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Inflationary Tensor Fossils and their implications

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Abstract

Inflation is a period during which the Universe expansion accelerated at very early times. Originally introduced to solve the fine tuning problems of the cosmological Hot Big Bang model, it has been a great success in explaining the origin of the small temperature anisotropies of the Cosmic Microwave Background (CMB). Actually the most accepted models of inflation are the so-called standard single-field models of slow-roll inflation. The quantum field theory (QFT) description of such models consists in the presence during inflation of one scalar field, the inflaton, which slowly rolls down an almost flat potential. At the beginning of inflation both the inflaton and the metric tensor have small oscillations around their background (i.e. their quantum fluctuations). During inflation these primordial perturbations are stretched by the accelerated expansion on very large (super-horizon) scales, where they get frozen. They form the seeds for the formation of primordial scalar (curvature) perturbations, associated to primordial density perturbation, which can explain the temperature anisotropies observed in the CMB. Another fundamental prediction of Inflation is the production of a stochastic background of tensor perturbations, corresponding to primordial gravitational waves. The statistics of the primordial perturbations predicted by the standard slow-roll models of Inflation is almost Gaussian. The small deviations from a Gaussian distributions of the primordial perturbations predicted by the standard single-field slow-roll models of Inflation cannot be observed at the moment given the sensitivity of the actual measurements. In the last years the WMAP and Planck satellites have constrained with increasing precision the level of primordial non-Gaussianity. The best constraints at present are those from the Planck measurements of the temperature (and polarization) CMB anisotropies. In this work we have studied a possible indirect way to detect the primordial stochastic background of Gravitational Waves which goes under the name "tensor fossils". The latter are primordial degrees of freedom that no longer interact or very weakly interact during late-time cosmic evolution. The only observational effect of an Inflation fossil might therefore be its imprint in the primordial curvature perturbation. Indeed the effect of these tensor fossils would entail a quadrupole perturbation in the mass distribution in the Universe. In order to measure their contribution it can be shown that it is possible to define a parametrization strictly connected both with the bispectrum, i.e., the scalar-scalar-tensor (fossil) three-point correlation function, and with the power spectrum (the two point correlation function) of the tensor (fossil) perturbations. A new model, Gaugid Inflation, has been studied. In this model the fields responsible for Inflation are three Abelian gauge vector fields with vacuum expectation values that manifestly break invariance under spatial rotations and translations. Imposing additional symmetries on the fields, allows us to restore, at the background level, the wanted isometries, and also to study the inflationary phase driven by these fields. Perturbing the fields we found that, besides the usual metric tensor perturbations, additional tensor degrees of freedom due to the gauge fields arise, which could play the role of tensor fossils. The original contribution of this work was the study of primordial perturbations for Gaugid Inflation in an extension of the original version proposed, and the finding that these new tensor degrees couple to the metric ones in a non-trivial way. This result affects the spectral index of the tensors. In our generalization we tried to add a manifestly parity-breaking term due to presence of time derivative of the gauge field. We expect this term to modify the power spectrum of gravitational waves, polarizing the primordial GW into left (L) and right (R) polarization states in the sense that the statistics of such L and R polarization states becomes different.

Contents

Introduction

Inflation is a period during which the Universe expansion accelerated, in the first moments of its life.
The success of this theory is due to the fact that it represents a powerful solution for the Hot Big Bang shortcomings: horizon, flatness and cosmic relics problems. Furthermore, Inflation is mainly considered because it provides a dynamical description for the origin of the temperature perturbations of the Cosmic Microwave Background (CMB), which is so far the most important observable that carries information about the very early Universe. These perturbations, as we will see, are strictly connected with the density perturbations, which are themselves the cause of the formation of the Large-Scale Structures. The simplest inflationary model, the so called *single-field slow-roll* model, is based on the hypothesis of the presence of a scalar field, the so-called inflaton, responsible of the accelerated expansion. In this model the potential of the scalar field is assumed to be approximately flat, according to the so called *slow-roll conditions*. During Inflation we can decompose the inflaton field into a background value, which is isotropic and homogeneous, and a small fluctuation around the background. We can do the same for the components of the metric tensor $g_{\mu\nu}$. The background dynamics of the inflaton is responsible for the accelerated expansion of the primordial Universe. Then the initial small fluctuations are stretched on superhorizon cosmological scales by the expansion of the Universe so that their amplitude gets "frozen". This happens because at a certain time the wavelength of a given oscillatory mode λ exits the comoving Hubble horizon $r_H(t)$, which provides a measure of the dimensions of the cosmological regions causally connected in the Universe at a fixed time t. At the end of Inflation, the inflaton decays into radiation through a process called *reheating* of the Universe, so that the standard Hot Big Bang phase can start.

Theoretical predictions show that the relevant types of dynamical perturbations during Inflation are two: scalar (curvature) perturbations, associated to primordial density perturbations, and primordial gravitational waves (GW). The former, after the reheating, remains frozen until the corresponding wavelength reenters into the comoving Hubble horizon, creating perturbations in the energy density of the radiation fluid. Then through this mechanism we explain essentially how in the Universe small perturbations that we observe in the CMB formed. The same mechanism explains the first seeds from which, via gravitational instability, the Large-Scale Structure of the Universe formed during the matter dominated epoch. The basic predictions of the slow-roll paradigm, like the approximate scale invariance and Gaussianity of the primordial power spectrum are in complete agreement with CMB and Large-Structures data. In the last years the Planck satellite has also put the strongest constraints on deviations from a pure Gaussian distribution of the primordial perturbations. Such constraints are compatible with a zero level of primordial non-Gaussianity as predicted by the slow-roll models, but there is still a window of almost two orders of magnitude unexplored, see e.g. [32, 43].

A common feature of *all* the inflationary models is the production of a stochastic background of gravitational waves, i.e. tensor degrees of freedom of the metric. Observing this background would be a *smoking gun* for the Inflation theory, because it would definitely confirm the theory. Furthermore a measure of the amplitude of the GW would allow to define also the energy scale at which the Inflation happened: this would be an extraordinary result for the understanding of the first moments of the Universe. Another important cosmological prediction is that the amplitude of the GW as predicted by the standard single-field slow-roll models of Inflation is much smaller compared to the scalar one. This constitutes a great challenge for a detection of the primordial gravitational waves; for this reason, in order to have indirect observation of such GW, some different ways have been proposed. One fundamental tool to investigate the primordial GW is provided by the so called *tensor fossils*, defined as "a hypothesized primordial degree of freedom that no longer interacts or very weakly interacts during late-time cosmic evolution. The only observational effect of an Inflation fossil might therefore be its imprint in the primordial curvature perturbation" [69]. These tensor fossils represent a fascinating indirect way to detect the primordial tensor perturbations: a tensor fossil would entail a quadrupole distortion in the CMB temperature perturbations and in the mass distribution in the Universe, as we will see. In this work we will also deal with the specific predictions of the single-field models about a tensor-scalar-scalar bispectrum, the Fourier transform for the three-point correlation function, between one gravitational wave (tensor) mode and two density perturbation (scalar) modes. It is indeed through this correlation that we will define the tensor fossils that will lead to a local power quadrupole. The tensor-scalar-scalar bispectrum satisfies a particular *consistency relation* (cr) that relates its functional dependence on the tensor wavenumber K and scalar wavenumbers k_1 and k_2 to the tensor and scalar power spectra in the so called *squeezed limit* ($K \ll k_1 \sim k_2$). An observation of violation of the consistency relation, both for scalar and tensor perturbations, would rule out all single-field models of inflation, which would be, of course, an extraordinary result.

In the last decade a new model which violates the consistency relations has been proposed: *Solid Inflation* [53]. In this theory the scalar fields which drive Inflation have background values which manifestly break the standard isometries, invariance under spatial translations and rotations. We will deeply study this model and its consequences for the predictions about the tensor fossils.

This Thesis focuses on a generalization of a new "solid-like" model of inflation: *Gaugid Inflation* [52]. In Gaugid Inflation the fields responsible for Inflation are three Abelian gauge vector fields, with vacuum expectation value (vev) that break invariance under spatial rotations and translations. Imposing additional (internal) symmetries on the fields allows us to restore, for the background, the wanted isometries, and also to study the inflationary phase driven by these fields. Perturbing the fields one finds that, besides the metric tensor perturbations, there are additional tensor degrees of freedom, due to the gauge fields, coupled to the metric ones in a non-trivial way. This is an interesting result: indeed the presence of an additional tensor degree of freedom will enhance the power spectrum of the gravitational waves, a prediction which is different from the standard inflationary models. Moreover, this new tensor degree of freedom could play the role of a tensor fossil.

The original contribution of this Thesis is the generalization was considering also the possibility of a parity-breaking term in the Lagrangian and in the vev of the field. This generalization is supported by previous works about possible parity-breaking signatures in the gravitational wave power spectrum and bispectrum, like [98]: a detection of a similar signature would give us essential informations about the fundamental physics of (very) high energy. Our aim was to find possible signatures at the level of power spectra: we have found that a parity breaking is possible in Gaugid Inflation both for scalars and tensors degree of freedom.

This Thesis is organized as follows.

In *Chapter 1* the physics of the isotropic and homogeneous background Universe is described. We explore the Hot Big Bang model and its problems, then we introduce Inflation as a powerful mechanism to solve them and describe the features of the simplest model, the single-field slow-roll.

In *Chapter 2* we outline the theory of cosmological perturbations and the methods involved to compute quantum correlations in a cosmological framework. We will focus on the evaluation of the scalar and tensor perturbations power spectra from Inflation. We will also briefly describe how to connect primordial (scalar) perturbations with the actual measurements on CMB.

In *Chapter 3* we define statistical correlators that describe the effects of primordial non-Gaussianity, like the bispectrum: we will describe a theoretical formalism to search for non-Gaussianities of the primordial perturbations in the slow-roll models of Inflation. We will introduce the role of the consistency relations for the bispectra of primordial perturbations and describe the observational consequences of a tensor fossil in the CMB and Large Scale Structure observations.

In *Chapter 4* the Solid Inflation model is reviewed. We will describe in detail the dynamics of this theory, its novelties with respect to the single-field slow-roll model and its predictions. We will then calculate what are the possible observable prediction for tensor fossils using this theory.

In *Chapter 5* the original contribution of this Thesis is developed. After reviewing Gaugid Inflation, we will generalize it in order to obtain new parity-breaking signatures for the gravitational waves observables. In this respect we have found new outcomes, such as the violation of parity both in the scalar and in the tensor sector, that whose consequences can be analyzed in more details in possible future works.

Chapter 1

Standard Cosmology

1.1 Cosmological Principle: homogeneity and isotropy of the Universe

Cosmology is the study of the properties, the composition and the dynamics of the Universe as a whole. In order to achieve this goal we need some practical (and supported by the experiments) assumptions about the background properties of the Universe. A large portion of modern cosmology is based on the *Cosmological Principle*, that is, the hypothesis that all the comoving observers in the Universe are equivalent. In other words, it can be formalized with two fundamental features:

Homogeneity of the Universe. Saying that the Universe is homogeneous means that there are non preferred locations: every location looks the same in every point of the space. Mathematically it can be expressed as the *translational invariance* of the quantities that describes the Universe. This is not a feature of all the spacetime points of the Universe, but it applies only to a "smeared-out" Universe averaged over cells of diameter 10^8 to 10^9 light years, which are large enough to include a sufficient number of cluster of galaxies. Homogeneity means, in simple words, that our point of observation is not privileged with respect to all the other points of the Universe, the so called *Copernican Principle*[1, 2, 3, 4];

Isotropy of the Universe. Isotropy means that the Universe looks the same on different directions of lines-of-sight of an observer. Wherever you look you always see he same things, on scale of the order of 100 Mpc, larger than the dimensions of the structures like galaxies and galaxy-clusters [1, 2, 3, 4].

The formulation of the Cosmological Principle given above allows us to consider the background metric as Friedmann-Lemaitre-Robertson-Walker (FLRW):

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega\right]\ ^1, \tag{1.1}$$

which describes a maximally symmetric and *expanding* (if $\dot{a}(t) > 0$) Universe. The metric in Eq. (1.1) is written from the point of view of a *comoving observer*[2] in polar coordinates

[1]Here we consider the usual convention that implies summation over repeated indices; greek letters running from 0 to 3 denote the four space-time components, latin letters running from 1 to 3 denote the spatial components.

[2]A comoving observer is an observer that sees the source of the geometry of the Universe homogeneous and isotropic.

($d\mathbf{x}^2 = (1-Kr^2)^{-1}dr^2 + r^2 d\Omega = (1-Kr^2)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$); K represents the curvature of the hypersurfaces at constant time t [5]. K can have three different values depending on the geometry of the Universe: we can clarify it calculating the proper distance at a fixed time t from the origin of the reference frame to a comoving object at fixed radial coordinate r ($d\Omega = 0$):

$$d(r,t) = \int_0^r ds = a(t)\cdot\int_0^r \frac{dr'}{\sqrt{1-Kr'^2}} = a(t)\cdot\begin{cases}\sin^{-1} r & \text{if } K=+1\\ \sinh^{-1} r & \text{if} K=-1\\ r & \text{if } K=0\end{cases} \tag{1.2}$$

In this coordinate system a comoving object has r time-independent, so the proper distance from us to a comoving object increases (or decreases) with $a(t)$, this clarifies also the meaning of the scale factor. $K = +1$ is the solution for a curved and *closed* Universe, $K = -1$ is the solution for a curved and *open* Universe and $K = 0$ is a Universe which is spatially flat (Euclidean). Observations show that the Universe we live is very close to being flat, or in other words it is compatible with the solution $K = 0$ [6, 7]. We will see in the next section which implication this measurement will have.

For this reason we will always assume a spatially-flat background:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{1.3}$$

In Eq. (1.3), t indicates the time with respect to a comoving observer, $a(t)$ is the usual scale factor. We can use instead the conformal time τ defined as $d\tau = dt/a$, obtaining the conformal metric

$$ds^2 = a^2(\tau)\left[-d\tau + \delta_{ij}dx^i dx^j\right]. \tag{1.4}$$

1.2 Dynamics in FLRW metric

We are now ready to study the background dynamics in the FLRW Universe. We begin by writing the Einstein field equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.5}$$

In General Relativity (GR) the dynamic variable is the metric itself [5, 8], describing the spacetime that changes in response to the presence of matter and energy. In Eq. (1.5), $G_{\mu\nu}$ is the Einstein tensor and contains the metric and its second order derivative; $T_{\mu\nu}$ is the *energy-momentum* tensor and describes the distribution of matter and energy. We will see later the possible forms of $T_{\mu\nu}$ under the assumption of a homogeneous and isotropic spacetime. $G_{\mu\nu}$ is the symmetric tensor defined by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}. \tag{1.6}$$

$R_{\mu\nu}$ is the *Ricci tensor*, the contraction of the *Riemann tensor*, while R is the *Ricci curvature scalar*, defined again as the contraction of the Ricci tensor:

$$R_{\mu\nu} = g^{\rho\sigma}R_{\mu\rho\nu\sigma} = R^{\rho}{}_{\mu\rho\nu}, \tag{1.7}$$

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{1.8}$$

We adopt the convention for which the Riemann tensor is defined as [5]

$$R^{\rho}{}_{\mu\sigma\nu} \equiv \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial\Gamma^{\rho}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\rho}_{\sigma\eta} - \Gamma^{\eta}_{\mu\sigma}\Gamma^{\rho}_{\nu\eta}. \tag{1.9}$$

The Riemann tensor contains the second-order derivatives of the metric w.r.t. the space-time coordinates via the *Christoffel symbols* defined as:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right). \tag{1.10}$$

These symbols measure the variation of the metric and are fundamental in GR for their connection with the motion of freely falling particles in a curved space through the geodesic equation:

$$\frac{d^2 x^\mu}{dt^2} - \Gamma^{\mu}_{\rho\sigma} \frac{d x^\rho}{dt} \frac{d x^\sigma}{dt} = 0 \tag{1.11}$$

With some trivial computations it can be shown that in the metric (1.3) the only Christoffel symbols which survive are

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij} = a^2 H \delta_{ij}, \tag{1.12}$$

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j = H\delta^i_j, \tag{1.13}$$

where $H = H(t) \equiv \dot{a}(t)/a(t)$ is the *Hubble parameter*.
Using these results in Eq. (1.7) we obtain:

$$R_{00} = -3\frac{\ddot{a}}{a}, \tag{1.14}$$

$$R_{0i} = R_{i0} = 0, \tag{1.15}$$

$$R_{ij} = a^2 \left(\frac{\ddot{a}}{a} + 2H^2 \right) \delta_{ij}. \tag{1.16}$$

Finally inserting these expressions into Eq. (1.8) we have

$$R = 6\left(\frac{\ddot{a}}{a} + H^2 \right). \tag{1.17}$$

The components of the Einstein tensor result to be

$$G_{00} = 3H^2, \tag{1.18}$$

$$G_{ij} = -a^2 \left(2\frac{\ddot{a}}{a} + H^2 \right) \delta_{ij}. \tag{1.19}$$

We have already said that in the right side of the Eq. (1.5) appears the energy-momentum tensor appears, which is the source of the geometry of the Universe. Saying that the Universe must be homogeneous and isotropic means requiring the homogeneity and isotropy also for the energy-momentum tensor. To define it we consider a set of observers with four-velocity

$$u^\mu = \frac{d x^\mu}{d\tau}, \tag{1.20}$$

with τ the proper time of the observer, so that $g_{\mu\nu}u^\mu u^\nu = -1$. For example, in the case of a perfect fluid the energy-momentum tensor becomes [5]

$$T_{\mu\nu} = \left[\rho(t) + p(t) \right] u_\mu u_\nu + p(t) g_{\mu\nu} \tag{1.21}$$

where $p = p(t)$ is the isotropic pressure of the fluid and $\rho = \rho(t)$ is its background matter-energy density. They depend only on time because of homogeneity and isotropy. Taking into account a comoving observer with null spatial velocity we obtain for the energy-momentum tensor of the cosmic fluid:

$$T_{00} = \rho(t), \qquad T_{0i} = T_{i0} = 0, \qquad T_{ij} = p(t) g_{ij} \tag{1.22}$$

Then substituting Eq. (1.22) into Eq. (1.5) we obtain two independent equations, the so-called Friedmann equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{1.23}$$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \tag{1.24}$$

Now we use a property of the stress-energy tensor

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1.25}$$

which can be interpreted as the *continuity equation* for the cosmic fluid. It can also be read as the consequence of the second Bianchi identity for the Riemann tensor and the metric compatibility, $\nabla_\mu g^{\rho\sigma} = 0$, required to have the Christoffel symbols (1.10)

$$\nabla_\eta R_{\mu\nu\rho\sigma} + \nabla_\rho R_{\mu\nu\sigma\eta} + \nabla_\sigma R_{\mu\nu\eta\rho} = 0, \tag{1.26}$$

which gives

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = \nabla_\mu G^{\mu\nu} = 0. \tag{1.27}$$

Eq. (1.25) gives a third Friedmann equation,

$$\dot{\rho} = -3H\left(p + \rho\right). \tag{1.28}$$

The three Friedmann equations define a system of non-independent first and second order derivative equations; Eq. (1.23) defines the dynamics, while Eq. (1.24) and Eq. (1.28) are just constraining equations. Indeed, deriving, for example Eq. (1.24) gives

$$H\left(\frac{\ddot{a}}{a} - H^2\right) = \frac{4\pi G}{3}\dot{\rho},$$

and substituting (1.28) we have

$$\frac{\ddot{a}}{a} = -4\pi G(\rho + p) + H^2.$$

Using again the expression for H^2 from (1.24) we resemble the first Friedmann equation, Eq. (1.23). These steps show the non-independence of the three Friedmann equations. Given that the independent variables in this system are three ($a(t)$, $\rho(t)$ and $p(t)$) but the independent equations are just two, we will add the *equation of state* of the cosmic fluid which connects the pressure to the energy density, $p = p(\rho)$. In general it can take the form

$$p = w\rho. \tag{1.29}$$

w represents a measure of the adiabatic speed of sound in the considered fluid. In general it can be variable, it can evolve with time, for example, but as a first, good, approximation we will consider it as a constant. We will see that for different kind of fluids it takes a

precise value.
Substituting Eq. (1.29) into Eq. (1.28) one obtains

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{1.30}$$

or

$$\rho \propto a^{-3(1+w)}. \tag{1.31}$$

Since the constant parameter w defines a certain kind of matter/energy, Eq. (1.31) clarifies the evolution of energy density with respect to the expansion of the Universe:

w=0 corresponds to a ordinary matter dominated Universe with null or negligible pressure, for which we have

$$\rho \propto a^{-3}; \tag{1.32}$$

w=1/3 corresponds to a radiation dominated Universe with density-pressure relation $p = \frac{1}{4}\rho$

$$\rho \propto a^{-4}; \tag{1.33}$$

w=-1 corresponds to a Universe dominated by a sort of exotic fluid with constant energy density (due to a *cosmological constant*)

$$\rho = \rho_0 = constant. \tag{1.34}$$

The dynamics of this kind of fluid will be of great interest for the theory of Inflation theory.

Now, substituting Eq. (1.31) into Eq. (1.24) we find the explicit time-dependent solution for the scale parameter, the energy density and the Hubble parameter in a flat Universe for a general fluid, assuming w is constant:

$$a(t) = a_0\left[1 + \frac{3}{2}(1+w)H_0(t-t_0)\right]^{\frac{2}{3(1+w)}}, \tag{1.35}$$

$$\rho(t) = \rho_0\left[1 + \frac{3}{2}(1+w)H_0(t-t_0)\right]^{-2}, \tag{1.36}$$

$$H(t) = H_0\left[1 + \frac{3}{2}(1+w)H_0(t-t_0)\right]^{-1}. \tag{1.37}$$

Here H_0 and a_0 are the Hubble and scale parameters defined at the time t_0, usually set as the present time. One can see that there exists a value for t such that the argument into the brackets is null

$$t_{BB} - t_0 = -\frac{2}{3H_0(1+w)}, \tag{1.38}$$

and if we define now the new time parameter

$$t' \equiv t - t_{BB} = t - t_0 + \frac{2}{3H_0(1+w)} \tag{1.39}$$

we obtain the following expressions for Eq. (1.35), Eq. (1.36) and Eq. (1.37)

$$a(t') = a_0\left[\frac{3}{2}(1+w)H_0 t'\right]^{\frac{2}{3(1+w)}}, \tag{1.40}$$

$$\rho(t') = \rho_0 \left[\frac{3}{2}(1+w)H_0 t'\right]^{-2}, \tag{1.41}$$

$$H(t') = \left[\frac{3}{2}(1+w)t'\right]^{-1}. \tag{1.42}$$

Setting $t' = 0$, or equivalently $t = t_{BB}$, in these equations we see that we obtain the following situation

$$a(t) \to 0, \qquad \rho(t) \to \infty, \qquad H(t) \to \infty, \tag{1.43}$$

namely, a configuration of the Universe with infinity energy density. We usually denote this event with the name *Big Bang*. In simple terms the Big Bang theory affirms that the Universe begin in a definite moment in the past from an initial configuration characterized by high energies, as we will see, high temperature (that's why we will refer to it as *Hot* Big Bang model). We will explore this model, its prediction and its consistency problems in the next section.

1.3 Hot Big Bang

Talking about the physical history of the Universe implies the knowledge about its *thermal* history, i.e. its evolution with the temperature, which means knowing also the variation of the temperature with respect to time (or, equivalently, $a(t)$, given Eq. (1.40)). How can we describe the thermal history of the Universe? We begin by defining the particle numerical density

$$n(T,\mu) = \frac{g}{(2\pi)^3}\int f(\mathbf{q},T,\mu)d^3q, \tag{1.44}$$

where $\mathbf{q}$ are the moments of the particles, g is the number of possible helicity states for each particle and μ is the chemical potential (that for primordial Universe can be considered negligible for all the particles)[3]. For the energy density we have

$$\rho(t,\mu) = \frac{g}{(2\pi)^3}\int E(\mathbf{q})f(\mathbf{q},T,\mu)d^3q,^4 \tag{1.45}$$

while the pressure is

$$p(T,\mu) = \frac{g}{(2\pi)^3}\int \frac{q^2}{3E(\mathbf{q})}f(\mathbf{q},T,\mu)d^3q. \tag{1.46}$$

Depending on the statistics of the particles (Fermions or Bosons), f will assume a different form and consequently also n, ρ and p. In the relativistic case ($T \gg m$) we have:

$$n(T) = \begin{cases} g\frac{\zeta(3)}{\pi^2}T^3 & \text{Bosons} \\ \frac{3}{4}g\frac{\zeta(3)}{\pi^2}T^3 & \text{Fermions} \end{cases}, \qquad \rho(T) = \begin{cases} g\frac{\pi^2}{30}T^4 & \text{Bosons} \\ \frac{7}{8}g\frac{\pi^2}{30}T^4 & \text{Fermions} \end{cases} \tag{1.47}$$

and $p = \frac{1}{3}\rho$ as can be expected for a relativistic fluid. Instead in the non-relativistic ($T \ll m$) case one finds that the number density is suppressed by a factor $\sim e^{-\frac{m}{T}}$.

[3]In general f, the distribution function of the particles over the energy or momentum states, should also depend on the particles' spatial components, but in a homogeneous and isotropic Universe we can directly mediate on the possible positions.

[4]Remember that $E(\mathbf{q}) = \sqrt{m^2+q^2}$.

In the early Universe the dominant component was the radiation, as we can easily see from the a-dependence of the various energy densities in Eqs. (1.33), (1.32) and (1.34), hence for primordial times we can consider just this ultra-relativistic fluid. This fluid was initially in a thermodynamic equilibrium, in the sense that the rate of interactions between the particles was much greater than the rate of expansion of the Universe, the Hubble parameter H. Considering the conservation of the entropy in an expanding Universe one can find the following relation between the temperature and the scale factor:

$$T \propto a^{-1}, \tag{1.48}$$

or analogously, using the cosmic redshift definition

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{a_o}{a_e} = \frac{1}{a_e}, \tag{1.49}$$

where λ_o is the *observed* wavelength, i.e. at the present time, and λ_e is the *emitted* wavelength, i.e. at the time of emission, we have

$$T \propto 1 + z. \tag{1.50}$$

Eq. (1.48) tells us that going backwards in time, the Universe becomes hotter (or, equivalently, that the Universe becomes colder as time passes); this is the reasons why we call the Big Bang *Hot*. In this sense we can briefly describe the evolution of the Universe through its thermal history.

At early epochs all the particle species were in thermal equilibrium. For very high energies, with values that cannot be achieved by the modern accelerators, we expect that there could have been a phase where the symmetry between matter and anti-matter was broken. We usually call this era as *baryogenesis*, an epoch baryons and antibaryons annihilated, and a little initial asymmetry ($10^9 + 1$ baryons for 10^9 antibaryons) created the actual observed matter.

For early epochs, i.e. high energies, the weak interaction and the electromagnetic interaction were unified in the *electroweak interaction*. At energies of order ~ 1 MeV, the energy at which the rate of interaction for the electroweak processes became smaller than the Hubble parameter, the neutrinos ceased to interact with the photons. This means that the neutrinos were not in thermodynamical equilibrium with the radiation anymore.

At the temperature of the order ~ 100 MeV, i.e. around ~ 100 s after the Big Bang, the temperature of the radiation was so high to break all the nuclear bonds: all the nucleus were decomposed into protons and neutrons. While the temperature goes down we reach a moment in which the boundary nuclear energies are higher than the temperature of the nucleons, or in other words their kinetic energies. In this moment the formation of the first fundamental nucleus happens: this phase is called *nucleosynthesis*. This phase was described by Gamow, see [9] and it is a fundamental prediction of the Big Bang theory. It predicts an abundance of Helium, ^{4}He which is not justified considering only the elements produced in the stellar nuclear reactions. The observation of an abundance of chemical elements in accordance with the nucleosynthesis prediction was one of the striking successes of the Big Bang theory.

Until now the Universe was dominated by the radiation. It is highlighted by the scale factor dependence of the various components of the Universe in (1.32), (1.33) and (1.34). For early times, $a \ll 1$, the dominant component was the radiation. This epoch is called *radiation dominated era*. This era continued until the matter component became dominant. Matching Eqs. (1.33) and (1.32), it follows that the energy density of the radiation fluid

became soon smaller than the energy density of the matter fluid. When this happened in the Universe, it started a new epoch, the *matter dominated era*. During this epoch all the large scale structures that we observe in the Universe, such as galaxies and clusters, started to form through gravitational instability.
After the radiation-matter equivalence happened one of the most important event in the Universe history. When the Universe was almost $\sim 10^6$ years old the temperature was higher than the bound energy of the Hydrogen, 13.6 eV. This means that the Universe was ionized and opaque to the radiation. The photons were not free to stream as they interacted with the electron through the Compton scattering. When the temperature went down the boundary energy of H, the electron could interact with the protons through the interaction

$$e^- + p^+ \rightarrow H + \gamma, \tag{1.51}$$

i.e. the first atoms of Hydrogen were created. At this point the interaction between photons ad electrons was ceases to be to dominant one and the photons were free to travel. This point in the time (more precisely it is not a point since this precess happened in a finite time) defines the *last scattering surface* and this epoch is the so called *Hydrogen recombination*. The radiation which leaves the last scattering surface was then cooled by the expansion of the Universe to the value of ~ 3 K, and now constitutes the Cosmic Microwave Background (CMB) we observe. The cosmic microwave background is a fundamental prediction of the Big Bang model and it represents a milestone of the history of Cosmology: its discovery, see [10], opened the way to a more detailed study of the first moments of the Universe. Today we have very-high resolution photos of this last scattering surface, as Fig. 2.1. We will see that the CMB represents a treasure trove of informations about the early epoch of the Universe [11].

This is a brief description of the Hot Big Bang model, we refer the reader to [1, 4, 2] for more (standard) details.

1.3.1 ΛCDM model

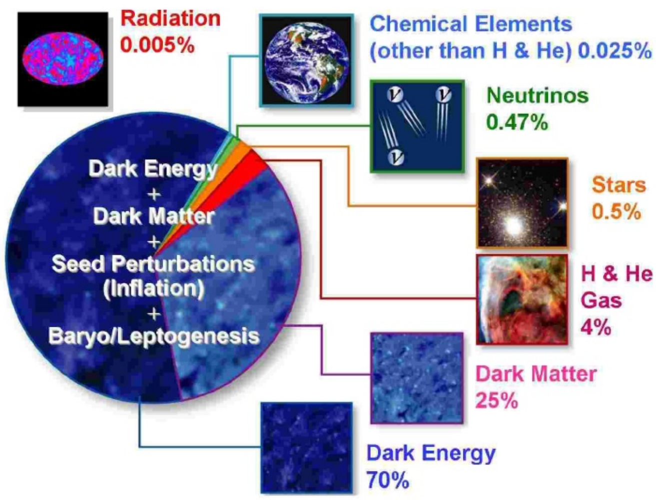

Figure 1.1: Actual composition of the Universe: here all the elements of the standard ΛCDM cosmological model are represented. The pie graphic illustrates the fraction of each component with respect to the total energy density of the Universe [12]. For the most precise measurements of the abundance of these elements see [7].

The last measurements on the Cosmic Microwave Background (CMB) [6, 7] allow a detailed description of the composition of the Universe. As we have explained above, at early times the Universe was dominated by its radiation component, then it passes through a matter-dominated era and now the dominant component is the *dark energy*, whose equation of state is constrained to be $w_{de} = -1.028 \pm 0.032$ (68% CL) [7], which is completely consistent with a cosmological constant. We have just mentioned above that the spatial curvature of the Universe is compatible with zero: a direct consequence of this measurement is that the actual density of the Universe is almost the critical one. The basis model to describe the cosmic evolution is the so-called ΛCDM model, which is depicted in in Fig. 1.1. Λ indicates the dark energy component, which is measured to be almost the 70% of the total energy density. This dark energy is also responsible for the actual acceleration of the expansion.
CDM means *Cold Dark Matter*, which is another fundamental component of our Universe, being almost the 25% of the total energy density. This component is called *dark* because it does not emit light and we can measure its presence only by indirect observations (e.g. galaxies rotation curves, large scale structure measurements, CMB). Its appellative *cold* means that it was non-relativistic at the time of decoupling: if the dark matter was *hot*, i.e. relativistic at the time of its decoupling, it would have had important consequences on the formation of the large scale structure of the Universe [13].
The baryonic matter, the galaxies, the stars, the planets and also us, is only a little percentage of the total composition of the Universe, meaning that the Universe we live in is almost *dark*, as we stressed above. In Fig. 1.2 it is possible to see all the six fundamental parameters (the first six of the list) that define with the ΛCDM model. The most striking characteristic of this model is, indeed, its unreasonable effectiveness for the description of the history of the Universe using just 6 parameters. Though many of the ingredients of the model remain highly mysterious from a fundamental physics point of view, ΛCDM is one of the most successful phenomenological models.

Parameter	TT+lowP 68 % limits	TT+lowP+lensing 68 % limits	TT+lowP+lensing+ext 68 % limits	TT,TE,EE+lowP 68 % limits	TT,TE,EE+lowP+lensing 68 % limits	TT,TE,EE+lowP+lensing+ext 68 % limits
$\Omega_b h^2$	0.02222 ± 0.00023	0.02226 ± 0.00023	0.02227 ± 0.00020	0.02225 ± 0.00016	0.02226 ± 0.00016	0.02230 ± 0.00014
$\Omega_c h^2$	0.1197 ± 0.0022	0.1186 ± 0.0020	0.1184 ± 0.0012	0.1198 ± 0.0015	0.1193 ± 0.0014	0.1188 ± 0.0010
$100\theta_{MC}$	1.04085 ± 0.00047	1.04103 ± 0.00046	1.04106 ± 0.00041	1.04077 ± 0.00032	1.04087 ± 0.00032	1.04093 ± 0.00030
τ	0.078 ± 0.019	0.066 ± 0.016	0.067 ± 0.013	0.079 ± 0.017	0.063 ± 0.014	0.066 ± 0.012
$\ln(10^{10} A_s)$	3.089 ± 0.036	3.062 ± 0.029	3.064 ± 0.024	3.094 ± 0.034	3.059 ± 0.025	3.064 ± 0.023
n_s	0.9655 ± 0.0062	0.9677 ± 0.0060	0.9681 ± 0.0044	0.9645 ± 0.0049	0.9653 ± 0.0048	0.9667 ± 0.0040
H_0	67.31 ± 0.96	67.81 ± 0.92	67.90 ± 0.55	67.27 ± 0.66	67.51 ± 0.64	67.74 ± 0.46
Ω_Λ	0.685 ± 0.013	0.692 ± 0.012	0.6935 ± 0.0072	0.6844 ± 0.0091	0.6879 ± 0.0087	0.6911 ± 0.0062
Ω_m	0.315 ± 0.013	0.308 ± 0.012	0.3065 ± 0.0072	0.3156 ± 0.0091	0.3121 ± 0.0087	0.3089 ± 0.0062
$\Omega_m h^2$	0.1426 ± 0.0020	0.1415 ± 0.0019	0.1413 ± 0.0011	0.1427 ± 0.0014	0.1422 ± 0.0013	0.14170 ± 0.00097
$\Omega_m h^3$	0.09597 ± 0.00045	0.09591 ± 0.00045	0.09593 ± 0.00045	0.09601 ± 0.00029	0.09596 ± 0.00030	0.09598 ± 0.00029
σ_8	0.829 ± 0.014	0.8149 ± 0.0093	0.8154 ± 0.0090	0.831 ± 0.013	0.8150 ± 0.0087	0.8159 ± 0.0086
$\sigma_8 \Omega_m^{0.5}$	0.466 ± 0.013	0.4521 ± 0.0088	0.4514 ± 0.0066	0.4668 ± 0.0098	0.4553 ± 0.0068	0.4535 ± 0.0059
$\sigma_8 \Omega_m^{0.25}$	0.621 ± 0.013	0.6069 ± 0.0076	0.6066 ± 0.0070	0.623 ± 0.011	0.6091 ± 0.0067	0.6083 ± 0.0066
z_{re}	$9.9^{+1.8}_{-1.6}$	$8.8^{+1.7}_{-1.4}$	$8.9^{+1.3}_{-1.2}$	$10.0^{+1.7}_{-1.5}$	$8.5^{+1.4}_{-1.2}$	$8.8^{+1.2}_{-1.1}$
$10^9 A_s$	$2.198^{+0.076}_{-0.085}$	2.139 ± 0.063	2.143 ± 0.051	2.207 ± 0.074	2.130 ± 0.053	2.142 ± 0.049
$10^9 A_s e^{-2\tau}$	1.880 ± 0.014	1.874 ± 0.013	1.873 ± 0.011	1.882 ± 0.012	1.878 ± 0.011	1.876 ± 0.011
Age/Gyr	13.813 ± 0.038	13.799 ± 0.038	13.796 ± 0.029	13.813 ± 0.026	13.807 ± 0.026	13.799 ± 0.021
z_*	1090.09 ± 0.42	1089.94 ± 0.42	1089.90 ± 0.30	1090.06 ± 0.30	1090.00 ± 0.29	1089.90 ± 0.23
r_*	144.61 ± 0.49	144.89 ± 0.44	144.93 ± 0.30	144.57 ± 0.32	144.71 ± 0.31	144.81 ± 0.24
$100\theta_*$	1.04105 ± 0.00046	1.04122 ± 0.00045	1.04126 ± 0.00041	1.04096 ± 0.00032	1.04106 ± 0.00031	1.04112 ± 0.00029
z_{drag}	1059.57 ± 0.46	1059.57 ± 0.47	1059.60 ± 0.44	1059.65 ± 0.31	1059.62 ± 0.31	1059.68 ± 0.29
r_{drag}	147.33 ± 0.49	147.60 ± 0.43	147.63 ± 0.32	147.27 ± 0.31	147.41 ± 0.30	147.50 ± 0.24
k_D	0.14050 ± 0.00052	0.14024 ± 0.00047	0.14022 ± 0.00042	0.14059 ± 0.00032	0.14044 ± 0.00032	0.14038 ± 0.00029
z_{eq}	3393 ± 49	3365 ± 44	3361 ± 27	3395 ± 33	3382 ± 32	3371 ± 23
k_{eq}	0.01035 ± 0.00015	0.01027 ± 0.00014	0.010258 ± 0.000083	0.01036 ± 0.00010	0.010322 ± 0.000096	0.010288 ± 0.000071
$100\theta_{s,eq}$	0.4502 ± 0.0047	0.4529 ± 0.0044	0.4533 ± 0.0026	0.4499 ± 0.0032	0.4512 ± 0.0031	0.4523 ± 0.0023

Figure 1.2: Table of all the fundamental parameters in ΛCDM cosmological model as measured by the Planck mission [6].

1.3.2 The problem of the initial conditions

The Big Bang model was universally accepted after some fundamental observations about the abundance of ^{4}He and CMB radiation [9, 10]. Despite this, the model presents some theoretical issues: we observe today that the Universe is the result of very unlikely initial conditions. In order to understand these problems we define an important concept, the one of *horizon*, as it is usually used in cosmology. Here we will use the notation adopted in [1].

Particle Horizon: from the metric given in Eq. (1.1) one can define the distance traveled by a light ray between two points of our spacetime. If we define

$$dl^2 = \frac{dr^2}{1 - Kr} + r^2 d\Omega^2$$

the comoving distance is given by

$$\chi_p(t) = \int_0^{l(t)} dl'$$

The isotropy of the space allows us to consider a fixed angle ($d\Omega = 0$), from ds^2 we have:

$$\frac{dt}{a(t)} = \frac{dr}{\sqrt{1 - Kr}},$$

hence the comoving distance results to be

$$\chi_p(t) = \int_0^{r(t)} \frac{dr}{\sqrt{1-Kr}} = \int_0^t \frac{dt'}{a(t')}. \tag{1.52}$$

It is known that in general, in cosmology, if we want a *physical* quantity one as to multiply the *comoving* quantity for the scale factor

$$\lambda_{phys} = a(t)\lambda_{com}.$$

Hence if we want the proper, or physical, distance travelled by a light ray we obtain the so-called *particle horizon*:

$$d_H = a(t) \int_0^t \frac{dt'}{a(t')}. \tag{1.53}$$

$d_H(t)$ defines the radius of a sphere centered in the observer within which are contained all the points that may have come into causal connection with the observer until the time t. If one point is outside this sphere then it cannot have been influenced in any way from the observer (and vice versa). More generally: if two points are separated by a distance greater than $d_H(t)$ then they have never been in causal connection.

We have seen that in the the standard FLRW models this relation between the scale factor and the time holds

$$a(t) \propto t^\alpha,$$

with $\alpha = \frac{2}{3(1+w)}$. So, looking at Eq. (1.53) we see that the integral converges if $\alpha < 1$, i.e. if $w > -1/3$ and, in particular, in this case assumes the form

$$d_H(t) = \frac{3(1+w)}{3w+1} t. \tag{1.54}$$

We also note that from the Friedmann equation (1.23) the condition on the value of w for the convergence of the particle horizon corresponds to $\ddot{a}(t) < 0$.
Hence, in a decelerated Universe the particle horizon exists.

Hubble horizon: be $\tau_H \equiv H^{-1}$ the *Hubble time*, the expansion characteristic time (remember that H can be interpreted as the expansion rate of the Universe). We can now define the *Hubble radius*, in natural units

$$R_H(t) = H^{-1}(t) = \tau_H, \tag{1.55}$$

which represents the distance travelled by a photon in a Hubble time. R_H tells us which are the points that have been in causal connection in a Hubble time.
The corresponding *comoving Hubble radius* is:

$$r_H(t) = \frac{R_H(t)}{a(t)} = (aH)^{-1} = \frac{1}{\dot{a}(t)}. \tag{1.56}$$

Like the particle horizon, the comoving Hubble radius represents a measure of the distance under which two points in the Universe are causally connected. The spherical region centered in the observer with radius $r_H(t)$ contains all the points with which the observer is causally connected at the time t. For this reason the comoving

Hubble radius is also called *comoving Hubble horizon.*
With a simple change of variable in Eq. (1.53) we can relate the particle horizon with the Hubble horizon

$$d_H(t) = a\int_0^t \frac{dt'}{a(t')} = a\int_0^a \frac{da'}{a'}\frac{1}{\dot{a}'} = a\int_0^a d(\log a) r_H,$$

i.e. the particle horizon is given by the logarithmic integral of the Hubble horizon.

Note that, despite their profound conceptual difference, the Hubble and the particle horizon have the same value, except for a factor of the order $\sim \mathcal{O}(1)$, compare the equations (1.42) and (1.54). Now, we have all the elements to understand the so called horizon and flatness problems of the standard Hot Big Bang model.

Horizon Problem

We have understood that a region of the Universe with a comoving characteristic length λ can be all causal connected only if $\lambda = r_H$, i.e. when the comoving Hubble horizon, becomes larger than the characteristic length.
This means that physical processes that occur at t cannot causally connect the region with dimension λ until $t \leq t_H$: this region will become causally connected when $t > t_H$, i.e. when $\lambda < r_H$. According to this argument if a region is large enough it gets in causal connection only recently, but this is not what we observed with the CMB. In theory all the photons that arrives to us from two different points of the last scattering surface with an angular distance higher than $\theta \simeq 1.6°$, which represents the angle subtended by the horizon at the time of last scattering, have become causally connected only recently [4].
In other words, in a matter- or radiation-dominated Universe no physical influence could have smoothed out initial inhomogeneities and brought points at at the redshift of the recombination that are separated by more than a few degrees to the same temperature. This is in contradiction with what we actually see. The COBE experiment [14] showed that the background radiation was nearly homogeneous ad isotropic, all the points had the same temperature with very small fluctuations of the order $\frac{\Delta T}{T} \sim 10^{-5}$. The instrument had an angular resolution of $\Delta\theta \sim 7°$, i.e. was observing regions causally disconnected (more precisely, regions that have become causally connected only in recent times). This is a problem: how it is possible that two regions, which were not able to talk to each other for their entire past evolution history, are observed to have the same properties (same temperature, same energy density)?
Another way to see the problem is the following: consider the metric in Eq. (1.1) and carry out the following change of coordinates:

$$d\tau = \frac{dt}{a},$$

$$r = f_K(\chi) = \begin{cases} \sinh\chi & \text{if K=-1} \\ \chi & \text{if K=0} \\ \sin\chi & \text{if K=+1} \end{cases} .$$

With this choice the metric becomes

$$ds^2 = a^2(\tau)\left[-d\tau^2 + d\chi^2 + f_K^2(\chi)d\Omega^2\right]$$

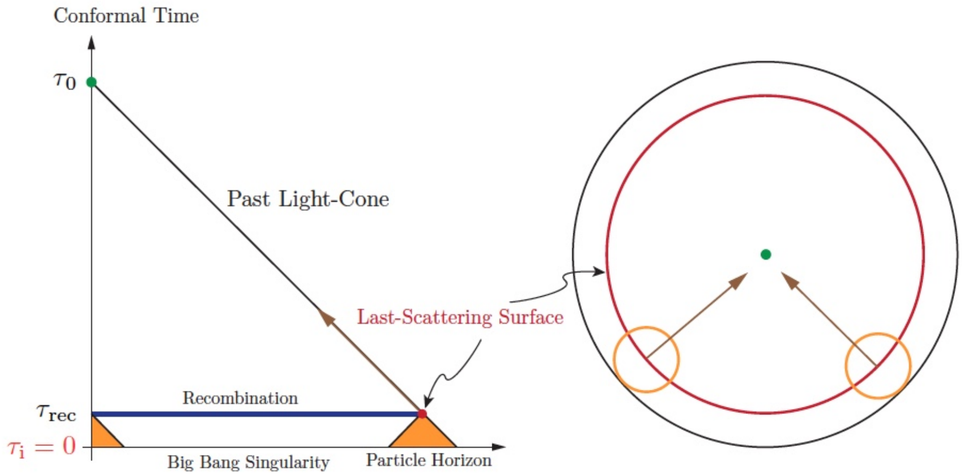

Figure 1.3: Conformal diagram of Big Bang cosmology. Here becomes clear that different patches of the Last Scattering Surface, in this scenario, are causally disconnected, contrary to what we observe today [15].

i.e., in this coordinate system the structure of the spacetime is analogous to that of Minkowski's, it is said to be *conformally flat*; if we look at the light cones of a light ray that travels with radial direction ($d\Omega^2 = 0$) we obtain exactly the same situation as in Minkowski spacetime:

$$d\tau = \pm d\chi.$$

In Fig. 1.3 we can see that the two light cone that leave a point at the recombination time, which are part of the last scattering surface (CMB), will never meet in the past with another light cone of the same constant-time hypersurface, because of the limit imposed by the Big Bang. In the right figure of Fig. 1.3 one can see a schematic representation of what we mentioned above: two light rays starting from different points of the last-scattering surface, which have never been in causal connection before, arrives at the observer in the center, us, showing almost the same properties. How is it possible if they have never had the possibility to communicate each other? Is there a dynamical process which explains what we observe today?

Flatness Problem

From recent observations and studies on the CMB [6, 7, 16] it results that our Universe is compatible with the solution of a *spatially flat Universe*, i.e. with a value of K, present in the FLRW metric, compatible with 0. Why we consider this as an issue within the standard Hot Big Bang model? In order to understand it we need to consider in the Friedmann equation (1.24) also the contribution deriving from the curvature K. It becomes

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}. \tag{1.57}$$

We can define the *critical density*

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{1.58}$$

as the density needed to have a Universe with null spatial curvature. We can also define the *density parameter* as the ratio between the energy density of the cosmic fluid at the time t and the critical energy density at the time t

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G}{3H^2}\rho(t). \tag{1.59}$$

Now substituting Eq. (1.59) in Eq. (1.57) we can rewrite the Friedmann equation as

$$1 = \Omega(t) - \frac{K}{a^2H^2}$$

or

$$\Omega(t) - 1 = \frac{K}{a^2H^2},$$

and, in the end, if we now remember the definition of the Hubble comoving horizon given in Eq. (1.56) we obtain

$$\Omega(t) - 1 = K r_H^2(t). \tag{1.60}$$

This equation describes the evolution of the density parameter with respect to time, and in particular we see that its evolution is connected with the Hubble horizon. This equation tells us one important thing: if the Universe is perfectly flat then $\Omega = 1$ exactly, but if $\Omega(t) \neq 1$, the difference between Ω and 1 will grows as fast as r_H does. In fact we have seen above that r_H during a matter- or radiation-dominated era grows while time passes; vice versa, going backwards in time the density parameter will be closer to 1. Today the measured value for $\Omega(t)$ is [7]

$$|\Omega_0 - 1| = 0.0007 \pm 0.0019 \quad (68\%).$$

Eq. (1.60) tells us that the value at the beginning of the Universe was much closer to 1 with respect to today. We take as reference time for the primordial Universe the Planck time $t_{Pl} \sim 10^{-44}$ s after the Big Bang. This is the time below which the modern quantum field theory description of the nature's law is incomplete. By a direct estimation we find [3, 15]:

$$|\Omega(t_{Pl}) - 1| \simeq |\Omega_0 - 1| \cdot 10^{-60}$$

or

$$|\Omega(t_{Pl}) - 1| \leq 10^{-62}$$

i.e., at the beginning of the Universe the spatial curvature needed to be extremely close to 1 (with one part over 10^{62}). This is a problem because the probability that the Universe was exactly spatially flat is null (is one configuration over infinite possibilities): this issue is also called *fine tuning* problem.

Unwanted Relics

According to several extensions of the Standard Model of particles (e.g. Grand Unification Theories GUT or string theories), in the early Universe at very high energies, the ones predicted in the Hot Big Bang model, various cosmological defects, such as magnetic monopoles, cosmic strings, domain walls and so on, could have been produced, which would still be present in the Universe, with an abundance that would overclose the Universe by many orders of magnitudes. These are called cosmological relicts, and they represent a problem, the so called problem of *unwanted relics* or *magnetic monopoles* problem, since what we observe today is an almost spatially-flat Universe. Historically the problem of the magnetic monopoles was the first theoretical evidence of the necessity to introduce Inflation in the primordial Universe [17, 18]. The Inflation provides a powerful solution also for this kind of problem. If we call X the relic particle hence its number density will have a behavior $n_X \sim a^{-3}$ where a is the scale factor. If we have a primordial era in which the expansion was (nearly) exponential, as Inflation is, $a \sim e^{Ht}$ and the number density is exponentially suppressed $n_X \sim e^{-3Ht}$. We refer the reader to Refs. [1, 19] for more details about other types of cosmological relicts.

1.3.3 A possible solution

Before giving a solution to the horizon and flatness problems we should spend some words about these issue of the primordial Universe. The flatness and the homogeneity of areas causally disconnected would not be so problematic if one assumes that the initial conditions of the Universe were exactly the flatness and the homogeneity and isotropy. So, why we do not accept the very high grade of order of the primordial Universe and solve these two problems before they even arise? Because if we consider all the possible Universes, the possibilities that a Universe started with a high level of order are very few (more precisely, they are null): one can even say that the initial conditions of the Universe were exactly those necessary to guarantee the existence of intelligent life. This statement goes under the name of *anthropic principle*, but it is not a real answer to our questions and not all the physicist accept it [1]. We are looking for a theory, a mechanism that explains these initial conditions dynamically: we will analyze the *Inflationary Universe* solution, the most accredited theory about the primordial Universe and the one which is up to now in complete agreement with different cosmological observations (we will see that besides solving the Big Bang puzzles, it will also explain the origin of the Large-Scale structure of the Universe).

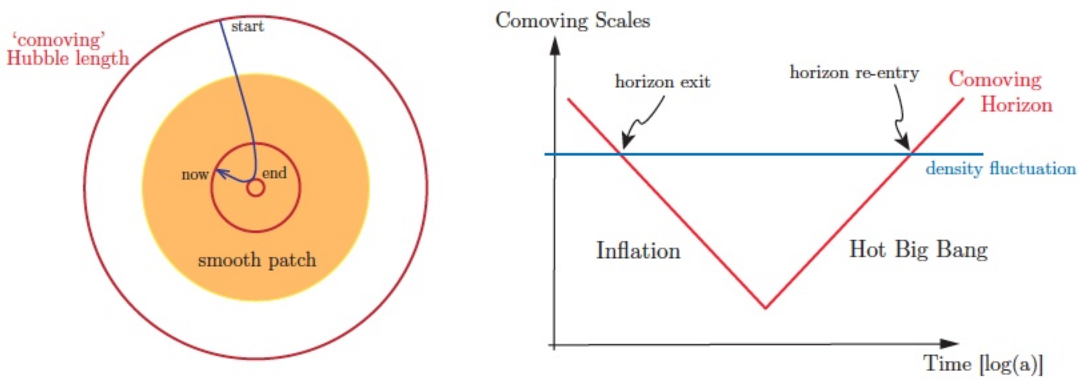

Figure 1.4: In the left figure we can see that, during the inflationary period, the comoving Hubble sphere has been shrunk, becoming smaller of the causally connected region at early times. On the right we can see the comoving length of the region considered that remains constant all along the cosmological evolution. At a certain time it exits the horizon, while Inflation occurs. Then it re-enters the horizon at late times [3, 15].

We can solve both the problem of causally-connected regions and the flatness one assuming that before the radiation-dominated era there was an epoch where $\dot{r}_H(t) < 0$, i.e. a primordial era during which the Hubble comoving horizon decreases instead of increasing as it usually happens in the standard FLRW Universes. Given this, the regions of dimension λ that entered the horizon only recently were already causally connected during inflation, see Fig. 1.4. In particular, assuming a decreasing Hubble horizon means assuming an epoch of accelerated expansion:

$$\dot{r}_H(t) = -\frac{\ddot{a}}{a^2} < 0 \;\Rightarrow\; \ddot{a}(t) > 0. \tag{1.61}$$

Hence we will call $Inflation$, in full generality, a primordial epoch of the Universe where the expansion is accelerated.
We have seen above that we need an accelerated expansion, Eq. (1.61), but looking at Eq. (1.23) we see that this corresponds to have a component with an effective pressure given by

$$p < -\frac{1}{3}\rho, \tag{1.62}$$

i.e. with a negative pressure, assuming ρ is positive. We have seen a particular case in the previous section which is compatible with the condition (1.62), the case of a Universe dominated by an exotic component with constant energy density and $w = -1$ exactly. In this case the Hubble scale factor is exactly constant in time while increases exponentially with the law

$$a(t) = a_\star e^{H(t-t_\star)} \tag{1.63}$$

where t_{star} is some reference time and $a_{star} \equiv a(t_\star)$. If the dominant component during Inflation had exactly $w = -1$ we would have an eternal exponentially expanding Universe. In order to have a finite inflationary period we need a primordial era described by a slowly varying Hubble parameter, as we will formalize later, which corresponds to have a quasi-*de Sitter Universe*

$$a(t) \propto e^{Ht}. \tag{1.64}$$

The condition of an accelerated expansion period is not sufficient to solve the horizon problem: we must impose also a condition on the duration of this accelerated expansion era. In fact Inflation needed to begin at a time t_i such that the comoving Hubble horizon at that time was larger than what we observe today: we need this condition to explain the high degree of homogeneity on regions of the Universe that, for us, have entered the horizon only recently. Imposing this condition means that the regions we observe today were already causally connected, then they exited the comoving Hubble horizon (because it was decreasing during inflation) and re-entered the horizon after inflation, showing homogeneity and isotropy. Quantitatively we can write

$$r_H(t_i) \geq r_H(t_0). \tag{1.65}$$

We see now that this equation can be interpreted as a condition on the total duration of Inflation.
A region of physical dimensions $a(t_i)r_H(t_i)$, where t_i is the time when Inflation begins, expands, during inflation, by a factor

$$Z \equiv \frac{a(t_f)}{a(t_i)}, \tag{1.66}$$

which can be rewritten as

$$Z = \frac{a(t_f)}{a(t_i)} = \exp\left\{\left[\int_{t_i}^{t_f} H(t)dt\right]\right\} = e^{N_{inf}}.$$

We have introduced the quantity N_{inf}, which is called the number of *e-foldings*. It gives a natural measure for the duration of the inflation

$$N_{inf} = \log\left[\frac{a(t_f)}{a(t_i)}\right] \tag{1.67}$$

Our question now is: which is the necessary e-foldings number to solve the horizon problem? The answer comes from Eq.(1.65): with some trivial calculations one finds

$$N_{inf} \geq \log\left(\frac{T_0}{H_0}\right) + \log\left(\frac{H_i}{T_f}\right) \tag{1.68}$$

where T_0 and H_0 are the temperature and the Hubble parameter at the present time, H_i is the Hubble parameter at the beginning of Inflation and T_f is the temperature at the end of Inflation. The first term on the right hand side of Eq. (1.68) is known experimentally, given that [7, 20]

$$\begin{aligned} H_0 &= (67.66 \pm 0.42)\text{km s}^{-1}\text{Mpc}^{-1}\ (68\%\,\text{CL}), \\ T_0 &= (2.72548 \pm 0.00057)\text{K}\ (95\%\,\text{CL}) \end{aligned} \tag{1.69}$$

and results to be

$$\log\left(\frac{T_0}{H_0}\right) \simeq 67,$$

while the second term depends on the model of Inflation considered and in particular it depends on how the Inflation ends. We will see that the inflationary period will end with a mechanism called *reheating*, a phase in which the scalar field decays into light particles which reheat the Universe so that the standard Hot Big Bang phase, in a radiation-dominated epoch, could start. This heuristic description of the end of Inflation allow us to estimate also the second term in the left hand side of Eq. (1.68). If we consider a constant Hubble parameter during all the inflationary period, i.e. $H_i \simeq H_{inf} \simeq$ const., we have, using Friedmann equation together with (1.47)

$$H_{inf}^2 = \frac{8\pi G}{3}\rho_r = \frac{8\pi G}{3}\frac{\pi^2}{30}gT^4\bigg|_{t_e},$$

where t_e means the time at which the end of Inflation occurs. In natural units we have $M_{Pl} = G^{-1/2} = T_{Pl}$, so that we can give an approximate measure of H_i

$$H_i^2 = H_{inf}^2 \simeq \frac{T_e^4}{T_{Pl}^2} \quad \rightarrow \quad \frac{H_i}{T_e} \simeq \frac{T_f}{T_{Pl}}.$$

Given that [21]

$$10^{-5} \leq \frac{T_e}{T_{Pl}} \leq 1,$$

we can estimate a range of possible values

$$\log\left(\frac{H_i}{T_f}\right) \simeq \log\left(\frac{T_e}{T_{Pl}}\right) \in [-11, 0].$$

Hence we obtain a range for the possible values of N_{inf}

$$N_{inf} > (56 - 67),$$

which means that the minimal number of e-folds is around 60, that corresponds to a total expansion during Inflation of a factor

$$\frac{a_f}{a_i} = e^{N_{inf}} \simeq 10^{26} - 10^{30}.$$

We can see that this minimal number of e-foldings, a measure of the expansion factor during inflation, solves also the flatness problem. In fact we have seen that the density parameter is strictly correlated to the comoving Hubble horizon through Eq. (1.60) and during an epoch dominated by radiation it tends to grow. But in an epoch with accelerated expansion the comoving Hubble horizon decreases, Eq. (1.61). If we have a de Sitter Universe, with a metric (1.79) which satisfies the condition (1.64), we have

$$\Omega(t) - 1 \propto r_H^2(t) = \frac{1}{a^2 H^2} \propto e^{-2Ht},$$

i.e. the inflationary phase exponentially suppresses the deviation of Ω from 1. In this sense we can describe Inflation as an *attractor mechanism.* The problem of the curvature of the Universe can be solved if we require that

$$\frac{\Omega_i^{-1} - 1}{\Omega_0^{-1} - 1} \geq 1, \tag{1.70}$$

i.e. if the density parameter at the beginning of Inflation is more distant from 1 with respect to today (for this computation it is more convenient using the difference between Ω^{-1} and 1). Using the definition of Eq. (1.67) and after some tedious but straightforward calculations we find that the minimum number of the e-foldings to solve the flatness problem turns out to be

$$N_{inf} \simeq 60 \sim 70,$$

the same as in the case of the horizon problem. Therefore we can state that solving the horizon problem is equivalent to solve the flatness problem. This result does not come as a surprise because of the dependence of the density parameter from the Hubble horizon. We have found a new dynamical way to solve these two problems; the inflationary scenario seems very attractive to us because it describes a mechanism which, whatever are the initial conditions (spatial curvature, inhomogeneities or anisotropies), attracts the Universe towards a FLRW solution, that is exactly what we were looking for. What are the physical implication of inflation? How does Inflation know when to stop? Is there new physics? These and other questions will be faced up in the following chapter.

1.4 Inflation

The first proposal to solve the problems we described in the previous chapter was the theory of cosmic Inflation [17, 18],[22]-[26]; this theory was born to solve principally the problem of the so-called *cosmic relics*, the appearing of new particles in the context of the spontaneous symmetry breaking of GUT theories [17, 18, 25, 26]. This powerful solution found in the early 80s was then studied and improved, until it became the *standard model* for cosmological inflation, with specific observable predictions that have been investigated in the last thirty years, and are still subject of active research.
In this chapter we will describe the dynamics of standard slow-roll models of inflation: we will see in detail the primordial (quantum) perturbations of the inflaton (a scalar field responsible of the inflationary era) and of the metric tensor. From this we will extract some observables starting from the definition of some gauge invariant variables, studying their statistics, and comparing them to the standard inflationary predictions. We will work with linear perturbations for the moment, making observational predictions about the Gaussian statistics of such perturbations. Indeed we will see that the some crucial observables will emerge from the non-Gaussianities of the primordial, inflationary, perturbations that will be analyzed in the next chapters.

1.4.1 Slow-Roll paradigm

In the previous chapter we described a kinematical process to solve the flatness and the horizon problems that contemplates the presence of a "fluid" with equation of state (1.62). There exist a dynamical process that can explain this negative pressure? We will see that the slow-roll paradigm has a very simple field theory description: it involves only the Einstein general theory of Relativity and a real scalar field, minimally coupled to the gravity [27]. The action we will consider is the following

$$S = \frac{M_{pl}^2}{2}\int \mathrm{d}\,x^4\sqrt{-g}\,R + \int \mathrm{d}\,x^4\sqrt{-g}\mathcal{L}_\phi[\phi, g_{\mu\nu}] + S_M, \tag{1.71}$$

with $g \equiv det\, g_{\mu\nu}$.
The first term in Eq.(1.71) is clearly the Hilbert-Einstein action [5], with R the scalar curvature; the second terms involves $\mathcal{L}_\phi$, which represents the Lagrangian density of a scalar field ϕ, whose dependence on the metric is explicitly highlighted by the presence of the metric $g_{\mu\nu}$; S_M represents the action of other fields and possible interactions with the scalar field (fermionic fields, gauge bosons and others) that, we will see, give in general negligible contribution during Inflation. We consider now the dynamics of a scalar field with Lagrangian density.

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{1.72}$$

Note that in general in the kinetic term one should use the covariant derivative instead of the partial derivative, but for a scalar field we have $\nabla_\mu\phi = \partial_\mu\phi$. $V(\phi)$ is the potential term that takes into account the ϕ mass term, auto-interactions of the field and and eventually interactions of ϕ with other fields that have been integrated out or vacuum loop-corrections.
We can associate a symmetric stress-energy tensor to our field, using the well-known definition given in General Relativity [5]

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}}, \tag{1.73}$$

where $\mathcal{L}$ represents the Lagrangian of any matter component. The functional derivative $\delta/\delta g^{\mu\nu}$ can be rewritten in terms of partial derivatives with respect to the metric

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} - \partial_\alpha\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu,\alpha}}\right) + \dots\right]. \tag{1.74}$$

The dots represent terms with higher order derivative with respect to $g^{\mu\nu}$[5].

After some trivial calculations we obtain, in the case of a minimally coupled scalar field

$$T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(-\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - V(\phi)\right). \tag{1.75}$$

[5]Terms like

$$\partial_\alpha\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu,\alpha}}\right)$$

are different from 0 if in the Lagrangian are present terms of the form $\xi\phi R^2$, i.e. when the scalar field is directly coupled with gravity. This is the case of a non-minimal coupling, appearing in scalar-tensor theories of gravity [28, 19].

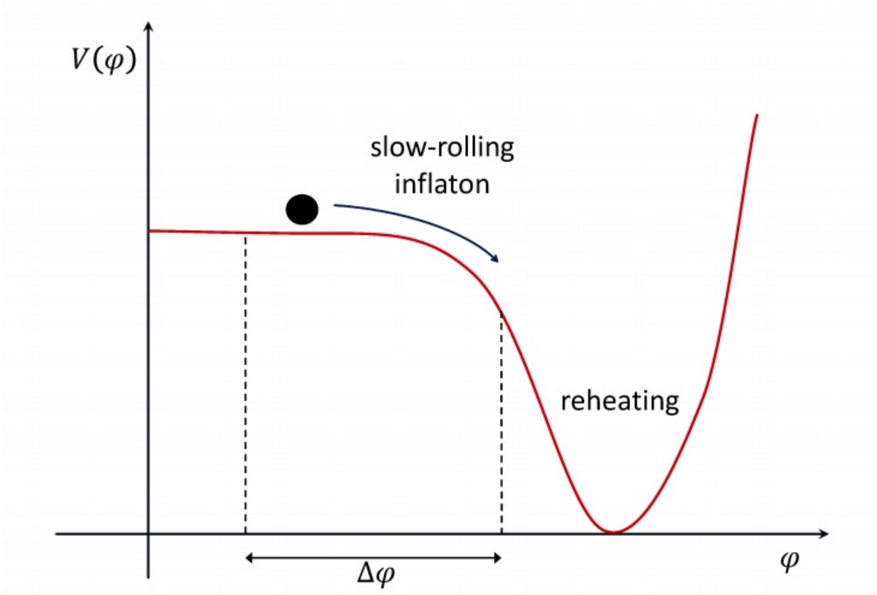

Figure 1.5: Example of inflationary potential with an almost flat region. While the inflaton slowly rolls on the flat region of the potential the slow-roll parameters respect the condition $\epsilon, |\eta| \ll 1$. When the inflaton arrives around the minimum of the potential these condition are violated and *reheating* starts [29], see Section 1.4.3.

In general the two fields $g_{\mu\nu}$ and ϕ will depend both on the space and time coordinates, but we have seen in the previous chapter that we observe an approximately homogeneous and isotropic Universe on large, cosmological, scales. These two properties can be translated, in the mathematical language, as the invariance of the metric under spatial translations and rotations (we will see how important are these two conditions), i.e., the (background) metric depends only on time. Given that the Inflation mechanism driven by ϕ has been introduced to explain the problem of initial conditions, also his background value, which we will indicate with ϕ_0, will depend only on time. For the metric, the background value is simply the Friedmann-Lemaitre-Robertson-Walker metric in (1.3), the solution for a homogeneous and isotropic Universe. Given this, we can split our fields, the metric and the inflaton, into two parts, the background value and the perturbation:

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t), \tag{1.76}$$

$$g_{\mu\nu}(\mathbf{x}, t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\mathbf{x}, t). \tag{1.77}$$

Both the perturbation will, in general, depend also on space coordinates.
This splitting is a good approximation in the case

$$\langle \delta\phi^2 \rangle \ll \phi_0^2(t) \tag{1.78}$$

and, from an observational point of view, we do expect that this property needs to be satisfied if we want to come up with a Universe where the temperature anisotropies of the CMB are very small $\Delta T/T \sim 10^{-5}$.

To understand the basics of the inflationary mechanism we begin studying the background dynamics, then we will add the (quantum) perturbations.

1.4.2 Background Dynamics of Inflation

The background dynamics is important because it describes the accelerated expansion during the inflationary period. In the following we will use the FLRW metric with null spatial curvature, i.e. the background solution for our Universe

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{1.79}$$

Our scalar field background will have the same properties, homogeneity and isotropy, so its stress-energy tensor will be of the form

$$T^{\phi}_{00} = \rho_{\phi_0}, \tag{1.80}$$

$$T^{\phi}_{ij} = p_{\phi_0} g_{ij}, \tag{1.81}$$

and using Eq. (1.75), we obtain

$$\rho_{\phi_0} = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0), \tag{1.82}$$

$$p_{\phi_0} = \frac{1}{2}\dot{\phi}_0^2 - V(\phi_0). \tag{1.83}$$

These results for the energy density and the pressure of the scalar field show us that it is possible to realize and inflationary epoch, thanks to the presence of the $-V$-term in the expression of the pressure. In fact, for certain values of the interaction potential it is possible to have a negative pressure satisfying the condition (1.62), i.e. accelerated expansion. The condition given in Eq. (1.62) translates, for our field, into

$$V > \dot{\phi}_0^2.$$

If we consider now a Universe dominated by the field ϕ with a potential much larger than its kinetic term

$$\frac{1}{2}\dot{\phi}_0^2 \ll V, \tag{1.84}$$

then from Eq. (1.82) and (1.83) we obtain

$$p_{\phi_0} \simeq -\rho_{\phi_0}. \tag{1.85}$$

In the previous chapter we derived the evolution equation for the scale factor a in the case of an exact equality in the previous expression: it represents the solution of a quasi de Sitter Universe. In our case, Eq (1.85) represents a quasi-de Sitter Universe with accelerated and nearly exponential expansion

$$a(t) \simeq a_0 e^{Ht}.$$

What does Eq. (1.84) represent from a dynamical point of view? It is telling to us that the field is moving very slowly with respect to its potential: this kind of motion is called *slow-roll* of the scalar field. We can see that this condition corresponds to have an almost flat potential for all the duration of Inflation. If we substitute Eq. (1.84) into Eq. (1.24) we obtain

$$H^2 = \frac{8}{3}\pi G\rho_\phi \simeq \frac{8}{3}\pi GV(\phi) \simeq const \tag{1.86}$$

where the last equalities holds for a quasi-de Sitter Universe.

We look now the evolution of the background of the scalar field: the equation of motion for a scalar field with a potential term is described by the Klein-Gordon equation

$$\Box\,\phi = \frac{\partial V}{\partial \phi}, \tag{1.87}$$

where, for a curved Universe, the box operator takes the form [19]

$$\Box\,\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right). \tag{1.88}$$

We obtain, after some trivial calculations, the equation of motion of $\phi_0(t)$

$$\ddot{\phi}_0 + 3\frac{\dot{a}}{a}\dot{\phi}_0 = -\frac{\partial V}{\partial \phi}. \tag{1.89}$$

The term $H\dot{\phi}_0$ is the true difference between the Minkowski and the FLRW metric: it takes into account the expansion of the Universe and represents a sort of friction term. Summing up, we have seen that a scalar field is sufficient to generate an inflationary epoch, but we need to constrain its potential (essentially its auto-interactions); now we want to define some model-independent parameters useful to describe, in a general way, the various models of Inflation. Then we will analyze also the perturbations of the scalar field.

