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World-sheet dynamics of EFT strings

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Abstract

String theory allows for huge classes of possible models admitting a gravitational low-energy effective description. However, not any effective field theory (EFT) of gravity admits a string theory UV completion. By using string theory as the main theoretical laboratory, the "Swampland program" aims at identifying the hallmarks distinguishing the "landscape" of gravitational EFTs that admit a consistent UV completion from the "swampland" of seemingly consistent EFTs that do not.

In this context, it has been recently realized that the perturbative regimes of UV complete four-dimensional $\mathcal{N} = 1$ (i.e. minimally supersymmetric) EFTs are strongly characterized by the presence of specific types of fundamental strings, called *EFT strings*. The latter generically support an 'internal' world-sheet sector, besides the universal 'center of mass' sector, and it is currently unknown how to couple it to the bulk supergravity sector in a supersymmetrically controlled way.

In this thesis, we illustrate how the universal sector of EFT strings can be coupled to four-dimensional $\mathcal{N} = 1$ supergravities by appropriately generalizing the superembedding formulation of superstrings. This provides a promising starting point to add the internal world-sheet sector. In this respect, we first study in depth the $\mathcal{N} = (0, 2)$ two-dimensional supergravity. We finally give a first example of inclusion of the internal sector, considering $\mathcal{N} = (0, 2)$ Fermi multiplets and $\mathcal{N} = (0, 2)$ chiral multiplets. In the latter case, we show that a generalization of the superembedding condition is needed, leaving the investigation of this interesting aspect for future work.

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Chapter 1

Introduction

The Standard Model (SM) is the most accurate theory that describes the interactions among elementary particles. Nevertheless, it is surely not the ultimate theory describing the reality at its very fundamental level. The reason is that it still presents some unsolved theoretical problems, like the so-called hierarchy problems, or the explanation of dark matter. Furthermore, the Standard Model leaves apart the gravitational interactions. Among the theories that have been proposed to go beyond the SM, the most promising one seems to be String Theory, according to which fundamental particles are nothing but different excitations of a one-dimensional object.

String theory started in the late 1960s as an attempt to organize and to explain the observed spectrum of hadrons and their interactions. However, together with the existence of a critical dimension[1], which is 26 for the bosonic string and 10 for the fermionic string, an undesired property of string theory was the appearance of a massless spin two particle, which is not present in the hadronic world. Therefore, string theory was ruled out as a possible theory of strong interactions, but the presence of a massless spin two particle led to regard it as a promising candidate for a complete theory of quantum gravity (QG).

There exist five different types of critical superstring theories: type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. Together with 11-dimensional M-theory, they are all linked among themselves by dualities, particular equivalences which relate different theories in different perturbative regimes (small-large or strong-weak dualities). These dualities suggest that these theories should be different manifestations of a unique theory, that is String theory.

Since superstring theories are 10-dimensional and M-theory is 11-dimensional, it is necessary to reduce the higher-dimensional string effective descriptions down to four dimensions in order to obtain relevant phenomenological, particle models. In order to get four-dimensional models, the unobserved 6/7 extra dimensions are assumed to be compactified on manifolds with an extremely small radius, and the structure of the compactification determines the structure of the four-dimensional theory.

The arbitrariness in compactifying higher dimensional theories down to four dimensions does not seem to allow for a systematic study of all the possible EFTs. Rather, one may start with a four-dimensional theory and inquire whether such a description is coherent with its quantum gravity ultraviolet (UV) completion. This approach has a bottom-up nature, and goes by the name of the *Swampland program* [1–3]; it is characterized by consistency criteria, within the four-dimensional description, which are called *Swampland conjectures* and aim at discriminating EFTs compatible with a complete theory of quantum gravity from those which do not admit such a UV completion.

[1]It is the dimension required by space-time Lorentz invariance of the quantized bosonic string (or superstring) theory.

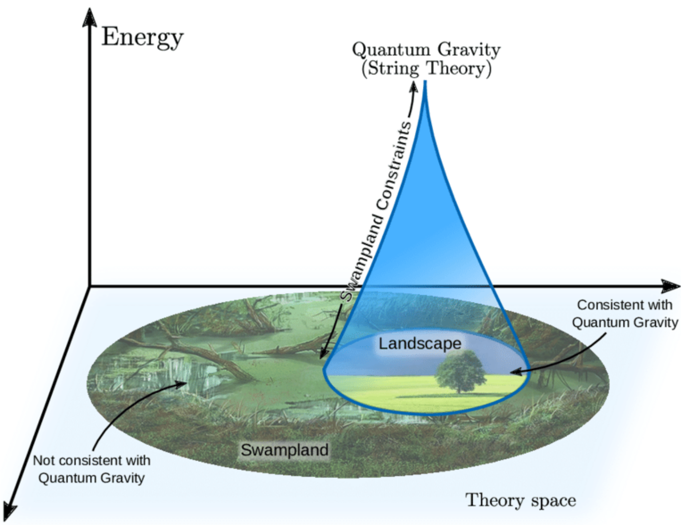

Figure 1.1: Among the (apparently) self-consistent 4-dimensional effective field theories, the string Landscape represents the sub-set of theories which admit a QG UV completion, while all the others are in the string Swampland.

The four-dimensional effective field theories originating from string theory are populated by *axions* a^i, which can be seen as zero gauge-forms subjected to the transformation

$$a^i \longrightarrow a^i + c^i \,, \tag{1.1}$$

where c^i are arbitrary constants. An axionic theory can be reformulated in terms of another, physically equivalent theory where the axions are replaced by gauge two-forms $\mathcal{B}_{2,i}$, as we will see in detail in Section 2.1.

Furthermore, typical EFTs admitting an UV QG completion are characterized by the presence of BPS objects. In the thesis work, we focus on *fundamental axionic strings*, which are fundamental in that they cannot be resolved into some smooth solitonic strings within a 4d EFT. Their tension satisfies

$$\Lambda^2 < \mathcal{T}_{\text{string}} < M_{\text{P}}^2 \,, \tag{1.2}$$

where Λ is the effective cut-off scale. These strings electrically couple to the two-form potentials $\mathcal{B}_{2,i}$, in turn related by an 'electromagnetic duality' to the axions. In this sense, it is said that BPS strings are 'magnetically' coupled to the axions, and this is the reason why are also called *axionic strings*. In particular, if we consider the dual axionic picture, in Section 2.1 we will show that, encircling the string, the axions undergo a shift set by the string charges e^i under the gauge two-forms $\mathcal{B}_{2,i}$: $a^i \longrightarrow a^i + e^i$.

In order to include the axionic strings in the EFT, it is necessary to have an axionic shift symmetry. However, one of the most widely accepted conjectures of the Swampland Program, which is the No Global Symmetry Conjecture [4, 5], states that exact global symmetries are not admitted in QG. Therefore, the above axionic shift symmetries must be understood as approximate global symmetries which are realised only at points of infinite distance in field space. This results in working close to the infinite distance boundary of the field space in the corresponding string theory models. As we will see in Section 2.4, the compatibility of this condition with the string backreaction motivates the restriction to the *EFT strings* [6–8], which are then a subclass of the BPS axionic strings, whose backreaction drives the surrounding scalar fields towards asymptotic field-space regions where the EFT admits a perturbative regime and corresponding axionic shift symmetries.

Throughout the work, we restrict to 4-dimensional EFTs preserving minimal ($\mathcal{N} = 1$) supersymmetry at the cut-off scale Λ: this supersymmetry can be spontaneously broken at lower energies, but our considerations will regard the EFT structure at energy scales of order Λ. Therefore, after having discussed in detail our motivations in Chapter 2, in Section 3.1 we will start from introducing $\mathcal{N} = 1$ supersymmetry to arrive at writing $\mathcal{N} = 1$ supergravity Lagrangians, by using the *superspace* *formalism*. Then, in Section 3.2.1 we will derive the supersymmetric action for the axions magnetically coupled to the strings, and its dual version, following [9, 10]. To this aim, we will use proper superfields which contain, among their components, the axions a^i and the gauge two-forms $\mathcal{B}_{2,i}$. They are the *chiral superfields* T^i, whose lowest components contain the axions a^i and the saxions s^i, which are related through a duality to the *real linear superfields* L_i, containing the gauge two-forms $\mathcal{B}_{2,i}$ and the dual saxions ℓ_i. The last ingredient to write the full action is the contribution of the BPS string electrically coupled to the gauge two-forms $\mathcal{B}_{2,i}$ with charges e^i. This part of the action is introduced in Section 4.1 and enjoys the so-called κ-*symmetry*, which is a local fermionic symmetry, which represents the way in which the bulk supersymmetry can be (partially) realized over the string worldsheet.

In the derivation of the string action, we use the so-called Green-Schwarz (GS) formulation discussed in [9], in which the string is described by the superspace embedding of the bosonic worldsheet, parametrized by two bosonic coordinates ξ^m, $m = 0, 1$, into the target superspace. The GS formulation allows for a description of the universal 'center of mass' sector, consisting of the bosonic and fermionic fields describing the string profile in the target superspace. However, from explicit UV completions (in which typically the axionic string corresponds to a brane wrapped on some internal cycle), we know that possible deformations of the internal configuration correspond to additional fields living on the worldsheet, which represent the so-called 'internal' degrees of freedom of the string. Such internal sector of the worldsheet theory allows for the anomaly cancellation, required to have a consistent EFT, which is discussed in [11]. Thus, we need to include this sector in the theory and describe its interaction with the dynamical background fields, but the GS formalism is not suitable for describing these degrees of freedom in a supersymmetrically controlled way.

An alternative approach which helps us to do this is the *superembedding approach* (see [12] and references therein). In this formulation, to realize local supersymmetry on the worldsheet, we extend the latter to an $\mathcal{N} = (0, 2)$ supersurface $\mathcal{M}_{2,2}$ parametrized by two bosonic coordinates ξ^m, $m = 0, 1$, and two real fermionic coordinates η^{+u}, $u = 1, 2$. This formalism provides the fermionic κ-symmetry of the GS formulation with a clear geometrical meaning of standard worldsheet local supersymmetry, thus giving a supersymmetric theory both in the superworldsheet of the string and the target superspace. Therefore, after having discussed $\mathcal{N} = (0, 2)$ supergravity in Section 3.3, in Section 4.2 we will introduce the superembedding formalism, by illustrating how it works in the case of $\mathcal{N} = 1$ superstrings [12]. At this point, we will be ready to implement this approach to our case of interest, and thus Section 4.3 will be dedicated to reformulate the theory for the BPS axionic strings in the superembedding formalism. Then, in Section 4.4, we will include in the theory $\mathcal{N} = (0, 2)$ Fermi superfields, thus providing a first example of inclusion of an internal sector to the world-sheet theory of an EFT string. Furthermore, we will see that an inconsistency arises when one tries to include $\mathcal{N} = (0, 2)$ chiral superfields.

Finally, in Chapter 5 we will draw our conclusions.

Chapter 2

The structure of $\mathcal{N} = 1$ effective field theories relevant to string compactifications

In this Chapter, we discuss how axions and axionic strings arise from string compactifications. In this respect, in Section 2.1 we show the key ingredients of the effective 4-dimensional theories we will consider throughout the thesis work. Then, in Section 2.3, we consider the explicit example of M-theory compactifications on G_2 manifolds. Finally, in Section 2.4 we discuss the so-called *EFT strings* and Section 2.5 is dedicated to show the main results related to them.

2.1 The UV origin of axions in $D = 4$ EFTs

As said in the Introduction, String theory admits different perturbative formulations, related to each other by dualities. They are given by the five superstring theories, which are Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$, and the 11-dimensional M-theory. The effective actions of superstring theories are 10-dimensional, while the low-energy description of M-theory is 11-dimensional supergravity, and they have a spectrum of massless fields which depends on the theory we are considering. In particular, all theories contain a massless graviton $g_{\mu\nu}$ and a dilaton ϕ, apart from M-theory which does not include the latter, while for the anti-symmetric tensor gauge fields we refer to Table 2.1. They can be represented as differential forms, and we use the notation according to which C_p is a p-form gauge field, given by

$$C_p = \frac{1}{p!} C_{\mu_1 \mu_2 \dots \mu_p} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \,. \tag{2.1}$$

Such anti-symmetric tensor fields couple to extended objects: as an example, the NS-NS B_2 field couples to the fundamental, or F1, strings, and their magnetic duals, the NS5-branes; the Ramond-Ramond fields C_{p+1} couple to the so-called Dp-branes, which are the hypersurfaces on which open F1 strings with Dirichlet boundary conditions end.

Theory	Antisymmetric tensor fields
Type I	C_2, A_1^i
Type IIA	B_2, C_1, C_3
Type IIB	B_2, C_0, C_2, C_4
Heterotic, $SO(32)$	B_2, A_1^i
Heterotic, $E_8 \times E_8$	B_2, A_1^i
M-theory	C_3

Table 2.1: Table showing the field content of the 5 superstring theories and M-theory, as far as the anti-symmetric tensor fields are concerned. Here, we use the notation according to which B_p are NS-NS sector fields, while C_p are R-R sector fields. Furthermore, A_1^i stand for Yang-Mills gauge fields.

They enter the action through their field-strength $F_{p+1} = \mathrm{d}C_p$[1], thus the resulting theory is invariant under the following transformation

$$C_p \longrightarrow C_p + \omega_p \,, \tag{2.2}$$

where ω_p is a closed p-form. This is a generalization of the ordinary Maxwell theory, which is retrieved in the case $p = 1$. In this case, we know that the photon, described by the 1-form gauge field $A_1 = A_\mu \mathrm{d}x^\mu$, couples to electrically charged particles through the interaction term

$$S_{\text{int}} = e \int A_1 \,, \tag{2.3}$$

where the integral is over the worldline of the particle and e is the electric charge, which, in 4 dimensions, is measured by integrating the electric field over the two-sphere S^2:

$$e = \int_{S^2} *F_2 \,, \tag{2.4}$$

with $F_2 = \mathrm{d}A_1$ and $*F_2$ the Hodge dual of the field-strength F_2[2].

Generalizing this scheme, a $(p+1)$-form gauge field C_{p+1} couples electrically to a p-brane and this interaction is described by

$$S_{\text{int}} = \mu_p \int C_{p+1} \,, \tag{2.5}$$

where the integration is over the world-volume of the brane, and μ_p is the p-brane charge, given by

$$\mu_p = \int_{S^{D-p-2}} *F_{p+2} \,, \tag{2.6}$$

since it is exactly $D-p-2$ the dimension required for a sphere to surround a p-brane in D dimensions.

The *magnetic dual* of a p-brane is a $(D-p-4)$-brane, and it carries a magnetic charge, denoted by μ_{D-p-4}, that is measured by computing the integral $\int F_{p+2}$ over a sphere S^{p+2}.

To make contact between string/M-theory and the 4-dimensional world of everyday experience, the typical procedure is *top-down* in nature and goes under the name of *Kaluza-Klein* (KK) *compactification*. One starts with the D-dimensional effective field theory, formulated in $\mathcal{M}_D$ and compactify it over an *internal* $(D-4)$-dimensional compact manifold X of size ℓ_c:

$$\mathcal{M}_D = \mathcal{M}_{1,3} \times X \,, \tag{2.7}$$

where $\mathcal{M}_{1,3}$ is the *external* four-dimensional manifold.

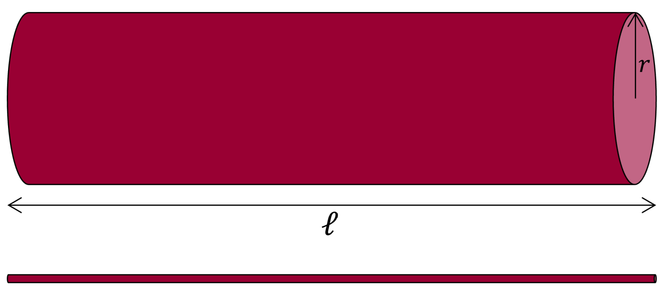

Figure 2.1: From far away a two-dimensional cylinder looks one-dimensional.

To better visualize the idea behind the KK compactification, we may consider the two cylinders of Fig. 2.1. Although the surface of the first cylinder is two-dimensional, if seen from a large distance

[1] Actually, their expression is generally more complicated. For example, $F^{\text{RR}}_{p+1} = \mathrm{d}C_p + H_3 \wedge C_{p-2}$. However, we may neglect these details for our discussion.

[2] For the definition of the Hodge dual of a p-form, see Appendix A.

(or, equivalently, if $r \ll \ell$), the cylinder looks effectively one-dimensional. We now have to imagine that the long dimension of the cylinder is replaced by our four-dimensional space-time and the short dimension by an appropriate six (for superstring theories), or seven-dimensional (for M-theory) compact manifold. Therefore, since the typical length of the internal manifold is very small, it cannot be seen at the energy scales we are able to probe, which are $E \ll 1/\ell_c$, and the world looks effectively four-dimensional. Nevertheless, even if the internal manifolds are invisible, their topological properties determine the particle content and structure of the four-dimensional theory, with its spectrum of fields determined by dimensionally reducing the higher-dimensional fields.

Axions generically arise in string theory compactifications as Kaluza-Klein (KK) zero modes of the masslss antisymmetric tensor gauge fields [13, 14]. We now qualitatively give the mechanism which provide the four-dimensional theories derived from string compactifications with axions and the corresponding strings, leaving a more quantitative analysis of a concrete example to Section 2.3.

Let us consider a D-dimensional theory and a gauge p-form C_p. In addition, let us introduce a basis of closed p-forms on the internal manifold X, $\omega_i \in H^p(X,\mathbb{Z})$, $i = 1,\dots,b_p(X)$, and also take them to be harmonic[3]. The KK expansion for the p-form C_p contains

$$C_p = a^i(x)\omega_i(y) + \dots \,, \tag{2.8}$$

where x and y represent the non-compact and compact coordinates, respectively. The coefficients of this linear combination, i.e. the $a^i(x)$, are four-dimensional (pseudo)scalar fields. Furthermore, if we assume for simplicity the absence of internal fluxes, the higher-dimensional gauge invariance of the antisymmetric tensor field action guarantees that no potential is generated at any order in perturbation theory, thus providing the theory for the scalar field with a global shift symmetry[4]. Therefore, the (pseudo)scalar fields under discussion are axions, which can be regarded as gauge 0-forms, subjected to the transformation

$$a^i \longrightarrow a^i + c^i \,, \tag{2.9}$$

with c^i being an arbitrary constant.

In 4 dimensions, an axionic theory can be reformulated in terms of another, physically equivalent theory where the axions a^i are replaced by gauge two-forms $\mathcal{B}_{2,i}$. We now illustrate how this duality works.

Henceforth, we will focus on four-dimensional EFTs preserving minimal $\mathcal{N} = 1$ supersymmetry at the UV cut-off scale Λ. In this context, let us consider a theory for a set of chiral superfields T^i, called *axionic multiplets*, whose lowest component contain two sets of real scalar fields, the axions a^i and the *saxions* s^i, namely

$$T^i\big| = t^i = a^i + is^i \,, \tag{2.10}$$

where the vertical line means that they are evaluated at $\theta = \bar\theta = 0$. The physics of theories containing a set of chiral multiplets is generically encoded in two scalar functions of the fields: the Kähler potential $K(T,\bar T)$ and the superpotential $W(T)$. In particular, let us focus on the part of the action describing the dynamics of the axions and the saxions, which acquires the form[5]

$$S = \frac{M_P^2}{2}\int \left(R * 1 - \mathcal{G}_{ij}(s)\mathrm{d}s^i \wedge *\mathrm{d}s^j - \mathcal{G}_{ij}(s)\mathrm{d}a^i \wedge *\mathrm{d}a^j\right) \,, \tag{2.11}$$

where $*1$ is the four-dimensional volume element, $*1 = e\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \equiv e d^4x$, and the field metric $\mathcal{G}_{ij}$ is defined in terms of the Kähler potential as

$$\mathcal{G}_{ij}(s) \equiv \frac{1}{2}\frac{\partial^2 K}{\partial s^i \partial s^j} \,, \tag{2.12}$$

[3]For the definition of cohomology group and harmonic p-form, see Appendix A.

[4]The shift symmetry is always broken by non-perturbative contributions, but such symmetry-breaking factors can be neglected in proper regions of field space, as we will see in Section 2.4.

[5]Here, we are neglecting possible contributions to the scalar potential arising from $W(T)$ and associated with non-perturbative effects. The reason of this assumption will be clearer in Section 2.4.1.

We consider the case in which $\mathcal{G}_{ij}$ only depends on the saxionic sector s^i, in order to preserve the invariance under axionic shifts $a^i \longrightarrow a^i + c^i$.

To formulate the dual description of (2.11) in terms of gauge two-forms $\mathcal{B}_{2,i}$, we first relax the assumptions that the one-forms $\mathrm{d}a^i$ are exact, but rather regard them as generic one-forms θ^i and add a 'dualizing term' to the action as follows

$$S = \frac{M_P^2}{2}\int \left(R * 1 - \mathcal{G}_{ij}(s)\mathrm{d}s^i \wedge *\mathrm{d}s^j - \mathcal{G}_{ij}(s)\theta^i \wedge *\theta^j\right) - \int \theta^i \mathcal{H}_{3,i}\,, \tag{2.13}$$

where, in the last term, we have introduced the field-strength of the gauge two-form $\mathcal{H}_{3,i} \equiv \mathrm{d}\mathcal{B}_{2,i}$. Two paths may be followed, one leading to the axionic formulation, described by (2.11), the other to a dual formulation where the gauge two-forms $\mathcal{B}_{2,i}$ replace the axions a^i. In details:

Axionic formulation If we integrate out $\mathcal{B}_{2,i}$, we obtain the relation

$$\mathrm{d}\theta^i = 0\,, \tag{2.14}$$

which is solved by requiring $\theta^i = \mathrm{d}a^i$ for a generic zero-form a^i, reducing the action to (2.11).

Dual formulation It is obtained by integrating out the one-forms θ^i. This leads to

$$M_P^2 * \theta^i = -\mathcal{G}^{ij}(s)\mathcal{H}_{3,j}\,, \tag{2.15}$$

where $\mathcal{G}^{ij}$ is the inverse of $\mathcal{G}_{ij}$ ($\mathcal{G}^{ij}\mathcal{G}_{jk} = \delta^i_k$). The relation (2.15) properly exchanges the one-forms $\mathrm{d}a^i$ with their Hodge-dual counterparts $\mathcal{H}_{3,i}$, so that providing the duality we were looking for. By plugging (2.15) into (2.13), we arrive at an action that now depends on the saxionic sector and the gauge two-forms $\mathcal{B}_{2,i}$:

$$S = \frac{M_P^2}{2}\int \left(R * 1 - \mathcal{G}_{ij}(s)\mathrm{d}s^i \wedge *\mathrm{d}s^j\right) - \frac{1}{2M_P^2}\int \mathcal{G}^{ij}(s)\mathcal{H}_{3,i} \wedge *\mathcal{H}_{3,j}\,. \tag{2.16}$$

This action is invariant under the gauge transformations $\mathcal{B}_{2,i} \longrightarrow \mathcal{B}_{2,i} + \mathrm{d}\alpha_{1,i}$, with $\alpha_{1,i}$ generic one-forms.

In Section 3.2 we will discuss in detail the supersymmetric generalisation of the dualization procedure discussed above. In the supersymmetric version, the axionic multiplets are replaced by *real linear multiplets* L_i, whose bosonic components are real scalars ℓ_i, the so-called *dual saxions*, and the gauge two-forms $\mathcal{B}_{2,i}$, dual to the axions. The latter are related to each other by (2.15), whereas the dual saxions are defined in terms of the saxions s^i as

$$\ell_i = -\frac{1}{2}\frac{\partial K}{\partial s^i}\,. \tag{2.17}$$

Let us stress that, in order for this dual description to be possible, the theory has to enjoy the axionic shift symmetry (2.9). In other words, since in the supersymmetric context the theory is specified by a Kähler potential K, this means that the latter must be invariant under the axionic shifts (2.9). In Section 2.3, we will see how this is realized in M-theory compactifications.

2.2 The UV origin of axionic strings in $D = 4$ EFTs

The relation $\mathrm{d}\theta^i = 0$ in (2.14) is modified if the so-called *axionic strings* are present in the theory. Generally speaking, the presence of extendend objects is ubiquitous in EFTs arising from string compactifications. In particular, BPS strings naturally arise in 4-dimensional $\mathcal{N} = 1$ theories with approximate shift symmetries. We focus our attention on *fundamental* strings, namely those satisfying

$$\Lambda^2 < \mathcal{T} < M_{\mathrm{P}}^2\,, \tag{2.18}$$

with Λ being the UV cut-off scale of the EFT, and consider them electrically coupled to the gauge two-forms $\mathcal{B}_{2,i}$. From the 4-dimensional point of view, they are structureless objects, i.e. they cannot be resolved into some smooth solitonic strings within the 4-dimensional EFT. For this reason, they correspond to fundamental localised objects in the theory.

A string spans a two-dimensional hypersurface $\mathcal{S}$ in the target four-dimensional space $\mathcal{M}_{1,3}$, which is determined by the embedding[6]

$$\xi^m \longmapsto \mathcal{S} : x^{\underline{m}} \equiv x^{\underline{m}}(\xi)\,, \tag{2.19}$$

where ξ^m, $m = 0, 1$, are two spacetime coordinates parametrizing the string worldsheet. The action describing a string minimally coupled to a set of gauge two-forms $\mathcal{B}_{2,i}$ is

$$S_{\text{string}} = -\int_{\mathcal{S}} \sqrt{-\det\gamma}\,\mathcal{T}_{\text{string}}(\ell) + e^i \int_{\mathcal{S}} \mathcal{B}_{2,i}\,, \tag{2.20}$$

where γ represents the induced metric over the string worldsheet

$$\gamma_{mn} = \frac{\partial x^{\underline{m}}}{\partial \xi^m}\frac{\partial x^{\underline{n}}}{\partial \xi^n} g_{\underline{mn}}\,. \tag{2.21}$$

The second, Wess-Zumino term expresses the minimal coupling of the string to the gauge two-forms $\mathcal{B}_{2,i}$, under which the string has charges e^i, whereas the first, Nambu-Goto term describes the motion of the string, encoding the kinetic terms of the string degrees of freedom, and it is invariant under reparametrizations of the worldsheet $\xi^m \longrightarrow \xi'^m(\xi)$. It further depends on the string tension $\mathcal{T}_{\text{string}}(\ell)$, its mass per unit length, which may generically depend on the dual saxions ℓ_i. In particular, in Section 3.2, we will show that, for a given set of charges e^i, $\mathcal{T}_{\text{string}}(\ell)$ is completely fixed by requiring the axionic strings to be $\frac{1}{2}$-BPS objects, namely preserving one half of the bulk supersymmetry, and its expression is [9]

$$\mathcal{T}_{\text{string}}(\ell) \equiv \mathcal{T}_{\mathbf{e}} = M_P^2 |e^i \ell_i|\,. \tag{2.22}$$

Therefore, in its dual version, the full action which describes the interaction of a string with the bulk fields is

$$\begin{aligned} S =& \frac{M_P^2}{2}\int \left(R * 1 - \mathcal{G}^{ij}\mathrm{d}\ell_i \wedge *\mathrm{d}\ell_j\right) - \frac{1}{2M_P^2}\int \mathcal{G}^{ij}\mathcal{H}_{3,i}\wedge *\mathcal{H}_{3,j} + \\ & - \int_{\mathcal{S}} \sqrt{-\det\gamma}\,\mathcal{T}_{\text{string}}(\ell) + e^i \int_{\mathcal{S}} \mathcal{B}_{2,i}\,. \end{aligned} \tag{2.23}$$

Let us see what are the effects of the coupling of the gauge two-forms to a string in the dual axion picture. In this respect, let us compute the equations of motion for the gauge two-forms $\mathcal{B}_{2,i}$ which originate from the action (2.23):

$$-\frac{1}{M_{\text{P}}^2}\mathrm{d}\left(\mathcal{G}^{ij} * \mathcal{H}_{3,j}\right) = e^i \delta_2(\mathcal{S})\,. \tag{2.24}$$

If we use the duality relation (2.15), we find what we anticipated previously, i.e. the relation $\mathrm{d}\theta^i = 0$ gets modified and becomes

$$\mathrm{d}\theta^i = e^i \delta_2(\mathcal{S})\,. \tag{2.25}$$

We can now integrate this equation over a disk $\mathcal{D}$, whose boundary $\mathcal{L} = \partial\mathcal{D}$ is a circle enclosing the string, and using that the one-form θ^i can be still written as $\theta^i = \mathrm{d}a^i$ everywhere but on the worldsheet $\mathcal{S}$, we obtain

$$\int_{\mathcal{D}} \mathrm{d}\theta^i = e^i \int_{\mathcal{D}} \delta_2(\mathcal{S}) \qquad \Longrightarrow \qquad \Delta a^i = e^i\,, \tag{2.26}$$

which tells that, encircling a string, an axion a^i is subjected to a *monodromy* transformation that shifts the axion a^i by the charge e^i of the string under the dual two-form $\mathcal{B}_{2,i}$. We will say that the axions a^i are 'magnetically' coupled to the string, since the latter is electrically coupled to the gauge two-forms $\mathcal{B}_{2,i}$, which, in turn, provide an alternative representation for the axions a^i, and call such a string an 'axionic string'.

[6]In what follows, the underlined indices refer to target-space indices, while not underlined ones correspond to world-sheet indices.

We now show how axionic strings arise from string compactifications. To this aim, let us consider the magnetic dual of the p-form $F_{p+1} = \mathrm{d}C_p$, which is given by $F_{D-p-1} = *F_{p+1}$[7]. Generally, from the equation of motion of F_{p+1}, we can always associate a $(D-p-2)$-form C_{D-p-2} to F_{D-p-1}, such that $F_{D-p-1} = \mathrm{d}C_{D-p-2}$. If we focus on the case $D=10$, the latter is an $(8-p)$-form. We now introduce a basis of $(6-p)$-forms, which we take to be harmonic, $\tilde{\omega}^i \in H^{6-p}(X,\mathbb{Z})$, $i=1,\ldots,b_{6-p}(X)$ dual to the basis ω_i introduced above, i.e. such that

$$\int_X \omega_i \wedge \tilde{\omega}^j = \delta_i^j \,. \tag{2.27}$$

By compactifying the theory on the internal space X, the expansion of C_{8-p} includes the following term:

$$C_{8-p} = \mathcal{B}_{2,i} \wedge \tilde{\omega}^i + \ldots \,. \tag{2.28}$$

This term arises from taking 2 external and $6-p$ internal indices. We know that C_{8-p} electrically couples to $(7-p)$-branes, and their minimal coupling is expressed by the interaction term in (2.5). Then, if we consider, in addition to the decomposition (2.28), a $(7-p)$-brane wrapping some internal cycle, namely whose worldvolume can be written as the product of a 2-dimensional surface $\mathcal{S}$ in the external space with a $(6-p)$-dimensional internal cycle $\mathcal{C}$, $\Gamma = \mathcal{S}\times\mathcal{C}$, we obtain that

$$\mu_{7-p}\int_\Gamma C_{8-p} = \int_{\mathcal{S}} \mathcal{B}_{2,i}\left(\mu_{7-p}\int_{\mathcal{C}}\tilde{\omega}^i\right) = e^i \int_{\mathcal{S}} \mathcal{B}_{2,i}\,, \tag{2.29}$$

where we have defined the charges e^i as

$$e^i \equiv \mu_{7-p}\int_{\mathcal{C}}\tilde{\omega}^i \,. \tag{2.30}$$

They are quantized since we are considering integral harmonic forms $\tilde{\omega}^i$, whose integral over internal cycles is quantized, by definition. Furthermore, μ_{7-p} is quantized due to a generalization of the Dirac quantization condition (valid for point-like charges in $D=4$) to the charges carried by a dual pair of p-branes. Therefore, the minimal coupling of C_{8-p} to a $(7-p)$-brane wrapping some internal $(6-p)$-cycle, gives us the *Wess-Zumino term*, which expresses the coupling of the gauge two-forms $\mathcal{B}_{2,i}$ to a four-dimensional string, whose world-sheet is $\mathcal{S}$, as in (2.20) and (2.23).

Let us discuss the 'magnetic' axionic description of these strings from the 10d viewpoint. For concreteness, we focus on the NS-NS B_2 field. In 10 dimensions, its magnetic dual is B_6. Similarly to before, according to how many indices we take to be internal or external, the two fields admit, in their expansions, the following terms

$$\begin{cases} B_2 = \mathcal{B}_{2,0} + a^i\omega_i + \ldots \\ B_6 = \mathcal{B}_{2,i}\wedge\tilde{\omega}^i + a^0\omega_6 + \ldots \end{cases}, \tag{2.31}$$

where ω_i and $\tilde{\omega}^i$ are the same as before, but with $p=2$, and ω_6 is an integral 6-form, such that

$$\int_X \omega_6 = 1 \,. \tag{2.32}$$

The B_2 field electrically couples to the F1-string, while the electric coupling of B_6 is with NS5-branes. Since they constitute a dual pair of fields, from their expansions we see that, in the 4-dimensional EFT:

- the axion a^0 is the magnetic dual of the $\mathcal{B}_{2,0}$ field. Therefore, the 10-dimensional fundamental string is the axionic string associated to the axion a^0 appearing in the expansion of B_6;
- the axions a^i are the magnetic duals of the gauge 2-forms $\mathcal{B}_{2,i}$, and consequently, in this case, the axionic strings correspond to NS5-branes wrapped around some internal 4-cycle $\mathcal{C}$.

[7] Here, as in Section 2.1, we are neglecting the possible quadratic corrections to this expression.

The scheme outlined here appears in several several string/M-theory compactifications. However, the above discussion does not take into account supersymmetry, which, as already mentioned, is crucial to fix the form of the tension as in (2.22). The requirement for the strings to be $\frac{1}{2}$-BPS objects impose additional conditions on the cycles around which higher-dimensional branes are wrapped in the internal compactification space. To discuss these topics, we now give an explicit example of how this kind of models arises from string/M-theory compactifications, by focusing on the case of M-theory compactified on G_2 manifolds.

2.3 An explicit example: M-theory on G_2 manifolds

In a smooth space-time, the low-energy effective description of M-theory is 11-dimensional supergravity. As already explained in the previous Section, to obtain a 4-dimensional EFT in M-theory compactifications, we need to consider 11d vacua of the form $\mathcal{M}_{1,10} = \mathcal{M}_{1,3} \times X$ where $\mathcal{M}_{1,3}$ is the well known flat Minkowski space-time, while X is the *internal space*, which is a compact 7-dimensional manifold characterized by a typical length scale ℓ_c which is very small with respect to the energy scales we are able to probe.

A class of 7-manifolds, used in M-theory compactifications, which allows us to obtain a 4-dimensional effective theory with $\mathcal{N} = 1$ supersymmetry is given by manifolds whose holonomy group is G_2. Therefore, in the following we will focus on this type of M-theory compactifications. Before describing how to compactify M-theory on such manifolds, we give a brief overview on useful concepts for the purposes of our goal.

2.3.1 G_2 manifolds

Let us start by introducing the notion of *holonomy*.

Consider an n-dimensional orientable Riemannian manifold (X, g), a point $p \in X$ and a vector $v \in T_pX$, tangent to X in p. The manifold is equipped with a Riemannian metric g, which naturally induces a connection, the Levi-Civita connection. The latter allows to define a parallel transport for tangent vectors along curves in X. One of the properties of parallel transport is that it preserves the norm of vectors, hence by transporting v along a closed loop γ in X, we obtain a vector v' with the same length, but in general a different orientation with respect to v and the two are related by a rotation

$$v' = Rv\,, \qquad R \in SO(n)\,. \tag{2.33}$$

On the other hand, the objects that are parallel transported may also be spinors. In this case, under parallel transport around a closed curve, the transformation of a spinor is an element of the covering group $Spin(n)$. By considering all possible closed curves in X, we obtain a set of spinors related to the original one by a matrix $U \in Spin(n)$. These rotations form a group called the *holonomy group* of X, denoted by $\mathrm{Hol}(X, g)$. Note that $\mathrm{Hol}(X, g)$ depends on the choice of the metric g, since it is built out of the parallel transport defined by the Levi-Civita connection. This means that the same manifold X with different metrics can have different holonomy groups. In the most general case, the holonomy group of an n-dimensional Riemannian manifold is $Spin(n)$, but in some special instances it can be a proper subgroup thereof:

$$\mathrm{Hol}(X, g) \subset Spin(n)\,. \tag{2.34}$$

In this case, (X, g) is called a manifold of *special holonomy*.

In 7 dimensions, the exceptional Lie group G_2 is a possible holonomy group, and we now explain why M-theory compactifications on G_2 manifolds lead to 4d EFTs with $\mathcal{N} = 1$ supersymmetry. First of all, we have to say how to determine the number of supersymmetry transformations preserved by compactification of M-theory on X. It turns out [15] that this number is equal to the number of covariantly constant spinors, i.e. the solutions of the equation

$$\nabla_M \eta = 0\,, \tag{2.35}$$

where η is an 11d Majorana spinor, ∇ is the covariant derivative containing the spin connection and $M = 0, 1, \ldots, 10$ is an index labelling the coordinates X^M on $\mathcal{M}_{1,10}$. Since we consider a space-time of the form $\mathcal{M}_{1,10} = \mathcal{M}_{1,3} \times X$, we can factorize the spinor η as

$$\eta(X^M) = \eta(x^\mu, y^A) = \zeta(x^\mu) \otimes \psi(y^A)\,, \tag{2.36}$$

where:

- ζ is a 4-component Majorana spinor on $\mathcal{M}_{1,3}$ with coordinates x^μ, $\mu = 0, \ldots, 3$;
- ψ is a spinor on X, with coordinates y^A, $A = 1, \ldots, 7$.

In our case, since we are considering $\mathcal{M}_{1,3}$ to be flat Minkowski space-time and it admits a basis of four independent covariantly constant spinors ζ, the number of preserved supersymmetry transformations is $4\mathcal{N}$, where $\mathcal{N}$ is the number of 7d spinors ψ solving the equation

$$\nabla^{(7)}_A \psi = 0\,, \tag{2.37}$$

where $\nabla^{(7)}$ is the 7d covariant derivative. To determine the number of solutions of (2.37), the idea is that of decomposing the spinor representation of $Spin(7)$ into irreducible representations of the holonomy group $\mathrm{Hol}(X, g)$ (in our case G_2) and look for singlets, on which $\mathrm{Hol}(X, g)$ acts trivially, hence leaving them invariant.

In the case of a G_2 manifold X, there is only one solution to (2.37), thus giving 4 conserved supercharges, which is exactly the number of preserved supersymmetry transformations necessary to have $\mathcal{N} = 1$ supersymmetry in 4 dimensions.

The presence of covariantly constant spinors typically allows one to define associated covariantly constant p-forms. In particular, a G_2 manifolds is characterized by the existence of one invariant 3-form φ, called *associative* form, with $\varphi_{ABC} = \psi^\dagger \Gamma_{ABC} \psi$[8], and one invariant 4-form, which is the 7-dimensional Hodge dual of φ and therefore denoted by $*\varphi$, called *coassociative* form.

Starting from φ, it is possible to reconstruct a G_2 holonomy metric g by means of the following formula

$$g_{AB} = \det(h)^{-\frac{1}{9}} h_{AB}\,, \tag{2.38}$$

with

$$h_{AB} = \frac{1}{144} \varphi_{ACD} \varphi_{BEF} \varphi_{GHI} \varepsilon^{CDEFGHI}\,. \tag{2.39}$$

One property of this metric is Ricci-flatness, i.e. it has vanishing Ricci tensor, $R_{AB} = 0$. Therefore, it automatically solves the 7-dimensional vacuum Einstein equations.

To conclude this introduction to G_2 manifolds, we illustrate an important property of the 3-form φ and its dual $*\varphi$.

Let us consider a closed p-form ω_p and a p-dimensional submanifold $S \subset X$. If we restrict the metric g on S, it defines a volume form vol_S. It is said that ω_p is a *calibration* if its pull-back on S satisfies[9]

$$\omega_p\big|_S = \alpha(x) \cdot vol_S\,, \qquad \text{with} \qquad \alpha(x) \le 1\,, \forall x \in S\,. \tag{2.40}$$

If $\alpha(x) = 1$, $\forall x \in S$, we say that the submanifold S is *calibrated* by ω_p, and we can then calculate its volume by integrating the corresponding calibration ω_p on S itself:

$$\mathrm{Vol}(S) = \int_S \omega_p\,. \tag{2.41}$$

Since ω_p is closed, by Stokes' theorem we have that

$$\int_{S-S'=\partial\Sigma} \omega_p = \int_\Sigma \mathrm{d}\omega_p = 0 \qquad \Longrightarrow \qquad \int_S \omega_p = \int_{S'} \omega_p\,, \tag{2.42}$$

[8]Here, $\Gamma_{ABC} = \Gamma_{[A}\Gamma_B\Gamma_{C]}$, where Γ_A are 7-dimensional Γ-matrices.

[9]For an introduction to calibrated geometries, see [16].

where S' belongs to the same homology class of S. On the other hand, since ω_p is a calibration, it satisfies (2.40) also when we restrict it on S', but with $\alpha(x)$ not equal to 1 everywhere on S', in general. Therefore, we have found that calibrated submanifolds have minimal volume in their homology class.

This is important since it can be shown that branes wrapped on internal cycles preserve some fraction of the bulk supersymmetry if and only if the cycles are calibrated. Thus, calibrated submanifolds are identified as supersymmetric cycles, and, in our case, the calibrations are exactly given by the associative 3-form φ and the coassociative 4-form $*\varphi$. We use this result when we will analyse the UV origin of BPS axionic strings in M-theory compactifications.

We now have all the ingredients to derive the structure of the 4d EFT arising from M-theory compactification on G_2 manifolds.

2.3.2 Axions from M-theory on smooth G_2 manifolds

At low energies and on a smooth space-time, M-theory is well approximated by 11-dimensional supergravity. In $D = 11$ supergravity, the bosonic degrees of freedom are given by the graviton and a 3-form C_3, whose field strength is denoted by $G_4 = \mathrm{d}C_3$. The part of the action associated to these fields is

$$S_{11} = \frac{2\pi}{\ell_M^9}\int\left(R*1 - \frac{1}{2}G_4\wedge *G_4 - \frac{1}{6}C_3\wedge G_4\wedge G_4\right), \tag{2.43}$$

where ℓ_M denotes the 11-dimensional Planck length. Let us study the massless scalar fields arising in the 4-dimensional theory, by first discussing the metric. Recall that we are considering a space-time of the form $\mathcal{M}_{1,10} = \mathcal{M}_{1,3}\times X$, thus the metric is decomposed in the following way:

$$\mathrm{d}\hat{s}^2 = G_{MN}\mathrm{d}X^M\mathrm{d}X^N = e^{2A}g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu + \ell_M^2\tilde{g}_{AB}(y)\mathrm{d}y^A\mathrm{d}y^B\,, \tag{2.44}$$

with the metric on X which can be constructed from the associative 3-form φ as in (2.38) and satisfies the vacuum Einstein equations. Here, ℓ_M^2 is introduced to make the 7d metric $\tilde{g}_{AB}$ and coordinates y^A dimensionless, while the expression for the Weyl rescaling factor e^{2A} is obtained by requiring that the 4d Einstein-Hilbert term arising from its 11d version has the canonical form

$$\frac{M_\mathrm{P}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R^{(4)}\,, \tag{2.45}$$

with $g \equiv \det g$. The dimensional reduction of the 11d Einstein-Hilbert term gives

$$\frac{2\pi}{\ell_M^9}\int \mathrm{d}^{11}X\sqrt{-G}\hat{R}^{(11)} \quad\longrightarrow\quad \frac{2\pi}{\ell_M^9}\ell_M^7 V_X\int \mathrm{d}^4x\sqrt{-g}e^{2A}R^{(4)} + \dots\,, \tag{2.46}$$

where V_X is the adimensional volume of the internal space, which can be expressed in terms of φ and $*\varphi$ as

$$V_X = \int_X \sqrt{\tilde{g}}\mathrm{d}^7y = \frac{1}{7}\int_X \varphi\wedge *\varphi\,, \tag{2.47}$$

with $\tilde{g}\equiv \det\tilde{g}$. Therefore, by comparing (2.45) and (2.46) we find that

$$e^{2A} = \frac{\ell_M^2 M_\mathrm{P}^2}{4\pi V_X}\,. \tag{2.48}$$

If we consider fluctuations of $\tilde{g}_{AB}(y)$

$$\tilde{g}_{AB}(y) + \delta\tilde{g}_{AB}(x,y)\,, \tag{2.49}$$

and require that Einstein equations are still satisfied, one can show [17] that the number of independent moduli obtained from the metric is $b_3(X)$, that is the number of harmonic 3-forms on X. Since the metric is built out of the associative 3-form φ, such moduli arising from the metric can be obtained by decomposing the latter. Let us then introduce a basis $\{\Sigma^i\}$, $i = 1,\dots,b_3(X)$ of the integral homology group $H_3(X,\mathbb{Z})$. In terms of this basis, each 3-cycle in X may be written as $\Sigma = m_i\Sigma^i$, with $m_i \in \mathbb{Z}$. Furthermore, let us introduce a dual basis $\{\omega_i\}$ of 3-forms, defined by requiring that

$$\int_{\Sigma^i} \omega_j = \delta^i_j\,. \tag{2.50}$$

This basis also represents a basis of the integral coohomology group $H^3(X,\mathbb{Z})$, which is the group of 3-forms whose integral on any 3-cycle Σ is always integer. Therefore, using the two bases and the duality relation (2.50), we obtain that, if we consider the integral of ω_i on a generic 3-cycle $\Sigma = m_i\Sigma^i$, it exactly gives the integer m_i:

$$\int_{\Sigma} \omega_i = m_i\,, \qquad m_i \in \mathbb{Z}\,. \tag{2.51}$$

We now perform a decomposition of the associative form φ in terms of the integral harmonic 3-form basis just introduced

$$\varphi = \sum_{i=1}^{b_3(X)} s^i(x)\omega_i(y)\,, \tag{2.52}$$

and we finally obtain what anticipated. Indeed, the notation for the coefficients of this decomposition is not accidental, since they are exactly the saxions of Section 2.1, which, utilising (2.50), can be written as

$$s^i = \int_{\Sigma^i} \varphi\,. \tag{2.53}$$

Similarly, one can decompose the 3-form field C_3 as[10]

$$C_3 = \sum_{i=1}^{b_3(X)} a^i(x)\omega_i(y) + \dots\,, \tag{2.54}$$

where a^i are the axions, which, as for the saxions, can be seen as integrals of C_3 on the basis of 3-cycles

$$a^i = \int_{\Sigma^i} C_3\,. \tag{2.55}$$

The reason why they are supersymmetric partners of the saxions lies in the fact that, in the 11d picture, the objects they come from, i.e. the metric and C_3, are in turn superpartners. Together, a^i and s^i form the lowest components of the axionic multiplets T^i mentioned in Secton 2.1, namely

$$t^i = a^i + is^i = \int_{\Sigma^i} (C_3 + i\varphi)\,, \qquad i = 1,\dots,b_3(X)\,. \tag{2.56}$$

Generically speaking, a 4d $\mathcal{N}=1$ supersymmetric EFT is characterized by a Kähler potential K for the scalar fields. In our case, its form was found in [18–20], and is given by

$$K = -3\log\left(\frac{V_X}{2\pi^2}\right)\,, \tag{2.57}$$

with V_X the adimensional volume of the internal space X, which is computed as in (2.47) in terms of φ and $*\varphi$. This result is important, because it tells us that the Kähler potential only depends on the metric and, if we recall (2.52) and (2.54), this implies that K only depends on the saxions s^i and not on the axions a^i. Therefore, K enjoys an axionic shift symmetry, and, as already said in Section 2.1, it is needed in order for the dualisation procedure to be possible.

