$  in a rectangle of width  $2\epsilon$  centered at the boundary,  $\int_{-\epsilon}^{\epsilon} \partial^0 T_{\mu 0} dx + \int_{-\epsilon}^{\epsilon} \partial^1 T_{\mu 1} dx = 0$ , then, in the absence of singular contributions along the line  $x = 0$ , the first integral vanishes in the limit  $\epsilon \rightarrow 0$  and from the second we get  $T_{\mu 1}(t, -\epsilon) = T_{\mu 1}(t, \epsilon)$ . This condition establish the conservation of the flux of the quantity "μ" along the boundary. This is coherent with the interpretation of  $T_{\mu\nu}$  as the flux of  $p_\mu$  along a surface of constant  $x_\nu$  [76].

2. Apart from the transmitted chiral fields, the quantum field theories on each side may have more chiral fields that are not transmitted, and each side may have further local fields that are not chiral. Thus both sides are extensions of quantum field theories of the form  $\mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{B}^{L,R}$ .
3. The boundary will be described by some extension containing both  $\mathcal{B}^L$  and  $\mathcal{B}^R$ , which will in general fail to be local. Since  $\mathcal{B}^L$  and  $\mathcal{B}^R$  describe respectively the theories on the left and right sides of the boundary, we would like to have  $[\mathcal{B}^L(\mathcal{O}_1), \mathcal{B}^R(\mathcal{O}_2)] = 0$  whenever  $\mathcal{O}_1$  lies on the spacelike left complement of  $\mathcal{O}_2$ . By conformal covariance, this requirement extends from regions lying strictly on opposite sides of the timelike defect to any pair of double cones with the same relative left-right ordering. If this "half-locality" condition is satisfied, we shall call this boundary a *defect*;
4. The extension satisfying the previous conditions may not be irreducible. Then, in view of proposition 4.1.1, we should look for a decomposition of such extension into irreducible components, i.e. each with trivial relative commutant with respect to  $\mathcal{A}_+ \otimes \mathcal{A}_-$ , as this would ensure that the extension contains the chiral algebras and, in particular, the full energy-momentum tensor: these describe the real elementary physical configurations. In this particular context, these irreducible extensions qualify as different defects.

This last observation will be the idea behind the universal construction studied by [35], which is the goal of this chapter.

According to the observation 3. we define in a precise way the meaning of "half locality".

**Definition 5.1.2.** For two nets of local algebras  $\mathcal{B}_i$ ,  $i = 1, 2$ , defined on the same Hilbert space, we say that  $\mathcal{B}_1$  is *left-local* with respect to  $\mathcal{B}_2$ , if

$$[\mathcal{B}_1(\mathcal{O}_1), \mathcal{B}_2(\mathcal{O}_2)] = 0$$

whenever  $\mathcal{O}_1$  is in the left causal complement of  $\mathcal{O}_2$ ; i.e.  $\mathcal{O}_1 \subset \mathcal{W}_{2,L}$ ,  $\mathcal{O}_2' = \mathcal{W}_{2,L} \cup \mathcal{W}_{2,R}$ , where  $\mathcal{W}_{2,L}, \mathcal{W}_{2,R}$  are the two wedges defined in figure 3.1. Right locality is defined in the same way.

Then we define a defect as follows:

**Definition 5.1.3.** [35] Consider two chiral nets  $\mathcal{A}_+$  and  $\mathcal{A}_-$ , and let  $\mathcal{A} := \mathcal{A}_+ \otimes \mathcal{A}_-$ . A *topological defect* between nets  $\mathcal{B}^L \supset \mathcal{A}$  and  $\mathcal{B}^R \supset \mathcal{A}$  is a transmissive boundary implemented by a relatively local extension  $\mathcal{D} \supset \mathcal{A}$  of finite index, such that  $\mathcal{B}^{L/R}$  are intermediate extensions and  $\mathcal{D}$  is left-local with respect to  $\mathcal{B}^R$  and right-local with respect to  $\mathcal{B}^L$ . In particular,  $\mathcal{B}^L$  is left-local with respect to  $\mathcal{B}^R$ . We call it a *boundary condition* if

$$\mathcal{D}(\mathcal{O}) = \mathcal{B}^L(\mathcal{O}) \vee \mathcal{B}^R(\mathcal{O})$$

holds, *simple* or *factorial* if  $\mathcal{D}(\mathcal{O})$  are factors and *irreducible* if  $\mathcal{A} \subset \mathcal{D}$  is an irreducible extension.

First of all note that the actual location of the defect is arbitrary, and it may be any timelike curve. For this reason it is called topological. Then, note that we specified a difference between general defects and boundary conditions, which satisfy equation (5.1.3): the reason is that on a boundary condition there are no extra degrees of freedom apart from the ones generated by the given local nets. In other words, "boundary conditions" refers to defects that do not contain any defect operator, or, using the previously defined terminology, junction. We will be mainly interested in boundary conditions, since a full description of general defects in this setup is still missing [74]. The recent approach to defects by [77] also admits such extra degrees of freedom, however their treatment is restricted to the chiral case. Nevertheless, as we are going to see, boundary conditions are interesting since they can be formulated as algebraic relations among the charged fields from  $\mathcal{B}^L$  and  $\mathcal{B}^R$ .

In the next sections we are going to build all the machinery needed to construct the non-local extensions implementing the defects.

## 5.2 Operations with Q-systems

We now introduce the operations needed to manipulate and combine different Q-systems.

We begin with the notion of a *reduced Q-system*. Starting from a projection  $P \in \text{Hom}(\Theta, \Theta)$  satisfying a suitable relation, one can construct a new Q-system describing a reduced extension. The interpretation of this extension depends on the additional properties of  $P$ . In particular, we will discuss *intermediate extensions*, which correspond to extensions lying between two given ones, and *central decompositions*, where, given an extension  $\mathcal{A} \subset \mathcal{B}$ , the reduced Q-system describes the extension  $\mathcal{A} \subset E\mathcal{B}$ , with  $E$  projecting onto one of the factors arising from the center of  $\mathcal{B}$ . This decomposition will play a central role in the classification of defects.

Intermediate extensions will also allow us to define the notion of a maximal local extension, which provides a more elegant algebraic characterization of defects.

Finally, we introduce the *braided product* of Q-systems, a multiplication that combines two extensions into a single one. Roughly speaking, the resulting extension contains the charged fields of both original extensions, with their mutual commutation relations determined by the braiding.

### 5.2.1 Reduced Q-system

Consider a Q-system  $(\Theta, W, X)$  and let  $P \in \text{Hom}(\Theta, \Theta)$  be a projection satisfying:

$$PXP = \Theta(P)XP = \Theta(P)PX \quad (5.3)$$

Then we can construct the following Q-system:

**Proposition 5.2.1.** [57] *Consider a Q-system  $(\Theta, W, X)$  and a projection  $P \in \text{Hom}(\Theta, \Theta)$  satisfying equation (5.3). Then the triple*

$$\begin{aligned} \Theta_P(\cdot) &:= S^\dagger \Theta(\cdot) S \prec \Theta \\ W_P &:= \dim(\Theta_P)^{-1/4} \cdot n_P^{1/2} S^\dagger W \\ X_P &:= \dim(\Theta_P)^{1/4} \cdot \Theta(n_P^{-1/2} S^\dagger) \cdot n_P^{-1/2} S^\dagger \cdot X S n_P^{1/2} \end{aligned}$$

where  $P = SS^\dagger$  and  $n_P = S^\dagger X^\dagger \cdot P\Theta(P) \cdot XS \in \text{Hom}(\Theta_P, \Theta_P)$  is a normalization intertwiner, is a Q-system called **reduced Q-system** if and only if  $W_P$  has the correct normalization  $W_P^\dagger W_P = W^\dagger X^\dagger \cdot \Theta(P)P \cdot XW = \dim(\Theta_P)$ .

The interesting part about reduced Q-systems is that they have a different interpretation depending on additional properties satisfied by  $P$ . In the following, we are not going to be interested in constructing such reduced Q-system, but rather in knowing the conditions for its existence.

First of all, we want to characterize *intermediate extensions*.

Let  $\mathcal{A} \subset \mathcal{B}_1 \subset \mathcal{B}$  be any intermediate extension with Q-system  $(\Theta_1, W_1, X_1)$ . Then  $\iota = \iota_2 \circ \iota_1$ , with  $\iota_1 : \mathcal{A} \rightarrow \mathcal{B}_1$ ,  $\iota_2 : \mathcal{B}_1 \rightarrow \mathcal{B}$ , and  $\Theta_1 = \bar{\iota}_1 \iota_1 \prec \bar{\iota} \iota = \Theta$ . Let  $P \in \text{Hom}(\Theta, \Theta)$  be the projection corresponding to  $\Theta_1 \prec \Theta$ . This can be defined starting from a projection  $E_2 = d_2^{-1} W_2 W_2^\dagger \in \text{Hom}(\Theta_2, \Theta_2)$  onto  $\text{id}_{\mathcal{B}_1} \prec \Theta_2$  so that  $P = \bar{\iota}_1(E_2)$ , which can be drawn as:

Which makes clear that  $P \in \text{Hom}(\Theta, \Theta)$  is indeed a projection onto  $\Theta_1 \prec \Theta$ .<sup>2</sup>

We can show that this projection satisfy the following relations:

$$PW = W, \quad PXP = P\Theta(P)X = \Theta(P)XP \quad (5.4)$$

---

<sup>2</sup>This can be checked also explicitly:  $\bar{\iota}_1(E_2)\Theta(A) = \bar{\iota}_1(E_2) \cdot \bar{\iota}_1 \bar{\iota}_2 \iota_2 \iota_1(A) = \bar{\iota}_1(E_2) \cdot \bar{\iota}_1 \Theta_2 \iota_1(A) = \bar{\iota}_1(E_2 \cdot \Theta_2 \iota_1(A)) = \bar{\iota}_1(\Theta_2 \iota_1(A) \cdot E_2) = \bar{\iota}_1(\Theta_2(\iota_1(A))) \cdot \bar{\iota}_1(E_2) = \Theta(A) \bar{\iota}_1(E_2)$ ,  $\forall A \in \mathcal{A}$ .

$$P = d_2^{-1} \begin{array}{c} \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \end{array}$$

I.e. it satisfies equation (5.3) and in addition  $PW = W$ . To prove the above properties we will need the graphical representations of  $W$  and  $X$  with respect to the standard rules for generic elements of the category, as done for representing  $P$  above. First we prove the additional property:

$$d_2^{-1} \begin{array}{c} \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \end{array} = \begin{array}{c} \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \end{array}$$

Where the  $d_2^{-1}$  factor simplified with the  $l_2$  bubble. Now we show that also (5.3) is satisfied:

$$\begin{array}{c} \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \end{array} = \begin{array}{c} \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \end{array} = \begin{array}{c} \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \bar{l}_1 \quad \bar{l}_2 \quad l_2 \quad l_1 \end{array}$$

where we have simplified the overall constant  $d_2^{-1}$  factor and the equalities are realized exploiting the isotopy invariance of the internal lines.

The following theorem states that also the converse is true:

**Proposition 5.2.2.** [57] *Let  $Q = (\Theta, W, X)$  be a  $Q$ -system defining the extension  $\mathcal{A} \subset \mathcal{B}$ . Let  $P \in \text{Hom}(\Theta, \Theta)$  be a projection satisfying equation (5.4), then the reduced  $Q$ -system  $Q_P = (\Theta_P, W_P, X_P)$*

of proposition 5.2.1 corresponds to the intermediate extension  $\mathcal{A} \subset \mathcal{B}_P \subset \mathcal{B}$  where  $\mathcal{B}_P = \mathcal{A} \cdot PV$ .

We can also say something about the locality of such intermediate extension:

**Lemma 5.2.3.** [35]  $\mathcal{B}$  is left-local with respect to  $\mathcal{B}_P$  if and only if  $P$  satisfies:

$$\Theta(P)X = \epsilon_{\Theta, \Theta} \cdot PX$$

$\mathcal{B}$  is right-local with respect to  $\mathcal{B}_P$  if and only if  $P$  satisfies the same relation with  $\epsilon_{\Theta, \Theta}$  replaced by  $\epsilon_{\Theta, \Theta}^\dagger$ .

Now we focus on the other case we are interested in: *central decomposition*.

Suppose that the extension  $\mathcal{B} \supset \mathcal{A}$  is characterized by a non-trivial center  $\mathcal{B}' \cap \mathcal{B} \neq \mathbb{C} \cdot \mathbf{1}$ . We know from lemma 4.2.5 that this could be the case if  $\dim(\text{Hom}(\text{id}_{\mathcal{A}}, \Theta)) > 1$ . Then every projection  $E \in \mathcal{B}' \cap \mathcal{B}$  gives rise to an inclusion<sup>3</sup>  $E\mathcal{A} \simeq \mathcal{A} \subset E\mathcal{B}$ . Now we want to characterize the corresponding embeddings  $E\mathcal{A} \rightarrow E\mathcal{B}$  in terms of the reduced Q-systems of proposition 5.2.1.

By defining the projection<sup>4</sup>  $P = \bar{\iota}(E) \in \text{Hom}(\Theta, \Theta)$  it is easy to see that it satisfies:

$$XP = \Theta(P)X = PX \tag{5.5}$$

which trivially implies also equation (5.3).

Indeed, consider the two equations

$$VE = \bar{\iota}(E)V \quad VE = EV$$

the first coming from the construction of the extension  $\mathcal{A} \subset \mathcal{B}$ , in particular it is the last relation in equation (4.2) with  $\iota(E) = E$  since  $E \in \mathcal{B}$ , and the second from the fact that  $E \in \mathcal{B}' \cap \mathcal{B}$ . Now we apply  $\bar{\iota}$  to both sides and we get:

$$\bar{\iota}(V)\bar{\iota}(E) = XP = \bar{\iota}\bar{\iota}(E)\bar{\iota}(V) = \Theta(P)X \quad \bar{\iota}(V)\bar{\iota}(E) = XP = \bar{\iota}(E)\bar{\iota}(V) = PX$$

Again, it turns out that also the converse is true:

**Proposition 5.2.4.** [57] Let  $Q = (\Theta, W, X)$  be a Q-system defining the extension  $\mathcal{A} \subset \mathcal{B}$ . Let  $P \in \text{Hom}(\Theta, \Theta)$  be a projection satisfying equation (5.5), then the reduced Q-system  $Q_P = (\Theta_P, W_P, X_P)$  of proposition 5.2.1 corresponds to the extension  $\mathcal{A}E \subset \mathcal{B}E$  where  $E \in \mathcal{B}' \cap \mathcal{B}$  and  $P = \bar{\iota}(E)$ .

## 5.2.2 Maximal local subtheories of non-local extensions

There are two distinguished maximal local subtheories contained in a non-local extension, namely the subtheories of  $\mathcal{B}(\mathcal{O})$  commuting with the algebras localized on the left/right causal wedges of  $\mathcal{O}$ ,  $\mathcal{O}' = \mathcal{W}^L \cup \mathcal{W}^R$ .

**Definition 5.2.1.** Let  $\mathcal{A} \subset \mathcal{B}$  be an extension. We define:

$$\mathcal{B}_{loc}^+(\mathcal{O}) := \mathcal{B}(\mathcal{W}_L)' \cap \mathcal{B}(\mathcal{O}) \quad \mathcal{B}_{loc}^-(\mathcal{O}) := \mathcal{B}(\mathcal{W}_R)' \cap \mathcal{B}(\mathcal{O})$$

**Lemma 5.2.5.** [35] The following properties hold:

1.  $\mathcal{B}_{loc}^\pm$  are Poincaré covariant isotonomous nets of von Neumann algebras.
2.  $\mathcal{B}$  is left-local with respect to  $\mathcal{B}_{loc}^+$  and right local with respect to  $\mathcal{B}_{loc}^-$ .

<sup>3</sup>Note that in principle the projected algebras are defined as  $E\mathcal{A}E$  and  $E\mathcal{B}E$ . However, since  $E \in \mathcal{B}' \cap \mathcal{B}$  and  $\mathcal{A} \subset \mathcal{B}$  we get  $E\mathcal{A}E = \mathcal{A}E^2 = \mathcal{A}E$ ; same for  $\mathcal{B}$ . Moreover, we write  $\mathcal{A} \simeq \mathcal{A}E$  since the two are isomorphic [57].

<sup>4</sup>The fact that  $P$  is a projection follows directly from the fact that  $E$  is a projection. Then, note that since  $E \in \mathcal{B}' \cap \mathcal{B}$  we have  $EA = AE \forall A \in \mathcal{A}$  and applying  $\bar{\iota}$  to both sides, recalling that  $\bar{\iota}(A) = \bar{\iota}(AW^\dagger)V = \Theta(AW^\dagger)X = \Theta(A)$ , we get  $P\Theta(A) = \Theta(A)P$ . Hence  $P \in \text{Hom}(\Theta, \Theta)$ .

3. The nets  $\mathcal{B}_{loc}^\pm$  are both local and intermediate between  $\mathcal{A}$  and  $\mathcal{B}$ :

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{B}_{loc}^\pm(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$$

4. The nets  $\mathcal{B}_{loc}^\pm$  are the maximal intermediate nets with properties stated in 2.

Of course, as a consequence of property 4., if  $\mathcal{B}$  is itself local, then  $\mathcal{B}_{loc}^\pm = \mathcal{B}$  and also the other extreme, i.e.  $\mathcal{B}_{loc}^\pm = \mathcal{A}$  may occur.

The following result shows how  $\mathcal{B}_{loc}^\pm$  are related to the construction of intermediate extensions arising from projections  $P \in \text{Hom}(\Theta, \Theta)$  introduced in the last section.

**Definition 5.2.2.** [35] The *left*  $C^+[Q]$  and *right*  $C^-[Q]$  *centers* of a  $Q$ -system  $Q = (\Theta, W, X)$  are the intermediate  $Q$ -systems associated to the intermediate extensions given by the projections:

$$\dim(\Theta)^{1/2} \cdot P^\pm := W^\dagger X^\dagger \Theta(\epsilon_{\Theta, \Theta}^\pm) X X \quad (5.6)$$

satisfying equation (5.4) and in addition

$$\Theta(P^+)X = \epsilon_{\Theta, \Theta} \cdot P^+ X \quad \Theta(P^-)X = \epsilon_{\Theta, \Theta}^\dagger \cdot P^- X \quad (5.7)$$

Therefore, as a consequence of lemma 5.2.3, the left (right) center  $C^+[Q]$  ( $C^-[Q]$ ) define an intermediate extension  $\mathcal{A} \subset \mathcal{B}_{P^+} \subset \mathcal{B}$  (resp.  $\mathcal{B}_{P^-}$ ) such that  $\mathcal{B}$  is left (right) local with respect to  $\mathcal{B}_{P^+}$  ( $\mathcal{B}_{P^-}$ ).

**Proposition 5.2.6.** [57] *The centers of a  $Q$ -system are maximal with respect to the property of equation (5.7), i.e. for every other projection  $P$  satisfying the first (second) relation of equation (5.7), one has  $PP^+ = P$  ( $PP^- = P$ ), i.e.  $P \leq P^+$  ( $P \leq P^-$ ).*

*Moreover, equation (5.7) entails  $\epsilon_{\Theta_P, \Theta_P} X_P = X_P$ , i.e. the left and right centers of a  $Q$ -system are commutative.*

This provides us with the desired connection:

**Proposition 5.2.7.** *The intermediate extensions associated with the left resp. right center of the  $Q$ -system for  $\mathcal{A} \subset \mathcal{B}$  are  $\mathcal{A} \subset \mathcal{B}_{loc}^+ \subset \mathcal{B}$  resp.  $\mathcal{A} \subset \mathcal{B}_{loc}^- \subset \mathcal{B}$ .*

*Proof.* By lemma 5.2.5,  $\mathcal{B}_{loc}^\pm$  are the maximal local intermediate extensions such that  $\mathcal{B}$  is left-local with respect to  $\mathcal{B}_{loc}^+$  and right-local with respect to  $\mathcal{B}_{loc}^-$ . By lemma 5.2.3, the associated projections are the maximal projections satisfying equation (5.7), but these properties characterize the center projections.  $\square$

This language allows us to rephrase the locality condition required at a boundary.

**Corollary 5.2.8.** *An extension  $\mathcal{D}$  of  $\mathcal{A}$  implements a defect between  $\mathcal{B}^L$  and  $\mathcal{B}^R$  if and only if*

$$\mathcal{B}^L \subset \mathcal{D}_{loc}^- \subset \mathcal{D} \supset \mathcal{D}_{loc}^+ \supset \mathcal{B}^R$$

*Proof.* By definition 5.2.7 we have  $\mathcal{D}_{loc}^{+/-} = \mathcal{D}(\mathcal{W}_{L/R})' \cap \mathcal{D}(\mathcal{O})$ , hence  $\mathcal{D}_{loc}^\pm(\mathcal{O}) \subset \mathcal{D}(\mathcal{O})$ . Then the defining properties of a defect, namely  $\mathcal{B}^{R/L}(\mathcal{O}) \subset \mathcal{D}(\mathcal{O})$  and left/right locality of  $\mathcal{D}$  with respect to  $\mathcal{B}^{R/L}$ , are equivalent to  $\mathcal{B}^{R/L}(\mathcal{O}) \subset \mathcal{D}_{loc}^{+/-}(\mathcal{O})$ .  $\square$

### 5.2.3 Braided products

In general, two extensions  $\mathcal{A} \subset \mathcal{B}_1$  and  $\mathcal{A} \subset \mathcal{B}_2$  with charged fields  $\Psi_{1,\rho}$  ( $\rho \prec \Theta_1$ ) and  $\Psi_{2,\rho}$  ( $\rho \prec \Theta_2$ ) are defined on different Hilbert spaces, that are both DHR representations of the underlying algebra  $\mathcal{A}$ . However, if we want to construct an extension  $\mathcal{D}$  implementing a defect, we need to be able to combine  $\mathcal{B}_1$  and  $\mathcal{B}_2$  in a single extension, which contains both as intermediate extensions.

**Proposition 5.2.9.** *Let  $Q_1 = (\Theta_1, W_1, X_1)$  and  $Q_2 = (\Theta_2, W_2, X_2)$  be two  $Q$ -systems. Then the triples*

$$(\Theta_1, W_1, X_1) \times^\pm (\Theta_2, W_2, X_2) := (\Theta = \Theta_1 \Theta_2, W = W_1 W_2, X^\pm = \Theta_1(\epsilon_{\Theta_1, \Theta_2}^\pm) X_1 \Theta_1(X_2))$$

*define a pair of  $Q$ -systems denoted  $Q_1 \times^\pm Q_2$ . We call them **braided product of  $Q$ -systems**, and denote the corresponding **braided product extensions** by  $\mathcal{A} \subset \mathcal{B}_1 \times^\pm \mathcal{B}_2 =: \mathcal{D}^\pm$ . Both product  $Q$ -systems  $Q_1 \times^\pm Q_2$  contain  $Q_1$  and  $Q_2$  as intermediate  $Q$ -systems, hence  $\mathcal{D}^\pm$  contains  $\mathcal{B}_1$  and  $\mathcal{B}_2$  as intermediate extensions:*

$$\mathcal{D}^\pm \supset \mathcal{B}_1 \supset \mathcal{A} \subset \mathcal{B}_2 \subset \mathcal{D}^\pm$$

*Moreover,  $\mathcal{D}^\pm$  can be characterized as the quotient of the free product of the algebras  $\mathcal{B}_1$  and  $\mathcal{B}_2$  by the relations  $\iota_1(\mathcal{A}) = \iota_2(\mathcal{A})$  and the commutation relations*

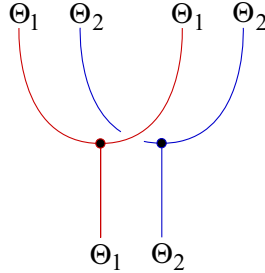
$$V_2 V_1 = \epsilon_{\Theta_1, \Theta_2}^\pm \cdot V_1 V_2 \quad (5.8)$$

*Proof.* We sketch the proof: the defining relations of  $Q$ -system for  $Q_1 \times^\pm Q_2$  can be shown by direct computation using the valid relations for both  $Q_1$  and  $Q_2$ . We give an example<sup>5</sup>: we prove that  $W^\dagger X^\pm = \mathbf{1}_{\mathcal{D}^\pm}$ :

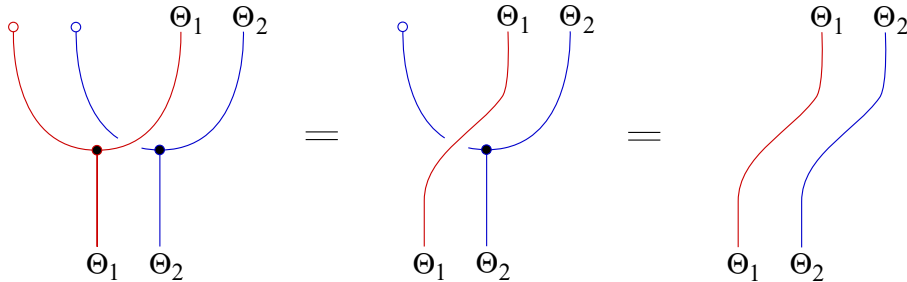
$$W^\dagger X^\pm = W_2^\dagger W_1^\dagger \Theta_1(\epsilon_{\Theta_1, \Theta_2}^\pm) X_1 \Theta_1(X_2) = W_2^\dagger \epsilon_{\Theta_1, \Theta_2}^\pm W_1^\dagger X_1 \Theta_1(X_2) = W_2^\dagger \epsilon_{\Theta_1, \Theta_2}^\pm \Theta_1(X_2)$$

where we used the fact that  $W_1^\dagger \in \text{Hom}(\Theta_1, \text{id}_{\mathcal{B}_1})$ . Now we use the compatibility of braiding with morphism of (3.23) (or equivalently equation (3.13) with  $\sigma_1 = \rho_1 = \Theta_1$ ,  $\sigma_2 = \text{id}$ ,  $\rho_2 = \Theta_2$ ,  $T_1 = 1$  and  $T_2 = W_2^\dagger$ ) to get  $W_2^\dagger \epsilon_{\Theta_1, \Theta_2}^\pm = \Theta_1(W_2^\dagger)$ , so that  $\Theta_1(W_2^\dagger) \Theta_1(X_2) = \Theta_1(W_2^\dagger X_2) = \mathbf{1}_{\mathcal{D}^\pm}$ .

It is easier to check these relations by using the graphical language we have developed. For example, by writing  $X^+$  as



the above relation can be easily proven (for  $Q_1 \times^+ Q_2$ ) as follows:



where we have used the properties of definitions 4.3.1 and 4.3.2 and we have used two colors to distinguish  $\Theta_1$  from  $\Theta_2$ .

Then the projections  $P_1 = d_2^{-1} \Theta_1(W_2 W_2^\dagger) \in \text{Hom}(\Theta, \Theta)$  and  $P_2 = d_1^{-1} (W_1 W_1^\dagger) \in \text{Hom}(\Theta, \Theta)$  satisfy equation (5.4), hence by proposition 5.2.2 they define intermediate extensions. For example we can show explicitly that  $P_{1,2} W = W$ :

$$P_1 W = d_2^{-1} \Theta_1(W_2 W_2^\dagger) W_1 W_2 = d_2^{-1} W_1 W_2 W_2^\dagger W_2 = W_1 W_2$$

<sup>5</sup>In the following we are going to suppress the inclusions  $\iota$  to enhance readability.

$$P_2 W = d_1^{-1} W_1 W_1^\dagger W_1 W_2 = W_1 W_2$$

Since  $P_1$  and  $P_2$  project respectively to  $\Theta_1$  and  $\Theta_2$ , we conclude that  $Q_1$  and  $Q_2$  are intermediate with respect to  $Q_1 \times^\pm Q_2$ . Hence  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are intermediate extensions with  $V_1 = \Theta_1(W_2^\dagger)V^\pm$  and  $V_2 = W_1^\dagger V^\pm$ . Conversely:

$$\begin{aligned} V_1 V_2 &= \Theta_1(W_2^\dagger)V^\pm W_1^\dagger V^\pm = \Theta_1(W_2^\dagger)\Theta(W_1^\dagger)V^\pm V^\pm = \\ &= \Theta_1(W_2^\dagger)\Theta_1\Theta_2(W_1^\dagger)X^\pm V^\pm = \Theta_1(W_2^\dagger)\Theta_1\Theta_2(W_1^\dagger)\Theta_1(\epsilon_{\Theta_1, \Theta_2}^\pm)X_1\Theta_1(X_2)V^\pm = \\ &= \Theta_1(W_2^\dagger)\Theta_2(W_1^\dagger)\epsilon_{\Theta_1, \Theta_2}^\pm X_1\Theta_1(X_2)V^\pm = \Theta_1(W_1^\dagger W_2^\dagger \epsilon_{\Theta_1, \Theta_2}^\pm)X_1\Theta_1(X_2)V^\pm = \\ &= \Theta_1(W_1^\dagger\Theta_1(W_2^\dagger))X_1\Theta_1(X_2)V^\pm = \Theta_1(W_1^\dagger)\Theta_1\Theta_1(W_2^\dagger)X_1\Theta_1(X_2)V^\pm = \\ &= \Theta_1(W_1^\dagger)X_1\Theta_1(W_2^\dagger)\Theta_1(X_2)V^\pm = V^\pm \end{aligned}$$

Hence  $\mathcal{D}^\pm$  is generated by  $\mathcal{A}$  together with  $V_1$  and  $V_2$ . Then the product  $V_2 V_1$  is given by:

$$\begin{aligned} V_2 V_1 &= W_1^\dagger V^\pm \Theta_1(W_2^\dagger)V^\pm = W_1^\dagger \Theta_1\Theta_2\Theta_1(W_2^\dagger)V^\pm V^\pm = \Theta_2\Theta_1(W_2^\dagger)W_1^\dagger X^\pm V^\pm = \\ &= \Theta_2\Theta_1(W_2^\dagger)W_1^\dagger \Theta_1(\epsilon_{\Theta_1, \Theta_2}^\pm)X_1\Theta_1(X_2)V^\pm = \Theta_2\Theta_1(W_2^\dagger)\epsilon_{\Theta_1, \Theta_2}^\pm W_1^\dagger X_1\Theta_1(X_2)V^\pm = \\ &= \Theta_2\Theta_1(W_2^\dagger)\epsilon_{\Theta_1, \Theta_2}^\pm \Theta_1(X_2)V^\pm = \epsilon_{\Theta_1, \Theta_2}^\pm \Theta_1\Theta_2(W_2^\dagger)\Theta_1(X_2)V^\pm = \\ &= \epsilon_{\Theta_1, \Theta_2}^\pm V^\pm = \epsilon_{\Theta_1, \Theta_2}^\pm V_1 V_2 \end{aligned}$$

□

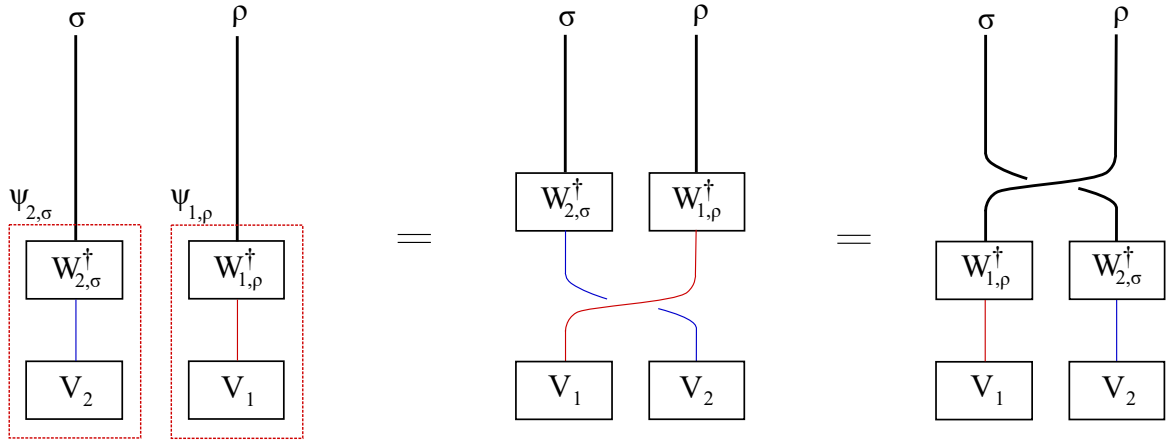
It is easy to see that equation (5.8), written in terms of the charged fields

$$\Psi_{1,\rho} = W_{1,\rho}^\dagger V_1 \quad \Psi_{2,\sigma} = W_{2,\sigma}^\dagger V_2,$$

reads:

$$\Psi_{2,\sigma}\Psi_{1,\rho} = \epsilon_{\rho,\sigma}^\pm \cdot \Psi_{1,\rho}\Psi_{2,\sigma} \quad (5.9)$$

Indeed:



where in the figure we have chosen the positive braiding.

