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Wilsonian Renormalization Group and Holography

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Introduction

The modern comprehension of renormalization in quantum field theory (QFT) is based on the Wilsonian approach to the renormalization group (RG) [1, 5]. This treatment develops the idea that a QFT can be considered as an effective field theory and so that it describes physics only in a certain range of energies. The behavior of the theory is described by a scale-dependent Wilsonian effective action which is obtained starting from the bare action defined at a UV cut-off scale Λ_0 and integrating out the field-modes with momentum between Λ_0 and a floating cut-off scale Λ, which represents the scale at which the theory is considered. In such a way the contribution of the integrated modes introduces the dependence on the floating scale into the Wilsonian effective action, which then describes the behavior of the theory at the energy scale Λ. Varying Λ from the UV cut-off Λ_0 down to the IR regime provides a flow in the space of the couplings, which represents the RG flow of the theory. In particular such flow is governed by a differential equation, called Wilson-Polchinski equation (WPE), which describes the evolution with the energy scale of the Wilsonian effective action and determines the running of the effective couplings of the theory.

More recently a new way to approach and to make computations (in particular at strong coupling) in quantum field theory has been provided by the AdS/CFT correspondence [17–19]. The latter is conjectured as a duality between a conformal field theory (CFT) and a quantum gravity theory defined on the anti-de Sitter space (AdS). However, in this work we will employ a more general definition of the duality, which determines the equivalence between a gravity theory (bulk theory) on a $d+1$-dimensional (asymptotic) AdS space and a field theory in d-dimensions which in general is non-conformal and hence have a non-trivial RG flow. Even though this description concerns also the non-conformal case, we will keep referring to it as the AdS/CFT correspondence.

From the computational point of view, there are two crucial aspects of this correspondence. The first is the fact that the strong coupling regime in the QFT-side corresponds to the weak coupling regime in the bulk side and viceversa. Moreover, the tree level regime of the bulk theory corresponds to the large N limit of the dual field theory. These facts have a fundamental importance because they enable one to study the strongly coupled regime of quantum field theory and, in particular, the RG flow in terms of the tree level gravity dual. Hence, in order to develop in detail the Wilsonian RG in this framework, it is important to study the large N limit of a quantum field theory.

The latter is a powerful approximation scheme first developed by 't Hooft [13]. This method applies, for example, to theories where fields are $N \times N$ matrices. Thanks to the fact that, when the fields are non-commutative, the interaction vertices of the theory have an ordered structure, one can associate to each Feynman diagram a unique Riemann surface. In such a way the N-dependence of every graph is related to the topology of the associated surface through its genus and the number of boundaries. For this reason, in the regime where the matrix size N is taken very large, the structure of the theory drastically simplifies because planar diagrams (graphs with null genus and minimum number of boundaries) represent the leading contribution, whereas all the other, non-planar, graphs are suppressed by powers of $1/N$.

In this framework one can apply the Wilsonian RG to the matrix theory and obtain the WPE in planar limit [16]. This equation has an interesting structure because the quantum term of the ordinary WPE, which is proportional to the second derivative with respect to the fields, reduces in the large N limit to a term proportional only to the first derivative and so the planar WPE is a Hamilton-Jacobi type equation. Moreover, one can see that the latter contains the contribution also of non-planar diagrams, which, thanks to the action of cut-off scale, are responsible of the production of multi-trace terms along the planar RG flow.

A Wilsonian RG approach has also been developed within the AdS/CFT correspondence [30, 31]. This method employs the Wilsonian treatment in the gravity side of the duality and it is developed exploiting the connection between the radial coordinate of the AdS space and the energy scale in the field theory side. Therefore one can obtain the Wilsonian effective action of the dual field theory and its RG equation in terms of the radial evolution of the bulk theory. In particular, taking the classical limit in the gravity side, one can determine the planar RG equation, which is a Hamilton-Jacobi type equation in the bulk.

For this reason one can define two different formulations which determine the RG equation for the Wilsonian effective action of a quantum field theory in the large N limit. The planar WPE represents the flow equation obtained in the QFT-side and it does not depend on the couplings of the theory, i.e. it holds true in both weak and strong coupling regime, whereas, in the holographic Wilsonian approach, the planar RG equation is obtained from the bulk side for a strongly coupled field theory. Therefore, by consistency with the AdS/CFT correspondence, one expects the two formulations to be equivalent. However, this is not true because, even though both equations are of Hamilton-Jacobi type, planar WPE is characterized by a cubic hamiltonian whereas that of the holographic one is at least quadratic.

This is an important structural mismatch, which seems to create a puzzling inconsistency. However, analyzing in depth the two formulations, we realize that the source of this mismatch can be attributed to the different definition of the theory in the two approaches. Indeed, in the WPE formulation the theory is explicitly defined in terms of the elementary fields, so that the cut-off scale is precisely defined inside the propagator. Instead, in the holographic formulation the field theory is described, through the AdS/CFT duality, only in terms of the composite operators dual to the bulk fields. Hence the implementation of the cut-off at the level of elementary fields is not understood because no information about the elementary structure of the theory is given.

We explore these aspects considering two explicit examples. First of all we describe the RG flow of a QFT which is dual to a free scalar field in the bulk. In holographic Wilsonian approach, studying the radial evolution of the scalar field in the bulk, we obtain the RG flow of the dual QFT, which is closed on a single- and a double- trace term. Instead, applying the WPE formulation to this case, we obtain a RG flow which has an extremely different structure since it is not closed, as in the holographic case, only on two terms, but it contains other multi-trace terms generated along the flow.

In the other example, generalizing some calculations already present in the literature [35–37], we define a different field theory procedure to derive the RG flow of a QFT, which is determined as a deformation of an abstract CFT defined only through the correlators of composite operators. From this computation we obtain a RG flow whose structure exactly matches with that obtained in the holographic treatment.

Therefore, thanks to this analysis, we suggest that the mismatch between the planar WPE and the RG equation in holographic Wilsonian treatment does not necessarily lead to an inconsistency, but is due to the different definition of the cut-off in the two formulations. In the future, a deeper study of the relation between elementary fields an composite operators would be of great importance in the understanding of this issue and of the nature of the AdS/CFT correspondence itself.

In chapter 1 we describe the Wilsonian approach to the renormalization group of a quantum field theory, focusing on the derivation of the Wilson-Polchinski equation and its features. In chapter 2 we describe the large N limit for general matrix theories and we derive the planar Wilson-Polchinski equation in the scalar case. In chapter 3 we describe the AdS/CFT correspondence in its standard definition. In chapter 4 we consider the AdS/CFT correspondence in the non-conformal regime of the boundary field theory and we describe the holographic Wilsonian RG approach. In chapter 5 we analyze the mismatch between the planar Wilson-Polchinski equation and the RG equation obtained through holographic Wilsonian method.

CHAPTER 1

Wilsonian Renormalization Group

Renormalization is one of the main aspects of quantum field theory. This procedure was introduced in the early 50's to remove the UV divergences arising from loop contributions to Feynman diagrams. In such a way higher orders in perturbative expansion are well defined and the theory is able to give more precise results, which are remarkably very close to experimental ones. This method, which we will call standard renormalization, implements a redefinition of fields ($\Phi \mapsto Z_\Phi \Phi$), masses ($m^2 \mapsto m^2 + \delta m^2$) and couplings ($\lambda \mapsto Z_\lambda \lambda$) to introduce in the action some counterterms ($\delta Z_\mathrm{I} = Z_{\Phi,\lambda} - 1, \delta m^2$), which make the generating functional finite, removing its divergent part. To do so, such couterterms have to be divergent as well, but we can however expand them in a formal perturbative loop counting series ($\delta Z_\mathrm{I} = \sum_k \delta Z_\mathrm{I}^{(k)}$), in which every term is divergent. In this way we can use perturbative method and renormalize the theory order by order in loop expansion, neglecting further terms as if they were smaller than the previous ones. This is deeply counterintuitive (how can we say that some infinite term is "smaller" than another infinite one and thus neglect it?) and also apparently meaningless from the physical point of view, but it perfectly works.

In this framework we can argue that the presence of UV divergences in the bare theory is due to the integration over all energy scales in the loop momentum. Nevertheless its physical meaning is still a bit obscure and a deeper understanding will be reached through further developments.

The concept of renormalizability of a theory, i.e. whether UV divergences can be removed by the redefinition of a finite number of coupling constants, naturally arises within standard renormalization. Indeed, considering the superficial degree of divergence D of a 1PI diagram, which is the total momentum power in the loop integral, the contribution of such graph is proportional to $\int dk\, k^{D-1}$ in the limit of large momenta, thus, if D is negative, the integral is convergent, otherwise it diverges. Through the analysis of the structure of the diagrams (number and type of internal and external legs and vertices) present, for example, in [2], one can obtain a relation of the type $D = f(E, V)$, which connects D to the number of external legs E and vertices V of its related graph by a function f. If the theory contains only couplings with mass dimension $d_i \geq 0$, one can show from such equation that D decreases as the number of external legs E grows. Thus there will be a particular value $\overline{E}$ of E such that $D < 0$ for $E > \overline{E}$. In such a way, even though one can build an infinite number of diagrams with the same number of external legs E, they are divergent only if $E \leq \overline{E}$. Since one can show that a vertex with $d_i \geq 0$ must have $n_i \leq \overline{E}$ external legs, making a redefinition of it, one can absorb the divergent part of all diagrams with n_i external legs. Therefore, one can renormalize a theory with all vertices with $d_i \geq 0$, namely absorb the divergent part of all diagrams, through the redefinition of all its parameters plus the renormalization of the fields. Since the number of all possible vertices with $d_i \geq 0$ is finite, renormalization requires a finite number of redefinitions. For this reason such theories are called renormalizable and they have a very simple form.

On the other hand a coupling with dimension $d_i < 0$ is called non-renormalizable. If a theory contains at least one non-renormalizable coupling it is called non-renormalizable. Indeed the presence of this type of couplings generates divergences in diagrams with $E > \overline{E}$. These terms can be renormalized only including in the theory other non-renormalizable vertices, which, in their turn, generates other new divergent terms. The latter require the introduction of other non-renormalizable vertices as the terms above, so that, in principle, the theory have to contain all possible non-renormalizable interactions to be fully renormalized. In such a way, technically speaking, the theory is renormalizable, but, since the non-renormalizable couplings are infinitely many, it requires the redefinition of an infinite number of parameters. For this reason a non-renormalizable theory is far less predictable than a renormalizable one. Hence, standard renormalization identifies the renormalizable theory as the best candidate for a fundamental theory of particles. This was confirmed by the large success of QED and Standard Model (SM), which are renormalizable theories, but the true reason of such success, as we will see in subsection 1.1.2, is different. Thus the question of why nature should be described by a renormalizable theory was answered, through standard renormalization, only at technical level.

Owing to these aspects, initially, renormalization was thought to be just an *ad hoc* technical tool to adjust QFT in order to obtain results consistent with experiments, but, subsequently, this idea has evolved. Indeed the understanding of this method had been greatly improved during the 70's and the 80's. First Wilson developed his approach based on the effective field theory [1], clarifying why UV divergences are expected to appear in loop diagrams and answering why at low energies physics is described by renormalizable theories. Then Polchinski formalized such treatment [5], deriving the exact renormalization group equation and proving the equivalence with standard renormalization.

1.1. Wilsonian approach

Wilson's treatment develops an important statement: field theories describe physics only in a certain limited range of energies. This is the starting point for a new way of understanding renormalization, where we can consider a QFT as an effective field theory.

An effective field theory (EFT) is a low-energy approximation of a more fundamental theory which may or may not be a field theory. It is strongly connected to the energy scale at which we investigate physics, so that its action can have different behavior at different scales. Owing to this, the principle of renormalizability loses the fundamental nature it has within standard renormalization and so, in principle, EFT may contain all non-renormalizable terms compatible with symmetries.