To see the origin of the dual saxions ℓ_i, we first introduce dual bases of harmonic 4-forms $\{\tilde{\omega}^i\}$ and 4-cycles $\{\Pi_i\}$, $i = 1,\dots,b_4(X)$, such that

$$\int_X \omega_i \wedge \tilde{\omega}^j = \delta^j_i\,, \qquad \int_{\Pi_i} \tilde{\omega}^j = \delta^j_i\,, \tag{2.58}$$

[10]Here, the ellipses stand for an additional term in the KK reduction of the 3-form field C_3, which gives origin to 4d Abelian gauge fields. However, we did not elaborate on them, since they are not relevant for the purposes of the thesis work.

where $\{\omega_i\}$ is the basis of harmonic 3-forms defined above. Furthermore, note that, since X is a 7-dimensional manifold and in general $b_p(X) = b_{n-p}(X)$ for an n-dimensional manifold, in our case $b_3(X) = b_4(X)$.

We can now decompose the coassociative form $*\varphi$ in terms of the 4-form basis $\{\tilde{\omega}^i\}$

$$\frac{1}{2V_X} * \varphi = \sum_{i=1}^{b_4(X)} \ell_i(x)\tilde{\omega}^i(y)\,, \tag{2.59}$$

and, analogously to the case of the axions and the saxions, the dual saxions ℓ_i are then the integrals of $*\varphi$ on the basis of 4-cycles $\{\Pi_i\}$.

2.3.3 BPS strings from M-theory on smooth G_2 manifolds

We now move to the analysis of axionic strings. According to the discussion of Section 2.1, since $p = 2$ and $D = 11$, the 3-form C_3 electrically couples to a 2-brane, called the M2-brane, and magnetically to a 5-brane, called the M5-brane. Axionic strings arise in the 4-dimensional EFT from M5-branes wrapped on 4-cycles. Therefore, let us start from the 11-dimensional bosonic action for an M5-brane[11]

$$S_{M5} = -T_{\rm M5}\int_{W_6}\sqrt{-h_6}\mathrm{d}^6\sigma + T_{\rm M5}\int_{W_6} A_6 + \dots\,, \tag{2.60}$$

where $T_{\rm M5} = 2\pi/\ell_M^6$ is the brane tension, W_6 is the 6-dimensional world-volume of the brane, parametrized by the coordinates σ^a, $a = 0,\dots,5$ and with induced metric $h_{6,ab}$, while A_6 is a 6-form potential, dual to the 3-form C_3, under which the M5-brane is charged. If we wrap the M5-brane around an internal 4-cycle, its world-volume W_6 can be written as $W_6 = \mathcal{S}\times\Pi$, where $\mathcal{S}$ represents the external 2-dimensional world-sheet of the string and Π the internal 4-cycle. We can now rewrite the two terms in (2.60):

- If we consider a metric G_{MN} with the form as in (2.44) and a calibrated Π (for which we have seen that the volume can be expressed as the integral of the coassociative form $*\varphi$), the first integral can be factorized into two pieces and becomes

$$\begin{aligned} -T_{\rm M5}\ell_M^4\int_{\mathcal{S}} e^{2A}\sqrt{-h}\mathrm{d}^2\sigma\int_{\Pi}\sqrt{h_4}\mathrm{d}^4\sigma &= -\frac{2\pi}{\ell_M^2}\mathrm{Vol}(\Pi)\int_{\mathcal{S}} e^{2A}\sqrt{-h}\mathrm{d}^2\sigma = \\ &= -\frac{2\pi}{\ell_M^2}\int_{\Pi}*\varphi\int_{\mathcal{S}} e^{2A}\sqrt{-h}\mathrm{d}^2\sigma\,. \end{aligned} \tag{2.61}$$

  By decomposing the 4-cycle Π as $\Pi = e^i\Pi_i$, $e^i\in\mathbb{Z}$, and the coassociative form $*\varphi$ as in (2.59), we can finally write the Nambu-Goto term of the M5-brane action as

$$-M_P^2 e^i\ell_i\int_{\mathcal{S}}\sqrt{-h}\mathrm{d}^2\sigma\,, \tag{2.62}$$

  where we have used (2.48) and (2.58). This is precisely the kinetic part of the action in (2.20), with string tension as in (2.22) in the case of $e^i\ell_i > 0$. It is worthwhile noting that this result has been obtained considering a calibrated 4-cycle Π, which, as seen previously, is the type of cycles necessary to obtain a BPS-string in the 4-dimensional theory.

- To rewrite the second term, we perform a Kaluza-Klein (KK) reduction of the 6-form gauge field A_6

$$A_6 = \frac{\ell_M^6}{2\pi}\sum_{i=1}^{b_4(X)}\mathcal{B}_{2,i}(x)\wedge\tilde{\omega}^i(y)\,. \tag{2.63}$$

[11] The ellipses stand for terms coming from the self-dual 2-form gauge field supported by the worldvolume of the M5-branes (see [21, 22] and references therein), which we neglect.

Here, $\mathcal{B}_{2,i}$ are the 2-form gauge fields dual to the axions a^i. The reason is that the former come from the decomposition of A_6, the latter from that of C_3, and A_6 is exactly the 6-form potential dual to the 3-form C_3. In analogy to the procedure for the Nambu-Goto term, from the interaction term in (2.60) we obtain

$$e^i \int_{\mathcal{S}} \mathcal{B}_{2,i}\,, \tag{2.64}$$

which is exactly the Wess-Zumino term appearing in (2.20).

To summarize, as an example of string/M-theory compactification, we have illustrated what is the structure of the 4d EFT obtained from compactification of M-theory on a G_2 manifold. This is an $\mathcal{N}=1$ supergravity theory, characterized by exactly those ingredients which we have introduced in Section 2.1.

We finally conclude this section by saying that the class of models arising from M-theory compactifications on smooth G_2 manifolds is not phenomenologically interesting, since it does not lead to chiral matter or non-abelian gauge fields. The reason is that M-theory is a nonchiral theory and compactification on a smooth manifold cannot lead to a chiral theory. To obtain a more realistic model, we should consider singular G_2 manifolds. However, a detailed analysis of this kind of compactification goes beyond the scope of our analysis, since this section is only intended to give an example of string/M-theory compactifications which give 4d EFTs like the ones we will focus on throughout the thesis work.

2.4 A special class of fundamental axionic strings: the EFT strings

In Section 2.1, we emphasized the fact that, in order for the dualization procedure to be possible, the theory should be invariant under the continuous shift symmetry (2.9). However, by one of the most known Swampland Conjectures, i.e. the No Global Symmetry Conjecture [4, 5], global symmetries are forbidden in a theory consistent with quantum gravity. As we will see in this Section, this means that the continuous shift symmetry (2.9) is only present in appropriate asymptotic field-space regions, where the axionic symmetries are understood as perturbatively preserved.

In particular, in Section 2.4 we will illustrate that, if we focus on a specific subclass of BPS axionic strings, the so-called *EFT strings* [6–8], it is sufficient to assume the existence of the perturbatively preserved axionic symmetry only at the starting point in the field space, since the backreaction induced by the axionic strings naturally drives the saxions s^i towards the proper asymptotic field-space regions, as one approaches the string itself.

Then, in Section 2.5.1 we discuss two Swampland conjectures related to the EFT strings, which are the Distant Axionic String Conjecture (DASC) and the Integral Scaling Conjecture (ISC) [6–8].

Finally, Section 2.5.2 shows how the consistency of an EFT with a standard coupling to the axionic sector requires the EFT string world-sheet theory to produce gauge and gravitational anomalies which cancel the anomaly inflow from the bulk on the worldsheet induced by the axionic couplings [11].

2.4.1 Characterization of the EFT strings

Let us consider a 4d $\mathcal{N}=1$ effective field theory for a set of chiral multiplets $\{T^i\}$, described by the following action

$$S = M_{\rm P}^2 \int \left(\frac{1}{2} R * 1 - K_{i\bar{\jmath}} \mathrm{d}t^i \wedge * \mathrm{d}\bar{t}^{\bar{\jmath}} \right)\,, \tag{2.65}$$

where only the bosonic part is taken into account. String-like solutions of this theory can be studied following the discussion of [23]. To find them, let us use as 4d coordinates $(t, x, z, \bar{z})$ and require 2d Poincaré invariance on (t, x). This implies that the fields t^i only depend on $(z, \bar{z})$. Furthermore, this also leads us to do the following metric ansatz

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + e^{2D} \mathrm{d}z \mathrm{d}\bar{z}\,, \tag{2.66}$$

where D only depends on $(z,\bar{z})$. If we now compute the equations of motion for the scalars t^i, we find

$$K_{\bar{\imath}j}\partial\bar{\partial}t^j + K_{\bar{\imath}jk}\partial t^j \wedge \bar{\partial}t^k = 0\,, \tag{2.67}$$

with $\partial \equiv \mathrm{d}z\frac{\partial}{\partial z}$. The simplest class of solutions correspond to holomorphic (anti-holomorphic) profiles, $\bar{\partial}t^i = 0$ ($\partial t^i = 0$). If we consider the holomorphic case, and look for solutions corresponding to strings located at $z = 0$ and associated to a monodromy of the form (2.26), that is

$$t^i \longrightarrow t^i + e^i\,, \qquad e^i \in \mathbb{Z}\,, \tag{2.68}$$

we get

$$t^i = t^i_0 + \frac{1}{2\pi i}e^i \log\left(\frac{z}{z_0}\right)\,, \tag{2.69}$$

where t^i_0, $z_0 \in \mathbb{C}$ are integration constants, while $\mathbf{e} = \{e^i\}$, $e^i \in \mathbb{Z}$ are the string charges. These are the solutions we are interested in, according to what we said in Section 2.1. It is useful to decompose t^i as $t^i = a^i + is^i$, and write $z = re^{i\theta}$ so that (2.69) becomes

$$s^i = s^i_0 + \sigma e^i\,, \qquad \text{with} \quad \sigma \equiv \frac{1}{2\pi}\log\left(\frac{r_0}{r}\right)\,, \tag{2.70}$$

$$a^i = \frac{\theta}{2\pi}\,e^i + \text{const}\,. \tag{2.71}$$

As suggested by the notation, the fields a^i are to be interpreted as the axions of Section 2.1, whereas s^i are their saxionic partners.

One may analyse how the saxionic coordinates s^i evolve as r changes, by either approaching or moving away from the string. In particular, we see that they develop a logarithmic singularity as they approach the string core, namely

$$s^i \longrightarrow e^i \cdot \infty \qquad \text{for} \quad r \longrightarrow 0\,. \tag{2.72}$$

To continue with our discussion, let us assume that the saxionic flow (2.70) takes us to a region of the field space where the Kähler potential only depends on the saxions and therefore exhibits a continuous axionic shift symmetry

$$a^i \longrightarrow a^i + c^i\,, \tag{2.73}$$

with c^i some arbitrary constants.

It is worthwhile to mention that the solution (2.69) is not unique: one might add an arbitrary linear combination of terms of the form

$$e^{2\pi i m_i t^i}\,, \qquad m_i \in \mathbb{Z}\,, \tag{2.74}$$

without spoiling the monodromy (2.68). These are non-perturbative contributions, generated by $\frac{1}{2}$-BPS instantons charged under the axionic symmetries, with charges m_i, and are typically the leading non-perturbative corrections. Their effect is that of breaking the continuous shift symmetry (2.73) to a discrete one, i.e. $a^i \to a^i + 1$.

Therefore, in order for the solution (2.69) to be physically sensible and restore the continuous axionic shift symmetry in an appropriate asymptotic field-space region, such terms must be negligible in a sufficiently large disk around the string core. By considering that

$$|e^{2\pi m_i t^i}| = e^{-2\pi m_i s^i} = e^{-2\pi m_i s^i_0}e^{-2\pi\sigma m_i e^i}\,, \tag{2.75}$$

we see that, if we start in the perturbative region, namely at a point in field space where $m_i s^i_0 \gg 1$, the instanton corrections remain negligible as $r \to 0$ (or, equivalently, $\sigma \to \infty$) if and only if the following positivity constraint is satisfied by the string and instanton charges:

$$e^i m_i \geq 0\,. \tag{2.76}$$

This additional requirement leads us to the definition of *EFT strings*, which are those axionic strings whose flows drive the EFT towards a region in which K displays a perturbative axionic shift symmetry and non-perturbative effects are suppressed.

What we said so far may be described in terms of some conic structures for the saxions s^i and the charges e^i, m_i. Before going on, let us recall that, as anticipated in Section 2.1, the BPS-ness of the solution implies that the string tension is given by

$$\mathcal{T}_{\mathbf{e}} = M_{\mathrm{P}}^2 \, |e^i \ell_i| \,, \tag{2.77}$$

where ℓ_i stand for the dual saxions that appear in the linear multiplet description of the axionic theory. In particular, let us focus on the case

$$e^i \ell_i > 0 \,, \tag{2.78}$$

which, from here on out, will be the one associated to the $\frac{1}{2}$-BPS strings, which preserve a given half of the bulk supersymmetry. Conversely, we call $\frac{1}{2}$-BPS anti-strings those satisfying $e^i\ell_i < 0$, and they preserve the opposite half of the bulk supersymmetry.

Let us denote by $\mathcal{C}_I$ the set of BPS instantons charges which measure the breaking of the perturbative axionic shift symmetry. According to (2.75), the perturbative regime is obtained by requiring the saxions to lie in the deep interior of the *saxionic cone* Δ, defined as

$$\Delta \equiv \{ s^i \in \mathbb{R} \,|\, \langle \mathbf{m}, \mathbf{s} \rangle > 0 \,, \forall \mathbf{m} \in \mathcal{C}_I \} \,, \tag{2.79}$$

where $\langle \mathbf{m}, \mathbf{s} \rangle \equiv m_i s^i$ represents the natural pairing between instanton charges and saxions. We now define the cone of dual saxions ℓ_i as

$$\mathcal{P} \equiv \left\{ \ell_i \in \mathbb{R} \,|\, \ell_i = -\frac{1}{2} \frac{\partial K}{\partial s^i} \bigg|_{\mathbf{s}\in\Delta} \right\} \,, \tag{2.80}$$

and the cone of BPS string charges, satisfying (2.78), as

$$\mathcal{C}_{\mathrm{S}} \equiv \mathcal{P}^{\vee} \cap E_{\mathbb{Z}} = \{ e^i \in \mathbb{Z} \,|\, e^i \ell_i > 0 \,, \forall \ell \in \mathcal{P} \} \,, \tag{2.81}$$

where $E_{\mathbb{Z}}$ is the integer lattice on which string charges take value, and $\mathcal{P}^{\vee}$ is the dual cone of $\mathcal{P}$. Its definition is the following: given a vector space V, its dual V^* and a subset $S \subset V$, the dual cone of S is [24]

$$S^{\vee} = \{ u \in V^* \,|\, \langle u, v \rangle \geq 0 \,, \forall v \in S \} \,. \tag{2.82}$$

At this point, from (2.76) we finally find that the exponential suppression is present for any $\mathbf{m} \in \mathcal{C}_I$ as $\sigma \to \infty$, i.e. as we approach the string, if and only if $\mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}$, with

$$\mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}} \equiv \overline{\Delta} \cap E_{\mathbb{Z}} = \{ e^i \in \mathbb{Z} \,|\, \langle \mathbf{m}, \mathbf{e} \rangle \geq 0 \,, \forall \mathbf{m} \in \mathcal{C}_I \} \,. \tag{2.83}$$

If we further require that $e^i\ell_i > 0$, namely that $\mathbf{e} \in \mathcal{C}_{\mathrm{S}}$, we get

$$\mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}} \subset \mathcal{C}_{\mathrm{S}} \quad \Longleftrightarrow \quad \overline{\Delta} \subset \mathcal{P}^{\vee} \,. \tag{2.84}$$

To sum up, the EFT strings are defined as those $\frac{1}{2}$-BPS strings with charge vector $\mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}$. Their definition implies that along an EFT string flow we can neglect the non-perturbative corrections breaking the axionic shift symmetries, and we are allowed to pass to the dual formulation of Section 2.1.

In conclusion, let us recall that in Section 2.3.3 we have seen that a 4-dimensional BPS axionic string of charges e^i arises from an M5-brane wrapped on the calibrated 4-cycle $\Pi = e^i \Pi_i$. This is an EFT string, i.e. $\mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}$, if the cohomology class of the Poincaré dual 3-cocycle $[\Pi]$[12] admits an associative 3-form $\varphi_{\mathbf{e}}$ (or a limit thereof, obtained by approaching the boundary of the corresponding saxionic cone) as representative [7].

[12]This cocycle is the 3-form related to the 4-cycle Π by Poincaré duality, which, generally speaking, is an isomorphism between $H^p(X)$ and $H_{d-p}(X)$. In this case, we have $d = 7$, $p = 3$.

2.5 EFT strings and the Swampland Program

As already said at the beginning of this Section, we now review the main results concerning EFT strings, which are the Swampland conjectures proposed in [6, 7] and the quantum gravitational constraints discussed in [11].

2.5.1 EFT strings and Swampland Conjectures

By standard quantum gravity arguments [4, 5], we only expect to realise global continuous symmetries at points of infinite distance in moduli space. As a result, 4d EFTs consistent with quantum gravity should map EFT string locations to points s^i_∞ at infinite distance in their moduli space, and this map is governed by the string charges. The *Distant Axionic String Conjecture* (DASC) [6, 7] claims that the reverse is also true, namely that *all* infinite field distance limits can be realised as an EFT string flow:

Conjecture 1 (Distant Axionic String Conjecture)**.** *Every infinite field distance limit of a 4d EFT consistent with quantum gravity can be realised as an RG flow UV endpoint of an EFT string.*

Another conjecture that deals with infinite distance limits is the Swampland Distance Conjecture (SDC) [25], which states that points at infinity in a moduli space correspond to points where an infinite tower of massless states appears. In particular, it predicts that an infinite tower of new light states arises whenever an asymptotic limit of infinite distance is taken, and its lightest mass m_* decreases exponentially as $e^{-\alpha\Delta\phi}$, where α is some constant and $\Delta\phi$ is the geodesic field distance. Conjecture 1 provides a natural cut-off scale, given by the EFT string tension, as we will explictly see momentarily. However, in general, this is a maximal cut-off scale, since there may be additional towers of states whose typical scale gets light faster than the EFT string tension. The so-called Integral Scaling Conjecture (ISC) [7, 8] elaborates on the relation between the EFT string tension $\mathcal{T}$ and the leading microscopic tower scale m_* [13]:

Conjecture 2 (Integral Scaling Conjecture)**.** *Along the asymptotic flow associated with an EFT string, its tension $\mathcal{T}$ goes to zero. The microscopic tower mass m_* then scales like*

$$m_*^2 \simeq M_\mathrm{P}^2 A \left(\frac{\mathcal{T}}{M_P^2}\right)^w , \qquad w \in \{1,2,3\} , \tag{2.85}$$

with A a coefficient not depending on the flowing fields.

The integer w is called *scaling weight* of the EFT string.

Let us now show some of the consequences of the two conjectures. First of all, one can derive, from a bottom-up perspective, the SDC in 4d EFTs, by means of the DASC and the Weak Gravity Conjecture (WGC) [26] for strings. The reasoning goes as follows: we first write the WGC for strings

$$M_\mathrm{P} \mathcal{Q}_\mathbf{e} \geq \gamma \mathcal{T}_\mathbf{e} , \tag{2.86}$$

where we refer to γ, which is a constant, as the *extremality factor* of the EFT string. On the other hand, $\mathcal{Q}_\mathbf{e}$ is the physical string charge, defined as

$$\mathcal{Q}_\mathbf{e} = M_\mathrm{P} \sqrt{\mathcal{G}_{ij} e^i e^j} , \tag{2.87}$$

with $\mathcal{G}_{ij}$ the (s)axion kinetic matrix, which can be read from (2.11) and is given by

$$\mathcal{G}_{ij} \equiv \frac{1}{2} \frac{\partial^2 K}{\partial s^i \partial s^j} . \tag{2.88}$$

[13] We report here its strongest version, proposed in [8], whereas in [7] the scaling weight w is free to be any positive integer.

We can convince ourselves that this is the proper definition of string charge, by making an analogy with a point particle coupled to the electromagnetic field A_μ. Indeed, in this case, the action reads

$$S_{\text{em}} = -m\int_\gamma \mathrm{d}s - \frac{1}{e^2}\int F\wedge *F + q\int_\gamma A\,, \tag{2.89}$$

where e is the electromagnetic coupling constant, $F=\mathrm{d}A$ is the electromagnetic field-strength and γ is the worldline of a particle with mass m and charge q in units of e. If we now want to obtain a canonically normalized kinetic term for the gauge field, we have to rescale $A\to eA$, thus obtaining

$$eq\int_\gamma A \tag{2.90}$$

for the interaction term. From (2.90), we see that the physical charge of the particle is $Q=eq$. In our case, as can be seen from (2.16), instead of $1/e^2$, there appears the gauge kinetic matrix $\mathcal{G}^{ij}$, therefore, by calling $\mathcal{G}_{ij}$ the inverse of $\mathcal{G}^{ij}$, the appropriate definition of the physical string charge is exactly given by (2.87).

Let us now consider the BPS saxionic flow (2.70). By also recalling the expression of the tension (2.77) and the dual saxions (2.17), we find that

$$\frac{\mathrm{d}\mathcal{T}_{\mathbf{e}}(\sigma)}{\mathrm{d}\sigma} = M_\mathrm{P}^2 e^i\frac{\mathrm{d}\ell_i}{\mathrm{d}\sigma} = M_\mathrm{P}^2 e^i\frac{\partial\ell_i}{\partial s^j}\frac{\partial s^j}{\partial\sigma} = -M_\mathrm{P}^2\left(\frac{1}{2}\frac{\partial^2 K}{\partial s^i\partial s^j}\right)e^ie^j = -\mathcal{Q}_{\mathbf{e}}^2<0\,, \tag{2.91}$$

where we have considered $\frac{1}{2}$-BPS strings, defined by (2.78). This relation shows that supersymmetry alone, which fixes the expression for the tension in terms of the string charges as in (2.77), leads to the conclusion that the tension of a string must decrease along its own flow, as we approach the string itself. In addition to that, this relation also allows us to compute the field space distance travelled by the saxionic flow in terms of the corresponding tension variation:

$$d_\sigma = \int_{\text{flow}}\sqrt{\mathcal{G}_{ij}\mathrm{d}s^i\mathrm{d}s^j} = \int_0^\sigma \mathrm{d}\sigma\sqrt{\mathcal{G}_{ij}e^ie^j} = \frac{1}{M_\mathrm{P}}\int_0^\sigma \mathcal{Q}_{\mathbf{e}}\mathrm{d}\sigma = \frac{1}{M_\mathrm{P}}\int_{\mathcal{T}_{\mathbf{e}}(\sigma)}^{\mathcal{T}_{\mathbf{e}}^0}\frac{1}{\mathcal{Q}_{\mathbf{e}}}\mathrm{d}\mathcal{T}_{\mathbf{e}}\,, \tag{2.92}$$

from which, by using the WGC bound (2.86), we get

$$d_\sigma \le \frac{1}{\gamma}\int_{\mathcal{T}_{\mathbf{e}}(\sigma)}^{\mathcal{T}_{\mathbf{e}}^0}\frac{\mathrm{d}\mathcal{T}_{\mathbf{e}}}{\mathcal{T}_{\mathbf{e}}} = \frac{1}{\gamma}\log\frac{\mathcal{T}_{\mathbf{e}}^0}{\mathcal{T}_{\mathbf{e}}(\sigma)}\,. \tag{2.93}$$

Therefore, the maximal EFT cut-off scale consistent with the existence of the string decreases as

$$\Lambda_{\text{max}}^2 = \mathcal{T}_{\mathbf{e}}(\sigma) \le \mathcal{T}_{\mathbf{e}}^0\exp\left(-\gamma d_\sigma\right)\,. \tag{2.94}$$

This result shows what we anticipated, namely that the WGC for strings, combined with the DASC, provides the exponential drop-off of the maximal EFT-breaking scale along every infinite distance limit in moduli space, as predicted by the SDC. Furthermore, it also gives a relation between the scaling weight w and the constant α appearing in the SDC. Indeed, (2.85) and (2.94) together imply the exponential drop-off of the leading microscopic scale m_*:

$$m_* \le m_*^0\exp\left(-\alpha d_\sigma\right)\,,\qquad \alpha = \frac{w\gamma}{2}\,. \tag{2.95}$$

Both Conjecture 1 and Conjecture 2 are supported by several string theory examples, as analysed in [7], and lead to interesting results. One of the most considerable is the direct relationship between the WGC for strings and the SDC provided by the DASC, as shown above. Furthermore, the two Conjectures are in agreement with the Emergent String Conjecture (ESC) [27], which claims that any equi-dimensional infinite distance limit[14] in the moduli space of a d-dimensional quantum gravity theory reduces to a weakly coupled string theory. By considering the ISC, this happens when the scaling weight w is equal to 1, since, in this case, the leading infinite tower of asymptotically massless states is provided by the EFT strings.

[14] This limit is distinguished from the so-called decompactification limit to a higher-dimensional theory, which appears, for instance, when the leading tower of asymptotically massless states is given by the Kaluza-Klein (KK) states, which become light as some of the directions in the compactification space of the higher-dimensional theory are taken to be large so that the total internal volume diverges.

2.5.2 EFT constraints from EFT strings

In Section 2.1, we saw how the presence of an axionic string of charges e^i and world-sheet $\mathcal{S}$ leads to the non-closure of the one-forms θ^i appearing in (2.13), which instead satisfy

$$\mathrm{d}\theta^i = e^i \delta_2(\mathcal{S})\,. \tag{2.96}$$

As already pointed out, this is the reason why axions undergo an integral shift

$$a^i \longrightarrow a^i + e^i\,, \tag{2.97}$$

set by the charges e^i, when encircling the string. In addition to this, (2.96) generates an anomaly inflow from the four-dimensional bulk to the string worldsheet. In particular, the anomaly inflow discussed in [11] is valid for all such theories with a standard coupling to the axionic sector. By this, we mean theories where the axions a^i couple to the gauge field strength F and the curvature two-form R via linear couplings of the form[15]

$$C_i \int a^i \,\mathrm{tr}\, F\wedge F + \tilde{C}_i \int a^i \,\mathrm{tr}\, R\wedge R\,. \tag{2.98}$$

The anomaly inflow induced by such couplings consists of a contribution localized on the string world-sheet $\mathcal{S}$. Therefore, since a consistent EFT must be anomaly free, we expect this contribution to be cancelled by a string world-sheet anomaly. In turn, this is generated by the world-sheet sector supported by the EFT strings.

Motivated by the analysis of explicit string theory realizations, it was proposed in [11] that such world-sheet sector admits a weakly-coupled non-linear sigma model (NLSM) description[16]. The NLSM includes:

- the universal 'center of mass' sector, which is described by the Green-Schwarz (GS) formulation. We will discuss the GS formalism for the universal sector of the EFT strings in Section 3.2. For the moment, let us only say that in this formalism the string is described by the embedding of its world-sheet, parametrized by two bosonic coordinates ξ^m, $m = 0, 1$, in the target superspace. Thus, this embedding is determined by the fields $x^{\underline{m}}(\xi)$, $\theta^{\underline{\mu}}(\xi)$ and $\bar{\theta}_{\underline{\dot\mu}}(\xi)$, which describe the bosonic and Grassmann spinor coordinates of the string in the target superspace, respectively[17]. As far as the bosonic part, we may use the reparametrization invariance of the string action to go locally in the so-called *static gauge*, which fixes the longitudinal directions $x^0 = \xi^0$, $x^3 = \xi^1$, while leaving $x^1(\xi)$ and $x^2(\xi)$ as the only bosonic physical fields which describe the dynamics of the string. Together, they can be regarded as the real and imaginary part of a complex scalar field, $u = x^1 + ix^2$. Furthermore, we will see that in the GS formalism the EFT string action also enjoys a local fermionic symmetry, called κ-symmetry, which is a manifestation of the fact that an $\mathcal{N} = (0, 2)$ local supersymmetry is preserved on the string world-sheet[18]. This implies that the complex scalar field u has a supersymmetric partner, which is given by the right-moving component of $\theta_{\underline{\mu}}(\xi) = (\theta_-(\xi), \theta_+(\xi))$, i.e. $\rho_+ \equiv \theta_+(\xi)$. Together, the two fields u and ρ_+ can be assembled to give the 'universal' $\mathcal{N} = (0, 2)$ two-dimensional chiral superfield

$$U = u + \sqrt{2}\theta^+\rho_+ - 2i\theta^+\bar{\theta}^+\partial_{++}u\,. \tag{2.99}$$

- an 'internal' sector, which we assume to be given by[19]:

[15] This assumption is not so restrictive since non-standard couplings to the axions, such as quadratic and cubic axion couplings to $F\wedge F$, although natural in non-minimal four-dimensional supergravity theories, are quite unusual in our context.

[16] Generally speaking, a strongly coupled subsector may also be present. However, it would not contribute to the anomaly cancellation and its dynamics would not interfere with the weakly coupled NLSM.

[17] Here, the underlined indices refer to target-space indices, while not underlined ones correspond to world-sheet indices.

[18] This will be manifest in Section 4.2, when we will discuss the superembedding formalism, which indeed provides the local fermionic κ-symmetry with a geometrical meaning of local world-sheet supersymmetry.

[19] For a definition of these $\mathcal{N} = (0, 2)$ superfields, see Section 4.4 and [28, 29].

1. n_C chiral multiplets;
2. n_N $U(1)_N$ charged Fermi multiplets;
3. n_F $U(1)_N$ neutral Fermi multiplets.

In the present thesis, we will discuss the case where the internal sector is given by a $U(1)_N$ neutral Fermi multiplet, as we will see in Section 4.4.

At this point, one can compute the 't Hooft anomaly associated with the world-sheet theory just described, and its comparison with the expression obtained by imposing the cancellation of the anomaly inflow leads to quantum gravity bounds on the axionic couplings and the possible ranks of the EFT gauge sector[20]. The main results are[21]:

$$\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle \in \mathbb{Z} \quad \forall \mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}\,, \tag{2.100a}$$
$$\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle \in 3\mathbb{Z}_{\geq 0} \quad \forall \mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}\,, \tag{2.100b}$$
$$4\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle \in 3\mathbb{Z}_{\geq 0} \quad \forall \mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}\,, \tag{2.100c}$$
$$r(\mathbf{e}) \leq r(\mathbf{e})_{\max} \equiv 2\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle - 2 \quad \forall \mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}\,, \tag{2.100d}$$

where $r(\mathbf{e})$ is the rank of the EFT gauge sector 'detected' by the EFT string, given by

$$r(\mathbf{e}) \equiv \mathrm{rank}\{\langle \mathbf{C}^{AB}, \mathbf{e}\rangle\} + \sum_{I|\langle \mathbf{C}^I, \mathbf{e}\rangle > 0} \mathrm{rk}(\mathfrak{g}_I)\,, \tag{2.101}$$

where C_i^{AB} appear in the axionic couplings to the gauge field strengths given by

$$C_i^{AB} \int a^i F_A \wedge F_B \tag{2.102}$$

Furthermore, the couplings in $\hat{\mathbf{C}}(\mathbf{e})$ appear in the following term localised on the string:

$$S_{\mathrm{N}} = -\frac{1}{24}\hat{C}_i(\mathbf{e}) \int_W h_1^i \wedge A_{\mathrm{N}}\,. \tag{2.103}$$

This term gives an additional contribution to the $U(1)_N$ anomaly inflow and arises from the fact that the pullback to $\mathcal{S}$ of $\delta_2(\mathcal{S})$ gives a non-vanishing finite term [11].

To recapitulate, we have seen how consistency conditions of the axionic strings are turned into non-trivial constraints on the four-dimensional effective field theory, which rule out otherwise consistent supergravity theories. In particular, (2.100d) gives a non-trivial bound on the rank of the EFT gauge sector detected by the EFT string, which may be relevant in phenomenological models. These bounds are derived considering that EFT strings support an additional 'internal' sector, in addition to the 'universal' one, which arises from deformations of the internal configuration of the compactification space. However, the above analysis does not take into account the full theory describing the interactions of this internal sector with the bulk fields. Having a way to study in a controlled way the full theory could be interesting to find possible new constraints on the $\mathcal{N}=1$ 4-dimensional EFT. Therefore, in the present thesis, we focus on this issue, by elaborating on how to describe the full theory in a supersymmetrically controlled way.

[20] To obtain such consistency conditions, we assume that the EFT string lattice $\mathcal{C}_S^{\mathrm{EFT}}$ is fully populated. This assumption is realized in large classes of string theory models (see [7]) and may be more generically motivated by invoking an EFT string version of the Completeness Hypothesis [30].

[21] Analogously to Section 2.4.1, $\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle = e^i \tilde{C}_i$ and so on.

Chapter 3

Supergravity theories in $D = 2$ and $D = 4$ dimensions

In this work, we focus on describing the physics of BPS strings in 4d $\mathcal{N} = 1$ EFTs. In particular, in Chapter 4 we will analyse the world-sheet theory of EFT strings, which preserve a local $\mathcal{N} = (0, 2)$ supersymmetry on their worldsheet. Therefore, the purpose of this Chapter is to discuss the main ingredients which will be needed in the following, which are $D = 4$ $\mathcal{N} = 1$ supergravity and $D = 2$ $\mathcal{N} = (0, 2)$ supergravity. The former is introduced in Section 3.1. This will give us the basics to build, in Section 3.2, the supergravity Lagrangian for the axionic multiplets T^i, introduced in Section 2.1, and its dual version. Finally, in Section 3.3 we discuss in detail $\mathcal{N} = (0, 2)$ supergravity, which will be necessary in the study of the world-sheet theory of the EFT strings.

3.1 $\mathcal{N} = 1$ supergravity theories in 4 dimensions

This Section is dedicated to the discussion of 4-dimensional $\mathcal{N} = 1$ supergravity theories. We start in Section 3.1.1 by introducing rigid $\mathcal{N} = 1$ superspace. Then, in Section 3.1.2 we build the minimal supergravity theory in 4 dimensions, namely the theory describing the graviton and the associated gravitino. We finally extend this theory to a supergravity also containing a set of chiral superfields in Section 3.1.3, by means of the *superspace formalism*.

3.1.1 Rigid $\mathcal{N} = 1$ superspace in 4 dimensions

Supersymmetry (SUSY) is a space-time symmetry mapping particles and fields of integer spin (bosons) into particles and fields of half integer spin (fermions), and viceversa. The generators Q schematically act as

$$\begin{cases} Q\,|Fermion\rangle = |Boson\rangle \\ Q\,|Boson\rangle = |Fermion\rangle \end{cases} . \tag{3.1}$$

Since it changes the spin of a particle, and hence its space-time properties, supersymmetry is a *space-time symmetry*.

The usual space-time Lagrangian formulation is not the most convenient one for describing supersymmetric field theories. This is because in ordinary space-time supersymmetry is not manifest. In fact, an extension of ordinary space-time, known as *superspace*, happens to be the most natural framework in which to formulate supersymmetric theories, at least for our purposes. Therefore, before considering $\mathcal{N} = 1$ supergravity theories in four dimensions, we briefly introduce the superspace formalism.

The basic idea of $\mathcal{N} = 1$ superspace is to enlarge the ordinary space-time with coordinates x^m, by adding $2+2$ anti-commuting Grassmann coordinates θ^α, $\bar{\theta}_{\dot{\alpha}}$, and obtain a eight coordinate superspace labelled by $(x^m, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$. A supersymmetry transformation with parameters ε^α, $\bar{\varepsilon}_{\dot{\alpha}}$ is a translation in superspace, given by the following infinitesimal variations of the superspace coordinates[1]

$$\begin{cases} \delta x^m = i(\theta\sigma^m\bar{\varepsilon} - \varepsilon\sigma^m\bar{\theta}) \\ \delta\theta^\alpha = \varepsilon^\alpha \\ \delta\bar{\theta}_{\dot{\alpha}} = \bar{\varepsilon}_{\dot{\alpha}} \end{cases}, \tag{3.2}$$

where the presence of the ε, $\bar{\varepsilon}$-depending piece in δx^m arises from the anticommutation relation

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m \,, \tag{3.3}$$

which, in words, expresses the fact that two subsequent supersymmetry transformations generate a space-time translation. The σ^m matrices appearing in the expression for δx^m are 2×2 matrices where $\sigma^0 = -\mathbb{1}$, while σ^i are the Pauli matrices[2].

Superfields are fields in superspace, namely functions of the superspace coordinates. Since the Grassmann coordinates anticommute, any product involving more than two θ's or two $\bar{\theta}$'s vanishes. Hence, the most general superfield $F = F(x, \theta, \bar{\theta})$ can be expanded as[3]

$$\begin{aligned} F(x,\theta,\bar{\theta}) = f(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) + \\ + \theta\sigma^\mu\bar{\theta} v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta} d(x) \,. \end{aligned} \tag{3.4}$$

The supersymmetry variation of a superfield by parameters ε, $\bar{\varepsilon}$ is represented as

$$\delta_{\varepsilon,\bar{\varepsilon}} F = F(x+\delta x, \theta+\delta\theta, \bar{\theta}+\delta\bar{\theta}) - F(x,\theta,\bar{\theta}) = (i\varepsilon Q + i\bar{\varepsilon}\bar{Q})F \,, \tag{3.5}$$

where Q and $\bar{Q}$ are the differential operators implementing the supersymmetry transformations

$$\begin{cases} Q_\alpha = -i\partial_\alpha - \sigma^m_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_m \\ \bar{Q}_{\dot{\alpha}} = +i\bar{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^m_{\beta\dot{\alpha}}\partial_m \end{cases}, \tag{3.6}$$

where $\partial_\alpha = \partial/\partial\theta^\alpha$, $\bar{\partial}_{\dot{\alpha}} = \partial/\partial\bar{\theta}^{\dot{\alpha}}$, $\partial_m = \partial/\partial x^m$.

Let us now see how to construct supersymmetric invariant actions. The basic point is that the integral over the superspace of an arbitrary superfield, i.e.

$$\int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} F(x,\theta,\bar{\theta}) \,, \tag{3.7}$$

is manifestly supersymmetric invariant. One can easily verify this property by observing that under supersymmetry transformations the integrand in eq. (3.7) transforms as a total space-time derivative plus terms which vanish because of the integration in $\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}$. Hence, supersymmetric invariant actions are built by integrating in superspace a properly defined superfield, which, in general, will be a product of superfields[4]. However, in the general expression for a superfield (3.4) there are too many field components to correspond to an irreducible representation of the supersymmetry algebra. Therefore, we now introduce two classes of constrained superfields, the *chiral superfields* and the *vector superfields*, which are relevant since they are the right superfields to describe matter and gauge fields, respectively.

A chiral superfield Φ is a superfield such that

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \,. \tag{3.8}$$

Analogously, an anti-chiral superfield Ψ is a superfield such that[5]

$$D_\alpha\Psi = 0 \,. \tag{3.9}$$

[1] As far as the conventions are concerned, we will follow [31].
[2] See Appendix B for further details.
[3] Throughout the thesis work, we use the convention according to which superfields are denoted by capital letters and their lowest components by the corresponding small letters.
[4] A product of superfields is still a superfield.
[5] Notice that, if Φ is chiral, $\bar{\Phi}$ is anti-chiral.

D_α and $\bar{D}_{\dot{\alpha}}$ are the covariant derivatives, defined as

$$\begin{cases} D_\alpha = \partial_\alpha + i\sigma^m_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_m \\ \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^m_{\beta\dot{\alpha}}\partial_m \end{cases} . \tag{3.10}$$

They satisfy the following anticommutation relations

$$\begin{aligned} &\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^m_{\alpha\dot{\alpha}}\partial_m \,, \\ &\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = \{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \,. \end{aligned} \tag{3.11}$$

In particular, they anticommute with the supersymmetry generators Q_α, $\bar{Q}_{\dot{\alpha}}$. This implies that the previous constraints (3.8) and (3.9) are invariant under supersymmetry transformations, because, if F is a superfield, $\delta_{\varepsilon,\bar{\varepsilon}}(D_\alpha F) = D_\alpha(\delta_{\varepsilon,\bar{\varepsilon}}F)$. To find the most general expression for a chiral superfield in terms of ordinary fields, we start from the observation that

$$y^m = x^m + i\theta\sigma^m\bar{\theta}, \qquad \bar{y}^m = x^m - i\theta\sigma^m\bar{\theta} \tag{3.12}$$

satisfy $D_\alpha\bar{y}^m = 0$, $\bar{D}_{\dot{\alpha}}y^m = 0$. Therefore, by also using $D_\alpha\bar{\theta}_{\dot{\beta}} = 0$, $\bar{D}_{\dot{\alpha}}\theta_\beta = 0$, it easily follows that, in this coordinate system, the chiral constraint (3.8) is solved by

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\,, \tag{3.13}$$

where $\phi(y)$ is a complex scalar field, $\psi(y)$ is a Weyl spinor, while $F(y)$ is an auxiliary field. Eq. (3.13) can be Taylor-expanded around x to obtain $\Phi(x,\theta,\bar{\theta})$. The expansion for $\bar{\Phi}$ is obtained from (3.13) by conjugation.

