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DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

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Derived Categories and Fourier-Mukai Transforms

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Introduction

The development of the theory of derived categories began with J.-L. Verdier's thesis in 1967 [Ver96][1] under A. Grothendieck. This was a successful attempt to give a proper context to the existing theory of hypercohomology of complexes, the framework with which we generalize the (co)homological theory of left (right) exact functors from sequences of objects in an abelian category $\mathcal{A}$ to a theory that can handle sequences of complexes.

The key insight is that a resolution of an object is a quasi-isomorphisms, i. e. a morphism that induces an isomorphisms in cohomology. Therefore the main idea is to build a new category, from the homotopy category $K(\mathcal{A})$, where all quasi-isomorphisms are formally inverted. This yields an identification between objects in $\mathcal{A}$ and all of their resolutions. Thereby the derived category adopts complexes from the beginning and the idea that an object is "made" of possibly simpler objects, i. e. objects whose cohomology have less degrees of complexity.

The epistemological justification for this procedure is the desire of a coherent description of semi-exact functors, which appear as naturally as profusely across all disciplines. By way of example, take $\mathcal{F} \otimes -$ or $\Gamma(X, -)$, in the context of sheaves of abelian groups over a topological space X. Their "naive" definitions should be applied only to special objects, namely the objects that lie in their respective adapted class (i. e. locally free and flasque sheaves, respectively). The reason is that we want to preserve relationships between those objects that might hold relevant information, e.g. kernels and cokernels of morphisms[2]. Therefore it is ideal to replace $\mathcal{F} \otimes -$ with $\mathcal{I}^\bullet \otimes -$, where $\mathcal{I}^\bullet$ is a complex of locally free sheaves, and extend the functor to handle complexes as well.

So to summarize, we <u>functorially</u> go through the following layers of successive abstractions,

$$\mathcal{A} \hookrightarrow C(\mathcal{A}) \longrightarrow K(\mathcal{A}) \xrightarrow{Q_{\mathcal{A}}} D(\mathcal{A}), \qquad \lambda : \mathcal{A} \to K(\mathcal{A})$$

where, at each layer we surgically modify our notion of what a relation between objects means. The toll we take for undertaking such transformations is—already at level of $K(\mathcal{A})$—that the category we land on is not abelian. Nonetheless the notion of exactness as we mean for complexes in $C(\mathcal{A})$ is inherited by the triangulated structure we can endow $K(\mathcal{A})$ and, consequently, $D(\mathcal{A})$ with.

It is worth underlining that the structures of triangulated and abelian categories are not necessarily related by inclusion. The former is not a weaker notion of the latter, as the intersection consists only of semisimple categories[3].

Among the many blatant achievements of this shift in perspective to the framework of derived categories, there is an overt description of the derived functor of a composition $G \circ F$, namely the following isomorphism holds

$$R(G \circ F) \simeq RG \circ RF,$$

as showed in Theorem 1.4.10. This result relived from the formal theory the weight of what previously had to be described through spectral sequences, although actual computations cannot avoid them. This trade-off will be manifest in the proof of Proposition 4.2.15 and Theorem 4.2.19.

[1] Only belatedly published in 1996

[2] Or, indeed, their sygyzies. See [GM03], III.1

[3] I.e. categories where all exact sequences split, e.g. $\mathrm{Vect}_k^{\mathrm{fin}}$, see [HJ10] 5.3

Since the 60s the theory of derived categories has permeated through many disciplines within mathematics and physics. Among those we list:

- Algebraic analysis, microlocal sheaf theory and their applications in symplectic topology.
- Homological mirror symmetry, which relates the derived category of coherent sheaves on a complex algebraic variety X to the derived Fukaya category of its mirror partner, a symplectic manifold Y.
- Algebraic geometry and the study of derived equivalences between algebraic varieties through Fourier-Mukai Transforms.

This work focuses on the last point of the list. Chapter 1 deals with the foundational aspects of derived categories as described in the above paragraphs. Here we lay down the stage where the later chapters build upon. The main references of this are [KS90; Sch23; Wei95; GM03].

Chapter 2 is devoted to instantiate the object of most interest, $D^b(X)$—the bounded derived category of coherent sheaves on a smooth projective variety X. This category is of considerable geometric interest, to quote A. Bondal and D. Orlov in their paper *Semiorthogonal decompositions for algebraic varieties* [BO95],

> This leads to the idea that the derived category of coherent sheaves might be reasonable to consider as an incarnation of the motive[4] of a variety.

Here the main references are the [Har77; Har06; Huy06] and J.-P. Serre seminal paper *Faisceaux algébriques cohérents* [Ser55].

In Chapter 3 we delve in the theory of Fourier-Mukai transforms, these functors were first introduced by S. Mukai in [Muk81]. Beyond the basic definitions and properties we put particular emphasis on equivalence criteria of functors of Fourier-Mukai type. The main sources here are again [Huy06] and T. Bridgeland's paper [Bri19]. At the end of the chapter we mention Orlov's famous Representability Theorem 3.3.11.

Lastly, Chapter 4 is split in three main sections:

1. The derived category of $\mathbb{P}^n$,
2. The Bondal-Orlov's Reconstruction Theorem,
3. Mukai's Theorem.

In the first section we discuss a result due to A. Beĭlinson in [Beĭ78], which lays bare the structure of the derived category of coherent sheaves on $\mathbb{P}^n$. This will be generated by the following exceptional sequence

$$\left\langle \mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n} \right\rangle.$$

The discussion follows A. Căldăraru's paper [Cal05].

The middle section exposes the most important result of this work, which is part of the modern approach to the study of algebraic varieties through the lens of their derived categories. Bondal-Orlov's Reconstruction Theorem 4.2.19 proves that is possible to exhibit an isomorphism $X \simeq Y$ between smooth projective varieties over an arbitrary field k from an equivalence $D^b(X) \simeq D^b(Y)$, provided the canonical or the anticanonical bundle of X $\omega_X^{\pm 1}$ is ample.

This follows the trail paved by well known results: in 1961, (P. Gabriel, [Gab62])

> Let X and Y be smooth projective varieties, then an equivalence $\mathrm{Coh}X \simeq \mathrm{Coh}Y$, induces an isomorphism $X \simeq Y$.

[4]"I formulated the notion of "motive" associated to an algebraic variety. By this term, I want to suggest that it is the "common motive" (or "common reason") behind this multitude of coho- mological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible." – A. Grothendieck, [Gro23]

And later in 1996[5] (A. L. Rosenberg, [Ros07])

> Let X, Y be quasi-separated schemes. If the categories QCohX and QCohY are equivalent, then $X \simeq Y$.

The previous theorems go under the framework of “reconstruction theorems”. Their main scope is to deal with the problem of rebuilding an object from either the information given about its invariants—therefore the underlying geometry, or its representation into another object. This section ends with the explicit description of the group of autoequivalences of $D^b(X)$,

$$\operatorname{Auteq} D^b(X) \simeq \operatorname{Aut} X \rtimes (\operatorname{Pic}(X) \oplus \mathbb{Z}).$$

Here the references are the article [BO01], [Huy06] and I. Dolgachev’s notes [Dol09].

Finally, the third section traces back the origins of the notion of Fourier-Mukai transform. We expound the proof of the theorem that sprang the whole field, namely

> Let A an abelian variety, $\hat{A}$ its dual and $\mathcal{P}$ the Poincaré bundle over $A \times \hat{A}$. Then the Fourier-Mukai functor
>
> $$\Phi_{\mathcal{P}} : D^b(A) \longrightarrow D^b(\hat{A})$$
>
> is a triangulated equivalence. Moreover,
>
> $$\Phi_{\mathcal{P}}^{A \to \hat{A}} \circ \Phi_{\mathcal{P}}^{\hat{A} \to A} \simeq \hat{\iota}^* \circ [-g]$$
>
> where $g = \dim A$ and $\hat{\iota}$ is the inverse map of $\hat{A}$.

The references for the general theory of abelian varieties are [Mil08; MRM08], as for the proof of Mukai’s Theorem, [Muk81] and [Huy06].

[5] See the expository paper [Bra14]

Vedi, in questi silenzi in cui le cose
s'abbandonano e sembrano vicine
a tradire il loro ultimo segreto,
talora ci si aspetta
di scoprire uno sbaglio di Natura,
il punto morto del mondo, l'anello che non tiene,
il filo da disbrogliare che finalmente ci metta
nel mezzo di una verità.

Eugenio Montale – Ossi di seppia

To whomever planted the first seed of curiosity
in my Orchard,
to the reckless forest of interests it became,
to my family and friends who nurtured the ground
and fostered its growth,
to each forked path I didn't take,
to the endless bliss it gave.

1 Derived Categories

We don't want definitions,
we want properties.

Sergey Shadrin

1.1 Motivation

The construction of the derived category addresses to two main desiderata:

1. A functor that expresses the association of an object in an abelian category $\mathcal{A}$ to its resolution in $C(\mathcal{A})$, the category of complexes made of objects in $\mathcal{A}$.
2. A proper way to identify an object or a complex of objects with a complex having same cohomology which is somehow computationally less cumbersome.

For (1) the naive attempt to construct a functor from $\mathcal{A}$ to $C(\mathcal{A})$ easily fails since for a fixed object neither its resolutions nor the maps induced on resolutions are unique. Somehow an hint that we might want to slightly change the target category $C(\mathcal{A})$ is given by the following key fact

> Every map between two object of $\mathcal{A}$ lift to a map of resolutions which is unique up to homotopy.

Therefore the association we want is indeed functorial only if the target is $K(\mathcal{A})$, the homotopy category of $\mathcal{A}$. In fact, the procedure outlined so far, describes how classical derived functors are usually defined: e.g. let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ a left exact functor between abelian categories, and $0 \longrightarrow X \longrightarrow I^\bullet$ an injective resolution of $X \in \mathcal{A}$, we denote $\mathcal{I}$ the full additive subcategory of $\mathcal{A}$ of injective objects, then the classical right derived functor is defined as the following composition:

$$\begin{array}{ccccccc} R^iF: \quad \mathcal{A} & \xrightarrow{\lambda} & K^+(\mathcal{I}) & \longrightarrow & K^+(\mathcal{B}) & \xrightarrow{H^i} & \mathcal{B} \\ & \searrow & \downarrow & \nearrow_{K^+F} & & & \\ & & K^+(\mathcal{A}) & & & & \end{array}$$

The diagram is commutative, λ is indeed a functor since any two resolution $I^\bullet, J^\bullet$ of the same object X are isomorphic in $K^+(\mathcal{I})$, for any morphism $X \longrightarrow Y$, $\lambda(X \longrightarrow Y)$ is unique up to homotopy, hence unique in $K^+(\mathcal{I})$[1]

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \xrightarrow{\lambda} & I^\bullet \\ & & \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle \wr} \\ 0 & \longrightarrow & X & \xrightarrow{\lambda} & J^\bullet \end{array}$$

But in general the homotopy category is no longer abelian, therefore we cannot speak of exact sequences.

[1] In the homotopy category of an abelian category the identity of an object is sent in the class of homotopic equivalences of that object

For (2) we adopt the notion of quasi-isomorphism (q.is). In fact, let $X \in \mathcal{A}$, we assume that exist an injective resolution on X

$$0 \longrightarrow X \xrightarrow{\epsilon} I^\bullet$$

We can build a morphism of complexes as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \epsilon} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \dots \end{array}$$

Therefore if we want to identify objects with their resolutions we have to invert quasi isomorphisms, so that we can fully shift our attention to complexes.

1.2 The triangulated structure of $K(\mathcal{A})$

Throughout this chapter we assume $\mathcal{A}$ to be and abelian category.

1.2.1 Triangles as generalized short exact sequences

Definition 1.2.1. A triangle in $C(\mathcal{A})$ is a diagram of the form

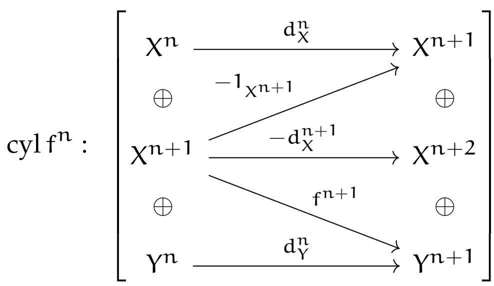
$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$

Definition 1.2.2. A triangle is called distinguished (d.t.) if it is isomorphic to one of the following diagrams (a choice is equivalent to the other),

$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow \mathrm{cone}(f)^\bullet \longrightarrow X^\bullet[1] \qquad \text{[Huy06]}$$

$$X^\bullet \longrightarrow \mathrm{cyl}(f)^\bullet \xrightarrow{f} \mathrm{cone}(f)^\bullet \xrightarrow{g} X^\bullet[1] \qquad \text{[GM03]}$$

Definition 1.2.3. The mapping cylinder of $X^\bullet \xrightarrow{f} Y^\bullet$ is a complex constructed in the following way:



$$d^n_{\mathrm{cyl}(f)} = \begin{bmatrix} d_X & -1_X & 0 \\ 0 & -d_X & 0 \\ 0 & f & d_Y \end{bmatrix}$$

Remark 1.2.4. The cone of f is the given by the last two rows on the LHS and the lower left 2×2 submatrix on the RHS.

Theorem 1.2.5. *The homotopy category $K(\mathcal{A})$ endowed with the shift functor [1] is triangulated. [KS90]*

Remark 1.2.6. For an morphism $X^\bullet \xrightarrow{f} Y^\bullet$ there exist the following commutative diagram in $C(\mathcal{A})$ with exact rows.

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & Y^\bullet & \longrightarrow & \mathrm{cone}(f)^\bullet & \longrightarrow & X^\bullet[1] \longrightarrow 0 \\
 & & & & \big\downarrow \alpha & & \big\| & & \\
0 & \longrightarrow & X^\bullet & \longrightarrow & \mathrm{cyl}(f)^\bullet & \longrightarrow & \mathrm{cone}(f)^\bullet & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow \beta & & & & \\
 & & X^\bullet & \xrightarrow[f]{} & Y^\bullet & & & &
\end{array}
\qquad
\alpha = \begin{bmatrix} 0 \\ 0 \\ 1_Y \end{bmatrix}
\qquad
\beta = \begin{bmatrix} 0 & f & 1_Y \end{bmatrix}
$$

This construction is functorial in f. Moreover we have $\beta\alpha = 1_Y$ and $\alpha\beta \sim 1_{\mathrm{cyl}(f)}$ (i.e. are homotopic), so that it is an isomorphism in $K(\mathcal{A})$ (i. e. an homotopic equivalence), in particular this implies that α and β are quasi-isomorphisms.

Lemma 1.2.7. *Any short exact sequence (SES) in $C(\mathcal{A})$ is quasi-isomorphic to the middle row of an appropriate diagram above, as given in the following diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{g} & Z^\bullet & \longrightarrow & 0 \\
 & & \big\| & & \big\uparrow \beta & & \big\uparrow \gamma=[0\ g] & & \\
0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \dots
\end{array}
$$

Remark 1.2.8. The previous Lemma states that every SES can be *completed* into a d.t. in $K(\mathcal{A})$, this shows how distinguished triangles can generalize the notion on short exact sequences, they will, as a matter of fact, replace the notion of exactness whenever a triangulated structure is available

Remark 1.2.9. Cohomological properties of distinguished triangles in $K(\mathcal{A})$ resemble those of short exact sequences, i. e. given a d.t.

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]$$

Then we obtain a the long exact sequence (LES) in cohomology

$$\cdots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow \cdots$$

Proof: Follows by definition of d.t., H^0 is a cohomological functor and axiom TR2 (rotation) of triangulated categories □

In the following we will adopt also the following shorthand notation for a d.t.

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \xrightarrow{+}$$

1.3 The construction of $D(\mathcal{A})$

Definition 1.3.1. A <u>null system</u> $\mathcal{N}$ is a family of objects of $\mathcal{A}$ that satisfy the following conditions

(N1) $0 \in \mathcal{N}$.

(N2) if $X^\bullet \in \mathcal{N}$ then $X^\bullet[1] \in \mathcal{N}$.

(N3) for any d.t.

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow k[1]^\bullet$$

such that $X^\bullet, Y^\bullet \in \mathcal{N}$, then $Z^\bullet \in \mathcal{N}$.

Recall that in a general triangulated category a null system gives rise to a multiplicative class of morphisms (as in [KS90])[2]

$$S(\mathcal{N}) := \left\{ X^\bullet \xrightarrow{f} Y^\bullet \mid f \text{ is embedded in a d.t. } X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow \underbrace{Z^\bullet}_{\in \mathcal{N}} \longrightarrow X^\bullet[1] \right\}$$

Remark 1.3.2. The family of objects in $K(\mathcal{A})$

$$\mathcal{N}_0 = \{X^\bullet \in K(\mathcal{A}) \mid H^n(X^\bullet) \simeq 0, \ \forall n \in \mathbb{Z}\}$$

is a null system. Conditions (N1) and (N2) clear, (N3) follows from the long exact sequence in cohomology.

Consequently we can construct its multiplicative system $S(\mathcal{N}_0)$. By exploiting once again the following LES

$$\cdots \longrightarrow \underset{\substack{\wr\Vert \\ 0}}{H^{i-1}(Z^\bullet)} \longrightarrow H^i(X^\bullet) \underset{\sim}{\longrightarrow} H^i(Y^\bullet) \longrightarrow \underset{\substack{\wr\Vert \\ 0}}{H^i(Z^\bullet)} \longrightarrow \cdots$$

Therefore $S(\mathcal{N}_0)$ can be described as the set of quasi-isomorphisms in $K(\mathcal{A})$, i. e.

$$S(\mathcal{N}_0) = \left\{ X^\bullet \underset{(q.is)}{\rightsquigarrow} Y^\bullet \mid X^\bullet, Y^\bullet \in K(\mathcal{A}) \right\}$$

Note that homotopic complexes give rise the same cohomology groups and if the $\text{cone}(f)^\bullet$ is acyclic (i. e. $H^i(\text{cone}(f)^\bullet) \simeq 0$ for all $i \in \mathbb{Z}$) then f is a q.is . We will implicitly adopt the following notation for quasi-isomporhisms

$$X^\bullet \overset{f}{\rightsquigarrow} Y^\bullet$$

Definition 1.3.3. The derived category of $\mathcal{A}$ is the localization [3]

$$D^*(\mathcal{A}) := K^*(\mathcal{A})/N^*(\mathcal{A})$$

where $* = \varnothing, b, +, -$ [4]. We will denote the localization functor as

$$Q \equiv Q_{\mathcal{A}} : K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$$

Remarks 1.3.4.

- Quasi-isomorphisms in $K(\mathcal{A})$ are isomorphisms is $D(\mathcal{A})$.
- Morphisms in $D(\mathcal{A})$ are denoted as triples $f = (X^\bullet, s, g)$ called roofs

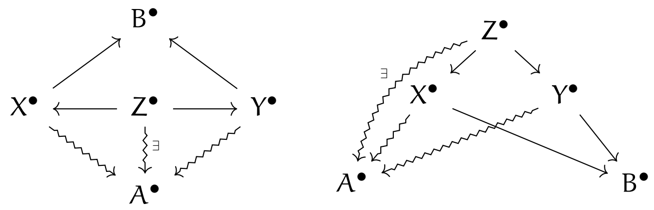
$$\begin{array}{ccc} & X^\bullet & \\ {\scriptstyle s}\swarrow\!\!\!\!\sim & & \searrow{\scriptstyle g} \\ A^\bullet & \overset{f}{\dashrightarrow} & B^\bullet \end{array}$$

  two roofs $(X^\bullet, s, g)$ and $(Y^\bullet, t, h)$ are equivalent if and only if exist $Z^\bullet$ and a q.is $Z^\bullet \overset{f}{\rightsquigarrow} A^\bullet$ making the following diagrams commute

[2] Or localizing class as in [GM03]

[3] The procedure of localizing categories here refers to the books [KS90] and [KS06]

[4] Unbounded, bounded, bounded below and bounded above, respectively



[KS90] [GM03]

- $D^*(\mathcal{A})$ are triangulated categories [KS06].

1.3.1 Generalized Objects

Recall in $C(\mathcal{A})$ can define the following truncation functors,

$$\begin{array}{lccccccccccc}
\tau^{\leq n}: & \cdots \longrightarrow & X^{n-1} & \longrightarrow & \ker d_X^n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\Big\downarrow (q.is) & & \| & & \downarrow & & \downarrow & & \| & \\
\tilde{\tau}^{\leq n}: & \cdots \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & \operatorname{im} d_X^n & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

$$\begin{array}{lccccccccccc}
\tau^{\geq n}: & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{coker} d_X^{n-1} & \longrightarrow & X^{n+1} & \longrightarrow \cdots \\
\Big\downarrow (q.is) & & \| & & \downarrow & & \downarrow & & \| & \\
\tilde{\tau}^{\geq n}: & \cdots \longrightarrow & 0 & \longrightarrow & \operatorname{im} d_X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow \cdots
\end{array}$$

Therefore the morphisms above are isomorphisms in $D(\mathcal{A})$

Remark 1.3.5. [KS90]. Let $X \in D(\mathcal{A})$[5], the following are distinguished triangles

- $\tau^{\leq n}X \longrightarrow X \longrightarrow \tau^{\leq n+1}X \xrightarrow{+}$
- $\tau^{\leq n-1}X \longrightarrow \tau^{\leq n}X \longrightarrow H^n(X)[-n] \xrightarrow{+}$
- $H^n(X)[-n] \longrightarrow \tau^{\geq n}X \longrightarrow \tau^{\geq n+1}X \xrightarrow{+}$

By employing the distinguished triangles above and the axioms of triangulated categories is possible to prove the following result.

Theorem 1.3.6. *[GM03]. There exists an equivalence of categories $\mathcal{A} \simeq D_0(\mathcal{A})$, between the abelian category $\mathcal{A}$ and the full subcategory $D_0(\mathcal{A}) \subset D(\mathcal{A})$ consisting of all complexes X with cohomology concentrated in degree 0, i. e. $H^i(X) \simeq 0$, $\forall i \neq 0$.*

This implies that the original category $\mathcal{A}$ lives inside $D(\mathcal{A})$ but its objects are identified up to quasi-isomorphic chain complexes.

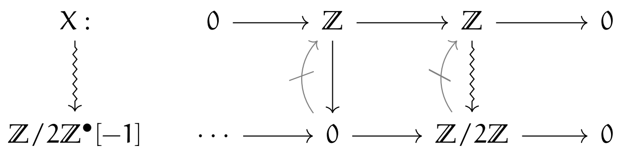
1.3.2 Morphisms in $D(\mathcal{A})$

Morphisms of a localized category do not behave nicely in general, this makes intuitive sense since we successively packed morphisms into classes when passing from $C(\mathcal{A})$ to $K(\mathcal{A})$ and then again when localizing the latter to get $D(\mathcal{A})$. Therefore we had to lose some "control" in order to make the pattern we wanted—i. e., the idea of generalized object—emerge. We delve into few examples that let us grasp the behaviour of such morphisms.

Remarks 1.3.7.

[5] We temporarily drop the bullet notation for complexes. It will be resumed if necessary, in order to avoid confusion

- It is possible to find morphisms in $D^*(\mathcal{A})$ that do <u>not</u> arise from chain maps in $C(\mathcal{A})$. Consider the following resolution in $\mathcal{A} = \mathbf{Ab}$



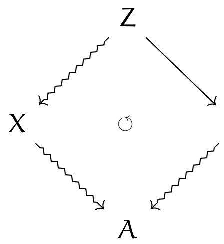
  so the inverse exists in $D(\mathbf{Ab})$ but $\mathrm{Hom}_{C(\mathbf{Ab})}(\mathbb{Z}/2\mathbb{Z}, X) = 0$.

- In general only the following relations hold true:

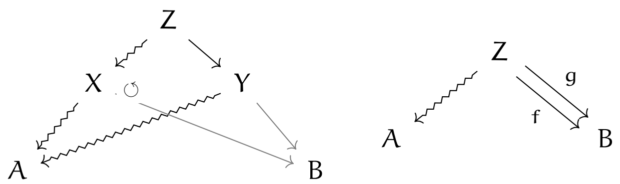
$$[f = 0 \text{ in } C(\mathcal{A})] \implies [f = 0 \text{ in } K(\mathcal{A})] \implies [f = 0 \text{ in } D(\mathcal{A})] \implies [H^n(f) = 0\ \forall n].$$

  all the implication are strict (cf. [GM03], [KS90])

- It is possible to add morphisms. Since $S(\mathcal{N}_0)$ is a multiplicative class, it satisfies the <u>Ore condition</u>:

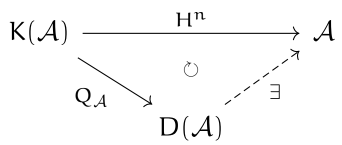


  then given two roofs $(X, s, f), (Y, t, g)$ in $D(\mathcal{A})$ we can construct the sum in the following way

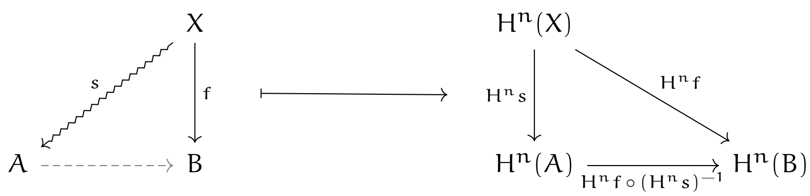


  Therefore we can replace the two roofs with the an equivalent one given by the sum of f and g in $K(\mathcal{A})$.

- Cohomology functors are well defined: there exists a unique functor that makes the following diagram commute



  We only need to define it on roofs, since quasi-isomorphisms induce isomorphisms in cohomology we have:



So morphisms in $D(\mathcal{A})$ can be tricky to work with, however in some cases it is possible to work with a special class of complexes for which it is available a simplified description.

Proposition 1.3.8. *Let* $A \rightsquigarrow B$ *a quasi-isomorphism and let* I (P) *a complex of injective (projective resp.) objects of* $\mathcal{A}$*. Then*

$$\mathrm{Hom}_{K^+(\mathcal{A})}(B, I) \xrightarrow{\sim} \mathrm{Hom}_{K^+(\mathcal{A})}(A, I)$$

$$\left(\textit{dually} \quad \mathrm{Hom}_{K^-(\mathcal{A})}(P, B) \xrightarrow{\sim} \mathrm{Hom}_{K^-(\mathcal{A})}(P, A) \right)$$

are isomorphisms.

proof (sketch): Since $\mathrm{Hom}_{K^+(\mathcal{A})}(-, I)$ is cohomological ([KS90]), by completing the morphism $A \longrightarrow B$ to a d.t. (in $K(\mathcal{A})$) and by the corresponding LES, we have

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(C, I) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(B, I) & \xrightarrow[\sim]{} & \mathrm{Hom}_{\mathcal{A}}(A, I) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(C[1], I) & \longrightarrow \cdots \\ \text{(claim)} & \wr\! \| & & & & & & \wr\! \| & \\ & 0 & & & & & & 0 & \end{array}$$

By definition of d.t. we know that C must be isomorphic to $\mathrm{cone}(f)$, since f is a q.is, C is acyclic, thus we only need to prove the following

<u>**Claim**</u> For any C acyclic in $C(\mathcal{A})$ (i. e. $C \rightsquigarrow 0$), holds

$$\mathrm{Hom}_{K(\mathcal{A})}(C, I) \simeq 0$$

The idea is to construct, for any map $f \in \mathrm{Hom}_{K(\mathcal{A})}(C, I)$, an homotopy to the 0 complex by successively killing the map $f^n : C^n \longrightarrow I^n$ at step n, exploiting the injectivity of I^n and pullbacks of the commuting squares (cf. [Huy06] for details). □

Now present the <u>key</u> result for the subsequent definition of derived functors between derived categories.

Corollary 1.3.9. *If* A *is an arbitrary complex in* $C^+(\mathcal{A})$ *and* I *a complex with injective terms, then*

$$\mathrm{Hom}_{K(\mathcal{A})}(A, I) \simeq \mathrm{Hom}_{D(\mathcal{A})}(A, I)$$

Proof: For every roof

$$\begin{array}{ccccc} & & B & & \\ & {}^{s}\swarrow\!\!\sim & & \searrow^{g} & \\ A & & \overset{f}{\dashrightarrow} & & I \end{array} \quad \in \mathrm{Hom}_{D(\mathcal{A})}(A, B)$$

there exists a unique morphism f given by the isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{K(\mathcal{A})}(A, I) & \xrightarrow[\sim]{-\circ s} & \mathrm{Hom}_{K(\mathcal{A})}(B, I) \\ g & \longmapsto & f \end{array}$$

By last proposition. □

The tag line is the following:

> Hom in $D(\mathcal{A})$ is what we are interested most, Hom in $K(\mathcal{A})$ is what we can compute.

Application

Theorem 1.3.10. *For any* $X, Y \in \mathcal{A}$ *, we have*[6]*:*

$$\mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet) \simeq \mathrm{Ext}^i_{\mathcal{A}}(X, Y), \ \forall i \geq 0$$

[6] We reintroduce bullets to avoid confusion, furthermore for $X \in \mathcal{A}$, then we will denote $X^\bullet$ the complex concentrated in degree 0, i. e. $0 \longrightarrow X \longrightarrow 0$ in $C(\mathcal{A})$

Proof: Let us consider an injective resolution[7] $0 \longrightarrow Y \longrightarrow I^\bullet$ i. e.