A process occurring at a certain energy accessible by EFT is influenced by the behavior of the theory at higher energies. Indeed it receives, at quantum level, UV contributions from the propagation of virtual quanta in loop integrals. Thus, since EFT provides physically relevant results in a low-energy regime where the process occurs, it has to contain such UV contributions in its action. In order to compute these corrections and so to describe the behavior of EFT at the energy of the process, Wilson thought of integrating out the contribution of the functional integral corresponding to energies higher than the one of the process. In such a way we obtain the action of the EFT defined precisely at the desired energy scale, so that it already contains the UV effects. Hence, applying this new approach to QFT, we can study how high-energy virtual quanta give contribution to lower energy processes.

To do so, consider a QFT as a theory with physical UV cut-off Λ_0. For energies above Λ_0 physics is described by an other theory (string theory, another QFT, ...), which is assumed to be fundamental, while below Λ_0 it is represented by a general (bare) action S_0 which contains in principle all possible interaction terms compatible with the symmetries of the theory. Now it is worth studying the behavior of the theory for energies $E \ll \Lambda_0$. Indeed, even though in general Λ_0 is not specified (it could be of the order of GeV as well as of the Planck mass), in many interesting cases it is far greater than energies accessible by experiments (for example the GUT scale $M_{\text{GUT}} \sim 10^{16}$ GeV), thus a comparison with experimental results makes sense if done with the effective behavior of QFT at low energies, i.e. $E \ll \Lambda_0$.

At this point we start considering the functional integral of a theory of a real scalar field ϕ in Euclidean d-dimensional space-time:

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \, e^{-S_0[\phi] + \int J\phi}, \tag{1.1.1}$$

where J is the source. The information about the UV cut-off Λ_0 has to be included in $\mathcal{Z}[J]$. There are two equivalent methods to do so and both will be used in this chapter.

The first way introduces Λ_0 directly in the functional measure $\mathcal{D}\phi$. Indeed, in the path integral formulation, every different momentum field-mode[1] ϕ_k is an independent integration variable, thus the functional measure can be written as $\mathcal{D}\phi = \prod_k d\phi_k$. Hence we introduce the cut-off to cut the modes with momentum larger than Λ_0, so that:

$$\mathcal{D}\phi|_{\Lambda_0} := \prod_{|k|<\Lambda_0} d\phi_k,$$

which is the truncated functional measure, while:

$$\mathcal{Z}[J] = \int \mathcal{D}\phi|_{\Lambda_0}\, e^{-S_0[\phi]+\int J\phi}, \tag{1.1.2}$$

which is the regulated functional integral.

The second way introduces the cut-off scale through the propagator, defining the latter as:

$$P_{\Lambda_0}(k) = \frac{K({k^2}/{\Lambda_0^2})}{(k^2+m^2)}, \tag{1.1.3}$$

where K is a $\mathcal{C}^\infty$ function which is equal to 1 for $k^2 < \Lambda_0^2$ while vanishes very rapidly for $k^2 > \Lambda_0^2$. In such a way, for momenta higher than Λ_0, the propagator becomes negligible, then the correspondent field-modes are non-propagating. Introducing a bracket notation defined as:

$$\langle f, g\rangle := \int \frac{d^dk}{(2\pi)^d}\, f(k)g(-k),$$

for every arbitrary functions on the momentum space f, g, the kinetic term (KT) becomes:

$$\mathrm{KT} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\, \phi_k\, P_{\Lambda_0}^{-1}(k)\, \phi_{-k} = \frac{1}{2}\left\langle \phi\, P_{\Lambda_0}^{-1}, \phi\right\rangle. \tag{1.1.4}$$

Hence, in the functional integral, field modes with momentum greater than Λ_0 are exponentially suppressed[2]. This approach is equivalent to the previous one, where we introduced the cut-off in the functional measure, but it is more useful for explicit calculations. As already said, in this chapter we will use both of them, depending on where one is more useful than the other.

Once defined the functional integral for a theory with UV cut-off, we focus on low energies considering a scale $\Lambda_{\mathrm{R}} \ll \Lambda_0$ which is assumed to be the greatest energy scale at which we can make experiments. In such a way the external source is set to be:

$$J_k = 0 \qquad \text{for} \quad |k| > \Lambda_{\mathrm{R}}. \tag{1.1.5}$$

In this approach the regulated functional integral suggests a way to investigate the influence of UV fluctuations of the fields on the behavior of the theory. Indeed, since the functional

[1] Obtained from the Fourier transform of $\phi(x)$.

[2] Indeed $Z \sim \exp(-P_{\Lambda_0}^{-1}\phi^2)$, therefore for momenta over Λ_0 the exponent goes to $-\infty$ and the exponential vanishes.

measure is the product of field-mode differentials at every momentum value, we can think of performing the integration over field-modes of momentum greater than a certain energy scale $\Lambda \leq \Lambda_0$. The term arising from this computation modifies the original bare action introducing corrections depending on the parameter Λ into the couplings. Therefore this procedure actually lowers the cut-off from Λ_0 to Λ, including the contributions of UV modes directly in the action. In this picture Λ is a floating cut-off which represents the energy scale at which we are considering the theory. The latter is described by a scale-dependent effective action, thus its behavior depends on the energy at which we are evaluating the observables. This is due to the fact that the UV fluctuations give a precise and computable contribution to processes of lower energy.

An instructive implementation of this procedure can be done in the general real scalar theory introduced before. The functional integral with the cut-off defined in the measure can be written as:

$$\mathcal{Z}[J] = \int \mathcal{D}\phi|_{\Lambda_0}\, e^{-\frac{1}{2}\left\langle \phi\, P^{-1}, \phi \right\rangle - S^{(0)}_{\text{int}}[\phi] + \langle J, \phi\rangle}, \tag{1.1.6}$$

where $P(k) = (k^2+m^2)^{-1}$ is the propagator of the scalar field and $S^{(0)}_{\text{int}}$ is the bare interaction term at Λ_0. Now, considering the floating cut-off $\Lambda \in [\Lambda_{\text{R}}, \Lambda_0]$, which represents the energy scale where the theory is considered, we can write $\phi = \phi_{\text{H}} + \phi_{\text{L}}$, with:

$$\phi_{\text{H}k} = \begin{cases} \phi_k & \text{for } \Lambda < k \leq \Lambda_0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_{\text{L}k} = \begin{cases} \phi_k & \text{for } k < \Lambda \\ 0 & \text{otherwise} \end{cases}. \tag{1.1.7}$$

Thus ϕ_{L} is the low-energy part of the field, while ϕ_{H} is the high-energy one. In order to integrate out ϕ_{H}-modes we can write:

$$\begin{aligned} \mathcal{Z}[J] &= \int \mathcal{D}\phi_{\text{H}} \mathcal{D}\phi_{\text{L}}\, e^{-\frac{1}{2}\left\langle (\phi_{\text{H}}+\phi_{\text{L}})\, P^{-1}, \phi_{\text{H}}+\phi_{\text{L}} \right\rangle - S^{(0)}_{\text{int}}[\phi_{\text{H}}+\phi_{\text{L}}] + \langle J, \phi_{\text{H}}+\phi_{\text{L}}\rangle} \\ &= \int \mathcal{D}\phi_{\text{L}}\, e^{-\frac{1}{2}\left\langle \phi_{\text{L}}\, P^{-1}, \phi_{\text{L}} \right\rangle + \langle J, \phi_{\text{L}}\rangle} \int \mathcal{D}\phi_{\text{H}}\, e^{-\frac{1}{2}\left\langle \phi_{\text{H}}\, P^{-1}, \phi_{\text{H}} \right\rangle - S^{(0)}_{\text{int}}[\phi_{\text{H}}+\phi_{\text{L}}]}, \end{aligned} \tag{1.1.8}$$

where $\langle J, \phi_{\text{H}}\rangle = 0$ by definition. The integral over ϕ_{H} represents contribution to the functional integral of the theory for energies higher than Λ. Performing it, we obtain a functional of the field-modes ϕ_{L}, which is a new interaction term replacing $S^{(0)}_{\text{int}}$, where bare parameters have been modified by Λ-dependent corrections, creating effective couplings. We call such functional Wilsonian action S^{Λ}_{int}, which reads:

$$e^{-S^{\Lambda}_{\text{int}}[\phi_{\text{L}}]} = \int \mathcal{D}\phi_{\text{H}}\, e^{-\frac{1}{2}\left\langle \phi_{\text{H}}\, P^{-1}, \phi_{\text{H}} \right\rangle - S^{(0)}_{\text{int}}[\phi_{\text{H}}+\phi_{\text{L}}]}. \tag{1.1.9}$$

The sum of the Wilsonian action with the kinetic term of the low-energy field forms the effective action S^{Λ}_{eff} of the theory at the scale Λ, so that:

$$\mathcal{Z}[J] = \int \mathcal{D}\phi|_{\Lambda}\, e^{-\frac{1}{2}\left\langle \phi\, P^{-1}, \phi \right\rangle - S^{\Lambda}_{\text{int}}[\phi] + \langle J, \phi\rangle} = \int \mathcal{D}\phi|_{\Lambda}\, e^{-S^{\Lambda}_{\text{eff}}[\phi] + \langle J, \phi\rangle}, \tag{1.1.10}$$

where we replaced ϕ_{L} with ϕ and $\mathcal{D}\phi_{\mathrm{L}}$ with $\mathcal{D}\phi|_{\Lambda}$. This procedure shows that a QFT is described by its bare action S_0 only at the cut-off energy Λ_0, whereas at another energy scale $\Lambda < \Lambda_0$ it is represented by the effective action $S^{\Lambda}_{\mathrm{eff}}$, which contains corrections coming from the high-energy fluctuations[3] of the fields. Varying the parameter Λ we obtain the effective action at different energies, so that, of course, we have $S^{\Lambda}_{\mathrm{eff}} \mapsto S_0$ for $\Lambda \mapsto \Lambda_0$. Hence the procedure described above produces a trajectory (or flow) of equivalent effective actions which describe the same theory at different energies. This is called Wilsonian Renormalization Group flow (WRG) and provides a new method of renormalization.

At this stage, the computation of correlation functions for a theory described by a bare action S_0 can be done using WRG. If their momenta are of the order of Λ_{R}, i.e. very small with respect to the cut-off scale, we have to compute such functions using the effective action obtained from S_0 by the integration down to Λ_{R} and perform loop integrals up to this scale. Indeed high-energy effects are already contained in the effective action and their corrections to correlation functions are introduced slowly within perturbation theory and without any divergence.

This method not only allows us to perform computations for renormalizable theories, but also for non-renormalizable ones. Hence it seems to provide a recipe wider than the one of standard renormalization. On the contrary, thanks to a deeper analysis, one can show that WRG is equivalent to the latter, even though it provides a wider comprehension of the renormalization method, as we will see in subsection 1.1.2.

1.1.1. Example: the ϕ^4 theory

To make these concepts more clear we study WRG in a simple example: the ϕ^4 theory in four space-time dimensions. Consider the bare action at the cut-off scale:

$$S_0 = \int d^4x \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right), \tag{1.1.11}$$

where ϕ is a real scalar field, whereas m^2 and λ are bare parameters.