Vector superfields satisfy the reality condition

$$V = \bar{V}\,. \tag{3.14}$$

In this case, their power series expansion in θ and $\bar{\theta}$ is

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& v(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta\theta\left[M(x) + iN(x)\right] + \\ & - \frac{i}{2}\bar{\theta}\bar{\theta}\left[M(x) - iN(x)\right] - \theta\sigma^m\bar{\theta}A_m(x) + i\theta\theta\bar{\theta}\left[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^m\partial_m\chi(x)\right] + \\ & - i\bar{\theta}\bar{\theta}\theta\left[\lambda(x) + \frac{i}{2}\sigma^m\partial_m\bar{\chi}(x)\right] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(x) + \frac{1}{2}\Box v(x)\right]\,, \end{aligned} \tag{3.15}$$

where the component fields v, D, M, N and A_m must all be real to satisfy (3.14). Imposing the reality condition, the vector component field A_m survives as a degree of freedom and becomes real, therefore this is the superfield which accomodates a $U(1)$ gauge field. We then introduce the supersymmetric version of gauge transformations, which acts on the vector superfield as

$$V \longrightarrow V + \Phi + \bar{\Phi}\,, \tag{3.16}$$

where Φ is a chiral superfield, and sends $A_m \longrightarrow A_m + \partial_m(2\,\mathrm{Im}\,\phi)$: this is exactly how an ordinary (abelian) gauge transformation acts on a vector field. From the transformations of the component fields of V under (3.16), one can show that properly choosing Φ one can gauge away v, M, N, χ. This gauge choice is called Wess-Zumino gauge[6] and, in this gauge, a vector superfield can be written as

$$V_{WZ} = -\theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}\lambda(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\,. \tag{3.17}$$

[6]Notice that the Wess-Zumino gauge is not supersymmetric. In other words, when working in this gauge, after a supersymmetry transformation, one has to do a compensating supersymmetric gauge transformation to come back to a vector superfield in the Wess-Zumino gauge.

We now define the supersymmetric generalization of the field strength, which is the gauge invariant object entering the action for the dynamics of vector superfields, given by

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V\,. \tag{3.18}$$

These superfields are chiral and gauge invariant. Hence, we can work in the Wess-Zumino gauge and find that W_α contains, among their components, the usual gauge (abelian) field strength $F_{mn} = \partial_m A_n - \partial_n A_m$, as expected.

We conclude this section by writing down the most general gauge invariant $\mathcal{N}=1$ supersymmetric model of matter fields coupled to gauge fields. To this end, we first promote the vector superfield V (3.14) to

$$V = V_a T^a\,, \qquad a = 1, \ldots, n \equiv \dim G\,, \tag{3.19}$$

where T^a are hermitian generators of a non-Abelian gauge group G and V_a are n vector superfields. Then, we define the finite version of the gauge transformation (3.16), given by

$$e^V \longrightarrow e^{i\bar{\Lambda}} e^V e^{-i\Lambda}\,, \tag{3.20}$$

where Λ is a chiral superfield. Under (3.20), the chiral superfield Φ (3.13) transforms as

$$\Phi \longrightarrow \Phi' = e^{i\Lambda}\Phi\,. \tag{3.21}$$

Finally, we generalize the super-field strength (3.18) as follows

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V}D_\alpha e^V\right)\,, \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\left(e^V \bar{D}_{\dot{\alpha}} e^{-V}\right)\,, \tag{3.22}$$

which reduces to (3.18) to first order in V. At this point, after redefining $V \longrightarrow 2gV$[7], the Lagrangian describing the coupling of matter superfields to gauge fields is [32]

$$S = \frac{1}{32\pi}\,\mathrm{Im}\left[\int \mathrm{d}^2\theta \mathcal{F}_{ab}(\Phi) W^{\alpha a} W^b_\alpha\right] + \int d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}e^{2gV}) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi})\,, \tag{3.23}$$

where: $K(\Phi, \bar{\Phi}e^{2gV})$, called *Kähler potential*, is a real function of the chiral superfields and determines the kinetic term of the action; $W(\Phi)$, called *superpotential*, is a holomorphic gauge invariant function of Φ and describes the interactions; $\mathcal{F}_{ab}(\Phi)$, called *generalized complex gauge coupling*, is a holomorphic function of the chiral superfields which transforms in such a way to preserve gauge invariance. The SUSY invariance of the second term of (3.23) has already been discussed (see (3.7)), while the SUSY invariance of the other terms is guaranteed by the chirality of W_α, $\mathcal{F}_{ab}(\Phi)$ and $W(\Phi)$[8].

3.1.2 The 4d $\mathcal{N}=1$ supergravity multiplet

After having introduced the necessary ingredients of supersymmetry, we now analyse *supergravity theories*. Supergravity can be defined in two independent ways that give the same result. It is a supersymmetric theory of gravity, and it is also a theory of local supersymmetry. Having a theory of local supersymmetry means that we need to make the constant parameter ε^α local. From gauge theory, we know that, if we want to make a global symmetry local, we need to introduce a gauge field for the symmetry. In the case of supersymmetry, since it acts on the spinorial index α, the gauge field would be a vector-spinor of spin $\frac{3}{2}$ denoted ψ^α_μ, which we call gravitino. The fact that we have a supersymmetric theory of gravity means that the gravitino must be transformed by supersymmetry into some gravity variable and the index structure suggests us that the latter should be something with only one curved index, namely the vielbein. Thus, the gravitino is at the same time the superpartner of the vielbein and the "gauge field of local supersymmetry". Moreover, due to the fact that we have

[7]This redefinition is performed in order to introduce explicitly the coupling constant g. For example, this leads to have, among the components of W_α, $F_{mn} = \partial_m A_n - \partial_n A_m - ig[A_m, A_n]$, which is the correct non-Abelian generalization for the field strength, with g explicitly appearing in the non-Abelian term.

[8]The chirality of $\mathcal{F}_{ab}(\Phi)$ and $W(\Phi)$ follows from the chirality of Φ and its holomorphicity.

spinors in the theory, we see that we need the *vielbein formalism*. The idea behind this is to consider a set of coordinates that is locally inertial, namely a set of coordinates so that locally we have that

$$g_{mn}(x) = e_m{}^a(x) e_n{}^b(x) \eta_{ab} \,, \tag{3.24}$$

where η_{ab} is the flat Minkowski metric, while $e_m{}^a(x)$ compose the so-called vielbein. The m, n indices (which are the so-called curved indices) are subject to the action of general coordinate transformations $x \longmapsto x'$,

$$e'_m{}^a(x') = \frac{\partial x^n}{\partial x'^m} e_n{}^a(x) \,, \tag{3.25}$$

while the a, b indices (which are called flat indices) are subject to the action of local Lorentz transformation $\Lambda^a{}_b(x)$:

$$e'_m{}^a(x) = \Lambda^a{}_b(x) e_m{}^b(x) \,. \tag{3.26}$$

Flat and curved indices are related by the vielbein itself. As an example, we can convert the constant γ matrices of the inertial frame into γ matrices in the curved frame by the action of the vielbein:

$$\gamma_m(x) = e_m{}^a(x) \gamma_a \,. \tag{3.27}$$

We define the Lorentz covariant derivative as

$$D_m \equiv \partial_m + \frac{1}{2} \omega_m{}^{ab} M_{ab} \,, \tag{3.28}$$

where M_{ab} are the (antisymmetric) generators of the Lorentz group $SO(1,3)$, which on vectors act as

$$M_{ab} X^c = 2 \delta^c_{[a} X_{b]} \,, \tag{3.29}$$

whereas their action on spinors is given by

$$M_{ab} \psi = \frac{1}{2} \gamma_{ab} \psi \,, \tag{3.30}$$

with $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$. On the other hand, $\omega_m{}^{ab}$ are a set of gauge fields, which together define the spin connection ω. Starting from the vielbein and the spin connection, we can define two sets of 2-forms, whose components are those of the torsion and the Riemann curvature[9]:

$$\begin{aligned} T^a &\equiv \frac{1}{2} T^a{}_{mn} dx^m \wedge dx^n \equiv de^a + \omega^a{}_b \wedge e^b \,, \\ R^a{}_b &\equiv \frac{1}{2} R^a{}_{bmn} dx^m \wedge dx^n \equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \,. \end{aligned} \tag{3.31}$$

The spin connection is determined by requiring that the vielbein is covariantly costant, i.e. $D_m e^a_n - \Gamma^p_{mn} e^a_p = 0$, and this condition leads to the following set of algebraic equations for ω:

$$D_{[m} e_{n]}{}^a = \frac{1}{2} T^a{}_{mn} \,. \tag{3.32}$$

A vanishing torsion implies that the spin connection can be fully computed starting from the vielbein and, in this case, its expression is

$$\omega_m{}^{ab}[e] = \frac{1}{2} e_{cm} \left(\Omega^{abc} - \Omega^{bca} - \Omega^{cab} \right) , \qquad \Omega_{abc} = e_a{}^m e_b{}^n \left(\partial_m e_{nc} - \partial_n e_{mc} \right) , \tag{3.33}$$

where $e_a{}^m$ is the inverse vielbein.

Let us use the vielbein formalism to write the supergravity Lagrangian. We start from the construction of the minimal supergravity theory in four dimensions, i.e. the theory describing only the graviton and the associated gravitino. Thus, it must contain a kinetic term for the graviton (the

[9] As reported in Appendix B, with the symbol '$\wedge$' we denote the wedge product, which will be understood henceforth.

Einstein-Hilbert Lagrangian $\mathcal{L}_{EH}$) and a kinetic term for the gravitino (the Rarita-Schwinger Lagrangian $\mathcal{L}_{RS}$). However, one can show [33] that a term quartic in the gravitino must be added to have a supersymmetric theory. In conclusion, the minimal supergravity Lagrangian in 4 dimensions is:

$$\begin{aligned}\mathcal{L}[e,\psi] &= \mathcal{L}_{EH}[e] + \mathcal{L}_{RS}[e,\psi] + \mathcal{L}_{\psi^4}[e,\psi] = \\ &= -\frac{1}{4}ee_a{}^m e_b{}^n R_{mn}{}^{ab}[\omega[e]] + \frac{1}{2}\varepsilon^{mnrs}\bar{\psi}_m\gamma_n\gamma_5 D_r\psi_s - \frac{1}{4}e(K_a{}^{ac}K_b{}^b{}_c + K^{abc}K_{cab})\,,\end{aligned} \tag{3.34}$$

where e stands for the determinant of the vielbein, $R_{mn}{}^{ab}[\omega[e]]$ are the coefficients of the curvature tensor, given by

$$R_{mn}{}^{ab}[\omega[e]] = 2\partial_{[m}\omega_{n]}{}^{ab} + 2\omega_{[m}{}^{ac}\omega_{n]c}{}^{b}\,, \tag{3.35}$$

while K is a quantity which depends quadratically on the gravitino:

$$K^{abc} = e^{bm}K^a{}_m{}^c = -ie^{bm}\left(\bar{\psi}^{[a}\gamma^{c]}\psi_m + \frac{1}{2}\bar{\psi}^a\gamma_m\psi^c\right)\,. \tag{3.36}$$

Note that e^a_m has 6 independent degrees of freedom: in fact, it has 16 components but we have the "gauge invariance" of general coordinate transformations and the local Lorentz invariance, which allow us to fix $4+6=10$ components of the vielbein. On the other hand, as far as the gravitino ψ^α_m is concerned, it has $4\cdot 4$ components, but we can use the local supersymmetry transformation, which acts on the gravitino as $\delta\psi_m = D_m\varepsilon$, to fix 4 components, thus we have 12 independent degrees of freedom.
In supersymmetry, particles are organized in supermultiplets, which contain an equal number of bosonic ad fermionic degrees of freedom, $n_B = n_F$. Therefore, we need 6 bosonic auxiliary degrees of freedom, and they are provided by two auxiliary fields, namely a vector field b_a and a complex scalar M, which, together with the vielbein e^a_m and the gravitino ψ^α_m, compose the so-called *supergravity multiplet*, which we will derive in the following.

3.1.3 4d $\mathcal{N}=1$ Supergravity in superspace

If we want to formulate locally supersymmetric theories in more involved cases, it is convenient the use of the superspace formalism. Since it is our case, we now give a generalisation of what we have just seen by using this formalism, including the presence of chiral fields in the theory [31].

Henceforth, we will denote flat superspace indices as $A = (a, \alpha, \dot{\alpha})$ and curved superspace indices as $M = (m, \mu, \dot{\mu})$. The basic objects used to build the supergravity action in the superspace formalism are the super-vielbein forms $E^A(z) = \mathrm{d}z^M E_M{}^A(z)$ and the super-spin connection $\Omega_A{}^B(z) = \mathrm{d}z^M\Omega_{MA}{}^B(z)$, where $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$[10]. The symmetry transformations are also generalised: we define the general coordinate transformations of superspace as parametrised by $\xi^M(z)$

$$z'^M = z^M + \xi^M(z)\,. \tag{3.37}$$

The parameter ξ may be written with either an Einstein or a Lorentz index, by means of the supervielbein

$$\xi^A = \xi^M E_M{}^A\,. \tag{3.38}$$

In the following, we will consider ξ^A as the field-independent transformation parameter, so that ξ^M will depend on the fields through the supervielbein. The lowest components of $\xi^a(z)| = \xi^a(x)$ characterize general coordinate transformations in the x-space, whereas $\xi^\alpha(z)| = \varepsilon^\alpha(x)$, $\xi_{\dot{\alpha}}(z)| = \bar{\varepsilon}_{\dot{\alpha}}(x)$ correspond to local supersymmetry transformations[11].

The generalization of the Lorentz transformations on superspace is described by $L_A{}^B(z)$, given by

$$L_A{}^B = \begin{pmatrix} L_a{}^b & 0 & 0 \\ 0 & L_\alpha{}^\beta & 0 \\ 0 & 0 & L^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}\,, \tag{3.39}$$

[10] For the superspace conventions on p-forms, see Appendix A.2.
[11] Here the vertical line means that the quantity is evaluated at $\theta = \bar{\theta} = 0$.

with the three components related through the σ-matrices as

$$\sigma^a_{\alpha\dot\alpha}\sigma^b_{\beta\dot\beta}L_{ab} = -2\varepsilon_{\alpha\beta}L_{\dot\alpha\dot\beta} + 2\varepsilon_{\dot\alpha\dot\beta}L_{\alpha\beta}\,. \tag{3.40}$$

We assume a diagonal form because we want spin to be preserved by super-Lorentz transformations. Analogously, since the spin connection is a connection (gauge field) for the local Lorentz transformations, the super-spin connection has a diagonal form:

$$\Omega_{MA}{}^B = \begin{pmatrix} \Omega_{Ma}{}^b & 0 & 0 \\ 0 & \Omega_{M\alpha}{}^\beta & 0 \\ 0 & 0 & \Omega_M{}^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}\,. \tag{3.41}$$

The lowest component of $L_{ab}(z)| = L_{ab}(x)$ gives the usual local Lorentz transformations. Their higher components, together with those of $\xi^a(z)$ and $\xi^\alpha(z)$, $\bar\xi_{\dot\alpha}(z)$, can be used to transform away some of the $\theta=\bar\theta=0$ components of the vielbein and the connection. In particular, we may use higher components of ξ^A to write $E_M{}^A(z)\big|$ as [31]

$$E_M{}^A(z)\big| = \begin{pmatrix} e_m{}^a(x) & \frac{1}{2}\psi_m{}^\alpha(x) & \frac{1}{2}\bar\psi_{m\dot\alpha}(x) \\ 0 & \delta_\mu{}^\alpha & 0 \\ 0 & 0 & \delta^{\dot\mu}{}_{\dot\alpha} \end{pmatrix}\,, \tag{3.42}$$

where $e_m{}^a$, $\psi_m{}^\alpha$ and $\bar\psi_{m\dot\alpha}$ describe the spin-2 graviton and the spin-$\frac{3}{2}$ gravitino, respectively. As far as the superconnection is concerned, we may use higher components of $L_A{}^B$ to transform away some of the components of $\Omega_{MA}{}^B(z)\big|$, i.e.

$$\begin{aligned} \Omega_{mA}{}^B(z)\big| &= \omega_{mA}{}^B(x)\,, \\ \Omega_{\mu A}{}^B(z)\big| &= \Omega^{\dot\mu}{}_A{}^B(z)\big| = 0\,. \end{aligned} \tag{3.43}$$

We now define the torsion components of curved superspace as the covariant exterior derivatives of the supervielbein components

$$T^A := \mathcal{D}E^A = \mathrm{d}E^A - E^B\Omega_B{}^A \equiv \frac{1}{2}E^BE^CT_{CB}{}^A\,, \tag{3.44}$$

and the Riemann curvature components in terms of the connection as

$$R^{AB} := \mathrm{d}\Omega^{AB} - \Omega^{AC}\wedge\Omega_C{}^B \equiv \frac{1}{2}E^CE^DR_{DC}{}^{AB}\,. \tag{3.45}$$

The supertorsion and the supercurvature are the only covariant tensors which may be constructed from the supervielbein and the superconnection. Higher derivatives lead to the so-called *Bianchi identities*, since $\mathrm{dd}=0$ [31]. At this point, one has to find proper constraints for the two tensors which reduce the number of component fields as much as possible and admit flat superspace as a particular solution. It turns out that

$$\begin{aligned} &T_{\alpha\dot\beta}{}^a = T_{\dot\beta\alpha}{}^a = -2i\sigma^a_{\alpha\dot\beta}\,, \\ &T_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = 0, \quad T_{\alpha\beta}{}^c = T_{\dot\alpha\dot\beta}{}^c = 0\,, \\ &T_{\underline{\alpha}b}{}^c = T_{a\underline{\beta}}{}^c = 0\,, \\ &T_{ab}{}^c = 0\,, \end{aligned} \tag{3.46}$$

are the proper constraints. Here $\underline{\alpha}$ denotes either α or $\dot\alpha$. Since it is going to be useful in Section 4.2, we only report the expression for T^a, which can be immediately obtained from the constraints (3.46):

$$T^a = -2i\sigma^a_{\alpha\dot\alpha}E^\alpha\wedge\bar E^{\dot\alpha}\,. \tag{3.47}$$

The constraints (3.46) allow for a consistent definition of the chiral superfield in the locally supersymmetric case (see (3.55)). Moreover, by means of them, we can express the superconnection via the supervielbein, as in general relativity. Solving the Bianchi identities subject to the constraints (3.46), one learns that all the components of the supertorsion and the supercurvature may be expressed in terms of the so-called 'main superfields' [31], which are the complex chiral superfield R and its conjugate $\bar{R}$, the vector superfield G_a and the chiral superfield $W_{\alpha\beta\gamma}$, totally symmetric in its indices. Furthermore, once chosen the gauge (3.42) and (3.43), we may try to express the lowest components of the main superfields in terms of $e_m{}^a$, $\psi_m{}^\alpha$ and $\bar{\psi}_{m\dot{\alpha}}$. However, it turns out that this cannot be performed for the lowest components of R and G_a, and this leads to the introduction of new component fields:

$$\begin{aligned} R(z)| &= -\frac{1}{6}M(x)\,, \\ G_a(z)| &= -\frac{1}{3}b_a(x)\,, \end{aligned} \tag{3.48}$$

which are exactly the auxiliary fields which equalize the number of bosonic and fermionic degrees of freedom within the supergravity multiplet.

We now define the proper generalization of gauge transformations, i.e. the *supergauge transformations*. They are a particular combination of general coordinate and structure group transformations of superspace with the property of mapping Lorentz tensors into Lorentz tensors and reducing to supersymmetry transformations in the limit of flat space. To be more precise, we start from the transformation law of a generic tensor field V^A

$$\delta V^A = -\xi^M \partial_M V^A + V^B L_B{}^A\,, \tag{3.49}$$

and we can see that if we impose $L_B{}^A$ to be given by

$$L_B{}^A = -\xi^C \Omega_{CB}{}^A\,, \tag{3.50}$$

we find

$$\delta V^A = -\xi^C \mathcal{D}_C V^A\,, \tag{3.51}$$

which is manifestly covariant under local Lorentz transformations. In (3.51), we have introduced $\mathcal{D}_C = E_C{}^M \mathcal{D}_M$, with

$$\begin{aligned} \mathcal{D}_M V_A &= \partial_M V_A - \Omega_{MA}{}^B V_B\,, \\ \mathcal{D}_M V^A &= \partial_M V^A + (-)^{mb} V^B \Omega_{MB}{}^A\,, \end{aligned} \tag{3.52}$$

if it acts on a covariant or contravariant vector, respectively[12].

Therefore, the transformation law for a tensor superfield V^A may be written as a supergauge transformation (3.51) together with an additional Lorentz transformation parametrized by $L_B{}^A$:

$$\delta V^A = -\xi^C \mathcal{D}_C V^A + V^A L_B{}^A\,. \tag{3.53}$$

Let us now consider a gauged supersymmetry transformation, parametrized by the lowest components $\xi^\alpha(z)| = \varepsilon^\alpha(x)$ and $\bar{\xi}_{\dot{\alpha}}(z)\big| = \bar{\varepsilon}_{\dot{\alpha}}(x)$, namely let us choose

$$\begin{aligned} \xi^a(z)| &= 0\,, \\ \xi^\alpha(z)| &= \varepsilon^\alpha(x)\,, \\ \bar{\xi}_{\dot{\alpha}}(z)\big| &= \bar{\varepsilon}_{\dot{\alpha}}(x)\,, \\ L_{AB}(z)| &= 0\,, \end{aligned} \tag{3.54}$$

in (3.53). In order to preserve the gauge (3.42) and (3.43), one can show [31] that this transformation must be accompanied by a field-dependent Lorentz transformation and an ε-dependent coordinate transformation in the x-space. These transformations represent the so-called *supergravity transformations*, and one can find the transformation laws of the components of the supergravity multiplet under

[12]Here, $(-)^{mb}$ is $+1$ if one or both of the indices M, B are bosonic, while it takes value -1 if both of them are fermionic.

them.
We now want to write down actions which are invariant under *supergravity transformations*. In particular, we are going to write the most general supergravity action for a set of chiral superfields Φ^i. To this aim, we define the chiral superfield Φ, which, in supergravity, is the superfield satisfying the following condition[13]

$$\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0\,, \tag{3.55}$$

which is a generalisation of the chirality condition of the global supersymmetric case, but with $\bar{D}_{\dot\alpha}$ substituted by the covariant derivative, which reduces to $\bar{\mathcal{D}}_{\dot\alpha} = \bar{E}_{\dot\alpha}{}^M\partial_M$ for scalars.

As seen in the global case, chiral superfields contain three component fields. Instead of defining them as the coefficient functions of the expansion in θ and $\bar\theta$, which are coordinate-dependent (since θ and $\bar\theta$ carry Einstein indices), we consider the following covariant components

$$A = \Phi|\,, \qquad \chi_\alpha = \frac{1}{\sqrt{2}}\,\mathcal{D}_\alpha\Phi|\,, \qquad F = -\frac{1}{4}\,\mathcal{D}^\alpha\mathcal{D}_\alpha\Phi|\;, \tag{3.56}$$

whose transformation laws under supergravity transformations are found from the supergauge transformation law of the superfield Φ:

$$\delta\Phi = -\xi^A\mathcal{D}_A\Phi\,. \tag{3.57}$$

One can introduce new Θ variables, carrying local Lorentz indices, so that the expansion coefficients of chiral superfields in these new variables are exactly the covariant components (3.56):

$$\Phi = A(x) + \sqrt{2}\Theta^\alpha\chi_\alpha(x) + \Theta^\alpha\Theta_\alpha F(x)\,. \tag{3.58}$$

In the Θ variables, we can rewrite the transformation law for a chiral multiplet in the following way:

$$\delta\Phi = -\eta^M(x,\Theta)\partial_M\Phi\,, \tag{3.59}$$

where the differential operator ∂_M acts on the spacetime coordinates x^m and the new variables Θ^α, while the transformations parameters, whose expansion is

$$\eta^M(x,\Theta) = \eta^M_{(1)}(x) + \Theta^\alpha\eta^M_{(1)\alpha}(x) + \Theta^\alpha\Theta_\alpha\eta^M_{(2)}(x)\,, \tag{3.60}$$

can be written in terms of the parameters $\varepsilon^\alpha(x)$ and $\bar\varepsilon_{\dot\alpha}(x)$ of the supergravity transformations. The variables Θ are useful in building actions invariant under supergravity transformations. Before seeing this, we first need to give the definition of *chiral density*. A chiral density Δ is a function of superspace, with transformation law

$$\delta\Delta = -\partial_M\left[\eta^M\Delta(-)^m\right]\,. \tag{3.61}$$

This is chosen so that the product of a chiral density and a chiral superfield is again a chiral density, i.e.

$$\delta\Delta\Phi = -\partial_M\left[\eta^M\Delta\Phi(-)^m\right]\,. \tag{3.62}$$

This property implies that we can build invariant actions from chiral superfields: if $g(\Phi)$ is a chiral function of Φ, we have

$$\delta\mathscr{L} = \delta\int \mathrm{d}^4x\mathrm{d}^2\Theta\Delta g(\Phi) = -\int \mathrm{d}^4x\mathrm{d}^2\Theta\partial_M\left[\eta^M\Delta g(\Phi)(-)^m\right] = 0\,. \tag{3.63}$$

An important chiral density, which we denote by $\mathcal{E}$, can be constructed by requiring that its lowest component is the determinant of the vielbein:

$$\mathcal{E}| = \frac{1}{2}e = \frac{1}{2}\det e_m{}^a\,. \tag{3.64}$$

The higher components of $\mathcal{E}$ are obtained from its lowest component and its transformation law, which is determined by the transformation law for $e_m{}^a$ but is also given by the definition (3.61) of chiral

[13] This is a consistent definition only under non-trivial restrictions on the supertorsion, i.e. $T_{\alpha\beta}{}^c = T_{\dot\alpha\dot\beta}{}^c = T_{\alpha\beta}{}^{\dot\gamma} = T_{\dot\alpha\dot\beta}{}^\gamma = 0$ [34], which are contained in the constraints (3.46).

density [31]. To summarize, we have just obtained that we can build an invariant action from a Lagrangian density as in (3.63), using the chiral density $\mathcal{E}$.
We are now ready to write the most general $\mathcal{N} = 1$ supergravity Lagrangian density[14] describing chiral superfields Φ^i coupled to gravity, which is ($\kappa^2 = 8\pi G_N = 1$):

$$\mathscr{L} = \int \mathrm{d}^2\Theta\, 2\mathcal{E} \left[-\frac{1}{8} \left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R \right) \Omega(\Phi, \bar{\Phi}) + W(\Phi) \right] + \text{h.c.}\,, \tag{3.65}$$

where $\Omega(\Phi, \bar{\Phi}) = -3e^{-\frac{1}{3}K(\Phi,\bar{\Phi})}$, with $K(\Phi, \bar{\Phi})$ being the ordinary Kähler potential and $W(\Phi)$, chiral and holomorphic in Φ^i, is the superpotential. The first term reproduces the Einstein and Rarita-Schwinger Lagrangians, and the usual Kähler potential for the chiral superfields in the low-energy limit[15], whereas as far as the second term is concerned, it represents the generalisation of the superpotential term of the flat case.

The detailed computation of the component expansion of (3.65) is discussed in Appendix C. The result for the bosonic components of the Lagrangian is

$$\begin{aligned} \mathscr{L} = &\frac{1}{6} e\Omega\mathscr{R} - e g^{mn}\Omega_{i\bar{\jmath}}\partial_m A^i \partial_n \bar{A}^{\bar{\jmath}} + e\Omega_{i\bar{\jmath}} F^i \bar{F}^{\bar{\jmath}} + \frac{1}{9} e\Omega M M^* - \frac{1}{3} e M^* \Omega_{\bar{\imath}} \bar{F}^{\bar{\imath}} - \frac{1}{3} e M \Omega_i F^i + \\ &- \frac{1}{9} e\Omega b^a b_a - \frac{i}{3} e (\Omega_i \partial_m A^i - \Omega_{\bar{\imath}} \partial_m \bar{A}^{\bar{\imath}}) b^a e_a{}^m - e M^* W - e M \bar{W} + e W_i F^i + e \bar{W}_{\bar{\imath}} \bar{F}^{\bar{\imath}}\,. \end{aligned} \tag{3.66}$$

The Lagrangian (3.66) enjoys a *Kähler-Weyl invariance*, which consists of the invariance under the combined action of a Kähler and a super-Weyl transformation. A *Kähler transformation* acts on the Kähler potential, the superpotential and the auxiliary fields as [10]

$$K(\Phi, \bar{\Phi}) \longrightarrow K(\Phi, \bar{\Phi}) + h(\Phi) + \bar{h}(\bar{\Phi})\,, \tag{3.67a}$$

$$W(\Phi) \longrightarrow e^{-h(\Phi)} W(\Phi)\,, \tag{3.67b}$$

$$M \longrightarrow e^{-\frac{2}{3}(h(\Phi) + \bar{h}(\bar{\Phi}))} M\,, \qquad F^i_\Phi \longrightarrow F^i_\Phi e^{-\frac{2}{3}(h(\Phi) + \bar{h}(\bar{\Phi}))}\,, \tag{3.67c}$$

where $h(\Phi)$ is an arbitrary holomorphic function of Φ^i. On the other hand, a *super-Weyl transformation* acts on the supervielbein as [35]

$$E^a_M \to e^{\Upsilon + \bar{\Upsilon}} E^a_M\,, \qquad E^\alpha_M \to e^{2\bar{\Upsilon} - \Upsilon} \left(E^\alpha_M - \frac{i}{4} E^a_M \sigma_a^{\alpha\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Upsilon} \right)\,, \tag{3.68}$$

with Υ an arbitrary chiral superfield. In particular, (3.68) implies that

$$e \longrightarrow e^{2\Upsilon + 2\bar{\Upsilon}} e\,, \quad \mathrm{d}^2\Theta\, 2\mathcal{E} \longrightarrow e^{6\Upsilon} \mathrm{d}^2\Theta\, 2\mathcal{E}\,. \tag{3.69}$$

It can be easily seen that the superspace Lagrangian (3.65) is invariant if a Kähler-Weyl transformation is performed, with $h(\Phi) = 6\Upsilon$.

The Lagrangian (3.65) may be rendered invariant under Kähler (3.67) and super-Weyl (3.68) transformations separately, while still arriving at the same component Lagrangian (3.66) when a proper gauge-fixing of the super-Weyl transformations is performed. This can be obtained with the so-called *super-Weyl invariant approach* [35–37]. The latter will be particularly useful in the following Section, therefore we provide a review on this formalism in Appendix D.

3.2 Axions in 4d Supergravity

In Section 2.1, we illustrated how to perform the dualization procedure between the axions a^i and the gauge two-forms $\mathcal{B}_{2,i}$. We also mentioned that such procedure admits a supersymmetric generalization. Since we eventually want to consider a four-dimensional $\mathcal{N} = 1$ supergravity theory, in Section 3.2.1 we study, as an intermediate step, the model described by (2.11) in a $\mathcal{N} = 1$ globally supersymmetric framework, to finally arrive at the supergravity action given by (3.87) in Section 3.2.2.

[14] We are considering theories with at most two derivatives acting on the various fields.
[15] This can be seen by restoring in the Lagrangian the factors κ^2 and working in the limit $\kappa \to 0$.

3.2.1 Axions and linear multiplets in global supersymmetry

As already seen in the previous section, in theories enjoying supersymmetry, fields are appropriately collected into multiplets. Therefore, we must introduce multiplets which contain, in their components, the gauge two-forms $\mathcal{B}_{2,i}$ and the axions a^i. We start by introducing the *real linear superfield* L, which, in rigid superspace, is a real superfield obeying the constraints

$$D^2 L = 0, \qquad \bar{D}^2 L = 0\,. \tag{3.70}$$

Its component expansion is

$$L = l + i\theta\eta - i\bar{\theta}\bar{\eta} - \frac{1}{2}\theta\sigma_m\bar{\theta}\varepsilon^{mnpq}\partial_n\mathcal{B}_{pq} + \frac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\eta + \frac{1}{2}\bar{\theta}^2\theta\sigma^m\partial_m\bar{\eta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box l\,, \tag{3.71}$$

where l is a real scalar, $\mathcal{B}_{mn}$ is a rank 2 antisymmetric tensor and η is a Weyl spinor. Thus, linear multiplets contain the three-form field strengths of gauge two-forms, making them a super-field strength completion thereof. We also recall that a chiral superfield Φ satisfies the constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$ and its expansion in components is

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F + i\theta\sigma^m\bar{\theta}\partial_m\phi - \frac{i}{\sqrt{2}}\theta^2\partial_m\psi\sigma^m\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi\,. \tag{3.72}$$

Here, ϕ and F are complex scalar fields, while ψ is a Weyl spinor.

We note that the real linear multiplet contains the same amount of propagating scalar fields as a chiral multiplet, namely, focusing on the bosonic sector:

- a real scalar (1 d.o.f.) and a gauge two-form (1 d.o.f.) for the former;
- a complex scalar for the latter (2 d.o.f.).

This suggests that there may exist a 'duality' which relates chiral and linear multiplets, and we now show it [9, 10].

Consider a supersymmetric theory with chiral superfields Φ^m, $m = 1, \dots, n$, which we will here treat as spectators, and some other chiral multiplets T^i, $i = 1, \dots, M$. Let us assume that the Kähler potential depends on T^i only via their imaginary parts:

$$K(\Phi, \bar{\Phi}; T) \equiv K(\Phi, \bar{\Phi}; \mathrm{Im}\, T)\,, \tag{3.73}$$

keeping generic the dependence on Φ^a. The Kähler potential is then invariant under the *axionic shifts*

$$T^i \quad\longrightarrow\quad T^i + c^i\,, \tag{3.74}$$

with real constant c^i and we identify the lowest components $\mathrm{Re}\, T^i| = \mathrm{Re}\, t^i \equiv a^i$ as *axions*, the only components influenced by the shift (3.74). For this reason, we refer to the multiplets T^i as *axionic multiplets*. We might trade the axions a^i with gauge two-forms $\mathcal{B}_{2,i}$ via the 'electro-magnetic duality' explained above. To implement this duality, we first relax the assumption that in (3.73) $\mathrm{Im}\, T^i$ are the imaginary parts of chiral superfields T^i and rather consider them real, unconstrained superfields U^i. Then, we start with the Lagrangian

$$\mathscr{L}_{\text{dual}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} K(\Phi, \bar{\Phi}; U) + 2\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, L_i U^i\,, \tag{3.75}$$

where L_i are linear multiplets. At this point, we have two possibilities:

Ordinary chiral formulation In this case, we need to integrate out the linear multiplets L_i from the Lagrangian (3.75). However, since they are constrained by (3.70), we first have to find an expression for the linear multiplets in terms of unconstrained spinorial superfields $\Psi_{i\alpha}$, $\bar{\Psi}_i^{\dot{\alpha}}$ which automatically solves the constraints (3.70). The expression is given by

$$L_i = D^\alpha \bar{D}^2 \Psi_{\alpha i} + \bar{D}_{\dot{\alpha}} D^2 \bar{\Psi}_i^{\dot{\alpha}}\,. \tag{3.76}$$

Then, the variations with respect to $\Psi_{i\alpha}$, $\bar{\Psi}_i^{\dot{\alpha}}$ of the Lagrangian (3.75) sets the following constraints on U^i:

$$\bar{D}^2 D^\alpha U^i = 0\,, \quad D^2 \bar{D}_{\dot{\alpha}} U^i = 0\,, \tag{3.77}$$

which are solved by

$$U^i = \frac{1}{2}(T^i - \bar{T}^i) = \operatorname{Im} T^i\,, \tag{3.78}$$

for arbitrary chiral multiplets T^i. Substituting (3.78) into (3.75), we get a Lagrangian solely depending on the chiral multiplets Φ^a and T^i

$$\mathscr{L}_{\text{chiral}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} K(\Phi, \bar{\Phi}; \operatorname{Im} T)\,. \tag{3.79}$$

Formulation with linear multiplets The alternative formulation where the axionic multiplets T^i are replaced by linear multiplets L_i is obtained by integrating out the real multiplets U^i from (3.75). This operation leads to

$$L_i = \frac{1}{2}\frac{\partial K}{\partial U^i}\,. \tag{3.80}$$

Once this is plugged inside (3.75), we get

$$\mathscr{L}_{\text{linear}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \left(K(\Phi, \bar{\Phi}; U) + \frac{\partial K}{\partial U^i} U^i \right) \equiv \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, F(\Phi, \bar{\Phi}; L)\,, \tag{3.81}$$

where $F(\Phi, \bar{\Phi}; L)$ is the Legendre transform of the Kähler potential $K(\Phi, \bar{\Phi}; \operatorname{Im} T)$.

The procedure outlined above can be followed also in the reverse direction: we may start with the Legendre transform (3.81) and come back to the ordinary formulation. The latter is the direction we will use when we implement the theory to a local supersymmetric one, and is achieved by considering the following Lagrangian

$$\mathscr{L}_{\text{dual}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} F(\Phi, \bar{\Phi}; L) - 2\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, L_i \operatorname{Im} T^i\,, \tag{3.82}$$

where L_i are understood to be real multiplets rather than linear multiplets.

The final step is to generalise the previous model, by formulating it in a $\mathcal{N} = 1$ locally supersymmetric framework, following [9].

3.2.2 Axions and linear multiplets in supergravity

As already mentioned at the end of Section 3.1, a useful technique to obtain supergravity Lagrangians is the super-Weyl invariant approach, which we introduce in Appendix D discussing theories with chiral multiplets. In addition to the latter, in this section we also couple linear multiplets.

The kinetic part of the super-Weyl invariant Lagrangian reads

$$\mathscr{L}_{\text{chiral}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \mathcal{K}(Z, \bar{Z}; \operatorname{Im} T)\,, \tag{3.83}$$

which, as before, depends on the sector T^i only through their imaginary parts. On the other hand, no assumptions are made on the dependence of the kinetic function on the other chiral multiplets $Z^a = U g^a(\Phi)$, $a = 1, \ldots, n+1$. As illustrated in Appendix D, U is the *super-Weyl compensator*: it is an unphysical, chiral superfield, which transforms as

$$U \longrightarrow e^{-6\Upsilon} U \tag{3.84}$$

under super-Weyl transformations. Conversely, $g^a(\Phi)$ are functions of the physical fields only and carry zero super-Weyl weight. Finally, consistency of super-Weyl transformation with the chirality of T^i

requires the axionic multiplets T^i to be inert under super-Weyl transformations, and the homogeneity properties of the kinetic function are

$$\mathcal{K}(\lambda Z, \bar{\lambda}\bar{Z}; \operatorname{Im} T) = |\lambda|^{\frac{2}{3}} \mathcal{K}(Z, \bar{Z}; \operatorname{Im} T)\,. \tag{3.85}$$

In supergravity, linear multiplets satisfy the covariant constraints

$$(\mathcal{D}^2 - 8\bar{\mathcal{R}})L_i = 0\,, \qquad (\bar{\mathcal{D}}^2 - 8\mathcal{R})L_i = 0\,, \tag{3.86}$$

and, as in the globally supersymmetric case, their bosonic components are a real scalar field l_i and a real field strength $\mathcal{H}_{3,i}$ of a gauge two form $\mathcal{B}_{2,i}$, which can be regarded as the electro-magnetic dual of the axions $\operatorname{Re} T^i| = a^i$. In the dual description where the axionic multiplets T^i are replaced with linear multiplets L_i, the kinetic Lagrangian (3.83) gets replaced by

$$\mathscr{L}_{\text{linear}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \mathcal{F}(Z, \bar{Z}; L)\,, \tag{3.87}$$

where $\mathcal{F}(Z, \bar{Z}; L)$, similarly to $F(\Phi, \bar{\Phi}; L)$ in (3.81) for the global case, is the Legendre transform of the kinetic function $\mathcal{K}(Z, \bar{Z}; \operatorname{Im} T)$. Explicitly, their relation is given by

$$\mathcal{K}(Z, \bar{Z}; \operatorname{Im} T) = \mathcal{F}(Z, \bar{Z}; L) - \mathcal{F}^i L_i\,. \tag{3.88}$$

where $\mathcal{F}^i = \partial \mathcal{F}/\partial L_i$. Unlike their chiral counterparts, the linear multiplets L_i are required to have super-Weyl weights $(\frac{1}{3}, \frac{1}{3})$, whence the homogeneity property of the kinetic function is

$$\mathcal{F}(\lambda Z, \bar{\lambda}\bar{Z}; |\lambda|^{\frac{2}{3}} L) = |\lambda|^{\frac{2}{3}} \mathcal{F}(Z, \bar{Z}; L)\,. \tag{3.89}$$

The connection between the ordinary formulation, in terms of chiral multiplets, with Lagrangian (3.83), and the dual formulation, in terms of linear multiplets, with Lagrangian (3.87), is established by the super-Weyl invariant Lagrangian

$$\mathscr{L}_{\text{dual}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \mathcal{F}(Z, \bar{Z}; L) - 2\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, L_i \operatorname{Im} T^i\,. \tag{3.90}$$

with L_i unconstrained real multiplets. The Lagrangian (3.90) represents the appropriate extension of (3.82) to the super-Weyl invariant context. The ordinary and the dual formulations are related as follows.

Ordinary chiral formulation To obtain this formulation, we need to integrate out the real multiplets L_i from (3.90). Varying the latter with respect to L_i, we get

$$\operatorname{Im} T^i = \frac{1}{2}\frac{\partial \mathcal{F}}{\partial L_i}\,. \tag{3.91}$$

The substitution of this relation in (3.90) gives back (3.83):

$$\begin{aligned} \mathscr{L}_{\text{dual}} &= \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E \left(\mathcal{F}(Z, \bar{Z}; L) - \frac{\partial \mathcal{F}}{\partial L_i} L_i\right) \\ &= \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \mathcal{K}(Z, \bar{Z}; \operatorname{Im} T) = \mathscr{L}_{\text{chiral}}\,. \end{aligned} \tag{3.92}$$

Formulation with linear multiplets Analogously to what done for L_i in (3.76), before integrating out the chiral multiplets T^i, we need to solve the chirality constraints $\mathcal{D}_\alpha \bar{T}^i = 0$, $\bar{\mathcal{D}}_{\dot{\alpha}} T^i = 0$ and re-express them as

$$\operatorname{Im} T^i = \frac{1}{2i}\left[(\bar{\mathcal{D}}^2 - 8\mathcal{R})\bar{\Xi}^i - (\mathcal{D}^2 - 8\bar{\mathcal{R}})\Xi^i\right]\,, \tag{3.93}$$

with Ξ^i unconstrained (complex) superfields. From the variation of (3.90) with respect to Ξ^i, one obtains

$$(\mathcal{D}^2 - 8\bar{\mathcal{R}})L_i = 0\,, \quad (\bar{\mathcal{D}}^2 - 8\mathcal{R})L_i = 0\,, \tag{3.94}$$

which tell that L_i are linear multiplets, retrieving (3.87).

We now focus on the dual formulation, described by the Lagrangian (3.87). In the super-Weyl invariant formalism, it turns out that their bosonic components are [9, 10]

$$
\begin{aligned}
e^{-1}\mathscr{L}_{\text{bos}} = &-\frac{1}{6}\tilde{\mathcal{F}}\,R - \mathcal{F}_{a\bar{b}}D_\mu z^a \bar{D}^\mu \bar{z}^{\bar{b}} + \frac{1}{4}\mathcal{F}^{ij}\partial_\mu l_i \partial^\mu l_j + \frac{1}{4\cdot 3!}\mathcal{F}^{ij}\mathcal{H}_{\mu\nu\rho,i}\mathcal{H}^{\mu\nu\rho}_j \\
&+ \left(\frac{i}{2\cdot 3!}\mathcal{F}^i_{\bar{a}}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma,i}\bar{D}_\mu \bar{z}^{\bar{a}} + \text{c.c.}\right) + \mathcal{F}_{a\bar{b}}F^a_Z \bar{F}^{\bar{b}}_Z\,,
\end{aligned}
\tag{3.95}
$$

where

$$
\tilde{\mathcal{F}} = \mathcal{F} - l_i\mathcal{F}^i\,, \qquad D_\mu z^a = \partial_\mu z^a + iA_\mu z^a\,, \tag{3.96}
$$

with

$$
A_\mu = \frac{3}{2(\tilde{\mathcal{F}} - \tilde{\mathcal{F}}^i l_i)}\left[i\left(\tilde{\mathcal{F}}_a\partial_\mu z^a - \bar{\tilde{\mathcal{F}}}_{\bar{a}}\partial_\mu \bar{z}^{\bar{a}}\right) + \frac{1}{3!}\tilde{F}^i\varepsilon_{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma,i}\right]. \tag{3.97}
$$

Before gauge fixing the super-Weyl invariance, we introduce the Legendre transform of the Kähler potential

$$
F(\varphi,\bar{\varphi};\ell) = K + 2\ell_i\,\mathrm{Im}\,t^i \equiv K + 2\ell_i s^i\,, \tag{3.98}
$$

where we have defined the *saxions* s^i as $s^i \equiv \mathrm{Im}\,t^i$ and introduced the *dual saxions* ℓ_i

$$
\ell_i \equiv -\frac{1}{2}\frac{\partial K}{\partial s^i}\,, \tag{3.99}
$$

related to the lowest components of the linear multiplets l_i as

$$
l_i = M_{\mathrm{P}}^2\,\ell_i\,. \tag{3.100}
$$

In (3.95), we may then gauge fix the super-Weyl invariance by setting

$$
u = M_{\mathrm{P}}^3\,e^{\frac{1}{2}\tilde{F}(\phi,\bar{\phi})}\,, \quad \text{with} \quad \tilde{F} = F - \ell_i F^i\,, \tag{3.101}
$$

However, we stress that gauge-fixing the super-Weyl invariance in presence of linear multiplets is a quite involved procedure, which is discussed in [9, 10]. It turns out that, after integrating out the fields F^a_Z, the bosonic components of the gauge-fixed Lagrangian reassemble into a rather simple form ($M_{\mathrm{P}} = 1$):

$$
\begin{aligned}
e^{-1}\mathscr{L}_{\text{bos}} = &\frac{1}{2}R - F_{m\bar{n}}\partial\phi^m\bar{\partial}\bar{\phi}^{\bar{n}} + \frac{1}{4}F^{ij}\left(\partial_\mu\ell_i\partial^\mu\ell_j + \frac{1}{3!}\mathcal{H}_{\mu\nu\rho,i}\mathcal{H}^{\mu\nu\rho}_j\right) + \\
&+ \left(\frac{i}{2\cdot 3!}F^i_{\bar{m}}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma,i}\partial_\mu\bar{\phi}^{\bar{m}} + \text{c.c.}\right).
\end{aligned}
\tag{3.102}
$$

3.3 $\mathcal{N} = (0,2)$ supergravity in $D = 2$ dimensions

As we will see in Chapter 4, the EFT strings, introduced in Section 2.4.1, preserve a local $\mathcal{N} = (0,2)$ supersymmetry on their world-sheet. Therefore, since we eventually want to study the world-sheet theory of such strings in a manifestly supersymmetric way, it is necessary to study $\mathcal{N} = (0,2)$ supergravity in $D = 2$ dimensions. To this aim, we now discuss in detail this theory, by following steps analogous to those already shown for the 4-dimensional case. In particular, we will impose supertorsion constraints in agreement with [38, 39], and in Section 3.3.5 we will arrive at writing the full component expansion of the supervielbein $e_M{}^A$ and the superconnection Ω_M.