We have just seen that the minimal condition we need to have cosmic Inflation is to have a potential term much larger than the kinetic one. In Eq. (1.86) it is fundamental to have an approximate equality, i.e. to have a potential that is *almost* constant, but not exactly constant. This ensures that the inflationary epoch will have an end. Eq. (1.86) also shows that the Hubble parameter must remain almost constant while the nearly exponential expansion. The variation of a generic quantity f in an expanding Universe with scale factor $a(t)$ can be estimated by the parameter

$$\epsilon_f = \frac{d\log f}{d\log a} = \frac{\dot{f}}{Hf}, \tag{1.90}$$

which gives a measure of the variation of f with respect the expansion of the Universe. Therefore we can define a similar parameter for the Hubble parameter, defining the first slow-roll parameter using (1.90)

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \tag{1.91}$$

Remind that the expression for the Hubble parameter $H = \dot{a}/a$ so its variation w.r.t. time is given by

$$\dot{H} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a} - H^2,$$

hence

$$\ddot{a} = aH^2\left(1 + \frac{\dot{H}}{H^2}\right).$$

Imposing an accelerated phase expansion we obtain:

$$\ddot{a} > 0 \Rightarrow -\frac{\dot{H}}{H^2} < 1$$

i.e., the Hubble parameter must vary slower than its squared value. For a quasi-de Sitter we want

$$-\frac{\dot{H}}{H^2} \ll 1$$

i.e.

$$\epsilon \ll 1. \tag{1.92}$$

This condition has been found without constraining our theory of Inflation or the form of the potential. However we will see that (1.92) could be translated into a condition for the potential of the scalar field using the Friedmann equations. As a first insight one can easily see that, deriving with respect to time Eq. (1.24) with Eq. (1.82) and Eq. (1.89), condition (1.92) corresponds automatically to the condition (1.84). The other request we need to satisfy is about the duration of Inflation. We have seen above that the horizon problem (but also the flatness problem) imposes to Inflation to last long enough to allow the causal connection of an area of the Universe large enough to justify what we see today. This means that the Hubble parameter must stay constant for at least the necessary time for the Universe to expand of a number of e-folds around $N \sim 60-70$. This is equivalent to ask that ϵ varies very slowly during the inflationary period. If we define $dN = d\ln a = Hdt$, we see that we can define the parameter

$$\eta \equiv \frac{d\ln\epsilon}{dN} = \frac{\dot{\epsilon}}{H\epsilon}. \tag{1.93}$$

For this parameter we will ask again

$$\eta \ll 1. \tag{1.94}$$

It is easy to see that this condition corresponds to [28]

$$\ddot{\phi} \ll 3H\dot{\phi}$$

Eq. (1.92) and (1.94) are the so called slow-roll conditions.

Related to the parameters above we can also define a new couple of parameters ϵ_V and η_V that take into account the form of the potential of the theory [30, 31]

$$\epsilon_V = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 \tag{1.95}$$

$$\eta_V = M_{pl}^2\frac{V''}{V}, \tag{1.96}$$

that can be rewritten also as

$$\eta_V = \frac{1}{3}\frac{V''}{H^2}. \tag{1.97}$$

Here with $'$ we mean a derivation of the potential w.r.t the scalar field $V' \equiv \frac{\partial V}{\partial\phi}$. In the slow-roll approximation it is easy to demonstrate that

$$\epsilon \simeq \epsilon_V, \quad \eta \simeq \eta_V - \epsilon_V. \tag{1.98}$$

Then, ϵ and η are adimensional parameters that will be different between the various theories because they depend on the kind of potential chosen. Eq. (1.95) and (1.96) show explicitly that the potential of the scalar field has to be rather flat relative to its height [32]. We will see, in particular when studying the theories of solid inflation, and similar ones (which are the focus of this Thesis), that the definitions (1.91) and (1.93) are more general. Indeed in other theories these two definitions, with their conditions, will ensure the possibility to have an inflationary period long enough [32].

The experimental bounds on the slow-roll parameters given by the Planck satellite are [21]

$$\epsilon_V < 0.0097 \text{ (95\%CL)}, \tag{1.99}$$

$$\eta_V = -0.010^{+0.0078}_{-0.0072} \text{ (68\%CL)}. \tag{1.100}$$

1.4.3 Reheating

What happened when the Inflation ended? We know that the observable results are in good agreement with the predictions given by the standard Hot Big Bang Model. Therefore we need a mechanism that allow the transition from the inflationary period to the standard FLRW Universe dominated by the radiation. This period of transition is the so-called *reheating*.

We know that Inflation holds as long as $\epsilon, |\eta| \ll 1$, so Inflation ends when $\epsilon, |\eta| \to 1$, when the potential that drives Inflation is not much flat as before. If the potential is of the form given in Fig. 1.5, we have that at the point of minimum σ for the potential it will acquire a mass and starts oscillating,

$$V'(\phi) \simeq V'(\sigma) + V''(\sigma)\phi = 0,$$

with $V'(\sigma) = 0$. The equation of motion for the scalar field becomes

$$\ddot{\phi} + 3H\dot{\phi} + V''(\sigma)\phi = 0.$$

This is the standard evolution equation for an oscillating field in an expanding Universe. A possible reheating mechanism is the following: while oscillating the field starts decaying into light an relativistic particles, giving rise to the standard radiation dominated epoch. To the evolution equation we need to add a term which takes into account the decays of the scalar field

$$\ddot{\phi} + (3H + \Gamma_\phi)\dot{\phi} + V'(\phi) = 0, \tag{1.101}$$

where Γ_ϕ is the decay rate of the inflaton. This expression can be manipulated and becomes [1]

$$\dot{\rho}_\phi + (3H + \Gamma_\phi)\rho_\phi = 0. \tag{1.102}$$

Inside Γ_ϕ there are information about the model adopted for the Inflation and the energies at which it occurs. These will not be described here. Eventually, the inflationary energy density is converted into standard model degrees of freedom and the Hot Big Bang starts.

1.4.4 A digression on the charm of the Inflation model

We conclude this chapter outlining a fundamental principle, which we will not completely describe, but we will mention it for its importance.

We have seen the problems arising in the Big Bang theory are solved by the slow-roll Inflation paradigm. We have found a dynamical process which explains the homogeneity, the isotropy and the flatness of our Universe, starting from the more general condition of anisotropy or curved Universe. We can easily see what we stated above by considering the Friedmann equation for the Hubble parameter

$$H^2 = \frac{8\pi G}{3}(\rho_\phi + \rho_r) - \frac{K}{a}^2.$$

Here we are also considering the contribution to the energy density due to the radiation and the term which takes into account the possible curvature of the Universe. During Inflation we have seen that the scalar field mimics an cosmological constant, so its energy density is essentially not diluted by the expansion of the Universe, while the radiation energy density and the curvature term have a behavior $\rho_r \propto a^{-4}$ and $K/a^2 \propto a^{-2}$. If we start with an expanding Universe, at a certain moment the inflaton will dominate over all the possible fluid components of the Universe.

This means that, starting from generic conditions, the Universe is *attracted* to an inflationary epoch: except for the scalar field, all the other components are wiped out. This is a qualitative formulation of the *cosmic no-hair theorem* [33, 34, 1].
The cosmic no-hair theorem guarantees that also anisotropic and inhomogeneous initial conditions will go through the homogeneous and isotropic solution, which therefore justifies in the previous sections the use of a FLRW metric. We can take, for example, a homogeneous but not isotropic Universe[6]. The metric in this case could take the form

$$ds^2 = -dt^2 + a_x^2(t)dx^2 + a_y^2(t)dy^2 + a_z^2(t)dz^2$$

with $a_x(t) \neq a_y(t) \neq a_z(t)$. In this solution we have a Universe that expands differently along the three directions x,y and z. It is possible to define a mean scale factor

$$\bar{a} = (a_x a_y a_z)^{1/3} = V^{1/3},$$

and we can write an equation analogous to the Friedmann equation

$$H^2 = \left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 = \frac{1}{2}\left(\frac{\dot{V}}{V}\right) = \frac{8\pi G}{3}(\rho_\phi + \rho_m + \rho_r + \dots) + F(a_x, a_y, a_z).$$

The dots inside the parenthesis denotes all the possible contribution to the energy density. The function F accounts for the effect of the anisotropic expansion on the mean expansion rate. This function can be, in general, very complicated, but its crucial feature is that for *all* the Bianchi models it scales at least as fast as $\bar{a}^{-2}$ [33]. Again, if the energy terms scales at most as $\bar{a}^{-2}$, like radiation ($\rho_r \propto a^{-4}$) and pressure-less matter ($\rho_r \propto a^{-3}$), at a certain moment the contribution of the inflaton will dominate, so the Inflation starts and all the anisotropies are wiped out because $\bar{a} \propto e^{Ht}$.
Wald studied a Universe dominated by an exact cosmological constant $\rho_\phi = \Lambda$ in presence of ordinary matter: under this condition the cosmic no-hair theorem is called *Wald theorem* [34]. It affirms that all the Bianchi models, except for the IX Bianchi Universe, with a positive cosmological constant become asymptotically a de Sitter Universe in a time $\tau_W \sim \left(\frac{8\pi G\Lambda}{3}\right)^{-1/2}$. For a more general version of the cosmic no-hair theorem see [35].

[6]These are the so-called *Bianchi models*.

Chapter 2

Cosmological perturbations

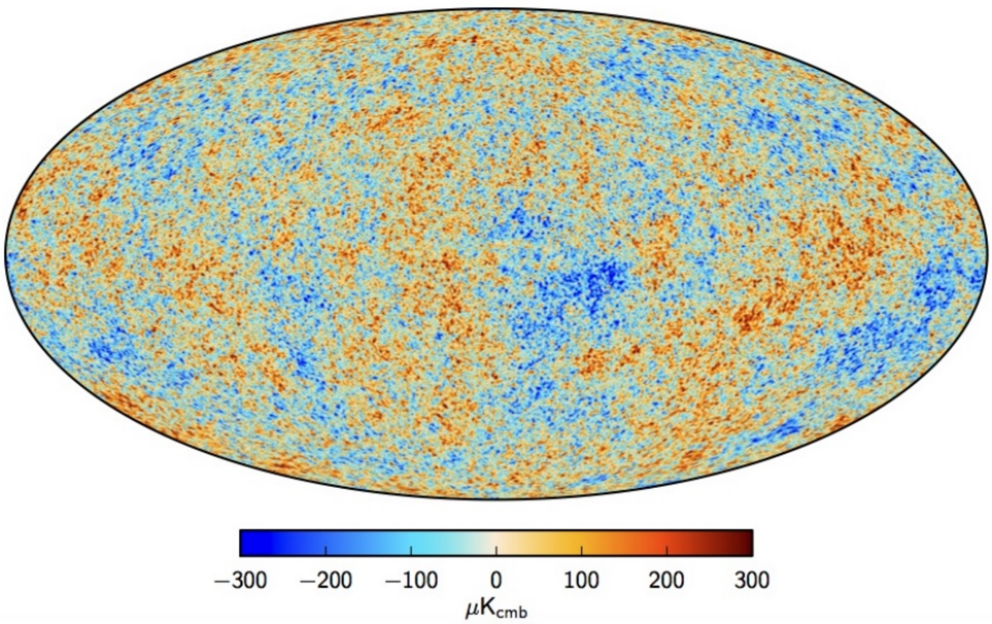

Figure 2.1: This is a photo of our Universe when it was almost 380000 years old. We see that, despite the assumption of homogeneity and isotropy, it presents small perturbations. This is an image based on the temperature measurements of the CMB, showing small anisotropies of the order $\frac{\Delta T}{T} \sim 10^{-5}$. These small perturbations are the seeds for the formation of the Large Scale Structure of the Universe [36].

The measurements on CMB show us a Universe which is neither homogeneous nor isotropic. It presents, indeed, small fluctuations in temperature, which are strictly connected with the primordial density perturbation [28]. This observational results does not surprise: the Universe as can be observed from anyone is not isotropic. Looking at the sky we can see structures of different shape, size and complexity, such as planets, stars, galaxies, clusters of galaxies and so on. The observed perturbation pattern would explain the inhomogeneities we easily see: more precisely it would explain the structure of the Universe at Large Scales. However, in the Chapter 1 we have seen that the inflationary model is a natural attractor mechanism: regardless the initial conditions of the Universe, after the Inflation it will be well described by (1.1) with null spatial curvature. An obvious question arises: which is the cause of the primordial fluctuations we observe in the CMB? The full treatise we gave in the previous chapter solves the Big Bang puzzles, but what about the perturbations? In this chapter we will see that the most powerful prediction of inflationary model is the production of perturbations due to (quantum) oscillations of the scalar field around its background value. Here we will study the perturbation theory in the context of General Relativity, perturbing both the scalar field and the metric, in accordance with the Einstein field equations. We will then calculate what are the possible observable predictions.

2.1 Perturbation theory in General Relativity

We know that the perturbations of the scalar field are connected with the density perturbation of the cosmic fluid [28]. Recall that the Einstein equations,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

affirm that a perturbation in the right-hand side, i.e. a perturbation in the stress-energy tensor, entails a perturbation in the left-hand side, i.e. the metric, and vice versa. This means that perturbing the scalar field implies that we also need to consider the metric perturbations. We can see that the case we are interested in necessarily needs a general relativistic approach: we will describe of to extend the Newtonian perturbative approach (Jeans instability) [37] to a more general case. In which limit it is allowed to use the non relativistic approach without doing large mistakes? To evaluate it we consider the linearly perturbed Poisson equation for the gravitational potential

$$\nabla^2\varphi = 4\pi G\delta\rho = 4\pi G\bar{\rho}\delta \tag{2.1}$$

with $\delta \equiv \delta\rho/\bar{\rho}$ and $\bar{\rho}$ is the mean value of the matter density. Here φ denotes the perturbation of the gravitational potential [5]. Using the Friedmann equation

$$H^2 = \frac{8}{3}\pi G\bar{\rho}$$

we obtain

$$\nabla^2\varphi = \frac{3}{2}H^2\delta.$$

We can define $\lambda_H \equiv H^{-1}$ as the physical Hubble radius (or length, or wavelength). If we indicate the characteristic length of the gravitational potential perturbation, i.e. the length in which the gravitational field varies significantly, with λ, to the first order we have:

$$\frac{\varphi}{c^2} \sim \left(\frac{\lambda}{\lambda_H}\right)^2\delta. \tag{2.2}$$

Note that here the light speed, c, has been explicated. It is necessary a relativistic approach in the case $\varphi/c^2 \sim 1$. From Eq. (2.2) we can extrapolate the conditions in which it is necessary to use the Einstein equations. We know that $\delta \sim 10^{-5}$ [36], hence there are three possibilities:

- $\lambda = \lambda_H$, i.e. when we consider scales of the order of the Hubble horizon;
- $\lambda \ll \lambda_H$, i.e. scales much smaller than the horizon;
- $\lambda > \lambda_H$, i.e. scales larger than the Hubble horizon.

The first two cases imply $\varphi/c^2 \ll 1$, so one can easily use the Newtonian approach in these two limits. In particular, in the second one, one can use the Newtonian approach both in the case of linear perturbation $\delta \lesssim 1$ and in the case of non-linear perturbation $\delta \gtrsim 1$, i.e. when the amplitude of the perturbation has the same order of the density mean value.
The third case is the most interesting. When we consider scales higher than the Hubble horizon we necessarily need to use the general relativistic approach. In particular we are interested in this case because the CMB is part of this case. We will give a treatise of perturbation theory in General Relativity.

Throughout this work we are assuming that our observable Universe can be approximately be described by a homogeneous and isotropic FLRW spacetime. Thus we are assuming that Eqs. (1.76) and (1.77) are valid, i.e. that the metric and the scalar field can usefully be decomposed into a homogeneous background (which dynamics we have already described) and inhomogeneous perturbations. The perturbations thus live on the background spacetime and it is this background spacetime which is used to split four-dimensional spacetime into spatial three-hypersurfaces, using a (3+1) decomposition [38, 39].
We start defining arbitrary perturbations of tensorial quantities and then proceed by decomposing vectors and tensors in "time" and "space" parts on the spatial hypersurfaces.

Any tensorial quantity, of our interest, can be split into a homogeneous background and an inhomogeneous perturbation

$$T(t,\mathbf{x}) = T_0(t) + \delta T(t,\mathbf{x}). \tag{2.3}$$

Here $T_0(t)$ represents the value of the tensor in the background spacetime (FLRW), $\mathcal{M}_0$, depending only on t for obvious reasons. T is the tensor in the perturbed spacetime $\mathcal{M}_\lambda$. The perturbation can be further expanded as a power series

$$\delta T(t,\mathbf{x}) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \delta T_n(t,\mathbf{x}), \tag{2.4}$$

where the subscript n denotes the order of the perturbations, and we explicitly include here the small parameter λ. In linear perturbation theory, for example, we only consider first-order terms, λ^1, and can neglect terms resulting from the product of two perturbations, which would necessarily be of order λ^2 or higher, which considerably simplifies the resulting equations. In the following sections we shall omit the small parameter λ whenever possible, as it is usually done to avoid the equations getting too cluttered.
Because the perturbations are small, i.e. $|\delta T| \ll |T_0|$, expanding the Einstein equations at linear order in perturbations approximates the full non-linear solution to very high accuracy

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \tag{2.5}$$

We will use these perturbed Einstein's equations to study the dynamics of the perturbations.

2.1.1 Gauge problem

We need now to consider a peculiarity arising in General Relativity. If we rewrite Eq. (2.4) as

$$\delta T = T - T_0,$$

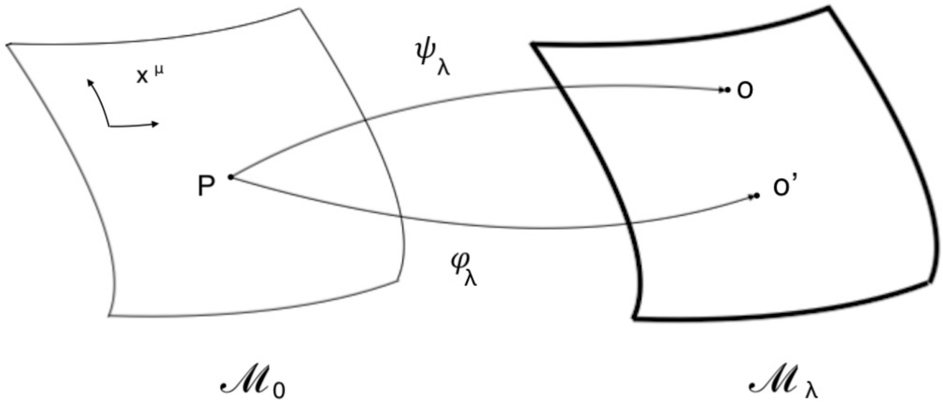

Figure 2.2: Passive approach in perturbation theory. The point P is mapped into two different point of the perturbed space through the definition of two different maps [38].

we see that the perturbation is defined, as usual, as the difference between the tensor in the perturbed spacetime and the background one. In Generale Relativity when we compare two tensors they need to be evaluated at the same point; here, instead, we are comparing two tensors which lie in two different spacetimes. It is necessary to connect these two spacetimes using a map, a diffeomorphism [38], which goes from the unperturbed spacetime $\mathcal{M}_0$ to $\mathcal{M}_\lambda$. We call this map ψ_λ

$$\begin{aligned} \psi_\lambda : \quad \mathcal{M}_0 &\longrightarrow \mathcal{M}_\lambda \\ \psi_\lambda(P) &\longrightarrow O \end{aligned} \tag{2.6}$$

where $P \in \mathcal{M}_0$ and $O \in \mathcal{M}_\lambda$, the meaning of the subscript λ will be clear further. It is obvious that ψ_λ is not the unique way one can use to go from $\mathcal{M}_0$ to $\mathcal{M}_\lambda$, but, as can be seen in the Fig. 2.2, one can choose a new map, φ_λ for example. A particular choice of the correspondence between $\mathcal{M}$ and $\mathcal{M}_\lambda$ is a *gauge choice*[1]. A change in the used map is what we usually call a *gauge transformation*. The liberty in the choose of the gauge is very useful tool: we can choose the gauge we prefer, to simplify the calculations or highlight different physical features. But it is not so easy. A consequence of this freedom in the choice is that changing the gauge entails different representations for the tensor T on the spacetime $\mathcal{M}_0$. If T and $\tilde{T}$ are two different representations of a tensor due to different gauge choices we have

$$\delta T = T - T_0$$
$$\delta \tilde{T} = \tilde{T} - T_0,$$

and, in general we will have $\delta\tilde{T} \neq \delta T$. This means that changing the gauge correspond to a variation in the perturbation side. This feature is known under the name of *gauge problem*. Why it is a problem? To understand it we follow [15].
Choosing a particular gauge could lead us to make a mistaken when studying perturbations. In fact the freedom of choice translates in an explicit "gauge dependence" of the perturbations. To demonstrate it we can consider an unperturbed homogeneous and isotropic Universe (as our background is). We have seen that the energy density in this

[1][40] stresses that the word *gauge* has a lot of different meanings, depending on the discipline, such as physics or mathematics, or the approach used. Here and further we will use the definition given above, without ambiguity.

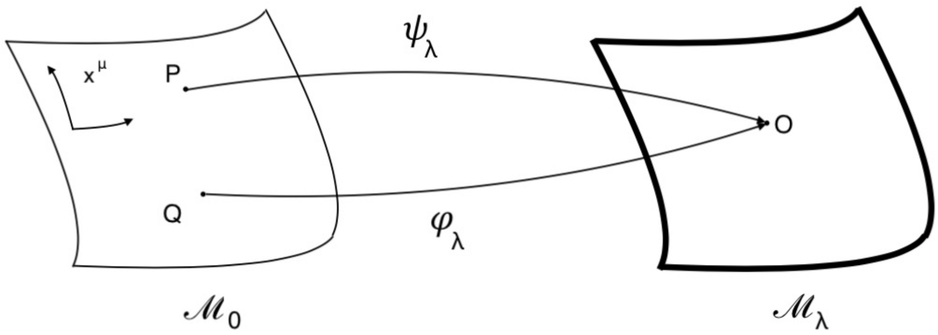

Figure 2.3: Active approach in perturbation theory. Two different points P of the starting spacetime are mapped into the same point O using, again, two different maps [38].

Universe is only a function of cosmic time $\rho(t, \mathbf{x}) = \rho(t)$. We can consider now a new time coordinate $\tilde{t} = t + \delta t(t, \mathbf{x})$. In general, the energy density in the new time slice will not be homogeneous, due to the $\mathbf{x}$ dependence of the new coordinate, $\tilde{\rho}(\tilde{t}, \mathbf{x}) = \rho(t(\tilde{t}, \mathbf{x}))$. The inhomogeneity of the density function implies the presence of new density perturbations, but our hypothesis was homogeneity and isotropy! These new *fictitious* perturbations in the energy density aren't indeed physical, but entirely due to the choice of new time coordinate. Similarly, we can remove a *real* perturbation in the energy density by choosing the hypersurface of constant time to coincide with the hypersurface of constant energy density[2]. Then the perturbation in the new coordinates disappear, $\delta\tilde{\rho} = 0$, although they are real inhomogeneities. To avoid this ambiguity we need to consider both the metric perturbation and the matter-energy perturbation. For example, choosing a gauge where metric perturbation are reabsorbed will make appear new perturbations in the matter-energy side and vice versa. Otherwise, if this does not happen we are sure that the perturbation are not real, but only due to our gauge choice. We will see that the standard approach to avoid such an ambiguity is to identify combinations of perturbations which are gauge-invariant quantities, that is to say quantities which are independent of the gauge transformation.

2.1.2 Gauge transformations

It is important to stress that a gauge transformation is different from a coordinate transformation. To understand it we can consider the background spacetime $\mathcal{M}_0$, supposing we have fixed the coordinate system x^μ. Fixing a gauge ψ_λ, i.e. a map between points means bringing into $\mathcal{M}_\lambda$ the coordinate fixed in $\mathcal{M}_0$. A gauge transformation corresponds to a different choice of the map $\mathcal{M}_0 \longrightarrow \mathcal{M}_\lambda$ once the background coordinates are fixed. A coordinates transformation affects both the background spacetime and the perturbed one.

Looking at Fig. 2.2 we can affirm that: ψ_λ identifies the point $P \in \mathcal{M}_0$ with the point $O \in \mathcal{M}_\lambda$ and will assign to O the same coordinate system x^μ. A new map (a new gauge choice) φ_λ will identify the same starting point P the point $O' \in \mathcal{M}_\lambda$, assigning to O' the coordinates of P. In this view, a gauge transformation corresponds to a coordinate change in the perturbed space $\mathcal{M}_\lambda$. This interpretation is called *passive view* [41]. To make it more clear we can proceed as follow. Given that the maps used are diffeomorphisms, they must be invertible: thus we can define the inverse map ψ_λ^{-1} as

[2]The choice of a particular gauge corresponds to the choice of spacelike hypersurfaces constant time t, *slicing* of spacetime, and of the timelike worldlines of constant $\mathbf{x}$, the *threading* of spacetime [15, 38, 39].

$$\psi_\lambda^{-1}: \quad \mathcal{M}_\lambda \longrightarrow \mathcal{M}_0$$
$$\psi_\lambda^{-1}(O) \longrightarrow P. \tag{2.7}$$

Given also that

$$\varphi_\lambda: \quad \mathcal{M}_0 \longrightarrow \mathcal{M}_\lambda$$
$$\varphi_\lambda(P) \longrightarrow O' \tag{2.8}$$

we can define the composition of this two maps, Ψ_λ, as a map that brings a point in $\mathcal{M}_\lambda$ to another point of the same spacetime: it can be interpreted as change of coordinates on $\mathcal{M}_\lambda$,

$$\Psi_\lambda \equiv \psi_\lambda^{-1} \circ \varphi_\lambda: \quad \mathcal{M}_\lambda \longrightarrow \mathcal{M}_\lambda$$
$$O \longrightarrow O', \tag{2.9}$$

with $\Psi_\lambda(O) = \varphi_\lambda\left(\psi_\lambda^{-1}(O)\right) = O'$.

For the *active view* we choose instead a point $O \in \mathcal{M}_\lambda$ and find the point P on $\mathcal{M}_0$ which maps to O under the gauge choice ψ_λ and the point Q, also on $\mathcal{M}_0$, which maps to O under the gauge choice φ_λ, see Fig. 2.3, $O = \psi_\lambda(P) = \varphi_\lambda(Q)$. With an analogous procedure, once fixed the coordinates x^μ on $\mathcal{M}_0$, we can construct a map from $\mathcal{M}_0$ into itself:

$$\Phi_\lambda \equiv \varphi_\lambda^{-1} \circ \psi_\lambda: \quad \mathcal{M}_0 \longrightarrow \mathcal{M}_0$$
$$P \longrightarrow Q \tag{2.10}$$

with $\Phi_\lambda(P) = \varphi_\lambda\left(\varphi_\lambda^{-1}(P)\right) = Q$. Again this gauge transformation can be interpreted as a univocal correspondence between different points of the background spacetime. Φ_λ is also called *infinitesimal point transformation*, see Fig. 2.6. In both the approaches we haven't changed the coordinates, which are still x^μ, here we are considering two different points of the same spacetime connected by a gauge transformation.

This allows us to calculate what are the tensor[3] transformation laws under a gauge

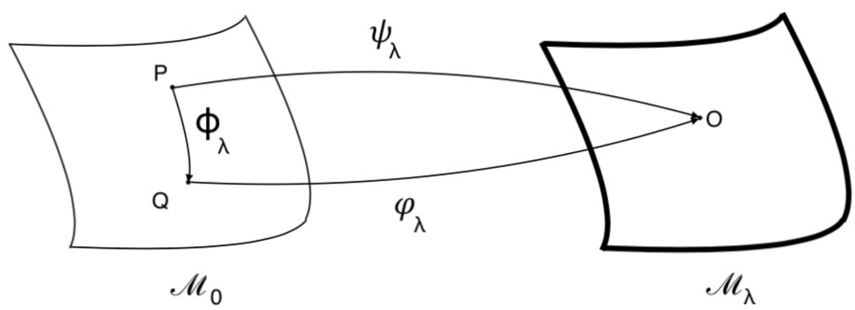

Figure 2.4: Gauge transformation in the active approach. Through the mapping Φ_λ it is possible to connect two different points of the same spacetime [38].

[3]For *tensor* here we indicate the most general mathematical object we will consider in our future calculations. Further we will write the transformation law for scalars, vectors and tensors.

transformation, in order to identify what are the possible gauge-invariants quantities we can define.

Consider now the coordinate system x^μ on $\mathcal{M}_0$ and the four-vector ξ^μ, which defines the parametric curve, through the parameter λ,

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \xi^\mu, \tag{2.11}$$

such that, given $P \in \mathcal{M}_0$, $x^\mu(\lambda = 0) = P$. Consider the point $Q \in \mathcal{M}_0$ at a parametric distance λ from P along the defined curve. P will simply have coordinates $x^\mu(P)$, while $x^\mu(Q)$ will be the coordinates of Q[4]. At the first order in the parameter λ we have

$$x^\mu(Q) = x^\mu(P) + \lambda\xi^\mu(x(P)) + o(\lambda^2). \tag{2.12}$$

This is a direct way to construct the map Φ_λ defined above a an infinitesimal point transformation, or active transformation, since we are working in the active approach. At the zeroth order one has $x^\mu(P) \simeq x(Q)$, hence $\xi^\mu(x(P)) \simeq \xi^\mu(x(Q))$, therefore we can write

$$x^\mu(P) = x^\mu(Q) - \lambda\xi^\mu(x(Q)). \tag{2.13}$$

We see that Eq. (2.13) recalls an usual (passive) coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x^\mu - \xi^\mu \quad (\lambda = 1),$$

where the new coordinate x' is represented by the P's coordinates. Hence we can introduce a new coordinate system[5] y^μ in which Q has coordinates

$$y^\mu \equiv x^\mu(P) \longrightarrow y^\mu(Q) = x^\mu(Q) - \lambda\xi^\mu(Q).$$

The choice of the point Q is completely arbitrary, so this relation must hold for each point in the spacetime $\mathcal{M}_0$

$$y^\mu(\lambda) = x^\mu - \lambda\xi^\mu. \tag{2.14}$$

We are now ready to write the general transformation law for a tensor under a gauge transformation. Given a tensor with m contravariant indices and n covariant indices on the space $\mathcal{M}_0$ with coordinates system x^μ, $T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x(P))$, we know that after a gauge transformation it will take the form $\tilde{T}^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x(P))$. Can we relate T with $\tilde{T}$? In the passive approach we have seen that the transformed tensor is not more than the starting tensor but evaluated in another point of the spacetime. We have defined the new coordinate system as $x^\mu(P) = y^\mu(Q)$, hence we can write

$$\begin{aligned}\tilde{T}^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x(P)) &= T'^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(y(Q)) \\ &= \frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial y^{\mu_m}}{\partial x^{m}}\frac{\partial x^{\beta_1}}{\partial y^{\nu_1}}\cdots\frac{\partial x^{\beta_n}}{\partial y^{\nu_n}}T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}(x(Q)).\end{aligned} \tag{2.15}$$

The first equality holds because we are going from the active view to the passive one; here $'$ indicates the tensor T in the new coordinate system. The second equality is the general transformation law for a tensor under a coordinates transformation[5]. For Eq. (2.14) we have

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu - \lambda\partial_\nu\xi^\mu$$

[4] We underline, again, that the coordinates of P and Q are defined in the same coordinate system.

[5] Here we are finally acting with a coordinates transformation.

and

$$\frac{\partial x^\mu}{\partial y^\nu} = \delta^\mu_\nu + \lambda \partial_\nu \xi^\mu,$$

because at the zeroth order $x^\mu \simeq y^\mu$. Eq. (2.15) becomes, at the first order in λ

$$\begin{aligned}
\tilde{T}^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}(x(P)) &= \left(\delta^{\mu_1}_{\alpha_1} - \lambda \partial_{\alpha_1} \xi^{\mu_1}\right) \dots \left(\delta^{\mu_m}_{\alpha_m} - \lambda \partial_{\alpha_m} \xi^{\mu_m}\right) \\
&\quad \left(\delta^{\beta_1}_{\nu_1} + \lambda \partial_{\nu_1} \xi^{\beta_1}\right) \dots \left(\delta^{\beta_n}_{\nu_n} + \lambda \partial_{\nu_n} \xi^{\beta_n}\right) T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}(x(Q)) \\
&= T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}(x(Q)) \\
&\quad - \lambda \left(\partial_{\alpha_1} \xi^{\mu_1} T^{\alpha_1 \mu_2 \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \dots + \partial_{\alpha_m} \xi^{\mu_m} T^{\mu_1 \dots \mu_{m-1} \alpha_m}{}_{\nu_1 \dots \nu_n}\right) \\
&\quad + \lambda \left(\partial_{\nu_1} \xi^{\beta_1} T^{\mu_1 \dots \mu_m}{}_{\beta_1 \nu_2 \dots \nu_n} + \dots + \partial_{\nu_n} \xi^{\beta_n} T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_{n-1} \beta_n}\right).
\end{aligned} \tag{2.16}$$

We can now expand in Taylor series $T(x(Q))$ around the point P using Eq. (2.12)[6]:

$$T(x(Q)) \simeq T(x(P)) + \lambda \xi^\nu \partial_\nu T. \tag{2.17}$$

Inserting Eq. (2.17) into Eq. (2.16) we have

$$\begin{aligned}
\tilde{T}^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}(x(P)) &= T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}(x(P)) + \lambda \xi^\rho \partial_\rho T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} \\
&\quad - \lambda \left(\partial_{\alpha_1} \xi^{\mu_1} T^{\alpha_1 \mu_2 \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \dots + \partial_{\alpha_m} \xi^{\mu_m} T^{\mu_1 \dots \mu_{m-1} \alpha_m}{}_{\nu_1 \dots \nu_n}\right) \\
&\quad + \lambda \left(\partial_{\nu_1} \xi^{\beta_1} T^{\mu_1 \dots \mu_m}{}_{\beta_1 \nu_2 \dots \nu_n} + \dots + \partial_{\nu_n} \xi^{\beta_n} T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_{n-1} \beta_n}\right).
\end{aligned} \tag{2.18}$$

An expert eye can easily recognize that the all the terms after the first in the right hand side of Eq. (2.18) is the explicit expression of the *Lie derivative* of the tensor T along the vector field that generates the gauge transformation ξ^μ[5]:

$$\begin{aligned}
(\mathcal{L}_\xi T)^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} &= \xi^\rho \partial_\rho T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} \\
&\quad - \left(\partial_{\alpha_1} \xi^{\mu_1}\right) T^{\alpha_1 \mu_2 \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \dots + \left(\partial_{\alpha_m} \xi^{\mu_m}\right) T^{\mu_1 \dots \mu_{m-1} \alpha_m}{}_{\nu_1 \dots \nu_n} \\
&\quad + \left(\partial_{\nu_1} \xi^{\beta_1}\right) T^{\mu_1 \dots \mu_m}{}_{\beta_1 \nu_2 \dots \nu_n} + \dots + \left(\partial_{\nu_n} \xi^{\beta_n}\right) T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_{n-1} \beta_n}.
\end{aligned} \tag{2.19}$$

All the quantities in Eq. (2.18) are evaluated at the same point P, which is arbitrary, hence

$$\tilde{T}^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} + \lambda \left(\mathcal{L}_\xi T\right)^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}, \tag{2.20}$$

for each point. Eq. (2.20) is the transformation law of a general tensor under a change of the gauge. More concisely it can be written as

$$\tilde{T} = T + \lambda \mathcal{L}_\xi T. \tag{2.21}$$

This equation is valid at the linear order, i.e. $\sim o(\xi) \sim o(\lambda)$). It is possible to extend this formula at higher order in ξ^μ, for example

$$\tilde{T} = \left[\exp\left(\lambda \mathcal{L}_\xi\right) T\right] = T + \lambda \mathcal{L}_\xi T + \frac{\lambda^2}{2} \mathcal{L}^2_\xi T + \dots$$

Here we need also to consider the second order expansion of the four-vector ξ

$$\xi^\mu = \sum_{r=1}^{\infty} \frac{\xi^\mu_r}{r!} = \xi^\mu_{(1)} + \frac{1}{2} \xi^\mu_{(2)} + \dots$$

[6]Here we are omitting, without loss of generality, all the indices.

All the calculations we will perform will be at the linear order in the perturbations, for a complete study of the perturbations at the second order look e.g. [38, 39, 42]. From now on we will consider only the $r = 1$ case.
We conclude this section writing the transformation law for the tensor perturbation. We have seen that a tensor can be divided into its background value plus a perturbation:

$$T = T_0 + \delta T, \tag{2.22}$$

$$\tilde{T} = T_0 + \delta \tilde{T}. \tag{2.23}$$

Using Eq. (2.21) we obtain, at the linear order

$$\delta \tilde{T} = \delta T + \mathcal{L}_\xi T_0. \tag{2.24}$$

We are now ready to study the cosmological perturbations.

2.1.3 Perturbed metric

Using the conformal time $d\tau = dt/a$, we can perturb the FLRW metric, using the following decomposition [15, 41]

$$g_{00} = -a^2(\tau)\left[1 + 2\sum_{r=1}^{\infty} \Psi^{(r)}(\tau, \mathbf{x})\right] = -a^2(\tau)\left[1 + 2\Psi(\tau, \mathbf{x})\right], \tag{2.25}$$

$$g_{0i} = g_{i0} = a^2(\tau)\sum_{r=1}^{\infty} \frac{\omega_i^{(r)}(\tau, \mathbf{x})}{r!} = a^2(\tau)\omega_i(\tau, \mathbf{x}), \tag{2.26}$$

$$\begin{aligned} g_{ij} &= a^2(\tau)\left\{\left[1 - 2\sum_{r=1}^{\infty} \frac{\Phi^{(r)}(\tau, \mathbf{x})}{r!}\right]\delta_{ij} + \sum_{r=1}^{\infty} \frac{\gamma_{ij}^{(r)}(\tau, \mathbf{x})}{r!}\right\} \\ &= a^2(\tau)\left[\left(1 - 2\Phi(\tau, \mathbf{x})\right)\delta_{ij} + \gamma_{ij}(\tau, \mathbf{x})\right]. \end{aligned} \tag{2.27}$$

Here we can easily recognize the background metric Eq.(1.4), the functions $\Psi^{(r)}$, $\omega^{(r)}$, $\Phi^{(r)}$ and $\gamma_{ij}^{(r)}$ represent the rth-order perturbations of the metric. In particular $\gamma_{ij}^{(r)}$ is a transverse and traceless tensor:

$$\partial^i \gamma_{ij} = \gamma_i{}^i = 0. \tag{2.28}$$

As we said previously, we will consider only the linear case with $r = 1$. It is useful to decompose these functions using the properties of the background metric. The symmetries of the FLRW metric allow a decomposition of the metric and the stress-energy perturbations into independent scalar (S), vector (V) and tensor (T) degrees of freedom, i.e. into objects with well-defined transformation under spatial rotations [29, 15]. This is useful because, at the linear order, the dynamics of the different degrees of freedom[7] is uncoupled. The functions Φ and Ψ are clearly scalar.
Exploiting the Helmholtz theorem, we can decompose each vector degree of freedom into a solenoidal part and a longitudinal part, the former is the vector one, the latter the scalar one:

$$\omega_i = \partial_i \omega^{\parallel} + \omega_i^{\perp}. \tag{2.29}$$

[7] Different means "with different transformation under spatial rotations"; this is the usual manner one uses to define *scalar*, *vector* and *tensor* in general relativity, that is, the transformation law of a certain object under a change of coordinates [8, 5]. The homogeneity and the isotropy of the background allow us to restrict just to the rotations.

$\omega_i^\perp$ is the solenoidal vector, meaning that $\partial^i \omega_i^\perp = 0$ and $\omega^\parallel$ is the longitudinal one. We can perform a similar decomposition of the traceless perturbation of g_{ij}

$$\gamma_{ij} = D_{ij}\gamma^\parallel + \partial_i \gamma_j^\perp + \partial_j \gamma_i^\perp + \gamma_{ij}^T, \tag{2.30}$$

where $\gamma^\parallel$ is a scalar function, $\gamma_i^\perp$ is a solenoidal vector field, and the *tensor part* h_{ij}^T is symmetric, solenoidal and traceless. D is defined as $D_{ij} = \partial_i \partial_j - \delta_{ij}\nabla^2/3$.

We perform now a gauge transformation of the metric tensor using Eq. (2.24). With the same procedure adopted above, we can decompose the vector field ξ^μ into its scalar and vector parts [43]:

$$\xi^0 = \alpha, \tag{2.31}$$

$$\xi^i = \partial^i \beta + d^i, \tag{2.32}$$

with $\partial_i d^i = 0$. The transformation law for the perturbations becomes, for the metric

$$\delta \tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \left(\mathcal{L}_\xi g^{(0)}\right)_{\mu\nu}.$$

After some trivial calculations we obtain

$$\tilde{\Psi} = \Psi + \alpha' + \mathcal{H}\alpha,, \tag{2.33a}$$

$$\tilde{\Phi} = \Phi - \frac{1}{3}\nabla^2\beta - \mathcal{H}\alpha, \tag{2.33b}$$

$$\tilde{\omega}^\parallel = \omega^\parallel - \alpha + \beta', \tag{2.33c}$$

$$\tilde{\omega}_i^\perp = \omega_i^\perp + d_i', \tag{2.33d}$$

$$\tilde{\gamma}^\parallel = \gamma^\parallel + 2\beta, \tag{2.33e}$$

$$\tilde{\gamma}_i^\perp = \gamma_i^\perp + d_i, \tag{2.33f}$$

$$\tilde{\gamma}_{ij}^T = \gamma_{ij}^T. \tag{2.33g}$$

In Eq. (2.33) $' \equiv d/d\tau$, and $\mathcal{H} = a'/a$ is the Hubble parameter in the conformal time. Here we immediately see a peculiarities that will be of great interest in the next. Eq. (2.33g) affirms that the tensorial part of the metric perturbation is a *gauge-invariant* quantity, at the linear order. Whatever the gauge choice is, the tensor perturbations will always be the same.
After having calculated the left hand side of Eq. (2.5), we can calculate its right hand side: the matter perturbations.

2.1.4 Matter perturbations

The stress-energy tensor for a generic fluid can be written as [5]

$$T_{\mu\nu} = (\rho + p_0)u_\mu u_\nu + p_0 g_{\mu\nu} + \sigma_{\mu\nu} \tag{2.34}$$

with respect to Eq. (1.21) we have added the anisotropic stress tensor, constrained as $u^\nu \sigma_{\mu\nu} = \pi_\mu^\mu = 0$. For a perfect fluid or a minimally coupled scalar field, it vanishes. The four-velocity, in the conformal metric, can be determined considering its normalization $u_\mu u^\mu = -1$:

$$u_\mu = a(-1 - \Psi, v_i), \quad \text{and} \quad u^\mu = a^{-1}(1 - \Psi, (v^i - \omega^i)). \tag{2.35}$$

The three-velocity v^i is the perturbation to the spatial velocity, given that the background value for the four-velocity of a comoving observer is $u^i = 0$. The three-velocity can be decomposed as usual

$$v^i = \partial^i v_{\parallel} + v^i_{\perp} \quad \text{with} \quad \partial_i v^i_{\perp} \tag{2.36}$$

We obtain, perturbing the stress-energy tensor[29, 15]

$$T^0_0 = -(\rho_0 + \delta\rho), \tag{2.37a}$$
$$T^i_i = 3(p_0 + \delta p) = 3p_0(1 + \sigma_L), \tag{2.37b}$$
$$T^0_i = -(\rho_0 + p_0)v_i, \tag{2.37c}$$
$$T^i_0 = (\rho_0 + p_0)(v^i - \omega^i), \tag{2.37d}$$
$$T^i_j = p_0\left[(1 + \sigma_L)\delta^i_j + \sigma^i_{T,\,j}\right]. \tag{2.37e}$$

We will never consider the anisotropic part of the stress-energy tensor for our cosmic fluid.

Applying again the transformation rule given in Eq. (2.24) we obtain

$$\delta\tilde{\rho} = \delta\rho + \alpha\rho_0', \tag{2.38a}$$
$$\delta\tilde{p} = \delta p - p_0', \alpha \tag{2.38b}$$
$$\tilde{v}_{\parallel} = v_{\parallel} - \mathcal{H}, \alpha - \alpha' \tag{2.38c}$$
$$\tilde{v}^i_{\perp} = v^i_{\perp} - \partial^i\beta' - (d^i)'. \tag{2.38d}$$

Here we have to do some considerations. Note that ρ is a scalar function, meaning that it is invariant under a change of coordinates (diffeomorphism). From Eq. (2.38a) we see that its perturbation varies under a gauge transformation. This highlights again the difference between a coordinate transformation and a gauge one.
From Eqs. (2.33) and (2.38) we clearly see that the transformation law of the perturbations, both metric and matter, are gauge dependent through the presence of the gauge parameter α, β and d^i. In particular the gauge is fixed when these parameter are fixed, that corresponds to the choice of certain hypersurfaces at constant time and the timeline worldline at constant $\mathbf{x}$ [15]. At the linear order fixing the gauge means fixing the value of two scalar and one vector [8] perturbations.

2.2 Quantum fluctuations during inflation

After the analysis of the dynamics of the background of the scalar field we consider now its perturbation given in Eq. (1.76). If we consider the complete field $\phi(\mathbf{x}, t)$ with also its space coordinates dependence, the full equation of motion Eq. (1.87) becomes[9]

$$\ddot{\phi}(\mathbf{x}, t) + 3H\dot{\phi}(\mathbf{x}, t) - a^{-2}\nabla^2\phi(\mathbf{x}, t) = -\frac{\partial V}{\partial\phi}, \tag{2.39}$$

where $\nabla^2 = \partial_i\partial^i$.
If we insert the background equation (1.89) and perturb to the first order, we obtain the equation of motion for the perturbations $\delta\phi(\mathbf{x}, t)$

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} = -\frac{\partial^2 V}{\partial\phi}\delta\phi \tag{2.40}$$

[8] With only two degree of freedom, due to $\partial_i d^i$.

[9] In this equation we are considering a perturbed (i.e. not homogenous or isotropic) scalar field in a background metric. To be rigorous both the metric and the scalar field must be perturbed, as we will see.

If we derive with respect to the time Eq. (1.89), we obtain the following equation for ϕ_0[10]

$$(\dot{\phi_0})^{\cdot\cdot} + 3H(\dot{\phi_0})^{\cdot} = -V''\dot{\phi_0}. \tag{2.41}$$

Here we can see that Eq.(2.40) and Eq.(2.41), respectively in the variable $\delta\phi$ and $\dot{\phi}_0$ are analogous except for the Laplacian term $a^{-2}\nabla^2\delta\phi$; in the limit of *large scale*, they are exactly analogous, and non independent as we will see shortly. Let's consider the two terms $3H\delta\dot{\phi} - a^{-2}\nabla^2\delta\phi$: if we perform a Fourier transform and we go from the coordinate space to the momentum space we have

$$3H\delta\dot{\phi} - a^{-2}\nabla^2\delta\phi \longrightarrow 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2\delta\phi_{\mathbf{k}}}{a^2},$$

we can make a qualitatively treatment of the perturbation profile on large scales. If $\delta\dot{\phi} \sim H\delta\phi$, since $H^{-1} = \tau$ the characteristic time of expansion, we have

$$3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2\delta\phi_{\mathbf{k}}}{a^2} \sim \left(3H^2 + \frac{k^2}{a^2}\right)\delta\phi_{\mathbf{k}}.$$

Considering the case $\frac{k^2}{a^2} \ll H$ correspond to neglect the second term, that means considering wavelength of physical dimension $\lambda_{phys} \gg H^{-1}$, since k is the wave number. This condition is equivalent to consider large scale lengths, or, more precisely, to consider wavelength larger than the Hubble horizon. This qualitatively process, that will become quantitative, is a sort of *smoothing* of the system, since this becomes homogeneous, and allow us to neglect the Laplacian term.
Now we solve the system of equations

$$\begin{cases} (\dot{\phi_0})^{\cdot\cdot} + 3H(\dot{\phi_0})^{\cdot} = -V''\dot{\phi_0} \\ \delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2\delta\phi}{a^2} = -\frac{\partial^2 V}{\partial\phi}\delta\phi \end{cases}$$

An accurate check shows that $\dot{\phi}_0$ and $\delta\phi$ are not independent. We can calculate the Wronskian function defined, in the case with variables x and y, as

$$W(x,y) = \dot{x}y - x\dot{y}, \tag{2.42}$$

and in the case $W = 0$, x and y are correlated, i.e. linearly dependent. In our case $x \to \dot{\phi}_0$, $y \to \delta\phi$, we have

$$W(\dot{\phi}_0, \delta\phi) = \ddot{\phi}_0\delta\phi - \dot{\phi}_0\delta\dot{\phi}$$

deriving this equation with respect to time we obtain

$$\dot{W} = -3HW \Rightarrow W \propto e^{-3Ht}.$$

This means that after a little transient ($t \sim H^{-1}$), the Wronskian goes to zero exponentially, and $\dot{\phi}$ and $\delta\phi$ becomes linearly dependent (remember that we are considering the large scale limit, or over-horizon limit). Mathematically we can write

$$\delta\phi(\mathbf{x},t) = -\delta\tau(\mathbf{x})\dot{\phi}_0(t). \tag{2.43}$$

The minus sign is purely conventional, while the dependence of $\delta\tau$ only on $\mathbf{x}$ reflect the null Laplacian at large scales.
Given Eq. (1.76), in force of Eq. (2.43) we can write

$$\phi(\mathbf{x},t) = \phi_0(t - \delta\tau(\mathbf{x})). \tag{2.44}$$

[10] Here we are considering $H \simeq const.$

We remember that Eq. (2.44) is the equation for the complete scalar field at large scale. This equation points out an important thing: in each point the scalar field assumes the same value of ϕ_0 but at different times, and this time displacement is due to the different space coordinate.

We look now for a general solution of Eq. (2.40). We perform a Fourier transform[11]

$$\delta\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\, k e^{i\mathbf{kx}} \delta\phi_{\mathbf{k}}, \tag{2.45}$$

where $\delta\phi_{\mathbf{k}} = \tilde{\delta\phi}(\mathbf{k}, t)$. The equation becomes now

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\dot{\phi}_{\mathbf{k}} + \frac{k^2\delta\phi}{a^2} = -V''\delta\phi_{\mathbf{k}}.$$

For the sake of simplicity we will omit, from now on, the pedice $\mathbf{k}$. Eq. (1.97), tells us that the mass term is negligible. In fact the potential V contains also the usual term $\frac{1}{2}m_\phi^2\phi^2$, and the condition cited above is equivalent to take a massless scalar field: $m_\phi^2 = V'' \simeq 0$. The equation now becomes

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \frac{k^2}{a^2}\delta\phi = 0.$$

We study this equation in two different regimes.

Small-scale regime $\lambda_{phys} \ll H^{-1}$

The wavelength we are consider are well under the horizon, and the friction term become negligible

$$\delta\ddot{\phi} + \frac{k^2}{a^2}\delta\phi = 0. \tag{2.46}$$

Eq.(2.46) is not more than the harmonic oscillator equation with a time dependent width

$$\omega_a = \frac{k^2}{a^2}$$

that decreases with time. On small scales the field fluctuates around its (false)vacuum value ϕ_0.

Large-scale regime $\lambda_{phys} \gg H^{-1}$

In this case the equation is

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} = 0.$$

Its exact solution is

$$\delta\phi = ae^{-3Ht} + b$$

with a and b constants. This means that almost immediately (in a time $t \gtrsim (3H)^{-1}$), the perturbation remains constant. Over the horizon the perturbation is *frozen* at its value at the time of its passing the Hubble horizon.

[11] Two notation: we are using here the Fourier transform because for linear perturbations the various modes ($\mathbf{k}$) evolve independently. The second concerns the $e^{i\mathbf{kx}}$ factor inside the Fourier transform. This terms implicitly means that we are considering the plane-wave decomposition, valid only with null spatial curvature. In the case of positive or negative curvature of the Universe we should use a generalization of the plane waves , the so-called *Helmholtz functions*, $Q_{\mathbf{k}}$, which satisfy the equation

$$\nabla_K^2 Q_{\mathbf{k}} + |\mathbf{k}|^2 Q_{\mathbf{k}} = 0$$

, where ∇_K^2 is the curved Laplacian [4].

The situation is schematically explained in Fig. 2.5.

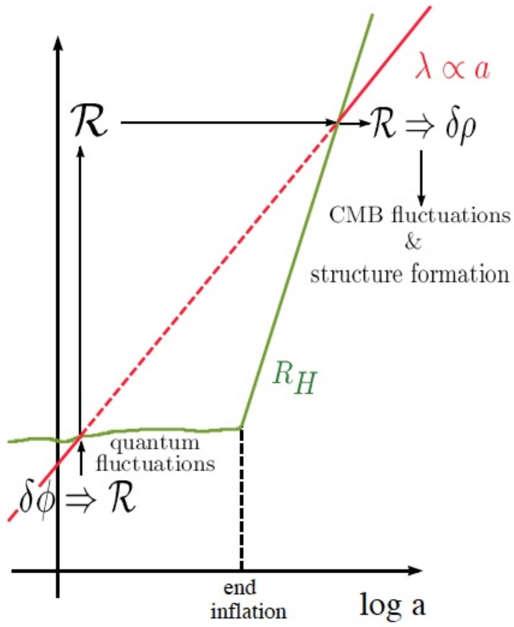

Figure 2.5: Stretching of the cosmological perturbation during the inflationary period. Quantum perturbations in the curvature, $\mathcal{R}$, are created during Inflation and their wavelengths, λ, are stretched from microscopic scales to astronomical scales during Inflation [43].

In the previous paragraphs we have referred to the perturbations of the scalar field as *quantum* perturbations. These same perturbations are strictly connected with the density perturbation that causes the observed anisotropy in the temperature in the CMB [28, 19]. This connection, that will be clear soon, arises a fundamental issue: the density perturbation causes (after the expansion of the Universe and the stretching of these perturbation) the collapse at the origin of the large-scale structure, like the galaxies and the clusters. But the latter are obviously classical objects! It is clear that if their formation origin resides in density perturbations whose origin in turn are the quantum fluctuations of the vacuum state during the inflationary period, there must have been a mutation of the profound nature of these perturbation at a certain time. There must have been a sort of phase transition from quantum to classical nature. Is there a mechanism that can explain this transition? This question is very attractive, but goes beyond the purpose of this work. We will bypass this question (unwillingly, with clear conscience) and quantize directly the perturbations of the scalar field, following [27, 15, 29].

2.2.1 Power-Spectrum

Before going on with the quantization of the perturbations we need to introduce a fundamental tool: the power spectrum. It represents an efficient way to characterize the properties of a field perturbations. In general one assumes that the perturbation is much smaller than the background value of the field, as we emphasized in Eq. (1.78). In particular we are implicitly assuming that all the perturbation fields are stochastic, i.e. $\langle \delta(\mathbf{x}, t) \rangle = 0$, where δ is a generic field perturbation. Another hypothesis one takes into account is that these fields has Gaussian statistics: this hypothesis simplifies our calculation, given that a Gaussian field is completely described by its mean value (that is 0 in our case) and its covariance, or its *two-point correlation function* defined as:

$$\chi(r) \equiv \langle \delta(\mathbf{x}), \delta(\mathbf{x}+\mathbf{r}) \rangle, \tag{2.47}$$

where angle brackets denote ensemble average. Given a galaxy in a random location, the correlation function describes the probability that another galaxy will be found within a given distance [37].
Note that if the background metric is homogeneous and isotropic as Eq. (1.1), χ depends only on the modulus of $\mathbf{r}$, r. For the same reasons we can expand in the Fourier space the perturbation field

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, e^{i\mathbf{kx}} \tilde{\delta}(\mathbf{k}), \tag{2.48}$$

here $\tilde{\delta}$ stands for the Fourier transform of δ. We can define now an efficient tool to characterize the statistical properties of a field perturbations: the power-spectrum. It is simply defined as

$$\langle \tilde{\delta}(\mathbf{k}), \tilde{\delta}(\mathbf{p}) \rangle = (2\pi)^2 \delta^{(3)}(\mathbf{k}+\mathbf{p}) \mathcal{P}_\delta(k). \tag{2.49}$$

We can see that the argument of $\mathcal{P}_\delta$ is just the modulus of the momentum: this is due to the isotropy of the background. The homogeneity is guaranteed by the three dimensional Dirac delta. We can see that, with this definition, $\mathcal{P}_\delta$ is nothing more than the Fourier transform of the two-point correlation function.

$$\begin{aligned}\chi(r) &= \langle \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{r})} \delta_{\mathbf{k}}, \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \rangle \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} \langle \tilde{\delta}_{\mathbf{k}}, \tilde{\delta}_{\mathbf{p}} \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathcal{P}(k)\end{aligned}$$

For a Gaussian statistics it can bee shown that [19] the three point correlation function vanishes[12]

$$\langle \tilde{\delta}_{\mathbf{k}_1} \tilde{\delta}_{\mathbf{k}_2} \tilde{\delta}_{\mathbf{k}_3} \rangle = 0$$

In general all the n-correlators vanish

$$\langle \tilde{\delta}_{\mathbf{k}_1} \dots \tilde{\delta}_{\mathbf{k}_n} \rangle = 0$$

if n is odd. The expression for the even-n can be rewritten in function of the two-point correlator; for the four-point function we have:

$$\langle \tilde{\delta}_{\mathbf{k}_1} \tilde{\delta}_{\mathbf{k}_2} \tilde{\delta}_{\mathbf{k}_3} \tilde{\delta}_{\mathbf{k}_4} \rangle = \langle \tilde{\delta}_{\mathbf{k}_1} \tilde{\delta}_{\mathbf{k}_2} \rangle \langle \tilde{\delta}_{\mathbf{k}_3} \tilde{\delta}_{\mathbf{k}_4} \rangle + \langle \tilde{\delta}_{\mathbf{k}_1} \tilde{\delta}_{\mathbf{k}_3} \rangle \langle \tilde{\delta}_{\mathbf{k}_2} \tilde{\delta}_{\mathbf{k}_4} \rangle + \langle \tilde{\delta}_{\mathbf{k}_1} \tilde{\delta}_{\mathbf{k}_4} \rangle \langle \tilde{\delta}_{\mathbf{k}_2} \tilde{\delta}_{\mathbf{k}_3} \rangle$$

which becomes

$$\langle \tilde{\delta}_{\mathbf{k}_1} \tilde{\delta}_{\mathbf{k}_2} \tilde{\delta}_{\mathbf{k}_3} \tilde{\delta}_{\mathbf{k}_4} \rangle = (2\pi)^6 \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2) \delta^{(3)}(\mathbf{k}_3+\mathbf{k}_4) \mathcal{P}_\delta(k_1) \mathcal{P}_\delta(k_2) + \text{permutations}.$$

Similar expressions hold for higher correlators. We have expressed what with mentioned before: if the perturbation have Gaussian statistics they are completely described by their two-point correlation function. We will see that are much more interesting the cases in which are present non-Gaussianities of the perturbation spectrum, which also contain a

[12] Here we are considering directly the Fourier space, the same argument holds also in the $\mathbf{x}$-space.

lot of physical information.

We define now the *dimensionless power-spectrum* $\Delta_\delta(k)$ as

$$\langle\tilde{\delta}(\mathbf{k}),\tilde{\delta}(\mathbf{k'})\rangle = \frac{2\pi^2}{k^3}\delta_D^{(3)}(\mathbf{k}-\mathbf{k'})\Delta_\delta(k). \tag{2.50}$$

Again, Δ is function only of the modulus of $\mathbf{k}$ (or $\mathbf{k}$', thanks to presence of the Dirac delta due to the homogeneity). Δ power spectrum measures the amplitude of the fluctuation at a given mode k. We see that $\mathcal{P}$ and Δ are strictly correlated

$$\Delta(k) = \frac{k^3}{2\pi^2}\mathcal{P}(k)$$

We can find, after some trivial calculations,

$$\langle\delta^2(\mathbf{x})\rangle = \int\frac{\mathrm{d}k}{k}\mathcal{P}_\delta(k) = \int \mathrm{d}(\ln k)\mathcal{P}_\delta(k). \tag{2.51}$$

Eq. (2.51) means that $\mathcal{P}_\delta$ represents the contribution to the variance per unit logarithmic interval in wave-number k.
We also define a new physical quantity to describe the slope of the power spectrum, the *spectral index*:

$$n_\delta(k) - 1 = \frac{\mathrm{d}\ln\Delta_\delta}{\mathrm{d}\ln k}. \tag{2.52}$$

In general the spectral index will depend on the considered scale; in the case of a constant value we have that the power spectrum[13] has a simple power law dependence on the considered scale:

$$\Delta(k) = \Delta(k_0)\left(\frac{k}{k_0}\right)^{n_s-1} \tag{2.53}$$

with k_0: pivot scale. We will see soon that the spectral index will depend on the slow-roll parameters. An interesting particular case is when the spectral index is exactly equal to 1: it means that the power spectrum of the considered field is scale invariant (Harrison-Zel'dovich power spectrum). The amplitude of the perturbation in this case is the same for each cosmological scale.
We are now ready to specify the form that the power-spectrum gets in a general case, when the stochastic field is a canonically quantized scalar field ϕ living in a curved spacetime, as the inflaton is.
We proceed, referring to Eq.(2.40), with the standard second quantization procedure of a new variable, the *physical* perturbation

$$\hat{\delta\phi} = a\delta\phi, \tag{2.54}$$

and using the conformal time defined as $d\tau \equiv a^{-1}dt$. Quantizing the new variable means promoting it to an operator through the definition of the two operators $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$:

$$\hat{\delta\phi}(\tau,\mathbf{x}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\left[u_k(\tau)a_\mathbf{k}e^{i\mathbf{kx}} + u_k^*(\tau)a^\dagger_\mathbf{k}e^{-i\mathbf{kx}}\right], \tag{2.55}$$

One vacuum choice, that will be clear in the next section, is that with the normalization of the u_k functions

$$u_k^*(\tau)u_k'(\tau) - u_k(\tau)u_k'^*(\tau) = -i, \tag{2.56}$$

[13]We refer to the dimensionless power spectrum simply with power spectrum.

with $' \equiv \frac{d}{d\tau}$.
The quantization rules descend directly from this choice

$$\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{p}}] &= [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{p}}] = 0, \\ [a_{\mathbf{k}}, a^\dagger_{\mathbf{p}}] &= \delta^{(3)}(\mathbf{k} - \mathbf{p}), \end{aligned} \tag{2.57}$$

where $\delta^{(3)}(\dots)$ is the usual three-dimensional Dirac delta.
From the redefinition of $\hat{\delta\phi}$ and Eqs. (2.55)-(2.57) we get

$$\langle \delta_{\phi_{\mathbf{k}_1}} \delta_{\phi_{\mathbf{k}_2}} \rangle = \frac{|u_k|^2}{a^2} \delta^{(3)}(\mathbf{k}_1 - \mathbf{k}_2)$$

which leads to the power spectrum

$$\Delta_{\delta\phi}(k) = \frac{k^3}{2\pi^2} |\delta\phi_k|^2 \tag{2.58}$$

2.2.2 Scalar perturbations in curved spacetime

In Eq. (2.55), $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ are, respectively, the annihilation and creation operators, defined as:

$$\begin{aligned} a_{\mathbf{k}} |0\rangle &= 0 \\ \langle 0| a^\dagger_{\mathbf{k}} &= 0 \end{aligned} \tag{2.59}$$

for all the possible values of $\mathbf{k}$. $|0\rangle$ is defined as the *free* vacuum state of the system, i.e. here we are considering only the mass term V'' of our theory, not also the interaction with other field and auto-interactions. Despite this assumption, it remains a fundamental ambiguity in the choice of the vacuum state due to the expansion of the background spacetime. In fact $u_k(\tau)$[14] are classical functions of time and in the limit of flat spacetime they are oscillating functions

$$u_k \simeq \frac{e^{-i\omega_k \tau}}{\sqrt{2k}} \tag{2.60}$$

with $\omega_k^2 = k^2 + m^2$. The ambiguity arises here. Eq.(2.60) is the solution to the Klein-Gordon equation in the flat spacetime, but so far we have considered this equation in an expanding Universe: this means that while the field is evolving, the background spacetime itself is varying. We can use here the *equivalence principle*: for small scales and short time intervals (with short we mean much shorter than the characteristic expansion time), we can locally approximate our spacetime to the Minkowski's, so that we can use the plane wave[5, 4] solution (2.60), i.e.

$$u_k(\tau) \xrightarrow{k \gg aH} \frac{e^{-i\omega_k \tau}}{\sqrt{2k}}. \tag{2.61}$$

This choice is called *Bunch-Davies vacuum* [44].
After all these considerations, and after some straightforward computations, we obtain an equation for the function $u_k(\tau)$

$$u_k''(\tau) + \left[k^2 - \frac{a''}{a} + \frac{\partial^2 V}{\partial \phi^2} a^2\right] u_k(\tau) = 0. \tag{2.62}$$

[14]Note that the function u depends only on the modulus of the moment $|\mathbf{k}| \equiv k$ and not on its direction: again, this is a consequence of the homogeneity and isotropy of the background.

The first thing we can notice is that the friction term $\dot{u}_k$ disappeared, leaving only a metric-dependent term of the form $\frac{a''}{a}$. Eq. (2.62) is the standard harmonic oscillator equation with a time-dependent angular frequency

$$\omega_k^2 = k^2 - \frac{a''}{a} + \frac{\partial^2 V}{\partial \phi^2} a^2 \tag{2.63}$$

We will solve Eq. (2.62) exactly in the case of a massless scalar field in a quasi-de Sitter Universe where we have $H \simeq const$, or

$$\dot{H} = -\epsilon H^2$$

with $\epsilon \ll 1$, so H varies very slowly. At the first order in the slow-roll parameter we can find the time dependence of the scale factor

$$a(\tau) = -\frac{1}{H(1-\epsilon)\tau}, \tag{2.64}$$

so that the term a''/a in the equation of motion becomes

$$\frac{a''}{a} = \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon\right).$$

Massless scalar field
This case will be very useful when studying the tensor perturbation of the metric. With *massless* scalar field we mean $V''(\phi) = 0$, hence the equation of motion is

$$u_k''(\tau) + \left[k^2 - \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon\right)\right] u_k(\tau) = 0$$

and defining $\nu^2 = 9/4 + 3\epsilon$ we have

$$u_k''(\tau) + \left[k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right] u_k(\tau) = 0. \tag{2.65}$$

At the first order the slow-roll parameter are constant, so we can consider also the index $\nu \simeq const$. We can define $x = -k\tau$ and the equation becomes

$$x^2 \frac{d^2}{dx^2} u(x) + \left[x^2 - (\nu^2 - \frac{1}{4})\right] u(x) = 0. \tag{2.66}$$

This allow us to solve the previous equation as a Bessel equation of the form

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0 \tag{2.67}$$

of which we know the exact solutions [43, 29]

$$u_k(\tau) = \sqrt{-\tau}\left[C_1(k) H_\nu^{(1)}(-k\tau) + C_2(k) H_\nu^{(2)}(-k\tau)\right], \tag{2.68}$$

where $H_\nu^{(i)}$ are the Hankel functions of i-th kind with index ν. The $C_i(k)$ are the integration constants depending only by k: they can be constrained using the Bunch-Davies vacuum choice given in Eq. (2.61). The subhorizon limit $k \ll aH$ in a quasi de Sitter spacetime becomes, thanks to Eq. (2.64)

$$-\tau k \gg 1$$

The behavior in the subhorizon limit for the Hankel functions is note

$$H_\nu^{(1)}(x) \xrightarrow{x\gg 1} \sqrt{\frac{2}{\pi x}} e^{i\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)} \tag{2.69}$$

that is the expected behavior for the function u_k in the subhorizon limit. This allow us to fix C_1 and C_2, using also the property $H_\nu^{(2)} = \left[H_\nu^{(1)}\right]^*$

$$u_k(\tau) \xrightarrow{x\gg 1} \sqrt{-\tau}\left[C_1(k)\sqrt{\frac{2}{-\pi k\tau}} e^{i\left(-k\tau-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)} + C_2(k) H_\nu^{(2)}(-k\tau \gg 1)\right].$$

In this result the second term inside the square brackets has negative frequency, so that we have to fix

$$C_2(k) = 0 \quad \text{and} \quad C_1(k) = \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)}\frac{\pi}{2}.$$

The exact solution is

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \sqrt{-\tau} H_\nu^{(1)}(-k\tau). \tag{2.70}$$

In order to calculate the power spectrum we consider the super-horizon limit $-k\tau \ll 1$. For the Hankel function we have

$$H_\nu^{(1)}(x) \xrightarrow{x\ll 1} \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\left(\nu-\frac{3}{2}\right)} \frac{\Gamma(\nu)}{\Gamma(3/2)} x^{-\nu} \tag{2.71}$$

where $\Gamma(z)$ is the Gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} \mathrm{d}x x^{z-1} e^{-x}.$$

This means that u_k in the super-horizon limit has the following behavior

$$u_k(\tau) \simeq 2^{\nu-\frac{3}{2}} e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{(-k\tau)^{\frac{1}{2}-\nu}}{\sqrt{2k}}.$$

For the amplitude of the scalar perturbations we have

$$|\delta\phi_k| = \frac{|u_k|}{a} = 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{H}{\sqrt{2k^3}} (-k\tau)^{\frac{3}{2}-\nu},$$

where we have used again Eq. (2.64). At the first order we have $\nu = \frac{3}{2} + \epsilon$, so we can write the expression for the amplitude at the first order in the slow-roll parameter

$$|\delta\phi_k| = \frac{H}{\sqrt{2k^3}} (-k\tau)^{-\epsilon} = \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{-\epsilon}. \tag{2.72}$$

Using Eq. (2.58) we obtain, for the power spectrum,

$$\Delta_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon}. \tag{2.73}$$

This is a fundamental result which shows that the power spectrum for a massless scalar perturbation in a quasi de Sitter space is almost scale invariant. The dependence on the scale k is $k^{-2\epsilon}$ and in the slow-roll limit this term is almost null. This is an important prediction for the standard single field slow-roll model for Inflation.

