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Theories of Massive Gravity

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"Matter tells space how to curve.
Space tells matter how to move."

J. A. Wheeler

Abstract

Einstein's General Relativity has been confirmed in numerous experiments and observations through the years, the latest being the detection of gravitational waves. However there still remain cosmological phenomena, such as the accelerating expansion of the Universe and the presence of Dark Matter, whose satisfactory explanation has not been found yet in the framework of the Standard Model of Elementary Particles and Cosmology. This motivated theorists to look for modifications of gravity aiming at gaining an alternative insight into the nature of Dark Energy and Dark Matter. This Master's Thesis studies internal consistency of the theories of Modified Gravity with a massive graviton, Massive Gravity and Bigravity, and discusses their strong points as well as their weaknesses in trying to interpret the Cosmological issues of fundamental importance.

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Introduction

Over one century has passed since the classical theory of gravitation, Einstein's General Relativity, was presented in 1915 in the Prussian Academy of Science. From that year to nowadays this theory has been confirmed with several evidences, such as the discovery of Mercury's perihelion precession, the deflection of starlight checked for the first time during an eclipse in 1919 or the recent detection of gravitational waves coming from the merging of two black holes.
Though the consistency of General Relativity has been shown in a lots of experiments with great precision, it is always challenging to make suppositions about modifications of this classical theory, which may shed an alternative light on the nature of still puzzling features of our Universe, such as the inflationary epoch, the actual accelerated expansion of the Universe, thought to be driven by a source of energy known as cosmological constant, and the presence of a type of matter invisible to all other forces than the gravitational one and thus called Dark Matter. It is not so unbelievable that at cosmological distances gravity itself could be different from what we experience at much smaller scales, in our Solar System.
In order to explain Dark Matter any suitable candidate competes with a plenty of various extra Standard Model particles, while so far there is no convincing explanation for the smallness of the observed cosmological constant. Unless both these issues eventually turn out to be resolved on the particle physics side, the severeness of the phenomenological problems challenges General Relativity as the ultimate and complete theory of gravity.
From a more general theoretical point of view looking for consistent deformations or generalizations of Einstein gravity can also tell us to which extent this theory is unique (or flexible).
In this Master Thesis, as a particular modification of gravity we will focus on the effects which arise when a mass is given to the graviton, the boson which mediates the gravitational fundamental force. Firstly, quadratic interactions responsible for furnish a mass will be put in the theory in the simplest and most obvious way, but at the end the correct models will proved to be Massive Gravity and Bigravity. From the particle physics perspective, the extension of the classical theory for gravitation performed adding a mass to the graviton seems rather natural and might be in a sense similar to the case of electroweak forces whose carriers acquire masses via the Brout-Englert-Higgs mechanism. However, an analogous mechanism which would give mass to the graviton is still unknown.
The graviton mass will take values in a very wide range according to the models of Modified Gravity analysed. All of them will be perfectly allowed in order to be consistent with observa-

tions (which, up to the sensitivity of the experiments, confirm General Relativity). For example, the graviton mass, in Massive Gravity, is bounded by General Relativity consistency to a very little value of $10^{-32}\frac{\mathrm{eV}}{c^2}$; the Bigravity massive mode, instead, is expected to lie in the window between $1\,\frac{\mathrm{TeV}}{c^2}$ and $66\,\frac{\mathrm{TeV}}{c^2}$.
Before addressing the challenging cosmological issues, a consistent modification of General Relativity with the addition of a mass for the graviton should be furnished. However this is surprisingly difficult to obtain and consistency analyses are complicated by the non-linearities of the original Einstein theory.
The first part of this Thesis is dedicated to solving these problems in building a massive theory of gravity. They will be faced following the historical developing process of the theory. The second part, instead, focuses on the predictive power of those modifications to Gravity for the Dark Energy and Dark Matter issues.
Firstly, we will proceed by extending the theory invariant under the linearised transformations of the diffeomorphism group with a quadratic mass term for the graviton. This was the approach presented for the first time by Fierz and Pauli in 1939 [19]. The linearised case, although at first sight seems to be convincing because it propagates the right number of degrees of freedom, will show a particular unphysical feature, that is the presence of a discontinuity between the massless limit of the massive theory and the free limit of GR, called the vDVZ discontinuity, due to the coupling to matter of the scalar degree of freedom of the massive graviton, discovered in 1970 independently by van Dam and Veltman [29], and by Zakharov [31]. The previously mentioned scalar mode arises naturally in the theory because in four dimensional spacetime a massive graviton propagates five degrees of freedom, three more than in the usual massless case.
The solution to this puzzle lies in the Vainshtein mechanism (found by the Russian physicist Vainshtein [28] in 1972), which works in an interacting theory when considering non-linearities and consists in the screening of the scalar degree of freedom by its own interactions, dominating over the linear terms in the massless limit. Moreover, this effect may also present particular features such as strong coupling and superluminalities.
When one tries to construct a non-linear theory and the formulation of the massive theory in this context, extending the Fierz-Pauli term in a non-linear way, the vDVZ effect does not manifest itself, but another problem seems to appear when adopting a naive way to proceed as Boulware and Deser showed in [8]. It is referred as Boulware-Deser ghost and it consists in the presence of a pathological degree of freedom. In field theory ghosts are fields whose kinetic term in the lagrangian has a wrong sign, leading to unstable configurations in classical physics as well as to states with negative norm whose existence thus violates unitarity in quantum physics. Creminelli, Nicolis, Papucci and Trincherini in [9] were able to prove, instead, that the additional degree of freedom in a Fierz-Pauli non-linear generalization is not removed although it is possible to add a wide range of non-linear interactions.
Then, we will explain how expressing the self-interactions for $g_{\mu\nu}$ in a reference frame given by the generic metric $\mathrm{f}_{\mu\nu}$ as a precise combination of traces of $\sqrt{g^{-1}\mathrm{f}}$ and other higher powers returns a consistent theory. The solution was proposed first by the three physicists de Rham,

Gabadadze and Tolley in [14] and it is called Massive Gravity. The theory propagates exactly five degrees of freedom, as expected from a massive graviton, and the massless limit is smooth. However, as we will discuss, there are some detectable effects which distinguish the massive case from General Relativity, outside a typical radius known as Vainshtein radius depending on the source mass and in an inverse way on the Planck mass and the graviton mass. The difference between massless and massive theory is, in particular, due to the presence of the unavoidable fifth force generated by the coupling of the scalar mode to matter sources which has been found to arise in the massive linearised version of General Relativity, but the screening mechanism reduces its effects, compared to canonical Einstein's Relativity, inside this typical distance.
In the vielbein formalism, a basis of one forms $e^a_\mu \, dx^\mu$ which, in a way, "diagonalizes" any metric, yielding to Minkowski metric in tangent space, the whole discussion simplifies and it will become easier to express the interaction terms (they will be an appropriate product between vielbeins of one type and of the other type) and even the equations of motion will be extracted in a simpler way. Hence Massive Gravity with vielbeins, besides being free from pathologies, is quite elegant and natural.
Then, we will pay attention to Bigravity, a theory similar to Massive Gravity but with both metrics being dynamical, discussing its self-consistency and, in a second moment, its peculiar properties, such as the capability in giving a self-contained explanation to Dark Matter without introducing particles outside the Standard Model, and the related possibility for the massive graviton to have a very huge mass, as already mentioned, without contradicting GR. An analysis of its decoupling limit will be carried out and the result will be that Bigravity reduces to Massive Gravity, justifying in this way the choice to start the exposition with the latter, as a first, simpler step of a theory of modified gravity, and also the possibility of deriving results which apply to Bigravity following the behaviour of Massive Gravity.
Having provided the evidences in order to prove that these modifications of General Relativity are ghost free, propagate the correct number of degrees of freedom and the massless limit is smooth, we will then focus on the Massive Gravity and Bigravity solutions.
The cosmological solutions will be obtained restricting ourselves to the case of proportional metrics, and both analogies and differences between them and that of the standard General Relativity case will be presented.
A Massive Gravity solution will be shown to perfectly admit an external arbitrarily large cosmological constant but at the same time to describe a nearly flat Universe, when the parameters of Massive Gravity are appropriately tuned compensating for that cosmological constant.
An other difference with respect to General Relativity are the self-accelerated solutions, which provide an accelerated expansion to the Universe only through the presence of the massive graviton.
In the Dark Matter issue, instead, the massive combination of the perturbations around a same zero order solution for $g_{\mu\nu}$ and $\mathrm{f}_{\mu\nu}$ can be potentially depicted as that constituent of the Universe.
That eigenstate of mass will be presented as a suitable candidate due to its feeble coupling to

Standard Model matter which, at tree level in perturbation theory, furnishes however the basis of an efficient production mechanism in primordial ages.
When relaxing the proportionality condition on the two metrics, Bigravity and its decoupling limit will be proved to admit black holes solutions as those solutions corresponding to a static and spherically symmetric sources of matter. One of the main features which makes these black holes different from those in General Relativity is that not all the metrics which couple to the solution are equivalently allowed: for example the second metric can be either de Sitter or Schwarzschild, but not Minkowski.
Furthermore, stability of all the above solutions will be investigated.

The material is arranged in the following way: after a brief review of the main features of General Relativity and its linearised realisation in chapter 1, chapter 2 deals with the Fierz-Pauli mass term, the Stückelberg trick in distributing the additional degrees of freedom of the massive graviton to a vectorial and a scalar field, and the vDVZ discontinuity. Then some attention is paid to an attempt in building a full diffeomorphism invariant theory with a mass in section 3 of chapter 2, but it will be plagued by a ghost field which will make it inconsistent. After having presented an equivalent reformulation of General Relativity in the vielbein language in section 1 of chapter 3, thus justifying the use of these one forms, in section 2 of the same chapter, consistent Massive Gravity is discussed mostly with the aid of vielbeins. Moreover the recovering of the metric formulation of Massive Gravity from the vielbein formulation is presented therein. The successive chapter 4 examines Bigravity, while the consistent decoupling limits of the massive theory are extracted in chapter 5, and immediately used to explain the Vainshtein screening mechanism in section 2 of that chapter. In the final chapter 6, cosmological predictions involving some solutions of Massive Gravity and Bigravity and their suitable properties, as well as their flaws, are investigated: section 1 deals with Dark Energy, section 2 with Dark Matter, and Black Holes are analysed in section 3.

Chapter 1

General Relativity and its free field limit

Gravity can be regarded as a gauge theory with associated gauge group being the group of diffeomorphisms. Its formulation relies on the observation that inertial and gravitational mass can not be distinguished, and on the equivalence principle, which follows from the previous remark and states that all accelerated reference frames are equivalent to one another, and hence the gravitational field is in complete correspondence with the acceleration in an accelerated reference frame. The equivalence principle has lead Einstein to notice that a consistent theory for gravitation should be invariant under diffeomorphisms, while only locally it should respect the Lorentz group of symmetry. The property of the masses, instead, is fundamental when trying to generalize the Newtonian law of gravity with the request that covariance is manifest: it is the spacetime itself and the metric $g_{\mu\nu}$ on it that rule the trajectories of the massive bodies, not the gravitational potential. Thus the fundamental gravitational field is given by the curvature of spacetime. Its coupling to the stress-energy tensor $\mathrm{T}_{\mu\nu}$ of matter, instead, furnishes its behaviour when a particular configuration of matter objects is chosen. As J.A. Wheeler said, "Matter tells space how to curve. Space tells matter how to move".
The covariant equations which generalise Newton's theory are, then, uniquely defined by the following expression, known as Einstein equation:

$$\mathrm{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathrm{R} = \frac{8\pi \mathrm{G}}{c^4} \mathrm{T}_{\mu\nu} \,, \tag{1.1}$$

in which G is the gravitational coupling constant and it is related to the Planck mass as $\mathrm{G} = \frac{1}{M_{Pl}}$, c is the speed of light, $\mathrm{R}_{\mu\nu}$ is known as the Ricci tensor, while R is the Ricci scalar ($\mathrm{R} := \mathrm{R}_{\mu\nu} g^{\mu\nu}$), and both of them are covariant objects built from the connection.

A connection is a way to parallel transport vectors along geodesics in a way which does not depend on the particular coordinate system, in the sense that on Riemann manifolds, rather than the case of topological flat spaces, the idea of derivative along tangent vectors is extended to a covariant differentiation, which does not change under the induced basis transformation of the tangent space when applying a coordinate chart transformation to the manifold. The covariant derivative and hence a connection could be obtained through the generalization of ∂_μ, applied to vectors of the curved space, to $\mathcal{D}_\mu$ defined through its action on a generic v^μ which belongs to the space, $\mathcal{D}_\mu v^\nu := \partial_\mu v^\nu + \Gamma_{\mu\ \rho}^{\ \nu} v^\rho$, where the $\Gamma_{\mu\ \rho}^{\ \nu}$ are called Christoffel symbols and are uniquely expressed in terms of the metric and its derivative due to Levi-Civita theorem, which is satisfied if the Riemann manifold has the metric $g_{\mu\nu}$ covariantly conserved (this is called the metric compatibility condition):

$$\mathcal{D}_\mu g_{\nu\rho} = 0\,, \tag{1.2}$$

and the torsion tensor is null.
The torsion tensor $T_{\mu\nu}$, if f is any smooth function, is defined by

$$\mathcal{D}_\mu \mathcal{D}_\nu f - \mathcal{D}_\nu \mathcal{D}_\mu f = -T_{\mu\ \nu}^{\ \rho} \mathcal{D}_\rho f\,.$$

In components, the torsion is $T_{\mu\ \nu}^{\ \rho} = \Gamma_{\mu\ \nu}^{\ \rho} - \Gamma_{\nu\ \mu}^{\ \rho}$. When it is zero, $\Gamma_{\mu\ \nu}^{\ \rho}$ is symmetric in its lower indices.
Then due to Levi-Civita theorem the connection coefficients are:

$$\Gamma_{\mu\ \nu}^{\ \rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\,. \tag{1.3}$$

The Christoffel symbols define the connection but above all they are involved in the construction of the Riemann tensor $\mathrm{R}_{\mu\nu\ \sigma}^{\ \ \rho}$ which, under appropriate contractions, gives back the Ricci tensor ($\mathrm{R}_{\mu\nu} := \mathrm{R}_{\rho\mu\ \nu}^{\ \ \rho}$) and the scalar curvature ($\mathrm{R} := \mathrm{R}_{\mu\nu} g^{\mu\nu}$):

$$\mathrm{R}_{\mu\nu\ \sigma}^{\ \ \rho} = \partial_\mu \Gamma_{\nu\ \sigma}^{\ \rho} - \partial_\nu \Gamma_{\mu\ \sigma}^{\ \rho} + \Gamma_{\nu\ \sigma}^{\ \alpha} \Gamma_{\alpha\ \mu}^{\ \rho} - \Gamma_{\mu\ \sigma}^{\ \alpha} \Gamma_{\alpha\ \nu}^{\ \rho}\,. \tag{1.4}$$

Due to its geometrical meaning, encoded in the measure of the extent to which the metric tensor is not locally isometric to that of Euclidean space, Riemann tensor satisfies an important identity known as Bianchi identity:

$$\mathcal{D}_\mu \mathrm{R}^{\alpha\nu\rho\sigma} + \mathcal{D}_\rho \mathrm{R}^{\alpha\nu\sigma\mu} + \mathcal{D}_\sigma \mathrm{R}^{\alpha\nu\mu\rho} = 0\,. \tag{1.5}$$

It implies exactly that extracting a covariant derivative from the left-hand side of equation (1.1) gives zero. A proof of the validity of this statement is derived in chapter 3.

Focusing on equations (1.1), they can be obtained by variational principle from the Hilbert-Einstein action $\mathcal{S}_{HE}$ plus an action for matter $\mathcal{S}_M$ with Lagrangian $\mathcal{L}_M$ (g denotes the determinant, $\mathrm{g} := \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta} g^\mu_{\ \alpha} g^\nu_{\ \beta} g^\rho_{\ \gamma} g^\sigma_{\ \delta}$):

$$\mathcal{S}_{HE} + \mathcal{S}_M = \int \mathrm{d}^4x\, \sqrt{-\mathrm{g}} \left(\frac{c^4}{16\pi\mathrm{G}} \mathrm{R}\,[g] + \mathcal{L}_M\right)\,, \tag{1.6}$$

where the stress-energy tensor $\mathrm{T}_{\mu\nu}$ is defined in General Relativity as

$$\mathrm{T}_{\mu\nu} := \frac{-2}{\sqrt{-\mathrm{g}}} \frac{\delta\left(\sqrt{-\mathrm{g}}\mathcal{L}_M\right)}{\delta g^{\mu\nu}}\,,$$

and the variation with respect to the metric $g^{\mu\nu}$ of $\sqrt{-\mathrm{g}}$ follows from the important identity which relates the logarithm of the determinant to the trace of the logarithm:

$$\log(\mathrm{g}) \;=\; \mathrm{tr}\left[\log g\right] \; . \tag{1.7}$$

Taking $\frac{\delta}{\delta g^{\mu\nu}}$ in $\sqrt{-\mathrm{g}}$ and using (1.7), one gets to:

$$\begin{aligned} \frac{\delta\sqrt{-\mathrm{g}}}{\delta\, g^{\mu\nu}} &= -\frac{1}{2\sqrt{-\mathrm{g}}}\,\mathrm{g}\,\frac{\delta \mathrm{tr}\left[\log g^{\rho\sigma}\right]}{\delta g^{\mu\nu}} \\ &= -\frac{\sqrt{-\mathrm{g}}}{2}\, g_{\mu\nu}\,, \end{aligned}$$

which is in perfect agreement with the second term in the LHS of (1.1).

Full diffeomorphism invariance usually means that applying a differentiable and invertible map, with inverse map differentiable as well, keeps the transformed lagrangian equivalent to the initial one (modulo total derivatives), and obviously it also preserves the equations of motion as well.
If x^μ is the chart used to cover the spacetime manifold, the diffeomorphism taken into account is denoted by $x^\mu \to x^{\mu\prime}$, then the Einstein-Hilbert action ($\mathcal{S}_{HE}$ in natural units, see (1.6)) invariance under the symmetry group of diffeomorphisms is easily derived from:

$$\begin{aligned} \mathrm{d}^4x' &= \left|\frac{\partial x'}{\partial x}\right|^4 \mathrm{d}^4x \\ g'_{\mu\nu}(x') &= g_{\rho\sigma}(x)\frac{\partial x^{\rho\prime}}{\partial x^\mu}\frac{\partial x^{\sigma\prime}}{\partial x^\nu} \;\to\; \sqrt{-\mathrm{g}'} = \left|\frac{\partial x'}{\partial x}\right|^4 \sqrt{-\mathrm{g}}\,. \end{aligned}$$

When focusing on a linearised version of the theory, correspondingly $g_{\mu\nu}$ could be thought as associated with a massless spin-2 particle in a Minkowski reference metric, in accordance with Wigner's classification of relativistic particles as representations of Poincaré group. In this sense, an analysis of the linearised diffeomorphism invariant theory of gravity corresponds exactly to discuss the properties of the massless spin-2 particle. Even giving a mass to the graviton is a quite natural theoretical successive step, in this context, and to this we will be dealt with in a while.
The linearised version of the symmetry group can be obtained by looking at the infinitesimal coordinate transformation, $x'^\mu = x^\mu + \xi^\mu(x)$. The metric tensor $g'_{\mu\nu}(x')$ in this chart is related to $g_{\mu\nu}(x)$ by:

$$\begin{aligned} g_{\mu\nu}(x) &= g'_{\mu\nu}(x') + \frac{\partial \xi^\lambda}{\partial x^\mu}\, g_{\lambda\nu}(x) + \frac{\partial \xi^\lambda}{\partial x^\nu}\, g_{\mu\lambda}(x) + o(\xi^2) \\ &= g'_{\mu\nu}(x) + \xi_\rho\frac{\partial g_{\mu\nu}(x)}{\partial x_\rho} + \frac{\partial \xi^\lambda}{\partial x^\mu}\, g_{\lambda\nu}(x) + \frac{\partial \xi^\lambda}{\partial x^\nu}\, g_{\mu\lambda}(x) + o(\xi^2)\,, \end{aligned}$$

and this implies that for $h_{\mu\nu}$, being the symmetric Lorentz tensor field associated to first order correction to flat spacetime, $g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu}$, the variation under a coordinate transformation, evaluated at the same point, is:

$$\begin{aligned} h_{\mu\nu}(x) + \eta_{\mu\nu} =& h'_{\mu\nu}(x) + \eta_{\mu\nu} + \xi^\rho \frac{\partial\left(h_{\mu\nu}(x) + \eta_{\mu\nu}\right)}{\partial x_\rho} \\ &+ \frac{\partial \xi_\mu}{\partial x_\rho}\left(h_{\rho\nu}(x) + \eta_{\rho\nu}\right) + \frac{\partial \xi_\nu}{\partial x_\rho}\left(h_{\mu\rho}(x) + \eta_{\mu\rho}(x)\right) + o(\xi^2) \\ =& h'_{\mu\nu}(x) + \eta_{\mu\nu} + \frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} + o(\xi^2)\,, \end{aligned} \tag{1.8}$$

where we stopped to first order in the expansion (neglecting terms involving derivatives h or $\partial\xi$ multiplying h).
The transformation law for $h_{\mu\nu}$, stopped at first order in ξ,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\,, \tag{1.9}$$

will be written, from now on, with the aid of the round parenthesis $(,)$, meaning that the condensed expression is actually including all possible symmetric contributions, divided per the factorial of the number of indices involved: $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$. This synthetic writing will be used everywhere, also in the case of square parenthesis $[,]$ which refer to the antisymmetric contributions, thus abandoning the extended expression.
Under the assumption that $h_{\mu\nu}$ should follow the transformation rule (1.9), then the most general kinetic term for this tensor respecting the local symmetry (1.8), Lorentz invariance and locality can only be:

$$\mathcal{L}^{\text{spin}-2}_{kin} = \frac{1}{8} h^{\mu\nu}\left(\Box h_{\mu\nu} - 2\partial_\alpha \partial_{(\mu} h^\alpha_{\ \nu)} + \partial_\mu \partial_\nu h - \eta_{\mu\nu}\left(\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta}\right)\right)\,. \tag{1.10}$$

A sketch of the proof of $\mathcal{L}^{\text{spin}-2}_{kin}$ invariance under (1.8) follows here:

$$\begin{aligned} \mathcal{L}^{\text{spin}-2\,\prime}_{kin} =& \mathcal{L}^{\text{spin}-2}_{kin} + \frac{h^{\mu\nu}}{8}\left(\Box\partial_{(\mu}\xi_{\nu)} - \partial_{(\mu}\Box\xi_{\nu)} - \partial^\alpha\partial_{(\mu}\partial_{\nu)}\xi_\alpha + \partial_{(\mu}\partial_{\nu)}\partial^\alpha\xi_\alpha\right) \\ &+ \frac{1}{16}\xi^\nu\left(\Box\partial^\mu h_{\mu\nu} - \Box\partial_\alpha h^\alpha_{\ \nu} - \partial_\nu\partial^\mu\partial_\alpha h^\alpha_{\ \mu} + \Box\partial_\nu h - \left(\Box\partial_\nu h - \partial_\alpha\partial_\beta\partial_\nu h^{\alpha\beta}\right)\right) \\ &- \frac{h}{8}\left(\Box\partial^\alpha\xi_\alpha - \Box\partial^\alpha\xi_\alpha\right) + \partial^\mu\left(...\right) + \text{other null contributions}\,. \end{aligned}$$

Then from integration by parts, since the total derivative contributions are nothing else than some additional pieces which return an equivalent lagrangian, $\mathcal{L}^{\text{spin}-2\,\prime}_{kin}$ really does coincide with $\mathcal{L}^{\text{spin}-2}_{kin}$.
The lagrangian (1.10) can be obtained directly from the Einstein Lagrangian restricting $\mathrm{R}[g]\sqrt{-\mathrm{g}}$ to the terms quadratic in $h_{\mu\nu}$. In fact:

$$\sqrt{-\mathrm{g}} = 1 + \frac{1}{2} h^\alpha_{\ \alpha}\,,$$

while the only first order term in the Christoffel symbols are:

$$\Gamma_{\sigma\ \mu}^{\ \rho} = \frac{1}{2}\left(\partial_\sigma h_\mu^{\ \rho} + \partial_\mu h^\rho_{\ \sigma} - \partial^\rho h_{\mu\sigma}\right)\,. \tag{1.11}$$

The Ricci tensor $R_{\mu\nu}$ is, thus:

$$\begin{aligned} R_{\rho\mu}{}^{\rho}{}_{\nu} &= \left(\partial_\rho \Gamma_{\mu}{}^{\rho}{}_{\nu} - \partial_\mu \Gamma_{\rho}{}^{\rho}{}_{\nu}\right) \\ &= \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_{\ \nu} + \partial_\rho\partial_\nu h_\mu^{\ \rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\rho h^\rho_{\ \nu} - \partial_\mu\partial_\nu h + \partial_\mu\partial^\rho h_{\rho\nu}\right) . \end{aligned}$$

This leads to the following expression for $\sqrt{-g}\,R$, noticing that $g^{\mu\nu}$ is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and stopping to first order in the perturbation:

$$\begin{aligned} \sqrt{-g}\,R[g] &= \left(1 + \frac{1}{2}h\right)\left(\eta^{\mu\nu} - h^{\mu\nu}\right)\frac{1}{2}\left(\partial_\nu\partial_\rho h_\mu^{\ \rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\mu\partial_\rho h^\rho_{\ \nu}\right) \\ &= -h^{\mu\nu}\left(\frac{1}{2}\partial_\rho\partial_\nu h_\mu^{\ \rho} - \frac{1}{2}\Box h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h + \frac{1}{2}\partial_\mu\partial^\rho h_{\rho\nu}\right) + \frac{1}{4}h\left(2\partial_\rho\partial_\nu h^{\nu\rho} - 2\Box h\right) , \end{aligned}$$

which is exactly (1.10) apart from a factor of $\frac{1}{4}$.
Therefore, the lagrangian discussed in the above lines gives a consistent theory for a symmetric tensor of rank 2, which propagates $2 = 10 - (4 \cdot 2)$ degrees of freedom (4 are related to diffeomorphism invariance which together with the gauge constraints remove 8 degrees of freedom).