Before focusing on $\mathcal{N} = (0,2)$ supergravity, let us introduce $\mathcal{N} = (0,2)$ supersymmetry in rigid superspace.

3.3.1 $\mathcal{N}=(0,2)$ rigid supersymmetry in $D=2$

Let us consider a $(0,2)$ superspace, with coordinates $z^M=(\xi^0,\xi^1,\theta^+,\bar\theta^+)$, where ξ^m, $m=0,1$, are two bosonic coordinates and θ^+ is a complex Grassmann coordinate. Furthermore, let us introduce the null coordinates $y^{\pm\pm}=\xi^0\pm\xi^1$, and the corresponding derivatives

$$\begin{cases}\partial_{++}=\frac{1}{2}(\partial_0+\partial_1)\\ \partial_{--}=\frac{1}{2}(\partial_0-\partial_1)\end{cases}\qquad \begin{cases}\partial_0=\partial_{++}+\partial_{--}\\ \partial_1=\partial_{++}-\partial_{--}\end{cases}\,. \tag{3.103}$$

where $\partial_m=\partial/\partial\xi^m$, $m=0,1$, and $\partial_{\pm\pm}=\partial/\partial y^{\pm\pm}$.

The fields are organized into superfields, and the most general superfield takes the form

$$\Phi(z)=\phi(x)+\theta^+\psi_+(x)+\bar\theta^+\bar\chi_+(x)+\theta^+\bar\theta^+\rho_{++}(x)\,. \tag{3.104}$$

The supersymmetry action on a superfield is given by

$$\delta_\zeta\Phi(z)=\left(\zeta^+Q_++\bar\zeta^+\bar Q_+\right)\Phi(z)\,, \tag{3.105}$$

where Q_+ and $\bar Q_+$ are the representations of the supersymmetry generators as differential operators in field space:

$$Q_+=\frac{\partial}{\partial\theta^+}-2i\bar\theta^+\partial_{++}\,,\qquad \bar Q_+=\frac{\partial}{\partial\bar\theta^+}-2i\theta^+\partial_{++}\,. \tag{3.106}$$

They satisfy

$$Q_+^2=0\,,\qquad \bar Q_+^2=0\qquad \{Q_+,\bar Q_+\}=-4i\partial_{++}\,. \tag{3.107}$$

We now introduce the supercovariant derivatives

$$D_+=\frac{\partial}{\partial\theta^+}+2i\bar\theta^+\partial_{++}\,,\qquad \bar D_+=\frac{\partial}{\partial\bar\theta^+}+2i\theta^+\partial_{++}\,, \tag{3.108}$$

which anti-commute with Q_+ and $\bar Q_+$, and obey the following anticommutation relations:

$$D_+^2=\bar D_+^2=0\,,\qquad \{D_+,\bar D_+\}=4i\partial_{++}\,. \tag{3.109}$$

Instead of the complex Grassmann coordinates θ^+ and $\bar\theta^+$, we may use the real fermionic coordinates η^{+q}, $q=1,2$, defined in terms of the former as

$$\begin{cases}\theta^+=\frac{1}{2\sqrt2}\left(\eta^{+1}+i\eta^{+2}\right)\\ \bar\theta^+=\frac{1}{2\sqrt2}\left(\eta^{+1}-i\eta^{+2}\right)\end{cases}\longrightarrow\qquad \begin{cases}\eta^{+1}=\sqrt2\left(\theta^++\bar\theta^+\right)\\ \eta^{+2}=-i\sqrt2\left(\theta^+-\bar\theta^+\right)\end{cases}\,. \tag{3.110}$$

In these new coordinates, the supercovariant derivatives become

$$D_{+q}=\partial_{+q}+\frac{i}{2}\eta^{+q}\partial_{++}\,, \tag{3.111}$$

where the relation between ∂_+, $\bar\partial_+$ and ∂_{+q} is

$$\begin{cases}\partial_+=\sqrt2\left(\partial_{+1}-i\partial_{+2}\right)\\ \bar\partial_+=\sqrt2\left(\partial_{+1}+i\partial_{+2}\right)\end{cases}\qquad \begin{cases}\partial_{+1}=\frac{1}{2\sqrt2}\left(\partial_++\bar\partial_+\right)\\ \partial_{+2}=\frac{i}{2\sqrt2}\left(\partial_+-\bar\partial_+\right)\end{cases}\,. \tag{3.112}$$

Starting from the supercovariant derivatives, we can compute the supervielbein, which will be the one associated to the flat case, by requiring that it corresponds to the cotangent basis dual to D_A:

$$\mathrm{d}z^M\partial_M=\mathrm{d}y^{++}\partial_{++}+\mathrm{d}y^{--}\partial_{--}+\mathrm{d}\eta^{+q}\partial_{+q}=e^AD_A=e^{++}D_{++}+e^{--}D_{--}+e^{+q}D_{+q}\,. \tag{3.113}$$

In the flat case, $D_{++}=\partial_{++}$ and $D_{--}=\partial_{--}$, thus we find that

$$e^{++}=\mathrm{d}y^{++}-\frac{i}{2}\mathrm{d}\eta^{+q}\eta^{+q},\qquad e^{--}=\mathrm{d}y^{--},\qquad e^{+q}=\mathrm{d}\eta^{+q}\,. \tag{3.114}$$

Now, by computing the supertorsion as $T^A = \mathrm{d}e^A$, we find that the only non-vanishing component is

$$T^{++} = \mathrm{d}e^{++} = \mathrm{d}\eta^{+q}\mathrm{d}\eta^{+r}\left(-\frac{i}{2}\delta_{qr}\right) = \frac{1}{2}\mathrm{d}z^M\mathrm{d}z^N T_{NM}{}^{++} \qquad \Longrightarrow \qquad T_{+q,+r}{}^{++} = -i\delta_{qr}\,. \quad (3.115)$$

The reason why we derived the value of the supertorsion in the rigid case is that our goal is to discuss $\mathcal{N} = (0,2)$ supergravity. As we will see momentarily, to do so, we need to impose supergravity constraints on the supertorsion, which must be compatible with the rigid case.

We are now ready to generalize the $\mathcal{N} = (0,2)$ rigid supersymmetry to the corresponding local supersymmetry in curved space.

3.3.2 2d $\mathcal{N} = (0,2)$ Supergravity in superspace

Let us start by considering an $\mathcal{N} = (2,2)$ supergeometry in a superspace with an $SO(1,1)$ tangent space group. In this case, the superspace has coordinates $z^M = (\xi^m, \theta^\mu, \theta^{\bar{\mu}})$, where θ^μ are two complex odd coordinates and $(\overline{\theta^\mu}) = \theta^{\bar{\mu}}$.

As we have seen in Section 3.1.3, to describe curved superspace geometry, one of the main ingredients is the *supervielbein* $e_M{}^A$, where $M = (m,\mu)$ is a curved superspace index and $A = (a,\alpha)$ a flat superspace index. The inverse supervielbein is denoted by $e_A{}^M$ and obey the following identities

$$e_A{}^M e_M{}^B = \delta_A{}^B\,, \qquad e_M{}^A e_A{}^N = \delta_M{}^N\,. \quad (3.116)$$

As we already know from the 4-dimensional case, we also need to introduce the analogue of the connection in superspace, called *superconnection*, given by $\Omega_A{}^B = \mathrm{d}z^M\Omega_{MA}{}^B$.

Generically, if we consider a p-superform $\phi_A{}^B$ with one covariant and one contravariant tangent space index, the action of the tangent space group on $\phi_A{}^B$ reads

$$\delta\phi_A{}^B = -L_A{}^C\phi_C{}^B + \phi_A{}^C L_C{}^B\,, \quad (3.117)$$

where $L_B{}^A = L\mathcal{M}_B{}^A$, with the superfield $L(z)$ being the boost parameter, and

$$\begin{aligned} \mathcal{M}_b{}^a &= -\varepsilon_b{}^a\,,\\ \mathcal{M}_\beta{}^\alpha &= \frac{1}{2}(\gamma_3)_\beta{}^\alpha\,,\\ \mathcal{M}_{\bar{\beta}}{}^{\bar{\alpha}} &= \frac{1}{2}(\gamma_3)_\beta{}^\alpha\,, \end{aligned} \quad (3.118)$$

where $\varepsilon_b{}^a = \eta_{bc}\varepsilon^{ca}$, with $\varepsilon^{01} = -\varepsilon^{10} = 1$ and $\eta_{bc} = \mathrm{diag}(-1,+1)$. The γ-matrices satisfy

$$\{\gamma^a,\gamma^b\} = 2\eta^{ab}\mathbb{1}\,, \quad (3.119)$$

and are chosen to be

$$\gamma^0 = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.120)$$

while

$$\gamma^3 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (\gamma^3)^2 = \mathbb{1}\,, \qquad \{\gamma^a,\gamma^3\} = 0\,. \quad (3.121)$$

Starting from the supervielbein $e_M{}^A$ and the superconnection $\Omega_B{}^A$, whose explicit form is

$$\Omega_B{}^A = \Omega\mathcal{M}_B{}^A = \mathrm{d}z^M\Omega_M\mathcal{M}_B{}^A\,, \quad (3.122)$$

we may define the supertorsion and the supercurvature as

$$\begin{aligned} T^A &= \frac{1}{2}e^C e^B T_{BC}{}^A = \mathcal{D}e^A = \mathrm{d}e^A + e^B\Omega_B{}^A\,,\\ R_A{}^B &= \mathrm{d}\Omega_A{}^B + \Omega_A{}^C\Omega_C{}^B\,, \end{aligned} \quad (3.123)$$

where $\mathcal{D}$ is the supergravity covariant derivative, whose action on a covariant and contravariant vector is given by

$$\begin{aligned} \mathcal{D}_M X^A &= \partial_M X^A + (-)^{mb} X^B \Omega_{MB}{}^A \,, \\ \mathcal{D}_M X_A &= \partial_M X_A - \Omega_{MA}{}^B X_B \,, \\ \mathcal{D}_B X^A &= E_B{}^M \mathcal{D}_M X^A \,, \\ \mathcal{D}_B X_A &= E_B{}^M \mathcal{D}_M X_A \,. \end{aligned} \tag{3.124}$$

Since we are dealing with an Abelian symmetry, the supercurvature is simply $R_A{}^B = \mathrm{d}\Omega_A{}^B = \mathrm{d}\Omega \mathcal{M}_A{}^B$. Indeed:

$$\Omega_A{}^C \Omega_C{}^B = \mathrm{d}z^M \mathrm{d}z^N \Omega_N \Omega_M \mathcal{M}_A{}^C \mathcal{M}_C{}^B \,. \tag{3.125}$$

Furthermore, we have that

$$\Omega_N \Omega_M \mathcal{M}_A{}^C \mathcal{M}_C{}^B = (-)^{mn} \Omega_M \Omega_N \mathcal{M}_A{}^C \mathcal{M}_C{}^B \,, \tag{3.126}$$

and then

$$\mathrm{d}z^M \mathrm{d}z^N \Omega_N \Omega_M \mathcal{M}_A{}^C \mathcal{M}_C{}^B = -\mathrm{d}z^M \mathrm{d}z^N \Omega_N \Omega_M \mathcal{M}_A{}^C \mathcal{M}_C{}^B = 0 \,. \tag{3.127}$$

The supertorsion and the supercurvature satisfy the *Bianchi identities*

$$\mathcal{D} T^A = e^B R_B{}^A \,, \qquad \mathcal{D} R_A{}^B = 0 \,. \tag{3.128}$$

Let us now label the fermion components as $\theta^\mu = (\theta^-, \theta^+)$. Since we are interested in considering $\mathcal{N} = (0, 2)$ supergravity, let us focus on the case in which $\theta^- = 0$. In this case, the components of $\mathcal{M}_B{}^A$ are

$$\begin{aligned} \mathcal{M}_b{}^a &= -\varepsilon_b{}^a \,, \\ \mathcal{M}_+{}^+ &= \frac{1}{2} \,, \\ \mathcal{M}_{\mp}{}^{\mp} &= \frac{1}{2} \,. \end{aligned} \tag{3.129}$$

In terms of the null coordinates $y^{\pm\pm} = \xi^0 \pm \xi^1$, we have that:

$$\varepsilon_{++}{}^a = \eta_{++b} \varepsilon^{ba} = \eta_{++--} \varepsilon^{--a} = -\frac{1}{2} \left(2\delta^{a,++} \right) = -\delta^{a,++} \,, \tag{3.130}$$

as can be explicitly seen by computing ε^{--a} as

$$\varepsilon^{--a} = \frac{\partial y^{--}}{\partial \xi^b} \frac{\partial y^a}{\partial \xi^c} \varepsilon^{bc} \,. \tag{3.131}$$

Conversely, $\varepsilon_{--}{}^a = \delta^{a,--}$. Analogously, we may rewrite $\mathcal{M}_+{}^+$ and $\mathcal{M}_{\mp}{}^{\mp}$ in the real Grassmann coordinates η^{+q} introduced in Section 3.3.1, and one can easily show that

$$\mathcal{M}_{+q}{}^{+r} = \frac{1}{2} \delta_{qr} \,. \tag{3.132}$$

Henceforth, we will use the indices $+q, +r, \ldots$ to indicate flat spinor indices, and $+u, +v, \ldots$ for curved spinor indices.

To reduce the number of component fields of $e_M{}^A$ and Ω_M, we proceed in analogy with the 4-dimensional case discussed in Section 3.1.3. In particular, in the next section we will perform a partial Wess-Zumino gauge-fixing, while in Section 3.3.4 we will impose constraints on the supertorsion and solve the Bianchi identities subject to such constraints. Finally, in Section 3.3.5 we will use the results of Section 3.3.3 and 3.3.4 to obtain the explicit expression for the component expansion of the supervielbein and the superconnection.

3.3.3 Gauge fixing

Under a general coordinate transformation in superspace, $z^M \to z'^M(z) = \zeta^M(z)$, whose η-expansion is given by

$$z'^M(z) = \zeta^M(z) = \zeta^{(0)M}(\xi) + \eta^{+u}\zeta^{(1)M}_{+u}(\xi) + \frac{1}{2}\eta^{+u}\eta^{+v}\zeta^{(2)M}_{+u+v}(\xi)\,, \qquad \zeta^{(2)M}_{+u+v}(\xi) = -\zeta^{(2)M}_{+v+u}(\xi)\,, \tag{3.133}$$

the supervielbein transforms as

$$e'_A{}^M(z') = e_A{}^N(z)\frac{\partial z'^M}{\partial z^N}\,. \tag{3.134}$$

We now use the higher components in the expansion (3.133) to simplify the expressions for the supervielbein components.

Let us consider the transformation under the full superspace diffeomorphisms of the components $e_{+q}{}^M$ of the inverse supervielbein:

$$\begin{aligned} e'_{+q}{}^m(z') &= e_{+q}{}^n\partial_n\zeta^{(0)m} + e_{+q}{}^{+u}\left(\zeta^{(1)m}_{+u} + \eta^{+v}\zeta^{(2)m}_{+u+v}\right)\,, \\ e'_{+q}{}^{+u}(z') &= e_{+q}{}^n\partial_n\zeta^{(0)+u} + e_{+q}{}^{+v}\left(\zeta^{(1)+u}_{+v} + \eta^{+w}\zeta^{(2)+u}_{+v+w}\right)\,. \end{aligned} \tag{3.135}$$

By expanding the inverse supervielbein components as

$$e_A{}^M(z) = e^{(0)M}_A + \eta^{+u}e^{(1)}_{+uA}{}^M + \frac{1}{2}\eta^{+u}\eta^{+v}e^{(2)}_{+u+vA}{}^M\,, \tag{3.136}$$

we obtain that

$$\begin{aligned} e'_{+q}{}^m &= \left[e^{(0)n}_{+q}\partial_n\zeta^{(0)m} + e^{(0)+u}_{+q}\zeta^{(1)m}_{+u}\right] + \\ &\quad + \eta^{+u}\left[e^{(1)}_{+u+q}{}^n\partial_n\zeta^{(0)m} - e^{(0)n}_{+q}\partial_n\zeta^{(1)m}_{+u} + e^{(0)+v}_{+q}\zeta^{(2)m}_{+v+u} + e^{(1)}_{+u+q}{}^{+v}\zeta^{(1)m}_{+v}\right] + \dots\,, \\ e'_{+q}{}^{+u} &= \left[e^{(0)n}_{+q}\partial_n\zeta^{(0)+u} + e^{(0)+v}_{+q}\zeta^{(1)+u}_{+v}\right] + \\ &\quad + \eta^{+v}\left[e^{(1)}_{+v+q}{}^n\partial_n\zeta^{(0)+u} - e^{(0)n}_{+q}\partial_n\zeta^{(1)+u}_{+v} + e^{(0)+w}_{+q}\zeta^{(2)+u}_{+w+v} + e^{(1)}_{+v+q}{}^{+w}\zeta^{(1)+u}_{+w}\right] + \dots\,. \end{aligned} \tag{3.137}$$

Therefore, we can choose ζ^M_{+q}, in order to set

$$e^{(0)m}_{+q} = 0\,, \qquad e^{(0)+u}_{+q} = \delta_{+q}{}^{+u}\,. \tag{3.138}$$

Analogously, ζ^M_{+u+v} may be employed to transform the antisymmetric parts of $e^{(1)}_{+u+q}{}^M$ to zero, namely

$$e^{(1)}_{+q+u}{}^m = e^{(1)}_{+u+q}{}^m\,, \qquad e^{(1)}_{+q+v}{}^{+u} = e^{(1)}_{+v+q}{}^{+u}\,. \tag{3.139}$$

A similar procedure may be used to eliminate some components of the superconnection $\Omega_A{}^B$, which we recall to be given by

$$\Omega_A{}^B = \Omega\mathcal{M}_A{}^B = \mathrm{d}z^M\Omega_M\mathcal{M}_A{}^B\,. \tag{3.140}$$

To this purpose, we start from the infinitesimal transformation of $\Omega_B{}^A$ under local Lorentz transformations, which is

$$\delta\Omega_B{}^A = -\mathrm{d}L_B{}^A - L_B{}^C\Omega_C{}^A + \Omega_B{}^C L_C{}^A\,. \tag{3.141}$$

In our case, since $SO(1,1)$ is Abelian, we have that $L_B{}^C\Omega_C{}^A = \Omega_B{}^C L_C{}^A$ and then

$$\delta\Omega_M = -\partial_M L\,. \tag{3.142}$$

The boost parameter $L(z)$ is

$$L(z) = L^{(0)}(\xi) + \eta^{+u}L^{(1)}_{+u}(\xi) + \frac{1}{2}\eta^{+u}\eta^{+v}L^{(2)}_{+u+v}(\xi)\,, \qquad L^{(2)}_{+u+v}(\xi) = -L^{(2)}_{+v+u}(\xi)\,, \tag{3.143}$$

where $L^{(0)}(\xi)$ is associated with the conventional Lorentz boosts, while $L^{(1)}_{+u}$, and $L^{(2)}_{+u+v}$ can be freely gauge fixed. Thus, by expanding Ω_M as

$$\Omega_M(z) = \Omega^{(0)}_M(\xi) + \eta^{+u}\Omega^{(1)}_{+u,M}(\xi) + \frac{1}{2}\eta^{+u}\eta^{+v}\Omega^{(2)}_{+u+v,M}(\xi)\,, \tag{3.144}$$

we find that

$$\delta\Omega_{+u} = -\partial_{+u}L = -L^{(1)}_{+u} - \eta^{+v}L^{(2)}_{+u+v}\,, \tag{3.145}$$

whence

$$\begin{aligned} \delta\Omega^{(0)}_{+u} &= -L^{(1,0)}_{+}\,, \\ \delta\Omega^{(1)}_{+v+u} &= -L^{(2)}_{+u+v} = L^{(2)}_{+v+u}\,. \end{aligned} \tag{3.146}$$

Therefore, we may set

$$\Omega^{(0)}_{+u} = 0\,, \tag{3.147}$$

while $L^{(2)}_{+v+u}$ can be used to gauge away the antisymmetric part of $\Omega^{(1)}_{+v+u}$:

$$\Omega^{(1)}_{+v+u} = \Omega^{(1)}_{+u+v}\,. \tag{3.148}$$

As a result, we remain with $\zeta^{(0)m}(\xi)$, $\zeta^{(0)+u}(\xi)$ and $L^{(0)}(\xi)$ as free parameters, corresponding to bosonic diffeomorphisms, local supersymmetry transformations and local Lorentz boosts, respectively.

To summarize, in this Section we have chosen a gauge in which the components of the inverse supervielbein and the superconnection are

$$\begin{aligned} \Omega_m(z) &= \omega_m(\xi) + \eta^{+u}\rho_{+u\,m}(\xi) + \frac{1}{2}\eta^{+u}\eta^{+v}\lambda_{+u+v\,m}(\xi)\,, \\ \Omega_{+u}(z) &= \eta^{+v}\rho_{+v\,+u}(\xi) + \frac{1}{2}\eta^{+v}\eta^{+w}\lambda_{+v+w\,+u}(\xi)\,, \qquad \rho_{+v+u}(\xi) = \rho_{+u+v}(\xi)\,, \\ e_a{}^m(z) &= e_a{}^m(\xi) + \eta^{+u}f_{+u\,a}{}^m(\xi) + \frac{1}{2}\eta^{+u}\eta^{+v}g_{+u+v,a}{}^m(\xi)\,, \\ e_a{}^{+u}(z) &= \chi_a{}^{+u}(\xi) + \eta^{+v}f_{+v\,a}{}^{+u}(\xi) + \frac{1}{2}\eta^{+v}\eta^{+w}g_{+v+w,a}{}^{+u}(\xi)\,, \\ e_{+q}{}^m(z) &= \eta^{+u}f_{+u+q}{}^m(\xi) + \frac{1}{2}\eta^{+u}\eta^{+v}g_{+u+v,+q}{}^m(\xi), \qquad f_{+u+q}{}^m(\xi) = f_{+q+u}{}^m(\xi)\,, \\ e_{+q}{}^{+u}(z) &= \delta_{+q}{}^{+u} + \eta^{+v}f_{+v\,+q}{}^{+u}(\xi) + \frac{1}{2}\eta^{+v}\eta^{+w}g_{+v+w,+q}{}^{+u}(\xi), \qquad f_{+v+q}{}^{+u}(\xi) = f_{+q+v}{}^{+u}(\xi)\,. \end{aligned} \tag{3.149}$$

In particular, the lowest components of $e_A{}^M$ are given by

$$e_A{}^M\big| = \begin{pmatrix} e_a{}^m & \chi_a{}^{+u} \\ 0 & \delta_{+q}{}^{+u} \end{pmatrix}\,, \tag{3.150}$$

and by exploiting $e_A{}^N e_N{}^B = \delta_A{}^B$, we find that

$$e_M{}^A\big| = \begin{pmatrix} e_m{}^a & -\chi_m{}^{+q} \\ 0 & \delta_{+u}{}^{+q} \end{pmatrix}\,, \tag{3.151}$$

where

$$\begin{aligned} e_a{}^m e_m{}^b &= \delta_a{}^b\,, \\ \chi_m{}^{+q} &= e_m{}^a\chi_a{}^{+u}\delta_{+u}{}^{+q}\,. \end{aligned} \tag{3.152}$$

For later convenience, we also give the complete component expansion of the supervielbein $e_M{}^A$:

$$
\begin{aligned}
e_m{}^a(z) &= e_m{}^a(\xi) + \eta^{+u} h_{+u\,m}{}^a(\xi) + \frac{1}{2}\eta^{+u}\eta^{+v}\ell_{+u+v,m}{}^a(\xi)\,, \\
e_m{}^{+q}(z) &= -\chi_m{}^{+q}(\xi) + \eta^{+u} h_{+u\,m}{}^{+q}(\xi) + \frac{1}{2}\eta^{+u}\eta^{+v}\ell_{+u+v,m}{}^{+q}(\xi)\,, \\
e_{+u}{}^a(z) &= \eta^{+v} h_{+v+u}{}^a(\xi) + \frac{1}{2}\eta^{+v}\eta^{+w}\ell_{+v+w,+u}{}^a(\xi)\,, \qquad h_{+v+u}{}^a(\xi) = h_{+u+v}{}^a(\xi)\,, \\
e_{+u}{}^{+q}(z) &= \delta_{+u}{}^{+q} + \eta^{+v} h_{+v\,+u}{}^{+q}(\xi) + \frac{1}{2}\eta^{+v}\eta^{+w}\ell_{+v+w,+u}{}^{+q}(\xi)\,, \qquad h_{+v+u}{}^{+q}(\xi) = h_{+u+v}{}^{+q}(\xi)\,,
\end{aligned}
\tag{3.153}
$$

where the symmetry of $h_{+u+v}{}^a$ and $h_{+u+v}{}^{+q}$ can be seen by considering that $e_A{}^N e_N{}^B = \delta_A{}^B$ and the symmetry of $f_{+u+q}{}^m$ and $f_{+q+v}{}^{+u}$.

3.3.4 Supergravity constraints

To reduce the number of component fields of $e_M{}^A$ and Ω_M, we impose the following constraints on $T_{AB}{}^C$ [12, 38, 39]:

$$
\begin{aligned}
&T_{+q,+r}{}^{++} = -i\delta_{qr} \qquad T_{+q,+r}{}^{+s} = 0\,, \\
&T_{AB}{}^{--} = 0 \qquad \forall A, B\,, \\
&T_{+q,++}{}^{A} = 0 \qquad \forall A\,, \\
&T_{++,--}{}^{++} = T_{+q,--}{}^{++} = 0\,.
\end{aligned}
\tag{3.154}
$$

They are compatible with the rigid case, where, as seen in (3.115), the only non-vanishing component is $T_{+q,+r}{}^{++} = -i\delta_{qr}$.

Let us recall that the supertorsion and the supercurvature satisfy the Bianchi identities (3.128), which, in components, become

$$
R_{[ABC\}}{}^D = \mathcal{D}_{[A}T_{BC\}}{}^D + T_{[AB|}{}^F T_{F|C\}}{}^D\,,
\tag{3.155}
$$

and

$$
\mathcal{D}_{[E}R_{AB\}} + T_{[EA|}{}^F R_{F|B\}} = 0\,,
\tag{3.156}
$$

respectively. Here, the square brackets denote generalised anti-symmetrization, which in our case is performed in three indices and is given by

$$
\begin{aligned}
A_{[ABC\}} = \frac{1}{3!}\Big(&A_{ABC} + (-)^{c(a+b)}A_{CAB} + (-)^{a(b+c)}A_{BCA} + \\
&-(-)^{ab}A_{BAC} - (-)^{bc}A_{ACB} - (-)^{ab+bc+ac}A_{CBA}\Big)\,.
\end{aligned}
\tag{3.157}
$$

We now solve these identities subject to the supertorsion constraints in (3.154), starting from (3.155).

Utilising $R_{[+q+r+s\}}{}^{+t} = \ldots$ we find

$$
R_{+q+r} = 0\,, \qquad \forall q, r = 1, 2\,.
\tag{3.158}
$$

The identity $R_{[a+q+r\}}{}^d = \ldots$ then yields

$$
\begin{aligned}
&T_{+1,--}{}^{+1} = T_{+2,--}{}^{+2} = 0\,, \\
&T_{+1,--}{}^{+2} + T_{+2,--}{}^{+1} = 0\,,
\end{aligned}
\tag{3.159}
$$

and if we use the latter results, together with (3.158), in the identity $R_{[a+q+r]}{}^{+s} = \ldots$, one obtains

$$
R_{+q,++} = 0 \qquad \forall q = 1, 2\,, \tag{3.160a}
$$
$$
T_{++,--}{}^{+1} = -2i\mathcal{D}_{+2}T_{+2,--}{}^{+1}\,, \tag{3.160b}
$$
$$
T_{++,--}{}^{+2} = -2i\mathcal{D}_{+1}T_{+1,--}{}^{+2}\,, \tag{3.160c}
$$
$$
R_{+1,--} = iT_{++,--}{}^{+1}\,, \tag{3.160d}
$$
$$
R_{+2,--} = iT_{++,--}{}^{+2}\,. \tag{3.160e}
$$

On the other hand, the $R_{[ab+q\}}{}^{+r} = \dots$ equation gives

$$\begin{aligned} &R_{++,--} = 2\mathcal{D}_{+1}T_{++,--}{}^{+1} = 2\mathcal{D}_{+2}T_{++,--}{}^{+2}\,, && (3.161a)\\ &\mathcal{D}_{+1}T_{++,--}{}^{+2} = \mathcal{D}_{++}T_{+1,--}{}^{+2}\,, && (3.161b)\\ &\mathcal{D}_{+2}T_{++,--}{}^{+1} = \mathcal{D}_{++}T_{+2,--}{}^{+1}\,, && (3.161c)\end{aligned}$$

while the other components of (3.155) do not give additional information and are identically satisfied, once the above relations are exploited. Let us now move to the second Bianchi identity.

Let us start from the following component:

$$\mathcal{D}_{[a}R_{b+q\}} + T_{[ab|}{}^{F}R_{F|+q\}} = 0\,, \tag{3.162}$$

from which we find

$$\begin{aligned} &\mathcal{D}_{+1}R_{++,--} = \mathcal{D}_{++}R_{+1,--}\,, && (3.163a)\\ &\mathcal{D}_{+2}R_{++,--} = \mathcal{D}_{++}R_{+2,--}\,. && (3.163b)\end{aligned}$$

Finally, the last condition arising from the Bianchi identities is found from the identity $\mathcal{D}_{[a}R_{+q+r]} + \dots = 0$ and tells us that

$$\mathcal{D}_{+1}R_{+2,--} + \mathcal{D}_{+2}R_{+1,--} = 0\,. \tag{3.164}$$

To sum up, we have found that, once we know $T_{+1,--}{}^{+2} = -T_{+2,--}{}^{+1}$, we can compute all the non-vanishing components of the supertorsion and the supercurvature.

Indeed, one can derive the expressions of $T_{++,--}{}^{+q}$ and $R_{+q,--}$ starting from the equations (3.160b), (3.160c), (3.160d), (3.160e) and requiring them to satisfy (3.161b), (3.161c) and (3.164). Finally, we may compute $R_{++,--}$ using (3.161a), together with (3.163a) and (3.163b).

What we obtained is in agreement with the results of [38]. Indeed, by working in the coordinates θ^+ and $\bar{\theta}^+$ and calling

$$\begin{aligned} &T_{+,--}{}^{+} = iT_{+1,--}{}^{+2} = -iT_{+2,--}{}^{+1} \equiv iG_{--}\,,\\ &T_{++,--}{}^{+} = \frac{1}{2\sqrt{2}}\left(T_{++,--}{}^{+1} + iT_{++,--}{}^{+2}\right) \equiv -\Sigma^+\,,\\ &R_{++,--} \equiv -\mathscr{R}\,,\end{aligned} \tag{3.165}$$

we find that the previous conditions arising from the Bianchi identities become

$$\begin{aligned} &\mathcal{D}_+ G_{--} = 2\bar{\Sigma}^+\,,\\ &\mathcal{D}_+\bar{\Sigma}^+ = 0\,,\\ &\mathscr{R} = \mathcal{D}_+\Sigma^+ + \bar{\mathcal{D}}_+\bar{\Sigma}^+\,,\\ &\mathcal{D}_+\mathscr{R} = 4i\mathcal{D}_{++}\bar{\Sigma}^+\,.\end{aligned} \tag{3.166}$$

3.3.5 Final results

We are now ready to derive the complete component expansion of the supervielbein and the superconnection. The strategy is to find the supervielbein and superconnection which reproduce the results of Section 3.3.4 for the supertorsion and the supercurvature, by using, for their η–expansions, the expression reported at the end of Section 3.3.3.

Let us recall that:

$$T_{BC}{}^{A} = (-)^{b(m+c)} e_C{}^{M} e_B{}^{N} T_{NM}{}^{A}\,, \tag{3.167}$$

with

$$\begin{aligned} T_{NM}{}^{A} &= \partial_N e_M{}^{A} - (-)^{nm}\partial_M e_N{}^{A} + (-)^{n(m+b)} e_M{}^{B}\Omega_{NB}{}^{A} - (-)^{mb} e_N{}^{B}\Omega_{MB}{}^{A} =\\ &= (-)^{n(m+b)} e_M{}^{B} e_N{}^{C} T_{CB}{}^{A}\,.\end{aligned} \tag{3.168}$$

In particular, we have that:

$$T_{NM}{}^{--} = (-)^{n(m+b)} e_M{}^B e_N{}^C T_{CB}{}^{--} = 0\,, \tag{3.169}$$

since $T_{CB}{}^{--} = 0$. Therefore, we have that:

$$\partial_N e_M{}^{--} - (-)^{nm}\partial_M e_N{}^{--} - (-)^{nm} e_M{}^{--}\Omega_N + e_N{}^{--}\Omega_M = 0\,. \tag{3.170}$$

Recalling the notation introduced in (3.153) and considering the cases $N = +u$, $M = +v$ and $N = n$, $M = +u$ in (3.170), one can derive the following conditions on the supertorsion and superconnection components:

$$\begin{aligned}
&h_{+v+u}{}^{--} = \ell_{+v+w,+u}{}^{--} = 0\,, && (3.171)\\
&\rho_{+v+u} = \lambda_{+v+w+u} = 0\,, && (3.172)\\
&h^{--}_{+um} = \ell_{+u+vm}{}^{--} = 0\,, && (3.173)\\
&e_n^{--}\rho_{+um} - e_m^{--}\rho_{+un} = 0\,, && (3.174)\\
&e_n^{--}\lambda_{+u+v\,m} - e_m^{--}\lambda_{+u+v\,n} = 0\,. && (3.175)
\end{aligned}$$

The equations (3.171) imply that $e_{+u}{}^{--} = 0$. From (3.172), it turns out that also $\Omega_{+u} = 0$, while (3.173) tells us that only the lowest component of $e_m{}^{--}$ is left, i.e. $e_m{}^{--}(z) = e_m{}^{--}(\xi)$.

We now move to consider eq. (3.168) with $A = +q$. In this case, the equation reads

$$\begin{aligned}
&\partial_N e_M{}^{+q} - (-)^{nm}\partial_M e_N{}^{+q} + \frac{1}{2}(-)^{n(1+m)} e_M{}^{+q}\Omega_N - \frac{1}{2}(-)^m e_N{}^{+q}\Omega_M = \\
&\quad = (-)^{nm}\left[e_M{}^{--}e_N{}^{++} - e_M{}^{++}e_N{}^{--}\right] T_{++,--}{}^{+q} + (-)^{nm}\left[e_M{}^{--}e_N{}^{+r} - (-)^n e_M{}^{+r}e_N{}^{--}\right] T_{+r,--}{}^{+q}
\end{aligned} \tag{3.176}$$

Let us consider the case $N = +u$, $M = +v$. Since we have just obtained that $e_{+u}{}^{--} = 0$ and $\Omega_{+u} = 0$, the r.h.s. is zero and only the first two terms of the l.h.s. are non-vanishing. This leads to

$$h_{+v+u}{}^{+q} = \ell_{+v+w,+u}{}^{+q} = 0 \quad\Longrightarrow\quad e_{+u}{}^{+q}(z) = \delta_{+u}{}^{+q}\,. \tag{3.177}$$

For convenience, before going on with the case $N = n$, $M = +u$, we consider (3.168) with $A = ++$, obtaining

$$\partial_N e_M{}^{++} - (-)^{nm}\partial_M e_N{}^{++} + (-)^{nm} e_M{}^{++}\Omega_N - e_N{}^{++}\Omega_M = -i(-)^{n(1+m)} e_M{}^{+q}e_N{}^{+q}\,. \tag{3.178}$$

By analysing (3.178) for $N = +u$, $M = +v$ and $N = n$, $M = +u$, we find

$$\begin{aligned}
&h_{+u+v}{}^{++} = -\frac{i}{2}\delta_{uv}\,,\\
&\ell_{+v+w+u}{}^{++} = 0\,,\\
&h_{+um}{}^{++} = -i\chi_m{}^{+q}\delta_{uq}\,,\\
&h_{+1m}{}^{+1} = h_{+2m}{}^{+2} = \frac{1}{2}\omega_m\,,\\
&h_{+2m}{}^{+1} = -h_{+1m}{}^{+2}\,,\\
&\ell_{+1+2m}{}^{++} = ih_{+2m}{}^{+1}\,,\\
&\rho_{+1m} = -2\ell_{+1+2m}{}^{+2}\,,\\
&\rho_{+2m} = 2\ell_{+1+2m}{}^{+1}\,.
\end{aligned} \tag{3.179}$$

At this point, to finish the discussion of (3.176), we define the component expansion of $T_{+1,--}{}^{+2}$ as

$$T_{+1,--}{}^{+2} = t_{--} + \eta^{+v}d_{+v,--} + \frac{1}{2}\eta^{+v}\eta^{+w}p_{+v+w,--}\,, \tag{3.180}$$

and we recall that in Section 3.3.4 we have shown that, starting from $T_{+1,--}{}^{+2}$, one can obtain all the other non-vanishing supertorsion and supercurvature components. Thus, after computing the inverse supervielbein, it turns out that (3.160b) leads to

$$T_{++,--}{}^{+1} = 2i\mathcal{D}_{+2}T_{+1,--}{}^{+2} = A_{+,--} + \eta^{+v}B_{+v+,--} + \frac{1}{2}\eta^{+v}\eta^{+w}C_{+v+w+,--}\,, \tag{3.181}$$

where

$$\begin{aligned}
A_{+,--} &= 2id_{+2,--}\,,\\
B_{+1+,--} &= 2ip_{+2+1,--}\,,\\
B_{+2+,--} &= -e_{++}{}^{m}\hat{\mathcal{D}}_m t_{--} - \chi_{++}{}^{+u}d_{+u,--}\,,\\
C_{+1+2+,--} &= \frac{i}{2}\chi_{++}{}^{+1}e_{++}{}^{m}\hat{\mathcal{D}}_m t_{--} + e_{++}{}^{m}\hat{\mathcal{D}}_m d_{+1,--} + e_{++}{}^{m}\rho_{+1m}t_{--}+\\
&\quad + \frac{i}{2}\chi_{++}{}^{+1}\chi_{++}{}^{+u}d_{+u,--} - h_{+1,++}{}^{+u}d_{+u,--} - \chi_{++}{}^{+u}p_{+u+1,--}\,,
\end{aligned} \tag{3.182}$$

and we have defined

$$\begin{aligned}
\hat{\mathcal{D}}_m t_{--} &= \partial_m t_{--} + \omega_m t_{--}\,,\\
\hat{\mathcal{D}}_m d_{+v,--} &= \partial_m d_{+v,--} + \omega_m d_{+v,--}\,.
\end{aligned} \tag{3.183}$$

Similarly, from (3.160c) we find that

$$T_{++,--}{}^{+2} = -2i\mathcal{D}_{+1}T_{+1,--}{}^{+2} = D_{+,--} + \eta^{+v}E_{+v+,--} + \frac{1}{2}\eta^{+v}\eta^{+w}F_{+v+w+,--}\,, \tag{3.184}$$

with

$$\begin{aligned}
D_{+,--} &= -2id_{+1,--}\,,\\
E_{+1+,--} &= e_{++}{}^{m}\hat{\mathcal{D}}_m t_{--} + \chi_{++}{}^{+u}d_{+u,--}\,,\\
E_{+2+,--} &= -2ip_{+1+2,--}\,,\\
F_{+1+2+,--} &= \frac{i}{2}\chi_{++}{}^{+2}e_{++}{}^{m}\hat{\mathcal{D}}_m t_{--} + e_{++}{}^{m}\hat{\mathcal{D}}_m d_{+2,--} + e_{++}{}^{m}\rho_{+2m}t_{--}+\\
&\quad + \frac{i}{2}\chi_{++}{}^{+2}\chi_{++}{}^{+u}d_{+u,--} - h_{+2,++}{}^{+u}d_{+u,--} - \chi_{++}{}^{+u}p_{+u+2,--}\,.
\end{aligned} \tag{3.185}$$

By recalling (3.160d), (3.160e) and (3.161a), we may also obtain the component expansion of $R_{+q,--}$ and $R_{++,--}$. The results are:

$$\begin{aligned}
R_{+1,--} &= iT_{++,--}{}^{+1} = iA_{+,--} + i\eta^{+v}B_{+v+,--} + \frac{i}{2}\eta^{+v}\eta^{+w}C_{+v+w+,--}\,,\\
R_{+2,--} &= iT_{++,--}{}^{+2} = iD_{+,--} + i\eta^{+v}E_{+v+,--} + \frac{i}{2}\eta^{+v}\eta^{+w}F_{+v+w+,--}\,,\\
R_{++,--} &= 2\mathcal{D}_{+1}T_{++,--}{}^{+1} = H_{++,--} + \eta^{+v}L_{+v++,--} + \frac{1}{2}\eta^{+v}\eta^{+w}M_{+v+w++,--}\,,
\end{aligned} \tag{3.186}$$

with

$$\begin{aligned}
H_{++,--} &= 2B_{+1+,--}\,,\\
L_{+v++,--} &= 2f_{+v+1}{}^{m}\tilde{\mathcal{D}}_m A_{+,--} + 2f_{+v+1}{}^{+u}B_{+u+,--} + 2C_{+1+v+,--}\,,\\
M_{+v+w++,--} &= 2f_{+v+1}{}^{m}\tilde{\mathcal{D}}_m B_{+w+,--} - 2f_{+w+1}{}^{m}\tilde{\mathcal{D}}_m B_{+v+,--} + f_{+v+1}{}^{m}\rho_{+wm}A_{+,--}+\\
&\quad - f_{+w+1}{}^{m}\rho_{+vm}A_{+,--} + 2g_{+v+w+1}{}^{m}\tilde{\mathcal{D}}_m A_{+,--} + 2g_{+v+w+1}{}^{+u}B_{+u+,--}+\\
&\quad - 2f_{+v+1}{}^{+u}C_{+u+w+,--} + 2f_{+w+1}{}^{+u}C_{+u+v+,--}\,,
\end{aligned} \tag{3.187}$$

and

$$\begin{aligned}
\tilde{\mathcal{D}}_m A_{+,--} &= \partial_m A_{+,--} + \frac{1}{2}\omega_m A_{+,--}\,,\\
\tilde{\mathcal{D}}_m B_{+w+,--} &= \partial_m B_{+w+,--} + \frac{1}{2}\omega_m B_{+w+,--}\,.
\end{aligned} \tag{3.188}$$

We are now ready to finish the discussion of (3.176). In particular, by considering the case $N = +u$, $M = m$, we find

$$\begin{aligned}
&h_{+1m}{}^{+1} = h_{+2m}{}^{+2} = \frac{1}{2}\omega_m\,, \\
&h_{+1m}{}^{+2} = -h_{+2m}{}^{+1} = i\ell_{+1+2m}{}^{++} = e_m{}^{--}t_{--}\,, \\
&\rho_{+1m} = -2\ell_{+1+2m}{}^{+2} = -2e_m{}^{--}d_{+2,--}\,, \\
&\rho_{+2m} = 2\ell_{+1+2m}{}^{+1} = 2e_m{}^{--}d_{+1,--}\,, \\
&\lambda_{+1+2m} = -ie_m{}^{--}e_{++}{}^{n}\hat{\mathcal{D}}_n t_{--} - ie_m{}^{--}\chi_{++}{}^{+u}d_{+u,--}\,.
\end{aligned} \tag{3.189}$$

Note that the expressions of ρ_{+um} and λ_{+1+2m} satisfy the equations (3.174) and (3.175) we have found before.

