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TESI DI LAUREA MAGISTRALE

COSMOLOGICAL PERTURBATIONS IN MIMETIC GRAVITY MODELS

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Dalla meta mai non toglier gli occhi.
Alessandro Manzoni

Contents

Introduction

The last century has demonstrated that Modern Cosmology is one of the most surprising sector of Physics. From the Einstein's proposal of General Relativity in 1915, our view point on the Universe around us has been changed continuously: the expansion of the Universe, the Big-Bang scenario, primordial nucleosynthesis, the discovery of the Cosmic Microwave Background radiation and the inflationary scenario are just a few of the main features.

At the end of 20th century, two other fundamental discoveries shocked scientists of all over the world: the dark components of the Universe, Dark Matter and Dark Energy. The ΛCDM model has its birth as a modification of General Relativity in which, in particular, the two new dark components are included. However, no explanations about the fundamental origins of Dark Matter and Dark Energy are firmly established yet, and they are just parameters of the model: one being a very small cosmological constant and the other the so-called Cold Dark Matter, that observations indicate to be cold, that is matter decoupled already in the non-relativistic regime, and collisionless, i.e. no interacting on cosmological scales with ordinary matter except for the gravitational force. For this reason the ΛCDM model, that defines the today Standard Model of Cosmology and that has been so successful at explaining all cosmological observations up to now, might have to be modified to give a more accurate description of these two new phenomena. This is the simple origin of the modified gravity models, that start by modifying in the proper way the laws of gravitation of Einstein's General Relativity.

In this Thesis, we study a very interesting attempt that recently has been introduced to explain the dark matter phenomenon, the so-called "Mimetic Dark Matter" proposal [1]. In this scenario, General Relativity is reformulated in terms of an auxiliary metric which is conformally related to the original physical metric, where the conformal factor is a certain function of the new auxiliary metric and the first derivative of a scalar field. In these variables, the theory provides a new degree of freedom of gravity which behaves as an irrotational perfect fluid without pressure, and it can mimic a cold Dark Matter component [8]. In a subsequent work [2] it was shown that, by introducing a potential for the new scalar field, one can mimic the gravitational behavior of almost any form of matter or energy, and also the Dark Energy contribution. So, in this scenario the dark components of the Universe, Dark Matter and Dark Energy, can be identified as manifestations of the same modification of gravity.

The original paper by Chamseddine and Mukhanov [1] started from the Hilbert-Einstein action to provide a theory where a new degree of freedom - the Mimetic Dark Matter - could mimic a cold dark matter component even in the absence of matter. It has been shown that in general Mimetic theories can be obtained from a given initial theory and applying to it a non-invertible disformal transformation, that is the fundamental property that characterizes a so-called "Mimetic" theory.
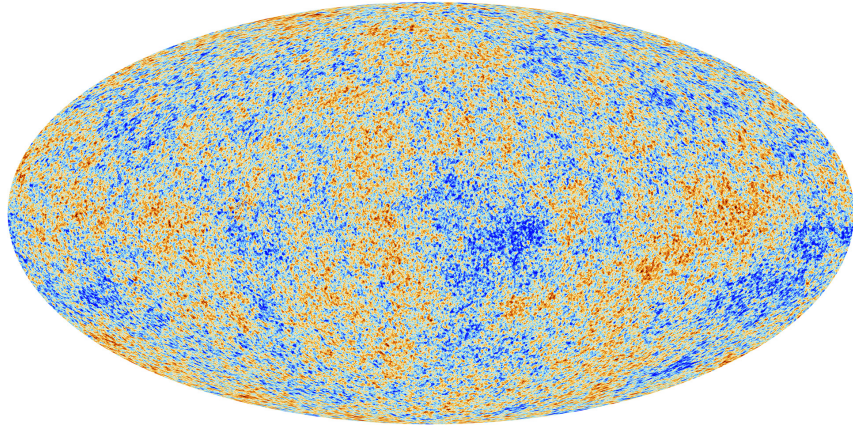
Figure 1: *Cosmic microwave background seen by Planck satellite [19].*

Recently, this Mimetic theory has been generalized [3] to very general scalar-tensor theories of gravity, and one of these scenarios is the Horndeski model [18], that is one the most general 4D covariant theory of scalar-tensor gravity that is derived from an action and gives rise to second-order equations of motion (in all gauges and in any background) for both the metric and the scalar field. In [3] the Horndeski model is studied in its Mimetic scenario, imposing the typical non-invertible disformal transformation. These scenarios are now ready for the study of their cosmological perturbations [4, 6].

This Thesis is going to focus on the latter aspect, that is to say the study of the cosmological perturbations in Mimetic gravity models. In particular for the Mimetic Horndeski models we are going to study the small-scale limit of the cosmological perturbations, identifying a Newtonian regime in which a generalized form of the Poisson equation can describe the behaviour of the gravitational potentials in presence of matter on small scales and that - as we will see in Chapter 3 - is expected to be experimentally well verified in the near future. The main purpose is the identification of a generalized form of Poisson equation to construct physical observables that could be measured to verify the Mimetic models. This will allow to make more detailed predictions and forecasts of some observational quantities for Mimetic gravity models of particular interest for Large Scale Structure surveys, like the forthcoming Euclid satellite mission will provide.

This Thesis is structured as follows:

Introduction at the beginning of this Thesis, we introduce the main important scientific features of Dark Matter and Dark Energy;

Chapter 1 in the first Chapter we briefly recall some of the most important features of Modern Cosmology, i.e. the Standard Model of Cosmology and the inflationary scenario;

Chapter 2 in Chapter two we present an introduction to perturbation theory applied to the cosmological perturbations;

Chapter 3 in the third Chapter there is an illustration of some of the main targets that one of the next ESA missions - the Euclid satellite - will study also in relation to models of Modified Gravity;

Chapter 4 in Chapter four we present the origins of the Mimetic Dark Matter model from the first proposal of Chamseddine and Mukhanov [1] and its extensions regarding the cosmological implications of the model [2];

Chapter 5 in Chapter five we discuss the generalization of the Mimetic Dark Matter model to a more general Mimetic gravity scenario. This promotion can be obtained by two ways: by using a disformal transformation or inserting the so called "Mimetic constraint" as a Lagrange multiplier;

Chapter 6 in Chapter six we present the most general case of the Mimetic gravity scenario applied to the Horndeski models, presenting also an introduction to the cosmological perturbations of the model;

Chapter 7 in the last Chapter, we discuss some analyses of the cosmological perturbations of Mimetic Horndeski models in the small-scale limit. In particular, we will use the Newtonian regime in the small-scale limit to verify the existence of a Poisson equation of the theory - in a generalized form - and its possible implications on the experimental parameters that one can analyze.

Conclusions finally, a brief review of this Thesis and future extensions of this work are presented.

Conventions

Finally, we set the notations used throughout this Thesis.

The theory of General Relativity will usually be denoted as GR, the Dark Matter components will be denoted with DM and the Dark Energy's ones with DE, the Cosmic Background Microwaves with CMB.

We choose the metric tensor $g_{\mu\nu}$ with the following signature $(-,+,+,+)$. We will use the dot above a function to indicate the derivative with respect to the cosmic time t while we will use the apostrophe ' to indicate the derivative with respect the conformal time η which is defined in the following way:

$$d\eta = \frac{dt}{a(t)}$$

With ∇_μ we call the covariant derivative built from the metric tensor $g_{\mu\nu}$. We will use the symbol $\phi_{,X}$ to indicate the ordinary derivative with respect to the coordinate x, and $\phi_{;X}$ the covariant one.

The Hubble rate H is defined as the ratio

$$H = \frac{\dot{a}}{a}$$

and we introduce also the conformal Hubble parameter $\mathcal{H} = aH$ which will be helpful to express some results while dealing with cosmological perturbations.

When dealing with the perturbations, we will generally use a number between parenthesis to indicate the order of the perturbation: for example $E^{(0)}$ is the first order part of the tensor E.

Dark Matter

One of the most shocking discoveries of the 20th century is that ordinary baryonic matter is not the dominant form of matter in the Universe. Another strange form of matter, called "Dark Matter" (DM), fills the Universe. However, despite the strong evidence for Dark Matter, its nature is pratically completely unknown and it is one of the greatest puzzles in Modern Cosmology.

Long time has been passed since the first discovery of DM effects by F. Zwicky in the early 1930's. Zwicky studied the Coma cluster of galaxies and, using observed doppler shifts in galactic spectra, was able to calculate the velocity dispersion of the galaxies. Zwicky calculated the average mass of more than a thousand of nebulae in the cluster and found that this result was completely different from that obtained using the standard M/L ratios: the latter measurement was approximately 2% of the former one. Zwicky did not know that a large part of the mass of nebulae was in the intracluster gas, a fact that lightly reduces the ratio between the two results. However, this was the first time that the effect of DM was measured: the majority of the mass of the Coma cluster was for some reason "missing" or non-luminous, and this problem will was called the "issue of missing mass".

More than fourty years later, Vera Rubin and her collaborators studied in a deep way the rotation curves of more than sixty isolated galaxies, and they made another discovery about the presence of DM. In fact, Rubin's result showed an extreme deviation from the prediction of the Newtonian gravity: the rotation curves for stars becomes flat at high radii, instead of following a Kleperian profile. The conclusion was astonishing: mass, unlike luminosity, is not concentrated near the center of spiral galaxies. Thus the light distribution in a galaxy is not at all a guide to mass distribution, and the hypothesis of a some other unknown form of matter that takes the place of the invisible one started to animate the scientific debate.

Today we have many other effects due to the presence of DM: one can remember - for example - the strong gravitational lensing effect and the weak one, in which the light path is modified due to the presence of large halos of DM. Some of these effects allows us to determine in very a accurate way the the total abundance of dark matter relative to normal baryonic matter in the Universe.

The most recent data of the *Planck* satellite [19] constrained the baryonic matter density to $\Omega_b = 0.0486 \pm 0.001$ (95% C.L.), and the total matter density to $\Omega_M = 0.308 \pm 0.012$.

Particle Physics has proposed several possible dark matter candidates, but up to now no one has been found to have all the good requested properties. Today DM is retained to be cold (that means that it has been decoupled after it became non-relativistic) to allow formation of structures, collisionless (i.e. with very small interaction rates, with vanishing pressure), and stable over a long period of time: such a candidate is referred to as a Weakly Interacting Massive Particle (WIMP).

There is another form of Dark Matter known in literature, the so-called "Hot Dark Matter" (HDM) [75], made by neutrinos with masses of up to a few electron volts. Initially, this was considered the most plausible dark matter model candidate: it provides a cosmological structure formation with a top-down formation scenario, in which superclusters of galaxies are the first objects to form, with galaxies and clusters forming through a process of fragmentation. These models were abandoned because if galaxies form sufficiently early to agree with observations, their distribution would be much more inhomogeneous than it is observed to be. Since 1984, the most successful structure formation models have been those in which most of the mass in the Universe is in the form of cold dark matter. But experimental data provides the existence of a lower limit on the HDM (i.e., light neutrino) contribution to the cosmological density. This was the reason for which mixed models with
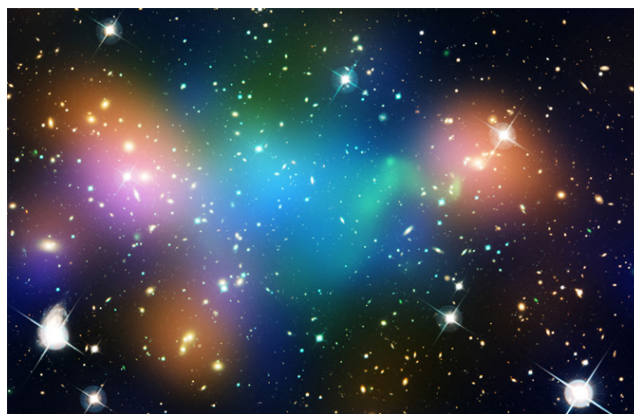
Figure 2: *A "false-colored" maps showing the concentration of starlight, hot gas, and dark matter in the cluster is superimposed to a natural-color image of the galaxies taken with NASA's Hubble Space Telescope. This composite image shows the distribution of dark matter, galaxies, and hot gas in the core of the merging galaxy cluster Abell 520, formed from a violent collision of massive galaxy clusters.*

both Cold and Hot Dark Matter (CHDM) were also proposed in the early 1990's.

Finally, another possibility in the description of Dark Matter is by introducing modified gravity models. These theories are possible generalizations of GR with extra gravitational degrees of freedom that can be used to describe a dust that behaves similarly to DM. One of this model is the Mimetic Dark Matter theory, that can explain the phenomenon of the cold dark matter that appears as a simple integration constant in studying the background solutions, and that will be explained in the following Chapters.

Dark Energy

The discovery of cosmic acceleration at the end of the 1990's and its possible explanation in terms of a cosmological constant, made Cosmology return to its roots when Einstein pubblicated his famous paper of 1917, that simultaneously inaugurated modern Cosmology and the history of the cosmological constant Λ. Einstein did not know that he had found the simplest model for the description of the Dark Energy that is used also today. The role of Λ in the first Einstein's proposal was to keep the Universe static: it is funny to think that nowdays the same term is used to describe the accelerated expansion of the Universe.

Observational data confirm that only a fraction of the total energy density in the Universe is composed by any form of matter. The most recent results of the *Planck* satellite [19] finds that $\Omega_M = 0.308 \pm 0.012$ and about 69% of the energy density of the Universe is composed of an unknown form of energy, generically called "Dark Energy" (DE), which brings the Universe close to the today critical density and is responsible for the recent phase of accelerated expansion of the Universe.

The first explanation that was given, as we mentioned, is the insertion of the cosmological constant Λ in the Einstein equations

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} - \Lambda\, g_{\mu\nu}\ .$$

Great debates were done about the nature of this cosmological constant, and also nowdays no clear answers are given.

However, as a matter of fact, today the best description of Dark Energy from an observational point of view can be done with the very constant Λ. It is inserted in the Einstein equations and it is treated as a parameter to interpret experimental data and to look for more details about its behaviour and its nature.

There are some very important aspects to explain about this constant. The first is that we do not know why it has a so small but non-zero value, and this is called "the cosmological constant problem". The second fact is that the value of Ω_Λ is also surprisingly close to another quantity, the present matter-energy density Ω_m. The fact the these two densities are almost equal implies that our epoch is a special one in the history of the Universe: this is called "the coincidence problem" and it could be just a coincidence, but of course in this case this would not be a satisfactory answer for Physics.

However, the description of DE via the cosmological constant is not the only way to approach the problem. The last years were characterized by a very deep study of General Relativity and of its limits. The clear task is to generalize GR - in a better way of the ΛCDM model - to include the dark components and, above all, the dark energy one. Thus, modified gravity models have been produced in a wide variety of flavours trying to interpret all data from the today Universe.

However, it is still difficult to probe the large number of modified gravity models (i.e., to distinguish them from a cosmological constant and eventually to distinguish among them) with the nowdays observational data, and no accurate selection can be done because they are generally in broad agreement with current constraints on the background Cosmology. Future precise measurements should provide stronger constraints analyzing experimental observables produced by cosmological perturbations.

One of the most important physical observables of cosmological perturbations is the CMB, that the *Planck* satellite [19] has studied in a deep way. It has obtained a measurement of the constant w in the equation of state as a function of red-shift: all tests on time varying $w(z)$ are compatible with the ΛCDM one $w = -1$. It has been possible to give a constraint on the equation of state for dark energy that is constrained to be $w = -1.006 \pm 0.045$ (95% C.L.) and is therefore compatible with a cosmological constant, as assumed in the base ΛCDM Cosmology.

Thus, the ΛCDM model represents the best fit of the today observations more than 20 years after it was introduced. For this reason, it is the so-called Standard Model of Cosmology.

Chapter 1

Modern Cosmology

The first clear elegant discussion on the evolution of the Universe and formation of structures based on physical laws has its basis on the proposal of General Relativity by Einstein in 1915. Soon after, in 1922, Friedmann found dynamical cosmological solutions by solving the Einstein field equations, and in 1929 Hubble observed the expansion of the Universe by the measurements of the redshifts of galaxies. At the end of 1946, Gamov and his collaborators showed that the Universe must begin in a very hot and dense state from the theory of nucleosynthesis. They announced that the present Universe should be filled with microwaves of black body radiation, that in 1965 Penzias and Wilson discovered as the Cosmic Microwave Background radiation (CMB). These are the main strong observational evidences that made scientists believe that the Universe has been continuously expanding from a very dense and hot state to the present condition, building the so called the "Big-Bang model".

The Big-Bang model could describe almost every feature observed at that time and it was considered for long time a sort of definitive model of Cosmology. When between 1980's and 1990's the dark components were discovered and studied, this model was gradually modified through the addition of the cosmological constant and the Dark Energy and Dark Matter components. From the beginning of the 2000's, the ΛCDM model has survived to a huge number of experimental tests [19, 23, 47, 48] and thus it has become the **Standard model of Cosmology**. This model also includes the initial conditions provided by inflation during the early Universe.

1.1   Short review of Standard Cosmology

1.1.1   Cosmological principle

The ΛCDM model is based on the cosmological principle [28], which states that the Universe is homogeneous and isotropic on large scales for a comoving observer. From an observational point of view this is an extremely nontrivial statement because, on small scales, the Universe looks rather inhomogeneous. So the cosmological principle is assumed to be valid on very large scales.

There are various observational evidences of the cosmological principle: one of the most important is the high level of isotropy of the CMB radiation, and the statistical distribution of matter in the Universe on large scales [19].

1.1.2 Einstein equations

The cosmological principle leads us to introduce in General Relativity a metric that can describe a homogeneous and isotropic Universe in a very general way: it is the Robertson-Walker metric, and has the following form:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],\qquad(1.1)$$

where $a(t)$ is the scale factor with t being the cosmic time. Here k is the spatial curvature constant and its value can be +1, 0 or -1 corresponding to closed, flat, and open Universes respectively.

The dynamics of an expanding Universe is determined by the Einstein equations, which relate the spacetime evolution to its energy and matter content. They can be expressed as

$$G_{\mu\nu} = 8\pi G\,T_{\mu\nu} - \Lambda\,g_{\mu\nu},\qquad(1.2)$$

with

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\,,\qquad(1.3)$$

and where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor of the matter component, $G_{\mu\nu}$ is the Einstein tensor and G the Newton gravitational constant. Λ is a cosmological constant originally introduced by Einstein and that was later recovered as the parameter describing the Dark Energy contribution.

For a homogeneous and isotropic Universe, Einstein equations reduce to the Friedmann equations

$$H^2 = \frac{8\pi G}{3}\rho + \frac{k}{a^2}\qquad(1.4)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right).\qquad(1.5)$$

The Bianchi's identity $\nabla_\nu G^{\mu\nu} = 0$ for the tensor $G_{\mu\nu}$ in (1.3) leads to the continuity equation

$$\dot{\rho} = -3H(\rho + p)\,,\qquad(1.6)$$

where $\rho(t)$ and $p(t)$ are respectively the energy density and the isotropic pressure.

Since equations (1.4), (1.5) and (1.6) are linearly dependent (the latter equation can be obtained from the other two), we need an extra equation to complete the set to specify all the three unknown variables ρ, p and a. For example, matter can be described by an ideal gas with an equation of state

$$p = w\rho\,,\qquad(1.7)$$

where w is a constant that specifies the equation of state of the fluid considered. From (1.6) it follows that the relation between ρ and the scale factor a is of the form

$$\rho \propto a^{-3(1+w)}\,.\qquad(1.8)$$

For dust matter - for example - the pressure is negligible $w = 0$, so it follows that

$$a \propto t^{2/3} \qquad \rho \propto a^{-3} \qquad \text{dust}\,,$$

while for radiation we have $w = 1/3$ and hence it follows that

$$a \propto t^{1/2} \qquad \rho \propto a^{-4} \qquad \text{radiation} .$$

Notice that the value for which $p = -\rho$ is $w = -1$ and can be assumed to describe a cosmological constant Λ. This value of w implies that the energy density of Λ is constant and does not depend on time, that is

$$\rho_\Lambda \equiv \text{const} .$$

Today, we have strong experimental evidences about this value: the *Planck* satellite has given a precise measurement of the value of $w = -1.006 \pm 0.045$, confirming the ΛCDM model of a cosmological constant as origin of the accelerated expansion of the Universe. Moreover, we can note that in equation (1.4) ρ can also be rewritten in terms of two stress-energy components $\rho = \rho_m + \rho_{DE}$, respectively non-relativistic matter (dominated by cold dark matter) and dark energy with equation of state $p = w(t)\rho$, which is the relevant case in the late Universe [21]. Observations restrict w to be very close to -1 in the present Universe [53], but this need not to be the case for all times and so it is taken as a function of time.

Notice, for later use, that the Friedmann equation (1.4) can be rewritten as

$$\Omega - 1 = \frac{k}{a^2 H^2} , \qquad (1.9)$$

where

$$\Omega \equiv \frac{\rho}{\rho_c}, \qquad \text{with} \qquad \rho_c \equiv \frac{3H^2}{8\pi G} . \qquad (1.10)$$

Here the density parameter Ω is the ratio of the energy density to the critical density ρ_c.

1.2 Problems of the Standard Hot Big-Bang model

Here we briefly summarize the main issues that affected the Standard Hot Big-Bang model, which lead to the introduction of the inflationary scenario.

1.2.1 The flatness problem

In the Standard Hot Big-Bang theory, the Universe is eternally decelerated

$$\ddot{a} < 0 ,$$

so $a^2 H^2 = \dot{a}^2$ in (1.9) always decreases: this indicates that Ω tends to shift away from unity with the expansion of the Universe. However, since present observations show that Ω is very close to one ($\Omega_0 = 1.00 \pm 0.04$ today [39]), in the past it should have been even closer to one. For example we should require $|\Omega - 1| < \mathcal{O}(10^{-20})$ at the epoch of nucleosynthesis. This is an extreme fine-tuning of initial conditions and is called the "flatness problem".

1.2.2 The horizon problem

CMB photons, which are propagating freely since they decoupled from matter at the epoch of last scattering, appear to be in thermal equilibrium at almost the same temperature

(remember that CMB measurements [19] reveal anisotropies of the order $\Delta T/T \simeq 10^{-5}$). The most natural explanation for this is that the Universe has indeed reached a state of thermal equilibrium through interactions between the different regions before the last scattering. This means that the cosmological scales we can now see must have been casually connected before the decoupling of radiation from matter. But this is not possible in the Standard Hot Big Bang model, where there is no possibility for the regions that became casually connected recently to interact before the last scattering because of the finite speed of light.

Notice that new regions of the Universe that appear from the cosmological horizon scale - and that can be observed with our telescopes - should not to be in causal connection if their angular distance is of order $1°$. Observationally, however, we see photons in wider regions of the sky at the same time, and they have almost the same temperature in all the CMB sky. For example the COBE satellite [23] had a $7°$ resolutions, seeing at the same time regions that surely were not in causal contact at the nucleosynthesis epoch. This is the so called "horizon problem".

1.2.3 The monopole problem

Particle physics predicts that a spontaneous symmetry breaking occurs in the primordial Universe, with high temperatures and high densities, there would be production of many unwanted relics such as monopoles, cosmic strings, and topological defects [32]. String theories also predict supersymmetric particles such as the gravitinos.

If these particles exist in the early stage of the Universe, the energy densities of them decrease as a matter component and these massive relics would be the dominant matter overclosing the Universe [7, 9]. This problem is generally called as the "monopole problem".

1.3 Inflation

1.3.1 From the standard Hot Big-Bang model to the inflationary proposal

As we have seen, Standard Cosmology cannot solve some cosmological problems that affect the standard Big-Bang scenario. Such issues can be solved if we consider an epoch of *accelerated* expansion in the early Universe, the so called **inflation** (see, e.g. [7, 28]). Let us see a brief resume of the most popular inflation models.

The basic idea of inflation were originally proposed by Guth [29] in 1981 and is now called *old inflation.* This corresponds to a de-Sitter inflation which makes use of a first-order phase transition of a scalar field from a false to the true vacuum state. However, one of the problems of this model is that inflation occurs just in some region of space and Universe becomes inhomogeneous because these bubbles of true vacuum cannot merge due to the strong expansion of the Universe.

The revised version was proposed by Linde and Albrecht-Steinhandt [30, 52] in 1982, and is dubbed as *new inflation.* In this model a second order phase transition of the scalar field and a slow-roll phase are introduced. Unfortunately this original scenario also suffers from a fine-tuning problem of spending enough time to lead to sufficient amount of inflation.

In 1983 Linde [31] considered an alternative version of the slow-roll inflation called *chaotic inflation,* in which initial conditions of scalar fields are chaotic. According to this model, our homogeneous and isotropic Universe may be produced in the regions where inflation occurs sufficiently.

Lots of variations and new models have been proposed up to now [7], but however, the specific model of inflation is not clear today: in the next section we will review the most important inflationary scenario that can describe the most important features of the phenomenon.

The inflationary scenario not only explains and solves the cosmological problems mentioned in the previous section, but it also provides an extremely appealing explanation for the formation of structure on large scales and the inhomogeneities of the CMB of order $\Delta T/T \simeq 10^{-5}$ through the generation of primordial perturbations.
Let us say that, nowdays, inflation is considered a central paradigm in Cosmology but there are still many aspects which are unknown, for example the potential of the inflaton scalar field. To solve these pending questions we can count on experimental data, whose precision is increasing greatly.
The inflationary predictions are found to be totally compatible with the measurements of the WMAP [48] and *Planck* satellites [19, 20]. The CMB angular spectrum is the most useful observable, in its shape are encoded large amounts of information even on the very early Universe. Its properties were studied with successful results by the *Planck* collaboration and new strong constraints on inflation were given.

The problems that affect the Standard Hot Big-Bang model lie in the fact that just a decelerating expansion of the Universe is considered. Thus, assume the existence of a primordial epoch with an accelerated expansion of the Universe, that is when the scale factor is such that

$$\ddot{a} > 0 \,.$$

From the relation (1.5), this corresponds to the condition ($a(t)$ is taken always positive)

$$-\frac{4\pi G}{3}(\rho + 3p) > 0 \qquad \Rightarrow \qquad p < -\frac{1}{3}\rho \,, \tag{1.11}$$

and

$$w < -\frac{1}{3} \,, \tag{1.12}$$

and that $\dot{a} = aH$ increases during inflation. Then the comoving Hubble radius, that is defined to be $r_H = \frac{c}{aH}$ (where we use $c = 1$ in natural units) and represents the last spherical surface of photons that reach us, goes like $r_H \sim (aH)^{-1}$ and decreases during the inflationary phase.

Let us to remember a useful quantity to describe the amount of inflation is the number of *e*-foldings, defined by

$$N \equiv \ln \frac{a_f}{a_i} = \int_{t_i}^{t_f} H dt \,, \tag{1.13}$$

where the subscripts i and f denote the quantities at the beginning and the end of the inflation, respectively. It turns out that the number of *e*-foldings is required to be at least $N \sim 70$ to solve the flatness and the horizon problem [7].

In the following, we recall how inflation can solve the previous cosmological problems thanks to this accelerated phase.

1.3.2   The flatness problem

Since the a^2H^2 term in (1.9) decreases during inflation, Ω rapidly approaches unity: inflations acts as an attractor of $\Omega \to 1$. When the inflationary period ends, the evolution of the Universe is followed by the conventional decelerated expansion and $|\Omega - 1|$ begins to increase. However, if the inflationary expansion occurs for a sufficiently long period and makes Ω very close to one [19], Ω stays of order unity even at the present epoch.

1.3.3   The horizon problem

Since during inflation we require that $\ddot{a} > 0$, this implies that - from the definition of the comoving Hubble radius -

$$\dot{r}_H(t) = -\frac{\ddot{a}}{\dot{a}^2} < 0 \; ,$$

that is r_H is decreasing. So the physical wavelength $\sim a$ grows faster than the Hubble radius during inflation and therefore it is pushed outside the Hubble radius during inflation. This means that regions where the causality works is stretched on scales much larger than the Hubble radius, thus solving the horizon problem: regions that now enter the horizon was really connected in the past.
In order to solve the horizon problem, it is thus required that

$$r_H(t_0) < r_H(t_i) \; , \tag{1.14}$$

that is, the today Hubble radius must be lower than that before inflation. According to this fact, it can be showed that the horizon and flatness problems can be solved if the Universe expands about e^{70} times during the inflationary period, thus $N \simeq 70$ [7].

1.3.4   The monopole problem

The accelerated expansion epoch in which consists the inflationary model "dilutes" the density of unwanted relics since the latter has a dependency as a^{-3}. This effect can be obtained if the inflationary period in sufficiently long: this implies that the contribution of these unobserved particles to the Universe density is negligible today, as experimentally observed.

1.4 Inflationary dynamics

Since Guth proposal of an inflationary epoch, the theory of inflation has been studied and developed with great efforts. Now the usual way to treat inflation is through a scalar field called the "inflaton" which - under specific conditions on its potential - acts like an effective cosmological constant.

Consider a homogeneous scalar field ϕ with Lagrangian density

$$\mathcal{L}_\phi = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \,, \tag{1.15}$$

where $V(\phi)$ is the potential energy. As we now see, it has the crucial role to lead to the exponential expansion of the Universe.
From the Lagrangian (1.15), one can get the action

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi \,,$$

which tells us that the energy-momentum tensor is

$$T_\phi^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g_{\mu\nu}}.$$

We will assume, on the basis of the small level of anisotropies of the CMB (of order $\Delta T/T \simeq 10^{-5}$), that the field ϕ can be decomposed as a background term ϕ_0 and a fluctuation $\delta\phi$

$$\phi(x,t) = \phi_0(t) + \delta\phi(x,t) \,.$$

So it follows that the background components of the energy-momentum tensor describe the energy density and the pressure density of the inflaton field, respectively, as

$$\rho = \frac{1}{2}\dot{\phi_0}^2 + V(\phi) \quad \text{and} \quad p = \frac{1}{2}\dot{\phi_0}^2 - V(\phi) \,. \tag{1.16}$$

Substituting (1.16) in (1.4), we get (neglection the term proportional to the k curvature, that becomes rapidly negligible due to its dependence on a^{-2})

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi_0}^2 + V(\phi)\right] \,, \tag{1.17}$$

and, from the generic Klein-Gordon equation in a curved spacetime, equation

$$\ddot{\phi_0} + 3H\dot{\phi_0} + V'(\phi) = 0 \,, \tag{1.18}$$

that describes the evolution of the background component.