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \wr\downarrow \varepsilon & & & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array}$$

Then

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet[i]) &\simeq \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, I^\bullet[i]) \\ &\simeq \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet[i]) \end{aligned}$$

Where the second isomporhisms comes from last Corollary. More explicitly let $f \in \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet[i])$

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow & \swarrow s & \downarrow f & \swarrow 0 & & \swarrow 0 & \\ I^{i-1} & \xrightarrow[d^{i-1}]{} & I^i & \xrightarrow[d^i]{} & I^{i+1} & \xrightarrow[d^{i+1}]{} & \cdots \end{array}$$

The diagram above is to be considered up to homotopy. Then we apply the covariant functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ to obtain the following LES

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(X, I^{i-1}) & \xrightarrow{d} & \mathrm{Hom}_{\mathcal{A}}(X, I^i) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(X, I^{i+1}) & \longrightarrow & \cdots \\ & & s & \xmapsto[(I)]{} & d^{i-1} \circ s = f & \xmapsto[(II)]{} & 0 & & \end{array}$$

Where exactness in the first slot tells us that [8]

$$f \in \mathrm{Im}(d^{i-1} \circ -) \iff f \sim 0 \text{ i. e. } f = (d^{i-1} \circ s + 0),$$

and exactness in the second slot means

$$f \in \mathrm{Ker}(d^i \circ -) \iff f \text{ is a chain map}$$

Therefore chain maps modulo homotopy are exactly:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(X^\bullet, I^\bullet[i]) \simeq \frac{\ker(d^i \circ -)}{\mathrm{Im}\, f(d^{i-1} \circ -)} &=: H^i(\mathrm{Hom}_{\mathcal{A}}(X, I^\bullet)) \\ &=: \mathrm{Ext}^i_{\mathcal{A}}(X, Y) \end{aligned}$$

□

Remark 1.3.11. [KS90] The construction given in the last theorem can be generalized to arbitrary complexes $X^\bullet, Y^\bullet \in C(\mathcal{A})$. We obtain the following isomorphisms

$$H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet)) \simeq \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet)$$

Example 1.3.12 (Computation)**.** Let us consider the following free (i. e. projective) resolution of $\mathbb{Z}/2\mathbb{Z}$ in **Ab**

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

In order to compute $\mathrm{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ we apply the contravariant functor $\mathrm{Hom}_{\mathbf{Ab}}(-, \mathbb{Z}/2\mathbb{Z})$ to the sequence above,

[7]Recall $\mathcal{A}$ is assumed to be abelian with enough injectives

[8]This has to be read as maps of complexes, with appropriate induced maps at degrees $i-1, i, i+1$

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Ab}}(A,B) \longrightarrow \mathrm{Hom}_{\mathbf{Ab}}(A,B) \longrightarrow \mathrm{Hom}_{\mathbf{Ab}}(A,B) \longrightarrow \mathrm{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

Thus we have[9]

$$\mathrm{Ext}^0_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathrm{Hom}_{D(\mathbf{Ab})}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$
$$\mathrm{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathrm{Hom}_{D(\mathbf{Ab})}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}[1])$$

Where the non trivial element in the latter corresponds to

$$\begin{array}{lccccccc}
\mathbb{Z}/2\mathbb{Z}^\bullet : & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
\quad\nwarrow\!\!\sim & & & & & \uparrow\wr & & \\
\qquad P^\bullet : & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\
\quad\swarrow & & & \downarrow & & & & \\
\mathbb{Z}/2\mathbb{Z}^\bullet : & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

1.4 Derived functors

We fix the full <u>additive</u> subcategory $\mathcal{I} \subset \mathcal{A}$ of all injective objects. Then $K^*(\mathcal{I})$ [10], the corresponding homotopy category, is well defined and triangulated. Therefore the inclusion of $\mathcal{I}$ in $\mathcal{A}$ yields a natural functor

$$\begin{array}{ccccc}
\mathcal{I} & \hookrightarrow & \mathcal{A} & & \\
\downarrow & & \downarrow & & \\
K^*(\mathcal{I}) & \hookrightarrow & K^*(\mathcal{A}) & \xrightarrow{Q_{\mathcal{A}}} & D^*(\mathcal{A})
\end{array}
\qquad i : K^*(\mathcal{I}) \dashrightarrow D^*(\mathcal{A})$$

Theorem 1.4.1. *Suppose $\mathcal{A}$ has enough injectives. Then*

$$K^+(\mathcal{I}) \xrightarrow[\sim]{i} D^+(\mathcal{A})$$

is an equivalence of categories.

proof (sketch):

(1) Since $\mathcal{I}$ is cogenerating[11], then it is possible to add a layer of abstraction and prove that every complex in $K^+(\mathcal{A})$ embeds quasi-isomorphically into a complex of injectives[12]

(2) $\mathcal{N}_{\mathcal{I}} \equiv \mathcal{N}^+(K(\mathcal{I})) := \mathcal{N}(K(\mathcal{A})) \cap K^+(\mathcal{I})$ is a null system.

(3) From the theory of localization of subcategories and the previous step, we obtain
$$K^+(\mathcal{I})/\mathcal{N}_{\mathcal{I}} \subset K^+(\mathcal{A})/\mathcal{N}_{\mathcal{A}} =: D^+(\mathcal{A})$$
Is a full embedding

[9] As complex concentrated in zero, shifted by 1: $\mathbb{Z}/2\mathbb{Z}[1]$

[10] For $* = \varnothing, b, +, -$

[11] Cf. [KS06]. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$. We say that $\mathcal{J}$ is cogenerating if for all X in $\mathcal{A}$, there exist $Y \in \mathcal{J}$ and a monomorphism $X \rightarrowtail Y$.

[12] Ibid.

(4) Let $I^\bullet \longrightarrow J^\bullet$ in $C^+(\mathcal{I})$ then[13]

$$I^\bullet, J^\bullet \rightsquigarrow 0 \implies f \sim 0$$

Therefore elements in $S(\mathcal{N}_\mathcal{I})$ are already isomorphisms:

$$K^+(\mathcal{I})/\mathcal{N}_\mathcal{I} = K^+(\mathcal{I})$$

(5) Any object of $D^+(\mathcal{A})$ is isomorphic to an object in $K^+(\mathcal{I})$

Observe, (2) and (3) set i to be fully-faithful, (5) is merely a restatement of (1) which in turn assure essential surjectivity. □

Proposition 1.4.2. *Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ any additive functor between abelian categories, then F naturally extends to a functor $K^*(\mathcal{A}) \xrightarrow{K^*} K^*(\mathcal{B})$. Furthermore, if F is exact, we have:*

1. *$K^*(F)$ maps q.is to q.is (in particular, acyclic to acyclic), so induces the following commutative diagram.*

$$\begin{array}{ccc} K^*(\mathcal{A}) & \xrightarrow{K^*(F)} & K^*(\mathcal{B}) \\ \downarrow Q_\mathcal{A} & & \downarrow Q_\mathcal{B} \\ D^*(\mathcal{A}) & \xrightarrow{D^*(F)} & D^*(\mathcal{B}) \end{array}$$

2. DF *maps distinguished triangles in distinguished triangles*[14] *(i. e. is triangulated*[15]*).*

Remark 1.4.3 (Important)**.** Last proposition doesn't hold true if F is not exact to and the naive extension (term-wise) of F to a functor $D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{B})$ doesn't make sense for obvious reasons.

But the good news is we can ask for exactness only on one side and still retrive a unique lift on the respective derived categories

Definition 1.4.4. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ left exact

$$\begin{array}{ccccc} K^+(\mathcal{I}) & \subset & K^+(\mathcal{A}) & \xrightarrow{K^+(F)} & K^+(\mathcal{B}) \\ \uparrow i^{-1} & \searrow i & \downarrow & & \downarrow Q_\mathcal{B} \\ & & D^+(\mathcal{A}) & \overset{RF}{\dashrightarrow} & D^+(\mathcal{B}) \end{array}$$

Then we define the right derived functor RF as the following composition

$$RF := Q_\mathcal{B} \circ K^+(F) \circ i^{-1}$$

Dually, if F is right exact, we define the left derived functor LF by replacing $K^+(\mathcal{A})$ with the homotopy category $K^-(\mathcal{P})$ of complexes made of projective objects.

Remarks 1.4.5.

1. Everything in this section can be dualized, i. e. if F is right exact, we define the left derived functor LF by replacing $K^+(\mathcal{A})$ with the category $K^-(\mathcal{P})$[16] and repeat the same arguments, provided that the categories have enough projectives.

[13] Cf. [GM03]. Recall that we already know (;;add ref) $f \sim 0 \implies f = 0$ in $D(\mathcal{A})$

[14] Cf. [GM03]. Any additive functor maps cones to cones, cylinders to cylinders, then to prove (b) is enough to apply (a)

[15] When working with derived categories "triangulated" and "exact" are used s synonyms labeling an additive functor

[16] Or even an adapted classes of object to a specific functor

2. The choice of a quasi inverse i^{-1} amounts to the choice of a q.is $X^\bullet \rightsquigarrow I^\bullet$, so that on objects we have:

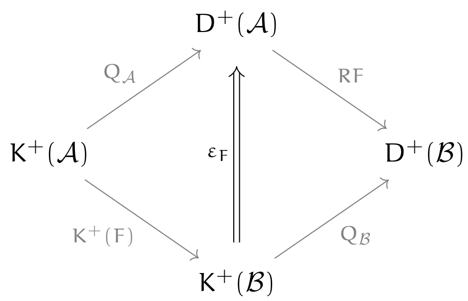
$$RF(X^\bullet) := K^+F(I^\bullet) \in D^+(\mathcal{B}).$$

3. This definition leads to some ambiguity, namely the choice of a quasi inverse i^{-1} is <u>not unique</u> .

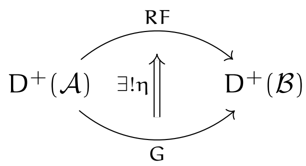
To atone point 3, we need the following proposition:

Proposition 1.4.6. *Let* RF *the right derived functor of* $F : \mathcal{A} \longrightarrow \mathcal{B}$ *left exact, then*

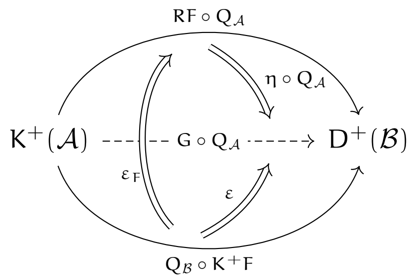
1. RF *is triangulated, i. e. sends distinguished triangles to distinguished triangles.*
2. *Exists a morphism of functors* $Q_\mathcal{B} \circ K^+(F) \xrightarrow{\varepsilon_F} RF \circ Q_\mathcal{A}$



satisfying the following universal property: for any $D^+(\mathcal{A}) \xrightarrow{G} D^+(\mathcal{B})$ *triangulated functor and any morphism* $Q_\mathcal{B} \circ K^+(F) \xrightarrow{\varepsilon} G \circ Q_\mathcal{A}$ *there exists a unique morphism of functor* η*:*



such that the internal triangle of morphisms of functors commute:



Proof:

1. Because RF is the following composition of triangulated functors:

$$RF : \; A \xrightarrow{i^{-1}} B \xrightarrow{KF} C \xrightarrow{Q_\mathcal{B}} D$$

where:

- i^{-1} is triangulated since $i^{-1} \dashv i$ and i is triangulated (cf. [Huy06]).
- KF is triangulated because F is exact on $\mathcal{I}$.
- $Q_\mathcal{B}$ is triangulated by definition.

2. Cf. [GM03]

□

Remarks 1.4.7.

- Last proposition determines RF up to unique isomorphism
- The derived functor can be defined as the couple $(\mathrm{RF}, \varepsilon_F)$ being the right localization of the functor $Q_B \circ K^+(F)$, that is, the representative of the following functor [17]

$$\begin{aligned} [D^+(\mathcal{A}), D^+(\mathcal{B})] &\longrightarrow \mathbf{Set} \\ G &\longmapsto \mathrm{Hom}_{[K^+(\mathcal{A}), K^+(\mathcal{B})]}(Q_{\mathcal{B}} \circ K^+(F), G \circ Q_{\mathcal{B}}) \end{aligned}$$

  Thus, last proposition is hidden behind all the machinery of the construction of the localization of functors.
- We could replace from the beginning of this section the category $\mathcal{I}$ with $\mathcal{I}_F$ the full additive subcategory of F−adapted[18] objects and repeat the same arguments above

Definition 1.4.8. Let RF the right derived functor of a left exact functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$. Then for any complexes $X^\bullet \in D^+(\mathcal{A})$ we define:

$$R^i F(X^\bullet) \coloneqq (H^i \circ \mathrm{RF})(X^\bullet).$$

Note that these correspond to the classical derived functors whenever $X^\bullet$ is a complex concentrated in degree 0

Remarks 1.4.9.

1. Since RF is triangulated and H^0 is cohomological, we obtain for a d.t.

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \xrightarrow{+}$$

   the associated long exact sequence

$$\cdots \longrightarrow R^i F(X^\bullet) \longrightarrow R^i F(Y^\bullet) \longrightarrow R^i F(Z^\bullet) \longrightarrow R^{i+1} F(X^\bullet) \longrightarrow \cdots$$

2. Recall that

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet) = \mathrm{tot}\left(\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{A}}(X^\bullet, Y^\bullet)\right).$$

   We can define

$$\mathrm{R}\,\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, -) : D^+(\mathcal{A}) \longrightarrow D^+(\mathbf{Ab})$$

   and set

$$\mathrm{Ext}^i_{\mathcal{A}}(X^\bullet, Y^\bullet) \coloneqq H^i\left(\mathrm{R}\,\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet)\right)$$

   So that we can generalize what we saw before: for $X^\bullet \in D^-(\mathcal{A})$ and $Y^\bullet \in D^+(\mathcal{A})$

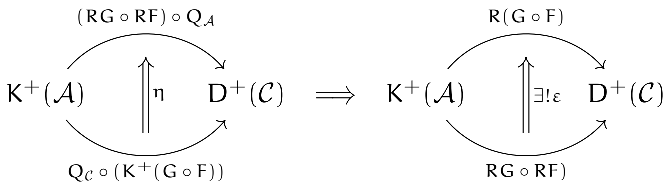
$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(X^\bullet, Y^\bullet) &\simeq \mathrm{Hom}_{\mathcal{A}}(X^\bullet, I^\bullet) \\ &\simeq \mathrm{Hom}_{\mathcal{A}}(X^\bullet, I^\bullet) \\ &\simeq \mathrm{Hom}_{\mathcal{A}}(X^\bullet, Y^\bullet) \end{aligned}$$

Theorem 1.4.10. *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be two left exact functors between abelian categories. Since $G \circ F$ is left exact, by the universal property of $(\mathrm{R}(G \circ F), \varepsilon_{G \circ F})$ we have a natural morphism of functor $\mathrm{RG} \circ \mathrm{RF} \xrightarrow{\varepsilon} \mathrm{R}(G \circ F)$.*

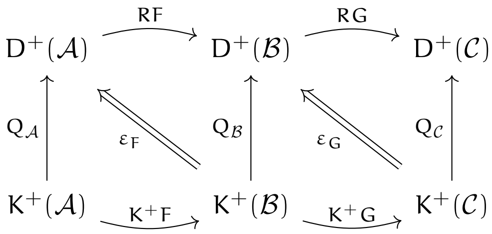
[17] Cf. [KS06]. We adopt the shorthand notation $[\mathcal{A}, \mathcal{B}]$ for the category of functors between two categories $\mathcal{A}$ and $\mathcal{B}$

[18] Also called F−injective if F is left exact, or F−projective for F right exact

Recall from Proposition 1.4.6 we have



Where η *is the horizontal composition of the following two morphism of functors:*



$$\begin{array}{c} \eta_{X^\bullet}: \; Q_{\mathcal{C}} \circ K^+G(K^+F(X^\bullet)) \\ \downarrow \\ K^+G \circ Q_{\mathcal{B}}(K^+F(X^\bullet)) \\ \downarrow f \\ K^+G \circ K^+F(Q_{\mathcal{A}}(X^\bullet)) \end{array}$$

Moreover if we assume that exists the full additive categories $\mathcal{I}_F \subset \mathcal{A}, \mathcal{I}_G \subset \mathcal{B}$*, where* $\mathcal{I}_F$ *is* $F-$*adapted and* $\mathcal{I}_G$ *is* $G-$*adapted such that* $F(\mathcal{I}_F) \subset \mathcal{I}_G$*. Then the morphism of functors* ε *is an isomorphism*

Proof: Note that the assumption $F(\mathcal{I}_F) \subset \mathcal{I}_G$ implies that $\mathcal{I}_F$ is also $(G \circ F)-$adapted. Now let $X^\bullet \in K^+(\mathcal{A})$, we know that exists a q.is X

$$\begin{array}{rcl} R(G \circ F)(X^\bullet) & \xrightarrow{\varepsilon_{X^\bullet}} & RG \circ (RF(X^\bullet)) \\ \simeq (K^+G \circ K^+F)(I^\bullet) & & \simeq RG \circ (RF(I^\bullet)) \\ & & \simeq K^+G \circ (K^+F(I^\bullet)) \end{array}$$

Is an isomorphism for all $X^\bullet \in K^+(\mathcal{A})$, therefore ε is an isomorphism. □

2 The Derived Category of Coherent Sheaves

I never saw a moor,
I never saw the sea;
Yet know I how the heather looks,
And what a wave must be.
Emily Dickinson

2.1 Preliminaries: Sheaves of Modules

The notion of sheaves of modules over a ringed space $(X, \mathcal{O}_X)$ allow us to refine our understanding of the geometry of the space[1] by making, broadly speaking, more functions or function-like objects available. Within this setting there are two particularly relevant notions, namely quasi-coherent and coherent sheaves, they will be the "non-local" analogous of the usual notions, respectively, of modules and of finitely generated modules over a ring.

Definition 2.1.1. Let $\mathcal{O}_X$ be a sheaf of rings on a topological space X, an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of abelian groups over X with the following additional requirement: for all $U \in X$ open, $\mathcal{F}(U)$[2] has the structure of an $\mathcal{O}_X(U)$-module compatible with restrictions, i. e. for $V \subseteq U$ in $\mathrm{Op}(X)$[3]

$$\begin{array}{ccccccccc} U & & \Gamma(U,\mathcal{O}_X) & \times & \Gamma(U,\mathcal{F}) & \xrightarrow{\text{action}} & \Gamma(U,\mathcal{F}) & & U \\ \downarrow & \overset{\mathcal{O}_X}{\longmapsto} & \downarrow{\scriptstyle \mathrm{res}^{\mathcal{O}}_{VU}} & & \downarrow & & {\scriptstyle \mathrm{res}^{\mathcal{F}}_{VU}}\downarrow & \underset{\mathcal{F}}{\longleftarrow\!\shortmid} & \downarrow \\ V & & \Gamma(V,\mathcal{O}_X) & \times & \Gamma(V,\mathcal{F}) & \xrightarrow{\text{action}} & \Gamma(V,\mathcal{F}) & & V \end{array} \tag{1}$$

Where $\mathrm{res}^{\mathcal{F}}_{VU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ denote the usual restriction of the sheaf $\mathcal{F}$. A morphism $\mathcal{F} \longrightarrow \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves such that for each open set $U \subseteq X$, $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism.

We now present few more constructions and known facts (cf. [Har77] II.5).

Definition 2.1.2. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules, we denote the group of morphisms from $\mathcal{F}$ to $\mathcal{G}$ by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})(\equiv \mathrm{Hom}(\mathcal{F},\mathcal{G})$ whenever it causes no harm). If $U \subseteq X$ is open, then the restriction $\mathcal{F}|_U$ is an $\mathcal{O}_X|_U$-module, where the restrictions have to be understood as functors $\mathrm{Op}_U^{\mathrm{op}} \longrightarrow$ **Rings**.

- The presheaf $U \longrightarrow \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, B\ \mathcal{G}|_U)$ is indeed a sheaf that we will call <u>sheaf Hom</u> and denote as

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \quad (\text{abbr. } \mathcal{H}om_X(\mathcal{F},\mathcal{G}))$$

- The sheaf associated[4] to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

[1]For "geometry" on a topological space X we mean what kind of structure sheaf we give to X
[2]We will also use $\Gamma(U,\mathcal{F})$ to denote the sections of a sheaf $\mathcal{F}$
[3]The notation Op_X stands for the poset category of all open sets of the topological space X
[4]The functor $(-)^a$ denotes the sheafification functor, we call $(\mathcal{F})^a$ the sheaf <u>associated</u> to a presheaf $\mathcal{F}$

is called the tensor product of $\mathcal{F}$ and $\mathcal{G}$, and it will be denoted

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \quad (\text{abbr. } \mathcal{F} \otimes_X \mathcal{G})$$

We now focus on particular examples of $\mathcal{O}_X$-modules

Definition 2.1.3. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.

- We say that $\mathcal{F}$ is a free sheaf of rank n, if there is an isomorphism $\mathcal{F} \simeq \mathcal{O}_X^{\oplus n}$.
- We shall say that $\mathcal{F}$ is locally free if there is an open cover of X, $U_{i\,i \in I}$ such that $\mathcal{F}|_{U_i}$ id free for all U_i. If X is connected then the rank of a locally free sheaf is well defined and it will be the same across all the open sets of the cover of X.
- A locally free sheaf of rank 1 is called invertible sheaf.

Example 2.1.4. Every vector bundle can be casted as a locally free sheaf, in fact the respective categories are equivalent (cf. [PM97], I.1.8)

Remark 2.1.5. The category of locally free sheaves is not abelian. The following bundle exemplify the problem

$$E = [0, 1] \times \mathbb{R} \xrightarrow{\pi} \mathbb{R}.$$

Then the kernel and the image of the map below are not locally free

$$\begin{aligned} E &\longrightarrow E \\ (x, t) &\longmapsto (x, xt) \end{aligned}$$

This problem is solved by encasing the category of locally free sheaves into *reasonable*, larger abelian categories, so that we are able to apply the tools of homological algebra. We'll soon discover which properties we would like to address as reasonable.

We first need to add further syntax to the language of sheaves of modules, in order to operate fully.

Definition 2.1.6. Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, $\mathcal{F}$ an $\mathcal{O}_X$-module and G an $\mathcal{O}_Y$-module, then

- $f_*\mathcal{F}$ is an $f_*\mathcal{O}_Y$-module, since we have the morphism of ringed spaces

  $$f^\# : \mathcal{O}_Y \longrightarrow f_* O_X$$

  as part of the datum coming along with f. There is a natural structure of $\mathcal{O}_Y$-module[5] on $f_*\mathcal{F}$, we will call this sheaf the direct image or pushforward of $\mathcal{F}$ along the morphism f.
- Likewise $f^{-1}G$ is an $f^{-1}\mathcal{O}_Y$-module, by adjointness of the pair $f^{-1} \dashv f_*$ in the category of ringed spaces, there is an unique morphism $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$. [6]

  Then in order to provide a structure of $\mathcal{O}_X$-module to $f^{-1}\mathcal{G}$ we rely on the following tensor product:

  $$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

  This will be called the inverse image or the pullback of $\mathcal{G}$ along f. [7]

Remark 2.1.7. $f^* \dashv f_*$ is an adjoint pair between the categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules.

Lemma 2.1.8. *(Cf. [Sta23], 17.10; ibid. 26.7). Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module, then the following are equivalent*

[5] The structure of an $\mathcal{O}_Y$-module is given by further composing the diagram (1) with the morphism $f^\#$

[6] Recall $f^{-1}\mathcal{G}(U) := \varinjlim_{V \supseteq f(U)} \mathcal{G}(U)$, as a presheaf

[7] Recall that if A, B are rings, given a ring homomorphism $A \xleftarrow{f} B$ and an A-module M, then M can be made also a B-module via the morphism f, thus $\mathcal{O}_X(U)$ is an $f^{-1}\mathcal{O}_Y$-module via the adjoint map to $f^\#$

- *[Gro60]: There exists a covering $\{U_\alpha\}$ of X such that on each open U_α, $\mathcal{F}|_{U_\alpha}$ fits the following exact sequence*

$$\mathcal{O}_X|_{U_\alpha}^{\oplus I_\alpha} \xrightarrow{\ \varphi\ } \mathcal{O}_X|_{U_\alpha}^{\oplus J_\alpha} \xrightarrow{\ \psi\ } \mathcal{F}|_{U_\alpha} \longrightarrow 0$$

  Where I_α and J_α may be infinite.[8]

- *[Har77]: For any affine open subscheme $\operatorname{Spec} A$ of X and any $f \in A$, the map induced by the universal property of the localization*

$$\Gamma(\operatorname{Spec} A, \mathcal{F})_f \xrightarrow{\ \sim\ } \Gamma(A_f, \mathcal{F})$$

  is an isomorphism.

Definition 2.1.9. Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module

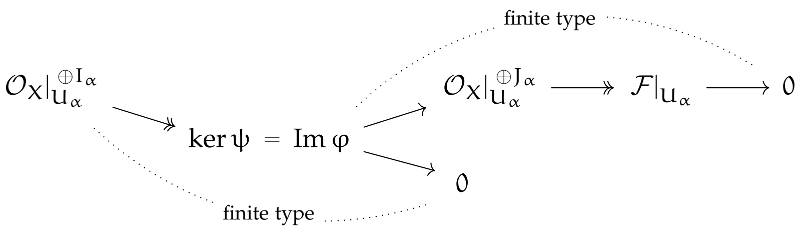
- $\mathcal{F}$ is of finite type if for any affine open $\operatorname{Spec} A = U$, the A-module $M = \Gamma(U, \mathcal{F})$ is finitely generated i. e. exists a surjective morphism

$$\mathcal{O}_X|_U^{\oplus n} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

  where n here is finite, thus $\mathcal{F}$ is locally generated by finitely many sections.

- $\mathcal{F}$ is quasi-coherent if any of the two equivalent conditions in Lemma 2.1.8 are met.

- $\mathcal{F}$ is coherent if It is quasi-coherent with I_α and J_α finite, i. e.



Theorem 2.1.10. *The categories of quasi-coherent sheaves* $\mathrm{QCoh}(X)$ *and of coherent sheaves* $\mathrm{Coh}(X)$ *are both abelian.*

$\mathcal{O}_X$-modules (abelian)	$\supseteq$	*Quasi-coherent sheaves (abelian)*	$\supseteq$	*Coherent sheaves (abelian)*	$\supseteq$	*locally free sheaves (not abelian)*

Remark 2.1.11. Given an affine scheme $\operatorname{Spec} R$ over a commutative ring R, then there is an equivalence of categories:

$$\begin{array}{ccc} \mathcal{F} & \longmapsto & \Gamma(\operatorname{Spec} R, \mathcal{F}) \\ \mathrm{QCoh}(\operatorname{Spec} R) & \underset{\sim}{\rightleftarrows} & R-\mathrm{Mod} \\ \widetilde{M} & \longleftarrow\!\shortmid & M \end{array}$$

Where $\widetilde{M}$ refers to the following construction: let $U \subseteq \operatorname{Spec} R$ then we define $\Gamma(U, \widetilde{M})$ to be the set of sections $s : U \longrightarrow \coprod_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that $s(\mathfrak{p}) \in M_\mathfrak{p}$ and are locally given by a fraction i. e., exists an open subset $U(\mathfrak{p}) \subseteq U$ such that $\forall q \in U(\mathfrak{p})$, $s(q) = \frac{m}{f}$ for $m \in M$ and $f \in R$[9]

[8] A sheaf satisfying (1) is also called locally presentable
[9] This construction is referred as the Espace Etalé, see for instance [Ten75]

2.2 Lack of Injectives and Consequences

We now expose the key environment that later results will inhabit.

Definition 2.2.1. Let X be a scheme, we define its derived category $D^b(X)$ to be the bounded derived category of the abelian category $\mathrm{Coh}(X)$, in symbols:

$$D^b(X) := D^b(\mathrm{Coh}(X))$$

Recall that, given k a field, an additive category $\mathcal{C}$ is said to be k-linear if every group $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is endowed with the structure of a k-vector space, compatible with the composition, i. e.

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$$

is a k-bilinear map for all $A, B, C \in \mathcal{C}$.

Moreover an additive functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between k-linear categories is called k-linear, if it is linear at the level of morphisms, i. e.

$$F : \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(FA, FB)$$

is k-linear for all $A, B \in \mathcal{C}$

Definition 2.2.2. Two schemes X and Y over a field k are called derived equivalent if there exists a k-linear triangulated (exact) equivalence

$$D^b(X) \xrightarrow{\sim} D^b(Y)$$

Recall from the previous chapter, for an abelian category $\mathcal{A}$ with enough injectives we have the following equivalence of triangulated categories

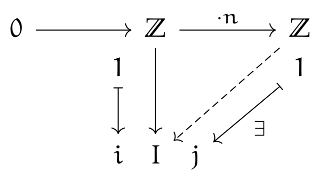
$$K^+(I) \underset{\sim}{\overset{i}{\hookrightarrow}} D^+(\mathcal{A})$$

Then for a left exact functor $F : A \longrightarrow B$ between abelian categories we could construct its right derived functor as

$$RF := Q_B \circ K(F) \circ i^{-1}$$

Remark 2.2.3. In general the category of coherent sheaves $\mathrm{Coh}\, X$ over a scheme X does not possess enough injectives

Example 2.2.4. Let us consider $X = \mathrm{Spec}(\mathbb{Z})$, then $\mathrm{Coh}\, X$ is equivalent to the category of finitely generated abelian groups, let I be an injective in such category. Fix $i \in I$ we have



Maps from $\mathbb{Z} \longrightarrow I$ are uniquely identified by where they send the unit. Since I is injective, the above diagram must commute, thus we have $nI = I$ for every $n \in \mathbb{N}$. This means that I is divisible, but there are no non trivial finitely generated divisible abelian groups.

Therefore in order to compute derived functors we need to go through a bigger category.

Proposition 2.2.5. *([Har06], II.7.18). On a Noetherian scheme X any quasi-coherent sheaf $\mathcal{F}$ admits a resolution*

$$I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

by quasi-coherent sheaves $I^{i \geqslant 0}$ which are injective in $\mathrm{Mod}(\mathcal{O}_X)$. Indeed, the injective hull in $\mathrm{Mod}(\mathcal{O}_X)$ of any quasi-coherent sheaf is quasi-coherent.