At this point, all supervielbein and superconnection components are completely fixed. Indeed, by using all the results found so far, we obtain

$$\begin{aligned}
\Omega_m(z) &= \omega_m - 2\eta^{+1}e_m{}^{--}d_{+2,--} + 2\eta^{+2}e_m{}^{--}d_{+1,--} + \\
&\quad + \eta^{+1}\eta^{+2}\left(-ie_m{}^{--}e_{++}{}^{n}\hat{\mathcal{D}}_n t_{--} - ie_m{}^{--}\chi_{++}{}^{+u}d_{+u,--}\right)\,, \\
\Omega_{+u}(z) &= 0\,, \\
e_m{}^{--}(z) &= e_m{}^{--}\,, \\
e_m{}^{++}(z) &= e_m{}^{++} - i\eta^{+1}\chi_m{}^{+1} - i\eta^{+2}\chi_m{}^{+2} - i\eta^{+1}\eta^{+2}e_m{}^{--}t_{--}\,, \\
e_m{}^{+1}(z) &= -\chi_m^{+1} + \frac{1}{2}\eta^{+1}\omega_m - \eta^{+2}e_m{}^{--}t_{--} + \eta^{+1}\eta^{+2}e_m{}^{--}d_{+1,--}\,, \\
e_m{}^{+2}(z) &= -\chi_m^{+2} + \eta^{+1}e_m{}^{--}t_{--} + \frac{1}{2}\eta^{+2}\omega_m + \eta^{+1}\eta^{+2}e_m{}^{--}d_{+2,--}\,, \\
e_{+u}{}^{--}(z) &= 0\,, \\
e_{+1}{}^{++}(z) &= -\frac{i}{2}\eta^{+1}\,, \\
e_{+2}{}^{++}(z) &= -\frac{i}{2}\eta^{+2}\,, \\
e_{+u}{}^{+q}(z) &= \delta_{+u}{}^{+q}\,,
\end{aligned} \tag{3.190}$$

where $\chi_{++}{}^{+u} = e_{++}{}^{m}\chi_m{}^{+q}\delta_{uq}$.

In (3.190), $e_m{}^{a}$, $\chi_m{}^{+q}$ and ω_m are the vielbein, the gravitino and the Lorentz connection, respectively. The latter is not an independent field, being related to the vielbein and the gravitino by

$$\begin{aligned}
&\partial_n e_m{}^{--} - \partial_m e_n{}^{--} = e_m{}^{--}\omega_n - e_n{}^{--}\omega_m\,, \\
&\partial_n e_m{}^{++} - \partial_m e_n{}^{++} + e_m{}^{++}\omega_n - e_n{}^{++}\omega_m = -i\chi_m^{+q}\chi_n^{+q}\,,
\end{aligned} \tag{3.191}$$

which are equations arising from the cases $N = n$, $M = m$ of (3.170) and (3.178). From (3.191), one obtains

$$\omega_n = e_{--}{}^{m}\left(\partial_n e_m{}^{--} - \partial_m e_n{}^{--}\right) + e_{++}{}^{m}\left(\partial_m e_n{}^{++} - \partial_n e_m{}^{++} - i\chi_m{}^{+q}\chi_n{}^{+q}\right)\,. \tag{3.192}$$

As far as t_{--} and $d_{+u,--}$ are concerned, the former enters the $(0,2)$ supergravity multiplet, together with the vielbein $e_m{}^{a}$ and the gravitino $\chi_m{}^{+q}$[16], while $d_{+u,--}$, $u = 1, 2$, are not independent and can be expressed as[17]

$$\begin{aligned}
d_{+1,--} &= \frac{i}{2}\left(\partial_{--}\chi_{++}{}^{+2} - \partial_{++}\chi_{--}{}^{+2} + \frac{1}{2}\chi_{++}{}^{+2}\omega_{--} - \frac{1}{2}\chi_{--}{}^{+2}\omega_{++} + t_{--}\chi_{++}{}^{+1}\right)\,, \\
d_{+2,--} &= -\frac{i}{2}\left(\partial_{--}\chi_{++}{}^{+1} - \partial_{++}\chi_{--}{}^{+1} + \frac{1}{2}\chi_{++}{}^{+1}\omega_{--} - \frac{1}{2}\chi_{--}{}^{+1}\omega_{++} + t_{--}\chi_{++}{}^{+2}\right)\,.
\end{aligned} \tag{3.193}$$

[16] Indeed, t_{--} gives exactly the bosonic degree of freedom which allows us to match the number of bosonic and fermionic degrees of freedom of the supergravity multiplet, as requested by a supersymmetric theory.

[17] The equations in (3.193) originate from the case $N = n$, $M = m$ of (3.178).

In the complex coordinates θ^+ and $\bar\theta^+ = \theta^{\mp}$, related to η^{+q} by the relations (3.110), (3.190) becomes

$$
\begin{aligned}
\Omega_m(z) &= \omega_m - 2i\theta^+ e_m{}^{--} d_{+,--} + 2i\theta^{\mp} e_m{}^{--} d_{\mp,--} + \\
&\qquad + 4\theta^+\theta^{\mp} e_m{}^{--}\left(e_{++}{}^n \hat{\mathcal{D}}_n t_{--} + \chi_{++}{}^+ d_{+,--} + \chi_{++}{}^{\mp} d_{\mp,--}\right) ,\\
\Omega_{+u}(z) &= 0 ,\\
e_m{}^{--}(z) &= e_m{}^{--} ,\\
e_m{}^{++}(z) &= e_m{}^{++} - 4i\theta^+\chi_m{}^{\mp} - 4i\theta^{\mp}\chi_m{}^+ + 4\theta^+\theta^{\mp} e_m{}^{--} t_{--} ,\\
e_m{}^{+}(z) &= -\chi_m{}^+ + \theta^+\left(\frac{1}{2}\omega_m + i e_m{}^{--} t_{--}\right) + i\theta^+\theta^{\mp} e_m{}^{--} d_{\mp,--} ,\\
e_m{}^{\mp}(z) &= -\chi_m{}^{\mp} + \theta^{\mp}\left(\frac{1}{2}\omega_m - i e_m{}^{--} t_{--}\right) + i\theta^+\theta^{\mp} e_m{}^{--} d_{+,--} ,\\
e_+{}^{--}(z) &= e_{\mp}{}^{--} = 0 ,\\
e_+{}^{++}(z) &= -2i\theta^{\mp} ,\\
e_{\mp}{}^{++}(z) &= -2i\theta^{+} ,\\
e_+{}^{+}(z) &= e_{\mp}{}^{\mp} = 1 ,\\
e_+{}^{\mp}(z) &= e_{\mp}{}^{+} = 0 .
\end{aligned}
\tag{3.194}
$$

Finally, the inverse supervielbein is

$$
\begin{aligned}
e_a{}^m(z) &= e_a{}^m + \frac{i}{2}\eta^{+1}\chi_a{}^{+1} e_{++}{}^m + \frac{i}{2}\eta^{+2}\chi_a{}^{+2} e_{++}{}^m + \frac{1}{4}\eta^{+1}\eta^{+2}\left(\chi_a{}^{+1}\chi_{++}{}^{+2} + \chi_{++}{}^{+1}\chi_a{}^{+2}\right) e_{++}{}^m ,\\
e_a{}^{+1}(z) &= \chi_a{}^{+1} + \eta^{+1}\left(\frac{i}{2}\chi_a{}^{+1}\chi_{++}{}^{+1} - \frac{1}{2}\omega_a\right) + \eta^{+2}\left(\frac{i}{2}\chi_a{}^{+2}\chi_{++}{}^{+1} + t_{--}\delta^{a,--}\right) + \\
&\qquad + \eta^{+1}\eta^{+2}\left(-d_{+1,--}\delta^{a,--} + \frac{1}{4}\chi_a{}^{+1}\chi_{++}{}^{+2}\chi_{++}{}^{+1} - \frac{i}{4}\chi_a{}^{+2}\omega_{++}\right) ,\\
e_a{}^{+2}(z) &= \chi_a{}^{+2} + \eta^{+1}\left(\frac{i}{2}\chi_a{}^{+1}\chi_{++}{}^{+2} - t_{--}\delta^{a,--}\right) + \eta^{+2}\left(\frac{i}{2}\chi_a{}^{+2}\chi_{++}{}^{+2} - \frac{1}{2}\omega_a\right) + \\
&\qquad + \eta^{+1}\eta^{+2}\left(-d_{+2,--}\delta^{a,--} + \frac{1}{4}\chi_{++}{}^{+1}\chi_a{}^{+2}\chi_{++}{}^{+2} + \frac{i}{4}\chi_a{}^{+1}\omega_{++}\right) ,\\
e_{+1}{}^m(z) &= \frac{i}{2}\eta^{+1} e_{++}{}^m - \frac{1}{4}\eta^{+1}\eta^{+2}\chi_{++}{}^{+2} e_{++}{}^m ,\\
e_{+2}{}^m(z) &= \frac{i}{2}\eta^{+2} e_{++}{}^m + \frac{1}{4}\eta^{+1}\eta^{+2}\chi_{++}{}^{+1} e_{++}{}^m ,\\
e_{+1}{}^{+1}(z) &= 1 + \frac{i}{2}\eta^{+1}\chi_{++}{}^{+1} + \frac{1}{4}\eta^{+1}\eta^{+2}\chi_{++}{}^{+1}\chi_{++}{}^{+2} ,\\
e_{+1}{}^{+2}(z) &= \frac{i}{2}\eta^{+1}\chi_{++}{}^{+2} - \frac{i}{4}\eta^{+1}\eta^{+2}\omega_{++} ,\\
e_{+2}{}^{+1}(z) &= \frac{i}{2}\eta^{+2}\chi_{++}{}^{+1} + \frac{i}{4}\eta^{+1}\eta^{+2}\omega_{++} ,\\
e_{+2}{}^{+2}(z) &= 1 + \frac{i}{2}\eta^{+2}\chi_{++}{}^{+2} + \frac{1}{4}\eta^{+1}\eta^{+2}\chi_{++}{}^{+1}\chi_{++}{}^{+2} ,
\end{aligned}
\tag{3.195}
$$

where $\omega_a = e_a{}^n \omega_n$. In the coordinates θ^+ and $\theta^{\bar{+}}$, it becomes

$$
\begin{aligned}
e_a{}^m(z) &= e_a{}^m + 2i\theta^+ \chi_a{}^{\bar{+}} e_{++}{}^m + 2i\theta^{\bar{+}} \chi_a{}^+ e_{++}{}^m + 4\theta^+\theta^{\bar{+}} (\chi_a{}^{\bar{+}} \chi_{++}{}^+ - \chi_a{}^+ \chi_{++}{}^{\bar{+}}) e_{++}{}^m \,, \\
e_a{}^+(z) &= \chi_a{}^+ + \theta^+ \left(2i\chi_a{}^{\bar{+}} \chi_{++}{}^+ - \frac{1}{2}\omega_a - it_{--}\delta^{a,--} \right) + 2i\theta^{\bar{+}} \chi_a{}^+ \chi_{++}{}^+ + \\
&\qquad + \theta^+\theta^{\bar{+}} \left(4\chi_a{}^+ \chi_{++}{}^+ \chi_{++}{}^{\bar{+}} - id_{\bar{+},--}\delta^{a,--} - i\chi_a{}^+ \omega_{++} \right) \\
e_a{}^{\bar{+}}(z) &= \chi_a{}^{\bar{+}} + 2i\theta^+ \chi_a{}^{\bar{+}} \chi_{++}{}^{\bar{+}} + \theta^{\bar{+}} \left(2i\chi_a{}^+ \chi_{++}{}^{\bar{+}} - \frac{1}{2}\omega_a + it_{--}\delta^{a,--} \right) + \\
&\qquad + \theta^+\theta^{\bar{+}} \left(4\chi_a{}^{\bar{+}} \chi_{++}{}^+ \chi_{++}{}^{\bar{+}} - id_{+,--}\delta^{a,--} + i\chi_a{}^{\bar{+}} \omega_{++} \right) \\
e_+{}^m(z) &= (2i\theta^{\bar{+}} + 4\theta^+\theta^{\bar{+}} \chi_{++}{}^{\bar{+}}) e_{++}{}^m \,, \\
e_{\bar{+}}{}^m(z) &= (2i\theta^+ - 4\theta^+\theta^{\bar{+}} \chi_{++}{}^+) e_{++}{}^m \,, \\
e_+{}^+(z) &= 1 + 2i\theta^{\bar{+}} \chi_{++}{}^+ + i\theta^+\theta^{\bar{+}} (\omega_{++} + 4i\chi_{++}{}^+ \chi_{++}{}^{\bar{+}}) \,, \\
e_{\bar{+}}{}^{\bar{+}}(z) &= 1 + 2i\theta^+ \chi_{++}{}^{\bar{+}} - i\theta^+\theta^{\bar{+}} (\omega_{++} - 4i\chi_{++}{}^+ \chi_{++}{}^{\bar{+}}) \,, \\
e_+{}^{\bar{+}}(z) &= 2i\theta^{\bar{+}} \chi_{++}{}^{\bar{+}} \,, \\
e_{\bar{+}}{}^+(z) &= 2i\theta^+ \chi_{++}{}^+ \,.
\end{aligned}
\tag{3.196}
$$

In conclusion, in this section we have studied $\mathcal{N} = (0, 2)$ supergravity in 2 dimensions, and the results we have obtained will be important, in Section 4.4, to incorporate the internal sector in the world-sheet theory of the EFT strings in a manifestly supersymmetric way.

Chapter 4

World-sheet theory of the EFT strings

In Section 2.2, we introduced the action for $\frac{1}{2}$-BPS axionic strings, and in Section 2.4.1 we illustrated that, if we consider a 4d $\mathcal{N} = 1$ effective field theory for a set of chiral multiplets and focus on the subclass of BPS axionic strings given by the *EFT strings*, their backreaction on the moduli of the theory is such that, close to the string core, the effective theory becomes weakly coupled. Then, in Section 2.5.1 we saw 2 conjectures related to the physics of EFT strings, while Section 2.5.2 discussed how theories with a standard coupling to the axionic sector must have, to be consistent, a string world-sheet theory producing an anomaly which cancels the anomaly induced by the axionic couplings.

In this Chapter, we focus on the world-sheet theory of the EFT strings. It is already known [9, 10] how to incorporate the universal sector of the EFT strings in a supersymmetric way in the target superspace. This is achieved by means of the Green-Schwarz (GS) formalism, and will be reviewed in Section 4.1. However, as already discussed in Section 2.5.2, models arising from string/M-theory compactifications tell us that, besides the universal sector, an additional sector supported on the EFT string is also present, and the Green-Schwarz formalism does not allow for the inclusion of this sector in the theory in a supersymmetrically controlled way. An alternative approach is represented by the *superembedding formalism* (see [12] and references therein), which will be introduced in Section 4.2 and provides a doubly supersymmetric theory, namely a supersymmetric theory both on the target superspace and the string world-sheet. This is useful in order to include the additional degrees of freedom of the EFT strings, described in terms of $\mathcal{N} = (0, 2)$ multiplets [11], where $\mathcal{N} = (0, 2)$ is the local supersymmetry preserved by such strings.

Before including the internal sector, in Section 4.3 we rewrite the world-sheet theory of the universal sector in the superembedding formulation. Finally, in Section 4.4 we start considering the internal sector of the theory, by introducing $\mathcal{N} = (0, 2)$ Fermi multiplets, and comment about the issues that arise when one considers the inclusion of $\mathcal{N} = (0, 2)$ chiral multiplets.

4.1 BPS axionic strings in the Green-Schwarz formalism

In this Section, we illustrate how to write down an action for the universal sector of the BPS axionic strings in a supersymmetric way in the target superspace, following the discussion of [9, 10].

First of all, let us recall that the action describing a string minimally coupled to a set of gauge two-forms $\mathcal{B}_{2,i}$ is (2.20), i.e.

$$S_{\text{string}} = -\int_{\mathcal{S}} \sqrt{-\det \gamma}\, \mathcal{T}_{\text{string}}(\ell) + e^i \int_{\mathcal{S}} \mathcal{B}_{2,i}\,, \tag{4.1}$$

where γ represents the induced metric over the string worldsheet, given by (2.21), while $\mathcal{T}_{\text{string}}(\ell)$ is completely fixed by requiring the axionic strings to be $\frac{1}{2}$–BPS objects, and its expression is [9]

$$\mathcal{T}_{\text{string}}(\ell) \equiv \mathcal{T}_{\mathbf{e}} = M_P^2 |e^i \ell_i|\,. \tag{4.2}$$

To extend (4.1) to the local supersymmetric case, we first consider its global version, namely we start from a flat target superspace. In the Green-Schwarz formalism, the fundamental string is described by the embedding of the string world-sheet $\mathcal{S}$, parametrized by two bosonic coordinates ξ^m, $m = 0, 1$, in the target superspace[1]:

$$\xi^m \longmapsto \mathcal{S} : z^{\underline{M}}(\xi) = (x^{\underline{m}}(\xi), \theta^{\underline{\mu}}(\xi), \bar{\theta}_{\underline{\dot{\mu}}}(\xi)) . \tag{4.3}$$

where $x^{\underline{m}}(\xi)$, $\theta^{\underline{\mu}}(\xi)$ and $\bar{\theta}_{\underline{\dot{\mu}}}(\xi)$ are the fields describing the bosonic and the Grassmann spinor coordinates of the string in the target superspace, respectively. Consequently, we define the induced metric on the string as

$$\gamma_{mn} \equiv \mathcal{E}^{\underline{a}}_m \mathcal{E}^{\underline{b}}_n \eta_{\underline{ab}} , \tag{4.4}$$

where we have introduced the pull-backs of the target superspace supervielbein of the global supersymmetric case

$$\begin{aligned} &\mathcal{E}^{\underline{a}}_m(\xi) \equiv \partial_m z^{\underline{M}}(\xi) \mathcal{E}^{\underline{a}}_{\underline{M}}(z(\xi)) , \\ &\mathcal{E}^{\underline{a}} = \mathrm{d}z^{\underline{M}} \mathcal{E}^{\underline{a}}_{\underline{M}}(z) = \mathrm{d}x^{\underline{a}} - i\mathrm{d}\theta\sigma^{\underline{a}}\bar{\theta} + i\theta\sigma^{\underline{a}}\mathrm{d}\bar{\theta} . \end{aligned} \tag{4.5}$$

Furthermore, it is necessary to promote the gauge two-forms $\mathcal{B}_{2,i}$ to *super-gauge two-forms* $\mathbf{B}_{2,i}$ whose lowest bosonic components just coincide with $\mathcal{B}_{2,i}$. The super-gauge-two-forms $\mathbf{B}_{2,i}$ are defined in terms of the linear multiplets L_i introduced in Section 3.2.1, and their expression is obtained starting from the closed super-field strength three-form

$$\begin{aligned} \mathbf{H}_{3,i} = \mathrm{d}\mathbf{B}_{2,i} &= 2i\mathcal{E}^{\underline{a}} \wedge \mathcal{E}^{\underline{\alpha}} \wedge \bar{\mathcal{E}}^{\underline{\dot{\alpha}}} (\sigma_{\underline{a}})_{\underline{\alpha\dot{\alpha}}} L_i \\ &- \mathcal{E}^{\underline{b}} \wedge \mathcal{E}^{\underline{a}} \wedge \mathcal{E}^{\underline{\alpha}} (\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} D_{\underline{\beta}} L_i - \mathcal{E}^{\underline{b}} \wedge \mathcal{E}^{\underline{a}} \wedge \bar{\mathcal{E}}^{\underline{\dot{\alpha}}} (\bar{\sigma}_{\underline{ab}})^{\underline{\dot{\beta}}}{}_{\underline{\dot{\alpha}}} \bar{D}_{\underline{\dot{\beta}}} L_i \\ &- \frac{1}{24} \mathcal{E}^{\underline{c}} \wedge \mathcal{E}^{\underline{b}} \wedge \mathcal{E}^{\underline{a}} \varepsilon_{\underline{abcd}} (\bar{\sigma}^{\underline{d}})^{\underline{\dot{\alpha}\alpha}} [D_{\underline{\alpha}}, \bar{D}_{\underline{\dot{\alpha}}}] L_i , \end{aligned} \tag{4.6}$$

where $\mathcal{E}^{\underline{\alpha}} = \mathrm{d}\theta^{\underline{\alpha}}$, $\bar{\mathcal{E}}^{\underline{\dot{\alpha}}} = \mathrm{d}\bar{\theta}^{\underline{\dot{\alpha}}}$. This is the unique closed super-three-form that can be constructed from the linear multiplets [40]. It can be shown that its lowest bosonic component is

$$\mathbf{H}_{3,i}| = \mathcal{H}_{3,i} = \mathrm{d}\mathcal{B}_{2,i} , \tag{4.7}$$

as it should.

We are now in the position to introduce the supersymmetric action for a fundamental string:

$$S_{\text{string}} = -\int_{\mathcal{S}} \mathrm{d}^2\xi |e^i L_i| \sqrt{-\det \gamma} + e^i \int_{\mathcal{S}} \mathbf{B}_{2,i} . \tag{4.8}$$

It reduces to (4.1) once we restrict to the bosonic components, defining the string tension[2]

$$\mathcal{T}_{\mathbf{e}} = |e^i l_i| , \tag{4.9}$$

where the scalar fields l_i are assumed to be evaluated over the string worldsheet $\mathcal{S}$.

Besides being invariant under worldsheet reparametrizations, the action (4.8) enjoys a local, fermionic κ-symmetry specified by the parameters $\kappa^{\underline{\alpha}}(\xi)$ and $\bar{\kappa}_{\underline{\dot{\alpha}}}(\xi) = \overline{(\kappa_{\underline{\alpha}}(\xi))}$, which acts on the embedding coordinates as

$$\delta z^{\underline{M}}(\xi) = \kappa^{\underline{\alpha}}(\xi) \mathcal{E}^{\underline{M}}_{\underline{\alpha}}(z(\xi)) + \bar{\kappa}_{\underline{\dot{\alpha}}}(\xi) \mathcal{E}^{\underline{M\dot{\alpha}}}(z(\xi)) , \tag{4.10}$$

which, more explicitly, reads

$$\begin{cases} \delta x^{\underline{m}}(\xi) = i\kappa\sigma^{\underline{m}}\bar{\theta}(\xi) - i\theta(\xi)\sigma^{\underline{m}}\bar{\kappa} \\ \delta\theta^{\underline{\mu}}(\xi) = \kappa^{\underline{\mu}}(\xi) \\ \delta\bar{\theta}_{\underline{\dot{\mu}}}(\xi) = \bar{\kappa}_{\underline{\dot{\mu}}}(\xi) \end{cases} . \tag{4.11}$$

[1] In what follows, the underlined indices refer to target-space indices, while not underlined ones correspond to world-sheet indices.

[2] Actually, as we will shortly see, the tension reported in (4.2) is recovered in the Einstein frame.

This is a symmetry of (4.8) provided that $\kappa_{\underline{\alpha}}(\xi)$ obeys the following projection conditions:

$$\kappa_{\underline{\alpha}} = \frac{e^i L_i}{|e^i L_i|}\Gamma_{\underline{\alpha}}{}^{\underline{\beta}}\kappa_{\underline{\beta}}\,, \tag{4.12}$$

where we have defined

$$\Gamma_{\underline{\alpha}}{}^{\underline{\beta}} \equiv -\frac{1}{\sqrt{-\det\gamma}}\varepsilon^{mn}\mathcal{E}_m^{\underline{a}}\mathcal{E}_n^{\underline{b}}(\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}}\,, \tag{4.13}$$

which satisfies

$$\Gamma_{\underline{\alpha}}{}^{\underline{\gamma}}\Gamma_{\underline{\gamma}}{}^{\underline{\beta}} = \delta_{\underline{\alpha}}{}^{\underline{\beta}}\,. \tag{4.14}$$

The world-sheet reparametrization invariance $\xi^m \to \xi'^m(\xi)$ can be used to go in the so-called *static gauge*, which fixes the longitudinal directions $x^0 = \xi^0$, $x^3 = \xi^1$, while leaving $x^1(\xi)$ and $x^2(\xi)$ as the only bosonic physical fields which describe the dynamics of the string. x^1 and x^2 can be regarded as the real and imaginary parts of a complex scalar field $u = x^1 + ix^2$. In this gauge, the κ-symmetry operator reduces to $\Gamma_{\underline{\alpha}}{}^{\underline{\beta}} = (\sigma_3)_{\underline{\alpha}}{}^{\underline{\beta}}$. Providing that we work in the case $e^i L_i > 0$, the projection condition on $\kappa_{\underline{\alpha}} = (\kappa_-, \kappa_+)$ then becomes $\kappa_{\underline{\alpha}} = (\sigma_3)_{\underline{\alpha}}{}^{\underline{\beta}}\kappa_{\underline{\beta}}$. The latter condition imposes $\kappa_+ = 0$, thus, by looking at (4.11) and considering that $\kappa^{\underline{\alpha}} = (\kappa^-, \kappa^+) = (\kappa_+, -\kappa_-)$[3], we find that the physical component of $\theta^{\underline{\mu}}(\xi) = (\theta^-(\xi), \theta^+(\xi)) = (\theta_+(\xi), -\theta_-(\xi))$ is the right-moving one, i.e. $\theta_+(\xi)$, while the other one can be set to zero by a κ-symmetry transformation. This is in agreement with what we said in Section 2.5.2, when we discussed the universal part of the world-sheet sector of the EFT strings. In particular, this tells us that a $\mathcal{N} = (0, 2)$ local supersymmetry is preserved on the string world-sheet, since we are left with $\kappa^+ = -\kappa_-$ as free parameter of the κ-symmetry, and then (4.8) describes a $\frac{1}{2}$-BPS string.

We now want to extend the action (4.8) for a string minimally charged under some gauge two-forms $\mathcal{B}_{2,i}$, with charges e^i, to the local supersymmetric case. To this aim, in analogy with Section 3.2.2, one may utilize the super-Weyl invariant approach [9, 10]. The final result is

$$S_{\text{string}} = -\int_{\mathcal{S}} \mathrm{d}^2\xi\, |e^i L_i|\sqrt{-\det\boldsymbol{\gamma}} + e^i\int_{\mathcal{S}} \mathbf{B}_{2,i}\,, \tag{4.15}$$

where $\mathbf{B}_{2,i}$ is the super-gauge two-form whose purely bosonic component is just the ordinary $\mathcal{B}_{2,i}$. The procedure to define it goes along the same path as before, but now the super-three-form field-strength, $\mathbf{H}_{3,i} = \mathrm{d}\mathbf{B}_{2,i}$, is a proper covariantization of (4.6):

$$\begin{aligned}\mathbf{H}_{3,i} &= 2iE^{\underline{a}} \wedge E^{\underline{\alpha}} \wedge \bar{E}^{\underline{\dot\alpha}}(\sigma_{\underline{a}})_{\underline{\alpha\dot\alpha}}L_i \\ &\quad - E^{\underline{b}} \wedge E^{\underline{a}} \wedge E^{\underline{\alpha}}(\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}}\mathcal{D}_{\underline{\beta}}L_i - E^{\underline{b}} \wedge E^{\underline{a}} \wedge \bar{E}^{\underline{\dot\alpha}}(\bar\sigma_{\underline{ab}})^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}}\bar{\mathcal{D}}_{\underline{\dot\beta}}L_i \\ &\quad - \frac{1}{24}E^{\underline{c}} \wedge E^{\underline{b}} \wedge E^{\underline{a}}\epsilon_{\underline{abcd}}\left[(\bar\sigma^{\underline{d}})^{\underline{\dot\alpha\alpha}}[\mathcal{D}_{\underline{\alpha}}, \bar{\mathcal{D}}_{\underline{\dot\alpha}}] + 8G^{\underline{d}}\right]L_i\,.\end{aligned} \tag{4.16}$$

Analogously to the global case, it can be seen that (4.15) reduces to (4.1) once we restrict to the bosonic components.

In the so-called *Einstein frame*, the string action reads

$$S_{\text{string}} = -M_{\mathrm{P}}^2\int_{\mathcal{S}} \mathrm{d}^2\xi\, |e^i L_i|\sqrt{-\det\boldsymbol{\gamma}} + e^i\int_{\mathcal{S}} \mathbf{B}_{2,i}\,, \tag{4.17}$$

where we have employed the change of variables (3.100), i.e. $l_i = M_{\mathrm{P}}^2\,\ell_i$, and which defines the physical string tension as

$$\mathcal{T}_{\mathbf{e}} = M_{\mathrm{P}}^2|e^i\ell_i|\,. \tag{4.18}$$

The action (4.17) naturally couples to the bulk action (3.87), which also contains the dynamics of the linear multiplets.

Even in this case, (4.15) is invariant under κ-symmetry transformations, parametrized by $\kappa_{\underline{\alpha}}$, $\bar\kappa^{\underline{\dot\alpha}}$ satisfying the projection conditions (4.12). Hence, the bulk supersymmetry, with local parameter

[3] This is obtained by raising and lowering the spinor indices with the antisymmetric tensor $\varepsilon^{\alpha\beta}$ of Appendix B.

$\epsilon_{\underline{\alpha}}(x)$ can be preserved, over the string worldsheet, only if the κ-parameters can be identified with $\epsilon_{\underline{\alpha}}(x)$[4], namely

$$\kappa_{\underline{\alpha}}(\xi) \equiv \epsilon_{\underline{\alpha}}(x)|_{\text{string}} \,. \tag{4.19}$$

Therefore, by following the same reasoning as the one done in the global supersymmetric case, which is still valid locally, the action (4.15) describes $\frac{1}{2}$-BPS strings, preserving only half of the bulk supersymmetry over their worldsheet. In particular, depending on the sign of $e^i L_i$, we may preserve a given half of the bulk supersymmetry or the opposite one. In agreement with the notation already used in 2.4.1, we call $\frac{1}{2}$-BPS strings those whose tension is

$$\mathcal{T}_{\text{string}} = M_{\text{P}}^2 e^i \ell_i \,, \tag{4.20}$$

and preserve a given half of the bulk supersymmetry. Those preserving the opposite half have $\mathcal{T}_{\text{string}} = -M_{\text{P}}^2 e^i \ell_i$ and are dubbed $\frac{1}{2}$-BPS anti-strings.

4.2 Superembedding formalism

In Section 4.3, we will show how to reformulate the theory for the universal sector of the fundamental axionic strings in the superembedding approach. Before discussing the implementation of this formalism to that case, in Section 4.2.1 we study how this works in the case of superstrings [12]. By 'superstring', we mean the supersymmetric extension of the bosonic string, discussed, for example, in [15, 41], and the reason for considering this case is that it will give us the possibility of introducing the formalism in an easier setup with respect to the one we are interested in[5].

4.2.1 Superembedding approach for $\mathcal{N} = 1$ superstrings

The purpose of this section is to write a doubly supersymmetric action for the $\mathcal{N} = 1$ superstring, namely a supersymmetric action both on the target superspace and the string world-sheet. However, as a preliminary step, let us introduce the superstring in the Green-Schwarz formulation, first considering the case of a flat Minkowski space-time.

Green-Schwarz formulation of $\mathcal{N} = 1$ superstrings

In analogy with what we already know from Section 4.1, the Green-Schwarz formulation describes the superstring as the superspace embedding of its bosonic two-dimensional world-sheet, parametrized by the coordinates ξ^m, $m = 0, 1$, in the target superspace. In particular, the $\mathcal{N} = 1$ Green-Schwarz action for a superstring in a flat D-dimensional space-time is [41, 42]

$$S = -\frac{T}{2} \int \mathrm{d}^2\xi \sqrt{-g} g^{mn} \mathcal{E}_m^{\underline{a}} \mathcal{E}_n^{\underline{b}} \eta_{\underline{ab}} + T \int B^{(2)}(\xi) \,, \tag{4.21}$$

where T is the string tension, $B^{(2)}(\xi)$ is the worldsheet pullback of the target superspace two-form

$$B^{(2)} = i \mathrm{d}x^{\underline{a}} \mathrm{d}\bar{\theta} \gamma_{\underline{a}} \theta \,, \tag{4.22}$$

whereas $\mathcal{E}_m^{\underline{a}}$ are the vector components of the pullback of the supervielbein in the flat target superspace[6]

$$\mathcal{E}^{\underline{a}} = \left(\mathrm{d}x^{\underline{m}} + i \mathrm{d}\bar{\theta} \gamma^{\underline{m}} \theta\right) \delta_{\underline{m}}^{\underline{a}} = \mathrm{d}\xi^m \left(\partial_m x^{\underline{m}} + i \partial_m \bar{\theta} \gamma^{\underline{m}} \theta\right) \delta_{\underline{m}}^{\underline{a}} \equiv \mathrm{d}\xi^m \mathcal{E}_m^{\underline{a}} \,, \tag{4.23a}$$

$$\mathcal{E}^{\underline{\alpha}} = \mathrm{d}\theta^{\underline{\mu}} \delta_{\underline{\mu}}^{\underline{\alpha}} = \mathrm{d}\xi^m \partial_m \theta^{\underline{\mu}} \delta_{\underline{\mu}}^{\underline{\alpha}} \equiv \mathrm{d}\xi^m \mathcal{E}_m^{\underline{\alpha}} \,. \tag{4.23b}$$

[4] Indeed, as we will see in Section 4.2, the superembedding formalism provides the local fermionic κ-symmetry with a clear geometrical meaning of local wordlsheet supersymmetry.

[5] Indeed, unlike the BPS axionic strings (see (4.15)), in this case the string tension is constant.

[6] In (4.22) and (4.23a), $\gamma^{\underline{a}}$ are the D-dimensional Dirac matrices. Furthermore, note that, if we focus on the case $D = 4$ and choose the representation given in Appendix B for the γ-matrices, the expression for $\mathcal{E}^{\underline{a}}$ coincides with (4.5).

The first term of (4.21) contains the inverse of the worldsheet metric $g_{mn}(\xi)$, i.e. $g^{mn}(\xi)$. If we describe the geometry of the worldsheet in terms of the zweibein $e^a(\xi) = \mathrm{d}\xi^m e_m{}^a(\xi)$, the worldsheet metric and its inverse may be expressed as

$$\begin{aligned} g_{mn}(\xi) &= e_m{}^a e_n{}^b \eta_{ab} = -\frac{1}{2}(e_m{}^{--}e_n{}^{++} + e_m{}^{++}e_n{}^{--})\,, \\ g^{mn}(\xi) &= \eta^{ab} e_a{}^m e_b{}^n = -2(e_{++}{}^m e_{--}{}^n + e_{--}{}^m e_{++}{}^n)\,, \end{aligned} \tag{4.24}$$

where $e_a{}^m$ is the inverse of $e_m{}^a$, and we are using the light-cone coordinates defined in Section 3.3.1. From (4.24), we also deduce that

$$\det g_{mn} \equiv g = -(\det e_m{}^a)^2\,, \qquad \det e_m{}^a = \frac{1}{2}\varepsilon^{mn} e_m{}^{--} e_n{}^{++}\,. \tag{4.25}$$

Therefore, by defining $e \equiv |\det e_m{}^a|$ we can rewrite the first term in (4.21) as

$$\begin{aligned} &-\frac{T}{2}\int \mathrm{d}^2\xi \sqrt{-g} g^{mn} \mathcal{E}_m^{\underline{a}} \mathcal{E}_n^{\underline{b}} \eta_{\underline{ab}} = T\int \mathrm{d}^2\xi\, e(e_{++}{}^m e_{--}{}^n + e_{--}{}^m e_{++}{}^n)\mathcal{E}_m^{\underline{a}} \mathcal{E}_n^{\underline{b}} \eta_{\underline{ab}} = \\ &= T\int \mathrm{d}^2\xi\, \varepsilon^{mn} e_m{}^{--} e_n{}^{++} \mathcal{E}_{--}^{\underline{a}} \mathcal{E}_{++}^{\underline{b}} \eta_{\underline{ab}}\,, \end{aligned} \tag{4.26}$$

where $\mathcal{E}_{--}^{\underline{a}} = e_{--}{}^m \mathcal{E}_m^{\underline{a}}$ and $\mathcal{E}_{++}^{\underline{b}} = e_{++}{}^m \mathcal{E}_m^{\underline{b}}$.

Starting from (4.21), we can obtain the dynamical equations of motion of the superstring by varying the action with respect to $x^{\underline{m}}(\xi)$ and $\theta^{\underline{\mu}}(\xi)$, and the two Virasoro constraints

$$\mathcal{E}_{--}^{\underline{a}} \mathcal{E}_{--\underline{a}} = \mathcal{E}_{++}^{\underline{a}} \mathcal{E}_{++\underline{a}} = 0\,, \tag{4.27}$$

obtained by varying the action with respect to the worldsheet metric.

It turns out that the action (4.21) is invariant under the global $\mathcal{N} = 1$ bulk supersymmetry transformations

$$\delta\theta = \varepsilon, \qquad \delta x^{\underline{a}} = i\bar{\varepsilon}\gamma^{\underline{a}}\theta\,, \tag{4.28}$$

only in $D = 3$, 4, 6 and 10 and, since we are eventually interested in considering the superembedding approach for the BPS axionic strings in 4 dimensions, we restrict to the case $D = 4$.

Besides the global $\mathcal{N} = 1$ bulk supersymmetry, (4.21) also has a local worldsheet fermionic symmetry, called κ-symmetry. In analogy with the case of the BPS axionic strings, the κ-parameter satisfies a projection condition, which leads to have a number of independent κ-transformations that is half the number of components of θ. Thus, in $D = 4$ κ-symmetry has 2 independent components, and we now see how the superembedding approach replaces the κ-symmetry transformations of the Green-Schwarz type superstring action with world-sheet local supersymmetry transformations, giving a theory with a manifest $\mathcal{N} = (0, 2)$ local world-sheet supersymmetry. We start the discussion, by first considering the case of flat target superspace.

Superembedding reformulation of $\mathcal{N} = 1$ superstrings: flat case

To have the worldsheet and target space supersymmetry manifest, the superembedding formulation should be constructed as a superfield theory on both the worldsheet superspace and the target superspace, the former being embedded into the latter. To construct a theory with these properties and be able to describe the dynamics of GS superstrings in this approach, we will shortly see that it is necessary to impose a condition on the embedding of the superworldsheet into the target superspace, which is the so-called *superembedding condition*, which was first found in [43] for superparticles and then proved to be generic to all known types of superbranes.

First of all, to realize local supersymmetry on the worldsheet, we extend the latter to a supersurface $\mathcal{M}_{2,2}$ parametrized by two bosonic coordinates $\xi^m = (\xi^0, \xi^1)$ and 2 fermionic coordinates η^{+u}, $u = 1, 2$. We denote the full set of superworldsheet coordinates by $z^M = (\xi^m, \eta^{+u})$. This is exactly the $\mathcal{N} = (0, 2)$ superspace introduced in Section 3.3, where we have studied in detail $\mathcal{N} = (0, 2)$ supergravity in 2 dimensions.

At this point, we impose the *superembedding condition* for the $\mathcal{N}=1$ superstring, which prescribes that the Grassmann component of the superworldsheet pullback of the flat target-space supervielbein vector component $\mathcal{E}^{\underline{a}}(Z)$ is zero. To write explicitly its expression, let us first write the proper extension of (4.23a), which is

$$\begin{aligned}\mathcal{E}^{\underline{a}}(Z(z)) &= \left(\mathrm{d}X^{\underline{m}} + i\mathrm{d}\bar{\Theta}\gamma^{\underline{m}}\Theta\right)\delta^{\underline{a}}_{\underline{m}} = \mathrm{d}z^M\left(\partial_M X^{\underline{m}} + i\partial_M\bar{\Theta}\gamma^{\underline{m}}\Theta\right)\delta^{\underline{a}}_{\underline{m}} = \\ &= e^A\left(D_A X^{\underline{m}} + iD_A\bar{\Theta}\gamma^{\underline{m}}\Theta\right)\delta^{\underline{a}}_{\underline{m}} \equiv e^A\mathcal{E}^{\underline{a}}_A\,,\end{aligned} \tag{4.29}$$

where $D_A = e_A{}^M\partial_M$ and $e_A{}^M$ represent the inverse supervielbein components, while

$$Z^{\underline{M}}(z^M) = Z^{\underline{M}}(\xi,\eta) = \left(X^{\underline{m}}(\xi,\eta), \Theta^{\underline{\mu}}(\xi,\eta)\right), \tag{4.30}$$

with $X^{\underline{m}}(\xi,\eta)$, $\Theta^{\underline{\mu}}(\xi,\eta)$ two world-sheet superfields, whose lowest components are the bosonic ($x^{\underline{m}}(\xi)$) and the Grassmann spinor coordinates ($\theta^{\underline{\mu}}(\xi)$) of the string in the target superspace. Therefore, the superembedding condition reads

$$\mathcal{E}^{\underline{a}}_{+q}(Z(z)) = D_{+q}X^{\underline{a}} + iD_{+q}\bar{\Theta}\gamma^{\underline{a}}\Theta = 0\,. \tag{4.31}$$

Taking the covariant derivative D_{+r} of (4.31) and summing the same expression with the exchange $r \leftrightarrow q$, one obtains

$$\begin{aligned}&D_{+r}D_{+q}X^{\underline{a}} + iD_{+r}(D_{+q}\bar{\Theta}\gamma^{\underline{a}}\Theta) + (r \leftrightarrow q) = \\ &= \{D_{+q}, D_{+r}\}X^{\underline{a}} + i\{D_{+q}, D_{+r}\}\bar{\Theta}\gamma^{\underline{a}}\Theta + 2iD_{+q}\bar{\Theta}\gamma^{\underline{a}}D_{+r}\Theta = 0\,,\end{aligned} \tag{4.32}$$

where $D_{+q}\bar{\Theta}\gamma^{\underline{a}}D_{+r}\Theta = D_{+r}\bar{\Theta}\gamma^{\underline{a}}D_{+q}\Theta$ because of symmetry properties of the γ–matrices [44]. If we now take into account the results of Section 3.3, we have that

$$\{D_{+q}, D_{+r}\} = i\delta_{qr}D_{++} \tag{4.33}$$

and then we get

$$\delta_{qr}\mathcal{E}^{\underline{a}}_{++} \equiv \delta_{qr}(D_{++}X^{\underline{a}} + iD_{++}\bar{\Theta}\gamma^{\underline{a}}\Theta) = -2D_{+q}\bar{\Theta}\gamma^{\underline{a}}D_{+r}\Theta \quad\Longrightarrow\quad \mathcal{E}^{\underline{a}}_{++} = -D_{+q}\bar{\Theta}\gamma^{\underline{a}}D_{+q}\Theta\,, \tag{4.34}$$

which, in turn, implies one of the Virasoro constraints, namely

$$\mathcal{E}^{\underline{a}}_{++}\mathcal{E}_{++\underline{a}} = 0\,, \tag{4.35}$$

because of the property of the γ-matrices

$$(\gamma_{\underline{a}})_{\underline{\mu}(\underline{\nu}}(\gamma^{\underline{a}})_{\underline{\rho\sigma})} = 0\,. \tag{4.36}$$

On the other hand, the superstring dynamical equations of motion and the second Virasoro constraint should be derived from a worldsheet superfield action in $\mathcal{N}=1$, $D=4$ target superspace. We start to construct this action by writing down the term which produces the superembedding condition (4.31):

$$S_0 = -i\int d^2\xi d^2\eta P^{+q}_{\underline{a}}\mathcal{E}^{\underline{a}}_{+q}\,, \tag{4.37}$$

where the superfield $P^{+q}_{\underline{a}}(z)$ is a Lagrange multiplier. However, this action is incomplete: it does not describe the fully fledged $\mathcal{N}=1$, $D=4$ superstrings, but the so-called null (or tensionless) superstrings, extended objects characterized by having zero tension and a degenerate worldsheet metric [45]. Let us then show how one can extend the null superstring action (4.37) to describe the standard $\mathcal{N}=1$ superstring.

We start by introducing an 'electromagnetic' field $A_m(\xi)$ on the superstring worldsheet and construct a worldsheet two-form which is invariant under the target space supersymmetry transformations (4.28). The appropriate two-form is

$$\mathcal{F}^{(2)} = e^{--}e^{++}\mathcal{E}^{\underline{a}}_{--}\mathcal{E}^{\underline{b}}_{++}\eta_{\underline{ab}} + B^{(2)} + \mathrm{d}A\,, \tag{4.38}$$

where $e^{++} = e^0 + e^1 = d\xi^m e_m{}^{++}(\xi)$ and $e^{--} = e^0 - e^1 = d\xi^m e_m{}^{--}(\xi)$ are the light-cone components of the zweibein $e^a(\xi) = d\xi^m e_m{}^a(\xi)$, while $B^{(2)}$ is the worldsheet pullback of the Wess-Zumino form (4.22).