Chapter 2

Linear theory with Fierz-Pauli mass term

We are interested now in endowing the graviton with a mass. Adding a mass term breaks local gauge invariance (1.8), but we would keep our theory Lorentz-invariant and try to avoid pathologies such as ghosts. These requirements restrict the possible forms of the graviton mass term to a single one, and this will be presented later in this paragraph.
First, let us state a criterion very useful in order to establish if the theory analysed is free from ghosts. Ghosts are fields with negative kinetic energy whose presence would lead to instabilities at the classical level and non-unitarity of the theory at the quantum level.
Such undesired behaviour may come up when one considers a theory with more than one time derivative of the examined field in the lagrangian, e.g.:

$$\mathcal{L}_{gh} = \frac{1}{2}\left(\ddot{\phi}\right)^2 . \tag{2.1}$$

We can rewrite this lagrangian in a form with two derivatives introducing the auxiliary field p.

$$\mathcal{L}_{gh} = p\ddot{\phi} - \frac{1}{2}p^2 = -\dot{p}\dot{\phi} - \frac{1}{2}p^2 . \tag{2.2}$$

The equation of motion of p is $p = \ddot{\phi}$, so substituting it back into (2.2) gives (2.1).
On the other hand, diagonalising now the lagrangian (2.2) with the choices $\Psi_+ = p + \phi$ and $\Psi_- = p - \phi$, so that

$$\mathcal{L}_{gh} = -\frac{1}{2}\left(\dot{\Psi}_+\right)^2 + \frac{1}{2}\left(\dot{\Psi}_-\right)^2 - \frac{1}{2}\left(\frac{\Psi_+ + \Psi_-}{2}\right)^2 ,$$

then clearly in $\mathcal{L}_{gh}$ one of the fields, Ψ_+, has the kinetic term with the wrong sign.
Thus from a kinetic lagrangian with a second order time derivative for the scalar field ϕ as (2.1) the superior degree of the derivative has made possible to split ϕ in two dynamical degrees of freedom, but one of them is a ghost, so that the corresponding theory is unphysical. This result is known as Ostrogradsky instability.

Another important requirement which should always be checked in order to make a field theory consistent is the conformity between the observable quantities in the case in which some interactions are initially switched on and then turned off and the observables for the other case, in which there is no interaction from the beginning. The physical system described should be exactly the same.

The problem to which we are interested, anticipated previously, is whether the graviton, in the linearised realization of the symmetry group, may have a mass. The form of the mass term in the lagrangian is restricted by the previously justified requirement of Lorentz invariance and absence of the ghost degrees of freedom; these are satisfied by a unique choice, that is:

$$\mathcal{L}_{mass} = -\frac{1}{8}m^2\left(h^{\mu\nu}h_{\mu\nu} - h^2\right), \tag{2.3}$$

and it is known as the Fierz-Pauli mass term. Any other combination of $h_{\mu\nu}$ and h would lead to the appearance of the ghost field as we will see in the next section.

1 Counting of degrees of freedom

The mass term (2.3) in the Lagrangian breaks diffeomorphism invariance. Thus the theory with lagrangian

$$\mathcal{L}_{kin}^{\text{spin}-2} + \mathcal{L}_{mass} \tag{2.4}$$

for a symmetric tensor field $h_{\mu\nu}$ will have more physical degrees of freedom than its massless counterpart. From Wigner's classification we know that in four-dimensional spacetime the spin-2 massive particle has 5 degrees of freedom hence we should show that on the mass shell $h_{\mu\nu}$ has 5 genuine degrees of freedom. The lagrangian (2.4) yields the following equations of motion:

$$\Box h_{\mu\nu} - 2\partial_\alpha\partial_{(\mu}h^\alpha_{\nu)} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\left(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}\right) = m^2\left(h_{\mu\nu} - h\eta_{\mu\nu}\right). \tag{2.5}$$

Acting on the equations of motion with ∂^μ we have

$$\Box\partial^\mu h_{\mu\nu} - \Box\partial_\alpha h^\alpha_\nu - \partial_\nu\partial_\alpha\partial^\mu h^\alpha_\mu - \partial_\nu\left(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}\right) = m^2\left(\partial^\mu h_{\mu\nu} - \partial^\nu h\right),$$

the left-hand side (Einstein) part vanishes due to Bianchi identity (1.5), and hence the RHS should be zero as well, i.e.

$$\partial^\mu h_{\mu\nu} = \partial_\nu h. \tag{2.6}$$

Taking now the trace of (2.5) we get:

$$\Box h - 2\partial^\mu\partial_\alpha h^\alpha_\mu + \Box h - 4\left(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}\right) = -3m^2 h. \tag{2.7}$$

The left hand side of (2.7) is zero due to equation (2.6). Thus we get

$$h = 0 \tag{2.8}$$

and hence, in view of (2.6), also

$$\partial^\mu h_{\mu\nu} = 0. \tag{2.9}$$

The condition of null trace (2.8) removes from $h_{\mu\nu}$ one degree of freedom and the transversality condition (2.9) removes 4 of them. Hence, the number of degrees of freedom of the massive graviton in 4 dimensions is reduced from 10 to 5, in accordance to Wigner classification.

Having set all the important features of the free theory, now it is time to focus on its coupling to matter sources, described through the stress-energy tensor $\mathrm{T}_{\mu\nu}$, considering the action built from the Lagrangian (2.4) plus a coupling of $h_{\mu\nu}$ to an external source, $\frac{1}{2M_{Pl}^2} h_{\mu\nu}\mathrm{T}^{\mu\nu}$:

$$\mathcal{L} \,=\, \mathcal{L}_{kin}^{\mathrm{spin-2}} + \mathcal{L}_{mass} + \frac{1}{M_{Pl}^2} h_{\mu\nu}\mathrm{T}^{\mu\nu}\,, \tag{2.10}$$

where M_{Pl} is the Planck mass, related to the gravitational constant via $M_{Pl} = \frac{1}{\sqrt{\mathrm{G}}}$.
Then the equations of motion obtained by varying $\mathcal{L}$ with respect to $h_{\mu\nu}$ are:

$$\frac{1}{2}\left(\Box h_{\mu\nu} \,-\, 2\partial_\alpha\partial_{(\mu}h^{\alpha}{}_{\nu)} \,+\, \partial_\mu\partial_\nu h \,-\, \eta_{\mu\nu}\left(\Box h \,-\, \partial_\alpha\partial_\beta h^{\alpha\beta}\right)\right) - \frac{1}{2}m^2\left(h_{\mu\nu} \,-\, h\eta_{\mu\nu}\right) \,=\, -\frac{1}{M_{Pl}^2}T_{\mu\nu}\,. \tag{2.11}$$

We would now like to see if in the massless limit $m \to 0$ the theory under consideration reduces to linearised General Relativity coupled to matter.

One simple way to answer the question is by checking the behaviour of the Ricci scalar. In classical General Relativity, the trace of (1.1) in natural units is:

$$-\,\mathrm{R} \,=\, \frac{1}{M_{Pl}^2}\mathrm{T}\,.$$

The Ricci scalar, according to Einstein's equations, is proportional to the trace of the stress-energy tensor. This also holds in the linearised massless theory described by (1.10).
In the linearised case with the Fierz-Pauli mass term, instead, (2.11) can be written also as:

$$\mathrm{R}_{\mu\nu} \,-\, \frac{1}{2}\eta_{\mu\nu}\mathrm{R} \,+\, \frac{m^2}{2}\left(h_{\mu\nu} \,-\, h\eta_{\mu\nu}\right) \,=\, \frac{1}{M_{Pl}^2}\mathrm{T}_{\mu\nu}\,.$$

While directly tracing (2.11) gives:

$$-\mathrm{R} - \frac{3}{2}m^2 h \,=\, \Box h \,-\, \partial_\alpha\partial_\beta h^{\alpha\beta} \,-\, \frac{3}{2}m^2 h \,=\, \frac{1}{M_{Pl}^2}\mathrm{T}\,, \tag{2.12}$$

which combines with the result coming from the calculation of ∂^μ on the equations of motion:

$$\partial^\mu h_{\mu\nu} \,-\, \partial_\nu h \,=\, \frac{2}{m^2 M_{Pl}^2}\partial^\mu \mathrm{T}_{\mu\nu}\,.$$

Now, assuming that as in the massless case, the source is conserved, i.e. $\partial^\mu \mathrm{T}_{\mu\nu} \,=\, 0$, we have $\partial^\mu h_{\mu\nu} - \partial_\nu h \,=\, 0$ and then from (2.12) we get:

$$\Box h \,-\, \partial_\alpha\partial^\alpha h \,-\, \frac{3}{2}m^2 h \,=\, \frac{1}{M_{Pl}^2}\mathrm{T}$$

$$h \,=\, -\frac{2}{3m^2 M_{Pl}^2}\mathrm{T}\,. \tag{2.13}$$

Equation (2.13) allows to rewrite the equation for the trace (2.12)

$$-\mathrm{R} - \frac{3}{2}m^2 h = -\mathrm{R} + \frac{1}{M_{Pl}^2}\mathrm{T} = \frac{1}{M_{Pl}^2}\mathrm{T}\,, \tag{2.14}$$

which implies that $\mathrm{R} = 0$ even in the presence of $\mathrm{T} \neq 0$ in contrast to Einstein gravity. Some suitable examples for $\mathrm{T}_{\mu\nu}$ with null trace exist in Nature. One of them can be found in the Maxwell theory for the free electromagnetic field with the field strength $F_{\mu\nu}$ which respects General Relativity by means of the equivalence principle, with lagrangian $\mathcal{L}_{\text{em}} = -\frac{1}{4}\sqrt{-\mathrm{g}}F_{\mu\nu}F^{\mu\nu}$. It can be proved that $\mathrm{T}_{\mu\nu} = F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F^2$.
However $\mathrm{R} = 0$ must hold choosing any possible stress-energy tensor, so the linearised massive theory of gravitation does not lead to the same result which comes from the linearised classical gravitational theory.

Another way to check if the massless limit of the massive theory fits the classical massless theory consists in checking the amplitude between two sources, mediated by the graviton propagator arising from each theory. From equations (2.11) one gets:

$$\begin{aligned}\left(\Box - m^2\right) h_{\mu\nu} =& \frac{2}{M_{Pl}^2}\left(-\mathrm{T}_{\mu\nu} + \frac{\eta_{\mu\nu}}{3}\left(\mathrm{T} + \frac{1}{m^2}\partial^\alpha\partial^\beta\mathrm{T}_{\alpha\beta}\right)\right)\\ &- \frac{2}{M_{Pl}^2}\left(-\frac{2}{m^2}\partial_\alpha\partial_{(\mu}\mathrm{T}^\alpha_{\nu)} + \frac{1}{3m^2}\left[\partial_\mu\partial_\nu\mathrm{T} + \frac{2}{m^2}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\mathrm{T}^{\alpha\beta}\right]\right)\,.\end{aligned}$$

Defining $\tilde{\eta}_{\mu\nu} := \eta_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu$, this can be reduced to:

$$\left(\Box - m^2\right) h_{\mu\nu} = -\frac{2}{M_{Pl}^2}\left[\tilde{\eta}_{\mu\,(\alpha}\tilde{\eta}_{\beta)\,\nu} - \frac{1}{3}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta}\right]\mathrm{T}^{\alpha\beta}\,,$$

which, in momentum space, is:

$$\begin{aligned}\left(p^2 - m^2\right) h_{\mu\nu} =& -\frac{2}{M_{Pl}^2}\left[\eta_{\mu(\alpha}\eta_{\beta)\nu} - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta} + \frac{2}{3m^4}p_\mu p_\nu p_\alpha p_\beta\right]\mathrm{T}^{\alpha\beta}\\ &- \frac{2}{M_{Pl}^2}\left[\frac{1}{m^2}\eta_{\mu(\alpha}p_{\beta)}p_\nu + \frac{1}{m^2}\eta_{\nu(\alpha}p_{\beta)}p_\mu - \frac{1}{3m^2}p_\mu p_\nu\eta_{\alpha\beta} - \frac{1}{3m^2}p_\alpha p_\beta\eta_{\mu\nu}\right]\mathrm{T}^{\alpha\beta}\,.\end{aligned} \tag{2.15}$$

As above, when taking the limit $m \to 0$, one should assume that the matter source is conserved, which entails that every term that combines a p_α or a p_β with a $\mathrm{T}^{\alpha\beta}$ is null: only $-\frac{2}{M_{Pl}^2}[\eta_{\mu(\alpha}\eta_{\beta)\nu} - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta} - \frac{1}{3m^2}p_\mu p_\nu\eta_{\alpha\beta}]\,\mathrm{T}^{\alpha\beta}$ still brings a contribution. Let us consider now the exchange amplitude

$$\int \mathrm{d}^4p\, h_{\mu\nu}\mathrm{T}'^{\mu\nu}\,, \tag{2.16}$$

in which $h_{\mu\nu}$ is defined in (2.15) and the term $\mathrm{T}^{\alpha\beta}\dfrac{\left(-\frac{1}{3m^2}p_\mu p_\nu\eta_{\alpha\beta}\right)}{p^2 - m^2}\mathrm{T}'^{\mu\nu}$ vanishes due to matter source conservation. Then (2.16) leads to the following graviton propagator:

$$-\frac{2}{M_{Pl}^2}\frac{\left[\eta_{\mu(\alpha}\eta_{\beta)\nu} - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta}\right]}{p^2 - m^2}\,. \tag{2.17}$$

Note that the propagator does not depend on the graviton mass.
In the massless case, instead, the same procedure can be applied choosing now a specific gauge, the so called De Donder gauge $\partial_\mu \left(\sqrt{-\mathrm{g}}\, g^{\mu\nu} \right) = 0$, to fix the diffeomorphism invariance. In our linearised case of a generic metric expanded around Minkowski space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the De Donder condition (given that $-\mathrm{g} = 1 + h^a{}_a$ and then $\sqrt{-\mathrm{g}} = 1 + \frac{1}{2} h^a{}_a$) becomes

$$\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^a{}_a = 0 . \tag{2.18}$$

Applying this relation to the equations of motion for $h_{\mu\nu}$ obtained from (2.10), in which obviously $\mathcal{L}_{mass}$ is set to zero, leads to the following equation for $h_{\mu\nu}$:

$$\Box h_{\mu\nu} = -\frac{2}{M_{Pl}^2} \left(\mathrm{T}_{\mu\nu} - \frac{1}{2} \mathrm{T} \eta_{\mu\nu} \right) ,$$

and the inverse of the propagator can be further expressed as:

$$\Box h_{\mu\nu} = -\frac{2}{M_{Pl}^2} \left(\eta_{\mu(\alpha} \eta_{\beta)\nu} - \frac{1}{2} \eta_{\mu\nu} \eta_{\alpha\beta} \right) \mathrm{T}^{\alpha\beta} .$$

Correspondingly, the massless graviton propagator in the momentum space is:

$$-\frac{2}{M_{Pl}^2} \frac{\left[\eta_{\mu(\alpha} \eta_{\beta)\nu} - \frac{1}{2} \eta_{\mu\nu} \eta_{\alpha\beta} \right]}{p^2} \tag{2.19}$$

As one can see, the propagators (2.17) and (2.19), do not coincide even when m is set to 0: this is known as Van Dam-Veltman-Zakharov discontinuity (vDVZ discontinuity) [4].
Physical implications of this effect can easily be found even in the non-relativistic regime. Focusing our attention on the exchange amplitude between two stress-energy tensors, (2.16), the choices $\mathrm{T}'_{\mu\nu} = \mathrm{diag}\,(M', 0, 0, 0)$ and $\mathrm{T}_{\mu\nu} = \mathrm{diag}\,(M, 0, 0, 0)$, representing two static stars, in the massless limit of the massive case lead to:

$$-\frac{2}{M_{Pl}^2} \frac{\left[\frac{1}{2} \mathrm{T}_\alpha^{\ \nu} \mathrm{T}'^{\ \alpha}_\nu + \frac{1}{2} \mathrm{T}_\beta^{\ \nu} \mathrm{T}'^{\ \beta}_\nu - \frac{1}{3} \mathrm{T}\mathrm{T}' \right]}{p^2} = -\frac{4}{3 M_{Pl}^2} \frac{M' M}{p^2} ,$$

while, obviously, the result for the massless gravity theory is found by substitution of the $\frac{1}{3}$ factor with $\frac{1}{2}$:

$$-\frac{1}{M_{Pl}^2} \frac{M' M}{p^2} .$$

Performing an inverse Fourier transform, and switching to spherical coordinates ($|p|$, θ, ϕ):

$$\begin{aligned}
-\frac{1}{(2\pi)^2} \int \mathrm{d}p_0 \mathrm{d}^3 p\, \frac{1}{p^2} e^{ipx} &= -\frac{1}{(2\pi)^2} \int \mathrm{d}p_0 \mathrm{d}^3 p\, \frac{1}{-p_0^2 + p^2} e^{-ip_0 x^0} e^{i\vec{p}\vec{x}} \\
&= -\frac{1}{(2\pi)^2} \int \mathrm{d}\,|p|\, \mathrm{d}\phi\, \mathrm{d}\cos\theta\ |p|^2 \left(\frac{1}{2\,|p|} \right) e^{i|p||x|\cos\theta} \\
&= -\frac{1}{(2\pi)^2} \int_0^{+\infty} \mathrm{d}\,|p|\ |p|^2 \left(\frac{2\pi}{2i\,|x|\,|p|^2} \right) e^{i|x||p|} - e^{-i|x||p|} \\
&= -\frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \mathrm{d}\,|p|\ |p|^2 \left(\frac{2\pi}{i\,|x|\,|p|^2} \right) e^{i|x||p|} \\
&= -\frac{1}{(2\pi)^2} \frac{4i\pi^2}{i\,|x|} = -\frac{1}{r} ,
\end{aligned}$$

it is clear that the result found in the inverse space involves directly the potentials ($G := \frac{1}{M_{Pl}^2}$ is Newton gravitational constant)

$$V_{\text{massive}} = -\frac{4G}{3}\frac{M'M}{r}\,, \quad V_{\text{massless}} = -G\frac{M'M}{r}\,,$$

and translates into a discrepancy in the value of the Massive Gravity potential, which differs from the Newtonian potential by a factor of $\frac{3}{4}$. This quite big difference would be undoubtedly noticed at the experiments.
If one tries, instead, to redefine the Newton constant, the discrepancy will then reappear in other observable quantities, such as the light bending. It will then be 25% smaller in the massive case than in the massless one, a too large value to make it compatible with current measurement of the light bending by the Sun [29], [31].

2 Stückelberg trick

A smart expedient to restore the underlying symmetry under diffeomorphisms is to introduce into the lagrangian the Stückelberg field χ_μ transforming in a way that keeps the linearised diffeomorphism symmetry valid,

$$\chi_\mu \to \chi_\mu - \xi_\mu\,.$$

Putting χ_μ into $\mathcal{L}_{mass}$ in this way:

$$\begin{aligned}\mathcal{L}_{mass} &= -\frac{1}{8}m^2\left(\left(h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}\right)^2 - (h + 2\partial_\alpha\chi^\alpha)^2\right)\\ \to \mathcal{L}'_{mass} &= -\frac{1}{8}m^2\left(\left(h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} + 2\partial_{(\mu}\chi_{\nu)} - 2\partial_{(\mu}\xi_{\nu)}\right)^2 - (h + 2\partial\xi + 2\partial\chi - 2\partial\xi)^2\right)\,,\end{aligned}$$

then automatically diffeomorphism invariance is respected.
One of the extents of the Stückelberg trick is also to move 3 of the 5 degrees of freedom of the massive graviton, which appear when the system loses its invariance under diffeomorphism, to a vector A^μ and a scalar π, while spin-2 boson $h_{\mu\nu}$ carries 2 degrees of freedom, as the corresponding massless tensor-2 field.
With the Stückelberg trick one can directly check whether the massive graviton scalar mode contains the ghost or not. In fact, let us further split the Stückelberg field into an effective gauge field A_μ and a scalar mode π, appropriately renormalized with some mass coefficients

$$\chi_\mu = \frac{1}{m}A_\mu + \frac{1}{m^2}\partial_\mu\pi\,. \tag{2.20}$$

This decomposition is invariant under the gauge transformation $A'_\mu = A_\mu + \frac{1}{m}\partial_\mu\varphi$, $\pi' = \pi - \varphi$. Equations (1.10) and (2.3) become:

$$
\begin{aligned}
&= \mathcal{L}_{kin}^{\text{spin}-2} - \frac{1}{8}m^2 \left(\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) + \frac{1}{m^2}\left((\partial_\mu A_\nu + \partial_\nu A_\mu)^2 - 4\left(\partial_\alpha A^\alpha\right)^2 \right) + \frac{4}{m^4}\left(\partial_\mu\partial_\nu\pi\right)^2 \right) \\
&\quad + \frac{1}{2m^2}\left(\Box\pi\right)^2 - \frac{1}{8}m^2\left(\frac{2}{m}h_{\mu\nu}\left(\partial^\mu A^\nu + \partial^\nu A^\mu\right) - \frac{4}{m}h\left(\partial_\alpha A^\alpha\right) + \frac{4}{m^4}h_{\mu\nu}\left(\partial^\mu\partial^\nu\pi\right) - \frac{4}{m^4}h\left(\Box\pi\right) \right) \\
&= \mathcal{L}_{kin}^{\text{spin}-2} - \frac{1}{8}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) - \frac{1}{8}F_{\mu\nu}^2 - \frac{1}{4}m\left(h_{\mu\nu} - h\eta_{\mu\nu}\right)\left(\partial^\mu A^\nu + \partial^\nu A^\mu\right) \\
&\quad - \frac{1}{2m^2}h_{\mu\nu}\left(\partial^\mu\partial^\nu\pi - \Box\pi\eta^{\mu\nu}\right) - \frac{1}{2m^2}\left(\left(\partial_\mu\partial_\nu\pi\right)^2 - \left(\Box\pi\right)^2 \right).
\end{aligned}
$$

We can further diagonalise the π term using the field redefinition $h_{\mu\nu} = \tilde{h}_{\mu\nu} + \eta_{\mu\nu}\pi$:

$$
\begin{aligned}
&= \mathcal{L}_{kin}^{\text{spin}-2}\left(\tilde{h}\right) - \frac{m^2}{8}\left(\tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu} - \tilde{h}^2 + 16\pi^2 + 8\tilde{h}\pi - 16\pi^2 \right) - \frac{1}{8}F_{\mu\nu}^2 - \frac{1}{2m^2}\pi\left(\Box\pi - 4\Box\pi\right) \\
&\quad - \frac{m}{4}\left(\partial^\mu A^\nu + \partial^\nu A^\mu\right)\left(\tilde{h}_{\mu\nu} + \eta_{\mu\nu}\pi - \tilde{h}\eta_{\mu\nu} - 4\pi\eta_{\mu\nu} \right) - \frac{1}{2m^2}\tilde{h}_{\mu\nu}\left(\partial^\mu\partial^\nu\pi - \Box\pi\eta^{\mu\nu}\right) - 8\tilde{h}\pi \\
&= \mathcal{L}_{kin}^{\text{spin}-2}\left(\tilde{h}\right) - \frac{m^2}{8}\left(\tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu} - \tilde{h}^2 \right) - \frac{1}{8}F_{\mu\nu}^2 - \frac{m}{4}\left(\tilde{h}_{\mu\nu} - \tilde{h}\eta_{\mu\nu} - 3\pi\eta_{\mu\nu} \right)\left(\partial^\mu A^\nu + \partial^\nu A^\mu\right) \\
&\quad - \frac{1}{2m^2}\tilde{h}_{\mu\nu}\left(\partial^\mu\partial^\nu\pi - \Box\pi\eta^{\mu\nu}\right) + \frac{3}{2m^2}\pi\Box\pi .
\end{aligned}
$$

So, the Stückelberg field χ_μ allows to distribute the degrees of freedom of one massive Lorentz rank-2 tensor between three fields. Still present as in General Relativity and propagating 2 degrees of freedom there is the helicity-2 mode, $\tilde{h}_{\mu\nu}$, then there is a Lorentz vector A_μ, which represents the helicity-1 mode and propagates 2 degrees of freedom because all its mixing terms can not give a $A_\mu A^\mu$ massive contribution even after a field redefinition, and last there is a scalar field for the helicity-0 mode, π, which propagates 1 degree of freedom. The total number of degrees of freedom is exactly 5.
From the above Lagrangian, we see, moreover, that the scalar π has the standard second-order kinetic term and hence does not develop any ghost degree of freedom.