Set $J = 0$ for simplicity and split the field into ϕ_{H} and ϕ_{L}:

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\phi_{\mathrm{H}}\mathcal{D}\phi_{\mathrm{L}}\, e^{-\int d^4x \left[\frac{1}{2}(\partial\phi_{\mathrm{L}}+\partial\phi_{\mathrm{H}})^2+\frac{1}{2}m^2(\phi_{\mathrm{L}}+\phi_{\mathrm{H}})^2+\frac{\lambda}{4!}(\phi_{\mathrm{H}}+\phi_{\mathrm{L}})^4\right]}\\ &= \int \mathcal{D}\phi_{\mathrm{L}}\, e^{-S_0[\phi_{\mathrm{L}}]}\int \mathcal{D}\phi_{\mathrm{H}} \exp\Bigg(-\int d^4x \left[\frac{1}{2}(\partial\phi_{\mathrm{H}})^2 + \frac{1}{2}m^2\phi_{\mathrm{H}}^2\right.\\ &\qquad\left. + \frac{\lambda}{4!}(\phi_{\mathrm{H}}^4 + 4\,\phi_{\mathrm{L}}^3\phi_{\mathrm{H}} + 4\,\phi_{\mathrm{L}}\phi_{\mathrm{H}}^3 + 6\,\phi_{\mathrm{L}}^2\phi_{\mathrm{H}}^2)\right]\Bigg),\end{aligned} \tag{1.1.12}$$

[3]when we refer to high-energy fluctuations we mean those associated to the modes of momentum larger than the scale we are considering.

where the term $\int d^4x\, \partial\phi_{\rm H}\partial\phi_{\rm L}$ does not appear because in momentum space it becomes $\int d^4k\, k^2\phi_{{\rm H}k}\phi_{{\rm L}-k}$ and so $\phi_{{\rm H}k}\phi_{{\rm L}-k} = 0$ by definition (1.1.7).

Now we have to integrate out $\phi_{\rm H}$ of the functional integral, in order to obtain the effective action $S^{\Lambda}_{\rm eff}$. We can consider the parameter λ small and use the perturbative method on it. Then we expand its exponential term and evaluate its contributions using Wick's theorem, as showed in [4] and [12]. Thus the part of the action involving $\phi_{\rm H}$, whose exponential is not expanded, is:

$$\int \mathcal{L}_0 = \frac{1}{2}\int d^4x \left[(\partial\phi_{\rm H})^2 + m^2\phi_{\rm H}^2\right] = \frac{1}{2}\int_{\Lambda\le|k|<\Lambda_0} \frac{d^4k}{(2\pi)^4}(k^2+m^2)\phi_{{\rm H}k}\phi_{{\rm H}-k}, \tag{1.1.13}$$

so that we can evaluate the correlation functions of high-energy modes using:

$$\left\langle \phi_{{\rm H}k}\phi_{{\rm H}p}\right\rangle_0 = \mathcal{N}_{\rm H}\int \mathcal{D}\phi_{\rm H}\, \phi_{{\rm H}k}\phi_{{\rm H}p}\, e^{-\int\mathcal{L}_0} = \frac{\widehat{\Theta}(k)}{k^2+m^2}\,(2\pi)^4\delta^4(k+p), \tag{1.1.14}$$

where $\widehat{\Theta}(k)$ is a Heaviside-like function which is 1 for $\Lambda < |k| \le \Lambda_0$ and vanishes elsewhere, whereas:

$$\mathcal{N}_{\rm H}^{-1} = \int \mathcal{D}\phi_{\rm H}\, e^{-\int\mathcal{L}_0}, \tag{1.1.15}$$

which is the normalization. Furthermore the exponential of the remaining part of the action is expanded in Taylor series:

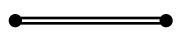
$$\int\mathcal{D}\phi_{\rm H}\left[1 - \frac{\lambda}{4!}\int d^4x\,(\phi_{\rm H}^4 + 4\,\phi_{\rm L}^3\phi_{\rm H} + 4\,\phi_{\rm L}\phi_{\rm H}^3 + 6\,\phi_{\rm L}^2\phi_{\rm H}^2) + O(\lambda^2)\right]e^{-\int\mathcal{L}_0}, \tag{1.1.16}$$

and we can evaluate its behavior focusing on the contribution of single terms in the expansion. Let us start with the first order term $\phi_{\rm L}^2\phi_{\rm H}^2$, which gives the contribution:

$$-\frac{\lambda}{4}\mathcal{N}_{\rm H}\int d^4x\,\phi_{\rm L}^2(x)\left\langle\phi_{\rm H}(x)\phi_{\rm H}(x)\right\rangle_0, \tag{1.1.17}$$

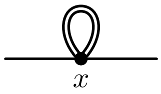
where $\mathcal{N}_{\rm H}$ is the normalization term which can be absorbed in the definition of the functional measure. Henceforth it will be neglected, so we can focus only on the computation of correlation functions.

The contribution (1.1.17) has a simple and clear diagrammatic description. We represent the propagator of high-energy modes with a double line and the low-energy field $\phi_{\rm L}$ with the usual single line:

$\langle\phi_{\rm H}(x)\phi_{\rm H}(y)\rangle_0 \quad = \quad x$  y (1.1.18)

$\phi_{\rm L}(x) \quad = \quad$  x (1.1.19)

In such a way, (1.1.17) can be represented as:

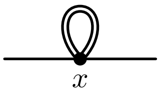
$$-\frac{\lambda}{4}\int d^4x\,\phi_{\mathrm{L}}^2(x)\,\langle\phi_{\mathrm{H}}(x)\phi_{\mathrm{H}}(x)\rangle_0 = -\frac{\lambda}{4}$$  (1.1.20)

where the internal line forms a loop because $\langle\phi_{\mathrm{H}}(x)\phi_{\mathrm{H}}(x)\rangle_0$ is evaluated at the same space-time point and it contains the integration on the high-energy range because it concerns only the UV field-modes ϕ_{H}, so:

$$\begin{aligned} &= \langle\phi_{\mathrm{H}}(x)\phi_{\mathrm{H}}(x)\rangle_0 = \int\frac{d^4k}{(2\pi)^4}\frac{\widehat{\Theta}(k)}{k^2+m^2} \\ &= \frac{1}{16\pi^2}\left[(\Lambda_0^2-\Lambda^2)-m^2\log\left(\frac{\Lambda_0^2+m^2}{\Lambda^2+m^2}\right)\right]. \end{aligned} \tag{1.1.21}$$

The explicit calculation of this integral, as well as of all the other relevant loop integrals we are dealing with in this chapter is performed in appendix A.

Therefore, since $\langle\phi_{\mathrm{H}}(x)\phi_{\mathrm{H}}(x)\rangle_0$ does not depend on x, we have:

$$-\frac{\lambda}{4}$$ 

$$= -\frac{\mu^2}{2}\int d^4x\,\phi_{\mathrm{L}}^2(x) = -\frac{\mu^2}{2}\int\frac{d^4k}{(2\pi)^4}\,\phi_{\mathrm{L}k}\phi_{\mathrm{L}-k}, \tag{1.1.22}$$

where:

$$\mu^2 = \frac{\lambda}{32\pi^2}\left[(\Lambda_0^2-\Lambda^2)-m^2\log\left(\frac{\Lambda_0^2+m^2}{\Lambda^2+m^2}\right)\right], \tag{1.1.23}$$

which, as we are going to show, is a correction to the mass term of S_0 generated by the high-energy fluctuations of the field coming from $\phi_{\mathrm{L}}^2\phi_{\mathrm{H}}^2$. Indeed, evaluating the second order correction arising from the same term:

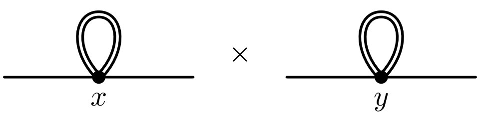
$$\int\mathcal{D}\phi_{\mathrm{H}}\left(\frac{\lambda^2}{32}\int d^4x\,d^4y\,\phi_{\mathrm{L}}^2(x)\phi_{\mathrm{L}}^2(y)\phi_{\mathrm{H}}^2(x)\phi_{\mathrm{H}}^2(y)\right)e^{-\int\mathcal{L}_0}, \tag{1.1.24}$$

we have two contributions:

$$\frac{\lambda^2}{32}\int d^4x\,d^4y\,\phi_{\mathrm{L}}^2(x)\phi_{\mathrm{L}}^2(y)\,\langle\phi_{\mathrm{H}}(x)\phi_{\mathrm{H}}(x)\rangle_0\,\langle\phi_{\mathrm{H}}(y)\phi_{\mathrm{H}}(y)\rangle_0 \tag{1.1.25}$$

$$\frac{\lambda^2}{16}\int d^4x\,d^4y\,\phi_{\mathrm{L}}^2(x)\phi_{\mathrm{L}}^2(y)\,\langle\phi_{\mathrm{H}}(x)\phi_{\mathrm{H}}(y)\rangle_0^{\;2}, \tag{1.1.26}$$

The term (1.1.25) is associated to the disconnected diagram:



(1.1.27)

which is the product of two connected diagrams (1.1.22), so it gives a contribution proportional to μ^4. Considering the third order correction arising from the same term, i.e. $\phi_{\rm L}^2\phi_{\rm H}^2$, we have, among all terms, a product of three diagrams (1.1.22). Going on to further orders we have the same behavior, therefore the sum of all these terms produces the exponential series, so that the action S_0 is modified with the term (1.1.22) to produce a Λ-dependent effective action:

$$S_{\rm eff}^{\Lambda} = \int d^4x \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(m^2+\mu^2)\,\phi^2 + \frac{\lambda}{4!}\,\phi^4 + \cdots\right). \tag{1.1.28}$$

The mass correction μ^2 in (1.1.23) introduces the dependence on the energy scale in the mass parameter. In such a way the effective mass $M^2 := m^2 + \mu^2$ at the scale Λ is:

$$M^2 = m^2 + \frac{\lambda}{32\pi^2}\left[(\Lambda_0^2 - \Lambda^2) - m^2 \log\left(\frac{\Lambda_0^2 + m^2}{\Lambda^2 + m^2}\right)\right]. \tag{1.1.29}$$

If we take $\Lambda \mapsto \Lambda_0$, the contribution μ^2 vanishes, so that $M^2 \mapsto m^2$ and the effective action reduces to S_0. Moreover, since we are mainly interested in cases where the cut-off scale Λ_0 is very high, it is worth determining the leading behavior of M^2 in the regime where $m^2 \ll \Lambda_0^2$. Thus we have $\mu^2 \sim \lambda(\Lambda_0^2 - \Lambda^2)$, so that:

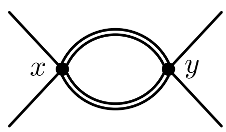
$$M^2 \sim m^2 + \frac{\lambda}{32\pi^2}(\Lambda_0^2 - \Lambda^2). \tag{1.1.30}$$

Now, being $\overline{m}^2 = m^2/\Lambda_0^2$ the dimensionless bare mass, we can define the dimensionless effective mass at the scale Λ as $\overline{M}^2 := M^2/\Lambda^2$. Thus its behavior is:

$$\overline{M}^2(\Lambda/\Lambda_0) \sim \left(\overline{m}^2 + \frac{\lambda}{32\pi^2}\right)\left(\frac{\Lambda_0}{\Lambda}\right)^2 - \frac{\lambda}{32\pi^2}. \tag{1.1.31}$$

This shows that $\overline{M}^2$ scales as $(\Lambda_0/\Lambda)^2$, so it becomes increasingly important at low energies (when $\Lambda \mapsto 0$) and, sending $\Lambda_0 \mapsto +\infty$, it reproduces the renormalization of the mass provided by standard renormalization [4].

The contribution (1.1.26) is a non-local term associated to the connected diagram:



(1.1.32)

Many non-local terms associated to connected diagrams, like (1.1.26), are produced when we consider orders beyond the first in the expansion of the functional integral over $\phi_{\rm H}$ (1.1.16). If we try to evaluate them in order to obtain their correction to S_0 we have to face the problem of integration over different space-time coordinates. Such integrals are not easily computable and, most importantly, contain products of the field $\phi_{\rm L}$ evaluated at different points, which cannot appear as corrections of a local bare action like S_0. To overcome this problem we can expand the non-local terms in a series of infinite local contributions with growing number of derivatives of the field. In such a way the corrections arising from them can be included in the action as an infinite number of derivative interactions. Let us explain more explicitly this method, considering the contribution (1.1.26) and its corresponding diagram (1.1.32).