Under the supersymmetry transformations (4.28), $B^{(2)}$ transforms as a total derivative:

$$\delta B^{(2)} = i\mathrm{d}\left[\left(\mathrm{d}x^{\underline{a}} - i\mathrm{d}\bar{\theta}\gamma^{\underline{a}}\theta\right)\bar{\theta}\gamma_{\underline{a}}\varepsilon\right] . \tag{4.39}$$

Thus, to cancel its variation in (4.38), we fix the variation of $A_m(\xi)$ to be

$$\delta A = -i\left(\mathrm{d}x^{\underline{a}} - i\mathrm{d}\bar{\theta}\gamma^{\underline{a}}\theta\right)\bar{\theta}\gamma_{\underline{a}}\varepsilon\,. \tag{4.40}$$

The vector field $A_m(\xi)$ should not be a new propagating worldsheet field since our aim is to describe the ordinary superstrings which do not carry such fields. Therefore, on the mass shell, it should be expressed in terms of superstring dynamical variables. This is achieved by assuming that $\mathcal{F}^{(2)}$ vanishes on the mass shell, i.e.

$$\mathrm{d}A = -e^{--}e^{++}\mathcal{E}^{\underline{a}}_{--}\mathcal{E}^{\underline{b}}_{++}\eta_{\underline{ab}} - B^{(2)}\,. \tag{4.41}$$

The latter equation implies that the field strength of $A_m(\xi)$ is not independent and hence does not describe new physical degrees of freedom.

Let us now consider the 2-form (4.38) given on the superworldsheet $\mathcal{M}_{2,2}$, in that all the quantities in its definition depend on $z^M = (\xi^m, \eta^{+u})$. In other words, let us promote the 2-form (4.38) to a 2-superform. One can show that such a 2-superform is closed on $\mathcal{M}_{2,2}$ when the superembedding condition (4.31) is satisfied [12], namely

$$\mathrm{d}\mathcal{F}^{(2)}\Big|_{\mathcal{M}_{2,2}} = 0\,. \tag{4.42}$$

We do not report the proof of this statement, but the strategy would be the same as the one we will utilize in Section 4.3.1 for the BPS axionic strings. The property (4.42) implies that the two-superform $\mathcal{F}^{(2)}$ can be written as

$$\mathrm{d}\mathcal{F}^{(2)} = \Omega^{+q}_{\underline{a}}\mathcal{E}^{\underline{a}}_{+q} + \Omega^{++\{qr\}}_{\underline{a}}D_{+r}\mathcal{E}^{\underline{a}}_{+q}\,, \tag{4.43}$$

where Ω are some three-superforms.

Let us now consider the following action

$$S = S_0 + S_T = -i\int \mathrm{d}^2\xi\mathrm{d}^2\eta P^{+q}_{\underline{a}}\mathcal{E}^{\underline{a}}_{+q} + T\int_{\mathcal{M}_2}\mathcal{F}^{(2)}\,, \tag{4.44}$$

where the second term is the integral of $\mathcal{F}^{(2)}$ over the two-dimensional slice of $\mathcal{M}_{2,2}$ such that $\eta^{+u} = 0$, that is the ordinary world-sheet $\mathcal{M}_2$.

By integrating out $P^{+q}_{\underline{a}}$ and considering (4.38), this equation reproduces the Green-Schwarz action (4.21), with the Nambu-Goto term expressed as in (4.26)[7]. Furthermore, it can be shown [12, 46] that (4.44) possesses local $\mathcal{N} = (0,2)$ world-sheet supersymmetry, although the second term is not a full superspace integral. Indeed, the variation of $\mathcal{F}^{(2)}$ under the worldsheet superdiffeomorphisms $z^M \longrightarrow z'^M = z^M + \delta z^M(\xi,\eta)$ reads[8]

$$\delta\mathcal{F}^{(2)} = \mathrm{d}(i_\delta\mathcal{F}^{(2)}) + i_\delta\mathrm{d}\mathcal{F}^{(2)}\,, \tag{4.45}$$

and then (up to boundary terms)

$$\delta S_T = T\int_{\mathcal{M}_2} i_\delta\mathrm{d}\mathcal{F}^{(2)}\,, \tag{4.46}$$

with $\mathrm{d}\mathcal{F}^{(2)}$ 'proportional' to the superembedding condition, as seen in (4.43). Therefore, once we consider the full action (4.44), the variation (4.46) can be compensated by a proper variation of the Lagrange multiplier of S_0.

[7]Note that the term with $A_m(\xi)$ is a total derivative, therefore it can be neglected, at least for closed strings.

[8]In this expression, i_δ represents the operation of interior product associated to δz^M [47].

One can also obtain a manifestly supersymmetric action, by writing S_T as a full superspace integral. To this aim, one constructs a Lagrange multiplier term [48] which produces the on-shell condition

$$\mathcal{F}^{(2)} = 0\,, \tag{4.47}$$

and the superstring action which includes such a term is

$$S = S_0 + S_T = -i\int \mathrm{d}^2\xi\mathrm{d}^2\eta P^{+q}_{\underline{a}}\mathcal{E}^{\underline{a}}_{+q} + \int \mathrm{d}^2\xi\mathrm{d}^2\eta P^{MN}\mathcal{F}^{(2)}_{MN}\,, \tag{4.48}$$

where[9]

$$S_T = \int \mathrm{d}^2\xi\mathrm{d}^2\eta P^{MN}\mathcal{F}^{(2)}_{MN} = \int \mathrm{d}^2\xi\mathrm{d}^2\eta P^{MN}\left[\mathcal{E}^{\underline{a}}_{[M}e^{++}_{N\}}\mathcal{E}_{++\underline{a}} + B^{(2)}_{MN} + \partial_{[M}A_{N\}}\right]. \tag{4.49}$$

The action (4.48) is invariant under the local transformations of the Lagrange multiplier P^{MN}

$$\delta P^{[MN\}} = \partial_L\Lambda^{[LMN\}}\,. \tag{4.50}$$

Indeed, the variation of S_T w.r.t. (4.50) is

$$\delta S_T = \int \mathrm{d}^2\xi\mathrm{d}^2\eta\Lambda^{[LMN\}}(\mathrm{d}\mathcal{F}^{(2)})_{LMN}\,, \tag{4.51}$$

where $\mathrm{d}\mathcal{F}^{(2)}$ is 'proportional' to the superembedding condition (see eq. (4.43)), thus the variation (4.51) is canceled by an appropriate variation of the Lagrange multiplier $P^{+q}_{\underline{a}}$ in (4.48), which is

$$\delta P^{+q}_{\underline{a}} = -\Lambda^{[LMN\}}\Omega^{+q}_{\underline{a},LMN} + \partial_P(e_{+r}{}^P\Lambda^{[LMN\}}\Omega^{++\{qr\}}_{\underline{a}}) \tag{4.52}$$

where the three-superforms Ω are the ones appearing in (4.43). This is analogous to what happens for (4.44).

The variation of the action (4.49) with respect to A_M gives

$$\partial_M P^{[MN\}} = 0\,, \tag{4.53}$$

and its generic solution is given by

$$P^{[MN\}} = \partial_L\tilde{\Lambda}^{[LMN\}}(z) + \frac{1}{2}\varepsilon_{qr}\eta^{+q}\eta^{+r}\delta^{[M}_{--}\delta^{N\}}_{++}T\,, \tag{4.54}$$

with T a constant. The first term may be set to zero by means of an appropriate local transformation (4.50), while from the second term of (4.54) we obtain the string tension as an integration constant and, upon the η-integration, the action reduces to the Green-Schwarz superstring action (4.21). Therefore, the worldsheet superfield action (4.48) describes the $\mathcal{N}=1$, $D=4$ superstrings.

Superembedding reformulation of $\mathcal{N}=1$ superstrings: curved case

The generalization of the superstring action (4.48) to describe a superstring propagating in curved target superspace is obtained through the following steps:

- we replace the flat bulk supervielbein with the curved one, i.e. $\mathcal{E}^{\underline{A}} \to E^{\underline{A}}$. In particular, the superembedding condition $\mathcal{E}^{\underline{a}}_{+q} = 0$ becomes $E^{\underline{a}}_{+q} = 0$;
- we consider $B^{(2)}(Z)$ as a two-form gauge superfield, whose leading component $B_{\underline{mn}}(X)$ is the Neveu-Schwarz gauge potential entering the supergravity multiplet;
- we introduce a dilaton superfield $\Phi(Z)$ coupling by redefining $\mathcal{F}^{(2)}$ as

$$\mathcal{F}^{(2)} = e^{\Phi}E^{\underline{a}}\wedge e^{++}E_{++\underline{a}} + B + \mathrm{d}A\,. \tag{4.55}$$

[9] As in Section 3.3, $[,\}$ denotes graded antisymmetrization of the superworldsheet indices: if one or both of the indices M, N are bosonic they are antisymmetrized, and if both of them are fermionic they are symmetrized.

The superstring action takes the form

$$S = S_0 + S_T = -i \int \mathrm{d}^2\xi \mathrm{d}^2\eta P_{\underline{a}}^{+q} E_{+q}^{\underline{a}} + T \int_{\mathcal{M}_2} \mathcal{F}^{(2)} \,, \tag{4.56}$$

or

$$S = S_0 + S_T = -i \int \mathrm{d}^2\xi \mathrm{d}^2\eta P_{\underline{a}}^{+q} E_{+q}^{\underline{a}} + \int \mathrm{d}^2\xi \mathrm{d}^2\eta P^{MN} \mathcal{F}^{(2)}_{MN} \,. \tag{4.57}$$

In analogy with the discussion done in the flat case, if $\mathrm{d}\mathcal{F}^{(2)}$ has the form

$$\mathrm{d}\mathcal{F}^{(2)} = \Omega_{\underline{a}}^{+q} E_{+q}^{\underline{a}} + \Omega_{\underline{a}}^{++\{qr\}} D_{+r} E_{+q}^{\underline{a}} \,, \tag{4.58}$$

where Ω are some super 3–forms, then (4.56) is invariant under worldsheet superdiffeomorphisms $z^M \to z'^M = z^M + \delta z^M(\xi, \eta)$, while (4.57) enjoys an invariance under the local transformation (4.50) of the Lagrange multiplier P^{MN}, which leads us to reproduce the generalization of the Green-Schwarz action (4.21) in a curved target space.

Performing the direct computation of $\mathrm{d}\mathcal{F}^{(2)}$, it turns out[10] that it has the form (4.58) (and then is closed on $\mathcal{M}_{2,2}$ when the superembedding condition is satisfied) if the superbackground satisfies the supergravity torsion constraint[11]:

$$T^{\underline{a}}_{\underline{\alpha\beta}} = -2i(C\gamma^{\underline{a}})_{\underline{\alpha\beta}} \,, \tag{4.59}$$

and the components of the field strength $H^{(3)} = \mathrm{d}B^{(2)}$ of the two-form gauge superfield are constrained as follows

$$H_{\underline{\alpha\beta a}} = 2ie^{\Phi(Z)}(C\gamma_{\underline{a}})_{\underline{\alpha\beta}}, \quad H_{\underline{\alpha\beta\gamma}} = 0 \,, \tag{4.60}$$

which are the same supergravity constraints required in the case of the Green-Schwarz formulation to consistently couple the superstrings to the supergravity background [40].

The procedure of integrating out the auxiliary superfield A_M is somehow universal, since it relies on the structure of the form $\mathcal{F} = ... + \mathrm{d}A$. The solution for $P^{[MN\}}$, thus, is always the same, provided that we can write $\mathrm{d}\mathcal{F}$ as in the form (4.58), i.e. that $\mathcal{F}^{(2)}$ is closed on $\mathcal{M}_{2,2}$ when the superembedding condition is satisfied. Therefore, we now move to the analysis of the BPS axionic strings, having in mind the procedure outlined in this section.

4.3 Superembedding reformulation of BPS axionic strings

In Section 4.1, we have discussed the GS formulation for the universal sector of the BPS axionic strings. Our purpose is now to extend the discussion of Section 4.2.1 to reformulate the theory of the BPS axionic strings in the superembedding formulation, and this will represent one of the main results of the thesis work.

In this respect, the GS superstring action (4.21) is replaced by (4.15), i.e.

$$S_{\text{string}} = -\int_{\mathcal{S}} \mathrm{d}^2\xi \, e^i L_i \sqrt{-\det \boldsymbol{\gamma}} + e^i \int_{\mathcal{S}} \mathbf{B}_{2,i} \,, \tag{4.61}$$

where we are considering the case $e^i L_i > 0$.

Therefore, we now want to find the two-superform $\mathcal{F}^{(2)}$, which will be of the form $\mathcal{F}^{(2)} = ... + \mathrm{d}A$, which reproduces the GS action (4.61). The proposed form for $\mathcal{F}^{(2)}$ is

$$\mathcal{F}^{(2)} = e^i L_i E^{\underline{a}} \wedge e^{++} E_{++\underline{a}} + e^i \mathbf{B}_{2,i} + \mathrm{d}A \,. \tag{4.62}$$

[10] In analogy with the flat case, we do not report the computation of the external differential of (4.55). However, in Section 4.3.2 we will perform such computation in the case of the BPS axionic strings in curved target space, which follows the same steps that should be done in the superstring case.

[11] Here and in (4.60), C stands for the charge conjugation matrix which can be used to raise and lower the spinor indices.

From the discussion given above on the superembedding formulation of $\mathcal{N}=1$ superstrings, we know that we need to check that $\mathcal{F}^{(2)}$ is closed on $\mathcal{M}_{2,2}$ when the superembedding condition $E^{\underline{a}}_{+q}=0$ is satisfied. This will guarantee that $\mathrm{d}\mathcal{F}^{(2)}$ has the form (4.58). Therefore, once done this, in principle one has to repeat the same steps of the derivation carried out in the case of $\mathcal{N}=1$ superstrings, and finally arrives at the conclusion that the action for a BPS axionic string in the superembedding approach is given by the same expression as before, i.e.

$$S=S_0+S_T=-i\int \mathrm{d}^2\xi\mathrm{d}^2\eta P^{+q}_{\underline{a}}E^{\underline{a}}_{+q}+\int_{\mathcal{M}_2}\mathcal{F}^{(2)}\,, \tag{4.63}$$

but with $\mathcal{F}^{(2)}$ given by (4.62). Note that we have written an action of the form (4.56), and not of the type (4.57). The reason is that the electric charge appearing in (4.61) is quantized, $e^i\in\mathbb{Z}$. However, the procedure outlined in the case of (4.48), which also holds for its curved extension (4.57), gives the superstring tension T as an integration constant. Conversely, in the case of BPS axionic strings with $\mathcal{F}^{(2)}$ given by (4.62), this integration constant would spoil the quantization of e^i, and this explains why we have chosen to write S as in (4.63).

Let us start with the proof of (4.42), by first discussing, in Section 4.3.1, the case in which the target superspace is the flat Minkowski space-time, and then extending the results to the general case of a curved target superspace in Section 4.3.2.

4.3.1 Superembedding approach for BPS axionic strings: flat case

In the flat target superspace, the expression (4.62) of $\mathcal{F}^{(2)}$ becomes[12]

$$\mathcal{F}^{(2)}=e^iL_i\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}}+e^i\mathbf{B}_{2,i}+\mathrm{d}A\,, \tag{4.64}$$

where the lowest component of the world-sheet superfield $\mathcal{E}^{\underline{a}}$ is given by (4.5). Furthermore, by definition of pullback and the superembedding condition $\mathcal{E}^{\underline{a}}_{+q}=0$, we find that $\mathcal{E}^{\underline{a}}$ can be expanded as

$$\mathcal{E}^{\underline{a}}=e^{--}\mathcal{E}^{\underline{a}}_{--}+e^{++}\mathcal{E}^{\underline{a}}_{++}+e^{+q}\mathcal{E}^{\underline{a}}_{+q}=e^{--}\mathcal{E}^{\underline{a}}_{--}+e^{++}\mathcal{E}^{\underline{a}}_{++}\,, \tag{4.65}$$

where we have used that if Ω is a p-form and Σ is a q-form, then $\mathrm{d}(\Omega\Sigma)=\Omega\mathrm{d}\Sigma+(-)^q\mathrm{d}\Omega\Sigma$ (the external differential acts from the right).

Let us start with the computation of $\mathrm{d}\mathcal{F}^{(2)}$:

$$\mathrm{d}\mathcal{F}^{(2)}=\mathrm{d}(e^iL_i\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}}+e^i\mathbf{B}_{2,i})=-\mathrm{d}(e^iL_i\mathcal{E}^{\underline{a}})e^{++}\mathcal{E}_{++\underline{a}}+e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}(e^{++}\mathcal{E}_{++\underline{a}})+e^i\mathrm{d}\mathbf{B}_{2,i}\,. \tag{4.66}$$

The first two terms in (4.66) can be expanded as

$$\begin{aligned}
&-\mathrm{d}(e^iL_i\mathcal{E}^{\underline{a}})e^{++}\mathcal{E}_{++\underline{a}}+e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}(e^{++}\mathcal{E}_{++\underline{a}})=\\
&=+e^i\mathrm{d}L_i\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}}-e^iL_i\mathrm{d}\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}}+e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}(e^{++}\mathcal{E}_{++\underline{a}})=\\
&=+e^i\mathrm{d}L_i\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}}-e^iL_i\mathrm{d}\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}}+e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}(\mathcal{E}_{\underline{a}}-e^{--}\mathcal{E}_{--\underline{a}})\,.
\end{aligned} \tag{4.67}$$

It turns out that

$$e^iL_i\mathrm{d}\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}}=0,\qquad e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}(e^{--}\mathcal{E}_{--\underline{a}})=0\,. \tag{4.68}$$

Before proving this, we report some relations which are useful for our aims. As already stated in the previous section, from the superembedding condition $\mathcal{E}^{\underline{a}}_{+q}=0$ one can derive the relation (4.34) for $\mathcal{E}^{\underline{a}}_{++}$. Thanks to this expression, we have seen that one can show that $\mathcal{E}^{\underline{a}}_{++}\mathcal{E}_{++\underline{a}}=0$, namely one of the two Virasoro constraints. We now prove again this, by rewriting the relation (4.34) for $\mathcal{E}^{\underline{a}}_{++}$ in the 2-component spinor notation, which is the notation used in Section 4.1. Furthermore, the reason why $\mathcal{E}^{\underline{a}}_{++}\mathcal{E}_{++\underline{a}}=0$ in this notation will be also useful to derive that other terms in $\mathrm{d}\mathcal{F}$ are vanishing.

[12] From now on, we drop the symbol $\wedge$ of wedge product.

Let us choose for the 4-dimensional γ-matrices the representation given in the Appendix B. Therefore:

$$\begin{aligned}\mathcal{E}^{\underline{a}}_{++} &= -D_{+q}\bar{\Theta}\gamma^{\underline{a}}D_{+q}\Theta = -D_{+q}\Theta^{\dagger}\gamma^0\gamma^{\underline{a}}D_{+q}\Theta = \\ &= -\begin{pmatrix} D_{+q}\Theta_\alpha \\ D_{+q}\bar{\Theta}^{\dot{\alpha}}\end{pmatrix}^{\dagger}\begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^{\underline{a}} \\ \bar{\sigma}^{\underline{a}} & 0\end{pmatrix}\begin{pmatrix} D_{+q}\Theta_\alpha \\ D_{+q}\bar{\Theta}^{\dot{\alpha}}\end{pmatrix} = \\ &= -\begin{pmatrix} -D_{+q}\Theta^\alpha & -D_{+q}\bar{\Theta}_{\dot{\alpha}}\end{pmatrix}\begin{pmatrix} 0 & \sigma^{\underline{a}} \\ \bar{\sigma}^{\underline{a}} & 0\end{pmatrix}\begin{pmatrix} D_{+q}\Theta_\alpha \\ D_{+q}\bar{\Theta}^{\dot{\alpha}}\end{pmatrix} = \\ &= \left(D_{+q}\bar{\Theta}\bar{\sigma}^{\underline{a}}D_{+q}\Theta + D_{+q}\Theta\sigma^{\underline{a}}D_{+q}\bar{\Theta}\right) = 2D_{+q}\Theta\sigma^{\underline{a}}D_{+q}\bar{\Theta}\,,\end{aligned} \tag{4.69}$$

where the last equality is true since $D_{+q}\Theta$ and $D_{+q}\bar{\Theta}$ are commuting quantities, being the product of two Grassmann-odd quantities. Therefore, we can write:

$$\delta_{qr}\mathcal{E}^{\underline{a}}_{++} = 4D_{+q}\Theta\sigma^{\underline{a}}D_{+r}\bar{\Theta} \quad\Longrightarrow\quad \mathcal{E}^{\underline{a}}_{++} = 4D_{+1}\Theta\sigma^{\underline{a}}D_{+1}\bar{\Theta} = 4D_{+2}\Theta\sigma^{\underline{a}}D_{+2}\bar{\Theta}\,, \tag{4.70}$$

and

$$\mathcal{E}^{\underline{a}}_{++}\mathcal{E}_{++\underline{a}} = 16D_{+1}\Theta\sigma^{\underline{a}}D_{+1}\bar{\Theta}D_{+1}\Theta\sigma_{\underline{a}}D_{+1}\bar{\Theta} = 16D_{+1}\Theta^{\underline{\alpha}}\sigma^{\underline{a}}_{\underline{\alpha\dot{\alpha}}}D_{+1}\bar{\Theta}^{\underline{\dot{\alpha}}}D_{+1}\Theta^{\underline{\beta}}\sigma_{\underline{a\beta\dot{\beta}}}D_{+1}\bar{\Theta}^{\underline{\dot{\beta}}} = 0\,, \tag{4.71}$$

because of the following relation of the σ–matrices

$$\sigma^m_{\alpha\dot{\alpha}}\sigma_{m\beta\dot{\beta}} = -2\varepsilon_{\beta\alpha}\varepsilon_{\dot{\beta}\dot{\alpha}}\,, \tag{4.72}$$

which can be obtained from[13]

$$\sigma^m_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_m = -2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}}\,. \tag{4.73}$$

We are now ready to prove the relations in (4.68). First of all, note that in flat target superspace:

$$\mathcal{E}^{\underline{a}} = \mathrm{d}X^{\underline{a}} - i\mathrm{d}\Theta\sigma^{\underline{a}}\bar{\Theta} + i\Theta\sigma^{\underline{a}}\mathrm{d}\bar{\Theta}\,, \tag{4.74}$$

which reduces to (4.5) if we consider the lowest components of $X^{\underline{a}}$ and $\Theta^{\underline{\alpha}}$. Therefore, we find that the torsion component $T^{\underline{a}}$ is

$$T^{\underline{a}} = \mathrm{d}\mathcal{E}^{\underline{a}} = -i\mathrm{d}\Theta\sigma^{\underline{a}}\mathrm{d}\bar{\Theta} - i\mathrm{d}\Theta\sigma^{\underline{a}}\mathrm{d}\bar{\Theta} = -2i\mathrm{d}\Theta\sigma^{\underline{a}}\mathrm{d}\bar{\Theta} = \mathrm{d}\Theta^{\underline{\alpha}}\mathrm{d}\bar{\Theta}^{\underline{\dot{\beta}}}(-2i(\sigma^{\underline{a}})_{\underline{\alpha\dot{\beta}}})\,. \tag{4.75}$$

Furthermore:

$$\begin{aligned}\mathrm{d}\Theta^{\underline{\alpha}} &= e^{++}D_{++}\Theta^{\underline{\alpha}} + e^{--}D_{--}\Theta^{\underline{\alpha}} + e^{+q}D_{+q}\Theta^{\underline{\alpha}}\,,\\ \mathrm{d}\bar{\Theta}^{\underline{\dot{\beta}}} &= e^{++}D_{++}\bar{\Theta}^{\underline{\dot{\beta}}} + e^{--}D_{--}\bar{\Theta}^{\underline{\dot{\beta}}} + e^{+q}D_{+q}\bar{\Theta}^{\underline{\dot{\beta}}}\,,\end{aligned} \tag{4.76}$$

and then

$$\begin{aligned}e^iL_i\mathrm{d}\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}} =& e^iL_i(e^{--}D_{--}\Theta^{\underline{\alpha}} + e^{+q}D_{+q}\Theta^{\underline{\alpha}})\\ &(e^{--}D_{--}\bar{\Theta}^{\underline{\dot{\beta}}} + e^{+r}D_{+r}\bar{\Theta}^{\underline{\dot{\beta}}})(-2i(\sigma^{\underline{a}})_{\underline{\alpha\dot{\beta}}})e^{++}\mathcal{E}_{++\underline{a}}\,.\end{aligned} \tag{4.77}$$

Thus, we have three terms which contribute: one 'proportional' to $e^{--}e^{+r}$, another to $e^{+q}e^{--}$ and the last one to $e^{+q}e^{+r}$. We first consider the latter:

$$\begin{aligned}&-2ie^iL_ie^{+q}D_{+q}\Theta^{\underline{\alpha}}e^{+r}D_{+r}\bar{\Theta}^{\underline{\dot{\beta}}}(\sigma^{\underline{a}})_{\underline{\alpha\dot{\beta}}}e^{++}\mathcal{E}_{++\underline{a}} \propto\\ &\propto e^{+q}e^{+r}\,\underbrace{D_{+q}\Theta\sigma^{\underline{a}}D_{+r}\bar{\Theta}}_{\propto\delta_{qr}\mathcal{E}^{\underline{a}}_{++}}\,e^{++}\mathcal{E}_{++\underline{a}} = 0\,,\end{aligned} \tag{4.78}$$

where we have used (4.71). Let us now consider the term with $e^{--}e^{+r}$:

$$\begin{aligned}&-2ie^iL_ie^{--}D_{--}\Theta^{\underline{\alpha}}e^{+r}D_{+r}\bar{\Theta}^{\underline{\dot{\beta}}}(\sigma^{\underline{a}})_{\underline{\alpha\dot{\beta}}}e^{++}\mathcal{E}_{++\underline{a}} \propto\\ &\propto e^{--}e^{++}(e^{+1}D_{--}\Theta\sigma^{\underline{a}}D_{+1}\bar{\Theta} + e^{+2}D_{--}\Theta\sigma^{\underline{a}}D_{+2}\bar{\Theta})\mathcal{E}_{++\underline{a}}\,.\end{aligned} \tag{4.79}$$

[13] Recall that $\bar{\sigma}^{m\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^m_{\beta\dot{\beta}}$.

Analogously, we can rewrite the $e^{+q}e^{--}$ term as

$$\begin{aligned}&-2ie^iL_ie^{+q}D_{+q}\Theta^{\underline{\alpha}}e^{--}D_{--}\bar{\Theta}^{\underline{\dot{\beta}}}(\sigma^{\underline{a}})_{\underline{\alpha\dot{\beta}}}e^{++}\mathcal{E}_{++\underline{a}}\propto\\&\propto e^{--}e^{++}(e^{+1}D_{+1}\Theta\sigma^{\underline{a}}D_{--}\bar{\Theta}+e^{+2}D_{+2}\Theta\sigma^{\underline{a}}D_{--}\bar{\Theta})\mathcal{E}_{++\underline{a}}\,.\end{aligned}\tag{4.80}$$

One can show that both terms are zero by properly choosing the expression for $\mathcal{E}_{++\underline{a}}$ in each piece and using the relation (4.72) of the σ–matrices. Indeed, in order to use (4.72), it is convenient to write $\mathcal{E}^{\underline{a}}_{++}=4D_{+1}\Theta\sigma^{\underline{a}}D_{+1}\bar{\Theta}$ in the first term of (4.79), and $\mathcal{E}^{\underline{a}}_{++}=4D_{+2}\Theta\sigma^{\underline{a}}D_{+2}\bar{\Theta}$ in its second contribution. The same happens for (4.80).

Let us now consider the second expression in (4.68), i.e.

$$e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}(e^{--}\mathcal{E}_{--\underline{a}})\propto e^{++}e^{--}e^{+q}D_{+q}\mathcal{E}_{--\underline{a}}\mathcal{E}^{\underline{a}}_{++}\,.\tag{4.81}$$

We observe that $e^{--}e^{+q}D_{+q}\mathcal{E}_{--\underline{a}}$ can be regarded as a component of the torsion $T^{\underline{a}}$ in the worldsheet supervielbein basis. Indeed, if we focus on the $e^{--}e^{+q}$ component, we have:

$$\begin{aligned}T^{\underline{a}}&=\mathrm{d}\mathcal{E}^{\underline{a}}\,,\\\mathrm{d}\mathcal{E}^{\underline{a}}&\longrightarrow\mathrm{d}(e^{--}\mathcal{E}_{--\underline{a}})=e^{--}e^{+q}D_{+q}\mathcal{E}_{--\underline{a}}\,,\\T^{\underline{a}}=-2i\mathrm{d}\Theta^{\underline{\alpha}}\mathrm{d}\bar{\Theta}^{\underline{\dot{\beta}}}\sigma^{\underline{a}}_{\underline{\alpha\dot{\beta}}}&\longrightarrow-2ie^{--}e^{+q}\left[D_{--}\Theta\sigma^{\underline{a}}D_{+q}\bar{\Theta}-D_{+q}\Theta\sigma^{\underline{a}}D_{--}\bar{\Theta}\right]\,,\end{aligned}\tag{4.82}$$

from which we obtain that

$$D_{+q}\mathcal{E}_{--\underline{a}}\propto D_{--}\Theta\sigma^{\underline{a}}D_{+q}\bar{\Theta}-D_{+q}\Theta\sigma^{\underline{a}}D_{--}\bar{\Theta}\,,\tag{4.83}$$

and then

$$e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}(e^{--}\mathcal{E}_{--\underline{a}})\propto e^{++}e^{--}e^{+q}\left[D_{--}\Theta\sigma^{\underline{a}}D_{+q}\bar{\Theta}-D_{+q}\Theta\sigma^{\underline{a}}D_{--}\bar{\Theta}\right]\mathcal{E}^{\underline{a}}_{++}\,.\tag{4.84}$$

The two terms have the same form as (4.79) and (4.80), respectively, therefore they also vanish. This completes the proof of (4.68), and then (4.66) becomes

$$\mathrm{d}\mathcal{F}^{(2)}=e^i\mathrm{d}L_i\mathcal{E}^{\underline{a}}e^{++}\mathcal{E}_{++\underline{a}}+e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}\mathcal{E}_{\underline{a}}+e^i\mathrm{d}\mathbf{B}_{2,i}\,,\tag{4.85}$$

where $\mathrm{d}\mathbf{B}_{2,i}$ is given by (4.6), and we report here its expression for convenience:

$$\begin{aligned}\mathrm{d}\mathbf{B}_{2,i}=\;&2i\mathcal{E}^{\underline{a}}\mathcal{E}^{\underline{\alpha}}\bar{\mathcal{E}}^{\underline{\dot{\alpha}}}(\sigma_{\underline{a}})_{\underline{\alpha\dot{\alpha}}}L_i+\\&-\mathcal{E}^{\underline{b}}\mathcal{E}^{\underline{a}}\mathcal{E}^{\underline{\alpha}}(\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}}D_{\underline{\beta}}L_i-\mathcal{E}^{\underline{b}}\mathcal{E}^{\underline{a}}\bar{\mathcal{E}}^{\underline{\dot{\alpha}}}(\bar{\sigma}_{\underline{ab}})^{\underline{\dot{\beta}}}{}_{\underline{\dot{\alpha}}}\bar{D}_{\underline{\dot{\beta}}}L_i+\\&-\frac{1}{24}\mathcal{E}^{\underline{c}}\mathcal{E}^{\underline{b}}\mathcal{E}^{\underline{a}}\varepsilon_{\underline{abcd}}(\bar{\sigma}^{\underline{d}})^{\underline{\dot{\alpha}\alpha}}\left[D_{\underline{\alpha}},\bar{D}_{\underline{\dot{\alpha}}}\right]L_i\,.\end{aligned}\tag{4.86}$$

Let us now analyse each line of (4.86).

The third line vanishes when the superembedding condition is satisfied, since $\mathcal{E}^{\underline{a}}$ only has two non-vanishing components and $e^{++}\wedge e^{++}=e^{--}\wedge e^{--}=0$.

The first line cancels the second term in (4.85). Indeed, if we recall the expression (4.75) of $T^{\underline{a}}$, we get

$$e^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}\mathcal{E}_{\underline{a}}=e^iL_i\mathcal{E}^{\underline{a}}T_{\underline{a}}=-2ie^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}\Theta^{\underline{\alpha}}\mathrm{d}\bar{\Theta}^{\underline{\dot{\alpha}}}\sigma_{\underline{a\alpha\dot{\alpha}}}\,.\tag{4.87}$$

Finally, by recalling that in flat superspace:

$$\mathcal{E}^{\underline{\alpha}}=\mathrm{d}\Theta^{\underline{\alpha}},\qquad\bar{\mathcal{E}}^{\underline{\dot{\alpha}}}=\mathrm{d}\bar{\Theta}^{\underline{\dot{\alpha}}}\,,\tag{4.88}$$

we get, for the first line of (4.86),

$$e^i\mathrm{d}\mathbf{B}_{2,i}(1)=+2ie^iL_i\mathcal{E}^{\underline{a}}\mathrm{d}\Theta^{\underline{\alpha}}\mathrm{d}\bar{\Theta}^{\underline{\dot{\alpha}}}\sigma_{\underline{a\alpha\dot{\alpha}}}\,,\tag{4.89}$$

which cancels (4.87).

Therefore, at this point, $\mathrm{d}\mathcal{F}^{(2)}$ is given by

$$\mathrm{d}\mathcal{F}^{(2)} = e^i \mathrm{d}L_i \mathcal{E}^{\underline{a}} e^{++}\mathcal{E}_{++\underline{a}} - e^i \mathcal{E}^{\underline{b}}\mathcal{E}^{\underline{a}}\mathcal{E}^{\underline{\alpha}}(\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} D_{\underline{\beta}} L^\Lambda - e^i \mathcal{E}^{\underline{b}}\mathcal{E}^{\underline{a}}\bar{\mathcal{E}}^{\underline{\dot\alpha}}(\bar\sigma_{\underline{ab}})^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} \bar D_{\underline{\dot\beta}} L^\Lambda \,. \tag{4.90}$$

By using the properties of the σ–matrices reported in Appendix B, one can show that:

$$\begin{aligned}
\eta_{\underline{ab}}\delta_{\underline{\alpha}}{}^{\underline{\beta}} + 2\sigma_{\underline{ab}\underline{\alpha}}{}^{\underline{\beta}} &= -\left(\sigma_{\underline{b}}\bar\sigma_{\underline{a}}\right)_{\underline{\alpha}}{}^{\underline{\beta}} ,\\
\eta_{\underline{ab}}\delta_{\underline{\alpha}}{}^{\underline{\beta}} - 2\sigma_{\underline{ab}\underline{\alpha}}{}^{\underline{\beta}} &= -\left(\sigma_{\underline{a}}\bar\sigma_{\underline{b}}\right)_{\underline{\alpha}}{}^{\underline{\beta}} ,\\
\eta_{\underline{ab}}\delta^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} - 2\bar\sigma_{\underline{ab}}{}^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} &= -\left(\bar\sigma_{\underline{a}}\sigma_{\underline{b}}\right)^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} ,\\
\eta_{\underline{ab}}\delta^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} + 2\bar\sigma_{\underline{ab}}{}^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} &= -\left(\bar\sigma_{\underline{b}}\sigma_{\underline{a}}\right)^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} .
\end{aligned} \tag{4.91}$$

Furthermore, we recall that:

$$\mathrm{d}L_i = \mathcal{E}^{\underline{A}} D_{\underline{A}} L_i = \mathcal{E}^{\underline{a}} D_{\underline{a}} L_i + \mathcal{E}^{\underline{\alpha}} D_{\underline{\alpha}} L_i + \bar{\mathcal{E}}_{\underline{\dot\alpha}} \bar D^{\underline{\dot\alpha}} L_i \,. \tag{4.92}$$

Let us now expand the three terms in (4.90):

$$\begin{aligned}
e^i \mathrm{d}L_i \mathcal{E}^{\underline{a}} e^{++}\mathcal{E}_{++\underline{a}} &= e^i(\mathcal{E}^{\underline{a}} D_{\underline{a}} L_i + \mathcal{E}^{\underline{\alpha}} D_{\underline{\alpha}} L_i + \bar{\mathcal{E}}_{\underline{\dot\alpha}} \bar D^{\underline{\dot\alpha}} L_i) e^{--}e^{++}\mathcal{E}_{++\underline{a}}\mathcal{E}^{\underline{a}}_{--} = \\
&= e^i(\mathcal{E}^{\underline{\alpha}} D_{\underline{\alpha}} L_i + \bar{\mathcal{E}}_{\underline{\dot\alpha}} \bar D^{\underline{\dot\alpha}} L_i) e^{--}e^{++}\mathcal{E}_{++\underline{a}}\mathcal{E}^{\underline{a}}_{--} = \\
&= e^i e^{+q} e^{--}e^{++}\mathcal{E}_{++\underline{a}}\mathcal{E}^{\underline{a}}_{--}\mathcal{E}^{\underline{\alpha}}_{+q} D_{\underline{\alpha}} L_i + \\
&+ e^i e^{+q} e^{--}e^{++}\mathcal{E}_{++\underline{a}}\mathcal{E}^{\underline{a}}_{--}\bar{\mathcal{E}}^{\underline{\dot\alpha}}_{+q} \bar D_{\underline{\dot\alpha}} L_i = \\
&= \frac{1}{2} e^i e^{+q} e^{--}e^{++}\left(\mathcal{E}^{\underline{a}}_{++}\mathcal{E}^{\underline{b}}_{--} + \mathcal{E}^{\underline{b}}_{++}\mathcal{E}^{\underline{a}}_{--}\right)\mathcal{E}^{\underline{\alpha}}_{+q}\left(\eta_{\underline{ab}}\right)\delta_{\underline{\alpha}}{}^{\underline{\beta}} D_{\underline{\beta}} L_i + \\
&+ \frac{1}{2} e^i e^{+q} e^{--}e^{++}\left(\mathcal{E}^{\underline{a}}_{++}\mathcal{E}^{\underline{b}}_{--} + \mathcal{E}^{\underline{b}}_{++}\mathcal{E}^{\underline{a}}_{--}\right)\bar{\mathcal{E}}^{\underline{\dot\alpha}}_{+q}\left(\eta_{\underline{ab}}\right)\delta^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} \bar D_{\underline{\dot\beta}} L_i \,, \\
\mathcal{E}^{\underline{b}}\mathcal{E}^{\underline{a}}\mathcal{E}^{\underline{\alpha}} &= (e^M e^N \mathcal{E}^{\underline{a}}_N \mathcal{E}^{\underline{b}}_M) e^{+q}\mathcal{E}^{\underline{\alpha}}_{+q} = \\
&= (e^{--}e^{++}\mathcal{E}^{\underline{a}}_{++}\mathcal{E}^{\underline{b}}_{--} + e^{++}e^{--}\mathcal{E}^{\underline{a}}_{--}\mathcal{E}^{\underline{b}}_{++}) e^{+q}\mathcal{E}^{\underline{\alpha}}_{+q} = \\
&= e^{+q}e^{--}e^{++}\left(\mathcal{E}^{\underline{a}}_{++}\mathcal{E}^{\underline{b}}_{--} - \mathcal{E}^{\underline{b}}_{++}\mathcal{E}^{\underline{a}}_{--}\right)\mathcal{E}^{\underline{\alpha}}_{+q} \,, \\
\mathcal{E}^{\underline{b}}\mathcal{E}^{\underline{a}}\bar{\mathcal{E}}^{\underline{\dot\alpha}} &= (e^{--}e^{++}\mathcal{E}^{\underline{a}}_{++}\mathcal{E}^{\underline{b}}_{--} + e^{++}e^{--}\mathcal{E}^{\underline{a}}_{--}\mathcal{E}^{\underline{b}}_{++}) e^{+q}\bar{\mathcal{E}}^{\underline{\dot\alpha}}_{+q} = \\
&= e^{+q}e^{--}e^{++}\left(\mathcal{E}^{\underline{a}}_{++}\mathcal{E}^{\underline{b}}_{--} - \mathcal{E}^{\underline{b}}_{++}\mathcal{E}^{\underline{a}}_{++}\right)\bar{\mathcal{E}}^{\underline{\dot\alpha}}_{+q} \,.
\end{aligned} \tag{4.93}$$

The combination of (4.91) and (4.93) allows us to rewrite (4.90) as

$$\begin{aligned}
\mathrm{d}\mathcal{F}^{(2)} = &- e^i e^{+q} e^{--}e^{++}\mathcal{E}^{\underline{a}}_{++}\mathcal{E}^{\underline{b}}_{--}\mathcal{E}^{\underline{\alpha}}_{+q}\left(\sigma_{\underline{a}}\bar\sigma_{\underline{b}}\right)_{\underline{\alpha}}{}^{\underline{\beta}} D_{\underline{\beta}} L_i + \\
&- e^i e^{+q} e^{--}e^{++}\mathcal{E}^{\underline{a}}_{++}\mathcal{E}^{\underline{b}}_{--}\bar{\mathcal{E}}^{\underline{\dot\alpha}}_{+q}\left(\bar\sigma_{\underline{a}}\sigma_{\underline{b}}\right)^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} \bar D_{\underline{\dot\beta}} L_i \,,
\end{aligned} \tag{4.94}$$

and we need to prove that it vanishes. To do this, we need to use (4.70), (4.72) and

$$\mathcal{E}^{\underline{\alpha}}_{+q} = D_{+q}\Theta^{\underline{\alpha}}, \qquad \bar{\mathcal{E}}^{\underline{\dot\alpha}}_{+q} = D_{+q}\bar\Theta^{\underline{\dot\alpha}} \,. \tag{4.95}$$

We then arrive at expressions very similar to those in (4.79) and (4.80), which vanish. Therefore, we have just proved that $\mathrm{d}\mathcal{F}^{(2)} = 0$ in the flat case when the superembedding condition is satisfied.

4.3.2 Superembedding approach for BPS axionic strings: curved case

Let us now extend the previous case to an EFT string propagating in curved target superspace. In this case, we start from an expression for the 2-superform $\mathcal{F}^{(2)}$ which is the same as before, but with the flat supervielbeins replaced with curved ones $\mathcal{E}^{\underline{a}}_{+q} \longrightarrow E^{\underline{a}}_{+q}$:

$$\mathcal{F}^{(2)} = e^i L_i E^{\underline{a}} e^{++} E_{++\underline{a}} + e^i \mathbf{B}_{2,i} + \mathrm{d}A \,. \tag{4.96}$$

Furthermore, we need to consider the appropriate covariantization of the expression for $\mathrm{d}\mathbf{B}_{2,i}$ considered in the flat case. It is given by (4.16), i.e.

$$\begin{aligned}\mathrm{d}\mathbf{B}_{2,i} = {} & 2iE^{\underline{a}}E^{\underline{\alpha}}\bar{E}^{\underline{\dot\alpha}}(\sigma_{\underline{a}})_{\underline{\alpha\dot\alpha}}L_i + \\ & - E^{\underline{b}}E^{\underline{a}}E^{\underline{\alpha}}(\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}}\mathcal{D}_{\underline{\beta}}L_i - E^{\underline{b}}E^{\underline{a}}\bar{E}^{\underline{\dot\alpha}}(\bar\sigma_{\underline{ab}})^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}}\bar{\mathcal{D}}_{\underline{\dot\beta}}L_i + \\ & - \frac{1}{24}E^{\underline{c}}E^{\underline{b}}E^{\underline{a}}\varepsilon_{\underline{abcd}}\left((\bar\sigma^{\underline{d}})^{\underline{\dot\alpha\alpha}}\left[\mathcal{D}_{\underline{\alpha}},\bar{\mathcal{D}}_{\underline{\dot\alpha}}\right] + 8G^{\underline{d}}\right)L_i\,.\end{aligned} \tag{4.97}$$

In analogy with the flat case, we now prove that $\mathrm{d}\mathcal{F}^{(2)} = 0$ when the superembedding condition $E_{+q}^{\underline{a}} = 0$ is satisfied.