Proposition 2.2.6. *(Cf. [Huy06], 3.5). Let* X *be a Notherian scheme. Then the functor induced by the inclusion*

$$D^b(X) \hookrightarrow D^b(\mathrm{QCoh}\,X)$$

defines an equivalence of categories between $D^b(X)$ *and the full triangulated subcategory* $D^b_{\mathrm{Coh}}(\mathrm{QCoh}\,X)$ *of bounded complexes of quasi-coherent sheaves with coherent cohomology.*

Remark 2.2.7. By the characterization of morphisms in derived category in the previous chapter we have

$$\mathrm{Ext}^i_{\mathrm{QCoh}\,X}(\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{D^b(X)}(\mathcal{F},\mathcal{G}[i])$$

for any $\mathcal{F},\mathcal{G}$ coherent sheaves. This can be extended to complexes $\mathcal{E}^\bullet,\mathcal{F}^\bullet$ of coherent sheaves. Cf. [Huy06] 2.57.

2.3 Duality and Dimension

2.3.1 Serre functors, Serre duality

Definition 2.3.1. Let $\mathcal{A}$ a k-linear category. A Serre functor is a k-linear equivalence $S : \mathcal{A} \longrightarrow \mathcal{A}$, such that for any two objects $A, B \in \mathcal{A}$ exists an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(A,B) \longrightarrow \mathrm{Ham}_{\mathcal{A}}(A,S(B))$$

of k-vector spaces, functorial in both slots. Then we will write the induced pairing as:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(\mathcal{B},S(A)) \times \mathrm{Hom}_{\mathcal{A}}(A,B) & \longrightarrow & k \\ (f,g) & \longmapsto & \langle f\,|\,g\rangle \end{array}$$

Remark 2.3.2. For any locally free sheaf M on X, the functor

$$\begin{array}{ccc} \mathrm{Coh}\,X & \longrightarrow & \mathrm{Coh}\,X \\ \mathcal{F} & \longmapsto & \mathcal{F}\otimes\mathcal{M} \end{array}$$

is exact. To see this, it is enough to work locally on stalks. Let:

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be an exact sequence in $\mathrm{Mod}(\mathcal{O}_X)$, then

$$0 \longrightarrow \mathcal{F}_1\otimes\mathcal{M} \xrightarrow{f} \mathcal{F}_2\otimes\mathcal{M} \xrightarrow{g} \mathcal{F}_3\otimes\mathcal{M}$$

Is in principle only right exact, but $\forall x \in X$, $\mathcal{M}_x \simeq (\mathcal{O}_X)_x^{\oplus n}$, therefore

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{F}_1\otimes\mathcal{M})_x & \longrightarrow & (\mathcal{F}_2\otimes\mathcal{M})_x & \longrightarrow & (\mathcal{F}_3\otimes\mathcal{M})_x & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & \wr\| & & \\ & & (\mathcal{F}_1)_x^{\oplus n} & \longrightarrow & (\mathcal{F}_2)_x^{\oplus n} & \longrightarrow & (\mathcal{F}_3)_x^{\oplus n} & & \end{array}$$

is exact. Then in particular $\mathcal{M}\otimes -$ induces a triangulated functor on the derived category of X

$$D^*(x) \xrightarrow{\mathcal{M}\otimes -} D^*(x) \qquad * = \pm, b$$

Now consider a smooth projective variety X over a field k, ω_X its canonical bundle.

Definition 2.3.3. Let X be a smooth projective variety of dimension n. Then we define S_X to be the following composition of triangulated functors

$$D^*(X) \xrightarrow[\omega_X\otimes -]{} D^*(X) \xrightarrow[[n]]{} D^*(X)$$

Theorem 2.3.4 (Serre's Duality). *(Cf. [Huy06], 3.12) Let X be a smooth projective variety over a field k. Then $S_x : D^b(X) \longrightarrow D^b(X)$ is a Serre functor, more explicitly: for any $\mathcal{E}^\bullet, F^\bullet \in D^b(x)$ we have an isomorphism*

$$\mathrm{Hom}_X(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \simeq \mathrm{Hom}_X(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[n])^\vee$$

Remark 2.3.5. In the some setting as above, we can actually retrieve more information

$$\begin{aligned}\mathrm{Ext}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) &\simeq \mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, F^\bullet[i]) \\ &\simeq \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet[i], \mathcal{E}^\bullet \otimes \omega_X[n])^\vee \\ &\simeq \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[n-i])^\vee \\ &\simeq \mathrm{Ext}^{n-i}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)^\vee\end{aligned}$$

All isomorphisms are functorial in both $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$

Serre functors can mold left adjoints into right adjoints and vice versa.

Theorem 2.3.6. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between k-linear categories that admit both Serre functors $S_{\mathcal{C}}, S_{\mathcal{D}}$, respectively. Assume F has a left adjoint $G \dashv F$ so $G : \mathcal{D} \longrightarrow \mathcal{C}$. Then*

$$H \coloneqq S_{\mathcal{C}} \circ G \circ S_{\mathcal{D}}^{-1} : \mathcal{D} \longrightarrow \mathcal{C}$$

is a (the) right adjoint to F.

Proof:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(FX, Y) &\simeq \mathrm{Hom}_{\mathcal{D}}(S_{\mathcal{D}}^{-1}Y, FX)^\vee \simeq \mathrm{Hom}_{\mathcal{C}}(GS_{\mathcal{D}}^{-1}Y, X)^\vee \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(X, S_{\mathcal{C}}GS_{\mathcal{D}}^{-1}Y) \coloneqq \mathrm{Hom}_{\mathcal{C}}(X, HY)\end{aligned}$$

□

2.3.2 Interlude: Homological Dimension

Definition 2.3.7. Let $\mathcal{A}$ be an abelian category, then we say that it has finite homological dimension if there exists an integer l such that $\mathrm{Hom}_{D(\mathcal{A})}(A, B[i]) = 0$ for all $A, B \in \mathcal{A}$ and $i > l$. If $\mathcal{A}$ has enough injectives then this is equivalent to require

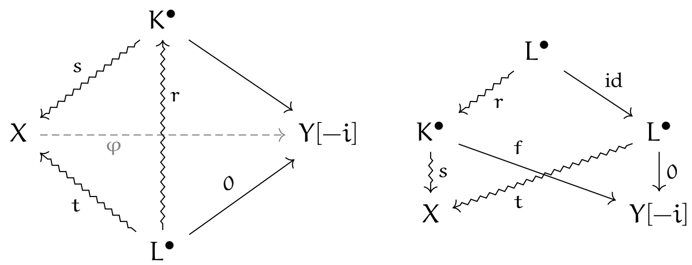
$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0 \quad \text{for all } A, B \in \mathcal{A},\ i > l$$

m such cases we say that $\mathcal{A}$ has homological dimension $\leqslant l$ and it will be denoted $\mathrm{dh}(\mathcal{A}) \leqslant l$

To some extent the complexity of the derived category $D(\mathcal{A})$ is measured (or at least captured) by its homological dimension.

Remark 2.3.8. $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0$ for $i < 0$

Proof: Consider the following diagram in $D(\mathcal{A})$, let $i > 0$



We can construct a complex $L^\bullet$ and quasi-isomorphisms t, s such that the diagram commutes, so that φ would be equivalent to the zero morphism.

So we set $L^\bullet = \tau^{\leqslant i-1} K^\bullet$, i. e.

$$L^\bullet : \quad \cdots \longrightarrow K^{i-3} \xrightarrow{d_K^{i-3}} K^{i-2} \xrightarrow{d_K^{i-2}} \ker d_K^{i-1} \longrightarrow 0 \longrightarrow \cdots$$

Then r is just the natural inclusion, $t^j = s^j$ for all $j < i$. For s is a quasi-isomorphism we have

$$H^0 (K^\bullet) = X \text{ and } H^j (K^\bullet) = 0 \quad \text{for all} \quad j \neq 0$$

Since $i > 0$ it is clear that both r and t ore quasi-isomorphisms. Then the commutativity of the two equivalent diagrams above follows immediately. □

Proposition 2.3.9. *(Cf. [Sch23]). Let $\mathcal{A}$ an abelian category of homological dimension* $\mathrm{dh}(\mathcal{A}) \leqslant 1$, *let* $X \in D^b(\mathcal{A})$. *Then :*

$$X^\bullet \simeq \bigoplus_j H^j (X^\bullet) [-j]$$

Proof: Call the amplitude of $X^\bullet$ the smallest integer k such that $H^j (X^\bullet) = 0$ for j not in an interval of length k. If $k = 0$, this means that exists i such that

$$X^\bullet \simeq H^i (X^\bullet) [-i] \quad \text{in } D^b(\mathcal{A})$$

We then proceed by induction on the amplitude: consider the following distinguished triangle

$$\tau^{\leq n-1} X \longrightarrow \tau^{\leq n} X \longrightarrow H^n(X)[-n] \xrightarrow{+} \tag{2}$$

as

$$\tau^{\leq n-1} X^\bullet \simeq \bigoplus_{j<n} H^j (X^\bullet) [-j]$$

where $X^\bullet$ is bounded.

Claim: the d.t. (2) splits.

To show this, it is enough to show that

$$\begin{aligned} &\mathrm{Hom}_{D^b(\mathcal{A})} \left(H^n(X^\bullet)[-n], \tau^{\leq n-1} X^\bullet[1] \right) \\ \simeq &\bigoplus_{j<n} \mathrm{Hom}_{D^b(\mathcal{A})} \left(H^n(X^\bullet), H^j (X^\bullet) [n-j+1] \right) = 0 \end{aligned}$$

But this follows since $n - j + 1 > 1$ for all $j < n$ and $\mathrm{dh}(\mathcal{A}) \leq 1$.

Now, to produce the splitting, simply notice that $\mathrm{Hom}_{D^b(\mathcal{A})} (-, Y^\bullet)$ is cohomological for all $Y^\bullet \in D^b(\mathcal{A})$, by applying it to (2) with $Y^\bullet = \tau^{\leq n-1} X^\bullet$ yields an exact sequence equivalent to the existence of a splitting. □

Now we move back to our previous setting to collect two important results.

Corollary 2.3.10. *Let $\mathcal{F}, \mathcal{G}$ coherent sheaves on a smooth projective variety X of dimension n. Then* $(\mathrm{h}(\mathrm{Coh}\, X) \leq n$

Proof: By Serre duality, for $n - i < 0$

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^{n-i} (\mathcal{G}, \mathcal{F} \otimes \omega_X)^\vee \simeq 0$$

□

Corollary 2.3.11. *Let C be a smooth projective curve, then any object in $D^b(C)$ is isomorphic to a direct sum $\oplus \mathcal{E}_i[i]$, where $\mathcal{E}_i$ are coherent sheaves on C*

Proof: Follows from the previous results and Serre duality □

2.4 Functors in Derived Geometry

Definition 2.4.1. A <u>thick</u> subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is a full abelian subcategory such that any extensions in $\mathcal{A}$ of objects of $\mathcal{C}$ is again in $\mathcal{C}$, i. e. for all short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

with $M', M'' \in \mathcal{C}$ then also $M \in \mathcal{C}$.

Proposition 2.4.2. *(Cf. [Huy06], 2.42). Let $A \subseteq B$ a thick subcategory, suppose that any object $A \in \mathcal{A}$ can be embedded in an object $A' \in \mathcal{A}$ injective as an object of $\mathcal{B}$. Then the natural inclusion induces an equivalence of triangulated categories. between the derived category $D^+(\mathcal{A})$ and the full triangulated subcategory of $D^+_A(\mathcal{B}) \subseteq D^+(\mathcal{B})$ of complexes with cohomology in $\mathcal{A}$.*

Corollary 2.4.3. *(Cf. [Huy06], 2.68). Suppose $F : K^+(\mathcal{A}k^+(\mathcal{B})$ is exact, then we know that (can be lifted to a triangulated functor*

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

Moreover, assume that $\mathcal{A}$ has enough injectives:

1. *Suppose $\mathcal{C} \subset \mathcal{B}$ is a thick subcategory with $R^iF(A) \in \mathcal{C}$ for all $A \in \mathcal{A}$, then RF take values in $D^+_{\mathcal{C}}(\mathcal{B})$, i. e.*
$$RF : D^+(\mathcal{A}) \longrightarrow D^+_{e}(\mathcal{B})$$
2. *If $RF(A) \in D^b(\mathcal{B})$ for any object $A \in \mathcal{A}$ then $RF(A) \in D^b(\mathcal{B})$ for any complex $A \in D^b(\mathcal{A})$, i. e. RF induces a triangulated functor:*

$$RF : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B})$$

2.4.1 Global Sections

Taking into account the previous preliminary results, let now X to be a Noetherian scheme. Then we know

$$\begin{aligned} \Gamma : \quad \operatorname{QCoh} X &\longrightarrow \operatorname{Vect}_k \\ \mathcal{F} &\longmapsto \Gamma(X, \mathcal{F}) \end{aligned}$$

is a left exact functor. Since QCoh X has enough injectives, we can grant the existence of its derived functor:

$$R\Gamma : \quad D^+(X) \longrightarrow D^+(\operatorname{Vect}_k)$$

we will denote the higher derived functors as

$$H^i(X, \mathcal{F}^\bullet) \coloneqq R^i\Gamma(\mathcal{F}^\bullet)$$

which for a complex concentrated in degree zero $\mathcal{F}$, these are just its i-th sheaf cohomology, for an arbitrary complex they go under the denomination of hypercohomology.

Since every complex of vector spaces splits, $\operatorname{dh}(\operatorname{Vect}_k) \leq 1$ and by the results of the previous section, we can conclude that

$$R\Gamma(\mathcal{F}^\bullet) \simeq \bigoplus_i H^i(X, \mathcal{F}^\bullet)[-i]$$

The following non trivial result will help us to route the desired lift of Γ

Theorem 2.4.4 (Grothendiek's Vanishing Theorem)**.** *(Cf. [Har77], III.2.7). Let X be a Noetherian topological space of dimension n. Then for all abelian sheaves $\mathcal{F}$ on X :*

$$H^i(X, \mathcal{F}) = 0$$

For all $i > n$

Theorem 2.4.5 (Serre). *(Cf. [Har77], II.5.19; [Ser55], II.3.44). Let $\mathcal{F} \in \operatorname{Coh} X$ on a projective scheme X over a field k. Then all cohomology groups $H^i(X, \mathcal{F})$ are of finite dimension.*

Remark 2.4.6. For $i = 0$ we obtain a left exact functor

$$\Gamma : \quad \operatorname{Coh} X \longrightarrow \operatorname{Vect}_k^{\text{fin}}$$

to the category of finite dimensional vector spaces. However, computing its right derived functor is trickier, since $\operatorname{Coh} X$ does not have enough injectives as already noted. But for X Noetherian scheme we can leverage the theory that rests upon us

$$\begin{array}{ccccc}
D^b(X) & \xrightarrow[\text{Prp. 2.2.6}]{\sim} & D^b_{\operatorname{Coh}}(\operatorname{QCoh} X) & \xrightarrow{R\Gamma} & D^b(\operatorname{Vect}_k) \\
 & & & \searrow^{\text{Thm. 2.4.5}} & \uparrow \\
 & & & & D^b_{\operatorname{Vect}_k^{\text{fin}}}(\operatorname{Vect}_k) \xrightarrow[\text{Prp. 2.4.2}]{\sim} D^b(\operatorname{Vect}_k^{\text{fin}}) \\
 & & & & \left[\operatorname{Vect}_k^{\text{fin}} \xhookrightarrow[\text{thick}]{} \operatorname{Vect}_k\right]
\end{array} \tag{3}$$

2.4.2 Direct Image

Let $f : x \longrightarrow Y$ be a morphism of Noetherian schemes. The direct image is a left exact functor

$$f_* : \operatorname{QCoh} X \longrightarrow \operatorname{QCoh} Y$$

So we may construct its derived functor as usual:

$$Rf_* : D^+(\operatorname{QCoh} X) \longrightarrow D^+(\operatorname{QCoh} Y)$$

we call higher direct images of a complex $\mathcal{F}^\bullet$, the following sheaves:

$$R^i f_*(\mathcal{F}^\bullet) := H^i(Rf_*(\mathcal{F}^\bullet))$$

In particular for any quasi-coherent sheaf $\mathcal{F}$ on X we obtain quasi-coherent sheaves $R^i f_* \mathcal{F}$. To further clarify the discussion, the following result aims to show what $R^i f_* \mathcal{F}$ on an affine open set.

Proposition 2.4.7. *(Cf. [Har77], II.8.5). Let x be a Noetherion scheme and $f : X \longrightarrow \operatorname{Spec} A$ a morphism of X to an affine scheme. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, we have*

$$R^i f_*(\mathcal{F}) \simeq \widetilde{H^i(X, \mathcal{F})}$$

Remark 2.4.8. Thus for a general morphism $\pi : X \to Y$ and $\mathcal{F} \in \operatorname{QCoh} X$, let $\operatorname{Spec} A \subseteq Y$ the sheaves

$$H^i(\pi^{-1}\widetilde{(\operatorname{Spec} A, \mathcal{F})})$$

patch together to form a quasi-coherent sheaf.

Remark 2.4.9. Therefore the Vanishing theorem applies so that

$$R^i f_* \mathcal{F} = 0 \text{ for } i > \dim X$$

and by ([Huy06], 2.68) we have that Rf_* induces a triangulated functor

$$Rf_* : D^b(\operatorname{QCoh} x) \longrightarrow D^b(\operatorname{QCoh} y)$$

We now list yet another known and important result of algebraic geometry.

Theorem 2.4.10 (Grothendieck Coherence Theorem)**.** *([Har77], II.8.8). Suppose $\pi : X \longrightarrow Y$ is a proper morphism [10] of locally Noetherian schemes. Then for any coherent sheaf $\mathcal{F}$ on X, $R^i\pi_*\mathcal{F}$ is coherent on Y.*

Therefore by repeating similar arguments as above, we can construct

$$Rf_* : \; D^b(X) \longrightarrow D^b(Y)$$

whenever f is proper.

Definition 2.4.11. A sheaf $\mathcal{F}$ is called flasque (or flabby) if for any open subset $U \subseteq X$, the restriction map $res_{U,X} : \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})$ is surjective.

Lemma 2.4.12. *Any flasque sheaf $\mathcal{F}$ om X is f_*-acyclic for any morphism $f : X \longrightarrow Y$. Moreover $f_*\mathcal{F}$ is again flasque.*

Furthermore if we consider a composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of two morphisms, then it holds in general

$$g_* \circ f_* = (g \circ f)_* \tag{4}$$

But, recall that by the last result of the previous chapter, in order to lift (4) to the derived categories we need an f_*-adapted class $\mathcal{I} \subset \mathrm{QCoh}\, X$ such that $f_*(\mathcal{I})$ is contained in an g_*-adapted class in $\mathrm{QCoh}\, Y$. Since $\mathrm{QCoh}\, X$ has enough injectives and injective sheaves are flasque by the lemma above, indeed wee can consider the following isomorphism of functors

$$R(f \circ g)_* \approx Rg_* \circ Rf_* : \quad \mathrm{QCoh}\, X \longrightarrow \mathrm{QCoh}\, Z$$

Remark 2.4.13. Let $f : X \longrightarrow Y$ a morphism of Noetherian schemes over a field k. Then the composition

$$X \xrightarrow{f} Y \longrightarrow \mathrm{Spec}\, k$$

yields

$$R\Gamma(Y, -) \circ Rf_* = R\Gamma(X, -)$$

Its Leray spectral sequence becomes

$$E_2^{p,q} = H^p\left(Y, R^q f_*\left(\mathcal{F}^\bullet\right)\right) \Rightarrow H^{p+q}\left(X, \mathcal{F}^\bullet\right)$$

2.4.3 Local Homs

Let $f \in \mathrm{QCoh}(X)$, where X is a Noetherian scheme. Then the functor

$$\mathcal{H}om_X(\mathcal{F}, -) : \quad \mathrm{QCoh}(X) \longrightarrow \mathrm{QCh}(X) \tag{5}$$

is left exact and $\mathcal{H}om_X(\mathcal{F}, \mathcal{E})$ is quasi-coherent $\mathcal{F}, \mathcal{E} \in \mathrm{QCoh}\, X$. To see this we can work locally on an U_α, since $\mathcal{H}om_X(\mathcal{F}, \mathcal{E})|_{U_\alpha} \simeq \mathcal{H}om_X\left(\mathcal{F}|_{U_\alpha}, \mathcal{E}|_{U_\alpha}\right)$ and $\mathcal{F} \in \mathrm{QCoh}\, X$, from

$$\mathcal{O}_X|_{U_\alpha} \longrightarrow \mathcal{O}_X|_{U_\alpha} \longrightarrow \mathcal{F}|_{U_\alpha} \longrightarrow 0$$

we apply $\mathcal{H}om_X\left(-, \mathcal{E}|_{U_\alpha}\right)$

$$0 \longrightarrow \mathcal{H}om_X\left(\mathcal{F}|_{U_\alpha}, \mathcal{E}|_{U_\alpha}\right) \longrightarrow \mathcal{H}om_X\left(\mathcal{O}_X|_{U_\alpha}^{\oplus J_\alpha}, \mathcal{E}|_{U_\alpha}\right) \longrightarrow \mathcal{H}om_X\left(\mathcal{O}_X|_{U_\alpha}^{\oplus I_\alpha}, \mathcal{E}|_{U_\alpha}\right)$$

[10] Separated, of finite type and universally closed, cf. [Sta23], 29.41

we can pull out the sums and since $\mathcal{H}om_X\left(\mathcal{O}_X|_{U_\alpha}, \mathcal{E}|_{U_\alpha}\right) \simeq \mathcal{E}|_{U_\alpha}$ we have

$$0 \longrightarrow \mathcal{H}om_X\left(\mathcal{F}|_{U_\alpha}, \mathcal{E}|_{U_\alpha}\right) \longrightarrow \bigoplus^{J_\alpha} \mathcal{E}|_{U_\alpha} \longrightarrow \bigoplus^{I_\alpha} \mathcal{E}|_{U_\alpha}$$

is exact, and we know that QCoh X is abelian, so it is closed under kernels and sums. Again, since QCoh X has enough injectives, we can build its derived functor

$$R\mathcal{H}om_X(\mathcal{F}^\bullet, -) : D^+(\mathrm{QCoh}\, X) \to D^+(\mathrm{QCoh}\, X) \tag{6}$$

By definition

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{E}) := R^i\,\mathcal{H}om(\mathcal{F}, \mathcal{E}) \in \mathrm{QCoh}\, X$$

Then we can restrict (5) to coherent sheaves on X (cf. [Huy06], 3.3; [Sta23] 17.22) along the same lines as above

$$\mathcal{H}om_X(\mathcal{F}, -) : \ \mathrm{Coh}\, X \longrightarrow \mathrm{Coh}\, X$$

In particular if $\mathcal{E}, \mathcal{F}$ are coherent on a Noetherian scheme X, also $\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{E})$, computed in the category QCoh X, are coherent. This is due to the existence of a locally free resolution of any coherent sheaf $\mathcal{F}$ over X, and by the following non trivial fact (cf. [Har77], II.5.2; ibid. III.6.8)

$$\begin{aligned} \mathcal{E}xt(\mathcal{F}, \mathcal{E})_x &= H^i\left(R\,\mathcal{H}om(\mathcal{F}, \mathcal{E})_x\right) \\ &\simeq \mathrm{Ext}^i_{(\mathcal{O}_X)_x}(\mathcal{F}_x, \mathcal{E}_x) \\ &= R^i\,\mathrm{Hom}_{(\mathcal{O}_X)_x}(\mathcal{F}_x, \mathcal{E}_x) \end{aligned}$$

by which the latter is finitely generated for $\mathcal{F}_x$ and $\mathcal{E}_x$ finitely generated, the restriction of (6) to the derived category of X is well defined

$$R\mathcal{H}om_X(\mathcal{F}, -) : \ D^+(X) \longrightarrow D^+(X)$$

If in addition we assume X smooth and projective [11] then we get

$$R\mathcal{H}om_X(\mathcal{F}, -) : \ D^b(X) \longrightarrow D^b(X)$$

as higher Ext's vanish. To summarize, we have:

$$\begin{array}{ccc} D^+(\mathrm{QCoh}\, X) & \xrightarrow[R\mathcal{H}om(\mathcal{F},-)]{} & D^+(\mathrm{QCoh}\, X) \\ \uparrow & & \uparrow \\ D^+(X) & \xrightarrow{\mathcal{F}\ \text{Coherent}} & D^+(X) \\ \uparrow & & \uparrow \\ D^b(X) & \xrightarrow{X\ \text{Smooth}} & D^b(X) \end{array}$$

Remark 2.4.14. We define the dual of a complex $\mathcal{F}^\bullet \in D^-(\mathrm{QCoh}\, X)$ as

$$\mathcal{F}^{\bullet\vee} := R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{O}_X)$$

[11]It is possible to relax the assumptions by letting X to be a regular scheme cf. [Sta23] 28.9

2.4.4 Tensor Product

Let $\mathcal{F}^\bullet \in K^-(\mathrm{Coh}\, X)$ we define the following exact functor

$$\mathcal{F}^\bullet \otimes - : \quad K^-(\mathrm{Coh}\, X) \longrightarrow K^-(\mathrm{Coh}\, X)$$

Where, given $\mathcal{E}^\bullet \in K^-(\mathrm{Coh}\, X)$ we define: $(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet)^i := \bigoplus_{p+q=i} \mathcal{F}^p \otimes \mathcal{E}^q, \quad d = d_\mathcal{F} \otimes 1_\mathcal{E} + (-1)^i 1_\mathcal{F} \otimes d_\mathcal{E}$ The class of locally free sheaves is adapted for this functor, thus we can construct its left derived functor

$$\mathcal{F}^\bullet \overset{L}{\otimes} - : \quad D^-(X) \longrightarrow D^-(X)$$

If additionally X is smooth, we an define

$$\mathcal{F}^\bullet \overset{L}{\otimes} - : \quad D^b(X) \longrightarrow D^b(X)$$

indeed any bonded complex of coherent sheaves is quasi isomorphic to a bounded complex of locally free sheaves and their tensor product is again bounded. We define

$$\mathcal{T}or_i(\mathcal{F}^\bullet, \mathcal{E}^\bullet) := H^{-i}(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet) \text{ for } \mathcal{F}^\bullet, \mathcal{E}^\bullet \in D^b(X)$$

Lastly, the following functorial isomorphisms hold

$$\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet \simeq \mathcal{E}^\bullet \overset{L}{\otimes} \mathcal{F}^\bullet$$
$$\mathcal{F}^\bullet \overset{L}{\otimes} (\mathcal{E}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet) \simeq (\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet) \overset{L}{\otimes} \mathcal{G}^\bullet$$

2.4.5 Inverse Image

Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, then consider the exact functor

$$f^{-1} : \ \mathrm{Mod}(\mathcal{O}_Y) \longrightarrow \mathrm{Mod}(f^{-1}\mathcal{O}_Y))$$

and the right exact functor

$$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} - : \ \mathrm{Mod}(f^{-1}\mathcal{O}_Y)) \longrightarrow \mathrm{Mod}(\mathcal{O}_X)$$

Their composition yields our definition of inverse image f^*, and we define its left derived functor as follows:

$$Lf^* := \left(\mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} (-)\right) \circ f^{-1} : \quad D^-(Y) \longrightarrow D^-(X)$$

If f is of finite Tor-dimension[12](e.g. f is flat or Y is regular) then we can consider the restriction to bounded complexes

$$Lf^* : \quad D^b(Y) \longrightarrow D^b(X)$$

Moreover when f is flat $Lf^* = f^*$.

2.4.6 Compatibilities

Let $f : X \longrightarrow Y$ be a proper morphism of schemes over a field k. We have the following natural isomorphisms

1. Projection formula : let $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in D^b(X)$

$$Rf_*(\mathcal{F}^\bullet) \overset{L}{\otimes} \mathcal{E}^\bullet \overset{\sim}{\longrightarrow} Rf_*(\mathcal{F}^\bullet \overset{L}{\otimes} Lf^*(\mathcal{E}^\bullet)$$

[12] Cf. [Sta23] 15.66

2. let $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in D^b(Y)$

$$Lf^*(\mathcal{F}^\bullet) \overset{L}{\otimes} Lf^*(\mathcal{E}^\bullet) \xrightarrow{\sim} Lf^*(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet)$$

3. <u>Pull-Push Adjunction</u>: $Lf^* \dashv Rf_*$, i.e. for any $\mathcal{E}^\bullet \in D^b(X)$ and $\mathcal{F}^\bullet \in D^b(Y)$, the following is an isomorphism

$$\mathrm{Hom}_X(Lf^*\mathcal{F}^\bullet, \mathcal{E}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_Y(\mathcal{F}^\bullet, Rf_*\mathcal{E}^\bullet)$$

4. <u>Hom-Tensor Adjunction</u>:

$$\begin{aligned} R\mathcal{H}om_X(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{E}^\bullet, \mathcal{G}^\bullet) &\simeq R\mathcal{H}om_X(\mathcal{F}^\bullet, R\mathcal{H}om_X(\mathcal{E}^\bullet, \mathcal{G}^\bullet)) \\ R\mathcal{H}om_X(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \overset{L}{\otimes} \mathcal{G}^\bullet &\simeq R\mathcal{H}om_X(\mathcal{F}^\bullet, \mathcal{E}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet) \end{aligned}$$

In particular

$$\begin{aligned} R\mathcal{H}om_X(\mathcal{F}^\bullet, \mathcal{E}^\bullet) &\simeq \mathcal{F}^{\bullet\vee} \overset{L}{\otimes} \mathcal{E}^\bullet \\ \mathcal{F}^\bullet &\simeq (\mathcal{F}^{\bullet\vee})^\vee \end{aligned}$$

3 Fourier-Mukai Transforms

```clojure
(defn rose [x]
  (with-out-str (print x "is a rose")))
(rose (rose (rose "rose")))
```

"rose is a rose is a rose is a rose"

Getrude Stein – Sacred Emily

rewritten as Clojure procedure

In past chapters we have been procedurally frosting new layers of abstraction to the already calcified theory underneath. We went from the homotopy category of an abelian category to its derived category, from varieties to the derived category of coherent sheaves on them. This chapter exemplifies what we mean by "studying geometry": probing the "space" with a suitable notion of "transformation" and examine the shards of what has been fixed by it. The provision of such transformations will inevitably add yet a new layer of abstraction to the body of the theory.