The fundamental requirement of $\ddot{a} > 0$ that describes inflation, and thus that $p < -1/3\rho$, imposes a constraint on pressure and density of the inflaton field ϕ. This is clear imposing it on (1.16) and obtaining

$$\dot{\phi_0}^2 \ll V(\phi) \,, \tag{1.19}$$

and that $p \simeq -\rho$. This equation indicates that the potential energy of the inflaton dominates over its kinetic energy: this phase is called "slow-roll" stage. Moreover, a flat potential of
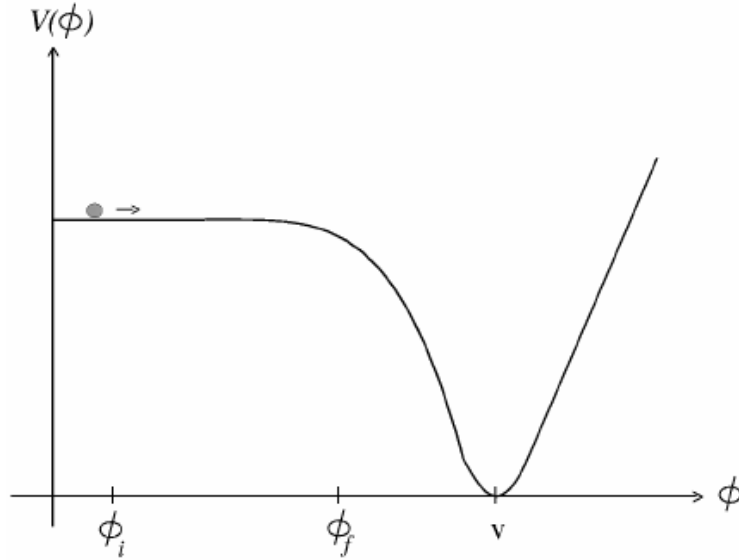
Figure 1.1: *An example of potential for a slow-roll inflation.*

the inflaton is required in order to lead to a sufficient amount of inflation. See Figure 1.1 for a sketch of a possible inflation potential. For this reason, we expect also that derivatives of the potential V', V'' are small in comparison with V and that they do not depend strongly on ϕ, and one can show that this implies

$$\ddot{\phi}_0 \ll 3H\dot{\phi}_0 \; . \tag{1.20}$$

So (1.19) and (1.20) give

$$H^2 \simeq \frac{8\pi G}{3} V(\phi) \; , \tag{1.21}$$

$$3H\dot{\phi} \simeq -V'(\phi) \; . \tag{1.22}$$

Defining the so-called slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \quad \text{and} \quad \eta \equiv \frac{1}{3}\frac{V''}{H^2} \; , \tag{1.23}$$

one can easily verify that the previous slow-roll approximations (1.19) and (1.20) imply

$$\epsilon \ll 1 \quad \text{and} \quad |\eta| \ll 1 \; . \tag{1.24}$$

The parameter ϵ must be much smaller than one in order to have a kinetic term negligible with respect to the potential, while the η parameter must be much smaller than one to guarantee that the potential is sufficiently flat to have the slow-roll phase of the field ϕ.
Notice that slow-roll parameters (1.24) only restrict the form of the potential $V(\phi)$, but not the properties of dynamical solutions which instead are constrained by (1.21) and (1.22). Thus, the (1.24) can be considered as necessary conditions to have a slow-roll phase, defining

a potential of a form such that the slow-roll inflation is made possible.

The inflationary phase ends when ϵ reaches one and the potential $V(\phi)$ is no more flat. The end of inflation implies the beginning of a later epoch in the Universe history: the reheating phase. This phase is characterized by a potential well in which the field oscillates around the minimum: the inflaton field acquires mass and decays in radiation, letting the Universe to start the usual standard radiation-dominated era.

1.4.1 Perturbations from inflation

Let us study some fundamental properties of the fluctuation $\delta\phi$ of the inflaton field.

From the generic Klein-Gordon equation in a curved spacetime in the Fourier space of the field $\phi(x,t)$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{k^2}{a^2}\phi = -\dot{V} \, ,$$

and for the fluctation $\delta\phi$ one finds

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k - \frac{k^2}{a^2}\delta\phi_k = 0 \, , \tag{1.25}$$

where we can ignore the term $V''(\phi)$ considering a massless field ϕ.

This equation leads to two very important behaviours of the scalar fluctuation $\delta\phi_k$. In fact, for subhorizon scales $k \gg aH$, in (1.25) the term $3H\delta\dot{\phi}_k$ can be neglected with respect to the one proportional to k, obtaining the equation of a harmonic oscillator with amplitude dependent on a.

In the superhorizon limit, instead, $k \ll aH$ so that the term proportional to k can be neglected and the solution is

$$\delta\phi_k \equiv \text{const} \, ,$$

that is, we have that the fluctuation freezes.

The really interesting physical meaning is that, during inflation, the scalar fluctuations freeze above the horizon and, at the end of the inflationary period, they will be transferred to the radiation fluid, inducing small density perturbations. As we will see in (2.7), on can show that the curvature perturbation on comoving ipersurfaces ζ is equal to (neglecting the metric perturbation)

$$\zeta = \delta N = H\delta t = -H\frac{\delta\phi}{\dot{\phi}} \, .$$

During the slow-roll phase of inflation, it yields

$$-H\frac{\delta\phi}{\dot{\phi}} = \frac{3H^2\dot{\phi}\delta\phi}{-3H\dot{\phi}^2} = -H\frac{V'\delta\phi}{-3H\dot{\phi}^2} = -H\frac{\delta\rho}{\dot{\rho}} \, ,$$

and, using the continuity equation for the energy density $\rho(t)$, also

$$-H\frac{\delta\phi}{\dot{\phi}} \simeq \frac{\delta\rho}{\rho} \, .$$

Thus, finally one can show that

$$\delta\phi|_{t_i} \equiv -\frac{H}{\dot{\phi}}\frac{\delta\rho}{\rho}\bigg|_{t_H} \, ,$$

that is, primordial scalar perturbations at time t_i induce small perturbations in the local energy density at end of inflation t_H. Such density fluctuations which grow because of gravitational collapse and ends up by building the large-scale structures we observe today in the Universe.

In the inflationary models, it is possible to provide an expectation for the power spectra of density perturbations and tensor perturbations (i.e. gravitational waves) at primordial epochs, that - in the case of a single slow roll scalar field - result to be

$$\Delta_{\frac{\delta\rho}{\rho}} = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \left(\frac{k}{aH}\right)^{n_s-1} , \tag{1.26}$$

$$\Delta_T = \frac{8}{m_{pl}^2\pi}H^2\left(\frac{k}{aH}\right)^{n_T} , \tag{1.27}$$

where the so called spectral indices turn out to be

$$n_s - 1 = 2\eta - 6\epsilon \quad \text{scalar spectral index} , \tag{1.28}$$

$$n_T - 1 = -2\epsilon \quad \text{tensorial spectral index} , \tag{1.29}$$

A fundamental quantity is the ratio between the value of the two power spectra amplitudes

$$r = \frac{\Delta_T}{\Delta_{\frac{\delta\rho}{\rho}}}$$

that is the tensor-to-scalar perturbation ratio. This quantity tells us in which proportion scalar and tensor perturbations were produced in the early Universe and it is fundamental to classify possible inflationary models [28]. Moreover, for single-field slow-roll inflation, the tensor to scalar ratio is linked to the tensor spectral index n_T by

$$r = 16\epsilon = -8n_T , \tag{1.30}$$

which is called "consistency relation". The lastest measurement of the *Planck* satellite [19] constrains the value of n_s to $n_s = 0.968 \pm 0.006$, confirming a red-tilted scalar spectral index and a strong deviation from the simple Harrison-Zel'dovich that considers $n_s \equiv 1$ and scale-independence for the amplitude of perturbations. The most challenging part from the experimental point of view is to increase the accuracy in the measure of n_T.

Planck has given also a constraint on the value of $r < 0.11$ (95% C.L.), compatible with previous results [20].

If the experiments confirm this relation (1.30), it would be an indisputable proof of the fact that inflation has actually been driven by a single slow-roll scalar field, otherwise it would mean that we need to consider alternative scenarios in which maybe there are more fields (the consistency relation would differ in this case). This relation, if verified in this form or in a more complicated one, would be a direct evidence of the inflationary model.

Chapter 2

Cosmological perturbation theory

Today we know that the observed Universe is not perfectly homogeneous and isotropic: matter is arranged in galaxies and cluster of galaxies, and there are large voids in the distribution of galaxies. The Standard Cosmology successfully describes many observational characteristics of our Universe: its expansion and consequent cooling, the abundances of light nuclei, the CMB. Even though these results outlined the effectiveness of this model, newer observations strongly reinforced the need for a further step: the presence of non-baryonic matter as dark matter, the structure of the Universe on large scales, the presence of anisotropies in the CMB indicating that the early Universe was not completely smooth.

Nevertheless, we note that every non-homogeneity on sufficiently large scales is affected by small fluctuations: for example, the CMB has anisotropies of the order of $\Delta T/T \simeq 10^{-5}$. For this reason, our modern Cosmology describes the physical Universe by the splitting of some physical observables in two different quantities: the background value of the given observable and its fluctuation, and the latter is assumed to be small with respect to the background (but not completely negligible).

So the perturbation in some quantity T is defined as the difference between the value T_{phys} that it has at a point in the perturbed physical spacetime and the value T_0 it has at the corresponding point in the background spacetime

$$\Delta T = T_{\text{phys}} - T_0 \ .$$

We start from a spatially homogeneous and isotropic FRW model as a background solution with simple properties, within which we can study the increasing complexity of inhomogeneous perturbations order by order.

Throughout the study of cosmological perturbations, we will encounter different types of perturbations, such as scalar, vector and tensor perturbation modes, which play different roles in the evolution of the Universe. These perturbations were produced during inflation as quantum fluctuations of the fields, and then they evolved.

We previously cited scalar fluctuations, but inflation also generates tensor fluctuations in the gravitational metric, the so-called gravity waves. These are not coupled to the density and so are not responsible for the large-scale structure of the Universe, but for example they induce perturbations in the CMB.

2.1   Gauge issue

One of the most important properties of perturbations is the so called "gauge issue" [11]. If we call $\mathcal{M}_0$ the background manifold with FRW metric and $\mathcal{M}_{phys}$ the manifold of the "real" physical Universe with little inhomogeneities and anisotropies, then a generic map φ is called a gauge if it links a point p in the background to the corresponding physical one $\phi(p)$ by adding a little perturbation $\delta\phi(p)$:

$$\phi(p): \quad \mathcal{M}_0 \to \mathcal{M}_{phys} \quad p \mapsto \phi(p)$$

$$\varphi: \quad \mathcal{M}_0 \to \mathcal{M}_{phys} \text{ with } p \to \varphi(p) = \phi(p) + \delta\phi(p) . \tag{2.1}$$

For example one can have

$$\rho_0(t) \to \rho_{phys}(x,t) = \rho_0(t) + \delta\rho(x,t) ,$$

where $\rho_0(t)$ can be for example the background value of the energy density.

A gauge transformation, let us call it ψ, is a change in the correspondence between background and physical points, keeping the background coordinates fixed. So, if ϕ_1 and ϕ_2 are two different gauge choices which associate two different points a and b respectively in $\mathcal{M}_{phys}$ to the same point p in $\mathcal{M}_0$, then ψ can be represented by
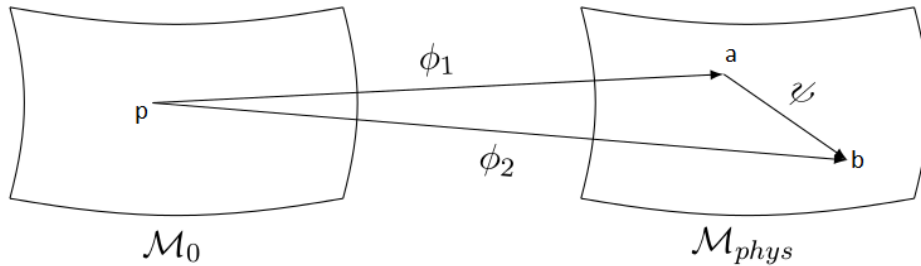
Figure 2.1: *A representation of the gauge issue.*

We know that Physics is invariant under gauge transformations, so we will need to make the appropriate gauge choice.

In this context, using gauge-independent variables which are independent by the choice of gauge, is helpful because it gives an exact physical interpretation in the sense that these variables represent the same physical quantity in each gauge. For example also in electromagnetism we encounter the same problem and it is clearly easier to work with the electric and magnetic fields rather than the gauge-dependent scalar and vector potentials.

2.2   Gauge and coordinates transformations

Besides the gauge choice problem, we have to distiguish between a gauge transformation and a coordinate transformation. The gauge transformation is a change in the correspondence between points in the physical and background spacetime, keeping the background coordinates fixed. This have to be distinguished from a coordinate transformation, which changes the labelling of points in the background and physical spacetime together. A choice of coordinates

defines a threading of spacetime into lines at fixed spatial coordinates and slicing into hypersurface at fixed time.

A gauge transformation induces a coordinate transformation in the physical spacetime, but it also changes the point in the background spacetime corresponding to the given point in the physical one. Thus, even if a quantity is a scalar under a coordinate trasformations, the value of its perturbation will not be invariant under a gauge transformation if the quantity is non-zero and position dependent in the background.

2.3 Lie dragging

The gauge issue arises in any approach to General Relativity that splits quantities into a background and a perturbation. We know from the study of General Relativity that solutions of the Einstein equations

$$G_{\mu\nu} = 8\pi G \, T_{\mu\nu} \tag{2.2}$$

are invariant under diffeomorphisms.

Consequently, if $g_{\mu\nu}$ is a solution for a particular choice of $T_{\mu\nu}$, acting with a diffeomorphism we find $\tilde{g}_{\mu\nu}$ which is a solution for $\tilde{T}_{\mu\nu}$. The mathematical relation between $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ is

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x) \, . \tag{2.3}$$

Now we consider an infinitesimal coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu - \xi^\mu \tag{2.4}$$

described by the functions ξ^μ.

Let us remember the laws of transformations of some quantities under a small change of coordinates:

$$\phi(x')' = \phi(x)$$

$$V'_\mu(x') = \frac{\partial x^\nu}{\partial x^{\mu'}} V_\mu(x)$$

$$V^{\mu'}(x') = \frac{\partial x^{\mu'}}{\partial x^\nu} V^\nu(x)$$

$$T^{\mu\nu'}(x') = \frac{\partial x^{\mu'}}{\partial x^\rho} \frac{\partial x^{\nu'}}{\partial x^\sigma} T^{\rho\sigma}(x)$$

$$T'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} T_{\rho\sigma}(x)$$

$$T^{\mu'}_{\ \nu}(x') = \frac{\partial x^{\mu'}}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x^{\nu'}} T^\rho_{\ \sigma(x)} \, .$$

Consider now the case of an infinitesimal transformation, the left term of (2.3) can be rewritten as

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}(x - \xi(x)) = \tilde{g}_{\mu\nu}(x) - \frac{\partial g_{\mu\nu}}{\partial \xi^\lambda} \xi^\lambda + \mathcal{O}(\xi^2) \, . \tag{2.5}$$

Using again (2.4) we can also rewrite the right part of (2.3):

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} = \delta^\rho_\mu - \frac{\partial \xi^\mu}{\partial x^\rho} \to \frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\mu_\rho + \frac{\partial x^\rho}{\partial \xi^\mu} + \mathcal{O}(\xi^2) \, , \tag{2.6}$$

hence

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \left(\delta^{\mu}_{\rho} + \frac{\partial x^{\rho}}{\partial \xi^{\mu}}\right)\left(\delta^{\sigma}_{\nu} - \frac{\partial \xi^{\sigma}}{\partial x^{\nu}}\right) \tag{2.7}$$

$$= g_{\mu\nu}(x) + \frac{\partial \xi^{\rho}}{\partial x^{\mu}}g_{\rho\nu}(x) + \frac{\partial \xi^{\sigma}}{\partial x^{\nu}}g_{\mu\sigma}(x) + \mathcal{O}(\xi^2) . \tag{2.8}$$

Putting together the equations (2.5) and (2.8) we find

$$\tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x) + \frac{\partial \xi^{\rho}}{\partial x^{\mu}}g_{\rho\nu}(x) + \frac{\partial \xi^{\sigma}}{\partial x^{\nu}}g_{\mu\sigma}(x) + \frac{\partial g_{\mu\nu}}{\partial \xi^{\lambda}}\xi^{\lambda} + \mathcal{O}(\xi^2) , \tag{2.9}$$

which is the expansion of the Lie derivative along the vector ξ^{μ} acting on $g_{\mu\nu}(x)$:

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathcal{L}_{\xi}g_{\mu\nu}(x) . \tag{2.10}$$

This equations means that the **Lie dragging** relates the metric tensor evaluated in the coordinate point x^{μ} with the transformed metric tensor under a diffeomorphism evaluated in the same coordinate point. Actually the relation (2.10) holds only at first order in ξ, however it can be generalized: if we take the function T (which can be a scalar, vector or tensor) and taking into account also the terms $\mathcal{O}(\xi^2)$ we get:

$$\tilde{T}(x) = \exp \mathcal{L}_{\xi}T(x) = T(x) + \mathcal{L}_{\xi}T(x) + \frac{1}{2}\mathcal{L}_{\xi}^2 T(x) + \dots \quad . \tag{2.11}$$

Since background quantities are not affected by gauge transformations, we can easily write

$$\delta\tilde{T}(x) = \tilde{T}(x) - T_0(x) = T(x) + \mathcal{L}_{\xi}T(x) + \frac{1}{2}\mathcal{L}_{\xi}^2 T(x) - T_0(x)$$

$$= \delta T(x) + \mathcal{L}_{\xi}T(x) + \frac{1}{2}\mathcal{L}_{\xi}^2 T(x) , \tag{2.12}$$

that is the relation between perturbations in different gauges up to second order from (2.11).

Note that the Lie derivative of the metric $g_{\mu\nu}$ simplifies because of the vanishing covariant derivative calculated of $g_{\mu\nu}$ itself, so that it becomes

$$\mathcal{L}_{\xi}g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} .$$

The Lie derivatives, moreover, can be generalized to arbitrary tensors as

$$\mathcal{L}_{\xi}s = s_{;\lambda}\xi^{\lambda}$$
$$\mathcal{L}_{\xi}V_{\mu} = V_{\lambda}\xi^{\lambda}_{;\mu} + \xi^{\lambda}V_{\mu;\lambda}$$
$$\mathcal{L}_{\xi}V^{\mu} = -V^{\lambda}\xi^{\mu}_{;\lambda} + \xi^{\lambda}V^{\mu}_{;\lambda}$$
$$\mathcal{L}_{\xi}T^{\mu\nu} = -T^{\lambda\nu}\xi^{\mu}_{;\lambda} - T^{\mu\lambda}\xi^{\nu}_{;\lambda} + T^{\mu\nu}_{;\lambda}\xi^{\lambda}$$
$$\mathcal{L}_{\xi}T^{\mu}_{\nu} = -T^{\lambda}_{\nu}\xi^{\mu}_{;\lambda} + T^{\mu}_{\lambda}\xi^{\lambda}_{\nu} + T^{\mu}_{\nu;\lambda}\xi^{\lambda} .$$

In general the effect of an infinitesimal coordinate transformation of any tensor T is that the new tensor equals the old one at the same coordinate point plus the Lie derivative $\mathcal{L}_{\xi}T$.

2.4   Active and passive approach

There are two mathematically equivalent approaches to calculate how perturbations change under a small gauge transformation: the active and the passive approach.

Active view: here perturbations change under mapping, where the map directly induces the transformation on the perturbed quantities. First we fix the coordinates x_0^μ on the background manifold $\mathcal{M}_0$. Any diffeomorphism $\Phi : \mathcal{M}_0 \to \mathcal{M}_{phys}$ with $\Phi : p \mapsto q_1$ induces a system of coordinates on the physical manifold $\mathcal{M}_{phys}$ via $\Phi : x_0^\mu \to x^\mu$.
For a given diffeomorphism Φ, we can define the perturbation δT of the generic function T (that can be scalar, vector or tensor) defined on $\mathcal{M}_{phys}$ as

$$\delta T(p) = T(p) - T_0(\Phi^{-1}(q_1)) \ .$$

A second diffeomorphism $\Psi : \mathcal{M}_0 \to \mathcal{M}_{phys}$ with $\Psi : p \mapsto q_2$ induces a new set of coordinates $\tilde{x}^\mu$ on $\mathcal{M}_{phys}$ via $\Psi : x_0^\mu \to \tilde{x}^\mu$ and a different $\delta\tilde{T}$:

$$\delta\tilde{T}(p) = \tilde{T}(p) - \tilde{T}_0(\Psi^{-1}(q_2)) \ ,$$

where $\tilde{T}$ is the value of T in the $\tilde{x}^\mu$ coordinates.
As it is clear, in this approach the gauge transformation $\delta T(p) \to \delta\tilde{T}(p)$ is generated by the change $\Phi \to \Psi$.
For this reason, we can change of function to a change of coordinates $x^\mu \to \tilde{x}^\mu$ on $\mathcal{M}_{phys}$. The gauge transformation can be seen as a one to one correspondence between different points on the background. In fact Φ sends a background point p to a point q in the physical manifold, for example $\Phi(p) = q$. Notice that Ψ is not the image of p, but rather it is the image of another point in the background, for example $\tilde{p}$. So we can write

$$\Phi(p) = q = \Psi(\tilde{p}) \ ,$$

which can be rewritten as

$$p = \Phi^{-1}(\Psi(\tilde{p})) = \mathcal{D}(\tilde{p}) \ ,$$

where the composite map $\mathcal{D}$ can be considered the origin of the change $\Phi \to \Psi$.

Passive view: in this approach, instead, we specify the relation between two coordinate systems directly and then calculate the change in the metric and matter variables when changing from one system to the other. First of all we choose some system of coordinates x^μ on the physical space-time manifold. The background is defined by assigning to all functions T on $\mathcal{M}_{phys}$ a background value $T_0(x^\mu)$ which is a fixed function of the coordinates. Therefore in a second coordinate system $\tilde{x}^\mu$ the background function $T_0(\tilde{x}^\mu)$ will have exactly the same functional dependence on $\tilde{x}^\mu$. The perturbation δT in the system of coordinates x^μ is defined as

$$\delta T(p) = T(x^\mu(p)) - T_0(x^\mu(p)) \ .$$

Similarly, in the second system of coordinates, the perturbation of T is

$$\delta\tilde{T}(p) = \tilde{T}(\tilde{x}^\mu(p)) - \tilde{T}_0(\tilde{x}^\mu(p)) \ .$$

Here $\tilde{T}(\tilde{x}^\mu(p))$ is the value of T in the new coordinate system at the same point p of $\mathcal{M}_{phys}$.
The transformation

$$\delta T(p) \to \delta\tilde{T}(p)$$

is called the gauge transformation associated with the change of variables $x^\mu \to \tilde{x}^\mu$ on the manifold $\mathcal{M}_{phys}$.

We can now apply the exponential map (2.11) to the function x^μ at the physical point q, to obtain the relation between the old coordinate system and the new one $\tilde{x}^\mu$:

$$\tilde{x}^\mu(q) = x^\mu(q) - \xi^\mu(q) + \dots \quad ,$$

where dots indicate higher order terms.

It can be seen, but it is above the scope of this Thesis, that these two approaches are completely equivalent and one can switch the description offered by an approach to the other. It is clear that the active approach relates amplitudes of the perturbations between the background manifold and the physical manifold. The passive one, instead, allows to connect the gauge transformation with the choice of the system of coordinates on $\mathcal{M}_{phys}$ in which the perturbations are described.

2.5 Cosmological perturbations

Remember that we are assuming that our Universe can be described at zero order by a homogeneous and isotropic flat Friedmann-Robertson-Walker space-time, which we can write as

$$ds^2 = a^2[-d\eta^2 + \delta_{ij} dx^i dx^j] \, ,$$

where $a = a(\eta)$ is the scale factor and η is the conformal time.

Under these hypothesis, we can now resume the consequences of the *Helmholtz theorem*, that describes how to decompose any tensor in scalar, vector and tensor parts. The reason for splitting the metric perturbations into scalars, vectors and tensors is that their governing equations decouple at linear order and hence we can solve each perturbation type separately. Notice, however, that this is no longer true at higher order in the perturbations.

We can write the perturbed metric tensor as

$$g_{00} = -a(\eta)^2 \left(1 + 2 \sum_{r=1}^{\infty} \frac{1}{r!} \psi^{(r)} \right) \, , \tag{2.13}$$

$$g_{0i} = g_{i0} = a(\eta)^2 \sum_{r=1}^{\infty} \frac{1}{r!} \omega_i^{(r)} \, , \tag{2.14}$$

$$g_{ij} = a(\eta)^2 \left(1 - 2 \sum_{r=1}^{\infty} \frac{1}{r!} \varphi^{(r)} \right) \delta_{ij} + \sum_{r=1}^{\infty} \frac{1}{r!} \chi_{ij}^{(r)} \, , \tag{2.15}$$

where $\chi^{(r)i}_{i} = 0$, r is the order of perturbation, ψ the lapse function, ω is the shift perturbation. Here and in what follows, latin indices are raised and lowered using δ_{ij} and δ^{ij} respectively. Thanks to Helmholtz theorem, we can define scalar, vector and tensor parts of perturbations, where scalar (longitudinal) parts are those related to a scalar potential, vector parts are related to tranverse (divergenceless or solenoidal) vector fields, and tensor parts to transverse and trace-free tensors.

Consider now a perfect fluid described by an energy-momentum tensor $T^{\mu\nu}$ given by

$$T^{\mu\nu} = p h^{\mu\nu} + \rho u^\mu u^\nu \, ,$$

where $h^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu}$.

We can write for the energy density

$$\rho = \rho_0 + \sum_{r=1}^{\infty} \frac{1}{r!}\delta\rho_{(r)} \ . \tag{2.16}$$

Analogously, we can write the four velocity u^{μ} of the matter as

$$u^{\mu} = \frac{1}{a}\left(\delta_0^{\mu} + \sum_{r=1}^{\infty} \frac{1}{r!}v_{(r)}^{\mu}\right) \ . \tag{2.17}$$

Because of the normalization condition for u^{μ} given by $u^{\mu}u_{\mu} = -1$, at any order the time component $v_{(r)}^0$ is related to the lapse function perturbation $\psi_{(r)}$. For example, at first order we have

$$v_{(1)}^0 = -\psi_{(1)} \ .$$

The velocity perturbation can also be split into a scalar and vector (vortical) part

$$v_{(r)}^i = \partial^i v_{(r)}^{\|} + v_{(r);\perp}^i \ ,$$

with $\partial_i v_{(r);\perp}^i = 0$.

We will consider also a scalar field with perturbation of this type

$$\varphi = \varphi_0 + \sum_{r=1}^{\infty} \frac{1}{r!}\delta\varphi_{(r)} \ . \tag{2.18}$$

Notice that in our case, the shift $\omega_i^{(r)}$ can be decomposed as

$$\omega_i^{(r)} = \partial_i \omega_{(r)}^{\|} + \omega_{i,\perp}^{(r)} \ ,$$

where $\omega_{i,\perp}^{(r)}$ is a solenoidal vector $\partial^i \omega_{i,\perp}^{(r)}$.

Finally, the traceless part of the spatial metric can be decomposed at any order as

$$\chi_{ij} = D_{ij}\chi^{(r)} + \partial_i\chi_j^{(r),\perp} + \partial_j\chi_i^{(r),\perp} + \chi_{ij}^{(r),T} \ ,$$

where $\chi_{ij}^{(r),T}$ is a suitable function, $\chi_{ij}^{(r),\perp}$ is a solenoidal vector fieldt and $\partial^i\chi_{ij}^{(r),\perp} = 0$. So we obtain

$$D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2 \ .$$

As we have seen, gauge transformations are described by the vectors $\xi_{(r)}$. We will use

$$\xi_{(r)}^0 = \alpha_{(r)} \ , \tag{2.19}$$

$$\xi_{(r)}^i = \partial^i\beta_{(r)} + d_{(r)}^i \ , \tag{2.20}$$

with $\partial_i d_{(r)}^i = 0$. Thus, the definition of two scalars and one vector it is sufficient to define a gauge transformation. If all these three quantity are not defined, there will be a residual gauge degree of freedom.

A gauge transformation - as previously discussed - allows to express the following quantities

$$\tilde{\psi} = \psi + \alpha' + H\alpha \tag{2.21}$$

$$\tilde{\omega}_i = \omega_i - \alpha_{,i} + \beta'_{,i} + d'_i \tag{2.22}$$

$$\tilde{\varphi} = \varphi - \frac{1}{3}\nabla^2\beta - H\alpha \tag{2.23}$$

$$\tilde{\chi}_{ij} = \chi_{ij} + 2D_{ij}\beta + d_{i,j} + d_{j,i} \ , \tag{2.24}$$

$$\tag{2.25}$$

and also

$$\delta\tilde{\rho} = \delta\rho + \rho'\alpha \ , \tag{2.26}$$

$$\tilde{v}^0 = v^0 - \alpha' - H\alpha \ , \tag{2.27}$$

$$\tilde{v}^i = v^i - \beta'^{,i} - d'^{,i} \ . \tag{2.28}$$

2.6 Examples of some gauges

In the previous sections, we have seen that the vector ξ^μ generating the gauge transformations involves two scalars α and β, and a divergence-less vector d_i: this holds at any order in perturbation theory.
In the following notes, we resume some of the most popular gauge choices.

Poisson gauge One of the most useful gauge. It is defined by the choice $\omega^\parallel = \chi^\parallel = \chi_i^\perp = 0$. This generalizes the so-called longitudinal or Newtonian gauge in which vector and tensor perturbations are not considered.
The name is due to the fact the one can easily obtain the Newtonian limit in the gauge.

Synchronous gauge It is defined by the choice $\psi = 0$. If we also take $\omega^\parallel = \omega_i^\perp = 0$ it is called synchronous and time orthogonal gauge. In this gauge, the proper time for observers at fixed spatial coordinates coincides with the cosmic time in the FRW background.

Comoving gauge It is defined by the choice $v_i = 0$ that implies $v^\parallel = v_i^\perp = 0$. If we also require orthogonality of the constant-η hypersurfaces to the 4-velocity, this gives $v^\parallel + \omega^\parallel = 0$ (zero momentum) and then the gauge is called comoving orthogonal gauge.

Spatially flat gauge This is defined by the condition that one selects spatial hypersurfaces on which the induced 3-metric of spatial hypersurfaces is left unperturbed by scalar or vector perturbations, which requires $\zeta = \chi^\parallel = \chi_i^\perp = 0$.

Uniform density gauge It is defined by the condition $\delta\rho = 0$.

Notice that, for example, the synchronous and uniform-density gauges have residual gauge freedom because they do not fix all three degrees of freedom.

2.7 Gauge invariant quantities

As we have seen, the gauge issue states that perturbations are affected by gauge transformations: when we change the gauge also the value of perturbations change. This problem can be overtaken in two ways. One can remember that, at the end of calculations, physical observables must be gauge invariant. However, usually it is better to start calculations using quantities that are gauge-invariant by definition.

Of course there are an infinite number of gauge-invariant variables, since any combination of gauge-invariant variables will also be gauge-invariant. Here we resume some of the most important ones at the linear order.