The following result is a natural consequence of the last proposition:

**Proposition 5.2.10.** [35] *Within  $\mathcal{D}^+ = \mathcal{B}_1 \times^+ \mathcal{B}_2$ ,  $\mathcal{B}_1$  is right-local w.r.t.  $\mathcal{B}_2$ , whereas within  $\mathcal{D}^- = \mathcal{B}_1 \times^- \mathcal{B}_2$ ,  $\mathcal{B}_1$  is left-local w.r.t.  $\mathcal{B}_2$ . Every irreducible extension containing both  $\mathcal{B}_1$  and  $\mathcal{B}_2$  as intermediate extensions such that  $\mathcal{B}_1$  is right-local (left-local) w.r.t.  $\mathcal{B}_2$ , is a quotient of  $\mathcal{D}^+$  ( $\mathcal{D}^-$ ).*

The braided product  $Q_1 \times^\pm Q_2$  of two irreducible Q-systems (i.e. such that  $\dim \text{Hom}(\text{id}, \Theta_i) = 1$ ,  $i = 1, 2$ ) will in general fail to be irreducible, because  $\dim \text{Hom}(\text{id}, \Theta_1\Theta_2) = \dim \text{Hom}(\Theta_1, \Theta_2) > 1$ , as explained in the end of section 4.2.<sup>6</sup>

<sup>6</sup>The isomorphism  $\text{Hom}(\text{id}, \Theta_1\Theta_2) \simeq \text{Hom}(\Theta_1, \Theta_2)$  comes from the Frobenius reciprocity relations valid for standard pairs, together with the fact that, by definition, the dual canonical endomorphism  $\Theta = \bar{\iota} \circ \iota$  is self-conjugate.

**Proposition 5.2.11.** *The relative commutant  $\mathcal{A}' \cap \mathcal{D}^\pm$  of the braided product extension  $\mathcal{D}^\pm$  is spanned by all operators of the form  $(\Psi_{1,\rho}^n)^\dagger \Psi_{2,\rho}^m$ , where  $\rho \prec \Theta_1$ ,  $\rho \prec \Theta_2$ , and  $n = 1, \dots, \dim \text{Hom}(\rho, \Theta_1)$ ,  $m = 1, \dots, \dim \text{Hom}(\rho, \Theta_2)$ . If both the  $Q$ -systems  $Q_1$  and  $Q_2$  are commutative (hence the corresponding extensions  $\mathcal{B}_i$ ,  $i = 1, 2$ , are local), then the relative commutant coincides with the center:*

$$\mathcal{A}' \cap \mathcal{D} = \mathcal{D}^{\pm'} \cap \mathcal{D}^\pm$$

*Proof.* In lemma 4.2.5, we have characterized the relative commutant by the elements  $\iota(H)V$  with  $H \in \text{Hom}(\Theta, \text{id})$ , where  $\Theta = \Theta_1 \Theta_2$  for a product  $Q$ -systems. By Frobenius reciprocity, we have  $\text{Hom}(\Theta_1 \Theta_2, \text{id}) = R_1^\dagger \Theta_1(\text{Hom}(\Theta_2, \Theta_1))$  where  $R_1 := X_1 W_1 \in \text{Hom}(\text{id}, \Theta_1 \Theta_1)$ . A basis of  $\text{Hom}(\Theta_2, \Theta_1)$  is given by  $W_{1,\rho} W_{2,\rho}^\dagger$  with  $W_{i,\rho} \in \text{Hom}(\rho, \Theta_i)$ ,  $i = 1, 2$ . Define the corresponding charged fields as

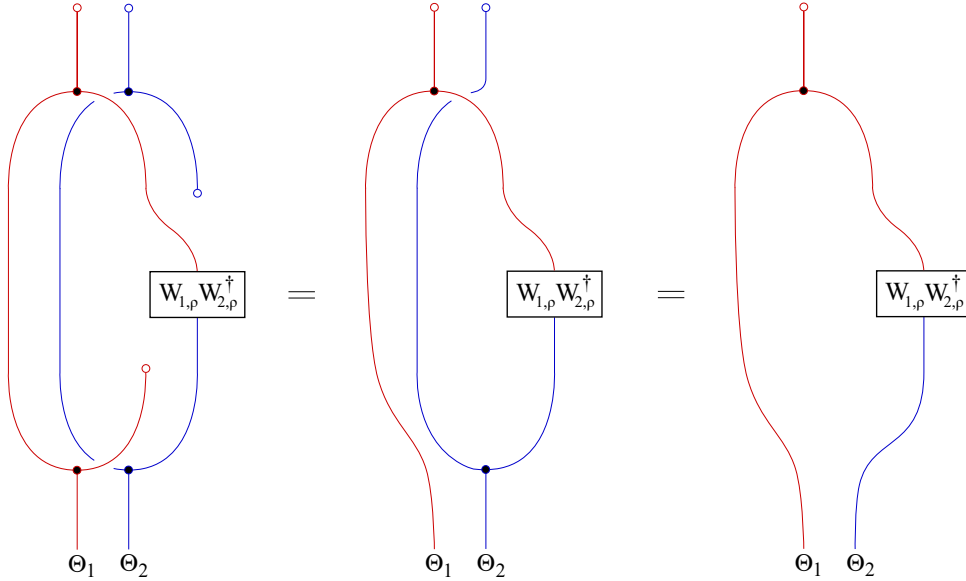
$$\Psi_{1,\rho} = W_{1,\rho}^\dagger \Theta_1(W_2)^\dagger V^\pm \quad \Psi_{2,\rho} = W_1^\dagger \Theta_1(W_{2,\rho})^\dagger V^\pm$$

where we have suppressed the symbols  $\iota$  for simplicity. Then

$$\begin{aligned} \Psi_{1,\rho}^\dagger \Psi_{2,\rho} &= (V^\pm)^\dagger \Theta_1(W_2) W_{1,\rho} W_1^\dagger \Theta_1(W_{2,\rho}^\dagger) V^\pm = W^\dagger (X^\pm)^\dagger V^\pm \Theta_1(W_2) W_{1,\rho} W_1^\dagger \Theta_1(W_{2,\rho}^\dagger) V^\pm = \\ &= R^\dagger \Theta(\Theta_1(W_2) W_{1,\rho} W_1^\dagger \Theta_1(W_{2,\rho}^\dagger)) V^\pm V^\pm = R^\dagger \Theta(\Theta_1(W_2) W_{1,\rho} W_1^\dagger \Theta_1(W_{2,\rho}^\dagger)) X^\pm V^\pm \end{aligned}$$

where  $R := X^\pm W \in \text{Hom}(\text{id}, \Theta)$ .

We want to show that this equals  $HV^\pm$  where  $H := R_1^\dagger \Theta_1(W_{1,\rho} W_{2,\rho}^\dagger)$ . Since the algebraic computation of this equality is quite involved, we use the graphical calculus to prove it<sup>7</sup>. Starting from  $R^\dagger \Theta(\Theta_1(W_2) W_{1,\rho} W_1^\dagger \Theta_1(W_{2,\rho}^\dagger)) X^\pm$  we have:



where on the right-hand side we have exactly the diagram representing  $R_1^\dagger \Theta_1(W_{1,\rho} W_{2,\rho}^\dagger) =: H$ . Hence:

$$\Psi_{1,\rho}^\dagger \Psi_{2,\rho} = R^\dagger \Theta(\Theta_1(W_2) W_{1,\rho} W_1^\dagger \Theta_1(W_{2,\rho}^\dagger)) X^\pm V^\pm = HV^\pm$$

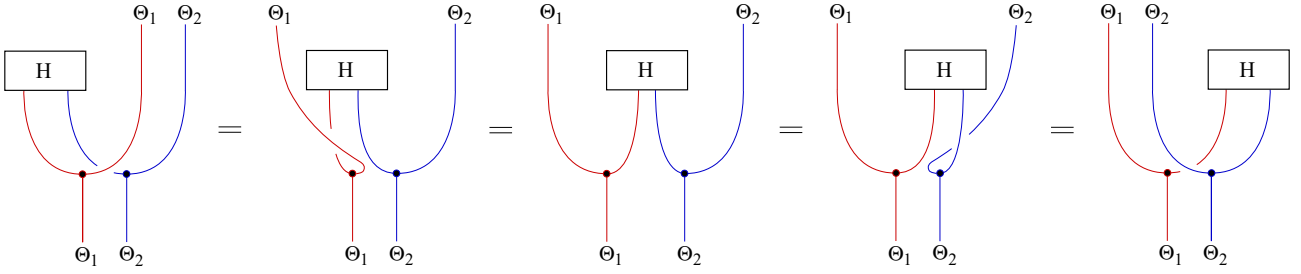
Since  $H \in \text{Hom}(\Theta_1 \Theta_2, \text{id})$  arises by Frobenius reciprocity from a basis  $W_{1,\rho} W_{2,\rho}^\dagger$  of  $\text{Hom}(\Theta_1, \Theta_2)$ , the operators  $\Psi_{1,\rho}^\dagger \Psi_{2,\rho}$  form a basis of  $\text{Hom}(\Theta_1 \Theta_2, \text{id}) V^\pm = \mathcal{A}' \cap \mathcal{D}^\pm$ .

In lemma 4.2.5, we have characterized the center by  $H \in \text{Hom}(\Theta, \text{id})$  satisfying the supplementary relation (4.5). For  $H \in \text{Hom}(\Theta_1 \Theta_2, \text{id})$  one computes:

$$HX^\pm = \Theta_1(H) \epsilon_{\Theta_1, \Theta_1}^\mp X_1 \Theta_1(X_2) = \Theta_1(H) X_1 \Theta_1(X_2) = \Theta_1(H) X_1 \Theta_1(\epsilon_{\Theta_2, \Theta_2}^\mp X_2) = \Theta_1 \Theta_2(H) X^\pm \quad (5.10)$$

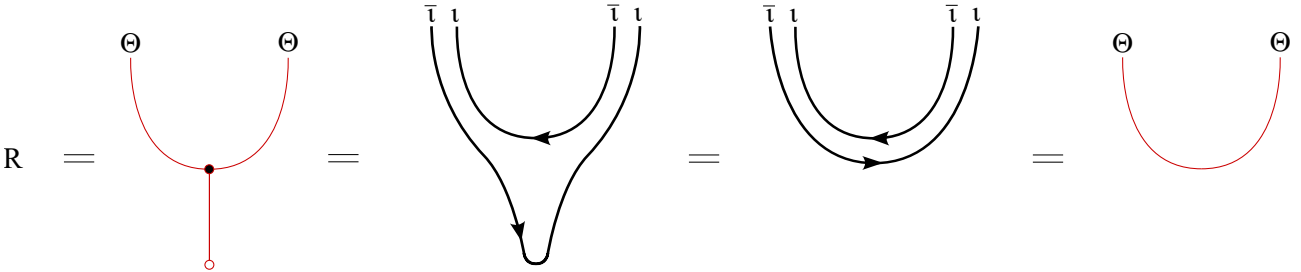
<sup>7</sup>All the graphical computations in this proof are referred to  $\mathcal{D}^+$ . The generalization to  $\mathcal{D}^-$  is trivial.

where we have used the hypothesis of both Q-systems being commutative (i.e.  $\epsilon_{\Theta_i, \Theta_i}^\pm X_i = X_i$ ,  $i = 1, 2$ ). The first and last equalities are easily proved with graphical calculus; in particular, the complete equation (5.10) is represented as follows:



Thus the supplementary condition of equation (4.5) for the product Q-system is satisfied by every  $H \in \text{Hom}(\Theta, \text{id})$ .  $\square$

Since the combination  $R := XW \in \text{Hom}(\text{id}, \Theta^2)$  we have defined in the last proof will come back several times, we give a precise graphical interpretation of it:



Thanks to the auto-duality of  $\Theta$ ,  $R$  and  $R^\dagger$ , which are normalized as  $R^\dagger R = \dim(\iota)^2 = \dim(\Theta)$ , can be seen as a standard pair for  $\Theta$ , and so they can be identified with the evaluation/co-evaluation morphisms associated to  $\Theta$ . However, we are not going to draw the arrows, as in principle, if we didn't invoke the auto-duality of  $\Theta$ , we would just have  $R \in \text{Hom}(\text{id}, \Theta\Theta)$ .

### 5.3 The universal construction

Consider a boundary between two *local* CFTs described by extensions  $\mathcal{A} := \mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{B}^L$  and  $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{B}^R$ , where  $\mathcal{A}_+$ ,  $\mathcal{A}_-$  are the chiral nets, by a pair of Q-systems  $Q^L = (\Theta^L, W^L, X^L)$  and  $Q^R = (\Theta^R, W^R, X^R)$  <sup>8</sup>.

Consider the braided product of the two Q-systems:

$$Q^L \times^\pm Q^R = (\Theta = \Theta^L \Theta^R, W = W^L W^R, X^\pm = \Theta^L(\epsilon_{\Theta^L, \Theta^R}^\pm) X^L \Theta^L(X^R))$$

and the corresponding extensions  $\mathcal{A} \subset \mathcal{D}^\pm$ . As a consequence of proposition 5.2.9 we have

$$\mathcal{D}^\pm \supset \mathcal{B}^L \supset \mathcal{A} \subset \mathcal{B}^R \subset \mathcal{D}^\pm$$

and the nets  $\mathcal{D}^\pm$  are generated by  $\mathcal{A}$  and two sets of charged fields

$$\Psi_{\sigma \otimes \rho}^L (\sigma \otimes \rho \prec \Theta^L) \quad \Psi_{\sigma' \otimes \rho'}^R (\sigma' \otimes \rho' \prec \Theta^R)$$

suppressing possible multiplicity indices. The braided product Q-system determines their commutation relations among each other as in equation (5.9):

$$\Psi_{\sigma' \otimes \rho'}^R \Psi_{\sigma \otimes \rho}^L = \epsilon_{\sigma \otimes \rho, \sigma' \otimes \rho'}^\pm \Psi_{\sigma \otimes \rho}^L \Psi_{\sigma' \otimes \rho'}^R \quad (5.11)$$

<sup>8</sup>The subscript "2D" on the local algebras is understood, since we are interested in two-dimensional theories. However, the results of this section are valid also for chiral theories where a boundary is just a point [35]. In the next chapter, our results will be based on more specific hypotheses.

Now recall that by equation (3.12),  $\epsilon_{\sigma \otimes \rho, \sigma' \otimes \rho'}^- = \mathbf{1}$  whenever  $\sigma \otimes \rho$  is localized to the spacelike left of  $\sigma' \otimes \rho'$ . Thus, the choice of  $\epsilon^-$  ensures that  $\mathcal{B}^L$  is left-local with respect to  $\mathcal{B}^R$ . Therefore, we consider the product

$$Q^L \times^- Q^R$$

and the extension  $\mathcal{D} := \mathcal{D}^-$ . We are ready to state the main theorem from [35]:

**Theorem 5.3.1** ("Universal construction"). *The extension  $\mathcal{D}$  of  $\mathcal{A}$  defined by the Q-system  $Q^L \times^- Q^R$  implements a boundary condition. It is universal in the sense that every irreducible boundary condition appears as a representation of  $\mathcal{D}$ .*

*Proof.* By proposition 5.2.9,  $\mathcal{D}$  is generated by  $\mathcal{B}^L$  and  $\mathcal{B}^R$ , and by proposition 5.2.10,  $\mathcal{B}^L$  is left-local with respect to  $\mathcal{B}^R$ . Hence, According to corollary 5.2.8,  $\mathcal{D}$  implements a defect.

The universal property follows because apart from the identification of the subalgebras  $\mathcal{A} \subset \mathcal{B}^L$  and  $\mathcal{A} \subset \mathcal{B}^R$ , the commutation relation equation (5.8), i.e., left locality, is the only additional relation among the generators of  $\mathcal{D}$  imposed by the product Q-system.  $\square$

By proposition 5.2.11, the extension  $\mathcal{A} \subset \mathcal{D}$  is not in general irreducible and since both  $\mathcal{B}^L$  and  $\mathcal{B}^R$  are assumed to be local, the relative commutant  $\mathcal{A}' \cap \mathcal{D}$  coincides with the center  $\mathcal{D}' \cap \mathcal{D}$ . The central decomposition gives irreducible representations of  $\mathcal{D}$ , restricting to representations of both  $\mathcal{B}^L$  and  $\mathcal{B}^R$  on a common Hilbert space.

Let

$$\Theta^L = \bigoplus_{\rho} Z_{\rho}^L \cdot \rho \quad \Theta^R = \bigoplus_{\rho} Z_{\rho}^R \cdot \rho$$

where each  $\rho$  is of the form  $\rho^+ \otimes \rho^-$ . By proposition 5.2.11, the dimension of the center is

$$\dim \text{Hom}(\Theta^L, \Theta^R) = \sum_{\rho} Z_{\rho}^L \cdot Z_{\rho}^R = \text{Tr}[(Z^L)^T Z^R] \quad (5.12)$$

and there are as many irreducible representations, implementing different boundary conditions.

**Lemma 5.3.2.** *Each minimal projection  $E \in \mathcal{D}' \cap \mathcal{D}$  defines an irreducible extension  $\mathcal{A} \simeq \mathcal{A}E \subset \mathcal{D}E$ . In particular, the representation  $\mathcal{D}E$  contains a unique vacuum vector.*

*Proof.* Every Q-system contains the identity at least with multiplicity one. By proposition 5.2.11, the number of minimal central projections is equal to the dimension of the relative commutant  $\mathcal{A}' \cap \mathcal{D}$ , which in turn is equal to the multiplicity of the identity in  $\Theta = \Theta^R \Theta^L$ . Since  $\Theta$  decomposes as the direct sum of the projected endomorphisms  $\Theta_E$ , it follows that  $\dim \text{Hom}(\text{id}, \Theta_E) = 1$  for each minimal projection  $E$ .<sup>9</sup> Therefore, the extension  $\mathcal{A}E \subset \mathcal{D}E$  is irreducible, and the vacuum representation of  $\mathcal{A}$  is contained exactly with multiplicity one.  $\square$

Combining these results with the universal construction we get:

**Corollary 5.3.3.** *The minimal central projections of the braided product between two local QFTs classify the irreducible boundary conditions.*

Note that this result remains valid even when  $\mathcal{B}^L$  and  $\mathcal{B}^R$  describe two different CFTs. In that case, the corresponding boundary conditions represent interfaces between the two theories, often referred to in the literature as *phase boundaries* [35]. Although our main interest lies in defects within a single CFT, we will retain the labels  $L/R$  on  $\mathcal{B}^{L/R}$  to emphasize that the two theories are embedded differently into  $\mathcal{D}$ . This distinction is in fact crucial: in the Ising model, we will see that the duality defect can be detected precisely because the theories on the two sides are allowed to be distinct, though isomorphic. Indeed, as discussed in section 2.5, the defect  $N$  relates the Ising model to its  $\mathbb{Z}_2$ -orbifold theory.

<sup>9</sup>From the central decomposition we have  $\mathbf{1}_{\mathcal{D}} = \sum_a E_a$ . The projections  $P_a = \bar{\iota}(E_a) \in \text{Hom}(\Theta, \Theta)$  which are used to construct the reduced Q-system, as show in section 5.2.1, satisfy  $\sum_a P_a = \mathbf{1}_{\mathcal{A}}$  and  $P_a P_b = P_b P_a = \delta_{ab}$ . For this reason  $\Theta$  decomposes as  $\Theta \simeq \bigoplus_a \Theta_{E_a}$ . Therefore  $\dim \text{Hom}(\text{id}, \Theta) = \sum_a \dim \text{Hom}(\text{id}, \Theta_{E_a})$ .

Before proceeding with the general strategy needed to compute boundary conditions, we want to note that the number of defects predicted within this framework coincide with the expected one. Indeed, the number of operators commuting with the energy-momentum tensor predicted by equation (2.16) agrees with the dimension of the center of equation (5.12), assuming that we have the same theory on both sides of the defect.

### 5.3.1 Computation of the boundary conditions

From proposition 5.2.11 we know that a basis for the center is given by

$$B_\rho = (\Psi_\rho^L)^\dagger \Psi_\rho^R$$

where  $\rho \prec \Theta^L$  and  $\rho \prec \Theta^R$ . In an irreducible representation, they are multiples of  $\mathbf{1}$ , thus, every representation comes with a characteristic set of relations between the charged fields  $\Psi_\rho^L$  and  $\Psi_\rho^R$ , corresponding to the boundary conditions. In presence of multiplicities for the charged fields we get a complex matrix of boundary conditions,  $\pi_a(B_\rho^{ij}) = \pi_a((\Psi_\rho^{i,L})^\dagger \Psi_\rho^{j,R})$ ; this matrix can be interpreted as a measure of the overlapping of the two embedded spaces of left and right charged fields<sup>10</sup>.

We now need to determine the values of the central operators  $B_\rho$  in each representation. To do so, we need to know the minimal central projections  $\{E_a\}_{a=1}^{\dim(\mathcal{D}' \cap \mathcal{D})}$  of  $\mathcal{D}$ . Then we will be able to expand the  $B_\rho$  as

$$B_\rho = \sum_a \pi_a(B_\rho) \cdot E_a \quad (5.13)$$

where  $\pi_a(B_\rho) \in \mathbb{C}$  and  $\pi_a(E_b) = \delta_{a,b} \mathbf{1}$ .

To accomplish this task, we should reformulate the problem, since working directly within the center  $\mathcal{D}' \cap \mathcal{D}$  can be rather difficult, involving the non-trivial commutation relations of charged fields.

We introduce a linear bijection [35]:

$$\chi : \text{Hom}(\Theta^R, \Theta^L) \rightarrow \mathcal{D}' \cap \mathcal{D}, \quad T \mapsto \iota(R^{L\dagger} \Theta^L(T)) \cdot V$$

Then one has the following:

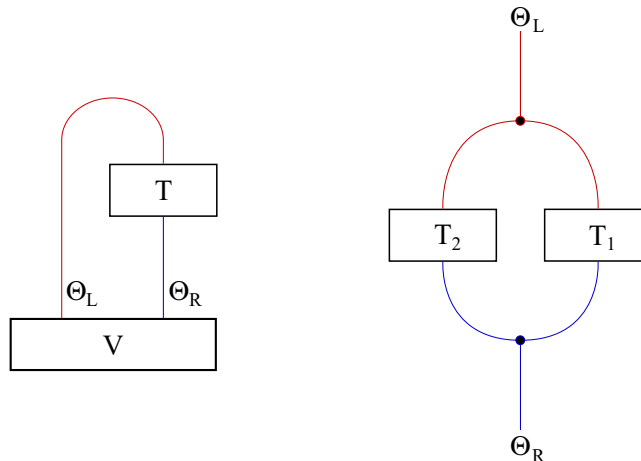
**Proposition 5.3.4.** *The above bijection satisfies*

$$\chi(T_1) \cdot \chi(T_2) = \chi(T_1 * T_2) \quad (5.14)$$

where the associative  $*$ -product in  $\text{Hom}(\Theta^R, \Theta^L)$  is defined as

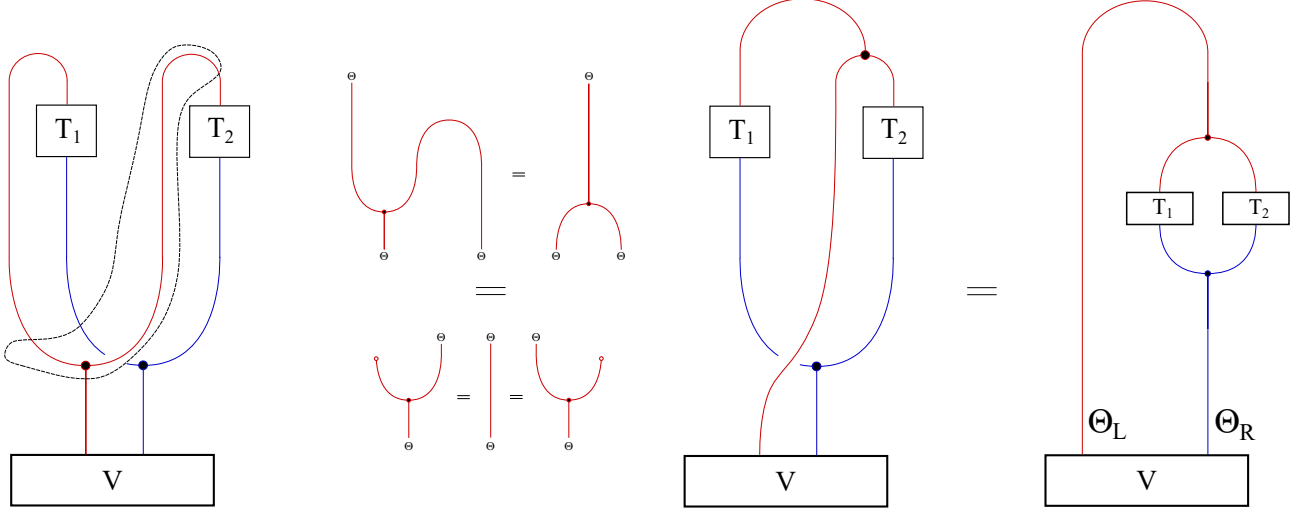
$$T_1 * T_2 := X^{L\dagger} \cdot \Theta^L(T_1) T_2 \cdot X^R = X^{L\dagger} \cdot T_2 \Theta^R(T_1) \cdot X^R$$

*Proof.* We can represent graphically the  $\chi$  and  $*$ -products respectively as:

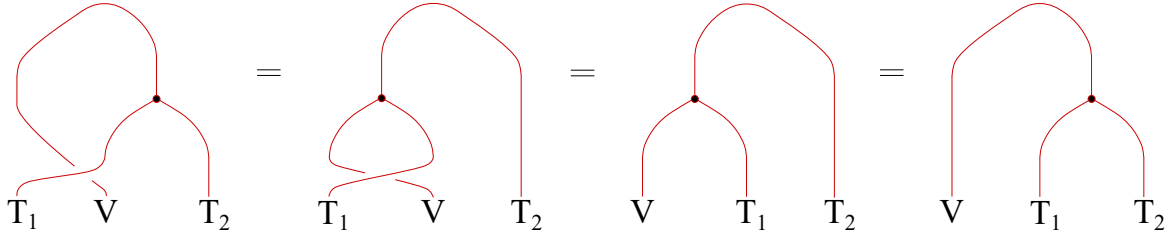


<sup>10</sup>In general, this relation is not invertible, i.e. we cannot express one field as a linear combination of the others; in this case the boundary condition is just a sesquilinear relation between the charged fields on both sides. See appendix C for more details.

and prove equation (5.14) starting from  $\chi(T_1) \cdot \chi(T_2)$ :

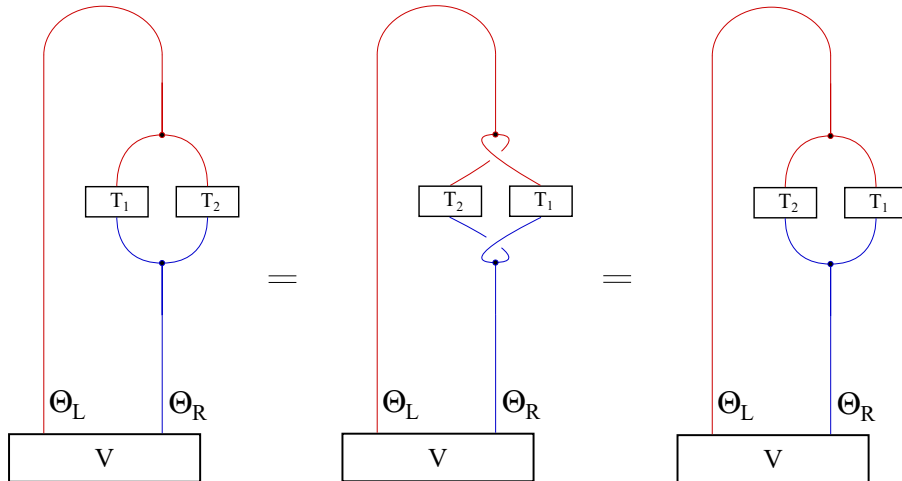


Where we have used the Frobenius property combined with the unit property of Q-systems within the part encircled by the dotted line<sup>11</sup>. The last equality is non-trivial and involves the following moves:

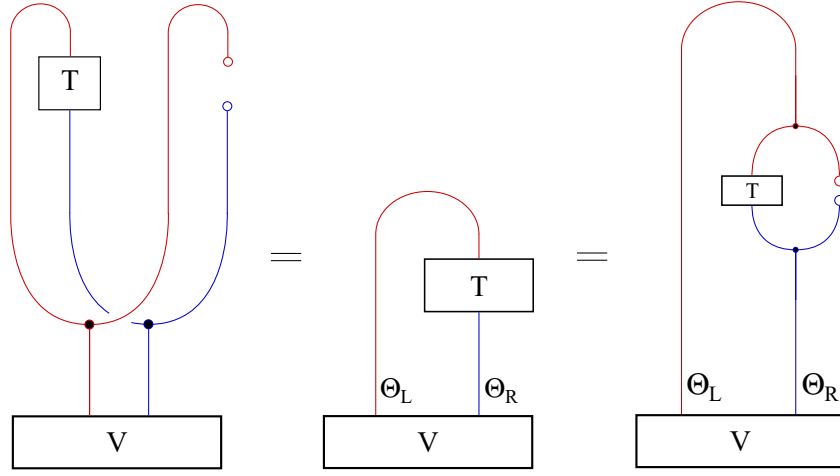


Where we have passed the braiding between  $\Theta^L$  and  $\Theta^R$  over  $T_1$ , denoted the morphism to which lines are attached on the bottom and we have used the associativity of the product together with the commutativity of the Q-system.

The  $*$ -product is also commutative, because the Q-systems  $Q^L$  and  $Q^R$  are commutative:



The unit is given by  $W^L W^{R\dagger}$ :



□

**Proposition 5.3.5.** *The above bijection satisfies*

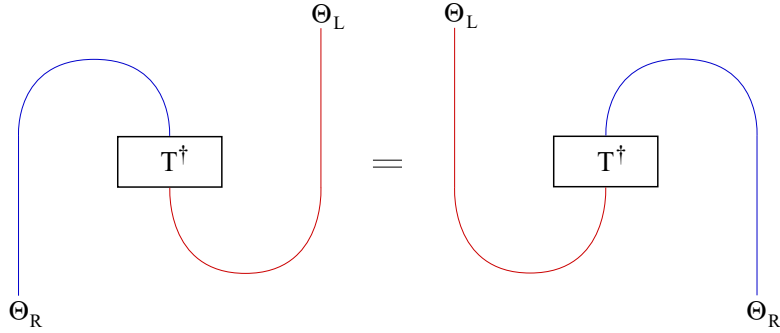
$$\chi(T)^\dagger = \chi(F(T^\dagger)) \quad (5.15)$$

where

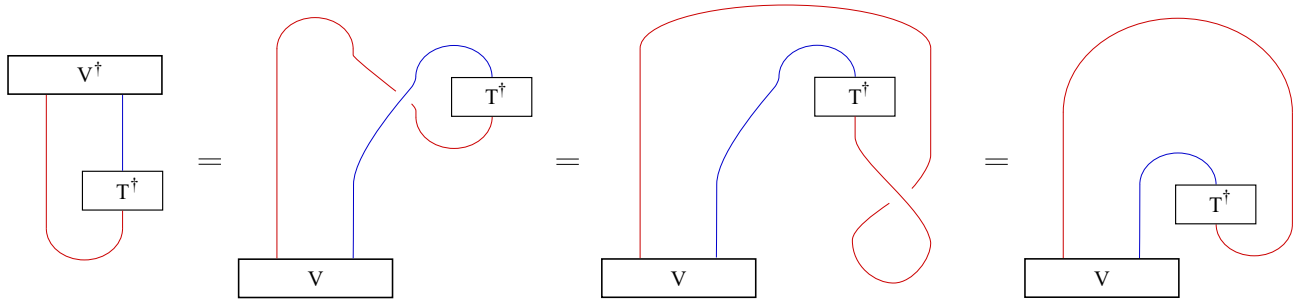
$$F : \text{Hom}(\Theta^L, \Theta^R) \rightarrow \text{Hom}(\Theta^R, \Theta^L), \quad T^\dagger \mapsto R^{R^\dagger} \Theta^R (T^\dagger R^L) = \Theta^L (R^{R^\dagger} T^\dagger) R^L$$

is called *Frobenius conjugation*

*Proof.* We can represent the Frobenius conjugation as:



Thus equation (5.15) is proved as follows:



Where we have used  $V^\dagger = W^\dagger X^\dagger V = R^\dagger V$  and the commutativity of the left Q-system. □

<sup>11</sup>The top-right theta appearing in the Frobenius property is composed with  $W^\dagger$ . Thus we have represented that part of the diagram with a "cap" as we do for  $R^\dagger = W^\dagger X^\dagger$  and applied the Frobenius property to that diagram. On the right-hand-side of the Frobenius property, the top-right leg ends again in the identity through  $W^\dagger$ ; therefore we have used the unit property to remove it.