Let us start by noticing that the third line of (4.97) is zero also in the curved case, due to the definition of pullback and the superembedding condition $E_{+q}^{\underline{a}} = 0$, which give

$$E^{\underline{a}} = e^{++}E_{++\underline{a}} + e^{--}E_{--\underline{a}}\,. \tag{4.98}$$

Before considering the other two lines, let us focus on the first term of $\mathrm{d}\mathcal{F}^{(2)}$, i.e.

$$\begin{aligned}\mathrm{d}(e^iL_iE^{\underline{a}}e^{++}E_{++\underline{a}}) &= -\mathrm{d}(e^iL_iE^{\underline{a}})e^{++}E_{++\underline{a}} + e^iL_iE^{\underline{a}}\mathrm{d}(e^{++}E_{++\underline{a}}) = \\ &= e^i\mathrm{d}L_iE^{\underline{a}}e^{++}E_{++\underline{a}} - e^iL_i\mathrm{d}E^{\underline{a}}e^{++}E_{++\underline{a}} + e^iL_iE^{\underline{a}}\mathrm{d}(e^{++}E_{++\underline{a}}) = \\ &= e^i\mathrm{d}L_iE^{\underline{a}}e^{++}E_{++\underline{a}} - e^iL_i\mathrm{d}E^{\underline{a}}e^{++}E_{++\underline{a}} + e^iL_iE^{\underline{a}}\mathrm{d}(E_{\underline{a}} - e^{--}E_{--\underline{a}}) = \\ &= e^i\mathrm{d}L_iE^{\underline{a}}e^{++}E_{++\underline{a}} - e^iL_i\mathrm{d}E^{\underline{a}}e^{++}E_{++\underline{a}} + e^iL_i\mathrm{d}E^{\underline{a}}E_{\underline{a}} - e^iL_iE^{\underline{a}}\mathrm{d}(e^{--}E_{--\underline{a}})\,.\end{aligned} \tag{4.99}$$

Before manipulating this expression, let us recall that, in order to reduce the number of component fields of the target-space supervielbein $E^{\underline{A}} = \mathrm{d}Z^{\underline{M}}E_{\underline{M}}{}^{\underline{A}}$ and superconnection $\Omega_{\underline{A}}{}^{\underline{B}} = \mathrm{d}Z^{\underline{M}}\Omega_{\underline{MA}}{}^{\underline{B}}$ to those contained in the 4d $\mathcal{N} = 1$ supergravity multiplet, in Section 3.1 we have imposed the supertorsion constraints (3.46), i.e.[14]

$$\begin{aligned}& T_{\underline{\alpha\dot\beta}}{}^{\underline{a}} = T_{\underline{\dot\beta\alpha}}{}^{\underline{a}} = -2i\sigma^{\underline{a}}_{\underline{\alpha\dot\beta}}\,, \\ & T_{\underline{\bar\alpha\bar\beta}}{}^{\underline{\bar\gamma}} = 0, \quad T_{\underline{\alpha\beta}}{}^{\underline{c}} = T_{\underline{\dot\alpha\dot\beta}}{}^{\underline{c}} = 0\,, \\ & T_{\underline{\bar\alpha b}}{}^{\underline{c}} = T_{\underline{a\bar\beta}}{}^{\underline{c}} = 0\,, \\ & T_{\underline{ab}}{}^{\underline{c}} = 0\,,\end{aligned} \tag{4.100}$$

In particular, we are interested in the form of $T^{\underline{a}}$, and from (4.100) we immediately see that it is

$$T^{\underline{a}} = -2iE^{\underline{\alpha}}\bar{E}^{\underline{\dot\alpha}}\sigma^{\underline{a}}_{\underline{\alpha\dot\alpha}}\,. \tag{4.101}$$

On the other hand, $T^{\underline{a}}$ is given by

$$T^{\underline{a}} = \mathcal{D}E^{\underline{a}} = \mathrm{d}E^{\underline{a}} + E^{\underline{B}}\Omega_{\underline{B}}{}^{\underline{a}} \quad\Longrightarrow\quad \mathrm{d}E^{\underline{a}} = T^{\underline{a}} - E^{\underline{B}}\Omega_{\underline{B}}{}^{\underline{a}}\,, \tag{4.102}$$

and therefore

$$e^iL_i\mathrm{d}E^{\underline{a}}E_{\underline{a}} = -2iE^{\underline{\alpha}}\bar{E}^{\underline{\dot\alpha}}E^{\underline{a}}\sigma_{\underline{a\alpha\dot\alpha}}e^iL_i - e^iL_iE^{\underline{B}}\Omega_{\underline{B}}{}^{\underline{a}}E_{\underline{a}}\,. \tag{4.103}$$

The first term cancels the first line of (4.97), therefore only the last term of this expression remains.

As far as the other term $\propto \mathrm{d}E^{\underline{a}}$ in (4.99) is concerned, we get

$$-e^iL_i\mathrm{d}E^{\underline{a}}e^{++}E_{++\underline{a}} = 2iE^{\underline{\alpha}}\bar{E}^{\underline{\dot\alpha}}e^{++}E_{++\underline{a}}\sigma^{\underline{a}}_{\underline{\alpha\dot\alpha}}e^iL_i + e^iL_iE^{\underline{B}}\Omega_{\underline{B}}{}^{\underline{a}}e^{++}E_{++\underline{a}}\,. \tag{4.104}$$

[14] Here, $\underline{\bar\alpha}$ denotes either $\underline{\alpha}$ or $\underline{\dot\alpha}$.

Therefore:

$$
\begin{aligned}
\mathrm{d}\mathcal{F}^{(2)} =&\ e^i \mathrm{d}L_i E^{\underline{a}} e^{++} E_{++\underline{a}} + 2iE^{\underline{\alpha}}\bar{E}^{\underline{\dot\alpha}} e^{++} E_{++\underline{a}} \sigma^{\underline{a}}_{\underline{\alpha\dot\alpha}} e^i L_i + e^i L_i E^{\underline{B}} \Omega_{\underline{B}}{}^{\underline{a}} e^{++} E_{++\underline{a}} + \\
&- e^i L_i E^{\underline{B}} \Omega_{\underline{B}}{}^{\underline{a}} E_{\underline{a}} - e^i L_i E^{\underline{a}} \mathrm{d}(e^{--} E_{--\underline{a}}) - e^i \Big[E^{\underline{b}} E^{\underline{a}} E^{\underline{\alpha}} (\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} \mathcal{D}_{\underline{\beta}} L_i + \\
&+ E^{\underline{b}} E^{\underline{a}} \bar{E}^{\underline{\dot\alpha}} (\bar\sigma_{\underline{ab}})^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} \bar{\mathcal{D}}_{\underline{\dot\beta}} L_i \Big] = \\
=&\ e^i \mathrm{d}L_i E^{\underline{a}} e^{++} E_{++\underline{a}} + 2iE^{\underline{\alpha}}\bar{E}^{\underline{\dot\alpha}} e^{++} E_{++\underline{a}} \sigma^{\underline{a}}_{\underline{\alpha\dot\alpha}} e^i L_i - e^i L_i E^{\underline{B}} \Omega_{\underline{B}}{}^{\underline{a}} e^{--} E_{--\underline{a}} + \\
&- e^i L_i E^{\underline{a}} \mathrm{d}(e^{--} E_{--\underline{a}}) - e^i \Big[E^{\underline{b}} E^{\underline{a}} E^{\underline{\alpha}} (\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} \mathcal{D}_{\underline{\beta}} L_i + E^{\underline{b}} E^{\underline{a}} \bar{E}^{\underline{\dot\alpha}} (\bar\sigma_{\underline{ab}})^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} \bar{\mathcal{D}}_{\underline{\dot\beta}} L_i \Big] \,.
\end{aligned}
\tag{4.105}
$$

Since the linear superfields L_i do not carry indices which transform under local Lorentz transformations, we have that $\mathcal{D}_{\underline{A}} L_i = E_{\underline{A}}{}^{\underline{M}} \partial_{\underline{M}} L_i$, and

$$
\begin{aligned}
\mathrm{d}L_i =&\ \mathrm{d}z^{\underline{M}} \partial_{\underline{M}} L_i = \mathrm{d}z^{\underline{M}} \underbrace{E_{\underline{M}}{}^{\underline{A}} E_{\underline{A}}{}^{\underline{N}}}_{\delta_{\underline{M}}{}^{\underline{N}}} \partial_{\underline{N}} L_i = E^{\underline{A}} \mathcal{D}_{\underline{A}} L_i = \\
=&\ E^{\underline{a}} \mathcal{D}_{\underline{a}} L_i + E^{\underline{\alpha}} \mathcal{D}_{\underline{\alpha}} L_i + \bar{E}_{\underline{\dot\alpha}} \bar{\mathcal{D}}^{\underline{\dot\alpha}} L_i \,.
\end{aligned}
\tag{4.106}
$$

Moreover, since the non-vanishing pullback components of $E^{\underline{a}}$ are only $E^{\underline{a}}_{--}$ and $E^{\underline{a}}_{++}$ when the superembedding condition is satisfied, we can rewrite the first term in (4.105) as

$$
e^i \mathrm{d}L_i E^{\underline{a}} e^{++} E_{++\underline{a}} = e^i (E^{\underline{\alpha}} \mathcal{D}_{\underline{\alpha}} L_i + \bar{E}_{\underline{\dot\alpha}} \bar{\mathcal{D}}^{\underline{\dot\alpha}} L_i) E^{\underline{a}} e^{++} E_{++\underline{a}} \,. \tag{4.107}
$$

Apart from the substitution of flat quantities with curved ones, this term and the last two in (4.105) are the same as the flat case, therefore, by following the same procedure as before, they can be rewritten as

$$
\begin{aligned}
&e^i \mathrm{d}L_i E^{\underline{a}} e^{++} E_{++\underline{a}} + e^i \Big[E^{\underline{b}} E^{\underline{a}} E^{\underline{\alpha}} (\sigma_{\underline{ab}})_{\underline{\alpha}}{}^{\underline{\beta}} \mathcal{D}_{\underline{\beta}} L^\Lambda + E^{\underline{b}} E^{\underline{a}} \bar{E}^{\underline{\dot\alpha}} (\bar\sigma_{\underline{ab}})^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} \bar{\mathcal{D}}_{\underline{\dot\beta}} L^\Lambda \Big] = \\
&= -e^i e^{+q} e^{--} e^{++} E^{\underline{a}}_{++} E^{\underline{b}}_{--} E^{\underline{\alpha}}_{+q} \left(\sigma_{\underline{a}} \bar\sigma_{\underline{b}}\right)_{\underline{\alpha}}{}^{\underline{\beta}} \mathcal{D}_{\underline{\beta}} L_i - e^i e^{+q} e^{--} e^{++} E^{\underline{a}}_{++} E^{\underline{b}}_{--} \bar{E}^{\underline{\dot\alpha}}_{+q} \left(\bar\sigma_{\underline{a}} \sigma_{\underline{b}}\right)^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} \bar{\mathcal{D}}_{\underline{\dot\beta}} L_i \,.
\end{aligned}
\tag{4.108}
$$

Let us now consider the following term:

$$
- e^i L_i E^{\underline{a}} \mathrm{d}(e^{--} E_{--\underline{a}}) = -e^i L_i E^{\underline{a}} \mathcal{D}(e^{--} E_{--\underline{a}}) + e^i L_i E^{\underline{a}} \Omega_{\underline{a}}{}^{C} e^{--} E_{--\underline{C}} \,. \tag{4.109}
$$

By considering $T_{\underline{a}} = \mathcal{D} E_{\underline{a}}$ and (4.101), we can replace the $e^{--}e^{+q}$ term of $\mathcal{D}(e^{--}E_{--\underline{a}})$ (which, due to the superembedding condition, is the only one contributing to the first term) with the $e^{--}e^{+q}$ term of $T_{\underline{a}}$, given by

$$
2ie^{--} e^{+q} \left[E^{\underline{\alpha}}_{--} \bar{E}^{\underline{\dot\alpha}}_{+q} + E^{\underline{\alpha}}_{+q} \bar{E}^{\underline{\dot\alpha}}_{--} \right] \sigma_{\underline{a\alpha\dot\alpha}} \,, \tag{4.110}
$$

therefore

$$
\begin{aligned}
-e^i L_i E^{\underline{a}} \mathrm{d}(e^{--} E_{--\underline{a}}) =&\ - 2ie^i L_i e^{++} E_{++\underline{a}} e^{--} e^{+q} \left[E^{\underline{\alpha}}_{--} \bar{E}^{\underline{\dot\alpha}}_{+q} + E^{\underline{\alpha}}_{+q} \bar{E}^{\underline{\dot\alpha}}_{--} \right] \sigma_{\underline{a\alpha\dot\alpha}} + \\
&+ e^i L_i E^{\underline{a}} \Omega_{\underline{a}}{}^{C} e^{--} E_{--\underline{C}} \,.
\end{aligned}
\tag{4.111}
$$

The final expression for $\mathrm{d}\mathcal{F}^{(2)}$ is

$$
\begin{aligned}
\mathrm{d}\mathcal{F}^{(2)} =&\ - e^i e^{+q} e^{--} e^{++} E^{\underline{a}}_{++} E^{\underline{b}}_{--} E^{\underline{\alpha}}_{+q} \left(\sigma_{\underline{a}} \bar\sigma_{\underline{b}}\right)_{\underline{\alpha}}{}^{\underline{\beta}} \mathcal{D}_{\underline{\beta}} L_i + \\
&- e^i e^{+q} e^{--} e^{++} E^{\underline{a}}_{++} E^{\underline{b}}_{--} \bar{E}^{\underline{\dot\alpha}}_{+q} \left(\bar\sigma_{\underline{a}} \sigma_{\underline{b}}\right)^{\underline{\dot\beta}}{}_{\underline{\dot\alpha}} \bar{\mathcal{D}}_{\underline{\dot\beta}} L_i + 2iE^{\underline{\alpha}} \bar{E}^{\underline{\dot\alpha}} e^{++} E_{++\underline{a}} \sigma^{\underline{a}}_{\underline{\alpha\dot\alpha}} e^i L_i + \\
&- e^i L_i E^{\underline{B}} \Omega_{\underline{B}}{}^{\underline{a}} e^{--} E_{--\underline{a}} - 2ie^i L_i e^{++} E_{++\underline{a}} e^{--} e^{+q} \left[E^{\underline{\alpha}}_{--} \bar{E}^{\underline{\dot\alpha}}_{+q} + E^{\underline{\alpha}}_{+q} \bar{E}^{\underline{\dot\alpha}}_{--} \right] \sigma_{\underline{a\alpha\dot\alpha}} + \\
&+ e^i L_i E^{\underline{a}} \Omega_{\underline{a}}{}^{C} e^{--} E_{--\underline{C}} \,.
\end{aligned}
\tag{4.112}
$$

The two terms containing superconnection components cancel each other, because of the fact that the only non-vanishing components of $\Omega_{\underline{CB}}{}^{\underline{a}}$ are those with $\underline{B} = \underline{b}$, therefore:

$$
-e^i L_i E^{\underline{B}} \Omega_{\underline{B}}{}^{\underline{a}} e^{--} E_{--\underline{a}} + e^i L_i E^{\underline{a}} \Omega_{\underline{a}}{}^{C} e^{--} E_{--\underline{C}} = -e^i L_i E^{\underline{b}} \Omega_{\underline{b}}{}^{\underline{a}} e^{--} E_{--\underline{a}} + e^i L_i E^{\underline{a}} \Omega_{\underline{a}}{}^{\underline{c}} e^{--} E_{--\underline{c}} = 0 \,. \tag{4.113}
$$

The remaining terms are vanishing if we find an expression for $E_{++\underline{a}}$ analogous to that of the flat case and use the relation (4.72) of the σ-matrices. To this aim, let us write the superembedding condition in the curved case, which reads

$$E_{+q}^{\underline{a}} = D_{+q} Z^{\underline{M}} E_{\underline{M}}^{\underline{a}} = 0 \,. \tag{4.114}$$

On the other hand, $E_{++}^{\underline{a}}$ is given by

$$E_{++}^{\underline{a}} = D_{++} Z^{\underline{M}} E_{\underline{M}}^{\underline{a}} \,. \tag{4.115}$$

If we take the covariant derivative D_{+r} of (4.114) and sum the same expression with the exchange $r \leftrightarrow q$, we obtain

$$D_{+r}(D_{+q} Z^{\underline{M}} E_{\underline{M}}^{\underline{a}}) + D_{+q}(D_{+r} Z^{\underline{M}} E_{\underline{M}}^{\underline{a}}) = 0 \,. \tag{4.116}$$

At this point, we consider that

$$D_{+q}(D_{+r} Z^{\underline{M}} E_{\underline{M}}^{\underline{a}}) = D_{+q} D_{+r} Z^{\underline{M}} E_{\underline{M}}^{\underline{a}} + (-)^{1+m} D_{+r} Z^{\underline{M}} D_{+q} E_{\underline{M}}^{\underline{a}} \,, \tag{4.117}$$

and the anticommutation relation (4.33), finding that

$$\delta_{qr} E_{++}^{\underline{a}} = i(-)^{1+m} \left[D_{+r} Z^{\underline{M}} D_{+q} E_{\underline{M}}^{\underline{a}} + D_{+q} Z^{\underline{M}} D_{+r} E_{\underline{M}}^{\underline{a}} \right] \,, \tag{4.118}$$

where $m = 0$ if $\underline{M}$ is a bosonic index, while $m = 1$ if $\underline{M}$ is a spinor index.

We now show that we can re-write (4.118) by considering the component $T^{\underline{a}}$ of the torsion, in particular its $e^{+q} e^{+r}$ component

$$T^{\underline{a}} \longrightarrow -2i e^{+q} e^{+r} E_{+q}^{\underline{\alpha}} \bar{E}_{+r}^{\dot{\underline{\alpha}}} \sigma_{\underline{\alpha}\dot{\underline{\alpha}}}^{\underline{a}} = -2i e^{+q} e^{+r} E_{+q} \sigma^{\underline{a}} \bar{E}_{+r} \,. \tag{4.119}$$

To do this, we start from the definition of the torsion (4.102), namely

$$T^{\underline{a}} = \mathrm{d}E^{\underline{a}} + E^{\underline{B}} \Omega_{\underline{B}}{}^{\underline{a}} = \mathrm{d}E^{\underline{a}} + E^{\underline{b}} \Omega_{\underline{b}}{}^{\underline{a}} \,. \tag{4.120}$$

The second term does not contribute to the $e^{+q} e^{+r}$ component because of the superembedding condition, therefore we only need to consider the first term, for which we have

$$\begin{aligned} \mathrm{d}E^{\underline{a}} = \mathrm{d}Z^{\underline{M}} \mathrm{d}Z^{\underline{N}} \partial_{\underline{N}} E_{\underline{M}}{}^{\underline{a}} \longrightarrow & \left(e^{+q} D_{+q} Z^{\underline{M}} \right) \left(e^{+r} D_{+r} E_{\underline{M}}{}^{\underline{a}} \right) = \\ & = (-)^{1+m} e^{+q} e^{+r} D_{+q} Z^{\underline{M}} D_{+r} E_{\underline{M}}{}^{\underline{a}} \,. \end{aligned} \tag{4.121}$$

Thus, from (4.119) and (4.121) we obtain

$$D_{+q} Z^{\underline{M}} D_{+r} E_{\underline{M}}{}^{\underline{a}} = (-)^m 2i E_{+q} \sigma^{\underline{a}} \bar{E}_{+r} \,, \tag{4.122}$$

which, if plugged into (4.118), gives

$$\delta_{qr} E_{++}^{\underline{a}} = 2 \left[E_{+r} \sigma^{\underline{a}} \bar{E}_{+q} + E_{+q} \sigma^{\underline{a}} \bar{E}_{+r} \right] \,. \tag{4.123}$$

Therefore, we obtain that $E_{++}^{\underline{a}}$ is given by

$$E_{++}^{\underline{a}} = 4 E_{+1} \sigma^{\underline{a}} \bar{E}_{+1} = 4 E_{+2} \sigma^{\underline{a}} \bar{E}_{+2} = \frac{1}{2} \left[2 E_{+q} \sigma^{\underline{a}} \bar{E}_{+q} + 2 E_{+q} \sigma^{\underline{a}} \bar{E}_{+q} \right] = 2 E_{+q} \sigma^{\underline{a}} \bar{E}_{+q} \,. \tag{4.124}$$

At this point, if we recall (4.113), we see that all the remaining terms in (4.112) contain $E_{++\underline{a}}$, and then, by plugging (4.124) into (4.112) and using the property (4.72) of the σ-matrices, we find that all of them are vanishing. Therefore, we have proven that, when the superembedding condition is satisfied, $\mathrm{d}\mathcal{F}^{(2)} = 0$ also in the generic case.

To summarize, we have shown that we can use the superembedding formalism, described in Section 4.2.1 for the case of $\mathcal{N} = 1$ superstrings, to write an action which is supersymmetric both on the superworldsheet of the EFT string and the target superspace. Its expression is

$$S = S_0 + S_T = -i \int \mathrm{d}^2\xi \mathrm{d}^2\eta P_{\underline{a}}^{+q} E_{+q}^{\underline{a}} + \int_{\mathcal{M}_2} \mathcal{F}^{(2)} \,, \tag{4.125}$$

with $\mathcal{F}^{(2)}$ given by (4.62), and where the first term of (4.125) imposes the superembedding condition $E_{+q}^{\underline{a}} = 0$, while the second term reproduces the Green-Schwarz action (4.61).

4.4 Inclusion of the internal world-sheet sector of EFT strings

In Section 4.2, we have constructed the action for BPS axionic strings in the superembedding formalism. However, this description only encodes the contribution arising from the universal sector of the world-sheet theory, while we know, from explicit UV completions, that also an 'internal' sector is present. In particular, in Section 2.5.2 we have said that such internal sector is assumed to be given by n_C chiral multiplets, n_N $U(1)_N$ charged Fermi multiplets and n_F $U(1)_N$ neutral Fermi multiplets.

Therefore, in this section we start from the result of Section 4.3.2 and include the internal degrees of freedom of the EFT strings. In particular, we consider the inclusion in the theory of a $U(1)_N$ neutral Fermi multiplet, by using the results for $\mathcal{N}=(0,2)$ supergravity obtained in Section 3.3.

A general $\mathcal{N}=(0,2)$ *Fermi multiplet* Λ_- obeys the relation [28]

$$\mathcal{D}_{\mp}\Lambda_- = \sqrt{2}E\,, \tag{4.126}$$

with the superfield E satisfying $\mathcal{D}_{\mp}E=0$[15]. Typically, $E=E(\Phi_i)$ is a holomorphic function of some $\mathcal{N}=(0,2)$ chiral superfields Φ_i[16]. However, since we want to consider an internal sector given by only one Fermi multiplet, we set $E=0$. Therefore, the r.h.s. of (4.126) is vanishing, while the l.h.s. is given by

$$\mathcal{D}_{\mp}\Lambda_- = e_{\mp}{}^M\left(\partial_M\Lambda_- - \Omega_M\mathcal{M}_-{}^A\Lambda_A\right)\,. \tag{4.127}$$

The expression of $\mathcal{M}_-{}^A$ can be read from (3.118). In particular

$$\mathcal{M}_\beta{}^\alpha = \frac{1}{2}(\gamma_3)_\beta{}^\alpha\,, \tag{4.128}$$

with $\gamma_3=\mathrm{diag}(-1,+1)$. Thus, the only non-vanishing component of $\mathcal{M}_-{}^A$ is $\mathcal{M}_-{}^-=-\frac{1}{2}$, whence

$$\mathcal{D}_{\mp}\Lambda_- = e_{\mp}{}^M\left(\partial_M\Lambda_- + \frac{1}{2}\Omega_M\Lambda_-\right) \stackrel{!}{=} 0\,. \tag{4.129}$$

Let us define the component expansion of Λ_- as

$$\Lambda_- = \lambda_- + \theta^+\alpha_{+,-} + \theta^{\mp}\beta_{\mp,-} + \theta^+\theta^{\mp}\gamma_{+\mp,-}\,. \tag{4.130}$$

By plugging this expansion in (4.129) and using the expressions of Ω_M and $e_{\mp}{}^M$ given in (3.194) and (3.196), we find that the component expansion of a Fermi multiplet Λ_- is

$$\Lambda_- = \lambda_- + \theta^+G + 2i\theta^+\theta^{\mp}(\mathscr{D}_{++}\lambda_- + \chi_{++}{}^+G)\,, \tag{4.131}$$

with

$$\mathscr{D}_{++}\lambda_- = \left(D_{++} + \frac{1}{2}\omega_{++}\right)\lambda_-\,, \tag{4.132}$$

and $D_{++}=e_{++}{}^m\partial_m$. Therefore, we see that the Fermi multiplet contains, among their components, a left-moving fermion, λ_-, and its supersymmetric partner, which is the complex scalar field G.

In general, to build supergravity Lagrangians within the superspace approach and write them in terms of the fields contained in the supermultiplets of the theory, the computation of the Berezinian (superdeterminant) of the supervielbein is needed. Therefore, let us start from its definition, which is

$$\tilde{e} \equiv \mathrm{sdet}\, e_M{}^A = \det\left(e_m{}^a - e_m{}^{+q}e^{-1}{}_{+q}{}^{+u}e_{+u}{}^a\right)\det\left(e_{+u}{}^{+q}\right)^{-1}\,. \tag{4.133}$$

From (3.190), we see that $e_{+u}{}^{+q}=\delta_{+u}{}^{+q}$, we have that $e^{-1}{}_{+q}{}^{+u}=\delta_{+q}{}^{+u}$ and $\det(e_{+u}{}^{+q})=1$. Therefore, in our case, its expression is given by

$$\tilde{e} = \det\left(e_m{}^a - e_m{}^{+u}e_{+u}{}^a\right) = \frac{1}{2}\varepsilon^{mn}e_m{}^{--}(e_n{}^{++} - 2i\theta^+\chi_n{}^{\mp} - 2i\theta^{\mp}\chi_n{}^+)\,. \tag{4.134}$$

[15] Here, we are using the notation $\theta^{\mp}\equiv\bar{\theta}^+$, therefore $\mathcal{D}_{\mp}\equiv\bar{\mathcal{D}}_+$.

[16] We will give the definition of an $\mathcal{N}=(0,2)$ chiral superfield and analyse its matter content later in this section.

We are now ready to consider the simplest contribution of a Fermi multiplet to the $\mathcal{N}=(0,2)$ supergravity action, which is taken to be

$$S_F = \frac{1}{4}\int \mathrm{d}^2\xi \mathrm{d}^2\theta\, \tilde{e}\,\bar{\Lambda}_-\Lambda_- \,. \tag{4.135}$$

By considering (4.134) and (4.131), we get

$$\int \mathrm{d}^2\theta\, \tilde{e}\,\bar{\Lambda}_-\Lambda_- = e\left[-|G|^2 + 2i(\bar{\lambda}_-\mathscr{D}_{++}\lambda_- - \mathscr{D}_{++}\bar{\lambda}_-\lambda_-)\right] , \tag{4.136}$$

where $e=\det(e_m{}^a)$. We see that G has no kinetic term and then is a purely auxiliary field. By plugging the equations of motion of G, we finally obtain

$$S_F = \frac{i}{2}\int \mathrm{d}^2\xi\, e\,\left(\bar{\lambda}_-\mathscr{D}_{++}\lambda_- - \mathscr{D}_{++}\bar{\lambda}_-\lambda_-\right) = i\int \mathrm{d}^2\xi\, e\,\left(\bar{\lambda}_- D_{++}\lambda_- - D_{++}\bar{\lambda}_-\lambda_-\right) . \tag{4.137}$$

Let us consider the second term:

$$-eD_{++}\bar{\lambda}_-\lambda_- = -ee_{++}{}^m\partial_m\bar{\lambda}_-\lambda_- = -\partial_m(ee_{++}{}^m\bar{\lambda}_-\lambda_-) + \partial_m(ee_{++}{}^m)\bar{\lambda}_-\lambda_- + e\bar{\lambda}_- D_{++}\lambda_- \,. \tag{4.138}$$

In particular:

$$\begin{aligned}\partial_m(ee_{++}{}^m) &= \frac{1}{2}\varepsilon^{pn}\partial_m(e_p{}^{--}e_n{}^{++}e_{++}{}^m) = \frac{1}{2}\varepsilon^{pn}\partial_m(e_p{}^{--}(e_n{}^a e_a{}^m - e_n{}^{--}e_{--}{}^m)) = \\ &= \frac{1}{2}\varepsilon^{pn}\partial_n e_p{}^{--} = \frac{1}{4}\varepsilon^{pn}(\partial_n e_p{}^{--} - \partial_p e_n{}^{--}) \,.\end{aligned} \tag{4.139}$$

On the other hand, from (3.191) we know that

$$\partial_n e_p{}^{--} - \partial_p e_n{}^{--} = e_p{}^{--}\omega_n - e_n{}^{--}\omega_p \,, \tag{4.140}$$

and then

$$\partial_m(ee_{++}{}^m) = \frac{1}{4}\varepsilon^{pn}(e_p{}^{--}\omega_n - e_n{}^{--}\omega_p) = \frac{1}{2}\varepsilon^{pn}e_p{}^{--}e_n{}^a\omega_a = \frac{1}{2}\varepsilon^{pn}e_p{}^{--}e_n{}^{++}\omega_{++} = e\omega_{++} \,. \tag{4.141}$$

Therefore, the final expression of (4.135) is given by

$$S_F = i\int \mathrm{d}^2\xi\, e\,\bar{\lambda}_-\mathscr{D}_{++}\lambda_- \,. \tag{4.142}$$

At this point, we consider the following action

$$\begin{aligned} S &= S_0 + S_T + S_F = \\ &= -i\int \mathrm{d}^2\xi \mathrm{d}^2\eta P^{+q}_{\underline{a}}E^{\underline{a}}_{+q} + \int_{\mathcal{M}_2}\mathcal{F}^{(2)} + \frac{1}{4}\int \mathrm{d}^2\xi\mathrm{d}^2\theta\,\tilde{e}\,\bar{\Lambda}_-\Lambda_- \,,\end{aligned} \tag{4.143}$$

where $\mathcal{F}^{(2)}$ is given by (4.96), i.e.

$$\mathcal{F}^{(2)} = e^i L_i E^{\underline{a}} e^{++}E_{++\underline{a}} + e^i\mathbf{B}_{2,i} + \mathrm{d}A \,. \tag{4.144}$$

This action describes the theory of a $\frac{1}{2}$-BPS string, minimally coupled to the gauge two-forms $\mathcal{B}_{2,i}$ with charges e^i, which, in addition to the fields parametrizing the string profile in the target superspace, contains an internal sector given by a Fermi multiplet Λ_-. The description is performed in a supersymmetric way both on the superworldsheet and the target superspace. Therefore, (4.143) provides a first example of supersymmetric world-sheet theory of an EFT string.

Finally, we report the expression of (4.143) in components:

$$S = \int \mathrm{d}^2\xi\varepsilon^{mn}\left[e^{--}_m e_n{}^{++}e^i L_i E^{\underline{a}}_{--}E_{++\underline{a}} + e^i\mathcal{B}_{mn,i} + \frac{i}{2}e_m{}^{--}e_n{}^{++}\bar{\lambda}_-\mathscr{D}_{++}\lambda_-\right] . \tag{4.145}$$

which is obtained by integrating out the Lagrange multiplier $P_{\underline{a}}^{+q}$ and using the superembedding condition to write $E^{\underline{a}}$ as in (4.98).

In Appendix E, we compute the energy-momentum tensor associated to (4.145) and show that the Virasoro constraint $E_{++}^{\underline{a}} E_{++\underline{a}} = 0$, which arises from the component $T_{++,++}$ of the energy-momentum tensor, remains unchanged in the case of the inclusion in the theory of a Fermi multiplet. This is important since, as seen in Section 4.2, the superembedding condition $E_{+q}^{\underline{a}} = 0$ automatically gives this constraint. Therefore, the superembedding condition is compatible with the equations of motion in presence of a Fermi multiplet on the world-sheet of the string. The same holds if, instead of considering only one Fermi multiplet, we consider a set of Fermi multiplets, $\{\Lambda_{-,i}\}$, $i = 1, \ldots, n_F$, described by the following action:

$$\tilde{S}_F = \frac{1}{4} \int \mathrm{d}^2\xi \mathrm{d}^2\theta \, \tilde{e} \, M^{ij} \bar{\Lambda}_{-,i} \Lambda_{-,j} \,, \tag{4.146}$$

where M^{ij} are the components of a constant $n_F \times n_F$ matrix.

Conversely, the previous analysis is not anymore true if we try to include *chiral multiplets*. In order to illustrate this point, we can focus on the simplest case in which only one (non-universal) chiral multiplet is present.

An $\mathcal{N} = (0, 2)$ *chiral multiplet* satisfies

$$\mathcal{D}_{\mp}\Phi = e_{\mp}{}^m \partial_m \Phi + e_{\mp}{}^+ \partial_+ \Phi + e_{\mp}{}^{\mp} \partial_{\mp} \Phi = 0 \,. \tag{4.147}$$

Its θ expansion can be obtained by using the results at the end of Section 3.3.5, and has the following form:

$$\Phi = \phi + \theta^+ \psi_+ + 2i\theta^+ \theta^{\mp} (D_{++}\phi + \chi_{++}{}^+ \psi_+) \,. \tag{4.148}$$

Therefore, the chiral multiplet is made up of a complex scalar field ϕ and a right-moving fermion, ψ_+[17].

Let us now consider the following action for the chiral multiplet:

$$S_{\text{ch}} = i \int \mathrm{d}^2\xi \mathrm{d}^2\theta \, \tilde{e} \bar{\Phi} \mathcal{D}_{--} \Phi \,. \tag{4.149}$$

If we focus on the the bosonic components of Φ, i.e. we set $\psi_+ = 0$ in (4.148), and we compute the component expansion of (4.149), we simply get

$$S_{\text{ch}} = - \int \mathrm{d}^2\xi \, e |D_a \phi|^2 \,, \tag{4.150}$$

where $D_a \phi = e_a{}^m \partial_m \phi$.

If we follow the procedure outlined in Appendix E, one finds that, in flat indices, the energy-momentum tensor associated to (4.150) is given by

$$\begin{aligned} T_{ab} = {} & 2\eta_{ab}(D_{++}\bar{\phi}D_{--}\phi + D_{--}\bar{\phi}D_{++}\phi) + \delta_{(b}{}^{--}(D_{a)}\bar{\phi}D_{--}\phi + D_{--}\bar{\phi}D_{a)}\phi) + \\ & + \delta_{(b}{}^{++}(D_{++}\bar{\phi}D_{a)}\phi + D_{a)}\bar{\phi}D_{++}\phi) \,. \end{aligned} \tag{4.151}$$

Thus, if we now add (4.150) to (4.145) and consider the full energy-momentum tensor, it turns out that $E_{++}^{\underline{a}} E_{++\underline{a}} \neq 0$, i.e. this is not anymore vanishing, due to the presence of the chiral multiplet, which gives a non-vanishing contribution to $T_{++,++}$, as can be seen from (4.151). This represents an obstruction in the inclusion of the chiral multiplets in the world-sheet theory, since the superembedding condition $E_{+q}^{\underline{a}} = 0$ leads to $E_{++}^{\underline{a}} E_{++\underline{a}} = 0$.

[17] Let us recall that in Section 4.1 we have seen that, in a local frame in which the string is locally stretched along the (x^0, x^3)-directions, the physical fields describing the string profile in the target superspace are the transversal embedding coordinates $x^1(\xi)$ and $x^2(\xi)$, which can be recasted as the real and imaginary part of a complex scalar field $u = x^1 + ix^2$, and the right-moving component of $\theta_{\underline{\mu}}(\xi)$, i.e. $\rho_+ \equiv \theta_+$. Therefore, the universal degrees of freedom of the string can be regarded as the components of an $\mathcal{N} = (0, 2)$ chiral superfield U, in agreement with what we said in Section 2.5.2.

Together with Section 4.3, this section contains the main results of the thesis work. In particular, we have included an $\mathcal{N} = (0,2)$ Fermi multiplet in the world-sheet theory of an EFT string, and its action is given by (4.143), or, in components, by (4.145). Then, we have moved to the case of an $\mathcal{N} = (0,2)$ chiral multiplet, and we have seen that its presence leads to a modification of the Virasoro constraint $E^{\underline{a}}_{++}E_{++\underline{a}} = 0$, which, on the other hand, is automatically deduced from the superembedding condition $E^{\underline{a}}_{+q} = 0$. This means that, in order for the superembedding approach to be applied also in presence of chiral multiplets, a proper generalization of the superembedding condition needs to be found. It would be interesting to elaborate on this aspect, but this is beyond the scope of the present work.

Chapter 5

Conclusions

In this work we have considered particular perturbative regimes of 4-dimensional $\mathcal{N} = 1$ EFTs arising from string/M-theory compactifications. An example of such effective theories has been discussed in Section 2.3, when we have considered M-theory compactifications on G_2 manifolds. Such theories contain a set of chiral superfields, denoted by T^i, whose lowest components are $T^i\big| = t^i = a^i + is^i$, with a^i and s^i real scalar fields. Furthermore, they are characterized by an invariance under the global transformations $a^i \to a^i + c^i$, where c^i are arbitrary constants, thus the fields a^i behave as axions, and their supersymmetric partners s^i are called saxions.

In this context, we have considered a class of solutions, associated with *fundamental axionic strings*, around which the axions undergo a monodromy transformation $a^i \to a^i + e^i$, where e^i are the string charges under some gauge 2-forms $\mathcal{B}_{2,i}$. Furthermore, these strings induce a backreaction flow of the saxionic fields, which drives them towards infinite distance limits in the field space.

In presence of the axionic shift symmetry, the theory of the axionic multiplets admit a dual version, where the axions a^i are replaced by the gauge 2-forms $\mathcal{B}_{2,i}$ under which the axionic strings are electrically charged, and the saxions s^i are traded for the dual saxions ℓ_i.

The axionic shift symmetry discussed so far is actually valid only at the perturbative level, being broken by non-perturbative instanton contributions to a discrete symmetry. This is in agreement with the No Global Symmetry Conjecture [4, 5], which claims that a theory consistent with quantum gravity can have no exact global symmetries. However, these non-perturbative corrections get exponentially suppressed along the saxionic flows if we consider a particular subclass of $\frac{1}{2}$-BPS axionic strings, given by the so-called *EFT strings*. Therefore, by definition, the EFT strings drive the scalar fields towards infinite distance field-space regions, where we can neglect the non-perturbative corrections breaking the axionic shift symmetries.

Associated to the EFT strings, in [6–8] two Swampland Conjectures are proposed, as discussed in Section 2.5.1, relating infinite distance field-space limits with EFT string flows and establishing a relation between the EFT string tension $\mathcal{T}$ and the scale m_* of the leading infinite tower of asymptotically massless states, predicted by the Swampland Distance Conjecture [25], which causes the EFT breakdown when considering infinite distance limits in field space. Furthermore, in Section 2.5.2 we have shown how theories with a standard coupling to the axionic sector are characterized by an anomaly inflow from the 4-dimensional bulk to the string world-sheet, which is cancelled by the 2-dimensional anomalies of the world-sheet theory itself [11]. The interesting result is that this leads to constraints on the 4-dimensional bulk theory, for example providing an upper bound on the rank of the EFT gauge sector detected by the EFT string, which may be relevant in the building of phenomenological models. These bounds are derived considering that, from explicit models arising from string/M-theory compactifications, we know that EFT strings support an additional 'internal' sector, in addition to the 'universal' one, which arises from deformations of the internal configuration of the compactification space. However, the analysis of [11] does not take into account the full theory describing the interactions of this internal sector with the bulk fields. Therefore, having a way to study in a controlled way the full theory could be interesting to find possible new constraints on the $\mathcal{N}=1$ 4-dimensional EFT.

In this context, the present thesis aims to describe the world-sheet theory of EFT strings in a supersymmetric way. Indeed, the Green-Schwarz (GS) formulation of the EFT strings [9, 10], discussed in Section 4.1, describes the universal sector of the world-sheet theory in a supersymmetric way in the target superspace. However, this formulation does not possess a manifest supersymmetry on the string world-sheet and, more importantly, is not suitable to include the internal sector of the world-sheet theory in a supersymmetrically controlled way.

In this respect, we have proposed to use an alternative formulation, namely the *superembedding formulation* (for a review see [12]). In this approach, the bosonic string world-sheet is extended to an $\mathcal{N}=(0,2)$ superworldsheet, embedded into the target superspace, and the fermionic κ-symmetry of the GS formulation is replaced by a local $\mathcal{N}=(0,2)$ local supersymmetry. To describe the embedding of the superworldsheet in the target superspace, we need to specify the geometrical properties of the former. To this aim, we have dedicated Section 3.3 to analyse in full detail $\mathcal{N}=(0,2)$ supergravity, finding the component expansion of the supervielbein $e_M{}^A$ and the superconnection Ω_M.

Then, after having introduced the superembedding approach in Section 4.2 by considering the example of $\mathcal{N}=1$ superstrings, in Section 4.3 we have used this formalism, until now applied to superbranes [12], to reformulate the theory of the universal sector of the BPS axionic strings.

Finally, in Section 4.4 we have combined the results of Section 3.3 and Section 4.3 to write a first example of manifestly supersymmetric action describing the world-sheet theory of an EFT string, with an internal sector given by an $\mathcal{N}=(0,2)$ Fermi multiplet. Furthermore, we have seen how the potential inclusion of an $\mathcal{N}=(0,2)$ chiral multiplet leads to an inconsistency, modifying the Virasoro constraints in a way which is incompatible with the superembedding condition. Together, Section 4.3 and Section 4.4 contain the main results of the thesis work.

Based on our results, one may further investigate the world-sheet theory of an EFT string, by considering a more intricate internal sector. For example, one could find a proper generalization of the superembedding condition which allows for the inclusion of a set of $\mathcal{N}=(0,2)$ chiral superfields to the internal sector. Furthermore, according to the general formulation with three-form potentials of [9], the two-form potentials $\mathcal{B}_{2,i}$, under which the axionic strings are electrically charged, could be gauged under some two-form gauge transformations. This effect makes some axionic strings 'anomalous', forcing them to be the boundary of membranes [9, 49, 50]. It would be interesting to extend the results of the thesis work to this kind of strings.

Appendix A

Conventions on differential forms

A.1 Differential forms in ordinary space-time

In generic D dimensions, we define the totally antisymmetric Levi-Civita symbol such that

$$\varepsilon^{01\dots D} = -\varepsilon_{01\dots D} = 1 \,. \tag{A.1}$$

In the differential basis $\{\mathrm{d}x^\mu\}$, with $\mu = 0, \dots, D-1$, a generic bosonic p-form ω_p has the following expansion

$$\omega_p = \frac{1}{p!}\omega_{\mu_1\dots\mu_p}\mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} \,. \tag{A.2}$$

The *exterior derivative* d acts on ω_p as

$$\mathrm{d}\omega_p = \frac{1}{p!}\partial_{[\sigma}\omega_{\mu_1\dots\mu_p]}\mathrm{d}x^\sigma \wedge \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} \,, \tag{A.3}$$

where the square brackets (brackets) denote antisymmetrization (symmetrization) with unit weight, e.g.

$$\begin{aligned} A_{(\alpha}B_{\beta)\gamma} &:= \frac{1}{2}\left(A_\alpha B_{\beta\gamma} + A_\beta B_{\alpha\gamma}\right) \,, \\ A_{[\alpha}B_{\beta]\gamma} &:= \frac{1}{2}\left(A_\alpha B_{\beta\gamma} - A_\beta B_{\alpha\gamma}\right) \,. \end{aligned} \tag{A.4}$$

A p-form is said to be *closed* if

$$\mathrm{d}\omega_p = 0 \,, \tag{A.5}$$

and *exact* if it can be written as

$$\omega_p = \mathrm{d}\omega_{p-1} \,, \tag{A.6}$$

with ω_{p-1} a globally defined $(p-1)$-form. Although an exact form is always closed, the opposite may not be true, in general. The space of closed p-forms on a manifold M is denoted by $C^p(M)$, whereas the space of exact p-forms on M by $Z^p(M)$. We can now define the pth *de Rham cohomology group* $H^p(M)$ as the quotient space

$$H^p(M) = C^p(M)/Z^p(M) \,. \tag{A.7}$$

In words, $H^p(M)$ is the space of closed forms in which two forms which differ by an exact form are considered to be equivalent. The dimension of $H^p(M)$ is called the *Betti number* and is denoted by $b_p(M)$.

In a similar way, but using the boundary operator δ instead of the exterior derivative d, we can define the *homology groups*. The boundary operator δ acts on submanifolds of M. If N is a submanifold of M, then δN is its boundary. Arbitrary linear combinations of submanifolds of dimension p are called p-chains and denoted by z_p. A chain is *closed* if it has no boundary, i.e. $\delta z_p = 0$, while is exact if it is a boundary. A closed chain z_p is also called a *cycle*. The simplicial homology group $H_p(M)$ is made of equivalence classes of p-cycles.

The Hodge-dual of a p-form ω_p is the $(D-p)$-form defined as

$$* \, \omega = -\frac{e}{(D-p)!p!} \varepsilon_{\mu_1 \ldots \mu_D} g^{\mu_1 \nu_1} \ldots g^{\mu_p \nu_p} \omega_{\nu_1 \ldots \nu_p} \mathrm{d}x^{\mu_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_D} \,, \tag{A.8}$$

where $g^{\mu_i \nu_i}$ are the components of the inverse metric tensor. The components $\omega^{\mu_1 \cdots \mu_p}$ are defined by raising the indices of the components of ω_p with the inverse $g^{\mu\nu}$ of the metric as usual.