Considering purely geometric quantities, only two scalar independent gauge-invariant quantities can be constructed from the metric tensor amplitudes alone

$$2\Psi \equiv 2\psi + 2\omega'^{\parallel} + 2H\omega^{\parallel} - \left(\chi''^{\parallel} + H\,\chi'^{\parallel}\right)$$

and

$$2\Phi_H \equiv -2\varphi - \frac{1}{3}\nabla^2\chi^{\parallel} + 2H\omega^{\parallel} - H\,\chi'^{\parallel}\,,$$

which in the gauge where $\omega^{\parallel} = \frac{1}{2}\chi'^{\parallel}$ would reduce to $\Psi = \psi$ and to $-\Phi_H = -\varphi - \frac{1}{6}\nabla^2\chi^{\parallel}$. Φ_H is usually called as the "Bardeen potential" [11].

Another scalar gauge invariant quantity is for example the velocity

$$v_S \equiv 2v^{\parallel} + \chi'^{\parallel}\,,$$

that describes the amplitude of the scalar shear of the fluid velocity.

An important gauge invariant quantity usually used in inflationary context is

$$\zeta = \hat{\varphi} - \mathcal{H}\frac{\delta\rho}{\rho'}\,,$$

with $\hat{\varphi} = \varphi + \frac{1}{6}\nabla^2\chi^{\parallel}$. This is called "curvature perturbation of uniform density spatial hypersurface" and it is used for example in connection with inflation because it is conserved on super-horizon scales and if non-adiabatic pressure perturbations are absent.

Considering vectorial quantities in term of geometric perturbations, instead, there is only the

$$\psi_i = \omega_i^{\perp} + \omega_i'^{\perp}\,,$$

which represent a "frame-dragging" term.

Another example is the matter velocity that implies

$$V_S^i \equiv v_{\perp}^i + \chi_{\perp}'^i\,,$$

that describes the amplitude of the rotational part of the fluid velocity.

With regards to the tensorial quantities, instead, at the first order all tensor perturbations are automatically gauge-invariant. In fact, separating the scalar, vector and tensor modes in the metric trasformation rules (2.21)-(2.24), one obtains that the transformation rule of the tensorial part at linear order is

$$\chi_{ij} \to \tilde{\chi}_{ij} \equiv \chi_{ij}\,,$$

and so the tensorial quantities are gauge invariant.

Chapter 3

Large-Scale Structure of the Universe
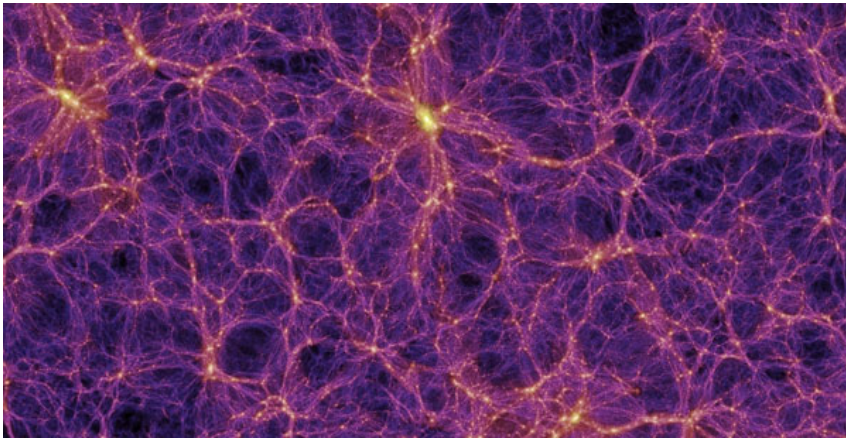
Figure 3.1: *A pictorial but very impressive representation of the Large-Scale Structures.*

In this Chapter we will present the main features that characterize modified gravity models. Then, we will discuss the relation between the possible predictions of modified gravity models and some of the main cosmological observables that can constrain them, recalling the definitions of some observables and some parameters have become standard to select the alternative gravity models. At the end of the chapter, we illustrate some aspects of the forthcoming Euclid satellite.

3.1 Modified gravity

We know today that the cosmological constant exists and it is of fundamental origin: thus, to understand the Physical reason of it, we have to reconsider the principles that are at the basis of the Standard Model of Cosmology. One of these is that gravity is well described by General Relativity on all scales: but we know that the presence of the cosmological

constant is a phenomenological evidence that some other physical mechanism have to be considered. For this reason, modifications of gravity could provide an interesting solution to the cosmological constant problem and, hopefully, to the dark matter presence.

But when we modify GR we immediately encounter some problems. There is a theorem, known as Lovelock's theorem, that proves that Einstein's equations are the only second-order local equations of motion for a metric derivable from the action in 4D. This indicates that if we modify GR, we need to have one or more of these: extra degrees of freedom, higher derivatives terms with respect to the metric in the equations of motion, higher dimensional spacetime or non-locality [55].
And once we introduce these extra ingredients into the theory beyond GR, we have to check the theoretical consistency of the model.
First we have to verify if solutions are stable. Consider the following simple Lagrangian of a scalar field ϕ that can be useful to illustrate several kinds of instabilities

$$\mathcal{L}_\phi = k_t \dot{\phi}^2 - k_x \phi_{,i}^2 - m^2 \phi^2 \ . \tag{3.1}$$

A first form of instability is the tachyonic one, that it is a hypothetical particle that always moves faster than light and it would violate causality. It arises when the scalar field has a negative mass squared $m^2 < 0$.
Another well-known instability arises when the time kinetic term of the scalar field has a wrong sign $k_t < 0$, and it is even more severe: the so called "ghost" instability, that can introduce two main problems. They make the theories ill-defined at the quantum level in the high energy/sub horizon regime, where the vacuum is unstable and decays instantaneously. To avoid the instability, it is required to introduce a non-Lorentz-invariant cut-off in the theory. Moreover, they create an instability already at the linearized level in perturbation theory [56].
If the theory satisfies the requirements for theoretical consistency, it also needs to pass observational tests. A century of observations have led to lots of small scale constraint, such as Solar System constraints [55]. For example the deflection angle θ of the light path due to the gravitational attraction of Sun is observed to be [62, 63]

$$\theta = (0.9999 \pm 0.0002) \times 1.75'' \ , \tag{3.2}$$

where $1.75''$ is the prediction of GR. Another example of very well measured quantity is the prediction of GR in time dilation Δt due to the effect of the Sun's gravitational field, the so called "Shapiro time delay" [64] (measured very accurately using the Cassini satellite [61])

$$\Delta t = (1.00001 \pm 0.00001) \times \Delta t_{GR} \ .$$

Any modified theory of gravity needs to satisfy these constraints on deviations from GR in the Solar System and in the small scale limit.
On cosmological scales, observations have reached a very high level of precision and at the moment, the standard ΛCDM model survives all these improved measurements [44, 65, 66]. For example, in Figure 3.2 we can see some Astrophysical data that confirm the presence of a cosmological constant Λ by the analysis of Hubble diagrams of supernovae [68]. The lastest data of the *Planck* mission confirm that the ΛCDM model describes the present equation of state of DE: the value of w is well-constrained by

$$w = -1.006 \pm 0.045 \ ,$$

that is perfectly compatible with the cosmological constant Λ in the ΛCDM model [19].
This implies that, in the background, the expansion of the Universe should look very similar to that of the ΛCDM model [67].
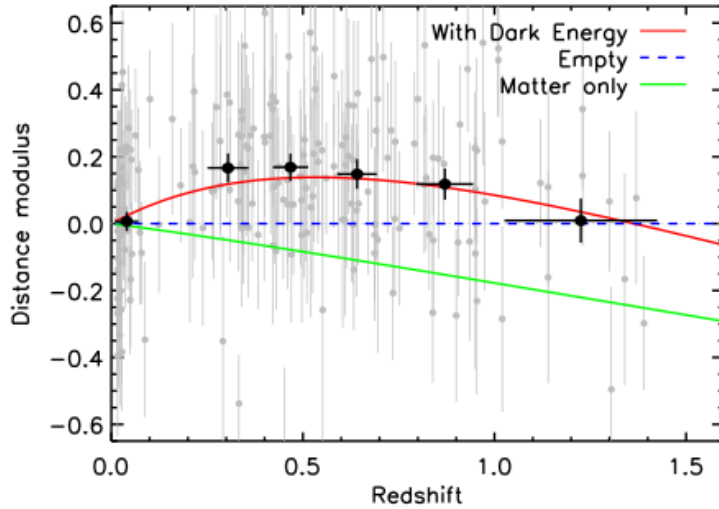
Figure 3.2: *Hubble diagram - supernova magnitude vs redshift - normalised to the predicted expansion history for an empty Universe (dashed line). Raw data are in grey, binned with 1 σ uncertainties in black. The red line is a flat model with 30% matter, 70% dark energy. The green line that is a poor fit to the data has 30% of the critical density of matter and zero dark energy [68].*

3.1.1 Examples

In this section, we briefly present some representative modified gravity models. For a more complete review of various modified gravity models, see [44, 45].

Brans-Dicke gravity

A first example of modified gravity model is the Brans-Dicke gravity [44, 55], given by the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \Big(\phi R - \frac{\omega_{BD}}{\phi} (\partial\phi)^2 \Big) + \int d^4x \sqrt{-g} \mathcal{L}_m. \tag{3.3}$$

where the ω_{BD} is the so called "Brans-Dicke parameter" and ϕ a scalar field. It is a theory that includes a scalar field non-minimally coupled to gravity. It is an example of a scalar-tensor theory where the gravitational constant G is not presumed to be constant and it is instead modified by the scalar field.

It is interesting to analyze the so called quasi-static approximation (QSA) of perturbations. It consists in a simplification of equations of motion when considering conditions of the growth of structure on subhorizon scales and when the time derivatives of the metric potentials are taken to be negligigle with respect to the spatial ones. Usually,it corresponds to the approximation under which the dominant contributions to the perturbation equations are those including k^2/a^2 and δ terms in Fourier space [38, 54, 69, 70]. Considering the scalar field perturbation expressed as $\phi = \phi_0 + \delta\phi$ and the metric perturbations of the section (2.5)

$$ds^2 = -(1 + 2\Psi)dt^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j \ ,$$

in the QSA limit and in the Poisson gauge, the equations of motion reduce to

$$\nabla^2 \Psi = 4\pi G\rho - \frac{1}{2}\nabla^2 \delta\phi , \qquad (3.4)$$

$$(3 + 2\omega_{BD})\nabla^2 \delta\phi = -8\pi G\rho , \qquad (3.5)$$

$$\Phi - \Psi = \delta\phi , \qquad (3.6)$$

where Φ and Ψ are the gravitational potentials. Note that the scalar field - that is non-minimally coupled to gravity - gives an effective anisotropic stress through its perturbations, modifying the relation between Φ and Ψ.

These equations can be rewritten as

$$\nabla^2 \Psi = 4\pi G\mu\rho \quad \text{and} \qquad \Psi = \eta^{-1}\Phi , \qquad (3.7)$$

where

$$\mu = \frac{4 + 2\omega_{BD}}{3 + 2\omega_{BD}}, \quad \eta = \frac{1 + \omega_{BD}}{2 + \omega_{BD}} . \qquad (3.8)$$

Notice moreover that we recover GR in the large ω_{BD} limit. Indeed, imposing the experimental small scale constraints such (3.2), we obtain $|\eta - 1| \simeq 10^{-5}$ and the constraint ω_{BD} is given by $\omega_{BD} \gg 10^4$: if we impose this constraint on the parameter of the model ω_{BD}, the theory is basically indistinguishable from GR on all scales.

In fact, this is one of the main problems of some modified gravity models: the Solar System tests are so stringent that, if they are imposed on the parameters of many models, their values become such that to reduce the theory to GR.

$f(R)$ **gravity**

One of the simplest and most popular extensions of GR and known example of modified gravity models is the so-called "$f(R)$ gravity", in which the action is given by some generic function $f(R)$ of the Ricci scalar R.

The Einstein-Hilbert action is generalised as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + \mathcal{S}_m. \qquad (3.9)$$

where G is a bare gravitational constant, and the observed one will be in general different. To recover General Relativity we have to impose $f(R) = R - 2\Lambda$ [44].

The $f(R)$ gravity theories turns to have equations of motion fourth order for the metric. However, it is possible to introduce a scalar field ψ and to make the equation of motion second order, thus the action becomes of the form

$$S = \int d^4x \sqrt{-g} \left(f(\psi) + (R - \psi)f'(\psi) \right) . \qquad (3.10)$$

Varying the action with respect to ψ, we obtain $(R - \psi)f''(\psi) = 0$. Notice that, if $f''(\psi) \neq 0$ and $R = \psi$, we recover the original action.

Notice moreover that, by defining $\phi \equiv f'(\psi)$ and $V = f(\psi) - \psi f'(\psi)$, the action can be written

$$S = \int d^4x \sqrt{-g} \left(\phi R - V(\phi) \right) , \qquad (3.11)$$

that is the same as the Brans-Dicke model with a potential. Comparing this with the action (3.3), we notice that the BD parameter is given by $\omega_{BD} = 0$. Thus, with regards to what we

said in the previous section, if we ignore the potential, this model is already excluded by the small scale constraints. However, the appropriate choice of the potential and consequently the form of the $f(R)$ function, can allow to include a screening mechanism known as the "chameleon" mechanism, that allows the theory to have a good small scale limit and to respect the Solar System observational constraint [55].

Outlook

In the last decades, we have seen the proposal of several modified gravity models. However, we still do not have a consistent theory that is a true alternative to ΛCDM.
Despite of this, one can determine some common properties of many different models of modified gravity. There are usually three regimes of gravity:

- There is a length scale above which gravity is considered to be modified and possible deviations from it could be identified in the large scale structure of the Universe. The cosmological constant - for example - could have its origin at these scales.

- Below the modification scale but at not such small scales, gravity is assumed to be modified due to an extra scalar degree of freedom. In this regime, for example, gravity can be described by scalar tensor gravity with a $O(1)$ Brans-Dicke parameter.

- On small scales, GR appears to be completely restored. In some modified gravity models, this can be produced with a non-linear interaction with the scalar field. This is often called a "screening mechanism" [44, 55]. This is essential to ensure that the theory passes the stringent Solar System constraints.

Every modified gravity model that would substitute or extend GR is requested to respect all the previous behaviour with respect to the different regimes of gravity at different scales.

3.2 Large-Scale structures

Today we know that GR is very well tested through the entire Solar System and, generally, at level of the entire galaxy. But in relation with the entire Universe, these are only small scales. In the previous Chapter, we have seen that modified gravity theories become significant at large scales. These scales deal with the Large Scale Structure (LSS) of the Universe, which for example refers to clusters and superclusters of galaxies, and their distribution.
Thus, in this Chapter we discuss the relation between some of the main LSS observables and their implications on modified gravity models.

3.2.1 Parameters

We now consider first order scalar perturbations on a flat background, where equations (2.13)-(2.15) simplifies in the Newtonian gauge and describe the spacetime with the metric

$$ds^2 = a^2(\eta) \left[-(1 + 2\Psi)d\eta^2 + [(1 - 2\Phi)\delta_{ij}dx^i dx^j] \right] . \tag{3.12}$$

It is well known [34] that in GR, at late time when matter dominates and the anisotropic stress is negligible, the two gravitational potential are equal

$$\Phi \equiv \Psi , \tag{3.13}$$

that is, the curvature perturbation Φ is equal to the Newtonian potential Ψ. Under these hypothesis, we introduce the Weyl potential

$$\Phi_+ \equiv \frac{\Phi + \Psi}{2} \ , \tag{3.14}$$

that affects relativistic particles. Thus, equation (3.13) tells us that Φ_+ is the same as the gravitational potential felt by non-relativistic particles. This particular feature is not generally verified in alternative theories of gravity, where the equivalence between Φ_+, Φ and Ψ is typically broken.

Moreover, notice that the equivalence of Φ and Ψ in GR has also another consequence. On sub-Hubble scales, the gravitational potential is related to the matter density perturbation by the Poisson equation [35, 14]

$$\nabla^2 \Psi = 4\pi G a^2 \rho \ \delta_m \ , \tag{3.15}$$

where $\delta_m = \frac{\delta \rho_m}{\rho_m}$ is the relative matter density.

It is customary to describe the evolution of density perturbation through the so called "growth factor" $D(a)$, defined as

$$\delta_m(k, a) = D(a)\delta(k, a_i) = D(a)\delta_i \ , \tag{3.16}$$

starting from an initial time a_i where the modes of interest are sub-Hubble horizon. Studying the power spectra, one can find also the following relation $P_{\text{eff}}(k, a) = D^2(a)P_\delta(k)$: that is, the growth factor enters directly in modifying the power spectra that we can measure with our instruments, and for this reason can be experimentally determined.

From the study of the growth factor it is possible to infer that, in alternative gravity theories in which the two potential Φ and Ψ differ, matter perturbations can be different in relation with Φ or Ψ. As a consequence, modifications of gravity generally introduce a time- and scale-dependent function between the gravitational potentials, that in Fourier space are tipically described with

$$\Phi = \tilde{\gamma}(a, k)\Psi \ , \tag{3.17}$$

where the function $\tilde{\gamma}(a, k)$ can be determined purely in a experimental way or can be verified from the theoretical predictions that a model offers. These modifications are expected to leave different patterns on the LSS of the Universe [14], above all because of their departure from GR predictions.

There are other two very important quantities that will be more directly probed by observations of galaxy redshifts and weak lensing effect. They are the effective gravitational constants G_{matter} and G_{light} that appear, respectively, in the Poisson equations for Ψ

$$\nabla^2 \Psi = 4\pi G_{\text{matter}} a^2 \rho \ \delta \tag{3.18}$$

and for Φ_+

$$\nabla^2 \Phi_+ = 4\pi G_{\text{light}} a^2 \rho \delta \ . \tag{3.19}$$

It is clear that their main task is to verify the correspondence between the gravitational constant measured in a modified gravity model and the GR one.

One can therefore define two other quantities

$$\mu = \frac{G_{matter}}{G} \quad \text{and} \quad \Sigma = \frac{G_{light}}{G} \ ,$$

defined by the following equations in Fourier space

$$k^2 \Psi = -4\pi G a^2 \rho \delta \, \mu(a,k) \qquad\qquad (3.20)$$

and

$$k^2(\Phi + \Psi) = -8\pi G a^2 \rho \delta \Sigma(a,k) \quad . \qquad\qquad (3.21)$$

Notice that these three function Σ, μ and $\tilde{\gamma}$ are related: inserting (3.17) in (3.21) one finds that

$$k^2 \Psi (1 + \tilde{\gamma}) = -4\pi G a^2 \rho \delta \cdot 2\Sigma(a,k) \quad ,$$

that provides the following relation

$$\mu = \frac{2\Sigma}{1 + \tilde{\gamma}} \qquad\qquad (3.22)$$

betweeen the parameters. Thus, providing any two of them is sufficient for solving for the evolution of cosmological perturbations.

3.3 Some LSS cosmological observables

In the study of modifications of gravity, we have the fundamental problem of verifying the theoretical model with the observational data. In the last decades, a huge number of theoretical models alternative to GR have been proposed, but the main effort that will be necessary in the near and long-term future is to understand what is to try to measure deviations from GR with the best possible precision. The first step will be to classify alternative gravity models and to construct some observables to select the models that respect observational data trying to increase the precision of the measurements and hence their selection power, leading eventually to a detection of deviation from GR.

3.3.1 Observables

A physical observable is the expansion history of the Universe $H(z)$. The latter however can be easily fitted by both scalar field dark energy models and also modifications of gravity models, or equivalently by any evolution of the effective dark energy equation of state parameter $w(z)$. Thus the evolution history of the Universe by itself cannot give us informations about the physical nature of the mechanism behind the accelerated expansion. Therefore it becomes necessary to look for modifications of gravity at the perturbation level in the physical quantities. The discrimination between models of dark energy and modified gravity can be obtained mainly with information on the growth of LSS at different scales and different redshifts.

It is possible to find two categories of cosmological observational probes [14] the geometrical probes and the structure formation probes. While the former provide a measurement of the Hubble function $H(z)$, the latter are a test of the gravitational theory in an almost Newtonian limit on subhorizon scales.
So in the following we present a brief summary of how dark energy or modified gravity effects can be detected through an example of a geometrical feature - the weak lensing - and an example of a structure formation probe via galaxy redshift surveys.

Weak lensing

A very powerful probe of structure growth is the weak lensing effect [36, 14], that is generated when the presence of any mass modifies the path of light passing near it. This effect rarely produces multiple images associated with the strong gravitational lensing: this is the clearest distinction between the weak and the strong lensing effects.

Astronomers think that the weak lensing regime is really more likely than the strong one, but in the weak regime the deflection is difficult to detect in a single background source. However, in some cases, the presence of the foreground mass can be detected, by way of a systematic alignment of background sources around the lensing mass.

Generally, the effect of gravitational lensing acts as a coordinate transformation that distorts the images of background objects (for example galaxies) near a foreground mass. The transformation can be split into two terms: the convergence and the shear. The convergence term increases the size of the background objects: it cannot be directly observable, but it can be statistically measured through the modifications that it induces on the galaxy number density. The shear term, instead, stretches objects tangentially around the foreground mass: detection of this misalignment can be done measuring the ellipticities of the background galaxies and construct a statistical estimate of their systematic alignment.

The convergence and the shear can be quantified through the definition of the magnification matrix. It is a 2×2 matrix that relates the true shape of a galaxy to its image. In the matrix, the convergence is defined as the trace of the matrix, whereas the shear is defined as the symmetric traceless part.

On small scales the shear and the convergence are not independent: they satisfy a consistency relation, and they contain therefore the same information on matter density perturbations. More precisely, the shear and the convergence are both related to the sum of the two Bardeen potentials, $\Psi + \Phi$, integrated along the photon trajectory.

At large scales however, various effects contribute to modify this consistency relation. Some of these effects are generated along the photon trajectory, whereas others are due to the perturbations of the galaxies redshift. These relativistic effects provide independent information on the two Bardeen potentials, breaking their degeneracy.

Redshift surveys

The study of wide-deep galaxy redshift surveys can give information on both the Hubble parameter $H(z)$ and the growth factor $D(z)$ through, e.g., measurements of the Baryon Acoustic Oscillations (BAO) and redshift-space distortions [14].

In Cosmology, BAO are regular and periodic fluctuations in the density of the visible baryonic matter of the Universe. In the same way that supernova provide a "standard candle" for astronomical observations, BAO matter clustering provides a "standard ruler" for length scales in Cosmology that is about ~ 490 million light years in today's Universe and that can be measured by looking at the LSS of matter.

With regard to the redshift space distortions, instead, if gravity is not modified, then a detection of the expansion rate $H(a)$ is directly linked to a unique prediction of the growth function $D(a)$. In fact, as we will see in the next section, it yields

$$\frac{d\log D(a)}{d\log a} \simeq \Omega_m(a)^\gamma ,$$

where

$$\Omega_m(a) = \frac{8\pi G \rho_m}{3H(a)^2} .$$
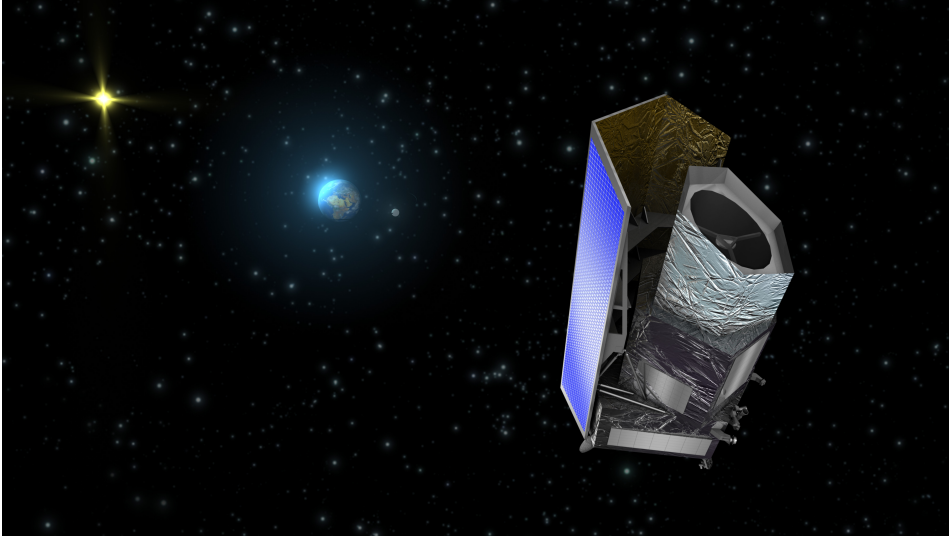
Figure 3.3: *A representation of the Euclid satellite.*

Departures from the expected growth rate can be a signal of modified gravity effects.
Since Euclid can measure directly BAO from measurement of the matter power spectrum,
we can obtain the expansion rate of the Universe $H(z)$. In addition, it can also measure the
cosmic growth history $D(z)$ studying the clustering of galaxies.

3.4   The Euclid satellite

Euclid is a satellite of the European Space Agency (ESA) mission that will be lauched in
2019 from Earth to get the L2 Lagrange point for a six-year mission [14].
The main task of Euclid is to find the physical reason of the accelerated expansion of the
Universe, the so called "Dark Energy", and eventually the identification of a more accurate
selection of modified gravity models.
Since clustering of galaxies and weak lensing effects are the most important observables of
Euclid and since they also depend on the properties of Dark Matter, Euclid will be capable
of discover more informations about the nature of Dark Matter as well.
Moreover, Euclid will improve the neutrino mass measurements, and will confirm some
observational data about the inflationary scenario.
Besides Cosmology, Euclid will cover a wide range of scientific topics and it will provide also
informations about galaxy evolution, galaxy structure, and planetary searches.

3.4.1   Telescope and general tasks

Euclid is a satellite whose main scope is to explore the expansion history of the Universe and
the evolution of cosmic structures. This is achieved by measuring shapes and redshifts of
galaxies over wide regions of the sky.
The most important instruments on the Euclid satellite are a 1.2 m telescope and three
imaging and spectroscopic instruments working in the visible and near-infrared wavelength
domains.

The two main observables [14] that Euclid will study are the reconstruction of clustering of galaxies out to a redshift $z \sim 2$ and the measurement of the pattern of light distortion from weak lensing to redshift $z \sim 3$. In addition it will be able to study - for example - the correlation between the CMB and the LSS, abundance and properties of galaxy clusters and strong lensing effect.

3.4.2 Parameters and some forecasts

In order to present some forecast of Euclid measurements [14, 34], let us remember that for a LSS and weak lensing survey the crucial quantities are the matter relative density δ_m and the gravitational potentials Ψ and Φ: we therefore focus on scalar perturbations in the Newtonian gauge with the metric (3.12).

The time evolution of the density field δ_m can be a sensitive probe of not only the expansion rate of the Universe, but also of its matter content. In a flat and matter dominated Universe, we have that δ_m evolves as $\delta_m \propto a$, where a is the scale factor. In different conditions, we can parametrize the departure from this relation by defining the growth rate f_D [74, 73, 57] as

$$f_D \equiv \frac{d \log D(a)}{d \log a} \simeq \Omega_m(a)^\gamma \ . \tag{3.23}$$

The standard growth index γ, in the presence of a cosmological constant as in the ΛCDM model, is found to be $\gamma_\Lambda \simeq 6/11 \simeq 0.545$ [14, 74].

In the case in which there are modifications to the ΛCDM model, the (3.15) will be modified and the growth rate (3.23) has a different index from γ_Λ and may become time and scale dependent. Therefore, being able to measure a deviation of γ from its predicted value in ΛCDM is really important in the confirmation or not of the ΛCDM model.

Euclid's forecasts with regards to the parameter γ show that it will be possible to discriminate any model with departures from GR which has a difference in γ greater that $\simeq 0.03$ (95% C.L.) [71].

Another interesting parameter Σ_0 will be studied in Euclid, not only for its physical meaning, but also for its insensitiveness on the γ parameter. The parameter Σ_0 comes from the expansion $\Sigma = 1 + a\Sigma_0$, where Σ is defined as the parameter in the following form of Poisson equation (3.21).

Together with Σ, also the μ and $\tilde{\gamma}$ parameters are studied in the Euclid mission. They are defined in a phenomenological way by the following equations (3.20) and (3.17). Some forecast for the parameters Σ and μ indicates that, for simple models for the redshift evolution of Σ and μ, both quantities will be measured up to 20% accuracy [72].

One of the most interesting parameter to measure would be the $\tilde{\gamma}$ one, because it is the relation between the two gravitational potentials Φ and Ψ and a value different from 1 would mean a departure from GR effect. However, the $\tilde{\gamma}$ parameter is the most difficult to measure with high accuracy: this is due to the fact that it is not directly probed by the observables and it is effectively derived from the other two parameters.

Parameters Σ and μ will be measured with greater precision by Euclid than in the past. There are some forecasts about the measurements of these two parameters studied within the so called Horndeski models [34]. Horndeski models are a generalization of GR, that entail large classes of modified gravity models. Every model proposes a specific relation for Σ and μ, and it is expected that observations of Euclid will help in the selection of models constraining these two parameters.

For example, it has been shown that Horndeski models would be completely ruled out if the measurement of $\Sigma - 1$ and $\mu - 1$ would be found of opposite sign at any redshift or scale [34]. A signal in this direction was given by *Planck* measurement [19] that indicate $\mu < 1$ and $\Sigma > 1$, but the statistical significance is not sufficient to make definite statements. However, this is an example of the power of some of these parameters in the selection of modified gravity models.

Chapter 4

Mimetic Dark Matter
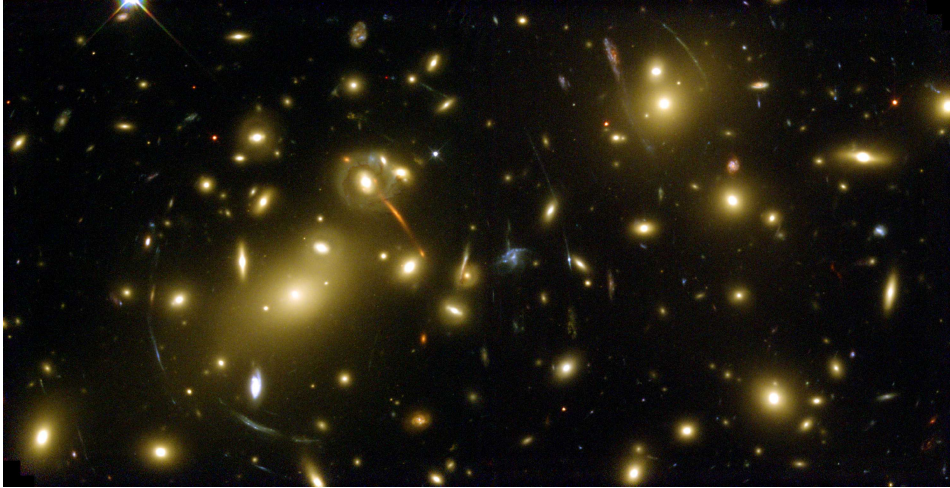
Figure 4.1: *Strong gravitational lensing as observed by the Hubble Space Telescope in Abell 1689 indicates the presence of dark matter.*

In this Chapter we describe the main aspects of the Mimetic scenario, originally proposed as a new theory from a redefinition of the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric and a scalar field ϕ. The fundamental structure of this theory is discussed in the original article of Chamseddine and Mukhanov [1]. In the second part of this Chapter, we present a deeper study on the cosmological implications of this theory proposed by Chamseddine, Mukhanov and Vikman in [2] the following year, in which very interesting properties of this theory appear. The lastest generalizations of the Mimetic scenario, applied to very general scalar-tensor theories, will be studied in the next Chapters.