Now let  $\Psi_\rho^L = W_\rho^{L\dagger} \Theta^L(W^{R\dagger})V$  and  $\Psi_\rho^R = W_\rho^{R\dagger} W^L V$  as in the previous section. For simplicity, we suppress possible multiplicity indices if these spaces are more than one-dimensional, and assume the charged field operators to be isometries, i.e.  $W_\rho^{Y\dagger} W_\rho^Y = d_Y / \dim(\rho)$ ,  $Y = L, R$ , in accordance with proposition 4.2.6, where  $d_Y = \sqrt{\dim(\Theta^Y)}$ .

By Frobenius reciprocity  $\text{Hom}(\rho, \Theta^Y) \simeq \text{Hom}(\bar{\rho}, \Theta^Y)$  so that we do not need to define an independent intertwiner  $W_{\bar{\rho}}$ . Instead, we set  $W_{\bar{\rho}}^Y := R_{\bar{\rho}}^\dagger (W_\rho^{Y\dagger} R^Y) = \Theta^Y(\tilde{R}_\rho^\dagger W_\rho^{Y\dagger}) R^Y \in \text{Hom}(\bar{\rho}, \Theta^Y)$  with the same normalization, where  $\tilde{R}_\rho^\dagger \in \text{Hom}(\rho\bar{\rho}, \text{id})$  and  $R_\rho^\dagger \in \text{Hom}(\bar{\rho}\rho, \text{id})$  are part of a standard pair for  $\rho$  and  $\bar{\rho}$ . This is represented as the snake diagrams (2.24) used to define dual morphisms in a fusion category<sup>12</sup>. Then  $\Psi_\rho^Y$  and  $B_{\bar{\rho}}^Y$  are the corresponding central operators.

The following proposition gives the algebraic relations among  $B_\rho$  in terms of the pair of Q-systems  $Q^L, Q^R$ .

**Proposition 5.3.6.** *The algebraic relations of the center  $\mathcal{D}' \cap \mathcal{D}$  of the braided product of two local extensions (corresponding to two commutative Q-systems) are*

$$B_\rho^\dagger = B_{\bar{\rho}}, \quad B_\rho \cdot B_\sigma = \sum_{\tau \prec \rho\sigma} f_{\rho\sigma}^\tau \cdot B_\tau$$

where

$$f_{\rho\sigma}^\tau = f_{\sigma,\rho}^\tau = \frac{\dim(\tau)^2}{d_L d_R} \cdot W_\tau^{L\dagger} \cdot ((W_\rho^L W_\rho^{R\dagger}) * ((W_\sigma^L W_\sigma^{R\dagger})) \cdot W_\tau^R$$

The expressions given for  $f_{\rho\sigma}^\tau$  are self-intertwiners  $\in \text{Hom}(\tau, \tau)$ , hence multiples of  $\mathbf{1}$ .

*Proof.* We give just the idea of the proof. The first relation follows from  $F(W_\rho^R W_\rho^{L\dagger}) = W_{\bar{\rho}}^L W_{\bar{\rho}}^{R\dagger}$  thanks to equation (5.15). The second relation is just the expansion of  $(W_\rho^L W_\rho^{R\dagger}) * ((W_\sigma^L W_\sigma^{R\dagger}))$  in terms of  $W_\tau^L W_\tau^{R\dagger}$ , with the appropriate normalization and by using equation (5.14).  $\square$

Knowing the algebra of the  $B_\rho$ 's, one can compute their decomposition into central projections. In particular now the task is to find  $I_a = \chi^{-1}(E_a)$ , which are the minimal projections in  $\text{Hom}(\Theta^R, \Theta^L)$  with respect to the  $*$ -product, and expand  $\chi^{-1}(B_\rho)$  in terms of  $I_a$ , i.e.

$$W_\rho^L W_\rho^{R\dagger} = \sum_a \pi_a(B_\rho) \cdot I_a \tag{5.16}$$

which is just the image of equation (5.13) under  $\chi^{-1}$ .

To do so, one in principle has to impose three conditions:

$$\begin{aligned} \text{self-adjointness: } I_a^\dagger &= F(I_a) \\ \text{idempotency: } I_a * I_{a'} &= \delta_{a,a'} \cdot I_a \\ \text{completeness: } \sum_a I_a &= W^L W^{R\dagger} \end{aligned} \tag{5.17}$$

Minimality is then automatically ensured if the number of  $I_a$  exhausts the dimension of the center  $\dim(\mathcal{D}' \cap \mathcal{D}) = \dim \text{Hom}(\Theta^R, \Theta^L) = \text{Tr}[(Z^L)^T Z^R]$  [35]. This is not an easy calculation to do in general, but it simplifies considerably for diagonal theories.

**Proposition 5.3.7.** *For two-dimensional theories, if  $\mathcal{A}_+ = \mathcal{A}_-$  is completely rational and both Q-systems  $Q^L$  and  $Q^R$  equal the canonical Q-system in the category  $\mathcal{C}^{DHR}(\mathcal{A}) \boxtimes \mathcal{C}^{DHR}(\mathcal{A})^{opp}$ , each sector is of the form  $\rho \otimes \bar{\rho}$ , there are no multiplicities, and the coefficients  $f_{\rho\sigma}^\tau$  simplify drastically:*

$$f_{\rho \otimes \bar{\rho} \sigma \otimes \bar{\sigma}}^{\tau \otimes \bar{\tau}} = \frac{\dim(\tau)}{\dim(\rho) \dim(\sigma)} \cdot N_{\sigma\rho}^\tau$$

<sup>12</sup>I.e. its diagram coincide with the graphical representation of the Frobenius conjugation, but with  $\rho$  as bottom line,  $\Theta^Y$  as top line and  $W_\rho^{Y\dagger}$  as intertwiner.

*Proof.* Recall that in the two-dimensional case we have  $W_{\rho \otimes \bar{\rho}}^{Y\dagger} W_{\rho \otimes \bar{\rho}}^Y = d_Y / \dim(\rho \otimes \bar{\rho}) = d_Y / \dim(\rho)^2$ ,  $Y = L, R$ . In particular, recall that for the canonical Q-system we have  $W_{\rho \otimes \bar{\rho}}^Y = \frac{\sqrt{d_Y}}{\dim(\rho)} T_\rho^Y$  where  $T_\rho^Y \in \text{Hom}(\rho \otimes \bar{\rho}, \Theta^Y)$  is an isometry. Now consider the definition of  $f_{\rho \otimes \bar{\rho} \sigma \otimes \bar{\sigma}}^{\tau \otimes \bar{\tau}}$ :

$$\begin{aligned} f_{\rho \otimes \bar{\rho} \sigma \otimes \bar{\sigma}}^{\tau \otimes \bar{\tau}} &= \frac{\dim(\tau \otimes \bar{\tau})^2}{d_L d_R} \cdot W_{\tau \otimes \bar{\tau}}^{L\dagger} \cdot ((W_{\rho \otimes \bar{\rho}}^L W_{\rho \otimes \bar{\rho}}^{R\dagger}) * ((W_{\sigma \otimes \bar{\sigma}}^L W_{\sigma \otimes \bar{\sigma}}^{R\dagger}))) \cdot W_{\tau \otimes \bar{\tau}}^R = \\ &= \frac{\dim(\tau)^4}{d_L d_R} \cdot W_{\tau \otimes \bar{\tau}}^{L\dagger} \cdot X^{L\dagger} \cdot W_{\sigma \otimes \bar{\sigma}}^L W_{\sigma \otimes \bar{\sigma}}^{R\dagger} \Theta^R(W_{\rho \otimes \bar{\rho}}^L W_{\rho \otimes \bar{\rho}}^{R\dagger}) \cdot X^R \cdot W_{\tau \otimes \bar{\tau}}^R \end{aligned}$$

Now it is easy to realize graphically that  $W_{\sigma \otimes \bar{\sigma}}^{R\dagger}$  and  $\Theta^R(W_{\rho \otimes \bar{\rho}}^L)$  commute, so that we can bring all the things labeled as R to the right. Then we can focus on that part, that is:

$$W_{\sigma \otimes \bar{\sigma}}^{R\dagger} \Theta^R(W_{\rho \otimes \bar{\rho}}^{R\dagger}) \cdot X^R \cdot W_{\tau \otimes \bar{\tau}}^R$$

The other half is resolved in the same way.

Now we use the definition of X for the canonical Q-system:

$$W_{\sigma \otimes \bar{\sigma}}^{R\dagger} \Theta^R(W_{\rho \otimes \bar{\rho}}^{R\dagger}) \left( d_R^{-1/2} \sum_{\rho', \sigma', \tau'} \left( \frac{d_{\rho'} d_{\sigma'}}{d_{\tau'}} \right)^{1/2} \Theta(T_{\sigma'}) T_{\rho'} \cdot \left( \sum_a t_a \otimes t_{\bar{a}} \right) \cdot T_{\tau'}^\dagger \right) W_{\tau \otimes \bar{\tau}}^R$$

Using the  $W_{\rho \otimes \bar{\rho}}^Y = \frac{\sqrt{d_Y}}{\dim(\rho)} T_\rho^Y$  we get:

$$d_R^{-1/2} \left( \frac{d_\rho d_\sigma}{d_\tau} \right)^{1/2} \frac{\sqrt{d_R}}{d_\rho} \frac{\sqrt{d_R}}{d_\sigma} \left( \sum_a t_a \otimes t_{\bar{a}} \right) \frac{\sqrt{d_R}}{d_\tau} = \frac{d_R}{d_\tau} \frac{1}{\sqrt{d_\tau d_\rho d_\sigma}} \left( \sum_a t_a \otimes t_{\bar{a}} \right)$$

Recall now that  $t_a \in \text{Hom}(\tau, \rho\sigma)$  are isometries, so that when we combine the two parts we get:

$$f_{\rho \otimes \bar{\rho} \sigma \otimes \bar{\sigma}}^{\tau \otimes \bar{\tau}} = \frac{\dim(\tau)^4}{d_L d_R} \cdot \frac{d_L d_R}{d_\tau^2} \frac{1}{d_\tau d_\rho d_\sigma} \left( \sum_b t_b \otimes t_{\bar{b}} \right)^\dagger \left( \sum_a t_a \otimes t_{\bar{a}} \right) = \frac{\dim(\tau)}{\dim(\rho) \dim(\sigma)} \cdot N_{\sigma\rho}^\tau$$

□

**Corollary 5.3.8.** *Under the hypothesis of the last proposition:*

$$\pi_\sigma(\Psi_{\rho \otimes \bar{\rho}}^{L\dagger} \Psi_{\rho \otimes \bar{\rho}}^R) = \frac{S_{\rho, \sigma} S_{0,0}}{S_{\rho,0} S_{0,\sigma}} \quad (5.18)$$

*Proof.* From the last proposition together with the algebra for the central operators, we see that  $\dim(\rho) B_\rho$  represent the fusion rules. This implies that the various irreducible representations are labelled by the irreducible chiral sectors  $\sigma \in \mathcal{C}^{DHR}(\mathcal{A})$ , and the equation

$$\pi_\sigma(B_\rho) \dim(\rho) \cdot \pi_\sigma(B_\gamma) \dim(\gamma) = \sum_{\tau \prec \rho\gamma} N_{\rho\gamma}^\tau \cdot \pi_\sigma(B_\tau) \dim(\tau)$$

has the above solution. □

Note that for diagonal theories charged fields do not have multiplicities: for this reason the "angles" or "overlapping values"  $\pi_\sigma(B_\rho)$  between the charged fields defined on the two sides of the defect can be either zero or a generic scalar number. Then recalling that charged fields are normalized to be isometries, when  $\pi_\sigma(B_\rho)$  is a phase, we can invert the above relation to get:

$$\Psi_{\rho \otimes \bar{\rho}}^R = \pi_\sigma(B_{\rho \otimes \bar{\rho}}) \cdot \Psi_{\rho \otimes \bar{\rho}}^L = \frac{S_{\rho, \sigma} S_{0,0}}{S_{\rho,0} S_{0,\sigma}} \cdot \Psi_{\rho \otimes \bar{\rho}}^L$$

When  $\pi_\sigma(B_\rho)$  is not a phase, the above equation is no longer valid, as the isometry condition for  $\Psi_{\rho\otimes\bar{\rho}}^R$  would not be satisfied. In particular, when  $\pi_\sigma(B_\rho) = 0$ , the two fields  $\Psi_{\rho\otimes\bar{\rho}}^R$  and  $\Psi_{\rho\otimes\bar{\rho}}^L$  have zero overlap, i.e. are independent.

Moreover, recalling equations (2.13) and (2.14) from chapter 2, we can rewrite equation (5.18) as:

$$\pi_\sigma(\Psi_{\rho\otimes\bar{\rho}}^{L\dagger}\Psi_{\rho\otimes\bar{\rho}}^R) = \frac{S_{\rho,\sigma}S_{0,0}}{S_{\rho,0}S_{0,\sigma}} = \frac{S_{\rho,\sigma}/S_{\rho,0}}{S_{0,\sigma}/S_{0,0}} = \frac{\langle\Psi_{\rho\otimes\bar{\rho}}|\hat{\mathcal{L}}_\sigma|\Psi_{\rho\otimes\bar{\rho}}\rangle}{\langle\mathcal{L}_\sigma\rangle} \quad (5.19)$$

where  $\hat{\mathcal{L}}_\sigma$  denotes the operator associated to the defect labeled by  $\sigma$  and describing its local action, and  $|\Psi_{\rho\otimes\bar{\rho}}\rangle$  is the primary state built acting on the vacuum with the primary operator generating the sector labeled by  $\rho$ . Now it is clear that we have found exactly the first term of equation (2.11), describing what happens when a defect is moved past a local operator!

It is not a surprise that the other terms are missing: as we have said, we are restricting our attention to boundary conditions, that are extensions of the type:

$$\mathcal{D}(\mathcal{O}) = \mathcal{B}^L(\mathcal{O}) \vee \mathcal{B}^R(\mathcal{O})$$

i.e. without additional degrees of freedom. Since the other terms in (2.11) involve junctions lying on our defect, these cannot appear in our treatment.

Finally note that the numerator of (5.19) is exactly the eigenvalue expected for the action of a Verlinde line on a local operator, found in the euclidean context in equation (2.13).

## 5.4 Defects of the relativistic 2D Ising model CFT

In the Ising model, both Q-systems are canonical and the extension  $\mathcal{D}$  is generated by  $\mathcal{A} = Vir_{1/2} \otimes Vir_{1/2}$  and the charged fields  $\Psi_{\psi\otimes\psi}^Y$  and  $\Psi_{\sigma\otimes\sigma}^Y$ ,  $Y = L, R$ . Furthermore, equation (5.11) for  $Q_{can}^L \times^- Q_{can}^R$ , together with the chiral braidings of equation (3.31), gives us the following commutation relations:

$$\begin{aligned} \Psi_{\psi\otimes\psi}^R \Psi_{\psi\otimes\psi}^L &= \Psi_{\psi\otimes\psi}^L \Psi_{\psi\otimes\psi}^R & \Psi_{\psi\otimes\psi}^R \Psi_{\sigma\otimes\sigma}^L &= (u \otimes u) \Psi_{\sigma\otimes\sigma}^L \Psi_{\psi\otimes\psi}^R \\ \Psi_{\sigma\otimes\sigma}^R \Psi_{\psi\otimes\psi}^L &= (u \otimes u) \Psi_{\psi\otimes\psi}^L \Psi_{\sigma\otimes\sigma}^R & \Psi_{\sigma\otimes\sigma}^R \Psi_{\sigma\otimes\sigma}^L &= ((rr^\dagger - itt^\dagger) \otimes ((rr^\dagger + itt^\dagger))) \Psi_{\sigma\otimes\sigma}^L \Psi_{\sigma\otimes\sigma}^R \end{aligned} \quad (5.20)$$

The center of  $\mathcal{D}$  is spanned by:

$$B_{1\otimes 1} = \mathbf{1} \quad B_{\psi\otimes\psi} = \Psi_{\psi\otimes\psi}^{L\dagger} \Psi_{\psi\otimes\psi}^R \quad B_{\sigma\otimes\sigma} = \Psi_{\sigma\otimes\sigma}^{L\dagger} \Psi_{\sigma\otimes\sigma}^R$$

To compute the minimal central projections, the hardest part would be to compute the coefficients  $f_{\rho\sigma}^\tau$ . For the Ising model, being a diagonal model, these coefficients are known from proposition 5.3.7. Then, since in this case the algebra of the  $B_\rho$ 's is rather simple, the minimal central projections can be computed directly using proposition 5.3.6.

Consider the algebra of the central elements:

$$B_{\psi\otimes\psi}^2 = \mathbf{1} \quad B_{\psi\otimes\psi} B_{\sigma\otimes\sigma} = B_{\sigma\otimes\sigma} = B_{\sigma\otimes\sigma} B_{\psi\otimes\psi} \quad B_{\sigma\otimes\sigma}^2 = \frac{1}{2}(\mathbf{1} + B_{\psi\otimes\psi})$$

First note that, since  $B_{\psi\otimes\psi}^2 = \mathbf{1}$ , the operator  $B_{\psi\otimes\psi}$  has two eigenvalues (possibly with multiplicities)  $\lambda = \pm 1$ . The associated projectors are

$$P_\pm = \frac{\mathbf{1} \pm B_{\psi\otimes\psi}}{2}$$

Instead, from  $B_{\sigma\otimes\sigma}^2 = \frac{1}{2}(\mathbf{1} + B_{\psi\otimes\psi})$  we see that  $B_{\sigma\otimes\sigma}$  has three eigenvalues:  $\lambda = 0, \pm 1$ . The zero eigenvalue is associated to  $P_-$ , whereas the remaining two must be proportional to  $\mathbf{1} \pm B_{\sigma\otimes\sigma}$ . However, these two combinations are mutually orthogonal, but they are not with respect to  $P_-$ . We can then use

the fact that  $P_- P_+ = 0$  together with the fact that when  $B_{\psi \otimes \psi}$  is restricted to the  $\lambda = +1$  eigenspace,  $P_+ = 1$ , so that the three combinations

$$E_1 = \frac{1}{2}(1 + B_{\psi \otimes \psi})\frac{1}{2}(1 + B_{\sigma \otimes \sigma}) \quad E_2 = \frac{1}{2}(1 + B_{\psi \otimes \psi})\frac{1}{2}(1 - B_{\sigma \otimes \sigma}) \quad E_3 = \frac{1 - B_{\psi \otimes \psi}}{2}$$

are orthogonal. In particular note that  $E_1 E_3 = 0 = E_2 E_3$  thanks to the fact that  $B_{\sigma \otimes \sigma}$  and  $B_{\psi \otimes \psi}$  commute. Since in this model all chiral sectors are self-conjugate, we also satisfy  $E_i^\dagger = E_i$ ,  $i = 1, 2, 3$ . Finally, we can check the completeness relation:

$$E_1 + E_2 + E_3 = \frac{1}{4}(1 + 2B_{\sigma \otimes \sigma} + B_{\psi \otimes \psi}) + \frac{1}{4}(1 - 2B_{\sigma \otimes \sigma} + B_{\psi \otimes \psi}) + \frac{1 - B_{\psi \otimes \psi}}{2} = 1$$

Hence  $E_i$ ,  $i = 1, 2, 3$ , are the three minimal central projections.

According to equation (5.13), we expand the central elements  $B_\rho$  in terms of the minimal projections and find their values in each irreducible representation.

$$B_{1 \otimes 1} = E_1 + E_2 + E_3 \quad B_{\psi \otimes \psi} = E_1 + E_2 - E_3 \quad B_{\sigma \otimes \sigma} = E_1 - E_2$$

Hence we have found:

$$\begin{aligned} \pi_1(B_{1 \otimes 1}) &= 1 & \pi_1(B_{\psi \otimes \psi}) &= 1 & \pi_1(B_{\sigma \otimes \sigma}) &= 1 \\ \pi_2(B_{1 \otimes 1}) &= 1 & \pi_2(B_{\psi \otimes \psi}) &= 1 & \pi_2(B_{\sigma \otimes \sigma}) &= -1 \\ \pi_3(B_{1 \otimes 1}) &= 1 & \pi_3(B_{\psi \otimes \psi}) &= -1 & \pi_3(B_{\sigma \otimes \sigma}) &= 0 \end{aligned}$$

It is easy to see that these values agree with the general formula given in corollary 5.3.8.

We can translate the above values into the following relations between charged fields:

1. *Trivial defect*  $\mathcal{I}$ :  $\Psi_{\psi \otimes \psi}^R = \Psi_{\psi \otimes \psi}^L$ ,  $\Psi_{\sigma \otimes \sigma}^R = \Psi_{\sigma \otimes \sigma}^L$
2.  $\mathbb{Z}_2$  *defect*  $\eta$ :  $\Psi_{\psi \otimes \psi}^R = \Psi_{\psi \otimes \psi}^L$ ,  $\Psi_{\sigma \otimes \sigma}^R = -\Psi_{\sigma \otimes \sigma}^L$
3. *Duality defect*  $N$ :  $\Psi_{\psi \otimes \psi}^R = -\Psi_{\psi \otimes \psi}^L$ ,

Note that for the duality defect we have just a condition on  $\Psi_{\psi \otimes \psi}$ . This is not a surprise: in our description we cannot have junctions along the defect, hence when we move the duality defect through  $\Psi_{\sigma \otimes \sigma}$  we identify only the first term of the resulting sum of equation (2.11). In this case that term is just 0. Nevertheless we still detect the duality defect through its action on  $\Psi_{\psi \otimes \psi}$ . Furthermore, the fact that we don't have any compatibility condition between  $\Psi_{\sigma \otimes \sigma}^L$  and  $\Psi_{\sigma \otimes \sigma}^R$  means that the two fields are independent. Since the two theories on the two sides of the defects may not be identical, but also isomorphic, we conclude that one of these two fields must be interpreted as the spin field  $\sigma$  while the other as the disorder field  $\mu$ . In other words, on one side we have the Ising CFT and on the other the *Ising*/ $\mathbb{Z}_2$  orbifold theory. This should not surprise since  $N$  is the duality defect that allows for the identification of these two, as seen in section 2.5.

To make this identification more explicit we can consider the fermionic extension of the Ising model, obtained by introducing the two operators:

$$\psi_+ := \sqrt{-i} \cdot \Psi_{\sigma \otimes \sigma}^{L\dagger}(u \otimes \mathbf{1})\Psi_{\sigma \otimes \sigma}^R \quad \psi_- := \sqrt{i} \cdot \Psi_{\sigma \otimes \sigma}^{L\dagger}(\mathbf{1} \otimes u)\Psi_{\sigma \otimes \sigma}^R$$

Indeed, they satisfy the following relations<sup>13</sup>:

$$\psi_\pm^\dagger = \psi_\pm \quad \psi_\pm^2 = \frac{1}{2}(1 - \Psi_{\psi \otimes \psi}^L \Psi_{\psi \otimes \psi}^R) \quad \psi_+ \psi_- = -\psi_- \psi_+ = \frac{i}{2}(\Psi_{\psi \otimes \psi}^R - \Psi_{\psi \otimes \psi}^L) \quad (5.21)$$

Moreover they are by construction chiral intertwiners:

$$\psi_+ A = (\psi \otimes \text{id})(A)\psi_+ \quad \psi_- A = (\text{id} \otimes \psi)(A)\psi_- \quad A \in \mathcal{A}$$

<sup>13</sup>Since the proofs of these relations involving the fermions are quite involved, they are showed in appendix D.

In particular they belong to the relative commutants  $(\mathbf{1} \otimes \mathcal{A})' \cap \mathcal{D}$  and  $(\mathcal{A} \otimes \mathbf{1})' \cap \mathcal{D}$  respectively; the latter are generated by  $(\mathcal{A} \otimes \mathbf{1}) \vee \psi_+$  and  $(\mathbf{1} \otimes \mathcal{A}) \vee \psi_-$  respectively.

Therefore, in the representations  $\pi_1$  and  $\pi_2$ ,  $\psi_{\pm} = 0$ , whereas in  $\pi_3$  they are a pair of left- and right-moving selfadjoint unitary Majorana fields implementing the chiral automorphism  $\psi$ .

The interesting fact is that we can compute

$$\sqrt{\pm i} \cdot \psi_{\pm} \Psi_{\sigma \otimes \sigma}^R = \frac{1}{2} (\Psi_{\sigma \otimes \sigma}^L - \Psi_{\psi \otimes \psi}^R \Psi_{\sigma \otimes \sigma}^L) \quad (5.22)$$

This vanishes in the representations  $\pi_1$  and  $\pi_2$  (since  $\Psi_{\psi \otimes \psi}^R = \Psi_{\psi \otimes \psi}^L$  and  $\Psi_{\psi \otimes \psi}^L \Psi_{\sigma \otimes \sigma}^L = \Psi_{\sigma \otimes \sigma}^L$ ), while it is equal to  $\Psi_{\sigma \otimes \sigma}^L$  in the representation  $\pi_3$ , giving us:

$$\Psi_{\sigma \otimes \sigma}^L = \sqrt{\pm i} \cdot \psi_{\pm} \Psi_{\sigma \otimes \sigma}^R \quad \Psi_{\sigma \otimes \sigma}^R = \sqrt{\mp i} \cdot \psi_{\pm} \Psi_{\sigma \otimes \sigma}^L$$

These are the AQFT analogue of the fusion rule (1.45), thus confirming the identification, in the representation  $\pi_3$ , of one of the two  $\Psi_{\sigma \otimes \sigma}^Y$ ,  $Y = L, R$ , with the disorder field  $\mu$ .<sup>14</sup>

Before ending this section we make an observation: note that no primary operator is left invariant by all boundary conditions. This may seem a coincidence due to the fact that in the Ising model we just have two primary operators (apart from the identity). However, as we show in appendix C, this is a structural result: if we look for fixed points in our net  $\mathcal{B} \supset \mathcal{A}$ , i.e. operators left invariant, with respect to all defects, we always end up with the Virasoro net  $\mathcal{A} = \text{Vir}_c \otimes \text{Vir}_c$ .

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<sup>14</sup>The normalization conventions used in the AQFT literature differ from the standard CFT conventions adopted above. In particular, the coefficient appearing in the short-distance product (1.45) of  $\psi$  and  $\sigma$  is convention-dependent and depends on the normalization chosen for the order, disorder and Majorana fields. Therefore the relation above should not be compared directly with the coefficient of the Euclidean OPE.

## Chapter 6

# The bridge with Euclidean CFT

The previous chapter classified topological defects using the relativistic framework of algebraic quantum field theory. However, defects in two-dimensional CFT are most commonly studied in the Euclidean formulation, where one uses holomorphic fields, modular invariance and categorical methods. The purpose of this chapter is to explain why these two apparently different approaches are in fact based on the same underlying algebraic structure. This is the conceptual bridge between conformal nets and vertex operator algebras.

We begin by introducing vertex operator algebras. A VOA provides an axiomatization of chiral CFT which is close to the Euclidean and field-theoretic point of view: states are related to fields through the operator-state correspondence, while the algebraic structure is encoded in the vertex operators and their mode expansions. Conformal nets, on the other hand, start from bounded operator algebras localized in intervals and emphasize locality, covariance and von Neumann algebra techniques. Although the two formalisms look very different, under suitable analytic assumptions a unitary VOA gives rise to a conformal net by smearing vertex operators, while the representation theories of rational VOAs and rational conformal nets are expected to describe the same chiral data.

The common language is category theory. The objects of the relevant category represent superselection sectors, the tensor product describes fusion, duals encode conjugate charges, and the braiding records the exchange statistics. In rational theories, these data organize into a modular tensor category. Within this category, the structures used to build full two-dimensional CFTs appear as algebra objects. In the operator-algebraic formulation these are precisely  $Q$ -systems, while in the Euclidean categorical formulation they are special Frobenius algebras.

The chapter then explains the role of the full center. Starting from a chiral algebra object, the full center produces the corresponding local two-dimensional theory. Physically, it selects a consistent pairing of left- and right-moving sectors and ensures the locality and modular invariance of the full CFT.

Finally, we discuss modules and bimodules. A module describes a boundary of a CFT, while a bimodule describes an interface between two theories. Intuitively, a bimodule is an object on which two algebra objects act simultaneously, one from the left and one from the right, in a compatible way; in the present context, these two actions describe how the degrees of freedom of the two theories are attached to the two sides of an interface. When the two theories coincide, bimodules become topological defects of that theory. Their relative tensor product reproduces the fusion of defects, and simple bimodules correspond to elementary defect lines. The categorical classification by bimodules can then be compared with the algebraic classification obtained in the previous chapter from the center of the braided-product extension. The final part of the chapter illustrates these ideas in the chiral Ising model, where the relevant algebra objects, modules and bimodules can be described explicitly.

This chapter is based on [35, 61, 78, 79].

## 6.1 Vertex operator algebras

Consider a vector space  $V$  over  $\mathbb{C}$ . We define the *formal series*  $a(z)$  as

$$a(z) := \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

where  $a_{(n)} \in \text{End}(V)$ . This series is called "formal" since  $a(z)$  is not a function of  $z$  resulting from the convergence of the series, but rather the power series itself, without concern for convergence.

**Definition 6.1.1.** A formal series  $a(z) := \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$  is called a *field*, if for every  $b \in V$  we have  $a_{(n)}b = 0$  for  $n$  sufficiently large.

Note that in this section we are not using the usual bra-ket notation since  $V$  is not a Hilbert space.

Before stating the formal axioms, it is useful to grasp the physical intuition. In Wightman's approach to quantum field theory, fields are operator-valued distributions. In two-dimensional conformal field theory, thanks to the relation with complex analysis, these distributions can be entirely encoded into formal power series. The vertex algebra axioms are the algebraic translation of the operator product expansion (OPE) and the state-field correspondence found in physics.

**Definition 6.1.2.** A *vertex algebra* is a complex vector space  $V$  together with a given vector  $\Omega \in V$ , called the *vacuum vector*, and an operator  $T \in \text{End}(V)$  called the *infinitesimal translation operator*, and a linear map from  $V$  to the space of fields on  $V$  (the *state-field correspondence*)

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

satisfying:

1. *Translation covariance:*  $[T, Y(a, z)] = \frac{d}{dz} Y(a, z)$ ;
2. *Vacuum:*  $T\Omega = 0$ ,  $Y(\Omega, z) = \mathbf{1}_V$ ,  $a_{(-1)}\Omega = a$ ;
3. *Locality:* for all  $a, b \in V$ ,  $(z - w)^N [Y(a, z), Y(b, w)] = 0$  for a sufficiently large non-negative integer  $N$ .

The fields  $Y(a, z)$ ,  $a \in V$ , are called *vertex operators*.

The locality axiom might look strange algebraically, but it is the direct translation of the physical requirement that observables must commute at spacelike separations. In the language of formal distributions, multiplying the commutator by  $(z - w)^N$  smoothly cancels the pole singularity that arises when the two coordinates approach each other.