Thus we focus on Fourier-Mukai Transforms, an instance of the broader subject of integral transforms between categories. The core idea of such transformations is that an object in the derived category of the product of the varieties conveys *almost* all functorial information there is to know between the derived categories of the two varieties.

In here we develop the basic theory and examples to get us acquainted to this new instrument. By the end of the chapter we will be able to spot already uncanny patterns brought out by this kind transforms, namely the criteria concerning equivalences of Fourier-Mukai type and Orlov's Theorem.

3.1 Definition and Examples

We will adopt the following conventions: Let X and Y be smooth projective varieties over a field. We have the projections

$$\pi_X : X \times Y \to X, \quad \pi_Y : X \times Y \to Y$$

Definition 3.1.1. Let $P \in D^b(X \times Y)$, the induced Fourier-Mukai transform is the composition of the following functors

$$\begin{array}{cccccccccc}
 & R\pi_{X*}(\pi_Y^*\mathcal{E} \overset{L}{\otimes} P) & \longleftarrow & \pi_Y^*\mathcal{E} \overset{L}{\otimes} P & \longleftarrow & \pi_Y^*\mathcal{E} & \longleftarrow & \mathcal{E} & \\
\Phi_P^{X \to Y} : & D^b(X) & \underset{\pi_X^*}{\overset{R\pi_{*X}}{\leftrightarrows}} & D^b(X \times Y) & \overset{(- \overset{L}{\otimes} P)}{\longleftrightarrow} & D^b(X \times Y) & \underset{R\pi_{Y*}}{\overset{\pi_Y^*}{\leftrightarrows}} & D^b(Y) & : \Phi_P^{X \leftarrow Y} \\
 & \mathcal{E} & \longmapsto & \pi_X^*\mathcal{E} & \longmapsto & \pi_X^*\mathcal{E} \underset{L}{\otimes} P & \longmapsto & R\pi_{Y*}(\pi_X^*\mathcal{E} \underset{L}{\otimes} P) &
\end{array}$$

The definition is symmetric, the same object P parametrizes two transforms:

$$\Phi_P^{X \to Y}(\mathcal{E}) := R\pi_{Y*}(\pi_X^*\mathcal{E} \overset{L}{\otimes} P), \qquad (\Phi_P^{X \leftarrow Y} \equiv) \Phi_P^{Y \to X}(\mathcal{E}) := R\pi_{X*}(\pi_Y^*\mathcal{E} \overset{L}{\otimes} P)$$

Note that projections π_X, π_Y are flat, therefore we don't need to derive their pullbacks π_X^*, π_Y^*[1]. We say P is the Fourier-Mukai kernel of Φ_P. Such varieties X and Y are called in the literature (e.g. [Huy06]) Fourier-Mukai partners when Φ_P is an equivalence.

For ease of reading we will denote $\Phi_P \equiv \Phi_P^{X \to Y}$ whenever the direction of the tranform does not arise to confusion and use the notation Φ_P^t to refer at the transform from the other direction.

Remark 3.1.2. Since Φ_P is a composition of triangulated (exact) functor, it is itself triangulated.

We list few examples in order to exploit how common this type of functor is. Recall the projection formula of last chapter:

$$f_*\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet \simeq f_*\left(\mathcal{E}^\bullet \otimes f^*\mathcal{F}^\bullet\right)$$

Examples 3.1.3.

1. Identity functor:

$$\mathrm{id} : D^b(X) \longrightarrow D^b(X)$$

can be casted as a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta$ where Δ is the diagonal in $X \times X$. In fact, let $i : X \xrightarrow{\sim} \Delta \subset X \times X$ we have $i_*\mathcal{O}_X = \mathcal{O}_\Delta$, then by the projection formula, we obtain

$$\begin{aligned}\Phi_{\mathcal{O}_\Delta}\left(\mathcal{E}^\bullet\right) &= \pi_*\left(\pi^*\mathcal{E}^\bullet \otimes i_*\mathcal{O}_X\right)\\ &= \pi_*\left(i_*\left(i^*\pi^*\mathcal{E}^\bullet \otimes \mathcal{O}_X\right)\right)\\ &= (\pi \circ i)_*\left((\pi \circ i)^*\mathcal{E}^\bullet \otimes \mathcal{O}_X\right)\\ &= \mathcal{E}^\bullet\end{aligned}$$

2. For a function $X \xrightarrow{f} Y$ we have the graph $X \xrightarrow{\Gamma_f} X \times Y$ where $\Gamma_f = \mathrm{id} \times f$. We have $\Gamma_{f*}\mathcal{O}_X = \mathcal{O}_{\Gamma_f}$ so similar to the identity case we get

$$\Phi_{\mathcal{O}_{\Gamma_f}}\left(\mathcal{E}^\bullet\right) = (\pi_Y \circ \Gamma_f)_*\left((\pi_X \circ \Gamma_f)^*\mathcal{E}^\bullet \otimes \mathcal{O}_X\right) = f_*\mathcal{E}^\bullet$$

We can reverse the roles of π_Y and π_X to get

$$\Phi_{\mathcal{O}_{\Gamma_f}}^{X \to Y} = f_* \quad , \quad \Phi_{\mathcal{O}_{\Gamma_f}}^{Y \to X} = f^*$$

In particular for $f : X \to \operatorname{Spec} k$ we have $f_* = \Gamma$, therefore taking global sections can be seen as a special case of the above Fourier-Mukai transform

3. Taking the shift of the diagonal gives the shift, we have

$$\Phi_{\mathcal{O}_\Delta[1]}\left(\mathcal{E}^\bullet\right) = \mathcal{E}^\bullet \otimes \mathcal{O}_X[1] = \mathcal{E}^\bullet[1]$$

4. The Serre functor $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X[n]$ is of Fourier-Mukai type with kernel $i_*\omega_X[n] \in D^b(X \times X)$, where $n = \dim X$ and i is the diagonal embedding above

5. The tensor product $\mathcal{F}^\bullet \otimes -$ is of Fourier-Mukai type, using the kernel $i_*\left(\mathcal{F}^\bullet\right)$ with $i : X \hookrightarrow X \times X$.

6. Let $P \in D^b(X \times Y)$ be flat over X of base filed k, $x \in X$ be a closed point and $k(x) \simeq k$ be its residue field. For the purpose of the example we can think of a family of sheaves $\{P_x\}_{x \in X}$ on Y parametrized by elements of X, or as a deformation of the sheaf P_{x_0} for a distinguished closed point x_0. Consider then the Fourier-Mukai transform $\Phi_P : D^b(X) \longrightarrow D^b(Y)$, we have

$$\Phi_P(k(x)) \simeq P_x$$

[1]Recall the pullback is defined as a tensor product

Where $k(x)$ is the skyscraper sheaf supported at the closed point x with stalk the base field k i. e.

$$(k(x))_y = \begin{cases} k & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

and $P_x := (P|_{\{x\}\times Y})$ as a sheaf on Y. To see this, we apply the definitions: $\Phi_P(k(x)) = R\pi_{Y*}(P \overset{L}{\otimes} \pi_X^* k(x))$. Looking more closely as the argument the derived pushforward we have that by flatness of P over X, the functor $\mathrm{Coh}(X) \ni \mathcal{G} \mapsto \pi_X^*\mathcal{G} \otimes P$ is exact, therefore its derived functor will be just the operation of applying the tensor product term-wise, but $k(x)$ is a complex concentrated in degree zero, so the input of the Fourier-Mukai transform is just a sheaf.

$$\begin{aligned} P \overset{L}{\otimes} \pi_X^* k(x) &\simeq P \otimes i_* i^* \mathcal{O}_{X\times Y} && (\text{where } i : \{x\} \times Y \hookrightarrow X \times Y) \\ &\simeq i_*(i^*\mathcal{O}_{X\times Y} \otimes i^* P) && (\text{Classical Projection Formula}) \\ &\simeq i^* i_*(P \otimes \mathcal{O}_{X\times Y}) && (\equiv P|_{\{x\}\times Y}) \end{aligned}$$

Then by applying $R\pi_{Y*}$, as i_* is also exact, we obtain

$$\Phi_P(k(x)) \simeq R(\pi_{Y*} \circ i_*)(i^*P) \simeq (\pi_Y \circ i)_*(i^*)P \simeq P_x \in \mathrm{Coh}(Y)$$

where the middle isomorphism follows by the fact that $\pi_Y \circ i : \{x\} \times Y \hookrightarrow Y$ is an isomorphism, therefore exact.

Example 6 exhibits the following philosophy: the evaluation of the Fourier-Mukai transform at the skyscraper sheaf with support on a closed point, $\Phi_P(k(x))$, has to be thought as the pairing of a Dirac delta function $\delta_x \in (C_c^\infty(\Omega))'$ at x and a test function $\varphi \in C_c^\infty(\Omega)$. Then

$$\langle \delta_x, \varphi \rangle = \int_\Omega \delta_x \varphi = \varphi(x) \qquad \approx \qquad P_x = \Phi_P(k(x))$$

This, of course, goes in strict analogy with the classical Fourier Transform of functional analysis, that is historically why S. Mukai described the functors in Definition 3.1.1 as "Fourier functors" in his renowned article [Muk81]. By further extending this analogy, we can see an actual pattern:

Classical Integral Transforms			Fourier-Mukai Transforms
Function over X	f	$\mathcal{F}$	Complex of coherent sheaves on X
Embedding into $X \times Y$	$f \times id_Y$	π_X^*	Pullback
Product with a kernel $K(x, y)$	$(- \cdot K)$	$(- \otimes P)$	Tensor product with P
Integration	$\int$	$R\pi_{Y*}$	Derived pushforward along fibers of Y

As a heuristic first approximation, the reason why last row of the table above should be sensible is given by the following isomorphism:

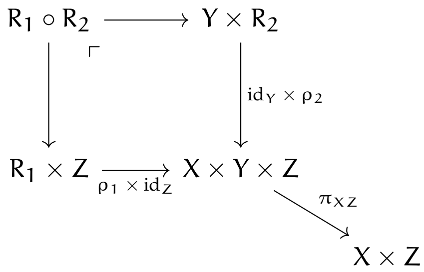
$$\Gamma(U, \pi_{Y*}(\mathcal{G})) \simeq \bigoplus_{x\in X} \Gamma(U \times \{x\}, \mathcal{G}_x), \qquad U \subseteq Y \text{ open}$$

which actually holds for any sheaf of modules $\mathcal{G}$ over $X \times Y$ under the assumption of equipping X with the discrete topology.

It is also possible to develop the theory of integral transforms for arbitrary categories, in this framework, the analogy between the two view crystallizes as just a matter of selecting the desired category. See [Dol09], chapter 3.

3.2 Adjoint Kernels and Composition

Definition 3.2.1. A correspondence over two objects X and Y in an arbitrary category $\mathcal{C}$ is a morphism $\rho : R \longrightarrow X \times Y$. If $\mathcal{C}$ admits pullbacks (or finite limits), it is possible to define a composition of correspondences in the following way: let $\rho_1 : R_1 \longrightarrow X \times Y$ $\rho_2 : R_2 \longrightarrow Y \times Z$.

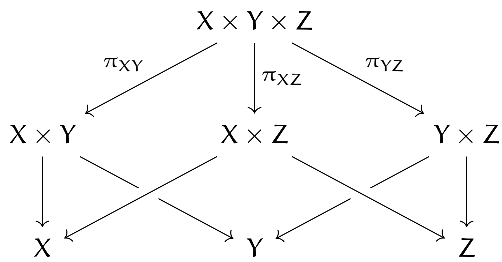


Where $R_1 \circ R_2$ denotes the categorical pullback of the square, then the composition is defined as $\rho_1 \circ \rho_2 \coloneqq R_1 \circ R_2 \longrightarrow X \times Z$ given by further composing with the projection π_{YZ}

As an immediate example in the category of sets we can consider two binary relations, $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$, then

$$R_1 \circ R_2 = \pi_{XZ}(\pi_{XY}^{-1}(R_1) \cap \pi_{YZ}^{-1}(R_2)) \subseteq X \times Z$$

Where the projections follow the diagram below:



Remark 3.2.2. Correspondences form indeed a category which can be thought as a generalization of the category **Rel** of binary relations.

Let $\Phi_{P_1}^{X \longrightarrow Y}, \Phi_{P_2}^{Y \longrightarrow Z}$ be two Fourier-Mukai transforms. Then we define [2]

$$P_1 \circ P_2 \coloneqq \pi_{XZ*}\left(\pi_{XY^*} P \otimes \pi_{YZ^*} Q\right) \in D^b(X \times Z).$$

Then,

$$\Phi_{P_2}^{Y \to Z} \circ \Phi_{P_1}^{X \to Y} \simeq \Phi_{P_1 \circ P_2}^{X \to Z}$$

This shows that the definition of the Fourier-Mukai transform is functorial also in the slot located by the kernel, i. e.

$$\begin{aligned} D^b(X \times Y) &\longrightarrow \left[D^b(X), D^b(Y)\right] \\ P &\longmapsto \Phi_P^{X \to Y}(-) \end{aligned}$$

is a functor targeting the category $\left[D^b(X), D^b(Y)\right]$ of functors between the derived categories of X and Y. Since $P_1 \circ P_2 = P_2 \circ P_1$ by the inherent symmetry of the categorical pullback, then the transform in the other direction is again parametrized by $P_1 \circ P_2$, i. e. $(\Phi_{P_1})^t \circ (\Phi_{P_2})^t \simeq (\Phi_{P_2} \circ \Phi_{P_1})^t \simeq \Phi_{P_1 \circ P_2}^{Z \to X}$

Now we focus on another essential property of the transform: left and right adjoints Fourier-Mukai transform are doomed to be of Fourier-Mukai type.

Definition 3.2.3. Let $P \in D^b(X \times Y)$, we define

$$P_L \coloneqq P^\vee \otimes \pi_Y^* \omega_Y[\dim Y], \quad P_R \coloneqq P^\vee \otimes \pi_X^* \omega_X[\dim X] \in D^b(X \times Y)$$

[2] In the literature it is also denoted as $P_1 \underset{Y}{\boxtimes} P_2$ (Cf. [Orl09])

Let $\Phi_{P_L}, \Phi_{P_R} : D^b(Y) \to D^b(X)$ be their corresponding Fourier-Mukai transforms.

Proposition 3.2.4. *([Huy06], 5.9). The Fourier-Mukai transforms $\Phi_{P_L}, \Phi_{P_R} : D^b(Y) \to D^b(X)$ are left, respectively right adjoint to Φ_P, i. e.*

$$\Phi_{P_L} \dashv \Phi_P \dashv \Phi_{P_R}$$

3.3 Equivalence Criteria and Orlov's Theorem

In order to explain how equivalences interweave with the notion of integral functors we need to introduce some technology first. This section closely follows the exposition found in [Bri19] and [Huy06].

Definition 3.3.1. A collection Ω of objects in a triangulated category $\mathcal{D}$ is a spanning class of $\mathcal{D}$ (or spans $\mathcal{D}$) if for all $A \in \mathcal{D}$ the following two conditions hold:

1. If $\mathrm{Hom}(A[i], \omega) = 0$ for all $\omega \in \Omega$ and all $i \in \mathbb{Z}$, then $A \simeq 0$.
2. If $\mathrm{Hom}(\omega, A[i]) = 0$ for all $\omega \in \Omega$ and all $i \in \mathbb{Z}$, then $A \simeq 0$.

Lemma 3.3.2. *([Huy06], 3.17). Let X be a smooth projective variety. Then the class of skyscraper sheaves of the form $k(x)$ with $x \in X$ a closed point, are a spanning class for the derived category $D^b(X)$.*

Proof: It is enough to prove that for any non-trivial $\mathcal{F}^\bullet \in D^b(X)$ there exist closed points $x_1, x_2 \in X$ and integers i_1, i_2 such that

$$\mathrm{Hom}\left(\mathcal{F}^\bullet, k\left(x_1\right)\left[i_1\right]\right) \neq 0 \neq \mathrm{Hom}\left(k\left(x_2\right), \mathcal{F}^\bullet\left[i_2\right]\right)$$

However, by applying Serre duality we obtain

$$\mathrm{Hom}\left(k(x), \mathcal{F}^\bullet\left[i_2\right]\right) \simeq \mathrm{Hom}\left(\mathcal{F}^\bullet, k(x)\left[\dim(X) - i_2\right]\right)^\vee .$$

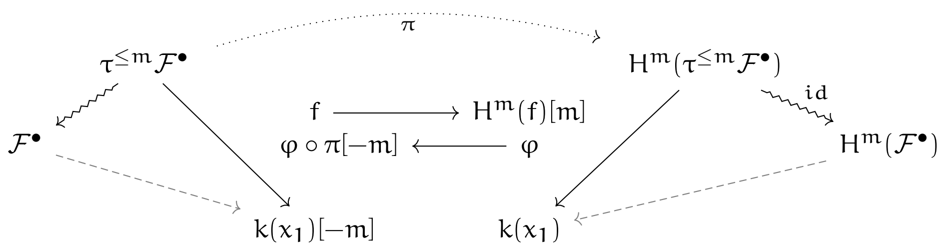
Therefore let x_1 be a closed point in $\mathrm{supp}\left(H^m(\mathcal{F}^\bullet)\right)$ where m is the maximal integer for which $\mathcal{H}^i := H^i(\mathcal{F}^\bullet) \neq 0$ for $i \in \mathbb{Z}$. Then there is a non-trivial map[3] in

$$0 \neq \mathrm{Hom}_{\mathcal{O}_{X,x_1}}\left((H^m(\mathcal{F}^\bullet))_{x_1}, k(x_1)\right) \simeq \mathrm{Hom}_{\mathcal{O}_X}\left(H^m(\mathcal{F}^\bullet), k(x_1)\right)$$

Since $(H^m(\mathcal{F}^\bullet))_{x_1}$ is a finite dimensional vector space over the residue field[4] $k(x_1)$, then we have

$$\mathrm{Hom}\left(\mathcal{F}^\bullet, k\left(x_1\right)[-m]\right) \stackrel{\star}{\simeq} \mathrm{Hom}\left(H^m(\mathcal{F}^\bullet), k\left(x_1\right)\right) \neq 0$$

where the isomorphism $(\star)$ is expounded by the following diagrams:



[3] Let $i_x\{x\} \hookrightarrow X$ the natural inclusion, recall that the stalk functor and $\mathrm{Mod}(\mathcal{O}_{X,x}) \ni A \mapsto i_{x,*}A \in \mathrm{Mod}(\mathcal{O}_X)$ are adjoint:

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, A) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*}A).$$

[4] A.k.a. the fiber at x_1

More explicitly, π is the canonical projection:

$$\begin{array}{ccccccccccc}
\mathcal{F}^\bullet : & \cdots & \longrightarrow & \mathcal{F}^{m-1} & \xrightarrow{d^{m-1}} & \mathcal{F}^m & \xrightarrow{d^m} & \mathcal{F}^{m+1} & \longrightarrow & \cdots \\
\uparrow\wr & & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \\
\tau^{\leq m}\mathcal{F}^\bullet : & \cdots & \longrightarrow & \mathcal{F}^{m-1} & \longrightarrow & \ker d^m & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow\pi & & \downarrow & & \\
H^m(\mathcal{F}^\bullet) : & \cdots & \longrightarrow & 0 & \longrightarrow & H^m(\mathcal{F}^\bullet) & \xrightarrow{d} & 0 & \longrightarrow & \cdots
\end{array}$$

Since morphisms in derived category are up to quasi-isomorphisms, by how we defined m, we can work with the truncated complex $\tau^{\leq m}\mathcal{F}^\bullet$. Thus we conclude the proof by taking $i_1 = -m$. □

Remark 3.3.3. The argument in the last bit of the proof above is actually a de facto strategy when working with complexes concentrated in degree zero in a derived category. So it is useful to generalize the passage to see clearly the pattern and explore its implications.

Let $\mathcal{A}$ be an abelian category, $A^\bullet \in D^b(\mathcal{A})$ we define:

$$i^+ := \max\{i : H^i(A^\bullet) \neq 0\} \quad \text{and} \quad i^- := \min\{i : H^i(A^\bullet) \neq 0\}$$

Then:

1. There are morphisms in $D^b(\mathcal{A})$

$$A^\bullet \xrightarrow{\varphi} H^{i^+}(A^\bullet)[-i^+]$$
$$H^{i^-}(A^\bullet)[-i^-] \xrightarrow{\psi} A^\bullet$$

   such that $H^{i^+}(\varphi) \simeq \mathrm{id}_{H^{i^+}(A^\bullet)}$ and $H^{i^-}(\psi) \simeq \mathrm{id}_{H^{i^-}(A^\bullet)}$

2. Let $B \in \mathcal{A}$, from the previous point, we obtain the following isomorphisms in $D^b(\mathcal{A}$

$$\mathrm{Hom}(H^{i^+}(A^\bullet), B) \simeq \mathrm{Hom}(A^\bullet, B[-i^+]) \tag{1}$$
$$\mathrm{Hom}(B, H^{i^-}(A^\bullet)) \simeq \mathrm{Hom}(B[-i^-], A^\bullet) \tag{2}$$

3. Let $H^i(A^\bullet) = 0$ for $i < m$, then there is a distinguished triangle in $D^b(\mathcal{A})$

$$H^m(A^\bullet)[-m] \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow H^m(A^\bullet)[-m+1]$$

   with

$$H^j(B^\bullet) \simeq \begin{cases} H^j(A^\bullet) & \text{if } j > m, \\ 0 & \text{if } j \leq 0. \end{cases}$$

Theorem 3.3.4. *([BO95], 1.1). Let X, Y be smooth, projective varieties, and let $\Phi : D^b(X) \to D^b(Y)$ be an integral transform. For $x \in X$ let $P_x = \Phi(k(x))$. Then Φ is fully faithful if, and only if, we have*

$$\mathrm{Hom}_{D^b(Y)}(P_x, P_y[i]) = \begin{cases} 0 & \textit{if } x \neq y \textit{ or } i \notin [0, \dim X] \\ k & \textit{if } x = y \textit{ and } i = 0. \end{cases}$$

The proof of last statement is already given and well expounded in multiple sources; see, for instance, the accounts given in [Bri19] 5.1 and [Huy06] 7.1. We will give, nevertheless, the proof of the theorem that will follow. This is to give us a chance to introduce few more crucial concepts and techniques that benefit our understanding of the whole theory of Fourier-Mukai transforms.

Definition 3.3.5. A triangulated category $\mathcal{D}$ is called decomposable if there exists two full subcategories $\mathcal{D}_1$ and $\mathcal{D}_2$, each containing objects non-isomorphic to the zero object, such that

1. any object X in $\mathcal{D}$ is isomorphic to the bi-product of an object A_1 from $\mathcal{D}_1$ and an object A_2 from $\mathcal{D}_2$;
2. $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2[i]) = \mathrm{Hom}_{\mathcal{D}}(A_2, A_1[i]) = 0$ for all $i \in \mathbb{Z}$ and all $A_1 \in \mathcal{D}_1$, $A_2 \in \mathcal{D}_2$.

Recall that the biproduct, or sum, of objects A, B in an additive category is an object which is both the product and the coproduct of A and B.

Remarks 3.3.6.

- The decomposition is stable with respect to the shift functor: let $A_i \in \mathcal{D}_i$ as in the definition above, then $A_i[r] \in \mathcal{D}_i$ for any $r \in \mathbb{Z}$. Indeed,

$$\mathrm{Hom}_{\mathcal{D}}(A_2, A_1[r+i]) = 0 = \mathrm{Hom}_{\mathcal{D}}(A_1, A_2[r+i])$$

  for all $i \in \mathbb{Z}$; then if $A_1[r]$ is the biproduct of $A \in \mathcal{D}_1$ and $B \in \mathcal{D}_2$ with $B \neq 0$, then there is a non-zero morphism $B \to A_1[r]$. Thus B must be a zero-object, and hence $A_1[r]$ is an object of $\mathcal{D}_1$.

- One can restate the condition about the biproduct by saying that for any object X in $\mathcal{D}$ there is a distinguished triangle $A_1 \to X \to A_2 \to A_1[1]$, where $A_i \in \mathcal{D}_i$. Since $A_1[1] \in \mathcal{D}_1$, the morphism $A_2 \to A_1[1]$ is the zero morphism. One can prove that this implies that the triangle splits, i. e. there is a section $A_2 \to B$. Applying the functors $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$, we obtain that B is the bi-product of A_1 and A_2.

- A triangulated category which is not decomposable is called, unsurprisingly enough, indecomposable

Example 3.3.7. ([Huy06], 3.10). If X is a scheme then $D(X)$ is indecomposable if and only if X is connected.

Proposition 3.3.8. *[Huy06]* 1.54. *Let* $F : \mathcal{D} \longrightarrow \mathcal{D}'$ *be a fully faithful exact functor between triangulated categories. Suppose that* $\mathcal{D}$ *contains objects not isomorphic to 0 and that* $\mathcal{D}'$ *is indecomposable. Then* F *is an equivalence of categories if and only if* F *has a left adjoint* $G \dashv F$ *and a right adjoint* $F \dashv H$ *such that for any object* $B \in \mathcal{D}'$ *one has:*

$$H(B) \simeq 0 \implies G(B) \simeq 0$$

Theorem 3.3.9 (Bridgeland). *[Bri19], 5.4; [Huy06], 7.11. Suppose* $\Phi_P : D^b(X) \longrightarrow D^b(Y)$ *is fully faithful. Then* Φ_P *is an equivalence if and only if*

$$\Phi_P(k(x)) \otimes \omega_Y \simeq \Phi_P(k(x))$$

for all closed points $x \in X$.

Proof: Assume first $\Phi_P(k(x)) \otimes \omega_Y \simeq \Phi_P(k(x))$. Let us abide to the following syntactic shortcuts for the adjoint transforms of last section

$$\begin{array}{ccccc} \Phi_{P_L} & \dashv & \Phi_P & \dashv & \Phi_{P_R} \\ \parallel\!\mid & & \parallel\!\mid & & \parallel\!\mid \\ G & & F & & H \end{array}$$

Let $H(\mathcal{F}^\bullet) \simeq 0$, then for all $i \in \mathbb{Z}$

$$\begin{aligned} \mathrm{Hom}(H(\mathcal{F}^\bullet), k(x)[i]) &\simeq \mathrm{Hom}(\mathcal{F}^\bullet, F(k(x))[i]) \\ &\simeq \mathrm{Hom}(\mathcal{F}^\bullet, F(k(x)) \otimes \omega_Y[i]) && \text{(by assumption)} \\ &\simeq \mathrm{Hom}(F(k(x)), \mathcal{F}^\bullet[\dim(Y) - i])^\vee && \text{(Serre duality)} \\ &\simeq \mathrm{Hom}(k(x), G(\mathcal{F}^\bullet)[\dim(Y) - i])^\vee = 0. \end{aligned}$$

We know by Lemma 3.3.2 the objects of the form $k(x)$ span $D^b(X)$, this suffices to see that $H(\mathcal{F}^\bullet) \simeq 0$. Thus we conclude by means of Proposition 3.3.8.

On the other hand, since F is an equivalence, $H \simeq G$ are both quasi-inverses so

$$k(x) = HF(k(x)) = GF(k(x))$$

furthermore,

$$GF(k(x)) \otimes \omega_X[\dim X] = S_X GF(k(x)) \simeq GS_Y F(k(x)) \simeq G(F(k(x)) \otimes \omega_Y[\dim Y])$$

Then by merging the last two lines, we obtain

$$GF(k(x)) \simeq GF(k(x)) \otimes \omega_X \simeq G(F(k(x)) \otimes \omega_Y)[\dim Y - \dim X]$$

Therefore $\dim X = \dim Y$ and the desired isomorphism follows. □

Remark 3.3.10. In the last proof we used the following fact.

Let $\mathcal{C}, \mathcal{D}$ k-linear categories, $S_\mathcal{C}, S_\mathcal{D}$ their Serre functors and $A \in \mathcal{C}, B \in \mathcal{D}$. If $F : \mathcal{D} \longrightarrow \mathcal{C}$ is an equivalence, then $F \circ S_\mathcal{D} = S_\mathcal{C} \circ F$. Indeed,

$$\begin{aligned}
\mathrm{Hom}_\mathcal{C}(A, S_\mathcal{C}(FB)) &\simeq \mathrm{Hom}_\mathcal{C}(FB, A)^\vee \\
&\simeq \mathrm{Hom}_\mathcal{D}(B, F^{-1}A)^\vee \\
&\simeq \mathrm{Hom}_\mathcal{D}(F^{-1}A, S_D B)^\vee \\
&\simeq \mathrm{Hom}_\mathcal{C}(A, F(S_D B))
\end{aligned}$$

Or simply, by exploitation of the universal property of Serre functors, $F^{-1}S_\mathcal{C}F$ is a Serre functor for $\mathcal{D}$, and hence has to be isomorphic to $S_\mathcal{D}$.

The analysis of whether or not a Fourier-Mukai transform is also an equivalence erupts in the following questions.