2.2.3 Gauge-invariant scalars

Until now we have a little cheated. In the previous section we have seen the evolution of the scalar perturbations in a quasi-de Sitter phase, as the Inflation predicts, without perturbing the metric in which the field lives. For the Einstein equations a perturbation in the stress-energy tensor, i.e. a perturbation of the scalar field, involves inevitably a perturbation in the metric side: this causes a variation in the motion equation of the scalar field. We consider now both the perturbation of the scalar field and the perturbation of the metric. While for the tensor perturbation we have seen that we can use a gauge invariant quantity, γ^T_{ij}, for the scalar perturbations we not defined a good quantity in this sense. There are various scalar perturbation, four just in the metric, what quantity we will use? We can see that we can define different gauge invariant scalar quantities exploiting the transformation law under a gauge transformation [40]. A good candidate to describe the scalar perturbation is the intrinsic spatial curvature on hypersurfaces of constant conformal time

$$^{(3)}R = \frac{4}{a^2}\nabla^2\hat{\Phi} \tag{2.74}$$

where we have defined the *curvature perturbation*[15]

$$\hat{\Phi} = \Phi + \frac{1}{2}\nabla^2\gamma_{\parallel}. \tag{2.75}$$

$\hat{\Phi}$ is not a gauge-invariant quantity, since under a transformation on constant time hypersurfaces $\tau \to \tau + \alpha$ we have $\hat{\Phi} \to \hat{\Phi} - \mathcal{H}\alpha$. For this reason we need a gauge-invariant scalar quantity which reduces, in some gauge choices, to the curvature perturbation. Considering the transformation law for the matter perturbation given in Eq. (2.38) we can define the following gauge invariant quantity

$$\zeta \equiv -\hat{\Phi} - \mathcal{H}\frac{\delta\rho}{\rho_0'}. \tag{2.76}$$

It is trivial to show the gauge invariance of ζ:

$$\tilde{\zeta} = -\tilde{\hat{\Phi}} - \frac{a'}{a}\frac{\tilde{\delta\rho}}{\rho_0'} = -\hat{\Phi} + \frac{a'}{a}\alpha - \frac{a'}{a}\left(\frac{\delta\rho}{\rho_0'} + \alpha\right) = -\hat{\Phi} - \frac{a'}{a}\frac{\delta\rho}{\rho_0'} = \zeta.$$

ζ is defined as the *gauge-invariant curvature perturbation of the uniform energy-density hyper-surfaces*.

In the case of a single scalar field we can define also the *gauge-invariant perturbation on comoving hyper-surfaces*

$$\mathcal{R} \equiv \hat{\Phi} + \frac{\mathcal{H}}{\phi'}\delta\phi. \tag{2.77}$$

[15] The meaning of this name becomes clear when performing a full calculation of the intrinsic spatial curvature on hypersurfaces of constant conformal time, which gives the result, in an unperturbed Universe

$$^{(3)}R = \frac{6}{a^2}K,$$

where K is the space curvature introduced in Eq. (1.1). This means that when we have $K = 0$ we have $^{(3)}R = 0$. In the perturbed Universe we have the general expression

$$^{(3)}R = \frac{6}{a^2}K + \frac{12K}{a^2}\hat{\Phi} + \frac{4}{a^2}\nabla^2\hat{\Phi},$$

which confirms what we said before.

The combinations $\mathcal{R}$ and ζ are related [15, 29]

$$-\zeta = \mathcal{R} + \frac{2\rho}{9(\rho + p)} \left(\frac{k}{aH}\right)^2 \Psi. \tag{2.78}$$

From this equation we see that in the super-horizon limit ($k \ll aH$), $\mathcal{R} = -\zeta$.

The importance of ζ

The gauge-invariant quantity ζ hold a fundamental role in the Inflation models because, thanks to its definition and properties on super-horizon scales, it allows to connect the energy-density perturbations $\delta\rho$ to the perturbation of the scalar field $\delta\phi$. Indeed we can see that for super-horizon scales it is constant in time. To understand it we have to consider the perturbed continuity equation which reads

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - 3(p_0 + \rho_0)\hat{\Phi}' + (p_0 + \rho_0)\nabla^2(V + \sigma) = 0, \tag{2.79}$$

where $\sigma = -\omega_{\|} + 1/2\gamma'_{\|}$ is the shear.

If we consider super-horizon scales we can neglect the contribution due to the Laplacian. We can choose now the uniform energy density gauge in which $\delta\rho = 0$. So

$$3\mathcal{H}\delta\rho + 3(p_0 + \rho_0)\zeta',$$

because in this gauge $\zeta = -\hat{\Phi}$. The previous equation gives an equation for the time evolution of ζ

$$\zeta' = -\frac{\mathcal{H}\delta p}{(p_0 + \rho_0)}\bigg|_{\delta\rho=0}. \tag{2.80}$$

Remind that the equation of state for a generic fluid is given by

$$\delta p = c_s^2\delta\rho + \delta p_{NA}$$

where δp_{NA} is the *non adiabatic* contribution. Hence we can write Eq. (2.80) as

$$\zeta' = -\frac{\mathcal{H}}{p_0 + \rho_0}\delta p_{NA}.$$

The non adiabatic contribution arises in the presence of a relative difference in the density perturbations of the different components which fill the Universe. If we have different components ρ_i, there are non adiabatic modes, or *entropy modes*, if

$$\frac{\mathcal{H}\delta\rho_i}{\rho_i} \neq \frac{\mathcal{H}\delta\rho_j}{\rho_j},$$

for $i \neq j$ [1]. Nowadays there are strong observational limits on a possible deviation from the adiabaticity condition for the expansion of the Universe. Furthermore during the Inflation we have seen that there is, in the simplest models, only one, scalar, component: relative differences are not permitted, simply because all the components are "swept away" by the Inflation. During inflation, on super-horizon scales δp_{NA}, so

$$\zeta' = 0 \quad \rightarrow \quad \zeta = \text{const}$$

We underline again that ζ is a gauge-invariant quantity, so this equation holds in any gauge.

The constance of ζ during Inflation and on scales much larger than the Hubble horizon allow to connect today observables to quantities strictly connected with the Inflation period. If we call with $t_H^{(1)}(k)$ the time at which the wavelength of mode k exit the horizon and $t_H^{(2)}(k)$ the time at which the same mode re-enters the horizon we have, look Fig. 1.4

$$\zeta_{t_H^{(1)}(k)} = \zeta_{t_H^{(2)}(k)}$$

that is, an observed mode which re-enters the horizon today (like the CMB modes) can give information about the inflationary epoch. Both $t_H^{(1)}(k)$ and $t_H^{(2)}(k)$ correspond to the condition $k = aH$, i.e. when the k mode has the same value of the Hubble horizon. For example we can consider a mode which passes the horizon in the radiation dominated epoch will be

$$\zeta_{t_H^{(2)}(k)} = \frac{1}{4}\frac{\delta\rho}{\rho}\bigg|_{t_H^{(2)}(k)} = \zeta_{t_H^{(1)}(k)} = \frac{H\delta\phi}{\dot{\phi}}\bigg|_{t_H^{(1)}(k)}.$$

We see again what we affirmed before: the temperature anisotropies of the CMB, connected with the density perturbations, are related to the fluctuations of the primordial scalar field. This situation is schematically represented in Fig. 2.5.

2.2.4 Power spectrum for scalar perturbations

We have seen that the evolution equation for the inflaton field is the Klein-Gordon equation:

$$\Box\phi = \frac{\partial V}{\partial\phi}$$

where the d'Alambertian is defined as $\Box \equiv D_\mu D^\mu$. After some manipulation it can be written in the more convenient form [19]

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right). \tag{2.81}$$

Perturbing the KG equation at the linear order using Eqs. (2.25), (2.26) and (2.27) we obtain [43, 15, 29]

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi + a^2\delta\phi\frac{\partial^2 V}{\partial\phi}a^2 + 2\Psi\frac{\partial V}{\partial\phi} - \phi_0'\left(\Psi' + 3\Phi' + \nabla^2\omega^{\parallel}\right) = 0. \tag{2.82}$$

We see that with respect to Eq. (2.40) there are more terms due to the metric perturbations to be considered.
To get a simpler equation of motion we introduce the *Sasaki-Mukhanov gauge-invariant variable* [45, 43]

$$\mathcal{Q}_\phi \equiv \delta\phi + \frac{\phi'}{\mathcal{H}}\hat{\Phi}. \tag{2.83}$$

As usual we introduce the field $\tilde{\mathcal{Q}}_\phi = a\mathcal{Q}_\phi$. The Klein-Gordon equation now reads[16], [43]

$$\tilde{\mathcal{Q}}_\phi'' + \left(k^2 - \frac{a''}{a} + \mathcal{M}_\phi^2 a^2\right)\tilde{\mathcal{Q}}_\phi = 0, \tag{2.84}$$

[16] We are clearly omitting some steps, the Eq. (2.84) is obtained after the second quantization of the field $\tilde{\mathcal{Q}}_\phi$, as we did in the previous section for the field $\delta\tilde{\phi}$.

where

$$\mathcal{M}_\phi^2 = \frac{\partial^2 V}{\partial \phi^2} - \frac{8\pi G}{a^3}\left(\frac{a^3}{H}\dot{\phi}^2\right)$$

is the effective mass of the inflaton field. At the first order in the slow-roll parameter it can be shown that

$$\frac{\mathcal{M}_\phi^2}{H^2} = 3\eta_V - 6\epsilon_V.$$

Eq. (2.84) is formally identical to Eq. (2.62), so we can perform the same procedure to solve it. We consider the case of a quasi-de Sitter Universe and a light scalar field, i.e. $\mathcal{M}_\phi^2 \ll H^2$ given that we are considering a slow-roll model. With these hypothesis the equation becomes identical[17]

$$\mathcal{Q}_\phi'' + \left(k^2 - \frac{\nu_\phi^2 - \frac{1}{4}}{\tau^2}\right)\mathcal{Q}_\phi = 0 \tag{2.85}$$

where we have defined $\nu_\phi = 3/2 + 3\epsilon - \eta$ at the first order in the slow-roll parameters. From the previous section we can conclude that on super-horizon scales and to the lowest order in the slow-roll parameters the inflaton fluctuations amplitude is

$$|\mathcal{Q}_\phi(k)| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\phi}. \tag{2.86}$$

We can see now that $\mathcal{Q}_\phi$ and $\mathcal{R}$ and ζ are connected:

$$\mathcal{R} = \frac{\mathcal{H}}{\phi'}\mathcal{Q}_\phi = -\zeta$$

so we can calculate the power spectrum of ζ using Eq. (2.58) and

$$|\zeta|^2 = |\mathcal{R}|^2 = \left(\frac{\mathcal{H}}{\phi'}\right)^2 |\mathcal{Q}_\phi|^2 = \left(\frac{H}{\dot{\phi}}\right)^2 \frac{H^2}{2k^3}\left(\frac{k}{aH}\right)^{3-2\nu_\phi}.$$

Finally, the power spectrum is

$$\Delta_\zeta(k) = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu_\phi}. \tag{2.87}$$

Again, Eq. (2.87) shows that curvature perturbations remain time-independent on super-horizon scales. The spectral index at the lowest order in slow-roll reads

$$n_\zeta - 1 = 3 - 2\nu_\phi = -6\epsilon + 2\eta. \tag{2.88}$$

Remind that we are interested to the value of the power spectrum when a certain mode crosses the horizon, i.e. $t_H^1(k)$ such as $k = aH$, so the power spectrum for ζ becomes

$$\Delta_\zeta(k) = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2\Bigg|_{t_H^1(k)}.$$

Here the k dependence is inside $t_H^1(k)$: each mode exit the horizon at a different time.

[17] We omit the ˜.

2.3 Gravitational Waves from Inflation

The inflationary scenario predicts also the production, in the early Universe, of a background of stochastic Gravitational Waves [46]. Here we will perform the full calculation to obtain the second-order action for the gravitational waves. The action we have to perturb is the following

$$S = \frac{M_{Pl}^2}{2} \int \mathrm{d}^4x \sqrt{-g}R + S_\phi \tag{2.89}$$

where S_ϕ is the one given in (1.71).
We will follow Refs. [29] and [47]. Perturbing Eq. (2.89), and manipulating the integral argument, we finally find that

$$S_T^{(2)} = \frac{M_{Pl}^2}{8} \int \mathrm{d}^4x a^3(t) \left[(\dot{\gamma}_{ij})^2 - a^{-2}(\partial\gamma_{ij})^2 \right]. \tag{2.90}$$

As already mentioned γ_{ij} is a gauge invariant quantity, so varying the action with respect to this quantity, we get the required equation of motion

$$\ddot{\gamma}_{ij} + 3H\dot{\gamma}_{ij} - a^{-2}\nabla^2\gamma_{ij} = 0. \tag{2.91}$$

It is now clear that tensor perturbations solve a wave equation (in an expanding Universe), hence the name *gravitational waves*. Exploiting the γ_{ij} properties, it is possible to decompose the field in the following form [29]

$$\gamma_{ij}(t, \mathbf{x}) = \sum_{\lambda=+,\times} \gamma^{(\lambda)}(t)\epsilon_{ij}^{(\lambda)}(\mathbf{x}). \tag{2.92}$$

$\epsilon_{ij}^{(+,\times)}$ is the polarization tensor whose properties reflect the γ ones:

$$\epsilon_{ij}^{(\lambda)} = \epsilon_{ji}^{(\lambda)}, \quad k^i\epsilon_{ij}^{(\lambda)} = 0 = \epsilon_{ii},$$

and $+, \times$ are the two GW polarization states [8].
If we use now the conformal time we obtain

$$S_T^{(2)} = \frac{M_{Pl}^2}{8} \int \mathrm{d}\tau \mathrm{d}^3x a^2(\tau) \left[\left(\gamma_{ij}'\right)^2 - (\partial\gamma_{ij})^2 \right]. \tag{2.93}$$

We see that a useful transformation to solve the motion equation for the tensor field in the conformal time is

$$\gamma_{ij} = \frac{\sqrt{2}}{aM_{Pl}} v_{ij}. \tag{2.94}$$

In terms of the new variable v_{ij} the action Eq. (2.93) becomes, after some integration by parts,

$$S_T^{(2)} = \frac{1}{4} \int \mathrm{d}\tau \mathrm{d}^3x \left[(v_{ij}')^2 + \frac{a''}{a}(v_{ij})^2 - (\partial v_{ij})^2 \right]. \tag{2.95}$$

If we perform now a Fourier transform, using also the decomposition Eq. (2.92),

$$v_{ij}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=+,\times} v_{\mathbf{k}}^{(\lambda)}(\tau)e_{ij}^{(\lambda)}(k)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.96}$$

Exploiting the orthogonality of polarization tensor

$$e_{ij}^{\lambda}e_{ij}^{\lambda'} = \delta^{\lambda\lambda'} \tag{2.97}$$

we can easily find the equation of motion for each mode $v_{\mathbf{k}}^{(\lambda)}$

$$v_{\mathbf{k}}^{\prime\prime(\lambda)} + \left(k^2 - \frac{a''}{a}\right) v_{\mathbf{k}}^{(\lambda)} = 0. \tag{2.98}$$

We see that the equation of motion for each mode resembles Eq. (2.62): this means that the each mode v can be described as a massless[18] scalar field, thanks to the absence of the term $V(\phi)''$. So we can treatise $v_{\mathbf{k}}$ in the same manner as $u_{\mathbf{k}}$, in particular in the case of a massless scalar field. After the standard second quantization of the field $v_{\mathbf{k}}^{(\lambda)}$

$$v_{\mathbf{k}}^{(\lambda)} = v_k(\tau) a_{\mathbf{k}}^{(\lambda)} + v_k^*(\tau) a_{-\mathbf{k}}^{(\lambda)\dagger} \tag{2.99}$$

The procedure now is analogous to that for the massless scalar field given above, the solution for $v_{\mathbf{k}}$ is

$$v_{\mathbf{k}} = \frac{\sqrt{\pi}}{2} e^{i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \sqrt{-\tau} H_\nu^{(1)}(-k\tau), \tag{2.100}$$

with $\nu \simeq 3/2 + \epsilon$. For large scales we obtain

$$v_{\mathbf{k}} = e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2}-\nu}. \tag{2.101}$$

The only difference resides into the calculation of the power spectrum, given that now we have two different polarization states for the tensor perturbations:

$$\Delta_\gamma(k) = \frac{k^3}{2\pi^2} \sum_\lambda \left|\gamma_{\mathbf{k}}^{(\lambda)}\right|^2, \tag{2.102}$$

so that on super-horizon scales we obtain the following power-spectrum

$$\Delta_\gamma(k) = \frac{8}{M_{Pl}^2} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon}. \tag{2.103}$$

We define the spectral index for the tensor perturbation as

$$n_\gamma = \frac{\mathrm{d}\log\Delta_\gamma(k)}{\mathrm{d}\log k}. \tag{2.104}$$

From Eq. (2.103) we see that it has the value $n_\gamma = -2\epsilon < 0$. In this case the power-spectrum is called *red*, while for $n_\gamma > 0$ it is called *blue*. We will see that some theories will present a blue tilt for the tensor power spectrum.

2.3.1 Consistency relations

In the considered inflationary scenario an interesting consistency relation holds for scalar and tensor power spectra. We have seen that they can be rewritten as

$$\begin{aligned} \Delta_\zeta(k) &= A_S \left(\frac{k}{k_0}\right)^{n_\zeta - 1} \\ \Delta_\gamma(k) &= A_T \left(\frac{k}{k_0}\right)^{n_\gamma} \end{aligned} \tag{2.105}$$

[18] It can be states that this field has an *effective* mass squared equal to a''/a [29].

where $k_0 = 0.002$ Mpc^{-1} is the pivot scale [48]. A_S and A_T represent the amplitude of the power spectra at the pivot scale k_0. It is useful to define the *tensor-to-scalar ratio*

$$r \equiv \frac{A_T}{A_S} \tag{2.106}$$

that yields the amplitude of the GW with respect to that of the scalar perturbations at some fixed pivot scale. Knowing that

$$A_S = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \quad \text{and} \quad A_T = \frac{8}{M_{Pl}^2}\left(\frac{H}{2\pi}\right)^2$$

and recalling that during Inflation holds $\dot{H} = -4\pi G\dot{\phi}^2$, we obtain

$$r = 16\epsilon \tag{2.107}$$

or

$$r = -8n_\gamma. \tag{2.108}$$

This is called *consistency relation* because it connects three different parameters and holds for each single field slow-roll model of Inflation [28]. This equality can be checked only with a measurement of the tensor power spectrum, i.e. not only of its amplitude, but also of its spectral index. Furthermore if this relation really holds true it means that it will be very hard to measure any scale dependence of the tensors, since a large spectral index would invalidate the consistency relation. At present we have only un upper bound on the tensor-to-scalar ratio: $r_{0.05} < 0.07$ at 95% CL [48], assuming the consistency relation.

2.3.2 Energy scale of inflation

We seen above that the amplitude of the GW has an expression

$$A_T = \frac{2}{\pi^2}\frac{H^2}{M_{Pl}^2},$$

and during Inflation we have $H^2 \simeq (3M_{Pl})^{-2}V(\phi)$, hence

$$A_T = \frac{2}{3\pi^2}\frac{V(\phi)}{M_{Pl}^4}.$$

From this equation we can define an energy scale for the Inflation as

$$E_{inf} = V^{1/4}, \tag{2.109}$$

so that the amplitude of the tensor power spectrum depends on the energy scale at which the Inflation acted. This is a fundamental feature for the single field slow-roll models: observing the GW would entail a confirm and a comprehension of the Inflation period. For this reason the GW are said to be a *smoking gun* for the Inflation. Using the note relation holding during Inflation and the consistency relation we can give an estimation of the scale energy [48]

$$E_{inf} = V^{1/4} = (1.88 \times 10^{16}\text{GeV})\left(\frac{r}{0.10}\right)1/4$$

We can conclude noting that a measurement of r provides the energy scale of Inflation. We have seen that a measurement of the primordial gravitational waves would be of crucial importance for the description of the primordial Universe [49]. Although it is much difficult [50], we will see that it is possible to define new quantities which will provide good indirect observables for the gravitational wave background.

2.3.3 CMB observations

We briefly introduce here a useful formalism to connect the anisotropies in temperature of the CMB and the primordial matter perturbations. Fig. 2.1 shows the small variation, point by point, in the temperature of the Cosmic Microwave Background radiation. In general we can parametrize the temperature field, which will depend on (conformal) time, position and the direction of the photon momentum

$$T(\tau,\mathbf{x},\hat{\mathbf{p}}) = T_0(\tau) + \Delta T(\tau,\mathbf{x},\hat{\mathbf{p}}) = T_0(\tau)\left(1+\Theta(\tau,\mathbf{x},\hat{\mathbf{p}})\right), \tag{2.110}$$

where T_0 represent the background black-body temperature, see (1.69), while we have defined $\Theta \equiv \Delta T/T$ the temperature perturbation field. Although this field is defined at all times and positions we can observe it only at the present time τ_0 and in our position $\mathbf{x}_0$: one can easily object that our observations (COBE, WMap, Planck) have been made over the last 30 years and the satellites are not exactly located on Earth. We stress that these excursions from the spacetime point $(\tau_0,\mathbf{x}_0)$ are negligible with respect to the scales over which the temperature varies (of the order of the Hubble time) [2]. Hence the only fundamental dependence of the perturbation field is the momentum direction $\hat{\mathbf{p}}$. To understand which information we can gain from the temperature of the photons from CMB we need to understand how the photons distribution function evolve while the Universe is expanding. We use the *Boltzmann equation*

$$\mathbb{L}[f] = \mathbb{C}[f] \tag{2.111}$$

where $\mathbb{L}$ is the Liouville operator which takes into account the evolution of the distribution function f and in a general metric takes the form [1]

$$\mathbb{L} = p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\rho_{\mu\nu} p^\mu p^\nu \frac{\partial}{\partial p^\rho}.$$

The right hand side of (2.111) represents the *collision* operator and contains all possible collision term, i.e. all the interaction of the particular particle species we are considering. We are interested in studying the photons distribution functions, therefore we need to know all the possible interactions with the other components of the Universe. We will not study the entire Boltzmann equation for all the components of the Universe, but we will mention some tools that will be useful also in Chapter 3. For a detailed, clear and exhaustive study of the Boltzmann equation of all the particle species see Ref. [2]. We know that the CMB radiation is a black-body radiation [36], hence there is a one to one relation between the temperature and the momentum of the photon. Hence we can use the temperature perturbation field to described the variation in energy for the photons: some equations which describes the evolution of the temperature perturbation field considering also matter perturbations, metric perturbations and so on, are found. One useful tool is the multipole expansion of the temperature perturbation

$$\Theta(\mathbf{k},\hat{\mathbf{p}}) = \sum_{l=1}^{\infty}(-i)^l(2l+1)\mathcal{P}_l(\hat{\mathbf{p}})\Theta_l(\mathbf{k}) \tag{2.112}$$

where $\mathcal{P}_l$ are the Legendre polynomials, solutions of the Legendre differential equation. Each multipole is defined as, inverting the previous relation

$$\Theta_l \equiv \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}\mathcal{P}_l(\mu)\Theta(\mu) \tag{2.113}$$

Exploiting the properties of the Legendre polynomials, the photon perturbations can be described either by Θ or by a whole hierarchy of moments, Θ_l. $l = 0$ corresponds to the monopole, i.e. a constant perturbation, $l = 1$ describes a dipole perturbation, $l = 2$ the quadrupole, $l = 3$ the octupole, and so on. The Boltzmann equations are written in terms of the first multipole Θ_l. Now we want to connect the observations on the CMB temperature to the theoretical Θ_l. Given that we are observing a spherical sky, it is useful to decompose the temperature perturbation field using spherical harmonics

$$\Theta(\tau, \mathbf{x}, \hat{\mathbf{p}}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{+l} a_{lm}(\tau, \mathbf{x}) Y_{lm}(\hat{\mathbf{p}}). \tag{2.114}$$

This expansion is the equivalent of a Fourier transform on the surface of a sphere. The subscripts l, m are conjugate to the real space unit vector $\hat{\mathbf{p}}$, just as the variable $\mathbf{k}$ is conjugate to the Fourier transform variable $\mathbf{x}$ in (2.45). The complete set of eigenfunctions, of the operator $i\nabla$, for the Fourier transform are $e^{i\mathbf{k}\cdot\mathbf{x}}$, here the complete set of eigenfunctions for expansion on the surface of a sphere are $Y_{lm}(\hat{\mathbf{p}})$. Now, in the expansion (2.114) the coefficients a_{lm} are of fundamental fundamental importance in cosmology. They contain all the information for the temperature field or, more precisely, they contain the information about the temperature perturbation field. Therefore we want to find the relation between the coefficients a_{lm} and Θ_l. We use the orthogonality property of spherical harmonics

$$\int_{\Omega} d\Omega Y_{lm}(\hat{\mathbf{p}}) Y^*_{l'm'}(\hat{\mathbf{p}}) = \delta_{ll'}\delta_{mm'}, \tag{2.115}$$

where Ω represents the solid angle spanned by $\hat{\mathbf{p}}$. We can invert the relation (2.114) multiplying for $Y^*_{lm})(\hat{\mathbf{p}})$ and integrating in the solid angle

$$a_{lm}(\tau, \mathbf{x}) = \int d\Omega Y^*_{lm}(\hat{\mathbf{p}}) \Theta(\tau, \mathbf{x}, \hat{\mathbf{p}}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \int d\Omega Y^*_{lm}(\hat{\mathbf{p}}) \Theta(\tau, \mathbf{k}, \hat{\mathbf{p}}), \tag{2.116}$$

in the second equality we have used the Fourier transform of the temperature perturbation field, since all the solution are in the momentum space. As for all the perturbative field, also a_{lm} are stochastic fields, which means that their mean value is zero and the first observable we can define is their variance

$$\langle a_{lm} \rangle = 0, \quad \langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'}\delta_{mm'} C_l, \tag{2.117}$$

where we have indicated with C_l the variance. It is very important to note that, for a given l, each a_{lm} has the same variance. For $l = 50$, for example, all the 101 $a_{50,m}$'s are drawn from the same distribution and when we measure these 101 coefficients we are sampling the distribution. Therefore this much information will give us a good handle on the underlying variance of the distribution. But if we measure the quadrupole, i.e. $l = 2$, we do not get very much information about the variance, since it has only 5 $a_{2,m}$. This fundamental uncertainty in the low-l variances is called *cosmic variance*, see Fig. 2.6.
We can now obtain an expression for C_l in terms of Θ_l. First we square (2.116) and take the expectation value

$$\langle a_{lm} a^*_{lm} \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} e^{i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x})} \int d\Omega d\Omega' Y^*_{lm}(\hat{\mathbf{p}}) Y_{lm}(\hat{\mathbf{p}}') \langle \Theta(\tau, \mathbf{k}, \hat{\mathbf{p}}) \Theta^*(\tau, \mathbf{k}', \hat{\mathbf{p}}') \rangle. \tag{2.118}$$

We need to evaluate now the expectation value $\langle \Theta(\tau, \mathbf{k}, \hat{\mathbf{p}}) \Theta^*(\tau, \mathbf{k}', \hat{\mathbf{p}}') \rangle$. This expectation value is complicated since it depends on two phenomena: the initial amplitude and phase of the perturbation, which depend on the Inflation mechanism; the evolution of the perturbations produces anisotropies, i.e., dependence on $\hat{\mathbf{p}}$. It is useful to separate these two phenomena rewriting the photon distribution as $\delta_{DM} \cdot (\Theta/\delta_{DM})$, where δ_{DM} is the Dark Matter overdensity, which does not depend on any direction vector [2]. We divide Θ into these two pieces because Θ/δ_{DM} can be found using the Boltzmann equation (2.111) and it has a particular feature: the ratio does not depend on the initial amplitude of the perturbations, so it can be removed from the averaging over the distribution. We have

$$\begin{aligned}\langle\Theta(\mathbf{k},\hat{\mathbf{p}})\Theta^*(\mathbf{k}',\hat{\mathbf{p}}')\rangle &= \langle\delta_{DM}(\mathbf{k})\delta^*_{DM}(\mathbf{k}')\rangle\frac{\Theta(\mathbf{k},\hat{\mathbf{p}})}{\delta_{DM}(\mathbf{k})}\frac{\Theta^*(\mathbf{k}',\hat{\mathbf{p}})}{\delta_{DM}(\mathbf{k}')} \\ &= (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\mathcal{P}_{DM}(k)\frac{\Theta(k,\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})}{\delta_{DM}(k)}\frac{\Theta^*(k,\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}')}{\delta_{DM}(k)},\end{aligned} \tag{2.119}$$

where $\mathcal{P}_{DM}$ is the dark matter power spectrum, and we have used the fact that the ratio Θ/δ_{DM} depends only on the magnitude of $\mathbf{k}$ and the dot product $\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}$. Now inserting Eqs. (2.112) and(2.119) into (2.118) we obtain, using the spherical harmonics properties,

$$C_l = \frac{2}{\pi}\int_0^\infty dk k^2\mathcal{P}_{DM}(k)\left|\frac{\Theta_l(k)}{\delta_{DM}(k)}\right|^2. \tag{2.120}$$

For a given l, then, the variance of a_{lm}, C_l, is an integral over all Fourier modes of the variance of Θ_l. Using the Boltzmann equation for all the modes Θ_l it is possible to plot the anisotropy spectrum today, see Fig. 2.6.

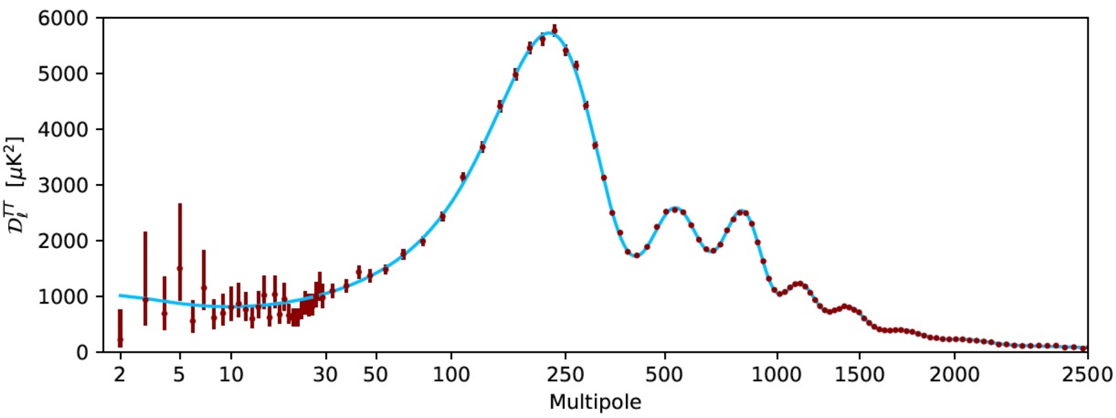

Figure 2.6: This is the latest plot for the temperature multipoles power spectrum, [11]. The position of each peak can indicate the history and the composition of the Universe. In the figure is plotted $\mathcal{D}_l^{TT} \equiv l(l+1)C_l/(2\pi)$, not exactly C_l. This is because for large scales, i.e. small l the dominant effect is the so called *Sachs-Wolfe effect* [51], which predicts $l(l+1)C_l \simeq const.$ when considering small multipoles. Note also that the error bars become larger when considering small ls: this is the effect of *cosmic variance* described above.

2.4 Beyond the standard slow-roll inflation

We conclude mentioning possible alternatives to the standard single-field slow-roll Inflation model. The possibilities for getting inflationary expansion are (maybe frustratingly) varied. Inflation is a paradigm, a framework for a theory of the early Universe, but it is not a unique theory. A large number of phenomenological models has been proposed with different theoretical motivations and observational predictions. In this thesis we will consider one of this class of models, the so-called *Gaugid Inflation* [52], a generalization of the *Solid Inflation*, [53] . However, in this short section we want mention the broader landscape of inflationary model-building (see also Ref. [30]). The simplest inflationary action Eq. (1.71) can be extended in a number of obvious ways

Non-minimal coupling to gravity
The action (1.71) is called minimally coupled in the sense that there is no direct coupling between the inflaton field and the metric. The inflaton interacts with the metric only through the measure of the integral, $\sqrt{-g}$, and the indices contraction in the kinetic term. In principle we could imagine a non-minimal coupling between the inflaton and the graviton, however, in practice, non-minimally coupled theories can be rewritten as minimally coupled theories by a field redefinition.

Modified gravity
Similarly, we could entertain the possibility that the Einstein-Hilbert part of the action is modified at high energies. However, the simplest examples for this UV modification of gravity, so-called $f(R)$ theories, can again be transformed into minimally coupled scalar field with potential $V(\phi)$. With the last Planck measurements it results that the modified gravity theory called $R^2 - inflation$ still survives [48].

Non-canonical kinetic term
The action (1.71) has a canonical kinetic term

$$\mathcal{L}_\phi = X - V(\phi), \qquad X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$$

Inflation can then only occur if the potential $V(\phi)$ is sufficiently flat. More generally, however, we could imagine that the high-energy theory has fields with non-canonical kinetic terms

$$\mathcal{L}_\phi = F(\phi, X) - V(\phi)$$

where $F(\phi, X)$ is some function of the inflaton field and its derivatives. For such an action it is possible that Inflation is driven by the kinetic term and occurs even in the presence of a steep potential.
These are the so-called K-$inflation$ models [54].

More than one field
If we allow more than one field to be dynamically relevant during inflation, then the possibilities for the inflationary dynamics (and the mechanism for the production of fluctuations) expand dramatically and the theory loses a lot of its predictive power.

Breaking the de Sitter spacetime isometries
Until now we have considered a background metric of a (quasi) de Sitter spacetime. We know that this choice was dictated by the request of having an isotropic and homogeneous background. We also know that these two features can be mathematically formalized as the invariance under spatial translations and three-dimensional rotations. One possibility is considering a vacuum state of the field which drives Inflation which manifestly breaks these symmetries but is symmetric under time diffeomorphism, i.e. is time independent while it depends on the coordinate

$$\langle\phi(x)\rangle = \alpha x.$$

To restore the wanted homogeneity and isotropy of the space one impose some internal symmetries. While simple in principle this procedure gives important and nontrivial outcomes that will be deepen in the next chapter.

Chapter 3

Probing Inflation: non-Gaussianities and Tensor Fossils

We have seen that the slow-roll Inflation model provides a very good description of the evolution of the Universe in its first moments. A small number of parameters is needed to define the standard cosmological model, just the first six parameters of in Fig. 1.2, but it is not enough. Knowing the relative abundances of the elements in the Universe does not tell us what these elements are, e.g. dark matter and dark energy. Also knowing the behavior of the power spectrum of the primordial perturbations does not tell us what are their possible sources. The power spectrum alone, indeed, is not sufficient to constrain the interactions of the field which drives Inflation during the primordial era: different Inflation models[1] can produce similar results for the power spectrum prediction [55, 43]. Resuming the slow-roll model we see that it makes the following assumptions:

Single Field
: We have assumed so far that there was only one quantum field responsible for driving Inflation and generating the seeds for large scale structures due to its fluctuations.

Canonical Kinetic Energy
: We have pointed out that the kinetic term used above has a canonical form, which means that the speed of propagation of fluctuations is equal to the speed of light.

Slow Roll
: The time evolution of the field was very slow compared to the Hubble time during Inflation.

Initial Vacuum State
: The quantum field was in the preferred adiabatic vacuum state, the *Bunch-Davies vacuum*.

Assuming *all* these properties for the scalar field implies that the quantum fluctuations of scalar field have an almost Gaussian statistics, i.e. their statistics is completely defined by the two-point correlation function [42, 56]. Hence we will have undetectable level of primordial non-Gaussianities for those models which respect the conditions listed above.

[1] The differences between the Inflation models can be of various types: different interactions, non-standard kinetic term, new physics at high energies, different symmetries.

This results does not surprise too much. Since the inflaton field is driving Inflation has a very flat potential, this means that the interaction terms must be suppressed, hence also the non-linearities eventually producing non-Gaussian features.
Detecting a certain amount of non-Gaussianity (or determining stringent bounds on it) would entail precise constraints on the interactions of the quantum field(s) responsible for inflation, i.e. it will it will crucially help in determining the right theory for the primordial Universe, [32, 43]. The simplest idea to calculate the non-Gaussianity is looking at the predictions on the three-point correlation function of the primordial perturbations generated during inflation, given that it would vanish in the Gaussian case. Indeed we have seen in Chapter 2 that a perturbation with a Gaussian statistics is completely defined by its 2-point correlator, while all the odd-point correlation functions are null. In presence of a non-Gaussian statistics we would have a non-zero 3-point function or, equivalently, a 4-point function which cannot be rewritten in terms of the 2-point function. Calculating a non-zero 3-point function is the first sign of a deviation from the "standard" model of Inflation.
In the following we will calculate what are the prediction for the bispectrum, the Fourier transform of the three-point function. We have already mentioned the importance of an observation of the primordial gravitational wave background, because it would provide a striking proof for Inflation. In this chapter we will briefly describe the importance of non-Gaussianities and the prediction for the standard inflationary theories. Then we will consider an interesting effect from a gravitational wave background: tensor fossils and a related quadrupole perturbation in the power spectrum of the galaxy clustering. We will review what are the consequences of a stochastic background of gravity waves both in the primordial Universe and at the present time.

3.1 Primordial non-Gaussianities

In the previous chapters we have seen the predictive power of the inflationary models. While the latest observations on the CMB have ruled out various Inflation models, there is still a sort of "theory degeneracy". There are still various models which fit the observational data at our disposal. A good understanding of the power spectrum of the theory of Inflation might not be sufficient to provide a good description of the *physics* of Inflation [55]. The non-Gaussianities would be crucial from this point of view. Different theories can make similar predictions for the power spectrum; the true difference might arise in the three-point function, or in its Fourier transform, the Bispectrum.

3.1.1 Bispectrum

In chapter 2 we have seen that in the case of a Gaussian field the three-point correlation function vanish. Therefore if we consider a non-Gaussian field, we have that the first correlator which manifests non-Gaussianity is the three-point function, $\langle\delta(x_1)\delta(x_2)\delta(x_3)\rangle$. For our purpose we are interested in the Fourier-space of the three-point function $\langle\delta_{\mathbf{k}_1}\delta_{\mathbf{k}_2}\delta_{\mathbf{k}_3}\rangle$. Analogously to Eq. (2.49) it is possible to exploit the homogeneity and isotropy of the background to parametrize the three-point function as follows

$$\langle\delta_{\mathbf{k}_1}\delta_{\mathbf{k}_2}\delta_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\mathcal{B}_\delta(k_1,k_2,k_3).$$

The momentum conservation is a consequence of the homogeneity, while the dependence of $\mathcal{B}_\delta$ only on the modulus of the three momenta derives from the isotropy. $\mathcal{B}_\delta$ is the so-called *bispectrum* of the perturbation field δ. We have mentioned above the importance of this new correlator and the consequences which arise if a non-zero value would be measured. In the following we will consider the bispectrum of the ζ perturbation, because of its importance in cosmology and because it is directly related to the to the CMB angular bispectrum [57, 43]:

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (2\pi)^2 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \mathcal{B}_{\zeta\zeta\zeta}(k_1, k_2, k_3). \tag{3.1}$$

In the formula (3.1) is not explicitly exposed the time at which we evaluate the correlator. It is implicitly assumed that we evaluate it in the large scale limit, which correspond to the time at the end of Inflation. The delta function ensuring the momentum conservation entails that the three momenta of the perturbation fields form a triangle: $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$. We will see that the shape of this triangle will be of fundamental importance. In general we can parametrize the bispectrum as

$$\mathcal{B}_{\zeta\zeta\zeta}(k_1, k_2, k_3) = S(k_1, k_2, k_3) \frac{\Delta_\zeta^2(k_*)}{k_1^2 k_2^2 k_3^2}, \tag{3.2}$$

where $\Delta_\zeta^2(k_*)$ is the dimensionless power spectrum, defined in Eq. (2.50), of ζ evaluated at a fixed scale k^*. The function S is adimensional and, in the case of scale-invariant bispectra, it is invariant under the rescaling of all the three momenta k_i. The function S also allows to define the momentum dependence of the bispectrum. In fact there are two types of the momentum dependence, the shape of the bispectrum and the running of the bispectrum:

The *shape* of the bispectrum is the dependence of the function S on the ratios of the momenta k_2/k_1 and k_3/k_1, while we fix the overall momentum $K = (k_1 + k_2 + k_3)/3$;

The *running* of the bispectrum is the dependence of S on the overall momentum K, while we take constant the ratios between the momenta.

It is possible to give a first evaluation of the amplitude of non-Gaussianities through the definition of a parameter which provides a measure for the non linearities of the primordial perturbations, the so called non-linearity parameter f_{NL}. It can be defined as the bispectrum in the equilateral configuration, $k1 = k2 = k3 = K$, normalized for the square of the power spectrum of the perturbation ζ evaluated at the momentum k. In formula it reads

$$f_{NL} = \frac{5}{18} \frac{\mathcal{B}_{\zeta\zeta\zeta}(K, K, K)}{\mathcal{P}_\zeta^2(K)}. \tag{3.3}$$

The factor 5/18 comes from the relation between the Bardeen's gravitational potential and ζ during the matter dominated epoch, see Ref. [57]. This dimensionless amplitude tells us if a particular shape of non-Gaussianity is detectable or not by the experiment. From the definition given above we can see that it depends on k. Moreover, if we substitute (3.3) into (3.2) we have

$$f_{NL} = \frac{5}{18} S(K, K, K)$$

i.e. f_{NL} corresponds to the shape function in the equilateral limit times a 5/18 factor. If the bispectrum is scale invariant in general we can extract the amplitude f_{NL} from the shape function S and parametrize the bispectrum of ζ as:

$$\mathcal{B}_{\zeta\zeta\zeta}(k_1, k_2, k_3) = \frac{18}{5} f_{NL} S(k_1, k_2, k_3) \frac{\Delta_\zeta^2(k_*)}{k_1^2 k_2^2 k_3^2},$$

where the shape function is normalized as $S(k,k,k) = 1$. The predictive power of the three-point function resides in the fact that different models of Inflation predict different shapes of the non-Gaussianities. Following [58, 57, 32, 43] we can find different examples of non-Gaussianities depending on the particular shape of the considered triangle:

Local shape of non-Gaussianity: a local non-Gaussian shape arises from a local (point by point in the real space) non-linear correction to the perturbation ζ_g, where the suffix g denotes that this perturbation coincides with the linear, Gaussian, perturbation ζ analyzed in the previous chapter. We can rewrite the new non-linear ζ as:

$$\zeta(\mathbf{x}) = \zeta_g(\mathbf{x}) + \frac{3}{5} f_{NL}^{loc} \left[\zeta_g^2(\mathbf{x}) - \langle \zeta_g^2(\mathbf{x})\rangle\right]. \tag{3.4}$$

As explained in Ref. [59, 43, 56], this non-linearity parameter is defined for the Bardeen Newtonian potential in matter dominance, which implies, at linear level

$$\Phi_g = \frac{3}{5}\zeta;$$

this explains the factor 3/5 in the definition. This type of non-Gaussianity is called *local* because the non-linear relation (3.4) is locally defined. In Eq. (3.4) already appears the amplitude of non-Gaussianity produced. The bispectrum in this case turns out to be:

$$\mathcal{B}^{loc}_{\zeta\zeta\zeta}(k_1, k_2, k_3) = \frac{6}{5} f_{NL}^{local} \left[\mathcal{P}_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_3)\right] \tag{3.5}$$

where, as usual, $\mathcal{P}_\zeta(k)$ is the power spectrum of the comoving curvature perturbation ζ. It is possible to verify, going in the equilateral limit, the Eq. (3.3). Considering the limit in which the bispectrum is (almost) scale invariant we obtain:

$$\mathcal{B}^{loc}_{\zeta\zeta\zeta}(k_1, k_2, k_3) = \frac{6}{5} f_{NL}^{loc} \frac{\Delta_\zeta^2(k_*)}{k_1^2 k_2^2 k_3^2}\left(\frac{k_1^2}{k_2 k_3} + \frac{k_2^2}{k_1 k_3} + \frac{k_3^2}{k_1 k_2}\right), \tag{3.6}$$

where Δ_ζ is the dimensionless power spectrum of the perturbation ζ in the limit in which we neglect the scale dependence.
Thus, the template for the local shape reads

$$S_{loc}(k_1, k_2, k_3) = \frac{1}{3}\left(\frac{k_1^2}{k_2 k_3} + \frac{k_2^2}{k_1 k_3} + \frac{k_3^2}{k_1 k_2}\right).$$

The signal of the local non-Gaussianities peaks in the so-called *squezeed* configuration of the triangle ($k_1 \ll k_2 \simeq k_3$) and it can arise in the case of multi-fields models of inflation, [43, 32, 58, 60].

Equilateral shape of non-Gaussianity: this is a shape which is peaked in the equilateral configuration $k_1 = k_2 = k_3 = K$ and it arises a consequence of several Inflation models such as k-Inflation [54], or in general models in which a non-canonical kinetic term is present [43, 32, 58, 60]. The corresponding shape function has the form:

$$S_{equil}(k_1, k_2, k_3)\left(\frac{k_1}{k_3} + 5\text{perms.}\right) - \left(\frac{k_1^2}{k_2 k_3} + 2\text{perms.}\right) - 2.$$

This kind of shape arise from considering higher derivative interactions in the action of several inflationary models of Inflation.

Orthogonal shape of non-Gaussianity: this is another shape of non-Gaussianity which arise, as the equilateral shape, from considering higher derivative interactions in inflationary models. The template associated to this shape is:

$$S_{ortho}(k_1,k_2,k_3) = -3.84\left(\frac{k_1^2}{k_2k_3} + 2\text{perms.}\right) + 3.94\left(\frac{k_1}{k_3} + 5\text{perms.}\right) - 11.10. \tag{3.7}$$

This non-Gaussianity is also present in single-field models of Inflation with a non-canonical kinetic term, or with general higher order-derivative interactions.

For more details about these shapes and the mechanism that can create such non-Gaussianities we refer the reader to Ref. [57, 43, 32, 58, 60]. In the same references the latest results of the Planck satellite (updated to year 2015) on primordial non-Gaussianities are exposed. The experimental constraints for the three amplitudes of primordial non-Gaussianities relative to the measurement of the CMB temperature anisotropies are the following

$$f_{NL}^{loc} = 2.5 \pm 5.7 \text{ (68\%CL)} \tag{3.8}$$

$$f_{NL}^{equil} = -16 \pm 70 \text{ (68\%CL)} \tag{3.9}$$

$$f_{NL}^{ortho} = -34 \pm 33 \text{ (68\%CL)} \tag{3.10}$$

Combining temperature and polarization data the Planck collaboration finds the following constraints

$$f_{NL}^{loc} = 0.8 \pm 5.0 \text{ (68\%CL)} \tag{3.11}$$

$$f_{NL}^{equil} = -4 \pm 43 \text{ (68\%CL)} \tag{3.12}$$

$$f_{NL}^{ortho} = -26 \pm 21 \text{ (68\%CL)} \tag{3.13}$$

These measurements represent the tightest constraints that have been obtained so far from CMB measurements (and for temperature CMB data the ultimate ones). As it can be seen these constraints are compatible with Gaussianity. However notice that there is still a very large window between the single-field predictions for f_NL (of the order of the slow-roll parameters [56, 42]) and the present error bars. So why is actually so important trying to reduce the errors for better constraining such non-Gaussianities? As we will see in detail in the next section, it is possible to compute at the leading order in slow-roll parameters the bispectrum of the gauge invariant perturbation ζ with a non linear extension of the slow-roll models of Inflation. We anticipate that the amplitude of such bispectrum is suppressed in the slow-roll limit: such a low level of primordial non-Gaussianity is (at present) impossible to measure. Therefore any signals of non-Gaussianities may come only from an extension or a modification of the slow-roll theories of Inflation which, for the moment, are the most accepted paradigms for describing Inflation. A violation of just one of the standard single field slow-roll (SFSR) conditions listed above would entail a high level of non-Gaussianity, $|f_{NL}| \gg 1$ [55]. Hence a detection of such a signal would rule out the simplest model of Inflation described in the previous chapter.
In particular, such non-Gaussianities can be signatures of possible modifications of the law of physics at the energy scales at which Inflation can take place that are no still achievable in the actual colliders. The reason is that contributions on non-Gaussianities from Inflation of the gauge invariant variable ζ arise essentially by auto-interaction terms of ζ and by interaction terms between ζ and the primordial gravitational waves γ_{ij} (or any other possible field present during inflation). These interactions in the standard slow-roll models of Inflation are suppressed. So signals of non-Gaussianity can be signatures of interaction terms between scalar perturbation and new fields associated to new degrees of freedom that could appear at high energies in a new physics scenario.

3.1.2 In-In Formalism

Now our aim is to perform an explicit computation of the bispectrum $\mathcal{B}_{\zeta\zeta\zeta}$, following [43, 56].
So far we have always used as observables of inflationary models the various n-point correlation functions

$$\langle \delta_1 \dots \delta_n \rangle,$$

where the brackets $\langle\rangle$ denotes the ensemble average. Now, following [32] and the exhaustive [61] we will briefly describe the *in-in formalism*, a useful formalism to calculate the various correlation functions.
In general we consider a perturbation $\delta(t, \mathbf{x})$, that must be quantized, as we have done so far for all the perturbations we have encountered (scalar, tensor) during Inflation. In general we are dealing with correlators of the type:

$$\langle \Omega | \, \delta(t, \mathbf{x}_1) \dots \delta(t, \mathbf{x}_n) \, |\Omega\rangle \, . \tag{3.14}$$

Here $|\Omega\rangle$ represents the vacuum state of our full theory, i.e. also considering the interactions of the field, not only the vacuum state of the free particle theory as it is used in QFT. We will deepen this argument in the following. It is clear from Eq. (3.14) that we are working in the *Heisenberg picture*, where only the operators evolve in time, and states do not. In order to explain the method we will work within the Hamiltonian formalism. We know indeed that the predictions in the Lagrangian formalism and in the Hamiltonian one must be equal[2]. We can decompose the Hamiltonian function into a quadratic part H_0 and some interaction terms H_{int} as:

$$H_{tot} = H_0 + H_{int}. \tag{3.15}$$

The quadratic part describes essentially the free evolution of the field δ. The fundamental step of the in-in formalism consists in switching from the Heisenberg picture to the *interaction picture*. The operator in the interaction picture, which we will indicate with $\delta^I(t)$[3], is related to the corresponding one in the Heisenberg picture at the time t, $\delta(t)$ by the relation [61]

$$\delta^I(t) = F(t, t_0)\delta(t)F^{-1}(t, t_0), \tag{3.16}$$

where we have defined

$$F(t, t_0) = T \exp\left\{ -i \int_{t_0}^{t} H^I_{int}(t')dt' \right\} \tag{3.17}$$

where T indicates the time-ordered operator. H^I_{int} is the interaction Hamiltonian in the interaction picture which coincides with the one in the Heisenberg picture. The time t_0 is the time in which we switch on the interaction $H_{int}(t)$.
If we insert (3.16) into (3.14) we have

$$\langle \Omega | \left[\bar{T} \exp\left\{ -i \int_{t_0}^{t} H^I_{int}(t')dt' \right\} \right] \delta^I(t, \mathbf{x}_1) \dots \delta^I(t, \mathbf{x}_n) \left[T \exp\left\{ -i \int_{t_0}^{t} H^I_{int}(t')dt' \right\} \right] |\Omega\rangle \, , \tag{3.18}$$

where $\bar{T}$ is now the anti-time-ordered operator.
In addition the relation between the Hamiltonian and the Lagrangian in the interaction picture is:

$$H^I_{int} = -L^I_{int}. \tag{3.19}$$

[2]In addition, we know that, except for derivative terms in the interaction part, we have $L_{int} = -H_{int}$.
[3]For the moment we will omit the space dependence since we are interested in the time evolution of the field.

In fact the Legendre transform which links Hamiltonian formalism to Lagrangian one reads like $H^I \sim (\dot{\delta}\pi_\delta - L)$, where π_δ is the conjugate momentum of the field δ. But the term $\dot{\delta}\pi_\delta$ is a quadratic term and, if we consider only the interaction terms, the equality (3.19) follows. We can extend this consideration also for the case where there is more than one field in the theory. Then, if we compute the interaction terms in the Lagrangian of the theory, we can compute perturbatively the correlator (3.18) by expanding the time(anti)-ordered exponentials. If we drop the expansion of the exponentials up to first order and we use Eq. (3.19), we obtain:

$$\langle \delta(t, \mathbf{x}_1) \dots \delta(t, \mathbf{x}_n) \rangle = i \int_{t_0}^{t} dt' \, \langle \Omega | \left[\delta^I(t, \mathbf{x}_1) \dots \delta^I(t, \mathbf{x}_n), L_{int} \right] | \Omega \rangle \,. \tag{3.20}$$

As a final consideration we should remark that the vacuum $|\Omega\rangle$ is the vacuum of the full theory, including also interaction terms in the theory. If we call $|0\rangle$ the vacuum of the theory whose action is dropped at quadratic order in the fields (which is the free vacuum of the theory), we would like to write $|\Omega\rangle$ as a function of $|0\rangle$. The reason is that $|0\rangle$ is the vacuum that we have introduced in Chapter 2 to quantize the primordial cosmological perturbations, and so we know how the creation and annihilation operators act on it.
In studying scattering processes in QFT in general the two vacuum states do not coincide due to vacuum fluctuations caused by the interactions. But in our case we are evaluating expectation values. These processes do not generate any non-trivial vacuum fluctuations through interactions. This is a direct consequence of the identity:

$$F^{-1} F = 1$$

where F is defined in Eq. (3.17).
For this reason we can replace $|\Omega\rangle$ with $|0\rangle$ in (3.20), see [61]. This fact is crucial for doing the computations. In fact the fields in the interaction picture evolve as in the free quadratic case. Thus, if we know the free solutions in terms of annihilation and creation operators a and $a^\dagger$ (which are the ones we have introduced in Chapter 2), we can do easily the contractions with the free vacuum state $|0\rangle$.

3.1.3 Computation of the bispectrum $\zeta\zeta\zeta$

Using the in-in formalism, now we want to evaluate the three-point function (3.1) in the single field slow-roll model exposed in [56]. We perform a three level computation, so we have

$$\langle \zeta(k_1, \tau_e) \zeta(k_2, \tau_e) \zeta(k_3, \tau_e) \rangle = -i \int_{\tau_i}^{\tau_e} d\tau' a \, \langle 0 | \left[\zeta(k_1, \tau_e) \zeta(k_2, \tau_e) \zeta(k_3, \tau_e), H_{int}(\tau') \right] | 0 \rangle \,. \tag{3.21}$$

The time at which we evaluate the correlators in this expression is $\tau_e = 0$, corresponding to the end of a Inflation and to the super-horizon limit, and the switching on of the interactions is kept when the fluctuation modes are on very sub-horizon scales, corresponding to $\tau_i \to \infty$. The Wick theorem guarantees that bra-ket contractions with the vacuum state are non zero only if the interaction Hamiltonian has the same number of fields of the operator on the left hand side of the commutator operator. For this reason to evaluate the right hand side of equation (3.21) we need to compute the Hamiltonian cubic in the field ζ and its derivatives. In order to describe the third order Hamiltonian, it is useful to define perturbations and the inflationary action in the Arnowitt-Deser-Misner (ADM) form [56, 62, 40]. In this formalism the metric tensor is described by

$$ds^2 = -(N^2 - N_i N^i) dt^2 + 2 N_i dx^i dt + a^2(t) h_{ij} dx^i dx^j \,, \tag{3.22}$$

where h_{ij} takes the form described in the previous chapter (2.27) and (2.30). The action in Eqs. (1.71) and (1.72) can be rewritten as

$$S = \frac{1}{2} \int d^4x \sqrt{h} \Big[N R^{(3)} + N (K_{ij} K^{ij} - K^2) + N^{-1} (\dot{\phi} - \partial_i N^i)^2 - N h^{ij} \partial_i \phi \partial_j \phi - N V \Big], \tag{3.23}$$

where we have put the reduced Planck mass equal to one for simplicity of notation. It will be easily restored by dimensional analysis. In addition $h = -\det(h_{ij})$ and K_{ij} is the extrinsic curvature tensor defined as

$$K_{ij} = \frac{1}{2N} \left[\nabla_i^{(3)} N_j + \nabla_j^{(3)} N_i - \dot{h}_{ij} \right], \quad K^2 = K_i^i \tag{3.24}$$

$\nabla_i^{(3)}$ is the covariant derivative computed with the three-metric h_{ij} instead of the full metric $g_{\mu\nu}$. $R^{(3)}$ is the curvature scalar computed with the three-metric.

The advantage of the ADM formalism is that it allows us to perform more easily a non-linear analysis, in particular as far as the count of the number of the propagating degrees of freedom is concerned. In fact the fields N and N_i are not dynamical and can be expanded in power of series of the dynamical degrees of freedom [56]. The zero order value of this expansion is fixed by the background, namely $N_{(0)} = 1, N^i_{(0)} = 0$. To find the other orders, we have to derive from the action (3.23) the Euler-Lagrange equations for N and N_i and then to solve them order by order. If we are interested to an expansion of the action at cubic order in the dynamical fields, we need to know the expressions of N and N_i only at first order. Indeed it turns out that the third order terms in N , N_i would be multiplying the constraint equations evaluated at zeroth order, i.e. they gives null contribution to the total action. The second order terms in N, N_i multiply the constraints evaluated to first order, which vanish due to the first order expressions for N and N_i, see Ref. [56].

We fix now the *spatially flat gauge*, in which all the scalar perturbations of the three-metric are removed, leaving only the scalar perturbation of the inflaton $\delta\phi$, besides the ones in N and N^i. Then one is free to remove also the vector perturbation remaining with a three-metric h_{ij} of the form

$$h_{ij} = a^2 [\delta_{ij} + \gamma_{ij}], \quad \gamma_i^i = \partial^i \gamma_{ij}, \tag{3.25}$$

together with the scalar Inflation perturbations $\delta\phi$. In the gauge (3.25) we can connect the perturbation of the inflaton $\delta\phi$ to the curvature perturbation ζ through the linear relation [40, 43]

$$\delta\phi = -\frac{\dot{\phi}}{H} \zeta. \tag{3.26}$$

Following the classical field theory approach, we should derive the equations of motion for the fields N and N_i by doing the functional derivatives $\delta S / \delta N$ and $\delta S / \delta N_i$ respectively and putting them equal to zero. After some calculations and throwing away all terms but the first order ones we find the solutions

$$N^{(1)} = \frac{\dot{\phi}_0}{2 H M_{Pl}^2} \delta\phi, \quad N_i^{(1)} = \frac{\partial_i \chi}{M_{Pl}^2}, \quad \chi = -a^2 \frac{\dot{\phi}_0}{2H^2} \partial^{-2} \left[\frac{d}{dt} \left(\frac{H \delta\phi}{\dot{\phi}_0} \right) \right], \tag{3.27}$$

where the Planck mass has been restored through a dimensional analysis. Using all these steps, we compute the cubic interaction Lagrangian for ζ_1 (which is the first order expression of ζ) at the leading order in the slow parameters and find [56]

$$L_{int} = \epsilon_V^2 M_{Pl}^2 \int d^3x \left[a^3 \zeta_1 \dot{\zeta_1}^2 + a \zeta_1 (\partial_i \zeta_1)(\partial^i \zeta_1) - 2 a^3 \dot{\zeta}_1 (\partial_i \partial^{-2} \dot{\zeta}_1)(\partial^i \zeta_1) \right]. \tag{3.28}$$

Given that we have found all the solutions in the conformal time we express the action with respect to the conformal time:

$$L_{int} = \epsilon_V^2 M_{Pl}^2 \int d^3x \left[a\zeta_1\zeta_1'^2 + a\zeta_1(\partial_i\zeta_1)(\partial^i\zeta_1) - 2a\zeta_1'(\partial_i\partial^{-2}\zeta_1')(\partial^i\zeta_1) \right]. \tag{3.29}$$

The interaction Hamiltonian is simply minus the interaction Lagrangian because it takes only the potential terms, as we have seen above. In these expressions we have omitted the suffix *int* for the fields, it is understood that we are evaluating them in the interaction picture. From now on to the end of the chapter we will use this convention.
If we insert the Fourier decomposition of the field ζ_1

$$\zeta_1(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \zeta_1(\tau, k) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.30}$$

into the Hamiltonian, we find

$$\begin{aligned} H_{int}(\tau) = -\frac{\epsilon_V M_{Pl}^2}{(2\pi)^6} \int d^3k\, d^3p\, d^3q\, \delta^{(3)}(\mathbf{k}+\mathbf{p}+\mathbf{q}) \Big[& a\zeta_1(k)\zeta_1'(p)\zeta_1'(q) \\ & - a(\mathbf{p}\cdot\mathbf{q})\zeta_1(k)\zeta_1(p)\zeta_1(q) - 2a\frac{(\mathbf{p}\cdot\mathbf{q})}{p^2}\zeta_1'(k)\zeta_1'(p)\zeta_1(q) \Big]. \end{aligned} \tag{3.31}$$

The Dirac delta comes from an integration $\int d^3x\, e^{i(\mathbf{k}+\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}$, where the integral is the one in the definition of the Hamiltonian. Putting Eq. (3.31) into Eq. (3.21) we obtain

$$\begin{aligned} \langle \zeta_1(k_1)\zeta_1(k_2)\zeta_1(k_3) \rangle = & i\frac{\epsilon^2 M_{Pl}^2}{(2\pi)^6} \int d^3K \delta^{(3)}(\mathbf{k}+\mathbf{p}+\mathbf{q}) \times \\ & \times \int_{-\infty}^{0} d\tau' a[\mathcal{A}_1(\tau') + \mathcal{A}_2(\tau') + \mathcal{A}_3(\tau')], \end{aligned} \tag{3.32}$$

where we have used the notation $d^3K = d^3k\, d^3p\, d^3q$ and the functions $\mathcal{A}_n$ stands for the contractions

$$\begin{aligned} \mathcal{A}_1 &= a \langle 0| [\zeta_1(k_1,\tau_e)\zeta_1(k_2,\tau_e)\zeta_1(k_3,\tau_e), \zeta_1(k,\tau')\zeta_1'(p,\tau')\zeta_1'(q,\tau')] |0\rangle \\ \mathcal{A}_2 &= -a(\mathbf{p}\cdot\mathbf{q}) \langle 0| [\zeta_1(k_1,\tau_e)\zeta_1(k_2,\tau_e)\zeta_1(k_3,\tau_e), \zeta_1(k,\tau')\zeta_1(p,\tau')\zeta_1(q,\tau')] |0\rangle \\ \mathcal{A}_3 &= -2a\frac{(\mathbf{p}\cdot\mathbf{q})}{p^2} \langle 0| [\zeta_1(k_1,\tau_e)\zeta_1(k_2,\tau_e)\zeta_1(k_3,\tau_e), \zeta_1'(k,\tau')\zeta_1'(p,\tau')\zeta_1(q,\tau')] |0\rangle \end{aligned} \tag{3.33}$$

From the Wick Theorem we know some rules that simplify the evaluation of these contractions. We have to sum over the terms obtained by doing all the possible bra-ket contractions with the vacuum states between couples of fields evaluated at different times. The terms in which at least one field remains uncontracted are vanishing. From the form of our interaction Hamiltonian we need to compute the following two contractions

$$\begin{aligned} &\langle 0| \zeta_1(k,\tau)\zeta_1(k',\tau') |0\rangle, \\ &\langle 0| \zeta_1(k,\tau)\zeta_1'(k',\tau') |0\rangle. \end{aligned} \tag{3.34}$$

Inserting the solution of ζ in the form (2.55) we can write the two contractions in term of the mode function

$$\begin{aligned} \langle 0| \zeta_1(k,\tau)\zeta_1(k',\tau') |0\rangle &= \langle 0| [f_k(\tau)a_k + f_k^*(\tau)a_{-k}^\dagger][f_{k'}(\tau')a_{k'} + f_{k'}^*(\tau')a_{-k'}^\dagger] |0\rangle \\ &= (2\pi)^3\delta(k+k')f_k(\tau)f_{k'}^*(\tau'), \end{aligned} \tag{3.35}$$

$$\begin{aligned}\langle 0|\,\zeta_1(k,\tau)\zeta_1'(k',\tau')\,|0\rangle &= \langle 0|\,[f_k(\tau)a_k + f_k^*(\tau)a_{-k}^\dagger]\frac{d}{d\tau}[f_{k'}(\tau')a_{k'} + f_{k'}^*(\tau')a_{-k'}^\dagger]\,|0\rangle \\ &= (2\pi)^3\delta(k+k')f_k(\tau)\frac{d}{d\tau}f_{k'}^*(\tau').\end{aligned} \tag{3.36}$$

Now we can compute the $\mathcal{A}_n$ functions

$$\begin{aligned}\mathcal{A}_1 =&(2\pi)^9 a\left[f_{k_1}(0)f_{k_2}(0)f_{k_3}(0)\left(\frac{d}{d\tau}f_{k_1}^*(\tau')\right)\left(\frac{d}{d\tau}f_{k_2}^*(\tau')\right)f_{k_3}^*(\tau') - c.c.\right] + \mathrm{perm}(k_i),\\ \mathcal{A}_2 =&-(2\pi)^9 a(\mathbf{k}_1\cdot\mathbf{k}_2)\left[f_{k_1}(0)f_{k_2}(0)f_{k_3}(0)f_{k_1}^*(\tau')f_{k_2}^*(\tau')f_{k_3}^*(\tau') - c.c.\right] + \mathrm{perm}(k_i).\\ \mathcal{A}_3 =&(2\pi)^9(-2a)\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{k_1^2}\times\\ &\times\left[f_{k_1}(0)f_{k_2}(0)f_{k_3}(0)\left(\frac{d}{d\tau}f_{k_1}^*(\tau')\right)f_{k_2}^*(\tau')\left(\frac{d}{d\tau}f_{k_3}^*(\tau')\right) - c.c.\right] + \mathrm{perm}(k_i).\end{aligned} \tag{3.37}$$

The permutations over the k_i come from all the different ways of contracting the fields and the minus complex conjugate comes from the commutator operator between fields in the expression of $\mathcal{A}_n$. Inserting the expression (3.37) into (3.32) we can write

$$\begin{aligned}\langle\zeta_1(k_1)\zeta_1(k_2)\zeta_1(k_3)\rangle =& i(2\pi)^3\delta(k_1+k_2+k_3)\epsilon_V M_{Pl}^2\times\\ &\times Im[I_1 - (k_1\cdot k_2)I_2 - 2\frac{(k_1\cdot k_2)}{k_1^2}I_3] + \mathrm{perm}(k_i),\end{aligned} \tag{3.38}$$

where the I_n are the integrals

$$I_1 = f_{k_1}(0)f_{k_2}(0)f_{k_3}(0)\int_{-\infty}^0 d\tau' a^2\left[\left(\frac{d}{d\tau}f_{k_1}^*(\tau')\right)\left(\frac{d}{d\tau}f_{k_2}^*(\tau')\right)f_{k_3}^*(\tau')\right],$$

$$I_2 = f_{k_1}(0)f_{k_2}(0)f_{k_3}(0)\int_{-\infty}^0 d\tau' a^2\left[f_{k_1}^*(\tau')f_{k_2}^*(\tau')f_{k_3}^*(\tau')\right],$$

$$I_1 = f_{k_1}(0)f_{k_2}(0)f_{k_3}(0)\int_{-\infty}^0 d\tau' a^2\left[\left(\frac{d}{d\tau}f_{k_1}^*(\tau')\right)f_{k_2}^*(\tau')\left(\frac{d}{d\tau}f_{k_3}^*(\tau')\right)\right].$$

In order to perform these integrals we need an analytic expression for the mode function $f_k(\tau)$. We have seen that it is sufficient to use the free mode function, i.e., we need to multiply $a(\tau)$ to the Bunch-Davies solution for the Sasaki-Mukhanov variable v_τ, which yields

$$f_k(\tau) = \frac{iH}{M_{Pl}\sqrt{2\epsilon_V k^3}}(1+ik\tau)e^{-ik\tau}, \tag{3.39}$$

where we have also restored the correct normalization factor for the variable $\zeta(x,\tau)$ [56, 32]. In such a computation (at lowest-order in the slow-roll parameters), we can use the de-Sitter approximation for the scale factor $a(\tau) = -1/(H\tau)$ with $H \simeq const$. The time derivative of f is

$$\frac{d}{d\tau}f_k(\tau) = \frac{iH}{M_{Pl}\sqrt{2\epsilon_V k^3}}k^2\tau e^{-ik\tau}.$$

To perform the computation of the I_n we are dealing with integrals of the type

$$I(n,K) = \int_{-\infty}^0 dx x^n e^{iKx} = (-1)^n\frac{\Gamma(n+1)}{(iK)^{n+1}} = (-1)^n\frac{n!}{(iK)^{n+1}}, \tag{3.40}$$

where the second equalities holds for natural numbers. Starting with I_1 we have

$$I_1 = -H_*^4 k_1^2 k_2^2\left(\prod_{i=1}^3\frac{1}{M_{Pl}^2 2\epsilon_V k_i^3}\right)\int_{-\infty}^0 d\tau'(1-ik_3\tau')e^{iK\tau'}. \tag{3.41}$$

Here the suffix $*$ indicates that the corresponding quantity is evaluated at horizon crossing time. This seems to create an ambiguity because we have three different modes that exit from the horizon at different conformal times. In order to solve this ambiguity we choose the time of horizon crossing of the momentum $K = k_1 + k_2 + k_3$, that corresponds to a time at which we are sure that all the three momenta have already left the horizon. In the de Sitter solution H is constant so that definition does not cause any problem. In order to evaluate the integral (3.41) we have to correct the oscillatory behavior at $\tau \to -\infty$ of the exponential. We achieve this by performing a Wick rotation of the real axis. We promote the real integration variable to a complex one and do the change $\tau'' = i\tau'$. The form (3.41) becomes now

$$I_1 = iH_*^4 k_1^2 k_2^2 \left(\prod_{i=1}^{3} \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \int_{-\infty}^{0} d\tau'' (1 - ik_3\tau'') e^{K\tau''}. \tag{3.42}$$

Using Eq. (3.40) we can integrate to obtain

$$I_1 = iH_*^4 k_1^2 k_2^2 \left(\prod_{i=1}^{3} \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \left(\frac{k_1^2 k_2^2}{K} + \frac{k_1^2 k_2^2 k_3}{K^2} \right). \tag{3.43}$$

We now evaluate

$$I_2 = -H_*^4 k_1^2 k_2^2 \left(\prod_{i=1}^{3} \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \int_{-\infty}^{0} \frac{d\tau'}{(\tau')^2} (1 - ik_1\tau')(1 - ik_2\tau')(1 - ik_3\tau') e^{iK\tau'}. \tag{3.44}$$

In computing some integrals in this equation we will make use again of (3.40), but we need also another type of integral, which reads

$$\begin{aligned}
\bar{I} &= \int_{-\infty}^{0} \frac{dx}{x^2} (1 - iKx) e^{iKx} \\
&= \int_{-\infty}^{0} \frac{dx}{x^2} e^{iKx} - iK \int_{-\infty}^{0} \frac{dx}{x} e^{iKx} \\
&= -\frac{e^{iKx}}{x}\bigg|_{-\infty}^{0} + iK \int_{-\infty}^{0} \frac{dx}{x} e^{iKx} - iK \int_{-\infty}^{0} \frac{dx}{x} e^{iKx} \\
&= \lim_{x\to 0} \left(-\frac{1}{x} e^{iKx} \right) = \lim_{x\to 0} \left[-\frac{\cos Kx}{x} \right] + \lim_{x\to 0} \left[-i\frac{\sin Kx}{x} \right].
\end{aligned} \tag{3.45}$$

The first limit in the last form gives a real divergent contribution to the integral. Fortunately it does not create any problem, because at the end we have to take only the imaginary parts of the integral we compute. The second limit is finite and pure imaginary

$$\lim_{x\to 0} \left[-i\frac{\sin Kx}{x} \right] = -iK$$

Therefore I_2 can be easily evaluated as

$$I_2 = iH_*^4 \left(\prod_{i=1}^{3} \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \left(K - \frac{k_1k_2 + k_2k_3 + k_1k_3}{K} - \frac{k_1k_2k_3}{K^2} \right). \tag{3.46}$$

The computation of I_3 is very similar to I_1 and it gives

$$I_3 = iH_*^4 (k_1^2 k_3^2) \left(\prod_{i=1}^{3} \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \left(\frac{1}{K} + \frac{k_2}{K^2} \right). \tag{3.47}$$

Inserting Eqs. (3.43), (3.46) and (3.47) into Eq. (3.38) we have, at the leading order in the slow-roll parameters

$$\begin{aligned}\langle\zeta_1(k_1)\zeta_1(k_2)\zeta_1(k_3)\rangle =&(2\pi)^3\delta(k_1+k_2+k_3)4\frac{H_*^4}{M_{Pl}^4}\frac{1}{\epsilon_V}\left(\prod_{i=1}^{3}\frac{1}{2k_i^3}\right)\times\\ &\left[\frac{k_1^2k_2^2+k_2^2k_3^2+k_1^2k_3^2}{K}+\frac{k_1^2k_2^2k_3+k_2^2k_3^2k_1+k_1^2k_3^2k_2}{K^2}\right].\end{aligned} \tag{3.48}$$

This is the three-point function for the variable ζ_1, which is the linear approximation of ζ. On super-horizon scales we can link the two variables by [56]

$$\zeta=\zeta_1+\alpha\zeta_1^2,\quad \alpha=\frac{1}{2}\frac{\ddot{\phi}}{\dot{\phi}H}+\frac{1}{4}\frac{\dot{\phi}^2}{H^2}\zeta_1^2\simeq-\frac{1}{2}\eta_V. \tag{3.49}$$

With the symbol $\simeq$ we mean that α is evaluated at first order in the slow-roll parameters. This means that when passing from ζ_1 to ζ then the three-point function has an additional contribution to the bispectrum. We find [56]

$$\begin{aligned}\langle\zeta(k_1)\zeta(k_2)\zeta(k_3)\rangle =&\langle\zeta_1(k_1)\zeta_1(k_2)\zeta_1(k_3)\rangle+2\alpha(2\pi)^3\delta(k_1+k_2+k_3)\times\\ &\times\left(P_\zeta(k_1)P_\zeta(k_2)+P_\zeta(k_2)P_\zeta(k_3)+P_\zeta(k_1)P_\zeta(k_3)\right),\end{aligned} \tag{3.50}$$

where $P_\zeta(k)=\frac{H_*}{M_{Pl}^2}\frac{1}{2\epsilon_V k^3}$ is the power spectrum for the curvature perturbation ζ on super-horizon scales. At the end we get the three-point correlator

$$\begin{aligned}\langle\zeta(k_1)\zeta(k_2)\zeta(k_3)\rangle =&(2\pi)^3\delta(k_1+k_2+k_3)\left(\sum_{i<j}P_\zeta(k_i)P_\zeta(k_j)\right)\times\\ &\times\left[-\eta_V+2\epsilon_V\left(\frac{k_1^2k_2^2k_3+k_2^2k_3^2k_1+k_1^2k_3^2k_2}{K^2\sum_i k_i^3}\right)\right]+2\epsilon_V\frac{\sum_{i>j}k_i^2k_j^2}{K\sum_i k_i^3}.\end{aligned} \tag{3.51}$$

We finally obtain the bispectrum

$$\begin{aligned}\mathcal{B}_\zeta(k_1,k_2,k_3)=&\left(\sum_{i<j}P_\zeta(k_i)P_\zeta(k_j)\right)\times\\ &\times\left[-\eta_V+2\epsilon_V\left(\frac{k_1^2k_2^2k_3+k_2^2k_3^2k_1+k_1^2k_3^2k_2}{K^2\sum_i k_i^3}\right)\right]+2\epsilon_V\frac{\sum_{i>j}k_i^2k_j^2}{K\sum_i k_i^3}.\end{aligned} \tag{3.52}$$

Now we want to match this result with the non-Gaussianities constrained by the Planck satellite in the CMB anisotropies [57]. In equation (3.52) the fractions that depend on the k_i in the square brackets are approximately of order $\sim\mathcal{O}(1)$ due to momentum conservation [56]. Then we can take as a good approximation

$$\mathcal{B}_\zeta\simeq\left(\sum_{i>j}P_\zeta(k_i)P_\zeta(k_j)\right)(4\epsilon_V-\eta_V). \tag{3.53}$$

This expression corresponds to the bispectrum of the local shape of Non-Gaussianity (3.6), so matching the two expressions we predict

$$(f_{NL}^{loc})_{slow-roll}\simeq\frac{10}{3}\epsilon_V-\frac{5}{6}\eta_V. \tag{3.54}$$

From the experimental constraints on the slow-roll parameters (see Eqs. (1.99) and (1.100)) it follows

$$(f_{NL}^{loc})_{slow-roll} \lesssim 10^{-2} \tag{3.55}$$

This value is very small and definitely compatible with the best constraints on local non-Gaussianity from the Planck experiment, see Eqs (3.11), (3.12) and (3.13), [57, 58]. We have therefore seen that a measurement of these non-Gaussianities would entail a high sensibility of the measurement instrument. We can now proceed through a more general treatment of the bispectrum function in single-field models for inflation, which go also beyond the standard single-filed slow-roll models whose primordial bispectrum we have discussed in this section. We will see that, in a certain limit, it is possible to find some consistency relations which connect the bispectrum and the power spectrum.
An alternative way to calculate the 3-point correlation function for the scalar perturbation, or its Fourier transform the bispectrum, is given in [42]. In this work the in-in formalism is not used, while a second-order perturbative expansion is performed. This method is more similar to the one used in Chapter 2 to calculate the power spectra. One expands the metric the scalar field up to the second order in the perturbations, finds the perturbed Einstein and Klein-Gordon equations and then provides an exact solution for the gauge invariant variables ζ and $\mathcal{R}$, up to the second order in perturbations and in the slow-roll approximation. With this method used in [42], the predictions on the non-Gaussianities for single-field slow-roll model of Inflation are in accordance with the ones of [56].