Note that translation covariance is equivalent to:

$$[T, a_{(n)}] = -n a_{(n-1)}, \quad a \in V, n \in \mathbb{Z}$$

From the field property we know that there is a smallest non-negative integer  $N$  such that  $a_{(n)}\Omega = 0$  for all  $n \geq N$ . Then:

$$0 = T a_{(N)}\Omega = [T, a_{(N)}]\Omega = -N a_{(N-1)}\Omega$$

Hence  $N = 0$ . Moreover we can replace the condition  $a_{(-1)}\Omega = a$  in the above definition with the stronger condition:

$$Y(a, z)\Omega|_{z=0} = a$$

As a consequence of the axioms, the *Borcherds (or Jacobi) identity* holds:

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} a_{(m+n-j)}b_{(k+j)}c - \sum_{j=0}^{\infty} (-1)^{j+n} \binom{n}{j} b_{(n+k-j)}a_{(m+j)}c$$

where  $a, b, c \in V$ ,  $m, n, k \in \mathbb{Z}$ . This identity encodes the associativity structure of the vertex algebra, together with the locality properties of the fields, and can be regarded as the mode expansion of the OPE relations among vertex operators [80].

**Definition 6.1.3.** A linear subspace  $W \subset V$  is called *vertex subalgebra*, if  $\Omega \in W$  and  $a_{(n)}b \in W$  for all  $a, b \in W$ ,  $n \in \mathbb{Z}$ .

A vertex subalgebra can be interpreted as a closed sector under the OPE: inserting fields from  $W$  cannot produce states outside  $W$ .

Note that:

$$Ta = Ta_{(-1)}\Omega = a_{(-1)}T\Omega + [T, a_{(-1)}]\Omega = a_{(-2)}\Omega$$

Hence a vertex subalgebra is always  $T$ -invariant and thus  $W$  inherits the structure of a vertex algebra. The intersection of a family of vertex subalgebras is again a vertex subalgebra. Then for every subset  $\mathcal{F} \subset V$  there is a smallest vertex subalgebra  $W(\mathcal{F})$  containing  $\mathcal{F}$ , the vertex subalgebra generated by  $\mathcal{F}$ . If  $W(\mathcal{F}) = V$  we say that  $V$  is *generated* by  $\mathcal{F}$ .

**Definition 6.1.4.** The subspace  $\mathcal{I} \subset V$  is an *ideal* if it is  $T$ -invariant and  $a_{(n)}b \in \mathcal{I}$  for every  $a \in V$ ,  $b \in \mathcal{I}$  and  $n \in \mathbb{Z}$ . A vertex algebra  $V$  is *simple* if every ideal is either  $\{0\}$  or  $V$ .

Ideals correspond to decoupled sectors of the theory: they represent states that are not dynamically connected to the rest of the VOA under operator insertions. So basically, the algebra is simple if there is no completely decoupled sector.

So far, a vertex algebra  $V$  only describes the vacuum sector of the theory, namely the states generated from the vacuum by the action of the holomorphic fields belonging to the chiral algebra. In the special case in which the chiral algebra coincides with the Virasoro algebra,  $V$  consists of the identity field together with all its descendants.

However, a full physical theory contains additional sectors. To describe them algebraically, one introduces representations of  $V$ , called  $V$ -modules. Each irreducible module corresponds to a different primary field, or equivalently to a different superselection sector of the chiral theory. These modules provide the chiral building blocks from which the generally non-holomorphic local fields of the full two-dimensional CFT are constructed.

**Definition 6.1.5.** A *module*  $M$  over the vertex algebra  $V$  is a complex vector space  $M$  together with a linear map:

$$V \ni a \mapsto Y^M(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-n-1}, \quad a_{(n)}^M \in \text{End}(M)$$

such that  $Y^M(\Omega, z) = \mathbf{1}_M$  and the Borcherds identity holds on  $M$ .

Note that every vertex algebra  $V$  becomes itself a  $V$ -module by setting  $Y^V(a, z) = Y(a, z)$ . This module is called the *adjoint module*. If the adjoint module is irreducible, then the algebra is simple. The converse is in general not true [61].

While the vertex algebra axioms capture the algebraic nature of quantum fields, they completely miss the geometric side of Conformal Field Theory. In CFT, the fields must transform covariantly under local conformal transformations. To implement this geometrically, we need the algebraic generator of these transformations: the energy-momentum tensor. This is why we must introduce the Virasoro algebra.

**Definition 6.1.6.** Consider a field  $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$  on the vector space  $V$ . If the endomorphisms  $\{L_n : n \in \mathbb{Z}\}$  satisfy the Virasoro algebra relations:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\mathbf{1}_V$$

with central charge  $c \in \mathbb{C}$ , then  $L(z)$  is called a *Virasoro field*. If  $V$  is a vertex algebra, we call  $\nu \in V$  a *Virasoro vector* if the corresponding vertex operator  $Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ ,  $L_n = \nu_{(n+1)}$ , is a Virasoro field.

Then the following nomenclature fixes the analogy.

**Definition 6.1.7.** We shall call a Virasoro vector  $\nu \in V$  a *conformal vector* if  $L_{-1} = T$  and  $L_0$  is diagonalizable in  $V$ . The corresponding vertex operator  $Y(\nu, z)$  is called *energy-momentum field* and  $L_0$  is called *conformal Hamiltonian*. A vertex algebra with a fixed conformal vector is called *conformal vertex algebra*.

One can show that for a conformal vertex algebra,  $V$  is simple if and only if its adjoint module is irreducible [61].

Now, let  $V$  be a conformal vertex algebra and  $Y(\nu, z)$  be the corresponding energy-momentum field. Set  $V_\alpha = \text{Ker}(L_0 - \alpha \mathbf{1}_V)$ ,  $\alpha \in \mathbb{C}$ . Since  $L_0$  is diagonalizable we can decompose  $V$  as follows:

$$V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha$$

**Definition 6.1.8.** A non-zero element  $a \in V_\alpha$  is called a *homogeneous element of conformal weight*  $h_a = \alpha$

For such elements we use the following convention:

$$a_n := a_{(n+h_a-1)} \quad n \in \mathbb{Z} - h_a + 1 \simeq \mathbb{Z} - h_a$$

so that:

$$Y(a, z) = \sum_{n \in \mathbb{Z} - h_a} a_n z^{-n-h_a}$$

And we have the following commutation relations [81]:

$$[L_0, a_n] = -n a_n$$

$$[L_{-1}, a_n] = (-n - h_a + 1) a_{n-1}$$

$$[L_1, a_n] = -(n - h_a + 1) a_{n+1} + (L_1 a)_{n+1}$$

for all homogeneous  $a \in V, n \in \mathbb{Z}$

**Definition 6.1.9.** A homogeneous vector  $a$  in a conformal vertex algebra  $V$  and the corresponding field  $Y(a, z)$  are called *quasi-primary* if  $L_1 a = 0$  and *primary* if  $L_n a = 0$  for every integer  $n > 0$ .

The vacuum vector is obviously a primary vector in  $V_0$ . The conformal vector  $\nu$  is a quasi-primary vector in  $V_2$  which cannot be primary if  $c \neq 0$ .

**Definition 6.1.10.** A conformal vertex algebra  $V$  is of *CFT type* if  $V_\alpha = \{0\}$  for  $\alpha \notin \mathbb{N}$  and  $V_0 = \mathbb{C}\Omega$ .

Hence, in a conformal vertex algebra of CFT type the vacuum is unique and all states have non-negative integer conformal dimension, reflecting a discrete and bounded-from-below energy spectrum.

Finally we have the following definition:

**Definition 6.1.11.** A *vertex operator algebra (VOA)* is a conformal vertex algebra such that the corresponding energy-momentum field  $Y(\nu, z)$  and homogeneous subspaces  $\text{Ker}(L_0 - \alpha \mathbf{1}_V)$  satisfy the following additional conditions:

1.  $V = \bigoplus_{n \in \mathbb{Z}} V_n$ ;
2.  $V_n = \{0\}$  for  $n$  sufficiently small;
3.  $\dim(V_n) < \infty$ .

If in addition  $V_0 = \mathbb{C}\Omega$ , then one can show that  $V_n = \{0\}$  for all  $n < 0$ ; i.e. the VOA is of CFT type [61].

## 6.2 From Vertex Operator Algebras to conformal nets

In the last section we have introduced the basic definition of VOA. It is clear that this approach to one-dimensional conformal field theories is rather different from the one given by conformal nets. However, this approach is closer to the usual treatment of holomorphic CFT.

In the following we are going to sketch the construction of a conformal net starting from a VOA. To do so we need a couple of definitions.

**Definition 6.2.1.** An *invariant bilinear form* on  $V$  is a bilinear form  $(\cdot, \cdot)$  on  $V$  satisfying

$$(Y(a, z)b, c) = (b, Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})c)$$

The transformation rule for the fields on the right-hand side is induced by the inversion of the complex plane  $z \mapsto 1/z$ .

Intuitively, an invariant bilinear form should be thought of as a pairing between states inserted at the origin and at infinity. Since these two points are exchanged by an inversion, moving a field from one entry of the pairing to the other requires transforming it accordingly under this conformal map.

In particular, the factor  $(-z^{-2})^{L_0}$  arises from the Weyl factor associated with the inversion, while the term  $e^{zL_1}$  accounts for the non-trivial transformation of descendant states.

**Definition 6.2.2.** An (antilinear) automorphism of a vertex algebra is a (anti)linear map  $\theta$  from  $V$  to itself such that  $\theta(a_{(n)}b) = \theta(a)_{(n)}\theta(b)$  for all  $a, b \in V$  and  $n \in \mathbb{Z}$ , and it satisfies  $\theta(\Omega) = \Omega$  and  $\theta(Ta) = T\theta(a)$  for all  $a \in V$ .

If  $V$  is a VOA and  $\theta(\nu) = \nu$ , then  $\theta$  is a VOA (anti)linear automorphism.

**Definition 6.2.3.** Let  $V$  be a VOA with conformal vector  $\nu$  and let  $\langle \cdot, \cdot \rangle$  be a scalar product on  $V$ , namely a positive-definite sesquilinear form (linear in the second variable). We say that it is normalized if  $\langle \Omega, \Omega \rangle = 1$  and invariant if there is a VOA antilinear automorphism  $\theta$  such that  $\langle \theta \cdot, \cdot \rangle$  is an invariant bilinear form on  $V$ . In this case we will say that  $\theta$  is a PCT operator associated with  $\langle \cdot, \cdot \rangle$ .

The inclusion of the antilinear automorphism  $\theta$  is necessary to bridge the gap between the geometric definition of VOA invariant bilinear form and the physical requirement of a positive-definite sesquilinear scalar product. In other words,  $\theta$  compensates for the conjugate-linearity of the first entry, restoring the bilinearity needed for algebraic consistency.

**Definition 6.2.4.** A *unitary VOA* is a pair  $(V, \langle \cdot, \cdot \rangle)$  where  $V$  is a VOA and  $\langle \cdot, \cdot \rangle$  is a normalized invariant scalar product on  $V$ .

A unitary VOA is basically a VOA where every vertex operator  $Y(a, z)$  has an adjoint, i.e. a field defined as  $Y^\dagger(a, z) := \sum_{n \in \mathbb{Z}} a_{(n)}^\dagger z^{n+1}$  where  $a_{(n)}^\dagger$  is the adjoint of  $a_{(n)}$  with respect to the scalar product  $\langle \cdot, \cdot \rangle$  [61].

Heuristically,  $\theta$  may be viewed as a remnant, within the purely chiral setting of a VOA, of the PCT structure that in a full two-dimensional CFT is encoded geometrically by the transformation  $(z, \bar{z}) \rightarrow (\frac{1}{\bar{z}}, \frac{1}{z})$ , allowing the purely holomorphic VOA formalism to carry a notion of Hermitian conjugation analogous to (1.17).

The vertex algebra  $L(c, 0)$  associated with the unitary representation of the Virasoro algebra with central charge  $c$  and lowest conformal weight 0 is a simple unitary VOA. The only property non-trivially verified is unitarity: consider the scalar product of the Hilbert space associated to such representation and  $\theta$  as simply the complex conjugation of complex scalars. Considering the conformal field  $Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$  we have:

$$L_0 \nu = 2\nu \quad L_1 \nu = 0$$

Then:

$$\langle \theta(Y(\nu, z)b), c \rangle = \langle Y(\nu, z)b, c \rangle = \langle b, Y(z^{-4}\nu, z^{-1})c \rangle = \langle b, z^{-4}Y(\nu, z^{-1})c \rangle$$

And if we expand the field we get:

$$\langle \sum_{n \in \mathbb{Z}} L_n z^{-n-2} b, c \rangle = \langle b, \sum_{n \in \mathbb{Z}} L_n z^{n-2} c \rangle \Rightarrow \langle L_n b, c \rangle = \langle b, L_{-n} c \rangle$$

So the general condition above reduces to the classical condition  $L_{-n}^\dagger = L_n$ , which is clearly satisfied by  $L(c, 0)$ , where the possible values of  $c$  are restricted to  $c(m) = 1 - \frac{6}{m(m+1)}$ ,  $m = 3, 4, \dots$  and  $c \geq 1$ , as discussed in section 1.2.7.

**Definition 6.2.5.** Let  $(V, \langle \cdot, \cdot \rangle)$  be a unitary VOA. We say that  $a \in V$  (or equivalently the corresponding field  $Y(a, z)$ ) satisfies *energy bounds* if there exist positive integers  $s, k$  and a constant  $M > 0$  such that, for all  $n \in \mathbb{Z}$  and all  $b \in V$ :

$$\|a_n b\| \leq M(|n| + 1)^s \|(L_0 + \mathbf{1}_V)^k b\|$$

If every  $a \in V$  satisfies energy bounds we say that  $V$  is *energy-bounded*.

The following proposition gives us a criterion to know when a simple unitary VOA is energy-bounded.

**Proposition 6.2.1.** *If  $V$  is a simple unitary vertex operator algebra generated by  $V_1$  and  $\mathcal{F} \subset V_2$  where  $\mathcal{F}$  is a family of quasi-primary  $\theta$ -invariant Virasoro vectors, then  $V$  is energy-bounded.*

Of course  $L(c, 0)$  is energy bounded since it is generated by the conformal vector  $\nu \in V_2$ .

Consider an energy-bounded unitary VOA. Now we start the construction of the conformal net associated to  $V$ : first we define the Hilbert space  $\mathcal{H} = \mathcal{H}_{(V, \langle \cdot, \cdot \rangle)}$  as the completion of  $V$  with respect to  $\langle \cdot, \cdot \rangle$ . For any  $a \in V$  and  $n \in \mathbb{Z}$ ,  $a_{(n)}$  is an operator on  $\mathcal{H}$  with dense domain  $V \subset \mathcal{H}$ . Due to the invariance of the scalar product it has also a densely defined adjoint and hence it is closable (see the definition in footnote 9 of chapter 3).

Now consider a smooth function  $f(z)$  on  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$  with fourier coefficients:

$$\hat{f}_n = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} f(e^{i\theta}) e^{-in\theta} = \oint_{S^1} \frac{dz}{2\pi i z} f(z) z^{-n}$$

For every  $a \in V$  we define the operator  $Y_0(a, f)$  with domain  $V$  by:

$$Y_0(a, f)b = \sum_{n \in \mathbb{Z}} \hat{f}_n a_n b \quad b \in V$$

One can show that sum converges in  $\mathcal{H}$  due to the energy bounds and hence  $Y_0(a, f)$  is a densely defined operator on  $\mathcal{H}$  [61]. From the invariance of the scalar product it follows that  $Y_0(a, f)$  has densely defined adjoint and hence it is closable. We call the closure of  $Y_0(a, f)$  *smeared vertex operator*  $Y(a, f)$ .

Now we can associate with every interval  $I \in \mathcal{I}$  a von Neumann algebra  $\mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}(I)$  generated by the operators  $Y(a, f)$ . Since these operators are generally unbounded, one has to clarify what is meant by ‘generated’, as a von Neumann algebra, by definition, consists only of bounded operators.

An operator  $A$  on  $\mathcal{H}$  with domain  $D(A)$  is said to commute with a bounded operator  $B \in \mathcal{B}(\mathcal{H})$  (and vice versa), denoted  $BA \subset AB$ , if  $BD(A) \subset D(A)$  and  $ABa = BAa$  for all  $a \in D(A)$ . If  $A$  is densely defined and closable and if  $A$  commutes with  $B \in \mathcal{B}(\mathcal{H})$ , then  $A^\dagger$  commutes with  $B^\dagger$ .

A closed densely defined operator  $A$  on  $\mathcal{H}$  is said to be *affiliated* with a von Neumann algebra  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  if  $A$  commutes with all operators in  $\mathcal{M}'$ . For any closed densely defined operator  $A$  on  $\mathcal{H}$  the set

$$\{B \in \mathcal{B}(\mathcal{H}) : BA \subset AB, B^\dagger A \subset AB^\dagger\}$$

is a von Neumann algebra and

$$W^*(A) := \{B \in \mathcal{B}(\mathcal{H}) : BA \subset AB, B^\dagger A \subset AB^\dagger\}'$$

is the smallest von Neumann algebra to which  $A$  is affiliated called the von Neumann algebra generated by  $A$ . If we have a set of unbounded operators, then  $W^*(\{A_i\}_{i \in \mathcal{I}})$  will be the smallest von Neumann algebra affiliated to all  $\{A_i\}_{i \in \mathcal{I}}$ <sup>1</sup>. The intuitive idea behind this definition is that a von Neumann algebra to which  $A$  is affiliated ‘sees’ the operator  $A$  as if it belonged to the algebra, even though  $A$  may be unbounded.

Then we define:

$$\mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}(I) := W^*(\{Y(a, f) : a \in V, f \in C^\infty(S^1), \text{supp}(f) \subset I\})$$

The net  $I \mapsto \mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}(I)$  clearly satisfies isotony. Moreover one can show that  $\Omega$  is cyclic for the von Neumann algebra

$$\mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}(S^1) := \bigvee_{I \in \mathcal{I}} \mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}(I)$$

and also conformal covariance is satisfied [35].

Locality is not guaranteed by construction. In particular we define the following:

**Definition 6.2.6.** A unitary vertex operator algebra  $(V, \langle \cdot, \cdot \rangle)$  is *strongly local* if it is energy-bounded and we have

$$\mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}(I) \subset \mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}(I')'$$

for all intervals  $I \in \mathcal{I}$ .

A criterion for strong locality is the following:

**Proposition 6.2.2.** [61] *Let  $V$  be a simple unitary vertex operator algebra generated by  $V_1 \cup \mathcal{F}$  where  $\mathcal{F} \subset V_2$  is a family of quasi-primary  $\theta$ -invariant Virasoro vectors, then  $V$  is strongly local.*

Again, this shows immediately that  $L(c, 0)$  is strongly local. However, there are many other more complex examples exposed in [61].

This construction seems to depend on the choice of the scalar product. However, the following conclusive result also addresses this problem:

**Theorem 6.2.3.** [61] *Let  $(V, \langle \cdot, \cdot \rangle)$  be a simple strongly local unitary VOA. Then the map  $I \mapsto \mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}(I)$  defines an irreducible conformal net  $\mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}$  on  $S^1$ . If  $\{\cdot, \cdot\}$  is another normalized invariant scalar product on  $V$  then  $(V, \{\cdot, \cdot\})$  is again strongly local and  $\mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}$  and  $\mathcal{A}_{(V, \{\cdot, \cdot\})}$  are isomorphic conformal nets.*

We say that  $\mathcal{A}_V := \mathcal{A}_{(V, \langle \cdot, \cdot \rangle)}$  is the irreducible conformal net associated with the strongly local unitary simple vertex operator algebra  $V$ . The irreducible conformal net associated to  $L(c, 0)$  is the Virasoro net  $\mathcal{A}_{L(c, 0)} = \text{Vir}_c$ , since in  $L(c, 0)$  the only non-trivial field is the energy-momentum field  $Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$  and its derivatives.

It is also possible to reconstruct the vertex operator algebra from the conformal net  $\mathcal{A}_V$ . The main difficulty lies in recovering pointlike fields  $Y(a, z)$  from the local von Neumann algebras  $\mathcal{A}(I)$ , which are defined only for intervals and do not carry an explicit dependence on the complex coordinate.

Since also this construction is very involved, we give just the idea [61]: consider a conformal net  $\mathcal{A}$  and let  $a \in \mathcal{H}$  be a quasi-primary vector of conformal weight  $h_a \in \mathbb{N}$ . For every  $f \in C^\infty(S^1)$  we can consider the vector  $a(f)$  defined as:

$$a(f) := \sum_{n \in \mathbb{N}} \hat{f}_{-n-h_a} \frac{1}{n!} L_{-1}^n a$$

where in the context of conformal nets,  $L_{-1}$  is the representation on the vacuum Hilbert space  $\mathcal{H}$  of the associated generator of the Virasoro algebra.

<sup>1</sup>Precisely [61]:  $W^*(\{A_i\}_{i \in \mathcal{I}}) = (\cup_{i \in \mathcal{I}} W^*(A_i))'' = (\cap_{i \in \mathcal{I}} W^*(A_i)')'$  since the commutant of the union equals the intersection of the commutants.

Then, with the help of Tomita–Takesaki theory, one can define for all  $I \in \mathcal{I}$  the operator-valued distributions

$$C^\infty(S^1) \ni f \mapsto Y_I(a, f), \quad Y_I(a, f) B \Omega = B a(f), \quad \text{supp}(f) \subset I, \quad B \in \mathcal{A}(I)'$$

These operators are called *Fredenhagen–Jörß smeared vertex operators or FJ fields*. As usual for distributions we can use the formal notation

$$Y_I(a, f) = \int_I \frac{dz}{2\pi i z} Y_I(a, z) f(z) z^{h_a}$$

Then we say that the family  $\{Y_I(a, z), z \in I : I \in \mathcal{I}\}$  is an FJ vertex operator or an FJ field.

It is not known if the FJ smeared vertex operators admit a common invariant domain. Hence we cannot extend the family of distributions  $\{Y_I(a, z), z \in I : I \in \mathcal{I}\}$  to a unique distribution  $\tilde{Y}(a, z)$ . In particular the FJ fields cannot in general be considered as quantum fields in the sense of Wightman [61].

**Theorem 6.2.4.** [61] *Let  $V$  be a simple unitary strongly local VOA and let  $\mathcal{A}_V$  be the corresponding irreducible conformal net. Then, for any quasi-primary vector  $a \in V$  we have  $Y_I(a, f) = Y(a, f)$  for all  $I \in \mathcal{I}$  and all  $f \in C^\infty(S^1)$ ,  $\text{supp}(f) \subset I$ , i.e. the smeared vertex operators of  $V$  coincide with the FJ smeared vertex operators of  $\mathcal{A}_V$ . In particular one can recover the VOA structure on  $V := \bigoplus_{n \in \mathbb{N}} \text{Ker}(L_0 - n\mathbf{1}_{\mathcal{H}})$  from the conformal net  $\mathcal{A}_V$ .*

In addition, in [61] the authors have identified a set of conditions for a generic irreducible conformal net  $\mathcal{A}$  necessary to construct a vertex operator algebra  $V$  such that  $\mathcal{A}_V = \mathcal{A}$ .

In particular, we say that for every  $a \in \mathcal{H}$  the corresponding FJ vertex operator  $Y_I(a, z) : I \in \mathcal{I}$  satisfies energy bounds if there exist a real number  $M > 0$  and positive integers  $k$  and  $s$  such that:

$$\|Y_I(a, f)b\| \leq M \left( \sum_{n \in \mathbb{Z}} (|n| + 1)^s |\hat{f}_n| \right) \|(L_0 + \mathbf{1}_{\mathcal{H}})^k b\|$$

Moreover we say that a family  $\mathcal{F} \subset \mathcal{H}$  of quasi-primary vectors generates  $\mathcal{A}$  if the corresponding FJ vertex operators generate the local algebras, i.e.

$$\mathcal{A}(I) = W^*(\{Y_{I_1}(a, f) : a \in \mathcal{F}, f \in C^\infty(I_1), I_1 \in \mathcal{I}, I_1 \subset I, \text{supp}(f) \subset I_1\})$$

**Theorem 6.2.5.** [61] *Let  $\mathcal{A}$  be an irreducible conformal net that is generated by a family of quasi-primary vectors  $\mathcal{F}$ . Assume<sup>2</sup>  $\theta\mathcal{F} = \mathcal{F}$  and that for every  $a \in \mathcal{F}$  the FJ vertex operator  $\{Y_I(a, z) : I \in \mathcal{I}\}$  satisfies energy bounds. Moreover, assume that  $\text{Ker}(L_0 - n\mathbf{1}_{\mathcal{H}})$  is finite-dimensional for all  $n \in \mathbb{N}$ . Then, the vector space  $V := \bigoplus_{n \in \mathbb{N}} \text{Ker}(L_0 - n\mathbf{1}_{\mathcal{H}})$  admits a VOA structure making  $V$  into a simple unitary strongly local VOA such that  $\mathcal{A}_V = \mathcal{A}$ .*

A direct correspondence between the representation theories of a vertex operator algebra (VOA) and a local conformal net is generally more difficult to establish, although significant recent progress has been made [82, 83]. However, for the simple models given by  $L(c, 0)$  and  $Vir_c$ , the representation theory in both frameworks reduces to the representations of the Virasoro algebra, making this identification seamless for our purposes.

To construct a consistent two-dimensional Euclidean CFT out of a VOA, simply combining two chiral theories and imposing modular invariance is usually insufficient; numerous examples exist of modular-invariant bilinear combinations of characters that fail to correspond to the partition function of any physical CFT [33]. Remarkably, the strategy developed for the relativistic setup remains valid in the Euclidean construction: one starts from the modular tensor category of representations of the two VOAs,  $\mathcal{C} = \text{Rep}(V_{L/R})$ , describing a chiral RCFT, and considers the product  $\mathcal{C} \boxtimes \mathcal{C}^{\text{opp}}$ . The missing

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<sup>2</sup> $\theta$  is a PCT operator associated to the conformal net which coincides with the modular conjugation operator  $J$  from Tomita–Takesaki theory restricted to the upper semicircle  $S_+^1$  [61].

data required to consistently glue the left- and right-moving chiral sectors together is precisely a special Frobenius algebra, or a Q-system [33, 84].

In summary, the key takeaways are:

- VOAs and conformal nets provide equivalent (under suitable hypotheses) axiomatizations of one-dimensional chiral CFTs.
- In both frameworks, the construction of a full 2d CFT requires additional data, captured in the RCFT setting by a Q-system (or special Frobenius algebra). Consequently, modular tensor categories provide a natural categorical language linking the Euclidean and relativistic formulations of RCFTs.
- for RCFTs, the classification of defects is identical in both approaches, with the total number of defects given by  $\text{Tr}[Z^T Z]$  in both cases.

Hence, it is natural to expect category theory to provide a unified description of defects, at least for certain classes of models. However, the derivation of defects in the relativistic theory relies heavily on causality, in contrast with the Euclidean approach, which is rooted in modular invariance.

In [33], the authors demonstrate that the correct categorical framework for defects is given by  $Q$ - $Q$  bimodules. A  $Q$ - $Q$  bimodule is an object within the category endowed with commuting left and right actions of a chiral Q-system  $Q$ , which encodes the two-dimensional theory. This structure naturally mirrors the physical behavior of a defect line partitioning the complex plane into two halves, each governed by a bulk theory associated with the chiral Q-system  $Q$ .<sup>3</sup> In the next section, we will follow the approach of [35] to show that this categorical framework is fully compatible with the algebraic classification established in the preceding chapter.

### 6.3 The categories of modules and bimodules

In this section we are going to use a notation adapted to  $\mathcal{C} = \mathcal{C}^{DHR}(\mathcal{A})$  with  $\mathcal{A}$  chiral net, since we are going to apply these notions directly to that category. In particular,  $Obj(\mathcal{C})$  are denoted with Greek letters, since they will represent DHR endomorphisms, and intertwiners, including the one defining the algebra objects, will be denoted with lowercase Latin letters.

We are going to extend the usual notion of a module over a ring to the general setting of tensor categories.

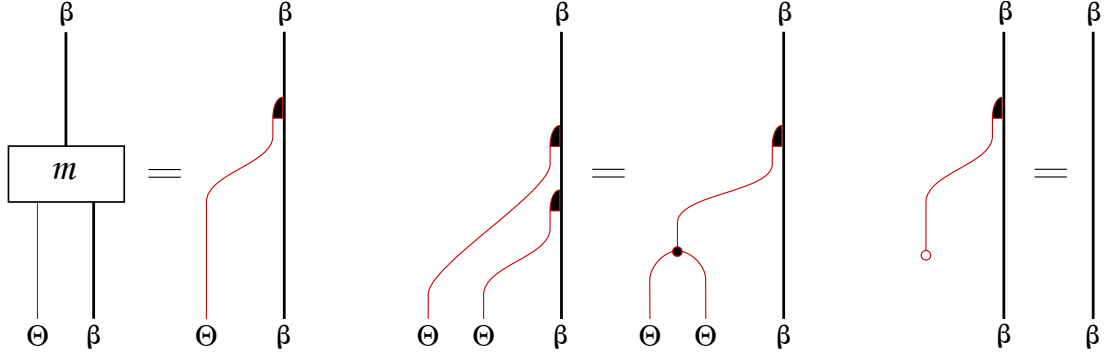
**Definition 6.3.1.** Let  $Q = (\Theta, w, x^\dagger)$  be an algebra in a tensor category  $\mathcal{C}$ . A (left)  $Q$ -module in  $\mathcal{C}$  is a pair  $\mathbf{m} = (\beta, m)$  of two data: an object  $\beta \in Obj(\mathcal{C})$  and a morphism of  $\mathcal{C}$  that specifies the action of  $Q$  on  $\beta$ ,  $m_\beta = m \in \text{Hom}(\Theta \otimes \beta, \beta)$ , called *representation morphism*. Further,  $m$  must satisfy the *representation properties*

$$m \circ (\text{id}_\Theta \otimes m) = m \circ (x^\dagger \otimes \text{id}_\beta) \quad m \circ (w \otimes \text{id}_\beta) = \text{id}_\beta$$

We can represent  $m$  and the representation properties respectively as:

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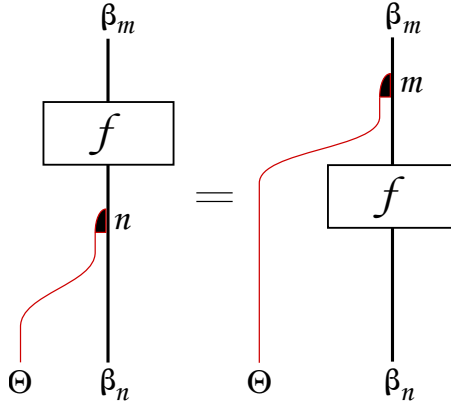
<sup>3</sup>As in the general algebraic treatment of defects, we will focus on defects within a single theory, which are described by  $Q - Q$ -bimodules. However, the same framework also describes interfaces between two CFTs, each associated with a particular Q-system. Such interfaces are classified by  $Q_1 - Q_2$ -bimodules [33, 85].



Note that in the above definition,  $Q$  is not required to be a Frobenius algebra, nor is required a dagger structure on  $\mathcal{C}$ . In particular, we remind that  $Q$ -systems are denoted  $Q = (\Theta, w, x)$  and are both algebras, denoted as triples  $(\Theta, w, x^\dagger)$ , and co-algebras, written as  $(\Theta, w^\dagger, x)$ .

Analogously, one can define right modules and co-modules over a co-algebra. Note that not every object needs to underlie some  $Q$ -module, and the same object can be a  $Q$ -module in different inequivalent ways.

**Definition 6.3.2.** For (left)  $Q$ -modules  $\mathbf{n}, \mathbf{m}$ , a  $\mathbf{n} - \mathbf{m}$ -intertwiner is a morphism  $f$  between  $\beta_n$  and  $\beta_m$  satisfying  $f \circ n = m \circ (\text{id}_\Theta \otimes f)$ . Pictorially:



Given an algebra  $Q$  in  $\mathcal{C}$ , we call  $\mathcal{C}_Q$  the category with  $Q$ -modules as objects and

$$\text{Hom}_Q(\mathbf{n}, \mathbf{m}) := \{f \in \text{Hom}(\beta_n, \beta_m) : f \circ n = m \circ (\text{id}_\Theta \otimes f)\}$$

as morphism spaces.