1. Do they naturally arise? What are the conditions to ensure that a given functor is of Fourier-Mukai Type? How can these conditions be sharpened?
2. If a functor is isomorphic to a Fourier-Mukai transform, is its kernel unique (up to isomorphism)?

Orlov's theorem, we are about to state, is an attempt to give a precise answer to all these enquiries.

Theorem 3.3.11 (Orlov)**.** *Let X and Y be two smooth projective varieties and let*

$$F : D^b(X) \longrightarrow D^b(Y)$$

be a fully faithful exact functor. If F admits right and left adjoint functors, then there exists an object $\mathcal{P} \in D^b(X \times Y)$ unique up to isomorphism such that F is isomorphic to $\Phi_\mathcal{P}$:

$$F \simeq \Phi_\mathcal{P}$$

Although this theorem sits almost too casually in this section, this should not make the reader lax about the depths of its proof—which we refrain to give—and the far reaching consequences in many fields. The proof employs the use of Postnikov systems and the Beĭlinson resolution of $\mathcal{O}_\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ in order to craft a suitable kernel for the given fully faithful functor.

4 Applications and Examples

> To think is to forget differences, generalize, make abstractions. In the teeming world of Funes, there were only details, almost immediate in their presence.
>
> Jorge Luis Borges – Funes the Memorious

4.1 The derived category of $\mathbb{P}^n$

This paragraph draws its content mostly from [Cal05], we will adopt most of the notations and the theory of dimension of triangulated categories (which we refrain to explore in full extent) from [Sta23][1], [Rou04] and [Orl08].

In order to elicit the architecture of the derived category of $\mathbb{P}^n$ we need to build up on some basic concepts that inherently come into play when a category is endowed with a triangulated structure.

Definition 4.1.1. Let $\mathcal{D}$ a triangulated category, we identify full subcategories of $\mathcal{D}$ with subsets of $\mathrm{Ob}(\mathcal{D})$; then we employ the usual abuse of notation where $A \in \mathcal{D}$ stands for $A \in \mathrm{Ob}(\mathcal{D})$. Let $\mathcal{A}, \mathcal{B}$ be full subcategories of $\mathcal{D}$. We define

$\mathcal{A}[a, b]$ will be the full subcategory of $\mathcal{D}$ consisting of all objects $A[-i]$ with $i \in [a, b] \cap \mathbb{Z}$ and $A \in \mathcal{A}$. Therefore it closed under the shift from left to right!

$\operatorname{smd} \mathcal{A}$ be the full subcategory of $\mathcal{D}$ consisting of all objects which are isomorphic to direct summands of objects of $\mathcal{A}$

$\operatorname{add} \mathcal{A}$ be the full subcategory of $\mathcal{D}$ of all objects which are isomorphic to directs sums of objects of $\mathcal{A}$

$\mathcal{B} \star \mathcal{A}$ the full subcategory of $\mathcal{D}$ consisting of all objects $X \in \mathcal{D}$ that fit into a distinguished triangle of the form

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A[1] \\ & & \cap & & & & \cap \\ & & \mathcal{B} & & \star & & \mathcal{A} \end{array}$$

Then we define, for $E \in \mathcal{D}$ viewed as a full subcategory

$$\langle E \rangle_1 := \operatorname{add} E[-\infty, \infty]$$
$$\downarrow$$
$$\langle E \rangle_n := \operatorname{smd}\left(\langle E \rangle_1 \star \langle E \rangle_{n-1}\right)$$
$$\downarrow$$
$$\langle E \rangle := \bigcup_n \langle E \rangle_n$$

Remarks.

[1] Chapter 05QI

- $\mathcal{A}^{\star n} := \mathcal{A} \star \cdots \star \mathcal{A}$ with $n \geq 1$ and $\star$ is associative
- Each $\langle E \rangle_n$ is a strictly full[2] additive subcategory of $\mathcal{D}$, closed under taking summands and shift, but does not necessarily preserve cones.
- $\langle E \rangle$ is strictly full, triangulated subcategory and it is the smallest subcategory of $\mathcal{D}$ containing the object E
- We can generalize $\langle E \rangle$ to multiple objects as follows

$$\langle E_1, \ldots, E_n \rangle := \langle E_1 \oplus \cdots \oplus E_n \rangle$$

Definition 4.1.2. Let $\mathcal{D}$ be a triangulated category and $E \in \mathcal{D}x$

1. We say E is a classical generator of $\mathcal{D}$ if $\langle E \rangle = \mathcal{D}$.
2. We say E is a strong generator of $\mathcal{D}$ if $\langle E \rangle_n = \mathcal{D}$ for some $n \geq 1$.
3. We say E is a weak generator or a generator of $\mathcal{D}$ if for any nonzero object K of $\mathcal{D}$ there exists an integer n and a nonzero map $E \longrightarrow K[n]$.

Let us untangle—only marginally so—the relationships among the definitions above

Remarks 4.1.3.

- If E is a classical generator, then E is a weak generator
- If $\mathcal{D}$ has a strong generator, then all its classical generators are strong.

We can now state the structure theorem for $D^b(\mathbb{P}^n)$.

Theorem 4.1.4. *The derived category of $\mathbb{P}^n$ is generated by*

$$\left\langle \mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n} \right\rangle$$

In order to prove it though, we need to collect the following result from [Beĭ78]

Proposition 4.1.5 (Beĭlinson). *Exists a resolution made of locally free sheaves of $\mathcal{O}_\Delta$, namely*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n) \boxtimes \Omega^n(n) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n+1) \boxtimes \Omega^{n-1}(n-1) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega^1(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_\Delta \to 0$$

Recall,

Definition 4.1.6. Let $\varphi : \mathcal{E} \longrightarrow \mathcal{O}_X$ be an $\mathcal{O}_X$-module map on a ringed space X, where we assume $\mathcal{E}$ to be locally free of rank n . The Koszul (chain) complex[3] $K_\bullet(\varphi)$ associated to φ is the complex of sheaves of commutative differential graded algebras defined as follows:

$$K_\bullet(\varphi) = \left\{ 0 \longrightarrow \wedge^n \mathcal{E} \xrightarrow{d_n} \wedge^{n-1} \mathcal{E} \longrightarrow \cdots \longrightarrow \wedge^1 \mathcal{E} \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow 0 \right\}$$

The differential $d_\bullet : K_\bullet(\varphi) \longrightarrow K_\bullet(\varphi)$ is the unique derivation such that $d_1(e) = \varphi(e)$ for all local sections e of $\mathcal{E} = K_1(\varphi)$. More explicitly, on a basis element of $\wedge^k \mathcal{E}$

$$d_k(e_1 \wedge \ldots \wedge e_k) = \sum_{i=1,\ldots,k} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_k$$

Remark 4.1.7. If $\mathcal{E} \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow 0$ is exact, then its Koszul complex is exact in $\mathrm{Mod}(\mathcal{O}_X)$ and is called Koszul resolution of $\mathcal{O}_X$ relative to φ. Now if we select a section

$$s \in \Gamma(X, \mathcal{E}) \simeq \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{E}) \simeq \mathrm{Hom}_X(\mathcal{E}^\vee, \mathcal{O}_X),$$

[2] it is a full subcategory and given $X \in \langle E \rangle$ any object of $\mathcal{D}$ which is isomorphic to X is also in $\langle E \rangle$.
[3] The Koszul complex is selfdual

we can build its Koszul complex $K_\bullet(s)$:

$$0 \longrightarrow \wedge^n \mathcal{E}^\vee \xrightarrow{d_n} \wedge^{n-1}\mathcal{E}^\vee \longrightarrow \cdots \longrightarrow \wedge^1 \mathcal{E}^\vee \xrightarrow{s} \mathcal{O}_X \longrightarrow 0$$

Where the differential is

$$d_k(t_1 \wedge \ldots \wedge t_k) = \sum_{i=1,\ldots,n} (-1)^{i+1} \varphi(t_i) t_1 \wedge \ldots \wedge \widehat{t_i} \wedge \ldots \wedge t_k ; \qquad t_i(e_j) := \delta_i^j$$

Then the image of s in $\mathcal{O}_X$ is a sheaf of ideals, those are in $1:1$ correspondence to closed subschemes of X. In fact, locally around $x \in X$, $s_x \in \mathcal{E}_x$ is represented by an tuple of regular functions $f_1, \ldots, f_n : U \longrightarrow \mathbb{A}^1$, for some open neighborhood $U \subseteq X$ of x. For such functions f_i , it makes sense to ask whether or not $f_i(x) = 0$. Then we say that $s(x) = 0$ if $f_i(x)$ for $i = 1, \ldots, n$. This does not depend on the open neighborhood U. The locus of such x's is closed[4]. We call such subscheme $Z(s)$, the <u>zero scheme</u> of s.

Then we say a section is <u>regular</u> at a point $x \in X$, if f_i is not a zero divisor in

$$\mathcal{E}_{X,x}/(f_1, \ldots f_{i-1})\mathcal{E}_{X,x};$$

then a section is regular if regular at every x.

The above notion of regularity is equivalent to require exactness of the <u>augmented</u> Koszul complex[5], i. e.

$$K_\bullet^+(s) := \left\{ 0 \longrightarrow \wedge^n \mathcal{E}^\vee \xrightarrow{d_n} \wedge^{n-1}\mathcal{E}^\vee \longrightarrow \cdots \longrightarrow \wedge^1 \mathcal{E}^\vee \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow \mathcal{O}_{Z(s)} \longrightarrow 0 \right\}$$

proof of Prop. 4.1.5 (Beĭlinson):

Let $\{y_0, \ldots, y_n\}$ be a basis for $\Gamma(\mathbb{P}, \mathcal{O}(1))$. Then consider Euler's exact sequence on $\mathbb{P}^n$

$$0 \longrightarrow \Omega^1 \longrightarrow \mathcal{O}(-1)^{\oplus n+1} \longrightarrow \mathcal{O} \longrightarrow 0.$$

Dualizing and twisting by -1, we obtain:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{n+1} \longrightarrow \mathcal{T}(-1) \longrightarrow 0$$

where $\mathcal{T}$ denotes the tangent sheaf on $\mathbb{P}^n$. Now, by applying the global section functor

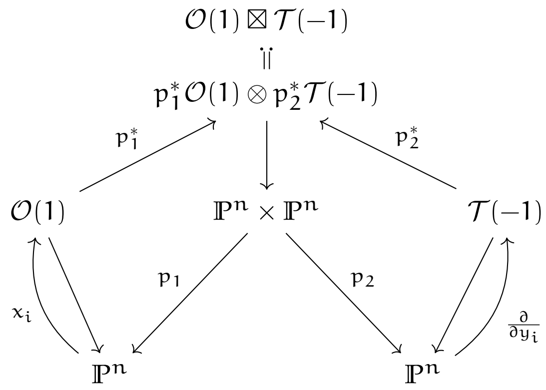
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\mathbb{P}^n, \mathcal{O}(-1)) & \longrightarrow & \Gamma(\mathbb{P}^n, \mathcal{O}^{n+1}) & \xrightarrow[\sim]{\pi} & \Gamma(\mathbb{P}^n, \mathcal{T}(-1)) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & 0 & & & & & & \end{array}$$

Then we can select for $\Gamma(\mathbb{P}, \mathcal{O}^{\oplus n+1})$ the dual basis[6] $\{y_0^\vee \ldots y_n^\vee\}$ to the chosen above, so we can denote the image of $y_i^\vee$ through π in $\Gamma(\mathbb{P}^n, \mathcal{T}(-1))$ as $\frac{\partial}{\partial y_i}$. Now, we chose a global section s of $\mathcal{O}(1) \boxtimes \mathcal{T}(-1)$ on $\mathbb{P}^n \times \mathbb{P}^n$, namely

$$s = \sum_{i=0}^{n} x_i \boxtimes \frac{\partial}{\partial y_i}$$

where $\{x_i\}$ and $\{y_i\}$ denote the coordinates on the first and second $\mathbb{P}^n$ respectively.

[4]Cf. [Har77].
[5]Cf. [FL13], [IV, §31]
[6]$\mathrm{Hom}(\Gamma(\mathbb{P}^n, \mathcal{O}^{\oplus n+1}), \mathbb{C}) \simeq \mathrm{Hom}(\mathbb{C}^{n+1}, \mathbb{C}) \simeq \Gamma(\mathbb{P}^n, \mathcal{O}(1))$



(1)

<u>*Claim*</u>: The zeroes of s lie precisely along $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$.

To this end, let us consider a coordinate patch of $\mathbb{P}^n \times \mathbb{P}^n$, say, where $x_0 \neq 0 \neq y_0$. Then in this affine patch we set $Y_j = y_j/y_0$ for $1 \leq j \leq n$. Therefore $\left\{\frac{\partial}{\partial Y_j}\right\}$ is a basis for $\mathcal{T}$ at each point of the patch. Then follows

$$dY_j = \frac{y_0 dy_j - y_j dy_0}{y_0^2}$$

and

$$\frac{\partial}{\partial y_j} = \sum_{i=1}^{n} dY_j(\partial/\partial y_j)\frac{\partial}{\partial Y_j}$$

thus we have

$$\frac{\partial}{\partial y_j} = \frac{1}{y_0}\frac{\partial}{\partial Y_j}, \qquad \frac{\partial}{\partial y_0} = -\sum_{j=1}^{n}\frac{y_j}{y_0^2}\frac{\partial}{\partial Y_j}$$

Now we can express s in this patch as follows

$$s = \sum_{i=0}^{n} x_i \boxtimes \frac{\partial}{\partial y_i} = \sum_{j=1}^{n} x_j \boxtimes \frac{1}{y_0}\frac{\partial}{\partial Y_j} - \sum_{j=1}^{n} x_0 \boxtimes \frac{y_j}{y_0^2}\frac{\partial}{\partial Y_j}$$

For all $0 \leq i \leq n$ and $1 \leq j \leq n$. Now in the patch $x_0 \neq 0 \neq y_0$ we can easily manipulate the above expression to find out when $s = 0$.

$$\begin{aligned}
0 = s &\iff x_i \boxtimes \frac{1}{y_0} - x_0 \boxtimes \frac{y_i}{y_0^2} = 0 \\
&\iff \frac{x_i}{x_0} \boxtimes 1 - 1 \boxtimes \frac{y_i}{y_0} = 0 \\
&\iff \frac{x_i}{x_0} = \frac{y_i}{y_0}
\end{aligned}$$

Therefore, in the patch, the zero scheme of s is exactly the diagonal. Since this procedure can be repeated for all affine patches of $\mathbb{P}^n \times \mathbb{P}^n$, we conclude $Z(s) = \Delta$.

So as to resolve the proof, let us construct the augmented Koszul complex $K_\bullet^+(s)$, let $\mathcal{E} = \mathcal{O}(1) \boxtimes \mathcal{T}(-1)$

$$0 \longrightarrow \wedge^n \mathcal{E}^\vee \xrightarrow{d_n} \wedge^{n-1}\mathcal{E}^\vee \longrightarrow \cdots \longrightarrow \wedge^1 \mathcal{E}^\vee \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

More explicitly $\mathcal{E}^\vee = \mathcal{O}(-1) \boxtimes \mathcal{T}^\vee(1)$ and $\wedge^p \mathcal{E}^\vee = \mathcal{O}_{\mathbb{P}^n}(-p) \boxtimes \Omega^p(p)$. This complex is exact and we call it <u>Beĭlison resolution</u>. □

Remark 4.1.8. We can split the resolution above into short exact sequences as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-n)\boxtimes\Omega^n(n) & \longrightarrow & \mathcal{O}(-n+1)\boxtimes\Omega^{n-1}(n-1) & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\ 0 & \longrightarrow & C_{n-1} & \longrightarrow & \mathcal{O}(-n+2)\boxtimes\Omega^{n-2}(n-2) & \longrightarrow & C_{n-2} & \longrightarrow & 0 \\ & & & & \vdots & & & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & \mathcal{O}\boxtimes\mathcal{O} & \longrightarrow & \mathcal{O}_\Delta & \longrightarrow & 0 \end{array} \tag{2}$$

Since short exact sequences lift to distinguished triangles (cf. Chapter 1), we see that $\mathcal{O}_\Delta$ can be reached by successively taking cones of the components of its resolution. In symbols

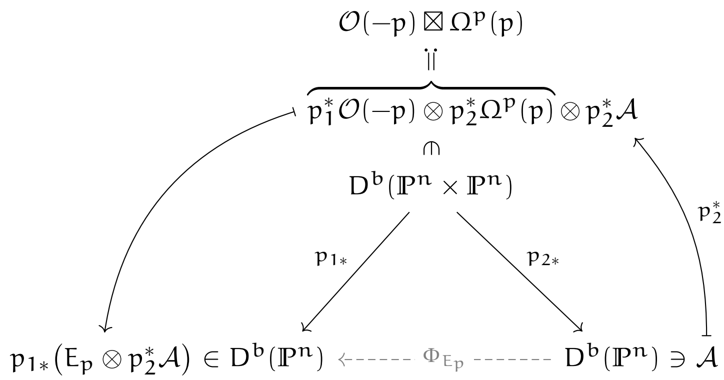
$$\mathcal{O}_\Delta \in \left\langle \mathcal{O}(-n)\boxtimes\Omega^n(n), \mathcal{O}(-n+1)\boxtimes\Omega^{n-1}(n-1), \dots, \mathcal{O}\boxtimes\mathcal{O} \right\rangle_n$$

We are now able to prove the structure theorem of the derived category of $\mathbb{P}^n$

Proof of Theorem 4.1.4:

Let us denote the Fourier-Mukai Transform $\Phi_E(\mathcal{A}) = p_{1*}(E \otimes p_2^*\mathcal{A})$ from the *second* $\mathbb{P}^n$ to the first. We employ the following syntactic shortcut $E_p = \mathcal{O}(-p)\boxtimes\Omega^p(p)$, throughout.

We picture this by regrafting diagram (1) in the derived categories, since all functors are meant to be derived, we neglect adding R's and L's to keep notations a bit lighter



(3)

We know that $\Phi_{(-)}(\mathcal{A})$ is a triangulated functor, thus, if we apply it to (2)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_{E_n}(\mathcal{A}) & \longrightarrow & \Phi_{E_{n-1}}(\mathcal{A}) & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\ 0 & \longrightarrow & C_{n-1} & \longrightarrow & \Phi_{E_{n-2}}(\mathcal{A}) & \longrightarrow & C_{n-2} & \longrightarrow & 0 \\ & & & & \vdots & & & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & \Phi_{\mathcal{O}\boxtimes\mathcal{O}}(\mathcal{A}) & \longrightarrow & \Phi_{\mathcal{O}_\Delta}(\mathcal{A}) & \longrightarrow & 0 \end{array}$$

Are all distinguished triangles. Therefore $\Phi_{\mathcal{O}_\Delta}(\mathcal{A}) = \mathcal{A}$ is generated by

$$\left\langle \Phi_{E_n}(\mathcal{A}), \Phi_{E_{n-1}}(\mathcal{A}), \dots, \Phi_{\mathcal{O}\boxtimes\mathcal{O}}(\mathcal{A}) \right\rangle_n$$

<u>*Claim*</u>: $\Phi_{E_i}(\mathcal{A}) = \Phi_{\mathcal{O}(-i)\boxtimes\Omega^i(i)}(\mathcal{A}) \in \langle \mathcal{O}(i) \rangle$

Let's see:

$$\begin{aligned} \Phi_{E_i}(\mathcal{A}) &= p_{1,*}(p_1^*\mathcal{O}(-i)\otimes p_2^*\Omega^i(i)\otimes p_2^*\mathcal{A}) \\ \text{(Projection formula)} \quad &= \mathcal{O}(-i)\otimes p_{1,*}(p_2^*(\Omega^i(i)\otimes\mathcal{A})) \\ (p_2 \text{ is open}) \quad &\overset{\star}{=} \mathcal{O}(-i)\otimes\Gamma(\mathbb{P}^n, \Omega^i(i)\otimes\mathcal{A})\otimes_{\mathbb{C}}\mathcal{O}_{\mathbb{P}^n} \\ &= \mathcal{O}(-i)\otimes\Gamma(\mathbb{P}^n, \Omega^i(i)\otimes\mathcal{A}) \end{aligned}$$

The passage from first to the second line is the reason why we chose the direction from right to left, cf. diagram (3), of the Fourier-Mukai Transform at the beginning.

To expand on $(\star)$, we know this holds in general for a sheaf of $\mathcal{O}_X$-modules F and open maps—such as projections $\pi_X, \pi_Y : X \times Y \longrightarrow X, Y$:

$$\Gamma(U, \pi_{X,*}(\pi_Y^* F)) = \Gamma(U \times Y, \pi_Y^* F) = \Gamma(Y, F) \otimes_{\mathbb{C}} \mathcal{O}_X$$

Now, we already know from chapter 2 that the derived global sections functors targets the derived category $D^b(\mathrm{Vect}_k^{\mathrm{fin}})$ which is of cohomological dimension ≤ 1[7], i. e.

$$R\Gamma(F) \simeq \bigoplus_i H^i R\Gamma(X, \mathcal{F})[-i]$$

Therefore $\Phi_{E_i}(\mathcal{A})$ is isomorphic to a complex which has zeroes as differentials and at position k

$$(\Phi_{E_i}(\mathcal{A}))^k \simeq H^k R\Gamma(X, \mathcal{A} \otimes \Omega^i) \otimes \mathcal{O}(-i) \simeq \mathcal{O}(-i)^{\oplus h_k}$$

where we denoted $h_k = \dim H^k R\Gamma(X, \mathcal{A} \otimes \Omega^i)$ as a vector space.

Now we notice direct sums is can be generated by shifts and a cones (of zero morphisms) so

$$\Phi_{E_i}(\mathcal{A}) \in \langle \mathcal{O}(i) \rangle$$

□

Remarks 4.1.9.

- By choosing the integral transform in 3, but in the other direction (namely from the first $\mathbb{P}^n$ to the second), the same exact argument can be adapted to prove that

$$\left\langle \Omega^n(n), \Omega^{n-1}(n-1), \ldots, \Omega^1(1), \mathcal{O} \right\rangle$$

  is a generating set for $D^b(\mathbb{P}^n)$

- Theorem 4.1.4 is sometimes (e.g. in [Dol09]) casted in the theory of semi-orthogonal decompositions of triangulated categories. The key concept is that the generating set in Prop. 4.1.5 form an exceptional sequence for $D^b(\mathbb{P}^n)$. By such sequence we mean objects $A_n, \ldots, A_0$ such that

$$\mathrm{Ext}^i(A_p, A_q) = 0 \quad \text{for all } \ i, \text{ if } p < q,$$

  and

$$\mathrm{Ext}^i(A_p, A_p) = \begin{cases} 0 & \text{if } i > 0 \\ k & \text{if } i = 0 \end{cases}$$

- Let $\mathcal{F}$ as a sheaf on $\Delta \simeq X$ but viewed as a sheaf on $X \times X$. If we tensor $\mathcal{F}$ with the Beĭlinson resolution in Prop. 4.1.5, we obtain a resolution of $\mathcal{F}$ in $X \times X$, which we can then cunningly pushforward with Rp_{2*} to obtain a complex quasi-isomorphic to $\mathcal{F}$. This will give rise to the following spectral sequence

$$E_1^{p,q} = H^p\left(\mathbb{P}^n, \mathcal{F}(r)\right) \otimes \Omega_X^{-q}(-q) \Longrightarrow E^{p+q} = \begin{cases} \mathcal{F} & p+q=0 \\ 0 & p+q \neq 0 \end{cases}$$

  and similarly, by using Rp_{1*}

$$E_1^{p,q} = H^p\left(\mathbb{P}^n, \Omega_X^{-q}(-q)\right) \otimes \mathcal{O}_X(q) \Longrightarrow E^{p+q} = \begin{cases} \mathcal{F} & p+q=0 \\ 0 & p+q \neq 0 \end{cases}$$

  which are called Beĭlinson spectral sequences

[7] Cf. Interlude: Homological Dimension

4.2 Reconstruction theorems

In this section we discuss Bondal-Orlov's Reconstruction Theorem, the proof presented here closely follows the original given in [BO01]. A the end we will discuss another way to prove the result in terms of Fourier-Mukai Transforms and the structure of the group of autoequivalences of a smooth projective variety with ample (anti-)canonical bundle.

Bondal-Orlov's theorem constitute the core of this thesis, in the course of its proof we will already be able to glimpse at the very structure of equivalences between the derived categories of coherent sheaves of the varieties. This gives, in our view, a basis for an epistemological justification of Orlov's Representability Theorem 3.3.11.

Using the techniques developed insofar we will see that it is possible to pin down objects in $D^b(X)$ that are "point-like" (though the terminology will be clear in few lines) which will be mapped to through the equivalence to point-like objects in $D^b(Y)$. But this correspondence of points is not enough to recover the Zariski topology of the varieties, in fact, such feat will be delivered by the reconstruction of a correspondence between invertible sheaves. To quote the authors of [BO01],

> Invertible sheaves help us 'glue' points together.

First we need to pick up some of basic definitions about sheaves, i. e. the concept of ampleness of a sheaf.

Definition 4.2.1. Let $\mathcal{F}$ be a sheaf over a scheme[8] X, we say that $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$ is generated by global sections if exists a family of sections $\{s_i\} \subset \Gamma(X, \mathcal{F})$ such that the germs $\{s_{i,x}\}$ generates $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module, for every $x \in X$. Equivalently, there is a surjection

$$\bigoplus_{i \in I} \mathcal{O}_X \twoheadrightarrow \mathcal{F}$$

i. e. $\mathcal{F}$ is the cokernel of a free sheaf.

Definition 4.2.2. Let X be a scheme. Then we call an invertible sheaf $\mathcal{L}$ on X <u>very ample</u> if exists a closed immersion $i : X \hookrightarrow \mathbb{P}^r$, for some $r \geq 1$, such that $i^*(\mathcal{O}_X(1)) \simeq \mathcal{L}$.

Definition 4.2.3. (cf. [Gro60], II, 4.5.5). Let X be a scheme, and let L be an invertible sheaf on X. We say L is ample if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer n_0 such that for every $n \geq n_0$ the sheaf $\mathcal{F} \otimes L^{\otimes n}$ is generated by its global sections (as an $\mathcal{O}_X$-module).

Proposition 4.2.4. *Let X as above, L an invertible sheaf on X, the following conditions are equivalent*[9]*:*

- *L is ample.*
- *For some $n \geq 0$, $L^{\otimes n}$ is very ample.*

if moreover X is proper, i. e. $X \longrightarrow \operatorname{Spec} k$ is proper, then the above are equivalent to

- *For every coherent sheaf $\mathcal{F}$ on X , there is an integer n_0 such that for all $n \geq n_0$ and $i > 0$,*

$$H^i(X, \mathcal{F} \otimes L^{\otimes n}) = 0.$$

Proposition 4.2.5. *Let X a scheme, L, M be invertible sheaves. Then:*

1. *If $n > 0$ is an integer, L is ample $\Longleftrightarrow$ $L^{\otimes n}$ is ample.*
2. *If L, M are ample, then $L \otimes M$ is ample.*
3. *If L is ample, M arbitrary, then $M \otimes L^{\otimes n}$ is ample for large enough n.*

Definition 4.2.6. Let $\mathcal{D}$ a k-linear derived category of some abelian category. Suppose $\mathcal{D}$ admits a Serre functor $S : \mathcal{D} \longrightarrow \mathcal{D}$. An object $P \in \mathcal{D}$ is called <u>point-like object</u> of codimension r if

[8] Here we consider only schemes of finite type over a algebraically closed field $k = \bar{k}$, as customary
[9] Cf. [Gro60], II, 4.4.2, 4.5.10; III, 2.6.1

1. $S(P) \simeq P[r]$.
2. $\mathrm{Hom}(P, P[i]) = 0$ if $i < 0$.
3. $\mathrm{Hom}(P, P) =: k(P)$ is a field.

An object satisfying only 3 is called simple, i. e. every endomorphism of P is invertible.

Remark 4.2.7. Since we assume Hom's to be finite dimensional, for a simple object P, $k(P)$ will be a finite field extension of k. Thus, if k is algebraically closed, $k(P) = k$.

Examples 4.2.8.

- Let X be a smooth projective variety of dimension d over k. Let $x \in X$ be a closed point. Then the skyscraper $k(x) \in D^b(X)$ is a point-like object of codimension d. Lets examine the points of the definition above:
  1. $S_X(k(x)) = k(x) \otimes \omega_X[\dim X] \simeq k(x)[d]$ since the isomorphism holds stalkwise.
  2. We have seen in Ch. 2 Rmk. 2.3.8 and Cor. 2.3.10; this actually holds for any sheaf $\mathcal{F} \in \mathrm{Coh} X$.
  3. There are many ways to see this, for instance we can leverage the adjunction between the stalk functor and the skyscraper (cf. footnote 3. of last Chapter):

  $$\mathrm{Hom}(k(x), k(x)) \simeq \mathrm{Hom}(k(x), i_{x,*}k(x)) \simeq \mathrm{Hom}_{\mathcal{O}_{X,x}}(k(x), k(x)) \simeq k(x).$$

- Assume $\omega_X \simeq \mathcal{O}_X$, e.g. if X is an abelian variety, K3 surface or a Calbi-Yau manifold. Then any closed resduced connected subvariety $i : Y \hookrightarrow X$ defines a point-like object in $D^b(X)$. Indeed the pushforward of structure sheaf on Y, $i_*\mathcal{O}_Y$ is a point like object of codimension $\dim X$. The first two properties in Def. 4.2.6 are trivial, as for simpleness

  $$\mathrm{Hom}_X(i_*\mathcal{O}_Y, i_*\mathcal{O}_Y) = \mathrm{Hom}_Y(i^*i_*\mathcal{O}_X, \mathcal{O}_Y) = \mathrm{Hom}_Y(\mathcal{O}_Y, \mathcal{O}_Y)$$

  which is a field.