4.1 Mimetic Dark Matter

In the original article of Mimetic Dark Matter [1], the authors start considering a physical metric $g_{\mu\nu}$ to be a function of a scalar field ϕ and an auxiliary metric $\tilde{g}_{\mu\nu}$, defined by

$$g_{\mu\nu} = (\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)\tilde{g}_{\mu\nu} \equiv P\,\tilde{g}_{\mu\nu}\,. \tag{4.1}$$

This redefinition of the physical metric is applied to the Hilbert-Einstein action with $S = S_{HE} + S_m$ where $S_m = -\frac{1}{2}\int d^4x\,\sqrt{-g}\mathcal{L}_m$ is action of matter and

$$S_{HE} = -\frac{1}{2}\int d^4x\,\sqrt{-g}R\,, \tag{4.2}$$

action of the physical metric, considering the physical metric as a function of a scalar field ϕ and the auxiliary metric $\tilde{g}_{\mu\nu}$.

Variation of the action is then given, similarly to GR, by

$$\delta S = \int d^4x \frac{\delta S}{\delta g_{\alpha\beta}}\delta g_{\alpha\beta}\,, \tag{4.3}$$

in which we remember the following definitions

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}\,,$$

$$G^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}}\,,$$

where $T^{\mu\nu}$ is the energy-momentum tensor and $G^{\mu\nu}$ is the Einstein tensor.

Taking into account also that

$$g = \det g_{\mu\nu} = e^{\mathrm{tr}(\log g_{\mu\nu})}\,,$$

and hence

$$\delta g = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu}\,.$$

This implies that

$$\delta\sqrt{-g} = -\frac{\delta g}{2\sqrt{-g}} = -\frac{gg^{\mu\nu}\delta g_{\mu\nu}}{2\sqrt{-g}} = -\frac{\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}}{2}\,,$$

and the action becomes

$$\delta S_{HE} = \int d^4x\frac{\delta S}{\delta g_{\alpha\beta}}\delta g_{\alpha\beta} = -\frac{1}{2}\int d^4x\sqrt{-g}\left((R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta})\delta g_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right).$$

The term proportional to $\delta R_{\alpha\beta}$ vanishes upon integration by virtue of Gauss' theorem since $\delta\Gamma$ vanishes at the boundary, while collecting the terms proportional to $\delta g_{\alpha\beta}$ one recovers the famous form of the Einstein tensor as

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\,.$$

So (4.3) becomes

$$\delta S = -\frac{1}{2}\int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta})\delta g_{\alpha\beta}\,. \tag{4.4}$$

The variation $\delta g_{\alpha\beta}$ can now be expressed in terms of the variation of the auxiliary metric $\delta \tilde{g}_{\alpha\beta}$ and the scalar field $\delta\phi$, and takes the form

$$
\begin{aligned}
\delta g_{\alpha\beta} &= P \delta \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} \delta P \\
&= P \delta \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} (\delta \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2 \tilde{g}^{\mu\nu} \delta(\partial_\mu \phi) \partial_\nu) \\
&= P \delta \tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta} (- \tilde{g}^{\kappa\mu} \tilde{g}^{\lambda\nu} \delta \tilde{g}_{\mu\nu} \partial_\kappa \phi \partial_\lambda \phi + 2 \tilde{g}^{\kappa\lambda} \partial_\kappa \delta\phi \partial_\lambda \phi) \\
&= P \delta \tilde{g}_{\mu\nu} (\delta^\mu_\alpha \delta^\nu_\beta - g_{\alpha\beta} g^{\kappa\mu} g^{\lambda\nu} \partial_\kappa \phi \partial_\lambda \phi) + \\
&\quad + 2 g_{\alpha\beta} g^{\kappa\lambda} \partial_\kappa \delta\phi \partial_\lambda \phi \,,
\end{aligned}
$$

which implies that (4.4) becomes

$$
\begin{aligned}
\delta S = -\frac{1}{2} \int d^4 x \sqrt{-g} (G^{\alpha\beta} - T^{\alpha\beta}) \cdot \\
\cdot \left[P \delta \tilde{g}_{\mu\nu} (\delta^\mu_\alpha \delta^\nu_\beta - g_{\alpha\beta} g^{\kappa\mu} g^{\lambda\nu} \partial_\kappa \phi \partial_\lambda \phi) + 2 g_{\alpha\beta} g^{\kappa\lambda} \partial_\kappa \delta\phi \partial_\lambda \phi \right] \,.
\end{aligned}
$$

Imposing the variational condition $\delta S \equiv 0$, the corresponding equations of motion become

$$
(G^{\mu\nu} - T^{\mu\nu}) - (G - T) g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \phi \partial_\beta \phi = 0 \tag{4.5}
$$

for the term with variation with respect to $\delta \tilde{g}_{\mu\nu}$, and

$$
\frac{1}{\sqrt{-g}} \partial_\kappa (\sqrt{-g} (G - T) g^{\kappa\lambda} \partial_\lambda \phi) = \nabla_\kappa ((G - T) \partial^\kappa \phi) = 0 \tag{4.6}
$$

after integration by parts of the term multiplied by $\partial_\kappa \delta\phi$.

At this point we see that the metric $g_{\mu\nu}$ of (4.1) is invariant with respect to the conformal transformation of the auxiliary metric $\tilde{g}_{\mu\nu}$, that is, $g_{\mu\nu} \to g_{\mu\nu}$ when $\tilde{g}_{\mu\nu} \to \Omega^2 \tilde{g}_{\mu\nu}$. Thus, in these equations we have reformulated the GR equations by a Weyl transformation in term of the auxiliary metric and the field ϕ. In fact, the auxiliary metric $\tilde{g}_{\mu\nu}$ does not appear in these equations by itself but only via the physical metric $g_{\mu\nu}$, while the scalar field ϕ enters the equations explicitly.

Mimetic constraint

One of the most important feature of Mimetic Gravity is the condition that characterizes it and that we now discuss. As it follows from (4.1), because of the simple relation

$$
g^{\mu\nu} = \frac{1}{P} \tilde{g}^{\mu\nu},
$$

the scalar field satisfies the **mimetic constraint equation**

$$
g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1 \,. \tag{4.7}
$$

More than in the first article [1], this condition will become very important in the following exposition of the theory and will be the fundamental property of a so called "Mimetic" theory. In fact, one of the reason for which the mimetic adjective has been used is that this theory can provide an explanation of the presence of dark matter as a simple modification of gravity. This point is clearly seen, and it will be studied in a deeper way in the following, if we take the trace of equations (4.5)

$$
(G - T)(1 - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) = 0 \,, \tag{4.8}
$$

and note that this equation is identically satisfied also if $G - T \neq 0$, thanks to the mimetic constraint (4.7). The $G - T$ term, if it was zero, it would impose the well-known Einstein's equations. But its free value, generically different from zero, allows us to find nontrivial solutions to the gravitational field equations even in absence of matter, i.e. when $T^{\mu\nu} \equiv 0$. We will see in the following that this feature is fundamental in the description of Dark Matter effect as a modification of gravity.

Mimetic fluid

It is important to understand the role of the scalar field in the Mimetic scenario. To this goal, we remember the expression of the energy momentum tensor for a perfect fluid

$$\tilde{T}^{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu} \,, \tag{4.9}$$

in which we call p the pressure, ε the energy density, and u^{μ} the 4-velocity with the normalization condition $u^{\mu}u_{\mu} = -1$.
This general tensor can be compared with a tensor that describe the behaviour of the scalar field ϕ. To see this, take equation (4.5) and write it in the following way

$$G^{\mu\nu} - T^{\mu\nu} = (G - T\ )g^{\mu\alpha}g^{\nu\beta}\partial_{\alpha}\phi\partial_{\beta}\phi \equiv \tilde{T}^{\mu\nu} \,. \tag{4.10}$$

The right hand side of the previous equation can be expressed as a tensor $\tilde{T}^{\mu\nu}$, so we obtain

$$G^{\mu\nu} - T^{\mu\nu} = \tilde{T}^{\mu\nu}.$$

Now we can take the tensor in (4.9) and set it to be pressureless. This implies that it becomes equivalent to $T^{\mu\nu} \equiv \tilde{T}^{\mu\nu}$ with the identifications of $\varepsilon \equiv G - T$ and $u^{\mu} \equiv \partial^{\mu}\phi$. The comparation allows us to say that the potential motion of this form of dust can be imitated by the scalar field, that in particular takes the form of a velocity potential. The condition of normalization of the 4-velocity indicated here is equivalent to impose the mimetic constraint of equation (4.7), a fact that confirm the previous statements about the role of ϕ.

In this presentation, the last equation that we have to express is the (4.6). It can be obtained by the condition $\nabla_{\mu}\tilde{T}^{\mu}{}_{\nu} \equiv 0$, that is the conservation law of the tensor $\tilde{T}^{\mu}{}_{\nu}$. Remember that the differentiation of the mimetic constraint equation (4.7) gives

$$\begin{aligned}
\nabla_{k}\left(g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right) &= g^{\mu\nu}\left((\nabla_{k}\partial_{\mu}\phi)\partial_{\nu}\phi + \partial_{\nu}\phi(\nabla_{k}\partial_{\mu}\phi)\right) \\
&= 2\partial^{\mu}\phi\nabla_{k}\partial_{\mu}\phi \\
&\equiv 0 \,,
\end{aligned} \tag{4.11}$$

that is $\partial^{\mu}\phi\nabla_{k}\partial_{\mu}\phi = 0$ and $\nabla_{k}\partial_{\mu}\phi = \nabla_{\mu}\partial_{k}\phi$ because ϕ is scalar. So, inserting these conditions in the conservation law, it becomes

$$\begin{aligned}
\nabla_{\mu}\tilde{T}^{\mu}{}_{\nu} &= \partial_{\nu}\phi\nabla_{\mu}((G - T\ )\partial^{\mu}\phi) + (G - T\ )\partial^{\mu}\phi\nabla_{\mu}\partial_{\nu}\phi \\
&= \partial_{\nu}\phi\nabla_{\mu}((G - T\ )\partial^{\mu}\phi) \\
&\equiv 0 \,,
\end{aligned}$$

that leds to equation (4.6) with $\partial_{\nu}\phi \neq 0$ for at least one index ν.

Dark Matter

Now we can find the explicit solution of the equation (4.6) working in synchronous coordinate system where the metric takes the form $ds^2 = d\tau^2 - \eta_{ij} dx^i dx^j$, with η the 3-dimensional metric.

We now consider hypersurfaces of constant ϕ to be the same as the hypersurfaces of constant time $\phi(x^\mu) \equiv \tau$. This condition implies that (4.7)

$$1 = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \partial^\mu \phi(\tau) \partial_\mu \phi(\tau) \equiv 1$$

is satisfied, and that equation (4.6) becomes

$$\nabla_\kappa((G - T)\partial^\kappa \phi) = \partial_0(\sqrt{\det \eta}(G - T)) = 0 \, , \tag{4.12}$$

Integrating in time, we obtain that the simple solution is

$$G - T = \frac{K(x_i)}{\sqrt{\det \eta}} \, ,$$

where $K(x_i)$ is a constant of integration depending only on spatial coordinates. We can see that in the particular case of a flat Robertson-Walker Universe with $\det \eta_{ij} = a^2(\tau)\eta_{ij}$, we have

$$G - T = \frac{K(x_i)}{a^3} \, .$$

This equation, as explained before, shows that we have a form of "dark matter" without dark matter, that is dark matter is imitated by the modification of gravity instead of the presence of a true term of matter: this new form of matter has been called **Mimetic Dark Matter**. Here the quantity of this Mimetic Dark Matter is determined by the constant of integration $K(x_i)$.

Dark matter with inflation

This model have to be modified if one consider the initial conditions of the inflationary cosmology, in particular with regards to the nowdays abundance of observed dark matter. In fact, in Chapter 1 we have seen that a constraint on the inflationary period is on its time duration or, equivalently, on its effectiveness of the scale factor of at least 70 e-folds. Thus, to keep the amount of Mimetic Dark Matter given by the constant of integration $C(x_i)$ not completely negligible today, we have to set these initial conditions which do not spoil inflation.

A simple example of how introduce a mechanism to avoid an excessive decay of the Mimetic Dark Matter energy density during the exponential expansion is by introducing a coupling of the field ϕ with a function of the inflaton field $F(\varphi)$ as

$$\phi \, F(\varphi) \, .$$

By this way, equation (4.6), always in the case of synchronous coordinates and a Robertson-Walker Universe with metric $ds^2 = -d\tau^2 + \eta_{ij} dx^i dx^j$ and $\det \eta_{ij} = a^2(\tau)\eta_{ij}$, simplifies to

$$\frac{1}{\sqrt{-g}} \partial_\kappa(\sqrt{-g} \varepsilon g^{\kappa\lambda} \partial_\lambda \phi) = \frac{1}{a^3} \partial_0(a^3 \varepsilon) \, , \tag{4.13}$$

where we have used $\varepsilon = G - T$ as in the previous section.

During inflation, the function $F(\varphi)$ of the inflaton field can be assumed to changing slowly

$F(\varphi) \simeq$ const with the field φ. This allows us to integrate the previous differential equation to get

$$\varepsilon \approx -\frac{F(\varphi)}{3H} + M \exp(-3Ht) \,,$$

where we have used M costant of integration that does not matter and the scale factor as

$$a = \frac{1}{H} \exp\left(Ht\right) \,.$$

When the inflation ends, the second term decays and we obtain the approximated solution

$$\varepsilon \approx -\frac{F(\varphi)}{3H} \,.$$

It is clear from this equation that the energy density of the Mimetic Dark Matter can be modified directly from the inflaton field by the function $F(\varphi)$. Thus, inhomogeneity of the inflaton can affect **Mimetic Dark Matter**, and the resulting perturbations will be similar to adiabatic perturbations in case of real cold dark matter.

4.2   Cosmology with Mimetic Dark Matter

In 2014 the following work of Chamseddine, Mukhanov and Vikman [2] proposes a generalization of the Mimetic Dark Matter model. This minimal extension of the model is pursued by introducing an arbitrary potential $V(\phi)$ and studying the cosmological solutions in this theory by selecting the more interesting form for the potential. In particular, we show in this section that, with the appropriate choice for the potential $V(\phi)$ for the mimetic non-dynamical scalar field, we can mimic nearly any gravitational properties of the normal matter and any cosmological solution.

4.2.1   Lagrange multiplier

The first very important feature, that will be used in next Chapters for the generalization of the theory, is the fact that the derivation of the Mimetic constraint in (4.7) suggests to use it in the original action (4.2) as a constraint by employing a Lagrange multiplier. To this scope, we can consider the following action

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}R + \mathcal{L}_m + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) \right] \ . \tag{4.14}$$

The equations of motion of the action (4.14) led to

$$G_{\mu\nu} - T_{\mu\nu} + 2\lambda\partial_\mu\phi\partial_\nu\phi = 0 \tag{4.15}$$

by differentiating with respect to $\delta g_{\mu\nu}$, and

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$$

by differentiating with respect the Lagrange multiplier λ. In the previous section, we have seen that these equations of motion produces the Mimetic Dark Matter component.

This feature of the theory will be studied in a deeper way in the following. However, we can use this introduction to show how, with the same procedure, a cosmological constant $\tilde{\lambda}$ can appear in the theory.

To this goal, consider now the extended Lagrangian

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}R + \mathcal{L}_m + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + \tilde{\lambda}(\nabla_\mu V^\mu - 1) \right] \ , \tag{4.16}$$

where V^μ is a generic vector. This implies that, as before, the equation of motions become

$$G_{\mu\nu} - T_{\mu\nu} + 2\lambda\partial_\mu\phi\partial_\nu\phi + g^{\mu\nu}\tilde{\lambda} = 0 \ , \tag{4.17}$$

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1 \ , \tag{4.18}$$

and also

$$\nabla_\mu V^\mu = 1 \ , \tag{4.19}$$

$$\partial_\mu\tilde{\lambda} = 0 \ .$$

The last equation means that $\tilde{\lambda}$ is constant, and so we can identify it with the cosmological constant $\tilde{\lambda} = \Lambda$ and it becomes a constant of integration as in the case of Mimetic Dark Matter.

In this theory, the Lagrange multiplier λ is determined from the trace of the Einstein equations (4.17)

$$2\lambda = -G + T - 4\Lambda \ , \tag{4.20}$$

while the metric $g_{\mu\nu}$ by

$$(G_{\mu\nu} - T_{\mu\nu}) - (G - T)\partial_\mu\phi\partial_\nu\phi + (g_{\mu\nu} - 4\partial_\mu\phi\partial_\nu\phi)\Lambda = 0 \ , \tag{4.21}$$

and the scalar field ϕ by

$$\frac{1}{\sqrt{-g}}\partial_\kappa(\sqrt{-g}(G - T\ )g^{\kappa\lambda}\partial_\lambda\phi) = \nabla_\kappa((G - T\ )\partial^\kappa\phi) = 0 \ . \tag{4.22}$$

In this simple model, it is clear that both Dark Matter and Dark Energy can be produced by a non-dynamical scalar (4.18) and vector field (4.19) with a minimal modification of General Relativity.

Obviously, modified gravity models are proposed to look for a more general expression of the Dark Energy: a cosmological constant is not a novelty in gravity models. For this reason, we now extend this model introducing a more general potential $V(\phi)$. In the Lagrangian the presence of a cosmological constant Λ can be neglected because its effect is to shift the potential by the same value, with no appreciable implications on Physics.

4.2.2 Potential for Mimetic Matter

We can now consider the theory with the action

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - V(\phi) + \mathcal{L}_m\right] \ , \tag{4.23}$$

where we know that variation with respect to λ gives the mimetic constraint (4.7), while varying with respect to $g^{\mu\nu}$ gives

$$G_{\mu\nu} - T_{\mu\nu} - 2\lambda\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi) = 0 \ , \tag{4.24}$$

in which the new term is given by the potential $V(\phi)$.

As before, the Lagrange multiplier can be expressed by the trace of (4.24) as

$$2\lambda = G - T - 4V \ , \tag{4.25}$$

and so equation (4.24) becomes

$$G_{\mu\nu} = (G - T - 4V)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V(\phi) + T_{\mu\nu} \ . \tag{4.26}$$

As in the previous section (but now with the potential term $V(\phi)$), by taking the covariant derivative ∇^ν of equation (4.26) we obtain

$$\nabla^\nu\left((G - T - 4V)\partial_\nu\phi\right) = -V'(\phi) \ , \tag{4.27}$$

where $V'(\phi) = \partial V/\partial\phi$ and where we have used the Bianchi identity $\nabla^\nu G_{\mu\nu} = 0$, the conservation law of the energy-momentum tensor $\nabla^\nu T_{\mu\nu} = 0$ and simplifications in (4.11).

Analogously to the previous section, we can compare equations (4.26) with the equivalent Einstein equations with an extra ideal fluid with the energy tensor

$$\tilde{T}_{\mu\nu} = (\tilde\varepsilon + \tilde p)\partial_\mu\phi\partial_\nu\phi - \tilde p g_{\mu\nu} \ .$$

However, in this case, we have to change the definitions of the pressure

$$\tilde{p} = -V \tag{4.28}$$

and energy density

$$\tilde{\varepsilon} = G - T - 3V \ , \tag{4.29}$$

so that we obtain

$$
\begin{aligned}
G_{\mu\nu} - T_{\mu\nu} &= \tilde{\varepsilon}\partial_\mu\phi\partial_\nu\phi + \tilde{p}\partial_\mu\phi\partial_\nu\phi - \tilde{p}g_{\mu\nu} \\
&= (\tilde{\varepsilon} + \tilde{p})\partial_\mu\phi\partial_\nu\phi - \tilde{p}g_{\mu\nu} \\
&= \tilde{T}_{\mu\nu} \ ,
\end{aligned}
$$

with the same previous conclusions about the role of the scalar field ϕ as velocity potential of the fluid.

Dark Energy

We now follow a similar procedure to the previous section in which we will find the role of the potential $V(\phi)$ and its implication on the cosmological solutions.
Consider now the Mimetic constraint

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1 \tag{4.30}$$

in a flat Robertson-Walker Universe with the metric $ds^2 = -dt^2 + a^2(t)\delta_{ik}dx^i dx^k$, assuming that ordinary matter is absent. A general solution of (4.30) is in this case

$$\phi = \pm t + A \ , \tag{4.31}$$

where A is a constant of integration that - without loss of generality - can be set to zero, obtaining the identification of the field ϕ with time

$$\phi = t \ .$$

Analogously to (4.12), we can take equation (4.26) that, remembering that the pressure and the energy density in (4.28) and (4.29) depend only on time, becomes

$$\frac{1}{a^3}\frac{d}{dt}(a^3(\tilde{\varepsilon} - V)) = -\dot{V} \ ,$$

where we took into account that in this particular case the relation $V' = \dot{V}$ holds. Now by integration in time, we have

$$\tilde{\varepsilon} = V - \frac{1}{a^3}\int a^3\dot{V}dt = \frac{3}{a^3}\int a^2 V da \ , \tag{4.32}$$

that express the energy density in terms of the potential V.
Moreover, remembering that the expression

$$\dot{\varepsilon} + 3H\varepsilon = \frac{1}{a^3}\partial_0(a^3\epsilon)$$

holds, the energy density and the pressure can be expressed in the form of the conservation law

$$\dot{\tilde{\varepsilon}} = -3H(\tilde{\varepsilon} + \tilde{p}) \ .$$

Here we note that in (4.32) the constant of integration determines the quantity of Mimetic Dark Matter, which decays as a^{-3}. Nevertheless, for a non-vanishing potential $V(\phi)$, an **extra contribution to Mimetic matter** is given by the potential $V(\phi)$. This additional component of Mimetic matter is described by the Lagrangian similarly to a cosmological constant as in (4.16), by adding a constant value of the potential $V(\phi)$. Therefore, the number of degrees of freedom in the system does not increase compared to the case of Mimetic dust, but it receives the contribution of two different components: one is the Mimetic Dark Matter, the other is given by the potential V.

We now derive the Friedmann equation for this theory, that will allow us to build a differential equation of the scale factor in relation with the potential V.

In General Relativity, the Friedmann equation is obtained from the $0 - 0$ component of the Einstein equation

$$G_{\mu\nu} = T_{\mu\nu} .$$

This equation, considering the only non-zero terms in the Ricci tensor and the Ricci scalar, leds to the well known

$$H^2 = \frac{8}{3}\pi G\rho ,$$

with G the gravitational constant and ρ energy density.

In this theory, instead, we set the reduced Planck mass to one and the form of the Friedmann equation changes because of the generalization

$$G_{\mu\nu} = T_{\mu\nu} + \tilde{T}_{\mu\nu}$$

in which, in this particular case with no matter $T_{\mu\nu} \equiv 0$ but $\tilde{T}_{\mu\nu} \neq 0$, we obtain

$$\begin{aligned} H^2 &= \frac{1}{3}\tilde{\varepsilon} \\ &= \frac{1}{a^3}\int a^2 V da . \end{aligned} \tag{4.33}$$

This equation, for a given $V(\phi) = V(t)$, could be solved for $a(t)$.

However, instead of solving this integral equation, we can reduce it to an ordinary differential equation. Multiplying equation (4.33) by a^3 and differentiating it with respect to time, it becomes

$$2\dot{H} + 3H^2 = V(t) . \tag{4.34}$$

If we introduce the new variable y as

$$y = a^{\frac{3}{2}} ,$$

then we obtain the linear differential equation

$$\ddot{y} - \frac{3}{4}V(t)y = 0 \ \ . \tag{4.35}$$

It can be solved in $y(t)$, or equivalently in $a(t)$, in function of the given potential $V(t)$. This equation, under the previous hypothesis, allows us easily to find cosmological solutions, as follows.

4.2.3   Cosmological solutions

In this section, we analize some simple examples of potentials to show how cosmological solutions can be expressed in the Mimetic theory.

First, we now consider the potential

$$V(\phi) = \frac{\alpha}{\phi^2} = \frac{\alpha}{t^2} \ ,$$

where α is a constant and we have done the identification $\phi = t$ as in the previous section. Equation (4.35) becomes

$$\ddot{y} - \frac{3\alpha}{4t^2} y = 0 \ , \tag{4.36}$$

whose general solution can be evaluated as

$$y = \begin{cases} m \ t^{\frac{1}{2}} \cos\left(\frac{1}{2}\sqrt{|1+3\alpha|} \ln t + n\right) & \text{for } \alpha < -1/3 \ , \\[2mm] m \ t^{\frac{1}{2}(1+\sqrt{1+3\alpha})} + n \ t^{\frac{1}{2}(1-\sqrt{1+3\alpha})} & \text{for } \alpha \geq -1/3 \ , \end{cases} \tag{4.37}$$

where m and n are constants of integration.

Here we can see some general behaviours of these solutions. Remember that the pressure is

$$\tilde{p} = \frac{\alpha}{\phi^2} \ . \tag{4.38}$$

For large negative $\alpha \ll 0$ (and so for large positive pressure), the solution osclillates in time due to the term proportional to $\cos(\ln t)$ and its amplitude grows in time because it is $\propto t^{\frac{1}{2}}$. So this case describes an oscillating flat Universe with singularities and oscillation growing with time.
In the case in which the value of α is $\alpha \geq -1/3$, it is possible to express an interesting relation that describes the equation of state of the mimetic matter.
Assuming that the constant is $m \neq 0$, one can show that the general solution for the scale factor is

$$a(t) = t^{\frac{1}{3}(1+\sqrt{1+3\alpha})} \left(1 + \frac{n}{m} t^{-\sqrt{1+3\alpha}}\right)^{2/3} \ ,$$

Substitute this solution in (4.33) and remembering the expression for the pressure (4.38), one can find the equation of state for the Mimetic Dark Matter of this theory

$$w = \frac{\tilde{p}}{\tilde{\varepsilon}} = -3\alpha \left(1 + \sqrt{1+3\alpha} \frac{1 - \frac{n}{m} t^{-\sqrt{1+3\alpha}}}{1 + \frac{n}{m} t^{-\sqrt{1+3\alpha}}}\right)^{-2} \ ,$$

that can depend on time in general or, in the limit of small and large t, to approach a constant.
Let us consider some relevant cases. When $\alpha = -1/3$ we have an equation of state described by $\tilde{p} = \tilde{\varepsilon}$ and $a \propto t^{1/3}$, while in the case $\alpha = -1/4$ it corresponds to ultra-relativistic fluid with $\tilde{p} = \frac{1}{3}\tilde{\varepsilon}$ at large time and $\tilde{p} = 3\tilde{\varepsilon}$ when $t \to 0$ if $n \neq 0$. The case in which α is very small then we have Mimetic dark matter with negligible pressure.
Finally, positive values of α leds to negative pressure, and if $\alpha \gg 1$ the equation of state approaches the cosmological constant with $\tilde{p} = -\tilde{\varepsilon}$.

This example is useful to understand that this model can describe almost every cosmological history in relation with the given potential V, introducing also the two dark components that are mimicked by the same potential V and the scalar field ϕ.

More general results can be found in [2], where a generalization of the previous case is presented: it is considered an arbitrary power law potential of the form $V(\phi) = \alpha\phi^n = \alpha t^n$. The potential can be inserted in (4.35), that allows for a very general solution in terms of Bessel functions in the variable y.

4.2.4 Mimetic Matter as quintessence

Quintessence is an hypothetical form of Dark Energy, governed by a scalar field, postulated as an explanation of the observation of an accelerating rate of expansion of the Universe, rather than due to a true cosmological constant. Quintessence differs from the cosmological constant in that it is dynamic, that is it changes over time. We now see an example that shows how the Mimetic model can describe the quintessence.

Differently from previous sections, we now consider a Robertson-Walker Universe in which a dominant component of matter is present: this means that $T^{\mu\nu} \neq 0$, and p and ϵ respect a generic equation of state of the form $p = w\varepsilon$ with w constant. As well-known, in this case the scale factor is $a \propto t^{\frac{2}{3(1+w)}}$. In this example we consider, for simplicity, the same potential as before, given by $V(\phi) = \frac{\alpha}{\phi^2}$.

Assuming that $\phi = t$, then the energy density of the Mimetic matter given by (4.32) is given by

$$
\begin{aligned}
\tilde{\varepsilon} &= V - \frac{1}{a^3}\int a^3\dot{V}\,dt \\
&= \frac{\alpha}{t^2} + \frac{2\alpha}{a^3}\int \frac{a^3}{t^3}\,dt \\
&= \frac{\alpha}{t^2} + \frac{2\alpha}{a^3}\frac{1+w}{2-2(1+w)}t^{\frac{2-2(1+w)}{1+w}} \\
&= \frac{\alpha}{t^2} - \frac{\alpha}{t^2}\frac{w+1}{w} \\
&= -\frac{\alpha}{wt^2}\ ,
\end{aligned}
\tag{4.39}
$$

setting the constant of integration in (4.32) to zero.
Remembering that $p = w\varepsilon$ and considering that the pression of the mimetic matter (4.28) is $\tilde{p} = -V = -\alpha/t^2$, it results that

$$
\tilde{p} \simeq w\tilde{\varepsilon}\ ,
$$

similarly to the matter equation of state: that is, the Mimetic matter imitates the equation of state of the dominant matter. However, since the total energy density (analogously as

seen before from (4.33)) is

$$\varepsilon = 3H^2$$
$$= 3\left(\frac{\dot{a}}{a}\right)^2$$
$$= 3\frac{2^2}{3^2(1+w)^2}\left(\frac{a}{at}\right)^2$$
$$= \frac{4}{3(1+w)^2}\frac{1}{t^2}$$
$$= \frac{4}{3(1+w)^2 t^2} \ , \tag{4.40}$$

comparing (4.39) with (4.40) one can see that Mimetic matter can be subdominant only if $\frac{\alpha}{w} \ll 1$.
This scenario can be generalized if one consider $\phi = t + t_0$ (so the previous constant A in (4.31) is taken non-zero and equal to t_0). Mimetic matter behaves as subdominant in the regime for $t < t_0$ and corresponds to a cosmological constant, whilst from $t > t_0$ it starts to behave similarly to a form of dominant matter.

4.2.5 Mimetic Matter as an inflaton

Mimetic matter can also be used to build inflationary solutions. Using equation (4.35), we can look for potentials with the desidered form

$$V(\phi) = \frac{4}{3}\frac{\ddot{y}}{y} \ . \tag{4.41}$$

Remembering that $y = a^{\frac{3}{2}}$, the appropriate potential can ipotetically get any behaviour of the scale factor during the inflationary period and after of it.
For example, a potential as

$$V(\phi) = \frac{\alpha\phi^2}{\exp(\phi) + 1} \ ,$$

with positive values of α, can describe inflation and it guarantees a graceful exit. At large negative values of t, one can find that the scale factor goes as $a \propto \exp(-\sqrt{\frac{\alpha}{12}}t^2)$, whilst it is proportional to $t^{2/3}$ for positive t.