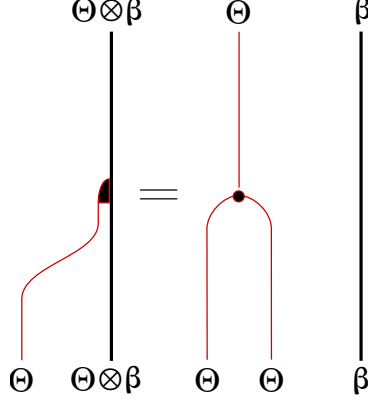
The category  $\mathcal{C}_Q$  inherits many properties from  $\mathcal{C}$  [33]. For example, when  $\mathcal{C}$  is modular and hence semisimple and when  $Q$  is special Frobenius, then the category  $\mathcal{C}_Q$  is semisimple. On the other hand, that  $\mathcal{C}$  is modular does not imply that  $\mathcal{C}_Q$  is modular. In fact, it does not even imply that  $\mathcal{C}_Q$  is a tensor category. However, a sufficient condition for  $\mathcal{C}_Q$  to be tensor is then that  $Q$  is a commutative algebra and has trivial twist, i.e.  $\theta_\Theta = \text{id}_\Theta$  [33].

A strategy to compute the modules over an algebra object is to study the induced modules.

**Definition 6.3.3.** For  $\beta \in \text{Obj}(\mathcal{C})$ , the *induced (left) module*  $\text{Ind}_Q(\beta)$  is the  $Q$ -module that is equal to  $\Theta \otimes \beta$  as an object in  $\mathcal{C}$  and has representation morphism  $x^\dagger \otimes \text{id}_\beta$ , i.e.

$$\text{Ind}_Q(\beta) := (\Theta \otimes \beta, x^\dagger \otimes \text{id}_\beta)$$

Pictorially:



Note that  $\text{Ind}_Q(\text{id}) = \mathbf{Q} = (\Theta, x^\dagger)$ , i.e. every  $Q$ -system can be seen as the induced  $Q$ -module by the tensor unit, here represented by the identity endomorphism.

The following proposition is particularly useful to compute induced modules.

**Proposition 6.3.1.** [33] *Let  $Q = (\Theta, w, x)$  be a Frobenius algebra in a tensor category  $\mathcal{C}$ ,  $\rho \in \text{Obj}(\mathcal{C})$  and  $\mathbf{m} = (\beta, m)$  a  $Q$ -module. Then there are natural isomorphisms:*

$$\text{Hom}(\rho, \beta) \simeq \text{Hom}_Q(\text{Ind}_Q(\rho), \mathbf{m}) \quad \text{Hom}(\beta, \rho) \simeq \text{Hom}_Q(\mathbf{m}, \text{Ind}_Q(\rho))$$

In particular:

$$\dim \text{Hom}(\rho, \beta) = \dim \text{Hom}_Q(\text{Ind}_Q(\rho), \mathbf{m}) \quad \dim \text{Hom}(\beta, \rho) = \dim \text{Hom}_Q(\mathbf{m}, \text{Ind}_Q(\rho))$$

This proposition is helpful because if  $\mathbf{m} = \text{Ind}_Q(\sigma)$  for  $\sigma \in \text{Obj}(\mathcal{C})$ , we get

$$\dim \text{Hom}(\rho, \Theta \otimes \sigma) = \dim \text{Hom}_Q(\text{Ind}_Q(\rho), \text{Ind}_Q(\sigma))$$

$$\dim \text{Hom}(\Theta \otimes \sigma, \rho) = \dim \text{Hom}_Q(\text{Ind}_Q(\sigma), \text{Ind}_Q(\rho))$$

Since the left-hand side can be easily computed, this relation can be used to determine whether an induced module is simple or not.

Then the following proposition tells us how induced modules are related to general modules:

**Proposition 6.3.2.** *If  $Q$  is a special Frobenius algebra in a semisimple tensor category  $\mathcal{C}$ , then every  $Q$ -module is a submodule of an induced module, and the category  $\mathcal{C}_Q$  of  $Q$ -modules is semisimple.*

**Corollary 6.3.3.** *The module  $\text{Ind}_Q(\rho_j)$  ( $j \in \mathcal{J}$ ) decomposes into simple modules  $\mathbf{m}_k$  ( $k \in \mathcal{K}$ ) according to:*

$$\text{Ind}_Q(\rho_j) = \bigoplus_{k \in \mathcal{K}} \dim \text{Hom}(\rho_j, \beta_k) \mathbf{m}_k$$

where  $\mathcal{J}$  labels simple objects and  $\mathcal{K}$  simple modules.

That  $\mathcal{C}_Q$  is semisimple means that every  $Q$ -module decomposes as a finite direct sum of simple  $Q$ -modules. Moreover, every simple  $Q$ -module appears as a subobject of an induced module  $\text{Ind}_Q(\rho)$ , where  $\rho$  is a simple object of  $\mathcal{C}$ . Since in our case  $\mathcal{C}$  is a modular tensor category, it contains only finitely many isomorphism classes of simple objects [33]. As a consequence, there are only finitely many isomorphism classes of simple  $Q$ -modules.

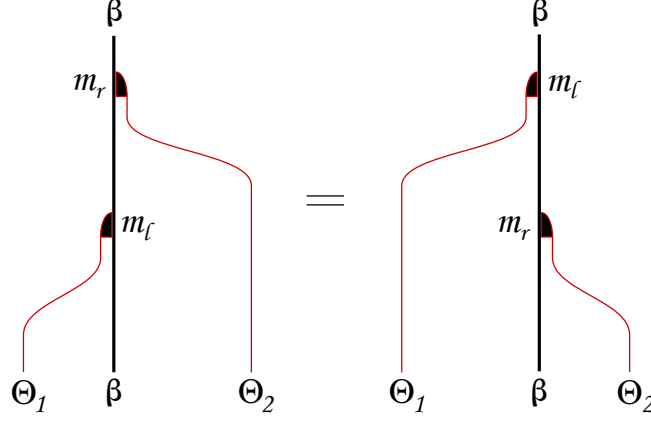
However, a defect separates the plane into two regions, each described by the same theory up to isomorphism. Therefore, a module is not sufficient to describe a defect, since it only encodes a one-sided action, while a defect requires both left and right actions together with their compatibility, i.e. a bimodule structure<sup>4</sup>.

<sup>4</sup>With the same reasoning, we can interpret  $Q$ -modules as describing boundaries of a CFT, where the action of the Frobenius algebra  $Q$  encodes the presence of the theory only on one side of the boundary [33].

**Definition 6.3.4.** Let  $Q_1 = (\Theta_1, w_1, x_1^\dagger)$  and  $Q_2 = (\Theta_2, w_2, x_2^\dagger)$  be algebra objects in a tensor category  $\mathcal{C}$ . Then a  $Q_1 - Q_2$ -bimodule is a triple  $\mathbf{m} = (\beta, m_l, m_r)$  consisting of an object  $\beta \in \text{Obj}(\mathcal{C})$  and two morphisms  $m_l \in \text{Hom}(\Theta_1 \otimes \beta, \beta)$  and  $m_r \in \text{Hom}(\beta \otimes \Theta_2, \beta)$ , such that  $(\beta, m_l)$  is a left  $Q_1$ -module and  $(\beta, m_r)$  is a right  $Q_2$ -module. Furthermore the actions  $m_l$  and  $m_r$  commute, i.e.

$$m_l \circ (\text{id}_{\Theta_1} \otimes m_r) = m_r \circ (m_l \otimes \text{id}_{\Theta_2})$$

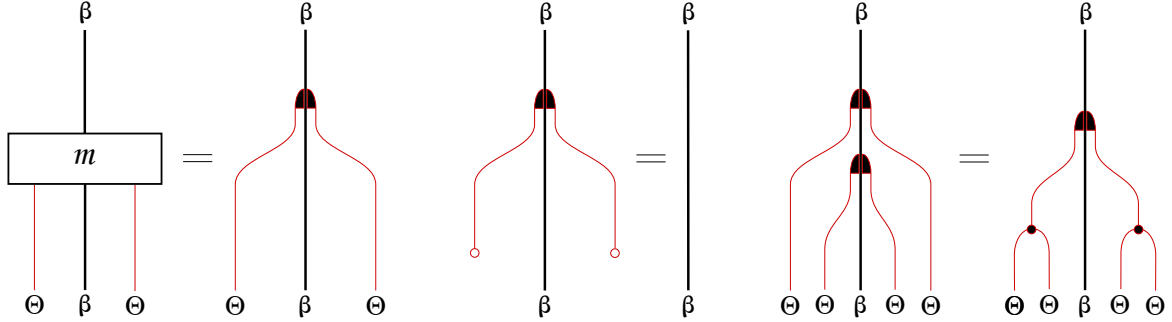
Pictorially:



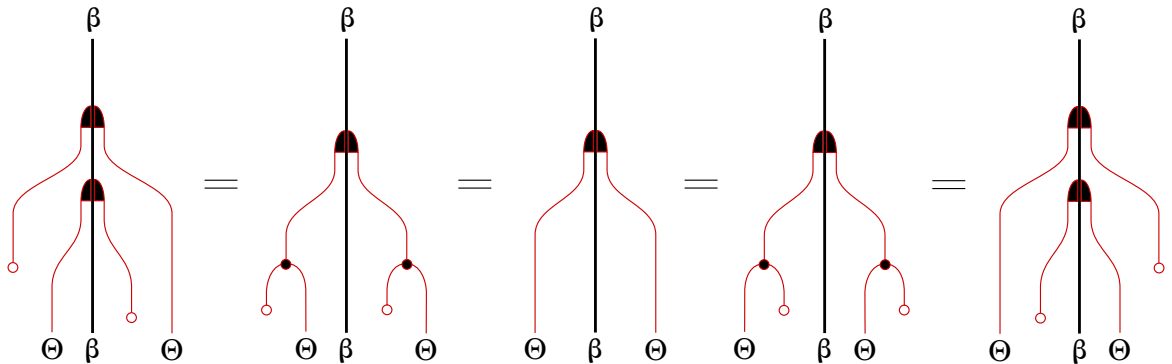
Note that, similarly to modules, the algebra  $Q$  itself can be seen as a  $Q - Q$ -bimodule with the algebra product as left/right actions.

In contrast to  $Q$ -modules, for any Frobenius algebra  $Q$  in a tensor category  $\mathcal{C}$ , the category  ${}_Q\mathcal{C}_Q$  of  $Q - Q$ -bimodules is again a tensor category. However, in general, the bimodule category does not have a braiding and a twist either [33].

Note that we can equivalently define a  $Q_1 - Q_2$ -bimodule as a pair  $\mathbf{m} = (\beta, m)$  where  $m \in \text{Hom}(\Theta_1 \otimes \beta \otimes \Theta_2, \beta)$  satisfy:



This is done simply by setting  $m := m_l \circ (\text{id}_{\Theta_1} \otimes m_r) = m_r \circ (m_l \otimes \text{id}_{\Theta_2})$ . On the other hand, given  $\mathbf{m} = (\beta, m)$ , the pairs  $\mathbf{m}_l = (\beta, (\text{id}_{\Theta_1} \otimes \text{id}_\beta \otimes w_2) \circ m)$  and  $\mathbf{m}_r = (\beta, (w_1 \otimes \text{id}_\beta \otimes \text{id}_{\Theta_2}) \circ m)$  are a left  $Q_1$ -module and a right  $Q_2$ -module with commuting actions:



One can show that if  $\mathcal{C}$  is a braided tensor category, then bimodules can equivalently be thought of as left  $(Q_1 \otimes Q_2^{opp})$ -modules, so that once one knows how to find modules, it is not necessary to develop new techniques to find bimodules [33].

The opposite algebra  $Q^{opp}$  is defined as follows:

**Proposition 6.3.4.** [33]

- $Q = (\Theta, w, x^\dagger)$  is an algebra if and only if  $Q^{opp} := (\Theta, w, x^\dagger \circ \epsilon_{\Theta, \Theta}^{-1})$  is an algebra;
- $Q = (\Theta, w, w^\dagger, x^\dagger, x)$  is a symmetric special Frobenius algebra if and only if  $Q^{opp} := (\Theta, w, w^\dagger, x^\dagger \circ \epsilon_{\Theta, \Theta}^{-1}, \epsilon_{\Theta, \Theta} \circ x)$  is an algebra;

Note that for a commutative algebra  $Q = Q^{opp}$ .

The tensor product of algebras is defined categorically as:

**Proposition 6.3.5.** [33]

- For any two algebras  $Q_1$  and  $Q_2$  the triple  $(\Theta_1 \otimes \Theta_2, x_{Q_1 \otimes Q_2}^\dagger, w_{Q_1 \otimes Q_2})$  with  $x_{Q_1 \otimes Q_2}^\dagger := (x_{Q_1}^\dagger \otimes x_{Q_2}^\dagger) \circ (\text{id}_{\Theta_1} \otimes \epsilon_{\Theta_1, \Theta_2}^{-1} \otimes \text{id}_{\Theta_2})$  and  $w_{Q_1 \otimes Q_2} = w_1 \otimes w_2$  furnishes an algebra structure on the object  $Q_1 \otimes Q_2$ .
- If  $Q_1, Q_2$  are symmetric special Frobenius, then also  $Q_1 \otimes Q_2$  is symmetric special Frobenius, with counit  $w_{Q_1 \otimes Q_2}^\dagger = w_1^\dagger \otimes w_2^\dagger$  and coproduct given by  $x_{Q_1 \otimes Q_2} := (\text{id}_{\Theta_1} \otimes \epsilon_{\Theta_1, \Theta_2} \otimes \text{id}_{\Theta_2}) \circ (x_{Q_1} \otimes x_{Q_2})$ .

Also the inverse convention of braiding is possible.

I.e. it corresponds to the braided product of two algebras.

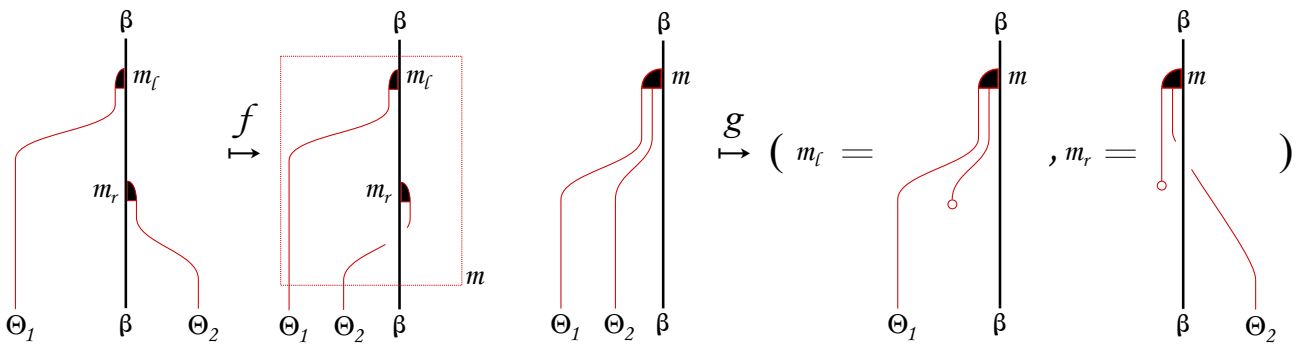
**Proposition 6.3.6.** [33] For  $Q_1, Q_2$  algebras in a braided tensor category, the mapping

$$f : (\beta, m_l, m_r) \mapsto (\beta, m_l \circ (\text{id}_{\Theta_1} \otimes m_r) \circ (\text{id}_{\Theta_1} \otimes \epsilon_{\beta, \Theta_2}^{-1}))$$

takes  $Q_1 - Q_2$ -bimodules to left  $Q_1 \otimes Q_2^{opp}$ -modules.  $f$  is invertible, with inverse  $g$  given by

$$g : (\beta, m) \mapsto (\beta, m \circ (\text{id}_{\Theta_1} \otimes w_2 \otimes \text{id}_\beta), m \circ (w_1 \otimes \epsilon_{\beta, \Theta_2}))$$

Pictorially:



We can define a product of bimodules. First, we need the following definition:

**Definition 6.3.5.** Let  $p \in \text{Hom}(\rho, \rho)$  be an idempotent,  $p \circ p = p$ . A triple  $(\sigma, e, r)$ , consisting of an object  $\sigma \in \text{Obj}(\mathcal{C})$  and morphisms  $e \in \text{Hom}(\sigma, \rho)$  and  $r \in \text{Hom}(\rho, \sigma)$ , is called a *retract* of  $p$  iff

$$e \circ r = p \quad r \circ e = \text{id}_\sigma$$

The subobject  $\sigma$  of  $\rho$  is then called the image of  $p$ , denoted  $\text{Im}(p)$ .

Consider now three special Frobenius algebras  $Q_1, Q_2$  and  $Q_3$  and let  $\mathbf{m}$  and  $\mathbf{n}$  be  $Q_1 - Q_2$ - and  $Q_2 - Q_3$ -bimodules respectively. Consider now the morphism  $p_{\mathbf{m}, \mathbf{n}} \in \text{Hom}(\beta_m \otimes \beta_n, \beta_m \otimes \beta_n)$  given by:

The diagram shows a pair of pants shape, which is a surface with three boundary components. Two vertical black lines on the left and right are labeled  $\beta_m$  at both the top and bottom. A red curve connects the two vertical lines, forming a 'V' shape that opens upwards. At the bottom vertex of this 'V' is a black dot labeled  $\Theta_2$ . A small red circle is located directly below  $\Theta_2$ . To the left of the diagram, the text  $p_{m,n} :=$  is written.

$$\mathbf{m} \otimes_{Q_2} \mathbf{n} := \text{Im}(p_{\mathbf{m}, \mathbf{n}}) \in {}_{Q_1} \mathcal{C}_{Q_3}$$

In [33] it is shown that the number of isomorphism classes of simple  $Q - Q$ -bimodules is given by  $\text{Tr}(Z^T Z)$ , where  $Z$  is the coupling matrix of the associated two-dimensional model, namely the multiplicity matrix relating left- and right-moving characters in the partition function. From the algebraic point of view, this is the same matrix appearing in the decomposition of the Hilbert space of the two-dimensional theory into irreducible left/right representations of the Virasoro algebra. Equivalently, if the two-dimensional theory is described by a  $Q$ -system,  $Z$  is the matrix encoding the decomposition of the object  $\Theta$  appearing in  $Q$ , as in equation (4.7).

This, in turn, requires a systematic procedure to construct a two-dimensional Q-system starting from a chiral one. Such a construction is provided by the *full center* of a Q-system.

<sup>5</sup>With respect to the  $Q_1$  left action of  $\mathbf{m}$  and the  $Q_3$  right action of  $\mathbf{n}$ .

## 6.4 Full center of Q-systems

Every Q-system in  $\mathcal{C}^{DHR}(\mathcal{A})$  lifts trivially to a Q-system in  $\mathcal{C} \boxtimes \mathcal{C}^{opp}$  by tensoring with the identity [35]. For each chiral Q-system, there exists an associated Q-system for the corresponding two-dimensional theory.

**Definition 6.4.1.** The *left/right full centers* of a chiral Q-system  $Q$  in  $\mathcal{C}^{DHR}(\mathcal{A})$  are given by

$$Z^\pm[Q] = C^\pm[(Q \otimes 1) \times^\pm Q_{can}]$$

which are Q-systems in  $\mathcal{C}^{DHR}(\mathcal{A} \otimes \mathcal{A}) := \mathcal{C}^{DHR}(\mathcal{A}) \boxtimes \mathcal{C}^{DHR}(\mathcal{A})^{opp}$

Therefore the full center is given by the following two steps:

1. Construction of a 2D non-local CFT as a braided product of a chiral extension with the canonical 2D local QFT;
2. Restriction to the respective<sup>6</sup> maximal local subtheory.

So the full center is an algebraic prescription to generate modular invariant two-dimensional CFTs from the chiral data.

The name full center can be seen as a generalization of the algebraic notion of center, since  $Z$  gives rise to the maximal local extension. In this sense, it is the maximal commutative (i.e. local) Q-system associated with the two-dimensional theory.

One can show that the two full centers are not independent [35]. In the sequel, “full center”  $Z[Q]$  will always mean the left full center  $Z^+[Q]$ .

**Proposition 6.4.1.** Denoting as  $1 = (\text{id}, \mathbf{1}, \mathbf{1})$  the trivial Q-system, the following relation holds:

$$Z[1] = Q_{can}$$

*Proof.* Of course  $1 \times^+ Q_{can} = Q_{can}$ . Now we have to show that  $C^+[Q_{can}] = Q_{can}$ . This is obviously true since the center of a Q-system gives the maximal local intermediate extension and  $Q_{can}$  is commutative.  $\square$

Since the trivial Q-system,  $Q = 1$ , always exists, all the results valid for full centers are also valid for canonically extended systems.

A useful side result is the following:

**Proposition 6.4.2.** [57] Let  $Q$  be an irreducible Q-system in a modular tensor category  $\mathcal{C}$ . Then all full centers have the same dimension:

$$d_{Z[Q]}^2 = \dim(\Theta^{Z[Q]}) = \sum_{\sigma, \tau} Z_{\sigma, \tau} \dim(\sigma) \dim(\tau) = d_{Q_{can}}^2 = \sum_{j \in \mathcal{J}} \dim(\rho_j)^2$$

where  $\mathcal{J}$  labels all the equivalence classes of simple objects of  $\mathcal{C}$ .

An important property of the full center involves the notion of Morita equivalence. This notion was originally formulated in ring theory: two algebras are said to be Morita equivalent if their categories of modules are equivalent. In this sense, Morita equivalence can be understood as a notion of “sameness” of algebras from the point of view of their representation theory.

In the more general setting of algebra objects in tensor categories, Morita equivalence is defined as follows [86]:

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<sup>6</sup>If in step 1. we choose  $\times^+$ , we are forced to choose the left center  $C^+$  and vice versa. Indeed, one can show that  $C^\pm[(Q \otimes 1) \times^\mp Q_{can}] = Q_{can}$  [35, 57].

**Definition 6.4.2.** Two algebras  $Q_1, Q_2$  over a tensor category  $\mathcal{C}$  are *Morita equivalent* if there exist a  $Q_1 - Q_2$ -bimodule  $\mathbf{m}$  and a  $Q_2 - Q_1$ -bimodule  $\mathbf{n}$  such that

$$\mathbf{m} \otimes_{Q_2} \mathbf{n} \simeq Q_1 \quad \mathbf{n} \otimes_{Q_1} \mathbf{m} \simeq Q_2$$

as bimodules.

These bimodules implement an equivalence between the corresponding module categories: for instance, if  $\mathbf{x}$  is a left  $Q_1$ -module, then  $\mathbf{n} \otimes_{Q_1} \mathbf{x}$  is a left module over  $Q_2$ . This provides a categorical generalization of classical Morita theory.

The main result is the following:

**Proposition 6.4.3.** [86] *If  $Q_1$  and  $Q_2$  are Morita equivalent algebras, then  $Z[Q_1]$  and  $Z[Q_2]$  are isomorphic.*

In the following we restrict ourselves to CFTs arising as full centers of chiral Q-systems. In this framework, Morita equivalent chiral algebras give rise to the same full two-dimensional CFT.

## 6.5 Bimodules-defects correspondence

Now we will expose the results from [35, 57] which relate the previous classification of defects to chiral bimodules. The main result is the following:

**Theorem 6.5.1.** [57] *Let  $\mathcal{A}$  be a completely rational chiral CFT net, and  $\mathcal{C} = \mathcal{C}^{DHR}(\mathcal{A})$  its (modular  $C^*$  tensor) DHR category. Let  $\mathcal{B}^Y$  ( $Y = L, R$ ) be two local chiral extensions of  $\mathcal{A} \otimes \mathcal{A}$  whose commutative Q-systems  $Q^Y = (\Theta^Y, W^Y, X^Y)$  arise as full centers of chiral Q-systems  $Q_1 = (\Theta_1, x_1, w_1)$  and  $Q_2 = (\Theta_2, x_2, w_2)$ . Then the minimal projections  $I_{\mathbf{m}} \in [\text{Hom}(\Theta^R, \Theta^L), *]$  (i.e., the solutions to the system (5.17)) are given by the operators*

$$I_{\mathbf{m}} = \frac{\dim(\beta)}{d_{Q_1}^2 d_{Q_2}^2 d_R^2} \cdot D_{R[\mathbf{m}]Z} \quad (6.1)$$

where  $\mathbf{m} = (\beta, m)$  runs over the equivalence classes of irreducible  $Q_1 - Q_2$ -bimodules. Consequently,  $E_{\mathbf{m}} = \chi(I_{\mathbf{m}})$  are the minimal central projections in  $\mathcal{D} \cap \mathcal{D}'$  that classify the irreducible boundary conditions according to corollary 5.3.3.

The proof is rather long and so we omit it, but can be found in [57]. Note that in the treatment of defects, this is the only theorem so far where modularity is assumed.

Now we have to define what are the operators  $D_{R[\mathbf{m}]Z}$ . Consider a  $Q_1 - Q_2$ -bimodule  $\mathbf{m} = (\beta, m)$  over the category  $\mathcal{C} = \mathcal{C}^{DHR}(\mathcal{A})$ . We can define the intertwiners:

$$D_{\mathbf{m}} := r_{\beta}^{\dagger} \bar{\beta} (\epsilon_{\Theta_1, \beta} \Theta_1 \beta (r_2^{\dagger}) m^{\dagger}) r_{\beta} \in \text{Hom}(\Theta_2, \Theta_1) \quad (6.2)$$

where, as usual,  $r_i = x_i w_i$ ,  $i = 1, 2$ , and  $r_{\beta} \in \text{Hom}(\text{id}, \bar{\beta} \beta)$  is part of a standard solution to the conjugacy relations for  $\beta$  and  $\bar{\beta}$  depends on  $\mathbf{m}$  only through its equivalence class as bimodule [35]. This intertwiner can be represented as:

In particular, these operators form a "representation" of the tensor category of bimodules [35, 57]:

**Proposition 6.5.2.** [57] *The operators  $D_{\mathbf{m}}$  satisfy the following properties:*

1.  $D_{\mathbf{m}}$  depends only on the unitary equivalence class of  $\mathbf{m} = (\beta, m)$ .
2.  $D_{\mathbf{m}}^\dagger = D_{\overline{\mathbf{m}}}$
3.  $D_{\mathbf{m}_1 \oplus \mathbf{m}_2} = D_{\mathbf{m}_1} + D_{\mathbf{m}_2}$
4. Denoting  $Q_i - Q_j$ -bimodules as  $\mathbf{m}_{Q_i Q_j}$ :

$$\begin{aligned} D_{\mathbf{m}_{Q_1 Q_2}} \cdot D_{\mathbf{m}_{Q_2 Q_3}} &= d_{Q_2} \cdot D_{\mathbf{m}_{Q_1 Q_2} \otimes_{Q_2} \mathbf{m}_{Q_2 Q_3}} = \\ &= d_{Q_2} \cdot \sum_{\mathbf{m}_{Q_1 Q_3}} \dim \text{Hom}(\mathbf{m}_{Q_1 Q_2} \otimes_{Q_2} \mathbf{m}_{Q_2 Q_3}, \mathbf{m}_{Q_1 Q_3}) \cdot D_{\mathbf{m}_{Q_1 Q_3}} \end{aligned}$$

Here the product on the left-hand side is the ordinary composition of intertwiners.

5.  $w_1^\dagger D_{\mathbf{m}} w_2 = \dim(\beta) =: \dim(\mathbf{m})$  for  $\mathbf{m} = (\beta, m)$ .

We have seen that a chiral Q-system in  $\mathcal{C}$  defines a Q-system  $Q \otimes 1$  in  $\mathcal{C} \boxtimes \mathcal{C}^{opp}$ . Similarly, a  $Q_1 - Q_2$ -bimodule  $\mathbf{m}$  in  $\mathcal{C}$  can be lifted to a  $Q_1 \otimes 1 - Q_2 \otimes 1$ -bimodule in  $\mathcal{C} \boxtimes \mathcal{C}^{opp}$ , denoted  $\mathbf{m} \otimes 1$ .

The notion of braided product of Q-systems gives rise to a braided product of bimodules [35], so that if we denote the full center of Q-systems as:

$$R[Q] := (Q \otimes 1) \times^+ Q_{can} \quad \Rightarrow \quad Z[Q] = C^+[R[Q]] = C^+[(Q \otimes 1) \times^+ Q_{can}]$$

we can define  $R[Q_1] - R[Q_2]$ -bimodules as

$$R[\mathbf{m}] := (\mathbf{m} \otimes 1) \times^+ \mathbf{Q}_{can}$$

where  $\mathbf{Q}_{can}$  is the canonical Q-system in  $\mathcal{C} \boxtimes \mathcal{C}^{opp}$  regarded as a  $Q_{can} - Q_{can}$ -bimodule. Since the full center  $Z[Q]$  is an intermediate Q-system with respect to  $R[Q]$ , we can define  $Z[Q_1] - Z[Q_2]$ -bimodules restricting  $R[\mathbf{m}]$  to the full centers  $Z[Q_1]$  and  $Z[Q_2]$ .

Then,  $D_{R[\mathbf{m}]/Z}$  in equation (6.1) is just equation (6.2) for the  $Z[Q_1] - Z[Q_2]$ -bimodule  $R[\mathbf{m}]|_Z$ , where  $\mathbf{m}$  is an irreducible  $Q_1 - Q_2$ -bimodule<sup>7</sup>.

As said earlier, the number of chiral bimodules between two Q-systems is known and equals the dimension of the center of equation (5.12). Thus, provided  $D_{R[\mathbf{m}]/Z}$  are linearly independent, they form a basis of  $\text{Hom}(\Theta^R, \Theta^L)$ .

In [57] it is shown, using the non-degeneracy property of the braiding in a modular category, that they are indeed linearly independent, and in fact orthogonal with respect to the scalar product:

$$(D_1, D_2) := R^{R^\dagger} \cdot D_1^\dagger D_2 \cdot R^R = R^{L^\dagger} \cdot D_2 D_1^\dagger \cdot R^L, \quad D_1, D_2 \in \text{Hom}(\Theta^R, \Theta^L)$$

Represented as<sup>8</sup>:

$$(D_1, D_2) = \begin{array}{c} \text{blue loop} \\ \boxed{D_1^\dagger} \\ \text{red line} \\ \boxed{D_2} \end{array} = \begin{array}{c} \text{red loop} \\ \boxed{D_2} \\ \text{blue line} \\ \boxed{D_1^\dagger} \end{array}$$

<sup>7</sup>Indeed, while  $D_{\mathbf{m}} \in \text{Hom}(\Theta_2, \Theta_1)$  we have  $D_{R[\mathbf{m}]/Z} \in \text{Hom}(\Theta^R, \Theta^L)$ .

where we are again using the blue color for  $\Theta^R$  in order to distinguish it from  $\Theta^L$ . In particular [57]:

$$(D_{R[\mathbf{m}_1|Z]}, D_{R[\mathbf{m}_2|Z]}) = d_1^2 d_2^2 d_{Q_{can}}^4 \cdot \delta_{\mathbf{m}_1, \mathbf{m}_2}$$

where  $d_1$  and  $d_2$  are the dimensions of the  $Q$ -systems  $Q_1$  and  $Q_2$  respectively.

Let's now denote  $T := W_\rho^L W_\rho^{R\dagger}$  the elements of the other basis of  $\text{Hom}(\Theta^R, \Theta^L)$ , where  $T$  is understood to include the label  $\rho$  and possible multiplicity indices. Recalling that  $W_\rho^{Y\dagger} W_\rho^Y = d_Y / \dim(\rho)$ ,  $Y = L, R$ , one finds:

$$(T, T') = \frac{d_{Q_{can}}^2}{\dim(\rho)} \cdot \delta_{T, T'} \quad (6.3)$$

where  $d_Y = d_{Q_{can}}$  thanks to proposition 6.4.2. Indeed:

$$(T, T') = R^{R\dagger} \cdot W_{\rho'}^R W_{\rho'}^{L\dagger} W_\rho^L W_\rho^{R\dagger} \cdot R^R = \frac{d_{Q_{can}}}{\dim(\rho)} \cdot \delta_{\rho', \rho} \cdot R^{R\dagger} \cdot W_{\rho'}^R W_\rho^{R\dagger} \cdot R^R$$

and the remaining term can be resolved using the cyclicity of the trace:

$$\text{Diagram 1} = \text{Diagram 2} = \frac{d_{Q_{can}}}{\dim(\rho)} \text{Diagram 3} = d_{Q_{can}} \quad (6.4)$$

Then we can normalize both bases and define a unitary matrix that allows one to pass from one basis to the other.