The field $\phi_{\rm L}(y)$ can be expanded in Taylor series around $y = x$:

$$\phi_{\rm L}(y) = \phi_{\rm L}(x) + \partial_\mu \phi_{\rm L}(x)(y-x)^\mu + \cdots \,, \tag{1.1.33}$$

where $\partial_\mu = \partial/\partial x^\mu$, so that:

$$\phi_{\rm L}^2(y) = \phi_{\rm L}^2(x) + 2\phi_{\rm L}(x)\partial_\mu \phi_{\rm L}(x)(y-x)^\mu + \cdots \,. \tag{1.1.34}$$

Now, calling $w = y - x$, the correlator $\langle \phi_{\rm H}(x)\phi_{\rm H}(y)\rangle_0$ can be expressed as $\langle \phi_{\rm H}(0)\phi_{\rm H}(w)\rangle_0$ for invariance under translations, so (1.1.26) becomes:

$$\frac{\lambda^2}{16}\int d^4x\, d^4w\, \phi_{\rm L}^2(x)\big(\phi_{\rm L}^2(x) + 2\phi_{\rm L}(x)\partial_\mu\phi_{\rm L}(x)w^\mu + \cdots\big)\, {\langle \phi_{\rm H}(0)\phi_{\rm H}(w)\rangle_0}^2. \tag{1.1.35}$$

This expansion produces an infinite number of local terms with growing number of derivatives, that can be written as:

$$-\frac{\zeta}{4!}\int d^4x\, \phi_{\rm L}^4 - \frac{A^\mu}{4!}\int d^4x\, \phi_{\rm L}^3\partial_\mu\phi_{\rm L} - \frac{B^{\mu\nu}}{4!}\int d^4x\, \phi_{\rm L}^2\partial_\mu\phi_{\rm L}\partial_\nu\phi_{\rm L} + \cdots \,, \tag{1.1.36}$$

where ζ, A, B, ... are UV corrections to S_0 proportional to λ^2 coming from the integrals over w in (1.1.35). Indeed, considering corrections of order higher than that of (1.1.26), we can reconstruct the exponential series of such terms, as we did for (1.1.22), so that (1.1.36) rises to the exponent and represents a new one-loop correction to S_0, coming from (1.1.32):

$$S_{\rm eff}^\Lambda = \int d^4x \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2+\mu^2}{2}\,\phi^2 + \frac{\lambda+\zeta}{4!}\,\phi^4 + \frac{A^\mu}{4!}\phi^3\partial_\mu\phi + \frac{B^{\mu\nu}}{4!}\phi^2\partial_\mu\phi\partial_\nu\phi + \cdots\right). \tag{1.1.37}$$

This is an important fact because $S_{\rm eff}^\Lambda$ now contains a series of non-renormalizable derivative interactions which are not present in the bare action at Λ_0.

Before analyzing the consequences of the presence of such new terms in the effective action, let us first focus on the first term in (1.1.36). It is a second order correction to the ϕ^4 vertex and it reads:

$$\frac{\zeta}{4!}\int d^4x\, \phi_{\rm L}^4(x) = -\frac{\lambda^2}{16}\int d^4x\, \phi_{\rm L}^4(x)\int d^4w\, {\langle \phi_{\rm H}(0)\phi_{\rm H}(w)\rangle_0}^2, \tag{1.1.38}$$

where:

$$\zeta = -\frac{3}{2}\,\lambda^2 \int d^4w\, \langle \phi_{\rm H}(0)\phi_{\rm H}(w)\rangle_0{}^2 = -\frac{3}{2}\,\lambda^2 \int \frac{d^4k}{(2\pi)^4}\frac{\widehat{\Theta}(k)}{(k^2+m^2)^2}. \tag{1.1.39}$$

Evaluating the integral (see appendix A for more details) one obtains:

$$\zeta = -\frac{3\lambda^2}{32\pi^2}\left[\log\left(\frac{\Lambda_0^2+m^2}{\Lambda^2+m^2}\right) + m^2\left(\frac{1}{\Lambda_0^2+m^2} - \frac{1}{\Lambda^2+m^2}\right)\right], \tag{1.1.40}$$

which is the UV correction to the coupling λ, so it provides the dependence on the energy scale of such parameter. Considering $\Lambda \mapsto \Lambda_0$ we must have back the bare coupling λ, indeed we have $\zeta \mapsto 0$. Moreover, considering the bare mass much lower than the cut-off $m \ll \Lambda_0$, as we did for μ^2, the leading behavior of ζ is:

$$\zeta \sim \frac{3\lambda^2}{16\pi^2}\log\left(\frac{\Lambda}{\Lambda_0}\right), \tag{1.1.41}$$

thus the total coupling evaluated at Λ reads:

$$\overline{\lambda}(\Lambda/\Lambda_0) \coloneqq \lambda + \zeta \simeq \lambda\left[1 + \frac{3\lambda}{16\pi^2}\log\left(\frac{\Lambda}{\Lambda_0}\right)\right]. \tag{1.1.42}$$

This relation is exactly the order λ^2 expansion of the running of $\overline{\lambda}$ calculated through the standard RG treatment [3, 4]:

$$\overline{\lambda}(\Lambda/\Lambda_0) = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2}\log(\frac{\Lambda}{\Lambda_0})} = \lambda\left[1 + \frac{3\lambda}{16\pi^2}\log\left(\frac{\Lambda}{\Lambda_0}\right)\right] + O(\lambda^3), \tag{1.1.43}$$

which stresses that WRG provides the same results for the scaling of renormalizable couplings as the standard RG treatment provided by Callan-Symanzik equation.

Hence WRG and standard renormalization give the same results for the renormalization of the mass parameter and the ϕ^4 coupling constant. The crucial difference stands in the fact that the effective action in WRG contains non-renormalizable interactions appearing along the RG flow.

Indeed we have seen, in (1.1.37), that the UV contribution coming from (1.1.32) not only produces the correction to the ϕ^4 vertex (1.1.38), but also a series of derivative interactions of four fields. Even though such terms are not present in the bare action, they arise along the flow influencing physical processes at lower energies.

Considering the behavior of these derivative vertices, one can note that $A^\mu = 0$ because, being it a four-vector, it would have introduced some privileged direction which breaks the invariance under rotations. Moreover, considering the double derivative term, we have that $B^{\mu\nu} = \delta^{\mu\nu}B(\Lambda,\Lambda_0)$ because it is a symmetric rank 2 tensor depending only on scalar parameters Λ and Λ_0, thus it has to be proportional to the only invariant symmetric rank 2 tensor, which in Euclidean signature is $\delta^{\mu\nu}$. In such a way, for dimensional reasons its behavior in the regime where $\Lambda_0^2, \Lambda^2 \gg m^2$ is:

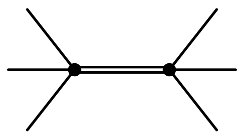
$$B(\Lambda, \Lambda_0) \sim \lambda^2 \left(\frac{1}{\Lambda_0^2} - \frac{1}{\Lambda^2} \right). \tag{1.1.44}$$

Going on with further derivative terms one obtains similar results and so this means that the dimensionless parameters associated to such couplings scale as some positive power of Λ/Λ_0 depending on their mass dimension, so they are suppressed with respect to $\overline{\lambda}$ in the low-energy regime.

In addition to these derivative terms introduced by (1.1.32), other non-renormalizable interactions are produced by terms different from $\phi_{\rm L}^2 \phi_{\rm H}^2$ in (1.1.16). For instance, consider $\phi_{\rm L}^3 \phi_{\rm H}$, its second order expansion reads:

$$\frac{\lambda^2}{72} \int d^4x \, d^4y \, \phi_{\rm L}^3(x) \phi_{\rm L}^3(y) \, \langle \phi_{\rm H}(x) \phi_{\rm H}(y) \rangle_0 \,, \tag{1.1.45}$$

which is associated to the diagram:



(1.1.46)

This contribution produces interactions of six fields, so that the first term in the Taylor expansion of $\phi_{\rm L}^3(y)$ around $y = x$ gives rise to a ϕ^6 vertex, whereas further orders generate derivative interactions.

Since (1.1.46) is a tree level diagram, it does not contain loop integrals. For this reason it is useful to write it in momentum space:

$$\frac{\lambda^2}{72} \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \frac{d^4p''}{(2\pi)^4} \frac{\widehat{\Theta}(p+p'+p'')}{(p+p'+p'')^2 + m^2} \times \\ \times \, \phi_{{\rm L}k} \phi_{{\rm L}k'} \phi_{{\rm L}p} \phi_{{\rm L}p'} \phi_{{\rm L}p''} \phi_{{\rm L}-k-k'-p-p'-p''} . \tag{1.1.47}$$

Such contribution depends on a parameter η of order λ^2, which reads:

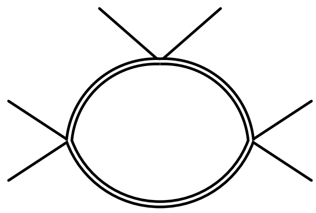
$$\eta = \frac{\lambda^2}{72} \frac{\widehat{\Theta}(p+p'+p'')}{(p+p'+p'')^2 + m^2} \tag{1.1.48}$$

where p, p', p'' are external momenta of (1.1.46). Hence this term depends on whether the transferred momentum $p+p'+p''$ in (1.1.46) is greater than Λ. Thus, since the external legs in such graph represent the field $\phi_{\rm L}$, their momentum is always lower than Λ by definition (1.1.7). Therefore the parameter η is vanishing.

For this reason we have to study other terms producing vertices with six fields. Considering the third order expansion of the term $\phi_{\rm L}^2\phi_{\rm H}^2$ in (1.1.16), we have:

$$-\frac{\lambda^3}{384}\int d^4x\, d^4y\, d^4z\, \phi_{\rm L}^2(x)\phi_{\rm L}^2(y)\phi_{\rm L}^2(z)\left\langle \phi_{\rm H}^2(x)\phi_{\rm H}^2(y)\phi_{\rm H}^2(z)\right\rangle_0 . \tag{1.1.49}$$

Expanding the correlator using Wick's theorem we have that the only contribution associated to a connected diagram is $8\,\langle\phi_{\rm H}(x)\phi_{\rm H}(y)\rangle_0\,\langle\phi_{\rm H}(y)\phi_{\rm H}(z)\rangle_0\,\langle\phi_{\rm H}(z)\phi_{\rm H}(x)\rangle_0$, which gives:



(1.1.50)