4.2.6 Cosmological perturbations

From this section, we will consider cosmological perturbations of Mimetic matter model. We can consider the metric perturbed in the Newtonian gauge as

$$ds^2 = -(1+2\Psi)dt^2 + a^2(1-2\Phi)\delta_{ij}dx^i dx^j \ ,$$

where Ψ is the Newtonian gravitational potential. We can take the mimetic constraint (4.7)

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$$

and perturbing the scalar field to first order we obtain

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$$
$$g^{\mu\nu}\left[(\partial_\mu\phi_0 + \partial_\mu\delta\phi)(\partial_\nu\phi_0 + \partial_\nu\delta\phi)\right] = 1$$
$$g^{\mu\nu}\left[\partial_\mu\phi_0\partial_\nu\phi_0 + \partial_\mu\delta\phi\partial_\nu\phi_0 + \partial_\mu\delta\phi\partial_\nu\delta\phi + \dots\right] = 1 .$$

From now on, we will consider the perturbation of the scalar field as $\phi = t + \delta\phi$. Thus, the first order perturbation of the previous equation becomes

$$g^{00}\left[1 + \dot{\delta\phi} + \dot{\delta\phi} + \dots\right] = 1$$
$$(1 - 2\Psi)\left[1 + 2\dot{\delta\phi}\right] = 1$$
$$-2\Psi + 2\dot{\delta\phi} = 0 ,$$

and so finally

$$\Psi = \dot{\delta\phi} . \tag{4.42}$$

The $0 - i$ components of linearized Einstein equations

$$G^{(1)}_{0i} = \tilde{T}^{(1)}_{0i}$$

can be found to be

$$\partial_i\left(\dot{\Psi} + H\Psi\right) = \frac{1}{2}(\tilde{\varepsilon} + \tilde{p})\partial_i\delta\phi . \tag{4.43}$$

Consider now that from (4.34)

$$2\dot{H} + 3H^2 = V$$
$$2\dot{H} + \tilde{\varepsilon} = -\tilde{p}$$

and hence $\tilde{\varepsilon} + \tilde{p} = -2\dot{H}$. Integrating in the spatial coordinates the previous (4.43)

$$\dot{\Psi} + H\Psi = -2\dot{H}\ \delta\phi$$

and substituting Ψ from (4.42) we obtain the following evolution equation for $\delta\phi$

$$\ddot{\delta\phi} + H\dot{\delta\phi} + \dot{H}\delta\phi = 0 . \tag{4.44}$$

The general solution of this differential equation is

$$\delta\phi = A(x_i)\frac{1}{a}\int a\, dt , \tag{4.45}$$

with $A(x_i)$ constant of integration. The corresponding gravitational potential from (4.43) is

$$\Psi = \dot{\delta\phi} = A\frac{d}{dt}(\frac{1}{a}\int a\, dt) = A\left(1 - \frac{H}{a}\int a\, dt\right) . \tag{4.46}$$

This is of the same form of a well-known general solution of the perturbation $\delta\phi$ in the case of large-scale ($k \ll aH$) cosmological perturbations when one can neglect the spatial derivative terms that are multiplied by the speed of sound for normal hydrodynamical fluid. So, in the general case, neglecting this term, one can assume that al large-scales the speed of sound vanishes. However, in our case the previous solution is valid for all scales, not only in the large-scale limit, and the vanishing speed of sound is a phemomenon that occurs at

all cosmological scales. Because of this, one can say that these perturbations behave as a dust with **vanishing speed of sound** even for Mimetic matter with non-vanishing pressure $\tilde{p} \neq 0$. So we cannot define the quantum fluctuations of Mimetic matter in the usual way because it would be $\delta\phi_k \equiv const$, and therefore the Mimetic inflation considered above would fail in explaining the large-scale structure formation as originated from quantum fluctuations.

A more detailed explanation of the solution proposed by Chamseddine and Mukhanov to the vanishing speed of sound is presented in the following lines.

4.2.7   Vanishing speed of sound

Here we give a brief explanation of why the vanishing sound speed can be a problem in the Mimetic theory.

It is well-known [58] that, to lowest order in the slow-roll parameters, the tensor-to-scalar ratio r is determined by a combination of the sound speed c_s and the slow-roll parameter ε

$$r = 16\varepsilon c_s \ . \tag{4.47}$$

In Chapter 1, we have shown that in the case of an inflation described by a single slow-rolling scalar field ϕ, the tensor to scalar ratio is given by

$$r = 16\varepsilon \ .$$

This means that, in the canonical inflation, we expect that $c_s \equiv 1$. This results in the well known consistency relation given by $r = -8n_T$.

The more general relation (4.47) is useful in the determination of c_s in non-canonical inflation scenarios, where c_s can be $c_s \neq 1$ and, generally, it can be smaller than one $c_s \ll 1$.

In non-canonical inflation, the value of c_s cannot only differ from 1, but it can change in time. In these models, in fact, it is often useful to introduce a new parameter s

$$s = \frac{\dot{c}_s}{Hc_s}$$

to describe the running of the sound of speed, that becomes a new observable of the inflationary Universe. The parameter s is constrained by the slow-roll conditions, so that it can be shown that $|s| \ll 1$.

Notice that, if we impose $c_s \equiv 0$ in the relation (4.47), it means by definition of r

$$r = \frac{\Delta_T}{\Delta_{\frac{\delta\rho}{\rho}}}$$

that the ratio between tensorial perturbations Δ_T and scalar perturbations is zero. This fact could be in relation with the blowing up of scalar perturbations and of primordial non-Gaussianity [58]. The study of CMB radiation allow us to provide constraints on the primordial non-Gaussianity, and in particular to construct a lower limit of the sound speed c_s.

Recently in [19], CMB anisotropies observations confirmed that there are lower values of c_s at least of $c_s > 0.021$ (95% C.L.). So, the speed of sound can be very small but not exactly vanishing.

Moreover, notice that a vanishing speed of sound for the field ϕ can produce problems in the quantization of same the field.

So, in the following section will we present an extension of the mimetic model which allows for the nontrivial speed of sound. This can be achieved by adding higher-order-derivative terms to the action without increasing the number of degrees of freedom in the system.

One can see that in this case the perturbations can have new observational features: in particular it is possible to strongly suppress the gravitational waves from inflation. It would be very interesting to analyze whether one can observationally distinguish Mimetic Inflation from other models.

4.2.8   Switching on a non-vanishing speed of sound

In the following, we will see that introducing an extra term in the main Lagrangian of the theory, we can produce a finite speed of sound, allowing us to study quantum cosmological perturbations of the model. We will see also that this finite sound speed can supress structures on small scales and have other interesting phenomenological consequences [24].

The vanishing speed of sound can be overhelmed in the following way. Consider the following extra term in the Lagrangian (4.23)

$$+\frac{1}{2}\gamma(\Box\phi)^2, \tag{4.48}$$

where γ is a constant and $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.

The action is

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}R + \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) - V(\phi) + \frac{1}{2}\gamma(\Box\phi)^2\right], \tag{4.49}$$

where we neglect the matter Lagrangian for the moment.

Varying with respect to the metric $g_{\mu\nu}$ we obtain

$$G_{\mu\nu} - \tilde{T}_{\mu\nu} = 0 , \tag{4.50}$$

where

$$\tilde{T}_{\mu\nu} = \left(V + \gamma(\phi_{,\beta}\chi^{,\beta} + \frac{1}{2}\chi^2)\right)\delta_{\mu\nu} + 2\lambda\phi_{,\nu}\phi_{,\mu} - \gamma(\phi_{,\nu}\chi_{,\mu} + \chi_{,\nu}\phi_{,\mu}) , \tag{4.51}$$

and $\chi = \Box\phi$.

The better way is now to solve equations (4.50) directly. Remember that the general solution of the mimetic constraint (4.7) in a Robertson-Walker Universe is $\phi = t + A$, with A constant of integration. Thus the relation $\chi = \Box\phi$ simplifies to

$$\chi = \Box\phi = \ddot{\phi} + 3H\dot{\phi} = 3H .$$

Now we can write down Einstein equations: the $0 - 0$ component reduces to

$$3H^2 = V + \gamma\dot{H} + \frac{9}{2}H^2 + 2\lambda - \gamma\dot{H} - \gamma\dot{H}$$

$$3H^2 = V - 3\gamma\dot{H} + \frac{9}{2}\gamma H^2 + 2\lambda$$

$$H^2 = \frac{V}{3} - \gamma\dot{H} + \frac{3}{2}\gamma H^2 + \frac{2}{3}\lambda , \tag{4.52}$$

while the $i - j$ component of the equations is

$$2\dot{H} + 3H^2 = V + \frac{3\gamma}{2}(2\dot{H} + 3H^2) \ , \tag{4.53}$$

where only the δ term survives, so that it becomes

$$2\dot{H} + 3H^2 = \frac{2}{2 - 3\gamma}V, \tag{4.54}$$

Notice that inserting equation (4.53) in (4.52), one gets

$$\lambda = (3\gamma - 1)\dot{H} \ . \tag{4.55}$$

Equations (4.34) and (4.54) differ only by a normalization factor of potential V in terms of γ, and for $\gamma = 0$ we return to (4.34): thus the modification of the potential is given by the extra term $(\Box\phi)^2$, and it is of order ~ 1.

This implies that the cosmological solutions derived before for homogeneous Universe are modified only of a tiny normalization factor. However, the main consequence of this extra term is on the large-scales: if fact, it affects the behavior of the short wave cosmological perturbations, as we now see.

Considering the field ϕ perturbed as

$$\phi = \phi_0 + \delta\phi = t + \delta\phi \ ,$$

the $0 - i$ component of the energy-momentum tensor (4.44) at first order in perturbation theory is

$$\begin{aligned}
\delta\tilde{T}_{i0} &= 2\lambda\phi_{,0}\phi_{,i} - \gamma(\phi_{,0}\chi_{,i} + \chi_{,0}\phi_{,i}) \\
&= 2\lambda\delta\phi_{,i} - 3\gamma(\dot{H}\delta\phi_{,i} - \delta\chi_{,i}) + \dots \quad ,
\end{aligned} \tag{4.56}$$

where the dots mean that we neglected next order terms. Consider that the term $\delta\chi$ is

$$\begin{aligned}
\delta\chi &= \delta(\Box\phi) \\
&= \delta(g^{\mu\nu}\partial_\mu\partial_\nu\phi) \\
&= \delta g^{\mu\nu}\,\partial_\mu\partial_\nu\phi + \Box\delta\phi \\
&= -4\dot{\Psi} - 6H\Psi + \ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{\Delta}{a^2}\delta\phi \\
&= -3\ddot{\delta\phi} - 3H\dot{\delta\phi} - \frac{\Delta}{a^2}\delta\phi \ ,
\end{aligned} \tag{4.57}$$

where the Δ is the Laplacian and where we used $\Phi = \dot{\delta\phi}$ as in (4.42).

Now, the (4.56) becomes

$$\ddot{\delta\phi} + H\dot{\delta\phi} - \frac{c_s^2}{a^2}\Delta\delta\phi + \dot{H}\,\delta\phi = 0 \ , \tag{4.58}$$

where

$$c_s^2 = \frac{\gamma}{2 - 3\gamma} \ . \tag{4.59}$$

As we said before, the effect on the small scales of the term (4.48) is that equation (4.59) is different from (4.44) only by the presence of the gradient terms multiplied by the speed of

sound c_s.

Let us solve the (4.59). In the Fourier space, (4.58) becomes

$$\delta\phi_k'' + \left(c_s^2 k^2 + \frac{a''}{a} - 2\left(\frac{a'}{a}\right)^2\right)\delta\phi_k = 0 \ , \tag{4.60}$$

where the $' = \frac{\partial}{\partial\eta}$ is the derivative with respect to the conformal time.

Consider now the large-scale limit with $c_s k\eta \gg 1$. In (4.2.8) time derivative terms can be neglected

$$\delta\phi_k'' + c_s^2 k^2 \delta\phi_k = 0 \ .$$

This equation can be solved with the plane wave equation to get

$$\delta\phi_k \propto e^{\pm ic_s k\eta} \ , \tag{4.61}$$

Instead, in the case of small scales with $c_s k\eta \ll 1$, the spatial term can be neglected $k^2 \ll a^2 H^2$ and one obtains the solution

$$\delta\phi = A\frac{1}{a}\int a^2 d\eta \ . \tag{4.62}$$

where we remember that $a\delta\eta = \delta t$ and $A(x^i)$ is a constant of integration depending only on the spatial coordinates.

Moreover, one can show that the $\delta\phi_k$ behaves like

$$\delta\phi_k \sim \sqrt{\frac{c_s}{\gamma}}\ k^{-3/2} \ ,$$

that implies a flat power-spectrum for $\delta\phi_k$ in the small scales limit. This means that, when k is very large and we analize the limit of small wavelenght, the fluctuation $\delta\phi_k$ approches zero.

The amplitude of quantum fluctuations A can be fixed by matching method. This implies to consider, in inflationary period when $t \simeq \frac{1}{H}$, the solutions (4.61) and (4.62) at $c_s k\eta \sim 1$, so that

$$A_k \sim \sqrt{\frac{c_s}{\gamma}}\ \frac{\tilde{H}}{k^{3/2}} \ , \tag{4.63}$$

where we denote $\tilde{H} = H|_{c_s k \sim Ha}$. This means that $\delta\phi_k$ oscillates with decreasing amplitude as k grows.

Therefore the typical amplitude of the gravitational potential

$$\Psi_k \sim \dot{\delta\phi}_k \sim a\delta\phi' \sim A_k$$

in comoving scales $\lambda \sim 1/k$ after inflation is

$$\Psi_\lambda \sim A_k\ k^{3/2} \sim \sqrt{\frac{c_s}{\gamma}}\ \tilde{H} \ , \tag{4.64}$$

and one obtains that

$$\text{for} \ \ c_s \ll 1 \ : \ \ \Psi_\lambda \sim \frac{\tilde{H}}{\sqrt{c_s}} \ . \tag{4.65}$$

On the other hand,

$$\text{for} \quad c_s \gg 1 \ : \quad \Psi_\lambda \sim \sqrt{c_s}\, \tilde{H} \ , \tag{4.66}$$

which is $\sqrt{c_s}$ enhanced with respect to the amplitude of the gravity waves $h_\lambda \sim \tilde{H}$. Notice that in this Mimetic model the scalar perturbations are always larger than the tensor perturbations.

4.3 Imperfect Dark Matter

Recently in [24] and [26] a generalization of the previous term $\frac{1}{2}\gamma(\Box\phi)^2$ has been proposed in which $\gamma = \gamma(\phi)$ is a function of the field ϕ and the action becomes

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}R + \frac{\lambda}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1\right) + \frac{\gamma(\phi)}{2}(\Box\phi)^2 \right] \ , \tag{4.67}$$

where we neglect the potential V.

This theory is called **Imperfect Dark Matter** (IDM in what follows). The Lagrange multiplier λ enforces the constraint $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$.

We now discuss briefly an important reason for which it is useful to introduce the term $\gamma(\phi)$ instead of the constant γ. We know that - almost at linear level, dark matter is well described by dust. However, this description should be modified in the non-linear regime: the reason is that dust develops caustic singularities, i.e., physical quantities such as the velocity dispersion and the energy density blow up at finite time [26]. The introduction of higher-derivative terms as (4.45) is useful to avoid this problem. Thus, in the large part of IDM evolution, one can assume that the function $\gamma(\phi)$ is constant. This condition implies that the model is invariant under a shift symmetry of the form $\phi \to \phi + b$, where b is a constant with respect to ϕ, and this fact gives origin to a Noether charge density that redshifts away as a^{-3}.

When the factor (4.45) is dominant at background level, the energy density of IDM is equal to the Noether charge density. Thus, in this regime the cosmological evolution of IDM and that of dust are similar.

The degeneracy gets broken when the γ-term becomes sensible at first order perturbations and it can produce a constant sound speed $c_s^2 \simeq \gamma$. This phenomenon sets a cut-off on the power spectrum at sufficiently small scales: below the sound horizon energy density perturbations cannot grow and one can assume that they are suppressed as predicted by CDM scenarios. In particular, setting $\gamma \sim 10^{-9}$, one can suppress the growth of structures with the comoving wavelength $\lesssim 100$ kpc. For those small values of the parameter γ, the linear evolution of IDM perturbations is analogous to that of CDM given that they start from the same initial conditions.

The main problem is that, if the shift symmetry is exact at all the times, the Noether charge density is exponentially reduced and IDM cannot be the dominant component of the invisible matter, constituting only a small $\mathcal{O}(\gamma)$ fraction of the overall DM during the dust dominated epoch.

However, in this scenario the issue can be solved considering that the shift symmetry must be broken at some regime, taking place at the early stages of the Universe, in the radiation dominated era. In IDM model, this is realized by promoting the constant γ to the function $\gamma(\varphi)$.

4.4 Why a simple redefinition is so important?

Here we give an explanation of the apparent paradox that a simple reparametrization of the metric $g_{\mu\nu}$ in terms of the scalar field ϕ of the form

$$g_{\mu\nu} = (\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)\tilde{g}_{\mu\nu} \equiv P\,\tilde{g}_{\mu\nu} \tag{4.68}$$

can lead to extra new solutions of equations of motion which differ from those of the original GR equations $G_{\mu\nu} = T_{\mu\nu}$, recalling the main arguments of [6].

As we have seen, the physical metric $g_{\mu\nu}$ in the theory (4.2) is conformally invariant with respect to the metric $\tilde{g}_{\mu\nu}$: that is, $g_{\mu\nu} \to g_{\mu\nu}$ when $\tilde{g}_{\mu\nu} \to \Omega^2\tilde{g}_{\mu\nu}$. Thus it results that this theory has local Weyl invariance with respect to the transformation of the metric: we can define such a transformation Δ_σ with an arbitrary function $\sigma(x)$

$$\Delta_\sigma g_{\mu\nu}(x) \equiv \sigma(x)\,g_{\mu\nu}(x)\,, \tag{4.69}$$

under which the action of the theory is invariant

$$\Delta_\sigma S[\,g_{\mu\nu}(\tilde{g}_{\mu\nu},\phi),\phi\,] = 0\,.$$

The degree of freedom that this invariance gives to the Mimetic theory has to be fixed by a conformal **gauge fixing** procedure. This implies that we require the term P in (4.68) to be fixed by a condition that, as we have seen in the first part of this Chapter, is the very Mimetic constraint

$$g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - 1 = 0\,. \tag{4.70}$$

The advantage to use this constraint as gauge condition is that it identifies the fundamental metric $\tilde{g}_{\mu\nu}$ with the physical one

$$g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}\,.$$

The Mimetic constraint (4.70) is thus reinterpreted as a gauge condition in the local gauge-invariant theory with the action $S[\,g_{\mu\nu}(\tilde{g}_{\mu\nu},\phi),\phi\,]$.

4.5 Caustic instabilities and ghosts

One of the main problems in modified gravity theories, that is the context in which the Mimetic model have to be considered, is the presence of instabilities in the theory. This is the theoretical feature that imposes to verify the stability of the model with respect to possible ghost modes: in the Mimetic model, this feature was not exhaustively considered in the first proposal [1, 2], and it has been studied in a deeper way in [6] that we now resume. Ghost modes arise in a theory when its kinetic term in the Lagrangian action is not positive definite and can dynamically evolve. So in [6] it has been shown that the theory is free of ghosts whenever the background satisfies positive energy condition $\varepsilon > 0$, where ε comes from $T_{\mu\nu} = \varepsilon u_\mu u_\nu$.

Thus the new dynamical degree of freedom of dark matter fluid is free of ghosts, but it can still suffer from the gravitational instability associated with caustic surfaces of the geodesic flow. This type of instability is due to formation of caustics, and it is inevitable for generic geodesic flow which is associated with the potential ϕ satisfying the mimetic constraint equation

$$g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - 1 = 0\,.$$

Field "dust" moves along geodesics - the characteristic curves of this equation - and forms caustic singularities because of its pressureless nature. The invariance under conformal transformations of the physical metric $g_{\mu\nu}$ given by (4.69), could allow us to think that a change of the gauge condition would be solve these instabilities. But, since in Mimetic theory the gauge is fixed by (4.70) to make the physical metric $g_{\mu\nu}$ to coincide with the metric $\tilde{g}_{\mu\nu}$, this problem cannot be circumvented by an alternative conformal gauge fixing and it remains a serious difficulty.

An alternative way to avoid these instabilities is proposed in [6], where an analogous conformal extension of the Einstein's theory is suggested by using a reparametrization of the physical metric in terms of the dynamical vector field u_μ

$$g_{\mu\nu} = -(g^{\alpha\beta})u_\alpha u_\beta)\tilde{g}_{\mu\nu} \ .$$

The action used is

$$S[\,g_{\mu\nu},\phi\,] = \int d^4x\, g^{1/2}\left(\frac{1}{2}\,R(\,g_{\mu\nu}) + L(g_{\mu\nu},\phi,\partial\phi,u_\mu) - \frac{\mu^2}{4}\,g^{\mu\alpha}g^{\nu\beta}\,F_{\mu\nu}F_{\alpha\beta}\right),$$

where

$$F_{\mu\nu} = \partial_\mu u_\nu - \partial_\nu u_\mu$$

and the F^2 term provides a kinetic term for u_μ and guarantees absence of ghosts among the components of this vector field. Here the parameter μ^2 have mass squared dimension and $L(g_{\mu\nu},\phi,\partial\phi,u_\mu)$ is a matter Lagrangian containing some direct coupling of the vector field to matter, $\partial L/\partial u_\mu \neq 0$.
This theory turns to be obviously Weyl invariant as the Mimetic theory and the gauge condition can be chosen analogously as

$$g^{\mu\nu}u_\mu u_\nu - 1 = 0 \ .$$

Proca nature of the vector field guarantees it from the negativeness of the kinetic term, and so it confirm the absence of ghost instabilities.

Notice also that, depending on the coupling of the vector field to matter in $L(g_{\mu\nu},\phi,\partial\phi,u_\mu)$, the potential part of this flow given by $F_{\mu\nu} = 0$ can modify the inflationary scenario. However, the potential vector field cannot describe the present dark matter because its density ε decays simultaneously with the inflaton field at the end of inflation: in fact its energy density is algebraically related to the inflaton ϕ in a relation of the following form

$$\varepsilon = -F\,\phi \ .$$

It can be shown that the role of dark matter can be played by the rotational part of the vector field which survives the decay of ϕ. The rotational part of the vector field might play the role of real dark matter and mimic its real adiabatic perturbations.

Chapter 5

Disformal method

This Chapter is devoted to propose the first step in the generalization of the Mimetic scenario via the disformal method applied to very general scalar-tensor theories [3, 13]. This procedure will encounter the original Mimetic proposal as a particular case, and another step in the generalization process will be presented in the next Chapter in which GR will be generalized to Horndeski models and the Mimetic scenario will be studied as far as its cosmological perturbations are concerned.

A first introduction in this generalization process is proposed originally by [15] and then rielaborated by [13], in which the mimetic constraint is acknowledged as a particular disformal transformation of GR. The authors showed that Einstein's theory is invariant under generic disformal transformations of the form

$$g_{\mu\nu} = A(\phi, w)\ell_{\mu\nu} + B(\phi, w)\partial_\mu\phi\partial_\nu\phi \, , \tag{5.1}$$

where

$$w \equiv \ell^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \, , \tag{5.2}$$

A^1 and B are arbitrary functions of the scalar field ϕ and w, $g_{\mu\nu}$ is the physical metric and $\ell_{\mu\nu}$ is the new auxiliary metric.

But the main point is that the authors of [13] showed that there exists a particular subset of the previous general case (5.1), such that the resulting equations of motion are no longer the general relativistic equations, but instead one finds the equations of motion of the so-called "Mimetic" dark matter model of [1].

5.1 Mimetic gravity from a disformal transformation

In this section, we will follow [3] performing a disformal transformation of the type (5.1) on a very general scalar-tensor theory and comparing the equations of motion that result with the ones of the Mimetic model of the previous Chapter.

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[1] We will assume $A > 0$ to preserve the Lorentzian signature as indicated in [10].

5.1.1 Disformal transformation method

The first step will be to consider a very general action and, computing its equations of motion, to verify on which conditions they can propose solutions. The action that we consider is

$$S = \int d^4x \sqrt{-g} \mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1} \dots \partial_{\lambda_p} g_{\mu\nu}, \phi, \dots \partial_{\lambda_1} \dots \partial_{\lambda_q} \phi] + S_m[g_{\mu\nu}, \phi_m] \ . \tag{5.3}$$

The Lagrangian density $\mathcal{L}$ can be function of the physical metric $g_{\mu\nu}$ or also of its derivatives L, and of the scalar field ϕ and its derivatives. The action for the matter field ϕ_m is

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}_m[g_{\mu\nu}, \phi_m] \ ,$$

where we assume ϕ_m to be uncoupled with ϕ.

The variation of the action with respect to the scalar field ϕ gives

$$\Omega_\phi = \frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta\phi} \quad ,$$

while its variation with respect to the metric $g_{\mu\nu}$ yields

$$E^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} \ ,$$

and

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} \quad ,$$

and finally with respect to ϕ_m gives

$$\Omega_m = \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\phi_m} \ .$$

This implies that we can write the total variation as

$$\delta S = \frac{1}{2} \int d^4x \sqrt{-g}(E^{\mu\nu} + T^{\mu\nu})\delta g_{\mu\nu} + \int d^4x \ \Omega_\phi \delta\phi + \int d^4x \ \Omega_m \delta\phi_m \ , \tag{5.4}$$

where equations of motion of matter implies $\Omega_m = 0$.
Now we consider a disformal transformation of the form (5.1). Taking its variation

$$
\begin{aligned}
\delta g_{\mu\nu} &= A\,\delta\ell_{\mu\nu} - \left(\ell_{\mu\nu}\frac{\partial A}{\partial w} + \partial_\mu\phi\partial_\nu\phi\frac{\partial B}{\partial w}\right)\left[(\ell^{\alpha\rho}\partial_\alpha\phi)\,(\ell^{\beta\sigma}\partial_\beta\phi)\,\delta\ell_{\rho\sigma} - 2\ell^{\rho\sigma}\,(\partial_\rho\phi)\,(\partial_\sigma\delta\phi)\right] \\
&\quad + \left(\ell_{\mu\nu}\frac{\partial A}{\partial\phi} + \partial_\mu\phi\partial_\nu\phi\frac{\partial B}{\partial\phi}\right)\delta\phi + B\left[(\partial_\mu\phi)(\partial_\nu\delta\phi) + (\partial_\nu\phi)(\partial_\mu\delta\phi)\right] \ ,
\end{aligned}
\tag{5.5}
$$

we can generalize the previous theory by means of the disformal transformation inserting (5.5) into (5.4). This fact produces new generalized Einstein equations of motion. In fact, variating the so modified action by $\ell_{\mu\nu}$ one obtains

$$A(E^{\mu\nu} + T^{\mu\nu}) = \left(P\frac{\partial A}{\partial w} + Q\frac{\partial B}{\partial w}\right)(\ell^{\mu\rho}\partial_\rho\phi)\,(\ell^{\nu\sigma}\partial_\sigma\phi), \tag{5.6}$$

and variation with respect to ϕ leds to

$$\frac{1}{\sqrt{-g}}\partial_\rho \left\{ \sqrt{-g}\,\partial_\sigma\phi \left[B(E^{\rho\sigma} + T^{\rho\sigma}) + \left(P\frac{\partial A}{\partial w} + Q\frac{\partial B}{\partial w} \right) \ell^{\rho\sigma} \right] \right\} - \frac{\Omega_\phi}{\sqrt{-g}} = \qquad (5.7)$$

$$= \frac{1}{2}\left(P\frac{\partial A}{\partial \phi} + Q\frac{\partial B}{\partial \phi} \right) . \qquad (5.8)$$

The new quantities define above are defined as

$$P \equiv (E^{\rho\sigma} + T^{\rho\sigma})\ell_{\rho\sigma}$$

and

$$Q \equiv (E^{\rho\sigma} + T^{\rho\sigma})\partial_\rho\phi\,\partial_\sigma\phi .$$

We can see that these two quantities form a 2-dimensional linear system. In fact, the metric equations of motion (5.6) can be contracted with the metric $\ell_{\mu\nu}$ and with $\partial_\mu\phi\partial_\nu\phi$, to find

$$P\left(A - w\frac{\partial A}{\partial w} \right) - Qw\frac{\partial B}{\partial w} = 0$$

$$\qquad (5.9)$$

$$P\,w^2\frac{\partial A}{\partial w} - Q\left(A - w^2\frac{\partial B}{\partial w} \right) = 0 .$$

The determinant of this system can be zero o non zero: this two cases are presented in the following.

Generic case

The simplest way to solve the condition on the determinant of the system (5.10) is by writing it in a matrix form, as

$$M\begin{pmatrix} P \\ Q \end{pmatrix} = 0 , \qquad \text{where} \qquad M = \begin{pmatrix} A - w\frac{\partial A}{\partial w} & -w\frac{\partial B}{\partial w} \\ w^2\frac{\partial A}{\partial w} & -A + w^2\frac{\partial B}{\partial w} \end{pmatrix} .$$

The determinant of the system is

$$\det M = \left(A - w\frac{\partial A}{\partial w} \right)\left(-A + w^2\frac{\partial B}{\partial w} \right) - \left(-w\frac{\partial B}{\partial w} \right)\left(w^2\frac{\partial A}{\partial w} \right) \qquad (5.10)$$

$$= -A^2 + wA\frac{\partial A}{\partial w} + w^2\frac{\partial B}{\partial w}A - w^3\frac{\partial B}{\partial w}\frac{\partial A}{\partial w} + w^3\frac{\partial B}{\partial w}\frac{\partial A}{\partial w} \qquad (5.11)$$

$$= w^2 A\frac{\partial}{\partial w}\left(B + \frac{A}{w} \right) . \qquad (5.12)$$

The most general solution of the system (5.10) is when $\det(M) \neq 0$, so that the only solution is $P = Q = 0$. Thus the equations of motion (5.6) and (5.8) reduce to

$$E^{\mu\nu} + T^{\mu\nu} = 0 ,$$

$$\Omega_\phi = 0 .$$

Notice that equations of motion of the system (5.10) in this generic case are the same of the original theory before doing any disformal transformation, that means tha the theory is generically invariant under these transformations.

Mimetic gravity

The most interesting case in when the determinant of the system (5.10) is zero $\det M \equiv 0$. In this case, as we now see, one obtains at the end an equation of the same form of the Mimetic constraint (4.7).

To see it, one can solve the differential equation in (5.11). This procedure leds to a constraint on the free function $B(\phi, w)$ that turns to be of the form

$$B(\phi, w) = -\frac{A(\phi, w)}{w} + b(\phi) , \tag{5.13}$$

where $b(\phi)$ is an integration constant depending on ϕ that we will consider non-zero for all ϕ.

The fact that the determinant of the system (5.10) is zero means that the two equations inside it are functionally dependent. This can be see easily inserting the solution (5.13) into the system (5.10), obtaining a relation between P and Q of the form $Q = wP$.