3.2 Consistency relations

3.2.1 Scalar consistency relation

In [56] some remarkable results about the bispectrum functions of the primordial perturbations, both scalars and tensors have been obtained. In the previous calculation we have considered the most general configuration for the three momenta k_i of the perturbation modes. In this section we will consider a particularly interesting case, the so called *squeezed* limit, in which one of the three momenta is much smaller than the other two. Thanks to the momentum conservation in this limit holds $k_3 \ll k_1 \sim k_2$. In this configuration the mode labeled by k_3 crosses the horizon much earlier than the other two modes. When the two modes $k_{1,2}$ cross the horizon ζ_3 is already constant and its only effect will be to make the comoving scales $k_{1,2}$ cross the horizon at a slightly earlier time. It is possible to give an intuitive estimate of this time shift [63]:

$$\delta t_* \simeq -\frac{\zeta_3}{H}. \tag{3.56}$$

In this limit we obtain the following relation [56]

$$\begin{aligned} \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle &\sim -\langle \zeta_{\mathbf{k}_3} \zeta_{-\mathbf{k}_3} \rangle' \frac{1}{H} \frac{d}{dt_*} \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_1} \rangle \\ &\sim -(n_{s*} - 1) \langle \zeta_{\mathbf{k}_3} \zeta_{-\mathbf{k}_3} \rangle \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle, \end{aligned} \tag{3.57}$$

where n_{s*} is the scalar spectral index and $(n_{s*} - 1)$ measures the deviation from a scale invariant power spectrum. Eq. (3.57) is the so-called *consistency relation* for the scalar perturbations. Despite the assumption of single-field slow-roll, it has been demonstrated that this relation holds in a more general case [59]. This relation, indeed, is a direct consequence of the invariance under space diffeomorphisms of the classical and quantum theory. In addition, one need not necessarily specify the form of the action as long as the

symmetry is in place and a locality requirement is satisfied. The relation (3.57) holds for *all* single-field models of inflation[4] [63].
This statement is of great importance in the studies about the primordial non-Gaussianities. In [64, 59] it is underlined that the presence of only one dynamical field entails that there is only one *clock of the Universe*: the time dependence of the scalar field which drives Inflation fixes the Hubble parameter and fluctuations of the inflaton are equivalent to a relative rescaling of the scale factor in different parts of the Universe. This implicite assumption we have made so far will be a fundamental difference between the "standard" model of Inflation (where here by standard we mean models which respect, in general, the standard isometries of a de Sitter spacetime) and the new proposal of Solid and Gaugid Inflation models, the latter being the main focus of the original results of this Thesis.
We have seen in the previous section that the theoretical prediction about the non-Gaussianities in single-field models of slow-roll Inflation is very small, much smaller of the present observable constraints. This implies an objective difficulty at the present in testing the validity or the violation of the consistency relation for the single-field models in (3.57) with an adequate precision with a measurement of the non-Gaussianity. On the other side, mathematics and physics share a feature which is very appreciated by the researchers: it is very difficult to demonstrate that a theory, or more specifically, a relation is true, but it is very simple to show that a law is false. There are infinite ways to violate a consistency relation, but there is only one way for it to be correct. A violation of the consistency relation would rule out the entire class of ("standard") single-field models of Inflation. For this reason a great work has been spent in searching new models which violate these conditions. The model we will focus on arose exactly with this intention.

3.2.2 Tensor consistency relation

In [56] some consistency relations involving also the tensor degrees of freedom in the squeezed limit have been similarly obtained. From now on we will concentrate on the these degrees of freedom, given their importance for the description of the early Universe, see Section 2.3. The most important three-point correlator is the one which involves two scalar and one graviton[5], in the squeezed limit $k_1 \ll k_2 \simeq k_3$, where k_1 represents the mode of the graviton and k_2, k_3 are the modes of the scalars. It takes the form [56]

$$\langle \gamma^s_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle' \sim -\langle \gamma_{\mathbf{k}^s_1} \gamma_{-\mathbf{k}^s_1} \rangle \epsilon^s_{ij}(\mathbf{k}_1) k^i_2 k^j_3 \frac{\partial}{\partial k^2_2} \langle \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle, \tag{3.58}$$

where $\epsilon^s_{ij}(\mathbf{k}_1)$ is the polarization tensor introduced in Eq. (2.92). The prime denotes the time when k_3 crosses the horizon. Analogously to the scalar case the squeezed limit corresponds to the physical situation in which when k_2, k_3 cross the horizon the tensor mode with k_1 is already frozen so that the fluctuations of ζ will be those that we expect in this deformed geometry. So the effect of a frozen gravitational wave on the scalar perturbation would be to modulate the power spectrum of the scalar perturbation when they cross the horizon. In [59] it is stressed out that this effect can be easily reabsorbed by a simple coordinate redefinition, entailing that every observer goes through the same history and scalar and tensor modes have physical meaning only when they reenter in the horizon. In [56] also the three-point functions for scalar-tensor-tensor and tensor-tensor-tensor, with their respective consistency relations have been computed.

[4] Note that it is a general feature of single-field model for Inflation and it is not required the slow-roll condition, see also [65]. For a critical review of the validity of these consistency relations see e.g. [32, 66].

[5] With graviton we mean the *particle* associated to the tensor fluctuations.

Another, more precise, expression for the tensor-scalar-scalar consistency relation is given in Refs. [63, 67]:

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\gamma^s_{\mathbf{K}}\rangle \xrightarrow{K\ll k_1\sim k_2} (2\pi)^3\delta^{(3)}(\mathbf{K}+\mathbf{k}_1+\mathbf{k}_2)\frac{1}{2}\frac{d\ln\mathcal{P}_\zeta}{d\ln k}\epsilon^s_{ij}(\mathbf{K})\hat{k}^i_1\hat{k}^j_2\mathcal{P}_\gamma(K)\mathcal{P}_\zeta(k), \tag{3.59}$$

where we have defined $\mathbf{k} = (\mathbf{k}_1 + \mathbf{k}_2)/2$. We will see that this consistency relation will allow us to define new observable quantities that allow to provide a new indirect way to detect primordial gravitational waves.
We conclude underlining again that an observation of the violation of the consistency relation, both for scalar and tensor perturbation, would rule out all single-field models of inflation, which, of course, would be an extraordinary result. Now, the question is: are there new observables which allow us a more precise reconstruction of what happened in the early Universe?
The answer is (in theory) yes. In the next sections we will study two possible indirect way to observe the gravitational wave background: the tensor fossils and the galaxy clustering.

3.2.3 Deviation from Statistical isotropy

We briefly describe now, what is the fundamental effect of a long-wavelength gravitational wave (GW) on the curvature perturbation power spectrum. Following [68] we will see that the effect of long-wavelength GW can be seen as a change of coordinates with respect to the standard cosmic frame. In this new frame homogeneity and isotropy hold, so the power spectrum resembles the standard form seen in Chapter 2. When going back to the cosmic frame, we will see, the net effect will be a departure from statistical isotropy. We begin with the standard FLRW perturbed metric. Given that at the first order the tensor and scalar perturbations are decoupled we will consider only the GW

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + \gamma_{ij})\right].$$

The frame in which the line element resembles this equation will be referred to as the Cosmic Frame (CF). Using the equivalence principle it is possible to define a coordinate transformation such that the spacetime appears locally Minkowski. The new coordinates can be defined as

$$\tilde{x}^\mu = \left(\delta^\mu_\nu + \frac{1}{2}\gamma^\mu_\nu\right)x^\nu \tag{3.60}$$

or, at the first order in γ_{ij}

$$x^\mu = \left(\delta^\mu_\nu - \frac{1}{2}\gamma^\mu_\nu\right)\tilde{x}^\nu. \tag{3.61}$$

Here we are considering $\gamma_{0\mu} = 0$. We easily see that this change of coordinates affects only the spatial part of the metric, given that

$$\tilde{\eta} = \left(\delta^0_\nu + \frac{1}{2}\gamma^0_\nu\right)x^\nu = \eta,$$

at the first order. For the spatial part we have

$$dx^i = d\tilde{x}^i - \frac{1}{2}\gamma^i_j d\tilde{x}^j - \frac{1}{2}\tilde{x}^j\partial_\mu\gamma^i_j d\tilde{x}^\mu,$$

so we obtain

$$ds^2 = a^2\left[-d\eta^2 + \delta_{ij}d\tilde{x}^i d\tilde{x}^j - \tilde{x}^j\partial_\mu h_{ij}d\tilde{x}^\mu d\tilde{x}^i\right]. \tag{3.62}$$

We see that in this metric the gravitational waves appears only with their derivative. It is said that in this coordinate system the GW are *gauged away.* This frame with metric (3.62) will be referred to as locally Friedmann frame (LFF), because in this coordinates the metric is locally that of an unperturbed FLRW Universe.
We have seen that on small scales Inflation generates scalar curvature perturbation. Now, for the equivalence principle, scalar perturbation on small scales cannot be affected by the long-wavelength tensor. For this reason we can assume statistical homogeneity and isotropy in the LFF, so that the power spectrum of scalar perturbation can be rewritten as a function of only the modulus of the wavenumber, $\tilde{\mathcal{P}}(\tilde{\mathbf{k}}) = \tilde{\mathcal{P}}(\tilde{k})$. Here we indicate the observed quantities in the LFF with a tilde. Going back to the CF using the Fourier transform of (3.61), i.e., $\tilde{k}_i \to k_i - k_j\gamma_i^j/2$ the power spectrum becomes

$$P(\mathbf{k}) = \tilde{P}(k) - \frac{k_i k_j \gamma^{ij}}{2k}\frac{d\tilde{P}}{dk} + \mathcal{O}(\frac{k_\gamma}{k}\gamma_{ij}, \gamma^2)\ ^6. \tag{3.63}$$

Hence breaking isotropy. This is a particular case of a more interesting class of observable effects we will investigate in the following section.

3.3 Fossils from primordial Universe

In the last few years, much strength has been devoted to the studies about new quantities to constrain the properties about the primordial gravitational waves background. Despite the difficulties of a detection of primordial gravitational waves we have seen that it is possible to have an indirect detection of it using the scalar perturbations. A possibility to detect the primordial gravitational waves we will deal with is provided by the fact that long-wavelength gravitational waves may give rise to an apparent local departure from statistical isotropy in the form of a power quadrupole, detectable with observations on the CMB. Another possibility is given by Eq. (3.59), from which we can see that the presence of a gravitational wave background, in the squeezed limit, entails a quadrupole distortion in the power spectrum of the curvature perturbation and, as we will see in the next section, in the matter perturbations characterizing the Large Scale Structure of the Universe. We can define now a new, intriguing, definition to define primordial fields whose effect is only an imprint on other observable quantities: *Inflation fossil.* These are defined as a hypothesized primordial degree of freedom that no longer interacts or very weakly interacts during late-time cosmic evolution, as metric tensor perturbations are [69, 70]. The only observational effect of an Inflation fossil might therefore be its imprint in the primordial curvature perturbation, as we have seen in the case of the squeezed limit of the $\zeta\zeta\gamma$ bispectrum. Inflation fossils can be those extra fields that are introduced in a variety of alternatives to the single-field slow-roll model. An example is given by the model we will study in the last chapter, Gaugid Inflation model [52]. In this model is predicted an additional tensor degree of freedom, which could play the role of the *tensor fossil.*
We are now ready to formalize the possible prediction for the tensor fossils. The expression (3.58) allow us to define a new correlation induced on the inflaton by the tensor degrees of freedom [69, 70]

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle|_{h_p(\mathbf{K})} = (2\pi)^3\delta^{(3)}(\mathbf{K}+\mathbf{k}_1+\mathbf{k}_2)f_p(\mathbf{k}_1,\mathbf{k}_2)h_p^*(\mathbf{K})\epsilon_{ij}^p(\mathbf{K})\hat{k}_1^i\hat{k}_2^j. \tag{3.64}$$

With h_p we mean a generic fossil field, which can be, in general, a scalar, a vector or a tensor field.

[6] Remember that here we are considering the squeezed limit, in which $k_\gamma \ll k$.

f_p is a "shape" function which is connected with the scalar-scalar-new field bispectrum as

$$\mathcal{B}_p(k_1, k_2, K) = P_p(K) f_p(\mathbf{k}_1, \mathbf{k}_2) \epsilon^p_{ij} k^i_1 k^j_2. \tag{3.65}$$

The subscript of the fossil field p indicates the nature of the field, while $\epsilon^p_{ij}(\mathbf{K})$ it polarization tensor, a symmetric 3×3 tensor. Due to its symmetry, ϵ_{ij} has six degrees of freedom, hence it can be decomposed into six orthogonal polarization states, which we can labeled with $p = \{+, \times, 0, z, x, y\}$, which satisfy the orthonormality condition $\epsilon^p_{ij}\epsilon^{p',ij} = 2\delta^{pp'}$. Each orthogonal state describes a different polarization for the perturbation which causes the departures from statistic homogeneity and isotropy [70]: they can be taken to be two scalar modes $\epsilon^0_{ij} \propto \delta_{ij}$ and $\epsilon^z_{ij} \propto K_i K_j - (K^2/3)\delta_{ij}$, two transverse-vector modes $\epsilon^{x,y}_{ij} \propto K^{(i}w^{j)}$, with $K^i w^i = 0$, and two transverse traceless modes, the *tensor* modes, ϵ_+ and $\epsilon_\times$.
To understand the meaning of these polarizations we can consider the case in which the mode $\mathbf{K}$ is taken to be in the $\hat{\mathbf{z}}$ direction. In this case the two tensor polarization are found to be $\epsilon^+_{xx} = -\epsilon^+_{yy} = 1$ and $\epsilon^\times_{xy} = \epsilon^\times_{yx} = 1$, with all other components are zero in both cases. In Fig. 3.1 it is clear that a distortion due to $\epsilon_\times$ or ϵ_+ generate a quadrupole displacement in the two point correlation function of the field we are interested in.
For the scalar mode we have $\epsilon^0_{ij} = \sqrt{2/3}\delta_{ij}$, whose normalization is given by the orthonormality condition. This perturbation represents just an isotropic modulation of the correlator function, as can be seen in FIg. 3.1. The other scalar takes the form $\epsilon^z_{ij} = \mathrm{diag}(-1, -1, 2)/\sqrt{3}$, that represents a stretching and compression along the $\hat{\mathbf{z}}$ direction. Both the scalar describe a distortion which is invariant under rotations around the $\hat{\mathbf{z}}$-axis, i.e. around the direction of $\mathbf{K}$.
For the vector modes we have $\epsilon^x_{xz} = \epsilon^x_{zx} = 1$ and $\epsilon^y_{yz} = \epsilon^y_{zy} = 1$, with all other components equal to zero. These vector modes represent, respectively, stretching and compression along the $\pm xz$ and $\pm yz$ directions. For the final chapter of this work we are interested in the possibile detection of a tensor fossil, hence we will concentrate only on tensor fields.

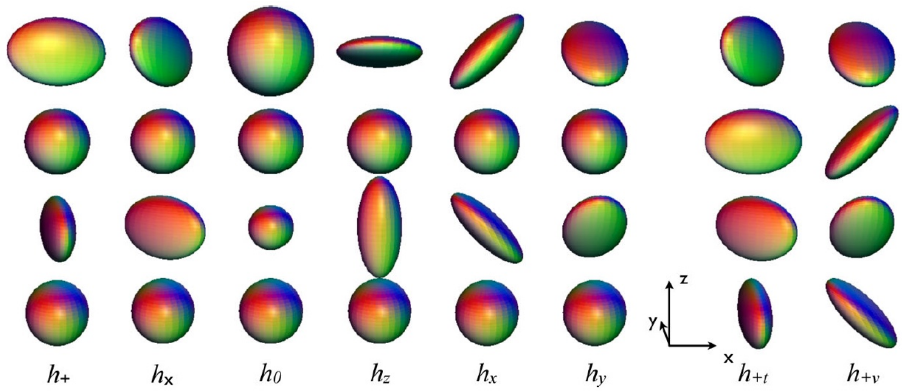

Figure 3.1: Distortion induced to an isotropic 2-point correlation function by correlation of the density (or any other scalar field) with a fossil field with different polarization, pointing in the $\hat{\mathbf{z}}$. The distortions to the sphere show the distortions of the two-point correlation function as one moves along the direction $\hat{\mathbf{z}}$ of the Fourier mode. Left: two scalar modes $(0, z)$, two vector modes (x, y) and the two tensor modes $(+, \times)$. In particular note the quadrupole distortion due to the tensor modes. Right: the circular polarizations of the tensor mode (h_{+t}) and vector mode (h_{+v}).[70].

Using the consistency relation in Eq. (3.59) and the new parametrization for the two-point correlator for the scalar in the presence of a long-wavelength tensor we have [67]

$$\begin{aligned}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_1}\rangle|_{\gamma_{\mathbf{K}}} =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\mathcal{P}_\zeta(k)\\ &+\int\frac{d^3\mathbf{K}}{(2\pi)^3}\sum_s(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{K})\frac{1}{2}\frac{d\ln\mathcal{P}_\zeta}{d\ln k}\mathcal{P}_\zeta(k)\times\\ &\times\gamma_s^*(\mathbf{K})\epsilon_{ij}^s(\mathbf{K})\hat{k}_1^i\hat{k}_2^j+\mathcal{O}((K/k)^2),\end{aligned} \tag{3.66}$$

where the possible polarizations are summed over. It is possible to observe the imprint left by these primordial fossils? Refs. [69, 70] provide a full treatment to connect the tensor modes in the observables connected with the CMB anisotropies in temperature. We will follow these references. Before focusing on it is important to underline that the specific functional form of the shape function f_p depends on the coupling of the new fossil field (which can be scalar, vector and tensor) with the inflaton. This is true in the case of a model which violates the relation (3.59), while for single-field models the shape of f_p does not depend on the coupling between the scalar and the tensor perturbations, since (3.59) is a sort of universal relation valid for all the single-field models, as we mentioned above. Because of the statistical isotropy we have that f_p will be te same for the two tensor polarization and the same for the two vector polarizations, i.e. $f_\times(\mathbf{k}_1,\mathbf{k}_2)=f_+(\mathbf{k}_1,\mathbf{k}_2)$ and $f_x(\mathbf{k}_1,\mathbf{k}_2)=f_y(\mathbf{k}_1,\mathbf{k}_2)$. It is possible to show that the same is not true for the two scalar polarizations. It is possible to define a new generic polarization s which merges the contributions due to the two scalar polarizations [70]. This means that the characterization of the contribution of the fossil fields is complete with the definition of one scalar, two vector and two tensor degrees of freedom.

3.3.1 Tensor fossils in CMB

Following [69] it is possible to calculate some estimators for the contribution of the fossil field to the observed CMB anisotropies in temperature. We can use the so-called TAM formalism (total-angular-momentum) to decompose the fields we are considering using spherical harmonics, see Ref. [71]. In practice we perform an expansion using spherical waves, which are general solution of the Helmholtz equation. This is useful to exploit the spherical symmetry of the background. The features of these wave functions is that they are eigenfunctions of the total angular momentum J and its third component M. In the following section we will briefly summarize the results of this method, then we will construct some estimators for the tensor fossils in the CMB.

Introduction to TAM formalism

Let us summarize the main results of this method. Later we will deal with the multipole expansion of the CMB temperature perturbation we have introduced in Section 2.3.3. For this reason we need a formalism which helps us to decompose the perturbations using spherical functions (remind that all the cosmological observation are performed on a spherical sky). We will describe here the decomposition of a scalar field as an introduction to this method, for more details see [71, 72]. Consider a scalar field which is solution of the Helmholtz equation $(\nabla^2+k^2)\phi(\mathbf{x})=0$. The most general solution can be written in terms of plane waves $e^{i\mathbf{k}\cdot\mathbf{x}}\equiv\Psi^{\mathbf{k}}(\mathbf{x})$

$$\phi(\mathbf{x})=\int\frac{d^3k}{(2\pi)^3}\phi_{\mathbf{k}}\Psi^{\mathbf{k}}(\mathbf{x}), \tag{3.67}$$

where we have indicated with $\phi_{\mathbf{k}}$ the Fourier transform of the scalar function. We want to decompose this solution into eigenfunctions of the angular momentum operator. We use the plane-wave expansion

$$\Psi^{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{lm} 4\pi i^l j_l(kr) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{n}}), \tag{3.68}$$

where j_l are the spherical Bessel functions of the first kind and Y_{lm} are the spherical harmonics. Here we have used the notation $\mathbf{x} \equiv r\hat{\mathbf{n}}$. Now we can define the total-angular-momentum (TAM) basis functions

$$\Psi^k_{lm}(\mathbf{x}) \equiv j_l(kr) Y_{lm}(\hat{\mathbf{n}}). \tag{3.69}$$

Now using the following properties of spherical harmonics and first kind Bessel function

$$\int k^2 dk j_l(kr) j_l(kr') = \frac{\pi}{2r^2}\delta(r - r'), \quad \sum_{lm} Y_{lm}(\hat{\mathbf{n}}) Y^*_{lm}(\hat{\mathbf{n}}') = \delta(\hat{\mathbf{n}} - \hat{\mathbf{n}}'),$$

we can write

$$\phi(\mathbf{x}) = \sum_{lm} 4\pi i^l \int \frac{k^2 dk}{(2\pi)^3} \phi_{lm}(k) \Psi^k_{lm}(\mathbf{x}), \tag{3.70}$$

where we have defined

$$\phi_{lm}(k) = \int d^3\mathbf{k} \left[4\pi i^l \Psi^k_{lm}(\mathbf{x})\right]^* \phi(\mathbf{x}). \tag{3.71}$$

These are the angular-momentum dependent coefficient we are interested in. We can decompose the ζ perturbation using (3.71). Furthermore we are dealing with scalar, vector and tensor fossil degrees of freedom that can be all incorporated into a symmetric traceless tensor field $h_{ab}(\mathbf{x})$, as we have seen in Section 3.3. We can expand this tensor field using the TAM formalism: the longitudinal mode $h^L_{JM}(K)$ describes a scalar fossil field, the two divergence-free vectorial modes $h^{VE}_{JM}(K)$ and $h^{VB}_{JM}(K)$ describe a transverse-vector fossil, and the two divergence-free tensorial modes $h^{TE}_{JM}(K)$ and $h^{TB}_{JM}(K)$ describe a transverse-tensorial fossil. We will indicate a generic mode of the fossil field with h^α_{JM}, where $\alpha = L, VE, VB, TE, TB$.
Using this expansion Eq. (3.64) becomes [69, 71]

$$\begin{aligned}\langle \zeta_{l_1 m_1}(k_1) \zeta_{l_2 m_2}(k_2) \rangle_{h^\alpha_{JM}(K)} =& [h^\alpha_{JM}(K)]^* f^\alpha_h(k_1, k_2, K)(4\pi)^3(-i)^{l_1+l_2+J} \times \\ &\times \frac{1}{k_1 k_2} \int d^3\mathbf{x} \left(\nabla^i \Psi^{k_1}_{(l_1 m_1)}(\mathbf{x})\right) \left(\nabla^j \Psi^{k_2}_{(l_2 m_2)}(\mathbf{x})\right) \Psi^{\alpha,K}_{(JM)ij}(\mathbf{x}),\end{aligned} \tag{3.72}$$

with α the polarization of the fossil field. Here $\Psi^k_{(lm)}(\mathbf{x})$ are the scalar TAM functions introduced above. With similar arguments one can introduce the analogous TAM wave function $\Psi^{\alpha,K}_{(JM)ab}(\mathbf{x})$ for tensor fields. We will connect the perturbations ζ_{lm} to a_{lm}, which are the coefficients for the temperature perturbation spherical expansion. We are now ready to build some estimators for the tensor fossils using the CMB temperature perturbations.

CMB estimators

We look now to the temperature map of the CMB. A long-wavelength tensor fossil would modulate the scalar perturbation, as it can be seen in Eq. (3.64) and 3.1. This means that it would be possible to see the effect of a tensor fossil also in the correlation between the harmonic modes of the CMB anisotropies. This effect translates into a (local) departure from statistical isotropy of the correlation of CMB temperature multipoles, i.e.:

$$\langle a^T_{l_1 m_1} a^{T*}_{l_2 m_2}\rangle_h = \langle a^T_{l_1 m_1} a^{T*}_{l_2 m_2}\rangle + \Delta_h. \tag{3.73}$$

Here $\langle a^T_{l_1 m_1} a^{T*}_{l_2 m_2}\rangle$ represents the "standard" correlator in absence of a fossil field and has the usual form

$$\langle a^T_{l_1 m_1} a^{T*}_{l_2 m_2}\rangle = C^{TT}_{l_1}\delta_{l_1 l_2}\delta_{m_1 m_2}, \tag{3.74}$$

while Δ_h represents the contribution of the fossil field which breaks isotropy. We want to write it in way that the TAM formalism introduced above can be useful. We can use another expansion for the temperature multipoles, which is the so-called bipolar spherical harmonics expansion (BiPoSH) [73]. With this formalism we have

$$\Delta_h = \sum_{JM} (-1)^{m_2} \langle l_1 m_1 l_2, -m_2 | JM\rangle A^{JM}_{l_1 l_2}, \tag{3.75}$$

where the $\langle l_1 m_1 l_2, -m_2 | JM\rangle$ denotes the Clebsch-Gordan coefficients. The $A^{JM}_{l_1 l_2}$ are the BiPoSH coefficients, given by

$$A^{JM}_{l_1 l_2} = (-1)^{l_1+l_2+M}\sqrt{2J+1}\sum_{m_1 m_2} \mathcal{W}^{l_1 l_2 J}_{m_1,-m_2,-M}\langle a^T_{l_1 m_1} a^{T*}_{l_2 m_2}\rangle_h. \tag{3.76}$$

The notation $\mathcal{W}^{l_1 l_2 l_3}_{m_1 m_2 m_3}$ indicates the Wigner-3j symbol. We know that the scalar perturbation sources the CMB anisotropies. Hence, in this formalism, we can connect the TAM coefficients ζ_{lm} with the temperature multipoles using the transfer function [69]

$$a^T_{lm} = \frac{1}{2\pi^2}(-i)^l \int k^2 dk g^T_l(k)\zeta_{lm}(k), \tag{3.77}$$

where $g_l(k)$ is the scalar radiation transfer function for the temperature.
Using Eqs. (3.72), (3.76) and (3.77) we obtain the modulation of the BiPoSH of the temperature due to a fossil wave in the TAM formalism:

$$\begin{aligned} A^{JM}_{l_1 l_2}|_{h^\alpha_{JM}(K)} = &-(-i)^J(-1)^{l_1+l_2+P(\alpha)} h^\alpha_{JM}(K)\frac{16}{\pi} \\ &\times\left(\frac{(2l_1+1)(2l_2+1)}{4\pi}\right)^{\frac{1}{2}} \int k_1^2 dk_1 g^T_{l_1}(k_1)\int k_2^2 dk_2 g^T_{l_2}(k_2) \\ &\times f^\alpha_h(k_1,k_2,K)\mathcal{I}^\alpha_{l_1 l_2 J}(k_1,k_2,K). \end{aligned} \tag{3.78}$$

The polarization takes different values according to the different nature of the fossil field [69, 71], $P(\alpha) = 0$ for the scalar mode and the E vector and tensor mode, $P(\alpha) = 1$ for the B vector and tensor mode. The integral functions are connected with the TAM coefficients

$$\begin{aligned} \mathcal{I}^\alpha_{l_1 l_2 J}(k_1,k_2,K) = &\left[\frac{4\pi}{(2l_1+1)(2l_2+1)(2J+1)}\right]^{\frac{1}{2}}\times \\ &\times\left[\int d^3\mathbf{x}\Psi^{L,k_1,a}_{(l_1 m_1)}(\mathbf{x})\Psi^{L,k_2,b}_{(l_2 m_2)}(\mathbf{x})\Psi^{\alpha,K}_{(JM)ab}(\mathbf{x})\right]/\mathcal{W}^{J l_2 l_1}_{M m_2 m_1}. \end{aligned} \tag{3.79}$$

Due to parity conservation, the E scalar vector and tensor modes induces on even-parity BiPoSHs, i.e. $J + l_1 + l_2 = n$, with n even, while vector and tensor fossils, which contain B-mode TAM waves, induce BiPoSHs with $J + l_1 + l_2 = r$, with r odd. Therefore vector and tensor fossils, but we are more interested in tensors, can be distinguished from scalar fossils from their signature in odd-parity. It is important to stress that much experiments will be devoted to the detection of the B mode of the polarization in the CMB. Nevertheless, with this treatment, vectors and tensors cannot be geometrically distinguished from each other, but we will see another method to distinguish them. This impossibility in distinguishing them can be understood as a loss of information. When observing the CMB we are watching a two-dimensional image, i.e. a projection of the background radiation. The lost information could be regained with three-dimensional surveys [70]. The power spectrum for the fossil field is simply given by

$$\langle h^{\alpha}_{JM}(\mathbf{K}) h^{\alpha'}_{J'M'}(\mathbf{K}')\rangle = P^{\alpha}_{h}\frac{(2\pi)^3}{K^2}\delta^{(3)}(\mathbf{K}-\mathbf{K}')\delta_{JJ'}\delta_{MM'}\delta_{\alpha\alpha'} \tag{3.80}$$

Note that in this formula we are assuming an almost scale invariant power spectrum. We are also assuming that at first order, all the possible fossil fields are uncorrelated: this is guaranteed by $\delta_{\alpha\alpha'}$.

We need now to define some quantities which provide good estimators for the fossil fields. From Eqs. (3.73) and (3.75) we see that we need to give an estimator which takes into account the coefficients $A^{JM}_{l_1 l_2}$. It is possible to give a statistical measurement of the imprint of the fossil through [69]

$$C^{J}_{l_1 l_2, l_3 l_4} = \frac{1}{2J+1}\left\langle \sum_{M=-J}^{+J} A^{JM}_{l_1 l_2}\left[A^{JM}_{l_3 l_4}\right]^* \right\rangle, \tag{3.81}$$

which is simply the average over all realizations of the fossil field. From Eq. (3.76) we see that the definition in Eq. (3.81) corresponds to the four-point correlations in the temperature map.

We assume, then, a phenomenological parametrization for the fossil field. The two parameters which describes it are the normalization P^{Z}_{h} of the power spectrum

$$\mathcal{P}^{Z}_{h}(K) = P^{Z}_{h}\tilde{\mathcal{P}}^{Z}_{h}(K) \tag{3.82}$$

and the normalization of the scalar-scalar-fossil bispectrum B^{Z}_{h}

$$f^{Z}_{h}(k_1,k_2,K) = B^{Z}_{h}\tilde{f}^{z}_{h}(k_1,k_2,K). \tag{3.83}$$

With the tilde we have indicated the fiducial-shape of the power spectrum and bispectrum. Here Z can represent scalar, vector and tensor fossil field, without distinction between E and B-modes. This is due to the statistical homogeneity, as we have mentioned above. Finally, with these parametrization, we can rewrite Eq. (3.76) as

$$A^{JM}_{l_1 l_2}|_{h^{\alpha}_{JM}(K)} = B^{Z}_{h} F^{J,\alpha}_{l_1 l_2}(K) h^{\alpha}_{JM}(K), \tag{3.84}$$

where now $\alpha \in Z$ and the coefficients $F^{J,\alpha}_{l_1 l_2}(K)$ can be obtained from Eq. (3.76)

$$\begin{aligned} F^{J,\alpha}_{l_1 l_2}(K) = &-(-i)^J(-1)^{l_1+l_2+P(\alpha)}\frac{16}{\pi}\left(\frac{(2l_1+1)(2l_2+1)}{4\pi}\right)^{1/2} \\ &\times\int k_1^2 dk_1 g^{T}_{l_1}(k_1)\int k_2^2 dk_2 g^{T}_{l_2}(k_2)\tilde{f}^{Z}_{h}(k_1,k_2,K) \\ &\times \mathcal{I}^{\alpha}_{l_1 l_2 J}(k_1,k_2,K). \end{aligned} \tag{3.85}$$

Eq. (3.84) is the equivalent for the CMB of (3.64). It represents the quadrupole distortion of the CMB temperature perturbation caused by a primordial tensor fossil. The distortion

is visible in (3.85) through its f_h dependence. This is an important results of the studies on the tensor fossils because we have found a possible observable effect connected to the CMB temperature perturbations, one of the most important observables of the last thirty years.
From Eq. (3.84) we explicitly see that the departure from statistical isotropy for the 2-point correlator of the CMB temperature is given, in this case, by the presence of a fossil field. If we define the reduced amplitude as $\mathcal{A}_h^Z \equiv P_h^Z(\mathcal{B}_h^Z)^2$, the BiPoSHs power spectra are then calculated

$$C^J_{l_1 l_2, l_3 l_4} = \frac{\mathcal{A}_h^Z}{(2\pi)^3} \sum_{\alpha \in Z} \int K^2 dK \tilde{\mathcal{P}}_h^Z(K) F^{J,\alpha}_{l_1 l_2}(K) \left[F^{J,\alpha}_{l_3 l_4}(K)\right]^* . \tag{3.86}$$

It is evident that to perform this calculation, the temperature transfer functions g_l^T play an fundamental role in this calculation, as shown in (3.85). It is possible now to define some estimator [69, 70] which are useful to determine the reduced amplitude experimentally. An estimator for the BiPoSH coefficients is

$$\widehat{A^{JM}_{l_1 l_2}} = \sum_{m_1 m_2} (-1)^{m_2} \langle l_1 m_1 l_2, m_2 | JM \rangle a^T_{l_1 m_1} a^{T*}_{l_2 m_2}, \tag{3.87}$$

while the estimators for the $C^J_{l_1 l_2, l_3 l_4}$ can be written as

$$\widehat{C^J_{l_1 l_2, l_3 l_4}} = \frac{1}{2J+1} \sum_{M=-J}^{J} \widehat{A^{JM}_{l_1 l_2}} \left[\widehat{A^{JM}_{l_3 l_4}}\right]^* - C^{TT}_{l_1} C^{TT}_{l_2} (\delta_{l_1 l_3} \delta_{l_2 l_4} + (-1)^{l_1 + l_2 + J} \delta_{l_1 l_4} \delta_{l_2 l_3}). \tag{3.88}$$

Defining

$$\mathcal{F}^{J,Z}_{l_1 l_2, l_3 l_4} \equiv \frac{1}{(2\pi)^3} \sum_{\alpha \in Z} \int K^2 dK \tilde{\mathcal{P}}_h^Z(K) F^{J,\alpha}_{l_1 l_2}(K) \left[F^{J,\alpha}_{l_3 l_4}(K)\right]^* , \tag{3.89}$$

we are ready to define an estimator for the reduced amplitude which will allow us to write down the statistical error. Given that the reduced amplitude is given by, see (3.86)

$$\mathcal{A}_h^Z = \frac{C^J_{l_1 l_2, l_3 l_4}}{\mathcal{F}^{J,Z}_{l_1 l_2, l_3 l_4}}, \tag{3.90}$$

we can write an estimator for the reduced amplitude for each combination of J and l_i, $i = 1, \ldots, 4$, using (3.90) and (3.88)

$$\widehat{\mathcal{A}^{J,Z}_{h, l_1 l_2, l_3 l_4}} = \frac{\widehat{C^J_{l_1 l_2, l_3 l_4}}}{\mathcal{F}^{J,Z}_{l_1 l_2, l_3 l_4}}. \tag{3.91}$$

We can use these estimator to construct another one for the total reduced amplitude, $\widehat{\mathcal{A}_h^Z}$. Treating the estimators in (3.91) as statistically independent estimators, we can combine them obtaining the inverse-variance-weighted estimator

$$\widehat{\mathcal{A}_h^Z} = \left[\sum_J \sum_{(l_1, l_2, l_3, l_4)} \widehat{\mathcal{A}^{J,Z}_{h, l_1 l_2, l_3 l_4}} \left\langle \left(\widehat{\mathcal{A}^{J,Z}_{h, l_1 l_2, l_3 l_4}}\right)^2 \right\rangle_0^{-1} \right] \cdot \left[\sum_J \sum_{(l_1, l_2, l_3, l_4)} \left\langle \left(\widehat{\mathcal{A}^{J,Z}_{h, l_1 l_2, l_3 l_4}}\right)^2 \right\rangle_0^{-1} \right]^{-1} . \tag{3.92}$$

The subscript 0 indicates that, at the zeroth order we can consider all the $\widehat{\mathcal{A}^{J,Z}_{h, l_1 l_2, l_3 l_4}}$ as uncorrelated modes. This feature is highlighted also by the sum over (l_{1, l_2, l_3, l_4}), which is performed over all the *independent* combinations of the multipoles [69]. The denominator of this expression is equal to the inverse variance of the reduced amplitude estimator, which is found to be

$$(\sigma_{\mathcal{A}}^{Z})^{-2} \equiv \left\langle \left[\widehat{\mathcal{A}_h^Z}\right]^2 \right\rangle_0^{-1} = \frac{1}{8}\sum_J \sum_{l_1 l_2 l_3 l_4} \frac{2J+1}{C_{l_1}^{TT} C_{l_2}^{TT} C_{l_3}^{TT} C_{l_4}^{TT}} \mathcal{F}^{J,Z}_{l_1 l_2, l_3 l_4} \left[\mathcal{F}^{J,Z}_{l_1 l_2, l_3 l_4}\right]^* . \tag{3.93}$$

A high signal-to-noise $\mathcal{S} = \mathcal{A}_h^Z / \sigma_{\mathcal{A}}^Z$ indicates a detectable BiPoSH from Inflation fossil. For numerical results about the detectability of these fossils see Ref. [69].

3.3.2 Quadrupole anisotropy in mass distribution

We conclude this section introducing also another visible effect due to the presence of a stochastic background of gravitational waves on large scales.
The correlations of primordial tensor perturbations with primordial scalar perturbations can have observational consequences for the mass distribution.
It has been shown that the presence of a fossil tensor field would entail a quadrupole anisotropy in mass distribution of the large scale structure [67, 70, 63]. This does not come as a surprise given the general formulas and arguments that we have described previously. It can be shown that in the case of a theory violating the consistency relations described above one has [63]

$$\mathcal{P}_\zeta(\mathbf{k}_S)|_{\gamma_p(\mathbf{k}_L)} = \mathcal{P}_\zeta(\mathbf{k}_S)\left[1 + \mathcal{Q}^p_{ij}(\mathbf{k}_L)\hat{k}^i_s \hat{k}^j_s\right], \tag{3.94}$$

with power quadrupole

$$\mathcal{Q}^p_{ij}(_L) = \frac{\mathcal{B}_{\not{c}r}(k_L, k_{,S}, k_S)}{\mathcal{P}_\gamma(k_L) P_\zeta(k_S)} \gamma^p_{ij}(_L), \tag{3.95}$$

where, again, $\mathcal{B}_{\not{c}r}(k_L, k_S, k_S)$ is the consistency-relation-violating part of the tensor-scalar-scalar bispectrum. The theory then predicts that this locally observed power quadrupole has variance [63]

$$\langle \mathcal{Q}^2 \rangle \equiv \langle \mathcal{Q}_{ij}\mathcal{Q}^{ij} \rangle = \frac{2}{\pi^2}\int_{k_L^{min}}^{k_S^{min}} k_L^2 dk_L \left[\frac{\mathcal{B}_{\not{c}r}(k_L, k_S, k_S)}{\mathcal{P}_\gamma(k_L)\mathcal{P}_\zeta(k_S)}\right]^2 \mathcal{P}_\gamma(k_L). \tag{3.96}$$

Here, the upper limit of integration, k_S^{min}, is the smallest wavenumber probed by the observations. The lower limit, k_L^{min}, corresponds to the longest wavelength gravitational wave mode produced during Inflation.
It is possible to provide some estimators to measure the effect of these tensor fossils in clustering of galaxies [70, 63]. Eqs. (3.64) and (3.95) provide an estimator for the fossil field for each mode ζ

$$\widehat{\gamma(\mathbf{K})} = \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\left[f_{\gamma^s}(\mathbf{k}_1, \mathbf{k}_2)\epsilon^s_{ij} k_1^i k_2^j\right]^{-1} = \frac{\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\mathcal{P}_\gamma(\mathbf{K})}{\mathcal{B}_{\gamma\zeta\zeta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{K})}. \tag{3.97}$$

The variance of this estimator can be evaluated considering the experimental formula for the observed power spectrum of a general field

$$\langle |\delta(\mathbf{k})|^2 \rangle = V P_\delta^{TOT}(k), \tag{3.98}$$

where V is the survey volume and P_δ^{TOT} is the total observed power spectrum, including both the signal power spectrum and the noise, given, in general, by

$$P^{TOT}(k) = P(k) + P_n(k). \tag{3.99}$$

The variance of (3.97) is given by

$$\text{var}\left(\widehat{\gamma(\mathbf{K})}\right) = 2VP_\zeta^{TOT}(k_1)P_\zeta^{TOT}(k_1)\left|f_{\gamma^s}(\mathbf{k}_1,\mathbf{k}_2)\epsilon^s_{ij}k_1^ik_2^j\right|^{-2}.$$

In order to have a good measure we have to find the estimator which minimize the variance: for this reason we have to sum (3.97) over all the possible mode pairs $\mathbf{k}_1$-$\mathbf{k}_2$ weighted by the variance

$$\widehat{\gamma(\mathbf{K})}_{min} = P^n_\gamma(K)\sum_{\mathbf{k}}\frac{f^*_{\gamma^s}(\mathbf{k},\mathbf{K}-\mathbf{k})\epsilon^s_{ij}k^i(K-k)^j}{2VP_\zeta^{TOT}(k)P_\zeta^{TOT}(|\mathbf{K}-\mathbf{k}|)}\zeta(\mathbf{k})\zeta(\mathbf{K}-\mathbf{k}), \tag{3.100}$$

where the noise power spectrum

$$P^n_\gamma(K) = \left[\sum_{\mathbf{k}}\frac{|f_\gamma(\mathbf{k},\mathbf{K}-\mathbf{k})\epsilon_{ij}k^i(K-k)^j|^2}{2VP_\zeta^{TOT}(k)P_\zeta^{TOT}(|\mathbf{K}-\mathbf{k}|)}\right]^{-1}, \tag{3.101}$$

is the variance with which the tensor estimator is measured. We can write the power spectrum splitting it into its amplitude and its *fiducial power spectrum*

$$P_\gamma(K) = A_\gamma P^f_\gamma(K). \tag{3.102}$$

The fiducial power spectrum gives only the scale-dependence of the power spectrum, for nearly scale-invariant power spectrum it will be

$$P^f(k) = k^{n-3}, \tag{3.103}$$

with $|n| \ll 1$. We are now ready to give an estimator for the fossil amplitude A_γ. Using (3.98) and the definition (3.102) we have

$$\widehat{A^{\mathbf{K}}_\gamma} = \left[P^f_\gamma(K)\right]^{-1}\left[V^{-1}|\widehat{\gamma(\mathbf{K})}|^2\right].$$

To unbias the estimator we have to subtract the noise

$$\widehat{A^{\mathbf{K}}_\gamma} = \left[P^f_\gamma(K)\right]^{-1}\left[V^{-1}|\widehat{\gamma(\mathbf{K})}|^2 - P^n_\gamma(K)\right]. \tag{3.104}$$

In this case the variance of this estimator is [70]

$$\text{var}\left(\widehat{A^{\mathbf{K}}_\gamma}\right) = 2\frac{\left[P^n_\gamma(K)\right]^2}{\left[P^f_\gamma(K)\right]^2} \tag{3.105}$$

As we did before for the γ estimator we can find the minimum-variance estimator for the amplitude

$$\widehat{A^{\mathbf{K}}_\gamma}_{min} = \sigma^2_\gamma\sum_{K,s}\frac{\left[P^f_\gamma(K)\right]^2}{\left[P^n_\gamma(K)\right]^2}\left(V^{-1}|\widehat{\gamma^s(\mathbf{K})}|^2 - P^n_{\gamma^s}(K)\right), \tag{3.106}$$

where the variance is

$$\sigma_\gamma^{-2} = \sum_{\mathbf{K},s}\frac{\left[P^f_{\gamma^s}(K)\right]^2}{2\left[P^n_{\gamma^s}(K)\right]^2}. \tag{3.107}$$

Note that in Eq. (3.106) we have to sum over the possible values for the tensor polarizations $s = +, \times$. We can now evaluate the smallest amplitude which can be detected with a given survey. Using the single-field slow-roll model prediction for the $\gamma\zeta\zeta$ bispectrum in Ref. [56] we have, in the squeezed limit $K \ll k_1 \sim k_2$

$$f_\gamma(\mathbf{k}_1, \mathbf{k}_2) = -\frac{3}{2}\frac{P_\zeta(k_1)}{k_1^2}. \tag{3.108}$$

We go from discrete to continue limit $\sum_{\mathbf{k}} \to V\int \frac{d^3\mathbf{k}}{(2\pi)^3}$ and we use now the following approximation for the total power spectrum. We define

$$\frac{P(k)}{P^{TOT}(k)} = \theta(k_{max} - k) = \begin{cases} 1 & \text{if} \quad k < k_{max} \\ 0 & \text{if} \quad k > k_{max} \end{cases}, \tag{3.109}$$

where θ is the Heaviside function. k_{max} is the UV cut to the integral which represents the largest wavenumber for which the power spectrum can be measured with high signal to noise. These hypothesis give a noise power spectrum for a tensor fossil

$$P_\gamma^n = \frac{20\pi^2}{k_{max}^3}. \tag{3.110}$$

We stress, again, that this is a prediction for the SFSR models of Inflation.
Evaluating the integral (3.107) gives

$$\sigma_\gamma^{-2} = 2V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{k_{max}^6}{800\pi^4 k^6} = \frac{k_{max}^6}{100\pi^3}\frac{V}{(2\pi)^3}\int_{k_{min}}^{k_{max}}\frac{dk}{k^4} \simeq \frac{1}{300\pi^3}\left(\frac{k_{max}}{k_{min}}\right)^6, \tag{3.111}$$

where k_{min} is the smallest wavenumber detectable in the survey. The factor 2 in the first step is given by the sum over the two polarizations of the gravitational wave. In the last step we have used the fact that the volume of the survey is given by $V = (2\pi)^3 k_{min}^{-3}$ and that $k_{max} \gg k_{min}$. We want to see what are the detectable tensor amplitudes at $> 3\sigma_\gamma$

$$3\sigma_\gamma = 30\pi\sqrt{3\pi}\left(\frac{k_{max}}{k_{min}}\right)^{-3}. \tag{3.112}$$

This result shows us that the smallest detectable GW power spectrum amplitude is inversely proportional to the Fourier modes present in the survey. If we consider a value for the tensor amplitude near the maximum amplitude $A_\gamma \simeq 2 \cdot 10^{-9}$, see Ref. [70, 63], a detection requires

$$\frac{k_{max}}{k_{min}} > 5200. \tag{3.113}$$

This request is beyond the actual reach of galaxy surveys. We will see that in solid Inflation the number of Fourier modes required for a detection of a tensor fossil power spectra is reduced, enhancing the possibilities of an observation. [63] provides a correction to the previous clustering fossils results obtained in [70], in particular they consider also the late time evolution of tensor fossils, solving some problems arising in the previous works. For a study of the effect of late-time evolution of tensor fossil see Ref. [63]. For other interesting effects of primordial gravitational waves on Large Scale Structures observations see, e.g. [74, 75, 76, 77]

Chapter 4

Solid Inflation

4.1 Why a new model of inflation?

Up to now we have studied the standard model of Inflation, the single-field slow-roll model. Despite the last observations have ruled out several inflationary models, see Ref. [21], there are still various models compatible with the data. It is useful to classify them in terms of an effective field theory description [78], where the behavior of the perturbations can be understood in terms of symmetries and symmetry breaking. In Chapter 2 we have seen that to have an inflationary period we need an almost flat potential for the field which drives Inflation. This property can be described as an approximate shift symmetry $\phi \to \phi + a$, where ϕ is the inflaton field. A small and controlled breaking of this symmetry ensures a slow roll inflaton evolution, which breaks time translational invariance. The broken time translation invariance is made explicit in the time dependence of all the background quantities for the description of the evolution of Universe. The breaking of time translation invariance also ensures that during Inflation we have a quasi-de Sitter background, which allows the end of the Inflation. Despite this symmetry breaking, the observed homogeneity and isotropy of the Universe on large scales require the assumption of invariance of the background value of the field under spatial translations and rotations. Most of the Inflation models have these features, hence we need to discriminate them using some univocal observation. In [53] it is proposed a new model of Inflation called *Solid Inflation*. In this model has been implemented a previous idea, called *Elastic Inflation* [79], for which the "fluid" which drives Inflation is composed by three scalar fields which have very particular properties: their background values are *time-independent* and *$\vec{x}$-dependent*. This means that the standard invariance under spatial translations and rotations are broken, while time translation is preserved. The homogeneity and the isotropy required to satisfy the Cosmological Principle are restored adding internal symmetries on the scalar fields. This simple procedure entails very interesting outcomes for various aspects: we will see that it predicts a blue tilt for the tensor perturbations, it breaks the consistency relation for the bispectra and it predicts interesting results for tensor fossils. For these reasons we will take into account this theory: it has observable predictions which are different from most of the (standard) single-field models of Inflation.

4.1.1 Solid on Minkowski

In [53] the field which drives Inflation manifestly breaks the standard isometries of a de Sitter spacetime. We have always considered a homogeneous and isotropic background, which means taking a background invariant under the spatial rotations and translations.

In other words all the quantities evaluated on the background are only time dependent, only the time diffeomorphism is broken. By the way, if we consider, for example, a scalar field with vev

$$\langle \phi \rangle = \alpha x \tag{4.1}$$

with $\alpha = const$, we are breaking the translations along x, while we are preserving the time diffeomorphism. In order to restore the homogeneity we can assume an internal shift symmetry of the scalar field, i.e.

$$\phi \to \phi + a \quad \text{with} \quad a = const, \tag{4.2}$$

so that the background (4.1) is invariant under a combination of x translation and shift transformation. To restore the isotropy we need to add an internal index, $I = 1, 2, 3$, for which (4.1) becomes

$$\langle \phi^I \rangle = \alpha x^I. \tag{4.3}$$

Now (4.3) breaks rotations, and to restore them we add another internal symmetry which involves the internal index I

$$\phi^I \to O^I_J \phi^J \quad \text{with} \quad O^I_J \in SO(3), \tag{4.4}$$

so that (4.3) is invariant under a combination of internal and spatial rotations. Eq. (4.2) with an internal index become simply

$$\phi^I \to \phi^I + a^I \quad \text{with} \quad a^I = const, \tag{4.5}$$

The index I indicates the internal coordinates: consider a fluid which fills all the space. We can describe this fluid by attaching to each volume element a three-dimensional label ϕ^I and following its trajectory as a function of the label and of time $\mathbf{x} = \mathbf{x}(\phi^I, t)$. Since at any given time the mapping between ϕ^I and $\mathbf{x}$ is invertible, the medium can be equivalently described by ϕ^I as a function of the comoving coordinates $\phi^I(t, \mathbf{x})$, and for our purpose it is more useful to work using ϕ as a function of the coordinates.. The mathematics under these assumption is very trivial: $(t, \mathbf{x}) = x^\mu$ are the usual coordinates and ϕ^I are three Lorentz scalars. We only need to build a theory for three Lorentz scalars respecting the symmetries (4.4) and(4.5) on a curved spacetime. We start with the low-energy limit, i.e., we will consider our solid on a Minkowski spacetime.
To construct our theory we need Lorentz scalar and shift-rotation invariant quantities. The shift invariance forces the field ϕ^I to appear only with its derivative, the Lorentz condition implies that these derivatives must be contracted. We can construct only one quantity with these features

$$B_{IJ} = \partial_\mu \phi_I \partial^\mu \phi_J. \tag{4.6}$$

From (4.6) we can construct the $SO(3)$ variables with which we will construct our action which are

$$[B], \quad [B^2], \quad [B^3],$$

where the square brackets stand for the trace of the matrix $[..] = Tr(..)$. It is useful to consider only one of these three invariants to keep track of the size of the matrix B, it will be fundamental when studying the background dynamics on the de Sitter spacetime. We can define the three variable we will consider for the solid action:

$$X = [B], \quad Y = \frac{[B^2]}{[B]^2}, \quad Z = \frac{[B^3]}{[B]^3}. \tag{4.7}$$

In this definition only the variable X is sensible to the stretching of the field, while Y and Z are defined in such a way they do not feel the expansion. The most general action we can construct with these blocks is

$$S = \int d^4x F(X, Y, Z) + \dots . \tag{4.8}$$

Note that in Eq. (4.8) we are using the flat Minkowski metric because we are considering the low-energy limit, for which we can omit, for the moment, the coupling with gravity. The dots indicate the higher energy contributions, like higher derivative terms of X, Y and Z.
The perturbations to the background (4.3) are

$$\phi^I = \alpha(x^I + \pi^I). \tag{4.9}$$

Inserting Eq. (4.9) into Eq. (4.6) we have the perturbed expression for the building block B_{IJ}

$$B_{IJ} = \alpha^2(\delta_{IJ} + \partial_I\pi_J + \partial_J\pi_I + \partial_\mu\pi_I\partial^\mu\pi_J). \tag{4.10}$$

For X we have

$$X = \delta_{IJ}B_{IJ} = \alpha^2[3 + 2\partial_I\pi_I + \dot{\pi}_I{}^2 + (\partial_i\pi_J)^2], \tag{4.11}$$

The second order action is [53, 80]

$$\begin{aligned} S^{(2)} = \int d^4x \Big[& -\frac{1}{3}F_X X \cdot \dot{\pi}_i^2 + \left(\frac{1}{3}F_X X + \frac{6}{27}(F_Y + F_Z)\right)(\partial_i\pi_j)^2 \\ & + \left(\frac{2}{9}F_{XX}X^2 + \frac{2}{27}(F_Y + F_Z)(\partial_i\pi_i)^2\right)\Big]. \end{aligned} \tag{4.12}$$

The perturbations π_i represent Goldstone bosons arising from breaking the global symmetries of a de Sitter space. These excitations are the analogue of the phonons in a real solid, this is the reason of the name of the model. We split now the excitations into longitudinal and transverse part,

$$\pi_i = \pi_i^L + \pi_i^T, \quad \epsilon_{ijk}\partial_j\pi_k^L = 0, \quad \partial_i\pi_i^T = 0, \tag{4.13}$$

so that the action becomes

$$\begin{aligned} S^{(2)} = \int d^4x \Big[& -\frac{1}{3}F_X X \cdot \left((\dot{\pi}_i^L)^2 + (\dot{\pi}_i^T)^2\right) \\ & + \left(\frac{1}{3}F_X X + \frac{6}{27}(F_Y + F_Z)\right)\left((\partial_i\pi_j^T)^2 + (\partial_i\pi_i^L)^2\right) \\ & + \left(\frac{1}{9}F_{XX}X^2 + \frac{2}{27}(F_Y + F_Z)\right)\left(\partial_i\pi_i^L\right)^2 \Big]. \end{aligned} \tag{4.14}$$

From Eq. (4.14) it is straightforward to calculate the longitudinal and transverse propagation speeds

$$c_L^2 = 1 + \frac{2}{3}\frac{F_{XX}X^2}{F_X X} + \frac{8}{9}\frac{F_Y + F_Z}{F_X X}, \quad c_T^2 = 1 + \frac{2}{3}\frac{F_Y + F_Z}{F_X X}, \tag{4.15}$$

in terms of which we can rewrite the action

$$S^{(2)} = -\frac{1}{3}\int d^4x F_X X \left[\dot{\pi}_i^2 - c_T^2(\partial_i\pi_j)^2 - (c_L^2 - c_T^2)(\partial_i\pi_i)^2\right]. \tag{4.16}$$

We impose now the condition of subluminal propagation for the longitudinal and transverse modes $0 < c_L^2 < 1$ and $0 < c_T^2 < 1$ we obtain the following constraints for the F derivatives

$$-\frac{3}{2} < \frac{F_Y + F_Z}{F_X X} < 0, \quad -\frac{3}{2} < \frac{F_{XX} X^2}{F_X X} < 2. \tag{4.17}$$

These conditions do not have an important role now. We will see their fundamental utility when studying the conditions for having an inflationary period.

4.2 Background dynamics

After the study of the subhorizon limit we want to see if this model predicts an era of (quasi) exponential expansion, an inflationary period. Which conditions may satisfy the fields?
We need now to reintroduce the coupling with gravity, given that we want to study our field in a (quasi) de Sitter Universe. We adopt the standard procedure of minimally coupling the field to the gravitational field. The definition (4.6) now is contracted with the generic metric $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$

$$B^{IJ} = g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J, \tag{4.18}$$

and the action (4.8) has in addition a $\sqrt{-g}$ term in the integral measure

$$S = \int d^4x\sqrt{-g}F(X, Y, Z). \tag{4.19}$$

Also the standard derivative in (4.6) should become covariant derivative, however, since we are dealing with scalar fields, we will not need to introduce any covariant derivative whatsoever. For the stress-energy tensor we use the standard Hilbert definition (1.73), and from Eqs. (4.18) and (4.19) we have, at the first order [53]

$$T_{\mu\nu} = -2\frac{\partial F}{\partial B_{IJ}}\partial_\mu\phi^I\partial_\nu\phi^J + g_{\mu\nu}F. \tag{4.20}$$

We can interpret now the coordinates x^I in (4.3) as the comoving FLRW coordinates, because now the de Sitter spacetime is invariant under the spatial translation and rotation and we want both the sides in (4.3) to be invariant under the same symmetries. For this reason we will not make difference between the internal index I and the coordinate index i. We can also choose the normalization of the comoving coordinates to set the α parameter to one

$$\langle\phi^i\rangle = x^i. \tag{4.21}$$

We are now ready to calculate the background dynamics. We start from the computation of the stress-energy momentum; using the standard perfect-fluid definition (1.21) and Eq. (4.21) we have

$$\rho = -F, \qquad p = F - \frac{2}{a^2}F_X. \tag{4.22}$$

We first underline that from the first equation of (4.22) F must be negative, in order to have a positive energy density and a stable theory. Note also that in the expression for the pressure appears only the derivative with respect to X. The reason resides in the definitions of Y and Z, Eq. (4.7). The only variable sensitive to the expansion of the Universe is X, since Y and Z are rescaled. Hence, since the pressure takes into account also the dilatation and the contraction of the Universe, when calculating the pressure we need only to consider the derivative of F w.r.t. X. For the general expression of the stress-energy tensor see Ref. [53, 80]. As a general result from the Friedmann equations (1.23) and (1.24) we obtain the following equation for the derivative of the Hubble parameter

$$\dot{H} = -4\pi G\rho(1+w),$$

where we have used the state equation $p = w\rho$. We see that, except for particular models with a cosmic fluid with $w < -1$, we have $\dot{H} < 0$, from which we obtain $F_X < 0$.
The Friedmann equations in the solid paradigm become

$$H^2 = -\frac{8\pi G}{3}F, \quad \frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(F - \frac{3}{a^2}F_X\right), \quad \dot{\rho} = 6HF_X, \tag{4.23}$$

from which we can define the slow-roll parameter (1.91), given that

$$\dot{H} = \frac{8\pi G}{a^2}F_X,$$

one obtains

$$\epsilon = \frac{3}{a^2}\frac{F_X}{F} = \frac{\partial \log F}{\partial \log X}, \tag{4.24}$$

where we have used the background values for $X \to 3/a^2(t)$, $Y \to 1/3$ and $Z \to 1/9$. The slow-roll condition imposes $\epsilon \ll 1$, hence, from Eq. (4.24) we need a very weak X-dependence for F in order to obtain an inflationary Universe. We have to do some clarifications. In this theory we do not have a field which *slowly rolls* along its potential along all the duration of the Inflation. We are, indeed, using the condition (1.92), which is more general than the single field slow roll models: it simply resumes the request of an accelerated expansion for the Universe (quasi-de Sitter condition).
We can see that the small X-dependence of F can be thought as an approximate symmetry with respect to scale transformation

$$\phi^I \to \lambda\phi^I, \quad \lambda = \text{const} \tag{4.25}$$

Under the transformation (4.25) B^{IJ} takes a multiplicative term λ^2 and the same argument hols for X, while Y and Z are insensitive to this internal transformation. If F depended only on Y and Z, the symmetry (4.25) would be exact, since they are both independent from rescaling of the field. Hence the smallness of F_X can be interpreted as an approximate invariance under (4.25). This not surprises too much since X is the only invariant which is sensible to the expansion of the Universe. During Inflation the energy of the solid should not change much if we dilate the solid.
The second "slow-roll" parameter given in (1.93) gives constraint also on the second order derivative of F with respect to X. We note that, using the background value of X,

$$H = \frac{\dot{a}}{a} = -\frac{1}{2}\frac{\dot{X}}{X},$$

and

$$\dot{\epsilon} = \dot{X}\left(\frac{F_X}{F} + \frac{XF_{XX}}{F} - \frac{XF_X^2}{F^2}\right).$$

We obtain

$$\eta = \frac{\dot{\epsilon}}{H\epsilon} = -2\left(1 + \frac{X^2F_{XX}}{XF_X} - \epsilon\right) \tag{4.26}$$

which must satisfy the *minimum e-folds* condition $\eta \ll 1$. From (4.26) we obtain a constraint equation for F_{XX}

$$\frac{X^2 F_{XX}}{X F_X} = -1 + \epsilon - \frac{1}{2}\eta, \tag{4.27}$$

which will allow us to consider $XF_{XX} \simeq -F_X$. With this new condition the expressions for the propagation velocities become, with respect to (4.15)

$$c_L^2 = \frac{1}{3} + \frac{8}{9}\frac{F_Y + F_Z}{XF_X}, \quad c_T^2 = 1 + \frac{2}{3}\frac{F_Y + F_Z}{F_X X}. \tag{4.28}$$

Manipulating these two expressions we find that the two speeds are connected by the relation, at all the orders in the slow-roll parameters

$$c_T^2 = \frac{3}{4}\left(1 + c_L^2 - \frac{2}{3}\epsilon + \frac{1}{3}\eta\right) \tag{4.29}$$

and, at the zeroth order,

$$c_T^2 = \frac{3}{4}(1 + c_L^2). \tag{4.30}$$

The subluminality conditions differs from (4.17), since (4.29) implies

$$c_L^2 < \frac{1}{3} + \frac{2}{3}\epsilon - \frac{1}{3}\eta \tag{4.31}$$

which reduces to

$$c_L^2 < \frac{1}{3} \tag{4.32}$$

The condition (4.31) entails some constraints on the derivative of F with respect to Y and Z:

$$-|F|\left(\frac{3}{4}\epsilon^2 - \frac{3}{8}\epsilon\eta\right) < F_Y + F_Z \leq \frac{3}{8}\epsilon|F|$$

where we have used the fact that F must be negative, so $XF_X = -\epsilon|F|$ from (4.24). We see that the left inequalities involves terms of order two in the slow-roll parameters. At the first order they can be neglected, and we can rewrite

$$0 < F_Y + F_Z \leq \frac{3}{8}\epsilon|F|. \tag{4.33}$$

This constraint has important outcomes on the predictions of the theory. We see that the derivative combination $F_Y + F_Z$ is suppressed by the slow-roll parameter. In [81] it is stressed out that the smallness of this combination can be interpreted as an insensibility of the solid which drives Inflation to the anisotropies of the geometry in which it resides. It is possible to satisfy Eq. (4.33) in two particular cases:

All the derivative of F are small. This statement can be formalized as

$$\frac{\partial F}{\partial B^{IJ}} B^{KL} \sim \epsilon F, \tag{4.34}$$

which can be interpreted as an approximate internal symmetry under diffeomorphisms

$$\phi^I \to \xi^I(\phi) \tag{4.35}$$

and not only under rescaling of the field. Eq. (4.34) means that the action we are considering does not depend too much on the fields which drive Inflation. It corresponds to have an inflationary period driven by a cosmological constant whose dynamics does not depend on the fields. Authors in [53] do not take into account this case, while we will consider exactly this case following [80].