To summarize, the linear theory fits perfectly the requirement of the absence of ghosts but it exhibits the vDVZ discontinuity. In 1972 [28] Vainshtein showed that the vDVZ problem was peculiar to the linear theory and could be solved in a non-linear formulation, in which the effect is screened by scalar mode interactions (see section 5). So the final conclusion is to extend the linear theory to a non-linear theory of massive gravity in order to avoid the discontinuity and its effects which we should be able to detect with our experimental sensitivity.

3 Towards the full non-linear diffeomorphism invariant theory

A straightforward implication deriving from the issues of the previous chapter is not that it is unavoidable to build our theory for gravitation in a non-Minkowskian reference metric $\tilde{f}_{\mu\nu}$, in fact a flat metric can be used, but that the interaction terms (as well as the kinetic terms) may

arise from the fluctuations around non-flat spacetime.
If $\tilde{f}_{\mu\nu}$ is the generic reference metric, the Stückelberg trick can act even in the non-linear context as, again, the expedient which allows to overcome full diffeomorphism symmetry breaking and hence allows to formally restore covariance. It works in the following sense: $\tilde{f}_{\mu\nu}$ has to be promoted to a covariant tensor, by putting in it the four Stückelberg fields ϕ^a transforming as scalars, $f_{\mu\nu} := \partial_\mu \phi^a \partial_\nu \phi^b \, \tilde{f}_{ab}$.
Then, splitting the degrees of freedom of the ϕ^a as $\phi^a = x^a - \frac{1}{M_{Pl}} \chi^a$ and choosing exactly the flat metric η_{ab} as reference metric f_{ab} leads to a $\eta_{\mu\nu}$ piece plus first order corrections to flat spacetime. χ^a can be further split into a vectorial part and a scalar contribution as in (2.20) (here $\Pi_{\mu\nu}$ is used as a short-hand notation for $\Pi_{\mu\nu} := \partial_\mu \partial_\nu \pi$):

$$\begin{aligned} f_{\mu\nu} &= \eta_{\mu\nu} - \frac{2}{M_{Pl}} \partial_{(\mu} \chi_{\nu)} + \frac{1}{M_{Pl}^2} \partial_\mu \chi^a \partial_\nu \chi^b \eta_{ab} \\ &= \eta_{\mu\nu} - \frac{2}{M_{Pl} m} \partial_{(\mu} A_{\nu)} - \frac{2\Pi_{\mu\nu}}{M_{Pl} m^2} + \frac{1}{M_{Pl}^2 m^2} \partial_\mu A^\alpha \partial_\nu A_\alpha + \frac{2}{M_{Pl}^2 m^3} \partial_\mu A^\alpha \Pi_{\nu\alpha} + \frac{1}{M_{Pl}^2 m^4} \Pi_{\mu\nu}^2; \end{aligned} \tag{2.21}$$

in this way the $h_{\mu\nu}$ of the linear theory has been promoted to a tensor, which is denoted by $H_{\mu\nu}$, for the full diffeomorphism invariant perturbed theory:

$$\begin{aligned} H_{\mu\nu} &= M_{Pl} \left(g_{\mu\nu} - f_{\mu\nu} \right) \\ &= h_{\mu\nu} + \frac{2}{m} \partial_{(\mu} A_{\nu)} + \frac{2}{m^2} \Pi_{\mu\nu} - \frac{1}{M_{Pl} m^2} \partial_\mu A^\alpha \partial_\nu A_\alpha - \frac{2}{M_{Pl} m^3} \partial_\mu A^\alpha \Pi_{\nu\alpha} - \frac{1}{M_{Pl} m^4} \Pi_{\mu\nu}^2 . \end{aligned}$$

Then the linearised diffeomorphism invariant massive lagrangian $\mathcal{L}_{mass}$ is promoted to a lagrangian for massive gravity which respects full diffeomorphism invariance by multiplying, in accordance with the minimal principle of covariance, the square root of minus the determinant of g, $\sqrt{-g}$, per a Fierz-Pauli term, in which now the indices are raised and lowered with the full metric $g^{\mu\nu}$. But following the works published on this subject, for example [3], it has been conventionally chosen to define a useful tensor $\mathbb{X}^\mu{}_\nu = g^{\mu\alpha} f_{\alpha\nu}$ and to express the theory in term of this tensor, considering also the following non-linear extension of the Fierz-Pauli mass term (2.3), as one of the simplest possible one (squared parenthesis meaning the trace of their argument):

$$\mathcal{L}_{mass} = -m^2 M_{Pl}^2 \sqrt{-g} \left(\left[(\mathbb{I} - \mathbb{X})^2 \right] - [\mathbb{I} - \mathbb{X}]^2 \right) . \tag{2.22}$$

In fact, it is $(\delta^\mu_\nu - \mathbb{X}^\mu_\nu) = \frac{1}{M_{Pl}} g^{\mu\rho} H_{\rho\nu}$, so (2.22) corresponds exactly to (2.3) in the non-linear case. Putting $\mathbb{X}^\mu_\nu = h^\mu_\nu$ in (2.22) leads, in fact, to:

$$\mathcal{L}_{mass} = -m^2 M_{Pl}^2 \left(16 - 8h + h^\mu{}_\alpha h^\alpha{}_\mu - 16 + 8h - h^2 \right) = -m^2 M_{Pl}^2 \left(h_\mu{}^\alpha h_\alpha{}^\mu - h^2 \right) ,$$

a quadratic lagrangian for the difference between $h_{\mu\nu}^2$ and the square of its trace, exactly as (2.3).

The linear Fierz-Pauli action for a massive theory of Gravity can be extended non-linearly in many other arbitrary ways. A very similar expression, which is not equivalent to (2.22) plus total derivative contributions, has been found by Boulware and Deser in [8] and it is presented here as an example of another mass term:

$$\mathcal{L}^{\star}_{mass} = -m^2 M^2_{Pl} \sqrt{-\mathrm{g}}\sqrt{\det \mathbb{X}} \left(\left[\left(\delta^\mu_\nu - (\mathbb{X}^\mu_\nu)^{-1} \right)^2 \right] - \left[\delta^\mu_\nu - (\mathbb{X}^\mu_\nu)^{-1} \right]^2 \right) . \tag{2.23}$$

Unfortunately, most of the generalizations of the mass term built in this way reveal the presence of the Boulware-Deser ghost.
Referring to (2.22) to be more specific, we will prove that the ghost appearance really takes place by analysing the behaviour of the helicity-0 mode π. In fact, neglecting all the other contributions, in the sense that the Stückelberg ξ^a is not given by $\xi^a = \frac{2A^a}{m} + \frac{\partial^a \pi}{m^2}$ but it is only $\xi^a = \frac{\partial^a \pi}{m^2}$, then $\mathbb{X}^\mu{}_\nu$ takes the form

$$\mathbb{X}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{2}{M_{Pl} m^2} \Pi^\mu{}_\nu + \frac{1}{M^2_{Pl} m^4} \Pi^\mu{}_\alpha \Pi^\alpha{}_\nu \, ,$$

so that (2.22) is now:

$$\begin{aligned} \mathcal{L}_{mass} = & -\frac{4\sqrt{-\mathrm{g}}}{m^2} \left([\Pi^2] - [\Pi]^2 \right) \\ & + \frac{4\sqrt{-\mathrm{g}}}{M_{Pl} m^4} \left([\Pi^3] - [\Pi]\,[\Pi^2] \right) - \frac{\sqrt{-\mathrm{g}}}{M^2_{Pl} m^6} \left([\Pi^4] - [\Pi^2]^2 \right) \end{aligned} \tag{2.24}$$

in which integration by parts makes the quadratic term a total derivative, but it does not help much with the quartic and cubic interactions. These bring up additional degrees of freedom which, due to Ostrogradsky theorem, always enter as ghosts, therefore one should look for different consistent non-linear extensions of massive gravity.

Before passing to the description of a consistent theory of massive gravity, let us point out that a theory for gravity in 5 dimensions, with the extra dimension compactified into a circle, in 4 dimensions may reproduce a theory of massive gravity. In fact, in 5 dimensions the rank-2 tensor of the massless theory has exactly the right number of degrees of freedom to describe a massive graviton in 4 dimensions. The way to get to this result is a kind of Kaluza-Klein dimensional reduction procedure.
The most important thing to notice is that the theory of Massive Gravity obtained in this way is a ghost free Massive Gravity, as it can be proved [15].

In the following chapter we will give instead the simplest and to our knowledge the most natural way of constructing the non-linear ghost-free generalization of the mass term, which instead of the metric will use the vielbein formalism. Now let us only note that a solution to the fundamental problem of ghosts was found only quite recently [21], in a somewhat brut force way constructing the interaction terms order by order in powers of the gravitation field, in such a way that all the higher derivative operators involving the helicity-0 mode $(\partial^2\pi)^n$ are total derivatives.

It has been shown that:

$$\mathcal{L}_{mass} = -m^2 M_{Pl}^2 \sqrt{-\mathrm{g}} \left(\left[\left(\mathbb{I} - \sqrt{g^{-1} f} \right)^2 \right] - \left[\mathbb{I} - \sqrt{g^{-1} f} \right]^2 \right) \tag{2.25}$$

is exactly the correct non-linear generalization of the mass term free of the ghosts. However, as anticipated, there is a more natural way to see that this is the desired non-linear generalization of the mass term, and it relies on the vielbein formulation of Gravity.

In the following section, after a quick translation of Einstein General Relativity from metric-based theory into a vielbein-based one, the self-interactions leading to a mass for the graviton will be built with those objects rather than $g_{\mu\nu}$ as in (2.25) and the result of the absence of the ghost degree of freedom is derived without too much difficulty.

Chapter 3

Consistent massive gravity

1 Vielbein formulation of General Relativity

The most straightforward way to get the consistent theory of Massive Gravity is to use the vielbein formulation of gravity. Vielbeins are a basis of one forms, $e^a_{\ \mu} dx^\mu$, which can make any generic metric flat:

$$g_{\mu\nu} = e^a_{\ \mu} e^b_{\ \nu} \eta_{ab} \,. \tag{3.1}$$

These one forms may be seen as a linear map from the tangent space of a Riemann manifold to Minkowski space, preserving the inner product.
An important identity which relates the determinants of $g_{\mu\nu}$ and $e^a_{\ \rho}$ and comes directly from (3.1) is extracted here because it will be used quite often in what follows.
Choosing the sign of the Levi-Civita symbol ϵ_{0123} as the negative one, so that $\epsilon^{0123} = 1$, then one important relation arises easily from the previous definition: if e stands for the determinant, i.e. $\mathrm{e} \equiv -\epsilon_{abcd}\epsilon^{\mu\nu\rho\sigma}\ e^a_{\ \mu} e^b_{\ \nu} e^c_{\ \rho} e^d_{\ \sigma}$, then in the tetrad basis g, the determinant of $g_{\mu\nu}$, and e are related by $\mathrm{g} = \mathrm{e}^2 \cdot (-1)$.
Notice that, moreover, the components of the one forms $e^a_{\ \mu}$ are the elements of a matrix which allows the passage from "curved" to "flat" indices, and viceversa for $e^\nu_{\ b}$.
For example:

$$\begin{aligned} V_\mu &\to V_a = e_a^{\ \mu}\, V_\mu \\ V_a &\to V_\mu = e_\mu^{\ a}\, V_a \,, \end{aligned}$$

where V_μ is a vector field.
This formalism is also known as Einstein-Cartan formalism.

In order to construct a minimal theory of ghost-free Massive Gravity formulated with the vielbeins, the first step is to convert the basic quantities of General Relativity, such as the connection, the torsion and the curvature from the usual metric formulation to the vielbein language. Moreover all the classical theory of Gravity, i.e. the Hilbert-Einstein action and its associated equations of motion should be revisited before taking into account any suggestion for

the interacting terms which give mass to the graviton.
With the vielbein one-forms, the equivalent notion of covariant derivative depends, now, on a particular one-form $\omega_{\mu}^{ab}\,\mathrm{d}x^{\mu}$, the so-called spin connection, which is the counterpart of the Christoffel symbols for the covariant derivative obtained from $g_{\mu\nu}$. Then in the Cartan formalism, neglecting from now on the wedge product $\wedge$ between forms, the covariant derivative acting on the vielbein itself is

$$\mathcal{D}e^a = \mathrm{d}e^a + \omega^a_{\ b} e^b .$$

The torsion tensor, now, is defined as

$$T^a := \mathcal{D}e^a . \tag{3.2}$$

Also the Riemann tensor $\mathrm{R}_{\mu\nu}{}^{\rho}{}_{\sigma}$, the Ricci tensor $\mathrm{R}_{\nu\sigma} := \mathrm{R}_{\mu\nu}{}^{\rho}{}_{\sigma}\, g_{\rho\alpha} g^{\alpha\mu}$ and the Ricci scalar $\mathrm{R} := \mathrm{R}_{\mu\nu} g^{\mu\nu}$ are expressed according to this spin-connection and the vielbeins.
This can be done by noticing that the curvature two-form R^{ab}, where $\mathrm{R}^{ab} := \frac{1}{2}\mathrm{R}^{ab}_{\ \ \mu\nu}\,\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}$, and defined by

$$\mathcal{D}\left(\mathcal{D}e^a\right) := \mathrm{R}^a_{\ b} e^b , \tag{3.3}$$

returns exactly $\mathcal{S}_{HE}$ of the classical Einstein theory for gravitation, as we will show after having developed expression (3.3) a bit. In fact, if one uses that applying two exterior derivatives on smooth functions gives zero, $\mathrm{d}\left(\mathrm{d}e^a\right) = 0$, then:

$$\begin{aligned}\mathcal{D}\left(\mathcal{D}e^a\right) &= \mathrm{d}^2 e^a + \mathrm{d}\omega^a_{\ b} e^b - \omega^a_{\ b}\mathrm{d}e^b\, \omega^a_{\ c}\mathrm{d}e^c + \omega^a_{\ c}\omega^c_{\ b} e^b \\ &= \mathrm{d}\omega^a_{\ b} e^b + \omega^a_{\ c}\omega^c_{\ b} e^b .\end{aligned}$$

That is,

$$\mathrm{R}^{ab} = \mathrm{d}\omega^{ab} + \omega^a_{\ c}\omega^{cb} . \tag{3.4}$$

Therefore Hilbert-Einstein action for Gravity, see equation (1.1), turns into a functional involving a scalar function for the curvature. In natural units it is:

$$\mathcal{S}_{HE} = \frac{1}{4}\int \epsilon_{abcd}\, \mathrm{R}^{ab} \wedge e^c \wedge e^d . \tag{3.5}$$

This action forms the basis of the description of the Hilbert-Einstein theory in the Einstein-Cartan formalism, and it is called Palatini action. The proof of the equivalence between Palatini action and Hilbert-Einstein action is given at the end of this section, as well as the one for the equations of motion arising from the metric or the vielbein formulation. Now we prefer to pay attention to the expressions assumed by minimizing the action and to their meaning.

First of all, it is possible to show that the condition of absence of torsion comes as a natural consequence of performing the variation with respect to ω^{ab} considered as an independent field.

In fact, varying the curvature two form (3.4) with respect to ω^{ab} gives:

$$\begin{aligned}
\delta\mathcal{S}_{HE} &= \epsilon_{abcd}\,\mathrm{d}\left(\delta\omega^{ab}\, e^c\, e^d\right) + \epsilon_{abcd}\,\delta\omega^{ab}\,\mathrm{d}\left(e^c\, e^d\right) + 2\delta\omega^{ab}\omega_{[b}^{\;\;f}\, e^c\, e^d\,\epsilon_{a]fcd} = 0 \\
\frac{\delta\mathcal{S}_{HE}}{\delta\omega^{ab}} &= 2\left(\epsilon_{abcd}\,(\mathrm{d}\,e^c)\; e^d + \omega_{[b}^{\;\;f}\, e^c\, e^d\,\epsilon_{a]fcd}\right) \\
\frac{\delta\mathcal{S}_{HE}}{\delta\omega^{ab}}\epsilon^{abik} &= \epsilon^{abik}\left(\epsilon_{abcd}\,\mathrm{d}e^c\, e^d + \omega_{[b}^{\;\;f}\, e^c\, e^d\,\epsilon_{a]fcd}\right) \\
&= -4\,\mathrm{d}e^{[i}\, e^{k]} - 2\cdot 3!\,\delta^{\,b}_{[f}\delta^{\,i}_{c}\delta^{\,k}_{d]}\,\omega_b^{\;\;f}\, e^c\, e^d \\
&= -4\left(\mathrm{d}e^{[i}\, e^{k]} + \omega_b^{\;\;[i\,|}\, e^b\, e^{|\,k]}\right) \\
&= 0\,.
\end{aligned}$$

This is exactly the condition for the absence of torsion (see definition of T^a in (3.2)):

$$T^i := \mathrm{d}e^i + \omega^i_{\;\;k}e^k = 0\,. \tag{3.6}$$

From (3.6) one can find the unique expression for ω in terms of e:

$$\omega^{ab}_\mu = \frac{1}{2}e^c_{\;\;\mu}\left(2e^{a\rho}e^{b\sigma}\partial_{[\rho}e_{\sigma]\,c} - 2e^\rho_{\;\;c}e^{a\sigma}\partial_{[\rho}e^b_{\;\;\sigma]} - 2e^{b\rho}e^\sigma_{\;\;c}\partial_{[\rho}e^a_{\;\;\sigma]}\right)\,. \tag{3.7}$$

Taking the variation with respect to e^d in (3.5) gives, instead, Einstein equations in the vielbein form:

$$\frac{\delta\mathcal{S}_{HE}}{\delta e^d} = \frac{1}{2}\epsilon_{abcd}\left(\mathrm{R}^{ab}\wedge e^c\right) = 0\,. \tag{3.8}$$

Now we would like to show that both the action and the equations of motion written for the classical Hilbert-Einstein theory of gravity with the vielbeins are completely equivalent to those of the theory written with the metric, as they are expected to be.

The fundamental thing which has to be noticed and used in the proofs that follow is that the Riemann tensor (1.4) and its contractions in the metric language are directly related to the expressions of these tensors in the Einstein-Cartan formalism. First of all, the Christoffel symbols $\Gamma_\mu{}^\rho{}_\nu(g)$, as stated in section 1, arise also from the condition of compatibility with the metric (1.2) which can be rewritten in the vielbein basis as:

$$\mathcal{D}_\mu\left(e^a_{\;\;\nu}e_{\rho a}\right) = \left(\mathcal{D}_\mu e^a_{\;\;\nu}\right)e_{\rho a} + e_\nu^{\;\;a}\left(\mathcal{D}_\mu e_{\rho a}\right) = 0\,,$$

and in the vielbein language the compatibility condition turns into:

$$\mathcal{D}_\mu e^a_{\;\;\nu} = \partial_\mu e^a_{\;\;\nu} + \Gamma_\mu{}^\rho{}_\nu e_\rho^{\;\;a} + \omega_\mu{}^a{}_b e^b_{\;\;\nu} = 0\,. \tag{3.9}$$

Thus, the Christoffel symbols depend on a derivative of the vielbein and on the spin connection:

$$\Gamma_\mu{}^\rho{}_\nu = -e^\rho_{\;\;a}\left(\partial_\mu e_\nu^{\;\;a} + \omega_\mu{}^a{}_b e^b_{\;\;\nu}\right)\,,$$

and this leads to the following equivalence between Riemann tensors, using expression (1.4) for the tensor written with the metric:

$$\begin{aligned}
\mathrm{R}_{\mu\nu,\,\rho}^{\;\;\;\;\;\;\sigma} &= -\,\partial_\mu\left(e^\sigma_{\;\;a}\left(\partial_\nu e_\rho^{\;\;a} + \omega_\nu{}^a{}_b e^b_{\;\;\rho}\right)\right) + \partial_\nu\left(e^\sigma_{\;\;a}\left(\partial_\mu e_\rho^{\;\;a} + \omega_\mu{}^a{}_b e^b_{\;\;\rho}\right)\right) \\
&\quad - e^\sigma_{\;\;a}\left(\partial_\mu e_\beta^{\;\;a} + \omega_\mu{}^a{}_b e^b_{\;\;\beta}\right)e^\beta_{\;\;c}\left(\partial_\nu e_\rho^{\;\;c} + \omega_\nu{}^c{}_d e^d_{\;\;\rho}\right) + \mu\leftrightarrow\nu \\
&= \left(\partial_\mu\omega_\nu{}^a{}_b - \partial_\nu\omega_\mu{}^a{}_b + \omega_{[\mu|}{}^a{}_c\,\omega_{|\nu]}{}^c{}_b\right)e^\sigma_{\;\;a}e^b_{\;\;\rho} = \mathrm{R}_{\mu\nu,\,b}^{\;\;\;\;\;\;a}\,e^\sigma_{\;\;a}e^b_{\;\;\rho}\,.
\end{aligned}$$

This means that the following relation holds:

$$\mathrm{R}_{\mu\nu,\,\rho}^{\ \ \ \sigma}(g) e_a^{\rho} e_\sigma^{b} \;=\; \mathrm{R}_{\mu\nu,\,a}^{\ \ \ b}(e)\,.$$

Now starting from $\mathcal{S}_{HE}$ in (3.5), let us give the proof of its equivalence with $\mathcal{S}_{HE}$ in (1.6) in natural units, $\mathcal{S}_{HE} \;=\; \int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\,\mathrm{R}[g]$.

$$\begin{aligned}
\frac{1}{4}\int \epsilon_{abcd}\mathrm{R}^{ab}\wedge e^c\wedge e^d \;&=\; \frac{1}{4}\int \frac{1}{2}\mathrm{R}^{ab}_{\ \ \mu\nu} e^c_{\ \rho} e^d_{\ \sigma}\,\epsilon_{abcd}\;\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\\
&=\; \frac{1}{4}\int \frac{1}{2}\mathrm{R}^{\alpha\beta}_{\ \ \mu\nu} e^a_{\ \alpha} e^b_{\ \beta} e^c_{\ \rho} e^d_{\ \sigma}\,\epsilon_{abcd}\;\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\\
&=\; -\frac{1}{4}\int\left(\frac{1}{2}\mathrm{R}^{\alpha\beta}_{\ \ \mu\nu}\epsilon_{\alpha\beta\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{e}\right)\mathrm{d}^4x\\
&=\; \int \mathrm{R}^{\alpha\beta}_{\ \ \mu\nu}\delta^\mu_{\ [\alpha}\delta^\nu_{\ \beta]}\,\sqrt{-\mathrm{g}}\,\mathrm{d}^4x\\
&=\; \int \sqrt{-\mathrm{g}}\,\mathrm{R}[g]\,\mathrm{d}^4x
\end{aligned}$$

The proof of the equivalence between the equations of motion (3.8), in the Einstein-Cartan formalism, and (1.1) without any coupling to a source follows here:

$$\begin{aligned}
\frac{1}{2}\epsilon_{abcd}\,\mathrm{R}^{ab}\,\wedge\,e^c \;&=\; \frac{1}{4}\mathrm{R}^{\rho\sigma}_{\ \ \mu\nu}\,e^a_{\ \rho}\,e^b_{\ \sigma}\,e^c_{\ \alpha}\,\epsilon_{abcd}\;\mathrm{d}x^\mu\;\wedge\;\mathrm{d}x^\nu\;\wedge\;\mathrm{d}x^\alpha\\
\frac{1}{2}\epsilon_{abcd}\,\mathrm{R}^{ab}\,\wedge\,e^c\wedge e^h \;&=\; \frac{1}{4}\epsilon_{abcd}\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{R}^{\rho\sigma}_{\ \ \mu\nu}\,e^a_{\ \rho}\,e^b_{\ \sigma}\,e^c_{\ \alpha}\,e^h_{\ \beta}\,\mathrm{d}^4x\\
&=\; -\mathrm{e}\cdot\frac{1}{4}\delta^h_{\ d}\,\epsilon^{\mu\nu\alpha\beta}\;\epsilon_{\rho\sigma\alpha\beta}\,\mathrm{R}^{\rho\sigma}_{\ \ \mu\nu}\,\mathrm{d}^4x\\
&=\; \mathrm{e}\cdot\frac{6}{4}\mathrm{R}^{\rho\sigma}_{\ \ \mu\nu}\,\delta^{[\mu}_{\ \rho}\delta^{\nu}_{\ \sigma}\delta^{\gamma]}_{\ \eta}\,e^h_{\ \gamma}e^\eta_{\ d}\,\mathrm{d}^4x\\
&=\; \frac{1}{2}\left(\mathrm{R}\,\delta^h_{\ d}\,+\,\mathrm{R}^{\nu\gamma}_{\ \ \mu\nu}\,e^h_{\ \gamma}e^\mu_{\ d}\,+\,\mathrm{R}^{\gamma\mu}_{\ \ \mu\nu}\,e^h_{\ \gamma}e^\nu_{\ d}\right)\sqrt{-\mathrm{g}}\,\mathrm{d}^4x\\
&=\; \left(\frac{1}{2}\mathrm{R}\,\delta^h_{\ d}\,-\,\mathrm{R}^h_{\ d}\right)\sqrt{-\mathrm{g}}\,\mathrm{d}^4x\,.
\end{aligned}$$

2 Massive terms in the vielbein formalism

The aim of this paragraph is to show that adding a specific mass contribution and other interacting terms to the Palatini action (3.5) respects all the requests for consistency. If f is the reference metric, $\mathrm{f}_{\mu\nu} \;=\; f^a_{\ \mu}\,f^b_{\ \nu}\,\eta_{ab}$, in which $f^a_{\ \mu}$ is a reference vielbein, taking into account also the possible linear interaction for f and a cosmological constant, the additional term which we are interested in is:

$$\mathcal{L}_{mass} \;=\; m^2\epsilon_{abcd}\left(c_2 f^a\wedge f^b\wedge e^c\wedge e^d + c_1 f^a\wedge e^b\wedge e^c\wedge e^d + c_0 e^a\wedge e^b\wedge e^c\wedge e^d\right)\,. \qquad (3.10)$$

It has to be checked whether this guess really gives a consistent Lagrangian in which $e^a_{\ \mu}$ propagates 5 degrees of freedom or not, and it is also important to verify if the massive lagrangian (3.10) corresponds to the metric lagrangian (2.25).