Remembering the previous identification of P and Q, equations of motion (5.6) and (5.8) become

$$E^{\mu\nu} + T^{\mu\nu} = \frac{P}{w} \left(\ell^{\mu\rho} \partial_\rho \phi\right) \left(\ell^{\nu\sigma} \partial_\sigma \phi\right) \tag{5.14}$$

$$\frac{1}{\sqrt{-g}} \partial_\rho \left(\sqrt{-g}\, b\, P\, \ell^{\rho\sigma} \partial_\sigma \phi\right) - \frac{\Omega_\phi}{\sqrt{-g}} = \frac{1}{2} P\, w \frac{db}{d\phi} . \tag{5.15}$$

Consider now that with the condition (5.13), the particular disformal transformation (5.1) is

$$g_{\mu\nu} = A(\phi, w)\, \ell_{\mu\nu} + \partial_\mu \phi\, \partial_\nu \phi \left(b(\phi) - \frac{A(\phi, w)}{w}\right) , \tag{5.16}$$

and that inverse metric transforms as

$$g^{\mu\nu} = \frac{\ell^{\mu\nu}}{A} + \frac{A - w\, b}{A\, b\, w^2} (\ell^{\mu\rho} \partial_\rho \phi) (\ell^{\nu\sigma} \partial_\sigma \phi) ,$$

and these equations can be used to write (5.15) in terms of $g_{\mu\nu}$ only.

To find a relation with the Mimetic constraint (4.7), consider that we can multiply the (5.14) for the metric $g_{\mu\nu}$, thus obtaining

$$g_{\mu\nu} \left(E^{\mu\nu} + T^{\mu\nu}\right) = \frac{P}{w}\, g_{\mu\nu} \left((\ell^{\mu\rho} \partial_\rho \phi) (\ell^{\nu\sigma} \partial_\sigma \phi)\right)$$

$$E + T = \frac{P}{w}\, A(\phi, w)\, \ell_{\mu\nu} + \partial_\mu \phi\, \partial_\nu \phi \left(b(\phi) - \frac{A(\phi, w)}{w}\right) \left((\ell^{\mu\rho} \partial_\rho \phi) (\ell^{\nu\sigma} \partial_\sigma \phi)\right)$$

$$= \frac{P}{w} \left(A + bw^2 - aw\right)$$

$$= \frac{P}{w}\, bw^2 ,$$

and hence $P = (E + T)/(bw)$. Notice that we used (5.2), $\partial^\mu \phi \equiv g^{\mu\rho} \partial_\rho \phi$ and the traces

$$E + T \equiv g_{\rho\sigma}(E^{\rho\sigma} + T^{\rho\sigma}) .$$

Finally, using (5.2) in the contraction of the relation $\ell^{\mu\rho} \partial_\rho \phi = bw \partial^\mu \phi$ with $\partial_\mu \phi$, we find

$$b(\phi)g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1 , \tag{5.17}$$

that is a very similar condition of the Mimetic constraint (4.7), where the function $b(\phi)$ is the term that generalizes it.

The constraint (5.17) leds to modify equations of motion as

$$E_{\mu\nu} + T_{\mu\nu} = (E + T)\, b(\phi)\, \partial_\mu \phi\, \partial_\nu \phi \ ,$$

$$\nabla_\rho \left[(E+T)b(\phi)\, \partial^\rho \phi \right] - \frac{\Omega_\phi}{\sqrt{-g}} = \frac{1}{2}(E+T)\frac{1}{b(\phi)}\frac{db(\phi)}{d\phi} \ . \tag{5.18}$$

These equations of motion illustrate that, by varyating the action (5.3) with respect to the original metric $g_{\mu\nu}$ produces different equation of motion instead of the last equations (5.18). This means that the new theory that is characterized by the generalized Mimetic constraint (5.17) and it defines the so called **Mimetic scenario** or Mimetic gravity of that theory.

5.2 Mimetic gravity from a Lagrange multiplier

In this section, we will show - following [3] - how the Mimetic equations of motion that result after transforming the theory (5.3) via a Mimetic disformal transformation of the type (5.16), can also be obtained by using a Lagrange multiplier in the main Lagrangian without performing any disformal transformation. This approah has also been used in the paper of [2], but here we show in a more general way the connection between this approach and the one presented before.
Mimetic theory can thus be obtained in two different and equivalent ways: via a Lagrange multiplier or via a disformal transformation, and these two methods are equivalent.

This second approach starts from (5.3) of the previous section with the following extra term

$$S_\lambda = S + \int d^4 x \sqrt{-g} \left[\lambda \left(b(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1 \right) \right], \tag{5.19}$$

where λ is a Lagrange multiplier field under variation of which one obtains the kinematical constraint

$$b(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1 = 0 \ . \tag{5.20}$$

The other equations of motion are obtained by variating with respect to ϕ

$$\Omega_\phi + \sqrt{-g}\frac{\lambda}{b(\phi)}\frac{db(\phi)}{d\phi} - 2\partial_\mu \left(\sqrt{-g}\lambda b(\phi) g^{\mu\nu} \partial_\nu \phi \right) = 0 \ , \tag{5.21}$$

with respect to $g_{\mu\nu}$

$$E^{\mu\nu} + T^{\mu\nu} - 2\lambda b(\phi)\partial^\mu \phi \partial^\nu \phi = 0 \ , \tag{5.22}$$

and finally with respect to ϕ_m

$$\Omega_m = 0 \ .$$

The Lagrange multiplier can be obtained by taking the trace of (5.22) and using (5.20)

$$2\lambda = E + T \ , \tag{5.23}$$

where E and T are the trace of the respective tensors), so that one can eliminate it from the equations of motion to obtain

$$b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1 = 0 \ , \tag{5.24}$$

$$\nabla_\mu \left[(E+T)b(\phi)\partial^\mu\phi \right] - \frac{\Omega_\phi}{\sqrt{-g}} = \frac{E+T}{2}\frac{1}{b(\phi)}\frac{db(\phi)}{d\phi} \ , \tag{5.25}$$

$$E^{\mu\nu} + T^{\mu\nu} = (E+T)b(\phi)\partial^\mu\phi\partial^\nu\phi, \tag{5.26}$$

$$\Omega_m = 0. \tag{5.27}$$

Notice that there equations of motion are identical to the Mimetic equations of motion in previous section, i.e. (5.17), (5.18) and the matter equation. This shows that Mimetic gravity can be formulated by action (5.19) also via the insertion of a Lagrange multiplier.

5.2.1 Independent equations of motion

Equations of motion of the Mimetic scenario found either via the disformal method or via the Lagrange multiplier inserction, are not linear independent. To see how one can be obtained from the others, consider the covariant derivative of (5.26) and use $\nabla_\mu T^{\mu\nu} = 0$ to obtain

$$\begin{aligned}
\nabla_\mu E^{\mu\nu} + \nabla_\mu T^{\mu\nu} &= \nabla_\mu \left[(E+T)b(\phi)\partial^\mu\phi\partial^\nu\phi \right] \\
\nabla_\mu E^{\mu\nu} &= \nabla_\mu \left[(E+T)b(\phi)\partial^\mu\phi \right] \partial^\nu\phi + (E+T)b(\phi)\partial^\mu\phi\nabla_\mu\partial^\nu\phi \ .
\end{aligned}$$

Now remember that the covariant derivative of the constraint $b(\phi)\partial^\mu\phi\partial_\mu\phi = 1$ gives

$$b(\phi)\nabla^\mu\nabla^\nu\phi\partial_\mu\phi = -\frac{1}{2}\frac{db(\phi)}{d\phi}\nabla^\nu\phi\partial_\mu\phi\partial^\mu\phi \ .$$

So the previous expression becomes

$$\nabla_\mu E^{\mu\nu} = \partial^\nu\phi \left[\nabla_\nu \left[(E+T)b(\phi)\partial^\mu\phi \right] - \frac{E+T}{2}\frac{1}{b(\phi)}\frac{db(\phi)}{d\phi} \right] \ . \tag{5.28}$$

It was shown by Horndeski [18] that

$$\sqrt{-g}\nabla_\mu E^{\mu\nu} = \Omega_\phi\nabla^\nu\phi \ .$$

Using this and the fact that $\partial^\nu\phi \neq 0$ at least for one index ν, we can simplify (5.28) to

$$\nabla_\mu \left[(E+T)b(\phi)\partial^\mu\phi \right] - \frac{\Omega_\phi}{\sqrt{-g}} = \frac{E+T}{2}\frac{1}{b(\phi)}\frac{db(\phi)}{d\phi} \ ,$$

that is the same equation as (5.25). Thus the independent equations of motion are (5.24), (5.26) and (5.27).

Chapter 6

Mimetic Horndeski models

In the following, we will discuss the widest generalization of the Mimetic scenario, applying it for concreteness to the Horndeski models. The Horndeski models are among the most general 4D covariant theory of scalar-tensor gravity that is derived from an action and that produces to second-order equations of motion (in all gauges and in any background) for both the metric and the scalar field. At the end of the Chapter, we will discuss the cosmological perturbations of these models.

6.1   Horndeski models

Scalar-tensor theories are probably the simplest, consistent and non trivial modification of General Relativity. They acquire an additional degree of freedom represented by a real scalar field, and they include many other theories of modified gravity. An example, as introduced in Chapter 3, is provided by the $f(R)$ theories, which are very particular scalar-tensor theories. Scalar-tensor theories constitute a generic theoretical context where one can test deviations from GR and - hopefully soon enough - conduct new gravity observational tests.
In a paper published in 1974, G.W. Horndeski [18] presented the most general scalar-tensor theory with second order field equations in four dimensions. Given the recently developed researches into modified gravity, we need to revisit Horndeski's work. Examples of scalar-tensor theories of modified gravity are Brans-Dicke gravity, galileon theories, GR [44]. Each of these models represent a special case of Horndeski's theory.

A very general action for scalar-tensor theories can be

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1}, \dots \partial_{\lambda_p} g_{\mu\nu}, \phi, \partial_{\lambda_1} \dots \partial_{\lambda_q} \phi] + S_m[g_{\mu\nu}, \phi_m] \, . \qquad (6.1)$$

The Lagrangian density $\mathcal{L}$ can be a function of the physical metric $g_{\mu\nu}$ or also of its derivatives, and of the scalar field ϕ and its derivatives. S_m denotes the action for some matter field ϕ_m which we assume that is coupled with $g_{\mu\nu}$ only.
We will only consider a particular subset of theories of the form (6.1) known as **Horndeski theory**, where the Lagrangian density $\mathcal{L}$ is given by Horndeski's Lagrangian density $\mathcal{L}_H$ as

$$S_H = \int d^4x \sqrt{-g}\mathcal{L}_H = \int d^4x \sqrt{-g} \sum_{n=0}^{3} \mathcal{L}_n \, , \qquad (6.2)$$

where

$$
\begin{aligned}
\mathcal{L}_0 &= K\left(X,\phi\right) , \\
\mathcal{L}_1 &= -G_3\left(X,\phi\right)\Box\phi , \\
\mathcal{L}_2 &= G_{4,X}\left(X,\phi\right)\left[\left(\Box\phi\right)^2 - \left(\nabla_\mu\nabla_\nu\phi\right)^2\right] + R\,G_4\left(X,\phi\right) , \\
\mathcal{L}_3 &= -\frac{1}{6}G_{5,X}\left(X,\phi\right)\left[\left(\Box\phi\right)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right] + G_{\mu\nu}\nabla^\mu\nabla^\nu\phi\,G_5\left(X,\phi\right) ,
\end{aligned}
$$

with

$$
X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi ,
$$
$$
\left(\nabla_\mu\nabla_\nu\phi\right)^2 = \nabla_\mu\nabla_\nu\phi\,\nabla^\mu\nabla^\nu\phi ,
$$
$$
\left(\nabla_\mu\nabla_\nu\phi\right)^3 = \nabla_\mu\nabla_\nu\phi\,\nabla^\mu\nabla^\rho\phi\,\nabla^\nu\nabla_\rho\phi .
$$

The subscript $,X$ denotes derivative with respect to X and in the following derivative with respect to ϕ will be denoted by a subscript $,\phi$. The functions K, G_3, G_4, G_5 of two variables, X and ϕ, define a particular Horndeski theory.

The Horndeski terms are also called generalized "galileons". The scalar field (or galileon) has the property of admitting a special symmetry in flat (nondynamical) spacetime for $G_2 \simeq G_3 \simeq X$ and $G_4 \simeq G_5 \simeq X^2$, which resembles the "Galilean symmetry", hence the name galileon. Galileon symmetry is broken for a curved background and for general choice of G_i. The term "generalized" refers to the fact that the functions G_i are arbitrary, in contrast to "covariant" galileon with fixed G_i [16].

General Relativity as a particular case

As said, Horndeski's theory can be reduced to GR if we impose the following constraints on the parameters in the Lagrangian

$$
\begin{aligned}
K &= 0 , \\
G_3 &= 0 , \\
G_5 &= 0 , \\
G_4 &\equiv \frac{1}{2} ,
\end{aligned}
$$

so that the action becomes

$$
S = -\frac{1}{2}\int d^4x\sqrt{-g}\mathcal{R} ,
$$

equivalent to the Hilbert-Einstein's action.

6.2 Mimetic Horndeski models

In this section we discuss the Mimetic scenario applied to the Horndeski theory as in [4]. As we have seen in the previous Chapter, this can be obtained either by the Lagrange multiplier method, or via the disformal method. We will now use the former approach, adding a Lagrange multiplier in (6.1) to impose the so-called Mimetic constraint. At the end of this Chapter, we will discuss the cosmological perturbations of this model.

The model

Consider the action (5.19) in the previous Chapter, that we remember here using from now on the scalar field ϕ

$$S \quad = \quad \int d^4x\sqrt{-g}\mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1}\ldots\partial_{\lambda_p}g_{\mu\nu}, \phi, \ldots\partial_{\lambda_1}\ldots\partial_{\lambda_q}\phi] + S_m[g_{\mu\nu}, \phi_m] \,. \quad (6.3)$$

Now we consider the Horndeski theory, in which the first contribution to the action (6.3) is given by the Lagrangian $\mathcal{L}_H$ given by (6.2).

Remembering that we are looking for the Mimetic scenario applied to the Horndeski theory, we add to the previous Lagrangian the Lagrange multiplier λ to enforce the Mimetic constraint, obtaining the action S_{MH} of **Mimetic Horndeski** expressed as

$$S_{MH} = \int d^4x\sqrt{-g}\mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1}\ldots\partial_{\lambda_p}g_{\mu\nu}, \phi, \ldots\partial_{\lambda_1}\ldots\partial_{\lambda_q}\phi] + S_m[g_{\mu\nu}, \phi_m]+$$

$$+ \int d^4x\sqrt{-g}\left[\lambda\left(b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1\right)\right] \quad (6.4)$$

$$= \int d^4x\sqrt{-g}\mathcal{L}_H + \int d^4x\sqrt{-g}\,\mathcal{L}_m + \int d^4x\sqrt{-g}\left[\lambda\left(b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1\right)\right] \,,$$

where with $\mathcal{L}_m$ we indicate the Lagrangian of matter.

Analogously to the previous Chapter, we can calculate the equations of motion by varying the action (6.4) with respect to λ, ϕ, $g_{\mu\nu}$ and ϕ_m. In particular, taking the trace of the equation obtained by variation with respect to the metric $g_{\mu\nu}$

$$E^{\mu\nu} + T^{\mu\nu} - 2\lambda b(\phi)\partial^\mu\phi\partial^\nu\phi = 0 \,,$$

we can express them without the Lagrange multiplier given by

$$2\lambda = E + T \,.$$

Therefore the equations of motion that results for the theory (6.4) read

$$b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1 = 0 \,, \quad (6.5)$$

$$\nabla_\mu\left[(E+T)b(\phi)\partial^\mu\phi\right] - \frac{\Omega_\phi}{\sqrt{-g}} = \frac{E+T}{2}\frac{1}{b(\phi)}\frac{db(\phi)}{d\phi} \,, \quad (6.6)$$

$$E^{\mu\nu} + T^{\mu\nu} = (E+T)b(\phi)\partial^\mu\phi\partial^\nu\phi \,, \quad (6.7)$$

$$\Omega_m = 0 \,, \quad (6.8)$$

where the meaning of the symbols are analogous to those of the previous Chapter:

$$\Omega_\phi = \frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta\phi} \,,$$

$$E^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} \,,$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} \,,$$

$$\Omega_m = \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\phi_m} \,.$$

However, from now on we will neglect for simplicity the matter term, i.e. $S_m = 0$: this matter contribution will be inserted in Section 6.3.4.

Independent equations of motion

In the previous Chapter dealing with Mimetic Dark Matter, we have seen that not all the equations of motion one get are independent. This happens similarly also for the more general case of Mimetic Horndeski. In fact, as presented in [4], equation (6.6) can be derived from the other equations. We will now show that the $0-0$ component of (6.7) can be derived from (6.5) and the remaining components of (6.7). In fact, take the constraint equation (6.5) and consider that

$$b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1 \ ,$$
$$= b(\phi)g^{00}(\phi')^2 + 2b(\phi)g^{0i}(\phi')\partial_i\phi + b(\phi)g^{ij}\partial_i\phi\partial_j\phi \ .$$

Multiplying both sides by $E + T$, we have

$$(E+T)b(\phi)g^{00}(\phi')^2 + 2(E+T)b(\phi)g^{0i}(\phi')\partial_i\phi + (E+T)b(\phi)g^{ij}\partial_i\phi\partial_j\phi =$$
$$= g^{00}(E_{00} + T_{00}) + 2g^{0i}(E_{0i} + T_{0i}) + g^{ij}(E_{ij} + T_{ij}) \ , \tag{6.9}$$

and, by using the other components of equations (6.7), that is

$$E_{ij} + T_{ij} = (E+T)b(\phi)\partial_i\phi\partial_j\phi \ , \tag{6.10}$$
$$E_{0i} + T_{0i} = (E+T)b(\phi)\phi'\partial_i\phi \ , \tag{6.11}$$

we can show that equation (6.9) simplifies to

$$(E+T)b(\phi)g^{00}(\phi')^2 = g^{00}(E_{00} + T_{00}) \ .$$

Because $g^{00} \neq 0$, equations in (6.11) together with the constraint equation (6.5) imply

$$E_{00} + T_{00} = (E+T)b(\phi)(\phi')^2.$$

which is the time component of (6.7).

However, in the next Chapter we will derive some of the reduntant equations of motion because they will be useful for some calculations in perturbed Mimetic Horndeski.

6.3 Cosmological perturbations

We now discuss the cosmological perturbations of Horndeski model, and then we will discuss them in framework of its Mimetic scenario. Remember that in this section and in the next one we will consider no matter $S_m = 0$. Moreover we will work in the Poisson gauge. Notice also that the tensor $E^{\mu\nu}$ introduced in the previous section will be the same for both Horndeski and Mimetic Horndeski gravity.

6.3.1 Linear perturbations in Horndeski models

In the following we derive cosmological perturbations of the Horndeski model defined by the action (6.2).

Since there are no matter sources, equations of motion reduce to

$$E_{\mu\nu} = 0 \,. \tag{6.12}$$

At zeroth-order in perturbation theory they simplify to

$$E_{\mu\nu}^{(0)} = 0 \,. \tag{6.13}$$

Remember that we denote with the superscript (0) the tensor $E^{\mu\nu}$ evaluated on the background. Notice, moreover, that we will use a compact notation for the tensor $E_{\mu\nu}^{(0)}$ to simplify the calculations.

The tensor $E_{\mu\nu}$ perturbed at first-order reads

$$E_{00}^{(1)} = f_1 \Phi' + f_2 \delta\phi' + f_3 \Psi + f_4 \delta\phi + f_5 \nabla^2 \Phi + f_6 \nabla^2 \delta\phi \,, \tag{6.14}$$

$$E_{ij}^{(1)} = \partial_i \partial_j \left(f_7 \Phi + f_8 \delta\phi + f_9 \Psi \right) + \delta_{ij} \Big(-f_7 \nabla^2 \Phi - f_8 \nabla^2 \delta\phi - f_9 \nabla^2 \Psi +$$

$$+ f_{10} \Phi'' + f_{11} \delta\phi'' + f_{12} \Phi' + f_{13} \delta\phi' + f_{14} \Psi' + f_{15} \Phi + f_{16} \delta\phi + f_{17} \Psi \Big), \tag{6.15}$$

$$E_{0i}^{(1)} = \partial_i \left(f_{18} \Phi' + f_{19} \delta\phi' + f_{20} \delta\phi + f_{21} \Psi \right), \tag{6.16}$$

where we denote with the superscript (1) the first-order perturbations of the tensor $E^{\mu\nu}$. The functions f_i with $i = 1, ..., 21$ are linear functions of the Horndeski functions K, G_3, G_4, G_5 in the Lagrangian $\mathcal{L}_H$ in (6.2). Functions f_i are functions of time only, their derivatives are evaluated on the background. Not all the functions are independent from each other: the explicit relations that give their dependence are given in Appendix B.

By taking the traceless part of the term $E_{ij}^{(1)} = 0$, one finds that

$$f_7 \Phi + f_8 \delta\phi + f_9 \Psi = 0 \,, \tag{6.17}$$

that is, the first part of $E_{ij}^{(1)}$ in (6.15) vanishes. This equations states that at least one of the fields is not a new dynamical degree of freedom. Moreover, it means that generally the anisotropic stress is not zero. Notice that, since we do not assume any constraint of the Mimetic theory, this equation will also be valid in Mimetic Horndeski.

Let us now summarize how many variables we have in this theory: we have three variables, $\delta\phi$, Ψ and Φ, and we have four equations of motion

$$f_1 \Phi' + f_2 \delta\phi' + f_3 \Psi + f_4 \delta\phi + f_5 \nabla^2 \Phi + f_6 \nabla^2 \delta\phi = 0, \tag{6.18}$$

$$f_7 \Phi + f_8 \delta\phi + f_9 \Psi = 0, \tag{6.19}$$

$$f_{10} \Phi'' + f_{11} \delta\phi'' + f_{12} \Phi' + f_{13} \delta\phi' + f_{14} \Psi' + f_{15} \Phi + f_{16} \delta\phi + f_{17} \Psi = 0, \tag{6.20}$$

$$f_{18} \Phi' + f_{19} \delta\phi' + f_{20} \delta\phi + f_{21} \Psi = 0. \tag{6.21}$$

These equations of motion are not all independent from each other, and it is possible to show that there are only three independent equations. In fact, equation (6.20) can be obtained from (6.19) and (6.21), using some of the identities for the f_i functions in Appendix B.

6.3.2 General Relativity as a particular case

As far as the functions f_i are concerned, that we will use in the following and that one can find in [4] and in the Appendix, there are some simplifications that can be obtained with the

previous conditions on the action, so that they become of the form indicated in the following table.

Function	GR-limit	Function	GR-limit
f_1	6H	f_{11}	0
f_2	0	f_{12}	-4H
f_3	0	f_{13}	0
f_4	0	f_{14}	-2H
f_5	-2	f_{15}	0
f_6	0	f_{16}	0
f_7	-1	f_{17}	$-2H^2 - 4H'$
f_8	0	f_{18}	-2
f_9	1	f_{19}	0
f_{10}	-2	f_{20}	0

Table 6.1: *GR-limit of functions f_i.*

However, we stress the fact that these simplifications are not valid in the Mimetic case, that is to say when considering the Mimetic constraint $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$. In fact, when we use the Mimetic approach, as shown in the previous Chapters we are doing a non invertible disformal transformation from GR to another new theory (in this case the Mimetic Horndeski), and it is not possible to return to GR simply imposing the previous constraints manually on the action.

6.3.3 Linear perturbations in Mimetic Horndeski

We now study in more details linear scalar perturbations in Mimetic Horndeski gravity. We assume a flat FLRW background and for the moment that there is no matter. As seen in Section 6.2, the independent equations of motion for the model reduce to

$$b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1 = 0$$
$$E_{\mu i} = Eb(\phi)\partial_\mu\phi\partial_i\phi.$$

At background level, they are

$$b_0(\phi_0')^2 = -a^2, \tag{6.22}$$
$$E_{ij}^{(0)} = 0, \tag{6.23}$$

while at first-order in perturbation theory the equations of motion become

$$2b_0\delta\phi' + \phi_0'b_{,\phi}\delta\phi - 2b_0\phi_0'\Psi = 0, \tag{6.24}$$
$$E_{ij}^{(1)} = 0, \tag{6.25}$$
$$E_{0i}^{(1)} = E^{(0)}b_0\phi_0'\partial_i\delta\phi, \tag{6.26}$$

where we denoted $b_0 \equiv b(\phi_0)$, $E^{(0)}$ is the zeroth-order trace of $E_{\mu\nu}$ and $b_{,\phi} \equiv b_{,\phi}(\phi_0)$.

The traceless part of equation (6.25) gives

$$f_7\Phi + f_8\delta\phi + f_9\Psi = 0, \tag{6.27}$$

while the trace of the same equation gives

$$f_{10}\Phi'' + f_{11}\delta\phi'' + f_{12}\Phi' + f_{13}\delta\phi' + f_{14}\Psi' + f_{15}\Phi + f_{16}\delta\phi + f_{17}\Psi = 0 , \tag{6.28}$$

and (6.26), using (6.22), implies

$$f_{18}\Phi' + f_{19}\delta\phi' + \left(f_{20} + \frac{a^2 E^{(0)}}{\phi_0'}\right)\delta\phi + f_{21}\Psi = 0.$$

Notice that, as we anticipated above, (6.27) is equal to (6.17) in the non-Mimetic case, with same conclusions.

Notice that the Horndeski's theory is invariant in form under a field redefinition, and in particular it is possible to re-define the field ϕ to set $b(\phi) = -1$. This implies that $b_{,\phi} = 0$ and so the first-order constraint simplifies to

$$\Psi = \frac{\delta\phi'}{\phi_0'} .$$

This fact will be important in the following to simplify the following computations.

At this point, the independent first-order equations of motion for the Mimetic Horndeski model that we will use from now on are

$$2b_0\delta\phi' + \phi_0' b_{,\phi}\delta\phi - 2b_0\phi_0'\Psi = 0, \tag{6.29}$$

$$f_7\Phi + f_8\delta\phi + f_9\Psi = 0, \tag{6.30}$$

$$f_{18}\Phi' + f_{19}\delta\phi' + \left(f_{20} + \frac{a^2 E^{(0)}}{\phi_0'}\right)\delta\phi + f_{21}\Psi = 0. \tag{6.31}$$

Notice that in this system of equations there are no spatial derivatives: this is a clear sign that there will be a zero speed of sound.

6.3.4   Linear perturbations in Mimetic Horndeski with matter

In this section we will recall the expressions for the linear scalar perturbations of the energy-momentum tensor of a generic fluid that may contain anisotropic stress. We will use the Poisson gauge. Therefore we will present the linear equations of motion for the cosmological perturbations around a FRW Universe in Mimetic Horndeski with a non zero matter term, assuming that there is no direct coupling between this matter fluid and the Mimetic scalar field ϕ.

The energy-momentum tensor

Remember that the general energy-momentum tensor of a fluid can be written as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + \pi_{\mu\nu} . \tag{6.32}$$

Here we assume that ρ is the energy density, p the pressure and u^μ is the 4-velocity (here the 4-velocity in this section is not related with the 4-velocity introduced earlier).
The anisotropic stress tensor $\pi_{\mu\nu}$ obeys $\pi_{\mu\nu}u^\mu = 0$ and $\pi^\mu_{\ \mu} = 0$. We consider that it is a

first-order quantity and that u^μ is defined so that $\pi_{00} = \pi_{0i} = 0$ (the so-called energy frame) [7, 57].

Neglecting vectorial and tensorial parts, the spatial part of $\pi_{\mu\nu}$ can be expressed as

$$\pi_{ij} = a^2 \left(\partial_i \partial_j \Pi - \frac{1}{3} \delta_{ij} \nabla^2 \Pi \right).$$

The 4-velocity is normalized with the condition $u_\mu u^\mu = -1$ and its perturbated expression is

$$u^0 = a^{-1}(1 - \Psi)$$
$$u^i = a^{-1} v^i \,,$$

where the first-order velocity can be expressed as $v^i = \delta^{ij} \partial_j v$.

At the background level, the energy-momentum tensor is given by

$$T_{00}^{(0)} = a^2 \rho_0, \quad T_{0i}^{(0)} = 0, \quad T_{ij}^{(0)} = a^2 p_0 \delta_{ij} \,, \tag{6.33}$$

with ρ_0 and p_0 the zeroth-order energy density and pressure respectively. The trace at zeroth-order is given by $T^{(0)} = -\rho_0 + 3p_0$.

The first-order components of the energy-momentum tensor are

$$T_{00}^{(1)} = a^2 \left(\delta\rho + 2\rho_0 \Psi \right), \quad T_{0i}^{(1)} = -a^2 \left(\rho_0 + p_0 \right) \partial_i v \,, \tag{6.34}$$

$$T_{ij}^{(1)} = a^2 \left((\delta p - 2p_0 \Phi)\delta_{ij} + \partial_i \partial_j \Pi - \frac{1}{3} \delta_{ij} \nabla^2 \Pi \right) \,, \tag{6.35}$$

where $\delta\rho$ and δP denote the energy density and pressure perturbations respectively. The trace of the first-order energy momentum tensor is $T^{(1)} = -\delta\rho + 3\delta p$.

Finally, we remember that the energy-momentum tensor respects the conservation law $\nabla^\mu T_{\mu\nu} = 0$ that at background level gives

$$\rho_0' + 3H(\rho_0 + p_0) = 0 \,, \tag{6.36}$$

and at first-order implies

$$\delta\rho' + 3H(\delta\rho + \delta p) - 3(\rho_0 + p_0)\Phi' + (\rho_0 + p_0)\nabla^2 v = 0 \,, \tag{6.37}$$

$$[(\rho_0 + p_0)v]' + \delta p + \frac{2}{3}\nabla^2 \Pi + 4H(\rho_0 + p_0)v + (\rho_0 + p_0)\Psi = 0 \,. \tag{6.38}$$

Mimetic Horndeski with matter

In this section we will use the fluid presented before to express the equations of motion of Mimetic Horndeski gravity. Remembering that they are given by (6.8), they imply

$$b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1 = 0,$$
$$E^{\mu\nu} + T^{\mu\nu} = (E + T)b(\phi)\partial^\mu\phi\partial^\nu\phi,$$
$$\nabla_\mu T^{\mu\nu} = 0 \,,$$

where the field equation can be neglected because of its dependence from the other ones, and where we used the equation $\nabla_\mu T^{\mu\nu} = 0$ instead of the equivalent $\Omega_m = 0$.