**Definition 6.5.1.** The passage from the normalized basis  $\frac{1}{d_1 d_2 d_{Q_{can}}} \cdot D_{R[\mathbf{m}|Z]}$  to the normalized basis  $\frac{\sqrt{\dim(\rho)}}{d_{Q_{can}}} \cdot T$  defines a unitary matrix:

$$S_{\mathbf{m}, T}^{Q_1 Q_2} := \frac{\sqrt{\dim(\rho)}}{d_1 d_2 d_{Q_{can}}} \cdot (D_{R[\mathbf{m}|Z]}, T) \quad (6.5)$$

called *generalized S-matrix*.

Then we can finally expand the elements  $T = W_\rho^L W_\rho^{R\dagger}$  in terms of the minimal central projections  $I_{\mathbf{m}}$  of equation (6.2) in order to compute the boundary conditions.

$$W_\rho^L W_\rho^{R\dagger} = T = \frac{d_{Q_{can}}}{\sqrt{\dim(\rho)}} \sum_{\mathbf{m}} S_{\mathbf{m}, T}^{Q_1 Q_2} \cdot \frac{1}{d_1 d_2 d_{Q_{can}}} \cdot D_{R[\mathbf{m}|Z]} = \frac{d_1 d_2 d_{Q_{can}}}{\dim(\beta) \sqrt{\dim(\rho)}} \sum_{\mathbf{m}} S_{\mathbf{m}, T}^{Q_1 Q_2} \cdot I_{\mathbf{m}}$$

Thus, by equation (5.16), the values of taken by the central operators  $B_\rho = \Psi_\rho^{L\dagger} \Psi_\rho^R$  in the irreducible representation labeled by  $\mathbf{m}$  are:

$$\pi_{\mathbf{m}}(B_\rho) = \frac{d_1 d_2 d_{Q_{can}}}{\dim(\beta) \sqrt{\dim(\rho)}} \cdot S_{\mathbf{m}, T}^{Q_1 Q_2}, \quad \mathbf{m} = (\beta, m), \quad T = W_\rho^L W_\rho^{R\dagger} \quad (6.6)$$

In summary:

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<sup>8</sup>Note that the equality between the two algebraic expressions for  $(D_1, D_2)$  admits a simple graphical interpretation: it follows from "sliding" one of the two intertwiners around the circle. Formally, this is a standard property of traces, namely the cyclicity of equation (2.26), which can be proved straightforwardly using the snake identities together with the definition of dual morphisms.

**Corollary 6.5.3.** *The irreducible boundary conditions, i.e., the irreducible representations of  $\mathcal{D} = \mathcal{B}_{2D}^L \times^- \mathcal{B}_{2D}^R$  are labelled by the chiral bimodules between the chiral  $Q$ -systems whose full centers give rise to the extensions  $\mathcal{B}_{2D}^L$  and  $\mathcal{B}_{2D}^R$ . The "angles"  $\pi_{\mathbf{m}}(\Psi_\rho^{L\dagger} \Psi_\rho^R)$  between left and right fields are given by the matrix elements of the generalized  $S$ -matrix as in equation (6.6).*

Recall now from proposition 6.4.1 that if  $Q_1 = Q_2 = (\text{id}, 1, 1) =: 1$ , then their full centers coincide with the canonical  $Q$ -system. By proposition 6.3.6, bimodules over  $Q=1$  are isomorphic to modules over  $Q \otimes Q^{opp} = 1 \otimes 1 = 1 = Q$ . Hence bimodules over 1 are isomorphic to left modules over 1. By proposition 6.3.1, each module over 1 induced by a simple object  $\rho$  is itself simple, since  $\dim \text{Hom}_1(\text{Ind}_1(\rho), \text{Ind}_1(\rho)) = \dim \text{Hom}(\rho, 1 \otimes \rho) = 1$ . Hence, the irreducible  $Q_1 - Q_2$ -bimodules  $\mathbf{m}$  of theorem 6.5.1 are labeled by simple objects  $\sigma$  of  $\mathcal{C}^{DHR}(\mathcal{A})$ . On the other hand, the isometries  $T = W_{\rho \otimes \bar{\rho}}^L W_{\rho \otimes \bar{\rho}}^{R\dagger}$  are labeled by  $\rho \otimes \bar{\rho}$  and the generalized  $S$ -matrix reduces to:

$$S_{\mathbf{m}, T}^{11} = S_{\text{Ind}_1(\sigma), \rho \otimes \bar{\rho}}^{11} = S_{\sigma, \rho}$$

where on the right-hand side we have the ordinary  $S$ -matrix describing the chiral CFT, since with these assumptions equation (6.6) collapses to match exactly equation (5.19):

$$\begin{aligned} \pi_{\mathbf{m}}(B_\rho) &= \frac{d_1 d_2 d_{Q_{can}}}{\dim(\beta) \sqrt{\dim(\rho)}} \cdot S_{\mathbf{m}, T}^{Q_1 Q_2} \quad \rightarrow \quad \pi_\sigma(B_\rho) = \frac{1 \cdot 1 \cdot d_{Q_{can}}}{\dim(\sigma) \sqrt{\dim(\rho \otimes \bar{\rho})}} \cdot S_{\sigma, \rho} = \\ &= \frac{d_{Q_{can}}}{\dim(\sigma) \dim(\rho)} \cdot S_{\sigma, \rho} = \\ &= \frac{S_{\sigma, \rho} S_{00}}{S_{\sigma, 0} S_{\rho, 0}} \end{aligned}$$

with  $S_{\rho, 0} = s_{\rho, 0} \cdot S_{00} = \dim(\rho)/d_{Q_{can}}$  where we have used equation (3.28) to write  $S_{00} = 1/d_{Q_{can}}$ <sup>9</sup>. Hence, we recover the result that, in diagonal theories, irreducible boundary conditions are labeled by simple chiral objects. Furthermore, their fusion rules coincide with those of the corresponding (induced) bimodules, which in turn match the fusion rules of chiral DHR endomorphisms. Thus, we reconfirm that Verlinde lines obey the same fusion algebra as primary operators.

Other special cases are the following:

- If  $T = W^L W^{R\dagger}$ , then:

$$\begin{aligned} S_{\mathbf{m}, \text{id}}^{Q_1 Q_2} &= \frac{\sqrt{\dim(\rho)}}{d_1 d_2 d_{Q_{can}}^3} \cdot (D_{R[\mathbf{m}]|Z}, W^L W^{R\dagger}) = \frac{\sqrt{\dim(\rho)}}{d_1 d_2 d_{Q_{can}}^3} \cdot \text{Tr}_{\Theta^R} [D_{R[\mathbf{m}]|Z}^\dagger W^L W^{R\dagger}] = \\ &= \frac{\sqrt{\dim(\rho)}}{d_1 d_2 d_{Q_{can}}^3} \cdot \text{Tr}_{\text{id}} [W^{R\dagger} D_{R[\mathbf{m}]|Z}^\dagger W^L] = \frac{\sqrt{\dim(\rho)}}{d_1 d_2 d_{Q_{can}}^3} \cdot W^{R\dagger} D_{R[\mathbf{m}]|Z}^\dagger W^L \end{aligned}$$

where we have used again the cyclicity of the trace.

Then a generalization of property 5. in proposition 6.5.2 to  $D_{R[\mathbf{m}]|Z}$  gives [57]:

$$S_{\mathbf{m}, \text{id}}^{Q_1 Q_2} = \frac{\sqrt{\dim(\rho)}}{d_1 d_2 d_{Q_{can}}^3} \cdot d_{Q_{can}}^2 \dim(\beta)$$

So that:

$$\pi_{\mathbf{m}}(B_{\text{id}}) = \frac{d_1 d_2 d_{Q_{can}}}{\dim(\beta) \sqrt{\dim(\rho)}} \cdot S_{\mathbf{m}, \text{id}}^{Q_1 Q_2} = \mathbf{1}$$

Therefore, as expected, every defect acts trivially on the identity operator.

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<sup>9</sup>Recall that  $d_{Q_{can}} = \sqrt{\dim(\Theta_{can})} = \sqrt{\sum_\rho \dim(\rho \otimes \bar{\rho})} = \sqrt{\sum_\rho \dim(\rho)^2}$  with  $\rho$  irreducible chiral endomorphism.

- If we have isomorphic theories on both sides of the boundary, i.e.  $Q_1 = Q_2 = Q$ , we can consider  $\mathbf{m} = Q$  as a  $Q - Q$ -bimodule. In this case, one can show that  $D_{R[\mathbf{m}||Z]} = d_1^2 d_{Q_{can}} \mathbf{1}_{\Theta_R}$  [57], so that:

$$S_{\mathbf{Q},T}^{QQ} = \frac{\sqrt{\dim(\rho)}}{d_1 d_2 d_{Q_{can}}^3} \cdot d_1^2 d_{Q_{can}}(\mathbf{1}, T) \stackrel{d_1=d_2=d_Q}{=} \frac{\sqrt{\dim(\rho)}}{d_Q^2 d_{Q_{can}}^3} \cdot d_Q^2 d_{Q_{can}} \cdot d_{Q_{can}} = \frac{\sqrt{\dim(\rho)}}{d_{Q_{can}}} \quad (6.7)$$

where we have used the fact that in this case, since  $\Theta^L = \Theta^R$ , the scalar product becomes

$$(\mathbf{1}, T) = \text{Tr}_{\Theta_R}[W_\rho^R W_\rho^{R\dagger}] = \text{Tr}_{\Theta_R}[W_\rho^{R\dagger} W_\rho^R] = \frac{d_R}{\dim(\rho)} \cdot \text{Tr}_\rho[\text{id}] = \frac{d_{Q_{can}} \dim(\rho)}{\dim(\rho)} = d_{Q_{can}}$$

where we have used again the cyclicity of the trace as in equation (6.4).

As a consequence, since  $d_Q^2 = d_\Theta = \dim(\beta)$ , we have  $\pi_Q(B_\rho) = \frac{d_Q^2}{\dim(\beta)} \cdot \mathbf{1} = \mathbf{1}$  for all  $\rho$ , meaning that we get exactly the identity for all the endomorphisms  $\rho$ . This implies that the trivial defect corresponds to the trivial bimodule.

One can also use the following proposition to compute the generalized S-matrix in more general situations.

**Proposition 6.5.4.** *Consider the assumptions of theorem 6.5.1, but now let  $Q^Y$ ,  $Y = A, B, C$ , be three  $Q$ -systems arising as full centers of chiral  $Q$ -systems  $Q_Y$ ,  $Q^Y := Z[Q_Y]$ ,  $Y = A, B, C$ . Furthermore, assume that  $\dim \text{Hom}(\Theta^Y, \rho) \in \{0, 1\}$  for all  $\rho \in \text{Obj}(\mathcal{C}^{DHR}(\mathcal{A}) \boxtimes \mathcal{C}^{DHR}(\mathcal{A})^{opp})$ ,  $Y = A, B, C$ , i.e. there are no multiplicities. Then:*

$$N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} = \sum_T \frac{d_{Q_{can}}}{\sqrt{\dim(\rho)}} \cdot S_{\mathbf{m}_1, T}^{AB} S_{\mathbf{m}_2, T}^{BC} \overline{S_{\mathbf{m}_3, T}^{AC}}, \quad T = W_\rho^A W_\rho^{C\dagger} \quad (6.8)$$

where  $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$  are  $Q_A - Q_B, Q_B - Q_C$  and  $Q_A - Q_C$ -bimodules respectively, and  $N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3}$  are the fusion coefficients:

$$\mathbf{m}_1 \otimes_{Q_B} \mathbf{m}_2 = \sum_{\mathbf{m}_3} N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} \cdot \mathbf{m}_3$$

*Proof.* First, we need to generalize property 4. of proposition 6.5.2 to the lifted intertwiners  $D_{R[\mathbf{m}||Z]}$ . For  $\mathbf{m}_1$  an irreducible  $Q_A - Q_B$ -bimodule and  $\mathbf{m}_2$  an irreducible  $Q_B - Q_C$ -bimodule, we use:<sup>10</sup>

$$D_{R[\mathbf{m}_1||Z]} \cdot D_{R[\mathbf{m}_2||Z]} = d_B^2 d_{Q_{can}} \sum_{\mathbf{m}_3} N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} D_{R[\mathbf{m}_3||Z]} \quad (6.9)$$

where  $d_B := d_{Q_B}$ . The product on the left-hand side is the ordinary composition of intertwiners.

As a consistency check, take  $\mathbf{m}_2 = Q_B$ , regarded as the trivial  $Q_B - Q_B$ -bimodule. As already observed  $D_{R[\mathbf{m}_2||Z]} = d_B^2 d_{Q_{can}} \mathbf{1}_{\Theta_R}$ . In general, one can show that  $\mathbf{m}_2 = Q_B$  acts as the unit of the fusion product of bimodules, i.e.  $\mathbf{m}_1 \otimes_{Q_B} \mathbf{m}_2 = \mathbf{m}_1$  [57]. In terms of the  $D_{R[\mathbf{m}||Z]}$  operators the product becomes:

$$D_{R[\mathbf{m}_1||Z]} \cdot D_{R[\mathbf{m}_2||Z]} = d_B^2 d_{Q_{can}} D_{R[\mathbf{m}_1||Z]}$$

which fixes the normalization in (6.9).

Now we use (6.5) to express equation (6.9) in the  $T$  basis. In particular, for a  $Q_A - Q_B$  bimodule  $\mathbf{m}$ :

$$D_{R[\mathbf{m}||Z]} = d_A d_B d_{Q_{can}} \sum_T \sqrt{\dim(\rho)} \overline{S_{\mathbf{m}, T}^{AB}} T$$

where the bar denotes complex conjugation of the corresponding matrix element.

<sup>10</sup>This equation appears in [35], but it contains a normalization error. The consistency check performed here should not be regarded as a proof of the formula, but rather as a way to identify the correct normalization.

Hence the left-hand side of (6.9) becomes:

$$d_A d_B^2 d_C d_{Q_{can}}^2 \sum_{T_1, T_2} \sqrt{\dim(\rho_1) \dim(\rho_2)} \overline{S_{\mathbf{m}_1, T_1}^{AB}} \overline{S_{\mathbf{m}_2, T_2}^{BC}} T_1 T_2$$

whereas the right-hand side becomes

$$d_B^2 d_{Q_{can}} d_A d_C d_{Q_{can}} \sum_{\mathbf{m}_3} N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} \sum_{T'_3} \sqrt{\dim(\rho'_3)} \overline{S_{\mathbf{m}_3, T'_3}^{AC}} T'_3$$

After simplifying the common factors:

$$\sum_{T_1, T_2} \sqrt{\dim(\rho_1) \dim(\rho_2)} \overline{S_{\mathbf{m}_1, T_1}^{AB}} \overline{S_{\mathbf{m}_2, T_2}^{BC}} T_1 T_2 = \sum_{\mathbf{m}_3} N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} \sum_{T'_3} \sqrt{\dim(\rho'_3)} \overline{S_{\mathbf{m}_3, T'_3}^{AC}} T'_3$$

We now take the scalar product with a fixed  $T_3$ . Using the normalization (6.3), this gives

$$\sum_{T_1, T_2} \sqrt{\dim(\rho_1) \dim(\rho_2)} \overline{S_{\mathbf{m}_1, T_1}^{AB}} \overline{S_{\mathbf{m}_2, T_2}^{BC}} (T_3, T_1 T_2) = \sum_{\mathbf{m}_3} N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} \overline{S_{\mathbf{m}_3, T_3}^{AC}} \frac{d_{Q_{can}}^2}{\sqrt{\dim(\rho_3)}}$$

Multiplying by

$$\frac{\sqrt{\dim(\rho_3)}}{d_{Q_{can}}^2} S_{\mathbf{m}_3, T_3}^{AC}$$

and summing over  $T_3$ , the unitarity of the generalized  $S$ -matrix yields

$$N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} = \sum_{T_1, T_2, T_3} d_{Q_{can}}^{-2} \sqrt{\dim(\rho_1) \dim(\rho_2) \dim(\rho_3)} \overline{S_{\mathbf{m}_1, T_1}^{AB}} \overline{S_{\mathbf{m}_2, T_2}^{BC}} S_{\mathbf{m}_3, T_3}^{AC} (T_3, T_1 T_2) \quad (6.10)$$

Let us now analyse the product  $T_1 T_2$ . This is simply the ordinary composition of intertwiners. In the present setting, the labels  $L/R$  are replaced by the labels  $A, B, C$  of the three Q-systems. Thus

$$T_1 \in \text{Hom}(\Theta^{Z[B]}, \Theta^{Z[A]}), \quad T_2 \in \text{Hom}(\Theta^{Z[C]}, \Theta^{Z[B]}), \quad T_3 \in \text{Hom}(\Theta^{Z[C]}, \Theta^{Z[A]})$$

where  $Z[Y] := Z[Q_Y]$ ,  $Y = A, B, C$ . In the multiplicity-free case, we may write

$$T_1 = W_{\rho_1}^A W_{\rho_1}^{B\dagger}, \quad T_2 = W_{\rho_2}^B W_{\rho_2}^{C\dagger}, \quad T_3 = W_{\rho_3}^A W_{\rho_3}^{C\dagger}$$

Therefore

$$T_1 T_2 = W_{\rho_1}^A \left( W_{\rho_1}^{B\dagger} W_{\rho_2}^B \right) W_{\rho_2}^{C\dagger}$$

Now  $W_{\rho_1}^{B\dagger} W_{\rho_2}^B \in \text{Hom}(\rho_2, \rho_1)$ . Since the sectors  $\rho_i$  are irreducible, Schur's lemma implies that this morphism vanishes unless  $\rho_1 \simeq \rho_2$ . With the chosen normalization of the  $W_\rho$ 's, we have  $W_\rho^{B\dagger} W_\rho^B = \frac{d_{Q_{can}}}{\dim(\rho)} \mathbf{1}_\rho$ . Hence<sup>11</sup>

$$T_1 T_2 = \frac{d_{Q_{can}}}{\dim(\rho_1)} \delta_{\rho_1, \rho_2} W_{\rho_1}^A W_{\rho_1}^{C\dagger}$$

Taking the scalar product with  $T_3$  and using again (6.3), we find

$$(T_3, T_1 T_2) = \frac{d_{Q_{can}}}{\dim(\rho_1)} \delta_{\rho_1, \rho_2} \left( T_3, W_{\rho_1}^A W_{\rho_1}^{C\dagger} \right) = \frac{d_{Q_{can}}^3}{\dim(\rho_1)^2} \delta_{\rho_1, \rho_2} \delta_{\rho_3, \rho_1}$$

Equivalently, in the multiplicity-free case,

$$(T_3, T_1 T_2) = d_{Q_{can}}^3 \dim(\rho)^{-2} \delta_{T_1, T_3} \delta_{T_2, T_3}.$$

<sup>11</sup>In the presence of multiplicities, one would still have  $W_{\rho_1}^{iB\dagger} W_{\rho_2}^{jB} \propto \delta_{\rho_1, \rho_2}$ , but with an additional Gram matrix  $g_\rho^{ij}$  on the multiplicity space. This matrix can be chosen diagonal by taking an orthonormal basis of  $\text{Hom}(\rho, \Theta^{Z[B]})$ . This, however, makes the derivation less straightforward.

Inserting this back into (6.10) gives:

$$N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} = \sum_T \frac{d_{Q_{can}}}{\sqrt{\dim(\rho)}} \cdot \overline{S_{\mathbf{m}_1, T}^{AB}} \overline{S_{\mathbf{m}_2, T}^{BC}} S_{\mathbf{m}_3, T}^{AC}$$

Finally, since the fusion coefficients are non-negative integers, we may take the complex conjugate of the whole expression and obtain the (6.8).  $\square$

Considering (6.7), the next corollary immediately follows:

**Corollary 6.5.5.** *With the assumption of the previous proposition, if  $B = C$  we find a "generalized Verlinde formula":*

$$N_{\mathbf{m}_1 \mathbf{m}_2}^{\mathbf{m}_3} = \sum_T \frac{S_{\mathbf{m}_1, T}^{AB} S_{\mathbf{m}_2, T}^{BB} \overline{S_{\mathbf{m}_3, T}^{AB}}}{S_{\mathbf{B}, T}^{BB}}, \quad T = W_\rho^A W_\rho^{B\dagger}$$

and the unitary matrices  $S_{\mathbf{m}, T}^{AB}$  diagonalize the fusion rules:

$$\sum_{\mathbf{m}_1 \mathbf{m}_3} \overline{S_{\mathbf{m}_1, T}^{AB}} N_{\mathbf{m}_1 \mathbf{m}}^{\mathbf{m}_3} S_{\mathbf{m}, T'}^{AB} = \frac{S_{\mathbf{m}, T}^{BB}}{S_{\mathbf{B}, T}^{BB}} \cdot \delta_{T, T'}$$

Actually, in [35] the authors came up with a stronger result, relating general defects to bimodules, whereas theorem 6.5.1 relates bimodules just to the boundary conditions.

Note first that by construction we have:

$$\mathcal{B}^L = \mathcal{D}_{loc}^- \quad \mathcal{D}_{loc}^+ = \mathcal{B}^R$$

Then the result is the following.

**Theorem 6.5.6.** [35] *Let  $\mathcal{B}^Y$ ,  $Y = L, R$ , be two local extensions of  $\mathcal{A} \otimes \mathcal{A}$  whose commutative  $Q$ -systems  $Q^Y = (\Theta^Y, W^Y, X^Y)$  arise as full centers of chiral  $Q$ -systems  $Q_1$  and  $Q_2$ , i.e.  $Q^L := Z[Q_1]$  and  $Q^R := Z[Q_2]$ . Then there is a one-to-one correspondence between factorial defects*

$$\mathcal{B}^L = \mathcal{D}_{loc}^- \subset \mathcal{D} \supset \mathcal{D}_{loc}^+ = \mathcal{B}^R$$

*and equivalence classes of (not necessarily irreducible)  $Q_1 - Q_2$ -bimodules. The defect is a boundary condition if and only if the bimodule is irreducible. In this case, the boundary condition is irreducible.*

Note that if the bimodule is not irreducible, then it is associated with a defect rather than with an elementary boundary condition. This is consistent with the interpretation of an algebraic defect  $\mathcal{D}$  as an extension encoding not only the defect itself, but also possible additional degrees of freedom corresponding to defect operators. Indeed, for a non-simple bimodule  $\mathbf{m}$ , the algebra  $\text{End}(\mathbf{m})$  is non-trivial, whereas for a simple bimodule Schur's lemma implies  $\text{End}(\mathbf{m}) \simeq \mathbb{C}$ .

Therefore, irreducible bimodules correspond to boundary conditions, while reducible bimodules describe superpositions of such boundary conditions together with non-trivial defect operator sectors.

## 6.6 Modules and algebra objects of the chiral Ising model

We have already seen that diagonal models can be seen as the full center with respect to the trivial algebra,  $Q = 1$ , and the corresponding simple bimodules are all induced by simple objects. Moreover we have already seen that in this case the generalized S-matrix reduces to the ordinary S-matrix, and the boundary conditions are the same found algebraically in the last chapter. However, at this point we have the tools to answer a natural question: is this the only structure we can have for the Ising model? Indeed, we cannot be sure that there are no other special Frobenius algebras which are not

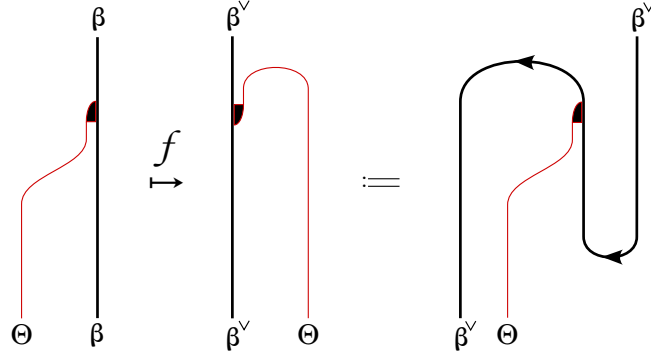
Morita equivalent to the identity compatible with this model. In that case, such Q-systems would describe a totally different CFT, with different defects. As we are going to see shortly, this is not the case. We will follow a strategy from [78].

First, we need to state two useful results. We begin with the following definition:

**Definition 6.6.1.** Given a left Q-module  $\mathbf{m} = (\beta, m)$  we can define a right Q-module  $\mathbf{m}^\vee = (\beta^\vee, f(m))$ , where the right-action  $f(m)$  is given by:

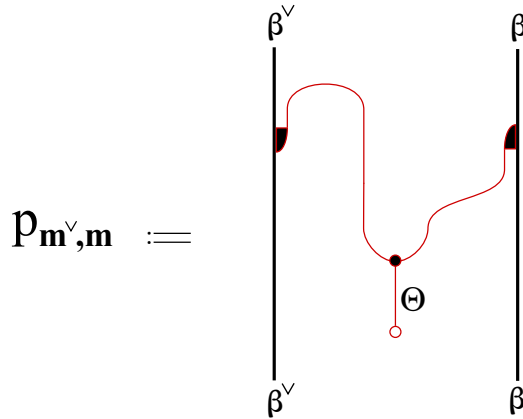
$$f(m) = (d_\beta \otimes \text{id}_{\beta^\vee}) \circ (\text{id}_{\beta^\vee} \otimes m \otimes \text{id}_{\beta^\vee}) \circ (\text{id}_{\beta^\vee} \otimes \text{id}_\Theta \otimes b_\beta)$$

Pictorially:



We recall that in our case right and left duality coincide and are denoted with the conjugation bar.

Recall that in general  $\mathcal{C}_Q$  is not a tensor category. However, it is possible to define a product between left- and right- modules following a similar procedure to the definition of the product of bimodules. Consider a simple symmetric special Frobenius algebra  $Q = (\Theta, w, x^\dagger)$  and a left Q-module  $\mathbf{m}$ . We define the morphism:



which can be shown to be an idempotent and to have a retract [78]. Then one defines:

$$\mathbf{m}^\vee \otimes_Q \mathbf{m} := \text{Im}(p_{\mathbf{m}^\vee, \mathbf{m}})$$

Then one can show the following<sup>12</sup>:

<sup>12</sup>In the following the Q-systems are assumed to be *simple*. This notion is defined in [57] as the requirement for the chiral Q-system to generate an extension  $\mathcal{N} \subset \mathcal{M}$  with  $\mathcal{M}$  being a factor. This is always the case, as we have seen in section 4.2, if  $\dim \text{Hom}(\text{id}, \Theta) = 1$ , but *may* fail when this condition is not satisfied. Simplicity has many equivalent conditions, namely that  $\dim \text{Hom}(\text{id} \otimes \text{id}, \Theta_{Z[Q]}) = 1$ , where  $Z[Q]$  is the full-center of the chiral Q-system  $Q$ , or  $\dim \text{Hom}(\text{id}, C^\pm[Q]) = 1$ , where  $C^\pm[Q]$  denote the left/right centers of  $Q$  [57]. These last two equivalent conditions are the ones used in [33] to define this property. Note that if  $Q$  is not simple, the full center  $Z[Q]$  describes an extension which is not a factor, which can be centrally decomposed as we have done for defects. Therefore, non-simple Q-systems describe the superposition of different CFTs; this is also discussed in [33]. Finally, one can introduce a notion of direct sum of Q-system so that a non-simple Q-system can be decomposed in a sum of simple Q-systems [78].

**Proposition 6.6.1.** [78] Let  $Q_1$  be a simple symmetric special Frobenius algebra in the modular tensor category  $\mathcal{C}$ . Given a left  $Q_1$ -module  $\mathbf{m}$  that is not a zero object, set  $Q_2 := \mathbf{m}^\vee \otimes_{Q_1} \mathbf{m}$ . Then  $Q_2$  can be endowed with the structure of a simple symmetric special Frobenius algebra such that  $Q_1 \simeq \mathbf{m} \otimes_{Q_2} \mathbf{m}^\vee$  and  $Q_2 \simeq \mathbf{m}^\vee \otimes_{Q_1} \mathbf{m}$  as  $Q_1 - Q_1$ - and  $Q_2 - Q_2$ -bimodules, respectively. In particular,  $Q_1$  and  $Q_2$  are Morita equivalent.

**Proposition 6.6.2.** [78] Every simple symmetric special Frobenius algebra  $Q_2$  is isomorphic to  $\mathbf{m}^\vee \otimes_{Q_1} \mathbf{m}$ , where  $Q_1$  is a haploid simple symmetric special Frobenius algebra and  $\mathbf{m}$  is a left  $Q_1$ -module.

Note that if in proposition 6.6.1 we consider  $Q_1 = 1$ , then every object of the form  $\beta^\vee \otimes \beta$  can be endowed with the structure of a simple symmetric special Frobenius algebra, which is Morita equivalent to the trivial  $Q$ -system.

We start from a generic algebra object for  $\mathcal{C}^{DHR}(\mathcal{A})$  given by:

$$\Theta = 1 \oplus \psi^{\oplus n_\psi} \oplus \sigma^{\oplus n_\sigma}$$

where  $n_\psi, n_\sigma \in \mathbb{N}$  and 1 does not have any multiplicity because, focusing on simple symmetric special Frobenius algebras, as a consequence of the above propositions, there is always a haploid representative ( $\dim \text{Hom}(1, \Theta) = 1$ ) in every Morita equivalence class.

Now we use proposition 6.3.1 to study the induced modules over this generic algebra. In particular, the following table contains the dimensions  $\dim \text{Hom}_Q(\text{Ind}_Q(U), \text{Ind}_Q(V)) = \dim \text{Hom}(U, \Theta \otimes V)$  for  $U, V = 1, \psi, \sigma$ :

|          | 1          | $\psi$     | $\sigma$   |
|----------|------------|------------|------------|
| 1        | 1          | $n_\psi$   | $n_\sigma$ |
| $\psi$   | $n_\psi$   | 1          | $n_\sigma$ |
| $\sigma$ | $n_\sigma$ | $n_\sigma$ | $1+n_\psi$ |

So we immediately know that both  $\text{Ind}_Q(1)$  and  $\text{Ind}_Q(\psi)$  are simple modules. This implies that  $n_\psi$  must be either 0 or 1 depending on whether these two modules are isomorphic or not. We now distinguish the two cases:

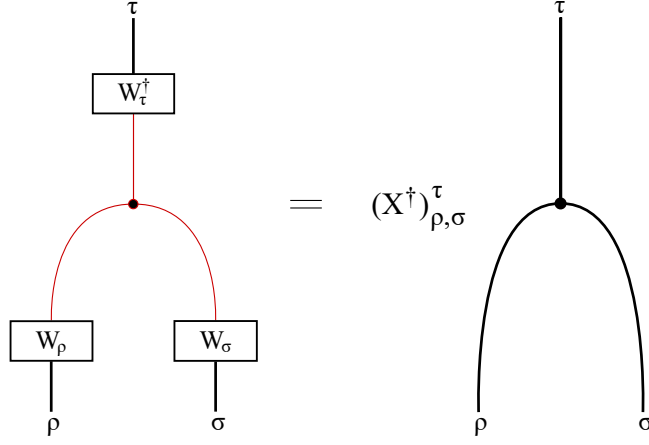
- Suppose that  $n_\psi = 0$ , i.e.  $\text{Ind}_Q(1)$  and  $\text{Ind}_Q(\psi)$  are not isomorphic. In this case also  $\text{Ind}_Q(\sigma)$  is simple. Then we must have  $n_\sigma = 0$ , thus fixing  $\Theta = 1$ .
- Now let  $n_\psi = 1$ , i.e.  $\text{Ind}_Q(1) \simeq \text{Ind}_Q(\psi)$ . Then  $\dim \text{Hom}_Q(\text{Ind}_Q(\sigma), \text{Ind}_Q(\sigma)) = 2$ , so it is reducible:  $\text{Ind}_Q(\sigma) \simeq \mathbf{m}_1 \oplus \mathbf{m}_2$  with  $\mathbf{m}_1, \mathbf{m}_2$  non-isomorphic simple  $Q$ -modules. Now we can have either  $n_\sigma = 1$ , meaning that exactly one among  $\mathbf{m}_1$  and  $\mathbf{m}_2$  is isomorphic to  $\text{Ind}_Q(1) \simeq \text{Ind}_Q(\psi)$ , or  $n_\sigma = 0$ .