In order to give the proof for the absence of the sixth degree of freedom of the $e^a_{\ \mu}$ vielbein, it is enlightening to start from the extraction of the equations of motion from $\mathcal{L} = M^2_{Pl}\,(\mathcal{L}_{HE} + \mathcal{L}_{mass})$.
Taking the variation of $\mathcal{L}$ with respect to e gives:

$$\frac{\delta \mathcal{L}_{mass}}{\delta e^d} = \epsilon_{abcd}\left(m^2(2c_2 f^a \wedge f^b \wedge e^c + 3c_1 f^a \wedge e^b \wedge e^c + 4c_0 e^a \wedge e^b \wedge e^c)\right) = -\frac{\delta \mathcal{L}_{HE}}{\delta e^d}\,, \tag{3.11}$$

which, passing from the differential forms to their components, turns into:

$$\begin{aligned}\left(\frac{1}{2}\mathrm{R}\delta^h_{\ d} - \mathrm{R}^h_{\ d}\right) &= -m^2\,\left(\frac{c_2}{2}\left(f^a_{\ b} f^b_{\ a} - f^a_{\ a} f^b_{\ b}\right)\delta^h_{\ d}\right) \\ -m^2\left(\frac{c_2}{2}\left(f^a_{\ d} f^h_{\ a} - f^a_{\ a} f^h_{\ d}\right) \right.&+ \left.\frac{c_2}{2}\left(f^h_{\ b} f^b_{\ d} - f^h_{\ d} f^b_{\ b}\right) + \frac{3c_1}{2}\left(f^a_{\ a}\delta^h_{\ d} - f^h_{\ d}\right) + 6c_0\,\delta^h_{\ d}\right)\,,\end{aligned} \tag{3.12}$$

where $f^a_{\ b} := f^a_{\ \mu} e^\mu_{\ b}$.
In fact, to get the RHS of (3.12) in the above form the following passages should be performed (in the second line we have to multiply by $e^h_{\ \beta}$ as it has been done in order to obtain the LHS from (3.5), moreover a division by a factor of 4 has to be done for the same reason):

$$\begin{aligned}&m^2\,\epsilon_{abcd}\,\left(2c_2\,f^a \wedge f^b \wedge e^c + 3c_1\,f^a \wedge e^b \wedge e^c\,+\,4c_0\,e^a \wedge e^b \wedge e^c\right) \\ &= m^2\,\epsilon_{abcd}\left(2c_2 f^a_{\ k} f^b_{\ w} e^k_{\ \mu} e^w_{\ \nu} e^c_{\ \rho} e^h_{\ \beta} + 3c_1 f^a_{\ j} e^j_{\ \mu} e^b_{\ \nu} e^c_{\ \rho} e^h_{\ \beta} + 4c_0\,e^a_{\ \mu} e^b_{\ \nu} e^c_{\ \rho} e^h_{\ \beta}\right)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\beta \\ &= m^2\,\left(2c_2\,f^a_{\ k}\,f^b_{\ w}\,\epsilon_{abcd}\epsilon^{kwch}\,+\,3c_1\,f^a_{\ j}\,\epsilon_{abcd}\epsilon^{jbch}\,+\,4c_0\,\epsilon_{abcd}\epsilon^{abch}\right)\,\sqrt{-\mathrm{g}}\,\mathrm{d}^4x \\ &= -\,6m^2\left(2c_2 f^a_{\ k} f^b_{\ w}\delta^{[k}_{\ a}\delta^{w}_{\ b}\delta^{h]}_{\ d} + 2c_1 f^a_{\ j}\delta^{[j}_{\ a}\delta^{h]}_{\ d} + 4c_0\delta^h_{\ d}\right)\,\sqrt{-\mathrm{g}}\,\mathrm{d}^4x \\ &= -\,m^2\,\sqrt{-\mathrm{g}}\left(2c_2\left(f^a_{\ a} f^b_{\ b} - f^a_{\ b} f^b_{\ a}\right)\delta^h_{\ d} + 2c_2\left(f^a_{\ d} f^b_{\ a} - f^a_{\ a} f^b_{\ d}\right)\delta^h_{\ b} + 2c_2\left(f^a_{\ b} f^b_{\ d} - f^a_{\ d} f^b_{\ b}\right)\delta^h_{\ a}\right)\,\mathrm{d}^4x \\ &\quad -\,6m^2\,\sqrt{-\mathrm{g}}\left(c_1\,f^a_{\ a}\delta^h_{\ d} - c_1 f^a_{\ d}\delta^h_{\ a} + 4c_0\delta^h_{\ d}\right)\,\mathrm{d}^4x \\ &= -\,m^2\sqrt{-\mathrm{g}}\,2c_2\left(\left(f^a_{\ a} f^b_{\ b} - f^a_{\ b} f^b_{\ a}\right)\delta^h_{\ d} + \left(\left(2f^{[a}_{\ \nu} e^\nu_{\ d]} + 2f^{(a}_{\ \nu} e^\nu_{\ d)}\right)\left(2f^{[h}_{\ \mu} e^\mu_{\ a]} + 2f^{(h}_{\ \mu} e^\mu_{\ a)}\right)\right)\right)\mathrm{d}^4x \\ &\quad -\,m^2\sqrt{-\mathrm{g}}\,2c_2\left(-f^a_{\ a}\left(2f^{[h}_{\ \mu} e^\mu_{\ d]} + 2f^{(h}_{\ \mu} e^\mu_{\ d)}\right) + \left(2f^{[h}_{\ \nu} e^\nu_{\ b]} + 2f^{(h}_{\ \nu} e^\nu_{\ b)}\right)\left(2f^{[b}_{\ \mu} e^\mu_{\ d]} + 2f^{[b}_{\ \mu} e^\mu_{\ d]}\right)\right)\mathrm{d}^4x \\ &\quad -\,m^2\sqrt{-\mathrm{g}}\,\left(-2c_2 f^b_{\ b}\left(2f^{[h}_{\ \nu} e^\nu_{\ d]} + 2f^{(h}_{\ \nu} e^\nu_{\ d)}\right) + 6c_1\left(f^a_{\ a}\delta^h_{\ d} - 2f^{[h}_{\ \nu} e^\nu_{\ d]} - 2f^{(h}_{\ \nu} e^\nu_{\ d)}\right) + 24c_0\delta^h_{\ d}\right)\mathrm{d}^4x\,.\end{aligned}$$

In order to match with the symmetry under the exchange of the h and d indices, which is respected by the LHS of (3.12), one should require that

$$f^{[a}_{\ \ \mu}\,e^{\mu\,b]}\,=\,0\,. \tag{3.13}$$

This constraint reduces to 10 the initial 16 degrees of freedom of the $e^a_{\ \mu}$ vielbein.

Other 4 constraints arise due to the identity

$$\mathcal{D}^\mu\left(\,\mathrm{R}_{\mu\nu}\,-\,\frac{1}{2}\,\mathrm{R}g_{\mu\nu}\right)\,=\,0\,. \tag{3.14}$$

In fact, this holds because the Riemann tensor satisfies, by construction,

$$\mathcal{D}_\alpha \mathrm{R}_{\mu\nu\rho\sigma}\,+\,\mathcal{D}_\rho \mathrm{R}_{\mu\nu\sigma\alpha}\,+\,\mathcal{D}_\sigma \mathrm{R}_{\mu\nu\alpha\rho}\,=\,0\,. \tag{3.15}$$

Contracting the μ and ρ indices with $g^{\mu\rho}$ leads to $\mathcal{D}_\alpha \mathrm{R}_{\nu\sigma} + \mathcal{D}^\rho \mathrm{R}_{\rho\nu\sigma\alpha} - \mathcal{D}_\sigma \mathrm{R}_{\nu\alpha} = 0$; then, multiplying by $g^{\nu\sigma}$ one obtains $\mathcal{D}_\alpha \mathrm{R} - \mathcal{D}^\rho \mathrm{R}_{\rho\alpha} - \mathcal{D}^\nu \mathrm{R}_{\nu\alpha} = 0$, which is exactly (3.14).

Now, given that (3.15), which, written in the vielbein formulation, is simply $\mathcal{D}\mathrm{R}^{ab} = 0$, with $\mathcal{D} := \mathrm{d}x^\mu \, \mathcal{D}_\mu$, should hold, then also the covariant derivative on the RHS of (3.12) is null. Applying $\mathrm{d}x^\mu \, \mathcal{D}_\mu$ to the variation with respect to e^d of an interaction term built from one vielbein of the f type (i.e. when we put c_2 and c_0 in (3.12) equal to zero) $f^a \wedge e^b \wedge e^c$, a simpler example which does not lose any general validity and choosing for simplicity $f^a_{\ \mu} = \delta^a_{\ \mu}$, in this case:

$$\begin{aligned}
\mathrm{d}x^\mu \, \mathcal{D}_\mu \left(f^a \wedge e^b \wedge e^c \right) \epsilon_{abcd} &= \\
\mathrm{d}x^\mu \omega_\mu{}^a{}_k \, \delta^k_{\ \nu} \, \mathrm{d}x^\nu \wedge e^b \wedge e^c \, \epsilon_{abcd} &= \\
\mathrm{d}^4x \, \epsilon^{\mu\nu\rho\sigma} \omega_\mu{}^a{}_\nu \, e^b{}_\rho \, e^c{}_\sigma \, \epsilon_{abcd} &= \\
-\sqrt{-\mathrm{g}} \, \mathrm{d}^4x \, \delta^\mu_{\ [a} \, \delta^\nu_{\ d]} \, \omega_\mu{}^a{}_\nu &= \\
\sqrt{-\mathrm{g}} \, \mathrm{d}^4x \, \left(\omega_d{}^a{}_a - \omega_a{}^a{}_d \right) &= \\
\sqrt{-\mathrm{g}} \, \mathrm{d}^4x \, \left(\omega_\mu{}^a{}_d \, e^\mu_{\ a} \right) &= 0
\end{aligned}$$

We thus find that Bianchi identity leads to the 4 conditions of the form

$$\omega_\mu{}^{ab} \, e^\mu_{\ a} = 0 \, . \tag{3.16}$$

These 4 equations further reduce the number of physical degrees of freedom of $e^a_{\ \mu}$ from 10 to 6. Now using (3.16) in the scalar curvature coming from the trace of the LHS of equation (3.12), one gets:

$$\begin{aligned}
\mathrm{R} &= 2 \left(\partial_{[\mu} \, \omega_{\nu]}{}^{ab} + \omega_{[\mu}{}^a{}_c \, \omega^c{}_{\nu]}{}^b \right) e^\mu_{\ a} e^\nu_{\ b} \\
&= 2 \left(\partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} \right) e^\mu_{\ a} \, e^\nu_{\ b} - 2\omega_\nu{}^a{}_c \, \omega_\mu{}^{cb} \, e^\mu_{\ a} e^\nu_{\ b} \\
&= 4 \, \partial_\mu \left(\omega_\nu{}^{ab} \, e^\nu_{\ b} \right) e^\mu_{\ a} - 2\omega_\nu{}^{ab} \, \partial_\mu \, e^\nu_{\ b} e^\mu_{\ a} - 2\omega_\nu{}^a{}_c \, \omega_\mu{}^{cb} \, e^\mu_{\ a} e^\nu_{\ b} \\
&= 4 \, \partial_\mu \left(\omega_\nu{}^{ab} \, e^\nu_{\ b} \right) e^\mu_{\ a} - 2\omega_\nu{}^{ab} \left(\Gamma_\mu{}^\nu{}_\rho e^\rho_{\ b} - \omega_{\mu b}{}^c e_c{}^\nu \right) e^\mu_{\ a} - 2\omega_\nu{}^a{}_c \, \omega_\mu{}^{cb} \, e^\mu_{\ a} e^\nu_{\ b} \\
&= 4 \, \partial_\mu \left(\omega_\nu{}^{ab} \, e^\nu_{\ b} \right) e^\mu_{\ a} + 2\omega_\nu{}^{ab} \, \omega_{\mu b}{}^c \, e_c{}^\nu \, e_a{}^\mu - 2\omega_\nu{}^a{}_c \, \omega_\mu{}^{cb} \, e^\mu_{\ a} e^\nu_{\ b} \\
&= 4 \, \partial_\mu \left(\omega_\nu{}^{ab} e^\nu_{\ b} \right) e^\mu_{\ a} = 0 \, .
\end{aligned}$$

In the second and third equality condition (3.16) is used, while in the fourth line the absence of torsion allow to express the partial derivative via the connection ω and the Christoffel symbols. On the other hand the Christoffel symbol is symmetric in the μ, ρ indices while ω is antisymmetric, so the only term with both ω and Γ cancels due to this contraction between indices. Eventually in the second to last line it is sufficient to relabel $b \leftrightarrow c$ in the second term to match the third. In the last line again condition (3.16) is used, so we get that if constraint (3.16) holds then R is identically zero, and so should be the RHS of the trace of (3.12).

Thus the fact that from Bianchi identity in the special case of a deformation arising only from $f^a \wedge e^b \wedge e^c \wedge e^d \, \epsilon_{abcd}$ and choosing the vielbein $f^a_{\ \mu} = \delta^a_{\ \mu}$ the Ricci scalar is null has the important consequence on the trace of the contribution to the equations of motion of $\mathcal{L}_{mass}$, which should be zero as well, i.e. $e^d \wedge \left(\delta^a_{\ \nu} \mathrm{d}x^\nu \wedge e^b_{\ \rho} \mathrm{d}x^\rho \wedge e^c_{\ \sigma} \mathrm{d}x^\sigma \, \epsilon_{abcd}\right) = 0$. Developing the calculations:

$$\begin{aligned} e^d_{\ \mu}\delta^a_{\ \nu}e^b_{\ \rho}e^c_{\ \sigma}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \, \epsilon_{abcd} &= \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd}\, e^d_{\ \mu}\delta^a_{\ \nu}e^b_{\ \rho}e^c_{\ \sigma} \\ &= \mathrm{d}^4x\sqrt{-\mathrm{g}}\, \epsilon^{kdbc}\epsilon_{abcd}\, \epsilon_{\alpha\mu\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\, e_k^{\ \alpha}\delta^a_{\ \nu} \\ &= -\mathrm{d}^4x\sqrt{-\mathrm{g}}\, 3!3!\delta^k_{\ a}\delta_\alpha^{\ \nu}\, e_k^{\ \alpha}\delta_a^{\ \nu} \\ &= 0\,, \end{aligned}$$

hence we found the ultimate condition necessary to remove the ghost degree of freedom, as in [16]:

$$e^a_{\ a} = 0\,. \tag{3.17}$$

After having proved that this vielbein formulation of Massive Gravity is not troubled with a ghost, and propagates the genuine 5 degrees of freedom of the massive graviton, in this paragraph we will show that the lagrangian (3.10) with $c_2 = c_0 = \frac{1}{2}$, $c_1 = -\frac{1}{2}$ is equivalent to the metric lagrangian (2.25).
Leaving apart the cosmological constant $4!\, c_0 \sqrt{-\mathrm{g}}\, m^2$, we can rewrite the other two terms of (3.10) as follows:

$$\begin{aligned} & m^2\, \epsilon_{abcd} \left(c_2 f^a_{\ g} e^g \wedge f^b_{\ h} e^h \wedge e^c \wedge e^d \,+\, 2c_1 f^a_{\ i} e^i \wedge e^b \wedge e^c \wedge e^d\right) \\ =\ & m^2\, \epsilon_{abcd} \left(c_2\, f^a_{\ g}\, f^b_{\ h}\, e^g_{\ \mu}\, e^h_{\ \nu} e^c_{\ \rho}\, e^d_{\ \sigma}\, \epsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x \,+\, c_1\, f^a_{\ i}\, e^i_{\ \mu}\, e^b_{\ \nu}\, e^c_{\ \rho}\, e^d_{\ \sigma}\, \epsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x\right) \\ =\ & m^2\, \mathrm{e}\, \epsilon_{abcd} \left(\epsilon^{ghcd}\ c_2 f^a_{\ g}\, f^b_{\ h} \,+\, \epsilon^{ibcd}\ c_1 f^a_{\ i}\right) \mathrm{d}^4x \\ =\ & -m^2\, \mathrm{e} \left(4\delta^g_{\ [a}\delta^h_{\ b]}\ c_2\, f^a_{\ g}\, f^b_{\ h} \,+\, 6\delta^i_{\ a}\ c_1\, f^a_{\ i}\right) \mathrm{d}^4x \\ =\ & -m^2\, \mathrm{e} \left(2\, c_2 \left(f^a_{\ a}\, f^b_{\ b} \,-\, f^a_{\ b}\, f^b_{\ a}\right) \,+\, 6c_1\, f^a_{\ a}\right) \mathrm{d}^4x \\ =\ & -2m^2\sqrt{-\mathrm{g}} \left(c_2 \left(f^a_{\ \rho}\, e^\rho_{\ a}\, f^b_{\ \sigma}\, e^\sigma_{\ b} \,-\, f^a_{\ \nu}\, e^\nu_{\ b}\, f^b_{\ \mu}\, e^\mu_{\ a}\right) \,+\, 3c_1 \left(f^a_{\ \rho}\, e^\rho_{\ a}\right)\right) \\ =\ & -2m^2\sqrt{-\mathrm{g}} \left(c_2 \left[\left(\sqrt{g^{-1}\mathrm{f}}\right)\right]^2 \,-\, c_2 \left[\left(\left(\sqrt{g^{-1}\mathrm{f}}\right)^\mu_{\ \nu}\right)^2\right] \,+\, 3c_1 \left[\sqrt{g^{-1}\mathrm{f}}\right]\right)\,, \end{aligned}$$

in which the last expression with the $g_{\mu\nu}$ metric ($(g^{-1})_{\mu\nu} = g^{\mu\nu}$) and the reference metric f can be obtained taking into account the previously found condition (3.13). Moreover, in order to better understand how the second to the last line is related to the expressions with the traces of the square root of the metric g and f, it is useful to define $\mathbb{Y}^\mu_{\ \nu} := e^\mu_{\ a} f^a_{\ \rho}$. In fact, using (3.13), the following relations hold:

$$\begin{aligned} \left(g^{-1}\mathrm{f}\right)^\mu_{\ \nu} &= e_a^{\ \mu} e^{\rho a} f_\rho^{\ b} f_{\nu b} \\ &= e^\mu_{\ a} e^{\rho b} f_\rho^{\ a} f_{\nu b} \\ &= \mathbb{Y}^{\mu\rho}\mathbb{Y}_{\rho\nu} = (\mathbb{Y})^{2\,\mu}_{\ \ \ \nu}\,, \end{aligned}$$

so $f^a_{\ \rho} e^\rho_{\ a} f^b_{\ \sigma} e^\sigma_{\ b} = [\mathbb{Y}]^2 = \left[\sqrt{g^{-1}\mathrm{f}}\right]^2$, while $f^a_{\ \nu} e^\nu_{\ b} f^b_{\ \mu} e^\mu_{\ a} = [\mathbb{Y}^2] = \left[\left(\sqrt{g^{-1}\mathrm{f}}^\mu_{\ \nu}\right)^2\right]$, and the term with the c_1 coefficient gives $f^a_{\ \rho} e^\rho_{\ a} = [\mathbb{Y}] = \left[\sqrt{g^{-1}\mathrm{f}}\right]$.

Thus we have seen that Massive Gravity in the vielbeins formulation has a nicer expression for the interaction term than that of the metric formulation. It has been shown to be free from ghost degrees of freedom.
In the following chapter we will present another theory for gravitation, called Bigravity, involving one massless graviton in interaction with a massive one, in which, moreover, both metrics are dynamic. Bigravity and Massive Gravity are related by the fact that the latter is the consistent decoupling limit of the former.

Chapter 4

Bigravity

Bigravity is, as it will be shown, a theory of gravity with a massive graviton interacting with a massless one (5 + 2 degrees of freedom in total). A good reason to consider this modification of gravity is that in this theory the mass of the graviton can be large enough to fit both as a candidate to be a source for Dark Matter (see section 2 of chapter 6) and to agree with experimental results, which confirm Einstein General Relativity up to detection sensitivity.
In terms of the vielbeins e and f (which are now both dynamic), with associated Planck masses M_g and M_f respectively, the lagrangian for Bigravity is:

$$\begin{aligned}\mathcal{L}_{bigrav} = & \epsilon_{abcd}\left(M_g^2 \mathrm{R}_g^{ab} \wedge e^c \wedge e^d + M_\mathrm{f}^2 \mathrm{R}_\mathrm{f}^{ab} \wedge f^c \wedge f^d + M_g^2 m^2 \mathrm{c}\,(f^a - e^a) \wedge (f^b - e^b) \wedge e^c \wedge e^d\right) \\ & + \epsilon_{abcd}\left(M_\mathrm{f}^2 m^2 \mathrm{c}'\,(e^a - f^a) \wedge (e^b - f^b) \wedge f^c \wedge f^d\right).\end{aligned} \tag{4.1}$$

Compared to $\mathcal{L}_{mass}$ of Massive Gravity, (3.10), here $c_0 = \mathrm{c}$, $c_1 = -2\mathrm{c}$ and $c_2 = \frac{\left(M_g^2 \mathrm{c} + M_\mathrm{f}^2 \mathrm{c}'\right)}{M_g^2}$. Written in this way, (4.1) is symmetric under the interchange of the e and the f vielbein, if also the corresponding masses M_g and M_f are interchanged. Apart from this symmetry, other relevant invariances owned by the Bigravity action are local Lorentz invariance and diffeomorphism invariance (but only for diagonal diffeomorphisms which do not mix the vielbein types). The former is responsible to reduce the number of degrees of freedom of each vielbein from 16 to 10 (symmetric vielbeins), while the latter removes a total of $2 \cdot 4 = 8$ degrees of freedom, but it leaves us also with two canonical standard Bianchi identities (3.16) for both the vielbein types which has to satisfy a common relation, so that the final result is the removal of $8-4 = 4$ degrees of freedom. Then $10 + 10 - 8 - 4 = 8$, but it is possible to show that through a Hamiltonian construction known as ADM decomposition the last constraint can be found (see [21]), and the theory really describes a massive graviton with 5 degrees of freedom, and a massless one, which propagates as usual 2 degrees of freedom.