Definitions 4.2.9.

- The support of a complex $F^\bullet \in D^b(X)$ is the union of all the supports of its cohomologies[10]. In other words $\mathrm{supp}\, F^\bullet$ is the closed subset of X defined by

  $$F := \bigcup_{i \in \mathbb{Z}} \mathrm{supp}(H^i(F^\bullet))$$

  Notice that for a complex concentrated in degree zero $\mathcal{F}$, the support as a sheaf[11] and of its complex trivially coincide.

- The homological dimension $\mathrm{dh}(\mathcal{F}^\bullet)$ of a non-zero $\mathcal{F}^\bullet$ is the smallest i such that $\mathcal{F}^\bullet$ is quasi-isomorphic to a complex of locally free sheaves of length $i+1$. For example, $\mathrm{dh}(\mathcal{F}^\bullet) = 0$ if and only if $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{E}[r]$, where $\mathcal{E}$ is a locally free sheaf.

Lemma 4.2.10. *Let $F^\bullet \in D^b(X)$ with $\mathrm{supp}\, F^\bullet = Z_1 \sqcup Z_2$ for some disjoint closed subsets $Z_1, Z_2 \subset X$. Then*

$$F^\bullet \simeq F_1^\bullet \oplus F_2^\bullet$$

for some non-zero objects $F_j^\bullet \in D^b(X)$ such that $\mathrm{supp}(F_j^\bullet) \subseteq Z_j$ for $j = 1, 2$.

Proof: We proceed by induction on the amplitude $\mathrm{amp}(F^\bullet) := i^+ - i^-$ of the complex $F^\bullet$[12]. As in Proposition 2.3.9, if $\mathrm{amp}(F^\bullet) = 0$ then by Theorem 1.3.6 $F^\bullet \simeq \mathcal{F} \in \mathrm{Coh} X$,

[10] Cohomology of the complex $F^\bullet$, not its sheaf cohomology
[11] i. e. $\mathrm{supp}\,\mathcal{F} = \bigcap \{U \in \mathrm{Op} X \mid \mathcal{F}|_U = 0\}^c$
[12] Notations as in Remark 3.3.3 of the previous chapter

i. e. (up to a shift) a coherent sheaf concentrated in degree zero. Then $\operatorname{supp} F^\bullet = \operatorname{supp} \mathcal{F} = Z_1 \sqcup Z_2$ we see then easily follows that[13]

$$\mathcal{F} \simeq \mathcal{F}_{Z_1} \oplus \mathcal{F}_{Z_2}.$$

Now let $\operatorname{amp} F^\bullet \geq 1$, $m = i^-$. Since we can complete the roof

$$\begin{array}{ccccc} & & \tau^{\leq m} F^\bullet & & \\ & \swarrow\!\!\sim & & \searrow & \\ H^m(F^\bullet)[-m] & & \dashrightarrow & & F^\bullet \end{array}$$

to a distinguished triangle in $D^b(X)$ (cf. Remark 3.3.3)

$$H^m(F^\bullet)[-m] \xrightarrow{\psi} F^\bullet \longrightarrow G^\bullet \coloneqq \operatorname{cone} \psi \longrightarrow H^m(F^\bullet)[-m+1]$$

So that, by the long exact sequence in cohomology

$$H^j(G^\bullet) \simeq \begin{cases} H^j(F^\bullet) & \text{if } j > i^-, \\ 0 & \text{if } j \leq 0. \end{cases}$$

We can apply the inductive hypothesis on both $H^m(F^\bullet)$ and $G^\bullet$, for all q and $j = 1, 2$

$$H^m(F^\bullet) = \mathcal{H}_1 \oplus \mathcal{H}_2 \;\text{ and }\; G^\bullet = G_1^\bullet \oplus G_2^\bullet$$

Such that $\operatorname{supp} \mathcal{H}_j$, $\operatorname{supp} H^q(G_j^\bullet) \subseteq Z_j$

Now since $H^{-q}(G_1^\bullet)$ and $\mathcal{H}_2$ are coherent sheaves with disjoint support, we have

$$\operatorname{Hom}_{D^b(X)}(H^q(G_1^\bullet), \mathcal{H}_2[p]) = \operatorname{Ext}^p(H^q(G_1^\bullet), \mathcal{H}_2) = 0\,, \quad \forall p \in \mathbb{Z}$$

This is clear for Ext^0 since $\operatorname{Hom}_{\operatorname{Coh} X}(H^q(G^\bullet), \mathcal{H})$. For higher Ext's observe that the roof $H^q(G_1^\bullet) \overset{s}{\leftsquigarrow} K^\bullet \longrightarrow \mathcal{H}_2[p]$ must be zero since s is a quasi-isomrphism, hence $\operatorname{supp} K^\bullet = \operatorname{supp} H^q(G_1^\bullet)$ and the morphism $K^\bullet \longrightarrow \mathcal{H}_2$, is stalkwise zero.

At this point if we apply Künneth spectral sequence

$$0 = \operatorname{Hom}(H^{-q}(G_1^\bullet), \mathcal{H}_u[p]) \implies \operatorname{Hom}(G_1^\bullet, \mathcal{H}_2[p+q])$$

To gain

$$\operatorname{Hom}(G_1^\bullet, \mathcal{H}_2[1-m]) = 0,$$

similarly

$$\operatorname{Hom}(G_2^\bullet, \mathcal{H}_1[1-m]) = 0.$$

Now choose $\mathcal{F}_j$ to complete the morphisms $G_j^\bullet \longrightarrow \mathcal{H}_j[1-m]$ to distinguished triangles, for $j = 1, 2$

$$F_j^\bullet \longrightarrow G_j^\bullet \longrightarrow \mathcal{H}_j[1-m] \longrightarrow F_j^\bullet[1]$$

Then we have the following diagram

$$\begin{array}{ccccccc} F_1^\bullet \oplus F_2^\bullet & \longrightarrow & G_1^\bullet \oplus G_2^\bullet & \longrightarrow & \mathcal{H}_1[1-m] \oplus \mathcal{H}_2[1-m] & \longrightarrow & F_1^\bullet[1] \oplus F_2^\bullet[1] \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \wr} & & \Big\downarrow{\scriptstyle \wr} & & \Big\downarrow{\scriptstyle h[1]} \\ F^\bullet & \longrightarrow & G^\bullet & \longrightarrow & \mathcal{H}[1-m] & \longrightarrow & F^\bullet[1] \end{array}$$

[13]The notation $\mathcal{F}_Z$ refers to the "cut by" procedure in standard sheaf theory, i. e. $\mathcal{F}_Z \coloneqq i_{Z*} i_Z^{-1} \mathcal{F}$ for any sheaf over a topological space X and any closed subset $Z \overset{i}{\hookrightarrow} X$. Stalkwise we have:

$$(\mathcal{F}_Z)_x \simeq \begin{cases} \mathcal{F}_x & \text{if } x \in Z \\ 0 & \text{otherwise} \end{cases}$$

where the arrow h comes from axiom TR3, moreover it is an isomorphism by the Five Lemma. Then by the long exact sequence in cohomology we have

$$H^m(F_j^\bullet) \simeq \mathcal{H}_j \quad \text{and} \quad H^q(F_j^\bullet) \simeq H^q(G_j^\bullet) \text{ for } q > m$$

In particular $\operatorname{supp} F_j^\bullet \subseteq Z_j$ □

Proposition 4.2.11. *Let X be a smooth projective variety over k. Suppose that $F^\bullet$ is a simple object of $D^b(X)$ with zero-dimensional support. If $\operatorname{Hom}(F^\bullet, F^\bullet[i]) = 0$ for $i < 0$, then*

$$F^\bullet \simeq k(x)[m]$$

for some closed point $x \in X$ and some integer m.

Proof: Since $\operatorname{supp} F^\bullet$ has dimension zero, it must be a finite, disjoint union of closed points in X. If $\operatorname{supp} F^\bullet$ is not a single point, then we may apply the previous Lemma 4.2.10 and we would have a non-trivial decomposition $F^\bullet \simeq F_1^\bullet \oplus F_2^\bullet$. Then the projection onto one of the components would yield a non invertible endomorphism.

Therefore $\operatorname{supp} F^\bullet$ is concentrated in a single point, thus all sheaves $H^q(F^\bullet)$ are supported in one closed point $x \in X$. The residue field is $k(x) \simeq A/\mathfrak{m}_x$ where $U = \operatorname{Spec} A$ is a neighborhood of x and since $\operatorname{supp} H^{i^-} = \operatorname{supp} H^{i^+} = \{x\}$, notations as above, we have $H^q(F^\bullet) \equiv \mathcal{H}^q$

$$0 \neq (\mathcal{H}^{i^+})_x \simeq M^+_{\mathfrak{m}_x} \quad \text{and} \quad M^-_{\mathfrak{m}_x} \simeq (\mathcal{H}^-)_x \neq 0$$

where M^+, M^- are finitely generated A-modules.

Then we can use the following fact from commutative algebra:

> For M a finitely generated module over a local Noetherian ring $(A, \mathfrak{m})$ and $\operatorname{supp} M = \{\mathfrak{m}\}$ there exists the following surjection π and injection i
>
> $$M \xrightarrow{\pi} A/\mathfrak{m} \xhookrightarrow{i} M$$
>
> Indeed, because A is Noetherian, there exists a filtration
>
> $$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$
>
> with $M_{i+1}/M_i \simeq A/\mathfrak{p}_i$, for $\mathfrak{p}_i \in \operatorname{supp} M$, see for instance [Sta23], 00LB. Since A is local, $\mathfrak{p}_i = \mathfrak{m}$ for all i. The sequence above gives rise to an injection i as well as to a canonical projection $M \twoheadrightarrow M/M_n \simeq A/\mathfrak{m}$

Therefore we have a non trivial composition

$$(\mathcal{H}^{i^+})_x \twoheadrightarrow k(x) \hookrightarrow (\mathcal{H}^{i^-})_x$$

Which extends to a non-trivial morphism of sheaves $\mathcal{H}^{i^+} \longrightarrow \mathcal{H}^{i^-}$ and completed to a distinguished triangle as above, by taking advantage of the composition of these two natural roofs

$$\begin{array}{ccccccc} & \tau^{\leq i^+} F^\bullet & & & & \tau^{\leq i^-} F^\bullet & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ F^\bullet & \overset{\varphi}{\dashrightarrow} & \mathcal{H}^{i^+}[-i^+] & \cdots\rightarrow & \mathcal{H}^{i^-}[-i^-] & \overset{\psi}{\dashrightarrow} & F^\bullet \end{array}$$

Which we shift to obtain

$$F^\bullet[i^+] \longrightarrow \mathcal{H}^{i^+} \xrightarrow{h} \mathcal{H}^{i^-} \longrightarrow F^\bullet[i^-]$$

Recall that from Remark 3.3.3, the maps φ and ψ induce the identity at the i^+-th and i^--th cohomology, respectively. Hence since h is non-trivial it must be $i^+ = i^- = m$, so $F^\bullet$ has zero amplitude, i. e. a shifted coherent sheaf supported in x, say $\mathcal{F}[m]$.

Now, $\mathcal{F} = \mathcal{H}^m \xrightarrow{h} \mathcal{H}^m$ is a non-trivial morphism given locally by

$$M \xrightarrow{\pi} A/\mathfrak{m} \xhookrightarrow{i} M$$

where $M^+ = M^- = M$. Since $\mathcal{F}$ is simple, h is invertible, in particular π and i are isomorphisms, and locally $M \simeq A/\mathfrak{m}$. Therefore $\mathcal{F} \simeq k(x)$ □

Proposition 4.2.12. *Let X be a smooth projective variety of dimension n with ample canonical or anti-canonical sheaf. Then any point-like objects of $D^b(X)$ are of the form $k(x)[m]$, i. e. shifts of skyscrapers supported at some closed point $x \in X$.*

Proof: We already know that the skyscraper sheaves $k(x)[m]$ are point-like objects, see Examples 4.2.8. Assume $P \in D^b(X)$ be a point-like object of codimension r, then

$$P \otimes \omega_X[\dim X] \simeq P[r]$$

implies[14]

$$H^j(P) \otimes \omega_X[n] \simeq H^j(P)[r]$$

non-trivially for $i^- \leq j \leq i^+$, this forces $n = r$. If we keep tensoring with ω_X we obtain

$$H^j(P) \equiv \mathcal{H}^j \simeq \mathcal{H}^j \otimes \omega_X^{\otimes t}, \quad t \geq 0$$

Where $\omega_X^{\otimes t}$ is very ample given ω_X ample. The same argument holds if the anti-canonical sheaf ω_X^{-1} were ample, we would have $\mathcal{H}^j \simeq \mathcal{H}^j \otimes \omega_X^{\otimes -t}$.

Let $\mathcal{L} = \omega^{\otimes \pm t}$ gives rise to a closed embedding $i : X \hookrightarrow \mathbb{P}^m$, for some m, such that $\mathcal{L} \simeq i^*\mathcal{O}_{\mathbb{P}^m}(1)$

$$i_*(\mathcal{H}^j \otimes \mathcal{L}) \simeq i_*\big(\mathcal{H}^j \otimes i^*\mathcal{O}_{\mathbb{P}^m}(1)\big) \simeq i_*\mathcal{H}^j \otimes \mathcal{O}_{\mathbb{P}^m}(1) \simeq (i_*\mathcal{H}^j)(1)$$

We may then assume[15] $X \simeq \mathbb{P}^m \simeq \operatorname{Proj} k[t_0, \dots, t_m]$ and $\mathcal{H}^j = \widetilde{M}$ for some graded module $M = \bigoplus_d M_d$ over $k[t_0, \dots, t_m]$. Notice that, by what we worked out before, twisting $\widetilde{M}$ produces no effect

$$\widetilde{M} \simeq \widetilde{M}(1) \simeq \widetilde{M}(2) \dots \tag{4}$$

Where $M(n)_d = M_{n+d}$. Let $P_M(T)$ be the Hilbert polynomial of M, which is defined to be $P_M(n) = \dim_k M_n$ for large enough n. We know[16] that $\deg P(T) = \operatorname{supp} \widetilde{M}$. Since (4), must have

$$P_M(n) = P_M(n+1) = \dots$$

Which is possible only if the polynomial has degree 0. Therefore $\operatorname{supp} \widetilde{M} = \operatorname{supp} \mathcal{H}^j$ is zero dimensional, thus we may apply Proposition 4.2.11. □

Remark 4.2.13. As we have seen in the proof above, ampleness plays a fundamental role in Proposition 4.2.12, and it fails when $\omega^{\pm 1}$ is not ample, see for instance Example 4.2.8.

We have shown a way to reconstruct points of a variety X, now realize line bundles on X as objects of $D^b(X)$

Definition 4.2.14. Let $\mathcal{D}$ be a triangulated category together with a Serre functor $S_{\mathcal{D}}$. An object $L \in \mathcal{D}$ is said to be invertible if for each point-like object $P \in \mathcal{D}$, there is an integer n_P (which depends also on L) such that

$$\operatorname{Hom}_{\mathcal{D}}(L, P[i]) = \begin{cases} k(P) & \text{if } i = n_P \text{ , and} \\ 0 & \text{otherwise} \end{cases}$$

[14] Locally free sheaves are adapted to the tensor product, i. e. $- \otimes \omega_X$ is exact and commutes with limits and colimits, cohomology is a cokernel

[15] The argument can be followed verbatim for X a closed subvariety of $\mathbb{P}^n$

[16] Cf. [Ser55], III, §6, par. 81, 6; [Vak23] 18.6.1; [Har77] 7.5.

Proposition 4.2.15 (Bondal, Orlov). *Let X be a smooth projective variety over k. Any invertible object $L^\bullet \in D^b(X)$ is of the form $\mathcal{L}[m]$ with $\mathcal{L} \in \mathrm{Coh}X$ a line bundle on X and $m \in \mathbb{Z}$.*

Conversely if we assume $\omega_X^{\pm 1}$ is ample, then for any line bundle $\mathcal{L}$ and any $m \in \mathbb{Z}$, the object $\mathcal{L}[m] \in D^b(X)$ is invertible.

Proof: Let us prove last part of the statement. Let $\mathcal{L}$ be a line bundle on X, P a point-like object in $D^b(X)$, by assumption, is of the form $k(x)[l]$. We want to show that $\mathcal{L}[m] \in D^b(X)$ is invertible: for $i \in \mathbb{Z}$

$$\begin{aligned}
\mathrm{Hom}_{D^b(X)}(\mathcal{L}[m], P[i]) &\simeq \mathrm{Hom}_{D^b(X)}(\mathcal{L}, k(x)[i+l-m]) \\
&\simeq \mathrm{Ext}^{i+l-m}(\mathcal{L}, k(x)) \\
&= R^{i+l-m}\,\mathrm{Hom}_{\mathrm{Coh}X}(\mathcal{L}, k(x)) \\
&= R^{i+l-m}\,\mathrm{Hom}_{\mathrm{Coh}X}(\mathcal{O}_X, \mathcal{L}^\vee \otimes k(x)) \\
&= R^{i+l-m}\Gamma(X, \mathcal{L}^\vee \otimes k(x)) \\
&= H^{i+l-m}\Gamma(X, \mathcal{L}^\vee \otimes k(x)) = 0
\end{aligned}$$

except for $i = m - l$ since $\mathcal{L}^\vee \otimes k(x)$ is flasque. Then we can set $n_p = m - l$.

The converse is a little involved, we won't use ampleness of $\omega_X^{\pm 1}$. Let $L^\bullet \in D^b(X)$ be an invertible object and $m = i^+$ maximal with $\mathcal{H}^m := H^m(L^\bullet) \not\simeq 0$.

Claim 1 $n_{k(x)} = -m$, for $x \in \operatorname{supp}\mathcal{H}^m$.
Indeed, by Remark 3.3.3 we have a non-trivial morphism $L^\bullet \longrightarrow H^m(L^\bullet)[-m]$, which induces the identity a the m-th cohomology and[17]

$$0 \not\simeq \mathrm{Hom}(\mathcal{H}^m, k(x_0)) \simeq \mathrm{Hom}(L^\bullet, k(x_0)[-m])$$

For any closed point[18] $x_0 \in \operatorname{supp}\mathcal{H}^m$. Therefore $n_{k(x_0)} = -m$ ♢

Claim 2 $\mathrm{Ext}^1(\mathcal{H}^m, k(x_0)) = 0$
We employ once again Künneth spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(H^{-q}(L^\bullet), k(x_0)[p]) \Longrightarrow \mathrm{Hom}(L^\bullet, k(x_0)[p+q])$$

to obtain at page 2

$$E_2^{1,-m} = \mathrm{Hom}(\mathcal{H}^m, k(x_0)[1]) = \mathrm{Hom}(L^\bullet, k(x_0)[1 + n_{k(x_0)}]) = 0$$

by definition of invertible object. ♢

Claim 3 $\mathcal{H}^m$ is a locally free $\mathcal{O}_X$-module.
Consider the Local-to-Global spectral sequence

$$E_2^{p,q} = H^p\left(X, \mathcal{E}xt^q(\mathcal{H}^m, k(x_0))\right) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{H}^m, k(x_0))$$

Which allow us to relate the global vanishing of $\Gamma(X, \mathcal{E}xt^1(\mathcal{H}^m, k(x_0))) = \mathrm{Ext}^1(\mathcal{H}^m, k(x_0) = 0$ to the local vanishing of $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)$. This spectral sequence makes sense because $R\Gamma \circ R\,\mathcal{H}om(\mathcal{F}^\bullet, -) \simeq R\,\mathrm{Hom}(\mathcal{F}^\bullet, -)$ and the image under $\mathcal{H}om(-, k(x))$ has support only in one point, therefore is flasque i. e. Γ-acyclic[19].

In particular we have

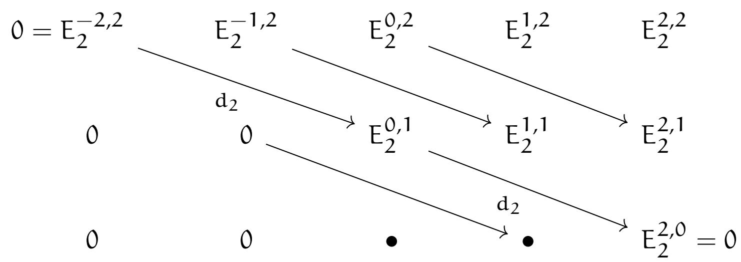
$$E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m, k(x_0)) = 0$$

As well as for negative cohomology,

$$E_2^{-2,2} = H^{-2}(X, \mathcal{E}xt^2(\mathcal{H}^m, k(x_0)) = 0$$

[17] See the proof of Lemma 3.3.2 for the construction of the isomorphism
[18] For instance one can use the same argument in the proof of Prop. 4.2.11 to construct a surjection $\mathcal{H}^m \twoheadrightarrow k(x_0)$
[19] i. e. higher $(q > 0)$ $R^q\Gamma(X, -) =: H^q(X, -)$ vanish!

We see that



Since

$$E_3^{0,1} = H^0\left(\cdots \longrightarrow 0 \longrightarrow E_2^{0,1} \longrightarrow 0 \longrightarrow \cdots \right) = E_2^{0,1}$$

Thus enforcing the same argument in the next pages of the spectral sequence, it yields $E_2^{0,1} = E_\infty^{0,1}$. But from claim 4.2 we know $E^1 = \operatorname{Ext}^1(\mathcal{H}^m, k(x_0)) = 0$, therefore

$$0 = E_\infty = E_2^{0,1} = H^0(X, \mathcal{E}xt^1(\mathcal{H}^m, k(x_0))$$

This means that $\mathcal{E}xt$ has no global sections, but we know it is globally generated[20], because it is supported on $\{x_0\}$. Therefore

$$\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)) = 0$$

Since $\mathcal{H}^m \in \operatorname{Coh} X$ we have

$$\operatorname{Ext}^1_{\mathcal{O}_{X,x_0}}(\mathcal{H}^m_{x_0}, k(x_0))\, \mathcal{E}xt^1(\mathcal{H}^m, k(x_0))_{x_0}$$

We can now use the following Lemma from commutative algebra.[21]

> **Lemma 4.2.16.** *Any finitely generated module* M *over a Noetherian local ring* $(A, \mathfrak{m})$ *with* $\operatorname{Ext}^1(M, A/\mathfrak{m}) = 0$ *is free*
>
> *Proof:* This can be found in [Mat70] 7.18, it proves the equivalent statement for Tor_1^A in Lemma 4 and then the equivalence to the assumption on Ext^1_A in Lemma 5. □

In view of the Lemmata above we have that $\mathcal{H}^m_{x_0}$ is a free $\mathcal{O}_{X,x_0}$-module. Recall that *free* is an open property,indeed to see this consider a non-empty affine neighborhood $x_0 \in U = \operatorname{Spec} A \subseteq \operatorname{supp} \mathcal{H}^m$, where the restrinction $\mathcal{H}^m|_U$ correspond to a finitely generated A-module M. Since A is Noetherian[22] we get an exact sequence of finitely genrated modules

$$0 \longrightarrow N \xrightarrow{f} A^{\oplus q} \xrightarrow{g} M \longrightarrow 0$$

This induces an exact sequence localized at $x_0 \in \operatorname{Spec} A$

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \| & & & & & & \\ 0 & \longrightarrow & N_{x_0} & \xrightarrow{f} & A^{\oplus q}_{x_0} & \xrightarrow{g} & M_{x_0} & \longrightarrow & 0 \\ & & & & & & \wr\| & & \\ & & & & & & \mathcal{H}^m_{x_0} & & \end{array}$$

Now, for a minimal set of generators $n_1, \ldots, n_l$ of N, since M_{x_0} is free, n_i restricts to zero in N_{x_0}. Therefore we can consider neighborhoods[23] $U_{n_i} \subseteq$

[20] The map $\mathcal{O}^{\oplus}_{X,x_0} \twoheadrightarrow \mathcal{E}xt_{x_0}$ is surjective since $\mathcal{E}xt$ is coherent and thus its stalk at x_0 is a finitely generated module.

[21] For a modern reference see §6.2 in [Mur06]

[22] Or by definition of a coherent sheaf

[23] Back at the sheaf theoretic formalism, if the germ n_{x_0} is zero in the stalk N_{x_0}, it must be zero also in a neighborhood

Spec A where $n_i = 0$. Since $x \in U_{n_i}$ for all $1 \leq i \leq l$ we have $N|_{\cap_i U_{n_i}} = 0$, hence $\mathcal{H}^m$ is free on $\bigcap_i U_{n_i}$.

Because X is irreducible, $\mathcal{H}^m$ is coherent and $\operatorname{supp} \mathcal{H}^m$ contains an open, dense subset of X, we have $\operatorname{supp} \mathcal{H}^m = X$. Thus $\mathcal{H}^m$ is locally free. $\diamondsuit$

Claim 4 $\mathcal{H}$ is a line bundle on X.
In *Claim 1* we proved there is a surjection $\mathcal{H}^m \twoheadrightarrow k(x)$ for any $x \in \operatorname{supp} \mathcal{H}^m$. Therefore

$$\operatorname{Hom}(L^\bullet, k(x)[-m]) \simeq \operatorname{Hom}(\mathcal{H}^m, k(x)) \neq 0$$

and from Definition 4.2.14 of invertible object, holds

$$n_{k(x)} = -m, \qquad \forall x \in X$$

i. e. $n_{k(x)}$ does not depend on x, if r is the rank of $\mathcal{H}^m$,

$$\begin{aligned} k(x) &\overset{\star}{\simeq} \operatorname{Hom}(L^\bullet, k(x)[-m]) \\ &\simeq \operatorname{Hom}(\mathcal{H}^m, k(x)) \\ &\simeq \operatorname{Hom}(\mathcal{O}_X^{\oplus r}, k(x)) \simeq k(x)^{\oplus r} \end{aligned}$$

Where in the first line $(\star)$ is justified stalkwise. Therefore $r = 1$. $\diamondsuit$

Claim 5 $L^\bullet$ is a sheaf
It is enough to show $\mathcal{H}^i = 0$ for $i < m$. Indeed, consider again the spectral sequence of *Claim 2*

$$\begin{aligned} E_2^{q,-m} &= \operatorname{Hom}(\mathcal{H}^m, k(x)[q]) \\ &= \operatorname{Ext}^q(\mathcal{H}^m, k(x)) \\ &\simeq H^q(X, \mathcal{H}om(\mathcal{H}^m, k(x))) = 0, \quad \forall q > 0 \end{aligned} \tag{4}$$

This is because $\mathcal{H}om(\mathcal{H}^m, k(x))$ is supported in a single point x, and hence is flasque.

Suppose that $i < m$. Then by Definition 4.2.14, we have

$$E^{-i} = \operatorname{Hom}(L^\bullet, k(x)[-i]) = 0\,, \qquad \forall x \in X$$

Now to show that $\mathcal{H}^i = 0$ is enough to show

$$E_2^{0,-i} = \operatorname{Hom}(\mathcal{H}^i, k(x)) = 0\,, \qquad \forall x \in X$$

4 Since $E^{-i} = 0$, we can just show that each of $E^{0,-i}$ persist up to the limit, i. e.

$$E_2^{0,-i} = E_\infty^{0,-i} := E^{-i}, \qquad i < m$$

By induction on i.
Anfang: $i = m-1$. We can visualize page 2 of the spectral sequence as follows

$$\begin{array}{ccccc} 0 = E_2^{-2,-m+2} & 0 & E_2^{0,-m+2} & E_2^{1,-m+2} & E_2^{2,-m+2} \\ 0 & 0 & E_2^{0,-m+1} & E_2^{1,-m+1} & E_2^{2,-m+1} \\ 0 & E_2^{-1,-m} & E_2^{0,-m} & 0 & E_2^{2,-m} = 0 \\ & \| & & & \\ & 0 & & & \end{array}$$

by (4), the row $q = -m$ is filled with zeroes except in position $(0, -m)$. Negative indexed columns corresponds to negative Ext's which vanish by 2.3.8, since both coherent sheaves.
Inductive step: $\mathcal{H}^i = 0, \ i_0 < i \leq m-1$. Then if we scroll the diagram above, up to row $i_0 + 1$, the induction hypothesis applies: $\mathcal{H}^{i_0+1} = 0$. We obtain

$$\cdots \longrightarrow 0 = E_2^{-2,(1-i_0)} \xrightarrow{d} E_2^{0,-i_0} \xrightarrow{d} E_2^{2,-i_0-1} = 0 \longrightarrow \cdots$$

Therefore we can repeat verbatim the argument in the Anfang. Hence

$$\mathcal{H}^i \equiv H^i(L^\bullet) = 0, \qquad \forall i \neq m.$$

So $L^\bullet$ must be a shift of some line bundle $\mathcal{L} \in \mathrm{Coh}X$.

□

Bondal-Orlov's reconstruction theorem employs a known result of algebraic geometry, which we present in the following form

Theorem 4.2.17. *([GW20], 13.47 and 13.48). Let X be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-modules on X. Consider the graded algebra $S := \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^{\otimes i})$, and its ideal $S_+ = \bigoplus_{i > 0} H^0(X, \mathcal{L}^{\otimes i})$. For each homogeneous elements $s \in H^0(X, \mathcal{L}^{\otimes i})$, $i > 0$, define*

$$X_s := \left\{ x \in X \ : \ s_x \notin m_x \mathcal{L}_x^{\otimes i} \right\}.$$

Then the following are equivalent.