This diagram produces interaction terms of six fields, so that, considering only the first term in the Taylor expansions of $\phi_{\rm L}^2(y)$ and $\phi_{\rm L}^2(z)$ around $y=x$, we have:

$$-\frac{\lambda^3}{48}\int d^4x\, \phi_{\rm L}^6(x)\int d^4y\, d^4z\,\langle\phi_{\rm H}(x)\phi_{\rm H}(y)\rangle_0\,\langle\phi_{\rm H}(y)\phi_{\rm H}(z)\rangle_0\,\langle\phi_{\rm H}(z)\phi_{\rm H}(x)\rangle_0\,, \tag{1.1.51}$$

which corresponds to a ϕ^6 vertex:

$$\frac{\xi}{6!}\int d^4x\,\phi_{\rm L}^6(x), \tag{1.1.52}$$

where the coupling is:

$$\begin{aligned}\frac{\xi}{6!} &= -\frac{\lambda^3}{48}\int d^4y\, d^4z\;\langle\phi_{\rm H}(x)\phi_{\rm H}(y)\rangle_0\,\langle\phi_{\rm H}(y)\phi_{\rm H}(z)\rangle_0\,\langle\phi_{\rm H}(z)\phi_{\rm H}(x)\rangle_0\\ &= -\frac{\lambda^3}{48}\int\frac{d^4k}{(2\pi)^4}\frac{\widehat{\Theta}(k)}{(k^2+m^2)^3},\end{aligned} \tag{1.1.53}$$

so that, evaluating the momentum integral (see appendix A), we have:

$$\xi=\frac{15\lambda^3}{16\pi^2}\left[\frac{1}{\Lambda_0^2+m^2}-\frac{1}{\Lambda^2+m^2}-\frac{m^2}{2}\left(\frac{1}{(\Lambda_0^2+m^2)^2}-\frac{1}{(\Lambda^2+m^2)^2}\right)\right]. \tag{1.1.54}$$

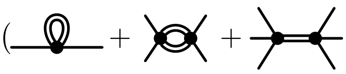
The leading behavior of ξ for $\Lambda_0^2\gg m^2$ and also $\Lambda^2\gg m^2$ is $\xi\sim\lambda^3(1/\Lambda_0^2-1/\Lambda^2)$. Defining the dimensionless parameter $\overline{\xi}:=\xi\Lambda^2$, we have that:

$$\overline{\xi}({}^{\Lambda}\!/_{\Lambda_0})\sim\frac{15\lambda^3}{16\pi^2}\left[\left(\frac{\Lambda}{\Lambda_0}\right)^2-1\right]. \tag{1.1.55}$$

This shows that at low energies also this term is suppressed with respect to $\overline{\lambda}$ and, since one can see that, as for (1.1.32), derivative terms are suppressed with respect to ξ of some positive power of Λ/Λ_0, also such terms are suppressed in the low-energy regime.

The production of these new interaction terms stresses that UV contributions of integrated modes generates all possible interaction terms along the RG flow, even though starting from a renormalizable bare action. This is one of the reasons why we can assert that a QFT at the cut-off scale contains all possible interaction terms.

Furthermore, this picture states that the UV corrections to the bare action coming from WRG can be seen as the sum of all connected diagrams of the bare theory in the interval of energies $[\Lambda, \Lambda_0]$:

$$S^{\Lambda}_{\text{eff}} = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \Big(\right.$$

$$\left. + \cdots \Big)\right]. \qquad (1.1.56)$$

This being true, one can see that, even though along the RG flow non-renormalizable terms are generated, at low energies the theory behaves as a renormalizable ϕ^4 theory because all new interactions are suppressed with respect to the mass term and the ϕ^4 vertex.

In the next subsection, we will show that, starting from a general bare theory at Λ_0, all non-renormalizable interactions are completely negligible at energies much lower than Λ_0, so that in such regime the theory behaves as a renormalizable one. In this sense the importance of renormalizability within WRG is restored.

1.1.2. WRG and the concept of renormalizability

In the beginning of the section 1.1 we stated that, at the scale Λ_0, QFT is described by a general bare action with all possible interactions, dropping any issue about renormalizability. Then we saw that along the RG flow the action is modified becoming scale-dependent, so that at low energies its behavior is different from that at high energies. This provides, through WRG, a new way of understanding and using renormalization.

In this picture calculations can be made even though the bare action is non-renormalizable, and this is possible because we made use of the concept of effective field theory. So one is driven to ask: why among all possible theories only renormalizable ones describe nature? The answer come from a simple analysis, explained, for example, in [2] and [4].

Consider an operator $\mathcal{O}_i$ of mass dimension d_i which represents an interaction term of a d-dimensional bare theory defined at Λ_0. Its coupling $g_i^{(0)}$ has dimension $d - d_i$, thus it can be written as a dimensionless parameter $\overline{g}_i^{(0)}$ multiplied by the factor $\Lambda_0^{d-d_i}$. Proceeding along the RG flow $g_i^{(0)}$ receives UV corrections coming from high energy fluctuations, thus it becomes a scale dependent effective coupling $g_i(\Lambda; \Lambda_0)$. Like for the bare couplings, we can write $g_i(\Lambda; \Lambda_0)$ as the dimensionless effective coupling $\overline{g}_i(\Lambda/\Lambda_0)$ multiplied by Λ^{d-d_i}. Considering physical processes occurring at energies much lower than the UV cut-off, i.e. of the order of $\Lambda_{\text{R}} \ll \Lambda_0$, the contribution coming from $\mathcal{O}_i$ produces a term proportional to $\overline{g}_i^{(0)}\Lambda_0^{d-d_i}$. The strength of such contribution is determined by $\int d^dx\, \mathcal{O}_i$ and, since the energy of the process is of order Λ_{R}, by dimensional analysis we have:

$$\int d^d x\, \mathcal{O}_i \sim \Lambda_{\mathrm{R}}^{d_i - d}.$$

This means that the effect of the interaction $\mathcal{O}_i$ on low-energy processes produces a contribution proportional to $(\Lambda_{\mathrm{R}}/\Lambda_0)^{d_i-d}$.

If $\mathcal{O}_i$ is a non-renormalizable operator with $d_i > d$, we have that the its contribution is suppressed by a factor $\Lambda_{\mathrm{R}}/\Lambda_0$ to a positive power and so it is completely negligible. For this reason the presence of non-renormalizable terms in the theory at the scale Λ_0 or, in general, at a scale Λ still much greater than Λ_{R}, has no effect on the behavior of physics at low energies.

This argument is very powerful because within a particle theory, such as the standard model, we can reasonably think that the cut-off scale Λ_0 is very high, for example of the order of the GUT scale or the Planck mass, so that the typical energy scale of the experiments in small compared to Λ_0. Then all experimental tests obviously reveal processes arising from renormalizable interactions only.

Moreover, studying the evolution of renormalizable and non-renormalizable couplings along the RG flow in the regime where couplings remain sufficiently small, a deeper analysis can be done. In this work we do not explicitly perform such analysis, provided by Polchinski in [5], but we just outline the main aspects. The first one is that the effective action evolved from the bare action defined at Λ_0 is equivalent, at low energies, to a renormalizable action. In particular, starting with a theory defined at Λ_0 by an initial surface on the space of parameters determined by the set of bare couplings $g_i^{(0)}$, the evolution along the RG flow necessarily reaches at low energies $\Lambda \ll \Lambda_0$ a stable IR surface parameterized only by renormalizable couplings, which is independent on both Λ_0 and the initial surface. Here non-renormalizable couplings are functions of the renormalizable ones, with the behavior $(\Lambda/\Lambda_0)^{d_i-d} \ll 1$, hence their influence is highly suppressed. Taking $\Lambda_0 \mapsto +\infty$ and keeping fixed the IR surface, non-renormalizable couplings reach a limit where they are strongly suppressed by negative powers of Λ_0. For this reason the Wilsonian treatment is connected with the concept of renormalizability, providing the identification between WRG and standard renormalization.

All these arguments stress that, whatever the explicit form of the action at Λ_0 is, a theory behaves as a renormalizable theory at low energies $\Lambda \ll \Lambda_0$. This explain why one expects the nature to be described by a renormalizable theory.

1.2. Wilson-Polchinski RG equation

In the previous section we have seen that WRG describes a flow of equivalent scale-dependent effective actions. This trajectory in the space of parameters is obtained from the functional integral integrating out modes of momentum larger than the scale Λ. Since the latter is introduced in $\mathcal{Z}$ to express a change of functional variables in order to write it in terms of the effective action, such functional is independent of Λ. This fact implies that the effective action has to produce, at every energy, the same scale-independent functional integral. This requires that such actions satisfy a constraint in the form of a differential equation, which describes their evolution along the flow. We can derive this equation imposing that $\partial_\Lambda \mathcal{Z} = 0$. This consists in the differential formalization of the WRG developed in [5] by Polchinski. See also [6] and [7] for further discussions.

To be more explicit, consider the scalar theory introduced above. This time the functional integral $\mathcal{Z}$ is expressed with the cut-off defined in the propagator:

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\left\langle \phi \, P_{\Lambda_0}^{-1}, \phi \right\rangle - S_{\text{int}}^{(0)}[\phi] + \langle J, \phi\rangle} = \int \mathcal{D}\phi \, e^{-S_0[\phi] + \langle J, \phi \rangle}. \tag{1.2.1}$$

When considering the integration of high-energy modes of the field, even the part of the propagator with momentum between Λ and Λ_0 is integrated, thus its contribution is included in the Wilsonian action S_{int}^Λ and the kinetic term propagates only up to Λ:

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\langle \phi \, P_\Lambda^{-1}, \phi\rangle - S_{\text{int}}^\Lambda[\phi] + \langle J, \phi\rangle} = \int \mathcal{D}\phi \, e^{-S_{\text{eff}}^\Lambda[\phi] + \langle J, \phi\rangle}. \tag{1.2.2}$$

Therefore the effective action can be written as:

$$S_{\text{eff}}^\Lambda[\phi] := \frac{1}{2}\left\langle \phi P_\Lambda^{-1}, \phi \right\rangle + S_{\text{int}}^\Lambda[\phi], \tag{1.2.3}$$

where P_Λ is the propagator with the cut-off scale lowered to Λ. So it is convenient to derive the evolution equation for S_{int}^Λ instead of S_{eff}^Λ. To do so consider the derivative of the functional integral with respect to Λ, which, as already said, has to be vanishing:

$$0 = \Lambda \partial_\Lambda \mathcal{Z} = \int \mathcal{D}\phi \left[-\frac{1}{2} \Lambda \partial_\Lambda \left\langle \phi \, P_\Lambda^{-1}, \phi \right\rangle - \Lambda \partial_\Lambda S_{\text{int}}^\Lambda \right] e^{-S_{\text{eff}}^\Lambda + \langle J, \phi\rangle} \tag{1.2.4}$$

Instead of performing calculations to obtain the desired equation, Polchinski guessed that the derivative of S_{int}^Λ is:

$$\Lambda \partial_\Lambda S_{\text{int}}^\Lambda = \frac{1}{2} \int d^d k (2\pi)^d \Lambda \partial_\Lambda P_\Lambda(k) \left[\frac{\delta S_{\text{int}}^\Lambda}{\delta \phi_k} \frac{\delta S_{\text{int}}^\Lambda}{\delta \phi_{-k}} - \frac{\delta^2 S_{\text{int}}^\Lambda}{\delta \phi_k \, \delta \phi_{-k}} \right]. \tag{1.2.5}$$

Substituting this equation in (1.2.4), we must prove that:

$$\begin{aligned} \frac{1}{2} \int d^d k (2\pi)^d \Lambda \partial_\Lambda P_\Lambda \int \mathcal{D}\phi \Big[-\frac{1}{(2\pi)^{2d}} P_\Lambda^{-2} \phi_k \, \phi_{-k} \\ + \frac{\delta S_{\text{int}}^\Lambda}{\delta \phi_k} \frac{\delta S_{\text{int}}^\Lambda}{\delta \phi_{-k}} - \frac{\delta^2 S_{\text{int}}^\Lambda}{\delta \phi_k \, \delta \phi_{-k}} \Big] e^{-S_{\text{eff}}^\Lambda + \langle J, \phi \rangle} = 0. \end{aligned} \tag{1.2.6}$$

To prove the identity above we can start by expressing the derivative of S^Λ_{int} in terms of the whole effective action:

$$\frac{\delta S^\Lambda_{\text{int}}}{\delta\phi_k} = \frac{\delta S^\Lambda_{\text{eff}}}{\delta\phi_k} - \frac{1}{(2\pi)^d} P_\Lambda^{-1} \phi_{-k}, \tag{1.2.7}$$

so that:

$$\frac{\delta^2 S^\Lambda_{\text{int}}}{\delta\phi_k\,\delta\phi_{-k}} = \frac{\delta^2 S^\Lambda_{\text{eff}}}{\delta\phi_k\,\delta\phi_{-k}} - \frac{1}{(2\pi)^d}\frac{\delta(P_\Lambda^{-1}\phi_{-k})}{\delta\phi_{-k}}, \tag{1.2.8}$$

$$\frac{\delta S^\Lambda_{\text{int}}}{\delta\phi_k}\frac{\delta S^\Lambda_{\text{int}}}{\delta\phi_{-k}} = \frac{\delta S^\Lambda_{\text{eff}}}{\delta\phi_k}\frac{\delta S^\Lambda_{\text{eff}}}{\delta\phi_{-k}} + \frac{1}{(2\pi)^{2d}} P_\Lambda^{-2}\phi_k\,\phi_{-k} - \frac{2}{(2\pi)^d} P_\Lambda^{-1}\phi_{-k}\frac{\delta S^\Lambda_{\text{eff}}}{\delta\phi_{-k}}. \tag{1.2.9}$$

Owing to the presence of the momentum integral in (1.2.6), some terms in (1.2.9) have been regrouped sending $k \mapsto -k$.