At zeroth-order, these equations of motion are

$$-a^{-2}b(\phi_0)(\phi_0')^2 = 1$$
$$E_{ij}^{(0)} = -a^2 p_0 \delta_{ij}$$
$$\rho_0' + 3H(\rho_0 + p_0) = 0 \,.$$

At first-order they read

$$2b_0\delta\phi' + \phi_0' b_{,\phi}\delta\phi - 2b_0\phi_0'\Psi = 0 \ , \tag{6.39}$$

$$f_7\Phi + f_8\delta\phi + f_9\Psi + a^2\Pi = 0 \ , \tag{6.40}$$

$$f_{10}\Phi'' + f_{11}\delta\phi'' + f_{12}\Phi' + f_{13}\delta\phi' + f_{14}\Psi' + f_{15}\Phi + f_{16}\delta\phi + f_{17}\Psi +$$
$$+ \frac{2}{3}a^2\nabla^2\Pi + a^2\left(\delta P - 2p_0\Phi\right) = 0 \ , \tag{6.41}$$

$$f_{10}\Phi' + f_{11}\delta\phi' + \left(f_{20} + \frac{a^2(E^{(0)} + T^{(0)})}{\phi_0'}\right)\delta\phi + f_{14}\Psi +$$
$$- a^2\left(\rho_0 + p_0\right)v = 0 \ , \tag{6.42}$$

$$\delta\rho' + 3H(\delta\rho + \delta p) - 3(\rho_0 + p_0)\Phi' + (\rho_0 + p_0)\nabla^2 v = 0 \ , \tag{6.43}$$

$$[(\rho_0 + p_0)v]' + \delta p + \frac{2}{3}\nabla^2\Pi + 4H(\rho_0 + p_0)v + (\rho_0 + p_0)\Psi = 0 \ . \tag{6.44}$$

As before, the third equation can be obtained from the others and the background ones: the set of independent equations of motion in Mimetic gravity with matter is then given by Eqs. (6.39), (6.40), (6.42), (6.43) and (6.44).

6.3.5 Beyond Horndeski models

Today, Horndeski theories are one of the main theoretical framework for scalar-tensor theories in which cosmological observations related to the present acceleration of the Universe are described. The Horndeski action (6.2) can include different models of modified gravity studied in the last years: for example it includes quintessence, k-essence and $f(R)$ models [44].

The theories "beyond Horndeski" that have recently been proposed are a new class of scalar-tensor theories of gravity that generalize Horndeski but that can lead to a different observational phenomenology. For this reason, they are of great interest for an extensive comparison of scalar-tensor theories with observations. The theories beyond Horndeski presented in [27] are also known as G^3 theories.

Despite possessing equations of motion with higher-order derivatives, one can see that the true propagating degrees of freedom obey well-behaved second-order equations and are free from instabilities or ghosts.

The G_3 theories can be described by linear combinations of the Lagrangians

$$L_2^\phi \equiv G_2(\phi, X) \ , \tag{6.45}$$

$$L_3^\phi \equiv G_3(\phi, X)\,\Box\phi \ , \tag{6.46}$$

$$L_4^\phi \equiv G_4(\phi, X)\,^{(4)}R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}) +$$
$$+ F_4(\phi, X)\epsilon^{\mu\nu\rho}{}_\sigma\,\epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'} \ , \tag{6.47}$$

$$L_5^\phi \equiv G_5(\phi, X)\,^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5,X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^\nu{}_\sigma) +$$
$$+ F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'} \ , \tag{6.48}$$

which depend on a scalar field ϕ (and its derivatives $\phi_\mu \equiv \nabla_\mu\phi$ and $\phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi$), on $X \equiv g^{\mu\nu}\phi_\mu\phi_\nu$, and on a metric $g_{\mu\nu}$ with respect to which matter is assumed to be minimally coupled; $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor.

Horndeski theories correspond to a subset of the above theories, subjected to the restricting conditions

$$F_4(\phi, X) = 0 \qquad \text{and} \qquad F_5(\phi, X) = 0,$$

which ensure that equations of motion are second-order. So G_3 theories contain two additional free functions with respect to the Horndeski ones.

These extensions of Horndeski theories have an interesting phenomenology: the scalar degree of freedom affects the sound speed of matter, even in the case of a minimal coupling.

In [4] there is a demonstration that even in a *Mimetic G^3* theory (without matter) at first-order around a flat FLRW background, the speed of sound of scalar perturbations is still exactly zero. This turns to be a general fact for Mimetic theories: the $0 - 0$ components of equations of motion at first-order does not contain any spatial derivatives. Because also the $i - j$ components do not contain spatial derivatives, the sound speed of the Mimetic G^3 model has to be zero as in the Mimetic Horndeski model. However, it is well-known that, inserting in the Lagrangian of a theory terms with higher-order derivatives and assuming an extra scalar degrees of freedom, then its Mimetic scenario may have a non-vanishing sound speed [2].

Chapter 7

Small-scale cosmological perturbations in Mimetic Horndeski models

In the following, we present a discussion on the small-scale limit of the cosmological perturbations within the Mimetic Horndeski model. In particular we will recall the canonical method to obtain the Poisson Equation in General Relativity: it will be important in the next sections, where we will show that the application of this method in Mimetic Horndeski seems not working.
We will show, moreover, that besides the classical method, the Poisson Equation can be derived in an alternative way. This different procedure will be applied to the Mimetic Horndeski theory, looking for a non-vanishing Laplacian term that can produce a generalized Poisson equation.

7.1   Poisson equation from General Relativity

We now describe a standard method to obtain the Poisson equation in General Relativity.

Consider the $0 - 0$ component of Einstein's equations

$$G_{00} = 8\pi G \, T_{00} \, , \tag{7.1}$$

and take the first-order perturbation around a spatially flat FRW Universe

$$G_{00}^{(1)} = 8\pi G \, T_{00}^{(1)} \, , \tag{7.2}$$

where the superscript (1) denotes the first-order perturbation of a given quantity.
We will consider that the theory is clear from anisotropy stress

$$\Pi \equiv 0 \, .$$

Moreover, we will work in the Newtonian gauge (see for details Section 2.6), where the geometric shear σ defined by $\sigma = -\omega^{\parallel} + \frac{\chi'^{\parallel}}{2}$ is identically zero

$$\sigma \equiv 0 \, ,$$

and where the the Bardeen potential $\Phi_H = \varphi + \frac{1}{6}\nabla^2\chi$ is equal to

$$\Psi = \varphi = \Phi_H \ .$$

The first term on the left-hand side of (7.2) becomes

$$\frac{1}{2}G^{(1)}_{00} = 3H\dot{\Psi} - \nabla^2\Psi \ , \tag{7.3}$$

while the right-hand side term is

$$4\pi G\, T^{(1)}_{00} = 4\pi G a^2 \rho_0(\delta + 2\Psi) \ .$$

So the $0 - 0$ component at first-order (7.2) becomes

$$3H(\dot{\Psi} + H\Psi) - \nabla^2\Psi = 4\pi G a^2 \rho_0 \delta \ . \tag{7.4}$$

The term in parenthesis $\dot{\Psi} + H\Psi$ can be obtained from the $0 - i$ component given by

$$G_{0i} = 8\pi G T_{0i} \ ,$$

where $T_{0i} = g_{0\mu}T^{\mu}{}_i = g_{00}T^0{}_i = -a^2 q_i$, and so

$$\partial_i(\dot{\Psi} + H\Psi) = -4\pi G a^2 q_i \ ,$$

with $q_i = (\rho_0 + p_0)\partial_i v$.
Now we can integrate the previous equation assuming that perturbations decay at infinity and obtaining

$$\dot{\Psi} + H\Psi = -4\pi G a^2(\rho_0 + p_0)v \ , \tag{7.5}$$

or also

$$v = -\frac{1}{4\pi G a^2(\rho_0 + p_0)}(\dot{\Psi} + H\Psi) \tag{7.6}$$

for later use.
Inserting (7.5) in (7.4), we obtain the **Poisson equation**

$$\nabla^2\Psi = 4\pi G a^2 \rho_0 \Delta \ , \tag{7.7}$$

where we have defined

$$\rho_0\Delta = \rho_0\delta - 3H(\rho_0 + p_0)v = \rho_0\delta + \dot{\rho}v \ ,$$

that is the so-called "comoving density perturbation".
Let us open a short parenthesis to show that the latter quantity is indeed a gauge-invariant quantity. In fact, remember from Section 2.6 that a generic gauge transformation is described by a vector ξ^μ such that it is transformed as

$$\xi^\mu = \left(\xi^0, \xi^i\right) \ ,$$

with

$$\xi^0 = \alpha \ ,$$
$$\xi^i = \partial^i\beta + d^i \ ,$$

in which α and β are scalars, d^i is a vector which obeys the condition $\partial_i d^i = 0$. Gauge transformations act on the following quantities as

$$\delta\rho \to \tilde{\delta\rho} = \delta\rho + \rho'\alpha \,,$$
$$v \to \tilde{v} = v - \beta' \,,$$
$$\omega \to \tilde{\omega} = \omega - \alpha + \beta' \,.$$

For the density contrast Δ previously defined we therefore find

$$
\begin{aligned}
\widetilde{\rho_0\Delta} &= \tilde{\delta\rho} + \rho_0'(\tilde{v} + \tilde{\omega}) \\
&= \delta\rho + \alpha\rho_0' + \rho_0'(v - \beta' + \omega - \alpha + \beta') \\
&= \delta\rho + \alpha\rho_0' + \rho_0'(v + \omega - \alpha) \\
&= \delta\rho + \rho_0'(v + \omega) \\
&= \rho_0\Delta \,.
\end{aligned}
$$

In the Newtonian gauge is $\omega \equiv 0$ by definition and so the previous argument remains true.

7.2 Poisson equation in Mimetic Horndeski

We will start from the Mimetic Horndeski equations of motion, that at first-order in perturbation theory with the energy-momentum tensor of a generic fluid included result in the following set of independent equations [4]

$$2b_0\delta\phi' + \phi_0' b_{,\phi}\,\delta\phi - 2b_0\phi_0'\Psi = 0 \tag{7.8}$$

$$f_7\Phi + f_8\delta\phi + f_9\Psi + a^2\Pi = 0 \tag{7.9}$$

$$f_{10}\Phi' + f_{11}\delta\phi' + \left(f_{20} + \frac{a^2(E^{(0)} + T^{(0)})}{\phi_0'}\right)\delta\phi + f_{14}\Psi - a^2(\rho_0 + p_0)v = 0 \tag{7.10}$$

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - 3(\rho_0 + p_0)\Phi' + (\rho_0 + p_0)\nabla^2 v = 0 \tag{7.11}$$

$$((\rho_0 + p_0)v)' + \delta p + \frac{2}{3}\nabla^2\Pi + 4\mathcal{H}(\rho_0 + p_0)v + (\rho_0 + p_0)\Psi = 0 \,, \tag{7.12}$$

where the reduced Planck mass is assumed to be one. Remember that we will use the apostrophe ' to indicate the derivative with respect to the conformal time η.
Notice that, in particular, equation (7.8) derives from the constraint equation

$$b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1 = 0 \,,$$

while equation (7.9) derives from the traceless term of the $i - j$ component, equation (7.10) comes from the $0 - i$ component and finally the last two equations (7.11) and (7.12) derive from the continuity equation for the energy-momentum tensor $T^{\mu\nu}$. Moreover, notice that the $i - j$ component gives origin to (6.41), that we rewrite here

$$f_{10}\Phi'' + f_{11}\delta\phi'' + f_{12}\Phi' + f_{13}\delta\phi' + f_{14}\Psi' + f_{15}\Phi + f_{16}\delta\phi + f_{17}\Psi + \frac{2}{3}a^2\nabla^2\Pi + a^2\,(\delta p - 2p_0\Phi) = 0 \tag{7.13}$$

for later use.
Thanks to the invariance in form of the Horndeski theory under a field redefinition, we can assume that $b(\phi) = -1$, such that the previous expression of the generalized Mimetic

constraint becomes similar to the original one $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$. This will be useful in the simplification of the next calculations.

In particular (7.8) becomes

$$\Psi = \frac{\delta\phi'}{\phi_0'} + \frac{b_{,\phi}}{2b_0}\delta\phi \ ,$$

and with the previous simplification $b(\phi) = -1$ and $b_{,\phi} = 0$ we obtain

$$\Psi = \frac{\delta\phi'}{\phi_0'} \ . \tag{7.14}$$

Velocity term

Take now (7.10) and write it in the following way - with the obvious definition of B -

$$f_{10}\Phi' + f_{11}\delta\phi' + B\delta\phi + f_{14}\Psi - a^2(\rho_0 + p_0)v = 0 \ ,$$

where the velocity v explicitely becomes

$$v = \frac{1}{a^2(\rho_0 + p_0)}\Big[f_{10}\Phi' + f_{11}\delta\phi' + B\delta\phi + f_{14}\Psi\Big] \ , \tag{7.15}$$

and, inserting (7.14), we have

$$v = \frac{1}{a^2(\rho_0 + p_0)}\Big[f_{10}\Phi' + (f_{11}\phi_0' + f_{14})\Psi + B\delta\phi\Big] \ .$$

The term in parenthesis can be simplified using the second identity of the functions f_i in B(18) in the Appendix B (that is to say $f_{14} - \mathcal{H}f_{10} + \phi_0'f_{11} = 0$), obtaining

$$v = \frac{1}{a^2(\rho_0 + p_0)}\Big[f_{10}\Phi' + f_{10}H\Psi + B\delta\phi\Big] \ . \tag{7.16}$$

Notice that one can check the relation for velocity (7.15) found before in comparison with the one in General Relativity. We will use the relations in the previous Chapter to reduce the f_i functions to their simpler value in the GR limit. In fact, we can consider the velocity in (7.15)

$$v = \frac{1}{a^2(\rho_0 + p_0)}\Big[f_{10}\Phi' + f_{11}\delta\phi' + B\delta\phi + f_{14}\Psi\Big] \ . \tag{7.17}$$

Notice that in this form, it has been obtained without using the Mimetic constraint. This fact allows us to look for the GR-limit and, using the simplifications $\delta\phi \equiv 0$ and $\Phi \equiv \Psi$ in the case of vanishing anisotropic stress, we obtain

$$v = \frac{1}{a^2(\rho_0 + p_0)}\Big[f_{10}\Phi' + f_{10}\mathcal{H}\Psi + B\delta\phi\Big] \tag{7.18}$$

$$= -\frac{2}{a^2(\rho_0 + p_0)}\Big[\Psi' + H\Psi\Big] \ , \tag{7.19}$$

that is the very form of (7.5) in GR, considering the reduced Planck mass.

7.2.1 GR-method applied to the Mimetic Horndeski

We will now see if and how we can apply the standard method used in GR as explained in Section 7.1 to the Mimetic Horndeski models. In particular, we will prove that the terms that in GR give origin to the $\nabla^2 \Psi$ which is at the origin of the Poisson equation, in this theory identically vanish in the $0-0$ Einstein equation.

Before showing this in details, let us make a preliminary check.
As we have seen in the case of the velocity (7.17), we can immediatly verify the correspondence of some terms of Horndeski theory (the general one, not its Mimetic scenario) with the ones of GR. For example, the term $E_{00}^{(1)}$ in (6.14)

$$E_{00}^{(1)} = f_1 \Phi' + f_2 \delta\phi' + f_3 \Psi + f_4 \delta\phi + f_5 \nabla^2 \Phi + f_6 \nabla^2 \delta\phi$$

in GR-limit - using the simplifications in Table 6.1 - becomes

$$E_{00}^{(1)} \to 6\mathcal{H}\Phi' - 2\nabla^2\Phi = 2(3\mathcal{H}\Psi' - \nabla^2\Psi) \,, \tag{7.20}$$

where we can see that there is the same term of (7.3), that corresponds to the Einstein tensor $G_{00}^{(1)}$.

Application of GR-method

From now on, we will consider the presence of a matter source. The equations of motion for the Mimetic Horndeski theory with matter are the following

$$b(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1 = 0 \,, \tag{7.21}$$

$$E_{\mu\nu} + T_{\mu\nu} = (E + T)b(\phi)\partial_\mu\phi\partial_\nu\phi \,, \tag{7.22}$$

$$\nabla_\mu T^{\mu\nu} = 0 \,. \tag{7.23}$$

To follow the canonical procedure to derive the Poisson equation, we consider the $0-0$ component of (7.22)

$$E_{00} + T_{00} = (E + T)b(\phi)\phi'^2$$

and we take its first-order perturbation

$$E_{00}^{(1)} + T_{00}^{(1)} = (E^{(1)} + T^{(1)})b_0\phi_0'^2 + 2(E^{(0)} + T^{(0)})b_0\phi_0'\delta\phi' \,,$$

where we used $b(\phi) = -1$.
The calculation of the first-order trace $E^{(1)}$ gives

$$E^{(1)} = -\frac{1}{a^2}(E_{00}^{(1)} - \delta^{ij}E_{ij}^{(1)} - 2\Psi E_{00}^{(0)} + 2\Phi\delta^{ij}E_{ij}^{(0)}) \,, \tag{7.24}$$

and we remember that from the zeroth-order equation of motion (6.22)

$$b_0\phi_0'^2 = -a^2 \,.$$

Using these simplifications, we obtain

$$\begin{aligned}
E_{00}^{(1)} + T_{00}^{(1)} &= (E^{(1)} + T^{(1)})b_0\phi_0'^2 + 2(E^{(0)} + T^{(0)})b_0\phi_0'\delta\phi' \\
&= -a^2(E^{(1)} + T^{(1)}) + 2(E^{(0)} + T^{(0)})b_0\phi_0'\delta\phi' \\
&= E_{00}^{(1)} - a^2 T^{(1)} + 2(E^{(0)} + T^{(0)})b_0\phi_0'\delta\phi' - \delta^{ij}E_{ij}^{(1)} - 2\Psi E_{00}^{(0)} + 2\Phi\delta^{ij}E_{ij}^{(0)}
\end{aligned}$$
$$\tag{7.25}$$

and we remain with

$$\cancel{E_{00}^{(1)}} + T_{00}^{(1)} = \cancel{E_{00}^{(1)}} - a^2 T^{(1)} + 2(E^{(0)} + T^{(0)})b_0\phi_0'\delta\phi' - \delta^{ij}E_{ij}^{(1)} - 2\Psi E_{00}^{(0)} + 2\Phi\delta^{ij}E_{ij}^{(0)} \ . \ (7.26)$$

The previous equation is very important because we note that the term $E_{00}^{(1)}$ cancels on the two sides of equation. This implies that the spatial derivatives in that term, that - as we have seen in the previous section - are at the origin of the $\nabla^2\Psi$ in the Poisson equation in GR as in (7.20), simply vanish.

This is the demonstration that the usual method applied in GR to extract the Poisson equation, in this theory is not straightforward. In what remains in the previous equation, we have all terms with only time derivatives in the fields.

7.3 Alternative method

In this section we will derive the Poisson equation in a different way with respect to the classical method that has been described in the previous section.

Remember the expression of the velocity (7.6). Remembering that in GR yields the relation $\Phi = \Psi$, we can rewrite it in the following way

$$\begin{aligned} v &= -\frac{1}{4\pi G a^2(\rho_0 + p_0)}(\Psi' + \mathcal{H}\Psi) \\ &= -\frac{1}{4\pi G a^3(\rho_0 + p_0)}(a\Psi)' \ , \end{aligned} \qquad (7.27)$$

where the term between parenthesis has been expressed as

$$\Psi' + \mathcal{H}\Psi = \frac{1}{a}(a\Psi)' \ . \qquad (7.28)$$

This method is based on the fact that we do not use the $0 - 0$ component of Einstein equations as in the standard method described in the previous Section. However, since the Einstein equations and the conservation law for the tensor $T^{\mu\nu}$ are linearly dependent, we can substitute the $0 - 0$ component of the former with the continuity equation $\nabla_\mu T^{\mu\nu} = 0$. In particular, we consider its first-order perturbation that gives equations (6.38) and the other one (6.37) that is

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - 3(\rho_0 + p_0)\Psi' + (\rho_0 + p_0)\nabla^2 v = 0 \ . \qquad (7.29)$$

We will consider from now on the small-scale limit $k \gg aH$, in which there are the two following simplifications.

The first simplification come from considering a perfect fluid of matter in non-relativistic regime and with negligible interactions, so that the pressure and its first-order perturbation are negligible: $p_0 \simeq 0$ and $\delta p \simeq 0$.

Moreover, the term $3(\rho_0 + p_0)\Psi' \simeq \rho_0\Psi'$ in (7.29) becomes negligible with respect to the term proportional to the Laplacian of v: in fact, from (7.29) and (7.27) one obtains

$$\rho_0\nabla^2 v \propto \frac{1}{Ga^2}\nabla^2\Psi' \ ,$$

and, since the latter in Fourier space becomes proportional to $\frac{k^2}{Ga^2}\Psi'$, it yields

$$\frac{k^2}{Ga^2}\Psi' \gg \rho\Psi' \ ,$$

and finally from (1.4) without curvature one obtains

$$k^2 \Psi' \gg a^2 \mathcal{H}^2 \Psi'$$

on small-scales ($k \gg a\mathcal{H}$).

Thus, equation (7.29) becomes

$$\delta\rho' + 3\mathcal{H}\delta\rho + \rho_0 \nabla^2 v = 0 \ ,$$

from which the term $\delta\rho' + 3\mathcal{H}\delta\rho$ can be rewritten as

$$\frac{1}{a^3} \left(\delta\rho a^3\right)' + \rho_0 \nabla^2 v = 0 \ . \tag{7.30}$$

Finally, inserting (7.27) in (7.30) one obtains

$$\left(\delta\rho a^3\right)' = \frac{1}{4\pi G} \nabla^2 \left(a\Psi\right)' \ ,$$

where, integrating in time, it becomes

$$\nabla^2 \Psi = 4\pi G \rho_0 a^2 \delta \ ,$$

that is the very **Poisson equation**.
Note that the $0-0$ component of Einstein equations has not been used in this calculation.

7.4 Poisson equation and modified gravity parameters in Mimetic Horndesky

From now on, we will consider non-relativistic and pressureless matter $p_0 \simeq 0$ and $\delta p \simeq 0$. We will assume also that the anisotropic stress is negligible $\Pi \equiv 0$. Therefore $\rho_0 + p_0 \simeq \rho_0$. Using relation (7.9), we find that

$$\delta\phi = - \left[\frac{f_9}{f_8} \Psi + \frac{f_7}{f_8} \Phi\right] \ .$$

Inserting this expression for $\delta\phi$ in the velocity (7.16), we obtain

$$\begin{aligned}
v &= \frac{1}{a^2 \rho_0} \left[f_{10} \Phi' + f_{10} \mathcal{H} \Psi + B \delta\phi\right] \\
&= \frac{1}{a^2 \rho_0} \left[f_{10} \Phi' + f_{10} \mathcal{H} \Psi - B \left(\frac{f_9}{f_8} \Psi + \frac{f_7}{f_8} \Phi\right)\right] \\
&= \frac{1}{a^2 \rho_0} \left[f_{10} \Phi' - B \frac{f_7}{f_8} \Phi + \left(f_{10} \mathcal{H} - B \frac{f_9}{f_8}\right) \Psi\right] \ . \tag{7.31}
\end{aligned}$$

Now we look for a relation between Φ and Ψ. Notice that from equation (7.9) one can also express Φ as a function of Ψ and $\delta\phi$

$$\Phi = - \left[\frac{f_9}{f_7} \Psi + \frac{f_8}{f_7} \delta\phi\right] \ .$$

We can use the relation (7.14) to obtain

$$\Phi = -\left[\frac{f_9}{f_7}\frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7}\delta\phi\right] \, , \tag{7.32}$$

that is, Φ is expressed as a function of $\delta\phi$ and its derivatives only.
Now we take the ratio between the previous (7.32) and (7.14), obtaining

$$\begin{aligned}
\frac{\Phi}{\Psi} &= \frac{-\left[\frac{f_9}{f_7}\frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7}\delta\phi\right]}{\frac{\delta\phi'}{\phi_0'}} \\
&= -\left[\frac{f_9}{f_7} + \frac{f_8}{f_7}\frac{\delta\phi}{\delta\phi'}\phi_0'\right] \\
&= -\left[\frac{f_9}{f_7} + \frac{f_8}{f_7}\frac{\phi_0'}{(\ln\delta\phi)'}\right] \\
&= \gamma(\delta\phi, \delta\phi', t) \, , \tag{7.33}
\end{aligned}$$

that is, the field Φ can be expressed in terms of the field Ψ and a function of $\delta\phi$ and its derivatives only. The behaviour of the function $\gamma(\delta\phi, \delta\phi', t)$ can be determined considering that the fields Φ and Ψ can be written in terms of $\delta\phi$ and its derivatives only in the equation (7.13) and using (7.32), determining an evolution equation for $\delta\phi$

$$\begin{aligned}
0 =& f_{10}\Phi'' + f_{11}\delta\phi'' + f_{12}\Phi' + f_{13}\delta\phi' + f_{14}\Psi' + f_{15}\Phi + f_{16}\delta\phi + f_{17}\Psi \\
=& - f_{10}\left[\frac{f_9}{f_7}\frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7}\delta\phi\right]'' + f_{11}\delta\phi'' - f_{12}\left[\frac{f_9}{f_7}\frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7}\delta\phi\right]' + f_{13}\delta\phi' + f_{14}\left(\frac{\delta\phi'}{\phi_0'}\right)' + \\
& - f_{15}\left[\frac{f_9}{f_7}\frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7}\delta\phi\right] + f_{16}\delta\phi + f_{17}\frac{\delta\phi'}{\phi_0'} \\
=& - f_{10}\left[\frac{f_9}{f_7}\frac{\delta\phi'}{\phi_0'}\right]'' - f_{10}\left[\frac{f_8}{f_7}\delta\phi\right]'' + f_{11}\delta\phi'' - f_{12}\left[\frac{f_9}{f_7}\frac{\delta\phi'}{\phi_0'}\right]' - f_{12}\left[\frac{f_8}{f_7}\delta\phi\right]' + f_{13}\delta\phi' + \\
& + f_{14}\left(\frac{\delta\phi'}{\phi_0'}\right)' - f_{15}\left[\frac{f_9}{f_7}\frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7}\delta\phi\right] + f_{16}\delta\phi + f_{17}\frac{\delta\phi'}{\phi_0'} \tag{7.34}
\end{aligned}$$

that can be solved for $\delta\phi$.[1]
Notice that equation (7.33) is of the form $\Psi = \gamma^{-1}\Phi$ and can be compared with the definition of the parameter $\tilde{\gamma}$ in (3.17) in Section 3.2.1. The phenomenological determination of the function $\tilde{\gamma}$ can be an experimental check of the Mimetic Horndeski theory.

---

[1] Alternatively, one can notice that equations (7.8), (7.9) and (7.13) form a system of three equations in the three variables Ψ, Φ and $\delta\phi$, from which one can single out an homogenous equation for Φ. Once the solution for Φ is found, the evolution of $\delta\phi$ can be obtained by integrating (7.39).

The velocity (7.31) can be rewritten using (7.33) as

$$
\begin{aligned}
v &= \frac{1}{a^2 \rho_0} \left[f_{10} \Phi' - B \frac{f_7}{f_8} \Phi + \left(f_{10} \mathcal{H} - B \frac{f_9}{f_8} \right) \Psi \right] \\
&= \frac{1}{a^2 \rho_0} \left[f_{10} \Phi' - B \frac{f_7}{f_8} \Phi + \left(f_{10} \mathcal{H} - B \frac{f_9}{f_8} \right) \gamma^{-1} \Phi \right] \\
&= \frac{1}{a^2 \rho_0} \left[f_{10} \Phi' + \left(f_{10} \mathcal{H} - B \frac{f_9}{f_8} - B \frac{f_7}{f_8} \right) \gamma^{-1} \Phi \right] \\
&= \frac{1}{a^2 \rho_0} \left[f_{10} \left(\Phi' + \left(\mathcal{H} - B \frac{f_9}{f_8 f_{10}} - B \frac{f_7 \gamma}{f_8 f_{10}} \right) \gamma^{-1} \Phi \right) \right] \\
&= \frac{1}{a^2 \rho_0} \left[f_{10} \left(\Phi' + \frac{(aF(\eta))'}{aF(\eta)} \Phi \right) \right] \ ,
\end{aligned}
$$

where we have defined the function $F(\eta)$ such that

$$
\frac{(aF(\eta))'}{aF(\eta)} \equiv \mathcal{H} - B \frac{f_9 + f_7 \gamma}{f_8 f_{10}} \ .
$$

Notice that the velocity can now be written as

$$
v = \frac{1}{a^3 \rho_0} f_{10} \frac{(aF(\eta)\Phi)'}{F(\eta)} \ , \tag{7.35}
$$

and it can be inserted in (7.30). Notice that the f_i functions are background quantities, and therefore they are defined by the model that one uses. Thus, defining the function $\frac{f_{10}}{F(\eta)}$, the velocity can be integrated in time to obtain a generalized form of Poisson equation.

7.4.1   Generalized Poisson equation

We can now write down a generalized Poisson equation for the Mimetic Horndeski theory.

Consider equation (7.11) (in which, as before, we can neglect the $3(\rho_0 + p_0)\Phi'$ term) in Fourier space and insert the velocity (7.35)

$$
\frac{1}{a^3} \left(\delta\rho a^3 \right)' = \frac{k^2}{a^3} f_{10} \frac{(aF(\eta)\Phi)'}{F(\eta)} \ , \tag{7.36}
$$

and thus

$$
\left(\delta\rho a^3 \right)' = k^2 \frac{f_{10}}{F(\eta)} (aF(\eta)\Phi)' \ , \tag{7.37}
$$

or equivalently

$$
\frac{F(\eta)}{f_{10}} \left(\delta\rho a^3 \right)' = k^2 (aF(\eta)\Phi)' \ . \tag{7.38}
$$

We can now use equations (7.9) and (7.14) to rewrite Φ in terms of $\delta\phi$

$$
\Phi = - \left[\frac{f_9}{f_7} \frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7} \delta\phi \right] \ . \tag{7.39}
$$

Therefore, equation (7.37) becomes

$$
\left(\delta\rho a^3 \right)' = -k^2 \frac{f_{10}}{F(\eta)} \left[aF(\eta) \left(\frac{f_9}{f_7} \frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7} \delta\phi \right) \right]' \ . \tag{7.40}
$$

Equation (7.40) allows us to obtain $\delta\rho$ as a functional, that we will call Δ, of the field $\delta\phi$ and its derivatives: this can be seen simply integrating in time (7.40) and finding that

$$\delta\rho = -\frac{k^2}{a^3} \int d\eta \, \frac{f_{10}}{F(\eta)} \left[aF(\eta) \left(\frac{f_9}{f_7} \frac{\delta\phi'}{\phi_0'} + \frac{f_8}{f_7} \delta\phi \right) \right]' \tag{7.41}$$

$$= -\frac{k^2}{a^3} \Delta[\delta\phi, \delta\phi', \delta\phi'', \eta] \ . \tag{7.42}$$

We will use this expression below.