- *$\mathcal{L}$ is ample.*
- *The collection of open sets X_s with $s \in S_+$ covers X, and the natural morphism*

$$X \hookrightarrow \operatorname{Proj} S$$

  is an open immersion.
- *The collection of open sets X_s, for $s \in S_+$ homogeneous, is a basis for the Zariski topology on X.*

Corollary 4.2.18. *([GW20], 13.75). Let X be a smooth projective variety over k. Let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L}$ or $\mathcal{L}^\vee$ is ample, then the natural morphism of k-schemes*

$$X \xrightarrow{\sim} \operatorname{Proj}\left(\bigoplus_n H^n(X, \mathcal{L}^{\otimes n}) \right)$$

is an isomorphism.

proof (sketch): As in the proof of Proposition 4.2.12, up to tensor powers of $\mathcal{L}$ we have at our disposal a closed immersion $X \overset{i}{\hookrightarrow} \mathbb{P}^n$ defined by the sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{F})$. Let $\mathcal{O}_X(1) := \mathcal{L} \simeq i^* \mathcal{O}_{\mathbb{P}^n}(1) \equiv \mathcal{O}(1)$. Let $\mathcal{I}_X$ the sheaf of ideals defined by X in $\mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$. Then $X \equiv i(X) = \operatorname{Proj} k[x_0, \ldots, x_n]/I$ where

$$I = \Gamma_*(\mathcal{I}_X) := \bigoplus_{t \in \mathbb{Z}} H^0(X, \mathcal{I}_X(t))$$

There is a natural morphism of graded algebras

$$\begin{aligned} k[x_0, \ldots, x_n] &\xrightarrow{\varphi} \bigoplus_{t \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(t)) \\ P(x_0, \ldots, x_n) &\longmapsto P(s_0, \ldots, s_n) \end{aligned}$$

It is easy to see that $\ker \varphi = I$. Then consider the following exact sequence

$$0 \longrightarrow \mathcal{I}_X \otimes \mathcal{O}(n) \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow 0$$

which we composed with $\mathrm{Hom}(\mathcal{O}_X, -)$ yields the usual long exact sequence in cohomology

$$\cdots \longrightarrow H^0(X, \mathcal{O}(n)) \longrightarrow H^0(X, \mathcal{O}_X(n)) \longrightarrow H^1(X, \mathcal{I}_X \otimes \mathcal{O}(n)) \longrightarrow \cdots$$

But by Grothendieck's Vanishing Theorem 2.4.4 applies to $\mathcal{O}(n) \otimes \mathcal{I}_X = \mathcal{I}_X(n)$ to obtain for large enough n

$$H^1(X, \mathcal{I}_X(n)) = 0$$

Therefore $H^0(X, \mathcal{O}(n)) \longrightarrow H^0(X, \mathcal{O}_X(n))$ is surjective and lift to φ. □

We are finally in position to prove the main theorem of this thesis

Theorem 4.2.19 (Bondal-Orlov's Reconstruction). *Let X and Y be smooth projective varieties over a field k, assume (anti)-canonical line bundle of X $\omega_X^{\pm 1}$ is ample. If there exists an exact equivalence $D^b(X) \xrightarrow[\sim]{F} D^b(Y)$, then $X \simeq Y$ as k-varieties. In particular, also $\omega_Y^{\pm 1}$ is ample.*

Proof: To deconstruct its complexity, we subdivide the proof in multiple steps.

Step 1: We shall assume $F(\mathcal{O}_X) = \mathcal{O}_Y$.

By definition of exact (triangulated) equivalence, point-like and invertible object. We must have

$$\begin{array}{ccc} \{\text{Point-like objects in } D^b(X)\} & \xleftrightarrow[\sim]{F} & \{\text{Point-like objects in } D^b(Y)\} \\ \Big\| & \text{(Prop. 4.2.12)} & \uparrow (*) \\ \mathfrak{p}_X := \{k(x)[m] \,:\, x \in X \text{ closed}, \; m \in \mathbb{Z}\} & & \{k(y)[m] \,:\, y \in Y \text{ closed}, \; m \in \mathbb{Z}\} =: \mathfrak{p}_Y \end{array} \tag{5}$$

$$\begin{array}{ccc} \{\text{Invertible objects in } D^b(X)\} & \xleftrightarrow[\sim]{F} & \{\text{Invertible objects in } D^b(Y)\} \\ \Big\| & \text{(Prop. 4.2.15)} & \downarrow (**) \\ \{\mathcal{L}[m] \,:\, \mathcal{L} \in \mathrm{Pic}\, X, \; m \in \mathbb{Z}\} & & \{\mathcal{L}[m] \,:\, \mathcal{L} \in \mathrm{Pic}\, Y, \; m \in \mathbb{Z}\} \end{array} \tag{6}$$

Where, $\mathfrak{p}_X$ (resp. $\mathfrak{p}_Y$) denotes the set of isomorphism classes of shifts of skyscrapers supported at closed points in X (resp. in Y), and $\mathrm{Pic}\, X$,$\mathrm{Pic}\, Y$ their respective Picard groups.

Since $\omega_X^{\pm 1}$ is ample, by last part of Prop. 4.2.15 we have that $\mathcal{O}_X$ is trivially an invertible object in $D^b(X)$. Because F is an equivalence, $F(\mathcal{O}_X)$ is an invertible object of $D^b(Y)$. By applying the first part of Prop. 4.2.15 in Y, we must have $F(\mathcal{O}_X) \simeq M[l]$ for some $M \in \mathrm{Pic}\, Y$ and $l \in \mathbb{Z}$, regardless of whether $\omega_Y^{\pm 1}$ is ample or not.

Now, if $F(\mathcal{O}_X) \neq \mathcal{O}_Y$ we can replace F with the following composition of equivalences

$$D^b(X) \xrightarrow{F} D^b(Y) \xrightarrow{M^\vee \otimes -} D^b(Y) \xrightarrow{[-l]} D^b(Y)$$

That we may still call F and it satisfies $F(\mathcal{O}_X) = \mathcal{O}_Y$. ♢

Step 2: F induces bijections $\mathfrak{p}_X \leftrightarrow \mathfrak{p}_Y$ and $\mathrm{Pic}\, X \leftrightarrow \mathrm{Pic}\, Y$

We shall prove the vertical inclusion $(*)$ in (5) is indeed a bijection between classes of isomorphisms, for the second $(**)$, follows immediately by Proposition 4.2.15.

From the bijection in the first row of (5)

$$\mathfrak{p}_X \simeq \{\text{Point-like objs. in } D^b(X)\} \simeq \{\text{Point-like objs. in } D^b(Y)\}.$$

Choose a closed point $y \in Y$, denote $x_y \in X$ the point that satisfies

$$F(k(x_y)[m_y]) \simeq k(y), \qquad \exists m_y \in \mathbb{Z} \tag{7}$$

Suppose there is a point-like object $P \in D^b(Y)$ which is not in the form $k(y)[m]$, because of the bijections above there is a unique closed point $x_P \in X$ such that $F(k(x_P)[m_p]) \simeq P$. Then $x_P \neq x_y$ for all closed points $y \in Y$. Therefore we must have for any closed $y \in Y$ and any integer m

$$\begin{aligned} \mathrm{Hom}\big(P, k(y)[m]\big) &\simeq \mathrm{Hom}\big(F(k(x_p)[m_p], k(y)[m]\big) \\ &\simeq \mathrm{Hom}\big(k(x_P)[m_p], k(x_y)[m_y+m]\big) \\ &\simeq \mathrm{Hom}\big(k(x_p), k(x_y)[m_y+m-m_p]\big) = 0 \end{aligned}$$

$k(x_P)$ and $k(x_y)$ are skyscrapers supported at different points, so

$$\mathrm{Ext}^i(k(x_P), k(x_y)) = 0$$

for all $i \in \mathbb{Z}$. But objects of the form $k(y)$ form a spanning class in $D^b(Y)$ (cf. Lemma 3.3.2), therefore $P \simeq 0$, which contradicts the assumption on P being a point-like object; indeed by Definition 4.2.6 $\mathrm{End}(P)$ is a field and $\mathrm{id}_P \neq 0$. So we can circle through diagram (5), i. e.

$$\mathfrak{p}_X \simeq \big\{\text{Point-like objs. in } D^b(X)\big\} \simeq \big\{\text{Point-like objs. in } D^b(Y)\big\} \simeq \mathfrak{p}_Y$$

To achieve the same feat but for (6), notice that by following the procedure in (7): for any closed point $x \in X$ exists a unique closed point $y_x \in Y$ such that $F(k(x)) \simeq k(y_x)[m_x]$. Since F is fully faithful and $F(\mathcal{O}_X) = \mathcal{O}_Y$, we have

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_X, k(x)) &\simeq \mathrm{Hom}(\mathcal{O}_Y, k(y_x)[m_x]) \\ &= R^{m_x}\,\mathrm{Hom}(\mathcal{O}_Y, k(y_x)) \\ &= R^{m_x}\Gamma(Y, k(y_x)) = H^{m_x}(Y, k(y_x)) \end{aligned}$$

Which is non-zero only if $m_x = 0$ because $k(y_x)$ is flasque[24]. Therefore, F maps skyscrapers to skyscrapers with no shift

$$F(k(x)) \simeq k(y_x)$$

This immediately implies a bijection $\mathrm{Pic}\,X \simeq \mathrm{Pic}\,Y$. In fact, from the bijections in (6) we were able to find for any $L \in \mathrm{Pic}\,X$ a unique $M \in \mathrm{Pic}\,Y$ and $m_L \in \mathbb{Z}$ such that $F(L) = M[m_L]$. So

$$\begin{aligned} \mathrm{Hom}(L, k(x)) &\simeq \mathrm{Hom}(F(L), F(k(x))) \\ &\simeq \mathrm{Hom}(M[m_L], k(x_y)) \\ &\simeq \mathrm{Hom}(M, k(y_x[-m_L])) \simeq \mathrm{Ext}^{-m_L}(M, k(y_x)) \end{aligned}$$

As above, this forces $m_L = 0$. ♢

Step 3: ω_Y is ample.

Since F is an equivalence and commute with Serre functors 3.3.10, by Step 1 $F(\mathcal{O}_Y) \simeq \mathcal{O}_Y$ in $D^b(Y)$. Since $\mathrm{dh}\,(\mathcal{O}_x) = n = \mathrm{dh}\,(F(\mathcal{O}_x)) = \dim Y$ we obtain that $\dim X = \dim Y$. So we have

$$\begin{aligned} F(\omega_X^k) &= F\big(S_X^k(\mathcal{O}_X)\big)[-kn] \\ &\simeq S_Y^k\big(F(\mathcal{O}_X)\big)[-kn] \\ &\simeq S_Y^k\big(\mathcal{O}_Y\big)[-kn] \\ &\simeq \omega_Y^k. \end{aligned}$$

and

$$\begin{aligned} H^0(X, \omega_X^i) &\simeq \mathrm{Hom}_X\,(\mathcal{O}_X, \omega_X^i) \\ &\simeq \mathrm{Hom}_Y\,(F(\mathcal{O}_X), F(\omega_X^i)) \\ &\simeq \mathrm{Hom}_Y\,(\mathcal{O}_Y, \omega_Y^i) \simeq H^0(Y, \omega_Y^i) \end{aligned}$$

[24]Perhaps also because $\mathcal{O}_Y$ is free and higher Ext's vanish!

for all i.

Let S be a quasi-compact scheme S over k. Consider line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on S and take

$$\alpha_p \in \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = H^0(X, \mathcal{L}_1^\vee \otimes \mathcal{L}_2)$$

for each $p \in S$ closed, define

$$\alpha_p^* := \mathrm{Hom}(\alpha, k(p)) : \mathrm{Hom}(\mathcal{L}_2, k(p)) \longrightarrow \mathrm{Hom}(\mathcal{L}_1, k(p))$$

Then $U_\alpha := \{p \in S \ : \ \alpha_p^* \neq 0\}$ is a Zariski open[25] subset of S (see for instance [GW20], Remark 13.46). Indeed this provides an homological description of the set of zeroes $Z(s)$ of a section s of an invertible sheaf $\mathcal{L}$, where $X_s = X \setminus Z(s)$.

In our setting, since F sends $k(x)$ to $k(y_x)$ for any $x \in Z(s)$. Then it maps

$$\begin{array}{c} (\mathcal{O}_X \xrightarrow{s} \mathcal{L} \longrightarrow k(x)) \\ \downarrow F \\ (\mathcal{O}_Y \xrightarrow{F(s)} F(\mathcal{L}) \longrightarrow k(y_x)) \end{array}$$

Therefore the bijection $\mathfrak{p}_X \xleftrightarrow{f} \mathfrak{p}_Y$ sends subsets of the form $X_s \subseteq X$ to subsets $Y_{F(s)} \subseteq Y$. Since among sets of the form X_t there are affine open subsets defining a base for the Zariski topology of X ($\omega^{\pm 1}$ is ample and 4.2.17), f establishes an homeomorphism between the sets of closed points of X and Y respectively. Recall that for any k-scheme S of finite type, we denote the set of closed points (or k-valued points) of S as

$$S_0 := \mathrm{Hom}_k(\mathrm{Spec}\, k, S),$$

we can reconstruct the scheme $(S, \mathcal{O}_S)$ up to isomorphism from its set of closed points $(S_0, \mathcal{O}_{S_0})$, where $\mathcal{O}_{S_0} = \iota^{-1}\mathcal{O}_X$ and $\iota : S_0 \hookrightarrow S$ the natural inclusion, cf. [GW20] 3.37.

Now fix a line bundle $\mathcal{L} \in \mathrm{Pic}\, X$. Recall that it follows from Proposition 4.2.17 that the collection of such U_α for $\alpha \in H^0(X, \mathcal{L}^{\otimes n})$, forms a basis for the Zariski topology if and only if either $\mathcal{L}$ or its dual $\mathcal{L}^\vee$ is ample.

Then the natural isomorphisms

$$H^0(X, \omega_X^{\pm \otimes i}) \simeq H^0(Y, \omega_Y^{\pm \otimes i})$$

give rise to a bijection between the following families of open subsets

$$\begin{array}{l} \mathcal{B}_X := \{U_\alpha \ : \ \alpha \in H^0(X, \omega_X^{\pm \otimes i})\} \\ \quad \wr| \\ \mathcal{B}_Y := \{U_\alpha \ : \ \alpha \in H^0(Y, \omega_Y^{\pm \otimes i})\} \end{array}$$

Therefore $\mathcal{B}_X$ is a basis for the Zariski topology of X, and restricts to a basis of the Zariski topology of X_0. The bijection f above is an homeomorphism, hence $\mathcal{B}_{Y_0}$ is a basis for Y_0 which then constructs the scheme Y. In virtue of Theorem 4.2.17 this implies that ω_Y is ample. $\diamondsuit$

Step 4: End game: $X \simeq Y$.

The product in the canonical ring

$$A(X) = \bigoplus_{i=0}^{\infty} H^0(X, \omega_X^i)$$

[25] Note that the germ f_x is invertible in $\mathcal{O}_{X,x}$ if and only if the residue class $f(x)$ of f in $k(x)$ is non-zero. See [GW20] section 7.11

can be expressed by the composition of $s_1 \in H^0(X, \omega_X^i), s_2 \in H^0(X, \omega_X^i)$,

$$s_1 \cdot s_2 = S_X^i(s_2)[-in] \circ s_1.$$

From the steps above we have that F defines an isomorphism of graded canonical rings $A(X) \to A(Y)$. By Step 3, $\omega_Y^{\pm 1}$ is ample, therefore we just need to apply Corollary 4.2.18. Since both (anti-)canonical bundles are ample:

$$X \simeq \operatorname{Proj} \bigoplus H^0(X, \omega_X^k) \simeq \operatorname{Proj} \bigoplus H^0(Y, \omega_Y^k) \simeq Y$$

□

Remarks 4.2.20.

- It is worth mentioning the proof given here works for arbitrary fields k.
- It is possible to employ an alternative argument[26] for the ampleness of ω_Y. Assume $k = \bar{k}$ is algebraically closed and let $\varphi : Y \longrightarrow \mathbb{P}_k^n$ be a k-morphism, $L = \varphi^*(\mathcal{O}(1))$ and $s_0, \dots, s_n$ its sections, denote $V \subseteq \Gamma(Y, L)$ the subspace spanned by all the $s_i := \varphi^*(y_i)$. Then φ is a closed immersion if and only if:
  - Elements of V separates points, i. e. for any two closed points $p, q \in Y$, there exists a $s \in V$ such that $s_p \in m_p L_p$ and $s_q \notin m_q L_q$ or viceversa, and
  - Elements of V separates tangent vectors, i. e. for any closed point $p \in Y$ the set $\{s \in V \ : \ s_p \in m_p L_p\}$ spans the k-vector space m_p/m_p^2.

  See [Har77] II.7.3.

We now review few facts about the Kodaira dimension of a variety.

Definition 4.2.21. Let X be a smooth projective variety and let $\mathcal{L} \in \operatorname{Pic}(X)$. The Kodaira dimension $\operatorname{kod}(X, \mathcal{L})$ of $\mathcal{L}$ on X is the integer m such that

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z} \\ \ell &\longmapsto h^0(X, \mathcal{L}^\ell) := \dim H^0(X, \mathcal{L}^\ell) \end{aligned}$$

grows like a polynomial of degree m for $\ell \gg 0$. By definition, we set $\operatorname{kod}(X, \mathcal{L}) = -\infty$ if $h^0(X, \mathcal{L}^\ell) = 0$ for all $\ell > 0$.

Remarks 4.2.22.

- If $\operatorname{kod}(X) = \dim X$, we say that X is of general type.
- The Kodaira dimension is a birational invariant, i. e. if X, Y smooth projective varieties birationally equivalent variety, then $\operatorname{kod} X = \operatorname{kod} Y$. See [CU06]

We shall give a different proof of Bondal-Orlov Reconstruction Theorem by means of Fourier-Mukai Transforms. From now on we make free use of Orlov's Representability Theorem 3.3.11, in order to prove the last results presented in this chapter.

Theorem 4.2.23 (Orlov)**.** *Suppose X and Y are smooth projective varieties with equivalent derived categories*

$$D^b(X) \xrightarrow{\sim} D^b(Y).$$

Then there exists a ring isomorphism $A(X) \simeq A(Y)$ between the respective canonical rings and, in particular, $\operatorname{kod}(X) = \operatorname{kod}(Y)$.

To prove Theorem 4.2.23 above we need the following technical Lemma, which highlights the categorical properties of composition of kernels

Lemma 4.2.24. *[Orl03], 2.1.7. Let X_1, X_2 and Y_1, Y_2 be smooth projective varieties over k. For each $i = 1, 2$, let $P_i \in D^b(X_i \times Y_i)$, and denote $P_1 \boxtimes P_2 \in D^b((X_1 \times Y_1) \times (X_1 \times Y_1))$ their external derived tensor product.*

[26] It can be found in [Huy06], 4.11

1. *Consider the induced Fourier-Mukai transforms $\Phi_{P_i} : D^b(X_i) \to D^b(Y_i)$, for $i = 1, 2$, and $\Phi_{P_1 \otimes P_2} : D^b(X_1 \times X_2) \longrightarrow D^b(Y_1 \times Y_2)$. Then there is an isomorphism*

$$\Phi_{P_1 \boxtimes P_2}(E_1^\bullet \boxtimes E_2^\bullet) \cong \Phi_{P_1}(E_1^\bullet) \boxtimes \Phi_{P_2}(E_2^\bullet),$$

   which is functorial in $E_i^\bullet \in D^b(X_i)$, for all $i = 1, 2$.

2. *If $\Phi_{P_i} : D^b(X_i) \longrightarrow D^b(Y_i)$ is an equivalence of categories, for $i = 1, 2$, then*

$$\Phi_{P_1 \boxtimes P_2} : D^b(X_1 \times X_2) \longrightarrow D^b(Y_1 \times Y_2)$$

   is also an equivalence of categories.

3. *For $R \in D^b(X_1 \times X_2)$, let $S = \Phi_{P_1 \boxtimes P_2}(\mathcal{R}) \in D^b(Y_1 \times Y_2)$. Then the following diagram commutes.*

$$\begin{array}{ccc} D^b(X_1) & \xleftarrow{\Phi_{P_1}} & D^b(Y_1) \\ {\scriptstyle \Phi_R}\downarrow & & \downarrow{\scriptstyle \Phi_S} \\ D^b(X_2) & \xrightarrow[\Phi_{P_2}]{} & D^b(Y_2) \end{array}$$

Proof of Theorem 4.2.23: By Orlov's Representability Theorem 3.3.11, there is $P \in D^b(X \times Y)$, unique up to isomorphism, so that F is of Fourier-Mukai type, i. e. $F \simeq \Phi_P$. In particular the left and right adjoints are isomorphic to Φ_P

$$\Phi_{P_L}^{Y \to X} \simeq \Phi_P^{X \to Y} \simeq \Phi_{P_R}^{Y \to X}.$$

Hence by uniqueness of the kernel,

$$P^\vee \otimes p_Y^* \omega_Y[n] =: P_L \simeq P_R := P^\vee \otimes p_X^* \omega_X[n]$$

where $n = \dim(X) = \dim(Y)$. We denote Q the kernel of the quasi inverse of Φ_P and $\sigma : X \times X \longrightarrow X \times X$ the permutation that swaps the factors. Notice that the compositions

$$\begin{array}{c} D^b(X) \xrightarrow{\Phi_P^{X \to Y}} D^b(Y) \xrightarrow{\Phi_Q^{Y \to X}} D^b(X) \\ \Phi_{\mathcal{O}_{\Delta_X}} \\ \wr\! \| \\ \Phi_{\mathcal{O}_{\sigma\Delta_X}} \\ D^b(X) \xrightarrow[\Phi_Q^{X \to Y}]{} D^b(Y) \xrightarrow[\Phi_P^{Y \to X}]{} D^b(X) \end{array}$$

are isomorphic to the identity, we know that by uniqueness of the kernel $P \circ Q := p_{13*}(p_{12}^* P \otimes p_{23}^* Q) \simeq \mathcal{O}_{\Delta_X}$, which is the kernel of $\mathrm{id}_{D^b(X)}$, likewise if we repeat the same argument swapping the functors, we obtain $Q \circ P \simeq \mathcal{O}_{\Delta_Y} \simeq \mathrm{id}_{D^b(Y)}$. Therefore if $\Phi_Q^{Y \to X}$ is an equivalence also its transpose

$$(\Phi_Q^{Y \to X})^t = \Phi_Q^{X \to Y} \tag{8}$$

is an equivalence. Denote

$$\begin{array}{rl} P \boxtimes Q := p_{13}^* \overset{L}{\otimes} p_{24}^* Q \in & D^b((X \times Y) \times (X \times Y)) \\ & \wr\| \\ & D^b((X \times X) \times (Y \times Y)) \end{array}$$

we can define a Fourier-Mukai transform

$$D^b(X \times X) \xrightarrow{\Phi_{P \boxtimes Q}} D^b(Y \times Y)$$

let

$$R := \Phi_{P\boxtimes Q}(i_*\omega_X^s) \in D^b(Y\times Y)) \tag{9}$$

where we denote $i : X \hookrightarrow X\times X$ the customary diagonal embedding of X. Then from Lemma 4.2.24 we have

$$\begin{array}{ccc} D^b(X) & \xleftarrow{\Phi_Q} & D^b(Y) \\ {\scriptstyle \Phi_{i_*\omega_X^s}}\downarrow & & \downarrow{\scriptstyle \Phi_R} \\ D^b(X) & \xrightarrow[\Phi_P]{} & D^b(Y) \end{array}$$

commutes. We know that $\Phi_{\omega_X^s}^{X\to X} = S_X[-s\dim X]$ and since any equivalence commutes with Serre functors S_X and S_Y (cf. Remark 3.3.10), we have

$$\begin{aligned} \Phi_R &\simeq \Phi_P \circ S_X^t[-tn] \circ \Phi_Q \\ &\simeq \Phi_P \circ \Phi_Q \circ S_Y^t[-tn] \\ &\simeq S_Y^t[-tn] \simeq \Phi_{j_*\omega_Y^t}^{Y\to Y} \end{aligned}$$

When $j : Y \hookrightarrow Y\times Y$ denotes the diagonal embedding of Y. Again, by uniqueness of the kernel we have $R \simeq j_*\omega_Y^t$, that is, from 9

$$\Phi_{P\boxtimes Q}(i_*\omega_X^t) = j_*\omega_Y^t, \quad \forall t \in \mathbb{Z}$$

We know by Lemma 4.2.24 and (8) that $\Phi_{P\boxtimes Q}$ is an exact equivalence. Therefore

$$\mathrm{Hom}_{D^b(X\times X)}(i_*\omega_X^s, i_*\omega_X^t) \simeq \mathrm{Hom}_{D^b(Y\times Y)}(j_*\omega_Y^s, j_*\omega_Y^t), \quad \forall s,t \in \mathbb{Z}.$$

Since the pushforward functor of the diagonal morphisms is exact, the have an isomorphism of vector spaces

$$\mathrm{Hom}_{D^b(X)}(\omega_X^s, \omega_X^t) \simeq \mathrm{Hom}_{D^b(Y)}(\omega_Y^s, \omega_Y^t), \quad \forall s,t \in \mathbb{Z}.$$

Then if we take $s = 0$, we obtain, for all t

$$H^0(X, \omega_X^t) \simeq H^0(Y, \omega_Y^t)$$

As in the last proof, this shows the isomorphism between the canonical rings. □

To further study how these equivalences are realized, we see that from Bondal-Orlov's reconstruction Theorem that this problem immediately reduces to the study of autoequivalences of the bounded derived category of a smooth projective variety, i. e. exact (triangulated) k-linear equivalences

$$D^b(X) \xrightarrow{\sim} D^b(X)$$

We will denote the set of all isomorphism classes of autoequivalences of $D^b(X)$

$$\mathrm{Auteq}\, D^b(X)$$

Remarks 4.2.25.

- All automorphisms $f \in \mathrm{Aut}\, X$ induce the autoequivalences[27]

$$\begin{array}{ccc} D^b(X) & \underset{f^*}{\overset{f_*}{\rightleftarrows}} & D^b(X) \end{array}$$

which are quasi-inverse of one other.

[27] These are for a plethora of reasons; first f_*, f^* are functors and as such they fix the identity and respect isomorphisms and compositions. Another, perhaps excessive, is Gabriel's Theorem

- The Picard Group $\operatorname{Pic} X$ embeds[28] in $\operatorname{Auteq} D^b(X)$, since any $\mathcal{L} \in \operatorname{Pic} X$ gives rise to an autoequivalence

$$D^b(X) \xrightarrow{\mathcal{L}\otimes -} D^b(X)$$

- The set of shift functors $[n]$ for $n \in \mathbb{Z}$ is a subgroup of $\operatorname{Auteq} D^b(X)$ naturally isomorphic to $\mathbb{Z}$

Corollary 4.2.26 (Bondal-Orlov). *Let X be a smooth projective variety with ample canonical or anti-canonical sheaf. Then any equivalence of derived categories $D^b(X) \to D^b(X)$ is a composition of f_*, where $f \in \operatorname{Aut} X$, a twist by an invertible sheaf, and the shift functor. Indeed, there is an isomorphism of groups*

$$\operatorname{Auteq} D^b(X) \simeq \operatorname{Aut} X \rtimes (\operatorname{Pic}(X) \oplus \mathbb{Z}).$$

Proof: It is clear that the composition of exact functors is exact and that the quasi-inverse of an exact equivalence is exact. Hence $\operatorname{Auteq} D^b(X)$ is indeed a group. Consider the three types of autoequivalences described in the above Remark 4.2.25, namely shifts, automorphisms of the variety and line bundle twists, they combine as follows

$$\operatorname{Aut} X \rtimes (\operatorname{Pic}(X) \oplus \mathbb{Z}) \leq \operatorname{Auteq} D^b(X)$$

In fact the group $\operatorname{Pix} X \oplus \mathbb{Z}$ is preserved under conjugation (hence normal) and meets trivially with $\operatorname{Aut} X$, because any non trivial element of the former does not fix $\mathcal{O}_X$ but elements in the latter do.

We now argue that all equivalences that map skyscrapers sheaves of closed points to themselves are exactly the one described above. To see this we use again Orlov's Theorem 3.3.11 for short[29]. Fix the Fourier-Mukai transform Φ_P, $P \in D^b(X \times X)$ that represents $F \in \operatorname{Auteq} D^b(X)$. First notice that from the proof of the Reconstruction Theorem 4.2.19, we know already that after a shift and a twist by a sheaf in $\operatorname{Pic} X$ (namely the pushforward of P along the projection onto X) we can assume $\Phi_P(\mathcal{O}_X)) = \mathcal{O}_X$. By assumption there is an isomorphism $X \xrightarrow{f} X$ such that $\Phi_P(k(x)) \simeq \Phi_P(k(f(x))))$ and $\operatorname{supp} P$ is the graph of f.

Then, we claim that P must be a sheaf concentrated in degree zero and of rank one. This follows from the claims in the proof of Proposition 4.2.15, we have indeed

$$\begin{aligned} R^i \operatorname{Hom}(\mathcal{O}_X, k(x)) &\simeq R^i \operatorname{Hom}(\mathcal{O}_X, \Phi_P(k(f(x))) \\ &\simeq R^i \operatorname{Hom}(\mathcal{O}_X, P_{f(x)}) \end{aligned}$$

is non zero only for $i = 0$ and

$$\begin{aligned} k(x) &= \Gamma(X, \mathcal{H}om(P_x, k(x))) \\ &= \operatorname{Hom}(P_x, k(x)) \\ &\simeq \operatorname{Hom}(H^0(P_x), k(x)) \\ &\simeq \operatorname{Hom}(\mathcal{O}_X^{\oplus r}, k(x)) \\ &\simeq k(x)^{\oplus r} \end{aligned}$$

thus $r = 1$. So

$$\Phi_P = \Phi_{\gamma_* \mathcal{L}} = f_* \circ (- \otimes L)$$

where $L \in \operatorname{Pic} X$ and

$$\begin{aligned} X &\xrightarrow{\gamma} X \times X \\ x &\longmapsto (x, f(x)) \end{aligned}$$

□

[28] As an injective group homomorphism

[29] A much longer proof would be required otherwise, that uses one of the core arguments needed in the proof Orlov's theorem, see [BO01], A.3 and [Huy06] 4.17

4.3 Abelian Varieties

In this section we will discuss the origin of the Fourier-Mukai transform, why they where considered in the first place and what was the result that motivated the theory behind. This will actually furnish a concrete example of a variety A with trivial canonical bundle such that $D^b(A) \simeq D^b(\hat{A})$ but $A \not\simeq \hat{A}$, thus Bondal-Orlov does not apply.