Thus the only possibilities are:

$$\Theta = 1 \quad \Theta = 1 \oplus \psi \quad \Theta = 1 + \psi + \sigma$$

Keeping, as always, the  $Q$ -system specific notation, to better study these algebras it is useful to express the product  $X^\dagger$  with respect to the basis  $\{W_\rho\}$  for the embedding spaces  $\text{Hom}(\rho, \Theta)$ , as well as dual basis  $\{W_\rho^\dagger\}$  for  $\text{Hom}(\Theta, \rho)$ ,  $\rho \prec \Theta$ , as compared to the bases of (2.28) describing the fusion of simple objects<sup>13</sup>:

<sup>13</sup>This procedure can be carried out in general; see, for example, [33]. However, since we will use it only in the present context, we specialize directly to the Ising model and simplify the notation accordingly. In particular, equation (6.11) is valid for general theories with  $N_{ij}^k \in \{0, 1\}$  and  $\dim \text{Hom}(\rho, \Theta) \in \{0, 1\}$ .



where  $(X^\dagger)_{\rho,\sigma}^\tau \in \mathbb{C}$ . It is easy to see that the associativity of the Q-system  $\mathcal{Q}$  is expressed as [33]:

$$(X^\dagger)_{\rho,\sigma}^\gamma (X^\dagger)_{\gamma,\tau}^\delta = \sum_{\varphi \prec \Theta} (X^\dagger)_{\sigma,\tau}^\varphi (X^\dagger)_{\rho,\varphi}^\delta F_{\varphi,\gamma}^{(\rho\sigma\tau)\delta} \quad (6.11)$$

where all the labels refer to subendomorphisms with respect to  $\Theta$ . One can show that the special property of the Q-system forces  $(X^\dagger)_{\rho,\rho}^1 \neq 0$  for all  $\rho \prec \Theta$  [33].

Now we consider this consistency constraint applied to the algebra given by  $\Theta = 1 \oplus \psi \oplus \sigma$ . We get the following three equations:

$$\begin{aligned} (X^\dagger)_{\psi,\sigma}^\sigma (X^\dagger)_{\sigma,\sigma}^\psi &= (X^\dagger)_{\sigma,\sigma}^1 (X^\dagger)_{\psi,1}^\psi F_{1,\sigma}^{(\psi\sigma\sigma)\psi} \\ (X^\dagger)_{\sigma,\psi}^\sigma (X^\dagger)_{\sigma,\sigma}^\psi &= (X^\dagger)_{\psi,\sigma}^\sigma (X^\dagger)_{\sigma,\sigma}^\psi F_{\sigma,\sigma}^{(\sigma\psi\sigma)\psi} \\ (X^\dagger)_{\sigma,\sigma}^1 (X^\dagger)_{1,\psi}^\psi &= (X^\dagger)_{\sigma,\psi}^\sigma (X^\dagger)_{\sigma,\sigma}^\psi F_{\sigma,1}^{(\sigma\sigma\psi)\psi} \end{aligned}$$

Since we want  $\mathcal{Q}$  to be special, we need to impose  $(X^\dagger)_{\rho,\rho}^1 \neq 0$  for all  $\rho = 1, \psi, \sigma$ . The only term of this form appearing in the above equations is  $(X^\dagger)_{\sigma,\sigma}^1$  and we normalize it to 1. Note that clearly we also have  $(X^\dagger)_{\rho,1}^\rho = (X^\dagger)_{1,\rho}^\rho = 1$  for all  $\rho$ .

Using  $F_{\sigma,1}^{(\sigma\sigma\psi)\psi} = 1 = F_{1,\sigma}^{(\psi\sigma\sigma)\psi}$  we get from the last equation:

$$1 \cdot 1 = (X^\dagger)_{\sigma,\sigma}^1 (X^\dagger)_{1,\psi}^\psi = (X^\dagger)_{\sigma,\psi}^\sigma (X^\dagger)_{\sigma,\sigma}^\psi$$

I.e.  $(X^\dagger)_{\sigma,\psi}^\sigma, (X^\dagger)_{\sigma,\sigma}^\psi \neq 0$ . Summing the first equation to the third we also find  $(X^\dagger)_{\psi,\sigma}^\sigma = (X^\dagger)_{\sigma,\psi}^\sigma$ .

On the other hand, since  $F_{\sigma,\sigma}^{(\sigma\psi\sigma)\psi} = -1$ , the second equation, upon using  $(X^\dagger)_{\psi,\sigma}^\psi \neq 0$ , gives  $(X^\dagger)_{\psi,\sigma}^\sigma = -(X^\dagger)_{\sigma,\psi}^\sigma$  and so we have a contradiction. Thus there is no associative multiplication on  $\Theta = 1 \oplus \psi \oplus \sigma$  that is also special.

Now we turn our attention to  $\Theta = 1 \oplus \psi$ . Since  $\psi \otimes \psi \simeq 1$ , the coefficient  $(X^\dagger)_{\psi,\psi}^1 \neq 0$  is the only non-trivial structure constant. As  $\text{Hom}(\psi \otimes \psi, 1)$  is one-dimensional, this coefficient amounts only to a choice of normalization and can therefore be set equal to 1. Consequently, the multiplication is unique.

Furthermore, as objects of  $\mathcal{C}^{DHR}(\mathcal{A})$ , we have  $1 \oplus \psi \simeq \sigma \otimes \sigma \simeq \sigma^\vee \otimes \sigma$ . Hence, by proposition 6.6.1, the object  $1 \oplus \psi$  can be endowed with the structure of a symmetric special Frobenius algebra, proving that the multiplication described above indeed exists. The same proposition also implies that  $1 \oplus \psi$  belongs to the same Morita class as the tensor unit algebra 1.

We conclude that the chiral Ising model admits a single Morita class of symmetric special Frobenius algebras. Therefore, no additional defect classifications arise beyond the one already obtained. In this sense, the classification of topological defects in the critical Ising model is complete.



# Conclusion

In summary, this thesis has explored the nature of topological defects and their realization in two-dimensional conformal field theories (CFTs). We began by examining these defects within the Euclidean theory, and subsequently developed the mathematical machinery required to translate these concepts into the relativistic framework.

Algebraic Quantum Field Theory (AQFT) served as a natural axiomatic framework for the step-by-step construction of a relativistic CFT. Relying on the underlying causal structure, we successfully constructed extensions of the net of observables to encode the presence of defects. A central tool in this procedure was the notion of a Q-system, which proved to be a symmetric special Frobenius algebra over the modular tensor category of the DHR endomorphisms of the net.

Parallel to the relativistic case, the Euclidean construction has also been axiomatized, primarily through the lens of Vertex Operator Algebras (VOAs). The equivalence between these approaches was outlined in general and explicitly proven for the unitary rational CFTs of interest. Ultimately, we demonstrated that modular tensor categories provide the natural and universal environment to describe both Euclidean and relativistic CFTs. Within this unified categorical framework, defects are elegantly described by bimodules.

Building upon the foundation laid in this work, there are several promising avenues for future research:

- *Non-diagonal models:* a natural extension involves studying a non-diagonal example, such as the three-state Potts model. While most of the theoretical instruments developed here are general enough to cover such examples, additional technical work is required. For instance, proposition 6.5.5 cannot be applied directly since the model does not satisfy the multiplicity-free condition.
- *Generalizing algebraic defects:* a comprehensive algebraic description of defects is still lacking, as our primary focus remained on boundary conditions. Although the description via bimodules inherently includes the notion of a defect-starting operator (see for example [85]), more algebraic work is necessary to fully characterize the defect net  $\mathcal{D}$  when it is not simply given by  $\mathcal{B}^L \vee \mathcal{B}^R$ .
- *Defect intersections:* we currently lack an algebraic understanding of the dynamics at the intersection of two defects (or boundary conditions). Addressing this clearly requires the definition of additional degrees of freedom residing on the defect junction, tightly connecting this issue to the aforementioned generalization of algebraic defects.
- *Spacelike defects:* in the AQFT framework, we have defined defects exclusively as timelike curves in Minkowski space. To complete the relativistic picture, a rigorous description of spacelike defects is also required.
- *Boundaries and interfaces:* finally, this work could be expanded by studying defects in the presence of CFT boundaries, or by adapting the developed algebraic framework to describe interfaces between distinct conformal field theories.

Ultimately, the results presented in this work clarify the relation between the Euclidean and relativistic formulations of defects in conformal field theories, providing a solid algebraic foundation for future developments as well as a unified environment for characterizing defects across different frameworks.



# Appendix

## A - Covariant Form of a Conserved Charge

Let  $M^{(d-1)}$  be parametrized by the coordinates  $\{y^a\}_{a=1}^{d-1}$  and let  $e_a^\mu = \frac{\partial x^\mu}{\partial y^a}$  be the corresponding tangent vectors. Then  $dx^\mu|_M = \frac{\partial x^\mu}{\partial y^a} dy^a = e_a^\mu dy^a$  and for  $j = j_\mu dx^\mu$  we get

$$\star j|_M = \frac{1}{(d-1)!} j^\mu \epsilon_{\mu\nu_1 \dots \nu_{d-1}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{d-1}}|_M = j^\mu \epsilon_{\mu\nu_1 \dots \nu_{d-1}} e_1^{\nu_1} \dots e_{d-1}^{\nu_{d-1}} dy^1 \wedge \dots \wedge dy^{d-1} = j^\mu n_\mu dM$$

with  $n_\mu := \epsilon_{\mu\nu_1 \dots \nu_{d-1}} e_1^{\nu_1} \dots e_{d-1}^{\nu_{d-1}} / \sqrt{\det(h)}$  and  $dM := \sqrt{\det(h)} dy^1 \wedge \dots \wedge dy^{d-1}$ , where  $h$  is the pull-back metric  $h_{ab} = e_a^\mu e_b^\nu \delta_{\mu\nu}$ . Using the antisymmetry properties of  $\epsilon$ , it is immediate that  $n_\mu e_a^\mu = 0$  for every  $a = 1, \dots, d-1$ , while  $n_\mu n^\mu = \det(e_a \cdot e_b) / \det(h) = 1$ . Thus,  $n_\mu$  is the unit normal to  $M$ , up to the sign fixed by the chosen orientation.

Moreover:

$$\begin{aligned} d \star j &= \frac{1}{(d-1)!} \partial_\rho j^\mu \epsilon_{\mu\nu_1 \dots \nu_{d-1}} dx^\rho \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{d-1}} = \\ &= \frac{1}{(d-1)!} \partial_\rho j^\mu \epsilon_{\mu\nu_1 \dots \nu_{d-1}} e^{\rho\nu_1 \dots \nu_{d-1}} dx^1 \wedge \dots \wedge dx^d = \\ &= \partial_\mu j^\mu dx^1 \wedge \dots \wedge dx^d = 0 \end{aligned}$$

Hence the conservation equation  $\partial_\mu j^\mu = 0$  is equivalent to  $d \star j = 0$ . The same relations generalize to a general metric by replacing the ordinary divergence with the covariant one.

## B - Computation of the bubble diagrams

We will not write the labels on the diagrams since junction spaces are one-dimensional, hence they do not need indices, and we use the conventions for drawing lines adopted in section 2.5 to distinguish the different TDLs.

We start by computing  $F^{(N\eta N)\eta}$ . The dotted circles indicate the use of the completeness relation (2.37). The only non-trivial entry is:

$$\text{Diagram 1} = \sqrt{2} \text{Diagram 2} - \text{Diagram 3} = -\sqrt{2} \text{Diagram 4} = \sqrt{2} F_{N,N}^{(N\eta N)\eta} \text{Diagram 5}$$

From this, we get  $F_{N,N}^{(N\eta N)\eta} = -1$ . Its cyclic permutation,  $F_{N,N}^{(\eta N \eta)N} = -1$ , is computed in the same way. Now we compute  $F^{(NNN)N}$ . As mentioned above, the first two entries are given by the completeness relation (2.37):  $F_{\mathcal{I},\mathcal{I}}^{(NNN)N} = F_{\mathcal{I},\eta}^{(NNN)N} = \frac{1}{\sqrt{2}}$ . The remaining entries are given by the following diagrams:

$$\begin{aligned}
& \text{Diagram} = \sqrt{2} \text{Diagram} - \text{Diagram} \xrightarrow{0} \\
& = -\sqrt{2} \text{Diagram} + \text{Diagram} = - \text{Diagram} = \sqrt{2} F_{\eta,\eta}^{(NNN)N}
\end{aligned}$$

where we have used (2.29) to resolve the bubbles. Note that the first diagram after the first equality goes to zero as a consequence of that rule.

$$\text{Diagram} = \sqrt{2} \text{Diagram} - \text{Diagram} = \sqrt{2} F_{\eta,I}^{(NNN)N}$$

From which we get:  $F_{\eta,\mathcal{I}}^{(NNN)N} = \frac{1}{\sqrt{2}}$  and  $F_{\eta,\eta}^{(NNN)N} = -\frac{1}{\sqrt{2}}$ .

Let us now see how to resolve the last term of the sum arising when inflating the  $N$  bubble on the torus:

$$\text{Diagram} = \text{Diagram} = \text{Diagram} = 1 = \text{Diagram} = \sqrt{2} \text{Diagram}$$

where in the first step we fuse the defects enclosed by the dotted lines, and then resolve the resulting bubbles using (2.29). The last term corresponds to an  $\eta$  defect wrapping the diagonal cycle of the torus; equivalently, it can be represented as a network of  $\eta$  defects wrapping the two fundamental cycles.

## C - The hypergroup of boundary conditions

In section 5.3 we have seen that the boundary conditions encode the local action of defects. Indeed, equation (5.19) coincides with the first term of equation (2.11), describing a defect moved past a local operator. This first term is graphically represented by a defect bubble encircling a local operator.

Here we discuss another interesting way of describing such local action, which is related to the language we have already developed, but will provide further insight on the structure of the algebraic theory with defects. Our main objective is to show that the observation made for the Ising model, i.e. that there are no other operators, apart from the identity, completely invariant under all defects, is not a coincidence.

We follow an observation from [73]. Consider a defect line encircling a local operator, as in figure C.1. Instead of shrinking the defect to act on the operator, we keep it at finite size and analyze the configuration on constant-time slices.

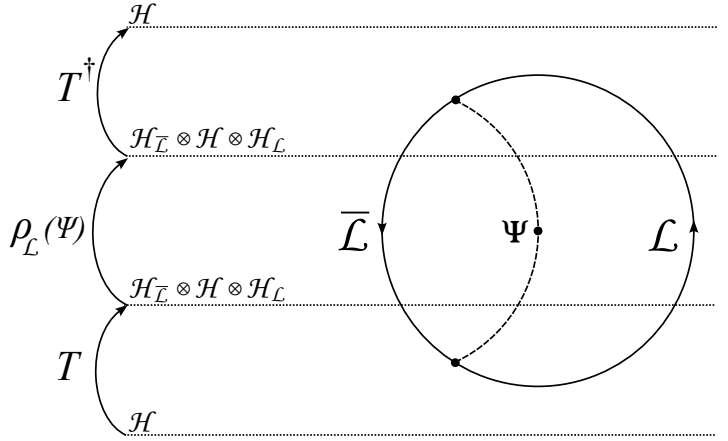


Figure C.1: Local action of the defect  $\mathcal{L}$  on the local field  $\Psi$ .

At early times, the state lies in the vacuum representation. As the time slice intersects the defect, the local operator acts on a state in a twisted representation associated with the defect.

The twisted representations live in the Hilbert space given by the tensor product  $\mathcal{H}_{\bar{\mathcal{L}}} \otimes \mathcal{H} \otimes \mathcal{H}_{\mathcal{L}}$ , where  $\mathcal{H}$  is the untwisted Hilbert space of the theory<sup>14</sup>. This can be understood by using corollary 2.2.6 to detach an identity defect from the original one and slide it across the local operator.

Finally, when the time slice exits the defect loop, the state returns to the vacuum sector.

Therefore, the entire process is described by a map acting on a generic primary operator  $\Psi$  like

$$\phi_{\rho_{\mathcal{L}}}(\Psi) = T^{\dagger} \rho_{\mathcal{L}}(\Psi) T \quad (\text{C.1})$$

where  $T \in \text{Hom}(\text{id}, \rho_{\mathcal{L}})$  is an isometry from the vacuum representation to the twisted representation  $\rho_{\mathcal{L}}$  associated to the defect  $\mathcal{L}$ . This might appear as another possible way to describe the defect, however, it is actually the exact same description we have already constructed.

Before formalizing this connection, we want to show that maps of the form above are special, in the sense that they are unital completely positive maps.

**Definition C.1.** An element  $A$  of a  $C^*$  algebra is positive, denoted  $A \geq 0$  if there exist a finite number of elements  $\{A_k\}_{k=0}^N$  such that  $A = \sum_{k=0}^N A_k^* A_k$ .

<sup>14</sup>Be careful not to confuse  $\mathcal{H}$  with the vacuum Hilbert space, which we have often denoted by the same symbol for simplicity. This Hilbert space corresponds to the “identity-twisted” Hilbert space, i.e. the full Hilbert space of the theory, where the vacuum sector is just one component.

**Definition C.2.** A map between two unital  $C^*$ -algebras  $\mathcal{A}_1$  and  $\mathcal{A}_2$ ,  $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ , is said to be a *unital completely positive (UCP) map* if it is:

- Unital:  $\phi(\mathbf{1}_{\mathcal{A}_1}) = \mathbf{1}_{\mathcal{A}_2}$ ;
- Positive: if  $X \in \mathcal{A}_1$  is positive,  $X \geq 0$ , then  $\phi(X) \geq 0$ ;
- Completely positive: the extended map  $\phi \otimes \text{id}_n : \mathcal{A}_1 \otimes M_n(\mathbb{C}) \rightarrow \mathcal{A}_2 \otimes M_n(\mathbb{C})$  is positive for every  $n \in \mathbb{N}$ .

**Theorem C.1** (Stinespring's dilation theorem). [87] *Let  $\mathcal{A}$  be a  $C^*$ -algebra with a unit,  $\mathcal{H}$  a Hilbert space and  $\mathcal{B}(\mathcal{H})$  be the bounded operators on  $\mathcal{H}$ . For every completely positive map*

$$\phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$$

*there exists a Hilbert space  $\mathcal{K}$  and a unital  $*$ -homomorphism*

$$\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$$

*such that*

$$\phi(A) = V^* \pi(A) V$$

*where  $V : \mathcal{H} \rightarrow \mathcal{K}$  is a bounded operator such that  $V^* V = \mathbf{1}$ . The converse is also true.*

Hence our heuristic considerations suggest that the local actions of defects are encoded in unital completely positive maps on the observable algebra. Now we want to show this clearly.

First we need the following definition:

**Definition C.3.** A (finite) *hypergroup*  $K$  is a (finite-dimensional) vector space with a distinguished basis  $k_i$  satisfying an associative multiplication

$$k_i k_j = \sum_l \lambda_{ij}^l \cdot k_l$$

with non-negative real coefficients  $\lambda_{ij}^l$  such that for each  $i, j$

$$\sum_l \lambda_{ij}^l = 1$$

There is a neutral element  $k_0 = 1$  and an involution  $i \leftrightarrow \bar{i}$  such that  $\lambda_{ij}^0 \neq 0$  if and only if  $j = \bar{i}$ .

A fusion algebra (properly normalized) is an example of hypergroup.

Now let  $\mathcal{N} \subset \mathcal{M}$  be an irreducible type III subfactor on a separable Hilbert space and let  $Q = (\Theta, W, X)$  be the associated Q-system. Assume that  $\Gamma = \iota \bar{\iota}$  is multiplicity-free. This holds, in particular, for commutative Q-systems [74].

Let  $P_a \in \text{Hom}(\Gamma, \Gamma)$  be projections onto  $\Gamma_a \prec \Gamma$  and choose isometries  $S_a \in \text{Hom}(\Gamma_a, \Gamma)$ , such that  $P_a = S_a S_a^\dagger$  and  $S_a^\dagger S_a = \mathbf{1}_{\mathcal{M}}$ .

Then we decompose the conditional expectation of equation (4.4), considered as a positive map of  $\mathcal{M}$  into itself, as:

$$\iota \circ \mu(\cdot) = d_Q^{-1} \iota(W^\dagger) \iota \bar{\iota}(\cdot) \iota(W) = d_Q^{-1} \iota(W^\dagger) \Gamma(\cdot) \iota(W) = d_Q^{-1} \iota(W^\dagger) \sum_a S_a \Gamma_a(\cdot) S_a^\dagger \iota(W)$$

where  $S_a^\dagger \iota(W) \in \text{Hom}(\iota, \Gamma_a \iota)$  are multiples of isometries:  $\iota(W^\dagger) S_a S_a^\dagger \iota(W) = \dim(\Gamma_a)/d_Q$ . Then in terms of isometries  $T_a = \sqrt{d_Q/\dim(\Gamma_a)} S_a^\dagger \iota(W)$  we have:

$$\iota \circ \mu(\cdot) = \frac{1}{d_Q^2} \sum_a \dim(\Gamma_a) \cdot T_a^\dagger \Gamma_a(\cdot) T_a =: \frac{1}{d_Q^2} \sum_a \dim(\Gamma_a) \cdot \phi_a(\cdot)$$

Finally, we can state the following<sup>15</sup>:

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<sup>15</sup>Most of the following propositions are proved by direct calculations. All these proofs are found in [74].

**Proposition C.2.** [74] *The unital completely positive maps*

$$\phi_a(\cdot) = T_a^\dagger \Gamma_a(\cdot) T_a : \mathcal{M} \rightarrow \mathcal{M}$$

are  $\mathcal{N}$ -bimodular maps (i.e.  $\phi_a(N_1 M N_2) = N_1 \phi_a(M) N_2 \quad \forall N_1, N_2$ ) with fixed points  $\mathcal{N}$ . In particular  $\phi_a$  form a hypergroup  $K_{\mathcal{N} \subset \mathcal{M}}$  under composition.

$$\phi_a \phi_b = \sum_c \lambda_{ab}^c \cdot \phi_c$$

where the sum runs over the subsectors  $\Gamma_c \prec \Gamma_a \Gamma_b$ .

Note that we are particularly interested in the fixed-point property of these maps. If we manage to relate the  $\phi_a$  to the local action of the defects, then we automatically show that what we have observed in the Ising model is not a coincidence.

Before doing so, we need to talk about matrix representations.

**Definition C.4.** A matrix representation of a hypergroup  $K$  is a homomorphism  $U$  of  $K$  into  $M_{m \times m}(\mathbb{C})$  such that  $U(\bar{k}) = U(k)^\dagger$ .

Consider the irreducible subrepresentation  $\rho \prec \Theta$  and the charged fields  $\Psi_\rho^i, i = 1, \dots, m_\rho = \dim \text{Hom}(\rho, \Theta)$

**Proposition C.3.** [74] *The positive maps  $\phi_a$  act linearly on the spaces  $\text{Hom}(\iota, \iota\rho)$ . The coefficients in*

$$\phi_a(\Psi_\rho^i) = \sum_j \Psi_\rho^j \cdot U_\rho(a)_{ji}$$

form irreducible matrix representations  $U_\rho$  of  $K_{\mathcal{N} \subset \mathcal{M}}$ . The representations  $\{U_\rho\}_{\rho \prec \Theta}$  are irreducible and pairwise inequivalent. They exhaust the irreducible representations of  $K_{\mathcal{N} \subset \mathcal{M}}$ .

Now we focus on the QFT application. For simplicity consider two isomorphic local extensions  $\mathcal{A} \subset \mathcal{B}^L$  and  $\mathcal{A} \subset \mathcal{B}^R$  where  $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$ . This means that the two Q-systems are the same,  $Q^L = Q^R$ . As in section 5.3, we construct the extension  $\mathcal{D}$  defined by the braided product Q-system  $Q^L \times^- Q^R$ .

The following result relates the central projections  $E_a$  to the action of the hypergroup  $K_{\mathcal{A} \subset \mathcal{B}}$  associated with the subfactor  $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$  on the charged fields.

**Proposition C.4.** [74] *The following expansion holds:*

$$B_\rho^{ij} = (\Psi_\rho^{i,L})^\dagger \Psi_\rho^{j,R} = \sum_a U_\rho(a)_{ij} \cdot E_a$$

which implies:

$$\pi_a((\Psi_\rho^{i,L})^\dagger \Psi_\rho^{j,R}) = U_\rho(a)_{ij} \tag{C.2}$$

Hence we see that the boundary conditions are related to the unitary representations of the hypergroup  $K_{\mathcal{A} \subset \mathcal{B}}$ .

Note that, in general, equation (C.2) does not imply a linear relation of the form

$$\pi_a(\Psi_\rho^{j,R}) = \sum_i \pi_a(\Psi_\rho^{i,L}) U_\rho(a)_{ij} ,$$

since the two sets of fields need not form bases<sup>16</sup>. In extreme cases, the operators  $\Psi_\rho^{i,L}$  and  $\Psi_\rho^{j,R}$  may even be linearly independent in a given representation, and the boundary condition is then merely a sesquilinear relation specifying the "overlapping" or "angles" between the two spaces of embedded charged field operators. In the particular case in which the matrix  $U_\rho(a)$  is unitary, however, one does obtain linear relations between the field operators.

<sup>16</sup>Although  $\mathcal{B}^L \simeq \mathcal{B}^R$ , the way they are embedded into  $\mathcal{D}$  may differ. The role of the boundary condition is to specify this difference.

To see why, consider the two vectors of charged fields<sup>17</sup>:

$$\Psi_\rho^R = (\Psi_\rho^{1,R}, \dots, \Psi_\rho^{Z_\rho,R}) \quad \Psi_\rho^L = (\Psi_\rho^{1,L}, \dots, \Psi_\rho^{Z_\rho,L})$$

where we are considering the same theory on both sides of the defect and  $Z_\rho$  is the multiplicity of  $\rho$  in both  $\Theta^L$  and  $\Theta^R$ . Now express  $\Psi_\rho^{j,R} = \sum_i \Psi_\rho^{i,L} \cdot U_\rho(a)_{i,j} + \chi_j$  where  $\chi$  is an orthogonal vector with respect to  $\Psi_\rho^{i,L}$ :  $\Psi_\rho^{L\dagger} \cdot \chi = 0$ . The isometry condition for  $\Psi_\rho^R$  reads:

$$\delta^{k,j} = \Psi_\rho^{k,R\dagger} \Psi_\rho^{j,R} = \left( \sum_l U_\rho(a)_{k,l}^\dagger \cdot \Psi_\rho^{l,L\dagger} + \chi_k^\dagger \right) \left( \sum_i \Psi_\rho^{i,L} \cdot U_\rho(a)_{i,j} + \chi_j \right)$$

Using the orthogonality between  $\chi$  and  $\Psi_\rho^L$  and the fact that also the  $\Psi_\rho^{i,L}$  are isometries:

$$\delta^{k,j} = \sum_i U_{k,i}^\dagger U_{i,j} + \chi_k^\dagger \chi_j$$

Hence if  $U$  is an isometry, this implies  $\chi = 0$ ; since this charged-field multiplicity space is finite-dimensional, then  $U$  is also unitary.

Suppose that  $U_\rho(a)$  is unitary. Then we can express the action of the defect as relations

$$\pi_a(\Psi_\rho^{j,R}) = \sum_i \pi_a(\Psi_\rho^{i,L}) U_\rho(a)_{ij} = \phi_a(\Psi_\rho^{j,L}) \quad (\text{C.3})$$

We know that this corresponds graphically to the sliding of the defect through  $\Psi_\rho^{i,L}$ , which results in the summation (2.11). In particular, in our theory only the first term of that sum is considered, as we cannot have junctions on the defect. Hence equation (C.3) shows that the defect bubble acts on the field  $\Psi_\rho^{j,L}$  as a UCP map, transforming it into the field  $\Psi_\rho^{j,R}$ . This map is characterized by a decomposition like the one we have originally guessed in equation (C.1).

This not only confirms and formalizes our initial ansatz, but as a consequence of proposition C.2 we know that these maps have  $\mathcal{A}$  as fixed point:

$$\mathcal{A} = \mathcal{B}^{K_{\mathcal{A} \subset \mathcal{B}}} := \{B \in \mathcal{B} : \phi_a(B) = B, \forall \phi_a \in K_{\mathcal{A} \subset \mathcal{B}}\}$$

i.e.  $\mathcal{A}$  is the *fixed point subnet*, or *orbifold net* with respect to the action of  $K_{\mathcal{A} \subset \mathcal{B}}$  on  $\mathcal{B}$ . Here  $\mathcal{B}$  can be either  $\mathcal{B}^L$  or  $\mathcal{B}^R$ .

Following our construction of two-dimensional CFTs,  $\mathcal{A}$  is just the tensor product of two Virasoro nets. Hence the orbifold of a full 2d CFT with respect to the hypergroup of boundary conditions, describing the local action of its defects, gives just the Virasoro net, i.e. the vacuum sector together with its descendants. In other words, we can conclude that in these theories *there is no primary operator that is invariant under all defects*, apart from the identity family. For this reason the hypergroup of boundary conditions is regarded as the minimal extension of the usual symmetry group of a theory, including also the action of non-invertible symmetries. Sometimes the elements of  $K_{\mathcal{A} \subset \mathcal{B}}$  are called *Quantum Operations* [71].

Finally, we state the following result from [74]:

**Proposition C.5.** [74] *Up to a trivial rescaling, this composition of boundary conditions is isomorphic to the hypergroup product of  $K_{\mathcal{A} \subset \mathcal{B}}$ .*

This is not a surprise after interpreting the elements of  $K_{\mathcal{A} \subset \mathcal{B}}$  as the maps describing the local actions of defects.

The generalization to the situation where  $\mathcal{B}^L$  and  $\mathcal{B}^R$  are different extensions of  $\mathcal{A}$  is also described in [74].

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<sup>17</sup>We omit here the representation  $\pi_a$  to simplify notation.

## D - Calculations of the fermionic extension

We now prove equations (5.22) and (5.21). To do so, we will use the ingredients developed for the Ising model, in particular: the application of the chiral endomorphism  $\sigma$  to the chiral intertwiners, i.e. equation (3.30); the fusion rules of the Ising primary fields of equation (4.13); the commutation relations between the left and right fields in equation (5.20).

First we show that  $\psi_{\pm}^{\dagger} = \psi_{\pm}$ . We do this for  $\psi_{+}$ , while the calculation for  $\psi_{-}$  is identical.

$$\psi_{+}^{\dagger} = (\sqrt{-i} \cdot \Psi_{\sigma \otimes \sigma}^{L\dagger}(u \otimes \mathbf{1}) \Psi_{\sigma \otimes \sigma}^R)^{\dagger} = \sqrt{i} \cdot \Psi_{\sigma \otimes \sigma}^{R\dagger}(u \otimes \mathbf{1}) \Psi_{\sigma \otimes \sigma}^L$$

where we have used  $u^{\dagger} = u$ . Now we use  $\Psi_{\sigma \otimes \sigma}^{R\dagger} = \sqrt{2}(r^{\dagger} \otimes r^{\dagger}) \Psi_{\sigma \otimes \sigma}^R$ :

$$\psi_{+}^{\dagger} = \sqrt{i} \cdot \sqrt{2}(r^{\dagger} \otimes r^{\dagger}) \Psi_{\sigma \otimes \sigma}^R(u \otimes \mathbf{1}) \Psi_{\sigma \otimes \sigma}^L = \sqrt{i} \cdot \sqrt{2}(r^{\dagger} \otimes r^{\dagger})(\sigma(u) \otimes \mathbf{1}) \Psi_{\sigma \otimes \sigma}^R \Psi_{\sigma \otimes \sigma}^L$$

Now we focus on  $(r^{\dagger} \otimes r^{\dagger})(\sigma(u) \otimes \mathbf{1})$ :

$$(r^{\dagger} \otimes r^{\dagger})(\sigma(u) \otimes \mathbf{1}) = (r^{\dagger} \otimes r^{\dagger})((tr^{\dagger} + rt^{\dagger}) \otimes \mathbf{1}) = t^{\dagger} \otimes r^{\dagger}$$

Where we recall that  $r^{\dagger}r = t^{\dagger}t = \mathbf{1}$  and  $r^{\dagger}t = t^{\dagger}r = 0$ .