Now let us come back to (7.36). By an integration by parts, one obtains

$$\int \beta(\eta) \left(\delta\rho a^3 \right)' d\eta = \beta(\eta)\delta\rho a^3 - \int \beta'(\eta)\delta\rho a^3 d\eta \ ,$$

where we defined $\beta(\eta) = F(\eta)/f_{10}$. Multiplying and dividing by $a^3\delta\rho$ the integral on the right-hand side, one can write

$$\beta(\eta)\delta\rho a^3 - a^3\delta\rho \frac{\int \beta'(\eta)\delta\rho a^3 d\eta}{a^3\delta\rho} \tag{7.43}$$

$$= \delta\rho a^3 \left[\beta(\eta) - \frac{\int \beta'(\eta)\delta\rho a^3 d\eta}{a^3\delta\rho} \right] \tag{7.44}$$

$$= \delta\rho a^3 \left[\beta(\eta) - \frac{\int \beta'(\eta)\Delta[\delta\phi, \delta\phi', \delta\phi'', \eta] d\eta}{\Delta[\delta\phi, \delta\phi', \delta\phi'', \eta]} \right] \ , \tag{7.45}$$

where we have used (7.42). Notice that there is no dependence on the wavenumber k, but only on time.

Finally, we can now take (7.45), (7.38) and (7.33) to write

$$k^2\Psi = -4\pi G_{\text{eff}}(\eta) \, \delta\rho a^2 \ , \tag{7.46}$$

that is the generalized Poisson equation in Mimetic Horndeski models we were looking for. In equation (7.46) the quantity $G_{\text{eff}}(\eta)$ is given by

$$G_{\text{eff}}(\eta) = -\frac{2 \left[\beta(\eta) - \frac{\int \beta'(\eta)\Delta[\delta\phi, \delta\phi', \delta\phi'', \eta] d\eta}{\Delta[\delta\phi, \delta\phi', \delta\phi'', \eta]} \right]}{\gamma(\delta\phi, \delta\phi', t) \ F(\eta)}$$

and plays the role of a modified gravitational constant (see the discussion around (3.18) in Section 3.2.1).

Moreover, notice that the definition of the effective gravitational constant G_{eff} allows us to do a comparison between G_{eff} and the experimental value of the gravitational constant G. This can be done by using equation (3.20) in Section 3.2.1, and defining a parameter μ such that

$$\mu = \frac{G_{\text{eff}}}{G} \ .$$

Finally, one can obtain also the Σ parameter in (3.20) using the relation (3.22) between the three parameters.

Conclusions

In this Thesis we discussed the so-called Mimetic gravity scenario, that allows us to provide new models of modified gravity based on a redefinition of the physical metric in terms on an extra scalar field ϕ and of an auxiliary metric. This theory, in its first proposal, turns to be very interesting because it can offer at the same time a description for Dark Matter and Dark Energy contributions, without appealing to the existence of new components in the Universe. These dark components arise naturally in the theory: the first one has its origin in the redefinition of the metric upon which the theory is based, and the latter is mimicked by a potential term that could, in general, describe almost every cosmology.

We showed that the Mimetic theory can be generalized to very general scalar-tensor theories. This generalization can be obtained in two ways: the first is the application to the physical metric of a particular type of disformal transformation, while the second procedure is by inserting the Mimetic constraint that characterizes the Mimetic theory in the main Lagrangian by using a Lagrange multiplier. This procedures promote the Mimetic theory to a more general Mimetic scenario.

Finally, we discussed the application of the Mimetic scenario to the Horndeski theory. This theory is a healthly generalization of General Relativity and includes many other theories of modified gravity as particular cases. We presented the cosmological perturbations of the Mimetic Horndeski theory and some applications.

Anyway, many aspects of the theory need to be analysed in a deeper way. A first step would be to understand the physical meaning of the vanishing speed of sound for scalar cosmological perturbations.

Another feature that will be important to analyse is the phenomenological tests of this model: new measurements of observables of the Large-Scale Structure of the Universe will confirm the theoretical predictions of the Mimetic model. This aspect will soon be studied in a deeper way with the forthcoming Euclid mission, that would provide some very important probes of the dark sector of the Universe. Thus, it would be very interesting to continue the analysis of this theory to produce forecasts and possible new observational tests to which apply next data of the Euclid satellite.

Appendix A

Background equations of motion

Here we recall the expressions for the tensor $E_{\mu\nu}$ on a flat FLRW background. These expressions are valid in the Horndeski and also in Mimetic Horndeski models. The non-vanishing components are

$$
\begin{aligned}
E_{00}^{(0)} &= -a^2 K - 6G_4\mathcal{H}^2 - 6G_{4,\varphi}\mathcal{H}\varphi_0' + (\varphi_0')^2\left(\frac{12G_{4,X}\mathcal{H}^2 - 9G_{5,\varphi}\mathcal{H}^2}{a^2} - G_{3,\varphi} + K_{,X}\right) + \\
&\quad + (\varphi_0')^3\left(\frac{5G_{5,X}\mathcal{H}^3}{a^4} + \frac{3G_{3,X}\mathcal{H} - 6G_{4,X\varphi}\mathcal{H}}{a^2}\right) + (\varphi_0')^4\frac{\left(6G_{4,XX}\mathcal{H}^2 - 3G_{5,X\varphi}\mathcal{H}^2\right)}{a^4} + \\
&\quad + (\varphi_0')^5\frac{G_{5,XX}\mathcal{H}^3}{a^6}, \tag{A.1}
\end{aligned}
$$

$$
\begin{aligned}
E_{ij}^{(0)} &= \delta_{ij}\Bigg[a^2 K + 2G_4\mathcal{H}^2 + 4G_4\mathcal{H}' + \varphi_0'\left(\frac{4G_{5,\varphi}\mathcal{H}\varphi_0'' - 4G_{4,X}\mathcal{H}\varphi_0''}{a^2} + 2G_{4,\varphi}\mathcal{H}\right) + \\
&\quad + (\varphi_0')^2\left(-G_{3,\varphi} + 2G_{4,\varphi\varphi} - \frac{3G_{5,X}\mathcal{H}^2\varphi_0''}{a^4} + \right. \\
&\quad \left. + \frac{-G_{3,X}\varphi_0'' + 2G_{4,X}(\mathcal{H}^2 - 2\mathcal{H}') + 2G_{4,X\varphi}\varphi_0'' + G_{5,\varphi}(2\mathcal{H}' - 3\mathcal{H}^2)}{a^2}\right) + \\
&\quad + (\varphi_0')^3\left(\frac{-4G_{4,XX}\mathcal{H}\varphi_0'' + 2G_{5,X\varphi}\mathcal{H}\varphi_0'' + 3G_{5,X}\mathcal{H}^3 - 2G_{5,X}\mathcal{H}\mathcal{H}'}{a^4} + \right. \\
&\quad \left. + \frac{G_{3,X}\mathcal{H} - 6G_{4,X\varphi}\mathcal{H} + 2G_{5,\varphi\varphi}\mathcal{H}}{a^2}\right) + \\
&\quad + (\varphi_0')^4\left(-\frac{G_{5,XX}\mathcal{H}^2\varphi_0''}{a^6} + \frac{4G_{4,XX}\mathcal{H}^2 - 3G_{5,X\varphi}\mathcal{H}^2}{a^4}\right) + \\
&\quad + \frac{G_{5,XX}\mathcal{H}^3(\varphi_0')^5}{a^6} + 2G_{4,\varphi}\varphi_0''\Bigg]. \tag{A.2}
\end{aligned}
$$

The zeroth-order trace $E^{(0)}$ can be easily computed from the previous equations by using $E^{(0)} = -a^{-2}E_{00}^{(0)} + a^{-2}\delta^{ij}E_{ij}^{(0)}$.

Appendix B

Explicit expressions of the f_i functions

In this Appendix we give the explicit expressions for the functions f_i, $i = 1, ..., 21$ defined in the main text. These expressions can be used for both the Horndeski and mimetic Horndeski models because no equations of motion were used.

They read

$$
\begin{aligned}
f_1 &= 12G_4\mathcal{H} + 6G_{4,\varphi}\varphi_0' + \frac{(18G_{5,\varphi}\mathcal{H} - 24G_{4,X}\mathcal{H})}{a^2}(\varphi_0')^2 + \\
&\quad + (\varphi_0')^3\left(\frac{6G_{4,X\varphi} - 3G_{3,X}}{a^2} - \frac{15G_{5,X}\mathcal{H}^2}{a^4}\right) + \\
&\quad + \frac{(6G_{5,X\varphi}\mathcal{H} - 12G_{4,XX}\mathcal{H})}{a^4}(\varphi_0')^4 - \frac{3G_{5,XX}\mathcal{H}^2}{a^6}(\varphi_0')^5,
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
f_2 &= -6G_{4,\varphi}\mathcal{H} + \varphi_0'\left(\frac{18G_{4,X}\mathcal{H}^2 - 18G_{5,\varphi}\mathcal{H}^2}{a^2} - 2G_{3,\varphi} + K_{,X}\right) + \\
&\quad + (\varphi_0')^2\left(\frac{15G_{5,X}\mathcal{H}^3}{a^4} + \frac{9G_{3,X}\mathcal{H} - 24G_{4,X\varphi}\mathcal{H}}{a^2}\right) + \\
&\quad + (\varphi_0')^3\left(\frac{36G_{4,XX}\mathcal{H}^2 - 21G_{5,X\varphi}\mathcal{H}^2}{a^4} + \frac{K_{,XX} - G_{3,X\varphi}}{a^2}\right) + \\
&\quad + (\varphi_0')^4\left(\frac{10G_{5,XX}\mathcal{H}^3}{a^6} + \frac{3G_{3,XX}\mathcal{H} - 6G_{4,XX\varphi}\mathcal{H}}{a^4}\right) + \\
&\quad + \frac{\left(6G_{4,XXX}\mathcal{H}^2 - 3G_{5,XX\varphi}\mathcal{H}^2\right)}{a^6}(\varphi_0')^5 + \frac{G_{5,XXX}\mathcal{H}^3}{a^8}(\varphi_0')^6,
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
f_3 &= -2a^2K + (\varphi_0')^2\left(\frac{18G_{5,\varphi}\mathcal{H}^2 - 18G_{4,X}\mathcal{H}^2}{a^2} + K_{,X}\right) + \\
&\quad + (\varphi_0')^3\left(\frac{18G_{4,X\varphi}\mathcal{H} - 6G_{3,X}\mathcal{H}}{a^2} - \frac{20G_{5,X}\mathcal{H}^3}{a^4}\right) + \\
&\quad + (\varphi_0')^4\left(\frac{21G_{5,X\varphi}\mathcal{H}^2 - 36G_{4,XX}\mathcal{H}^2}{a^4} + \frac{G_{3,X\varphi} - K_{,XX}}{a^2}\right) + \\
&\quad + (\varphi_0')^5\left(\frac{6G_{4,XX\varphi}\mathcal{H} - 3G_{3XX}\mathcal{H}}{a^4} - \frac{11G_{5,XX}\mathcal{H}^3}{a^6}\right) + \\
&\quad + (\varphi_0')^6\frac{\left(3G_{5,XX\varphi}\mathcal{H}^2 - 6G_{4,XXX}\mathcal{H}^2\right)}{a^6} - \frac{G_{5,XXX}\mathcal{H}^3(\varphi_0')^7}{a^8},
\end{aligned}
\tag{B.3}
$$

$$f_4 = -a^2 K_{,\varphi} - 6G_{4,\varphi}\mathcal{H}^2 - 6G_{4,\varphi\varphi}\mathcal{H}\varphi_0' + (\varphi_0')^2 \left(\frac{12G_{4,X\varphi}\mathcal{H}^2 - 9G_{5,\varphi\varphi}\mathcal{H}^2}{a^2} - G_{3,\varphi\varphi} + K_{,X\varphi} \right) +$$

$$+ (\varphi_0')^3 \left(\frac{5G_{5,X\varphi}\mathcal{H}^3}{a^4} + \frac{3G_{3,X\varphi}\mathcal{H} - 6G_{4,X\varphi\varphi}\mathcal{H}}{a^2} \right) + \frac{(6G_{4,XX\varphi}\mathcal{H}^2 - 3G_{5,X\varphi\varphi}\mathcal{H}^2)}{a^4} (\varphi_0')^4 +$$

$$+ \frac{G_{5,XX\varphi}\mathcal{H}^3}{a^6} (\varphi_0')^5, \tag{B.4}$$

$$f_5 = -4G_4 + \frac{(4G_{4,X} - 2G_{5,\varphi})(\varphi_0')^2}{a^2} + \frac{2G_{5,X}\mathcal{H}(\varphi_0')^3}{a^4}, \tag{B.5}$$

$$f_6 = 2G_{4,\varphi} + \frac{\varphi_0'(4G_{5,\varphi}\mathcal{H} - 4G_{4,X}\mathcal{H})}{a^2} + (\varphi_0')^2 \left(\frac{2G_{4,X\varphi} - G_{3,X}}{a^2} - \frac{3G_{5,X}\mathcal{H}^2}{a^4} \right) +$$

$$+ \frac{(2G_{5,X\varphi}\mathcal{H} - 4G_{4,XX}\mathcal{H})}{a^4} (\varphi_0')^3 - \frac{G_{5,XX}\mathcal{H}^2(\varphi_0')^4}{a^6}, \tag{B.6}$$

$$f_7 = -2G_4 + (\varphi_0')^2 \left(\frac{G_{5,X}\varphi_0''}{a^4} + \frac{G_{5,\varphi}}{a^2} \right) - \frac{G_{5,X}\mathcal{H}}{a^4} (\varphi_0')^3, \tag{B.7}$$

$$f_8 = 2G_{4,\varphi} + (\varphi_0')^2 \left(\frac{(G_{5,X\varphi} - 2G_{4,XX})\varphi_0''}{a^4} + \frac{2G_{5,X}\mathcal{H}^2 - G_{5,X}\mathcal{H}'}{a^4} + \frac{G_{5,\varphi\varphi} - 2G_{4,X\varphi}}{a^2} \right) +$$

$$+ (\varphi_0')^3 \left(\frac{2G_{4,XX}\mathcal{H} - 2G_{5,X\varphi}\mathcal{H}}{a^4} - \frac{G_{5,XX}\mathcal{H}\varphi_0''}{a^6} \right) + \frac{G_{5,XX}\mathcal{H}^2}{a^6} (\varphi_0')^4 +$$

$$- \frac{2G_{5,X}\mathcal{H}}{a^4}\varphi_0'\varphi_0'' + \frac{(2G_{5,\varphi} - 2G_{4,X})}{a^2}\varphi_0'', \tag{B.8}$$

$$f_9 = 2G_4 + \frac{(G_{5,\varphi} - 2G_{4,X})}{a^2} (\varphi_0')^2 - \frac{G_{5,X}\mathcal{H}}{a^4} (\varphi_0')^3, \tag{B.9}$$

$$f_{11} = 2G_{4,\varphi} + \frac{(4G_{5,\varphi}\mathcal{H} - 4G_{4,X}\mathcal{H})}{a^2}\varphi_0' + \left(\frac{2G_{4,X\varphi} - G_{3,X}}{a^2} - \frac{3G_{5,X}\mathcal{H}^2}{a^4} \right) (\varphi_0')^2 +$$

$$+ \frac{(2G_{5,X\varphi}\mathcal{H} - 4G_{4,XX}\mathcal{H})}{a^4} (\varphi_0')^3 - \frac{G_{5,XX}\mathcal{H}^2(\varphi_0')^4}{a^6}, \tag{B.10}$$

$$f_{12} = -8G_4\mathcal{H} + \varphi_0' \left(\frac{(4G_{4,X} - 4G_{5,\varphi})\varphi_0''}{a^2} - 4G_{4,\varphi} \right) + (\varphi_0')^2 \left(\frac{6G_{5,X}\mathcal{H}\varphi_0''}{a^4} + \frac{4G_{4,X}\mathcal{H}}{a^2} \right) +$$

$$+ (\varphi_0')^3 \left(\frac{(4G_{4,XX} - 2G_{5,X\varphi})\varphi_0''}{a^4} + \frac{2G_{5,X}\mathcal{H}' - 4G_{5,X}\mathcal{H}^2}{a^4} + \frac{4G_{4,X\varphi} - 2G_{5,\varphi\varphi}}{a^2} \right) +$$

$$+ (\varphi_0')^4 \left(\frac{2G_{5,XX}\mathcal{H}\varphi_0''}{a^6} + \frac{4G_{5,X\varphi}\mathcal{H} - 4G_{4,XX}\mathcal{H}}{a^4} \right) - \frac{2G_{5,XX}\mathcal{H}^2}{a^6} (\varphi_0')^5, \tag{B.11}$$

$$
\begin{aligned}
f_{13} = {}& 2G_{4,\varphi}\mathcal{H} + \\
& + \varphi_0'\left(\frac{2G_{4,X}(3\mathcal{H}^2 - 2\mathcal{H}') - 2G_{5,\varphi}(3\mathcal{H}^2 - 2\mathcal{H}')}{a^2} + \right. \\
& + \varphi_0''\left(\frac{6G_{4,X\varphi} - 2G_{3,X}}{a^2} - \frac{6G_{5,X}\mathcal{H}^2}{a^4}\right) - 2G_{3,\varphi} + 4G_{4,\varphi\varphi} + K_{,X}\Big) + \\
& + (\varphi_0')^2\left(\frac{\varphi_0''(10G_{5,X\varphi}\mathcal{H} - 16G_{4,XX}\mathcal{H})}{a^4} + \frac{9G_{5,X}\mathcal{H}^3 - 6G_{5,X}\mathcal{H}\mathcal{H}'}{a^4} + \right. \\
& + \frac{3G_{3,X}\mathcal{H} - 16G_{4,X\varphi}\mathcal{H} + 6G_{5,\varphi\varphi}\mathcal{H}}{a^2}\Big) + \\
& + (\varphi_0')^3\left(\frac{18G_{4,XX}\mathcal{H}^2 - 4G_{4,XX}\mathcal{H}' - 15G_{5,X\varphi}\mathcal{H}^2 + 2G_{5,X\varphi}\mathcal{H}'}{a^4} + \frac{2G_{4,X\varphi\varphi} - G_{3,X\varphi}}{a^2} + \right. \\
& + \varphi_0''\left(\frac{2G_{4,XX\varphi} - G_{3,XX}}{a^4} - \frac{7G_{5,XX}\mathcal{H}^2}{a^6}\right)\Big) + \\
& + (\varphi_0')^4\left(\frac{\varphi_0''(2G_{5,XX\varphi}\mathcal{H} - 4G_{4,XXX}\mathcal{H})}{a^6} + \frac{8G_{5,XX}\mathcal{H}^3 - 2G_{5,XX}\mathcal{H}\mathcal{H}'}{a^6} + \right. \\
& + \frac{G_{3,XX}\mathcal{H} - 6G_{4,XX\varphi}\mathcal{H} + 2G_{5,X\varphi\varphi}\mathcal{H}}{a^4}\Big) + \\
& + (\varphi_0')^5\left(\frac{4G_{4,XXX}\mathcal{H}^2 - 3G_{5,XX\varphi}\mathcal{H}^2}{a^6} - \frac{G_{5,XXX}\mathcal{H}^2\varphi_0''}{a^8}\right) + \frac{G_{5,XXX}\mathcal{H}^3(\varphi_0')^6}{a^8} + \\
& + \frac{\varphi_0''(4G_{5,\varphi}\mathcal{H} - 4G_{4,X}\mathcal{H})}{a^2},
\end{aligned} \tag{B.12}
$$

$$
\begin{aligned}
f_{14} = {}& -4G_4\mathcal{H} - 2G_{4,\varphi}\varphi_0' + (\varphi_0')^2\frac{(8G_{4,X}\mathcal{H} - 6G_{5,\varphi}\mathcal{H})}{a^2} + (\varphi_0')^3\left(\frac{5G_{5,X}\mathcal{H}^2}{a^4} + \frac{G_{3,X} - 2G_{4,X\varphi}}{a^2}\right) + \\
& + (\varphi_0')^4\frac{(4G_{4,XX}\mathcal{H} - 2G_{5,X\varphi}\mathcal{H})}{a^4} + \frac{G_{5,XX}\mathcal{H}^2(\varphi_0')^5}{a^6},
\end{aligned} \tag{B.13}
$$

$$
\begin{aligned}
f_{16} = {}& a^2 K_{,\varphi} + 2G_{4,\varphi}\mathcal{H}^2 + 4G_{4,\varphi}\mathcal{H}' + \varphi_0'\left(\frac{\varphi_0''(4G_{5,\varphi\varphi}\mathcal{H} - 4G_{4,X\varphi}\mathcal{H})}{a^2} + 2G_{4,\varphi\varphi}\mathcal{H}\right) + \\
& + (\varphi_0')^2\left(\frac{2G_{4,X\varphi}(\mathcal{H}^2 - 2\mathcal{H}') + G_{5,\varphi\varphi}(2\mathcal{H}' - 3\mathcal{H}^2)}{a^2} + \varphi_0''\left(\frac{2G_{4,X\varphi\varphi} - G_{3,X\varphi}}{a^2} - \frac{3G_{5,X\varphi}\mathcal{H}^2}{a^4}\right) + \right. \\
& - G_{3,\varphi\varphi} + 2G_{4,\varphi\varphi\varphi}\Big) + (\varphi_0')^3\left(\frac{\varphi_0''(2G_{5,X\varphi\varphi}\mathcal{H} - 4G_{4,XX\varphi}\mathcal{H})}{a^4} + \right. \\
& + \frac{3G_{5,X\varphi}\mathcal{H}^3 - 2G_{5,X\varphi}\mathcal{H}\mathcal{H}'}{a^4} + \frac{G_{3,X\varphi}\mathcal{H} - 6G_{4,X\varphi\varphi}\mathcal{H} + 2G_{5,\varphi\varphi\varphi}\mathcal{H}}{a^2}\Big) + \\
& + (\varphi_0')^4\left(\frac{4G_{4,XX\varphi}\mathcal{H}^2 - 3G_{5,X\varphi\varphi}\mathcal{H}^2}{a^4} - \frac{G_{5,XX\varphi}\mathcal{H}^2\varphi_0''}{a^6}\right) + \\
& + \frac{G_{5,XX\varphi}\mathcal{H}^3(\varphi_0')^5}{a^6} + 2G_{4,\varphi\varphi}\varphi_0'',
\end{aligned} \tag{B.14}
$$

$$
\begin{aligned}
f_{17} &= -4G_4\mathcal{H}^2 - 8G_4\mathcal{H}' + \varphi_0'\left(\frac{\varphi_0''(16G_{4,X}\mathcal{H} - 16G_{5,\varphi}\mathcal{H})}{a^2} - 4G_{4,\varphi}\mathcal{H}\right) + \\
&\quad + (\varphi_0')^2\left(+2G_{3,\varphi} - 4G_{4,\varphi\varphi} - K_{,X} + \frac{-10G_{4,X}\mathcal{H}^2 + 12G_{4,X}\mathcal{H}' + 12G_{5,\varphi}\mathcal{H}^2 - 8G_{5,\varphi}\mathcal{H}'}{a^2} + \right. \\
&\quad \left. + \varphi_0''\left(\frac{18G_{5,X}\mathcal{H}^2}{a^4} + \frac{4G_{3,X} - 10G_{4,X\varphi}}{a^2}\right)\right) + \\
&\quad + (\varphi_0')^3\left(\frac{\varphi_0''(28G_{4,XX}\mathcal{H} - 16G_{5,X\varphi}\mathcal{H})}{a^4} + \frac{12G_{5,X}\mathcal{H}\mathcal{H}' - 18G_{5,X}\mathcal{H}^3}{a^4} + \right. \\
&\quad \left. + \frac{-4G_{3,X}\mathcal{H} + 22G_{4,X\varphi}\mathcal{H} - 8G_{5,\varphi\varphi}\mathcal{H}}{a^2}\right) + \\
&\quad + (\varphi_0')^4\left(\frac{-26G_{4,XX}\mathcal{H}^2 + 4G_{4,XX}\mathcal{H}' + 21G_{5,X\varphi}\mathcal{H}^2 - 2G_{5,X\varphi}\mathcal{H}'}{a^4} + \frac{G_{3,X\varphi} - 2G_{4,X\varphi\varphi}}{a^2} + \right. \\
&\quad \left. + \varphi_0''\left(\frac{11G_{5,XX}\mathcal{H}^2}{a^6} + \frac{G_{3,XX} - 2G_{4,XX\varphi}}{a^4}\right)\right) + \\
&\quad + (\varphi_0')^5\left(\frac{\varphi_0''(4G_{4,XXX}\mathcal{H} - 2G_{5,XX\varphi}\mathcal{H})}{a^6} + \frac{2G_{5,XX}\mathcal{H}\mathcal{H}' - 11G_{5,XX}\mathcal{H}^3}{a^6} + \right. \\
&\quad \left. + \frac{-G_{3,XX}\mathcal{H} + 6G_{4,XX\varphi}\mathcal{H} - 2G_{5,X\varphi\varphi}\mathcal{H}}{a^4}\right) + \\
&\quad + \left(\varphi_0'\right)^6\left(\frac{G_{5,XXX}\mathcal{H}^2\varphi_0''}{a^8} + \frac{3G_{5,XX\varphi}\mathcal{H}^2 - 4G_{4,XXX}\mathcal{H}^2}{a^6}\right) + \\
&\quad - \frac{G_{5,XXX}\mathcal{H}^3}{a^8}\left(\varphi_0'\right)^7 - 4G_{4,\varphi}\varphi_0'', \tag{B.15}
\end{aligned}
$$

$$
\begin{aligned}
f_{20} &= -2G_{4,\varphi}\mathcal{H} + \left(\frac{6G_{4,X}\mathcal{H}^2 - 6G_{5,\varphi}\mathcal{H}^2}{a^2} - 2G_{3,\varphi} + 2G_{4,\varphi\varphi} + K_{,X}\right)\varphi_0' + \\
&\quad + \left(\frac{3G_{5,X}\mathcal{H}^3}{a^4} + \frac{3G_{3,X}\mathcal{H} - 10G_{4,X\varphi}\mathcal{H} + 2G_{5,\varphi\varphi}\mathcal{H}}{a^2}\right)(\varphi_0')^2 + \frac{\left(6G_{4,XX}\mathcal{H}^2 - 4G_{5,X\varphi}\mathcal{H}^2\right)}{a^4}(\varphi_0')^3 + \\
&\quad + \frac{G_{5,XX}\mathcal{H}^3}{a^6}(\varphi_0')^4 \;. \tag{B.16}
\end{aligned}
$$

The functions f_i obey the following identities:

$$
f_{10} = f_{18}, \qquad f_{11} = f_{19}, \qquad f_{21} = f_{14}, \qquad f_9 = -\frac{f_{10}}{2}, \qquad \frac{f_{10}' - f_{12}}{f_{10}} = -2\mathcal{H},
$$

$$
f_{11}(f_{10}' - f_{12}) + f_{10}(f_{13} - f_{20} - f_{11}') = 0, \qquad f_{14} - \mathcal{H}f_{10} + \varphi_0'f_{11} = 0, \tag{B.17}
$$

$$
f_{17} - \frac{f_{15}f_9}{f_7} - \frac{f_{14}}{f_{10}}(f_{12} - f_{10}') - f_{14}' + 3E_{ij}^{(0)} + E_{00}^{(0)} + \alpha E_{ij}^{(0)} = 0, \tag{B.18}
$$

$$
\left(f_{16} - \frac{f_8 f_{15}}{f_7} - \frac{f_{20}}{f_{10}}(f_{12} - f_{10}') - f_{20}'\right)\varphi_0' + E_{00}^{(0)'} + \mathcal{H}(3E_{ij}^{(0)} + E_{00}^{(0)}) + \beta E_{ij}^{(0)} = 0, \tag{B.19}
$$

$$
\left(\frac{\mathcal{H}}{\varphi_0'}\right)' f_{10} + \left(\frac{\varphi_0''}{(\varphi_0')^2} - \frac{\mathcal{H}}{\varphi_0'}\right) f_{14} - f_{20} - a^2\frac{E^{(0)}}{\varphi_0'} + 4\frac{E_{ij}^{(0)}}{\varphi_0'} = 0, \qquad 2E_{ij}^{(0)} = -f_{15}, \tag{B.20}
$$

where

$$
\begin{aligned}
\alpha &= -2 - 2\frac{f_9}{f_7} = -\frac{2(\varphi_0')^2 G_{4,X}}{a^2 G_4} + \\
&\quad + \frac{2(\varphi_0')^2\left(\left(a^2 G_{5,\varphi} - \mathcal{H}G_{5,X}\varphi_0'\right)\left(2a^2 G_4 - G_{4,X}(\varphi_0')^2\right) + G_{5,X}\left(a^2 G_4 - G_{4,X}(\varphi_0')^2\right)\varphi_0''\right)}{a^2 G_4\left(2a^4 G_4 - a^2 G_{5,\varphi}(\varphi_0')^2 + G_{5,X}(\varphi_0')^2\left(\mathcal{H}\varphi_0' - \varphi_0''\right)\right)} \;, \tag{B.21}
\end{aligned}
$$

and

$$\begin{aligned}
\beta &= -2\varphi_0'\frac{f_8}{f_7} = \frac{2\varphi_0'\left(a^4 G_{4,\varphi} + G_{4,XX}(\varphi_0')^2\left(\mathcal{H}\varphi_0' - \varphi_0''\right) - a^2\left(G_{4,X\varphi}(\varphi_0')^2 + G_{4,X}\varphi_0''\right)\right)}{a^4 G_4} + \\
&\quad + \frac{2\varphi_0'}{a^4 G_4\left(2a^4 G_4 - a^2 G_{5,\varphi}(\varphi_0')^2 + G_{5,X}(\varphi_0')^2\left(\mathcal{H}\varphi_0' - \varphi_0''\right)\right)} + \\
&\quad \times \Bigg[-G_{4,XX}G_{5,X}(\varphi_0')^4\left(-\mathcal{H}\varphi_0' + \varphi_0''\right)^2 + a^6\Big(\left(G_{4,\varphi}G_{5,\varphi} + G_4 G_{5\varphi\varphi}\right)(\varphi_0')^2 + 2G_4 G_{5,\varphi}\varphi_0''\Big) + \\
&\quad + a^2(\varphi_0')^2\left(\mathcal{H}\varphi_0' - \varphi_0''\right)\Big(G_4\mathcal{H}G_{5,XX}\varphi_0' + \left(G_{4,X\varphi}G_{5,X} + G_{4,XX}G_{5,\varphi}\right)(\varphi_0')^2 + G_{4,X}G_{5,X}\varphi_0''\Big) + \\
&\quad + a^4\varphi_0'\bigg(\varphi_0'\Big(G_4 G_{5,X}\left(2\mathcal{H}^2 - \mathcal{H}'\right) - \mathcal{H}\left(G_{4,\varphi}G_{5,X} + 2G_4 G_{5,X\varphi}\right)\varphi_0' - G_{4,X\varphi}G_{5,\varphi}(\varphi_0')^2\Big) + \\
&\quad + \Big(-2G_4\mathcal{H}G_{5,X} + \left(G_{4,\varphi}G_{5,X} + G_4 G_{5,X\varphi} - G_{4,X}G_{5,\varphi}\right)\varphi_0'\Big)\varphi_0''\bigg)\Bigg],
\end{aligned}$$

(B.22)

The expressions for the other functions f_i with $i = 10, 15, 18, 19, 21$ can be found from the previous identities.

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