The term $\epsilon_{abcd}\left(f^a \wedge f^b \wedge f^c \wedge e^d\right)$ is the only one which has not been characterised yet, but it is known that it has to correspond to one of the contributions appearing in the series written in (4.2), but which can also make sense in Massive Gravity. It is, in fact:

$$
\begin{aligned}
&\left(f^a_{\ \mu}\mathrm{d}x^\mu \wedge f^b_{\ \nu}\mathrm{d}x^\nu \wedge f^c_{\ \rho}\mathrm{d}x^\rho \wedge e^d_{\ \sigma}\mathrm{d}x^\sigma\right)\epsilon_{abcd}\\
&=\epsilon_{abcd}\left(f^a_{\ g}f^b_{\ h}f^c_{\ l}e^g_{\ \mu}e^h_{\ \nu}e^l_{\ \rho}e^d_{\ \sigma}\right)\epsilon^{\mu\nu\rho\sigma}\,\mathrm{d}^4x\\
&=\mathrm{e}\,\epsilon^{ghld}\epsilon_{abcd}\left(f^a_{\ g}f^b_{\ h}f^c_{\ l}\right)\mathrm{d}^4x\\
&=-\sqrt{-\mathrm{g}}\left(f^a_{\ a}f^b_{\ b}f^c_{\ c}+f^a_{\ b}f^b_{\ c}f^c_{\ a}+f^a_{\ c}f^b_{\ a}f^c_{\ b}-f^a_{\ a}f^b_{\ c}f^c_{\ b}-f^a_{\ b}f^b_{\ a}f^c_{\ c}-f^a_{\ c}f^b_{\ b}f^c_{\ ;a}\right)\mathrm{d}^4x\\
&=-\sqrt{-\mathrm{g}}\left(\left[\sqrt{g^{-1}\mathrm{f}}\right]^3-3\left[\left(\sqrt{g^{-1}\mathrm{f}^\mu_{\ \nu}}\right)^2\right]\left[\sqrt{g^{-1}\mathrm{f}}\right]+2\left[\left(\sqrt{g^{-1}\mathrm{f}^\mu_{\ \nu}}\right)^3\right]\right)\mathrm{d}^4x\,.
\end{aligned}
$$

The generalization of Lagrangian (4.1) (plus the contribution of matter fields ψ) can be written in term of the metric $g_{\mu\nu}$ and $f_{\mu\nu}$ introducing five coefficients α_n, polynomials related to c and c′ in (4.1):

$$
\begin{aligned}
\mathcal{L}_{bigrav} = &\frac{M_g^2}{2}\sqrt{-\mathrm{g}}\,\mathrm{R}\,[g]+\frac{M_\mathrm{f}^2}{2}\sqrt{-\mathrm{f}}\,\mathrm{R}\,[\mathrm{f}]+\frac{1}{4}m^2M_g^2\sqrt{-\mathrm{g}}\left(\sum_{n=0}^{4}\alpha_n\mathcal{L}_n\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right]\right)\\
&+\sqrt{-\tilde{\mathrm{g}}}\mathcal{L}^{(matter)}_{\tilde{g}}\left(\tilde{g}_{\mu\nu},\,\psi_{\tilde{g}}\right)\,,
\end{aligned}
\tag{4.2}
$$

where a possible dynamics is given only to a strictly defined combination of the metrics ($\tilde{g}$) depending on only two parameters, a and b. In fact this coupling to matter is bounded, as usual, to not develop an Ostrogradsky instability and to not propagate a Boulware-Deser ghost, as it is explained for example in [24]. In that paper, the combination of $g_{\mu\nu}$ and $\mathrm{f}_{\mu\nu}$ is proved to be $\tilde{g}^\mu_{\ \nu} = a\,g_{\mu\rho}\left[(1+b\mathcal{K})\right]$, where $\mathcal{K}$ is exactly the tensor appearing in (4.2) and it is defined as $\mathcal{K}^\mu_{\ \nu}\left[g,\,\mathrm{f}\right] := \delta^\mu_{\ \nu} - \left(\sqrt{g^{-1}\mathrm{f}}\right)^\mu_{\ \nu}$. In the condensed expression in the sum, the $\mathcal{L}_n$'s are used in order to indicate polynomials built by contracting $d-n$ indices of two Levi-Civita tensors, the other n indices being contracted with $\mathcal{K}^\mu_{\ \nu}$, which are:

$$
\begin{cases}
\mathcal{L}_0\left[\mathcal{K}\right] = \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\beta} = -4!\\
\mathcal{L}_1\left[\mathcal{K}\right] = \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\prime\nu\alpha\beta}\,\mathcal{K}^{\mu\prime}_{\ \mu}\\
\mathcal{L}_2\left[\mathcal{K}\right] = \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\prime\nu\prime\alpha\beta}\,\mathcal{K}^{\mu\prime}_{\ \mu}\mathcal{K}^{\nu\prime}_{\ \nu}\\
\mathcal{L}_3\left[\mathcal{K}\right] = \epsilon^{\mu\nu\alpha\beta}\,\epsilon_{\mu\prime\nu\prime\alpha\prime\beta}\,\mathcal{K}^{\mu\prime}_{\ \mu}\mathcal{K}^{\nu\prime}_{\ \nu}\mathcal{K}^{\alpha\prime}_{\ \alpha}\\
\mathcal{L}_4\left[\mathcal{K}\right] = \epsilon^{\mu\nu\alpha\beta}\,\epsilon_{\mu\prime\nu\prime\alpha\prime\beta\prime}\,\mathcal{K}^{\mu\prime}_{\ \mu}\mathcal{K}^{\nu\prime}_{\ \nu}\mathcal{K}^{\alpha\prime}_{\ \alpha}\mathcal{K}^{\beta\prime}_{\ \beta}
\end{cases}
$$

The proposed metric lagrangian for Bigravity in (4.2), due to the coupling to matter fields, thus renounces to the exchange symmetry between the metrics $g_{\mu\nu}$ and $\mathrm{f}_{\mu\nu}$, privileging $g_{\mu\nu}$ with respect to the other. Moreover it has, hence, 5 parameters in addition to the masses M_g and M_f, while $\mathcal{L}_{mass}$ in (2.25) is obtained by a suitable choice of 3 coefficients plus a mass. This is not an inconsistency because the non-dynamical reference metric f in Massive Gravity could also produce an avoidable constant and a tadpole, which in fact are not taken into account.
Let us mention that the interaction terms in (4.2) written in the synthetic expression as a sum of polynomials can be used also in Massive Gravity, if one assumes $\mathrm{f}_{\mu\nu}$ to be a non-dynamical reference metric and allows the tadpole and the other interactions of higher order ($n = 3,\,4$).

Now, we would like to notice that (4.2) is truly describing two gravitons, one massive and the other massless, but, as it is, it involves a superposition of mass eigenstates. In order to see this, one has to work in the linear approximation, that is:

$$\begin{cases} g_{\mu\nu} = \eta_{\mu\nu} + \dfrac{1}{M_g}\delta g_{\mu\nu} \\ \mathrm{f}_{\mu\nu} = \eta_{\mu\nu} + \dfrac{1}{M_\mathrm{f}}\delta \mathrm{f}_{\mu\nu} \end{cases}$$

Let us consider only the quadratic order in the perturbation around flat spacetime, put the cosmological constants and tadpole terms to zero ($\alpha_0 = 0 = \alpha_1$), and set $\alpha_2 = -\frac{1}{2}$. Then it turns out that the following linear combination of $\delta g_{\mu\nu}$ and $\delta \mathrm{f}_{\mu\nu}$, called $h_{\mu\nu}$, is a massive mode of mass m_eff, and $l_{\mu\nu}$ is massless:

$$h_{\mu\nu} := M_\mathrm{eff}\left(\frac{1}{M_\mathrm{f}}\delta \mathrm{f}_{\mu\nu} - \frac{1}{M_g}\delta g_{\mu\nu}\right),$$
$$l_{\mu\nu} := M_\mathrm{eff}\left(\frac{1}{M_\mathrm{f}}\delta g_{\mu\nu} + \frac{1}{M_g}\delta \mathrm{f}_{\mu\nu}\right),$$

where

$$M_\mathrm{eff}^2 := \left(\frac{1}{M_g^2} + \frac{1}{M_\mathrm{f}^2}\right)^{-1} \tag{4.3}$$

$$m_\mathrm{eff}^2 := m^2\frac{M_g^2}{M_\mathrm{eff}^2} = m^2\frac{(M_\mathrm{f}^2 + M_g^2)}{M_\mathrm{f}}. \tag{4.4}$$

Then the linearised action for Bigravity is:

$$\begin{aligned}
\mathcal{S}^{(2)}_{bigrav} &= \int \mathrm{d}^4x \left[\frac{1}{8}\delta g^{\mu\nu}\left(\Box\delta g_{\mu\nu} - 2\partial_\alpha\partial_{(\mu}\delta g^\alpha_{\nu)} + \partial_\mu\partial_\nu\delta g - \eta_{\mu\nu}(\Box\delta g - \partial^\alpha\partial^\beta\delta g_{\alpha\beta}\right)\right] \\
&+ \int \mathrm{d}^4x \left[\frac{1}{8}\delta \mathrm{f}^{\mu\nu}\left(\Box\delta \mathrm{f}_{\mu\nu} - 2\partial_\alpha\partial_{(\mu}\delta \mathrm{f}^\alpha_{\nu)} + \partial_\mu\partial_\nu\delta \mathrm{f} - \eta_{\mu\nu}(\Box\delta \mathrm{f} - \partial^\alpha\partial^\beta\delta \mathrm{f}_{\alpha\beta}\right)\right] \\
&+ \int \mathrm{d}^4x \left[-\frac{1}{8}m_\mathrm{eff}^2\left(h_{\mu\nu}^2 - h^2\right)\right], \\
&= \int \mathrm{d}^4x \left[\frac{1}{8}h^{\mu\nu}\left(\Box h_{\mu\nu} - 2\partial_\alpha\partial_{(\mu}h^\alpha_{\nu)} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta})\right)\right] \\
&+ \int \mathrm{d}^4x \left[\frac{1}{8}l^{\mu\nu}\left(\Box l_{\mu\nu} - 2\partial_\alpha\partial_{(\mu}l^\alpha_{\nu)} + \partial_\mu\partial_\nu l - \eta_{\mu\nu}(\Box l - \partial_\alpha\partial_\beta l^{\alpha\beta})\right)\right] \\
&+ \int \mathrm{d}^4x \left[-\frac{1}{8}m_\mathrm{eff}^2 h^{\mu\nu}(\delta^\alpha_\mu\delta^\beta_\nu - \eta^{\alpha\beta}\eta_{\mu\nu})h_{\alpha\beta} - \frac{1}{\sqrt{M_g^2 + M_\mathrm{f}^2}}\left(M_\mathrm{f}h_{\mu\nu} - M_g l_{\mu\nu}\right)T^{\mu\nu}\right].
\end{aligned} \tag{4.5}$$

Thus both $h_{\mu\nu}$ and $l_{\mu\nu}$ possess, as kinetic term, the linearised Hilbert-Einstein action already found in chapter 1, arising from the linear approximation in the Ricci scalar. There is only one massive mode, $h_{\mu\nu}$, which shows a Fierz-Pauli mass term in the linearised limit, while $\delta\mathrm{f}_{\mu\nu}$, that corresponds exactly to the above combination of $h_{\mu\nu}$ and $l_{\mu\nu}$, couples to matter sources, $\mathrm{T}^{\mu\nu}$.

However in this coupling the massive mode $h_{\mu\nu}$ enters with a coefficient of $\frac{M_{\rm f}}{\sqrt{M_g^2+M_{\rm f}^2}}$, depending on $\frac{m}{m_{\rm eff}}$ in which m can be taken as small as one wishes, even zero in the decoupling limit of Bigravity. Thus the conclusion is that the massive mode has no interactions other than the gravitational ones, and this property is typical of Dark Matter.

Massive Gravity, to which Bigravity reduces when taking one of the two equivalent metric to be $\eta_{\mu\nu}$, has already been proved to be free from ghosts. In the next chapter this decoupling limit and the relevant interactions at the suitable scale are analysed in order to establish if the vDVZ discontinuity is effectively relevant at every energy scale.

Chapter 5

Vainshtein mechanism

The coupling of the scalar mode of the massive graviton to matter has been shown, in section 2, to generate a pathology in the behaviour of the massless limit of the massive linear theory, which does not reduce to the massless theory. What we are going to show now is that in the non-linear theory there is a screening effect that, in the vicinity of the matter, removes the effect of the scalar field. Before investigating it, we will describe the possible decoupling limits of the theory. These would become quite important because the critical issue of the vDVZ discontinuity, as already stated, is due to the couplings of the scalar mode π to matter, and it has to be proved that at the relevant scale the scalar couplings are appropriately modified.

1 Decoupling limits

In Massive Gravity the interactions of the helicity-2 mode $h_{\mu\nu}$ and the scalar mode π do not arise at the Planck scale, as one might expect, but at a lower scale given by a combination of the Planck scale and the graviton mass. Considering the potential term for Massive Gravity and focusing on the case in which the decoupled theory is built about Minkowski metric, i.e. $M_{Pl}^2 m^2 \sqrt{-\mathrm{g}}\,\mathcal{L}_n\left[\mathcal{K}\left[g,\,\eta\right]\right]$, then generic interactions between the canonically normalised modes can be characterized by the mass scales as in the following lagrangian:

$$\begin{aligned}\mathcal{L}_{j,k,l} &= m^2 M_{Pl}^2 \left(\frac{h}{M_{Pl}}\right)^j \left(\frac{\partial A}{m M_{Pl}}\right)^{2k} \left(\frac{\partial^2 \pi}{m^2 M_{Pl}}\right)^l = \\ &= \Lambda_{j,k,l}^{-4+(j+4k+3l)} h^j \left(\partial A\right)^{2k} \left(\partial^2 \pi\right)^l ,\end{aligned}$$

where j, k and $l \in \mathbb{N}$ and the scales $\Lambda_{j,k,l}$ are defined as:

$$\Lambda_{j,k,l} = \left(m^{2k+2l-2} M_{Pl}^{j+2k+l-2}\right)^{\frac{1}{(j+4k+3l-4)}} .$$

Then, the first interaction, involving a triple power of $\partial^2\pi$, might appear at $\Lambda_{j=0,\,k=0,\,l=3} = \left(M_{Pl} m^4\right)^{1/5} := \Lambda_5$, with $m < \Lambda_5 < M_{Pl}$. But we already know that the appearance of the second derivative of any field with respect to time means that the theory is afflicted by the

Ostrogradsky instability.
At Λ_5 scale and above, the dangerous contributions are approximated by:

$$\mathcal{L}_n \sim \frac{\left(\partial^2\pi\right)^n}{M_{Pl}^{n-2} m^{2(n-1)}}\,,$$

with n varying from 2 to 4.
However, in the consistent theory of Massive Gravity it turns out that the terms at each order are combined into total derivatives of the form:

$$\begin{aligned}
\mathcal{L}^{(2)}_{\text{der}} &= [\Pi]^2 - \left[\Pi^2\right] \\
\mathcal{L}^{(3)}_{\text{der}} &= [\Pi]^3 - 3\,[\Pi]\left[\Pi^2\right] + 2\left[\Pi^3\right] \\
\mathcal{L}^{(4)}_{\text{der}} &= [\Pi]^4 - 6\left[\Pi^2\right][\Pi]^2 + 8\left[\Pi^3\right][\Pi] + 3\left[\Pi^2\right]^2 - 6\left[\Pi^4\right]\,.
\end{aligned}$$

Therefore no interaction really appears at scale below Λ_3, which identifies the full set of interactions $\left(\partial^2\pi\right)^l$, with l varying from 3 to $+\infty$, and it is precisely $\Lambda_{j=0,\,k=0,\,l\to\infty} := \Lambda_3 = (M_{Pl}m^2)^{1/3}$. Switching on the helicity-2 and the helicity-1 modes, instead, will imply higher scales.

The decoupling limit of Massive Gravity (with standard Hilbert-Einstein lagrangian plus other interactions of type $\frac{1}{4}\,m^2 M_g^2\,\sqrt{-\mathrm{g}}\left(\sum_{n=0}^{4}\alpha_n\mathcal{L}_n\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right]\right)$) is consistently extracted by taking

$$m \to 0\,,\quad M_{Pl}\to\infty,\quad \Lambda_3 \text{ fixed}\,,$$

which implies that the Hilbert-Einstein term takes its linearised expression (1.10), while the other interacting terms become

$$\mathcal{L}^{\text{dec}-\Lambda_3}_{mass} = \frac{1}{8}h^{\mu\nu}\left(2\alpha_2\,X^{(1)}_{\mu\nu} + \frac{2\alpha_2+3\alpha_3}{\Lambda_3^3}X^{(2)}_{\mu\nu} + \frac{\alpha_3+4\alpha_4}{\Lambda_3^6}X^{(3)}_{\mu\nu}\right)\,, \tag{5.1}$$

where the correct normalization should be $\alpha_2 = 1$ and the $X^{(n)}$ are polynomials of Π:

$$\begin{cases}
X^{(1)\,\mu}_{\ \ \mu'}\,[\Pi] \;=\; \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu'\nu'\alpha\beta}\Pi^{\nu'}_{\ \nu} \\
X^{(2)\,\mu}_{\ \ \mu'}\,[\Pi] \;=\; \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu'\nu'\alpha'\beta}\Pi^{\alpha'}_{\ \alpha}\Pi^{\nu'}_{\ \nu} \\
X^{(3)\,\mu}_{\ \ \mu'}\,[\Pi] \;=\; \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu'\nu'\alpha'\beta'}\Pi^{\beta'}_{\ \beta}\Pi^{\alpha'}_{\ \alpha}\Pi^{\nu'}_{\ \nu} \\
X^{(4)\,\mu}_{\ \ \mu'}\,[\Pi] \;=\; 0\,.
\end{cases} \tag{5.2}$$

These expressions are transverse, i.e. $\partial^{\mu'}X^{(n)\,\mu}_{\ \ \mu'} = 0$, in fact, checking for $n=1$,

$$\partial^{\mu'}\left(-\delta^{\mu}_{\ \mu'}\Pi^{\nu}_{\ \nu'} + \Pi^{\mu}_{\ \mu'}\right) \;=\; -\partial^{\mu}\Box\pi \;+\; \partial^{\mu'}\partial^{\mu}\partial_{\mu'}\pi \;=\; 0\,,$$

therefore the Λ_3-decoupling limit of Massive Gravity is free of any ghost-like pathologies. Moreover, (5.1) can be further diagonalised with a suitable choice of $\tilde{h}_{\mu\nu}$: it is possible to prove, thus, that in this way one gets the Galileon Lagrangians $\mathcal{L}^{(n)}_{(\text{Gal})}\left[\pi\right]$ [12], which typically involves a $\left(\partial\pi\right)^2$ term multiplying, according to the n index, the usual contractions between two Levi-Civita tensors and $n-2$ Π tensors. Those terms will be fundamental in the discussion of the Vainshtein mechanism.

2 Vainshtein radius for static and spherically symmetric source

From the decoupling limit analysis it has been shown that π is responsible for interactions of the type $(\partial\pi)^2[\Pi]^n$ or $(\partial\pi)^2[\Pi^n]$, called galileons [26] because they enjoy the special global symmetry:

$$\pi \to \pi + c + v_\mu x^\mu\,,$$

where c and v_μ are constant parameters.
Hence it is quite meaningful and without loss of generality to work with

$$\mathcal{L}^{\rm dec} \;=\; -\frac{1}{2}(\partial\pi)^2 - \frac{1}{\Lambda_3^3}(\partial\pi)^2\Box\pi + \frac{1}{M_{Pl}}\pi\mathrm{T}\,. \tag{5.3}$$

The decoupling limit of Massive Gravity (5.1) resembles a galileon but presents also few peculiarities. First, the coupling $h^{\mu\nu}X^{(3)}_{\mu\nu}$, which can not be removed with a local field redefinition, could be proved to allow no stable static spherically symmetric configuration unless $\alpha_3+4\alpha_4=0$, a tuning which sets to zero this coupling term in (5.1). It is particularly deprecable that we are obliged to drop it out, but we are interested exactly in the configuration generated by approximately spherically symmetric massive objects, like the one created by the Sun in the Solar System.
Second, when the cubic galileon is present also the quartic galileon should necessarily be there and one can not prevent the appearance of a new coupling to matter, $\partial_\mu\pi\partial_\nu\pi\,\mathrm{T}^{\mu\nu}$, typically absent in other Galileon theories. Its effect is to bound the values of α, where $\alpha = -\left(1+\frac{3}{2\alpha_3}\right)$, to be strictly positive, but also to give as Vainshtein solution close to the source a cosmological solution which does not depend on the distance r in this regime.

Having discussed how the galileon lagrangian (5.3) arises in the decoupling limit of Massive Gravity, let us now analyse the Vainshtein mechanism in the case of a point like spherically symmetric and static source. Its stress-energy tensor is:

$$\mathrm{T}_0 \;=\; -M\delta^3(r) \;=\; -M\frac{\delta(r)}{4\pi r^2}\,. \tag{5.4}$$

Let us study the equations of motion for the scalar mode π_0 sourced by (5.4):

$$\begin{aligned}
0 \;&=\; \Box\pi_0 + \frac{2}{\Lambda_3^3}\partial^\mu(\partial_\mu\pi_0\Box\pi_0) - \frac{1}{\Lambda_3^3}\Box(\partial\pi_0)^2 + \frac{\mathrm{T}_0}{M_{Pl}} \\
&=\; \Box\pi_0 + \frac{2}{\Lambda_3^3}(\Box\pi_0)^2 - \frac{2}{\Lambda_3^3}(\partial_\mu\partial_\nu\pi_0)^2 + \frac{\mathrm{T}_0}{M_{Pl}} \\
&=\; \frac{1}{r^2}\partial_r\left[r^3\left(\frac{\pi_0'(r)}{r} + \frac{1}{\Lambda_3^3}\left(\frac{\pi_0'(r)}{r}\right)^2\right)\right] - \frac{M}{4\pi M_{Pl}}\frac{\delta(r)}{r^2} \\
&=\; r^3\left(\frac{\pi_0'(r)}{r} + \frac{1}{\Lambda_3^3}\left(\frac{\pi_0'(r)}{r}\right)^2\right) - \frac{M}{4\pi M_{Pl}}\,.
\end{aligned}$$

so if the Vainshtein (or strong coupling) radius $r_\star$ is defined as:

$$r_\star = \frac{1}{\Lambda_3}\left(\frac{M}{4\pi M_{Pl}}\right)^{1/3}\,, \tag{5.5}$$

the dominant terms in the equation of motion for π_0, taking into account the different regimes, are respectively:

$$\text{for}\, r >> r_\star, \quad \pi_0'(r) \sim \frac{M}{4\pi M_{Pl}}\frac{1}{r^2}$$
$$\text{for}\, r << r_\star, \quad \pi_0'(r) \sim \sqrt{\frac{M}{4\pi M_{Pl}}\frac{\Lambda_3^{3/2}}{\sqrt{r}}}\,.$$

so in the first case one recovers a Newton square law for the force mediated by π_0, which means that the theory with a mass can be detected only at distances larger than $r_\star$, and its linearised version works.
Conversely, close to the source, the ratio between the force exerted in the massive theory for gravitation, which is implemented by a sort of fifth force contribution from the scalar mode and corresponds to distances $r << r_\star$, and the Newtonian force of the standard General Relativity case (the newtonian approximation is valid because the scales are large enough to be sufficiently away from the massive body):

$$\frac{F^{(\pi)}_{r<<r_\star}}{F_{Newt}} \sim \left(\frac{r}{r_\star}\right)^{3/2} << 1\,;$$

when considering a quartic galileon coming from (5.1) this is further suppressed and goes as $\left(\frac{r}{r_\star}\right)^2$. For a graviton mass of the order of the Hubble parameter today H_0 (i.e. $\Lambda_3 = (1000\,km)^{-1}$, $m = \left(\Lambda_3 M_{Pl}^{-\frac{1}{3}}\right)^{\frac{3}{2}} \simeq 1.25\cdot 10^{-32}\,\mathrm{eV}$) then the gravitational force exerted by the Sun at the position of the Earth in a massive theory is suppressed, compared to standard Newtonian force, by 16 orders of magnitude in the quartic Galileon (12 orders in the case of the cubic Galileon). Thus, the extra force mediated by π is negligible and deviations from General Relativity are extremely small, but detectable, at least in the case of the cubic Galileon.

Now, let us consider fluctuations around the solution discussed above. Following the behaviour of the perturbation to the second order, $\pi = \pi_0 + \phi$ and $\mathrm{T} = \mathrm{T}_0 + \delta\mathrm{T}$, the lagrangian which we are interested in is

$$\mathcal{L}^{(2)} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{\Lambda_3^3}(\partial\phi)^2(\Box\pi_0) - \frac{2}{\Lambda_3^3}\partial_\nu\phi\partial^\nu\pi_0(\Box\phi) + \frac{1}{M_{Pl}}\phi\delta\mathrm{T}\,.$$

It can be checked that $\mathcal{L}^{(2)}$ corresponds to the following expression:

$$\begin{aligned}\mathcal{L}^{(2)} &= -\frac{1}{2}Z_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + \frac{1}{M_{Pl}}\pi\mathrm{T}\\ Z_{\mu\nu} &= \eta_{\mu\nu} + \frac{2}{\Lambda_3^3}\left(\Box\pi_0\eta_{\mu\nu} - \partial_\mu\partial_\nu\pi_0\right)\,.\end{aligned} \tag{5.6}$$

Even if $Z_{\mu\nu}$ does not behave like $Z_{\mu\nu} \sim Z\eta_{\mu\nu}$, the idea of canonically normalizing the fluctuations as $\hat{\phi} = \sqrt{Z}\phi$ means that the coupling of the fluctuations to matter can arise at a scale which is very different from the Planck scale, $M_{Pl}\sqrt{Z} >> M_{Pl}$, and this allows to conclude that the coupling to matter, which depends on the inverse of this quantity, is thus very suppressed.