Derivative of the same order. We can consider $F_Y \simeq -F_Z \sim F$, with an underlying symmetry which forces the combination $F_Y + F_Z$ to vanish. This case simplifies a lot the calculations, especially when we will calculate the third order action and the bispectra. In [80] the fact that this assumption entails non negligible errors on the observable predictions is highlighted. We will deepen this argument in Section 4.4.

It is possible to introduce other two slow-roll parameters which involve the variation of the propagation speed of the longitudinal and transverse modes:

$$s \equiv \frac{\dot{c}_L}{c_L H}, \tag{4.36}$$

$$u \equiv \frac{\dot{c}_T}{c_T H}. \tag{4.37}$$

From the two definitions (4.28) we see that these two parameters must be small, given the small X-dependence of F during Inflation [53]. The two speeds are almost constant while Inflation acts.

Finally we mention how the possible UV divergencies can be avoided in Solid Inflation. The phonons enter the action with its derivative, see Eq. (4.12) and the following expressions for the perturbed action. This means that the nonlinear interactions becomes strong at energies greater than the scale

$$E_{strong} \simeq F^{1/4}(\epsilon^3 c_L^9)^{1/4}.$$

The perturbation theory is under control when this energy scale is much lesser than the Hubble parameter, $E_{strong} \gg H$, this ensures that exists a finite window of subhorizon scales in which the theory is weakly coupled. Give the first Friedmann equation in Eq. (4.23), we can rewrite the stability condition as

$$\epsilon c_L^3 \gg \left(\frac{H}{M_{Pl}}\right)^{\frac{2}{3}}.$$

This condition can be satisfied easily with sufficiently small H, where with "sufficiently small " we means smaller than the Planck mass.

4.3 Perturbations in Solid Inflation

We are now ready to introduce gravity in our model and to study the perturbation pattern arising from the breaking of the standard symmetries. We will make use of the metric (3.22), where the metric elements are

$$g_{00} = -N^2, \quad g_{0i} = g_{i0} = N_i, \quad g_{ij} = h_{ij}, \tag{4.38}$$

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = g^{i0} = \frac{N_i}{N^2}, \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}. \tag{4.39}$$

Now the action becomes [53]

$$S = \int d^4x N\sqrt{h}\left\{\frac{M_{Pl}^2}{2}\left[R^{(3)} + N^{-2}(E_{ij}E^{ij} - E^2)\right] + F(X,Y,Z)\right\}. \tag{4.40}$$

We can easily find the constraint equations for N and N_i by varying the action, obtaining

$$\frac{M_{Pl}^2}{2}\left[R^{(3)} - N^{-2}(E_{ij}E^{ij} - E^2)\right] + F(X,Y,Z) + N\frac{\partial F(X,Y,Z)}{\partial N} = 0, \tag{4.41}$$

$$\frac{M_{Pl}^2}{2}\nabla_i\left[N^{-1}(E^i_j - \delta^i_j E)\right] + N\frac{\partial F(X,Y,Z)}{\partial N^j} = 0. \tag{4.42}$$

Using the definition in (4.18) it is possible to calculate the derivative of F with respect to N and N_i

$$B^{IJ} = -\frac{1}{N^2}\left(\dot{\phi}^I - N^k\partial_k\phi^I\right)\left(\dot{\phi}^J - N^k\partial_k\phi^J\right) + h^{kl}\partial_k\phi^I\partial_l\phi^J \tag{4.43}$$

For the moment it is useful to work in the so-called Spatially Flat Slicing Gauge (SFSG), defined in [53] as

$$\phi^I = x^I + \pi^I, \quad h_{ij} = a(t)^2 exp(\gamma_{ij}), \quad N = 1 + \delta N, \tag{4.44}$$

where γ_{ij} is, as usual, transverse and traceless $\partial_i\gamma_{ij} = \gamma_{ii} = 0$. Now we spilt the vector perturbation into transverse and longitudinal parts, such as [53]

$$\pi^i = \frac{\partial_i}{\sqrt{-\nabla^2}}\pi_L + \pi^i_T, \quad \text{and} \quad N^i = \frac{\partial_i}{\sqrt{-\nabla^2}}N_L + N^i_T, \tag{4.45}$$

with $\partial_i\pi^i_T = \partial_i N^i_T = 0$. It is useful to note that in our treatise there is not difference between the internal indices I and the spatial ones i. This is because both π^i and N^i transform as vectors under $SO(3)$; from now on we will not distinguish between spatial and internal indices.

We can find a simplified expression for B^{ij} calculating (4.43) in the SFSG at the first order in the perturbations

$$B^{ij} = a(t)^{-2}\left(\delta^{ij} + \partial^i\pi^j + \partial^j\pi^i - \gamma^{ij}\right). \tag{4.46}$$

We have now to solve the constraint equations (4.41) and (4.42) for δN, N_L and N^i_T. It is useful to solve them in the Fourier space, finding [53]

$$\begin{aligned}
\delta N(t,\mathbf{k}) &= -\frac{a^2\dot{H}}{kH}\frac{\dot{\pi}_L - \dot{H}\pi_L/H}{1 - 3\dot{H}a^2/k^2} \\
N_{L(t,\mathbf{k})} &= \frac{-3a^2\dot{H}\dot{\pi}_L/k^2 + \dot{H}\pi_L/H}{1 - 3a^2\dot{H}/k^2} \\
N^i_T(t,\mathbf{k}) &= \frac{\dot{\pi}^i_T}{1 - k^2/4a^2\dot{H}}.
\end{aligned} \tag{4.47}$$

In Chapter 2 we have seen that it is possible to define some gauge invariant quantities which provides good estimators for the scalar perturbations. Looking at the definitions of ζ and $\mathcal{R}$ given in (2.76) and (2.77) we obtain the following expressions in the Fourier space [81] and in the mentioned gauge

$$\zeta = -H\frac{\delta\rho}{\rho} = \frac{1}{3}\partial_i\pi^i_L, \qquad \mathcal{R} = -Hv_\parallel = -\frac{k}{3H\epsilon}\frac{\dot{\pi}_L + H\epsilon\pi_L}{1 + k^2/3a^2H^2\epsilon}. \tag{4.48}$$

Note that in the Fourier space ζ becomes simply

$$\zeta = -\frac{k}{3}\pi_L. \tag{4.49}$$

It is easy to see that now the relation between ζ and $\mathcal{R}$ is

$$\mathcal{R} = \frac{1}{H\epsilon}\frac{\dot{\zeta} + H\epsilon\zeta}{1 + k^2/3a^2H^2\epsilon}. \tag{4.50}$$

We see that in solid Inflation the two quantities do not coincide anymore in the super-horizon limit, as they did in the SFSR model. In addition neither ζ nor $\mathcal{R}$ is conserved in time. This is in contrast with the standard predictions of the SFSR models and they are peculiar features which make the studied model much more appealing.
We are now ready to study the two- and three-point functions and see the observable predictions of solid Inflation. We will proceed with the standard steps we have used so far: we will find the solution to the free equation of motion of the mode we are interested in (scalar, vector, tensor) and then we will use it to calculate the power spectrum and the bispectrum.
We can find the second order action by inserting equations (4.47) into (4.40)

$$S^{(2)} = S^{(2)}_\gamma + S^{(2)}_T + S^{(2)}_L,$$

with

$$S^{(2)}_\gamma = \frac{M^2_{Pl}}{4}\int d^4x a^3\left[\frac{1}{2}\dot{\gamma}^2_{ij} - \frac{1}{2a^2}(\partial_m\gamma_{ij})^2 + 2\dot{H}c^2_T\gamma^2_{ij}\right], \tag{4.51}$$

$$S^{(2)}_T = M^2_{Pl}\int dt\frac{d^3k}{(2\pi)^3}a^3\left[\frac{k^2/4}{1-k^2/4a^2\dot{H}}|\dot{\pi}^i_T|^2 + \dot{H}c^2_T k^2|\pi^i_T|^2\right], \tag{4.52}$$

$$S^{(2)}_L = M^2_{Pl}\int dt\frac{d^3\mathbf{k}}{(2\pi)^3}a^3\left[\frac{k^2/3}{1-k^2/3a^2\dot{H}}\left|\dot{\pi}_L - \frac{\dot{H}}{H}\pi_L\right|^2 + \dot{H}c^2_L k^2|\pi_L|^2\right] \tag{4.53}$$

We can notice that in the action for the tensor modes (4.51) it is present a mass term γ^2_{ij} which will depend on the slow-roll parameter and on the parameter c_T of the theory. In the expressions (4.52) and (4.53) we clearly see that are present non-trivial k-dependent terms, which arise due to particular symmetries breaking pattern of the theory. Indeed the would translate into a spatially non-local structure in real space.

Time dependence of background quantities

Before going on with the equation of motion of the scalar, vector and tensor modes we write the explicit (conformal) time dependence of the background quantities, such as $a(\tau)$, $H(\tau)$ and $\epsilon(\tau)$ and the two propagation speeds c_L and c_S. Recalling (4.24) we easily see that

$$\frac{d}{d\tau}\left(\frac{1}{aH}\right) = -1 + \epsilon,$$

and integrating we have

$$\frac{1}{aH} = -(1-\epsilon_c)\tau + \mathcal{O}(\epsilon^2). \tag{4.54}$$

Here we have indicated with c a particular value for the conformal time τ_c, chosen as the conformal time at which the longest mode of observational relevance today exits the horizon, i.e. $|c_{L,c}k_{min}\tau_c| \simeq |c_{L,c}\tau_c H_{today}| = 1$, see Appendix A of [53]. The integration constant is chosen by demanding $a(\tau) \gg a(\tau_c)$, for $\tau/\tau_c \to 0$. From Eq. (4.54) we obtain the time-dependence of the scale factor

$$a(\tau) = a_c\left(\frac{\tau}{\tau_c}\right)^{-1-\epsilon_c} + \mathcal{O}(\epsilon^2). \tag{4.55}$$

For the Hubble parameter we have

$$H(\tau) = \frac{a'}{a^2} = -\frac{1+\epsilon_c}{a_c\tau_c}\left(\frac{\tau}{\tau_c}\right)^{\epsilon_c} + \mathcal{O}(\epsilon^2). \tag{4.56}$$

Using the definition of the second slow-roll parameter (4.26) we have for ϵ

$$\epsilon(\tau) = \epsilon_c\left(\frac{\tau}{\tau_c}\right)^{-\eta_c} + \mathcal{O}(\epsilon^3). \tag{4.57}$$

Finally, for the two velocities we have

$$c_L(\tau) = c_{L,c}\left(\frac{\tau}{\tau_c}\right)^{-s_c} + \mathcal{O}(\epsilon^2), \quad c_L = c_{T,c}\left(\frac{\tau}{\tau_c}\right)^{-u_c} + \mathcal{O}(\epsilon^2), \tag{4.58}$$

where s_c and u_c are the two slow-roll parameters defined in Eqs. (4.36) and (4.37).

4.3.1 Two-Point functions

Tensor modes

We start calculating the tensor mode functions decomposing them into the two polarization $+, \times$

$$\gamma_{ij}(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{s=+,\times}\epsilon^s_{ij}(\mathbf{k})\gamma^s(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.59}$$

with $\epsilon^s_{ij}(\mathbf{k})\epsilon^{p,*}_{ij}(\mathbf{k}) = 2\delta^{sp}$ and $\epsilon_{ii} = k_i\epsilon_{ij} = 0$. Substituting (4.59) into (4.51) and exploiting the ϵ_{ij} properties we obtain

$$S^{(2)}_\gamma = \frac{M^2_{Pl}}{4}\int dt\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{s=+,\times}a^3\left[\dot\gamma_s^2 + \left(4\dot H c_T^2 - \frac{k^2}{a^2}\right)\gamma_s^2\right]. \tag{4.60}$$

Using the conformal time we have

$$S^{(2)}_\gamma = \frac{M^2_{Pl}}{4}\int d\tau\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_s\left[2a^2\gamma_s'^2 - \left(4\epsilon a^4H^2c_T^2 + a^2\right)\gamma_s^2\right], \tag{4.61}$$

where we have used Eq. (4.24) and the notation $' \equiv d/d\tau$. Following the same path we have undertaken for the SFSR model, we promote the field γ_s to an operator, such as

$$\gamma^s(\tau,\mathbf{k}) = \gamma^s_{cl}(\tau,\mathbf{k})a^s(\mathbf{k}) + \gamma^s_{cl}(\tau,\mathbf{k})^*a^{s\dagger}(-\mathbf{k}). \tag{4.62}$$

Here γ^s_{cl} are the *classical* solution to the equation of motion derived from (4.61), $a^{s\dagger}$ and a^s are the usual creation and annihilation operators, obeying the commutation relation

$$[a^s(\mathbf{k}), a^{s'\dagger}(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta^{ss'}. \tag{4.63}$$

The equation of motion extracted from (4.61) varying the action is

$$\gamma_{cl}'' + 2aH\gamma_{cl}' + (4\epsilon a^2H^2c_T^2 + k^2)\gamma_{cl} = 0. \tag{4.64}$$

We can rewrite the coefficients of this equation making explicit their time dependence using Eqs. (4.55), (4.56), (4.57) and (4.58):

$$\gamma_{cl}'' - \frac{2+2\epsilon_c}{\tau}\gamma_{cl}' + \left(k^2 + \frac{4\epsilon_c c_{T,c}}{\tau^2}\right)\gamma_{cl} = 0. \tag{4.65}$$

It manifest that, using the variable $z = -k\tau$ and manipulating (4.65) it resembles the form of (2.67), for which the general solution is

$$\gamma_{cl}(\tau, \mathbf{k}) = (-\tau)^{3/2+\epsilon_c} \left[A H^{(1)}_{\nu_T}(-k\tau) + B H^{(2)}_{\nu_T}(-k\tau) \right], \tag{4.66}$$

with $\nu_T = \frac{3}{2} + \epsilon_c - \frac{4}{3} c^2_{T,c}\epsilon_c$, and H_ν are the Hankel functions we have encountered in Chapter 2. Following the same steps performed in Chapter 2 (using Bunch-Davies vacuum for the sub-horizon limit) we obtain the solution for the tensor mode on the super-horizon limit, at the first order in the slow-roll parameter

$$\gamma^s_{cl}(\tau, \mathbf{k}) \xrightarrow{-k\tau \to 0^+} k^{-\frac{3}{2}} \left(\frac{\tau}{\tau_c} \right)^{\frac{4c^2_{T,c}\epsilon_c}{3}} (-k\tau_c)^{c^2_{L,c}\epsilon_c} \left(\frac{iH_c}{M_{Pl}} + \mathcal{O}(\epsilon) \right). \tag{4.67}$$

We are finally ready to obtain the two-point function for the gravitational waves. We consider the limit at which two tensor modes are well outside the horizon:

$$\begin{aligned} \langle \gamma^s(\tau, \mathbf{k}_1) \gamma^p(\tau, \mathbf{k}_2) \rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \delta^{sp} |\gamma_{cl}(\tau, \mathbf{k}_1)|^2 \\ &\xrightarrow{-k\tau \to 0^+} (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \delta^{sp} \frac{H_c^2}{M_{Pl}^2} \frac{1}{k_1^3} \frac{(\tau/\tau_c)^{8c^2_{T,c}\epsilon_c/3}}{(-k_1\tau_c)^{-2c^2_{L,c}\epsilon_c}}, \end{aligned} \tag{4.68}$$

so that the power spectrum becomes

$$\mathcal{P}_\gamma = \frac{2H_c^2}{M_{Pl}^2} \frac{1}{k_1^3} \frac{(\tau/\tau_c)^{8c^2_{T,c}\epsilon_c/3}}{(-k_1\tau_c)^{-2c^2_{L,c}\epsilon_c}}. \tag{4.69}$$

From this expression we can easily read off the tensor tilt to the first order in slow-roll, defined in (2.104)

$$n_T = 2c_{L,c}2\epsilon_c. \tag{4.70}$$

This is a fundamental outcome of the theory. From Eq. (2.103) we saw that for SFSR model the power spectrum should be red tilted. In this case we clearly see that the prediction for solid Inflation is a *blue*-tilted tensor power spectrum. We will see, after the calculation of the ζ power spectrum, that the standard slow-roll consistency relation (2.108) is violated in this theory. We will also see that will be violated also the consistency relations associated to the three-point functions [80, 63].

Scalar perturbations

For the scalar perturbations we must be careful. We have always considered the curvature perturbation ζ in order to have observable predictions, but in this case we have two potentially good candidates for the observations, ζ and $\mathcal{R}$, since we have seen that they are no more equal on super-horizon scales. We will calculate the power spectrum of the curvature perturbation since it can be shown that when considering the end of the inflationary period, the reheating era, ζ is a continue function, see [53]. We proceed in the same way as before: as a first step we decompose the field of interest in terms of creation and annihilation operators

$$\zeta(t, \mathbf{k}) = \zeta_{cl}(t, \mathbf{k}) b(\mathbf{k}) + \zeta_{cl}(t, \mathbf{k})^* b^\dagger(-\mathbf{k}), \tag{4.71}$$

with the usual commutation relation $[b(\mathbf{k}), b^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$. We can then rewrite the action (4.53) as a function of ζ, using (4.48) and the conformal time:

$$S^{(2)}_\zeta = M_{Pl}^2 \int d\tau \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{3a^2}{1 + k^2/3\epsilon a^2 H^2} \left| \zeta' + \epsilon a H \zeta \right|^2 - 9\epsilon H^2 c_L^2 a^4 |\zeta|^2 \right]. \tag{4.72}$$

Extracting and solving the equation of motion for ζ is quite lengthy, due to the time dependence of the various coefficients, but it is possible to use a simple calculation trick to find the scalar mode function. Using the expression of $\mathcal{R}$ in (4.48) we can rewrite the action for the scalar mode in terms of the other gauge invariant quantity and obtain a new equation for the evolution of the scalar modes which simplifies the calculations [53]

$$\frac{1}{H}\dot{\mathcal{R}}_{cl}(t,\mathbf{k}) + (3+\eta-2\epsilon)\mathcal{R}_{cl}(t,\mathbf{k}) = -3c_L^2\zeta_{cl}(t,\mathbf{k}). \tag{4.73}$$

Deriving in time this expression we obtain, at the first order in the slow-roll parameters

$$\dot{\zeta}_{cl} = -\frac{\ddot{\mathcal{R}}_{cl}}{3Hc_L^2} - \left(\frac{9-5\epsilon+3\eta-2s}{9c_L^2}\right)\dot{\mathcal{R}}_{cl} + \frac{sH}{c_L^2}\mathcal{R}_{cl} \tag{4.74}$$

We can eliminate ζ in the expression (4.50) and obtain a second-order derivative equation for $\mathcal{R}_{cl}$

$$\ddot{\mathcal{R}}_{cl} + (3+\eta-2s)\dot{\mathcal{R}}_{cl} + a^{-2}\left[3a^2H^2(\epsilon + c_L^2\epsilon - 2s) + k^2c_L^2\right]\mathcal{R}_{cl} = 0$$

where we have used the definition (4.36). This equation becomes, using the conformal time

$$\mathcal{R}_{cl}'' + aH(2+\eta-2s)\mathcal{R}_{cl}' + \left[3a^2H^2(\epsilon + c_L^2\epsilon - 2s) + c_L^2k^2\right]\mathcal{R}_{cl} = 0. \tag{4.75}$$

Using the explicit time dependence of the various parameters we can find an equation of motion which resembles a Bessel equation

$$\tau^2\mathcal{R}_{cl}'' - \tau(2-\eta_c+2s_c-\epsilon_c)\mathcal{R}_{cl}' + \left[c_L^2k^2\tau^2 + (3\epsilon_c - 6s_c + 3c_{L,c}^2\epsilon_c)\right]\mathcal{R}_{cl} = 0. \tag{4.76}$$

using the new rescaled variable $\sigma(\tau) = (-\tau)^{\frac{\alpha}{2}}\mathcal{R}_{cl}$[1], with $\alpha = -3-\eta_c+2s_c-2\epsilon_c$ we have

$$\tau^2\sigma'' + \tau\sigma' + \left[c_L^2k^2\tau^2 - \frac{9}{4} - \frac{3}{2}\eta_c - 3s_c + 3c_{L,c}^2\epsilon_c\right]\sigma = 0$$

defining the new "time variable" $y = -c_Lk(1+s_c)\tau$ and $\nu_S = \frac{1}{2}(3+5s_c-3c_{L,c}^2\epsilon_c+\eta_c)$ at the first order we have

$$y^2\frac{d^2\sigma}{dy^2} + y\frac{d\sigma}{dy} + (y^2-\nu_S^2)\sigma = 0 \tag{4.77}$$

which has exactly the same form of Eq. (2.67). We know the exact solution and the relation between σ and $\mathcal{R}_{cl}$, so

$$\mathcal{R}_{cl}(\tau,\mathbf{k}) = (-\tau)^{-\alpha}\left[CH_{\nu_S}^{(1)}\big(-c_Lk\tau(1+s_c)\big) + DH_{\nu_S}^{(2)}\big(-c_Lk\tau(1+s_c)\big)\right]. \tag{4.78}$$

In order to restore the correct initial condition, i.e. to use the Bunch-Davies vacuum we have to canonically normalize the scalar mode π_L. Looking at (4.53) we can define the canonically normalized field and its behavior on small scales

$$v(\tau,\mathbf{k}) = \sqrt{2}\left[\frac{M_{Pl}^2a^2k^2}{3\left(1+\frac{k^2}{3a^2H^2\epsilon}\right)}\right]^{\frac{1}{2}}\pi_L \xrightarrow{-k\tau\to\infty} \sqrt{2\epsilon}M_{Pl}Ha^2\pi_L. \tag{4.79}$$

The action of this new variable resembles exactly the one of the harmonic oscillator, hence we can use the normalization induced by the Bunch-Davies vacuum behavior. We are now

[1] Remind that the time goes from $-\infty$ to 0, so this definition does not create any mathematical problem.

ready to give the behavior of the scalar modes in the subhorizon limit, remind the two definitions in (4.48)

$$\lim_{\tau\to-\infty}\zeta_{cl}(\tau,\mathbf{k}) = -\frac{kv_{cl}}{3\sqrt{2\epsilon}M_{Pl}Ha^2} = -\sqrt{\frac{k}{4\epsilon c_L}}\frac{e^{-i(1+s_c)c_Lk\tau}}{3M_{Pl}Ha^2} \tag{4.80}$$

and

$$\lim_{\tau\to-\infty}\mathcal{R}_{cl}(\tau,\mathbf{k}) = -\frac{a^2H^2}{k}\frac{d}{d\tau}\left(\frac{v_{cl}}{H}\right) = i\sqrt{\frac{c_L}{4\epsilon k}}\frac{e^{-i(1+s_c)c_Lk\tau}}{M_{Pl}a} \tag{4.81}$$

Knowing the behavior of the Henkel function in the subhorizon limit given in (2.69) we can fix the two parameters in (4.78), setting $D = 0$ and

$$C = -i\sqrt{\frac{\pi}{8\epsilon_c}}\frac{c_{L,c}H_c}{M_{Pl}}(-\tau)^{s_c-\epsilon_c-\eta_c/2}\left(1+\frac{1}{2}s_c-\epsilon_c\right)e^{i(\eta_c+5s_c-2c_{L,c}^2\epsilon_c)\pi/4}+\mathcal{O}(\epsilon^{3/2}). \tag{4.82}$$

We have now the exact solution for $\mathcal{R}_{cl}$, hence we can find the exact solution for ζ_{cl} substituting it into Eq. (4.73). We need to use the Hankel function property

$$\frac{d}{dy}H_\nu^{(1)}(y) = \frac{\nu_S}{y}H_{\nu_S}^{(1)}(y) - H_{\nu_S+1}^{(1)}(y)$$

where $y = -c_Lk\tau(1+s_c)$. Eq. (4.73) gives

$$\begin{aligned}\zeta_{cl}(\tau,\mathbf{k}) = -i\sqrt{\frac{\pi}{2}}e^{i\frac{\pi}{2}(\nu_S+\frac{1}{2})}\frac{c_{L,c}H_c(1-s_c-2\epsilon_c)}{3M_{Pl}\sqrt{4\epsilon_c c_{L,c}^5k^3}}\left(\frac{\tau}{\tau_c}\right)^{\epsilon_c+\frac{\eta_c}{2}+\frac{5}{2}s_c}\\ \times\left[y^{5/2}H_{\nu_S+1}^{(1)}(y)+c_{L,c}^2\epsilon_c y^{3/2}H_{\nu_S}^{(1)}(y)\right].\end{aligned} \tag{4.83}$$

It is important to note that this solution We are interested in the super-horizon limit. Using (2.71) we obtain, for $\mathcal{R}_{cl}$ and ζ_{cl}

$$\lim_{-k\tau\to0^+}\zeta_{cl}(\tau,\mathbf{k}) = \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c_{L,c}^2\epsilon_c}(-c_{L,c}k\tau_c)^{c_{L,c}^2\epsilon_c-\frac{5}{2}s_c-\frac{1}{2}\eta_c}\left(\frac{H_c}{\sqrt{4\epsilon_c}M_{Pl}c_{L,c}^{5/2}k^{3/2}}+\mathcal{O}(\epsilon^{1/2})\right), \tag{4.84}$$

$$\lim_{-k\tau\to0^+}\mathcal{R}_{cl}(\tau,\mathbf{k}) = \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c_{L,c}^2\epsilon_c-2s_c}(-c_{L,c}k\tau_c)^{c_{L,c}^2\epsilon_c-\frac{5}{2}s_c-\frac{1}{2}\eta_c}\left(-\frac{H_c}{\sqrt{4\epsilon_c}M_{Pl}c_{L,c}^{1/2}k^{3/2}}+\mathcal{O}(\epsilon^{1/2})\right). \tag{4.85}$$

At first glance we can note that neither ζ nor $\mathcal{R}$ is conserved on scales larger than the horizon, even if their time dependence is suppressed by the slow-roll parameter $\zeta,\mathcal{R}\sim\tau^{c_{L,c}\epsilon_c}$; remind that also the longitudinal mode propagation speed presents a small time dependence, which have to be taken into account when describing the time evolution of the two gauge invariant quantities. We also notice that the two quantities are not equal anymore, but are proportional, on large scales

$$\mathcal{R}\simeq -c_L^2(\tau)\zeta. \tag{4.86}$$

We can finally calculate the two point function of ζ and its power spectrum

$$\begin{aligned}\langle\zeta(\tau,\mathbf{k}_1)\zeta(\tau,\mathbf{k}_2)\rangle &= (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)|\zeta_{cl}(\tau,\mathbf{k}_1)|^2\\ &\xrightarrow{-k\tau\to0^+}(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\frac{H_c^2}{4\epsilon_c c_{L,c}^5M_{Pl}^2}\frac{1}{k_1^3}\frac{(\tau/\tau_c)^{8c_{L,c}^2\epsilon_c/3}}{(-c_{L,c}k_1\tau_c)^{5s_c-2c_{L,c}^2\epsilon_c+\eta_c}}.\end{aligned} \tag{4.87}$$

The power spectrum becomes

$$\mathcal{P}_\zeta(k) = \frac{H_c^2}{4\epsilon_c c_{L,c}^5 M_{Pl}^2}\frac{1}{k^3}\frac{(\tau/\tau_c)^{8c_{L,c}^2\epsilon_c/3}}{(-c_{L,c}k\tau_c)^{5s_c-2c_{L,c}^2\epsilon_c+\eta_c}} \tag{4.88}$$

from which we can extract the scalar index at the first order in the slow-roll parameters

$$n_S - 1 \simeq 2\epsilon_c c_{L,c}^2 - 5s_c - \eta_c. \tag{4.89}$$

We can calculate the tensor-to-scalar-ratio (2.106), which gives, considering Eqs. (4.69) and (4.88),

$$r \sim \epsilon c_L^5. \tag{4.90}$$

We clearly see that the consistency relation (2.108) is not satisfied anymore, since there is an extra $\sim c_L^4$ factor. This is a fundamental outcome of the theory which makes observable prediction which differs from the ones of single field slow roll theories.

4.4 Three-Point Functions

We have seen that new features appears at the level of the power spectrum predictions. We calculate now the third order Lagrangian in order to obtain the observable predictions of Solid Inflation for the non-Gaussianities, the relation between the $\gamma\zeta\zeta$ bispectrum and the ζ power spectrum, i.e. we want to test the consistency relation found in [56], and, in the next section, the predicted tensor fossils.
Finding the third order action is a lengthy work, for this reason the authors in [53] make a fine tuning hypothesis on the shape of the function F. We have seen that the request of subluminality for the subhorizon limit of the theory can be translated in the condition (4.33) on the derivative of F with respect to Y and Z. [53] assumes $F_Y, F_Z \simeq F$ in the limit $F_Y + F_Z \simeq 0$ in order to simplify the calculations for the third order action. In this section we will briefly review the results of [53] and [82] for the bispectra in this particular case. We will then proceed with the generalization of the calculations done in [80]. The results of this article will be used to calculate the prediction for the tensor fossils in solid Inflation.

We want to calculate the non-Gaussianities arising from this new model; we have learnt that the three-point function is given by the in-in formalism [61], schematically, using (3.20)

$$\langle\zeta(\tau)^3\rangle = i\int_{-\infty}^{\tau} d\tau' \langle 0|\left[\zeta(\tau)^3, \mathcal{L}_{\rm int}(\tau')\right]|0\rangle\,. \tag{4.91}$$

We have seen that it is not necessary to calculate the entire interaction Lagrangian, since the only commutator which are different from zero are the ones with the same number and the same kind of field, e.g. three scalar fields on the left and two scalar fields and one derivative of a scalar field on the right. Considering the limit $F_Y = -F_Z$ it is found that the third order Lagrangian with three scalar fields is [53, 82]

$$\mathcal{L}_{\pi\pi\pi} = a^3H^2M_{Pl}^2\frac{F_Y}{F}\left[\frac{7}{81}(\partial\pi)^3 - \frac{1}{9}\partial\pi\partial_i\pi^j\partial_j\pi^i - \frac{4}{9}\partial\pi\partial_j\pi^k\partial_j\pi^k + \frac{2}{3}\partial_j\pi^i\partial_j\pi^k\partial_k\pi^i\right]. \tag{4.92}$$

Here we are simplifying the notation, removing the subscript L and denoting π_L simply by π.

Without going into the detailed calculations (we will be more detailed in the following), the three-point function for ζ is [53]

$$\begin{aligned}\langle\zeta(\tau_e,\mathbf{k}_1)\zeta(\tau_e,\mathbf{k}_2)\zeta(\tau_e,\mathbf{k}_3)\rangle =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\times\\ &\times\frac{3}{32}\frac{F_Y}{F}\frac{H_c^4}{M_{Pl}^4}\frac{1}{\epsilon^3c_L^{12}}\left(\frac{\tau_e}{\tau_c}\right)^{4c_T^2\epsilon}\frac{Q(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)U(k_1k_2k_3)}{k_1^3k_2^3k_3^3},\end{aligned} \tag{4.93}$$

where

$$\begin{aligned}Q(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=&\frac{7}{81}k_1k_2k_3-\frac{5}{27}\left(k_1\frac{(\mathbf{k}_2\cdot\mathbf{k}_3)^2}{k_2k_3}+k_2\frac{(\mathbf{k}_1\cdot\mathbf{k}_3)^2}{k_1k_3}+k_3\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1k_2}\right)\\ &+\frac{2}{3}\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{k}_3)(\mathbf{k}_3\cdot\mathbf{k}_1)}{k_1k_2k_3}\end{aligned} \tag{4.94}$$

and

$$\begin{aligned}U(k_1,k_2,k_3)=&\frac{2}{k_1k_2k_3(k_1+k_2+k_3)^3}\Big[3\left(k_1^6+k_2^6+k_3^6\right)+20k_1^2k_2^2k_3^2\\ &+18\left(k_1^4k_2k_3+k_1k_2^4k_3+k_1k_2k_3^4\right)+12\left(k_1^3k_2^3+k_3^3k_1^3+k_2^3k_3^3\right)\\ &+9\left(k_1^5k_2+5\text{perms}\right)+12\left(k_1^4k_2^2+5\text{perms}\right)+18\left(k_1^3k_2^2k_3+5\text{perms}\right)\Big].\end{aligned} \tag{4.95}$$

The corresponding amplitude for non-Gaussianities f_{NL} from the three-point function (4.93) results to be *huge* [53]

$$f_{NL}\simeq-\mathcal{O}(1)\cdot\frac{F_Y}{F}\frac{1}{\epsilon c_L^2}. \tag{4.96}$$

As stressed out in [80], taking $F_Y\simeq F$, $c_L^2\simeq 1/3$ and $\epsilon\sim$ few percents, compatible with the value in (1.99), f_{NL} takes an absolute value $\gtrsim 100$, which is clearly too high for the actual observational constraints [57], see Eqs. (3.8)-(3.13). For this reason we will follow the generalization and the correction made in [80].

4.4.1 Non-Gaussianities

The third order Lagrangian obtained in [53] was found with the assumption of large derivative of the function F, i.e. $F_Y\sim F_Z\sim F$. As a result the authors considered only the leading terms $|F_{Y,Z}/F|=1+\mathcal{O}(\epsilon)$, neglecting all that terms that are of order of the slow-roll parameter ϵ. It can be naively seen that this induces errors on the calculation of f_{NL} of the order of unity. The leading value of f_{NL} from Eq. (4.96) is $f_{NL}\sim F_Y/\epsilon F\sim 1/\epsilon$, hence a correction to the Lagrangian of the order $\sim\epsilon$ would induce correction of order unity on f_{NL}, which are discarded. A symmetry argument also holds in disfavor of [53]. There is no profound reason for assuming $F_Y+F_Y\simeq 0$ while preserving the constraint equation (4.33), while, we have seen, the assumption of small derivatives of F can be explained with an approximate symmetry (4.35). More physically it provides a description of an inflationary period driven by a cosmological constant, which does not sound so strange as it seems. We will calculate the three-point functions $\langle\zeta\zeta\zeta\rangle$ and $\langle\gamma\zeta\zeta\rangle$ considering F_Y and F_Z as free parameters of the theory. We will see that are also present other corrections from performing a more precise computation of the bispectra.

The full calculation of the $\zeta\zeta\zeta$ bispectrum is performed in detail in Ref. [80]. The full three-point function is given by

$$\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = \langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle_{lead} + \langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle_{(1)} + \langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle_{(2)} \tag{4.97}$$

The leading contribution is due to the computation of the three-point function using the leading order Lagrangian

$$\begin{aligned}\mathcal{L}^{\text{lead}}_{\pi\pi\pi} = a^3H^2M_{Pl}^2\Bigg[&\frac{F_Y}{F}\left(-\frac{16}{27}(\partial\pi)^3 + \frac{8}{9}\partial\pi\partial_i\pi^j\partial_j\pi^i + \frac{4}{3}\partial\pi\partial_i\pi^j\partial_i\pi^j - \frac{4}{3}\partial_j\pi^i\partial_j\pi^k\partial_k\pi^i\right)\\ &+\frac{F_Z}{F}\left(-\frac{55}{81}(\partial\pi)^3 + \partial\pi\partial_i\pi^j\partial_j\pi^i + \frac{16}{9}\partial\pi\partial_i\pi^j\partial_i\pi^j - 2\partial_j\pi^i\partial_j\pi^k\partial_k\pi^i\right)\Bigg].\end{aligned} \tag{4.98}$$

Taking the first order solution for the mode function from (4.83)

$$\zeta_{cl}(\tau,k) = \frac{H}{2M_{Pl}\sqrt{\epsilon c_L^5 k^3}}\left(1 + ic_L - \frac{1}{3}c_L^2k^2\tau^2\right)e^{-ic_Lk\tau} \tag{4.99}$$

$\langle\zeta^3\rangle_{lead}$ has the form

$$\begin{aligned}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle_{lead} =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\\ &\times\frac{3}{32}\left(\frac{H}{M_{Pl}}\right)^4\frac{1}{\epsilon^3c_L^{12}}\frac{1}{k_1^3k_1^3k_2^3}U(k_1,k_2,k_3)\mathcal{F}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\end{aligned} \tag{4.100}$$

wit U given in (4.95) and

$$\mathcal{F}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = Q_Y(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\frac{F_Y}{F} + Q_Z(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\frac{F_Z}{F},$$

with

$$Q_Y(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv -\frac{16}{27}k_1k_2k_3 + \frac{20}{27}\left(\frac{k_1}{k_2k_3}(\mathbf{k}_2\cdot\mathbf{k}_3)^2 + 2\text{ perm.}\right) - \frac{4}{3}\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_2\cdot\mathbf{k}_3)}{k_1k_2k_3} \tag{4.101}$$

$$Q_Z(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv -\frac{55}{81}k_1k_2k_3 + \frac{25}{27}\left(\frac{k_1}{k_2k_3}(\mathbf{k}_2\cdot\mathbf{k}_3)^2 + 2\text{ perm.}\right) - 2\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_2\cdot\mathbf{k}_3)}{k_1k_2k_3} \tag{4.102}$$

Note that the expression (4.100) coincides with the one found in [53], Eq. (4.93), when $F_Z = -F_Y$ and $Q = Q_Y - Q_Z$.

If we consider the two derivative F_Y and F_Z of order $\sim\epsilon F$ we are not allowed to neglect the order ϵ terms in the Lagrangian. $\langle\zeta^3\rangle_{(1)}$ is the terms calculated using the subleading order Lagrangian, obtained considering the following solid relations

$$F_{XX} = -\frac{a^4}{9}\epsilon F, \quad F_{XXX} = \frac{2a^6}{27}\epsilon F, \quad (F_{XZ}+F_{XY}) = \mathcal{O}(\epsilon^2)$$

which is

$$\mathcal{L}^{sub}_{\pi\pi\pi} = \epsilon a^3H^2M_{Pl}^2\left(-\frac{8}{27}(\partial\pi)^3 + \frac{2}{3}\partial\pi\partial_i\pi^j\partial_i\pi^j\right). \tag{4.103}$$

The correction to the three-point function takes the same form of (4.100)

$$\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle_{(1)} = (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{3}{32}\left(\frac{H}{M_{Pl}}\right)^4\frac{\epsilon}{\epsilon^3c_L^{12}}\frac{1}{k_1^3k_2^3k_3^3}U(k_1,k_2,k_3)\bar{Q}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \tag{4.104}$$

with

$$\bar{Q}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv -\frac{8}{27}k_1k_2k_3 + \frac{2}{9}\left(\frac{k_1}{k_2k_3}(\mathbf{k}_2\cdot\mathbf{k}_3)^2 + 2\text{ perm}\right),$$

There are other two corrections: the first one comes from considering the full solution for the ζ mode function which is [81]

$$\begin{aligned}\zeta(\tau,k) = &-i\sqrt{\frac{\pi}{2}}\frac{c_{L,c}H_c}{2M_{Pl}\sqrt{\epsilon_c}}(-\tau)^{3/2}\left[1+\left(\epsilon_c+\frac{\eta_c}{2}-s_{L,c}\right)\ln\frac{\tau}{\tau_c}\right]\\ &\times\left[-\frac{\epsilon_c}{3}H^{(1)}_{\nu_S}(y)+\frac{k\tau}{3c_L}(1-\epsilon_c)H^{(1)}_{\nu_S+1}(y)\right]\end{aligned} \tag{4.105}$$

Considering the super-horizon limit of Eq. (4.105) we can write

$$\zeta \xrightarrow{-kc_L\tau\to 0} (-c_Lk\tau)^{-A}(1+B\ln(-c_Lk\tau)), \tag{4.106}$$

with

$$A = \frac{\eta}{2}+\frac{5}{2}s - c_L^2\epsilon, \quad B = \epsilon+\frac{\eta}{2}-s. \tag{4.107}$$

The structure of in-in integrals for this case is exactly the same as the leading order one, the only exception is the additional factor $(B-A)\ln(-c_Lk\tau_e)$. We obtain

$$\begin{aligned}\langle\zeta_{\mathbf{k}_1}(\tau_e)\zeta_{\mathbf{k}_2}(\tau_e)\zeta_{\mathbf{k}_3}(\tau_e)\rangle_{(2)} =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{9}{32}\left(\frac{H}{M_{Pl}}\right)^4\frac{(B-A)\ln(-c_Lk\tau_e)}{\epsilon^3c_L^{12}}\\ &\frac{1}{k_1^3k_2^3k_3^3}U(k_1,k_2,k_3)\mathcal{F}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3).\end{aligned} \tag{4.108}$$

The last correction is due to the order ϵ correction for the Hankel function for the small argument limit (super-horizon scales) and has expression [80]

$$\begin{aligned}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle_{(3)} =&(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{9}{16}\left(\frac{H}{M_{Pl}}\right)^4\frac{\epsilon\ln(-k_3c_L\tau_e)}{\epsilon^3c_L^{12}}\\ &\frac{1}{k_1k_2k_3}\bar{U}(k_1,k_2,k_3)\mathcal{F}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3),\end{aligned} \tag{4.109}$$

where

$$\bar{U}(k_1,k_2,k_3) \equiv \frac{k_1^2}{k_2k_3}+\frac{k_2^2}{k_1k_3}+\frac{k_3^2}{k_1k_2}.$$

The complete three-point function can be written as

$$\langle\zeta^3(\tau_e)\rangle = \langle\zeta^3(\tau_e)\rangle_{lead}+\langle\zeta^3(\tau_e)\rangle_{(1)}+\langle\zeta^3(\tau_e)\rangle_{(2)}+\langle\zeta^3(\tau_e)\rangle_{(3)}, \tag{4.110}$$

so that the complete result for the bispectrum is

$$\mathcal{B}_{\zeta\zeta\zeta}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{3H^4}{32M_{Pl}^4}\frac{U(k_1,k_2,k_3)}{\epsilon^3c_L^{12}}\frac{Q_{eff}}{k_1^3k_2^3k_3^3}, \tag{4.111}$$

where Q_{eff} is

$$Q_{eff} = \epsilon\bar{Q}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+\left(\frac{F_Y}{F}Q_Y+\frac{F_Z}{F}Q_Z\right)\left(1+(B-A)\sum_i N_{k_i}+2\epsilon\sum_i N_{k_i}\frac{\bar{U}}{U}\right). \tag{4.112}$$

Note that $N_{k_i} = \ln(-c_L k_i \tau_e)$, which represents the number of e-folds when the mode k_i has left the horizon. It can easily be seen that when considering $F_Y = -F_Z$ and $F_{Y,Z}/F \simeq 1$ Eq. (4.111) resembles the result of [53], Eq. (4.93).
We can evaluate the squeezed limit for the three-point function of the mode ζ to test the Maldacena consistency relation. Using the following relations, which hold in the limit $k_3 \to 0, \mathbf{k}_1 \simeq -\mathbf{k}_2$

$$U = \frac{15k_1}{k_3}, \qquad \bar{U} = \frac{2k_1}{k_3},$$

$$\bar{Q} = \frac{2k_1^2 k_3}{27}(6c^2 - 1), \quad Q_Y = \frac{4k_1^2 k_3}{27}(c^2+1), \quad Q_Z = \frac{4k_1^2 k_3}{81}(5-3c^2),$$

where $c \equiv \cos\theta$ and θ is angle between k_1 and k_3, we obtain

$$\begin{aligned} \mathcal{B}^{\text{sq}}_{\zeta\zeta\zeta} = &\frac{5}{96}\left(\frac{H}{M_{Pl}}\right)^4 \frac{1}{\epsilon^3 c_L^{12}} \frac{1}{k_1^3 k_3^3}\Bigg\{2\epsilon(6c^2-1) \\ &+ \left(4\frac{F_Y}{F}(c^2+1) + \frac{4}{3}\frac{F_Z}{F}(5-3c^2)\right)\left[1 + (B-A)(N_{k_3}+2N_{k_1}) + \frac{4\epsilon}{15}(N_{k_3}+2N_{k_1})\right]\Bigg\}. \end{aligned} \tag{4.113}$$

Using the first order solution for the scalar power spectrum in (4.88) and defining

$$\begin{aligned} G(\theta) =& 2\epsilon(6c^2-1) \\ &+ \left(4\frac{F_Y}{F}(c^2+1) + \frac{4}{3}\frac{F_Z}{F}(5-3c^2)\right)\left[1 + (B-A)(N_{k_3}+2N_{k_1}) + \frac{4\epsilon}{15}(N_{k_3}+2N_{k_1})\right], \end{aligned} \tag{4.114}$$

we can rewrite (4.113) as

$$\mathcal{B}^{\text{sq}}_{\zeta\zeta\zeta} = \frac{5}{6}\frac{G(\theta)}{\epsilon c_L^2}\mathcal{P}_\zeta(k_1)\mathcal{P}_\zeta(k_3). \tag{4.115}$$

Note that, except for tuned values for the parameters of the theory, the consistency relation (3.57) is violated in this theory, also because of the explicit angular dependence inside (4.114), which is that of a pure quadrupole. This result emerges also in the hypothesis of [53], see Ref. [82]. Eq. (4.115) generalizes the result found in Ref. [63]. Following [80] it is possible to evaluate the non-Gaussianities in this limit. The amplitude of non-Gaussianities is given by (3.5), which can be evaluated in the squeezed limit

$$\begin{aligned} f_{NL}^{sq} =& \frac{25}{36c_L^2}(6c^2-1) + \frac{25}{18\epsilon c_L^2}\left[\frac{F_Y}{F}(1+c^2) + \frac{F_Z}{F}(5-3c^2)\right] \\ &\times\left[1 + \frac{4\epsilon(2N_{k_1}+N_{k_3})}{15} + \frac{(B-A)}{4N_{k_1}} - 3N_{k_3}\right]. \end{aligned} \tag{4.116}$$

The first term in this expression has very particular features. It is (almost) model independent, which means that it depends on the form of $F(X,Y,Z)$ only through c_L. It comes from the subleading order terms of the Lagrangian (4.103). The other terms have the same form of [53, 82]. If we consider a particular limit, for which $F_Y = F_Z = 0$ we have an $F(X)$ theory. For this model, considering $c_L = \frac{1}{3}$ we obtain

$$f_{NL}^{sq} = \frac{25}{12}(6\cos^2\theta - 1). \tag{4.117}$$

This result is very interesting. It shows that for all the $F(X)$ theories of solid Inflation the amplitude of the non-Gaussianities has a universal form. Therefore, this amplitude is consistent with the bounds from the Planck observations [57], as opposed to the results of [53]. In the end, we note that the expression (4.117), but also (4.116), is anisotropic, due to its angular dependence.

4.4.2 Testing the $\gamma\zeta\zeta$ Consistency Relation

In this section we will calculate the scalar-scalar-bispectrum. In the previous chapter we have seen its importance. For SFSR models it assumes the form of a consistency relation in the squeezed limit, when the long mode is tensorial. We will calculate the leading order bispectrum and test the consistency relation (3.59), then we will calculate the observable prediction for the tensor fossil in this theory. For the reason described above we will follow [80] for the full calculation, which we report here. It is fundamental, for simplification of the calculation, to see that the leading contribution to the bispectrum is due to the matter sector, and the contributions from the metric perturbations are sub-leading [80]. For this reason we only need to calculate the $\pi\pi\gamma$ terms deriving from varying F.
At the first order in the slow-roll parameters the action involving two scalar modes π and one tensor mode γ is [80]

$$\begin{aligned}\mathcal{L}_{\gamma\pi\pi} = a^3\Bigg[&-\frac{1}{3}F_X X\gamma_{kj}\partial_k\pi^i\partial_j\pi^i - \frac{4}{9}F_{XX}X^2\gamma_{ij}\partial_i\pi^j\partial_k\pi^k \\ &+ F_Y\left(\frac{8}{9}\gamma_{ij}\partial_i\pi^j\partial_k\pi^k - \frac{4}{9}\gamma_{ij}\partial_i\pi^k\partial_j\pi^k - \frac{2}{9}\gamma_{ij}\partial_k\pi^i\partial_k\pi^j - \frac{4}{9}\gamma_{ij}\partial_i\pi^k\partial_k\pi^j\right) \\ &+ F_Z\left(\frac{32}{27}\gamma_{ij}\partial_i\pi^j\partial_k\pi^k - \frac{5}{9}\gamma_{ij}\partial_i\pi^k\partial_j\pi^k - \frac{1}{3}\gamma_{ij}\partial_k\pi^i\partial_k\pi^j - \frac{2}{3}\gamma_{ij}\partial_i\pi^k\partial_k\pi^j\right)\Bigg]\end{aligned} \tag{4.118}$$

We have to Fourier transform the gravitational waves

$$\gamma_{ij}(\tau,\mathbf{x}) = \int_k e^{i\mathbf{k}\cdot\mathbf{x}}\sum_s \epsilon^s_{ij}(\mathbf{k})\gamma^s_{\mathbf{k}}(\tau), \tag{4.119}$$

where ϵ^s_{ij} is the polarization tensor, which, in flat gauge, satisfies

$$k_i\epsilon^s_{ij}(\mathbf{k}) = 0 = \epsilon^s_{ii}(\mathbf{k}), \quad \epsilon^s_{ij}(\mathbf{k})\epsilon^{s'\,*}_{ij}(\mathbf{k}) = 2\delta^{ss'}. \tag{4.120}$$

For the scalar mode we will use the first order solution (4.99), the relation (4.49) and

$$\zeta(\tau,\mathbf{x}) = \int_k e^{i\mathbf{k}\cdot\mathbf{x}}\zeta_{\mathbf{k}}(\tau).$$

Now we write the interaction Hamiltonian having in mind the following fundamental relations

$$\frac{F_X X}{F} = \epsilon, \quad \frac{F_{XX}X}{F} = -1 + \mathcal{O}(\epsilon), \quad F = -3M_{Pl}^2H^2,$$

which are the relations found in the study of the background dynamics. The interaction Hamiltonian results to be

$$\begin{aligned}H_{\gamma\zeta\zeta}(\tau) = -9M_{Pl}^2a^3H^2\int_{p_1p_2p_3}&(2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\sum_s\epsilon^s_{ij}(\mathbf{p}_1)\gamma^s_{\mathbf{p}_1}\zeta_{\mathbf{p}_2}\zeta_{\mathbf{p}_3} \\ &\times\Bigg[\epsilon\hat{p}_{2i}\hat{p}_{3j}\hat{p}_{2l}\hat{p}_{3l} - \frac{4}{3}\epsilon\hat{p}_{2i}\hat{p}_{2j} \\ &+\frac{F_Y}{F}\left(-\frac{8}{3}\hat{p}_{2i}\hat{p}_{2j} + \frac{10}{3}\hat{p}_{2i}\hat{p}_{3j}\hat{p}_{2p}\hat{p}_{3p}\right) \\ &+\frac{F_Z}{F}\left(-\frac{32}{9}\hat{p}_{2i}\hat{p}_{2j} + \frac{14}{3}\hat{p}_{2i}\hat{p}_{3j}\hat{p}_{2p}\hat{p}_{3p}\right)\Bigg],\end{aligned} \tag{4.121}$$

where we have used the notation $\mathbf{k}_i = k_i \hat{k}_i$. The three-point function now takes the form

$$
\begin{aligned}
&\langle \gamma^s_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle_{\tau_e} = \\
&= i9 M_{Pl}^2 \sum_{s'} \int_{-\infty}^{\tau_e} d\tau a^4 H^2 \int_{p_1 p_2 p_3} (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \times \Bigg[\epsilon \hat{p}_{2i} \hat{p}_{3j} \hat{p}_{2l} \hat{p}_{3l} - \frac{4}{3} \epsilon \hat{p}_{2i} \hat{p}_{2j} \\
&+ \frac{F_Y}{F} \left(-\frac{8}{3} \hat{p}_{2i} \hat{p}_{2j} + \frac{10}{3} \hat{p}_{2i} \hat{p}_{3j} \hat{p}_{2p} \hat{p}_{3p} \right) + \frac{F_Z}{F} \left(-\frac{32}{9} \hat{p}_{2i} \hat{p}_{2j} + \frac{14}{3} \hat{p}_{2i} \hat{p}_{3j} \hat{p}_{2p} \hat{p}_{3p} \right) \Bigg] \epsilon^{s'}_{ij}(\mathbf{p}_1) \\
&\times \left[\langle \gamma^s_{\mathbf{k}_1}(\tau_e) \zeta_{\mathbf{k}_2}(\tau_e) \zeta_{\mathbf{k}_3}(\tau_e) \gamma^{s'}_{\mathbf{p}_1}(\tau) \zeta_{\mathbf{p}_2}(\tau) \zeta_{\mathbf{p}_3}(\tau) \rangle - c.c. \right]
\end{aligned}
\tag{4.122}
$$

It remains to solve the time integral. We use the firs order relation for the conformal time $aH\tau = -1$ and the Wick theorem to write the correlator. The procedure is standard, we quantize the tensor mode

$$
\gamma^s_{\mathbf{k}}(\tau) = \gamma^s_{cl}(\tau, k) a_{\mathbf{k}} + \gamma^{s'*}_{cl}(\tau, k) a^\dagger_{-\mathbf{k}} \equiv \gamma^{s+}_{\mathbf{k}}(\tau) + \gamma^{s-}_{\mathbf{k}}(\tau),
\tag{4.123}
$$

where we will use for γ^s_{cl} the first order solution, which is, looking (4.66),

$$
\begin{aligned}
\gamma^s_{cl}(\tau, k) &= \sqrt{\frac{\pi}{2}} \frac{H}{M_{Pl} \sqrt{k^3}} (-k\tau)^{\frac{3}{2}} H^{(1)}_{\frac{3}{2}}(-k\tau) \\
&= i \frac{H}{M_{Pl} \sqrt{k^3}} (1 + ik\tau) e^{-ik\tau}.
\end{aligned}
\tag{4.124}
$$

The commutation relation between the creation and annihilation operators $a^\dagger$, a, induces the following commutation relation for the tensor modes

$$
[\gamma^{s+}_{\mathbf{k}}(\tau), \gamma^{s'-}_{\mathbf{p}}(\tau')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{p}) \delta^{ss'} \gamma_{cl}(\tau, k) \gamma^*_{cl}(\tau', p).
\tag{4.125}
$$

Now, the contraction between two tensor modes is defined as

$$
\overbrace{\gamma^s_{\mathbf{k}}(\tau) \gamma^{s'}_{\mathbf{p}}}(\tau') = \langle 0 | \gamma^s_{\mathbf{k}}(\tau) \gamma^{s'}_{\mathbf{p}}(\tau') | 0 \rangle = [\gamma^{s+}_{\mathbf{k}}(\tau), \gamma^{s'-}_{\mathbf{p}}(\tau')].
\tag{4.126}
$$

for which we have, using the Wick theorem

$$
\begin{aligned}
&\langle \gamma^s_{\mathbf{k}_1}(\tau_e) \zeta_{\mathbf{k}_2}(\tau_e) \zeta_{\mathbf{k}_3}(\tau_e) \gamma^{s'}_{\mathbf{p}_1}(\tau) \zeta_{\mathbf{p}_2}(\tau) \zeta_{\mathbf{p}_3}(\tau) \rangle = \\
&\gamma^s_{\mathbf{k}_1}(\tau_e) \zeta_{\mathbf{k}_2}(\tau_e) \zeta_{\mathbf{k}_3}(\tau_e) \gamma^{s'}_{\mathbf{p}_1}(\tau) \zeta_{\mathbf{p}_2}(\tau) \zeta_{\mathbf{p}_3}(\tau) + \gamma^s_{\mathbf{k}_1}(\tau_e) \zeta_{\mathbf{k}_2}(\tau_e) \zeta_{\mathbf{k}_3}(\tau_e) \gamma^{s'}_{\mathbf{p}_1}(\tau) \zeta_{\mathbf{p}_2}(\tau) \zeta_{\mathbf{p}_3}(\tau) \\
&= (2\pi)^9 \delta^{ss'} \delta(\mathbf{k}_1 + \mathbf{p}_1) \gamma_{cl}(\tau_e, k_1) \zeta_{cl}(\tau_e, k_2) \zeta_{cl}(\tau_e, k_3) \zeta^*_{cl}(\tau, p_2) \zeta^*_{cl}(\tau, p_3) \gamma^*_{cl}(\tau, p_1) \times \\
&\left[\delta(\mathbf{k}_2 + \mathbf{p}_2) \delta(\mathbf{k}_3 + \mathbf{p}_3) + \delta(\mathbf{k}_2 + \mathbf{p}_3) \delta(\mathbf{k}_3 + \mathbf{p}_2) \right].
\end{aligned}
\tag{4.127}
$$

The correlator (4.122) becomes

$$
\begin{aligned}
&\langle \gamma^s_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = 18 M_{Pl}^2 (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \epsilon^s_{ij}(\mathbf{k}_1) \mathcal{M}_{ij}(\mathbf{k}_2, \mathbf{k}_3) \times \\
&\left[i\gamma_{cl}(\tau_e, k_1) \zeta_{cl}(\tau_e, k_2) \zeta_{cl}(\tau_e, k_3) \int_{-\infty}^{\tau_e} d\tau a^4 H^2 \gamma^*_{cl}(\tau, p_1) \zeta^*_{cl}(\tau, p_2) \zeta^*_{cl}(\tau, p_3) - c.c. \right]
\end{aligned}
\tag{4.128}
$$

with

$$
\mathcal{M}_{ij}(\mathbf{k}_2, \mathbf{k}_3) = \left(\epsilon + \frac{10}{3} \frac{F_Y}{F} + \frac{14}{3} \frac{F_Z}{F} \right) \hat{k}_{2i} \hat{k}_{3j} \hat{k}_{2l} \hat{k}_{3l} - \frac{2}{3} \left(\epsilon + 2 \frac{F_Y}{F} + \frac{8}{3} \frac{F_Z}{F} \right) \left(\hat{k}_{2i} \hat{k}_{2j} + \hat{k}_{3i} \hat{k}_{3j} \right).
\tag{4.129}
$$

The expression inside the square brackets becomes

$$\frac{H^4}{16\epsilon^2 c_L^{10} M_{Pl}^6} \frac{1}{k_1^3 k_2^3 k_3^3} (-2Im[I_1])$$

Where

$$I_1 = \int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau \tau^{-4} (1 - ik_1\tau)(1 - i\bar{k}_2\tau - \frac{1}{3}\bar{k}_2^2\tau^2)(1 - i\bar{k}_3\tau - \frac{1}{3}\bar{k}_3^2\tau^2) e^{iK_t\tau} \tag{4.130}$$

where we have used the notation $\bar{k}_i = c_L k_i$, $K_t = k_1 + \bar{k}_2 + \bar{k}_3$ and $H, \epsilon = const$, $aH\tau = -1$. The integral can be solved exactly, the result is

$$\begin{aligned} I_1 = \Bigg\{ &-\left[\frac{1}{3\tau_e^3} + \frac{1}{6\tau_e}\left(3k_1^2 + \bar{k}_2^2 + \bar{k}_3^2\right)\right] - \frac{iK_t}{9}\left(4k_1^2 + \bar{k}_2^2 + \bar{k}_3^2 - k_1\bar{k}_2 - \bar{k}_2\bar{k}_3 - k_1\bar{k}_3\right) \\ &- \frac{i\bar{k}_2\bar{k}_3}{9K_t^2}\left[\bar{K}_t\left(3k_1\bar{k}_2 + 3k_1\bar{k}_2 + \bar{k}_2\bar{k}_3\right) + k_1\bar{k}_2\bar{k}_3\right] + i\frac{k_1^3}{3}\int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau \frac{e^{iK_t\tau}}{\tau}\Bigg\} \end{aligned} \tag{4.131}$$

Note that we have used the fact that $K_t\tau_e \ll 1$ to expand the exponential function $e^{iK_t}\tau_e = 1 + iK_t\tau_e + \mathcal{O}(K_t^2\tau_e^2)$.
Before going on, we have to solve the integral in the second line

$$I = \int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau \frac{e^{iK_t\tau}}{\tau}.$$

It is possible to solve this integral promoting the variable τ from to real to complex

$$I = Re\left[\int_{-\infty(1-i\epsilon)}^{x_e} dx \frac{e^{iK_t x}}{x}\right]$$

with x complex. Performing a Wick rotation $x \to ix$ and a rescaling of the variable $z = K_t x$ we obtain

$$I = Re\left[\int_{-i\infty(1-i\epsilon)}^{iK_t\tau_e} dz \frac{e^{-z}}{z}\right].$$

The integral inside the square brackets is related with the *exponential integral*, defined as

$$E_1(z) \equiv \int_z^\infty dx \frac{e^{-x}}{x}, \tag{4.132}$$

for which is valid the following expansion

$$E_1(z) = -\gamma_M - \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k \cdot k!}, \tag{4.133}$$

where $\gamma_M \simeq 0.577$ is the Eulero-Mascheroni constant.
In our case we have

$$I = -Re\left[E_1(iK_t\tau_e)\right]$$

and, since we are interested in the super-horizon limit $K_t\tau_e \sim 0$, we finally have

$$I = Re[\gamma_M + \ln iK_t\tau_e] = Re[\gamma_M + \ln|K_t\tau_e| + i\pi/2] = \gamma_M + \ln|K_t\tau_e| = \gamma_M + N_{K_t}$$

where N_{K_t} is the number of e-folds since the scale corresponding to K_t crossed the horizon until the end of Inflation. Plugging the imaginary part of I_1 (4.131) into (4.128) we can finally write the expression for the three-point function

$$\langle \gamma^s_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle_{\tau_e} = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \frac{H^4}{4M_{Pl}^4 \epsilon^2 c_L^{10}} \frac{1}{k_1^3 k_2^3 k_3^3} \epsilon^s_{ij}(\mathbf{k}_1) \mathcal{M}_{ij}(\mathbf{k}_2, \mathbf{k}_3)$$
$$\Bigg\{ K_t \left(4k_1^2 + \bar{k}_2^2 + \bar{k}_3^2 - k_1\bar{k}_2 - \bar{k}_2\bar{k}_3 - k_1\bar{k}_3\right) \tag{4.134}$$
$$+ \frac{\bar{k}_2\bar{k}_3}{K_t^2} \left[\bar{K}_t \left(3k_1\bar{k}_2 + 3k_1\bar{k}_2 + \bar{k}_2\bar{k}_3\right) + k_1\bar{k}_2\bar{k}_3\right] + 3k_1^3(\gamma_M + N_{K_t}) \Bigg\}$$

hence the bispectrum becomes

$$\mathcal{B}_{\gamma\zeta\zeta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{H^4}{4M_{Pl}^4 \epsilon^2 c_L^{10}} \frac{1}{k_1^3 k_2^3 k_3^3} \epsilon^s_{ij}(\mathbf{k}_1) \mathcal{M}_{ij}(\mathbf{k}_2, \mathbf{k}_3) \Bigg\{ K_t \left(4k_1^2 + \bar{k}_2^2 + \bar{k}_3^2 - k_1\bar{k}_2 - \bar{k}_2\bar{k}_3 - k_1\bar{k}_3\right)$$
$$+ \frac{\bar{k}_2\bar{k}_3}{K_t^2} \left[\bar{K}_t \left(3k_1\bar{k}_2 + 3k_1\bar{k}_2 + \bar{k}_2\bar{k}_3\right) + k_1\bar{k}_2\bar{k}_3\right] + 3k_1^3(\gamma_M + N_{K_t}) \Bigg\}. \tag{4.135}$$

Note that this is the result for the bispectrum of one tensor mode and two scalar ones in the general configuration of the momenta. Neither [80] nor [82] reported this general result, while they take directly the squeezed limit. Their result could be taken as a check for the correctness of our result. We are interested in the possibility of violating the CR seen in the previous chapter, so we need to calculate the squeezed limit of (4.135). Here we want the long mode to be the tensor one, i.e. $k_L \equiv k_1 \ll k_2 \sim k_3 \equiv k_S$. In this limit the function $\mathcal{M}_{ij}$ in Eq. (4.129) becomes

$$\mathcal{M}_{ij} \xrightarrow{k_1 \ll k_2, k_3} \left(-\frac{\epsilon}{3} + \frac{2}{3}\frac{F_Y}{F} + \frac{10}{9}\frac{F_Z}{F}\right) \hat{k}_{2i}\hat{k}_{2j} \tag{4.136}$$

while the function inside the braces in (4.135) becomes $\frac{5}{2}k_2^3$. Plugging (4.136) into (4.135) we obtain the squeezed limit of the bispectrum

$$\mathcal{B}_{\gamma\zeta\zeta}(k_1, k_2, k_3) \xrightarrow{k_1 \ll k_2, k_3} \frac{5}{8} \left(\frac{H}{M_{Pl}}\right)^4 \frac{1}{\epsilon^2 c_L^7} \frac{1}{k_1^3 k_2^3} \left(-\frac{\epsilon}{3} + \frac{2}{3}\frac{F_Y}{F} + \frac{10}{9}\frac{F_Z}{F}\right) \epsilon^s_{ij}(\mathbf{k}_1) \hat{k}_{2i}\hat{k}_{2j}. \tag{4.137}$$

This expression is in agreement with the result of [80]. We can now rewrite the expression for the squeezed limit of the bispectrum as a function of the power spectrum of the tensor and the scalar modes

$$\mathcal{B}^{\text{sq}}_{\gamma\zeta\zeta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{5}{2} \frac{1}{\epsilon c_L^2} \mathcal{P}_\gamma(k_1) \mathcal{P}_\zeta(k_2) \left(-\frac{\epsilon}{3} + \frac{2}{3}\frac{F_Y}{F} + \frac{10}{9}\frac{F_Z}{F}\right) \epsilon^s_{ij}(\mathbf{k}_1) \hat{k}_{2i}\hat{k}_{2j} \tag{4.138}$$

Taking $F_Z = -F_Y$ and neglecting the $\epsilon/3$ term we have accordance with the result of [82]. We underline that except for particular conditions on the parameters of the theory, this expression violates the consistency relation (3.59)[2]. Eq. (4.138) shows a fundamental outcome of this theory: the squeezed limit of the tensor-scalar-scalar bispectrum departures from the standard expression of SFSR models. This is what makes so fascinating this theory and a test of the consistency relation would rule out one or the other model. We will use this formula for the predictions about the tensor fossil coming from this model.

[2]See Ref. [63] for the study of the violation of tensor-scalar-scalar consistency relation in Solid Inflation using the hypothesis of [53]

4.5 Tensor fossil in Solid Inflation

We have seen in the previous chapter that the a long tensor mode can affect the power spectrum of the scalar field. We have called this effect *tensor fossil* and we have seen how important is the Maldacena consistency relation for the bispectrum $\gamma\zeta\zeta$. Now we are in presence of a theory which violates the mentioned CR, hence this fossil effect can be large. Using the parametrization (3.64) we clearly see that the effect of a long tensor mode is a quadrupole remodulation. Through the parametrization of Ref. [63] it is possible to estimate the effect of this quadrupole, using (3.96). Using the notation of [67]

$$\langle \mathcal{Q}^2 \rangle \equiv \frac{8\pi}{15} \langle \mathcal{Q}_{ij} \mathcal{Q}^{ij} \rangle = \frac{16}{15\pi} \int_{k_L^{min}}^{k_S^{min}} k_L^2 dk_L \left[\frac{\mathcal{B}_{\gamma\zeta\zeta}(k_L, k_S, k_S)}{\gamma(k_L)\mathcal{P}_\zeta(k_S)} \right]^2 \mathcal{P}_\gamma(k_L), \tag{4.139}$$

where $\mathcal{Q}_{ij}$ is the power quadrupole defined in Eq. (3.95). Here k_L represents the momentum of the long-wavelength mode, the tensor one, and k_S the momentum of the short-wavelength modes, the scalar ones. We remind that the upper limit of integration, k_S^{min}, is the smallest wavenumber probed by the observations while the lower limit, k_L^{min}, corresponds to the longest wavelength gravitational wave mode produced during Inflation. We can calculate exactly the integral using the bispectrum calculated in the squeezed limit (4.138) and the first order power spectrum (4.69), obtaining

$$\langle \mathcal{Q}^2 \rangle = \frac{20}{3\pi} \frac{1}{\epsilon^2 c_L^4} \left(-\frac{1}{3} + \frac{2}{3}\frac{F_Y}{F} + \frac{4}{9}\frac{F_Z}{F} \right)^2 \left(\frac{H}{M_{Pl}} \right)^2 \ln\left(\frac{k_S^{min}}{k_L^{min}} \right). \tag{4.140}$$

We can define the parameter

$$\mathcal{K} \equiv -\frac{5}{3} \frac{1}{\epsilon c_L^2} \left(-\frac{1}{3}\epsilon + \frac{2}{3}\frac{F_Y}{F} + \frac{10}{9}\frac{F_Z}{F} \right), \tag{4.141}$$

so that Eq. (4.140) becomes

$$\langle \mathcal{Q}^2 \rangle = \frac{12}{5\pi} \mathcal{K}^2 \left(\frac{H}{M_{Pl}} \right)^2 \ln\left(\frac{k_S^{min}}{k_L^{min}} \right). \tag{4.142}$$

We know that the quadrupole distortion in (3.94) must be a perturbative effect, hence its perturbative nature imposes some limit on $\langle \mathcal{Q}^2 \rangle$. Given that the maximum theoretical observable wavelength today is the comoving Hubble horizon we have $k_S^{min} = H_0$, imposing $\langle \mathcal{Q}^2 \rangle < 1$ we have

$$\frac{12}{5\pi} \mathcal{K}^2 \left(\frac{H}{M_{Pl}} \right)^2 \left| \ln\left(\frac{k_L^{min}}{H_0} \right) \right| < 1. \tag{4.143}$$

For single-filed inflationary models all the above relations hold with $\mathcal{K} = 1$. In this work we want to evaluate the departure from the standard prediction, i.e. we will concentrate on that part of the bispectrum which violates the consistency relations, see Ref. [63]. It is equivalent to replace $\mathcal{K}$ with $\mathcal{K} - 1$ in (4.142).
As seen in the previous chapter it is possible to construct an estimator for the detection of the primordial GW from the quadrupole correction on the power spectrum. Following the same steps of [70] exposed in Chapter 3 we find a variance for the amplitude of gravitational waves

$$3\sigma_\gamma = 30\pi\sqrt{3\pi}\mathcal{K}^{-2} \left(\frac{k_{max}}{k_{min}} \right)^{-3}, \tag{4.144}$$

where k_{max} and k_{min} comes respectively from the UV and IR cut-off on momentum integral, they represent the minimum and maximum detectable scales. Note that with respect to the result for SFSR models found in [70], see Eq. (3.112) there is an extra $\mathcal{K}^{-2}$ factor. Eq. (4.144) can give an estimation on the possibility of measuring the primordial gravitational waves. More precisely it gives an estimate of the possible GW amplitudes detectable at $> 3\sigma$: a smaller variance entails a higher chance of detection. So, to estimate the detectability limits for solid Inflation we have to bound the parameter $\mathcal{K}$. Its bounds can be found using the subluminality condition (4.33), which can be rewritten as, using $c_L^2 = 1/3$,

$$\frac{5}{3} - \frac{20}{9}\frac{F_Y}{\epsilon|F|} < \mathcal{K} < \frac{15}{4} - \frac{20}{9}\frac{F_Y}{\epsilon|F|}. \tag{4.145}$$

Up to now we have made use of the hypothesis $F_{Y,Z} \sim \epsilon F$, and we have seen that in this case the predicted non-Gaussianities are in good agreement with the bounds from the Planck measurements [57]. For this reason we can bound F_Y as

$$-\epsilon < \frac{F_Y}{|F|} < \epsilon. \tag{4.146}$$

A smaller variance arises in the case of a higher $\mathcal{K}$, hence if we saturate the value of F_Y to the case $F_Y = -\epsilon|F|$, (4.145) becomes

$$\frac{35}{9} < \mathcal{K} < \frac{215}{36}. \tag{4.147}$$

The maximum value that $\mathcal{K}$ can assume is $\mathcal{K} \simeq 6$, for which we obtain, in the case of a tensor amplitude near to its maximum value $A_T \simeq 2.2 \times 10^{-9}$,

$$\frac{k_{max}}{k_{min}} > 1550, \tag{4.148}$$

which means that the signal is detectable at 3σ if the galaxy survey under consideration has $\frac{k_{max}}{k_{min}} > 1550$. In Chapter 3 we have seen that the observable predictions were more pessimistic on the possibility of a detection of a tensor fossil from a SFSR model, see Eq. (3.113) and compare it with (4.148). This is a very interesting feature of the Solid Inflation: it predicts a detectable of primordial tensor fossils. In [63] it has been shown that this possibility arises also in the hypothesis of [53], i.e. $F_{Y,Z} \sim F$. A detection of these kind of tensor in the reach of the future experiment, 21-cm surveys or EUCLID, not only would confirm the Inflation model but it would rule out all the SFSR model, in favor of solid-like inflationary models.