Now, using these relations, (1.2.6) becomes:

$$\begin{aligned}\frac{1}{2}\int d^dk(2\pi)^d\Lambda\partial_\Lambda P_\Lambda \int\mathcal{D}\phi\Big[&\frac{\delta^2 e^{-S^\Lambda_{\text{eff}}}}{\delta\phi_k\,\delta\phi_{-k}} + \frac{2}{(2\pi)^d}\frac{\delta\big(P_\Lambda^{-1}\phi_{-k}\,e^{-S^\Lambda_{\text{eff}}}\big)}{\delta\phi_{-k}} \\ &- \frac{1}{(2\pi)^d}\frac{\delta(P_\Lambda^{-1}\phi_{-k})}{\delta\phi_{-k}}\,e^{-S^\Lambda_{\text{eff}}}\Big]\,e^{\langle J,\,\phi\rangle} = 0,\end{aligned} \tag{1.2.10}$$

where we have regrouped some terms, such as:

$$\left(\frac{\delta S^\Lambda_{\text{eff}}}{\delta\phi_k}\frac{\delta S^\Lambda_{\text{eff}}}{\delta\phi_{-k}} - \frac{\delta^2 S^\Lambda_{\text{eff}}}{\delta\phi_k\,\delta\phi_{-k}}\right)e^{-S^\Lambda_{\text{eff}}} = \frac{\delta^2 e^{-S^\Lambda_{\text{eff}}}}{\delta\phi_k\,\delta\phi_{-k}}, \tag{1.2.11}$$

$$\begin{aligned}&\left(\frac{1}{(2\pi)^d}\frac{\delta(P_\Lambda^{-1}\phi_{-k})}{\delta\phi_{-k}} - \frac{2}{(2\pi)^d}P_\Lambda^{-1}\phi_{-k}\frac{\delta S^\Lambda_{\text{eff}}}{\delta\phi_{-k}}\right)e^{-S^\Lambda_{\text{eff}}} \\ &\qquad = \frac{2}{(2\pi)^d}\frac{\delta\big(P_\Lambda^{-1}\phi_{-k}\,e^{-S^\Lambda_{\text{eff}}}\big)}{\delta\phi_{-k}} - \frac{1}{(2\pi)^d}\frac{\delta(P_\Lambda^{-1}\phi_{-k})}{\delta\phi_{-k}}\,e^{-S^\Lambda_{\text{eff}}}.\end{aligned} \tag{1.2.12}$$

At this point, the last term in the integrand of (1.2.10) is field-independent, thus it can be reabsorbed by a redefinition of the functional measure. Therefore such equation can be written as:

$$\frac{1}{2}\int d^dk(2\pi)^d\Lambda\partial_\Lambda P_\Lambda\int\mathcal{D}\phi\frac{\delta}{\delta\phi_{-k}}\left[\left(\frac{\delta}{\delta\phi_k} + \frac{2}{(2\pi)^d}P_\Lambda^{-1}\phi_{-k}\right)e^{-S^\Lambda_{\text{eff}}}\right]e^{\langle J,\,\phi\rangle} = 0, \tag{1.2.13}$$

where the exponential of the source term can be included in the functional derivative because it does not give any contribution. Indeed the result of its derivation is proportional to J and we immediately realize that the product $\Lambda\partial_\Lambda P_\Lambda(k) \times J(k)$ vanishes[4]. Hence we have the functional integral of a functional derivative with respect to the same argument, which is zero by definition. This proves (1.2.6), hence the Wilsonian action satisfies the evolution equation:

$$\Lambda\partial_\Lambda S^\Lambda_{\text{int}} = \frac{1}{2}\int d^dk(2\pi)^d\Lambda\partial_\Lambda P_\Lambda(k)\left[\frac{\delta S^\Lambda_{\text{int}}}{\delta\phi_k}\frac{\delta S^\Lambda_{\text{int}}}{\delta\phi_{-k}} - \frac{\delta^2 S^\Lambda_{\text{int}}}{\delta\phi_k\,\delta\phi_{-k}}\right], \tag{1.2.14}$$

[4] $\partial_\Lambda P_\Lambda$ is different from zero only for $|k| > \Lambda$, while J vanishes for momenta over Λ_{R}.

which is called Wilson-Polchinski RG equation (WPE). Its solution, with initial condition $S_{\rm int}^{\Lambda_0} = S_{\rm int}^{(0)}$, provides the evolution of the Wilsonian action along the RG flow. In such a way we can know the behavior of the theory at all the energy scale. Moreover, we can expand $S_{\rm int}^{\Lambda}$ in power series of the field, so that, deriving the RG equations for the couplings, we can determine their evolution with the energy scale, i.e. their running. This is very powerful and useful, in particular, to prove the renormalizability of a theory by the derivation of its UV limit ($\Lambda_0 \mapsto \infty$), as showed by Polchinski in [5] and also discussed, for example, in [8–12].

1.2.1. Wilsonian action and the generating functional

The Wilsonian action $S_{\rm int}^{\Lambda}$ arises from the functional integration over high-energy modes of the bare interaction term and also over the high-momentum part (between the scales Λ and Λ_0) of the kinetic term. Thus we can write it as:

$$e^{-S_{\rm int}^{\Lambda}[\phi]} = \int \mathcal{D}\widetilde{\phi}\, e^{-\frac{1}{2}\left\langle \widetilde{\phi}\, {P_{\Lambda}^{\Lambda_0}}^{-1}, \widetilde{\phi} \right\rangle - S_{\rm int}^{(0)}[\widetilde{\phi}+\phi]}, \tag{1.2.15}$$

where:

$$P_{\Lambda}^{\Lambda_0}(k) := \frac{K(k^2/\Lambda_0^2) - K(k^2/\Lambda^2)}{k^2+m^2}, \tag{1.2.16}$$

which is the propagator with a double cut-off, which selects propagating modes only in the interval of energies $[\Lambda, \Lambda_0]$.

This way of writing $S_{\rm int}^{\Lambda}$ through $P_{\Lambda}^{\Lambda_0}$ is equivalent to the definition with the cut-off in the functional measure expressed by (1.1.9). Indeed the field $\phi_{\rm H}$ is non vanishing only for momenta between Λ and Λ_0, thus its kinetic term $\langle \phi_{\rm H}\, P^{-1},\, \phi_{\rm H} \rangle$ is equivalent to $\left\langle \widetilde{\phi}\, {P_{\Lambda}^{\Lambda_0}}^{-1},\, \widetilde{\phi} \right\rangle$ in (1.2.15).

Considering the properties of the Wilsonian action, we note, from (1.1.56), that $S_{\rm int}^{\Lambda}$ is related to Feynman diagrams carrying information only about high energies. We give here a general proof of this fact, showing that $S_{\rm int}^{\Lambda}$ is actually the generator of connected and amputated diagrams of the bare theory with the propagator $P_{\Lambda}^{\Lambda_0}$ with double cut-off.

Consider the generating functional of connected diagrams for such theory:

$$e^{-\mathcal{F}[J]} = \int \mathcal{D}\widetilde{\phi}\, e^{-\frac{1}{2}\left\langle \widetilde{\phi}\, {P_{\Lambda}^{\Lambda_0}}^{-1}, \widetilde{\phi} \right\rangle - S_{\rm int}^{(0)}[\widetilde{\phi}] + \left\langle J, \widetilde{\phi} \right\rangle}, \tag{1.2.17}$$

where we have used $\widetilde{\phi}$ to name the field for a reason that will be clear later.
Now we redefine the source as:

$$J_k := {P_{\Lambda}^{\Lambda_0}}^{-1}(k)\, \phi_k, \tag{1.2.18}$$

so that $\mathcal{F}$ becomes a functional of the new source ϕ, and we define:

$$\widetilde{\mathcal{F}}[\phi] = \mathcal{F}[{P_{\Lambda}^{\Lambda_0}}^{-1}\phi] + \frac{1}{2}\left\langle \phi\, {P_{\Lambda}^{\Lambda_0}}^{-1},\, \phi \right\rangle, \tag{1.2.19}$$

which is a deformed generating functional equal to $\mathcal{F}$ plus a term proportional to ϕ^2. In the following we will show that $\widetilde{\mathcal{F}}$ satisfies WPE. Indeed the addition of the term quadratic in ϕ allows us to write the deformed functional as:

$$e^{-\widetilde{\mathcal{F}}[\phi]} = \int \mathcal{D}\widetilde{\phi}\, e^{-\frac{1}{2}\left\langle (\widetilde{\phi}-\phi)\, {P_\Lambda^{\Lambda_0}}^{-1},\, \widetilde{\phi}-\phi \right\rangle - S^{(0)}_{\text{int}}[\widetilde{\phi}]} := \int \mathcal{D}\widetilde{\phi}\, e^{-S^{\Lambda_0}_{\Lambda}[\widetilde{\phi},\,\phi]}, \tag{1.2.20}$$

so that, when we take the derivative with respect to Λ we obtain:

$$\Lambda\partial_\Lambda\, e^{-\widetilde{\mathcal{F}}} = \int \mathcal{D}\widetilde{\phi}\Big(-\frac{1}{2}\Lambda\partial_\Lambda \left\langle (\widetilde{\phi}-\phi)\, {P_\Lambda^{\Lambda_0}}^{-1},\, \widetilde{\phi}-\phi \right\rangle \Big)\, e^{-S^{\Lambda_0}_{\Lambda}[\widetilde{\phi},\,\phi]}. \tag{1.2.21}$$

Recalling from (1.1.3) and (1.2.16) that the dependence on Λ in the propagator is introduced by the function $K(k^2/\Lambda^2)$, the derivative of the double cut-off propagator with respect to Λ is $\Lambda\partial_\Lambda P_\Lambda^{\Lambda_0} = -\Lambda\partial_\Lambda P_\Lambda$. Thus:

$$\begin{aligned}\Lambda\partial_\Lambda\, e^{-\widetilde{\mathcal{F}}} = -\frac{1}{2}\int d^dk (2\pi)^d \Lambda\partial_\Lambda P_\Lambda \\ \times \int \mathcal{D}\widetilde{\phi} \frac{1}{(2\pi)^{2d}} {P_\Lambda^{\Lambda_0}}^{-2} (\widetilde{\phi}-\phi)_k\, (\widetilde{\phi}-\phi)_{-k}\, e^{-S^{\Lambda_0}_{\Lambda}[\widetilde{\phi},\,\phi]}.\end{aligned} \tag{1.2.22}$$