$Z_{\mu\nu}$ for our case (5.6) has a peculiar property for small radii, $r << r_\star$, which becomes evident when performing the following calculation:

$$\begin{aligned}
Z_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu =& -\left(1+\frac{2}{\Lambda_3^3}\left(2\frac{\pi_0'(r)}{r}+\pi_0''(r)\right)\right)\mathrm{d}t^2+\left(1+4\frac{\pi_0'(r)}{r\Lambda_3^3}\right)\mathrm{d}r^2 \\
&+\left(1+\frac{2}{\Lambda_3^3}\left(\frac{\pi_0'(r)}{r}+\pi_0''(r)\right)\right)r^2\mathrm{d}\Omega^2 \\
Z_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu =& \,3\left(\frac{r_\star}{r}\right)^{3/2}\left(-\mathrm{d}t^2+\frac{4}{3}\mathrm{d}r^2+\frac{1}{3}r^2\mathrm{d}\Omega^2\right)+\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,.
\end{aligned}$$

The modes propagating along the radial direction have a superluminal phase and group velocity $c_r^2 = \frac{4}{3} > 1$. This fact has been a source of many questions in these kinds of gravity theories, which are related to Massive Gravity decoupling limit but do not exactly correspond to it, unless something further is required.

It is known that group velocity, which represents the speed at which the envelop of the signal propagates, could be superluminal without violating causality, because it is the wavefront velocity (high frequency limit of the phase velocity) that carries the information of the signal. Hence it is the behaviour of the latter which should be inquired in order to infer if superluminalities are really present.

In a Galileon theory of type

$$\mathcal{L} = \pi\sum_{n=1}^{4}\frac{c_{n+1}}{\Lambda_3^{3(n-1)}}\mathcal{L}_n(\Pi)\,, \tag{5.7}$$

where $\mathcal{L}_n$ are the appropriate contractions built from (5.2), taking π to be a plain wave in the x^1 direction plus small perturbation $\delta\pi$:

$$\pi_0(x^\mu) = F(x^1-t)+\delta\pi(x^\mu)\,, \tag{5.8}$$

it is possible to show that the perturbation travels with a velocity v

$$v = \frac{1-\dfrac{12c_3}{\Lambda_3^3}F''(x^1-t)}{1+\dfrac{12c_3}{\Lambda_3^3}F''(x^1-t)}\,.$$

This leads to a superluminal velocity in the case in which $12c_3F'' < -\Lambda_3^3$ for the "toy" model with lagrangian (5.7); however this is the group velocity and in order to infer something about the (a)causality one needs to calculate the front velocity, so one-loop corrections must be taken into account. So far this is yet to be determined [10].

In Massive Gravity, instead, a perturbative analysis on the static and spherically symmetric sourced solution would eventually prove that the modes along all the directions are subluminal and it is yet unclear if this result is due to the accidental specific case or a consistent property of Massive Gravity. Some massive gravity solutions, not related to the Vainshtein mechanism here presented, have been thought to admit superluminal propagation, but then they have been discarded because they were proved to describe only a confined region of space and time or to lie beyond the validity domain of the theory.

Another aspect of the Vainshtein mechanism which has to be discussed deeper concerns the graviton mass. In fact the Vainshtein screening effect relies on interactions which are important at a low energy scale $\Lambda << M_{Pl}$, quite lower than the typical scale of GR.
Moreover, the Galileon operators have mass dimension larger than 4, so in a traditional Effective Field Theory viewpoint these are irrelevant and hence the theory is non-renormalizable. But in order for the Vanishtein mechanism to be successful the irrelevant operator should dominate over the others, the so-called marginal operators in EFT, because only the couplings of type (5.1) are taken into account, so one may wonder whether or not it is possible to use the effective field description within the strong coupling region without going outside the regime of validity of the theory.
A non-renormalization theorem which assures that in a Galileon theory the Galileon operators do not get renormalized comes to our aid.
Hence it is possible to find a regime for the theory in which the operators generated by quantum corrections (which are not the ones obtained from a one-loop and higher loops analysis of the Galileons due to the non-renormalization theorem) are irrelevant and the leading interactions are then brought by the Galileons. This also implies that the scale Λ_3, which could not be renormalized, can be set to an arbitrary small value without running issues.
When beyond the decoupling limit Λ_3, instead, operators of the form $h^2\left(\partial^2\pi\right)^n$ are expected to spoil the non-renormalization theorem; however these operators are M_{Pl} suppressed, leading also to the suppression of the quantum corrections to the graviton mass:

$$\delta m^2 \lesssim m^2\left(\frac{m}{M_{Pl}}\right)^{\frac{2}{3}} .$$

In order for the massive theory to be viable, then, the graviton mass in Massive Gravity ought to be tuned to extremely small values.

Chapter 6

Cosmology

One of the main motivations for developing modified theories of gravity is to make a somewhat alternative insight into the nature of the main cosmological issues.
For instance, the modification of gravity by the introduction of a mass term is a correction to the behaviour of the metric tensor field in the infrared, which means that the late time acceleration of the Universe could be theoretically explained without assuming the existence of an alternative form of energy named Dark Energy.
There are two principal ways in which massive theories of gravity could be useful for addressing the cosmological constant problem. On the one hand, by weakening gravity in the infrared, they may weaken the sensitivity of the solutions which depends on already existing large cosmological constant. This is the idea behind screening or degravitating solutions.
On the other hand, a condensate of massive gravitons could act like a source for the present unnatural little cosmological constant, due to the small but technically natural value of the graviton mass. This is the idea of self-accelerating solution.
Bimetric theories, instead, possess desired features to give a self-contained explanation to Dark Matter, as we will discuss in section 2.
In section 3 the possibility for Massive (Bi-)Gravity to admit Schwarzschild solutions to spherically symmetric and static sources of matter is investigated and a discussion on the stability of those solutions follows.

1 Dark Energy from Massive Gravity

As previously stated at the beginning of this section, there are two technical ways which allow to relate massive gravitational theories to a cosmological constant: the idea of degravitating or screening solutions, in the decoupling limit, and the idea of self-acceleration due to a bunch of massive gravitons.

In the Λ_3-decoupling limit we know from the previous chapter (subsection 1) that the lagrangian for Massive Gravity coupled to an external source,

$$\mathcal{L}_{HE} + \mathcal{L}_{mass} + \frac{1}{M_{Pl}} h_{\mu\nu} T^{\mu\nu},$$

with $\mathcal{L}_{mass}$ given in (2.25), reduces to (5.1), which we rewrite in the following way:

$$\mathcal{L}[h_{\mu\nu}, \pi] = \mathcal{L}_{kin}^{\text{spin}-2} + h^{\mu\nu} \sum_{n=1}^{3} \frac{a_n}{\Lambda_3^{3(n-1)}} X_{\mu\nu}^{(n)} [\Pi] \,, \tag{6.1}$$

where the $X^{(n)}$ were defined in (5.2).
What we are going to show first is that some values of the scalar mode, chosen if we assume that the metric ought to be of a de Sitter type, prevent the geometry from being not flat. A de Sitter metric $g_{\mu\nu}^{\text{dS}}$ can be defined, in a Friedmann-Lemaître-Robinson-Walker (FLRW) context, by the condition that the Hubble constant, cosmological relevant parameter function of the scale factor $a(t)$ and its first time derivative, $H := \frac{\dot{a}}{a}$, is really constant in time. By FLRW context we mean a solution to Einstein equations representing an expanding Universe, with expansion regulated by the scale factor $a(t)$. If for c is meant the speed of light and for k we denote a parameter, taking values in the set $\{-1, 0, +1\}$, which represents the curvature of the space (closed, flat or opened space), in spherical coordinates the FLRW metric corresponds to:

$$\mathrm{d}s^2_{\text{FLRW}} = -c^2\mathrm{d}t^2 + a(t)^2 \left(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \right) .$$

Then we will also show that a self-accelerated solution in the decoupling limit exists even when the cosmological constant is absent, and depends on the value of the graviton mass.
We refer to [13] for the whole discussion.
The variations with respect to π and $h^{\mu\nu}$ of (6.1) are, respectively:

$$\begin{cases} 0 = a_1 \left(\Box h - \partial_\mu \partial_\nu h^{\mu\nu}\right) + 2\frac{a_2}{\Lambda_3^3} \left(\Box h \, \Box\pi + \partial_\mu \partial_\alpha h^{\mu\nu} \, \Pi^\alpha_{\ \nu} + \partial^\alpha \partial_\nu h^{\mu\nu} \Pi_{\alpha\nu} - \partial_\mu \partial_\nu h^{\mu\nu} \Box\pi - \Box h \Pi_{\mu\nu}\right) \\ \qquad - 2\frac{a_2}{\Lambda_3^3} \left(\partial^\alpha \partial^\beta h \Pi_{\alpha\beta}\right) + 3\frac{a_3}{\Lambda_3^6} \left(\partial_\alpha \partial_\beta h^{\mu\nu} \epsilon_\mu^{\ \alpha\rho\sigma} \epsilon_\nu^{\ \beta\gamma\delta} \Pi_{\rho\gamma} \Pi_{\sigma\delta}\right) \\ 0 = \frac{1}{4} \left(\Box h_{\mu\nu} - 2\partial_\alpha \partial_{(\mu} h^\alpha_{\ \nu)} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \left(\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta}\right)\right) + \sum_{n=1}^{3} \frac{a_n}{\Lambda_3^{3(n-1)}} X_{\mu\nu}^{(n)} [\Pi] + \frac{1}{M_{Pl}} T_{\mu\nu}. \end{cases} \tag{6.2}$$

Now, guessing a self-accelerated solution for this system of equations, the metric which we are interested in should only follow from an expansion around $|x| = \sqrt{\eta_{\mu\nu} x^\mu x^\nu}$ of a de Sitter type of metric with Hubble parameter H:

$$\mathrm{d}s^2_{\text{dS}} = e^{-\frac{H^2|x|^2}{2}} \eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \simeq \left[1 - \frac{1}{2}H^2|x|^2\right] \eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \,.$$

So, under this assumption, the ansatz for $h_{\mu\nu}$ is:

$$h_{\mu\nu} = -\frac{1}{2} M_{Pl} \, H^2 |x|^2 \eta_{\mu\nu} \,,$$

while the ansatz for the scalar mode π, bounded to respect the same symmetries as those respected by $h_{\mu\nu}$, i.e. isotropy and homogeneity, if q_0 is the current deceleration parameter $q_0 = \frac{\ddot{a}a}{\dot{a}^2}$, leads to:

$$\pi = \frac{1}{2} q_0 \Lambda_3^3 |x|^2 + c\Lambda_3 \,.$$

In order to be a source of dark energy, the stress-energy tensor $T_{\mu\nu}$ should be

$$T_{\mu\nu} = -\lambda \eta_{\mu\nu} \,, \quad \lambda > 0 \,.$$

Using the ansatz for $h_{\mu\nu}$, its free kinetic linear term is simply:

$$\frac{1}{4}\left(\Box h_{\mu\nu} - 2\partial_\alpha \partial_{(\mu} h^\alpha_{\ \nu)} + \partial_\mu \partial_\nu h - \eta_{\mu\nu}\left(\Box h - \partial_\alpha \partial_\beta h^{\alpha\beta}\right)\right) = \left(-1 + \frac{1}{2} - 1 + 4 - 1\right) H^2 \eta_{\mu\nu} = \frac{3}{2} H^2 \eta_{\mu\nu} \,.$$

The equations of motion for the helicity-0 and helicity-2 fields (6.2) can be recast now in the following form:

$$\begin{cases} \left(-\frac{a_1}{2} + 2a_2 q_0 + 3a_3 q_0^2\right) H^2 = 0 \\ 2q_0 \Lambda_3^3 \left(a_1 + a_2 q_0 + a_3 q_0^2\right) = M_{Pl} H^2 - \dfrac{\lambda}{3M_{Pl}} \,. \end{cases} \tag{6.3}$$

Hence the geometry can remain flat (i.e. $H = 0$) or approximately flat despite the presence of a cosmological constant. In fact when $H = 0$, then the first equation in (6.3) is trivially satisfied, while the second leads to

$$a_1 q_0 + a_2 q_0^2 + a_3 q_0^3 = -\frac{\lambda}{6\Lambda_3^3 M_{Pl}} \,,$$

which has always at least one real root when a_3 is present. These are the degravitating solutions, or the screening solutions.

In order to infer something about their stability, one has to consider small perturbations around the corresponding de Sitter background, denoted with a b superscript:

$$h_{\mu\nu} = h^b_{\mu\nu} + \chi_{\mu\nu} \,, \quad \pi = \pi^b + \frac{\phi}{2\left(a_1 + 2a_2 q_0 + 3a_3 q_0^2\right)} \,, \quad T_{\mu\nu} = -\lambda \eta_{\mu\nu} + \tau_{\mu\nu} \,.$$

The lagrangian for the perturbations is then:

$$\mathcal{L}_{\rm pert-degr} = \mathcal{L}^{\rm spin-2}_{kin}(\chi_{\mu\nu} - \phi \eta_{\mu\nu}) + \frac{3}{2} \phi \Box \phi + \frac{1}{M_{Pl}} \chi^{\mu\nu} \tau_{\mu\nu} \,. \tag{6.4}$$

The first two terms are the kinetic terms for χ and ϕ, while the third describes an interaction between χ and the first order perturbed stress-energy tensor.

The most interesting property of (6.4) is that the helicity-0 fluctuations ϕ decouples from matter sources.

When the kinetic term for ϕ is positive, then the solution is stable. This happens when

$$2\frac{a_1 + 2a_2 q_0 + 3a_3 q_0^2}{q_0 - 1} > 0 \,.$$

A second branch of solutions of de Sitter type (i.e. $H = \text{const} \neq 0$) is obtained from (6.3) and only exists if

$$a_2^2 \geq 3a_1 a_3 \,,$$

and its stability can be analysed as in the case of degravitating solutions by looking at fluctuations around the background configuration:

$$h_{\mu\nu} = h^b_{\mu\nu} + \chi_{\mu\nu} \,, \quad \pi = \pi^b + \phi \,, \quad T_{\mu\nu} = -\lambda \eta_{\mu\nu} + \tau_{\mu\nu} \,.$$

The resulting action is of the form:

$$\mathcal{L}_{\text{pert}-\text{self}-\text{acc}} = \mathcal{L}_{kin}^{\text{spin}-2}(\chi_{\mu\nu}) + \frac{6H_{\text{dS}}^2 M_{Pl}}{\Lambda_3^3}\,(a_2 + 3a_3 q_0)\,\phi\Box\phi + \frac{1}{M_{Pl}}\chi^{\mu\nu}\tau_{\mu\nu} \,, \tag{6.5}$$

so stability is assured when one of these conditions is achieved (setting $a_1 = -1/2$ and $\tilde{\lambda} := \frac{\lambda}{6M_{Pl}\Lambda_3^3}$):

$$\begin{cases} a_2 < 0 \text{ and } -\dfrac{2a_2^2}{3} \leq \dfrac{1 - 3a_2\tilde{\lambda} - (1 - 2a_2\tilde{\lambda})^{3/2}}{3\tilde{\lambda}^2} \,, \\ a_2 < \dfrac{1}{2\tilde{\lambda}} \text{ and } a_3 > \dfrac{1 - 3a_2\tilde{\lambda} + (1 - 2a_2\tilde{\lambda})^{3/2}}{3\tilde{\lambda}} \,, \\ a_2 < \dfrac{1}{2\tilde{\lambda}} \text{ and } a_3 > -\dfrac{2}{3}a_2^2 \,. \end{cases}$$

This means that there are self-accelerating solutions in which the Hubble constant has magnitude proportional to $\frac{\Lambda_3^3}{M_{Pl}} = m^2$, the graviton mass, even if $\lambda = 0$, i.e. no dark energy taken into account. In this sense a bunch of massive gravitons can give rise to accelerating solutions.
However it seems that, beside the fact that the range of the solutions is so wide that can accommodate a plenty of values for the cosmological constant, this range remains too small to significantly change the Old Cosmological Constant problem, i.e. the disagreement between measured values of the vacuum energy density (the small value of the cosmological constant) and its theoretical value suggested by quantum field theory [10].

2 Massive graviton mode as a Dark Matter candidate

In section 4 it has been shown that the bimetric theory propagates $5+2=7$ degrees of freedom, belonging respectively to a massive graviton $h_{\mu\nu}$ and a massless one $l_{\mu\nu}$. This is explicit when considering the quadratic order lagrangian in equation (4.5).
What we are going to study now, following [**?**], is whether the hypothesis that the massive graviton could be a suitable candidate for Dark Matter is consistent.
The relevant features for a good candidate able to take the role of Dark Matter are its very feeble interactions with Standard Model matter and its actual value, often given in terms of the product between the critical density for the Universe, $\Omega_{\text{DM},0}$, and h_0, which is defined as one percent of the actual value of the Hubble constant: $\Omega_{\text{DM},0}h_0^2 \simeq 0.26$. This abundance should follow from a specific model of production of Dark Matter during inflation.

Before starting the discussion, let us consider for the sake of simplicity only Bigravity backgrounds for which $g^0_{\mu\nu} = \mathrm{f}^0_{\mu\nu} = \bar{g}_{\mu\nu}$. Then the perturbations around this background are given as in section 4, with M_{eff} defined in (4.4):

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \frac{M_{\text{eff}}}{M_g}\left(\frac{1}{M_{\mathrm{f}}} l_{\mu\nu} - \frac{1}{M_g} h_{\mu\nu}\right) \\ \mathrm{f}_{\mu\nu} &= \bar{g}_{\mu\nu} + \frac{M_{\text{eff}}}{M_{\mathrm{f}}}\left(\frac{1}{M_g} l_{\mu\nu} + \frac{1}{M_{\mathrm{f}}} h_{\mu\nu}\right) . \end{aligned} \tag{6.6}$$

From now on, we will use $\alpha := \frac{M_{\mathrm{f}}}{M_g}$ instead of the ratio between the two masses. Moreover, M_g is related to the reduced Planck mass via

$$M^2_{Pl} = M^2_g \left(1 + \alpha^2\right) .$$

The zero-order bimetric vacuum equations, i.e. the equations obtained when $g_{\mu\nu} = \bar{g}_{\mu\nu} = \mathrm{f}_{\mu\nu}$, are:

$$\begin{cases} \mathrm{R}_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}\mathrm{R}(\bar{g}) + \Lambda_g \bar{g}_{\mu\nu} = 0 \\ \mathrm{R}_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}\mathrm{R}(\bar{g}) + \Lambda_{\mathrm{f}} \bar{g}_{\mu\nu} = 0 . \end{cases}$$

The system is solved only if $\Lambda_g = \Lambda_{\mathrm{f}}$, where the Λ's are defined by the following relations:

$$\begin{cases} \Lambda_g \bar{g}_{\mu\nu} = \left(-\frac{2M^2_g}{\sqrt{-\mathrm{g}}}\frac{\partial\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}]}{\partial g^{\mu\nu}}\right)\Bigg|_{g_{\mu\nu}=\bar{g}_{\mu\nu}} \\ \Lambda_f \bar{g}_{\mu\nu} = \left(-\frac{2M^2_g}{\alpha^2\sqrt{-\mathrm{f}}}\frac{\partial\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}]}{\partial \mathrm{f}^{\mu\nu}}\right)\Bigg|_{\mathrm{f}_{\mu\nu}=\bar{g}_{\mu\nu}} \end{cases} .$$

Expanding now the Bimetric Lagrangian (4.2) in the massive ($h_{\mu\nu}$) and massless mode ($l_{\mu\nu}$) as in (6.6), skipping mass-dependent coefficient therein because the final result will not be affected by them, i.e. $\bar{g}_{\mu\nu} \to \bar{g}_{\mu\nu} + l_{\mu\nu} + h_{\mu\nu} := G_{\mu\nu} + h_{\mu\nu}$, the potential becomes:

$$\begin{aligned} M^2_g\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}] \to M^2_g\left(\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}]\right)\Bigg|_{\mathrm{f}=g=G} &+ M^2_g\left[\frac{\partial\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}]}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial h^{\rho\sigma}}\right]\Bigg|_{\mathrm{f}=g=G} h^{\rho\sigma} \\ &+ M^2_g\left[\frac{\partial\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}]}{\partial \mathrm{f}^{\mu\nu}}\frac{\partial \mathrm{f}^{\mu\nu}}{\partial h^{\rho\sigma}}\right]\Bigg|_{\mathrm{f}=g=G} h^{\rho\sigma} . \end{aligned} \tag{6.7}$$

This expression can be reduced to a simpler form as follows. Bianchi identity for the Hilbert-Einstein contribution in (4.2), if $V_{\mu\nu}$ is used in place of

$$-\frac{2M^2_g}{\sqrt{-\mathrm{g}}}\frac{\partial(\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}])}{\partial g^{\mu\nu}} := V_{\mu\nu}$$

and $\tilde{V}_{\mu\nu}$ denotes the same quantity for the $\mathrm{f}_{\mu\nu}$ metric, implies that the following equality for the potential term should hold:

$$\sqrt{-\mathrm{g}}\, V^\rho_\sigma + \sqrt{-\mathrm{f}}\, \tilde{V}^\rho_\sigma = \sqrt{-\mathrm{g}}\, V \delta^\rho_\sigma , \tag{6.8}$$

where $V := V^{\mu}_{\mu}$.
It follows that the first piece of (6.7) is

$$M_g^2\sqrt{-\mathrm{G}}V \;=\; M_g^2\left(\Lambda_g+\alpha^2\Lambda_{\mathrm{f}}\right)\sqrt{-\mathrm{G}}\,.$$

Using now

$$\begin{cases} \left.\dfrac{\partial\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}]}{\partial g^{\mu\nu}}\right|_{\mathrm{f}=g=G} = \Lambda_g G_{\mu\nu}\left(-\dfrac{\sqrt{-\mathrm{G}}}{2M_g^2}\right)\,, & \dfrac{\partial g^{\mu\nu}}{\partial h^{\rho\sigma}} \;=\; \dfrac{\alpha}{M_{Pl}}2\delta^{\mu}_{(\rho}\delta^{\nu}_{\;\sigma)}\,,\\ \left.\dfrac{\partial\sqrt{-\mathrm{g}}\mathcal{L}[\sqrt{g^{-1}\mathrm{f}}]}{\partial \mathrm{f}^{\mu\nu}}\right|_{\mathrm{f}=g=G} = \Lambda_{\mathrm{f}} G_{\mu\nu}\left(-\dfrac{\sqrt{-\mathrm{G}}}{2M_g^2}\alpha^2\right)\,, & \dfrac{\partial \mathrm{f}^{\mu\nu}}{\partial h^{\rho\sigma}} \;=\; -\dfrac{1}{\alpha M_{Pl}}2\delta^{\mu}_{(\rho}\delta^{\nu}_{\;\sigma)}\,, \end{cases}$$

the second and the third piece in (6.7) can be written as:

$$-\frac{M_g^2\alpha}{M_{Pl}}\left(\frac{\sqrt{-\mathrm{G}}}{M_g^2}\right)\left(\Lambda_g-\Lambda_{\mathrm{f}}\right)G_{\mu\nu}h^{\mu\nu} \;=\; 0\,, \tag{6.9}$$

because $\Lambda_g \;=\; \Lambda_{\mathrm{f}}$.
If (6.9) was non-zero, it would imply that in perturbation theory for Bimetric Gravity the decay of the massive mode into the massless one can really take place. Conversely, at first order in perturbation the massive and the massless sector of the theory are independent from one another, and even a higher order expansion of the Bigravity potential leads to a very close result: a single massive mode can not decay to any arbitrary number of massless modes.
In fact a generic interacting term of type $\delta g^k\,\delta\mathrm{f}^n$, according to (6.6), can be parametrized as:

$$\begin{aligned} \delta g^k\,\delta\mathrm{f}^n &\sim \sum_{s=0}^{k}\sum_{r=0}^{n}\frac{\alpha^{s-r}}{M_{Pl}^{k+n}}\left(l^{k+n-s-r}h^{s+r}\right)\\ &\sim \sum_{s=0}^{k}\sum_{r=0}^{n}\frac{\alpha^{s-r}}{M_{Pl}^{k+n}}E^{k+n} \sim \frac{E^{k+n}}{M_{Pl}^{k+n}}\left(\alpha^{-n}+\alpha^{-n+1}+\,...\,+\alpha^{k}\right)\,, \end{aligned}$$

in which E is the field energy value chosen, each one of the metrics carrying, in dimensional analysis, the dimension of mass. In order for the perturbative approach to be consistent, E should be smaller than $\alpha\,M_{Pl}$, so if one parametrises the energy as $E := \alpha^{1+q}M_{Pl}\,,\ \alpha << 1\,,\ q\in\mathbb{R}_{>0}$, the Bigravity potential, taking into account that for $\alpha << 1$ the most enhanced vertices come from pure $\delta\mathrm{f}_{\mu\nu}$ terms, is approximated by:

$$\begin{aligned} V_{(m)} &\sim \sum_{k=0}^{m}\delta g^k\delta\mathrm{f}^{m-k} \sim \alpha^{qm}\left(1+\alpha+\alpha^2+...+\alpha^{2m}\right) \sim V^{(0)}_{(m)}+V^{(1)}_{(m)}+V^{(2)}_{(m)}+\,...\,+V^{(2m)}_{(m)}\\ V^{(j)}_{(m)} &:= V^{(0)}_{(m)}+V^{(1)}_{(m)}+V^{(2)}_{(m)}+\,...\,+V^{(2m)}_{(m)} \end{aligned} \tag{6.10}$$

Care must be taken in order to arrange the perturbative expansion with parameter α, in fact if we consider q given by the inverse of an integer p, $q=\frac{1}{p}$, then $V^{(j)}_{(m)} = V^{(0)}_{(m+jp)}$ because α appears $q\cdot m+j$ times.