We have seen that Solid Inflation have very interesting outcomes which will be verified with the future observations and analysis, see e.g [83] for an analysis of the perspectives of future detection of primordial tensor modes with interferometers, like LISA, or [84] for a complete and detailed analysis of the possible constraints on Inflation coming from galactic surveys, like EUCLID.
However, as we will see, the predictions for the tensor power spectrum are not so distant from the single-field predictions. For this reason we will try to generalize the solid paradigm. A possible generalization is considering the case of an inflationary period driven by a *supersolid*, [85]. In this theory in addition to the three scalar fields which break space diffeomorphisms, is present a fourth field which breaks the time diffeomorphism coupled to the solid in a non-trivial way. Another possibility is searching for a solid which is not constructed with scalar fields but with vector gauge fields: this is the case of Gaugid Inflation, [52], and it is this theory that we will study and generalize int his Thesis.

Chapter 5

Gaugid Inflation

In the previous Chapter we have seen the original features coming from the model of Solid Inflation. Its relevance is due to the non-standard approach to Inflation using the spontaneous-symmetry breaking mechanism, which entails very interesting and intriguing outcomes. Nevertheless we have seen that the prediction of this model for the power spectrum amplitude of the gravitational waves is similar to the prediction of single-field models. Indeed their amplitude is of the same order, see Eqs. (2.103) and (4.68)

$$A_\gamma^{SF} \sim A_\gamma^{Solid} \sim \frac{H^2}{M_{Pl}^2}. \tag{5.1}$$

This is a very interesting feature of the gravitational waves, which arises also in more general scenarios, see [86]. Therefore for the gravitational waves, at the level of the power spectrum, the model of Solid Inflation does not give any particular novelty. However, as we will see, there is the possibility to generalize the solid paradigm in order to obtain new outcomes for the gravitational waves. The study of the possible symmetry-breaking patterns preserving the homogeneity and isotropy is deepen in [87].
In this final chapter we will study and generalize one of these new models, the so called *Gaugid Inflation* [52], which is a sort of generalization of the solid paradigm. In Gaugid Inflation the fields which drive Inflation are three vector Abelian fields A_μ^I with a vev which manifestly breaks the spatial translations and rotations, as for Solid Inflation. The possibility of using a (Abelian or non-Abelian) gauge field as the responsible for Inflation has been studied, e.g., in Refs. [88, 89, 90, 91] (see also [92, 93, 94]). The fundamental difference is the choice of the background. In all the previous studies a vacuum expectation value that depends only on time has been used, i.e. breaking the time diffeomorphisms. In our case, we will choose a particular *solid-like* configuration for the vev of the gauge field A_μ^I. In [52] the authors take the expectation value

$$\langle A_\mu^I \rangle = \varepsilon_{Ijk}\delta_\mu^j x^k, \tag{5.2}$$

where $I = 1, 2, 3$ is the internal index in the vector, $\mu = 0, 1, 2, 3$ is the Lorentz index and $i, j, k = 1, 2, 3$ are the spatial indices. This configuration is called, in analogy with the electromagnetic four-potential A_μ, *magnetic gaugid*, because, as we will see, it describes three mutually orthogonal homogeneous "magnetic" fields. Here we are not really considering an electromagnetic field in a particular configuration, but anyway, following the notation of [95], we will keep the electromagnetic notation.
Using this electromagnetic notation we can write another possible expression for the vev of the vector fields

$$\langle A_\mu^I \rangle = \delta_\mu^0 x^I \tag{5.3}$$

which can be called *electric gaugid* with an analogous reasoning [87]. In general we can think of a vev expression as a combination of (5.2) and (5.3). In [52] this possibility is considered but the expression for the vev used is only the magnetic one, for parity-preserving reasons which will be clear later. In this work we will consider also a term of the kind (5.3) in order to generalize the results of [52]. We focus in particular on the predictions for the generation of primordial gravitational waves, given that also in the original model very interesting results arise in this respect. Indeed when studying the perturbation dynamics of this theory a new tensor degree of freedom for the cosmological perturbations of the gaugid vector fields arises, which we will indicate with E_{ij}. The coupling with this new tensor degree of freedom will enhance the power spectrum amplitude for the gravitational waves γ, giving an original result for their power spectrum. Moreover, in Chapter 3 we have studied the inflationary tensor fossils and their cosmological implications. The new tensor field E_{ij} could represent the fossil field we have described in Chapter 3, given that it would leave no traces except for a remodulation of the scalar power spectrum. Studying the possible predictions for the tensor fossils from this theory can be the object for future work.

Here, in the first section we will describe the Gaugid Inflation and its predictions; then we will introduce the generalization mentioned above, its consequences and possible outcomes for the gravitational waves.

5.1 Review of Gaugid Inflation

In this section we will review the most important features and outcomes of Gaugid Inflation as obtained in [52]. We will describe the hypothesis of this new model, we will calculate the second-order action and the two-point functions predicted for the scalar and the tensor perturbation modes.

5.1.1 Extension of the solid paradigm

The basic idea behind the Gaugid Inflation model is the same of Solid Inflation. We want a field with a vev which breaks the standard isometries of a de Sitter spacetime. In this case we will not consider a triad of scalar fields ϕ^I but we will consider a triplet of $U(1)$ Abelian gauge fields A^I_μ. As for Solid Inflation, to restore the homogeneity and isotropy of the space we impose additional symmetries on the internal index of the gauge field

$$A^I_\mu \to A^I_\mu \partial_\mu \chi^I, \qquad A^I_\mu \to R^I_J A^J_\mu, \tag{5.4}$$

where χ^I are three gauge parameters and R^I_J is a $SO(3)$ matrix. We define the usual antisymmetric field strength with which we will build up the Lagrangian

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu. \tag{5.5}$$

We will build our Lagrangian constructing some building blocks which will be invariant under Lorentz and $SO(3)$ transformation[1], i.e. we want it to be a function of $SO(3) \times SO(3,1)$ invariants built out of $F^I_{\mu\nu}$. In order to have $SO(3,1)$ invariants all the Lorentz indices must be contracted, like $F^I_{\mu\nu}F^{J\mu\nu}$. To construct all the possible Lorentz invariant combinations we must define the dual of the field strength

$$\widetilde{F}^I_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma} F^I_{\rho\sigma}, \tag{5.6}$$

[1]Note that the shift symmetry in (5.4) implies that the field must appear with its derivatives, i.e. we have to use the field strength to write down our Lagrangian.

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor completely antisymmetric. There are four possible Lorentz invariant independent terms that we can construct with $F^I_{\mu\nu}$ [52]

$$Y^{IJ} = F^I_{\mu\nu}F^{J\mu\nu}, \quad \widetilde{Y}^{IJ} = F^I_{\mu\nu}\widetilde{F}^{J\mu\nu}, \quad U^{IJK} = F^{I\,\nu}_{\mu}F^{J\,\sigma}_{\nu}F^{K\,\mu}_{\sigma}, \quad \widetilde{U}^{IJK} = F^{I\,\nu}_{\mu}F^{J\,\sigma}_{\nu}\widetilde{F}^{K\,\mu}_{\sigma}. \tag{5.7}$$

Terms like $\widetilde{F}^2$ are not present given that $\widetilde{F}^I_{\mu\nu}\widetilde{F}^{J\mu\nu} = F^I_{\mu\nu}F^{J\mu\nu}$. Now we are ready to write down the possible $SO(3)$ invariants using (5.7). There is a total of 11 invariants

$$\begin{aligned}
&X = F^I_{\mu\nu}F^{I\,\mu\nu} = [Y],\\
&I_1 = \frac{[\widetilde{Y}]}{[Y]}, \quad I_2 = \frac{[Y^2]}{[Y]^2}, \quad I_3 = \frac{[\widetilde{Y}^2]}{[Y]^2}, \quad I_4 = \frac{[Y\widetilde{Y}]}{[Y]^2},\\
&I_5 = \frac{[Y^3]}{[Y]^3}, \quad I_6 = \frac{[Y^2\widetilde{Y}]}{[Y]^3}, \quad I_7 = \frac{[\widetilde{Y}^3]}{[Y]^3},\\
&I_8 = \frac{[Y^3\widetilde{Y}]}{[Y]^4}, \quad I_9 = \frac{U^{IJK}\varepsilon_{IJK}}{[Y]^{3/2}}, \quad I_{10} = \frac{\widetilde{U}^{IJK}\varepsilon_{IJK}}{[Y]^{3/2}}.
\end{aligned} \tag{5.8}$$

Here the square brackets represent the trace of a matrix and the Lorentz indices are contracted using the general metric $g_{\mu\nu}$. As for Solid Inflation the only variable which will be affected by the expansion of the Universe will be only X, since all the other variables are rescaled with the right power of X. The most general gaugid Lagrangian we can write is thus a function of all the eleven variables in (5.8)

$$\mathcal{L} = -Z(X, I_1, \ldots, I_{10}). \tag{5.9}$$

5.1.2 Background solutions with *magnetic* configuration

We will consider now the background solution for gaugid inflation, considering the vacuum configuration in (5.2)

$$\langle A^I_\mu \rangle = \varepsilon_{Ijk}\delta^j_\mu x^k.$$

This is the so called *magnetic* configuration. Indeed, working with the "electric" and "magnetic" fields $\vec{E}^I$ and $\vec{B}^I$, defined using the "four-potential"

$$\begin{aligned}
B_{Ij} &= \varepsilon_{jkl}\partial_k A_{Il}\\
E_{Ij} &= \partial_j A_{I0} - \partial_0 A_{Ij}
\end{aligned} \tag{5.10}$$

we can see that Eq. (5.2) provides the description of a constant magnetic field. Indeed taking (5.10) and using (5.2) we have the background values for the fields

$$\begin{aligned}
\langle E^I_j \rangle &= 0,\\
\langle B^I_j \rangle &= -2\delta^I_j,
\end{aligned} \tag{5.11}$$

i.e. a constant magnetic field and a null electric field. We can see that we can consider this configuration since it is a solution to the equation of motion in the background, i.e. considering the FLRW metric in the case of a spatially flat Universe

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{5.12}$$

To verify this let us consider a general configuration of the electric and magnetic fields $E^I_j(t, \mathbf{x})$, $B^I_j(t, \mathbf{x})$. The only way to be compatible with homogeneity and isotropy for the fields is to be only time dependent, i.e. $E^I_j(t, \mathbf{x}) = f_1(t)\delta^I_j$, $B^I_j(t, \mathbf{x}) = f_2(t)\delta^I_j$. Before constraining the two time-dependent functions f_1 and f_2 with the equation of motion we can use the Bianchi identities for the field strength, which reads

$$\nabla_\mu F^I_{\nu\rho} + \nabla_\nu F^I_{\rho\mu} + \nabla_\rho F^I_{\mu\nu} = 0, \tag{5.13}$$

where ∇_μ stands for the covariant derivative. Considering $\mu = 0$, $\nu = i$ and $\rho = j$ and the FLRW metric (5.12), the Bianchi identity in this case becomes

$$\dot{B}^I_j + \varepsilon_{jkl}\partial_k E^I_l = 0, \tag{5.14}$$

which, in the case of only time-dependent functions, automatically gives $f_2 = const.$. In other words, for a homogeneous and isotropic background the configuration (5.11) for the magnetic field is a solution. For the electric field the reasoning is different, given that its form depends on the chosen dynamics. To see it we can consider a simplified form of the Lagrangian (5.9), $\mathcal{L} = -P(X)$. In this case the action for the system becomes

$$S_A = -\int d^4x a^3(t) P(X),$$

and varying it with respect to the field A^I_μ one obtains that the equation of motion is simply

$$\partial_\mu \left(a^3 \frac{\partial \mathcal{L}}{\partial \partial_\mu A^I_\nu} \right) = 0.$$

This equation finally becomes

$$\partial_\mu (a^3 P'(X) F^{I\,\mu\nu}) = 0, \tag{5.15}$$

where $'$ indicates a derivative w.r.t. X, which simplifies to

$$\frac{d}{dt}(aP'(X) f_1(t)) = 0. \tag{5.16}$$

This equation describes the dynamics of an electric gaugid. At this point this Thesis and the authors in [52] take different paths. Here we will review the work of [52], where no electric gaugid, for the background, is considered, while later on we will consider also the possibility of a non-vanishing contribution due to an electric field.
One can argue that the magnetic gaugid is always a solution of the equations of motion as long as the Lagrangian respects parity P, see [52]. Indeed, inspection of the building blocks in Eq. (5.7) makes it clear that parity forces the Lagrangian to depend only on even powers of the electric field. Therefore, around a magnetic gaugid configuration with vanishing electric fields, the Lagrangian is automatically stationary with respect to variations of the electric fields. Moreover on homogeneous backgrounds at hand, the action is obviously stationary with respect to variations of the B^I_j as these are pure space derivatives. Hence the magnetic gaugid FRW Universe is always a solution when P is respected. In our generalization of Gaugid Inflation we will consider also the possibility of a parity breaking Lagrangian, i.e. we will consider also the electric component of the vev of the field A^I_μ, see Section 5.2.1.

We are now ready to study the background dynamics with the choice for the vacuum (5.2). First of all notice that the only background value of the variables (5.8) which depends on the metric is that of X, since

$$\langle X \rangle \equiv \bar{X} = \frac{24}{a^4(t)}. \tag{5.17}$$

Using (1.73) for the stress-energy tensor we obtain, in the most general case (5.9), for the background

$$T_{\mu\nu} = -g_{\mu\nu} Z + 4 Z_X F^I_{\mu\alpha} F^{I\,\alpha}_{\nu} \tag{5.18}$$

where we have used the notation $Z_X = \partial Z / \partial X$. In (5.18) we have considered only the X-derivative of the Lagrangian since it is the only variable which feels the expansion of the Universe. Using the stress-energy tensor of a perfect fluid we have

$$\begin{aligned} \rho &= Z, \\ p &= -Z + \frac{4}{3} \bar{X} Z_X, \end{aligned} \tag{5.19}$$

where we used (5.17) and the fact that the background values for the field strength are

$$\begin{aligned} F^I_{0\mu} &= 0, \\ F^I_{jk} &= -2\varepsilon_{Ijk}. \end{aligned} \tag{5.20}$$

The Friedmann equations in this case become

$$\begin{aligned} H^2 &= \frac{8\pi G}{3} Z, \\ \frac{\ddot{a}}{a} &= \frac{8\pi G}{3} Z \left(1 - 2\frac{\bar{X} Z_X}{Z}\right), \end{aligned} \tag{5.21}$$

which allow us to write down the slow-roll parameter[2] which ensures the nearly constant behavior of H during inflation

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{2}{3} \frac{\bar{X} Z_X}{M^2_{Pl} H^2} = 2\frac{\bar{X} Z_X}{Z} = 2\frac{\partial \ln Z}{\partial \ln X} \ll 1. \tag{5.22}$$

The condition (5.22) is in perfect agreement with what we found for the Solid Inflation, Eq. (4.24), i.e. we can interpret the first slow-roll condition as a weak dependence of the Lagrangian for gaugid Inflation on the expansion of the Universe, implicitly expressed by X. Now we can evaluate the expression for the second slow-roll parameter. We use the relation on the background (5.17) to obtain

$$H = -\frac{1}{4} \frac{\dot{\bar{X}}}{\bar{X}}.$$

The condition on the second slow-roll parameter, $\eta \ll 1$ ensures that the Inflation holds for an enough long time and in this theory we have

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} = -4 \left[1 - \frac{\epsilon}{2} + \frac{\bar{X} Z_{XX}}{Z_X}\right]. \tag{5.23}$$

We obtain a useful relation for the second order derivative of Z

$$\frac{\partial \ln Z_X}{\partial \ln X} = -1 + \frac{\epsilon}{2} - \frac{\eta}{4}, \tag{5.24}$$

which can be rewritten as

$$\frac{\bar{X}^2 Z_{XX}}{\bar{X} Z_X} = -1 + \mathcal{O}(\epsilon, \eta), \tag{5.25}$$

[2]As said in Chapter 4, here there is nothing which is *rolling* but we will use the same expressions used in the slow-roll approximation for simplicity.

which we will use throughout to express Z_{XX} in terms of Z_X. All the derivatives w.r.t. X are evaluated on the background value $\bar{X}$.
From now on we will consider a simplified Lagrangian which preserves parity for the arguments mentioned above. The action we will focus on is the following

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - P(X) - (27M_1^4 + 18M_2^4)I_2 + 72M_2^4 W\right], \tag{5.26}$$

where we have defined

$$W \equiv \frac{F_{\alpha}^{I\,\beta}F_{\beta}^{I\,\gamma}F_{\gamma}^{J\,\delta}F_{\delta}^{J\,\alpha}}{X^2} = \frac{1 + I_2 + I_3}{4}. \tag{5.27}$$

Here the coefficients M_1^4 and M_2^4 are parameters with the dimension of a mass.
Extending this Lagrangian to a more general parity preserving one of the form (5.9) is straightforward, but it is not expected to introduce any qualitative novelty compared to the minimal case we study in this first section. Instead (as an original computation contained in this Thesis) we will add new terms when studying the case of a breaking-parity Lagrangian and we will see what the effects will be.

5.1.3 Perturbing the Gaugid

We are now ready to study the perturbations in Gaugid Inflation using as background the chosen vacuum (5.2). Up to now we have distinguished between internal and spatial indices, but, given the imposed symmetries (5.4) and the magnetic configuration, we are allowed to make no distinction between spatial and internal indices. From now on we will indicate both with lower-case letters and we will treat the internal index as a spatial index.
The perturbed gauge field can be written, in general, as

$$A_{i\mu} = \langle A_{i\mu}\rangle + a_{i\mu}, \tag{5.28}$$

where $a_{i\mu}$ is the perturbation field given by a general 3×4 matrix. Therefore it can be decomposed into a 3-vector a_{i0} and a 3-tensor a_{ij}. We will see that the action will not involve time derivatives on a_{i0}, hence their equations of motion are equivalent to constrain equations and all the propagating degrees of freedom reside in a_{ij}. It is useful to decompose the perturbation field into scalar, vector and tensor fluctuation modes, according to what we have explained in Chapter 2. More precisely, using helicity representations

$$a_{ij} = \alpha\delta_{ij} + \partial_i\partial_j S + \partial_i S_j + \partial_j S_i + E_{ij} + \varepsilon_{ijk}\left(V_k + \frac{\partial_k T}{\sqrt{-\partial^2}}\right), \tag{5.29}$$

where $\partial^2 = \delta^{ij}\partial_i\partial_j$. We see that, without fixing any gauge, there are 9 degrees of freedom: α, S and T are helicity scalars, S_i and V_i are transverse ($\partial_i S^i = \partial_i V^i = 0$) helicity vectors and E_{ij} is a symmetric, transverse and traceless ($E_i^i = \partial_i E^{ij} = 0$) helicity tensor mode. The fields $A_{i\mu}$ form a $U(1)$ triplet, hence a $U(1)^3$ gauge redundancy is present, which is given by $\delta a_{ij} = \partial_j\xi_i \equiv \partial_j(\xi_i^T + \partial_i\xi)$, where ξ_i^T is the transverse part of the gauge parameter. This means that we can fix three parameters, two using ξ_i^T and one using ξ. We can eliminate S_i and S, in order to have

$$a_{ij} = \alpha\delta_{ij} + E_{ij} + \varepsilon_{ijk}\left(V_k + \frac{\partial_k T}{\sqrt{-\partial^2}}\right). \tag{5.30}$$

We can use the parity transformation

$$A_{i0}(t,\mathbf{x}) \to A_{i0}(t,-\mathbf{x}), \quad A_{ij}(t,\mathbf{x}) \to -A_{ij}(t,-\mathbf{x}) \tag{5.31}$$

to classify our perturbations, since parity is preserved by vacuum (5.2). The perturbation field transforms as $a_{ij}(t,\mathbf{x}) \to -a_{ij}(t,-\mathbf{x})$, therefore the presence of the Levi-Civita symbol in (5.30) ensures that T and V_k are *polar* fields, i.e.with parity $(-1)^h$, where h is the helicity of the field, while α and E_{ij} are *axial* fields, i.e. with $(-1)^{h+1}$. The polar excitations T and V_k are the exact analogs of the longitudinal and transverse modes of an ordinary solid, and therefore they are the same perturbations present in Solid Inflation [53], see Sec. 4.3. This is a confirmation of what we said above: this theory is a generalization of Solid Inflation. We see that the novelty appears already. The two modes α and E_{ij} are new outcomes of the theory, and in particular, as we will see, the tensor field E_{ij} is the main source of novel results for this model. It represents an extra tensor degree of freedom, which can have non trivial effects when evaluating tensor fossils in this model. Before starting with the computation we decompose also a_{i0}

$$a_{i0} = \frac{\partial_i \chi}{\sqrt{-\partial^2}} + B_i. \tag{5.32}$$

We first perform the subhorizon calculations in order to find the conditions for the stability and the subluminality of the theory (5.26). In the small scale limit we can work with a Minkowski metric, i.e., we can neglect the coupling with gravity. Inserting (5.28) into $F_{i\mu\nu}$ we can define the perturbation of the field strength

$$F_{i\mu\nu} = \langle F_{i\mu\nu} \rangle + f_{i\mu\nu}, \tag{5.33}$$

where we have defined

$$\langle F_{i\mu\nu} \rangle = \partial_\mu \langle A_{i\nu} \rangle - \partial_\nu \langle A_{i\mu} \rangle = -2\varepsilon_{i\mu\nu} \tag{5.34}$$

and

$$f_{i\mu\nu} \equiv \partial_\mu a_{i\nu} - \partial_\nu a_{i\mu}. \tag{5.35}$$

We expand the action (5.26) up to second order in perturbations.
The matrices $Y_{ij} = F_{i\mu\nu} F_j^{\mu\nu}$ and its dual $\tilde{Y}_{ij} = F_{i\mu\nu} \tilde{F}_j^{\mu\nu}$ at second order in the perturbations, are

$$Y_{ij} = 8\delta_{ij} - 2\varepsilon_{ilm} f_{jlm} - 2\varepsilon_{jlm} f_{ilm} - 2 f_{ik0} f_{jk0} + f_{ilm} f_{jlm}, \tag{5.36}$$

$$\tilde{Y}_{ij} = -4 f_{ij0} - 4 f_{ji0} - \varepsilon_{lmn} \left(f_{i0l} f_{jmn} + f_{j0n} f_{ilm} \right). \tag{5.37}$$

We can see that in this configuration $\tilde{Y}_{ij}$ arises only at the level of the perturbations. On the other hand in our computations with an electric vacuum we will see that it has also a non vanishing zeroth order background value.
From (5.36) X becomes

$$X = F_{i\mu\nu} F^{i\mu\nu} = 24 - 4\varepsilon_{ijk} f_{ijk} + f_{i\mu\nu} f_i^{\mu\nu}. \tag{5.38}$$

In order to calculate I_2 and I_3, the two building blocks into the Lagrangian, we have to calculate the two contractions $[Y^2] = Y_{ij} Y_{ij}$, $[\tilde{Y}^2] = \tilde{Y}_{ij} \tilde{Y}_{ij}$ up to second order in perturbations and then we have to multiply for expression of X^{-2} expanded up to the second-order. All the expressions and the computations are in Appendix A. In the following we report only the final result for I_2 and I_3 in the magnetic gauge:

$$I_2 = \frac{1}{576} \left[192 + \frac{64}{3} f_{ijk} f_{ijk} - 32 f_{iik} f_{jjk} + \frac{64}{3} f_{ijk} f_{jik} \right], \tag{5.39}$$

$$I_3 = \frac{1}{576}\left[32 f_{ij0} f_{ij0} + 32 f_{ij0} f_{ji0}\right]. \tag{5.40}$$

The second-order perturbed expression for the kinetic term is

$$P(X)\big|_{(2)} = \frac{\bar{X}P_X}{24} f_{i\mu\nu} f_i^{\mu\nu} - \frac{\bar{X}P_X}{72}(\varepsilon_{ijk} f_{ijk})^2. \tag{5.41}$$

Inserting these expressions into (5.26) we obtain the second order action in Minkowski spacetime

$$\begin{aligned} S^{(2)} = \int d^4x \Bigg[& -\frac{\bar{X}P_X}{24} f_i^{\mu\nu} f_{i\mu\nu} + \frac{\bar{X}P_X}{72}(\varepsilon_{ijk} f_{ijk})^2 \\ & - M_1^4 \left(f_{ijk} f_{ijk} + f_{ijk} f_{jik} - \frac{3}{2} f_{iik} f_{jjk} \right) + M_2^4 \left(f_{ij0} f_{ij0} + f_{ij0} f_{ji0} \right) \Bigg]. \end{aligned} \tag{5.42}$$

In this work we have performed all the calculations of [52] and we have obtained the same result. Expanding the various $f_{i\mu\nu}$ using (5.30) we can write down the second order action for all the perturbation modes

$$\begin{aligned} S^{(2)} = \int d^4x \Bigg[& \left(\frac{\bar{X}P_X}{4} + 6M_2^4 \right) \dot{\alpha}^2 - \frac{\bar{X}P_X}{6} (\partial\alpha)^2 - \left(\frac{\bar{X}P_X}{6} + 4M_2^4 \right) \frac{\partial^2 \chi}{\sqrt{-\partial^2}} \dot{\alpha} \\ & - \left(\frac{\bar{X}P_X}{12} + 2M_2^4 \right) \chi \partial^2 \chi + \frac{\bar{X}P_X}{6} \dot{T}^2 - \left(2M_1^4 - \frac{\bar{X}P_X}{18} \right) (\partial T)^2 + \frac{\bar{X}P_X}{6} \dot{V}_k^2 \\ & - \left(\frac{\bar{X}P_X}{12} + \frac{3}{2} M_1^4 \right) (\partial V_k)^2 - \frac{\bar{X}P_X}{6} \varepsilon_{ijk} \partial_j B_i \dot{V}_k + \left(\frac{\bar{X}P_X}{12} + M_2^4 \right) (\partial B_k)^2 \\ & + \left(\frac{\bar{X}P_X}{12} + 2M_2^4 \right) \dot{E}_{ij}^2 - \left(\frac{\bar{X}P_X}{12} + 3M_1^4 \right) (\partial E_{ij})^2 \Bigg]. \end{aligned} \tag{5.43}$$

We see, as anticipated, that the fields χ and B_i, i.e. the fields which characterize a_{i0}, are non dynamical since their time derivative does not appear. Therefore we can obtain their expressions in terms of the other fields as

$$B_i = \frac{\bar{X}P_X}{\bar{X}P_X + 12M_2^4} \varepsilon_{ijk} \partial^{-2} \partial_i \dot{V}_k, \quad \chi = -\frac{\dot{\alpha}}{\sqrt{-\partial^2}}, \tag{5.44}$$

and inserting these solution back into the action yields the final expression

$$S^{(2)} = S_\alpha^{(2)} + S_T^{(2)} + S_V^{(2)} + S_E^{(2)} \tag{5.45}$$

with

$$S_\alpha^{(2)} = \int d^4x \left[\left(\frac{\bar{X}P_X}{6} + 4M_2^4 \right) \dot{\alpha}^2 - \frac{\bar{X}P_X}{6} (\partial\alpha)^2 \right], \tag{5.46}$$

$$S_T^{(2)} = \int d^4x \left[\frac{\bar{X}P_X}{6} \dot{T}^2 - \left(2M_1^4 - \frac{\bar{X}P_X}{18} \right) (\partial T)^2 \right], \tag{5.47}$$

$$S_V^{(2)} = \int d^4x \left[\frac{\bar{X}P_X(\bar{X}P_X + 24M_2^4)}{12(\bar{X}P_X + 12M_2^4)} \dot{V}_k^2 - \left(\frac{\bar{X}P_X}{12} + \frac{3}{2} M_1^4 \right) (\partial V_k)^2 \right], \tag{5.48}$$

$$S_E^{(2)} = \int d^4x \left[\left(\frac{\bar{X}P_X}{12} + 2M_2^4 \right) \dot{E}_{ij}^2 - \left(\frac{\bar{X}P_X}{12} + 3M_1^4 \right) (\partial E_{ij})^2 \right]. \tag{5.49}$$

From these equations we can easily extrapolate the propagation speeds of the fluctuation modes and impose the subluminality for the different modes

$$\begin{aligned} c_\alpha^2 &= \frac{\bar{X}P_X}{\bar{X}P_X + 24M_2^4}, & c_T^2 &= \frac{36M_1^4 - \bar{X}P_X}{3\bar{X}P_X}, \\ c_V^2 &= \frac{(\bar{X}P_X + 18M_1^4)(\bar{X}P_X + 12M_2^4)}{\bar{X}P_X(\bar{X}P_X + 24M_2^4)}, & c_E^2 &= \frac{\bar{X}P_X + 36M_1^4}{\bar{X}P_X + 24M_2^4}. \end{aligned} \tag{5.50}$$

We note that the speed of sound of the scalar α can be rewritten in terms of c_T^2 and c_E^2: $c_\alpha^2 = c_E^2/(3c_T^2 + 2)$. Imposing $0 < c_i^2 < 1$, with $i = \alpha, T, V, E$ we obtain the conditions on the kinetic term and on the parameters of the theory M_1^4 and M_2^4 in order to have a stable theory

$$\frac{\bar{X}P_X}{36} < M_1^4 < \frac{\bar{X}P_X}{9}, \quad M_2^4 > \frac{3M_1^4\bar{X}P_X}{2\bar{X}P_X - 36M_1^4}. \tag{5.51}$$

Note that w.r.t. the results of [52] the higher bound of M_1^4 presents a little difference, in the original article they have $\bar{X}P(X)/18$ while we have $\bar{X}P(X)/9$. To find these conditions we have considered that the product $\bar{X}P_X$ must be positive. This condition has profound reasons. If we consider the simplest model with Lagrangian $\mathcal{L} = -P(X)$ we would have, using the Friedmann equations (5.21),

$$\bar{X}P_X = \frac{3}{4}(\rho + p).$$

The second member must remain positive in order to satisfy the *Null Energy Condition* [96]: this condition, if we write the state equation $p = w\rho$, consist in requiring that $w > -1$. Given that the expression for the variation rate of the Hubble parameter is $\dot{H} = -4\pi G\rho(1+w)$ it is equivalent to require an ever-decreasing Hubble rate, $\dot{H} < 0$. For more details about the Null Energy Condition and its possible violations see Ref. [96]. For these reasons we will take, together with the conditions (5.51), also $XP_X > 0$.
Recall that the only variable sensible to the expansion of the Universe is X, hence the “minimal” model one can consider is that with only the kinetic term. So, why do we not consider the simplest model $\mathcal{L} = -P(X)$? We do not use this model because perturbing it one finds that all the fluctuation modes have exactly the speed of light except for the T mode, which would have an imaginary speed of sound, i.e. a UV instability. One can verify this statement by imposing $M_1 = 0$ in (5.50). Note also that the model with $M_1^4 \neq 0$ and $M_2^4 = 0$ would entail some problems: it would predict superluminal modes. Hence we can affirm that (5.26) (with the conditions (5.51)) is the “minimal” magnetic Gaugid theory to avoid instabilities.

After all these premises we are ready to switch on gravity in order to study the cosmological perturbations predicted by the Gaugid Inflation. We will use, as before, the ADM formalism for the metric

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{5.52}$$

where N and N_i are the usual lapse and shift non-dynamical variables. We can fix the gauge using the diffeomorphism invariance: in particular for our studies we will use the *spatially flat slicing gauge* (SFSG) where the 3-metric is perturbed only by the helicity 2-mode (i.e. by the tensor perturbations)

$$g_{ij} = a^2(t)\exp[\gamma_{ij}], \tag{5.53}$$

where γ_{ij} is the transverse traceless metric perturbation satisfying

$$\gamma^i_i = \partial_i \gamma^{ij} = 0. \tag{5.54}$$

The lapse and shift function are not affected by this choice of gauge and they can be integrated out using their equations of motion. Therefore SFSG does not affect the gaugid degrees of freedom. The calculation to obtain the second order actions for the different modes is lengthy and we leave all the details in Appendix A. The second order action including mixing with gravity is [52]

$$S^{(2)} = S^{(2)}_\alpha + S^{(2)}_T + S^{(2)}_V + S^{(2)}_{GW} \tag{5.55}$$

where

$$S^{(2)}_\alpha = \int d^4x a^3 \frac{3c_T^2 + 2}{4c_E^2} M_{Pl}^2 H^2 \epsilon \left[a^2 \dot{\alpha}^2 - c_\alpha^2 (\partial\alpha)^2\right], \tag{5.56}$$

$$S^{(2)}_T = \frac{M_{Pl}^2}{4} \int dt \int \frac{d^3\mathbf{k}}{(2\pi)^3} a^3 \left[\frac{k^2/3}{1 + k^2 3a^2 H^2 \epsilon} |\dot{T}_\mathbf{k} + \epsilon H T_\mathbf{k}|^2 - \epsilon H^2 c_T^2 k^2 |T_\mathbf{k}|^2\right], \tag{5.57}$$

$$S^{(2)}_V = M_{Pl}^2 \int dt \int \frac{d^3\mathbf{k}}{(2\pi)^3} a^3 \left[\frac{k^2/16}{1 + k^2/16a^2 \mathcal{N}_V^2} |\dot{V}^i_\mathbf{k}|^2 - c_V^2 \mathcal{N}_V^2 k^2 |V^i_\mathbf{k}|^2\right], \tag{5.58}$$

$$\begin{aligned} S^{(2)}_{GW} = \int d^4x a^3 \Bigg\{ &\frac{M_{Pl}^2}{8} \left(\dot{\gamma}_{ij}^2 - \frac{1}{a^2}(\partial\gamma_{ij})^2 - 3(c_T^2 + 1) H^2 \epsilon \gamma_{ij}\gamma_{ij}\right) \\ &+ \frac{(3c_T^2 + 2)}{8} M_{Pl}^2 H^2 \epsilon \left(a^2 c_E^{-2} \dot{E}_{ij}^2 - (\partial E_{ij})^2\right) + \frac{3}{4}(c_T^2 + 1) M_{Pl}^2 H^2 \epsilon \varepsilon_{ijk} \partial_k E_{il} \gamma_{lj} \Bigg\}, \end{aligned} \tag{5.59}$$

where we have defined

$$\mathcal{N}_V^2 \equiv \frac{2 + 3c_T^2}{4(2 + 3c_T^2 + c_E^2)} H^2 \epsilon. \tag{5.60}$$

Here we have just integrated out the non-dynamical lapse and shift functions. After the expansion $N = 1 + \delta N$, $N_i = \partial_i \psi + N_i^T$ one finds that the solutions, in Fourier space, contains only the scalar and the vector fields T and V_i, as for Solid Inflation [53]

$$\delta N = -\frac{a^2 \dot{H}}{2kH} \frac{\dot{T} - \dot{H}T/H}{1 - 3a^2 \dot{H}/k^2}, \quad \psi = -\frac{a^2}{2k} \frac{3a^2 \dot{H}\dot{T}/k^2 - \dot{H}T/H}{1 - 3a^2 \dot{H}/k^2}, \quad N_i^T = \frac{\dot{V}_i}{1 - k^2/4a^2 \dot{H}}. \tag{5.61}$$

We immediately note the first non-trivial outcome of this theory: we have just said above about the importance of a new tensor degree of freedom, E_{ij}. In (5.59) we explicitly see that the metric tensor perturbation and the one due to the gaugid mixes with a terms $\sim \partial E \gamma$. We will see later the importance of this new coupling between the two tensor fields when studying the tensor power spectrum.
We also note that the action for the scalar mode T, which is connected, as we will see, to the ζ perturbation, (5.57) is identical to the one of the longitudinal scalar mode in Solid Inflation in (4.53). For the vector degree of freedom we found the same analogy in the case of absence of the new tensor degree of freedom, i.e. imposing $c_E^2 = 0$, compare (5.58) with (4.52). For this reason, as we will see, the prediction for the power spectrum of the scalar field will be the same as in Solid Inflation. We will see that the true novelty comes from the tensor sector, for which we will provide a full description of the calculations.

Time dependence of background quantities

As for solid Inflation we need to define some new "slow-roll" parameters of this theory and describe the time-dependence of background quantities. We switch to conformal time, defining the integration constant such that $a \to \infty$ when $\tau \to 0$, i.e.

$$aH = -\frac{1+\epsilon_c}{\tau}, \tag{5.62}$$

where we use the notation $\epsilon_c = \epsilon(\tau_c)$, see (5.22). We set τ_c to be the time corresponding to the longest CMB mode exiting the horizon during inflation, so that all modes of phenomenological interest cross the horizon at $|\tau| \le |\tau_c|$. It follows that $N_e^{min} = \ln\big(\tau_c/\tau_f\big)$ where τ_f stands for the end of Inflation and hence N_e^{min} is the minimum number of e-folds for Inflation. We can now define the new slow-roll parameter

$$\epsilon_\gamma = \frac{3}{2}(c_T^2+1)\epsilon, \qquad \epsilon_E = \frac{3c_T^2+2}{c_E^2}\epsilon, \tag{5.63}$$

and their respective "η"s, which define their time dependence

$$\eta_\gamma = \frac{\dot{\epsilon}_\gamma}{\epsilon_\gamma H}, \qquad \eta_E = \frac{\dot{\epsilon}_E}{\epsilon_E H}. \tag{5.64}$$

All the ϵ parameters have the following time dependence

$$\epsilon = \epsilon_c\left(\frac{\tau}{\tau_c}\right)^{-\eta_c}, \quad \epsilon_\gamma = \epsilon_{\gamma,c}\left(\frac{\tau}{\tau_c}\right)^{-\eta_{\gamma,c}}, \quad \epsilon_E = \epsilon_{E,c}\left(\frac{\tau}{\tau_c}\right)^{-\eta_{E,c}}. \tag{5.65}$$

We can also define the parameters $s_E = \frac{\dot{c}_E}{c_E H}$ and $s_T = \frac{\dot{c}_T}{c_T H}$, which measure the time dependence of the two speeds of propagation c_E and c_T through the relation

$$c_E = c_{E,c}\left(\frac{\tau}{\tau_c}\right)^{-s_{E,c}}, \quad c_T = c_{T,c}\left(\frac{\tau}{\tau_c}\right)^{-s_{T,c}}. \tag{5.66}$$

These parameters can be rewritten in terms of the other slow-roll parameters

$$s_E = \frac{1}{2}\left[\frac{2\epsilon_\gamma\eta_\gamma - \epsilon\eta}{2\epsilon_\gamma - \epsilon} - \eta_E\right], \quad s_T = \frac{\epsilon_\gamma(\eta_\gamma - \eta)}{2\epsilon_\gamma - 3\epsilon}. \tag{5.67}$$

Note that the new ϵ_i give a measure of the two graviton masses. In terms of these new parameters the subluminality and stability conditions (5.51) can be rewritten

$$\epsilon > 0, \qquad \frac{3}{2}\epsilon < \epsilon_\gamma < 2\epsilon, \qquad \epsilon_E > \frac{\epsilon\epsilon_\gamma}{2\epsilon - \epsilon_\gamma}. \tag{5.68}$$

5.1.4 Scalar perturbations

As we mentioned above the action for the T mode is the same as π_L in solid Inflation. We now perturb the stress-energy tensor in order to see that this scalar field is the one which enters in the definition of ζ and $\mathcal{R}$, the comoving and uniform energy-density curvature perturbations respectively, introduced in Chapter 2. The other scalar degree of freedom α plays no role, as we will see soon. The stress energy tensor can be easily written looking at the action (5.26)

$$T^\mu_\nu = -\delta^\mu_\nu\left[P(X) + (27M_1^4 + 18M_2^4)I_2 - 72M_2^4 W\right] + 4P_X F^\mu_{I\,\alpha}F_{I\nu}{}^\alpha. \tag{5.69}$$

The perturbed *scalar* quantities are defined as

$$T^0_0 = -(\rho + \delta\rho), \quad T^0_i = \partial_i \delta q, \quad T^i_j = \delta^i_j (p + \delta p) + \sigma^i_j, \tag{5.70}$$

where σ^i_j is the anisotropic stress. Using the perturbed expressions for I_2 and I_3, one finds [52]

$$\begin{aligned} \delta\rho &= -M^2_{Pl} H^2 \epsilon \sqrt{-\partial^2} T, \\ \delta q &= M^2_{Pl} H^2 \epsilon \left(2\psi - a^2 \frac{\dot{T}}{\sqrt{-\partial^2}} \right), \\ \sigma^i_j &= M^2_{Pl} H^2 \epsilon \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \partial^2 \right) \frac{T}{\sqrt{-\partial^2}}, \end{aligned} \tag{5.71}$$

where ψ is given in (5.61). The two gauge-invariant variables $\mathcal{R}$ and ζ are defined, in general, using the notation of (2.27), as

$$\mathcal{R} = \Phi - \frac{H}{\rho + p} \delta q, \qquad \zeta = \Phi - \frac{\delta\rho}{3(\rho + p)}. \tag{5.72}$$

In the spatially flat slicing gauge $\Phi = 0$ and they take the forms (in Fourier space)

$$\mathcal{R}_{\mathbf{k}} = \frac{k}{6H\epsilon} \frac{\dot{T}_{\mathbf{k}} + H\epsilon T_{\mathbf{k}}}{1 + k^2/3a^2H^2\epsilon}, \qquad \zeta_{\mathbf{k}} = \frac{k}{6} T_{\mathbf{k}}. \tag{5.73}$$

Since the action and the equation of motions for T is identical to the scalar field π_L then we can proceed with the identical path undertaken in Section 4.3.1. Therefore the predicted scalar power spectrum takes the same form of (4.88)

$$\mathcal{P}_\zeta(k) = \frac{H^2_c}{4\epsilon_c c^5_{L,c} M^2_{Pl}} \frac{1}{k^3} \frac{(\tau/\tau_c)^{8c^2_{L,c}\epsilon_c/3}}{(-c_{L,c} k \tau_c)^{5s_c - 2c^2_{L,c}\epsilon_c + \eta_c}}, \tag{5.74}$$

and the scalar spectral index is

$$n_S - 1 \simeq 2\epsilon_c c^2_{L,c} - 5s_c - \eta_c. \tag{5.75}$$

5.1.5 Tensor perturbations

The main novelty of Ref. [52] is indeed the new arising tensor degree of freedom E_{ij}, Eq.(5.59). Usually, as we have seen in Sections 2.3 and 4.3.1 (where we have studied the gravitational waves arising in the single-field slow-roll and in the solid Inflation models), the calculations for the tensore degrees of freedom, are the simplest. However in this case the computations become definitely more involved. Hence we will dedicate this section to the complete calculation of the tensor power spectrum using the action (5.59). It is convenient to switch to canonically normalized fields

$$\gamma_{ij} = \frac{2}{aM_{Pl}} \gamma^c_{ij}, \quad E_{ij} = \frac{2}{a^2 M_{Pl} H \sqrt{\epsilon_E}} E^c_{ij}, \tag{5.76}$$

where we have used the definition in (5.63). If we use the conformal time we obtain the action

$$\begin{aligned} S^{(2)}_{GW} = \frac{1}{2} \int d^3x d\tau \Bigg[& \left(\gamma^{c\,\prime}_{ij}\right)^2 - \left(\partial \gamma^c_{ij}\right)^2 + \frac{1}{\tau^2} (2 + 3\epsilon - 2\epsilon_\gamma) \gamma^c_{ij} \gamma^c_{ij} \\ & + \left(E^{c\,\prime}_{ij}\right)^2 - c^2_E (\partial E^c_{ij})^2 + \frac{1}{\tau^2} \left(6 + 5\epsilon + \frac{5}{2} \eta_E \right) E^c_{ij} E^c_{ij} \\ & - \frac{4}{\tau} \frac{\epsilon_\gamma}{\sqrt{\epsilon_E}} \varepsilon_{ijk} \partial_k E^c_{il} \gamma^c_{lj} \Bigg], \end{aligned} \tag{5.77}$$

where we have used the new slow-roll parameters in Eqs. (5.64) and (5.65) and the prime denotes a derivative with respect to conformal time. Note that we have used the relation (5.62) to integrate out some boundary terms. We now perform a Fourier transform of the fields[3]

$$\gamma^c_{ij}(\tau,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{s=\pm} \epsilon^s_{ij}\gamma_s(\tau,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.78}$$

$$E^c_{ij}(\tau,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{s=\pm} \epsilon^s_{ij}E_s(\tau,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.79}$$

Given the helicity operator $s_{\parallel ij} \equiv i\hat{k}_l\varepsilon_{lij}$, the polarization eigenstates $\epsilon^s_{ij}(\mathbf{k})$ are defined by $[s_\parallel, \epsilon^\pm] = \pm 2\epsilon^\pm$. This property can be rewritten in a more useful expression [97]

$$\varepsilon_{ijl}\partial_l\left[\epsilon^s_{mj}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\right] = sk\epsilon^s_{im}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},$$

which can be rewritten as

$$ik_l\varepsilon_{ijl}\epsilon^s_{mj}(\mathbf{k}) = sk\epsilon^s_{im}(\mathbf{k}), \tag{5.80}$$

where $s = \pm$. Eq. (5.80) implies the usual properties for the polarization tensor

$$k_i\epsilon^s_{ij} = \epsilon^s_{ii} = 0, \quad \epsilon^{s*}_{ij}(\mathbf{k}) = \epsilon_{ij}(-\mathbf{k}), \quad \epsilon^s_{ij}(\mathbf{k})\epsilon^{s'}_{ij}(\mathbf{k})^* = 2\delta^{ss'}.$$

In particular the hermiticity property implies $\gamma^\dagger_s(\tau,\mathbf{k}) = \gamma_s(\tau,-\mathbf{k})$. Now the action reads

$$\begin{aligned} S^{(2)}_{\gamma E} = \frac{1}{2}\sum_{s=\pm}\int d\tau\frac{d^3\mathbf{k}}{(2\pi)^2}\Bigg\{&\gamma'_s(\mathbf{k})\gamma'_s(\text{-}\mathbf{k}) - \left(k^2 - \frac{2+3\epsilon-2\epsilon_\gamma}{\tau^2}\right)\gamma_s(\mathbf{k})\gamma_s(\text{-}\mathbf{k}) \\ &+ E'_s(\mathbf{k})E'_s(\text{-}\mathbf{k}) - \left(c_E^2k^2 - \frac{6+5\epsilon+5\eta_E/2}{\tau^2}\right)E_s(\mathbf{k})E_s(-\mathbf{k}) \\ &- \frac{2Bk}{\tau}\left[\gamma_+(\mathbf{k})E_+(-\mathbf{k}) - \gamma_-(\mathbf{k})E_-(-\mathbf{k})\right]\Bigg\} \end{aligned} \tag{5.81}$$

where we have integrated in $\mathbf{x}$ using the Dirac delta definition

$$\delta^{(3)}(\mathbf{k}+\mathbf{p}) = \int \frac{d^3x}{(2\pi)^3}e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{x}}$$

and we have exploited the orthonormality property of the polarization tensors (we keep the time dependence of the modes implicit). Note that the last line comes from the property of the helicity polarization property (5.80). In (5.81) we have defined the quantity

$$B \equiv -2\frac{\epsilon_\gamma}{\sqrt{\epsilon_E}} = B_c\left(\frac{\tau}{\tau_c}\right)^{-\eta_\gamma+\eta_E/2}. \tag{5.82}$$

Note that in the action (5.81) modes with different helicity do not mix, hence they can be treated separately. We have four equation of motion that can be written as

$$\frac{d^2}{dz^2}\gamma_\pm + \left(1 - \frac{2+3\epsilon-2\epsilon_\gamma}{z^2}\right)\gamma_\pm \pm \frac{B}{z}E_\pm = 0, \tag{5.83}$$

$$\frac{d^2}{dz^2}E_\pm + \left(c_E^2 - \frac{6+5\epsilon+5\eta_E/2}{z^2}\right)E_\pm \pm \frac{B}{z}\gamma_\pm = 0, \tag{5.84}$$

[3]Note that here we are using the notation $\pm$ to indicate the helicity states, instead of the standard $+, \times$. Of course this dose not change the physics, however the helicity polarization states will be useful when studying the parity properties of the tensor fields.

where $z = -k\tau$. We can verify that the invariance under parity transformation translates into invariance of the system (5.83)-(5.84) under the parity transformation

$$\gamma_\pm(\mathbf{k}) \to \gamma_\mp(-\mathbf{k}) \quad E_\pm(\mathbf{k}) \to -E_\mp(-\mathbf{k}). \tag{5.85}$$

The system (5.83)-(5.84) is composed by two second order differential equations, in which the two tensor functions γ and E are coupled through a source term in each equation. This means that the quantization procedure is more delicate with respect to the previous cases we have studied, since we have to consider also the correlation between the two modes.

As a first step we collect the fields of a given polarization into two doublets $\phi_{\pm\alpha} = (\gamma_\pm, E_\pm)^T$, with $\phi_{\pm 1} = \gamma_\pm$ and $\phi_{\pm 2} = E_\pm$. The system of equations can be then rewritten in the synthetic form

$$\frac{d^2}{dz^2}\phi_{\pm\alpha}(z) + M_{\alpha\beta}(z)\phi_{\pm\beta} = 0, \tag{5.86}$$

where $M_{\alpha\beta}$ is the symmetric matrix

$$M_{\alpha\beta}(z) = \begin{pmatrix} 1-(2+3\epsilon-2\epsilon_\gamma)/z^2 & \pm B/z \\ \pm B/z & c_E^2 - (6+5\epsilon+5\eta_E/2)/z^2 \end{pmatrix}. \tag{5.87}$$

The equation (5.87) has two independent solutions, see Appendix B in Ref. [52], so that each doublet can be expanded as

$$\phi_{\pm\alpha} = \phi^{(1)}_{\pm\alpha} + \phi^{(2)}_{\pm\alpha},$$

where

$$\phi^{(n)}_{\pm\alpha} = f^{(n)}_{\pm\alpha}(\tau,k)a_{\pm n}(\mathbf{k}) + f^{(n)}_{\pm\alpha}(\tau,k)^* a^\dagger_{\pm n}(-\mathbf{k}). \tag{5.88}$$

Here $a_{\pm n}$ and $a^\dagger_{\pm n}$ are the corresponding annihilation and creation operators respectively. The mode functions $f^{(1)}_\pm$ and $f^{(2)}_\pm$ solve the system in Eqs. (5.83)-(5.84). They can be chosen such that $f^{(n)}_{-\alpha} = P_{\alpha\beta} f^{(n)}_{+\beta}$ where $P_{\alpha\beta} = \mathrm{diag}(1,-1)$ represents parity. We can impose now the canonical commutation relations for the doublet ϕ

$$\begin{aligned} &[\phi_{\pm\alpha}(\tau,\mathbf{k}), \phi_{\pm\beta}(\tau,\mathbf{q})] = 0 = [\phi'_{\pm\alpha}(\tau,\mathbf{k}), \phi'_{\pm\beta}(\tau,\mathbf{q})], \\ &[\phi_{\pm\alpha}(\tau,\mathbf{k}), \phi'_{\pm\beta}(\tau,\mathbf{q})] = (2\pi)^3 i\delta^{(3)}(\mathbf{k}+\mathbf{q})\delta_{\alpha\beta}. \end{aligned} \tag{5.89}$$

For the choice of vacuum we have to be precise. The two solutions $f^{(1)}$ and $f^{(2)}$ can be, in general, correlated at a given time. To measure the correlation between two function we have introduced in Section 2.2 the Wronskian function in Eq. (2.42). We can generalize the normalization condition (2.56) to our theory through the following vacuum choice

$$W\big(f^{(m)}_\pm(\tau,k), f^{(n)}_\pm(\tau,k)\big) \equiv f^{(m)}_{\pm\alpha}(\tau,k)\frac{d}{d\tau}f^{(n)}_{\pm\alpha}(\tau,k)^* - \frac{d}{d\tau}f^{(m)}_{\pm\alpha}(\tau,k)f^{(n)}_{\pm\alpha}(\tau,k)^* = i\delta_{mn}, \tag{5.90}$$

Given that the Wronskian is constant for all the functions which solve Eq. (5.86), see [52], the vacuum choice (5.90) corresponds to the request that the two solutions $f^{(1)}$ and $f^{(2)}$ are correlated at *any time*. The commutation relations between the annihilation and creation operators are found to be

$$[a_{\pm m}(\mathbf{k}), a_{\pm n}(\mathbf{p})] = [a^\dagger_{\pm m}(\mathbf{k}), a^\dagger_{\pm n}(\mathbf{p})] = 0, \quad [a_{\pm m}(\mathbf{k}), a^\dagger_{\pm n}(\mathbf{p})] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{mn}. \tag{5.91}$$

Using (5.90) we can make the following choice for the early time modes ($-k\tau \to \infty$)

$$f^{(1)}_{\pm}(z \to \infty) = \left(\frac{e^{iz}}{\sqrt{2k}}, 0\right)^T, \quad f^{(2)}_{\pm}(z \to \infty) = \left(0, \pm\frac{e^{-ik\int d\tau c_E(\tau)}}{\sqrt{2c_E(\tau)k}}\right)^T, \tag{5.92}$$

we recall that the time dependence of c_E is expressed in (5.66). The configuration (5.92) correspond, as anticipated, to purely gravitational and purely gaugid Bunch-Davies excitations. Now the two-point function for the canonically normalized graviton with "+" polarization reads

$$\langle \gamma^c_+(\tau, \mathbf{k})\gamma^c_+(\tau, \mathbf{k}')\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\sum_n \left|f^{(n)}_{+1}(\tau, k)\right|^2 \equiv (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\mathcal{P}^+_\gamma, \tag{5.93}$$

where we have defined $\mathcal{P}^+_\gamma$ as the power spectrum of "+" polarization graviton field. For the $-$ polarization an analogous relation holds, moreover $\mathcal{P}^+_\gamma = \mathcal{P}^-_\gamma$. Using (5.76) we can find the expression for the power spectrum

$$\mathcal{P}_\gamma = \left(\frac{2}{aM_{Pl}}\right)^2 \left(\mathcal{P}^+_\gamma + \mathcal{P}^-_\gamma\right) = 2\left(\frac{2}{aM_{Pl}}\right)^2 \mathcal{P}^+_\gamma. \tag{5.94}$$

Now we have to find the solutions to (5.86), $f^{(1)}_\pm$ and $f^{(2)}_\pm$, imposing the initial conditions (5.92). We will see that the contribution of the first solution $f^{(1)}$ to the amplitude of the tensor power spectrum is at most of the order of the standard single field result (5.1), while the contribution of the second one $f^{(2)}$, as we will see, is parametrically large. We will then solve Eq. (5.86) separately in two cases[4]: the *small* mode is the one with initial condition $\left(\gamma_+ \xrightarrow{z\to\infty} e^{iz}/\sqrt{2z}, E_+ \xrightarrow{z\to\infty} 0\right)$ while the *large* mode is the one with initial condition $\left(\gamma_+ \xrightarrow{z\to\infty} 0, E_+ \xrightarrow{z\to\infty} e^{-ik\int d\tau c_E(\tau)}/\sqrt{2c_E(\tau)k}\right)$.

Large mode

We need to solve now the system (5.83)-(5.84) using the initial conditions

$$\gamma_+ \xrightarrow{z\to\infty} 0, \quad E_+ \xrightarrow{z\to\infty} \frac{e^{-ik\int d\tau c_E(\tau)}}{\sqrt{2c_E(\tau)k}}. \tag{5.95}$$

Recall that at first order in slow-roll parameters we have

$$\int d\tau c_E(\tau) = -(1+s_{E,c})c_E\tau. \tag{5.96}$$

The non-trivial solution of γ_+ is entirely due to the mixing with E_+ (in the absence of this mixing γ_+ would vanish at all times with the given initial conditions). To solve (5.84) we can consider perturbative arguments: the difficulty of the system resides in the coupling between γ and E, which is regulated by the coefficient B. This factor goes like $B \sim \epsilon_\gamma/\epsilon_E^{1/2}$ and we can affirm, looking at the stability conditions rewritten in terms of the new slow-roll parameters (5.68), that it is at most of order $\epsilon^{1/2}$, i.e. very small, but it can also be smaller[5]. This means that E_+ evolves practically independently of γ_+ and we can approximate (5.84) to

$$\frac{d^2}{dz^2}E_+ + \left(c_E^2 - \frac{6+5\epsilon+5\eta_E/2}{z^2}\right)E_+ = 0. \tag{5.97}$$

[4]Here we use only the +-polarization solution because the −-polarization can easily be found with a parity transformation.

[5]If the values of ϵ_γ and ϵ_E are saturated respectively to their maximum and minimum value, one finds that B is exactly 0.

If we define now $x = (1+s_{E,c})c_E z$ we have, at first order in the slow-roll parameters

$$x^2\frac{d^2}{dx^2}E_+ + \left[x^2 - \left(\nu_E^2 - \frac{1}{4}\right)\right]E_+ = 0, \tag{5.98}$$

where we have defined $\nu_E = \frac{5}{2} + \frac{12}{5}s_{E,c} + \epsilon_c + \frac{1}{2}\eta_{E,c}$. This equation resembles (2.66), for which we know the exact solution

$$E_+(\tau,k) = \sqrt{x}\left[C_1 H^{(1)}_{\nu_E}\big(-k\tau c_E(1+s_{E,c})\big) + C_2 H^{(2)}_{\nu_E}\big(-k\tau c_E(1+s_{E,c})\big)\right]. \tag{5.99}$$

The two constants C_1 and C_2 can be found by imposing the initial condition (5.95), which, using the asymptotic behavior of Hankel functions (2.69), imposes $C_2 = 0$ and

$$C_1 = (c_E z)^{-1/2}\sqrt{\frac{\pi z}{4k}}e^{i(2\nu_E+1)\pi/4}. \tag{5.100}$$

Therefore, the solution for E_+ is

$$E_+(\tau,k) = \left(1+\frac{s_{E,c}}{2}\right)\sqrt{\frac{\pi z}{4k}}e^{i(2\nu_E+1)\pi/4}H^{(1)}_{\nu_E}\big((1+s_{E,c})c_E z\big), \tag{5.101}$$

where, we remind, $z = -k\tau$. We will verify that (5.101) is a good solution for the system (5.83)-(5.84) and that our approximation of an evolution of the mode E_+ independent from γ_+ is justified. Taking the limit $z \gg 1$ we know that at the leading order in the slow-roll parameters $E_+ \to e^{ic_{E,c}z}/\sqrt{2c_{E,c}k}$. The equation (5.83) suggest a possible ansatz for γ_+ in this limit

$$\gamma_+ \xrightarrow{z\gg 1} A\frac{e^{ic_{E,c}z}}{z}. \tag{5.102}$$

Indeed, in this limit we have

$$\frac{d^2}{dz^2}\gamma_+ \xrightarrow{z\gg 1} -c_{E,c}^2\gamma_+. \tag{5.103}$$

Plugging (5.102) and (5.103) and taking the small-scales limit of E_+ fixes the coefficient A

$$\gamma_+ \xrightarrow{z\gg 1} -\frac{B_c}{(1-c_{E,c}^2)\sqrt{2c_{E,c}k}}\frac{e^{ic_{E,c}z}}{z}. \tag{5.104}$$

Now inserting this small-scale solution into (5.84) (now including also the source term from γ_+), one can easily show that there is no significant backreaction on the early-time dynamics of E_+: the correction to the zeroth-order solution (5.101) scales as $\delta E_+/E_+|_{z\gg 1} = \mathcal{O}(B_c^2/z)$.

We now evaluate the late time evolution. Using (2.71) we have, from (5.101),

$$E_+ \xrightarrow{z\ll 1} -\frac{3}{2\sqrt{kc_{E,c}^5 z_c^{5s_{E,c}}}}\frac{1}{z^{2+\epsilon_c+\eta_{E,c}/2}}, \tag{5.105}$$

where $z_c \equiv -k\tau_c$. Substituting this expression into (5.83) and using the explicit time dependence of B, (5.82), we have

$$\frac{d^2}{dz^2}\gamma_+ - \frac{2+3\epsilon-2\epsilon_\gamma}{z^2}\gamma_+ - \frac{3B_c\left(2kc_{E,c}^5\right)^{-1/2}}{z_c^{\eta_{E,c}/2+5s_{E,c}/2-\eta_{\gamma,c}}}\frac{1}{z^{3+\epsilon_c+\eta_{\gamma,c}}} = 0. \tag{5.106}$$

It is simple to find a solution for this equation. As a first step one could neglect the source term (i.e. the third term) in (5.106) and consider the solution of the homogeneous differential equation. We can then find the particular solution considering a power-law solution $\gamma_+ = Az^\alpha$ and solve the constraint equations for the two parameters A and α. The full solution will be a linear combination of these solutions

$$\gamma_+ = \frac{3B_c\left(2kc_{E,c}^5\right)^{-1/2}}{2\epsilon_{\gamma,c}z_c^{\eta_{E,c}/2+5s_{E,c}/2}}z^{-1-\epsilon_c} + C_+z^{\alpha_+} + C_-z^{\alpha_-}, \tag{5.107}$$

with

$$\alpha_\pm = \frac{1 \pm (3 + 2\epsilon_c - 4\epsilon_{\gamma,c}/3)}{2}, \tag{5.108}$$

where we have used (5.65). $C_\pm$ are two integration constants that can be fixed using perturbative arguments [52]

$$C_+ = \mathcal{O}(1), \qquad C_- = -\frac{3B_c\left(2kc_{E,c}^5\epsilon_{\gamma,c}^2\right)^{-1/2}}{2z_c^{\eta_{E,c}/2+5s_{E,c}/2}}. \tag{5.109}$$

We will not consider the C_+ mode because it decays outside the horizon. Finally, the solution for the metric tensor mode in the *large mode* hypothesis and in the limit $z \ll 1$ is

$$\gamma_+ = -\frac{3\left(2kc_{E,c}^5\epsilon_{E,c}\right)^{-1/2}}{z_c^{\eta_{E,c}/2+5s_{E,c}/2}}\frac{1}{z^{1+\epsilon_c}}\left(1 - z^{2\epsilon_{\gamma,c}/3}\right). \tag{5.110}$$

We can note that in the limit $\epsilon_\gamma \to 0$, that is the limit in which the two tensor modes are decoupled (the coupling terms in (5.83)-(5.84) go like $\sim \epsilon_\gamma$), γ_+ vanishes. This argument shows that in this model the effect of a *large mode* contribution to the power spectrum is exclusively given by the presence of the gaugid graviton E_{ij}. Now we can calculate the contribution to the power spectrum coming out from this solution. For γ_+ we have, using the notation of (5.93)

$$\mathcal{P}_+(k) = \left|f_+^{(2)}\right|^2 = |\gamma_+(k)|^2 = \frac{9\left(2kc_{E,c}^5\epsilon_{E,c}\right)^{-1}}{z_c^{\eta_{E,c}+5s_{E,c}}}\frac{\left(1 - z^{2\epsilon_{\gamma,c}/3}\right)^2}{z^{2(1+\epsilon_c)}}. \tag{5.111}$$

For the physical mode, (5.76), we therefore find

$$\mathcal{P}_\gamma(k) = \frac{36H^2}{M_{Pl}^2c_{E,c}^5\epsilon_{E,c}}\frac{1}{k^3}\frac{1}{z_c^{\eta_{E,c}+5s_{E,c}}}z^{-2\epsilon_c}\left(1 - z^{2\epsilon_{\gamma,c}/3}\right)^2 \tag{5.112}$$

where we have a two factor from the $\mathcal{P}_+ = \mathcal{P}_-$. As we will see later this is the fundamental result of this model. We can immediately see that the amplitude for the *large* mode is enhanced w.r.t. the standard single-field predictions by a factor $c_{E,c}^{-5}\epsilon_{E,c}^{-1}$. We will see that this is the dominant contribution to the power spectrum, and it represents the true novelty of this model [52].
We can calculate now the power spectrum for the *small* mode.

Small mode

We can evaluate now the contribution of $f_+^{(1)}$ in (5.93) solving (5.83)-(5.84) using the initial conditions

$$\gamma_+ \xrightarrow{z\to\infty} \frac{e^{iz}}{\sqrt{2k}}, \quad E_+ \xrightarrow{z\to\infty} 0. \tag{5.113}$$

As for the large mode we perform a perturbative expansion in the small mixing parameter B, expecting that the backreaction of E_+ in (5.83) would be negligible with respect to the full mode γ. We expect indeed a gaugid tensor mode of the order $E_+ \sim \mathcal{O}(B)$ as for γ_+ in the large mode calculations, see e.g. (5.104) and (5.107). Neglecting the mixing term Eq. (5.83) becomes

$$\frac{d^2\gamma_+}{dz^2} + \left(1 - \frac{2 + 3\epsilon - 2\epsilon_\gamma}{z^2}\right)\gamma_+ = 0. \tag{5.114}$$

The solution, after matching the asymptotic behavior of the solution and of the Hankel functions, reads

$$\gamma_+ = e^{i(2\nu_\gamma+1)\pi/4}\left(\frac{\pi z}{4k}\right)^{1/2} H^{(1)}_{\nu_\gamma}(z) + \delta\gamma_+, \tag{5.115}$$

where $\nu_\gamma = \frac{3}{2} + \epsilon + \frac{2}{3}\epsilon_\gamma$. In(5.115) we have indicated with $\delta\gamma_+$ the possible backreaction due to the mixing term; it is expected to be of order $\delta\gamma_+ \sim \mathcal{O}(B^2)$. We can can give an estimation of this backreaction solving Eq. (5.84) in the limit $z \gg 1$, for which we have

$$E_+ \xrightarrow{z\gg1} \frac{B}{\sqrt{2k}\left(1 - c_E^2\right)}\frac{e^{iz}}{z}.$$

Plugging this solution into the equation for γ_+ one finds

$$\delta\gamma_+ \simeq -i\frac{B^2}{2\sqrt{2k}\left(1 - c_E^2\right)}\frac{e^{iz}}{z},$$

i.e. we are allowed to neglect the backreaction at early times.

Unfortunately, for the late-time dynamics we cannot consider the same argument. From (5.115) and using (2.71) we see that γ_+ behaves, at late times, as

$$\gamma_+ = \frac{i}{\sqrt{2k}}\frac{1}{z^{1+\epsilon+2\epsilon_\gamma/3}}. \tag{5.116}$$

Plugging this expression into the equation for E_+ one can find a solution analogue to (5.107)

$$E_+\Big|_{z\ll1} = i\frac{B}{6\sqrt{2k}}\frac{1}{z^{\epsilon_c - 2\epsilon_{\gamma,c}/3}} + D_+ z^{\lambda_+} + D_- z^{\lambda_-}, \tag{5.117}$$

with

$$\lambda_\pm = \frac{1 \pm (5 + 2\epsilon_c + \eta_{E,c})}{2}, \tag{5.118}$$

and $D_\pm$ are two integration constants expected to be of order $D_\pm \sim B$ [52]. The first two terms in this solution gives negligible contributions in the $z \ll 1$ limit w.r.t. the third one which scales as $z^{-2-\epsilon_c-\eta_{E,c}/2}$. If we consider only this dominant term we can evaluate the $\delta\gamma_+$ contribution to (5.115), using (5.83)

$$\frac{d^2\delta\gamma_+}{dz^2} - (2 + 3\epsilon - 2\epsilon_\gamma)\frac{\delta\gamma_+}{z^2} + \frac{BD_-}{z^{3+\epsilon_c-+\eta_{E,c}/2}} = 0, \tag{5.119}$$

in the late time limit. The exact solution to this equation is

$$\delta\gamma_+ = -\frac{BD_-}{2\epsilon_{\gamma,c} + 3\eta_{E,c}/2}\frac{1}{z^{1+\epsilon_c+\eta_{E,c}/2}} + c_1 z^{-1-\epsilon_c+2\epsilon_{\gamma,c}/3} + c_2 z^{2+\epsilon_c-2\epsilon_{\gamma,c}/3}. \tag{5.120}$$

The third term is negligible in this limit and the constant of integration c_1 wold be of the same order of the first term coefficient. If we analyze the amplitude of this correction we have

$$\left.\frac{\delta\gamma_+}{\gamma_+}\right|_{z\ll 1} \sim \frac{BD_-}{\epsilon_\gamma} \sim \frac{B^2}{\epsilon_\gamma} = 4\frac{\epsilon_\gamma}{\epsilon_E}, \tag{5.121}$$

where in the last equality we have used the expression of B, (5.82). Note that this contribution can be considered small, justifying our perturbative method, only if $\epsilon_E \gg \epsilon_\gamma$, which is admitted by the stability conditions (5.68); otherwise the effects from the backreaction of the gaugid tensor mode for the late time dynamics of γ_+ are $\mathcal{O}(1)$.
If we can neglect the backreaction we have that the solution for the γ_+ mode in the superhorizon limit is (5.116), which provides a power spectrum, using (5.94),

$$\mathcal{P}_\gamma = \frac{4H^2}{M_{Pl}^2}\frac{(-k\tau)^{-2\epsilon_c-2\epsilon_{\gamma,c}/3}}{k^3}. \tag{5.122}$$

For the sake of precision we consider now also the limit in which the backreaction is not negligible. We will see that also considering the case $\epsilon_E \sim \epsilon_\gamma$ one finds an amplitude for the gravitational waves power spectrum of the order of (5.1). First of all we fix the integration constant D_- following what we said in the previous section, see (5.109),

$$D_- = -i\frac{B}{6\sqrt{2k}}.$$

Now we can correct the solution for the tensor mode using (5.116) and the dominant mode in (5.120), obtaining

$$\gamma_+ = \frac{i}{\sqrt{2k}}z^{-1-\epsilon_c-2\epsilon_{\gamma,c}/3} + \frac{iB^2}{6\sqrt{2k}(2\epsilon_{\gamma,c}+3\eta_{E,c}/2)}z^{-1-\epsilon_c-\eta_{E,c}/2}$$

which can be approximated to

$$\gamma_+ \simeq \frac{i}{\sqrt{2k}}z^{-1-\epsilon_c}\left(1+\frac{\epsilon_{\gamma,c}}{2\epsilon_{E,c}}\right) \simeq \frac{4i}{3\sqrt{2k}}z^{-1-\epsilon_c}, \tag{5.123}$$

where we have neglected some $\sim \mathcal{O}(\epsilon_i,\eta_i)$ terms and used the exact expression for B (5.82) in the limit $\epsilon_\gamma \sim \epsilon_E$. Using this solution the predicted power spectrum (5.94) becomes

$$\mathcal{P}_\gamma = \frac{64H^2}{9M_{Pl}^2}\frac{(-k\tau)^{-2\epsilon_c}}{k^3}, \tag{5.124}$$

i.e. very similar to (5.122). We can conclude that in the *small mode* case we have predictions on the power spectrum that are of the order of the standard inflationary models

$$A_\gamma^{small} \simeq \frac{H^2}{M_{Pl}^2},$$

while for the *large mode*, Eq. (5.112), we have an enhancement by a factor $c_E^5\epsilon_E$ due to the coupling between the metric tensor mode γ_{ij} and the gaugid one E_{ij}.
For this reason we will consider only the contribution to the power spectrum coming from the large mode, since it will be the dominant component.