Therefore we can obtain the desired result if we prove that the functional integral in (1.2.22) is equal to:

$$\left(\frac{\delta\widetilde{\mathcal{F}}}{\delta\phi_k}\frac{\delta\widetilde{\mathcal{F}}}{\delta\phi_{-k}} - \frac{\delta^2\widetilde{\mathcal{F}}}{\delta\phi_k\,\delta\phi_{-k}} \right) e^{-\widetilde{\mathcal{F}}} = \frac{\delta^2 e^{-\widetilde{\mathcal{F}}}}{\delta\phi_k\,\delta\phi_{-k}}.$$

Performing the derivatives we obtain:

$$\begin{aligned}\frac{\delta e^{-\widetilde{\mathcal{F}}}}{\delta\phi_k} &= \int \mathcal{D}\widetilde{\phi} \frac{1}{(2\pi)^d} {P_\Lambda^{\Lambda_0}}^{-1} (\widetilde{\phi}-\phi)_{-k}\, e^{-S^{\Lambda_0}_{\Lambda}[\widetilde{\phi},\,\phi]}, \\ \frac{\delta^2 e^{-\widetilde{\mathcal{F}}}}{\delta\phi_k\,\delta\phi_{-k}} &= -\int \mathcal{D}\widetilde{\phi} \frac{1}{(2\pi)^d} {P_\Lambda^{\Lambda_0}}^{-1} \delta^d(0)\, e^{-S^{\Lambda_0}_{\Lambda}[\widetilde{\phi},\,\phi]} \\ &\quad + \int \mathcal{D}\widetilde{\phi} \frac{1}{(2\pi)^{2d}} {P_\Lambda^{\Lambda_0}}^{-2} (\widetilde{\phi}-\phi)_k\, (\widetilde{\phi}-\phi)_{-k}\, e^{-S^{\Lambda_0}_{\Lambda}[\widetilde{\phi},\,\phi]},\end{aligned}$$

where the first term in the second derivative is a field independent term, then we can neglect it as we did for that in (1.2.10). The remaining term is exactly the functional integral in (1.2.22), so $\widetilde{\mathcal{F}}$ satisfies the Wilson-Polchinski RG equation:

$$\Lambda\partial_\Lambda \widetilde{\mathcal{F}} = \frac{1}{2}\int d^dk (2\pi)^d \Lambda\partial_\Lambda P_\Lambda(k) \left[\frac{\delta\widetilde{\mathcal{F}}}{\delta\phi_k}\frac{\delta\widetilde{\mathcal{F}}}{\delta\phi_{-k}} - \frac{\delta^2\widetilde{\mathcal{F}}}{\delta\phi_k\,\delta\phi_{-k}} \right], \tag{1.2.23}$$

with boundary conditions: $\widetilde{\mathcal{F}}[\phi] = S^{(0)}_{\text{int}}[\phi]$ for $\Lambda = \Lambda_0$. Indeed in this limit ${P_\Lambda^{\Lambda_0}}^{-1}$ diverges, thus $e^{-\widetilde{\mathcal{F}}}$ vanishes everywhere but for $\widetilde{\phi} = \phi$, where it is equal to the bare interaction term.

At this point $\widetilde{\mathcal{F}}$ and S^{Λ}_{int} satisfy the same equation with the same boundary conditions, then they must be equal up to some constant term that is completely irrelevant.

This fact introduces an important relation:

$$S^{\Lambda}_{\text{int}}[\phi] = \mathcal{F}[{P^{\Lambda_0}_{\Lambda}}^{-1}\phi] + \frac{1}{2}\left\langle \phi\, {P^{\Lambda_0}_{\Lambda}}^{-1},\, \phi \right\rangle, \tag{1.2.24}$$

which states that the Wilsonian action is equal to the generating functional of connected diagrams of the bare theory with double cut-off, with source ${P^{\Lambda_0}_{\Lambda}}^{-1}\phi$. Then, for $n > 2$:

$$\left.\frac{\delta^n S^{\Lambda}_{\text{int}}[\phi]}{\delta\phi_{k_1}\cdots\delta\phi_{k_n}}\right|_{\phi=0} = \prod_{i=1}^{n} {P^{\Lambda_0}_{\Lambda}}^{-1}(k_i)\left.\frac{\delta^n\mathcal{F}[J]}{\delta J_{k_1}\cdots\delta J_{k_n}}\right|_{J=0}. \tag{1.2.25}$$

Hence, since connected and amputated diagrams are obtained by connected ones cutting the external line propagators, (1.2.25) actually states that S^{Λ}_{int} is the generator of connected and amputated diagrams.

This is a very important feature of WRG and it will be used in the next chapter to derive the evolution equation for large matrix models.

WRG is a vast sector of QFT which, as already said, greatly improved the knowledge and comprehension of renormalization. In this chapter we mainly developed the aspects connected to the RG flow equation because they will be very important in the next stages of this thesis work.

CHAPTER 2

The Large N Limit for Matrix Models

In this chapter we analyze a very important approximation scheme that can be applied to a certain type of QFT's: the large N limit.

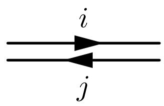
This method, first developed by 't Hooft in [13], produces important results for particular theories where fields are interpreted as $N \times N$ matrices and couplings are rearranged to be proportional to the size N of the fields ('t Hooft couplings). Indeed, considering the matrix size N going to infinity, the behavior of the theory drastically simplify because it receives contribution only from a certain class of Feynman graphs, called planar diagrams. This is possible thanks to the fact that the dependence on N of Feynman graphs is connected to their topology. Thus, in this chapter, we will study the topological properties of diagrams and we will show that planar diagrams are the leading contributions, whereas all the other graphs are suppressed by factors of order $1/N$ or higher. After that we will study WRG of a QFT in the large N expansion to show that the Wilson-Polchinski RG equation has a particular and interesting form in such limit.

2.1. Topological classification of Feynman diagrams

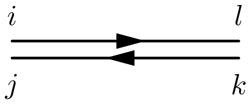
Consider a theory whose fields are $N \times N$ matrices $M_{ij}(x)$, where $i, j = 1, \cdots, N$ are the matrix indices, while other possible space-time and flavor indices are neglected for simplicity. In such a way products of the fields are non-commutative, so that the structure of the interaction terms made of these products depends on how field indices are contracted in them.

In general these matrix fields are thought to transform in the adjoint representation of a certain non-abelian algebra, such as gauge fields in Yang-Mills theory, so that their matrix structure arises from a symmetry principle. This fact requires that the action is written in terms of traces of the fields and their derivatives, so that every index is contracted and the whole functional is invariant under such symmetry transformations. The most important example is the $U(N)$ Yang-Mills theory, where gauge fields A_μ, and so also the field strength $F_{\mu\nu}$, are $N \times N$ matrices transforming in the adjoint representation of $U(N)$, so that the lagrangian of the theory has the form $\mathcal{L}_{\rm YM} \sim -{\rm tr}\, F_{\mu\nu}F^{\mu\nu}$.

The most important consequence of the non-commutative structure of the products of fields is that the vertices of the theory are not invariant under all permutations of external legs, but, thanks to the presence of the traces in the action, only under cyclic ones. This means that exchanging the order of some lines of a vertex, even though attached to other parts of a graph, one could produce a different diagram, i.e. a different contribution. This is due to the fact that field indices of the external lines of a vertex can be contracted inside a diagram in many different ways. For this reason it is useful to introduce a new notation, called double-line notation, where the field is represented by a double line:

$M_{ij}(x) =$  x (2.1.1)

Each line is associated to a field index and, if it is attached to another line carrying another index, it produces the saturation of such indices:

 $\sim \delta_{il}\,\delta_{jk}$ (2.1.2)

This is due to the fact that the structure of the kinetic term, neglecting the derivatives, is proportional to ${\rm tr}\,|M|^2 = \sum_{a,b} M_{ab}{M^\dagger}_{ba}$. Thus, taking the second derivative with respect to M in order to obtain the propagator, one has:

$$\frac{\partial}{\partial M_{ij}}\frac{\partial}{\partial {M^\dagger}_{kl}}\sum_{a,b} M_{ab}{M^\dagger}_{ba} \sim \delta_{il}\,\delta_{jk}.$$

In this way one can keep track of how indices are contracted into the diagrams. For example, consider the next-to-leading order correction to the free energy of the ϕ^4 scalar theory. If ϕ is a commutative field the action is that considered in subsection 1.1.1, thus we have the contribution of three equivalent diagrams (Figure 2.1.1) which arise from all possible contractions of the lines of the vertex. They are equivalent because in a commutative theory one can always rearrange the lines with a permutation in order to reach the same form for all diagrams.

If ϕ is a $N \times N$ matrix, the lagrangian contains traces of the field, thus it reads:

$$\mathcal{L} \sim {\rm tr}\left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right).$$

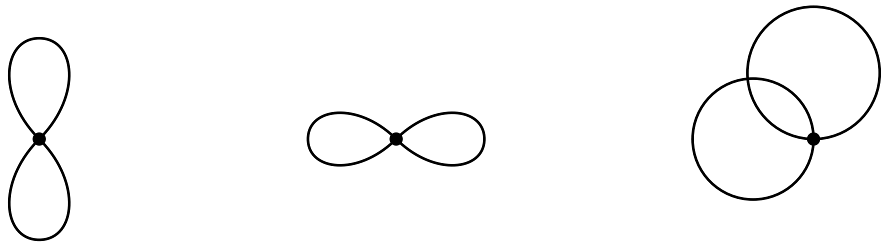


Figure 2.1.1. *The three equivalent diagrams representing the first order correction to the free energy of the commutative ϕ^4 theory*

Using the double-line notation, the graphs have the form shown in Figure 2.1.2, where diagram (c) is different from (a) and (b) because it contains different contractions of indices. Indeed, as we will see in detail in the next section, every diagram of a matrix theory contains a factor proportional to some power of the matrix size of N coming from the propagators and the vertices, which depends on the explicit form of the action. Moreover, since every line inside the diagram carries saturation of the indices, a closed line produces a factor $\sum_i \delta_{ii} = N$. Now, diagrams in Figure 2.1.2 have the same number of propagators and vertices, so the factor coming from them is the same, but (a) and (b) contain three closed lines whereas (c) only one. Therefore, taking the ratio of the contributions of such diagrams we note that the latter is suppressed with respect to (a) and (b) by a factor $1/N^2$.

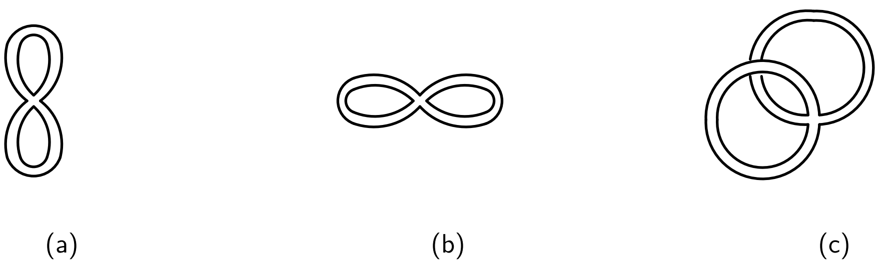


Figure 2.1.2. *The three diagrams representing the first order correction to the free energy of the ϕ^4 matrix theory*

This example stresses that it is possible to understand whether a diagram is subleading with respect to another one simply through the analysis of its structure of propagators, vertices and closed internal lines, even though the explicit form of the action is unknown. Thus one is led to ask whether it is possible to generalize this argument and to determine "a priori" the correct behavior of every graph. The answer is yes and it involves a topological analysis which allows us to make a classification of Feynman diagrams.