In any dimension D, we get the following useful identity

$$\begin{aligned} \omega \wedge * \omega &= -\frac{e}{p!(D-p)!} \omega_{\nu_1 \ldots \nu_p} \varepsilon_{\mu_1 \ldots \mu_D} \omega^{\mu_1 \cdots \mu_p} \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_p} \wedge \mathrm{d}x^{\mu_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_D} \\ &= \frac{e}{p!} \, \omega^{\mu_1 \ldots \mu_p} \omega_{\mu_1 \ldots \mu_p} \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^D \equiv e \, \omega \lrcorner \, \omega \,, \end{aligned} \tag{A.9}$$

and we have

$$* \, (* \omega_p) = -(-)^{p(D-p)} \omega_p \,. \tag{A.10}$$

We define the Laplace operator as

$$\Delta_p \equiv \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger = (\mathrm{d} + \mathrm{d}^\dagger)^2 \,, \tag{A.11}$$

where the operator $\mathrm{d}^\dagger$ acts as follows when applied to a p-form ω_p

$$\mathrm{d}^\dagger \omega_p = (-1)^{Dp+D+1} * \mathrm{d}\omega_p \,. \tag{A.12}$$

A p-form is called *harmonic* if and only if

$$\Delta_p \omega_p = 0 \,. \tag{A.13}$$

One can show that harmonic p-forms are in one-to-one correspondence with the elements of the group $H^p(M)$.

A.2 Differential forms in $D = 4$ superspace

Let us denote the superspace coordinates by $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$. These coordinates obey the following multiplication law:

$$z^M z^N = (-)^{nm} z^N z^M \,, \tag{A.14}$$

where n is a function of N which takes the values 0 or 1, depending on whether N is a vector or spinor indices, respectively, and the same holds for m.

The exterior product is defined as

$$\mathrm{d}z^M \wedge \mathrm{d}z^N = -(-)^{nm} \mathrm{d}z^N \wedge \mathrm{d}z^M \,, \tag{A.15}$$

which reduces to the definition in ordinary space-time in the case $N = n$, $M = m$. With this definition, a p-form has the following extension in superspace:

$$\Omega_p = \mathrm{d}z^{M_1} \wedge \cdots \wedge \mathrm{d}z^{M_p} W_{M_p \ldots M_1}(z) \,, \tag{A.16}$$

where the indices are labeled in such a way that there is always an even number of indices between those being summed. At this point, we introduce the exterior derivative, which maps a p-form into a $(p+1)$-form, giving

$$\mathrm{d}\Omega_p = \mathrm{d}z^{M_1} \wedge \cdots \wedge \mathrm{d}z^{M_p} \wedge \mathrm{d}z^N \frac{\partial}{\partial z^N} W_{M_p \ldots M_1}(z) \,, \tag{A.17}$$

and have the following properties:

$$\begin{aligned} \mathrm{d}(\Omega + \Sigma) &= \mathrm{d}\Omega + \mathrm{d}\Sigma \,, \\ \mathrm{d}(\Omega\Sigma) &= \Omega \mathrm{d}\Sigma + (-)^q \mathrm{d}\Omega\Sigma \,, \\ \mathrm{d}\mathrm{d} &= 0 \,, \end{aligned} \tag{A.18}$$

where Σ is a q-form. In this work, we usually drop the symbol $\wedge$ for exterior multiplication, since this does not lead to ambiguities.

Appendix B

Conventions in spinor algebra and useful properties

Throughout the thesis work, we use the 'mostly plus metric' $\eta_{mn} = \mathrm{diag}(-1,1,1,1)$. The σ-matrices σ^m are

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.1}$$

They have the index structure $\sigma^m_{\alpha\dot{\alpha}}$. Once we introduce the antysymmetric tensors $\varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}}$ and $\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}}$ as

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{B.2}$$

we can raise the indices of the σ-matrices in the following way:

$$\bar{\sigma}^{m\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^m_{\beta\dot{\beta}}. \tag{B.3}$$

The σ-matrices satisfy the following relations:

$$\begin{aligned} (\sigma_a\bar{\sigma}_b + \sigma_b\bar{\sigma}_a)_\alpha{}^\beta &= -2\eta_{ab}\delta_\alpha{}^\beta, \\ (\bar{\sigma}_a\sigma_b + \bar{\sigma}_b\sigma_a)^{\dot{\alpha}}{}_{\dot{\beta}} &= -2\eta_{ab}\delta^{\dot{\alpha}}{}_{\dot{\beta}}. \end{aligned} \tag{B.4}$$

Furthermore, we introduce the matrices σ_{ab} and $\bar{\sigma}_{ab}$, given by:

$$\begin{aligned} \sigma_{\underline{ab}\underline{\alpha}}{}^{\underline{\beta}} &= \frac{1}{4}\left(\sigma_{\underline{a\alpha\dot{\alpha}}}\bar{\sigma}_{\underline{b}}^{\underline{\dot{\alpha}\beta}} - \sigma_{\underline{b\alpha\dot{\alpha}}}\bar{\sigma}_{\underline{a}}^{\underline{\dot{\alpha}\beta}}\right), \\ \bar{\sigma}_{\underline{ab}}{}^{\underline{\dot{\alpha}}}{}_{\underline{\dot{\beta}}} &= \frac{1}{4}\left(\bar{\sigma}_{\underline{a}}^{\underline{\dot{\alpha}\alpha}}\sigma_{\underline{b\alpha\dot{\beta}}} - \bar{\sigma}_{\underline{b}}^{\underline{\dot{\alpha}\alpha}}\sigma_{\underline{a\alpha\dot{\beta}}}\right). \end{aligned} \tag{B.5}$$

Finally, recalling that

$$\{\gamma^a, \gamma^b\} = -2\eta^{ab}\mathbb{1}, \tag{B.6}$$

and looking at the relations in (B.4), we see that we can choose a representation for the 4-dimensional Dirac γ-matrices given by

$$\gamma^{\underline{a}} = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}, \qquad \sigma^a = (-\mathbb{1}, \sigma^i), \qquad \bar{\sigma}^0 = \sigma^0 = -\mathbb{1}, \qquad \bar{\sigma}^i = -\sigma^i. \tag{B.7}$$

Appendix C

Derivation of (3.66)

In this Appendix, we want to prove the result for the bosonic components of the Lagrangian (3.65) reported in (3.66).

To this aim, we start by writing the expansion in components of $\mathcal{E}$, R, $\Omega(\Phi,\bar{\Phi})$ and $W(\Phi)$, whose bosonic part is given by the following expressions

$$\begin{aligned} 2\mathcal{E} &= e(1-\Theta\Theta M^*)\,, \\ R &= -\frac{1}{6}M + \Theta\Theta\left[\frac{1}{12}\mathscr{R} - \frac{1}{9}MM^* - \frac{1}{18}b^a b_a + \frac{i}{6}e_a{}^m\mathcal{D}_m b^a\right]\,, \\ \Omega(\Phi,\bar{\Phi}) &= \Omega(\phi,\bar{\phi}) + \Theta\Theta\Omega_i F^i + \bar{\Theta}\bar{\Theta}\Omega_{\bar{\imath}}F^{\bar{\imath}} + \Theta\Theta\bar{\Theta}\bar{\Theta}\Omega_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}}\,, \\ W(\Phi) &= W(\phi) + \Theta\Theta W_i F^i\,, \end{aligned} \tag{C.1}$$

where we have denoted the partial derivatives with respect to ϕ^i or $\bar{\phi}^{\bar{\jmath}}$ by Ω_i and $\Omega_{\bar{\jmath}}$, respectively. Henceforth, $\Omega \equiv \Omega(\phi,\bar{\phi})$ and $W \equiv W(\phi)$.

We now compute the $\Theta\Theta$-term of the two pieces in (3.65). Let us start from the second term, i.e.

$$2\mathcal{E}W(\Phi) = e(1-\Theta\Theta M^*)(W+\Theta\Theta W_i F^i) \simeq \Theta\Theta(-eM^*W + eW_iF^i)\,, \tag{C.2}$$

where, from now on, '$\simeq$' is used to indicate when we consider only the $\Theta\Theta$-term.

As far as the first term is considered, we start by noticing that $\left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)\Omega(\Phi,\bar{\Phi})$ is chiral since $\left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)$ is the covariant generalization of the chiral projection operator $\bar{D}\bar{D}$, therefore it can be expanded as

$$\begin{aligned} \left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)\Omega(\Phi,\bar{\Phi}) = \left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)\Omega(\Phi,\bar{\Phi})\big| + \Theta^\alpha\mathcal{D}_\alpha\left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)\Omega(\Phi,\bar{\Phi})\big| +& \\ -\frac{1}{4}\Theta\Theta\mathcal{D}^\alpha\mathcal{D}_\alpha\left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)\Omega(\Phi,\bar{\Phi})\big|\,,& \end{aligned} \tag{C.3}$$

and then

$$2\mathcal{E}\left[-\frac{1}{8}\left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)\Omega(\Phi,\bar{\Phi})\right] \simeq \Theta\Theta\left[\frac{1}{32}e\mathcal{D}^\alpha\mathcal{D}_\alpha\left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)\Omega(\Phi,\bar{\Phi})\big| + \frac{1}{8}eM^*\left(\bar{\mathcal{D}}\bar{\mathcal{D}}-8R\right)\Omega(\Phi,\bar{\Phi})\big|\right]\,. \tag{C.4}$$

Therefore, we start computing the terms we need, i.e.

$$\begin{aligned} &\bar{\mathcal{D}}\bar{\mathcal{D}}\Omega(\Phi,\bar{\Phi})\big|\,, \\ &-8R\Omega(\Phi,\bar{\Phi})\big|\,, \\ &\mathcal{D}^\alpha\mathcal{D}_\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\Omega(\Phi,\bar{\Phi})\big|\,, \\ &-8\mathcal{D}^\alpha\mathcal{D}_\alpha\left[R\Omega(\Phi,\bar{\Phi})\right]\big|\,. \end{aligned} \tag{C.5}$$

We have:

$$
\begin{aligned}
\bar{\mathcal{D}}\bar{\mathcal{D}}\Omega(\Phi,\bar\Phi)\big| &= \bar{\mathcal{D}}_{\dot\alpha}\left(\frac{\partial\Omega}{\partial\bar\Phi^{\bar\imath}}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath}\right)\bigg| = \left(\frac{\partial^2\Omega}{\partial\bar\Phi^{\bar\jmath}\partial\bar\Phi^{\bar\imath}}\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath} + \frac{\partial\Omega}{\partial\bar\Phi^{\bar\imath}}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath}\right)\bigg| = \frac{\partial\Omega}{\partial\bar\Phi^{\bar\imath}}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath}\bigg|\,,\\
-8R\Omega(\Phi,\bar\Phi)\big| &= -8\left(\frac{1}{6}M\right)\Omega = \frac{4}{3}M\Omega\,,\\
-8\mathcal{D}^\alpha\mathcal{D}_\alpha(R\Omega)\big| &= -8\left(\mathcal{D}^\alpha\mathcal{D}_\alpha R\Omega - \mathcal{D}^\alpha R\mathcal{D}_\alpha\Omega + \mathcal{D}^\alpha R\mathcal{D}_\alpha\Omega + R\mathcal{D}^\alpha\mathcal{D}_\alpha\Omega\right)\big| = -8\left(\mathcal{D}^\alpha\mathcal{D}_\alpha R\Omega + R\mathcal{D}^\alpha\mathcal{D}_\alpha\Omega\right)\big|\,,\\
\mathcal{D}^\alpha\mathcal{D}_\alpha\bar{\mathcal{D}}\bar{\mathcal{D}}\Omega(\Phi,\bar\Phi)\big| &= \mathcal{D}^\alpha\mathcal{D}_\alpha\left(\frac{\partial^2\Omega}{\partial\bar\Phi^{\bar\jmath}\partial\bar\Phi^{\bar\imath}}\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath} + \frac{\partial\Omega}{\partial\bar\Phi^{\bar\imath}}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath}\right)\bigg| =\\
&= \mathcal{D}^\alpha\Big(\Omega_{k\bar\jmath\bar\imath}\mathcal{D}_\alpha\Phi^k\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath} + \Omega_{\bar\jmath\bar\imath}\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath} - \Omega_{\bar\jmath\bar\imath}\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\mathcal{D}_\alpha\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath}+\\
&\quad + \Omega_{j\bar\imath}\mathcal{D}_\alpha\Phi^j\bar{\mathcal{D}}^2\bar\Phi^{\bar\imath} + \Omega_{\bar\imath}\mathcal{D}_\alpha\bar{\mathcal{D}}^2\bar\Phi^{\bar\imath} \overset{(a)}{=}\\
&\overset{(a)}{=} \left(\Omega_{\bar\jmath\bar\imath}\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\mathcal{D}^\alpha\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath} - \Omega_{\bar\jmath\bar\imath}\mathcal{D}^\alpha\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\mathcal{D}_\alpha\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath} + \Omega_{j\bar\imath}\mathcal{D}^2\Phi^j\bar{\mathcal{D}}^2\bar\Phi^{\bar\imath} + \Omega_{\bar\imath}\mathcal{D}^2\bar{\mathcal{D}}^2\bar\Phi^{\bar\imath}\right)\Big| \overset{(b)}{=}\\
&\overset{(b)}{=} \left(+2\Omega_{\bar\jmath\bar\imath}\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\mathcal{D}^\alpha\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\imath} + \Omega_{j\bar\imath}\mathcal{D}^2\Phi^j\bar{\mathcal{D}}^2\bar\Phi^{\bar\imath} + \Omega_{\bar\imath}\mathcal{D}^2\bar{\mathcal{D}}^2\bar\Phi^{\bar\imath}\right)\Big|\,,
\end{aligned}
\tag{C.6}
$$

where in (a) we have only reported the non-vanishing terms, while in (b) we have used the fact that

$$
\mathcal{D}^\alpha = \varepsilon^{\alpha\beta}\mathcal{D}_\beta, \qquad \mathcal{D}_\alpha = \varepsilon_{\alpha\gamma}\mathcal{D}^\gamma, \qquad \varepsilon^{\alpha\beta}\varepsilon_{\alpha\gamma} = -\delta^\beta{}_\gamma\,. \tag{C.7}
$$

We now report the lowest components needed to compute the above four terms:

$$
\begin{aligned}
\bar{\mathcal{D}}^2\bar\Phi^{\bar\imath}\big| &= -4\bar F^{\bar\imath}\,,\\
\mathcal{D}^2\Phi^i\big| &= -4F^i\,,\\
\mathcal{D}^\alpha\mathcal{D}_\alpha R\big| &= -\frac{1}{3}\mathscr{R} + \frac{4}{9}MM^* + \frac{2}{9}b^ab_a - \frac{2}{3}ie_a{}^m\mathcal{D}_mb^a\,,\\
\mathcal{D}^\alpha\mathcal{D}_\alpha\Omega\big| &= \left(\Omega_{ji}\mathcal{D}^\alpha\Phi^j\mathcal{D}_\alpha\Phi^i + \Omega_i\mathcal{D}^\alpha\mathcal{D}_\alpha\Phi^i\right)\big| = \Omega_i\mathcal{D}^2\Phi^i\big| = -4\Omega_iF^i\,,\\
\mathcal{D}_\alpha\bar{\mathcal{D}}_{\dot\alpha}\bar\Phi^{\bar\jmath}\big| &= -2i\sigma^a_{\alpha\dot\alpha}e_a{}^m\partial_m\bar A^{\bar\jmath} \implies \mathcal{D}^\alpha\bar{\mathcal{D}}^{\dot\alpha}\bar\Phi^{\bar\jmath}\big| = -2i\bar\sigma^{a\dot\alpha\alpha}e_a{}^m\partial_m\bar A^{\bar\jmath}\,,\\
\mathcal{D}^2\bar{\mathcal{D}}^2\bar\Phi^{\bar\imath}\big| &= 16e_a{}^m\mathcal{D}_me^{an}\partial_n\bar A^{\bar\imath} + \frac{32}{3}ib^ae_a{}^m\partial_m\bar A^{\bar\imath} + \frac{32}{3}M^*\bar F^{\bar\imath}\,.
\end{aligned}
\tag{C.8}
$$

We now have all the ingredients to write down the component expansion for the Lagrangian (3.65):

$$
\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3 + \text{h.c.}\,, \tag{C.9}
$$

with

$$
\begin{aligned}
\mathscr{L}_1 &= \frac{1}{32}e\mathcal{D}^\alpha\mathcal{D}_\alpha\left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R\right)\Omega(\Phi,\bar\Phi)\big| =\\
&= \frac{1}{32}e\Big[2\Omega_{\bar\jmath\bar\imath}(-2i\sigma^a_{\alpha\dot\alpha}e_a{}^m\partial_m\bar A^{\bar\jmath})(-2i\bar\sigma^{b\dot\alpha\alpha}e_b{}^n\partial_n\bar A^{\bar\jmath}) + \Omega_{j\bar\imath}(-4F^j)(-4\bar F^{\bar\imath})+\\
&\quad +\Omega_{\bar\imath}\left(16e_a{}^m\mathcal{D}_me^{an}\partial_n\bar A^{\bar\imath} + \frac{32}{3}ib^ae_a{}^m\partial_m\bar A^{\bar\imath} + \frac{32}{3}M^*\bar F^{\bar\imath}\right)+\\
&\quad -8\left(-\frac{1}{3}\Omega\mathscr{R} + \frac{4}{9}\Omega MM^* + \frac{2}{9}\Omega b^ab_a - \frac{2}{3}i\Omega e_a{}^m\mathcal{D}_mb^a - \frac{1}{6}M(-4\Omega_iF^i)\right) =\\
&= -\frac{1}{4}e\Omega_{\bar\jmath\bar\imath}\sigma^a_{\alpha\dot\alpha}\bar\sigma^{b\dot\alpha\alpha}e_a{}^me_b{}^n\partial_m\bar A^{\bar\jmath}\partial_n\bar A^{\bar\jmath} + \frac{1}{2}e\Omega_{j\bar\imath}F^j\bar F^{\bar\imath} + \frac{1}{2}e\Omega_{\bar\imath}e_a{}^m\mathcal{D}_me^{an}\partial_n\bar A^{\bar\imath}+\\
&\quad + \frac{1}{3}ieb^ae_a{}^m\Omega_{\bar\imath}\partial_m\bar A^{\bar\imath} + \frac{1}{3}eM^*\Omega_{\bar\imath}\bar F^{\bar\imath} + \frac{1}{12}e\Omega\mathscr{R} - \frac{1}{9}e\Omega MM^* - \frac{1}{18}e\Omega b^ab_a + \frac{1}{6}ie\Omega e_a{}^m\mathcal{D}_mb^a - \frac{1}{6}eM\Omega_iF^i =\\
&= \frac{1}{2}e\Omega_{\bar\jmath\bar\imath}\partial_m\bar A^{\bar\jmath}\partial^m\bar A^{\bar\jmath} + \frac{1}{2}e\Omega_{j\bar\imath}F^j\bar F^{\bar\imath} + \frac{1}{2}e\Omega_{\bar\imath}e_a{}^m\mathcal{D}_me^{an}\partial_n\bar A^{\bar\imath}+\\
&\quad + \frac{1}{3}ieb^ae_a{}^m\Omega_{\bar\imath}\partial_m\bar A^{\bar\imath} + \frac{1}{3}eM^*\Omega_{\bar\imath}\bar F^{\bar\imath} + \frac{1}{12}e\Omega\mathscr{R} - \frac{1}{9}e\Omega MM^* - \frac{1}{18}e\Omega b^ab_a + \frac{1}{6}ie\Omega e_a{}^m\mathcal{D}_mb^a - \frac{1}{6}eM\Omega_iF^i\,,
\end{aligned}
\tag{C.10}
$$

where we have used the following relations

$$\sigma^{a}_{\alpha\dot{\alpha}}\bar{\sigma}^{b\dot{\alpha}\alpha} = -2\eta^{ab}, \qquad e_a{}^m e^{an} = g^{mn}\,. \tag{C.11}$$

On the other hand:

$$\mathscr{L}_2 = \frac{1}{8}eM^*\left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R\right)\Omega(\Phi,\bar{\Phi})\big| = \frac{1}{8}eM^*\left(-4\Omega_{\bar{\imath}}\bar{F}^{\bar{\imath}} + \frac{4}{3}M\Omega\right) = -\frac{1}{2}e\Omega_{\bar{\imath}}M^*\bar{F}^{\bar{\imath}} + \frac{1}{6}e\Omega M^*M\,, \tag{C.12}$$

and

$$\mathscr{L}_3 = -eM^*W + eW_iF^i\,. \tag{C.13}$$

Therefore, by adding the h.c., we finally obtain

$$\begin{aligned}
\mathscr{L} = {}& \frac{1}{6}e\Omega\mathscr{R} + \frac{1}{2}e\Omega_{\bar{\jmath}\bar{\imath}}\partial_m\bar{A}^{\bar{\jmath}}\partial^m\bar{A}^{\bar{\imath}} + \frac{1}{2}e\Omega_{ji}\partial_m A^j\partial^m A^i + e\Omega_{j\bar{\imath}}F^j\bar{F}^{\bar{\imath}} + \\
&+ \frac{1}{2}e\Omega_{\bar{\imath}}e_a{}^m\mathcal{D}_m e^{an}\partial_n\bar{A}^{\bar{\imath}} + \frac{1}{2}e\Omega_i e_a{}^m\mathcal{D}_m e^{an}\partial_n A^i + \frac{1}{3}ieb^a e_a{}^m(\Omega_{\bar{\imath}}\partial_m\bar{A}^{\bar{\imath}} - \Omega_i\partial_m A^i) + \\
&+ \frac{1}{3}eM^*\Omega_{\bar{\imath}}\bar{F}^{\bar{\imath}} + \frac{1}{3}eM\Omega_iF^i - \frac{2}{9}e\Omega MM^* - \frac{1}{9}e\Omega b^a b_a - \frac{1}{6}eM^*\Omega_{\bar{\imath}}\bar{F}^{\bar{\imath}} - \frac{1}{6}eM\Omega_iF^i + \\
&- \frac{1}{2}e\Omega_{\bar{\imath}}M^*\bar{F}^{\bar{\imath}} - \frac{1}{2}e\Omega_iMF^i + \frac{1}{3}e\Omega M^*M - eM^*W - eM\bar{W} + eW_iF^i + e\bar{W}_{\bar{\imath}}\bar{F}^{\bar{\imath}} = \\
= {}& \frac{1}{6}e\Omega\mathscr{R} + \frac{1}{2}e\Omega_{\bar{\jmath}\bar{\imath}}\partial_m\bar{A}^{\bar{\jmath}}\partial^m\bar{A}^{\bar{\imath}} + \frac{1}{2}e\Omega_{ji}\partial_m A^j\partial^m A^i + \\
&+ \frac{1}{2}e\Omega_{\bar{\imath}}e_a{}^m\mathcal{D}_m e^{an}\partial_n\bar{A}^{\bar{\imath}} + \frac{1}{2}e\Omega_i e_a{}^m\mathcal{D}_m e^{an}\partial_n A^i + \\
&+ e\Omega_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}} + \frac{1}{9}e\Omega MM^* - \frac{1}{3}eM^*\Omega_{\bar{\imath}}\bar{F}^{\bar{\imath}} - \frac{1}{3}eM\Omega_iF^i + \\
&- \frac{1}{9}e\Omega b^a b_a - \frac{i}{3}e(\Omega_i\partial_m A^i - \Omega_{\bar{\imath}}\partial_m\bar{A}^{\bar{\imath}})b^a e_a{}^m - eM^*W - eM\bar{W} + eW_iF^i + e\bar{W}_{\bar{\imath}}\bar{F}^{\bar{\imath}}\,.
\end{aligned} \tag{C.14}$$

We now show that, up to a total derivative

$$\frac{1}{2}e\Omega_{ji}\partial_m A^j\partial^m A^i + \frac{1}{2}e\Omega_i e_a{}^m\mathcal{D}_m e^{an}\partial_n A^i \simeq -\frac{1}{2}eg^{mn}\Omega_{i\bar{\jmath}}\partial_m A^i\partial_n\bar{A}^{\bar{\jmath}}\,. \tag{C.15}$$

We have that:

$$e\Omega_i e_a{}^m\mathcal{D}_m\left(e^{an}\partial_n A^i\right) = e\Omega_i e_a{}^m e^{an}\mathcal{D}_m\partial_n A^i = e\Omega_i g^{mn}\left(\partial_m\partial_n A^i - \Gamma_{mn}{}^p\partial_p A^i\right)\,, \tag{C.16}$$

where

$$\Gamma_{mn}{}^p = \frac{1}{2}g^{p\ell}\left(\partial_m g_{\ell n} + \partial_n g_{\ell m} - \partial_\ell g_{mn}\right)\,, \tag{C.17}$$

and we have used $\mathcal{D}_m e^{an} = 0$. If we consider the first term in of the r.h.s., we find that:

$$\begin{aligned}
e\Omega_i g^{mn}\partial_m\partial_n A^i &= \partial_m\left(e\Omega_i g^{mn}\partial_n A^i\right) - \partial_m(eg^{mn})\Omega_i\partial_n A^i - eg^{mn}\Omega_{ij}\partial_m A^j\partial_n A^i - eg^{mn}\Omega_{i\bar{\jmath}}\partial_m\bar{A}^{\bar{\jmath}}\partial_n A^i \simeq \\
&\simeq -\partial_m(eg^{mn})\Omega_i\partial_n A^i - eg^{mn}\Omega_{ij}\partial_m A^j\partial_n A^i - eg^{mn}\Omega_{i\bar{\jmath}}\partial_m\bar{A}^{\bar{\jmath}}\partial_n A^i\,,
\end{aligned} \tag{C.18}$$

and then the second term cancels the first term of (C.15). At this point, let us consider that:

$$g^{mn}\Gamma_{mn}{}^p = g^{p\ell}\partial^m g_{\ell m} - \frac{1}{2}g^{mn}\partial^p g_{mn}\,, \tag{C.19}$$

and that

$$\partial_m e = \partial_m\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\ell p}\partial_m g_{\ell p}\,. \tag{C.20}$$

Therefore:

$$
\begin{aligned}
&-\frac{1}{2}\partial_m(eg^{mn})\Omega_i\partial_n A^i-\frac{1}{2}e\Omega_i g^{mn}\Gamma_{mn}{}^p\partial_p A^i=\\
&\quad=-\frac{1}{2}\partial_m e g^{mn}\Omega_i\partial_n A^i-\frac{1}{2}e\Omega_i\partial_m g^{mn}\partial_n A^i-\frac{1}{2}e\Omega_i g^{p\ell}\partial^m g_{\ell m}\partial_p A^i+\frac{1}{4}e\Omega_i g^{mn}\partial^p g_{mn}\partial_p A^i=\\
&\quad=-\frac{1}{4}\sqrt{-g}g^{\ell p}\partial_m g_{\ell p}g^{mn}\Omega_i\partial_n A^i-\frac{1}{2}e\Omega_i\partial_m g^{mn}\partial_n A^i+\frac{1}{2}e\Omega_i\partial^m g^{p\ell}g_{\ell m}\partial_p A^i+\frac{1}{4}e\Omega_i g^{mn}\partial^p g_{mn}\partial_p A^i=\\
&\quad=-\frac{1}{4}e\Omega_i g^{\ell p}\partial^n g_{\ell p}\partial_n A^i-\frac{1}{2}e\Omega_i\partial_m g^{mn}\partial_n A^i+\frac{1}{2}e\Omega_i\partial_\ell g^{\ell p}\partial_p A^i+\frac{1}{4}e\Omega_i g^{mn}\partial^p g_{mn}\partial_p A^i=\\
&\quad=-\frac{1}{4}e\Omega_i g^{mn}\partial^p g_{mn}\partial_p A^i-\frac{1}{2}e\Omega_i\partial_m g^{mn}\partial_n A^i+\frac{1}{2}e\Omega_i\partial_m g^{mn}\partial_n A^i+\frac{1}{4}e\Omega_i g^{mn}\partial^p g_{mn}\partial_p A^i=0\,.
\end{aligned}
\tag{C.21}
$$

An analogous procedure can be done for the h.c. of (C.15). Therefore, the final expression of (C.14) is

$$
\begin{aligned}
\mathscr{L}=&\frac{1}{6}e\Omega\mathscr{R}-eg^{mn}\Omega_{i\bar{\jmath}}\partial_m A^i\partial_n\bar{A}^{\bar{\jmath}}+e\Omega_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}}+\frac{1}{9}e\Omega MM^*-\frac{1}{3}eM^*\Omega_{\bar{\imath}}\bar{F}^{\bar{\imath}}-\frac{1}{3}eM\Omega_i F^i+\\
&-\frac{1}{9}e\Omega b^a b_a-\frac{i}{3}e(\Omega_i\partial_m A^i-\Omega_{\bar{\imath}}\partial_m\bar{A}^{\bar{\imath}})b^a e_a{}^m-eM^*W-eM\bar{W}+eW_iF^i+e\bar{W}_{\bar{\imath}}\bar{F}^{\bar{\imath}}\,.
\end{aligned}
\tag{C.22}
$$

Appendix D

Super-Weyl invariant Lagrangians

Here we give the procedure to construct super-Weyl invariant Lagrangians in supergravity [35–37], by focusing on the theory with chiral multiplets considered in Section 3.1.

This approach allows us to write manifestly supersymmetric invariant actions, and the components of super-Weyl invariant Lagrangians, up to some proper covariantizations, have a striking resemblance to those of globally supersymmetric Lagrangians.

Let us consider a set of N dimensionless chiral multiplets Φ^m, whose bosonic components are

$$\Phi^m = \{\phi^m, F_\Phi^m\}\,, \quad \text{with} \quad m = 1, \ldots, n\,, \tag{D.1}$$

where ϕ^m are the lowest component complex scalar fields and F_Φ^m are the highest component auxiliary complex scalar fields. At the core of the super-Weyl invariant formalism is the introduction of an unphysical, chiral compensator U, which we choose to transform as

$$U \to e^{-6\Upsilon} U \tag{D.2}$$

under super-Weyl transformations. We recall that these act on the super-vielbeins as [35]

$$E_M^a \to e^{\Upsilon+\bar{\Upsilon}} E_M^a\,, \quad E_M^\alpha \to e^{2\bar{\Upsilon}-\Upsilon}\left(E_M^\alpha - \frac{i}{4} E_M^a \sigma_a^{\alpha\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Upsilon}\right). \tag{D.3}$$

where (a, α) are flat superspace indices, $M = (m, \mu)$ are curved indices and Υ is an arbitrary chiral superfield parameterizing the super-Weyl transformation. Instead, the dimensionless superfields Φ^m are invariant under super-Weyl tranformations. Combining Φ^m and the compensator U, we introduce new chiral superfields Z^a transforming as U under super-Weyl transformations:

$$Z^a = \{z^a,\, F_Z^a\} \quad \text{with} \quad a = 1, \ldots, n+1\,, \tag{D.4}$$

where z^a and F_Z^a are understood to be functions of the components of Φ^m. In order to isolate the physical fields, we assume that we can single out the compensator U as

$$Z^a = U g^a(\Phi)\,, \tag{D.5}$$

where g^a are functions of the physical fields only and are inert under super-Weyl transformations.

The most general supergravity Lagrangian that we can build out of the Z^a multiplets is

$$\mathscr{L} = \int \mathrm{d}^4\theta\, E\, \mathcal{K}(Z, \bar{Z}) + \left(\int \mathrm{d}^2\Theta\, 2\mathcal{E}\, \mathcal{W}(Z) + \text{c.c.}\right)\,, \tag{D.6}$$

where $\mathcal{K}(Z, \bar{Z})$ is the kinetic potential and $\mathcal{W}(Z)$ the superpotential. Additionally, however, we require that they satisfy the following homogeneity conditions

$$\mathcal{K}(\lambda Z, \bar{\lambda}\bar{Z}) = |\lambda|^{\frac{2}{3}} \mathcal{K}(Z, \bar{Z})\,, \qquad \mathcal{W}(\lambda Z) = \lambda \mathcal{W}(Z)\,, \tag{D.7}$$

with λ an arbitrary chiral superfield.

In order to recover the ordinary Kähler potential $K(\Phi,\bar\Phi)$ and superpotential $W(\Phi)$, we isolate the compensator U as

$$\mathcal{K}(Z,\bar Z) = -3|U|^{\frac{2}{3}} e^{-\frac{1}{3}K(\Phi,\bar\Phi)}\,, \qquad \mathcal{W}(Z) = U\,W(\Phi)\,, \tag{D.8}$$

where

$$K(\Phi,\bar\Phi) \equiv -3\log\left[-\frac{1}{3}\mathcal{K}(g(\Phi),\bar g(\bar\Phi))\right]\,, \qquad W(\Phi) \equiv \mathcal{W}(g(\Phi))\,. \tag{D.9}$$

Such homogeneity properties of $\mathcal{K}$ and $\mathcal{W}$ make the Lagrangian (D.6) manifestly invariant under super-Weyl transformations. In particular, (D.3) implies that

$$E \to e^{2\Upsilon+2\bar\Upsilon}E\,, \quad \mathrm{d}^2\Theta\, 2\mathcal{E} \to e^{6\Upsilon}\mathrm{d}^2\Theta\, 2\mathcal{E}\,. \tag{D.10}$$

Indeed, the Lagrangian (D.7) is also independently invariant under Kähler transformations. This is due to the fact that the split (D.5) is not unique, since we may perform the following redefinition

$$U \to e^{h(\Phi)}U\,, \quad g^a(\Phi) \to e^{-h(\Phi)}g^a(\Phi)\,. \tag{D.11}$$

with $h(\Phi)$ an arbitrary holomorphic function of Φ^i. By using the definitions (D.9) and the homogeneity conditions (D.7), it can be easily seen that this redefinition exactly corresponds to an ordinary Kähler transformation

$$K(\Phi,\bar\Phi) \to K(\Phi,\bar\Phi) + h(\Phi) + \bar h(\bar\Phi)\,, \quad W(\Phi) \to e^{-h(\Phi)}W(\Phi)\,. \tag{D.12}$$

The bosonic components of the Lagrangian (D.6) acquire a very simple form

$$e^{-1}\mathscr{L}_{\rm bos} = -\frac{1}{6}\mathcal{K}\,R - \mathcal{K}_{a\bar b}D_\mu z^a \bar D^\mu \bar z^b + \mathcal{K}_{a\bar b} f^a \bar f^b + (\mathcal{W}_a f^a + \text{c.c.})\,. \tag{D.13}$$

Here we have redefined

$$f^a \equiv \bar M z^a - F^a_Z\,, \tag{D.14}$$

and introduced the $U(1)$-covariant derivatives

$$D_\mu z^a = \partial_\mu z^a + iA_\mu z^a\,, \qquad \text{with} \quad A_\mu = \frac{3i}{2\mathcal{K}}(\mathcal{K}_a\partial_\mu z^b - \mathcal{K}_{\bar b}\partial_\mu \bar z^b)\,. \tag{D.15}$$

The auxiliary fields f^a may be easily integrated out from (D.13), leading to the Lagrangian

$$e^{-1}\mathscr{L}_{\rm bos} = -\frac{1}{6}\mathcal{K}\,R - \mathcal{K}_{a\bar b}D_\mu z^a \bar D^\mu \bar z^b - \mathcal{K}^{a\bar b}\mathcal{W}_a\bar{\mathcal{W}}_{\bar b}\,. \tag{D.16}$$

In order to pass to the Einstein frame, we isolate the compensator $u \equiv U|_{\theta=\bar\theta=0}$ and split the index $a = (0,m)$, with 0 associated to the compensator u and m to the physical fields ϕ^m, i.e. we set $z^m = (u, u\phi^m)$. At this point, we gauge-fix the super-Weyl invariance by setting

$$u = M_{\rm P}^2\, e^{\frac{1}{2}K(\phi,\bar\phi)} \quad \Rightarrow \quad \mathcal{K} = -3M_{\rm P}^2\,. \tag{D.17}$$

Then, the kinetic matrix $\mathcal{K}_{a\bar b}$ splits as

$$\mathcal{K}_{a\bar b} = e^{-K}\begin{pmatrix} -\frac{1}{3} & \frac{K_{\bar n}}{3} \\ \frac{K_m}{3} & K_{m\bar n} - \frac{1}{3}K_m K_{\bar n}\end{pmatrix}\,. \tag{D.18}$$

As far as the last term of (D.16) is concerned, the inverse of $\mathcal{K}_{\bar b a}$ is needed, and is given by

$$\mathcal{K}^{\bar b a} = e^{K}\begin{pmatrix} -3 + K^{\bar i j}K_j K_{\bar i} & K^{\bar i m}K_{\bar i} \\ K^{\bar n j}K_j & K^{\bar n m}\end{pmatrix}\,. \tag{D.19}$$

Therefore, we finally arrive at[1]

$$e^{-1}\mathscr{L}_{\rm bos} = \frac{1}{2}R - K_{m\bar n}\partial_\mu\phi^m\partial^\mu\bar\phi^{\bar n} - e^K\left(K^{\bar n m}D_m W \bar D_{\bar n}\bar W - 3|W|^2\right)\,, \tag{D.20}$$

with a canonically normalized Einstein-Hilbert term and where the last term is nothing but the well-known Cremmer et al. potential [51].

[1] For simplicity, in the following formulas we will set $M_{\rm P} = 1$ (an eventual dependence on the Planck mass may be easily reinstated by dimensional analysis).

Appendix E

Energy-momentum tensor of (4.145)

In this Appendix, we prove that the Virasoro constraint $E_{++}^{\underline{a}}E_{++\underline{a}}=0$ remains unchanged in the presence of an $\mathcal{N}=(0,2)$ Fermi multiplet with action (4.135). To this aim, let us start from the action (4.145) rewritten as[1]

$$S=2\int \mathrm{d}^2\xi\,\sqrt{-g}\left[e^i L_i E_{--}^{\underline{a}}E_{++\underline{a}}+\frac{i}{2}\left(\bar\lambda_- D_{++}\lambda_- - D_{++}\bar\lambda_-\lambda_-\right)\right]+e^i\int \mathrm{d}^2\xi\varepsilon^{mn}\mathcal{B}_{mn,i}\,, \tag{E.1}$$

and consider a variation of the world-sheet metric:

$$\begin{aligned}\delta S=&-\int \mathrm{d}^2\xi\,\sqrt{-g}\delta g^{mn}g_{mn}\left[e^i L_i E_{--}^{\underline{a}}E_{++\underline{a}}+\frac{i}{2}\left(\bar\lambda_- D_{++}\lambda_- - D_{++}\bar\lambda_-\lambda_-\right)\right]+\\&+2\int \mathrm{d}^2\xi\,\sqrt{-g}\delta e_{++}{}^m\left[e^i L_i E_{--}^{\underline{a}}E_{m\underline{a}}+\frac{i}{2}\left(\bar\lambda_-\partial_m\lambda_- - \partial_m\bar\lambda_-\lambda_-\right)\right]+\\&+2\int \mathrm{d}^2\xi\,\sqrt{-g}\delta e_{--}{}^m\left(e^i L_i E_m^{\underline{a}}E_{++\underline{a}}\right)\,,\end{aligned} \tag{E.2}$$

where we have used that

$$\delta\sqrt{-g}=-\frac{1}{2}\sqrt{-g}\delta g^{mn}g_{mn}\,. \tag{E.3}$$

Up to a local Lorentz transformation, we can associate a variation of the inverse vielbein $\delta e_a{}^m$ to a given δg^{mn}, considering that

$$\delta g^{mn}=\delta(e_a{}^m e_b{}^n\eta^{ab})=2e_a{}^{(m}\delta e_b{}^{n)}\eta^{ab}\,. \tag{E.4}$$

A possible choice is

$$\delta e_a{}^m=\frac{1}{2}\delta g^{mn}e_n{}^c\eta_{ac}\,. \tag{E.5}$$

Indeed:

$$\delta g^{mn}=2e_a{}^{(m}\delta e_b{}^{n)}\eta^{ab}=e_a{}^{(m}\delta g^{n)\ell}e_\ell{}^c\eta_{bc}\eta^{ab}=e_a{}^{(m}\delta g^{n)\ell}e_\ell{}^a=\delta_\ell{}^{(m}\delta g^{n)\ell}=\delta g^{(mn)}=\delta g^{mn}\,. \tag{E.6}$$

Therefore, from (E.5) we find that

$$\delta e_{++}{}^m=-\frac{1}{4}\delta g^{mn}e_n{}^{--}\,,\qquad \delta e_{--}{}^m=-\frac{1}{4}\delta g^{mn}e_n{}^{++}\,, \tag{E.7}$$

[1] We have used that $\sqrt{-g}=e=\frac{1}{2}\varepsilon^{mn}e_m{}^{--}e_n{}^{++}$, and the expression (4.137) for the component expansion of S_F, which is manifestly real.

and then (E.2) becomes

$$\begin{aligned}
\delta S = & -\int \mathrm{d}^2\xi \,\sqrt{-g}\delta g^{mn} g_{mn} \left[e^i L_i E^{\underline{a}}_{--} E_{++\underline{a}} + \frac{i}{2}\left(\bar\lambda_- D_{++}\lambda_- - D_{++}\bar\lambda_-\lambda_-\right)\right] + \\
& -\frac{1}{2}\int \mathrm{d}^2\xi \,\sqrt{-g}\delta g^{mn} e_{(n}{}^{--} \left[e^i L_i E^{\underline{a}}_{--} E_{m)\underline{a}} + \frac{i}{2}\left(\bar\lambda_- \partial_{m)}\lambda_- - \partial_{m)}\bar\lambda_-\lambda_-\right)\right] + \\
& -\frac{1}{2}\int \mathrm{d}^2\xi \,\sqrt{-g}\delta g^{mn} e_{(n}{}^{++} \left(e^i L_i E^{\underline{a}}_{m)} E_{++\underline{a}}\right) = \\
= & -\frac{1}{2}\int \mathrm{d}^2\xi \,\sqrt{-g}\delta g^{mn} \left\{ 2 g_{mn} \left[e^i L_i E^{\underline{a}}_{--} E_{++\underline{a}} + \frac{i}{2}\left(\bar\lambda_- D_{++}\lambda_- - D_{++}\bar\lambda_-\lambda_-\right)\right] + \right. \\
& \left. + e_{(n}{}^{--} \left[e^i L_i E^{\underline{a}}_{--} E_{m)\underline{a}} + \frac{i}{2}\left(\bar\lambda_- \partial_{m)}\lambda_- - \partial_{m)}\bar\lambda_-\lambda_-\right)\right] + e_{(n}{}^{++} \left(e^i L_i E^{\underline{a}}_{m)} E_{++\underline{a}}\right) \right\} .
\end{aligned} \tag{E.8}$$

If we now define the energy-momentum tensor as

$$\delta S \equiv -\frac{1}{2}\int \mathrm{d}^2\xi \,\sqrt{-g}\delta g^{mn} T_{mn} \,, \tag{E.9}$$

we conclude that the variation with respect to the world-sheet metric gives $T_{mn} = 0$, with

$$\begin{aligned}
T_{mn} = & \, 2 g_{mn} \left[e^i L_i E^{\underline{a}}_{--} E_{++\underline{a}} + \frac{i}{2}\left(\bar\lambda_- D_{++}\lambda_- - D_{++}\bar\lambda_-\lambda_-\right)\right] + \\
& + e_{(n}{}^{--} \left[e^i L_i E^{\underline{a}}_{--} E_{m)\underline{a}} + \frac{i}{2}\left(\bar\lambda_- \partial_{m)}\lambda_- - \partial_{m)}\bar\lambda_-\lambda_-\right)\right] + e_{(n}{}^{++} \left(e^i L_i E^{\underline{a}}_{m)} E_{++\underline{a}}\right) .
\end{aligned} \tag{E.10}$$

In flat indices, it becomes

$$\begin{aligned}
T_{ab} = e_a{}^m e_b{}^n T_{mn} = & \, 2 \eta_{ab} \left[e^i L_i E^{\underline{a}}_{--} E_{++\underline{a}} + \frac{i}{2}\left(\bar\lambda_- D_{++}\lambda_- - D_{++}\bar\lambda_-\lambda_-\right)\right] + \\
& + \delta_{(b}{}^{--} \left[e^i L_i E^{\underline{a}}_{--} E_{a)\underline{a}} + \frac{i}{2}\left(\bar\lambda_- D_{a)}\lambda_- - D_{a)}\bar\lambda_-\lambda_-\right)\right] + \delta_{(b}{}^{++} \left(e^i L_i E^{\underline{a}}_{a)} E_{++\underline{a}}\right) .
\end{aligned} \tag{E.11}$$

At this point, let us consider the component $a = b = ++$. In this case, since $\eta_{++,++} = 0$, only the last term is non-vanishing:

$$T_{++,++} = e^i L_i E^{\underline{a}}_{++} E_{++\underline{a}} \,, \tag{E.12}$$

and this completes our proof. Indeed, $T_{++,++} = 0$ tells us that the Virasoro constraint $E^{\underline{a}}_{++} E_{++\underline{a}} = 0$ remains unchanged in the presence of an $\mathcal{N} = (0, 2)$ Fermi multiplet Λ_-.

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