We can rewrite (5.112) in a more familiar version and we evaluate it at τ_f

$$\mathcal{P}(k) = \frac{16H_c^2}{M_{Pl}^2 c_{E,c}^5 \epsilon_{E,c}} \frac{\epsilon_{\gamma,c}^2 N_e^2}{k^3 (-k\tau_c)^{2\epsilon_c + \eta_{E,c} + 5s_{E,c}}}, \tag{5.125}$$

where we have used the fact that $N_e = \ln\left(-k\tau_f\right)$ represents the number of e-folds when the mode k has left the horizon from the end of Inflation. Using the dimensionless power spectrum (2.50) we can finally write

$$\Delta_\gamma = \frac{8}{\pi^2} \frac{H_c^2}{M_{Pl}^2} \frac{\epsilon_{\gamma,c}^2 N_e^2}{c_{E,c}^5 \epsilon_{E,c}} \frac{1}{(-k\tau_c)^{2\epsilon_c + \eta_{E,c} + 5s_{E,c}}} \tag{5.126}$$

Eq. (5.126) is the main result of this section. We see that the "standard" amplitude H^2/M_{Pl}^2 is multiplied by a factor $(\epsilon_{\gamma,c}^2 N_e^2)/(c_{E,c}^5 \epsilon_{E,c})$ which depends on the parameters of the theory. Moreover, we can consider two different limits for the parameter ϵ_γ since (5.68) entails only a lower bound for it [52]: $\epsilon_\gamma N_e \gtrsim 1$ and $\epsilon_\gamma N_e \ll 1$. In the first case the amplitude and the tensor tilt are

$$\begin{aligned} A_\gamma &= \frac{18}{\pi^2} \frac{H_c^2}{M_{Pl}^2} \frac{1}{c_{E,c}^5 \epsilon_{E,c}}, \\ n_\gamma &= -2\epsilon_c - \eta_{E,c} - 5s_{E,c}, \end{aligned} \tag{5.127}$$

while in the second case

$$\begin{aligned} A_\gamma &= \frac{8}{\pi^2} \frac{H_c^2}{M_{Pl}^2} \frac{\epsilon_{\gamma,c}^2}{c_{E,c}^5 \epsilon_{E,c}}, \\ n_\gamma &= -\frac{2}{N_e} - 2\epsilon_c - \eta_{E,c} - 5s_{E,c}. \end{aligned} \tag{5.128}$$

We see that in the first case, the most interesting, (5.127) ensures that the power spectrum amplitude arising in the standard inflationary models is enhanced by a large factor $c_{E,c}^{-5} \epsilon_{E,c}^{-1}$, i.e. this theory predicts an abundance of gravitational waves. However, we are not capable to give a prediction about the sign of the tensor power spectrum since it (and in particular its sign) depends on much parameters of the theory, which can have a large set of possible values, see (5.67) and (5.68). However the possibility of a *blue* tilt is not excluded, giving a possibility of detection with future experiments, [83, 84].

We can calculate now the tensor-to-scalar ratio in the $\epsilon_\gamma \lesssim N_e^{-1}$

$$r \simeq \frac{c_T^5}{c_E^3} \left(\epsilon_\gamma N_e\right)^2. \tag{5.129}$$

One can see that r is extremely sensitive the the scalar speed and can be highly suppressed if the phonons are subluminal, i.e. $c_T < 1$. On the other hand, the stability and subluminality conditions (5.68) do allow for a large tensor-to-scalar ratio.

5.2 From *Magnetic* to *Electromagnetic* Gaugid

5.2.1 Parity violation in Cosmology

This section contains the main original results of this Thesis work.
Here we will consider a generalization of Gaugid Inflation in which we will consider a

more general vacuum expectation value for the gauge field $A_{I\mu}$. In [52] a theory which is manifestly invariant under parity transformation is considered. In the literature, see, e.g. [98] and References therein, various inflationary models in which the parity transformation is not preserved have been discussed. In particular in [98] is presented the case of a modified gravity theory (with a gravitational Chern-Simons term) and its implications for the inflationary power spectrum and bispectrum have been analyzed. In this kind of theories a (small) violation of parity in the tensor power spectrum arises. The effects on CMB of a possible parity violation has been studied in [99] (for the latest analysis see [100] and Refs. therein for other previous analysis): having a fundamental theory which violates parity entails a polarization of primordial gravitational waves into chiral-eigenstates, the so-called left (L) and right (R) polarization states and the possibility to detect such parity breaking signatures looking at CMB polarization.

We have seen that in [52] a vacuum state (5.2) which preserves the parity of the theory is chosen, and the phenomenological action (5.26) is chosen so that it is invariant under parity transformation. It can easily be shown that the original theory preserves parity. Three $U(1)$ gauge fields must transform under parity as [52]

$$A_{i0}(t,\mathbf{x}) \to A_{i0}(t,-\mathbf{x}), \quad A_{ij}(t,\mathbf{x}) \to -A_{ij}(t,-\mathbf{x}). \tag{5.130}$$

These transformation rules are respected by the vacuum (5.2), as one can easily verify. But using the new choice of vacuum (5.3) we have $A_{i0}(t,\mathbf{x}) \to -A_{i0}(t,-\mathbf{x})$, clearly violating the parity transformation laws in (5.130).

For this reason we will consider as vacuum expectation value of our theory a combination of a *magnetic* component, i.e. (5.2), and an *electric* component, i.e. (5.3)

$$\langle A_{I\mu}\rangle = \epsilon_{Ijk}\delta^j_\mu x^k + \mathrm{g}\delta^0_\mu x_I. \tag{5.131}$$

The parameter g in general can be time dependent: indeed we have seen in section 5.1.2 that, taking a homogeneous and isotropic background, the only possible expression for the electric field is $E_{Ij} = \mathrm{g}(t)\delta_{Ij}$, with $\mathrm{g}(t)$ such that

$$\frac{d}{dt}(aP'(X)\mathrm{g}) = 0, \tag{5.132}$$

with $P'(X) \equiv \partial P/\partial X$. Note that here we would have to consider a more general Lagrangian, such as $\mathcal{L} = -Z(X, I_1, I_2, I_3)$ since I_1, I_2, I_3, are the terms that we will consider. This means that the equation of motion (5.132) could have, in general, other terms due to the presence of a time dependent background of I_1. The background value of I_1 was null in the original theory, but in our case, using a different vacuum choice, it can have a small time dependence. In particular, one finds that the background value of I_1 using the vacuum (5.131) is

$$I_1 = -\frac{4\mathrm{g}}{4 - \mathrm{g}^2 a^2(t)}. \tag{5.133}$$

This means that we will work in the hypothesis of switching on a small electric configuration where the parameter g in the vacuum (5.131) is perturbatively small. This allows us, as a zeroth-order approximation, to neglect the term I_1, (5.133), as far as the background dynamics is concerned.

We want to see now whether a background configuration with a constant electric and magnetic field exists, i.e. $\mathrm{g} = const.$, which means

$$\frac{d}{dt}(aP'(X)) = 0. \tag{5.134}$$

This condition necessarily implies that the kinetic term has a non-trivial time dependence. We will see that the background value of X in the electromagnetic configuration will scale as $\propto a^{-2}$, which means that a possible solution to (5.134) is having a kinetic term of the type $P(X) \propto X^{3/2}$. Possibilities of this type resemble closely the scenario of the so called *k-inflation* models for scalar driven inflationary models, [54]. Another possibility to have a constant electric field configuration is that of considering a standard kinetic term which is coupled to another (scalar field) in such a way to give rise at the background level to the expression $P(X) = c(t)X$, with $c(t) \propto a(t)^{-1}$. This possibility resembles the case when we consider suitable couplings between a scalar field and the kinetic term of a vector field, such as the one presented in [95] and Refs. therein. Here we will consider a general case and $\mathrm{g} = const.$ We could impose $\mathrm{g} = 1$ but we leave it to use the predictions of [52] as a check for our theory when $\mathrm{g} = 0$. We will study in detail the outcomes of this hypothesis in particular as far as the gravitational waves are concerned. We will see that a new coupling term in the tensor action appears, that will entail new non-trivial results. As a consequence of violation of parity we will also see that there are novelties also in the scalar side. There will be a coupling between the two scalar modes of the theory α and T, see Eq. (5.30), also in the subhorizon limit, which could also imply a violation of parity for the scalar power spectrum. However we are more interested in the possible violation of parity in the gravitational wave sector, so we will not consider the scalar fluctuations in performing the calculations when mixing our field with gravity: we leave this calculation for future works.

5.2.2 Inflation

Here we will see what are the conditions that our theory must satisfy in order to provide an inflationary period. Using a FLRW, now all the contraction of the Lorentz indices, not the internal ones, in (5.8) are performed using $g_{\mu\nu}$. See Appendix A for the new components of the field strength $F_{i\mu\nu}$.
First of all we notice that, in a FLRW metric, the background value of X changes using (5.131)

$$\langle X \rangle \equiv \bar{X} = -6a^{-2}(4a^{-2} - \mathrm{g}^2). \tag{5.135}$$

We now use again the stress-energy definition (5.18) with the general Lagrangian $\mathcal{L} = -Z(X, I_1, \ldots, I_{10})$ and obtain

$$\begin{aligned} \rho &= Z + 12\mathrm{g}^2 a^{-2}, \\ p &= -Z - 4\mathrm{g}^2 a^{-2} Z_X + 32 a^{-4} Z_X. \end{aligned} \tag{5.136}$$

Hence the Friedmann equations become

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\left(Z + 12\mathrm{g}^2 a^{-2}\right), \\ \frac{\ddot{a}}{a} &= \frac{8\pi G}{3}\left(Z - 48 a^{-4} Z_X\right), \end{aligned} \tag{5.137}$$

so the slow-roll condition (5.22) becomes

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{8\mathrm{g}^2 a^{-2} Z_X + 32 a^{-4} Z_X}{Z + 12\mathrm{g}^2 a^{-2} Z_X} = 2\frac{\bar{X} Z_X}{Z} \ll 1, \tag{5.138}$$

where in the last equality we have considered the limit $a \gg 1$. This means that, as before, the request of an inflationary period translates into a weak X dependence of the

Lagrangian. In this limit we can calculate also η. Using

$$\frac{\dot{\bar{X}}}{\bar{X}} = -2H, \tag{5.139}$$

we obtain

$$\eta = 2\left[-1 + \frac{\bar{X}^2 Z_{XX}}{\bar{X} Z_X} + \epsilon\right] \ll 1 \tag{5.140}$$

which means

$$\frac{\bar{X}^2 Z_{XX}}{\bar{X} Z_X} = 1 + \frac{\eta}{2} - \epsilon = 1 + \mathcal{O}(\epsilon, \eta). \tag{5.141}$$

We will exploit this relation in order to perform some perturbative calculations.

In the end, the theory we will consider is the same of [52], but to Eq. (5.26) we will add the simplest parity breaking term I_1 given in (5.8)

$$S = \int \mathrm{d}^4 x \sqrt{-g}\left[\frac{M_{Pl}^2}{2} R - P(X) + \mathrm{s} M^4 I_1 - 27 M_1^4 I_2 + 18 M_2^4 I_3\right]. \tag{5.142}$$

We will see that changing the vacuum expectation value from the case of Eq. (5.2) to Eq. (5.131) and adding the term I_1, will change significantly the perturbative expansion of the action up to the second-order in the perturbations.

5.2.3 Perturbations in the subhorizon limit

We perform now the perturbative expansion up to second order in the perturbations to study the stability and subluminality of this theory. We will use the same notation introduced in 5.1.3, since the new choice for the vacuum does not change the perturbative expansion of the gauge field $A_{i\mu}$. The perturbed field strength will have the form

$$F_{i\mu\nu} = -2\varepsilon_{i\mu\nu} + \mathrm{g}(\delta^0_\nu \delta_{i\mu} - \delta^0_\mu \delta_{i\nu}) + f_{i\mu\nu}, \tag{5.143}$$

with $f_{i\mu\nu}$ defined in (5.35). Using the perturbed expressions for I_1, I_2 and I_3 the we have computed in the Appendix A the perturbed action up to second order in the Minkowski limit is

$$S^{(2)} = S_S^{(2)} + S_V^{(2)} + S_T^{(2)}, \tag{5.144}$$

where the subscripts S, V, T stands for Scalar, Vector and Tensor and

$$S_S^{(2)} = \int d^4x \Bigg\{ \left(\frac{XP_X}{4} - \frac{36\mathrm{g}^2}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{96(4\mathrm{g}^4 + 33\mathrm{g}^2 + 4)}{(4-\mathrm{g}^2)^3} M_2^4 + \frac{4\mathrm{g}(4-5\mathrm{g}^2)}{(4-\mathrm{g}^2)^3} \mathrm{s}M^4 \right) \dot{\alpha}^2$$
$$+ \left(-\frac{XP_X}{6} + \frac{4\mathrm{g}^2(4\mathrm{g}^4 + \mathrm{g}^2 - 68)}{3(4-\mathrm{g}^2)^4} M_2^4 + \frac{8\mathrm{g}(20 - 23\mathrm{g}^2 + 5\mathrm{g}^4)}{9(4-\mathrm{g}^2)^3} \mathrm{s}M^4 \right) (\partial\alpha)^2$$
$$- \left(\frac{XP_X}{6} + \frac{64(4\mathrm{g}^4 + 33\mathrm{g}^2 + 4)}{(4-\mathrm{g}^2)^3} M_2^4 + \frac{8\mathrm{g}(4-5\mathrm{g}^2)}{3(4-\mathrm{g}^2)^3} \mathrm{s}M^4 \right) \dot{\alpha} \frac{\partial^2 \chi}{\sqrt{-\partial^2}}$$
$$+ \left(\frac{72\mathrm{g}}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{96\mathrm{g}(3\mathrm{g}^4 + 8\mathrm{g}^2 + 48)}{(4-\mathrm{g}^2)^4} M_2^4 - \frac{16(4+\mathrm{g}^2)^2}{(4-\mathrm{g}^2)^2} \mathrm{s}M^4 \right) \dot{\alpha} \frac{\partial^2 T}{\sqrt{-\partial^2}}$$
$$+ \left(-\frac{XP_X}{12} + \frac{8\mathrm{g}^2 M_1^4}{(4-\mathrm{g}^2)^2} + \frac{32(4\mathrm{g}^6 + 16\mathrm{g}^4 - 129\mathrm{g}^2 - 4)}{(4-\mathrm{g}^2)^3} M_2^4 + \frac{4\mathrm{g}(4+3\mathrm{g}^2)}{9(4-\mathrm{g}^2)^3} \mathrm{s}M^4 \right) \chi \partial^2 \chi$$
$$+ \left(\frac{32\mathrm{g} M_1^4}{(4-\mathrm{g}^2)^2} + \frac{16\mathrm{g}(13\mathrm{g}^4 - 88\mathrm{g}^2 + 208)}{3(4-\mathrm{g}^2)^4} M_2^4 - \frac{16(4+\mathrm{g}^2)^2}{(4-\mathrm{g}^2)^2} \mathrm{s}M^4 \right) T \partial^2 \chi$$
$$+ \left(\frac{XP_X}{6} + \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3} M_2^4 - \frac{8\mathrm{g}}{3(4-\mathrm{g}^2)^2} \mathrm{s}M^4 \right) \dot{T}^2$$
$$+ \left(\frac{XP_X}{18} - \frac{32 M_1^4}{(4-\mathrm{g}^2)^2} + \frac{16\mathrm{g}^2(3\mathrm{g}^4 + 64\mathrm{g}^2 + 80)}{3(4-\mathrm{g}^2)^4} M_2^4 + \frac{2\mathrm{g}(96 - 56\mathrm{g}^2 + 5\mathrm{g}^4)}{9(4-\mathrm{g}^2)^3} \mathrm{s}M^4 \right) (\partial T)^2 \Bigg\}, \tag{5.145}$$

$$S_V^{(2)} = \int d^4x \Bigg\{ \left(\frac{XP_X}{6} + \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3} M_2^4 - \frac{8\mathrm{g}}{3(4-\mathrm{g}^2)^2} \mathrm{s}M^4 \right) \left(\dot{V}_k\right)^2$$
$$+ \left(-\frac{XP_X}{12} - \frac{24 M_1^4}{(4-\mathrm{g}^2)^2} + \frac{4\mathrm{g}^2(\mathrm{g}^4 + 8\mathrm{g}^2 - 48)}{(4-\mathrm{g}^2)^4} M_2^4 + \frac{\mathrm{g}(96 - 56\mathrm{g}^2 + 5\mathrm{g}^4)}{9(4-\mathrm{g}^2)^3} \mathrm{s}M^4 \right) (\partial V_k)^2$$
$$+ \left(-\frac{XP_X}{6} - \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3} M_2^4 + \frac{8\mathrm{g}}{3(4-\mathrm{g}^2)^2} \mathrm{s}M^4 \right) \varepsilon_{ijk} \partial_j B_i \dot{V}_k - \frac{8\mathrm{g}(3M_1^4 + 2M_2^4)}{(4-\mathrm{g}^2)^2} \partial_i V_j \partial_i B_j$$
$$+ \left(\frac{XP_X}{12} - \frac{6\mathrm{g}^2}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{16(4+3\mathrm{g}^2)}{(4-\mathrm{g}^2)^3} M_2^4 - \frac{4\mathrm{g}}{3(4-\mathrm{g}^2)^2} \mathrm{s}M^4 \right) (\partial B_k)^2 \Bigg\}, \tag{5.146}$$

and

$$S_T^{(2)} = \int d^4x \Bigg\{ \left(\frac{XP_X}{12} - \frac{12\mathrm{g}^2}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{32(4+\mathrm{g}^2)}{(4-\mathrm{g}^2)^3} M_2^4 - \frac{4\mathrm{g}}{3(4-\mathrm{g}^2)^2} \mathrm{s}M^4 \right) \dot{E}_{ij}^2$$
$$+ \left(-\frac{XP_X}{12} - \frac{48 M_1^4}{(4-\mathrm{g}^2)^2} - \frac{8\mathrm{g}^2(\mathrm{g}^4 + 32\mathrm{g}^2 + 48)}{(4-\mathrm{g}^2)^4} M_2^4 + \frac{\mathrm{g}(96 - 56\mathrm{g}^2 + 5\mathrm{g}^4)}{9(4-\mathrm{g}^2)^3} \mathrm{s}M^4 \right) (\partial E_{ij})^2 . \Bigg\} \tag{5.147}$$

Despite the messy appearance of this action we can clearly highlight some new outcomes. The only effect of the new term I_1 in the action in (5.146) and (5.147) is only a renormalization of the coefficients of the dynamical terms, i.e. a renormalization of the speeds of propagation. As we will see, the real noteworthy effect of the new term I_1 appears only in the scalar action. First we have computed the solution for the dynamical for the non-dynamical field χ introduced in (5.32)

$$\chi = \frac{\mathcal{A}}{2\mathcal{B}} \frac{\dot{\alpha}}{\sqrt{-\partial^2}} - \frac{\mathcal{C}}{2\mathcal{B}} T, \tag{5.148}$$

where

$$\mathcal{A} = \frac{XP_X}{6} + \frac{64(4g^4+33g^2+4)}{(4-g^2)^3}M_2^4 + \frac{8g(4-5g^2)}{3(4-g^2)^3}sM^4, \tag{5.149}$$

$$\mathcal{B} = -\frac{XP_X}{12} + \frac{8g^2M_1^4}{(4-g^2)^2} + \frac{32(4g^6+16g^4-129g^2-4)}{(4-g^2)^3}M_2^4 + \frac{4g(4+3g^2)}{9(4-g^2)^3}sM^4, \tag{5.150}$$

$$\mathcal{C} = \frac{32gM_1^4}{(4-g^2)^2} + \frac{16g(13g^4-88g^2+208)}{3(4-g^2)^4}M_2^4 - \frac{16(4+g^2)^2}{(4-g^2)^2}sM^4. \tag{5.151}$$

We see that with respect to the result of [52] here we have an extra term depending on the scalar degree of freedom T, which disappears when neglecting parity-breaking terms, i.e. when $g = 0$. This means that we will have, also in the quadratic action, a coupling due to the presence of I_1 between the two scalar fields α and T. Indeed substituting (5.148) into (5.145) we obtain

$$\begin{aligned} S_S^{(2)} = \int d^4x\Bigg\{ &\left(\frac{XP_X}{4} - \frac{36g^2}{(4-g^2)^2}M_1^4 + \frac{96(4g^4+33g^2+4)}{(4-g^2)^3}M_2^4 + \frac{4g(4-5g^2)}{(4-g^2)^3}sM^4 + \frac{\mathcal{A}^2}{4\mathcal{B}}\right)\dot{\alpha}^2 \\ &+ \left(-\frac{XP_X}{6} + \frac{4g^2(4g^4+g^2-68)}{3(4-g^2)^4}M_2^4 + \frac{8g(20-23g^2+5g^4)}{9(4-g^2)^3}sM^4\right)(\partial\alpha)^2 \\ &+ \left(\frac{72g}{(4-g^2)^2}M_1^4 + \frac{96g(3g^4+8g^2+48)}{(4-g^2)^4}M_2^4 - \frac{16(4+g^2)^2}{(4-g^2)^2}sM^4 + \frac{\mathcal{AC}}{2\mathcal{B}}\right)\dot{\alpha}\frac{\partial^2 T}{\sqrt{-\partial^2}} \\ &+ \left(\frac{XP_X}{6} + \frac{128g^2}{(4-g^2)^3}M_2^4 - \frac{8g}{3(4-g^2)^2}sM^4\right)\dot{T}^2 \\ &+ \left(\frac{XP_X}{18} - \frac{32M_1^4}{(4-g^2)^2} + \frac{16g^2(3g^4+64g^2+80)}{3(4-g^2)^4}M_2^4 + \frac{2g(96-56g^2+5g^4)}{9(4-g^2)^3}sM^4 + \frac{\mathcal{C}^2}{4\mathcal{B}}\right)(\partial T)^2\Bigg\}, \end{aligned} \tag{5.152}$$

i.e. we can see that a term $\dot{\alpha}\partial^2 T$ appears, sign of a coupling of the fields. This implies a parity violation since in 5.1.3 we have seen that α is an *axial* field, i.e. with parity $P_\alpha = -1$, while T is a *polar* field, i.e. with parity $P = +1$. Therefore, the new term has a parity -1, suggesting a possible violation of parity. Although very interesting, we will not consider this effect in our studies. We will instead focus on the new outcomes for the new tensor degree of freedom E_{ij} using the *electromagnetic* vacuum. Since in this case the only effect of I_1 is a renormalization of the coefficients, we will not explicitly account for it in order to have simplified expressions.

We can now give a zeroth order measure for the two scalars speeds of sound neglecting the coupling term

$$\begin{aligned} c_\alpha^2 = &\big\{27(4-g^2)^4XP_X - 216g^2(4g^4+g^2-68)M_2^4 \\ &-144g(4-g^2)(20-23g^2+5g^4)sM^4\big\}/\big\{2(4-g^2)^4XP_X - 288(4-g^2)^2g^2M_1^4 \\ &+768(4-g^2)(4g^4+33g^2+4)M_2^4 + 32g(4-g^2)(4-5g^2)sM^4 + 8(4-g^2)^4\frac{\mathcal{A}^2}{4\mathcal{B}}\big\} \end{aligned} \tag{5.153}$$

$$\begin{aligned} c_T^2 = &\big\{-(4-g^2)^4XP_X + 576(4-g^2)^2M_1^4 - 96g^2(3g^4+64g^2+80)M_2^4 \\ &-4(4-g^2)g(96-56g^2+5g^4)sM^4 - 9(4-g)^2\frac{\mathcal{C}^2}{2\mathcal{B}}\big\}/\big\{3(4-g^2)^4XP_X \\ &+2304g^2(4-g^2)M_2^4 - 48(4-g^2)^2gsM^4\big\} \end{aligned} \tag{5.154}$$

Now, neglecting the I_1 terms, we obtain the two speeds of sound for the vector and tensor perturbation modes,

$$\begin{aligned} c_V^2 = &\left\{XP_X(4-\mathrm{g}^2)^3\left[1+2\mathcal{N}-\mathcal{G}^2\right]\right. \\ &+48\left[\mathrm{g}^2(\mathrm{g}^2+12)+32\mathrm{g}^2\mathcal{N}+4(4-\mathrm{g}^2)\mathcal{G}-4(4+3\mathrm{g}^2)\mathcal{G}^2\right]M_2^4 \\ &\left.+72(4-\mathrm{g}^2)\left[(4-\mathrm{g}^2+4\mathcal{G}+\mathrm{g}^2\mathcal{G}^2)\right]M_1^4\right\}/\left\{XP_X(4-\mathrm{g}^2)^3[2+\mathcal{N}^2]\right. \\ &\left.-72\mathrm{g}^2(4-\mathrm{g}^2)\mathcal{N}^2M_1^4+192[8\mathrm{g}^2+(4+3\mathrm{g}^2)\mathcal{N}^2]M_2^4\right\}, \end{aligned} \tag{5.155}$$

$$c_E^2 = \frac{XP_X(4-\mathrm{g}^2)^4+576(4-\mathrm{g}^2)^2M_1^4+96\mathrm{g}^2(\mathrm{g}^4+32\mathrm{g}^2+48)M_2^4}{XP_X(4-\mathrm{g}^2)^4-144\mathrm{g}^2(4-\mathrm{g}^2)^2M_1^4+384(4-\mathrm{g}^2)(4+\mathrm{g}^2)M_2^4}, \tag{5.156}$$

where $\mathcal{N}$ and $\mathcal{G}$ are defined in Appendix A. We see that the expressions (5.153), (5.154), (5.155) and (5.156) are very messy and do not give us much information about the subluminality of this theory, which are $0 < c_\alpha^2 < 1$, $0 < c_T^2 < 1$, $0 < c_V^2 < 1$ and $0 < c_E^2 < 1$.

5.3 Second order Lagrangian including mixing with gravity

We are now ready to switch on the gravity. We will make use, as usual, of the ADM formalism, Eq. (5.52), and of the spatially flat gauge, defined in (5.53) and (5.54). In this case the covariant components for the field strength are

$$F_{ij0} = -F_{i0j} = \mathrm{g}\delta_{ij} + f_{ij0}, \quad F_{ijk} = -2\varepsilon_{ijk} + f_{ijk}, \tag{5.157}$$

while the contravariant ones are

$$\begin{aligned} F_i^{\ 0j} &= \frac{\mathrm{g}}{N^2}\delta_i^j - \frac{h^{jk}f_{i0k}}{N^2} - 2\frac{\varepsilon_{ikl}h^{jl}N^k}{N^2} + \frac{N^kh^{jl}f_{ikl}}{N^2}, \\ F_i^{\ jk} &= -\frac{\mathrm{g}}{N^2}\left(N^j\delta_i^k - N^k\delta_i^j\right) - \frac{f_{il0}}{N^2}\left(N^jh^{kl} - N^kh^{jl}\right) \\ &- 2\varepsilon_{ilm}h^{km}h^{jl} + h^{jl}h^{km}f_{ilm} + \frac{2}{N^2}\left(\varepsilon_{ilm}N^kN^mh^{jl} + \varepsilon_{ilm}N^jN^lh^{km}\right) \\ &- \frac{f_{ilm}}{N^2}\left(N^kN^mh^{jl} + N^jN^lh^{km}\right). \end{aligned} \tag{5.158}$$

Given that we are interested in the linear equation of motions, and that we are interested only to the tensor degrees of freedom, we can consider from now on $N = 1$ and $N_i = N^i = 0$ to simplify our calculations. The action one obtains is quite involved, but one can see that the only $f_{i\mu\nu}$ terms which contribute to the tensor action are (here we write only the tensor perturbation)

$$f_{ij0} = -f_{i0j} = -\dot{E}_{ij}, \quad f_{ijk} = \partial_j E_{ik} - \partial_k E_{ij}. \tag{5.159}$$

After some lengthy calculations, see Appendix A, the action for the two perturbations γ_{ij} and E_{ij} turns out to be

$$
\begin{aligned}
S^{(2)}_{GW} = \int \mathrm{d}^4 x\sqrt{-g}\Bigg\{ & \frac{M_{Pl}^2}{8}\left(\dot\gamma_{ij}^2 - a^{-2}\left(\partial\gamma_{ij}\right)^2\right) \\
&+\left(-\frac{XP_X}{3(4a^{-2}-\mathrm{g}^2)} - \frac{12\mathrm{g}^2M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} + \frac{32a^{-2}(4a^{-6}-\mathrm{g}^2a^{-4}+2\mathrm{g}^2)M_2^4}{(4a^{-2}-\mathrm{g}^2)^3}\right)\dot E_{ij}^2 \\
&+\left(\frac{a^{-2}XP_X}{3(4a^{-2}-\mathrm{g}^2)} - \frac{48a^{-4}M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} + \frac{8\mathrm{g}^2a^{-4}(3\mathrm{g}^4+16a^{-4}\mathrm{g}^2-64a^{-4})M_2^4}{3(4a^{-2}-\mathrm{g}^2)^4}\right)(\partial E_{ij})^2 \\
&+\left(-2\frac{a^{-2}XP_X}{3(4a^{-2}-\mathrm{g}^2)} - \frac{3(4a^{-2}+\mathrm{g}^2)^2M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} + \frac{128\mathrm{g}^2a^{-4}(5a^{-2}-\mathrm{g}^2)M_2^4}{(4a^{-2}-\mathrm{g}^2)^2}\right)\gamma_{ij}\gamma_{ij} \\
&+\left(4\frac{a^{-2}XP_X}{3(4a^{-2}-\mathrm{g}^2)} + \frac{24a^{-2}(4a^{-2}+\mathrm{g}^2)M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} + \frac{16\mathrm{g}^2a^{-4}(28a^{-2}-3\mathrm{g}^2)M_2^4}{(4a^{-2}-\mathrm{g}^2)^3}\right)\varepsilon_{ijk}\gamma^{kl}\partial_jE_{il} \\
&+\left(2\mathrm{g}\frac{XP_X}{3(4a^{-2}-\mathrm{g}^2)} - \frac{12\mathrm{g}(4a^{-2}+\mathrm{g}^2)M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} - \frac{128\mathrm{g}^3a^{-4}M_2^4}{(4a^{-2}-\mathrm{g}^2)^2}\right)\gamma_{ij}\dot E_{ij}\Bigg\}.
\end{aligned}
\tag{5.160}
$$

A detailed perturbative calculation show a particular feature, useful for future studies. In the gravitational waves action can also appear another term coming from the I_1 expansion, see Appendix A. It is a term of the form $\varepsilon_{ijk}\gamma^{kl}\partial_l E_{jm}$ but it gives a null contribution in the integral, because it can be rewritten in terms of a total spatial *covariant* derivative. Indeed

$$\varepsilon_{ijk}\gamma^{kl}\partial_l E_{jm} = \partial_l(\varepsilon_{ijk}\gamma^{kl}E_{jm}) - \varepsilon_{ijk}\partial_l\gamma^{kl}E_{jm}.$$

Here the second term of the right-hand-side is null for the choice of the gauge. The first one can be rewritten substituting the partial derivative with the covariant derivative. Indeed

$$\nabla_l V^l = \partial_l V^l + \Gamma^l_{lk}V^k$$

where we have defined $V^k \equiv \varepsilon_{ijk}\gamma^{kl}E_{jm}$. But $\Gamma^l_{kl} \sim \gamma^2$, so that the term with the Christoffel symbol is automatically negligible since it is of order $\sim \mathcal{O}(\gamma^3 E)$. Therefore

$$\nabla_l V^l = \partial_l V^l,$$

i.e., contribution of $\varepsilon_{ijk}\gamma^{kl}\partial_l E_{jm}$ to the equation of motion is null. This term could be considered in a higher order perturbative expansion, with new possible parity-breaking terms.

One can easily see that imposing $\mathrm{g} = 0$ in (5.160) one founds the action (5.59), this is a check of the goodness of our computations. Moreover, in (5.160) a new non-trivial term of the form $\gamma_{ij}\dot E_{ij}$ is present in the gravitons action. This term will reappear in the equations of motion for γ and E as a new coupling term different from $\varepsilon_{ijk}\gamma_{kl}\partial_j E_{il}$. Moreover in the magnetic configuration, when expanding the two gravitons using (5.78) and (5.79), we have seen that modes with different helicity do not mixed, hence they could be treated separately. This was a direct consequence of the presence of the Levi-Civita symbol in the coupling term and of the property (5.80). In this case we can qualitatively see that the equation of motions of γ_{ij} and E_{ij} violate parity. If we use the decomposition in the helicity eigenstates (5.79) we can write

$$
\begin{aligned}
A_1\gamma''_\pm + B_1\gamma_\pm \pm C_1E_\pm + D_1E'_\pm = 0,\\
A_2E''_\pm + B_2E_\pm \pm C_2\gamma_\pm + D_2\gamma'_\pm = 0,
\end{aligned}
\tag{5.161}
$$

where A_i, b_i, C_i, D_i, with $i = 1, 2$ will be, in general, some time-dependent coefficients and a prime denotes a derivative w.r.t. the conformal time.
Using the parity transformation (5.85) we see that the system (5.161) becomes

$$\begin{aligned} A_1\gamma''_{\mp} + B_1\gamma_{\mp} \mp C_1 E_{\mp} - D_1 E'_{\mp} = 0, \\ A_2 E''_{\mp} + B_2 E_{\mp} \mp C_2\gamma_{\mp} - D_2\gamma'_{\mp} = 0. \end{aligned} \tag{5.162}$$

We can see that the systems (5.162) and (5.161) are not equivalent, and the term which manifestly violates the parity is due to the coupling $\gamma_{ij}\dot{E}_{ij}$. We underline that this term was not present in [52] and it represents the original contribution of this work. This further complicates the calculations for the tensor degrees of freedom, but we expect that this new term could give very interesting outcomes. In particular we need to calculate the explicit expressions of A_i, b_i, C_i, D_i, and then find the solutions of the system (5.161) in order to calculate the power spectrum for the gravitational waves. We leave for future works these calculations concerning this generalization of Gaugid model.

Conclusions

In this Thesis various aspects of Inflation have been investigated. First of all we have introduced the inflationary scenario as a solution of the classical problems of the hot Big Bang model: the horizon, the flatness and the "unwanted relics" problems. Then we have introduced the single-field slow-roll models of Inflation to study the production and evolution of the primordial perturbations during Inflation. We have performed the computation of the inflationary power spectra for scalar curvature and tensor perturbations and, always remaining in the context of the slow-roll models, we have made a computation of primordial non-Gaussianities provided by the scalar bispectrum (i.e. the Fourier counterpart of the three-point correlation function) of the curvature (density) perturbations. In order to do the computations we have defined and used the so called in-in formalism [61], the natural extension of the diagrammatic Feynman rules in a dynamical spacetime. We have reobtained the well-known and important result that the scalar bispectrum is suppressed when the slow-roll parameters are small and therefore the non-Gaussianities predicted by the slow-roll models of Inflation are very small [56, 42]: this is fully consistent with the observational constraints on non-Gaussianity provided by the Planck satellite. Such constraints are compatible with a zero level of primordial non-Gaussianity as predicted by the slow-roll models, but there is still a window of almost two orders of magnitude which does not rule out a priori the possibility to find out profiles of non-Gaussianity in the next future [43].

We have also presented the expressions for the scalar-scalar-scalar and tensor-scalar-scalar bispectra arising in single-field inflationary models in the limit in which one perturbation mode has momentum much smaller than the other two, the so-called *squeezed limit.* This configuration is very interesting because in this case the perturbation mode labeled by k_3 crosses the horizon much earlier that the other two modes, with new interesting physical consequences. For the scalar-scalar-scalar bispectrum we have that, in the squeezed limit, the following expression [56]

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle \xrightarrow{k_3\ll k_1\sim k_2} -(n_s-1)\langle \zeta_{\mathbf{k}_3}\zeta_{-\mathbf{k}_3}\rangle\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\rangle$$

holds. In this expression n_s is the scalar spectral index and (n_s-1) measures the deviation from a scale invariant (scalar) power spectrum. This relation between the bispectrum and the power spectra in the squeezed limit holds in *all* single-field inflationary models, since it is a consequence of the invariance under space diffeomorphisms of the classical and quantum theory. For these reasons we will refer to it as *consistency relation* for the scalar perturbations.

Actually there is an objective difficulty in verifying this consistency relation since it would entail a high precision measurement of the level of non-Gaussianity. On the other hand, it is easier to demonstrate, or observe, that a relation is false w.r.t. showing that it is true. A violation of the consistency relation would rule out the entire class of "standard" single-field models of Inflation. For this reason this consistency relation has been studied, and a great work has been spent in searching new models which violate these conditions. In particular in this Thesis we have focused on the the *tensor consistency relation.* We have studied the Gravitational Wave sector because of its fundamental importance in shedding a light into the physics of the the Early Universe. In this case the perturbation mode with smaller momentum is the tensor one and a relation similar to the scalar one holds in the squeezed limit [63, 67]

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \gamma^s_{\mathbf{K}} \rangle \xrightarrow{K \ll k_1 \sim k_2} (2\pi)^3 \delta^{(3)}(\mathbf{K} + \mathbf{k}_1 + \mathbf{k}_2) \frac{1}{2} \frac{d \ln \mathcal{P}_\zeta}{d \ln k} \epsilon^s_{ij}(\mathbf{K}) \hat{k}^i_1 \hat{k}^j_2 \mathcal{P}_\gamma(K) \mathcal{P}_\zeta(k),$$

where $\mathbf{k} = (\mathbf{k}_1 + \mathbf{k}_2)/2$ and $\mathcal{P}_\zeta$, $\mathcal{P}_\gamma$ are respectively the scalar and the tensor power spectra. In particular this consistency relation allow us to define a new observable quantity useful to provide a new indirect way to detect primordial gravitational waves: tensor fossils. These are defined as a hypothesized primordial degrees of freedom that no longer interacts or very weakly interacts during late-time cosmic evolution, as metric tensor perturbations are. The only observational effect of an Inflation fossil might therefore be its imprint in the primordial curvature perturbation, as we have seen in the case of the squeezed limit of the $\zeta\zeta\gamma$ bispectrum. The tensor consistency relation written above allows to define a new correlation induced on the inflaton by the tensor degrees of freedom [70]

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \rangle|_{h_p(\mathbf{K})} = (2\pi)^3 \delta^{(3)}(\mathbf{K} + \mathbf{k}_1 + \mathbf{k}_2) f_p(\mathbf{k}_1, \mathbf{k}_2) h^*_p(\mathbf{K}) \epsilon^p_{ij}(\mathbf{K}) \hat{k}^i_1 \hat{k}^j_2.$$

In this Thesis we have described what are the physical observable effects of the presence of a primordial fossil field: we have described its effect on CMB observations and on the mass distribution of Large Scale Structures. In both the cases the presence of a tensor fossil would entail a quadrupole distortion. Observing this quadrupole would be a very interesting result since it could be a sign of the presence of a gravitational wave background, as predicted by the Inflation models.
We have then considered a new model which violates the standard scalar and tensor consistency relations: *Solid Inflation* [53]. In this model three scalar fields are responsible for Inflation. These three scalar fields have a very particular property: their background values are *time-independent* and *$\vec{x}$-dependent.* This means that the standard invariance under spatial translations and rotations are broken, while time translation is preserved. We have recalled the procedure according to which, adding internal symmetries on the scalar fields, homogeneity and isotropy can be restored for the background. This simple procedure entails very interesting outcomes, such as a blue tilt predicted for the tensor perturbations and a violation of the consistency relations described above. In Chapter 4 we have performed an original computation of the $\gamma\zeta\zeta$ bispectrum calculated in a general configuration of the three momenta, while in the literature such a computation has been done only in the squeezed limit. The result we obtained is

$$\begin{aligned}\mathcal{B}_{\gamma\zeta\zeta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) =& \frac{H^4}{4 M^4_{Pl} \epsilon^2 c^{10}_L} \frac{1}{k^3_1 k^3_2 k^3_3} \epsilon^s_{ij}(\mathbf{k}_1) \mathcal{M}_{ij}(\mathbf{k}_2, \mathbf{k}_3) \Big\{ K_t \left(4k^2_1 + \bar{k}^2_2 + \bar{k}^2_3 - k_1 \bar{k}_2 - \bar{k}_2 \bar{k}_3 - k_1 \bar{k}_3 \right) \\ &+ \frac{\bar{k}_2 \bar{k}_3}{K^2_t} \left[\bar{K}_t \left(3k_1 \bar{k}_2 + 3k_1 \bar{k}_2 + \bar{k}_2 \bar{k}_3 \right) + k_1 \bar{k}_2 \bar{k}_3 \right] + 3k^3_1 (\gamma_M + N_{K_t}) \Big\},\end{aligned}$$

where $\mathbf{k}_1$ is the momentum of the gravitational wave while $\mathbf{k}_2$ and $\mathbf{k}_3$ are those of the scalar perturbations. Here $\mathcal{M}_{ij}$ is a function of the parameters of the theory and of the momenta $\mathbf{k}_2$ and $\mathbf{k}_3$. We have then calculated the squeezed limit of this expression and we have used it to reobtain observable prediction in the tensor fossil sector for Solid Inflation [80].

Finally in the last Chapter of this Thesis we have considered a new model, a generalization of Solid Inflation: *Gaugid Inflation* [52]. In this model the fields which drive Inflation are three vector Abelian fields with a vacuum expectation value which manifestly breaks the spatial translations and rotations. It is a generalization of the Solid paradigm. The background considered in the original work [52] for the vector field $A_{i\mu}$ is of the *magnetic* type

$$\langle A_{i\mu} \rangle = \varepsilon_{ijk} \delta^j_\mu x^k,$$

where i is the *internal* (spatial) index and μ is a Lorentz index. We have reviewed the observable predictions of this theory using this particular choice of background. A noteworthy result of this theory is the appearance of a new tensor degree of freedom, E_{ij}, due to the gauge fields, besides the metric tensor perturbations γ_{ij}. This field could play the role of a tensor fossil. The standard predictions on the amplitude of the tensor power spectrum are

$$A^{SF}_\gamma \sim A^{Solid}_\gamma \sim \frac{H^2}{M^2_{Pl}},$$

while the presence of this new tensor field would enhance the tensor power spectrum amplitude, giving

$$A_\gamma \sim \frac{H^2}{M^2_{Pl}} \frac{1}{\epsilon_E c^5_E}.$$

Here ϵ_E and c_E are two parameters of the theory that might be lesser than 1, i.e. the factor $(\epsilon_E c^5_E)^{-1}$ enhances the standard results for the tensor power spectrum amplitude.
As an original contribution we have explored the possibility of adding at the background level an additional configuration, compatible with the hypothesis of the theory, i.e.

$$\langle A_{i\mu} \rangle = \delta^0_\mu x_i.$$

This in turns allows to also consider an additional parity-breaking term in the Lagrangian of this theory which might have new observable results in the gravitational wave sector. Using this generalization we have obtained two interesting results. At the quadratic action level a possibility of violating parity for the scalar perturbations of the theory arises due to the appearance of new coupling terms in the quadratic action.
Another intriguing result we have obtained concerns the gravitational waves: we have found a new coupling term between the gauge tensor perturbation E_{ij} and the metric one γ_{ij} of the form $\gamma_{ij}\dot{E}_{ij}$. The complete action for the tensor degrees of freedom leads to the system of equations of motion

$$A_1\gamma''_\pm + B_1\gamma_\pm \pm C_1 E_\pm + D_1 E'_\pm = 0,$$
$$A_2 E''_\pm + B_2 E_\pm \pm C_2\gamma_\pm + D_2\gamma'_\pm = 0,$$

where the coefficients A_i, B_i, C_i, D_i, with $i = 1, 2$ are, in general, time-dependent coefficients, and a prime denotes a derivative w.r.t. the conformal time. Thesis system of differential equations is not invariant under parity transformation for the two polarization states ($\pm$) of the tensor modes, as we have verified. This is a very interesting result that could open to new possibilities.
In any case our theoretical computations have been just explorative. In order to have an observable prediction we need to solve the system of equation presented above and find the power spectrum for the gravitational waves. As a possible future extension of this work one could find the third-order action in order to calculate the bispectra for this theory and see the possible predictions about tensor fossils using also the new tensor field E_{ij}.

Appendix A

Calculations for electro-magnetic Gaugid Inflation

A.1 Subhorizon limit

The expression for the vev in our generalized theory is

$$\langle A_{i\mu} \rangle = \varepsilon_{ijk}\delta^j_\mu x^k + \mathrm{g}\delta^0_\mu x_i, \tag{A.1}$$

so that we can write the perturbed field as

$$A_{i\mu} = \langle A_{i\mu} \rangle + a_{i\mu} = \varepsilon_{ijk}\delta^j_\mu x^k + \mathrm{g}\delta^0_\mu x_i + a_{i\mu}. \tag{A.2}$$

The perturbed field strength will be

$$F_{i\mu\nu} \equiv \partial_\mu A_{i\nu} - \partial_\nu A_{i\mu} = -2\varepsilon_{i\mu\nu} + \mathrm{g}(\delta^0_\nu \delta_{i\mu} - \delta^0_\mu \delta_{i\nu}) + f_{i\mu\nu}, \tag{A.3}$$

where we have defined

$$f_{i\mu\nu} \equiv \partial_\mu a_{i\nu} - \partial_\nu a_{i\mu}. \tag{A.4}$$

The dual of the field strength is

$$\tilde{F}^{\mu\nu}_i \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{i\rho\sigma} = -\varepsilon^{\mu\nu jk}\varepsilon_{ijk} + \mathrm{g}\varepsilon^{\mu\nu i0} + \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}f_{i\rho\sigma}. \tag{A.5}$$

Let us calculate the building blocks of our theory: we start with $Y_{ij} = F_{i\mu\nu}F^{\mu\nu}_j$. Here for the calculations we will use the Minkowski metric, given that we want to evaluate the perturbations dynamics in the subhorizon limit, in which we can neglect the coupling with gravity.
We obtain

$$\begin{aligned} Y_{ij} =& 2(4 - \mathrm{g}^2)\delta_{ij} \\ &- 2\varepsilon_{ilm}f_{jlm} - 2\varepsilon_{jlm}f_{ilm} - 2\mathrm{g}f_{ij0} - 2\mathrm{g}f_{ji0} - 2f_{ik0}f_{jk0} \\ &+ f_{ilm}f_{jlm}, \end{aligned} \tag{A.6}$$

while $\tilde{Y}_{ij} \equiv F_{i\mu\nu}\tilde{F}^{\mu\nu}_j$ is

$$\begin{aligned} \tilde{Y}_{ij} =& -8\mathrm{g}\delta_{ij} \\ &- 4f_{ij0} - 4f_{ji0} + \mathrm{g}\varepsilon_{ilm}f_{jlm} + \mathrm{g}\varepsilon_{jlm}f_{ilm} \\ &- \varepsilon_{lmn}\left(f_{i0l}f_{jmn} + f_{j0n}f_{ilm}\right). \end{aligned} \tag{A.7}$$

We calculate now the terms we are interested in in Eq. (5.8)

$$
\begin{aligned}
X \equiv [Y] =& 6(4-\mathrm{g}^2) \\
&- 4\varepsilon_{ijk} f_{ijk} - 4\mathrm{g} f_{ii0} \\
&+ f_{ijk} f_{ijk} - 2 f_{ij0} f_{ij0},
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
\tilde{X} \equiv [\tilde{Y}] =& -24\mathrm{g} \\
&- 8 f_{ii0} + 2\mathrm{g}\varepsilon_{ijk} f_{ijk} \\
&- 2\varepsilon_{lmn} f_{i0l} f_{imn}.
\end{aligned}
\tag{A.9}
$$

Let us consider I_2, defined as

$$
I_2 = \frac{[Y^2]}{X^2}.
\tag{A.10}
$$

We have

$$
\begin{aligned}
[Y^2] =& 12(4-\mathrm{g}^2)^2 \\
&- 16(4-\mathrm{g}^2)\varepsilon_{ijk} f_{ijk} - 16\mathrm{g}(4-\mathrm{g}^2) f_{ii0} \\
&- 16(2-\mathrm{g}^2) f_{ij0} f_{ij0} + 8\mathrm{g}^2 f_{ij0} f_{ji0} + 16\mathrm{g}(\varepsilon_{ilm} f_{jlm} f_{ij0} + \varepsilon_{ilm} f_{jlm} f_{ji0}) \\
&- 32 f_{iik} f_{jjk} + 4(12-\mathrm{g}^2) f_{ijk} f_{ijk},
\end{aligned}
\tag{A.11}
$$

and

$$
\begin{aligned}
X^2 =& 36(4-\mathrm{g}^2)^2 \\
&- 48(4-\mathrm{g}^2)\varepsilon_{ijk} f_{ijk} - 48\mathrm{g}(4-\mathrm{g}^2) f_{ii0} \\
&+ 4(20-3\mathrm{g}^2) f_{ijk} f_{ijk} - 64 f_{ijk} f_{jik} + 16\mathrm{g}^2 (f_{ii0})^2 - 24(4-\mathrm{g}^2) f_{ij0} f_{ij0} + 32\mathrm{g}\varepsilon_{ijk} f_{ijk} f_{ll0}.
\end{aligned}
\tag{A.12}
$$

We have to perform a perturbative expansion up to the second order, hence we have

$$
\begin{aligned}
X^{-2} = \frac{1}{36(4-\mathrm{g}^2)^2}\Big[1 &+ \frac{4\varepsilon_{ijk} f_{ijk}}{3(4-\mathrm{g}^2)} + \frac{4\mathrm{g} f_{ii0}}{3(4-\mathrm{g}^2)} \\
&+ \frac{(4+\mathrm{g}^2) f_{ijk} f_{ijk}}{3(4-\mathrm{g}^2)^2} + \frac{4\mathrm{g}^2 (f_{ii0})^2}{3(4-\mathrm{g}^2)^2} - \frac{16 f_{ijk} f_{jik}}{3(4-\mathrm{g}^2)^2} \\
&+ \frac{2 f_{ij0} f_{ij0}}{3(4-\mathrm{g}^2)} + \frac{8\mathrm{g}\varepsilon_{ijk} f_{ijk} f_{ll0}}{3(4-\mathrm{g}^2)^2}\Big]
\end{aligned}
\tag{A.13}
$$

The result for I_2 in the subhorizon limit is

$$
\begin{aligned}
I_2 = \frac{1}{36(4-\mathrm{g}^2)^2}\Big[12(4-\mathrm{g}^2)^2 &+ \frac{64}{3} f_{ijk} f_{ijk} - 32 f_{iik} f_{jjk} + \frac{64}{3} f_{ijk} f_{jik} \\
&- \frac{32}{3}\mathrm{g}\varepsilon_{ijk} f_{ijk} f_{ll0} + 16\mathrm{g}(\varepsilon_{ilm} f_{jlm} f_{ij0} + \varepsilon_{ilm} f_{jlm} f_{ji0}) \\
&- \frac{16}{3}\mathrm{g}^2 (f_{ii0})^2 + 8\mathrm{g}^2 f_{ij0} f_{ij0} + 8\mathrm{g}^2 f_{ij0} f_{ji0}\Big]
\end{aligned}
\tag{A.14}
$$

Now we calculate I_3, defined as

$$
I_3 = \frac{[\tilde{Y}]^2}{X^2}.
\tag{A.15}
$$

For the numerator we have

$$\begin{aligned}[\tilde{Y}^2] =&192\mathrm{g}^2 + 128\mathrm{g}f_{iio} - 32\mathrm{g}^2\varepsilon_{ijk}f_{ijk}\\ &+ 32f_{ij0}f_{ij0} + 32f_{ij0}f_{ji0}\\ &- 16\mathrm{g}\varepsilon_{ilm}f_{jlm}f_{ij0} - 48\mathrm{g}\varepsilon_{ilm}f_{jlm}f_{ji0} + 8\mathrm{g}^2f_{ijk}f_{ijk} - 8\mathrm{g}^2f_{iik}f_{jjk}\end{aligned} \tag{A.16}$$

Using (A.13) we obtain the expression for I_3 in the subhorizon limit

$$\begin{aligned}I_3 = \frac{1}{36(4-\mathrm{g}^2)^2}[192\mathrm{g}^2 &+ \frac{32\mathrm{g}^2(4+\mathrm{g}^2)}{4-\mathrm{g}^2}\varepsilon_{ijk}f_{ijk} + \frac{128\mathrm{g}^2(4+\mathrm{g}^2)}{4-\mathrm{g}^2}f_{ii0}\\ &+ \frac{8\mathrm{g}^2(3\mathrm{g}^4+32\mathrm{g}^2+16)}{3(4-\mathrm{g}^2)^2}f_{ijk}f_{ijk} - \frac{512\mathrm{g}^2(2+\mathrm{g}^2)}{3(4-\mathrm{g}^2)^2}f_{ijk}f_{jik}\\ &+ \frac{256\mathrm{g}^2(8+\mathrm{g}^2)}{3(4-\mathrm{g}^2)^2}(f_{ii0})^2 - 8\mathrm{g}^2f_{iik}f_{jjk}\\ &+ \frac{128\mathrm{g}(\mathrm{g}^4+4\mathrm{g}^2+16)}{3(4-\mathrm{g}^2)^2}\varepsilon_{ijk}f_{ijk}f_{ll0} - 48\mathrm{g}\varepsilon_{klm}f_{ik0}f_{ilm}\\ &- 16\mathrm{g}\varepsilon_{klm}f_{ki0}f_{ilm} + 32f_{ij0}f_{ji0} + \frac{32(4+3\mathrm{g}^2)}{(4-\mathrm{g}^2)}f_{ij0}f_{ij0}]\end{aligned} \tag{A.17}$$

We only need to calculate I_1, defined as

$$I_1 = \frac{[\tilde{Y}]}{X} \tag{A.18}$$

Note that this term was not calculated in the original article [52]. Let us first consider its denominator, using (A.8)

$$\begin{aligned}X^{-1} = \frac{1}{6(4-\mathrm{g}^2)}[1 &+ \frac{2\varepsilon_{ijk}f_{ijk}}{3(4-\mathrm{g}^2)} + \frac{2\mathrm{g}f_{ii0}}{3(4-\mathrm{g}^2)}\\ &+ \frac{3\mathrm{g}^2+4}{18(4-\mathrm{g}^2)^2}f_{ijk}f_{ijk} - \frac{16f_{ijk}f_{jik}}{9(4-\mathrm{g}^2)^2}\\ &+ \frac{f_{ij0}f_{ij0}}{3(4-\mathrm{g}^2)} + \frac{4\mathrm{g}^2(f_{ii0})^2}{9(4-\mathrm{g}^2)^2} + \frac{8\mathrm{g}\varepsilon_{ijk}f_{ijk}f_{ll0}}{9(4-\mathrm{g}^2)^2}],\end{aligned} \tag{A.19}$$

and for I_1 we have

$$\begin{aligned}I_1 = \frac{1}{6(4-\mathrm{g}^2)}[-24\mathrm{g} &- \frac{2\mathrm{g}(4+\mathrm{g}^2)}{(4-\mathrm{g}^2)}\varepsilon_{ijk}f_{ijk} - \frac{8(4+\mathrm{g}^2)}{4-\mathrm{g}^2}f_{ii0}\\ &+ \frac{4\mathrm{g}(4-5\mathrm{g}^2)}{3(4-\mathrm{g}^2)}f_{ijk}f_{ijk} + \frac{16\mathrm{g}(4+\mathrm{g}^2)}{3(4-\mathrm{g}^2)^2}f_{ijk}f_{jik}\\ &- \frac{8\mathrm{g}}{4-\mathrm{g}^2}f_{ij0}f_{ij0} - \frac{16\mathrm{g}(3\mathrm{g}^2-4)}{3(4-\mathrm{g}^2)^2}(f_{ii0})^2\\ &- \frac{4(4+\mathrm{g}^2)^2}{3(4-\mathrm{g}^2)}\varepsilon_{ijk}f_{ijk}f_{ll0} + 2\varepsilon_{ijk}f_{li0}f_{ljk}].\end{aligned} \tag{A.20}$$

The action we will use to generalize the results of Gaugid Inflation is:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - P(X) + \mathrm{s}M^4I_1 - 27M_1^4I_2 + 18M_2^4I_3\right]. \tag{A.21}$$

Here we have indicated with c the coefficient of the new term I_1 and with M^4 the energy scale at which the effect of this new term becomes interesting. The value of c can be chosen in order to simplify the calculations, but from now on we will always leave it a parameter of the theory.

Eq. (A.21) at the second order, using the Minkowski metric, becomes

$$\begin{aligned}
S^{(2)} = \int \mathrm{d}^4 x \Bigg\{ & -\frac{XP_X}{24} f_i^{\mu\nu} f_{i\mu\nu} + \frac{XP_X}{72}(\varepsilon_{ijk} f_{ijk})^2 + \frac{\mathrm{s}M^4}{6(4-\mathrm{g}^2)}\Bigg[\frac{4\mathrm{g}(4-5\mathrm{g}^2)}{3(4-\mathrm{g}^2)} f_{ijk}f_{ijk} \\
& + \frac{16\mathrm{g}(4+\mathrm{g}^2)}{3(4-\mathrm{g}^2)^2} f_{ijk}f_{jik} - \frac{8\mathrm{g}}{4-\mathrm{g}^2} f_{ij0}f_{ij0} - \frac{16\mathrm{g}(3\mathrm{g}^2-4)}{3(4-\mathrm{g}^2)^2}(f_{ii0})^2 + 2\varepsilon_{ijk} f_{li0} f_{ljk} \\
& - \frac{4(4+\mathrm{g}^2)^2}{3(4-\mathrm{g}^2)} \varepsilon_{ijk} f_{ijk} f_{ll0}\Bigg] - \frac{27M_1^4}{36(4-\mathrm{g}^2)^2}\Bigg[\frac{64}{3} f_{ijk}f_{ijk} - 32 f_{iik}f_{jjk} - \frac{32}{3}\mathrm{g}\varepsilon_{ijk} f_{ijk} f_{ll0} \\
& + \frac{64}{3} f_{ijk} f_{jik} + 16\mathrm{g}(\varepsilon_{ilm} f_{jlm} f_{ij0} + \varepsilon_{ilm} f_{jlm} f_{ji0}) - \frac{16}{3}\mathrm{g}^2 (f_{ii0})^2 + 8\mathrm{g}^2 f_{ij0}f_{ij0} + 8\mathrm{g}^2 f_{ij0} f_{ji0}\Bigg] \\
& + \frac{18M_2^4}{36(4-\mathrm{g}^2)}\Bigg[\frac{32\mathrm{g}^2(4+\mathrm{g}^2)}{4-\mathrm{g}^2}\varepsilon_{ijk} f_{ijk} + \frac{128\mathrm{g}^2(4+\mathrm{g}^2)}{4-\mathrm{g}^2} f_{ii0} - 8\mathrm{g}^2 f_{iik} f_{jjk} \\
& + \frac{8\mathrm{g}^2(3\mathrm{g}^4+32\mathrm{g}^2+16)}{3(4-\mathrm{g}^2)^2} f_{ijk}f_{ijk} - \frac{512\mathrm{g}^2(2+\mathrm{g}^2)}{3(4-\mathrm{g}^2)^2} f_{ijk} f_{jik} - 48\mathrm{g}\varepsilon_{klm} f_{ik0} f_{ilm} \\
& + \frac{256\mathrm{g}^2(8+\mathrm{g}^2)}{3(4-\mathrm{g}^2)^2}(f_{ii0})^2 + \frac{128\mathrm{g}(\mathrm{g}^4+4\mathrm{g}^2+16)}{3(4-\mathrm{g}^2)^2}\varepsilon_{ijk} f_{ijk} f_{ll0} \\
& - 16\mathrm{g}\varepsilon_{klm} f_{ki0} f_{ilm} + 32 f_{ij0} f_{ji0} + \frac{32(4+3\mathrm{g}^2)}{(4-\mathrm{g}^2)} f_{ij0} f_{ij0}\Bigg]\Bigg\}
\end{aligned} \tag{A.22}$$

Now we consider the decomposition of our perturbation field $a_{i\mu}$. In (5.30) and (5.32) we have seen that, with a particular gauge choice, we can write

$$a_{i0} = \frac{\partial_i \chi}{\sqrt{-\partial^2}} + B_i, \tag{A.23}$$

and

$$a_{ij} = \alpha\delta_{ij} + E_{ij} + \varepsilon\left(\frac{\partial_k T}{\sqrt{-\partial^2}} + V_k\right). \tag{A.24}$$

One can immediately note that in the expression for the field strength will never appear the time derivatives of the fields χ and B_i. This means that their time derivatives will never appear in the perturbed action (A.22). These fields will not be dynamical and their equations of motion will only be two constraint equations.
For the terms that appear in (A.22) we have

$$f_{ij0} f_{ij0} = (\partial_i B_j)^2 - 2\varepsilon_{ijk}\partial_j B_i \dot{V}_k - \chi\partial^2\chi - 2\dot{\alpha}\frac{\partial^2\chi}{\sqrt{-\partial^2}} + 3\dot{\alpha}^2 + (\dot{E}_{ij})^2 + 2(\dot{V}_k)^2 + 2\dot{T}^2 \tag{A.25}$$

$$f_{ij0} f_{ji0} = 2\varepsilon_{ijk}\partial_j B_i \dot{V}_k - \chi\partial^2\chi - 2\dot{\alpha}\frac{\partial^2\chi}{\sqrt{-\partial^2}} + 3\dot{\alpha}^2 + (\dot{E}_{ij})^2 - 2(\dot{V}_i)^2 - 2\dot{T}^2 \tag{A.26}$$

$$f_{iik} f_{jjk} = 4(\partial_i\alpha)^2 + (\partial_i V_j)^2 \tag{A.27}$$

$$f_{ijk} f_{ijk} = 4(\partial_i\alpha)^2 + 2(\partial E_{ij})^2 + 2(\partial_i V_j)^2 + 4(\partial_i T)^2 \tag{A.28}$$

$$f_{ijk}f_{jik} = 2(\partial_i\alpha)^2 + (\partial E_{ij})^2 + (\partial_i V_j)^2 - 2(\partial_i T)^2 \tag{A.29}$$

$$(f_{ii0})^2 = 9\dot{\alpha}^2 - 6\dot{\alpha}\frac{\partial^2\chi}{\sqrt{-\partial^2}} - \chi\partial^2\chi \tag{A.30}$$

$$\varepsilon_{ijk}f_{ijk}f_{ll0} = 4(T\partial^2\chi + 3\dot{\alpha}\frac{\partial^2 T}{\sqrt{-\partial^2}}) \tag{A.31}$$

$$\varepsilon_{ilm}f_{jlm}f_{ij0} = 4\alpha\frac{\partial^2\dot{T}}{\sqrt{-\partial^2}} - \chi\partial^2 T + 2\partial_i V_j\partial_i B_j \tag{A.32}$$

$$\varepsilon_{ilm}f_{jlm}f_{ji0} = \chi\partial^2 T \tag{A.33}$$

$$\tag{A.34}$$

If we substitute now all these expression back into the action we obtain the messy expression

$$\begin{aligned}
S^{(2)} = \int \mathrm{d}^4x\Bigg\{ &\left(\frac{XP_X}{4} - \frac{36\mathrm{g}^2}{(4-\mathrm{g}^2)^2}M_1^4 + \frac{96(4\mathrm{g}^4+33\mathrm{g}^2+4)}{(4-\mathrm{g}^2)^3}M_2^4 + \frac{4\mathrm{g}(4-5\mathrm{g}^2)}{(4-\mathrm{g}^2)^3}\mathrm{s}M^4\right)\dot{\alpha}^2 \\
&+\left(-\frac{XP_X}{6} + \frac{4\mathrm{g}^2(4\mathrm{g}^4+\mathrm{g}^2-68)}{3(4-\mathrm{g}^2)^4}M_2^4 + \frac{8\mathrm{g}(20-23\mathrm{g}^2+5\mathrm{g}^4)}{9(4-\mathrm{g}^2)^3}\mathrm{s}M^4\right)(\partial\alpha)^2 \\
&+\left(-\frac{XP_X}{6} - \frac{64(4\mathrm{g}^4+33\mathrm{g}^2+4)}{(4-\mathrm{g}^2)^3}M_2^4 - \frac{8\mathrm{g}(4-5\mathrm{g}^2)}{3(4-\mathrm{g}^2)^3}\mathrm{s}M^4\right)\dot{\alpha}\frac{\partial^2\chi}{\sqrt{-\partial^2}} \\
&+\left(\frac{72\mathrm{g}}{(4-\mathrm{g}^2)^2}M_1^4 + \frac{96\mathrm{g}(3\mathrm{g}^4+8\mathrm{g}^2+48)}{(4-\mathrm{g}^2)^4}M_2^4 - \frac{16(4+\mathrm{g}^2)^2}{(4-\mathrm{g}^2)^2}\mathrm{s}M^4\right)\dot{\alpha}\frac{\partial^2 T}{\sqrt{-\partial^2}} \\
&+\left(-\frac{XP_X}{12} + \frac{8\mathrm{g}^2M_1^4}{(4-\mathrm{g}^2)^2} + \frac{32(4\mathrm{g}^6+16\mathrm{g}^4-129\mathrm{g}^2-4)}{(4-\mathrm{g}^2)^3}M_2^4 + \frac{4\mathrm{g}(4+3\mathrm{g}^2)}{9(4-\mathrm{g}^2)^3}\mathrm{s}M^4\right)\chi\partial^2\chi \\
&+\left(\frac{32\mathrm{g}M_1^4}{(4-\mathrm{g}^2)^2} + \frac{16\mathrm{g}(13\mathrm{g}^4-88\mathrm{g}^2+208)}{3(4-\mathrm{g}^2)^4}M_2^4 - \frac{16(4+\mathrm{g}^2)^2}{(4-\mathrm{g}^2)^2}\mathrm{s}M^4\right)T\partial^2\chi \\
&+\left(\frac{XP_X}{6} + \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3}M_2^4 - \frac{8\mathrm{g}}{3(4-\mathrm{g}^2)^2}\mathrm{s}M^4\right)\dot{T}^2 \\
&+\left(\frac{XP_X}{18} - \frac{32M_1^4}{(4-\mathrm{g}^2)^2} + \frac{16\mathrm{g}^2(3\mathrm{g}^4+64\mathrm{g}^2+80)}{3(4-\mathrm{g}^2)^4}M_2^4 + \frac{2\mathrm{g}(96-56\mathrm{g}^2+5\mathrm{g}^4)}{9(4-\mathrm{g}^2)^3}\mathrm{s}M^4\right)(\partial T)^2 \\
&+\left(\frac{XP_X}{6} + \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3}M_2^4 - \frac{8\mathrm{g}}{3(4-\mathrm{g}^2)^2}\mathrm{s}M^4\right)(\dot{V}_k)^2 \\
&+\left(-\frac{XP_X}{12} - \frac{24M_1^4}{(4-\mathrm{g}^2)^2} + \frac{4\mathrm{g}^2(\mathrm{g}^4+8\mathrm{g}^2-48)}{(4-\mathrm{g}^2)^4}M_2^4 + \frac{\mathrm{g}(96-56\mathrm{g}^2+5\mathrm{g}^4)}{9(4-\mathrm{g}^2)^3}\mathrm{s}M^4\right)(\partial V_k)^2 \\
&+\left(-\frac{XP_X}{6} - \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3}M_2^4 + \frac{8\mathrm{g}}{3(4-\mathrm{g}^2)^2}\mathrm{s}M^4\right)\varepsilon_{ijk}\partial_j B_i\dot{V}_k - \frac{8\mathrm{g}(3M_1^4+2M_2^4)}{(4-\mathrm{g}^2)^2}\partial_i V_j\partial_i B_j \\
&+\left(\frac{XP_X}{12} - \frac{6\mathrm{g}^2}{(4-\mathrm{g}^2)^2}M_1^4 + \frac{16(4+3\mathrm{g}^2)}{(4-\mathrm{g}^2)^3}M_2^4 - \frac{4\mathrm{g}}{3(4-\mathrm{g}^2)^2}\mathrm{s}M^4\right)(\partial B_k)^2 \\
&+\left(\frac{XP_X}{12} - \frac{12\mathrm{g}^2}{(4-\mathrm{g}^2)^2}M_1^4 + \frac{32(4+\mathrm{g}^2)}{(4-\mathrm{g}^2)^3}M_2^4 - \frac{4\mathrm{g}}{3(4-\mathrm{g}^2)^2}\mathrm{s}M^4\right)\dot{E}_{ij}^2 \\
&+\left(-\frac{XP_X}{12} - \frac{48M_1^4}{(4-\mathrm{g}^2)^2} - \frac{8\mathrm{g}^2(\mathrm{g}^4+32\mathrm{g}^2+48)}{(4-\mathrm{g}^2)^4}M_2^4 + \frac{\mathrm{g}(96-56\mathrm{g}^2+5\mathrm{g}^4)}{9(4-\mathrm{g}^2)^3}\mathrm{s}M^4\right)(\partial E_{ij})^2\Bigg\}
\end{aligned} \tag{A.35}$$