The ultimate result of the analysis, due to the correct rearrangement of the expansion of the Bigravity potential in power of α, is that the $k = m - 1$ terms in (6.10), which include in particular the $h\,l^2$ and the $h\,l^3$ vertices, do not appear. Any interaction of the type of a decay of the massive graviton into massless gravitons is not allowed.
Instead, vertices which couple h and l for $m = 3, 4$ always exist; the coupling constants are collected in [?].

Referring to chapter 4 for its more specific analysis on Bigravity massive and massless mode coupling to Standard Modern matter, there it has been pointed out that $h_{\mu\nu}$ in the expression (4.5) couples very weakly to SM matter. The coupling coefficient can be rewritten as $\frac{m}{m_{\rm eff}}$, and this ratio can be very small because it involves the bare coefficient appearing in the lagrangian, a parameter of the theory, and the effective mass, the actually measured quantity.
In spite of this weak probability that massive gravitons and SM particles interact at first perturbative order, when passing to higher orders this process will happen more likely. To be precise, a claim is that, at tree level, perturbation theory makes possible for the Standard Model particles to decay into massive gravitons (Dark Matter).
In particular, this can be seen as an efficient production mechanism for Dark Matter, and it is a very good tip since any other proposed mechanism has not given the correct abundance so far.
In a canonical early Universe cosmology viewpoint, Universe is considered to be made of a primordial thermal bath in which different species of particles are in equilibrium and interact through annihilation and inverse annihilation processes. Abundances depend on the relativistic or non-relativistic state of the species when they stop to interact, at $\Gamma < H$, where Γ is the interaction rate, while H is the Hubble parameter. In fact the equilibrium distribution is suppressed by a Boltzmann factor $e^{-\frac{m}{T}}$ when $m > T$, i.e. when the particle ceases to be relativistic, so that if the distribution freezes-out later or earlier, abundance can respectively be damped or not.
In the DM production context, freeze-out can not even start: thermal equilibrium, $\Gamma := n_{\rm DM} < \sigma v >\sim \Omega_{\rm DM}\frac{H^2}{m_{\rm eff}} >> H$, where $m_{\rm eff}$ is defined in (4.4), is never achieved if $m_{\rm eff} >> H$.
However the Universe can be filled with Dark Matter supposing that a freeze-in mechanism takes place. For freeze-in mechanism it is meant that the produced particles interact so feebly that they do not ever reach thermal equilibrium with the bath. In fact, in this scenario, it happens that a pair of Standard Model particles SM, still in equilibrium with the thermal bath, can annihilate in two massive gravitons, exchanging either a massless graviton or a massive one, but the opposite reaction does not counterbalance the process because the heavy spin-2 boson remains below its thermal abundance.
Both the combinations of vertices, $SM\,SM \to l \to h\,h$ and $SM\,SM \to h \to h\,h$, has the same total strength equal to $\frac{1}{M_{Pl}}\frac{1}{M_{Pl}}$ because in the latter expression the α factors cancel (the first factor, from the $SM\,SM \to h$ vertex, is $\frac{\alpha}{M_{Pl}}$, while the coefficient for the $h \to h\,h$ vertex is $\frac{1}{\alpha M_{Pl}}$).
Then the thermalized cross section $< \sigma v >$ can be proved to go as $\frac{T^2}{M_{Pl}^4}$. An estimate of the total DM abundance in radiation domination can be obtained by matching the observed DM

abundance Ω_{DM} via freeze-in:

$$m_{\text{eff}} \approx \frac{\Omega_{DM} M_{Pl}^3}{\Omega_b T_{\text{RH}}^3} m_p \eta_b$$

where m_p is the proton mass, Ω_b the abundance of baryons, and $\eta_b \approx 10^{-9}$ the baryon asymmetry. Since the scale of inflation can not be too high in order to avoid overproduction of tensor modes (not observed in the CMB), this implies that the heavy spin-2 mass will be constrained to the range,

$$1\,\frac{\text{TeV}}{c^2} \lesssim m_{\text{eff}} \lesssim 10^{11}\,\frac{\text{GeV}}{c^2}\,.$$

A more stringent bound is obtained when analysing the possible decays for the massive graviton. In fact the opened decay channels of the massive gravitons, carrying no Standard Model quantum numbers, are those with $m_X \leq \frac{m_{\text{eff}}}{2}$ but with the channel $m_X = 0$ forbidden. This limit intersects the bound on its lifetime: Dark Matter has to be stable on cosmological timescales of $\tau = 13.8\,Gyr$, which leads to the upper bound $\alpha^{2/3}\,m_{\text{eff}} < 0.13\,\frac{\text{GeV}}{c^2}$.
Till $m_{\text{eff}} \approx 6.6 \cdot 10^6$ GeV, both the requirements are satisfied.
Now using the (non)observation of SM particle fluxes due to DM decay in different channels and the decay width dependence on α and m_{eff}, as well as the consistency of the perturbative approach only in the case $m_{\text{eff}} \leq \alpha M_{Pl}$, the combined use of the criteria for α and m_{eff} implies that the graviton mass, if measured, should lie in the interval

$$1\,\frac{\text{TeV}}{c^2} \lesssim m_{\text{eff}} \lesssim 66\,\frac{\text{TeV}}{c^2}\,.$$

This is enough to respect the actual values of DM in the Universe, produced in a freeze-in scenario, and also to avoid any violation of the requirements listed.
A quite narrow mass range for heavy spin-2 DM is one of the strongest predictions we could infer from the discussion: a measured DM mass within this narrow range would be a good indication in support of the model.

3 Black Holes solutions

Until now we have paid attention to Massive Gravity solutions in which $\mathrm{f}_{\mu\nu}$ and $g_{\mu\nu}$ are proportional to one another, seeking for an accelerating Universe or a Dark Matter description.
Other types of solutions, in particular the ones for a static and spherically symmetric configurations can be found in Bigravity and thus in its decoupling limit of Massive Gravity.
In the case of Einstein's General Relativity these solutions are called Schwarzschild metrics and correspond to the cosmological objects known as black holes of Schwarzschild radius r_{BH}, which is commonly twice the mass M of the spherical body divided per the square of the Planck mass, $r_{\text{BH}} = 2\frac{M}{M_{Pl}^2}$:

$$\mathrm{d}s_{\text{BH}}^2 \;=\; -\left(1-\frac{r_{\text{BH}}}{r}\right)\mathrm{d}t^2 + \left(\frac{1}{1-\frac{r_{\text{BH}}}{r}}\right)\mathrm{d}r^2 \;+ r^2\,\mathrm{d}\Omega^2 \tag{6.11}$$

This solution exhibits a horizon, which is a boundary within which the black hole's escape velocity is greater than the speed of light, as well as a singularity (a specific point in the chosen

set of coordinates at which the curvature is infinite) at $r = r_{\mathrm{BH}}$. Any other coordinate system would lead to this description at points labelled with those different coordinates.
When working with two metrics instead of one, these typical features of black holes do not show themselves as easily as in General Relativity. To get Schwarzschild solutions one should overcome some critical issues, such as the fact that it is not possible to make the starting ansatz that both $g_{\mu\nu}$ and $\mathrm{f}_{\mu\nu}$ are diagonal.
There are propositions which rule the properties of the second metric when the first one satisfies some specific requirements, as stated in [18]:

1) *When the killing vector $\epsilon = \partial_t$ for g, defined by the condition $\mathcal{D}(g)_{(\mu}\epsilon_{\nu)} = 0$ and in this case infinitesimal generator of the time-translations, is null at $r = r_H$, then, if both $g_{\mu\nu}$ and $\mathrm{f}_{\mu\nu}$ are diagonal and describe smooth geometries at r_H, ∂_t is also a killing vector for f, which shares the same translation invariance in the time dimension as $g_{\mu\nu}$.*

2) *Let ϵ be a killing vector for $g_{\mu\nu}$ and $\mathrm{f}_{\mu\nu}$ and let $g_{\mu\nu}$ have a killing horizon (which is the variety defined by the condition that the norm of the killing vector is null) with non-vanishing surface gravity k ($\epsilon^\mu \mathcal{D}_\mu \epsilon^\nu := k\epsilon^\nu$). Suppose further that both geometries are regular and possess the reflection isometry in the (t, ϕ) plane (i.e. they are stationary, which means that $g_{\mu t}$ and also $\mathrm{f}_{\mu t}$ components are null). Then the horizon is also a horizon for $\mathrm{f}_{\mu\nu}$.*

As will be obtained and justified in the following lines, these propositions imply the explanation of the particular form, for the solutions $g_{\mu\nu}$ and $\mathrm{f}_{\mu\nu}$ in the spherically symmetric and static case. Their meaning will be clarified once having showed that starting from non-bidiagonal metrics will necessarily lead to bidiagonal metrics and to a particular specific pair of them.

In a Bigravity theory the action, following the work of Volkov [30], can be given by:

$$\begin{aligned} \mathcal{S}_{bigr} =& \frac{1}{16\pi \mathrm{G}_g}\int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\,\mathrm{R}(g) + \frac{1}{16\pi \mathrm{G}_\mathrm{f}}\int \mathrm{d}^4x\,\sqrt{-\mathrm{f}}\,\mathrm{R}(\mathrm{f}) + \mathcal{S}_g^{(m)} + \mathcal{S}_\mathrm{f}^{(m)} \\ &- \frac{\mathrm{m}^2}{16\pi(\mathrm{G}_g+\mathrm{G}_\mathrm{f})}\int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\left(\sum_{n=0}^{4} b_n \mathcal{L}_n\left[\mathcal{K}\left[g,\mathrm{f}\right]\right]\right), \end{aligned} \tag{6.12}$$

where the $\mathcal{L}_n$ are the usual interaction terms for Bigravity as in section 4 but the b_n's coefficients now, unlike the α_n, leads to the following normalizations:

$$\begin{cases} \mathcal{L}_0\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= 1 \\ \mathcal{L}_1\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= \left[\sqrt{g^{-1}\mathrm{f}}\right] \\ \mathcal{L}_2\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= \frac{1}{2!}\left(\left[\sqrt{g^{-1}\mathrm{f}}\right]^2 - \left[\left(\sqrt{g^{-1}\mathrm{f}}\right)^2\right]\right) \\ \mathcal{L}_3\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= \frac{1}{3!}\left(\left[\sqrt{g^{-1}\mathrm{f}}\right]^3 - 3\left[\left(\sqrt{g^{-1}\mathrm{f}}\right)^2\right]\left[\sqrt{g^{-1}\mathrm{f}}\right] + 2\left[\left(\sqrt{g^{-1}\mathrm{f}}\right)\right]^3\right) \\ \mathcal{L}_4\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= \frac{1}{4!}\left(\left[\sqrt{g^{-1}\mathrm{f}}\right]^4 - 6\left[\left(\sqrt{g^{-1}\mathrm{f}}\right)^2\right]\left[\sqrt{g^{-1}\mathrm{f}}\right]^2 + 8\left[\sqrt{g^{-1}\mathrm{f}}\right]\left[\left(\sqrt{g^{-1}\mathrm{f}}\right)^3\right]\right) \\ &\quad + \frac{1}{4!}\left(3\left[\left(\sqrt{g^{-1}\mathrm{f}}\right)^2\right]^2 - 6\left[(\sqrt{g^{-1}\mathrm{f}})^4\right]\right)\,. \end{cases}$$

The equations of motion which come from the Bimetric theory with action (6.12), if $-\frac{2}{\sqrt{-\mathrm{g}}}\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{-\mathrm{g}}\sum_{n=1}^{4} b_n\mathcal{L}_n\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right]\right) := \tau_{\mu\nu}(g)$ and the similar term variated with respect to f is called $\tau_{\mu\nu}(\mathrm{f})$, are:

$$\begin{cases} \mathrm{R}(g)^{\mu}_{\ \nu} - \frac{1}{2}g^{\mu\rho}g_{\rho\nu}\,\mathrm{R}(g) - m^2\frac{\sqrt{G_g^2+G_\mathrm{f}^2}}{G_g}\tau^{\mu}_{\ \nu}(g) = \frac{2}{\sqrt{-\mathrm{g}}}\frac{\delta}{\delta g^{\rho\nu}}\left(\mathcal{S}_g^{(m)}\right)g^{\rho\mu} \\ \mathrm{R}(\mathrm{f})^{\mu}_{\ \nu} - \frac{1}{2}\mathrm{f}^{\mu\rho}\mathrm{f}_{\rho\nu}\,\mathrm{R}(\mathrm{f}) - m^2\frac{\sqrt{G_g^2+G_\mathrm{f}^2}}{G_\mathrm{f}}\tau^{\mu}_{\ \nu}(\mathrm{f}) = \frac{2}{\sqrt{-\mathrm{f}}}\frac{\delta}{\delta \mathrm{f}^{\rho\nu}}\left(\mathcal{S}_\mathrm{f}^{(m)}\right)\mathrm{f}^{\rho\mu} \end{cases} \tag{6.13}$$

Then it is possible to look for the static solutions with spherical symmetry beginning with a diagonal $g_{\mu\nu}$ whose spatial part is written in spherical coordinates, while for f one is forced to keep $\mathrm{f}_{0r} = \mathrm{f}_{r0} \neq 0$:

$$\begin{cases} \mathrm{f}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu &= -\left(aQ\,\mathrm{d}t + cN\,\mathrm{d}r\right)^2 + \left(cQ\mathrm{d}t - bN\mathrm{d}r\right)^2 + u^2R^2\mathrm{d}\Omega^2 \\ g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu &= -Q^2\,\mathrm{d}t^2 + N^2\,\mathrm{d}r^2 + R^2\,\mathrm{d}\Omega^2\,, \end{cases} \tag{6.14}$$

where Q, N, R, a, b, c and u are a priori functions of r and t.
With these initial metrics, $\sqrt{g^{-1}\mathrm{f}}$ becomes the matrix with elements:

$$\left(\sqrt{g^{-1}\mathrm{f}}\right)^{\mu}_{\ \nu} = \begin{bmatrix} a & \frac{cN}{Q} & 0 & 0 \\ -\frac{cQ}{N} & b & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix},$$

whose eigenvalues are $\lambda_{0,1} = \frac{1}{2}\left[a + b \pm \sqrt{(a-b)^2 - 4c^2}\right]$, $\lambda_2 = \lambda_3 = u$. This implies that, after having calculated the second, third and fourth power of $\sqrt{g^{-1}\mathrm{f}}$, the interaction term $\mathcal{U}$ can be explicitly found, by summing the eigenvalues, to be:

$$\begin{cases} \mathcal{L}_0\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= 1 \\ \mathcal{L}_1\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= 2u + a + b \\ \mathcal{L}_2\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= u^2 + ab + 2ub + 2au + c^2 \\ \mathcal{L}_3\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= u(au + bu + 2ab + 2c^2) \\ \mathcal{L}_4\left[\mathcal{K}\left[g,\,\mathrm{f}\right]\right] &= u^2(ab + c^2)\,. \end{cases}$$

In the expression of the equations of motion for $\mathcal{S}_{bigr}$ in (6.12) it is useful to notice that the variation with respect to $g_{\mu\nu}$ of the interaction, $\tau^{\mu}_{\nu}(g)$, can be written in the following form:

$$\begin{aligned} \tau^{\mu}_{\nu}(g) = &(b_1\mathcal{L}_0 + b_2\mathcal{L}_1 + b_3\mathcal{L}_2 + b_4\mathcal{L}_3)\sqrt{g^{\mu\alpha}\mathrm{f}_{\alpha\nu}} + (b_3\mathcal{L}_0 + b_4\mathcal{L}_1)(\sqrt{g^{-1}\mathrm{f}}^{\,3})^{\mu}_{\nu} \\ &- (b_2\mathcal{L}_0 + b_3\mathcal{L}_1 + b_4\mathcal{L}_2)(\sqrt{g^{-1}\mathrm{f}}^{\,2})^{\mu}_{\nu} - b_4\mathcal{L}_0(\sqrt{g^{-1}\mathrm{f}}^{\,4})^{\mu}_{\nu} - \sum_{n=0}^{4}\mathcal{L}_n\delta^{\mu}_{\nu}\,, \end{aligned} \tag{6.15}$$

and an analogous expression can be obtained also in the case of the variation with respect to $\mathrm{f}_{\mu\nu}$.

The $(t,\,r)$ component of $\tau(g)$, which is computed using the above expression (6.15), is found to be:

$$\tau^{t}_{r}(g) = \frac{cN}{Q}\left[b_1 + 2b_2u + b_3u^2\right]\,.$$

Correspondingly the $(r,\,t)$ component differs only by a constant factor and a sign:

$$\tau^{r}_{t}(g) = -\frac{cQ}{N}\left[b_1 + 2b_2u + b_3u^2\right]$$

But we do want to seek some specific static solutions, so the radial flux of energy should be null, and even these components of $\tau(g)^{\mu}_{\nu}$ tensor should manifest themselves, i.e.:

$$\tau^{t}_{r}(g) = 0 = \tau^{r}_{t}(g)\,. \tag{6.16}$$

Leaving apart the $c = 0$ branch which we are not interested in, this condition is fulfilled by requiring that:

$$b_1 + 2b_2u + b_3u^2 = 0 \tag{6.17}$$

which implies that u is actually a constant depending on the parameters of the theory rather than a function of r and t.

Another meaningful implication which is obtained directly from definition (6.15) imposing condition (6.17) is that the $(t,\,t)$ component of the $\tau(g)$ tensor in (6.15) is

$$\tau^{t}_{t}(g) = \tau^{r}_{r}(g) = -(b_0 + 2b_1u + b_2u^2) + b_4((a+b)^2(1-c^2))\,,$$

in which one should set $c = \pm 1$ in order to obtain, at the end, a tensor proportional to the identity tensor.

Moreover the Bianchi identity in the case in which $\mathcal{S}^{(m)}_{g}$ is null,

$$\partial_\rho\tau^{\rho}_{\sigma}(g) + \Gamma^{\ \alpha}_{\sigma\ \beta}\tau^{\beta}_{\alpha}(g) - \Gamma^{\ \alpha}_{\alpha\ \beta}\tau^{\beta}_{\sigma}(g) \;=\; 0\,, \tag{6.18}$$

reduces in this case to the requirement that $\tau^t_{\ t}(g) - \tau^\theta_{\ \theta}(g)$ should vanish.
In fact, the only non-zero Christoffel symbols for the g metric, if the dot stands for a derivative with respect to time, while $'$ stands for a derivative with respect to r, are:

$$\Gamma^{\ t}_{r\,r} = -\frac{N\dot{N}}{Q^2}\,, \qquad \Gamma^{\ t}_{t\,t} = \frac{\dot{Q}}{Q}\,, \qquad \Gamma^{\ t}_{r\,t} = \frac{Q'}{Q}\,, \qquad \Gamma^{\ t}_{\theta\,\theta} = \frac{R\dot{R}}{Q^2}\,,$$

$$\Gamma^{\ r}_{t\,t} = \frac{Q'Q}{N^2}\,, \qquad \Gamma^{\ r}_{r\,r} = \frac{N'}{N}\,, \qquad \Gamma^{\ r}_{t\,r} = \frac{\dot{N}}{N}\,, \qquad \Gamma^{\ r}_{\theta\,\theta} = -\frac{RR'}{N^2}\,,$$

$$\Gamma^{\ t}_{\phi\,\phi} = \frac{R\dot{R}\sin^2(\theta)}{Q^2}\,, \quad \Gamma^{\ r}_{\phi\,\phi} = \frac{RR'\sin^2(\theta)}{R^2}\,, \qquad \Gamma^{\ \theta}_{\theta\,t} = -\frac{\dot{R}}{R}\,, \qquad \Gamma^{\ \theta}_{\theta\,r} = \frac{R'}{R}\,,$$

$$\Gamma^{\ \phi}_{\phi\,t} = \frac{\dot{R}}{R}\,, \qquad \Gamma^{\ \phi}_{\phi\,r} = \frac{R'}{R}\,, \qquad \Gamma^{\ \phi}_{\phi\,\theta} = \cot(\theta)\,, \quad \Gamma^{\ \theta}_{\phi\,\phi} = -\frac{\sin(2\theta)}{4}\,,$$

The τ conservation, being $\tau^t_{\ r}(g) = \tau^r_{\ t}(g) = 0$, brings us to:

$$\begin{cases} 0 = \partial_t \tau^t_{\ t} + \Gamma^{\ r}_{t\,r}\tau^r_{\ r} + \Gamma^{\ \theta}_{t\,\theta}\tau^\theta_{\ \theta} + \Gamma^{\ \phi}_{t\,\phi}\tau^\phi_{\ \phi} - \Gamma^{\ r}_{t\,r}\tau^t_{\ t} - \Gamma^{\ \theta}_{t\,\theta}\tau^t_{\ t} - \Gamma^{\ \phi}_{t\,\phi}\tau^t_{\ t} \\ 0 = \partial_r \tau^r_{\ r} + \Gamma^{\ t}_{r\,t}\tau^t_{\ t} + \Gamma^{\ \theta}_{r\,\theta}\tau^\theta_{\ \theta} + \Gamma^{\ \phi}_{r\,\phi}\tau^\phi_{\ \phi} - \Gamma^{\ t}_{t\,r}\tau^r_{\ r} - \Gamma^{\ \theta}_{r\,\theta}\tau^r_{\ r} - \Gamma^{\ \phi}_{r\,\phi}\tau^r_{\ r} \\ 0 = \partial_\theta \tau^\theta_{\ \theta} + \Gamma^{\ \phi}_{\theta\,\phi}\tau^\phi_{\ \phi} - \Gamma^{\ \phi}_{\theta\,\phi}\tau^\theta_{\ \theta} \\ 0 = \partial_\phi \tau^\phi_{\ \phi}\,, \end{cases}$$

where it is implicit that we refer to $\tau^\mu_{\ \nu}(g)$ every time, so (g) is omitted.
Then, noticing also that $\tau^\theta_{\ \theta}$ has to be equal to $\tau^\phi_{\ \phi}$ because the $(\theta,\,\theta)$ and the $(\phi,\,\phi)$ components of $\sqrt{g^{-1}\mathrm{f}}$ are equal, and using $\tau^t_{\ t} = \tau^r_{\ r}$, the above set of equations is:

$$\begin{cases} 0 = \partial_t \tau^t_{\ t} + 2\frac{\dot{R}}{R}\tau^\theta_{\ \theta} - 2\frac{\dot{R}}{R}\tau^t_{\ t} \\ 0 = \partial_r \tau^t_{\ t} + 2\frac{R'}{R}\tau^\theta_{\ \theta} - 2\frac{R'}{R}\tau^t_{\ t} \\ 0 = \partial_\theta \tau^\theta_{\ \theta} = \partial_\phi \tau^\theta_{\ \theta}\,. \end{cases}$$

From the last line it is possible to deduce that $\tau^\theta_{\ \theta}$ is a function of t and r only, while for the first two equations to be satisfied simultaneously only $\tau^t_{\ t} = \tau^\theta_{\ \theta}$ is allowed.
On the other hand, by direct computation from (6.15), $\tau^\theta_{\ \theta}$ is:

$$\tau^\theta_{\ \theta} = -b_0 - b_1(a+u+b) - b_2(ab+ub+au+c^2) - b_3(aub+uc^2)\,,$$

and the difference $\tau^t_{\ t} - \tau^\theta_{\ \theta}$, imposing condition (6.17):

$$\tau^t_{\ t} - \tau^\theta_{\ \theta} = (b_2 + b_3 u)[(u^2 - ua - ub + ab + c^2]\,.$$