Definition 4.3.1. An <u>abelian variety</u> A is a projective connected algebraic group over a field k. In other words a k-scheme endowed with morphisms

- $m : X \times X \longrightarrow X$ (the group law),
- $\iota : X \longrightarrow X$ (the inverse morphism),
- $e : \operatorname{Spec} k \longrightarrow X$ (the unit/identity k-valued point).

Such that the following diagrams commute

(associativity)

$$\begin{array}{ccc} A \times A \times A & \xrightarrow{\text{id} \times m} & A \times A \\ {\scriptstyle m \times \text{id}}\downarrow & & \downarrow{\scriptstyle m} \\ A \times A & \xrightarrow[m]{} & A \end{array}$$

(unit)

$$\begin{array}{ccc} A \times \operatorname{Spec} k & \xrightarrow{\text{id} \times e} & A \times A \\ \wr\Big| & & \Big\downarrow{\scriptstyle m} \\ A & \xrightarrow{\text{id}} & A \\ \wr\Big| & & \Big\uparrow{\scriptstyle m} \\ \operatorname{Spec} k \times A & \xrightarrow{e \times \text{id}} & A \times A \end{array}$$

(inverse)

$$\begin{array}{ccccc} & & A \times A & & \\ & {\scriptstyle (\text{id},\iota)}\nearrow & & \searrow{\scriptstyle m} & \\ A & \longrightarrow & \operatorname{Spec} k & \xrightarrow{e} & A \\ & {\scriptstyle (\iota,\text{id})}\searrow & & \nearrow{\scriptstyle m} & \\ & & A \times A & & \end{array}$$

An <u>homomorphism</u> $\varphi : A_1 \longrightarrow A_2$ between two abelian varieties A_1, A_2 ia a morphism which is also a group homomorphism. If φ is surjective and its kernel is finite, then it is called an <u>isogeny</u>, its degree is defined to be the order of the kernel $K_\varphi := \varphi^{-1}(e_2)$

The group law on abelian varieties is usually written additively, so for $a, b \in A$ we write $m(a, b) = a + b$, $\iota(a) = -a$ and $e = 0 \in A$ for the unit.

Then for any closed point $a \in A$ we define the <u>translation morphism</u>, given by

$$\begin{array}{ccc} A & \xrightarrow{t_a} & A \\ b & \longmapsto & a + b \end{array}$$

and as well the 'multiplication by n' morphism $n : A \longrightarrow A$ as $a \mapsto n \cdot a$.

We list a series of relevant known facts about abelian varieties, to further enquiry see [MRM08] and [Mil08]

Remarks 4.3.2.

- Any abelian variety is smooth and the underlying group is commutative.
- If $k = \mathbb{C}$, then the associated complex manifold is a compact complex Lie group, which is isomorphic to a complex torus $\mathbb{C}^g/\Lambda$.
- The cotangent bundle Ω_A of an abelian variety A is trivial, and so must be its canonical bundle $\omega_A \simeq \mathcal{O}_A$

- (See-saw principle_). Let X be an irreducible complete variety and T an integral scheme, $P \in \operatorname{Pic}(X \times T)$. If $L_t \coloneqq L|_{X\times\{t\}}$. Then exists a line bundle M on T such that $L \simeq \pi_T^* M$
- Suppose $L \in \operatorname{Pic} A$ then we have

$$m^*L \simeq \pi_1^*L \otimes \pi_2^*L \Longleftrightarrow t_a^*L \simeq L \text{ for all } a \in A$$

Definition 4.3.3. Let A be an abelian variety. Then we define

$$\operatorname{Pic}^0 A \coloneqq \{L \in \operatorname{Pic} A \ : \ t_a^*L \simeq L \text{ for all } a \in A\}$$

More generally we can define the Picard functor Pic_A^0 between the category of varieties over k, Var_k and Set, which on objects is

$$S \longmapsto \operatorname{Pic}_A^0(S)\{M \in \operatorname{Pic}(S \times A) \ : \ M_s \in \operatorname{Pic} A \text{ for every closed } s \in S\} / \sim .$$

Where, $M \sim M'$ if exists a line bundle L on S such that $M \otimes \pi_S^* L \simeq M'$. This functor is contravariant, i. e. for $f : T \longrightarrow S$,

$$\operatorname{Pic}_A^0(f) = (f \times \mathrm{id}_A)^* : \operatorname{Pic}_A^0(S) \longrightarrow \operatorname{Pic}_A^0(T)$$

The dual of an abelian variety can be introduced as a solution to the problem of representing the Picard functor. It is a general fact that when A is projective then Pic_A^0 is representable by an algebraic group $\operatorname{Pic} A$ and its connected component containing the origin will be denoted by $\operatorname{Pic}^0 A$, with underlying as in Definition 4.3.3. $\operatorname{Pic}^0 A$ represents line bundles whose first Chern class vanishes. From now on the algebraic group $\operatorname{Pic}^0 A$ will be denoted $\hat{A}$ and called the dual abelian variety of A, see [MRM08] III.13.

Theorem 4.3.4. *Let A be an abelian variety then there is a uniquely determined line bundle $\mathcal{P}$ on $A \times \hat{A}$ called the Poincaré bundle such that:*

- $\mathcal{P}|_{A\times\{\alpha\}} \in \operatorname{Pic}^0(A)$ *for all* $\alpha \in \hat{A}$, *and*
- $\mathcal{P}|_{\{e\}\times\hat{A}}$ *is trivial.*

Remarks 4.3.5.

- For $L \in \operatorname{Pic}^0 A$ and $n \in \mathbb{Z}$, we have

$$n^*L \simeq L^{\otimes n}$$

- We can identify $A \xrightarrow{\rho} \hat{\hat{A}}$ and the Poincaré bundle $\mathcal{P}$ of A corresponds to the Poincaré bundle $\hat{\mathcal{P}}$ of $\hat{A}$ through the composition

$$A \times \hat{A} \xrightarrow{\rho\times\mathrm{id}} \hat{\hat{A}} \times \hat{A} \xrightarrow[\sim]{\sigma} \hat{A} \times \hat{\hat{A}}$$

  where σ is the transposition that swaps the factors.

Lemma 4.3.6. *Let $\mathcal{O}_A \neq L \in \operatorname{Pic}^0 A$. Then $H^i(A, L) = 0$ for all i.*

Proof: Suppose $s \in H^0(A, L) \neq 0$, then it induces $\iota^* s \in H^0(A, i^*L) \neq 0$. They both vanish at their zero schemes $Z(s)$ and $Z(i^*s)$ (cf. [Har77], II.7.7) respectively, and so does their tensor product. But $s \otimes \iota^* s \in H^0(A, L \otimes i^*L) \simeq H^0(A, \mathcal{O}_A)$ shows that $s \otimes \iota^* s$ is constant, hence a contradiction.

Now suppose ℓ is minimal with $H^\ell(A, L) \neq 0$. Then apply Künneth formula to $m^* \simeq \pi_1^* L \otimes \pi_2^* L$, to obtain

$$H^\ell(A \times A, m^*L) \simeq \bigoplus_{i+j=\ell} H^i(A, L) \otimes H^j(A, L).$$

Since

$$A \xrightarrow{(\mathrm{id},e)} A \times A \xrightarrow{m} A, \qquad m \circ (\mathrm{id},e) = \mathrm{id}$$

we have an injection

$$H^\ell(A, L) \longrightarrow H^\ell(A \times A, m^*L)$$

But this yields a contradiction by the minimality of k and $H^0(A, L) = 0$, we must have

$$H^\ell(A \times A, m^*L) = 0$$

□

Theorem 4.3.7. *([Muk81], 2.2). Let $\mathcal{P}$ be the Poincaré bundle on $A \times \hat{A}$. Then the functor*

$$\Phi_{\mathcal{P}} : D^b(A) \longrightarrow D^b(\hat{A})$$

is a triangulated equivalence. Moreover, the composition

$$D^b(\hat{A}) \xrightarrow{\Phi_{\mathcal{P}}} D^b(A) \xrightarrow{\Phi_{\mathcal{P}}} D^b(\hat{A})$$

is isomorphic to $\hat{\iota}^* \circ [-g]$, where $g = \dim A$

Proof: We employ 3.3.4 and 3.3.9. Pick $\alpha, \beta \in \hat{A}$. Then $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ are line bundles in $\mathrm{Pic}^0 A$. We have, for $\alpha \neq \beta$

$$\mathrm{Hom}(\mathcal{P}_\alpha, \mathcal{P}_\beta[i]) \simeq H^i(A, \mathcal{P}_\alpha \otimes \mathcal{P}_\beta) \simeq 0$$

for all i by Lemma 4.3.6, and

$$\mathrm{Hom}(P_\alpha, P_\alpha) = H^0(A, \mathcal{O}_A) = k.$$

Therefore $\Phi_{\mathcal{P}}$ is fully faithful and, because the canonical bundles of A and $\hat{A}$ are trivial, we have

$$\Phi_P(k(\alpha)) \otimes \omega_{\hat{A}} \simeq \Phi_P(k(\alpha))$$

Hence, $\Phi_{\mathcal{P}}$ is an equivalence.

Now, let us consider the following diagram

$$\begin{array}{ccccc} & & \hat{A} \times A \times \hat{A} & & \\ & \swarrow^{\pi_{12}} & \downarrow^{\pi_{13}} & \searrow^{\pi_{23}} & \\ \hat{A} \times A & & \hat{A} \times \hat{A} & & A \times \hat{A} \end{array}$$

the kernel of the composition $\Phi_{\mathcal{P}}^{A \to \hat{A}} \circ \Phi_{\mathcal{P}}^{\hat{A} \to A}$ is

$$\mathcal{P} \circ \mathcal{P} = \pi_{13*}(\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{P})$$

Then as an application of the See-saw principle we have that $\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P} \simeq (\mathrm{id}_A \times \hat{m})^*\mathcal{P}$ (cf. [Huy06] 9.13). Since $\hat{m}$ is flat we can use the following base change

$$\begin{array}{ccc} \hat{A} \times A \times \hat{A} & \xrightarrow{\mathrm{id}_A \times \hat{m}} & \hat{A} \times A \\ \downarrow{\scriptstyle \pi_{13}} & & \downarrow{\scriptstyle \pi_1} \\ \hat{A} \times \hat{A} & \xrightarrow[\hat{m}]{} & \hat{A} \end{array}$$

Therefore, putting all pieces together

$$\begin{aligned} \mathcal{P} \circ \mathcal{P} &= \pi_{13*}(\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{P}) \\ &\simeq \pi_{13*}(\mathrm{id}_A \times \hat{m})^* \mathcal{P} \\ &\simeq \hat{m}^* \pi_{1*} \mathcal{P} \end{aligned}$$

Claim 1: $\pi_1\mathcal{P} \simeq k(\hat{0})[-g] \equiv k(\hat{e})[-g]$

Consider the spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p(\mathcal{O}_{\hat{A}}, R^q\pi_{1*}\mathcal{P}) \Longrightarrow \operatorname{Ext}^{p+q}(\mathcal{O}_{\hat{A}}, \pi_1\mathcal{P}) \tag{10}$$

We have

$$\begin{aligned}
\operatorname{Ext}^{p+q}\left(\mathcal{O}_{\hat{A}}, \pi_{1*}\mathcal{P}\right) &\simeq \operatorname{Hom}\left(\mathcal{O}_{\hat{A}}, \pi_{1*}\mathcal{P}[p+q]\right) \\
&\simeq \operatorname{Hom}\left(\Phi_{\mathcal{P}}(k(e)), \Phi_{\mathcal{P}}(\mathcal{O}_A)[p+q]\right) \\
&\simeq \operatorname{Hom}\left(k(e), \mathcal{O}_A[p+q]\right) && (\Phi_{\mathcal{P}} \text{ fully-faithful}) \\
&\simeq \operatorname{Hom}\left(\mathcal{O}_A, k(e)[-p-q] \otimes \omega_A[g]\right)^{\vee} && (\text{Serre duality}) \\
&\simeq H^{g-p-q}(A, k(s))^{\vee}
\end{aligned}$$

Now to pin down $\operatorname{supp} \pi_{1*}\mathcal{P}$ we investigate the cohomology of $\mathcal{P}$ along fibers[30] of $\pi_1 : \hat{A} \times A \longrightarrow \hat{A}$. But from Lemma 4.3.6 we know, for $\alpha \in \hat{A}$, $\pi^{\leftarrow}(\alpha) = \{\alpha\} \times A$,

$$H^i(A, \underbrace{P|_{\{\alpha\}\times A}}_{\substack{\cap \\ \operatorname{Pic}^0(A)}}) = 0$$

for all i, and yields (by definition of the Poincaré bundle) non trivial cohomology only when $\alpha = \hat{e} \equiv \hat{0}$.

Therefore every $R^q\pi_*\mathcal{P}$ has support concentrated in $\{\hat{e}\}$, hence are flasque and the spectral sequence (10) above must collapse at page 2 where all $E_2^{p,q}$ are zero except for $p = 0$. Hence

$$\operatorname{Ext}^0(\mathcal{O}_{\hat{A}}, R^q\pi_1\mathcal{P}) \simeq \operatorname{Hom}(\mathcal{O}_{\hat{A}}, R^q\pi_1\mathcal{P}) \simeq H^{g-q}(A, k(e))^{\vee}$$

which is zero for all $q \neq 0$, thus $\pi_*\mathcal{P} \simeq k(\hat{e})[-g]$ as expected.

Claim 2: $\mathcal{P} \circ P \simeq \mathcal{O}_{\Gamma_{\hat{\iota}}}$ where $\Gamma_{\hat{\iota}}$ is the graph of $\hat{\iota}$.

To see this we employ the same argument as in the previous claim, let $(\alpha, \beta) \in \hat{A} \times \hat{A}$ then

$$\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P}|_{\pi_{13}^{\leftarrow}(\alpha,\beta)} \simeq \mathcal{P}_\alpha \otimes \mathcal{P}_\beta \in \operatorname{Pic}^0 A$$

thus, as for Lemma 4.3.6, we must have for an $i \in \mathbb{Z}$

$$H^i(A, \mathcal{P}_\alpha \otimes \mathcal{P}_\beta) \neq 0 \Longleftrightarrow \mathcal{P}_\alpha \otimes \mathcal{P}_\beta \simeq \mathcal{O}_A$$

hence if and only if $\alpha = -\beta$, so $\operatorname{supp}(\mathcal{P} \circ \mathcal{P}) \subseteq \Gamma_{\hat{\iota}}$. From the previous claim we have

$$\mathcal{P} \circ \mathcal{P} \simeq \hat{m}^*\pi_{1*}\mathcal{P} \simeq \hat{m}^*k(\hat{0})[-g] \simeq \mathcal{O}_{\Gamma_{\hat{\iota}}}.$$

Finally,

$$\Phi_{\mathcal{P}\circ\mathcal{P}}^{\hat{A}\leftarrow\hat{A}} \simeq \Phi_{\mathcal{O}_{\Gamma_{\hat{\iota}}}[-g]}^{\hat{A}\leftarrow\hat{A}} \simeq \iota^* \circ [-g]$$

□

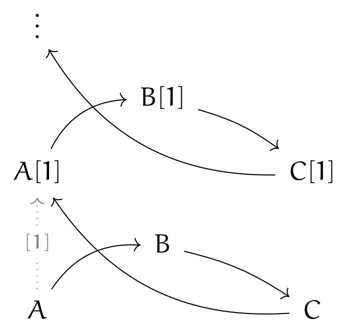
[30] Recall $\mathcal{F}_x = (\pi_*\mathcal{F})_x$, and see [Har77] III.2.10

Appendix

Triangulated Categories

Definition 4.0.1. Let $\mathcal{D}$ be an additive category. The triangulated structure on $\mathcal{D}$ is encoded by specifying the following data

1. An additive equivalence
$$T : \mathcal{D} \longrightarrow \mathcal{D}$$
called the <u>shift functor</u>. We will write $X[n]$ to denote $T^n(X)$ and $f[n]$ for $T^n(f)$. Now we define <u>triangles</u> in $\mathcal{D}$ to be the following diagram
$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$
that is,



Morphisms among triangles are given by commutative diagrams
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

2. A class of <u>distinguished triangles</u> satisfying the axioms TR1-TR4 below

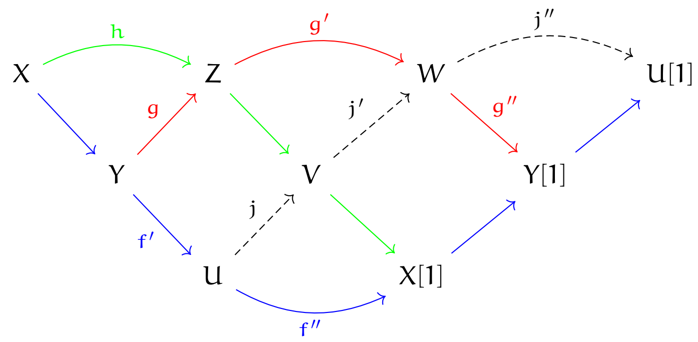
TR1
- $X \xrightarrow{id} X \longrightarrow 0 \longrightarrow X[1]$ is a distinguished triangle for every $X \in \mathcal{D}$
- Any triangle isomorphic to a distinguished one is itself distinguished
- Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle $X \xrightarrow{u} Y \longrightarrow Z \longrightarrow X[1]$

TR2 *(Rotation)*. A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$

TR3 For any commutative diagram of distinguished triangle with vertical arrows f and g
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

can be completed (not necessarily uniquely) to a morphism of triangles by a morphism h.

TR4 Let $h = g \circ f$, then given the distinguished triangles (f, f', f''), (g, g', g''), (h, h', h'') then exists a distinguished triangle (j, j', j'') that makes the following diagram commutative.



Remarks 4.0.2.

- Triangulated categories need not to be abelian in general.
- An abelian triangulated category is semisimple, cf. [HJ10].

Definition 4.0.3. A functor $H : \mathcal{D} \longrightarrow \mathcal{A}$ from a triangulated category D into an abelian category $\mathcal{A}$ is called <u>cohomological functor</u> if it is additive and the sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z),$$

in $\mathcal{A}$, is exact for any distinguished triangle

$$X \longrightarrow Y \xrightarrow{g} Z \xrightarrow{l} X[1]$$

in $\mathcal{D}$. By axiom TR2 we have that if H is a cohomological functor, then the sequence

$$H(X[i]) \xrightarrow{H(f[i])} H(Y[i]) \xrightarrow{H(g[i])} H(Z[i]) \xrightarrow{H(l[i])} H(X[i+1])$$

is exact in $\mathcal{A}$

Definition 4.0.4. Let $\mathcal{C}$ and $\mathcal{D}$ triangulated categories. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be <u>triangulated (exact)</u> if for any distinguished triangle in $\mathcal{C}$

$$X \longrightarrow Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

there are isomorphisms $\left\{F(A[1]) \xrightarrow[\sim]{\varphi_A} F(A)[1]\right\}_{A \in \mathcal{A}}$ such that

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\varphi_X \circ F(h)} F(X)[1]$$

is a distinguished triangle in $\mathcal{D}$

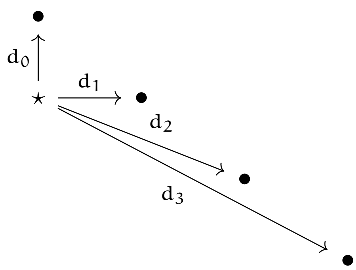
Spectral Sequences

Definition 4.0.1. Let $\mathcal{A}$ be an abelian category. A spectral sequence of consists of the following data:

1. A family of objects $E_r^{p,q}$ for $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ and $r \geq 0$.
2. Differentials:
$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$
such that for consecutive$d_r^{\bullet,\bullet} \circ d_r^{\bullet,\bullet} = 0$, for all r.
3. Isomorphisms:
$$E_{r+1}^{p,q} \simeq \frac{\ker d_r^{p,q}}{\operatorname{Im} d_r^{p-r,q+r-1}} = H^0(E_r^{p+\bullet r,q-\bullet r+\bullet}).$$
4. For any (p,q) there exists an $r_0(p,q) \equiv r_0$ such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for all $r \geq r_0$. In particular, $E_r^{p,q} \simeq E_{r_0}^{p,q}$ and we will denote such object $E_\infty^{p,q}$.
5. A decreasing filtration
$$\cdots \subset F^{p+1}E^n \subset F^pE^n \subset F^{p-1}E^n \subset \cdots \subset F^0E^n \coloneqq E^n$$
such that
$$\bigcap F^pE^n = 0 \text{ and } \bigcup F^pE^n = E^n$$
and isomorphisms
$$E_{r_0}^{p,q} \eqqcolon E_\infty^{p,q} \simeq F^pE^{p+q}/F^{p+1}E^{p+q}.$$

Remarks 4.0.2.

- The integer r marks the "pages" of the spectral sequence.
- The directions of the differentials are visually understood as follows



- When (4) holds for all p, q , we say that the spectral sequence collapses at page r_0.
- If $E_\infty^{p,q} = 0$, for all p, q, then $E^{p+q} = 0$. Follows from 5.
- If we are given objects on a page, say $r \geq 1$, then the next page is fully determined by the previous, up to isomorphism. Therefore we often introduce the spectral by writing
$$E_r^{p,q} \implies E^{p+q}$$
- To instantiate concretely a spectral sequence—and hance have a glimpse of its usefulness—lets assume all objects $E_r^{p,q}$ are finite dimensional vector spaces. Let $r_0 = 2$, then all the differentials in page 2 vanish and we must have for all $p, q \in \mathbb{Z}$
$$E_2^{p,q} \simeq E_\infty^{p,q} \simeq F^pE^{p+q}/F^{p+1}E^{p+q}$$

Therefore

$$F^pE^{p+q} = E_2^{p,q} \oplus F^{p+1}E^{p+q} \simeq E_2^{p,q} \oplus E_2^{p+1,q-1} \oplus F^{p+2}E^{p+q} = \cdots$$

So $F^pE^n = \bigoplus_{k\geq 0} E_2^{p+k,q-k}$, for $n = p+q$,

$$E^n = \bigcup F^pE^n \simeq \bigoplus_{k\in\mathbb{Z}} E_2^{k,n-k}.$$

Definition 4.0.3. A double complex $L^{\bullet,\bullet}$ is given by the following data: $(L^{p,q}, d_I^{p,q}, d_{II}^{p,q})$ i.e. a collection of objects $L^{p,q}$ and morphisms

$$d_I^{p,q} : L^{p,q} \longrightarrow L^{p+1,q} \quad \text{and} \quad d_{II}^{p,q} : L^{p,q} \longrightarrow L^{p,q+1}$$

satisfying the relations

$$d_I^2 = 0, \quad d_{II}^2 = 0, \quad d_Id_{II} + d_{II}d_I = 0.$$

So we have the following diagram where each square commutes

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L^{p-1,q} & \xrightarrow{d_I^{p-1,q}} & L^{p,q} & \xrightarrow{d_I^{p,q}} & L^{p+1,q} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d_{II}^{p-1,q}} & & \downarrow{\scriptstyle d_{II}^{p,q}} & & \downarrow{\scriptstyle d_{II}^{p+1,q}} & & \\
\cdots & \longrightarrow & L^{p-1,q+1} & \xrightarrow{d_I^{p-1,q+1}} & L^{p,q+1} & \xrightarrow{d_I^{p,q+1}} & L^{p+1,q+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d_{II}^{p-1,q+1}} & & \downarrow{\scriptstyle d_{II}^{p,q+1}} & & \downarrow{\scriptstyle d_{II}^{p+1,q+1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The associated total complex $L^\bullet := \operatorname{tot} L^{\bullet,\bullet}$ is defined by

$$L^n := \bigoplus_{p+q=n} L^{p,q}, \qquad d^n = d_I^{p,q} + (-1)^p d_{II}^{p,q}.$$

Then we can define the standard filtration on the total complex $L^\bullet$ as follows, n=p+q

$$F^pL^k := L^{p,q} \oplus L^{p+1,q-1} \oplus L^{p+2,q-2} \oplus \cdots \oplus L^{p+q,0} \oplus \cdots = \bigoplus_{q\geq p} L^{n-q,p}$$

and satisfies $d_I(F^pL^n) \subset F^pL^{n+1}$.

Assuming L is up-left bounded, we can visualize the filtration as follows

$$\begin{array}{ccc}
F^0L^2 & = & L^{0,2} \oplus L^{1,1} \oplus L^{2,0} \\
 & & \uparrow \\
F^1L^2 & = & L^{1,1} \oplus L^{2,0} \\
 & & \uparrow \\
F^2L^2 & = & L^{2,0}
\end{array}$$

More generally

Definition 4.0.4. A filtered complex is a complex $L^\bullet$ together with a decreasing filtration

$$\cdots \subset F^kL^n \subset\subset F^{k-1}L^n \subset \cdots \subset F^0L^n := L^n, \qquad \forall n,$$

satisfying $d^n(F^kL^n) \subset F^nK^{n+1}$ for all n.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & L^{n-1} & \longrightarrow & L^n & \longrightarrow & L^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & F^kL^{n-1} & \longrightarrow & F^kL^n & \longrightarrow & F^kL^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & F^{k+1}L^{n-1} & \longrightarrow & F^{k+1}L^n & \longrightarrow & F^{k+1}L^{n+1} & \longrightarrow & \cdots
\end{array}$$

Now given a double complex $L^{\bullet,\bullet}$, consider its standard filtration $\{F^kL^n\}_k$, defined above, of its total complex $L^\bullet = \mathrm{tot}(L^{\bullet,\bullet})$. We call

$$\mathrm{gr}^k L^n := F^kL^n/F^{k+1}L^n = L^{n-k,k}$$

the associated graded objects to the filtration. Note that they form a complex $\mathrm{gr}^k(L^\bullet)$ and $H^\ell(\mathrm{gr}^k(L^\bullet)) = H^{\ell-k}(L^{\bullet,k})$.

We will write $H^n_I(L^{\bullet,\bullet})$ for the complex given by $(H^n(L^{\bullet,q}))_{q\in\mathbb{Z}}$ and analogously $H^n_{II}(L^{\bullet,\bullet}) := (H^n(L^{p,\bullet}))_{p\in\mathbb{Z}}$.

Proposition 4.0.5. *([GM03], III.7.5). Let $L^{\bullet,\bullet}$ be a double complex such that $L^{n-k,k} = 0$ for $|k| \gg 0$. Then there is a spectral sequence:*

$$E^{p,q} := H^p_{II}H^q_I(L^{\bullet,\bullet}) \implies H^{p+q}(L^\bullet).$$

Definition 4.0.6. Let $A^\bullet \in K^+(\mathcal{A})$. A Cartan-Eilenberg resolution of $A^\bullet$ is a double complex $C^{\bullet,\bullet}$ equipped with a morphisms of complexes $A^\bullet \longrightarrow C^{\bullet,0}$ satisfying

- $C^{i,j} = 0$ for $j < 0$.
- The sequences
  $$A^n \longrightarrow C^{n,0} \xrightarrow{g} C^{n,1} \longrightarrow \cdots$$
  are injective resolutions of A^n, and the induced sequences
  $$\begin{array}{ccccccc}
  \ker(d^n_A) & \longrightarrow & \ker(d^{n,0}_I) & \longrightarrow & \ker(d^{n,1}_I) & \longrightarrow & \cdots \\
  \mathrm{Im}(d^n_A) & \longrightarrow & \mathrm{Im}(d^{n,0}_I) & \longrightarrow & \mathrm{Im}(d^{n,1}_I) & \longrightarrow & \cdots \\
  H^n(A^\bullet) & \longrightarrow & H^n_I(C^{\bullet,0}) & \longrightarrow & H^n_I(C^{\bullet,1}) & \longrightarrow & \cdots
  \end{array}$$
  are injective resolutions of $\ker(d^n_A), \mathrm{Im}\, d^n_A$ and $H^n(A^\bullet)$ respectively.
- All short exact sequences
  $$0 \longrightarrow \ker(d^{i,j}_I) \longrightarrow C^{i,j} \longrightarrow \mathrm{Im}(d^{i,j}_I) \longrightarrow 0$$
  split.

Proposition 4.0.7. *If $\mathcal{A}$ has enough injectives, then any $A^\bullet \in K^+(\mathcal{A})$ admits a Cartan-Eilenberg resolution.*

Theorem 4.0.8. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories and $\mathsf{F} : \mathsf{K}^+(\mathcal{A}) \longrightarrow \mathsf{K}^+(\mathcal{B})$ and $\mathsf{G} : \mathsf{K}^+(\mathcal{B}) \longrightarrow \mathsf{K}^+(\mathcal{C})$ be exact functors. Suppose $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and the image under F of a complex $I^\bullet \in \mathsf{K}^+(\mathcal{A})$ of injectives of $\mathcal{A}$ is contained in an G-adapted triangulated subcategory $\mathcal{I}_\mathsf{G} \subseteq \mathsf{K}^+(\mathcal{B})$. Then for any complex $\mathsf{A} \in \mathsf{D}^+(\mathcal{A})$, we have the following spectral sequence*

$$E_2^{p,q} \coloneqq R^p G(R^q F(A^\bullet)) \implies R^{p+q}(G \circ F)(A^\bullet) =: E^{p+q}.$$

Proof: See Proposition 1.4.10. □

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