Then:

$$(r^{\dagger} \otimes r^{\dagger})(\sigma(u) \otimes \mathbf{1}) \cdot \Psi_{\sigma \otimes \sigma}^R \Psi_{\sigma \otimes \sigma}^L = (t^{\dagger} \otimes r^{\dagger}) \cdot [(rr^{\dagger} - itt^{\dagger}) \otimes (rr^{\dagger} + itt^{\dagger})] \Psi_{\sigma \otimes \sigma}^L \Psi_{\sigma \otimes \sigma}^R = -i(t^{\dagger} \otimes r^{\dagger}) \Psi_{\sigma \otimes \sigma}^L \Psi_{\sigma \otimes \sigma}^R$$

Now we write  $\Psi_{\sigma \otimes \sigma}^L = \sqrt{2} \Psi_{\sigma \otimes \sigma}^{L\dagger}(r \otimes r)$  and  $-i\sqrt{i} = \sqrt{i}$ :

$$\begin{aligned} \psi_{+}^{\dagger} &= \sqrt{-i} \cdot \sqrt{2}(t^{\dagger} \otimes r^{\dagger}) \Psi_{\sigma \otimes \sigma}^L \Psi_{\sigma \otimes \sigma}^R = \sqrt{-i} \cdot 2(t^{\dagger} \otimes r^{\dagger}) \Psi_{\sigma \otimes \sigma}^{L\dagger}(r \otimes r) \Psi_{\sigma \otimes \sigma}^R = \\ &= \sqrt{-i} \cdot 2 \Psi_{\sigma \otimes \sigma}^{L\dagger}(\sigma(t^{\dagger}) \otimes \sigma(r^{\dagger}))(r \otimes r) \Psi_{\sigma \otimes \sigma}^R \end{aligned}$$

Finally:

$$(\sigma(t^{\dagger}) \otimes \sigma(r^{\dagger}))(r \otimes r) = \frac{1}{2}[(u^{\dagger}r^{\dagger} - u^{\dagger}t^{\dagger}) \otimes (r^{\dagger} + t^{\dagger})](r \otimes r) = \frac{1}{2}(u \otimes \mathbf{1})$$

Reinserting in the equation above we find:

$$\psi_{+}^{\dagger} = \sqrt{-i} \cdot 2 \Psi_{\sigma \otimes \sigma}^{L\dagger}(\sigma(t^{\dagger}) \otimes \sigma(r^{\dagger}))(r \otimes r) \Psi_{\sigma \otimes \sigma}^R = \sqrt{-i} \cdot \Psi_{\sigma \otimes \sigma}^{L\dagger}(u \otimes \mathbf{1}) \Psi_{\sigma \otimes \sigma}^R = \psi_{+}$$

Now we prove  $\psi_{+}^2 = \frac{1}{2}(\mathbf{1} - \Psi_{\psi \otimes \psi}^L \Psi_{\psi \otimes \psi}^R)$ . Again, the same calculations are valid for  $\psi_{-}$ .

Starting from the definition:

$$\psi_{+}^2 = \psi_{+}\psi_{+} = \psi_{+}\psi_{+}^{\dagger} = (\sqrt{-i} \cdot \Psi_{\sigma \otimes \sigma}^{L\dagger}(u \otimes \mathbf{1}) \Psi_{\sigma \otimes \sigma}^R)(\sqrt{i} \cdot \Psi_{\sigma \otimes \sigma}^{R\dagger}(u \otimes \mathbf{1}) \Psi_{\sigma \otimes \sigma}^L)$$

Let's focus on the middle term:

$$\begin{aligned} \Psi_{\sigma \otimes \sigma}^R \Psi_{\sigma \otimes \sigma}^{R\dagger} &= \sqrt{2} \cdot \Psi_{\sigma \otimes \sigma}^R(r^{\dagger} \otimes r^{\dagger}) \Psi_{\sigma \otimes \sigma}^R = \sqrt{2} \cdot (\sigma(r^{\dagger}) \otimes \sigma(r^{\dagger})) \Psi_{\sigma \otimes \sigma}^R \Psi_{\sigma \otimes \sigma}^R = \\ &= \sqrt{2} \cdot \frac{1}{2}[(r^{\dagger} + t^{\dagger}) \otimes (r^{\dagger} + t^{\dagger})] \frac{1}{\sqrt{2}}[(r \otimes r) + (t \otimes t) \Psi_{\psi \otimes \psi}^R] = \\ &= \frac{1}{2}(\mathbf{1} + \Psi_{\psi \otimes \psi}^R) \end{aligned} \tag{D.4}$$

Reinserting back in the equation above:

$$\psi_{+}^2 = \Psi_{\sigma \otimes \sigma}^{L\dagger}(u \otimes \mathbf{1}) \frac{1}{2}(\mathbf{1} + \Psi_{\psi \otimes \psi}^R)(u \otimes \mathbf{1}) \Psi_{\sigma \otimes \sigma}^L$$

Now we use  $\Psi_{\psi \otimes \psi}^R(u \otimes \mathbf{1}) = (\psi(u) \otimes \mathbf{1}) \Psi_{\psi \otimes \psi}^R = (-u \otimes \mathbf{1}) \Psi_{\psi \otimes \psi}^R$  together with  $u^2 = 1$  to get:

$$\psi_{+}^2 = \frac{1}{2} \Psi_{\sigma \otimes \sigma}^{L\dagger}(\mathbf{1} - \Psi_{\psi \otimes \psi}^R) \Psi_{\sigma \otimes \sigma}^L = \frac{1}{2} \Psi_{\sigma \otimes \sigma}^{L\dagger}(\Psi_{\sigma \otimes \sigma}^L - (u \otimes u) \Psi_{\sigma \otimes \sigma}^L \Psi_{\psi \otimes \psi}^R)$$

where we have also used the commutation relation between  $\Psi_{\sigma\otimes\sigma}^L$  and  $\Psi_{\psi\otimes\psi}^R$ . Now we focus on the two sub-problems. First:

$$\begin{aligned}\Psi_{\sigma\otimes\sigma}^{L\dagger}(u\otimes u)\Psi_{\sigma\otimes\sigma}^L &= \sqrt{2}(r^\dagger\otimes r^\dagger)(\sigma(u)\otimes\sigma(u))\Psi_{\sigma\otimes\sigma}^L\Psi_{\sigma\otimes\sigma}^L = \\ &= \sqrt{2}(r^\dagger\otimes r^\dagger)[(tr^\dagger+rt^\dagger)\otimes(tr^\dagger+rt^\dagger)]\frac{1}{\sqrt{2}}[(r\otimes r)+(t\otimes t)\Psi_{\psi\otimes\psi}^L] = \\ &= \Psi_{\psi\otimes\psi}^L\end{aligned}\quad (\text{D.5})$$

The other term is computed analogously, giving  $\Psi_{\sigma\otimes\sigma}^{L\dagger}\Psi_{\sigma\otimes\sigma}^L = \mathbf{1}$ . Then we find:

$$\psi_+^2 = \frac{1}{2}\Psi_{\sigma\otimes\sigma}^{L\dagger}(\Psi_{\sigma\otimes\sigma}^L - (u\otimes u)\Psi_{\sigma\otimes\sigma}^L\Psi_{\psi\otimes\psi}^R) = \frac{1}{2}(\mathbf{1} - \Psi_{\psi\otimes\psi}^L\Psi_{\psi\otimes\psi}^R)$$

Now we prove that  $\psi_+\psi_- = -\psi_-\psi_+$ .

$$\psi_+\psi_- = \psi_+\psi_-^\dagger = (\sqrt{-i}\cdot\Psi_{\sigma\otimes\sigma}^{L\dagger}(u\otimes\mathbf{1})\Psi_{\sigma\otimes\sigma}^R)(\sqrt{-i}\cdot\Psi_{\sigma\otimes\sigma}^{R\dagger}(\mathbf{1}\otimes u)\Psi_{\sigma\otimes\sigma}^L)$$

We use again equation (D.4):

$$\begin{aligned}\psi_+\psi_- &= -i\cdot\Psi_{\sigma\otimes\sigma}^{L\dagger}(u\otimes\mathbf{1})\Psi_{\sigma\otimes\sigma}^R\Psi_{\sigma\otimes\sigma}^{R\dagger}(\mathbf{1}\otimes u)\Psi_{\sigma\otimes\sigma}^L = \\ &= -i\cdot\Psi_{\sigma\otimes\sigma}^{L\dagger}(u\otimes\mathbf{1})\frac{1}{2}(\mathbf{1}+\Psi_{\psi\otimes\psi}^R)(\mathbf{1}\otimes u)\Psi_{\sigma\otimes\sigma}^L = \\ &= \frac{-i}{2}\cdot\Psi_{\sigma\otimes\sigma}^{L\dagger}((u\otimes u) - (u\otimes u)\Psi_{\psi\otimes\psi}^R)\Psi_{\sigma\otimes\sigma}^L = \\ &= \frac{-i}{2}\cdot(\Psi_{\sigma\otimes\sigma}^{L\dagger}(u\otimes u)\Psi_{\sigma\otimes\sigma}^L - \Psi_{\sigma\otimes\sigma}^{L\dagger}(u^2\otimes u^2)\Psi_{\sigma\otimes\sigma}^L\Psi_{\psi\otimes\psi}^R) = \\ &= \frac{-i}{2}\cdot(\Psi_{\psi\otimes\psi}^L - \Psi_{\psi\otimes\psi}^R) = \frac{i}{2}\cdot(\Psi_{\psi\otimes\psi}^R - \Psi_{\psi\otimes\psi}^L)\end{aligned}$$

Where we have also used again equation (D.5) and  $\Psi_{\sigma\otimes\sigma}^{L\dagger}\Psi_{\sigma\otimes\sigma}^L = \mathbf{1}$ .

Then:

$$\psi_-\psi_+ = (\psi_+\psi_-)^\dagger = \left(\frac{i}{2}\cdot(\Psi_{\psi\otimes\psi}^R - \Psi_{\psi\otimes\psi}^L)\right)^\dagger = -\frac{i}{2}\cdot(\Psi_{\psi\otimes\psi}^R - \Psi_{\psi\otimes\psi}^L) = -\psi_+\psi_-$$

Note that we have chosen the plus branch of the square root when multiplying the two  $\sqrt{-i}$  factors, the other choice would give an opposite overall sign in  $\psi_+\psi_-$ . However, the minus sign in the relative exchange of  $\psi_+$  with  $\psi_-$  appears anyway.

Finally we prove that  $\sqrt{i}\cdot\psi_+\Psi_{\sigma\otimes\sigma}^R = \frac{1}{2}(\Psi_{\sigma\otimes\sigma}^L - \Psi_{\psi\otimes\psi}^R\Psi_{\sigma\otimes\sigma}^L)$  Starting from the definitions:

$$\begin{aligned}\sqrt{i}\cdot\psi_+\Psi_{\sigma\otimes\sigma}^R &= \Psi_{\sigma\otimes\sigma}^{L\dagger}(u\otimes\mathbf{1})\Psi_{\sigma\otimes\sigma}^R\Psi_{\sigma\otimes\sigma}^R = \\ &= \sqrt{2}(r^\dagger\otimes r^\dagger)\Psi_{\sigma\otimes\sigma}^L(u\otimes\mathbf{1})\frac{1}{\sqrt{2}}[(r\otimes r)+(t\otimes t)\Psi_{\psi\otimes\psi}^R] = \\ &= (r^\dagger\otimes r^\dagger)\Psi_{\sigma\otimes\sigma}^L[(r\otimes r) - (t\otimes t)\Psi_{\psi\otimes\psi}^R] = \\ &= (r^\dagger\otimes r^\dagger)[(\sigma(r)\otimes\sigma(r))\Psi_{\sigma\otimes\sigma}^L - (\sigma(t)\otimes\sigma(t))\Psi_{\sigma\otimes\sigma}^L\Psi_{\psi\otimes\psi}^R] = \\ &= (r^\dagger\otimes r^\dagger)[(\sigma(r)\otimes\sigma(r))\Psi_{\sigma\otimes\sigma}^L - (\sigma(t)\otimes\sigma(t))(u\otimes u)\Psi_{\psi\otimes\psi}^R\Psi_{\sigma\otimes\sigma}^L] =\end{aligned}$$

where we have used  $u = rr^\dagger - tt^\dagger$  to resolve the multiplication involving  $u\otimes\mathbf{1}$ . Using  $\sigma(r) = \frac{1}{\sqrt{2}}(r+t)$  and  $\sigma(t) = \frac{1}{\sqrt{2}}(ru - tu)$  we find:

$$\sqrt{i}\cdot\psi_+\Psi_{\sigma\otimes\sigma}^R = \frac{1}{2}(\Psi_{\sigma\otimes\sigma}^L - \Psi_{\psi\otimes\psi}^R\Psi_{\sigma\otimes\sigma}^L)$$

that is the desired result.

# Bibliography

- [1] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory* (Graduate texts in contemporary physics). New York, NY: Springer, 1997. DOI: 10.1007/978-1-4612-2256-9. [Online]. Available: <https://cds.cern.ch/record/639405>.
- [2] C. Bachas, “The many faces of conformal field theory,” lecture notes, Università degli studi di Padova (2025).
- [3] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory: With Applications to String Theory*. Lecture Notes in Physics, Jan. 2009, vol. 779, ISBN: 978-3-642-00449-0. DOI: 10.1007/978-3-642-00450-6.
- [4] D. Tong, *Lectures on String Theory*, 2009. arXiv: 0908.0333 [hep-th].
- [5] D. Cariolato, “Topological defects in conformal field theory,” M.S. thesis, Università degli studi di Padova, 2023.
- [6] D. Petrovic, “Modularity constraint on topological defect lines in two-dimensional conformal field theories,” M.S. thesis, Uppsala University, 2025.
- [7] C. Hamer and W. Zheng, *Series expansion methods for strongly interacting lattice models*. Cambridge University Press, Mar. 2011, ISBN: 978-0-511-27988-1, 978-0-521-14359-2, 978-0-521-84242-6.
- [8] B. Schroer and T. T. Truong, “The Order / Disorder Quantum Field Operators Associated to the Two-dimensional Ising Model in the Continuum Limit,” *Nucl. Phys. B*, vol. 144, pp. 80–122, 1978. DOI: 10.1016/0550-3213(78)90499-6.
- [9] R. Volpato, “Lecture notes on non-invertible symmetries,” Università degli studi di Padova.
- [10] S.-H. Shao, “What’s Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetries,” in *Theoretical Advanced Study Institute in Elementary Particle Physics 2023: Aspects of Symmetry*, Aug. 2023. arXiv: 2308.00747 [hep-th].
- [11] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, “Topological defect lines and renormalization group flows in two dimensions,” *Journal of High Energy Physics*, vol. 2019, pp. 1–85, 2018. [Online]. Available: <https://api.semanticscholar.org/CorpusID:119045896>.
- [12] L. Bhardwaj et al., “Lectures on generalized symmetries,” *Phys. Rept.*, vol. 1051, pp. 1–87, 2024. DOI: 10.1016/j.physrep.2023.11.002. arXiv: 2307.07547 [hep-th].
- [13] N. Carqueville, M. Del Zotto, and I. Runkel, “Topological defects,” Nov. 2023. DOI: 10.1016/B978-0-323-95703-8.00098-7. arXiv: 2311.02449 [math-ph].
- [14] S. Schäfer-Nameki, “Ictp lectures on (non-)invertible generalized symmetries,” *Physics Reports*, vol. 1063, pp. 1–55, 2024, ISSN: 0370-1573. DOI: <https://doi.org/10.1016/j.physrep.2024.01.007>. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0370157324000310>.

- [15] J. McGreevy, “Generalized Symmetries in Condensed Matter,” *Ann. Rev. Condensed Matter Phys.*, vol. 14, pp. 57–82, 2023. DOI: 10.1146/annurev-conmatphys-040721-021029. arXiv: 2204.03045 [cond-mat.str-el].
- [16] R. Volpato, “Vertex algebras, topological defects, and Moonshine,” Dec. 2024. arXiv: 2412.21141 [hep-th].
- [17] R. Angius, S. Giaccari, and R. Volpato, “Topological defects in K3 sigma models,” *Journal of High Energy Physics*, vol. 2024, no. 7, 2024, Art. no. 111. DOI: 10.1007/JHEP07(2024)111.
- [18] C.-M. Chang, J. Chen, and F. Xu, “Topological defect lines in two dimensional fermionic cfts,” *SciPost Phys.*, vol. 15, p. 216, 2023. DOI: 10.21468/SciPostPhys.15.5.216.
- [19] G. Maggiorotti, “Non-invertible duality defects and lattice symmetries in 2d cfts,” M.S. thesis, University of Bologna, 2025.
- [20] B. Haghighat and Y. Sun, “Topological defect lines in bosonized Parafermionic CFTs,” *Adv. Theor. Math. Phys.*, vol. 28, no. 6, 2024. DOI: 10.4310/atmp.241031012317. arXiv: 2306.16555 [hep-th].
- [21] R. E. Behrend, P. A. Pearce, V. B. Petkova, and J.-B. Zuber, “Boundary conditions in rational conformal field theories,” *Nucl. Phys. B*, vol. 570, pp. 525–589, 2000. DOI: 10.1016/S0550-3213(99)00592-1. arXiv: hep-th/9908036.
- [22] D. Aasen, R. S. K. Mong, and P. Fendley, “Topological defects on the lattice: I. the ising model,” *Journal of Physics A: Mathematical and Theoretical*, vol. 49, no. 35, p. 354 001, Aug. 2016. DOI: 10.1088/1751-8113/49/35/354001.
- [23] D. Aasen, P. Fendley, and R. S. K. Mong, “Topological Defects on the Lattice: Dualities and Degeneracies,” Aug. 2020. arXiv: 2008.08598 [cond-mat.stat-mech].
- [24] B. Gabai, V. Gorbenko, J. Qiao, B. Zan, and A. Zhabin, “Quantum groups as global symmetries,” *JHEP*, vol. 08, p. 087, 2025. DOI: 10.1007/JHEP08(2025)087. arXiv: 2410.24142 [hep-th].
- [25] R. Thorngren and Y. Wang, “Fusion category symmetry. Part I. Anomaly in-flow and gapped phases,” *JHEP*, vol. 04, p. 132, 2024. DOI: 10.1007/JHEP04(2024)132. arXiv: 1912.02817 [hep-th].
- [26] Y.-H. Lin, M. Okada, S. Seifnashri, and Y. Tachikawa, “Asymptotic density of states in 2d CFTs with non-invertible symmetries,” *JHEP*, vol. 03, p. 94, 2023. DOI: 10.1007/JHEP03(2023)094. arXiv: 2208.05495 [hep-th].
- [27] E. P. Verlinde, “Fusion Rules and Modular Transformations in 2D Conformal Field Theory,” *Nucl. Phys. B*, vol. 300, pp. 360–376, 1988. DOI: 10.1016/0550-3213(88)90603-7.
- [28] D. Scherl, “Fusion categories,” lecture notes, <https://zmp.math.uni-hamburg.de/ZMP-SS20-Scherl.pdf>, 2020.
- [29] A. Marín-Salvador, “Continuous Tambara-Yamagami tensor categories,” Mar. 2025. arXiv: 2503.14596 [math.QA].
- [30] T. Leinster, *Basic Category Theory* (Cambridge Studies in Advanced Mathematics). Cambridge University Press, 2014.
- [31] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor Categories* (Mathematical surveys and monographs). American Mathematical Society, 2017, ISBN: 9781470437411. [Online]. Available: <https://books.google.it/books?id=9sC7vQEACAAJ>.

- [32] S. Mac Lane, *Categories for the Working Mathematician* (Graduate Texts in Mathematics), 2nd. New York: Springer-Verlag, 1998, vol. 5.
- [33] J. Fuchs, I. Runkel, and C. Schweigert, “TFT construction of RCFT correlators 1. Partition functions,” *Nucl. Phys. B*, vol. 646, pp. 353–497, 2002. DOI: 10.1016/S0550-3213(02)00744-7. arXiv: hep-th/0204148.
- [34] P. Etingof, D. Nikshych, and V. Ostrik, “On fusion categories,” *Annals of mathematics*, vol. 162, pp. 581–642, Mar. 2002. arXiv: math/0203060.
- [35] M. Bischoff, Y. Kawahigashi, R. Longo, and K.-H. Rehren, “Phase boundaries in algebraic conformal QFT,” *Commun. Math. Phys.*, vol. 342, no. 1, pp. 1–45, 2016. DOI: 10.1007/s00220-015-2560-0. arXiv: 1405.7863 [math-ph].
- [36] G. Mack and V. Schomerus, “Conformal Field Algebras With Quantum Symmetry From the Theory of Superselection Sectors,” *Commun. Math. Phys.*, vol. 134, pp. 139–196, 1990. DOI: 10.1007/BF02102093.
- [37] B. E. Blackadar, “Operator Algebras: Theory of C\*-Algebras and von Neumann Algebras,” 2005. [Online]. Available: <https://api.semanticscholar.org/CorpusID:118420887>.
- [38] R. Longo and K.-H. Rehren, “Nets of subfactors,” *Rev. Math. Phys.*, vol. 7, pp. 567–598, 1995. DOI: 10.1142/S0129055X95000232. arXiv: hep-th/9411077.
- [39] J. Fröhlich and F. Gabbiani, “Braid statistics in local quantum theory,” *Rev. Math. Phys.*, vol. 2, pp. 251–354, 1991. DOI: 10.1142/S0129055X90000107.
- [40] J. Bockenhauer, “Localized endomorphisms of the chiral Ising model,” *Commun. Math. Phys.*, vol. 177, pp. 265–304, 1996. DOI: 10.1007/BF02101894. arXiv: hep-th/9407079.
- [41] K.-H. Rehren, “Algebraic conformal quantum field theory in perspective,” *Advances in Algebraic Quantum Field Theory*, Jan. 2015. DOI: 10.1007/978-3-319-21353-8\_8.
- [42] Y. Tanimoto, “Introduction to algebraic quantum field theory,” Nagoya University, lecture notes (2023).
- [43] C. J. Fewster and K. Rejzner, “Algebraic Quantum Field Theory – an introduction,” Apr. 2019. arXiv: 1904.04051 [hep-th].
- [44] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*. Academic Press / American Mathematical Society, 1983–1992, vol. I–IV.
- [45] D. Buchholz and K. Fredenhagen, “Locality and the Structure of Particle States,” *Commun. Math. Phys.*, vol. 84, p. 1, 1982. DOI: 10.1007/BF01208370.
- [46] K. Fredenhagen, “Superselection sectors,” lecture notes, Hamburg University (1995).
- [47] P. Marchetti, “Introduction to abelian anyons in planar systems,” *Journal of Physics A: Mathematical and Theoretical*, vol. 58, no. 30, p. 303001, Jul. 2025. DOI: 10.1088/1751-8121/adeb16.
- [48] F. Shokrieh, “Basic von Neumann algebra theory,” lecture notes (2017).
- [49] R. Longo and F. Xu, “Topological sectors and a dichotomy in conformal field theory,” *Communications in Mathematical Physics*, vol. 251, no. 2, pp. 321–364, 2004. DOI: 10.1007/s00220-004-1063-1.
- [50] K. Fredenhagen, K.-H. Rehren, and B. Schroer, “Superselection Sectors with Braid Group Statistics and Exchange Algebras. 1. General Theory,” *Commun. Math. Phys.*, vol. 125, p. 201, 1989. DOI: 10.1007/BF01217906.

- [51] R. Longo, “Index of subfactors and statistics of quantum fields. I,” *Commun. Math. Phys.*, vol. 126, pp. 217–247, 1989. DOI: 10.1007/BF02125124.
- [52] J. Sorce, “Notes on the type classification of von Neumann algebras,” *Rev. Math. Phys.*, vol. 36, no. 02, 2024. DOI: 10.1142/S0129055X24300024. arXiv: 2302.01958 [hep-th].
- [53] H. Araki, *Mathematical Theory of Quantum Fields*. Oxford University Press, Oct. 1999, ISBN: 9780198517733. DOI: 10.1093/oso/9780198517733.001.0001. [Online]. Available: <https://doi.org/10.1093/oso/9780198517733.001.0001>.
- [54] H. Halvorson and M. Muger, “Algebraic quantum field theory,” in *Philosophy of physics*, J. Butterfield and J. Earman, Eds. 2007, pp. 731–864. DOI: 10.1016/B978-044451560-5/50011-7. arXiv: math-ph/0602036.
- [55] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, “Correspondences of ribbon categories,” *Adv. Math.*, vol. 199, pp. 192–329, 2006. DOI: 10.1016/j.aim.2005.04.007. arXiv: math/0309465.
- [56] E. Rowell, R. Stong, and Z. Wang, “On Classification of Modular Tensor Categories,” *Commun. Math. Phys.*, vol. 292, no. 2, pp. 343–389, 2009. DOI: 10.1007/s00220-009-0908-z. arXiv: 0712.1377 [math.QA].
- [57] M. Bischoff, Y. Kawahigashi, R. Longo, and K. Rehren, *Tensor Categories and Endomorphisms of von Neumann Algebras: with Applications to Quantum Field Theory* (SpringerBriefs in Mathematical Physics). Springer International Publishing, 2015, ISBN: 9783319143019. [Online]. Available: <https://books.google.it/books?id=1v4qBgAAQBAJ>.
- [58] K. Fredenhagen, K.-H. Rehren, and B. Schroer, “Superselection sectors with braid group statistics and exchange algebras. 2. Geometric aspects and conformal covariance,” *Rev. Math. Phys.*, vol. 4, no. spec01, pp. 113–157, 1992. DOI: 10.1142/S0129055X92000170.
- [59] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics. 1*. Springer-Verlag, 1987.
- [60] P. Marchetti, “Notes on non-perturbative quantum field theory,” notes taken by A. Grossutti, Università degli Studi di Padova.
- [61] S. Carpi, Y. Kawahigashi, R. Longo, and M. Weiner, “From vertex operator algebras to conformal nets and back,” *Memoirs of the American Mathematical Society*, vol. 254, Mar. 2018. DOI: 10.1090/memo/1213.
- [62] J. von Neumann, “On rings of operators. reduction theory,” *Annals of Mathematics*, vol. 50, p. 401, 1949. [Online]. Available: <https://api.semanticscholar.org/CorpusID:124439084>.
- [63] S. J. Summers, “Tomita-Takesaki modular theory,” Nov. 2003. arXiv: math-ph/0511034.
- [64] K. Fredenhagen, “On the Modular Structure of Local Algebras of Observables,” *Commun. Math. Phys.*, vol. 97, p. 79, 1985. DOI: 10.1007/BF01206179.
- [65] U. Haagerup, “Conne’s bicentralizer problem and uniqueness of the injective factor of type III<sub>1</sub>,” *Acta Mathematica*, vol. 158, pp. 95–148, 1987. DOI: 10.1007/BF02392257. [Online]. Available: <https://doi.org/10.1007/BF02392257>.
- [66] D. Buchholz, K. Fredenhagen, and C. D’Antoni, “The Universal Structure of Local Algebras,” *Commun. Math. Phys.*, vol. 111, p. 123, 1987. DOI: 10.1007/BF01239019.
- [67] L. Giorgetti, “Braided Actions of DHR Categories and Reconstruction of Chiral Conformal Field Theories,” Ph.D. dissertation, Gottingen U., Gottingen U., 2016. DOI: 10.53846/goediss-5913.

- [68] P. Bantay, “The Frobenius-Schur indicator in conformal field theory,” *Phys. Lett. B*, vol. 394, pp. 87–88, 1997. DOI: 10.1016/S0370-2693(96)01662-0. arXiv: hep-th/9610192.
- [69] Y. Kawahigashi and R. Longo, “Classification of local conformal nets: Case  $c < 1$ ,” *Annals Math.*, vol. 160, pp. 493–522, 2004. arXiv: math-ph/0201015.
- [70] R. Longo, “A duality for Hopf algebras and for subfactors. 1.,” *Commun. Math. Phys.*, vol. 159, pp. 133–150, 1994. DOI: 10.1007/BF02100488.
- [71] M. Bischoff, S. Del Vecchio, and L. Giorgetti, “Quantum operations on conformal nets,” *Rev. Math. Phys.*, vol. 35, no. 04, pp. 7–30, 2023. DOI: 10.1142/S0129055X23500071. arXiv: 2204.14105 [math.OA].
- [72] S. Doplicher and J. E. Roberts, “Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics,” *Commun. Math. Phys.*, vol. 131, pp. 51–107, 1990. DOI: 10.1007/BF02097680.
- [73] M. Okada and Y. Tachikawa, “Noninvertible symmetries act locally by quantum operations,” *Phys. Rev. Lett.*, vol. 133, p. 191602, 19 Nov. 2024. DOI: 10.1103/PhysRevLett.133.191602.
- [74] M. Bischoff and K.-H. Rehren, “The hypergroupoid of boundary conditions for local quantum observables,” *Adv. Stud. Pure Math.*, vol. 80, pp. 32–42, 2019. DOI: 10.2969/aspm/08010023. arXiv: 1612.02972 [math-ph].
- [75] M. Bischoff, Y. Kawahigashi, and R. Longo, “Characterization of 2D rational local conformal nets and its boundary conditions: the maximal case,” *Annals of Mathematics*, vol. 130, pp. 493–522, Oct. 2014. arXiv: 1410.8848 [math-ph].
- [76] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 2019.
- [77] A. Bartels, C. L. Douglas, and A. Henriques, “Conformal nets III: fusion of defects,” *American Mathematical Society*, vol. 258, no. 1237, Oct. 2019. arXiv: 1310.8263 [math.OA].
- [78] J. Fuchs, I. Runkel, and C. Schweigert, “TFT construction of RCFT correlators. 2. Unoriented world sheets,” *Nucl. Phys. B*, vol. 678, pp. 511–637, 2004. DOI: 10.1016/j.nuclphysb.2003.11.026. arXiv: hep-th/0306164.
- [79] J. Fuchs, I. Runkel, and C. Schweigert, “TFT construction of RCFT correlators IV: Structure constants and correlation functions,” *Nucl. Phys. B*, vol. 715, pp. 539–638, 2005. DOI: 10.1016/j.nuclphysb.2005.03.018. arXiv: hep-th/0412290.
- [80] C. A. Keller, “Introduction to vertex operator algebras,” lecture notes, ETH Zurich (2017).
- [81] V. Kac, *Vertex algebras for beginners*. 2nd ed. Providence, 1996.
- [82] Y. Kawahigashi, “Conformal Field Theory, Vertex Operator Algebras and Operator Algebras,” in *International Congress of Mathematicians*, 2018, pp. 2597–2616. DOI: 10.1142/9789813272880\_0152. arXiv: 1711.11349 [math-ph].
- [83] S. Carpi, “Representations of conformal nets and vertex operator algebra modules,” Jul. 2016.
- [84] J. Fuchs, I. Runkel, and C. Schweigert, “Conformal correlation functions, Frobenius algebras and triangulations,” *Nucl. Phys. B*, vol. 624, pp. 452–468, 2002. DOI: 10.1016/S0550-3213(01)00638-1. arXiv: hep-th/0110133.
- [85] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, “Duality and defects in rational conformal field theory,” *Nucl. Phys. B*, vol. 763, pp. 354–430, 2007. DOI: 10.1016/j.nuclphysb.2006.11.017. arXiv: hep-th/0607247.

- [86] L. Kong and I. Runkel, “Morita classes of algebras in modular tensor categories,” *Advances in Mathematics*, vol. 219, no. 5, pp. 1548–1576, 2008, ISSN: 0001-8708. DOI: <https://doi.org/10.1016/j.aim.2008.07.004>. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0001870808001928>.
- [87] W. F. Stinespring, “Positive functions on  $C^*$ -algebras,” *Proc. Am. Math. Soc.*, vol. 6, no. 2, pp. 211–216, 1955. DOI: 10.1090/s0002-9939-1955-0069403-4.