It can be proved that the constraint $b_2 + b_3 u = 0$ is too much limiting for the Bigravity theory, which ceases to be general enough in this case. Thus it is the choice

$$(u-a)(u-b) + c^2 = 0\,, \tag{6.19}$$

which has to be taken.
The final result is that $\tau(g)$ is a tensor proportional to the identity tensor, $\tau^{\mu}_{\ \nu}(g) = -(b_0+2b_1u+b_2u^2)\,\delta^{\mu}_{\ \nu}$.
These consequences, arising from condition (6.16), lead to the same conclusions for the $\tau^{\mu}_{\ \nu}(\mathrm{f})$ tensor: its $(t,\,t)$ component is equal to its $(r,\,r)$ component and it amounts to $-\frac{(b_2+2b_3u+b_4u^2)}{u^2}$.
Bianchi identity for the Riemann tensor of the $\mathrm{f}_{\mu\nu}$ metric gives the same result as in the g case, that is $\tau^{t}_{\ t} = \tau^{\theta}_{\ \theta}$.
The important outcome behind this is that, although one is forced to start with a non-diagonal ansatz metric $\mathrm{f}_{\mu\nu}$, then the equations of motion (6.13) for $\mathrm{f}_{\mu\nu}$ fully decouple from the $g_{\mu\nu}$ ones and, apart from a scaling, are completely analogous:

$$\begin{cases} \mathrm{R}(g)^{\mu}_{\ \nu} - \dfrac{1}{2}g^{\mu\rho}g_{\rho\nu}\,\mathrm{R}(g) + m^2\dfrac{\sqrt{G_g^2+G_\mathrm{f}^2}}{G_g}\,(b_0+2b_1u+b_2u^2)\,\delta^{\mu}_{\ \nu} = \dfrac{2}{\sqrt{-\mathrm{g}}}\dfrac{\delta}{\delta g^{\rho\nu}}\left(\mathcal{S}_g^{(m)}\right)g^{\rho\mu} \\ \mathrm{R}(\mathrm{f})^{\mu}_{\ \nu} - \dfrac{1}{2}\mathrm{f}^{\mu\rho}\mathrm{f}_{\rho\nu}\,\mathrm{R}(\mathrm{f}) + m^2\dfrac{\sqrt{G_g^2+G_\mathrm{f}^2}}{G_\mathrm{f}}\left(\dfrac{b_2+2b_3u+b_4u^2}{u^2}\right)\delta^{\mu}_{\ \nu} = \dfrac{2}{\sqrt{-\mathrm{f}}}\dfrac{\delta}{\delta \mathrm{f}^{\rho\nu}}\left(\mathcal{S}_\mathrm{f}^{(m)}\right)\mathrm{f}^{\rho\mu} \end{cases}$$

Thus the initial requirement on $g_{\mu\nu}$, i.e. that the radial flux of energy is set to zero, leads necessarily not only to a diagonal form of $\tau(g)^{\mu}_{\ \nu}$, but furthermore to a proportionality with respect to the identity tensor. These results apply also to $\tau(\mathrm{f})^{\mu}_{\ \nu}$, and then the equations of motion fully resemble standard General Relativity ones for a static and spherically symmetric distribution of matter.
Possible solutions which can be extracted from these equations are then exactly of the Schwarzschild type of GR. In fact, it can be checked that:

$$\mathrm{d}s_g^2 = -D\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{D} + r^2\mathrm{d}\Omega^2\,,\ \ D = 1 - \frac{r_g}{r} - \frac{m^2\sqrt{G_g^2+G_\mathrm{f}^2}\,(b_0+2b_1u+b_2u^2)\,r^2}{3G_g}$$
$$\mathrm{d}s_\mathrm{f}^2 = -\Delta(U)\mathrm{d}T^2 + \frac{\mathrm{d}U^2}{\Delta(U)} + U^2\mathrm{d}\Omega^2\,,\ \ \Delta = 1 - \frac{m^2\sqrt{G_g^2+G_\mathrm{f}^2}\,(b_2+2b_3u+b_4u^2)\,U^2}{3u^2\,G_\mathrm{f}} \tag{6.20}$$

where, under condition (6.19),

$$U = ur\,,\ \ T = ut - u\int\frac{D-\Delta}{D\Delta}\,\mathrm{d}r\,.$$

These are a set of solutions perfectly consistent in Bigravity and describing a Schwarzschild-(anti)-de-Sitter family of solutions of GR.
The particular pair of solutions is justified by the propositions in the beginning of the chapter, once the bidiagonal form is shown to follow from the requirements on $\tau(g)$ and $\tau(\mathrm{f})$.

3.1 Stability of Black Hole solutions

In the previous paragraph a Schwarzschild metric coupled to a de Sitter spacetime (6.20) have been found to be one of the spherically symmetric and static solutions to Bigravity in the generic

non-bidiagonal case.
Now one may ask whether Black Holes solutions in Bigravity are stable or not. Following the behaviour of a tensor perturbation around the solution, whose degrees of freedom are not separated in pure General Relativity tensor, vector and scalar mode, in the sense that:

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + h_{\mu\nu}(g)\,, \quad \mathrm{f}_{\mu\nu} \longrightarrow \mathrm{f}_{\mu\nu} + h_{\mu\nu}(\mathrm{f})$$

can lead to some enlightening outcomes. Here the tensor perturbation has been called $h_{\mu\nu}$ with an abuse of notation, having nothing to do with the helicity-2 mode of the full theory.
In order to check both metrics in a simpler way, let us work under the assumption that $g_{\mu\nu}$ is a Schwarzschild metric with different Schwarzschild radius $r_g := 2MG$ rather than a de Sitter solution, so that again both metrics still belong to the non-bidiagonal branch of solutions. This is an allowed solution to Bigravity complying with the proposition 1 and 2 of section 3 of chapter 6. Moreover we claim that, since we are mainly interested in the f metric, it is physically meaningful to extrapolate what is achieved with the calculations for one of the Schwarzschild metrics to the Schwarzschild-(anti)-de-Sitter initial case explored previously.
Hence, although the first order perturbed equations are reduced to less complicated ones, the analysis of the stability is not compromised.
Before starting the whole procedure, it is useful to introduce the ingoing Eddington-Finkelstein coordinates $(v,\, r,\, \theta,\, \phi)$ in which $v := t + r^\star$, while $r^\star$ is a specific radial coordinate for black holes known as tortoise coordinate:

$$r^\star = r + r_g \ln\left|\frac{r}{r_g} - 1\right| .$$

With these coordinates, $\mathrm{d}t = \mathrm{d}v - \dfrac{1}{1-\frac{r_g}{r}}\mathrm{d}r$ and the Schwarzschild black hole solution is not diagonal, but is described by the metric:

$$\mathrm{d}s^2_{BH} \;=\; g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \;=\; -\left(1 - \frac{r_g}{r}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2 \mathrm{d}\Omega^2 .$$

In the Eddington-Finkelstein coordinates we shall check whether unstable ($\Omega > 0$) spherically-symmetric modes exist, to which we can assign the form:

$$h^{\mu\nu}(g) = e^{\Omega v} \begin{bmatrix} h^{vv}_{(g)}(r) & h^{vr}_{(g)}(r) & 0 & 0 \\ h^{vr}_{(g)}(r) & h^{rr}_{(g)}(r) & 0 & 0 \\ 0 & 0 & \frac{h^{\theta\theta}_{(g)}(r)}{r^2} & 0 \\ 0 & 0 & 0 & \frac{h^{\theta\theta}_{(g)}(r)}{r^2 \sin^2(\theta)} \end{bmatrix} , \tag{6.21}$$

and a similar expression can be written for $h_{\mu\nu}(f)$.
The restrictions on the τ tensor for the interactions, which is still in (6.15), put as in section 3 of this chapter, i.e. the only non-diagonal elements are null:

$$\tau^{r}_{\ v}(g) = \tau^{r}_{\ v}(g) = -\tau^{r}_{\ v}(f) = 0\,,$$

leads to the following expression for the first order perturbation of the interaction tensor $\delta\tau_{\mu\nu}(g)$, which appears upon having extracted the variation with respect to $h^{\mu\nu}(g)$ in Bigravity action (6.12), [5]:

$$\delta\tau^{\mu}_{\nu}(g) \;=\; \frac{e^{\Omega v}\mathcal{A}(r_g - r_{\rm f})}{4r}\begin{bmatrix} 0 & 0 & 0 & 0 \\ h^{\theta\theta}_{(-)} & 0 & 0 & 0 \\ 0 & 0 & \frac{h^{vv}_{(-)}}{2} & 0 \\ 0 & 0 & 0 & \frac{h^{vv}_{(-)}}{2} \end{bmatrix} \;=\; -\delta\tau^{\mu}_{\nu}({\rm f})$$

in which the minus subscript refers to the difference

$$h^{\mu\nu}_{(-)} \;=\; h^{\mu\nu}(g) - h^{\mu\nu}({\rm f})\,,$$

while the r_g and $r_{\rm f}$ are the Schwarzschild radii which can be associated to each metric, and $\mathcal{A}$ is a constant built from combinations of the Bigravity lagrangian parameters.

Even at first perturbative order the covariant derivative of $\delta\tau(g)$, $\mathcal{D}_\mu\delta\tau^{\mu}_{\nu}(g)$ necessarily is zero due to the Bianchi identity for the Hilbert-Einstein perturbed lagrangian. Previously it has been used in finding further constraints on τ which at the end lead us to the Black Holes solution allowed in Bigravity.

When the covariant derivative is built from the zero-order $g_{\mu\nu}$ metric in the Schwarzschild case with Eddington-Finkelstein coordinates:

$$\mathcal{D}(g)_\mu\delta\tau^{\mu}_{\nu} \;:=\; \partial_\mu\delta\tau^{\mu}_{\nu} + \Gamma_{\alpha\,\nu}^{\;\rho}\delta\tau^{\alpha}_{\rho} - \Gamma_{\sigma\,\beta}^{\;\beta}\delta\tau^{\sigma}_{\nu}\,,$$

and taking into account that the only non-zero Christoffel symbols are

$$\Gamma_{v\,v}^{\;v} = \frac{r_g}{2r^2} = -\Gamma_{v\,r}^{\;r}\,; \qquad \Gamma_{\theta\,\theta}^{\;v} = -r = \frac{\Gamma_{\phi\,\phi}^{\;v}}{\sin^2\theta}\,; \qquad \Gamma_{\phi\,\phi}^{\;r} = -r\left(1-\frac{r_g}{r}\right) = \frac{\Gamma_{\theta\,\theta}^{\;r}}{\sin^2\theta}\,;$$

$$\Gamma_{\theta\,r}^{\;\theta} = \frac{1}{r} = \Gamma_{\phi\,r}^{\;\phi}\,; \qquad \Gamma_{\phi\,\phi}^{\;\theta} = -\sin\theta\cos\theta = -\sin^2\theta\,\Gamma_{\phi\,\theta}^{\;\phi}\,; \qquad \Gamma_{v\,v}^{\;r} = \left(1-\frac{r_g}{r}\right)\frac{r_g}{2r^2}\,, \tag{6.22}$$

then the only non trivial equations are:

$$\begin{cases} \dfrac{e^{\Omega v}}{4}\left(\left(\dfrac{h^{\theta\theta}_{(-)}}{r}\right)' + 2\dfrac{h^{\theta\theta}_{(-)}}{r^2}\right) \;=\; 0 \;=\; \mathcal{D}_\mu\delta\tau^{\mu}_{v}\,, \\ \dfrac{e^{\Omega v}}{4r}\dfrac{h^{vv}_{(-)}}{r} \;=\; 0 \;=\; \mathcal{D}_\mu\delta\tau^{\mu}_{r}\,. \end{cases} \tag{6.23}$$

so $h^{\theta\theta}_{(-)}$ and $h^{vv}_{(-)}$ are:

$$h^{\theta\theta}_{(-)} \;=\; \frac{c_0}{r}\,,\;\; h^{vv}_{(-)} \;=\; 0\,, \tag{6.24}$$

in which c_0 is an integration constant.

The conditions found for the perturbations on the metric tensor are essentially different from those which descend from the Bianchi identity in the linear theory of General Relativity with a mass, chapter 2,

$$\partial_\mu h^{\mu}{}_{\nu} = 0 = h\,.$$

The quite different expression (6.24) implied by the Bianchi identity will have interesting implications on the analysis of the stability of the solutions.
First of all, as it is stated in [5], it is not possible to write down a unique equation for $h^{\mu\nu}$ starting from the couple of equations of motion in Bigravity for f and g: the trick to get the full solution is to follow the behaviour of the gauge-dependent, GR-like, part and that of the other particular solution $h^{\mu\nu}_{(m)}$,

$$h^{\mu\nu}(g,\,\mathrm{f}) \;=\; h^{\mu\nu}_{\mathrm{GR}}(g,\,\mathrm{f}) + h^{\mu\nu}_{(m)}(g,\,\mathrm{f})\,.$$

The particular solution can be proved to have a single non-zero component for each metric perturbation:

$$h^{rr}_{(m)}(g) \;=\; \frac{\mathcal{A}(r_g - r_\mathrm{f})\,e^{\Omega v}}{4\Omega}\,m^2\,h^{\theta\theta}_{(-)}\,, \tag{6.25}$$

and $h^{rr}_{(m)}(\mathrm{f})$ is minus $h^{rr}_{(m)}(g)$ multiplied with the ratio of their Planck masses.
Conversely, $h^{\mu\nu}_{\mathrm{GR}}$ is pure gauge, hence it can be derived from the formal expression of the symmetrization of the Lie derivative $2\mathcal{D}^{(\mu}\xi^{\nu)} := -h^{\mu\nu}_{\mathrm{GR}}$ in which ξ, in order to reproduce $h_{\mu\nu}$ in (6.21), is of the type $\xi^\mu = e^{\Omega v}\left(\xi^0(r)\,,\ \xi^1(r)\,,\ 0\,,\ 0\right)$. In virtue of the conditions (6.24), $\xi(\mathrm{f})$ and $\xi(g)$ are related by:

$$\xi^0(\mathrm{f}) = \xi^0(g) + c_1\,,\ \ \xi^1(\mathrm{f}) = \xi^1(g) + \frac{c_0}{2}\,.$$

This implies that the $(v,\,v)$, the $(v,\,r)$, the $(r,\,r)$ and the $(\theta\,,\theta) \;=\; \sin^2\theta(\phi\,,\phi)$ components of $h^{\mu\nu}(\mathrm{f})$, given that the Christoffel symbols in the f metric are obtained switching r_g to r_f in the $\Gamma(g)$ listed in (6.22), and that:

$$\mathrm{f}_{\mu\nu} \;=\; \begin{bmatrix} -\left(1-\frac{r_\mathrm{f}}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}\,, \qquad \mathrm{f}^{\mu\nu} \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & \left(1-\frac{r_\mathrm{f}}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}\,,$$

corresponds to:

$$\begin{cases} h^{vv}(\mathrm{f}) = -\,2\mathrm{f}^{vr}\left(\partial_r\xi^v + \Gamma_{r\ \mu}^{\ v}\xi^\mu\right) = -2e^{\Omega v}\left(\xi^0(g)\right)' \\ h^{rr}(\mathrm{f}) = -\,2\mathrm{f}^{rr}\left(\partial_r\xi^r + \Gamma_{r\ \mu}^{\ r}\xi^\mu\right) - 2\mathrm{f}^{rv}\left(\partial_v\xi^r + \Gamma_{v\ \mu}^{\ r}\xi^\mu\right) = \\ \qquad -\,2e^{\Omega v}\left(\left(1-\frac{r_\mathrm{f}}{r}\right)\left[(\xi^1(g))' - \frac{r_\mathrm{f}}{2r^2}\left(\xi^0(g)+c_1\right)\right] + \Omega(\xi^1(g)+\frac{c_0}{2}) - \frac{r_\mathrm{f}}{2r^2}(\xi^1(g)+\frac{c_0}{2})\right) \\ \qquad -\,2e^{\Omega v}\left(\left(1-\frac{r_\mathrm{f}}{r}\right)\frac{r_\mathrm{f}}{2r^2}(\xi^0(g)+c_1)\right) \\ h^{rv}(\mathrm{f}) = -\,\mathrm{f}^{rr}\left(\partial_r\xi^v + \Gamma_{r\ \mu}^{\ r}\xi^\mu\right) - \mathrm{f}^{rv}\left(\partial_v\xi^v + \Gamma_{v\ \mu}^{\ v}\xi^\mu\right) - \mathrm{f}^{vr}\left(\partial_r\xi^r + \Gamma_{r\ \mu}^{\ r}\xi^\mu\right) = \\ \qquad -\,e^{\Omega v}\left(\left(1-\frac{r_\mathrm{f}}{r}\right)(\xi^0(g))' + \left(\Omega+\frac{r_\mathrm{f}}{2r^2}\right)(\xi^0(g)+c_1) + (\xi^1(g))' - \frac{r_\mathrm{f}}{2r^2}\left(\xi^0(g)+c_1\right)\right) \\ h^{\theta\theta}(\mathrm{f}) \;=\; -\,2\mathrm{f}^{\theta\theta}\left(\partial_\theta\xi^\theta + \Gamma_{\theta\ \mu}^{\ \theta}\xi^\mu\right) \;=\; -2\frac{1}{r^2}\frac{1}{r}\left(\xi^1(g)+\frac{c_0}{2}\right)e^{\Omega v} \end{cases}$$

We have the freedom to choose entirely the $h^{\mu\nu}_{\text{GR}}(\text{f})$ entries, due to gauge fixing, putting it to zero, $h^{\mu\nu}(\text{f}) = 0$, then ξ^0, ξ^1, c_0 and c_1 begin to be related to each other:

$$\begin{cases} 0 = \left(\xi^0(g)\right)' \\ 0 = \left(1 - \frac{r_\text{f}}{r}\right)\left(\xi^1(g)\right)' + \left(\Omega - \frac{r_\text{f}}{2r^2}\right)\left(\xi^1(g) + \frac{c_0}{2}\right) \\ 0 = \Omega\xi^0(g) + \Omega c_1 + \left(1 - \frac{r_\text{f}}{r}\right)(\xi^0(g))' + (\xi^1(g))' \\ 0 = \left(\xi^1(g) + \frac{c_0}{2}\right) \ , \end{cases}$$

so that

$$\xi^0(g) = -c_1\,, \quad \xi^1(g) = -\frac{c_0}{2}\,, \quad \Omega = \frac{r_\text{f}}{2r^2}\,. \tag{6.26}$$

In the end $h^{\mu\nu}_{\text{GR}}(g)$ assumes the form:

$$h^{\mu\nu}_{\text{GR}}(g) = e^{\Omega v} \begin{bmatrix} 0 & \Omega c_1 & 0 & 0 \\ \Omega c_1 & c_0\left(\Omega - \frac{r_g}{2r^2}\right) & 0 & 0 \\ 0 & 0 & c_0 r^{-3} & 0 \\ 0 & 0 & 0 & c_0 \frac{1}{\sin^2\theta r^3} \end{bmatrix} . \tag{6.27}$$

Finally, what emerges from the analysis is that (6.27) plus (6.25), both in the g and in the f case, where only (6.25) is present, give rise to a tensorial perturbation to the Schwarzschild black hole which does not develop pathologies at the Schwarzschild radius $r = r_g$. Therefore the non-bidiagonal black holes do not have unstable modes: the physical perturbations are regular when $\Omega = i\omega$ and describe non-growing ingoing waves, so they confirm the stability of the Schwarzschild solution in Bigravity.

Instead, the case $\Omega = 0$, which seems to be excluded due to the presence of a $\frac{1}{\Omega}$ factor in (6.25), can in reality be considered if one takes also $c_0 \sim \Omega$: then the only nonvanishing contribution is $h^{rr} \sim \frac{1}{r}$, which describes, anyhow, the same Schwarzschild original solution with different Schwarzschild radii. This means that, contrary to what was claimed at the beginning, the case $\Omega = 0$ does not describe a true perturbation but only a sort of rescailing of the original solution, which does not add further informations about the stability. Therefore it is reasonable to exclude the existence of other branches of solutions close to this family.

Chapter 7

Concluding remarks

Massive Gravity and Bigravity have been proposed in the context of Theoretical Physics only quite recently in order to give an alternative theory of gravitation. They consist in giving a mass to the mediator of the gravitational force, the graviton.
These theories have been formulated with the aim of giving possible explanations of fundamental cosmological issues such as the nature of Dark Energy and Dark Matter.

In the beginning of the thesis, which focused on the issues which appear when one tries to generalise the theory of General Relativity by introducing a mass to the graviton, basic problems concerning the propagation of ghost degrees of freedom and/or the presence of a discontinuity between the massless limit of the massive theory and the massless theory have been considered. These problematic aspects show themselves in the linear theory with a mass (Fierz-Pauli mass term) and even in some of its non-linear extensions.
The physically consistent formulation of a massive gravity model was found in the framework of Massive Gravity and Bigravity. In chapters 3 and 4, these theories are presented as perfectly consistent in the more elegant vielbein formalism, propagating exactly the 5 degrees of freedom of a spin-2 particle with a mass whose coupling via the Stückelberg scalar degree of freedom π_0 to the stress energy tensor is suppressed by Vainshtein mechanism. But the fundamental question is: do they really represent a good alternative to GR, giving a better answer to its open questions and at the same time reducing to it at observable cosmological scales?

The pros to these theories were presented using the equations of motion and their subsequent solutions, which were analysed starting from section 1 of chapter 6. When the helicity-2 tensor $h_{\mu\nu}$ and the scalar π_0 are chosen in order to be of Friedmann-Lemaître-Robertson-Walker type, and in particular to describe an expanding de Sitter universe, then two significant solutions are really allowed. On the one side, there is the self-accelerating solution: it describes an accelerating Universe even in the absence of the usual cosmological constant, with the acceleration given by the parameters of the interacting terms in Massive Gravity lagrangian. In the ΛCDM model, the most widely accepted cosmological model, the cosmological constant Λ refers instead to the stress-energy tensor of a hypothetical fluid with negative pressure taken as responsible for the actual accelerated expansion, while CDM denotes the assumption that the actual abundance of

Dark Matter is due to a production mechanism which took place when it was already cold (Cold Dark Matter).
On the other side, in Massive Gravity one can also find the so-called screening or degravitating solution in which massive graviton interactions are able to screen the effects of a large cosmological constant, which can be an a priori external input to the theory, leading thus to an approximately flat (at first order in a perturbation expansion) Universe, despite the presence of that large Λ.
Stability of these degravitating and self-accelerating branches of solutions has also been checked and they both have been found to be stable.

In section 2 of chapter 6, one combination of the first order perturbation to the proportional metrics, which are a class of Bigravity solutions, corresponding to a mass eigenstate, is presented as a suitable candidate for Dark Matter. In fact it is proved to interact very weakly with the stress-energy tensor of Standard Model matter.
Limiting us only to the first orders in perturbation theory, the analysis performed on the massless and massive eigenstates of the perturbation of the background $g_{\mu\nu}$ and $\mathrm{f}_{\mu\nu}$ solution has also allowed to prove that there is no vertex between a massive and a massless mode: these two sectors do not communicate. Going to tree level, then, the massive or massless mode mediated decay of a couple of Standard Model particles into a couple of massive graviton is really allowed. This is an effective mechanism of production for Dark Matter.
The mass value of the Dark Matter candidate can be constrained using crossed bounds coming from the right arrangement of the various perturbation orders and from considerations on the freeze-in mechanism of production, which has to reproduce the current abundance of Dark Matter. The values are obtained in a narrow window, $1\,\frac{\mathrm{TeV}}{c^2} \lesssim m_{\mathrm{eff}} \lesssim 66\,\frac{\mathrm{TeV}}{c^2}$. This implies that if future measurements of the Dark Matter mass will fall into this range, the Bigravity explanation for Dark Matter will begin to be considered more seriously considered as one of the correct cosmological models.

Black Holes solutions were shown to be allowed in Bigravity and Massive Gravity, but under some stringent hypothesis on the second coupled metric, such as the fact that it can not be of any possible type, but for example of de Sitter type or of Schwarzschild type.

Thus in these simple cases the solutions to the strongly coupled system of equations of Massive (Bi-)Gravity have been shown to reproduce the metrics which solve Einstein's equations in General Relativity. This is only a little step in the quest of viability for the theories: the check with General Relativity is performed with a plethora of simplifying assumptions and the aspects which differentiates them from the predictions of Einstein's theory match in a so perfect way with the actual observations that one should think that there is too much freedom in choosing the parameters, in the sense that they can be adapted to many different cases.
So other cross checks and further steps, such as the study of issues of coupling massive gravity to Standard Model matter, have to be performed before one can conclude that these generalized models of gravitation with a massive graviton are viable physical theories. Even their quantization has not been sufficiently discussed in the literature yet, and in general further work is

required to reveal whether one of these fascinating models can be promoted to a fully fledged theory of Nature.

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