Here we can see some new peculiarities. The presence of the new term for the vacuum expectation value and for the action translates into a new coupling term $\dot{\alpha}\partial^2 T$ between the two perturbative scalar degrees of freedom of this theory. This is exactly what we expected since α is an axial field, i.e. with odd parity $P_\alpha = -1$, while T is a polar field, i.e. with even parity $P_T = +1$. A coupling between these two fields would entail a possible parity breaking signature for the scalar perturbations, exactly what we were looking for. Moreover we see that the presence of I_1, traced back by the coefficient c, does not change the equation of motion for the tensor field, i.e. does not add any new term, but it just renormalizes the coefficients of $\dot{E}_{ij}^2$ and $(\partial E_{ij})^2$. For this reason, we will consider the limit c = 0 in order to simplify all the later calculations.
Now we go back to (A.35). Considering the case s = 0 we have

$$
\begin{aligned}
S^{(2)} = \int \mathrm{d}^4x \Bigg\{ & \left(\frac{XP_X}{4} - \frac{36\mathrm{g}^2}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{96(4\mathrm{g}^4 + 33\mathrm{g}^2 + 4)}{(4-\mathrm{g}^2)^3} M_2^4 \right) \dot{\alpha}^2 \\
& + \left(-\frac{XP_X}{6} + \frac{4\mathrm{g}^2(4\mathrm{g}^4 + \mathrm{g}^2 - 68)}{3(4-\mathrm{g}^2)^4} M_2^4 \right) (\partial\alpha)^2 + \left(-\frac{XP_X}{6} - \frac{64(4\mathrm{g}^4 + 33\mathrm{g}^2 + 4)}{(4-\mathrm{g}^2)^3} M_2^4 \right) \dot{\alpha} \frac{\partial^2 \chi}{\sqrt{-\partial^2}} \\
& + \left(\frac{72\mathrm{g}}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{96\mathrm{g}(3\mathrm{g}^4 + 8\mathrm{g}^2 + 48)}{(4-\mathrm{g}^2)^4} M_2^4 \right) \dot{\alpha} \frac{\partial^2 T}{\sqrt{-\partial^2}} \\
& + \left(-\frac{XP_X}{12} + \frac{8\mathrm{g}^2 M_1^4}{(4-\mathrm{g}^2)^2} + \frac{32(4\mathrm{g}^6 + 16\mathrm{g}^4 - 129\mathrm{g}^2 - 4)}{(4-\mathrm{g}^2)^3} M_2^4 \right) \chi \partial^2 \chi \\
& + \left(\frac{32\mathrm{g} M_1^4}{(4-\mathrm{g}^2)^2} + \frac{16\mathrm{g}(13\mathrm{g}^4 - 88\mathrm{g}^2 + 208)}{3(4-\mathrm{g}^2)^4} M_2^4 \right) T\partial^2\chi + \left(\frac{XP_X}{6} + \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3} M_2^4 \right) \dot{T}^2 \\
& + \left(\frac{XP_X}{18} - \frac{32 M_1^4}{(4-\mathrm{g}^2)^2} + \frac{16\mathrm{g}^2(3\mathrm{g}^4 + 64\mathrm{g}^2 + 80)}{3(4-\mathrm{g}^2)^4} M_2^4 \right) (\partial T)^2 \\
& + \left(\frac{XP_X}{6} + \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3} M_2^4 \right) \left(\dot{V}_k\right)^2 + \left(-\frac{XP_X}{12} - \frac{24 M_1^4}{(4-\mathrm{g}^2)^2} + \frac{4\mathrm{g}^2(\mathrm{g}^4 + 8\mathrm{g}^2 - 48)}{(4-\mathrm{g}^2)^4} M_2^4 \right) (\partial V_k)^2 \\
& + \left(-\frac{XP_X}{6} - \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3} M_2^4 \right) \varepsilon_{ijk} \partial_j B_i \dot{V}_k - \frac{8\mathrm{g}(3M_1^4 + 2M_2^4)}{(4-\mathrm{g}^2)^2} \partial_i V_j \partial_i B_j \\
& + \left(\frac{XP_X}{12} - \frac{6\mathrm{g}^2}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{16(4 + 3\mathrm{g}^2)}{(4-\mathrm{g}^2)^3} M_2^4 \right) (\partial B_k)^2 \\
& + \left(\frac{XP_X}{12} - \frac{12\mathrm{g}^2}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{32(4 + \mathrm{g}^2)}{(4-\mathrm{g}^2)^3} M_2^4 \right) \dot{E}_{ij}^2 \\
& + \left(-\frac{XP_X}{12} - \frac{48 M_1^4}{(4-\mathrm{g}^2)^2} - \frac{8\mathrm{g}^2(\mathrm{g}^4 + 32\mathrm{g}^2 + 48)}{(4-\mathrm{g}^2)^4} M_2^4 \right) (\partial E_{ij})^2 \Bigg\}.
\end{aligned}
\tag{A.36}
$$

If we calculate now the equations of motion for the non-dynamical degrees of freedom χ and B_i we can eliminate them from the action. For B_i we have

$$
\begin{aligned}
B_i =& \frac{96\mathrm{g}(4-\mathrm{g}^2)(3M_1^4 + 2M_2^4)}{XP_X(4-\mathrm{g}^2)^3 - 72\mathrm{g}^2(4-\mathrm{g}^2)M_1^4 + 192(4+3\mathrm{g}^2)M_2^4} V_i \\
&+ \frac{2XP_X(4-\mathrm{g}^2)^3 + 256\mathrm{g}^2 M_2^4}{XP_X(4-\mathrm{g}^2)^3 - 72\mathrm{g}^2(4-\mathrm{g}^2)M_1^4 + 192(4+3\mathrm{g}^2)M_2^4} \varepsilon_{ijk} \partial^{-2} \partial_j \dot{V}_k \\
\equiv& \mathcal{G} V_i + \mathcal{N} \varepsilon_{ijk} \partial^{-2} \partial_j \dot{V}_k,
\end{aligned}
\tag{A.37}
$$

where we have defined

$$\begin{aligned}
\mathcal{G} &= \frac{96\mathrm{g}(4-\mathrm{g}^2)(3M_1^4+2M_2^4)}{XP_X(4-\mathrm{g}^2)^3-72\mathrm{g}^2(4-\mathrm{g}^2)M_1^4+192(4+3\mathrm{g}^2)M_2^4},\\
\mathcal{N} &= \frac{2XP_X(4-\mathrm{g}^2)^3+256\mathrm{g}^2M_2^4}{XP_X(4-\mathrm{g}^2)^3-72\mathrm{g}^2(4-\mathrm{g}^2)M_1^4+192(4+3\mathrm{g}^2)M_2^4}.
\end{aligned} \tag{A.38}$$

For χ we obtain

$$\begin{aligned}
\chi = &-\left[\frac{2XP_X(4-\mathrm{g}^2)^3+768(4-\mathrm{g}^2)(4\mathrm{g}^4+33\mathrm{g}^2+4)M_2^4}{XP_X(4-\mathrm{g}^2)^3-8\mathrm{g}^2(4-\mathrm{g}^2)M_1^4-32(4\mathrm{g}^6+16\mathrm{g}^4-129\mathrm{g}^2-4)M_2^4}\right]\frac{\dot\alpha}{\sqrt{-\partial^2}}\\
&+4\mathrm{g}\left[\frac{32(4-\mathrm{g}^2)^2M_1^4+16(13\mathrm{g}^4-88\mathrm{g}^2+208)M_2^4}{XP_X(4-\mathrm{g}^2)^4-96\mathrm{g}^2(4-\mathrm{g}^2)^2M_1^4-256(4-\mathrm{g}^2)(4\mathrm{g}^6+16\mathrm{g}^4-129\mathrm{g}^2-4)M_2^4}\right]T\\
\equiv&\mathcal{H}\frac{\dot\alpha}{\sqrt{-\partial^2}}+\mathcal{S}T,
\end{aligned} \tag{A.39}$$

where

$$\begin{aligned}
\mathcal{H} &= -\frac{2XP_X(4-\mathrm{g}^2)^3+768(4-\mathrm{g}^2)(4\mathrm{g}^4+33\mathrm{g}^2+4)M_2^4}{XP_X(4-\mathrm{g}^2)^3-8\mathrm{g}^2(4-\mathrm{g}^2)M_1^4-32(4\mathrm{g}^6+16\mathrm{g}^4-129\mathrm{g}^2-4)M_2^4},\\
\mathcal{S} &= \frac{4\mathrm{g}\left[32(4-\mathrm{g}^2)^2M_1^4+16(13\mathrm{g}^4-88\mathrm{g}^2+208)M_2^4\right]}{XP_X(4-\mathrm{g}^2)^4-96\mathrm{g}^2(4-\mathrm{g}^2)^2M_1^4-256(4-\mathrm{g}^2)(4\mathrm{g}^6+16\mathrm{g}^4-129\mathrm{g}^2-4)M_2^4}.
\end{aligned} \tag{A.40}$$

For the scalar modes we have

$$\begin{aligned}
S^{(2)} = \int \mathrm{d}^4x\Bigg\{&\Bigg[\Bigg[\frac{XP_X}{4}-\frac{36\mathrm{g}^2}{(4-\mathrm{g}^2)^2}M_1^4+\frac{96(4\mathrm{g}^4+33\mathrm{g}^2+4)}{(4-\mathrm{g}^2)^3}M_2^4+\left(\frac{XP_X}{6}+\frac{64(4\mathrm{g}^4+33\mathrm{g}^2+4)}{(4-\mathrm{g}^2)^3}M_2^4\right)\mathcal{H}\\
&+\left(\frac{XP_X}{12}-\frac{8\mathrm{g}^2M_1^4}{(4-\mathrm{g}^2)^2}-\frac{32(4\mathrm{g}^6+16\mathrm{g}^4-129\mathrm{g}^2-4)}{(4-\mathrm{g}^2)^3}M_2^4\right)\mathcal{H}^2\Bigg]\dot\alpha^2\\
&+\left(-\frac{XP_X}{6}+\frac{4\mathrm{g}^2(4\mathrm{g}^4+\mathrm{g}^2-68)}{3(4-\mathrm{g}^2)^4}M_2^4\right)(\partial\alpha)^2\\
&+\Bigg[\frac{72\mathrm{g}}{(4-\mathrm{g}^2)^2}M_1^4+\frac{96\mathrm{g}(3\mathrm{g}^4+8\mathrm{g}^2+48)}{(4-\mathrm{g}^2)^4}M_2^4\\
&+\left(\frac{72\mathrm{g}}{(4-\mathrm{g}^2)^2}M_1^4+\frac{96\mathrm{g}(3\mathrm{g}^4+8\mathrm{g}^2+48)}{(4-\mathrm{g}^2)^4}M_2^4\right)\mathcal{S}\\
&+2\left(-\frac{XP_X}{12}+\frac{8\mathrm{g}^2M_1^4}{(4-\mathrm{g}^2)^2}+\frac{32(4\mathrm{g}^6+16\mathrm{g}^4-129\mathrm{g}^2-4)}{(4-\mathrm{g}^2)^3}M_2^4\right)\mathcal{R}\mathcal{S}\\
&+\left(\frac{32\mathrm{g}M_1^4}{(4-\mathrm{g}^2)^2}+\frac{16\mathrm{g}(13\mathrm{g}^4-88\mathrm{g}^2+208)}{3(4-\mathrm{g}^2)^4}M_2^4\right)\mathcal{H}\Bigg]\dot\alpha\frac{\partial^2T}{\sqrt{-\partial^2}}\\
&+\left(\frac{XP_X}{6}+\frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3}M_2^4\right)\dot T^2\\
&+\Bigg[\frac{XP_X}{18}-\frac{32M_1^4}{(4-\mathrm{g}^2)^2}+\frac{16\mathrm{g}^2(3\mathrm{g}^4+64\mathrm{g}^2+80)}{3(4-\mathrm{g}^2)^4}M_2^4\\
&+\left(\frac{XP_X}{12}-\frac{8\mathrm{g}^2M_1^4}{(4-\mathrm{g}^2)^2}-\frac{32(4\mathrm{g}^6+16\mathrm{g}^4-129\mathrm{g}^2-4)}{(4-\mathrm{g}^2)^3}M_2^4\right)\mathcal{S}^2\\
&-\left(\frac{32\mathrm{g}M_1^4}{(4-\mathrm{g}^2)^2}+\frac{16\mathrm{g}(13\mathrm{g}^4-88\mathrm{g}^2+208)}{3(4-\mathrm{g}^2)^4}M_2^4\right)\mathcal{S}\Bigg](\partial T)^2\Bigg\},
\end{aligned} \tag{A.41}$$

while for vector modes

$$
\begin{aligned}
S_V^{(2)} = \int d^4x \Bigg\{ &+ \Bigg[\frac{XP_X}{6} + \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3} M_2^4 \\
&+ \left(\frac{XP_X}{12} - \frac{6\mathrm{g}^2}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{16(4+3\mathrm{g}^2)}{(4-\mathrm{g}^2)^3} M_2^4 \right) \mathcal{N}^2 \Bigg] \left(\dot{V}_k\right)^2 \\
&+ \Bigg[-\frac{XP_X}{12} - \frac{24M_1^4}{(4-\mathrm{g}^2)^2} + \frac{4\mathrm{g}^2(\mathrm{g}^4+8\mathrm{g}^2-48)}{(4-\mathrm{g}^2)^4} M_2^4 \\
&+ \left(-\frac{XP_X}{6} - \frac{128\mathrm{g}^2}{(4-\mathrm{g}^2)^3} M_2^4 \right) \mathcal{N} - \frac{8\mathrm{g}(3M_1^4+2M_2^4)}{(4-\mathrm{g}^2)^2} \mathcal{G} \\
&+ \left(\frac{XP_X}{12} - \frac{6\mathrm{g}^2}{(4-\mathrm{g}^2)^2} M_1^4 + \frac{16(4+3\mathrm{g}^2)}{(4-\mathrm{g}^2)^3} M_2^4 \right) \mathcal{G}^2 \Bigg] (\partial V_k)^2 \Bigg\}.
\end{aligned}
\tag{A.42}
$$

The action for E_{ij} remains the same since, at the linear order, it is cannot be coupled to a scalar or a vector degree of freedom.
Now we can calculate the speed of propagation of the various mode, but the only two we can calculate are the vector and the tensor ones, since for the scalars we have a two coupled equations

$$
\begin{aligned}
c_V^2 = \big\{ &XP_X(4-\mathrm{g}^2)^3 \left[1+2\mathcal{N}-\mathcal{G}^2\right] \\
&+ 48\left[\mathrm{g}^2(\mathrm{g}^2+12) + 32\mathrm{g}^2\mathcal{N} + 4(4-\mathrm{g}^2)\mathcal{G} - 4(4+3\mathrm{g}^2)\mathcal{G}^2\right] M_2^4 \\
&+ 72(4-\mathrm{g}^2)\left[(4-\mathrm{g}^2+4\mathcal{G}+\mathrm{g}^2\mathcal{G}^2)\right] M_1^4 \big\} / \big\{ XP_X(4-\mathrm{g}^2)^3[2+\mathcal{N}^2] \\
&- 72\mathrm{g}^2(4-\mathrm{g}^2)\mathcal{N}^2 M_1^4 + 192[8\mathrm{g}^2 + (4+3\mathrm{g}^2)\mathcal{N}^2] M_2^4 \big\}.
\end{aligned}
\tag{A.43}
$$

These two expressions are very messy . It is possible that inserting a specific value for g we would get more useful expressions in order to write the subluminality and stability condition for our theory.

$$
c_E^2 = \frac{XP_X(4-\mathrm{g}^2)^4 + 576(4-\mathrm{g}^2)^2 M_1^4 + 96\mathrm{g}^2(\mathrm{g}^4+32\mathrm{g}^2+48) M_2^4}{XP_X(4-\mathrm{g}^2)^4 - 144\mathrm{g}^2(4-\mathrm{g}^2)^2 M_1^4 + 384(4-\mathrm{g}^2)(4+\mathrm{g}^2) M_2^4}
\tag{A.44}
$$

A.2 Mixing with gravity

Here we go to the superhorizon limit, the most interesting one, and hence we need to consider also the gravity. We will repeat the same calculations of the previous section but using a curved metric in the usual ADM formalism, see (5.52). The covariant components for the field strength $F_{i\mu\nu}$ are

$$
\begin{aligned}
F_{i00} &= 0, \\
F_{i0j} &= -\mathrm{g}\delta_{ij} + f_{i0j}, \\
F_{ij0} &= \mathrm{g}\delta_{ij} + f_{ij0}, \\
F_{ijk} &= -2\varepsilon_{ijk} + f_{ijk},
\end{aligned}
\tag{A.45}
$$

while the contravariant ones $F_i^{\mu\nu}$ are

$$\begin{aligned}
F_i^{00} =&0,\\
F_i^{0j} =&\frac{\mathrm{g}}{N^2}\delta_i^j - \frac{h^{jk} f_{i0k}}{N^2} - 2\frac{\varepsilon_{ikl} h^{jl} N^k}{N^2} + \frac{N^k h^{jl} f_{ikl}}{N^2},\\
F_i^{jk} =&-\frac{\mathrm{g}}{N^2}\left(N^j\delta_i^k - N^k\delta_i^j\right) - \frac{f_{il0}}{N^2}\left(N^j h^{kl} - N^k h^{jl}\right)\\
&-2\varepsilon_{ilm}h^{km}h^{jl} + h^{jl}h^{km}f_{ilm} + \frac{2}{N^2}\left(\varepsilon_{ilm}N^k N^m h^{jl} + \varepsilon_{ilm}N^j N^l h^{km}\right)\\
&-\frac{f_{ilm}}{N^2}\left(N^k N^m h^{jl} + N^j N^l h^{km}\right).
\end{aligned} \tag{A.46}$$

We consider the spatially flat gauge (5.53) and (5.54). Given that we are at the linear order and we are interested only in the tensor perturbations γ_{ij} and E_{ij}, we can also consider $N = 1$ and $N_i = N^i = 0$, and we obtain, for Y_{ij}

$$\begin{aligned}
Y_{ij} =&2a^{-2}(4a^{-2} - \mathrm{g}^2)\delta_{ij}\\
&+2a^{-2}(4a^{-2} + \mathrm{g}^2)\gamma_{ij}\\
&+2\mathrm{g}a^{-2}(f_{i0j} + f_{j0i}) - 2a^{-4}(\varepsilon_{ilk}f_{jlk} + \varepsilon_{jlk}f_{ilk})\\
&+2\mathrm{g}a^{-2}(\gamma_j^l f_{i0l} + \gamma_i^k f_{j0k}) + 4a^{-4}(\varepsilon_{inm}\gamma^{ms}f_{jns} + \varepsilon_{jnm}\gamma^{ms}f_{ins})\\
&-4a^{-4}(\delta_{ij}\gamma_{lk}\gamma^{lk} - 2\gamma_{ik}\gamma_j^k) - 2a^{-2}f_{ik0}f_{jk0} + a^{-4}f_{ilm}f_{jlm}.
\end{aligned} \tag{A.47}$$

Taking the trace of (A.47)

$$\begin{aligned}
X = [Y] =&6a^{-2}(4a^{-2} - \mathrm{g}^2)\\
&-4\mathrm{g}a^{-2}f_{ii0} - 4a^{-4}\varepsilon_{ijk}f_{ijk}\\
&-4\mathrm{g}a^{-2}\gamma^{ij}f_{ij0} + 8a^{-4}\varepsilon_{ijk}\gamma^{kl}f_{ijl} - 4a^{-4}\gamma_{ij}\gamma^{ij}\\
&+a^{-4}f_{ijk}f_{ijk} - 2a^{-2}f_{ij0}f_{ij0},
\end{aligned} \tag{A.48}$$

and for $[Y^2]$ we have

$$\begin{aligned}
[Y^2] =&12a^{-4}(4a^{-2} - \mathrm{g}^2)^2\\
&-16\mathrm{g}a^{-4}(4a^{-2} - \mathrm{g}^2)f_{ii0} - 16a^{-6}(4a^{-2} - \mathrm{g}^2)\varepsilon_{ijk}f_{ijk}\\
&-128\mathrm{g}a^{-6}\gamma^{ij}f_{ij0} + 32a^{-6}(4a^{-2} - \mathrm{g}^2)\varepsilon_{imn}\gamma^{ms}f_{ins}\\
&-16a^{-6}(4a^{-2} + \mathrm{g}^2)\varepsilon_{ilk}\gamma^{ij}f_{jlk} + 4a^{-4}\mathrm{g}^2(12a^{-2} + \mathrm{g}^2)\gamma_{ij}\gamma^{ij}\\
&-16a^{-4}(2a^{-2} - \mathrm{g}^2)f_{ij0}f_{ij0} + 8\mathrm{g}^2a^{-4}f_{ij0}f_{ji0} + 8\mathrm{g}a^{-6}(\varepsilon_{ilk}f_{jlk}f_{ij0} + \varepsilon_{ilk}f_{jlk}f_{ji0})\\
&+4a^{-6}(12a^{-2} - \mathrm{g}^2)f_{ijk}f_{ijk} - 32a^{-8}f_{iik}f_{jjk}.
\end{aligned} \tag{A.49}$$

For the dual components $\tilde{F}_i^{\mu\nu}$ we have

$$\tilde{F}_i^{00} = 0, \tag{A.50}$$

$$\tilde{F}_i^{0j} = -2\delta_i^j + \frac{1}{2}\varepsilon^{jlk}f_{ilk}, \tag{A.51}$$

$$\tilde{F}_i^{j0} = -\tilde{F}_i^{0j}, \tag{A.52}$$

$$\tilde{F}_i^{jk} = -\mathrm{g}\varepsilon_i{}^{jk} + \varepsilon^{jkl}f_{i0l}, \tag{A.53}$$

which lead to

$$
\begin{aligned}
\tilde{Y}_{ij} =& - 8\mathrm{g}a^{-4}(\delta_{ij} + \gamma_{ij}) \\
&+ \mathrm{g}a^{-4}\left(\varepsilon_{ilm}f_{jlm} + \varepsilon_{jlm}f_{ilm}\right) - 4a^{-6}\left(f_{ij0} + f_{ji0}\right) \\
&- 4\mathrm{g}a^{-4}\delta_{ij}\gamma_{lm}\gamma^{lm} - 8\mathrm{g}a^{-4}\gamma_{im}\gamma_j^m - \mathrm{g}a^{-4}\left(\varepsilon_{ils}\gamma^{ms}f_{jlm} + \varepsilon_{jks}\gamma^{ls}f_{ikl}\right) \\
&+ 4a^{-6}(\gamma_i^k f_{jk0} + \gamma_j^k f_{ik0}) + a^{-6}\left(\varepsilon_{klm}f_{ik0}f_{jlm} + \varepsilon_{klm}f_{jk0}f_{ilm}\right).
\end{aligned}
\tag{A.54}
$$

Taking the trace of (A.54) we have

$$
\begin{aligned}
[\tilde{Y}] \equiv \tilde{X} =& - 24\mathrm{g}a^{-4} \\
&+ 2\mathrm{g}a^{-4}\varepsilon_{ijk}f_{ijk} - 8a^{-6}f_{ii0} \\
&- 20\mathrm{g}a^{-4}\gamma_{ij}\gamma^{ij} - 2\mathrm{g}a^{-4}\varepsilon_{ils}\gamma^{ms}f_{ilm} + 8a^{-6}\gamma^{ij}f_{ij0} \\
&+ 2a^{-6}\varepsilon_{klm}f_{ik0}f_{ilm},
\end{aligned}
\tag{A.55}
$$

while for $[\tilde{Y}^2]$ we obtain

$$
\begin{aligned}
[\tilde{Y}^2] =& 192\mathrm{g}^2a^{-8} - 32\mathrm{g}^2a^{-8}\varepsilon_{ijk}f_{ijk} + 128\mathrm{g}a^{-10}f_{ii0} + 256\mathrm{g}^2a^{-8}\gamma_{ij}\gamma^{ij} \\
&+ 32\mathrm{g}^2a^{-8}\varepsilon_{ils}\gamma^{ms}f_{ilm} - 32\mathrm{g}^2a^{-8}\varepsilon_{ilm}\gamma^{ij}f_{jlm} \\
&+ 32a^{-12}f_{ij0}f_{ij0} + 32a^{-12}f_{ij0}f_{ji0} + 8\mathrm{g}^2a^{-8}(f_{ijk}f_{ijk} - f_{iik}f_{jjk}) \\
&- 48\mathrm{g}a^{-10}\varepsilon_{ilm}f_{jlm}f_{ji0} - 16\mathrm{g}a^{-10}\varepsilon_{ilm}f_{jlm}f_{ij0}.
\end{aligned}
\tag{A.56}
$$

In order to calculate I_2 e I_3 we have to compute X^2 and then take its inverse

$$
\begin{aligned}
X^2 =& 36a^{-4}(4a^{-2} - \mathrm{g}^2)^2 \\
&- 48\mathrm{g}a^{-4}(4a^{-2} - \mathrm{g}^2)f_{ii0} - 48a^{-6}(4a^{-2} - \mathrm{g}^2)\varepsilon_{ijk}f_{ijk} \\
&+ 96a^{-6}(4a^{-2} - \mathrm{g}^2)\varepsilon_{ijk}\gamma^{kl}f_{ijl} - 48\mathrm{g}a^{-4}(4a^{-2} - \mathrm{g}^2)\gamma^{ij}f_{ij0} - 48a^{-6}(4a^{-2} - \mathrm{g}^2)\gamma_{ij}\gamma^{ij} \\
&+ 16\mathrm{g}^2a^{-4}(f_{ii0})^2 - 64a^{-8}f_{ijk}f_{jik} + 4a^{-6}(20a^{-2} - 3\mathrm{g}^2)f_{ijk}f_{ijk} \\
&- 24a^{-4}(4a^{-2} - \mathrm{g}^2)f_{ij0}f_{ij0} + 32\mathrm{g}a^{-6}\varepsilon_{ijk}f_{ijk}f_{ll0},
\end{aligned}
\tag{A.57}
$$

which gives, at the second perturbative order

$$
\begin{aligned}
X^{-2} =& \frac{1}{A^2}\Bigg[1 + \frac{4\mathrm{g}f_{ii0}}{3(4a^{-2} - \mathrm{g}^2)} + \frac{4a^{-2}\varepsilon_{ijk}f_{ijk}}{3(4a^{-2} - \mathrm{g}^2)} - \frac{8a^{-2}\varepsilon_{ijk}\gamma^{kl}f_{ijl}}{3(4a^{-2} - \mathrm{g}^2)} \\
&+ \frac{4\mathrm{g}\gamma^{ij}f_{ij0}}{3(4a^{-2} - \mathrm{g}^2)} + \frac{4a^{-2}\gamma^{ij}\gamma_{ij}}{3(4a^{-2} - \mathrm{g}^2)} + \frac{4\mathrm{g}^2(f_{ii0})^2}{3(4a^{-2} - \mathrm{g}^2)^2} - \frac{16a^{-4}f_{ijk}f_{jik}}{3(4a^{-2} - \mathrm{g}^2)^2} \\
&+ \frac{a^{-2}(4a^{-2} + \mathrm{g}^2)}{3(4a^{-2} - \mathrm{g}^2)^2}f_{ijk}f_{ijk} + \frac{2f_{ij0}f_{ij0}}{3(4a^{-2} - \mathrm{g}^2)} + \frac{8\mathrm{g}a^{-2}\varepsilon_{ijk}f_{ijk}f_{ll0}}{3(4a^{-2} - \mathrm{g}^2)^2}\Bigg],
\end{aligned}
\tag{A.58}
$$

where we have defined

$$
A = 6a^{-2}(4a^{-2} - \mathrm{g}^2).
$$

Using (A.49) and (A.58) we can write I_2

$$I_2 \equiv \frac{[Y^2]}{X^2} = \frac{1}{A^2}\Bigg[12a^{-4}(4a^{-2}-\mathrm{g}^2)^2 - 16\mathrm{g}a^{-4}(4a^{-2}+\mathrm{g}^2)\gamma^{ij}f_{ij0}$$
$$-16a^{-6}(4a^{-2}+\mathrm{g}^2)\varepsilon_{ilk}\gamma^{ij}f_{jlk} + 4a^{-4}(4a^{-2}+\mathrm{g}^2)^2\gamma_{ij}\gamma^{ij} - \frac{16}{3}\mathrm{g}^2a^{-4}(f_{ii0})^2$$
$$-\frac{32}{3}\mathrm{g}a^{-6}\varepsilon_{ijk}f_{ijk}f_{ll0} + 8\mathrm{g}^2a^{-4}f_{ij0}f_{ij0} + 8\mathrm{g}^2a^{-4}f_{ij0}f_{ji0}$$
$$+8\mathrm{g}a^{-6}(\varepsilon_{ilm}f_{jlm}f_{ij0} + \varepsilon_{ilm}f_{jlm}f_{ji0}) - 32a^{-8}f_{iik}f_{jjk} + \frac{64}{3}a^{-8}f_{ijk}f_{ijk} + \frac{64}{3}f_{ijk}f_{jik}\Bigg]. \tag{A.59}$$

Note that imposing g = 0 we obtain the same expression of [52].
Using (A.56) and (A.58) we obtain I_3

$$I_3 = \frac{1}{A^2}\Bigg\{192\mathrm{g}^2a^{-8} + \frac{128\mathrm{g}a^{-8}[\mathrm{g}^2(2-a^{-2})+4a^{-4}]}{4a^{-2}-\mathrm{g}^2}f_{ii0} + \frac{32\mathrm{g}^2a^{-8}(4+\mathrm{g}^2)}{4a^{-2}-\mathrm{g}^2}\varepsilon_{ijk}f_{ijk}$$
$$-32\mathrm{g}^2a^{-8}\varepsilon_{ilm}\gamma^{ij}f_{jlm} + \frac{32\mathrm{g}^2a^{-8}(20a^{-2}-\mathrm{g}^2)}{4a^{-2}-\mathrm{g}^2}\varepsilon_{ijk}\gamma^{kl}f_{ijl} + \frac{256\mathrm{g}^3a^{-8}}{4a^{-2}-\mathrm{g}^2}\gamma^{ij}f_{ij0}$$
$$+\frac{256\mathrm{g}^2a^{-8}(5a^{-2}-\mathrm{g}^2)}{4a^{-2}-\mathrm{g}^2}\gamma_{ij}\gamma^{ij} + \frac{256\mathrm{g}^2a^{-8}(\mathrm{g}^2+8a^{-2})}{3(4a^{-2}-\mathrm{g}^2)}(f_{ii0})^2 + 32a^{-10}f_{ij0}f_{ji0}$$
$$+\frac{32a^{-6}[4a^{-6}+(4-a^{-4})\mathrm{g}^2]}{4a^{-2}-\mathrm{g}^2}f_{ij0}f_{ij0} + \frac{128\mathrm{g}a^{-8}[\mathrm{g}^4+4\mathrm{g}^2a^{-2}(2-a^{-2})+16]}{3(4a^{-2}-\mathrm{g}^2)^2}\varepsilon_{ijk}f_{ijk}f_{ll0}$$
$$-8\mathrm{g}^2a^{-8}f_{iik}f_{jjk} - 48\mathrm{g}a^{-10}\varepsilon_{ilm}f_{jlm}f_{ji0} - 16\mathrm{g}a^{-10}\varepsilon_{ilm}f_{jlm}f_{ij0}$$
$$-\frac{512\mathrm{g}^2a^{-10}(\mathrm{g}^2+2a^{-2})}{3(4a^{-2}-\mathrm{g}^2)^2}f_{ijk}f_{jik} + \frac{8\mathrm{g}^2a^{-8}(3\mathrm{g}^4+32\mathrm{g}^2a^{-2}+16a^{-4})}{3(4a^{-2}-\mathrm{g}^2)^2}f_{ijk}f_{ijk}\Bigg\}. \tag{A.60}$$

Again one can verify that imposing g = 0 one obtains the same result of the original article.
We will use this comparison as a check for the correctness of our calculations.
We only need to calculate the new term I_1. In order to compute it we calculate X^{-1}, i.e.

$$X^{-1} = \frac{1}{A}\Bigg[1 + \frac{2\mathrm{g}f_{ii0}}{3(4a^{-2}-\mathrm{g}^2)} + \frac{2a^{-2}\varepsilon_{ijk}f_{ijk}}{3(4a^{-2}-\mathrm{g}^2)} + \frac{2\mathrm{g}\gamma^{ij}f_{ij0}}{3(4a^{-2}-\mathrm{g}^2)} - \frac{4a^{-2}\varepsilon_{ijk}\gamma^{kl}f_{ijl}}{3(4a^{-2}-\mathrm{g}^2)}$$
$$+\frac{2a^{-2}\gamma_{ij}\gamma^{ij}}{3(4a^{-2}-\mathrm{g}^2)} + \frac{a^{-2}(4a^{-2}+3\mathrm{g}^2)}{18(4a^{-2}-\mathrm{g}^2)^2}f_{ijk}f_{ijk} - \frac{16a^{-4}f_{ijk}f_{jik}}{9(4a^{-2}-\mathrm{g}^2)^2} \tag{A.61}$$
$$+\frac{f_{ij0}f_{ij0}}{3(4a^{-2}-\mathrm{g}^2)} + \frac{4\mathrm{g}^2(f_{ii0})^2}{9(4a^{-2}-\mathrm{g}^2)^2} + \frac{8\mathrm{g}a^{-2}\varepsilon_{ijk}f_{ijk}f_{ll0}}{9(4a^{-2}-\mathrm{g}^2)^2}\Bigg].$$

Finally, using (A.55) and (A.61) we can write down the expression for I_1 in the superhorizon limit

$$I_1 = \frac{1}{A}\Bigg[- 24\mathrm{g}a^{-4} - \frac{8a^{-4}(2\mathrm{g}^2 - \mathrm{g}^2a^{-2} + 4a^{-4})}{4^{-2} - \mathrm{g}^2} f_{ii0} - \frac{2\mathrm{g}a^{-4}(4a^{-2} + \mathrm{g}^2)}{4a^{-2} - \mathrm{g}^2}\varepsilon_{ijk}f_{ijk}$$
$$- \frac{8a^{-4}(2\mathrm{g}^2 + \mathrm{g}^2a^{-2} - 4a^{-4})}{4a^{-2} - \mathrm{g}^2}\gamma^{ij}f_{ij0} + \frac{2\mathrm{g}a^{-4}(12a^{-2} + \mathrm{g}^2)}{4a^{-2} - \mathrm{g}^2}\varepsilon_{ijk}\gamma^{kl}f_{ijl}$$
$$- \frac{4\mathrm{g}a^{-4}(24a^{-2} - 5\mathrm{g}^2)}{4a^{-2} - \mathrm{g}^2}\gamma_{ij}\gamma^{ij} - \frac{8\mathrm{g}a^{-4}}{4a^{-2} - \mathrm{g}^2} f_{ij0}f_{ij0} - \frac{16\mathrm{g}a^{-4}(2\mathrm{g}^2 + \mathrm{g}^2a^{-2} - 4a^{-4})}{3(4a^{-2} - \mathrm{g}^2)^2}(f_{ii0})^2$$
$$- \frac{4a^{-2}(\mathrm{g}^4 + 12\mathrm{g}^2a^{-2} - 4a^{-4}\mathrm{g}^2 + 16a^{-6})}{3(4a^{-2} - \mathrm{g}^2)^2}\varepsilon_{ijk}f_{ijk}f_{ll0} - 2a^{-6}\varepsilon_{klm}f_{ik0}f_{ilm}$$
$$+ \frac{16\mathrm{g}a^{-6}(4a^{-2} + \mathrm{g}^2)}{3(4a^{-2} - \mathrm{g}^2)^2} f_{ijk}f_{jik} + \frac{4\mathrm{g}a^{-6}(4a^{-2} - 5\mathrm{g}^2)}{3(4a^{-2} - \mathrm{g}^2)^2} f_{ijk}f_{ijk}\Bigg] \tag{A.62}$$

Now we can write, for the sake of completeness, the perturbed action up to the second order when considering the mixing between the gaugid and gravity, considering only the tensor degree of freedom

$$S^{(2)} = \int \mathrm{d}^4x\sqrt{-g}\Bigg\{\frac{M_{Pl}^2}{2}R - P(X) + \frac{\mathrm{s}M^4}{6a^{-2}(4a^{-2} - \mathrm{g}^2)}\Bigg[- \frac{8a^{-4}(2\mathrm{g}^2 + \mathrm{g}^2a^{-2} - 4a^{-4})}{4a^{-2} - \mathrm{g}^2}\gamma^{ij}f_{ij0}$$
$$+ \frac{2\mathrm{g}a^{-4}(12a^{-2} + \mathrm{g}^2)}{4a^{-2} - \mathrm{g}^2}\varepsilon_{ijk}\gamma^{kl}f_{ijl} - \frac{4\mathrm{g}a^{-4}(24a^{-2} - 5\mathrm{g}^2)}{4a^{-2} - \mathrm{g}^2}\gamma_{ij}\gamma^{ij} - \frac{8\mathrm{g}a^{-4}}{4a^{-2} - \mathrm{g}^2}f_{ij0}f_{ij0}$$
$$- \frac{16\mathrm{g}a^{-4}(2\mathrm{g}^2 + \mathrm{g}^2a^{-2} - 4a^{-4})}{3(4a^{-2} - \mathrm{g}^2)^2}(f_{ii0})^2 - \frac{4a^{-2}(\mathrm{g}^4 + 12\mathrm{g}^2a^{-2} - 4a^{-4}\mathrm{g}^2 + 16a^{-6})}{3(4a^{-2} - \mathrm{g}^2)^2}\varepsilon_{ijk}f_{ijk}f_{ll0}$$
$$- 2a^{-6}\varepsilon_{klm}f_{ik0}f_{ilm} + \frac{16\mathrm{g}a^{-6}(4a^{-2} + \mathrm{g}^2)}{3(4a^{-2} - \mathrm{g}^2)^2}f_{ijk}f_{jik} + \frac{4\mathrm{g}a^{-6}(4a^{-2} - 5\mathrm{g}^2)}{3(4a^{-2} - \mathrm{g}^2)^2}f_{ijk}f_{ijk}\Bigg]$$
$$- \frac{27M_1^4}{A^2}\Bigg[- 16\mathrm{g}a^{-4}(4a^{-2} + \mathrm{g}^2)\gamma^{ij}f_{ij0} - 16a^{-6}(4a^{-2} + \mathrm{g}^2)\varepsilon_{ilk}\gamma^{ij}f_{jlk}$$
$$+ 4a^{-4}(4a^{-2} + \mathrm{g}^2)^2\gamma_{ij}\gamma^{ij} - \frac{16}{3}\mathrm{g}^2a^{-4}(f_{ii0})^2 - \frac{32}{3}\mathrm{g}a^{-6}\varepsilon_{ijk}f_{ijk}f_{ll0}$$
$$+ 8\mathrm{g}^2a^{-4}f_{ij0}f_{ij0} + 8\mathrm{g}^2a^{-4}f_{ij0}f_{ji0} + 8\mathrm{g}a^{-6}(\varepsilon_{ilm}f_{jlm}f_{ij0} + \varepsilon_{ilm}f_{jlm}f_{ji0})$$
$$- 32a^{-8}f_{iik}f_{jjk} + \frac{64}{3}a^{-8}f_{ijk}f_{ijk} + \frac{64}{3}a^{-8}f_{ijk}f_{jik}\Bigg]$$
$$+ \frac{18M_2^4}{A^2}\Bigg[- 32\mathrm{g}^2a^{-8}\varepsilon_{ilm}\gamma^{ij}f_{jlm} + \frac{32\mathrm{g}^2a^{-8}(20a^{-2} - \mathrm{g}^2)}{4a^{-2} - \mathrm{g}^2}\varepsilon_{ijk}\gamma^{kl}f_{ijl} + \frac{256\mathrm{g}^3a^{-8}}{4a^{-2} - \mathrm{g}^2}\gamma^{ij}f_{ij0}$$
$$+ \frac{256\mathrm{g}^2a^{-8}(5a^{-2} - \mathrm{g}^2)}{4a^{-2} - \mathrm{g}^2}\gamma_{ij}\gamma^{ij} + \frac{256\mathrm{g}^2a^{-8}(\mathrm{g}^2 + 8a^{-2})}{3(4a^{-2} - \mathrm{g}^2)}(f_{ii0})^2$$
$$+ 32a^{-10}f_{ij0}f_{ji0} + \frac{32a^{-6}[4a^{-6} + (4 - a^{-4})\mathrm{g}^2]}{4a^{-2} - \mathrm{g}^2}f_{ij0}f_{ij0}$$
$$+ \frac{128\mathrm{g}a^{-8}[\mathrm{g}^4 + 4\mathrm{g}^2a^{-2}(2 - a^{-2}) + 16]}{3(4a^{-2} - \mathrm{g}^2)^2}\varepsilon_{ijk}f_{ijk}f_{ll0}$$
$$- 8\mathrm{g}^2a^{-8}f_{iik}f_{jjk} - 48\mathrm{g}a^{-10}\varepsilon_{ilm}f_{jlm}f_{ji0} - 16\mathrm{g}a^{-10}\varepsilon_{ilm}f_{jlm}f_{ij0}$$
$$- \frac{512\mathrm{g}^2a^{-10}(\mathrm{g}^2 + 2a^{-2})}{3(4a^{-2} - \mathrm{g}^2)^2}f_{ijk}f_{jik} + \frac{8\mathrm{g}^2a^{-8}(3\mathrm{g}^4 + 32\mathrm{g}^2a^{-2} + 16a^{-4})}{3(4a^{-2} - \mathrm{g}^2)^2}f_{ijk}f_{ijk}\Bigg]\Bigg\} \tag{A.63}$$

If we consider only the gaugid and the metric gravitational waves, the only non zero terms for $f_{i\mu\nu}$ are

$$\gamma^{ij} f_{ij0} = -\gamma_{ij}\dot{E}_{ij}, \tag{A.64}$$

$$\varepsilon_{ilm}\gamma^{ij} f_{jlm} = 2\varepsilon_{ilm}\gamma_{ij}\partial_l E_{jm}, \tag{A.65}$$

$$\varepsilon_{ijk}\gamma^{kl} f_{ijl} = \varepsilon_{ijk}\gamma_{kl}\partial_j E_{il}, \tag{A.66}$$

$$f_{ij0}f_{ij0} = \dot{E}_{ij}^2, \tag{A.67}$$

$$f_{ijk}f_{ijk} = 2(\partial E_{ij})^2, \tag{A.68}$$

$$f_{ijk}f_{jik} = (\partial E_{ij})^2. \tag{A.69}$$

We have

$$\begin{aligned}
S^{(2)} = \int \mathrm{d}^4 x\sqrt{-g}\Bigg\{ & \frac{M_{Pl}^2}{2}R - P(X) \\
&+ \frac{\mathrm{s}M^4}{6a^{-2}(4a^{-2}-\mathrm{g}^2)}\Bigg[\frac{8a^{-4}(2\mathrm{g}^2+\mathrm{g}^2a^{-2}-4a^{-4})}{4a^{-2}-\mathrm{g}^2}\gamma^{ij}\dot{E}_{ij} \\
&- \frac{4\mathrm{g}a^{-4}(24a^{-2}-5\mathrm{g}^2)}{4a^{-2}-\mathrm{g}^2}\gamma_{ij}\gamma^{ij} - \frac{8\mathrm{g}a^{-4}}{4a^{-2}-\mathrm{g}^2}\dot{E}_{ij}^2 \\
&+ \frac{16\mathrm{g}a^{-6}(4a^{-2}+\mathrm{g}^2)}{3(4a^{-2}-\mathrm{g}^2)^2}(\partial E_{ij})^2 + \frac{8\mathrm{g}a^{-6}(4a^{-2}-5\mathrm{g}^2)}{3(4a^{-2}-\mathrm{g}^2)^2}(\partial E_{ij})^2\Bigg] \\
&- \frac{27M_1^4}{A}\Bigg[16\mathrm{g}a^{-4}(4a^{-2}+\mathrm{g}^2)\gamma^{ij}\dot{E}_{ij} + 4a^{-4}(4a^{-2}+\mathrm{g}^2)^2\gamma_{ij}\gamma^{ij} + 8\mathrm{g}^2a^{-4}\dot{E}_{ij}^2 \\
&+ 8\mathrm{g}^2a^{-4}\dot{E}_{ij}^2 + \frac{128}{3}a^{-8}(\partial E_{ij})^2 + \frac{64}{3}a^{-8}(\partial E_{ij})^2 - 32a^{-6}(4a^{-2}+\mathrm{g}^2)\varepsilon_{ilm}\gamma_{ij}\partial_l E_{jm})\Bigg] \\
&+ \frac{18M_2^4}{A}\Bigg[64\mathrm{g}^2a^{-8}\varepsilon_{ijk}\gamma^{kl}\partial_j E_{il} + \frac{32\mathrm{g}^2a^{-8}(20a^{-2}-\mathrm{g}^2)}{4a^{-2}-\mathrm{g}^2}\varepsilon_{ijk}\gamma^{kl}\partial_j E_{il} - \frac{256\mathrm{g}^3a^{-8}}{4a^{-2}-\mathrm{g}^2}\gamma^{ij}\dot{E}_{ij} \\
&+ \frac{256\mathrm{g}^2a^{-8}(5a^{-2}-\mathrm{g}^2)}{4a^{-2}-\mathrm{g}^2}\gamma_{ij}\gamma^{ij} + 32a^{-10}\dot{E}_{ij}^2 + \frac{32a^{-6}[4a^{-6}+(4-a^{-4})\mathrm{g}^2]}{4a^{-2}-\mathrm{g}^2}\dot{E}_{ij}^2 \\
&- \frac{512\mathrm{g}^2a^{-10}(\mathrm{g}^2+2a^{-2})}{3(4a^{-2}-\mathrm{g}^2)^2}(\partial E_{ij})^2 + \frac{16\mathrm{g}^2a^{-8}(3\mathrm{g}^4+32\mathrm{g}^2a^{-2}+16a^{-4})}{3(4a^{-2}-\mathrm{g}^2)^2}(\partial E_{ij})^2\Bigg]\Bigg\},
\end{aligned} \tag{A.70}$$

or

$$\begin{aligned}
S_{GW}^{(2)} = \int \mathrm{d}^4 x\sqrt{-g}\Bigg\{ & \frac{M_{Pl}^2}{2}R - P(X) \\
&+ \Bigg(-\frac{4\mathrm{sg}a^{-2}M^4}{3(4a^{-2}-\mathrm{g}^2)^2} - \frac{12\mathrm{g}^2M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} + \frac{32a^{-2}(4a^{-6}-\mathrm{g}^2a^{-4}+2\mathrm{g}^2)M_2^4}{(4a^{-2}-\mathrm{g}^2)^3}\Bigg)\dot{E}_{ij}^2 \\
&+ \Bigg(\frac{8\mathrm{sg}a^{-4}(6a^{-2}-19\mathrm{g}^2)M^4}{9(4a^{-2}-\mathrm{g}^2)^2} - \frac{48a^{-4}M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} + \frac{8\mathrm{g}^2a^{-4}(3\mathrm{g}^4+16a^{-4}\mathrm{g}^2-64a^{-4})M_2^4}{3(4a^{-2}-\mathrm{g}^2)^4}\Bigg)(\partial E_{ij})^2 \\
&+ \Bigg(-\frac{2\mathrm{sg}a^{-2}(24a^{-2}-5\mathrm{g}^2)M^4}{3(4a^{-2}-\mathrm{g}^2)^2} - \frac{3(4a^{-2}+\mathrm{g}^2)^2M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} + \frac{128\mathrm{g}^2a^{-4}(5a^{-2}-\mathrm{g}^2)M_2^4}{(4a^{-2}-\mathrm{g}^2)^2}\Bigg)\gamma_{ij}\gamma_{ij} \\
&+ \Bigg(\frac{24a^{-2}(4a^{-2}+\mathrm{g}^2)M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} + \frac{16\mathrm{g}^2a^{-4}(28a^{-2}-3\mathrm{g}^2)M_2^4}{(4a^{-2}-\mathrm{g}^2)^3}\Bigg)\varepsilon_{ijk}\gamma^{kl}\partial_j E_{il} \\
&+ \Bigg(\frac{4\mathrm{s}a^{-2}(2\mathrm{g}^2+\mathrm{g}^2a^{-2}-4a^{-4})M^4}{3(4a^{-2}-\mathrm{g}^2)^2} - \frac{12\mathrm{g}(4a^{-2}+\mathrm{g}^2)M_1^4}{(4a^{-2}-\mathrm{g}^2)^2} - \frac{128\mathrm{g}^3a^{-4}M_2^4}{(4a^{-2}-\mathrm{g}^2)^2}\Bigg)\gamma_{ij}\dot{E}_{ij}\Bigg\}.
\end{aligned} \tag{A.71}$$

We perturb now $P(X)$. For X we have

$$X = \bar{X} + \delta X,$$

where

$$\bar{X} = 6a^{-2}(4a^{-2} - \mathrm{g}^2),$$

and

$$\begin{aligned}\delta X = &- 4\mathrm{g}a^{-2} f_{ii0} - 4a^{-4}\varepsilon_{ijk} f_{ijk} \\ &- 4\mathrm{g}a^{-2}\gamma^{ij} f_{ij0} + 8a^{-4}\varepsilon_{ijk}\gamma^{kl} f_{ijl} - 4a^{-4}\gamma_{ij}\gamma^{ij} \\ &+ a^{-4} f_{ijk} f_{ijk} - 2a^{-2} f_{ij0} f_{ij0}.\end{aligned}$$

Hence we can expand $P(X)$

$$P(X) = P(\bar{X} + \delta X) = P(\bar{X}) + P_X \delta X + \frac{1}{2} P_{XX} \delta X^2$$

We can immediately note an important thing. Here we are treating the case of the gravitational waves, hence it is enough to calculate the contribution due to this term for the gravitational waves equation of motion. We see that the term δX^2 appears but it gives no contributions for the gravitational waves at this order. Indeed we have

$$\delta X^2 = (-4\mathrm{g}a^{-2} f_{ii0} - 4a^{-4}\varepsilon_{ijk} f_{ijk})^2 + O(f^4, (\gamma f)^2, \gamma^2),$$

i.e.

$$\delta X^2 = 16\mathrm{g}^2 a^{-4} (f_{ii0})^2 + 14a^{-8}(\varepsilon_{ijk} f_{ijk})^2 + 32\mathrm{g}a^{-6}\varepsilon_{ijk} f_{ijk} f_{ll0}.$$

Using the relations

$$(\varepsilon_{ijk} f_{ijk})^2 = 2 f_{ijk} f_{ijk} - 4 f_{ijk} f_{jik} \tag{A.72}$$

$$\varepsilon_{ilm}\varepsilon_{ikn} f_{jlm} f_{jkn} = 2 f_{ijk} f_{ijk} \tag{A.73}$$

$$\varepsilon_{ilm}\varepsilon_{jkn} f_{jlm} f_{ikn} = 2 f_{ijk} f_{ijk} - 4 f_{iik} f_{jjk} \tag{A.74}$$

we can see that any term of δX^2 gives a contribution for the tensors.
We can then write the first term of the series expansion as

$$P_X \delta X = X P_X \frac{\delta X}{X} = X P_X \frac{\delta X}{X_0}(1 - \frac{\delta X}{X_0}) = X P_X \frac{\delta X}{X_0} + \beta \delta X^2 = X P_X \frac{\delta X}{X_0},$$

hence

$$P(X)^{(2)} = \frac{X P_X}{6a^{-2}(4a^{-2} - \mathrm{g}^2)}(4\mathrm{g}a^{-2}\gamma^{ij}\dot{E}_{ij} + 8a^{-4}\varepsilon_{ijk}\gamma_{kl}\partial_j E_{il} - 4a^{-4}\gamma_{ij}\gamma^{ij} + 2a^{-4}(\partial E_{ij})^2 - 2a^{-2}\dot{E}_{ij}^2).$$

Finally, in order to simplify our calculations, if we do not consider, for the reasons mentioned in the previous section, the corrective terms due to I_1 (i.e. s = 0) we obtain

$$
\begin{aligned}
S^{(2)}_{GW} = \int \mathrm{d}^4\, x\sqrt{-g}\Bigg\{ & \frac{M^2_{Pl}}{8}\left(\dot{\gamma}^2_{ij} - a^{-2}\left(\partial\gamma_{ij}\right)^2\right) \\
& + \left(- \frac{XP_X}{3(4a^{-2} - \mathrm{g}^2)} - \frac{12\mathrm{g}^2 M_1^4}{(4a^{-2} - \mathrm{g}^2)^2} + \frac{32a^{-2}(4a^{-6} - \mathrm{g}^2 a^{-4} + 2\mathrm{g}^2)M_2^4}{(4a^{-2} - \mathrm{g}^2)^3}\right)\dot{E}^2_{ij} \\
& + \left(\frac{a^{-2}XP_X}{3(4a^{-2} - \mathrm{g}^2)} - \frac{48a^{-4}M_1^4}{(4a^{-2} - \mathrm{g}^2)^2} + \frac{8\mathrm{g}^2 a^{-4}(3\mathrm{g}^4 + 16a^{-4}\mathrm{g}^2 - 64a^{-4})M_2^4}{3(4a^{-2} - \mathrm{g}^2)^4}\right)(\partial E_{ij})^2 \\
& + \left(- 2\frac{a^{-2}XP_X}{3(4a^{-2} - \mathrm{g}^2)} - \frac{3(4a^{-2} + \mathrm{g}^2)^2 M_1^4}{(4a^{-2} - \mathrm{g}^2)^2} + \frac{128\mathrm{g}^2 a^{-4}(5a^{-2} - \mathrm{g}^2)M_2^4}{(4a^{-2} - \mathrm{g}^2)^2}\right)\gamma_{ij}\gamma_{ij} \\
& + \left(4\frac{a^{-2}XP_X}{3(4a^{-2} - \mathrm{g}^2)} + \frac{24a^{-2}(4a^{-2} + \mathrm{g}^2)M_1^4}{(4a^{-2} - \mathrm{g}^2)^2} + \frac{16\mathrm{g}^2 a^{-4}(28a^{-2} - 3\mathrm{g}^2)M_2^4}{(4a^{-2} - \mathrm{g}^2)^3}\right)\varepsilon_{ijk}\gamma^{kl}\partial_j E_{il} \\
& + \left(2\mathrm{g}\frac{XP_X}{3(4a^{-2} - \mathrm{g}^2)} - \frac{12\mathrm{g}(4a^{-2} + \mathrm{g}^2)M_1^4}{(4a^{-2} - \mathrm{g}^2)^2} - \frac{128\mathrm{g}^3 a^{-4}M_2^4}{(4a^{-2} - \mathrm{g}^2)^2}\right)\gamma_{ij}\dot{E}_{ij}\Bigg\}.
\end{aligned}
\tag{A.75}
$$

Here we see that we have a new term, $\gamma_{ij}\dot{E}_{ij}$ due to the choice of a parity-breaking vacuum. This term is exactly what we was looking for: it violates the parity since it mixes the different helicity states equations of motion.

Bibliography

[1] E. W. Kolb, M. S. Turner, *The Early Universe*, Addison-Wesley publishing company, 1988.

[2] S. Dodelson, *Modern Cosmology*, Academic Press, Amsterdam, 2003.

[3] P. Coles, F. Lucchin, *Cosmology. The Origin and Evolution of the Cosmic Structure*, John Wiley & sons, 2002.

[4] S. Weinberg, *Cosmology*, Oxford, UK: Oxford Univ. Pr. (2008).

[5] S. Weinberg,*Gravitation and Cosmology*, John Wiley and Sons, Canada, 1972, ISBN 0-471-92567-5.

[6] Planck Collaboration, P. A. R. Ade et al., *Planck 2015 results. XIII. Cosmological parameters*, (2015), 1502.01589v3.

[7] Planck Collaboration, N. Aghanim et al., *Planck 2018 results. VI. Cosmological parameters*, (2018).

[8] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*,W. H. Freeman, San Francisco, 1973.

[9] R. A. Alpher, H. Bethe and G. Gamow, *The Origin of Chemical Elements*, Phys. Rev. D 73 (1948) 803.

[10] A. A. Penzias and R. W. Wilson, *A Measurement of Excess Antenna Temperature at 4080 Mc/s*, ApJ. 142 (1965) 419.

[11] Planck Collaboration, Y. Akrami at al., *Planck 2018 results. I. Overview, and the cosmological legacy of Planck*, 2018, [arXiv:1807.06205v1].

[12] E. W. Kolb, *Cosmology and the Unexpected*, ArXiv.0709.3102 (2007)

[13] J. R. Primack, *The Nature of Dark Matter*, [arXiv:astro-ph/0112255].

[14] G. F. Smoot et al., *Structure in the COBE differential Microwave Radiometer first-year maps*, ApJ, 396, L1(1992).

[15] D. Baumann, *TASI Lectures on Inflation*, arXiv:0907.5424 [hep-th].

[16] Planck Collaboration, P. A. R. Ade et al., *Planck 2015 results. XVI. Isotropy and statistics of the CMB* (2015), 1506.07135v2.

[17] A. H. Guth, *Inflationary Universe: A possible solution to the horizon and flatness problems* Phys. Rev. D23 (1981) 347.

[18] D. Kazanas, *Dynamics of the Universe and spontaneous symmetry breaking*, Astrophys. J. Lett. 241, L59 (1980).

[19] A. R. Liddle and D. H. Lyth, *The primordial density perturbation*, Cambridge University Press, 2009.

[20] D. J. Fixsen, *The temperature of the cosmic microwave background*, 2009, ApJ, 707, 916.

[21] Planck Collaboration, Y. Akrami et al., *Planck 2018 results. X. Constraints on inflation*, 2018.

[22] R. Brout, F. Englert, and E. Gunzig, *The creation of the Universe as a quantum phenomenon*, Ann. Phys. (N.Y.) 115, 78 (1978).

[23] A. A. Starobinsky, *A new type of isotropic cosmological models without singularity*, Phys. Lett. B 91, 99 (1980).

[24] K. Sato, *First-order phase transition of vacuum and the expansion of the Universe*, Mon. Not. R. Astron. Soc. 195, 467 (1981).

[25] A. R. Linde, *A new inflationary Universe scenario: a possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems*, Phys. Lett. B108, 389 (1982).

[26] A. Albrecht and P. J. Steinhardt, *Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking*, Phys. Rev. Lett. 48, 1220 (1982).

[27] N. D. Birrel, P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics, 1982.

[28] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, Cambridge University Press, June, 2000.

[29] M. C. Guzzetti, N. Bartolo, M. Liguori and S. Matarrese, *Gravitational waves from inflation* Riv. Nuovo Cim. 39 (2016) 399 495.

[30] D. H. D. H. Lyth and A. A. Riotto, *Particle Physics Models of Inflation and the Cosmological Density Perturbation*, Phys. Rept. 314 (June, 1999) 1146.

[31] A. R. Liddle, P. Parsons and J. D. Barrow, *Formalising the Slow-Roll Approximation in Inflation*, Phys. Rev. D50 (1994) 7222.

[32] X. Chen, *Primordial Non-Gaussianities from Inflation Models* Adv. Astron. 2010 (2010) 638979.

[33] R. M. Wald, *General Relativity*, Chicago, University of Chicago Press, 1984.

[34] R. M. Wald, *Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant*, Phys. Rev. D 28, 2118 (1983).

[35] A. Maleknejad, M. M. Sheikh-Jabbari, *Revisiting Cosmic No-Hair Theorem for Inflationary Settings*, Phys. Rev. D 85, 123508 (2012).

[36] Planck Collaboration, P. A. R. Ade et al., *Planck 2015 results. I. Overview of products and scientific results*, (2015), 1502.01582v2.

[37] P. J. E. Peebles, *The Large-Scale Structure of the Universe* (Princeton University Press, New Jersey, 1972), ISBN 08137-9.

[38] K. A. Malik, D. R. Matravers, *A Concise Introduction to Perturbation Theory in Cosmology.* arXiv:0804.3276, 2008.

[39] K. A. Malik and D. Wands, *Cosmological perturbations*, Phys. Rept. 475, 1 (2009).

[40] J. M. Bardeen, *Gauge-invariant cosmological perturbations*, Phys. Rev. D22 (1980) 1882.

[41] S. Matarrese, S. Mollerach, M. Bruni, *Relativistic second-order perturbations of the Einstein-de Sitter Universe*, Phys. Rev. D58 (1998).

[42] V. Acquaviva, N. Bartolo, S. Matarrese and A. Riotto, *Second-Order Cosmological Perturbations from Inflation*, Nucl. Phys. B667 (2003) 119, astro-ph/0209156.

[43] N. Bartolo, E. Komatsu, S. Matarrese, A. Riotto, *Non-Gaussianity from Inflation, Theory and Observations*, 2004, Phys. Rept., 402, 103.

[44] T. S. Bunch and P. C. W. Davies, *Quantum field theory in de Sitter space: renormalization by point-splitting* Proc. Roy. Soc. A 360 (1978) 117.

[45] M. Sasaki, Prog. Theor. Phys., *Large Scale Quantum Fluctuations in the Inflationary Universe*, 76 (1986) 1036.

[46] A. A. Starobinskii, *Spectrum of relict gravitational and the early state of the Universe*, JETP Lett. 30 (1979) 682, [Pisma Zh. Eksp. Teor. Fiz.30,719(1979)].

[47] M. S. Wang, *Primordial Gravitational Waves from Cosmic Inflation*, 2017.

[48] Planck Collaboration, P. A. R. Ade et al., *Planck 2015 results. XX. Constraints on inflation*, (2017), 1502.02114v2.

[49] D. Baumann *et al.*, *CMBPol Mission Concept Study: Probing Inflation with CMB Polarization*, arXiv:0811.3919.

[50] BICEP2 Collaboration, P. Ade et al., *BICEP2 I: Detection of B-mode Polarization at Degree Angular Scales*, arXiv:1403.3985.

[51] R. K. Sachs, A. M. Wolfe, *Perturbations of a Cosmological Model and Angular Variations of the Microwave Background*, 1967, ApJ, 147, 73.

[52] F. Piazza, D. Pirtskhalava, R. Rattazzi, and O. Simon, *Gaugid Inflation* JCAP 1711 no. 11,(2017) 041.

[53] S. Endlich, A. Nicolis, J. Wang, *Solid Inflation* 2013, J. Cosmol. Astropart. Phys., 10, 11.

[54] C. Armendariz-Picon, T. Damour, V. Mukhanov, *k-inflation*, Phys. Lett. B 458, (1999) 209.

[55] E. Komatsu et al., *Non-Gaussianity as a Probe of the Physics of the Primordial Universe and the Astrophysics of the Low Redshift Universe*, (2009), 0902.4759.

[56] J. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, JHEP 05 (2003) 013.

[57] Planck Collaboration, P. A. R. Ade et al., *Planck 2015 results. XVII. Constraints on primordial non-Gaussianity*, (2015), 1502.01592v2.

[58] Planck Collaboration, P. A. R. Ade et al., *Planck 2013 results. XXIV. Constraints on primordial non-Gaussianity*, (2013), 1303.5084v2.

[59] P. Creminelli, M. Zaldarriaga, *Single field consistency relation for the 3-point function*, JCAP, 0410, 006 (2004).

[60] D. Babich, M. Zaldarriaga, P. Creminelli, *The shape of non-Gaussianities*, M. 2004, JCAP, 0408, 009.

[61] S. Weinberg, *Quantum Contributions to Cosmological Correlations*, Phys. Rev. D72, 043514 (2005), hep-th/0506236.

[62] R. Arnowitt, S. Deser, C. W. Misner, *The Dynamics of General Relativity* arXiv:gr-qc/0405109, 2004.

[63] E. Dimastrogiovanni, M. Fasiello, D. Jeong, M. Kamionkowski, *Inflationary tensor fossils in large-scale structure* (2015).

[64] D. S. Salopek and J. R. Bond, *Nonlinear evolution of long wavelength metric fluctuations in inflationary models*, Phys. Rev. D42 (1990) 3936-3962.

[65] A. Gruzinov, *Consistency relation for single scalar inflation*, Phys. Rev. D 71, 027301 (2005).

[66] X. Chen, H. Firouzjahi, M. H. Namjoo, and M. Sasaki, *A Single Field Inflation Model with Large Local Non-Gaussianity*, (2013), 1301.5699.

[67] L. Dai, D. Jeong, M. Kamionkowski, *Anisotropic imprint of long-wavelength tensor perturbation on cosmic structure*, Phys. Rev. D 88 (2013) 4, 043507.

[68] K. W. Masui and U.-L. Pen, *Primordial gravity waves fossils and their use in testing inflation*, Phys. Rev. Lett. 105, 161302 (2010).

[69] L. Dai, D. Jeong and M. Kamionkowski, *Seeking Inflation Fossils in the Cosmic Microwave Background*, Phys. Rev. D 87, no. 10, 103006 (2013)

[70] D. Jeong and M. Kamionkowski, *Clustering Fossils from the Early Universe*, Phys. Rev. Lett. 108 (25), 2012.

[71] L. Dai, M. Kamionkowski, D. Jeong, *Total Angular Momentum Waves for Scalar, Vector, and Tensor Fields*, Phys. Rev. D 86 (2012) 125013.

[72] L. Dai, D. Jeong and M. Kamionkowski, *Wigner-Eckart theorem in cosmology: Bispectra for total-angular-momentum waves*, [arXiv:1211.6110].

[73] A. Hajian, T. Souradeep and N. J. Cornish, *Statistical Isotropy of the Wilkinson Microwave Anisotropy Probe Data: A Bipolar Power Spectrum Analysis*, Astrophys. J. 618, L63 (2004).

[74] D. Jeong, F. Schmidt and C. M. Hirata, *Large-scale clustering of galaxies in general relativity*, Phys. Rev. D **85**, 023504 (2012).

[75] D. Jeong and F. Schmidt, *Large-Scale Structure with Gravitational Waves I: Galaxy Clustering*, Phys. Rev. D 86, 083512 (2012).

[76] F. Schmidt, D. Jeong, *Large-Scale Structure with Gravitational Waves II: Shear*, Phys. Rev. D 86, 083513 (2013).

[77] F. Schmidt, E. Pajer, M. Zaldarriaga, *Large-Scale Structure with Gravitational Waves III: Tidal Effects*, Phys. Rev. D 89, 083507.

[78] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan and L. Senatore, *Effective Field Theory for Inflation*, JHEP 0803, 014 (2008).

[79] A. Gruzinov, *Elastic Inflation*, Phys. Rev. D 70 (2004) 063518.

[80] M. Akhshik, *Clustering Fossils in Solid Inflation*, JCAP 1505 (2015) 043.

[81] N. Bartolo, S. Matarrese, M. Peloso, A. Ricciardone, *Anisotropy in solid inflation*, JCAP, 1308, 022 (2013).

[82] S. Endlich, B. Horn, A. Nicolis, J. Wang, *The squeezed limit of the solid Inflation three-point function*, Phys. Rev. D 90(6):063506, 2014.

[83] N. Bartolo et al., *Science with the space-based interferometer LISA. IV: Probing Inflation with gravitational waves*, JCAP 1612, 026 (2016).

[84] Euclid Theory Working Group Collaboration, L. Amendola et al., *Cosmology and Fundamental Physics with the Euclid Satellite*, Living Rev.Rel. 16 (2013) 6.

[85] A. Ricciardone and G. Tasinato, *Primordial gravitational waves in supersolid inflation*, arXiv:1611.04516.

[86] P. Creminelli, J. Gleyzes, J. Norena and F. Vernizzi, *Resilience of the standard predictions for primordial tensor modes*, Phys. Rev. Lett. 113 (2014) 231301.

[87] A. Nicolis, R. Penco, F. Piazza, and R. Rattazzi, *Zoology of condensed matter: Framids, ordinary stuff, extra-ordinary stuff*, JHEP 06 (2015) 155.

[88] A. Maleknejad, M. Sheikh-Jabbari, and J. Soda, *Gauge Fields and Inflation*, arXiv:1212.2921 [hep-th].

[89] A. Maleknejad, M.M. Sheikh-Jabbari, *Non-Abelian gauge field inflation*, Phys. Rev. D 84 (2011) 043515.

[90] P. Adshead and M. Wyman, *Chromo-Natural Inflation: Natural Inflation on a Steep Potential with Classical Non-Abelian Gauge Fields*, Phys.Rev.Lett. 108 (2012) 261302.

[91] A. Agrawal, T. Fujita, E. Komatsu, *Tensor Non-Gaussianity from Axion-Gauge-Fields Dynamics: Parameter Search*, arXiv:1802.09284.

[92] N. Bartolo, E. Dimastrogiovanni, S. Matarrese, and A. Riotto, *Anisotropic bispectrum of curvature perturbations from primordial non-Abelian vector fields*, J. Cosmol. Astropart. Phys. 10 (2009) 015.

[93] N. Bartolo, E. Dimastrogiovanni, S. Matarrese, and A. Riotto, *Anisotropic Trispectrum of Curvature Perturbations Induced by Primordial Non-Abelian Vector Fields*, J. Cosmol. Astropart. Phys. 11 (2009) 028.

[94] E. Dimastrogiovanni, N. Bartolo, S. Matarrese, and A. Riotto, *Non-Gaussianity and statistical anisotropy from vector field populated inflationary models*, Adv. Astron. 2010, 752670 (2010).

[95] N. Bartolo, S. Matarrese, M. Peloso, and A. Ricciardone, *Anisotropic power spectrum and bispectrum in the $f(\phi)F^2$ mechanism*, Phys. Rev. D 87, 023504 (2013).

[96] S. Dubovsky, T. Gregoire, A. Nicolis and R. Rattazzi, *Null energy condition and superluminal propagation* JHEP 0603, 025 (2006).

[97] J. Soda, H. Kodama and M. Nozawa, *Parity Violation in Graviton Non-gaussianity*, JHEP 1108, 067 (2011).

[98] N. Bartolo and G. Orlando, *Parity breaking signatures from a Chern-Simons coupling during inflation: the case of non-Gaussian gravitational waves*, JCAP 1707, 034 (2017).

[99] A. Lue, L. Wang and M. Kamionkowski, *Cosmological signature of new parity-violating interactions*, Physical Review Letters 83 (aug, 1999) 1506-1509.

[100] M. Gerbino, A. Gruppuso, P. Natoli, M. Shiraishi, A. Melchiorri, *Testing chirality of primordial gravitational waves with Planck and future CMB data: no hope from angular power spectra*, JCAP 1607, no. 07, 044 (2016).