Indeed, thanks to the double-line notation, one can think that the first two graphs in Figure 2.1.2 have the same topology because they can be drawn on a flat sheet without superpositions of lines (first graph in Figure 2.1.3). Instead, diagram (c) can be drawn only on a torus (second graph in Figure 2.1.3) because it contains a superposition of two lines, which makes its topology different from the previous ones.

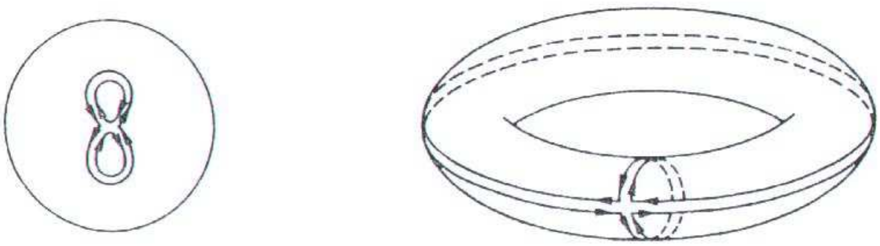


Figure 2.1.3. *First order diagrams of the free energy of the ϕ^4 matrix theory embedded in the associated surfaces*

This preliminary analysis suggests that there could be a connection between Feynman diagrams and surfaces, which enables us to organize graphs according to some topological property of the related surfaces, like for example the number of handles. In such a way one could define the topology of a diagram as the topology of its associated surface. This is true for any non-commutative theory, where lines and vertices are ordered up to cyclic permutations, and it is made evident by the double line notation.

To remark this important fact, in the following, we will develop the analysis for general theories, stating a general proposition and exploring its consequences, whereas in the next section we will focus on a concrete example of a scalar matrix theory to clarify these facts through explicit computations.

The connection between Feynman diagrams and surfaces is defined by a proposition which expresses that each connected Feynman graph corresponds to a unique Riemann surface:

Proposition:

Every connected diagram is associated to a Riemann surface whereon it can be drawn without any superposition of lines.

This allows us to translate properties of Riemann surfaces into their associated diagrams. Hence we can define the genus of a graph as the genus[1] of its corresponding surface. For example, the first graph in Figure 2.1.3, which represents both diagrams 2.1.2a and 2.1.2b, has null genus because it is associated to the sphere, or equivalently to a flat sheet[2]. The second one has, instead, genus 1 because it has to be drawn on a torus, which is the surface with unitary genus, i.e. with one handle. Thus graphs with null genus will be called planar diagrams, whereas graphs with non-vanishing genus will be non-planar diagrams.

At this point it is useful to limit our discussion only to vacuum diagrams[3] for simplicity. The results we will henceforth obtain can however be extended to diagrams with external

[1]The genus of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without making the resultant manifold disconnected. It is equal to the number of handles on it.

[2]One typically prefers to consider compact surfaces like the sphere instead of open ones, like a flat sheet, because topological properties are more easily formulated on them.

[3]Diagrams without external legs.

legs, as we will see later. Thus, considering connected vacuum diagrams, we can use the triangulation of Riemann surfaces applied to them. In such a way we can identify such diagrams as geometric figures drawn on the associated surfaces, where the edges are the propagator lines, the faces are the internal closed lines and the vertices correspond to the vertices of the diagram. Therefore we can apply Euler's formula to these graphs:

$$V - P + I = \chi, \tag{2.1.3}$$

where V, P and I are respectively the number of vertices, propagators and closed lines, whereas $\chi = 2 - 2g$ is the Euler characteristic and g is the genus of the surface.

By this identification we can provide a classification of vacuum diagrams based on the genus. Indeed, even though the power of N multiplying every diagram, which is called weight factor, depends on the explicit form of the action, one can show that the ratio of the contribution of two diagrams respectively of genus g_1 and g_2 is proportional to $1/N^{2(g_1 - g_2)}$.

In order to prove that, consider a general matrix theory where fields M are $N \times N$ matrices. Depending on the definition of the fields, the kinetic term may contain an arbitrary power α of N such that KT $\sim N^\alpha \,\mathrm{tr}\, |\partial M|^2$ and every propagator line carries a factor $1/N^\alpha$. For simplicity, the interaction term $\mathcal{L}_{\text{int}}[M]$ is set to contain only single-trace couplings:

$$\mathcal{L}_{\text{int}}[M] = \sum_{n=3}^{+\infty} g_n \mathrm{tr}\, |M|^n. \tag{2.1.4}$$

Other derivative or multi-trace couplings are neglected because their behavior is the same as the considered couplings and they would only represent an unnecessary complication to our proof.

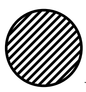
Now consider a generic vacuum diagram with P propagators, V vertices and I closed lines. Calling V_p the number of vertices associated to the interaction proportional to tr M^p, we have:

$$V = V_3 + V_4 + \cdots = \sum_{p=3}^{+\infty} V_p. \tag{2.1.5}$$

Every vertex of p fields in the diagram is attached to p propagators, which, in their turn, are attached each one to another vertex. So, summing the total number of vertex legs in the graph we count every propagator twice. Thus we have:

$$P = \frac{1}{2}(3\, V_3 + 4\, V_4 + \cdots) = \sum_{p=3}^{+\infty} \frac{p}{2}\, V_p. \tag{2.1.6}$$

The contribution of the diagram is:


VPI

$$\sim g_3^{V_3} g_4^{V_4} \cdots N^{I - \alpha P} = N^{I - \alpha P} \prod_{p=3}^{+\infty} g_p^{V_p}, \tag{2.1.7}$$

because every vertex give a factor equal to its coupling, whereas every propagator carries a factor $N^{-\alpha}$ and, since every line in the diagram carries a saturation of indices, every closed line produces $\sum_i \delta_{ii} = N$, so we have the term $N^{I-\alpha P}$.

Using (2.1.3), (2.1.5) and (2.1.6), this contribution can be rearranged. Indeed the number of closed lines is equal to:

$$I = 2 - 2g + P - V = 2 - 2g + \sum_{p=3}^{+\infty} \left(\frac{p}{2} - 1 \right) V_p,$$

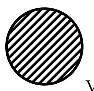
so we have:

$$N^{I-\alpha P} \prod_{p=3}^{+\infty} g_p^{V_p} = N^{2-2g-\alpha \sum_p \frac{p}{2} V_p} \prod_{p=3}^{+\infty} \left(g_p N^{\frac{p}{2}-1} \right)^{V_p} = N^{2-2g} \prod_{p=3}^{+\infty} \left(g_p N^{\frac{p}{2}-1-\frac{p}{2}\alpha} \right)^{V_p}.$$

At this point, we can rewrite the theory defining the so called 't Hooft coupling λ_p of the interaction $\operatorname{tr} M^p$ as:

$$\lambda_p := g_p \, N^{\frac{p}{2}-1-\frac{p}{2}\alpha}. \tag{2.1.8}$$

This being done, the diagram in (2.1.7) becomes:



$$_{\text{VPI}} \sim \frac{1}{N^{2g-2}} \prod_{p=3}^{+\infty} \lambda_p^{V_p}. \tag{2.1.9}$$

This shows that, writing the theory through 't Hooft couplings, the contribution of every vacuum diagram of genus g is proportional to $1/N^{2g-2}$. This feature is independent of the explicit form of the action, i.e. the precise value of α, because the definition of 't Hooft couplings (2.1.8) acts balancing the contribution of every vertex with that of every propagator so that the whole weight factor turns out to be independent of α. Therefore this proves that the ratio of the contribution of two diagrams of genus g_1 and g_2 is $1/N^{2(g_1-g_2)}$ and allows us to make a classification of vacuum diagrams according to their genus.

This feature of matrix theories defined through 't Hooft couplings leads to a very interesting behavior in a particular regime: the large N limit. This consists in taking $N \mapsto +\infty$ whereas 't Hooft couplings λ_p are kept fixed, so that the original couplings are switched off. Thus, thanks to the presence of the weight factor, the contribution of diagrams with non-vanishing genus becomes negligible compared to that of planar diagrams because it is subleading at least of a factor $1/N^2$. Then in this limit the theory becomes simpler, described only by planar graphs ($g = 0$) and so more easily computable.

For example, consider the free energy of the theory, which is defined as:

$$e^{-\mathcal{F}} := \int \mathcal{D}M \, e^{-S[M]}, \tag{2.1.10}$$

where $S[M]$ is the action. $\mathcal{F}$ contains the contribution of all connected vacuum diagrams of the theory at every order. So we can expand it according to the genus of its diagrams:

$$\mathcal{F} = \sum_g \frac{\mathcal{F}^{(g)}}{N^{2g-2}}, \tag{2.1.11}$$

where $\mathcal{F}^{(g)}$ is the contribution coming from all genus g connected diagrams, hence, it is weighted with the factor $1/N^{2g-2}$. When N is sent to infinity, keeping λ_p fixed, only $\mathcal{F}^{(0)}$ survives, because only planar diagrams contribute to the theory, so the value of the free energy is given only by few simple graphs.

(a) (b)

Figure 2.1.4. *Two one loop connected diagrams of the ϕ^4 matrix theory*

All these discussions concern only vacuum diagrams. When considering graphs with external legs, the correspondence with Riemann surfaces is still valid, but some complications are introduced by the presence of the external lines.

Indeed, consider for example the two diagrams with two external legs of the ϕ^4 matrix theory shown in Figure 2.1.4. In order to evaluate their contribution, we can note that both graphs have the same number of propagators and vertices, but in (a) there is a closed line (internal circle) which produces a factor N due to the saturation of the indices, whereas in (b) there is none. Thus (b) is subleading with respect to (a) of a factor $1/N$ even if both of them have sphere topology (vanishing genus) and so this fact completely disagrees with the predictions of the topological analysis performed for vacuum diagrams.

This particular behavior can be understood considering the associated Riemann surfaces. Indeed as long as dealing with vacuum diagrams, we can use the identification with geometric figures and compute their weight factor through Euler's formula (2.1.3), whereas, when there are external legs, such identification is broken because of the presence of the latter. In order to restore it we have to consider Riemann surfaces with boundaries[4], where the external legs will be attached. Thus a diagram with external legs is associated to a Riemann surface with boundaries, such that, drawing the graph on the latter, every external line is attached to one boundary. In such a way a diagram with external legs can be identified with a geometric figure where such lines are thought as part of the associated boundaries, so we can use

[4]The boundary of a subset S of a topological space X is the set of points which can be approached both from S and from the outside of S. For example a disk is of dimension 2 and its boundary is the 1-dimensional circle.

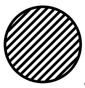
Euler's formula where Euler's characteristic generalizes to:

$$\chi = 2 - 2g - b, \tag{2.1.12}$$

where b is the number of boundaries of the surface.

Since external lines are treated as part of boundaries, this identification holds true for amputated diagrams, i.e. graphs where external legs are not propagators. Later, considering the connected correlation functions, we will see that the analysis we are going to perform still holds true for non-amputated diagrams, even though we have to include the contribution of the propagators in the external lines.

Now we can study the behavior of the weight factor of diagrams with external legs considering the general theory used in the case of vacuum diagrams. Recalling that every propagator carries a factor $1/N^{\alpha}$ and the interaction term is that in (2.1.4), where 't Hooft couplings are (2.1.8), the contribution of a connected amputated diagram with r external legs, V vertices, P propagators and I closed lines, associated to a surface with genus g and b boundaries, is:



$$_{r;\,\mathrm{VPI}} \sim N^{I-\alpha P} \prod_{p=3}^{+\infty} \left(\lambda_p N^{1-\frac{p}{2}+\frac{p}{2}\alpha}\right)^{V_p} = N^{I-\alpha P+\sum_p \left(1-\frac{p}{2}+\frac{p}{2}\alpha\right)V_p} \prod_{p=3}^{+\infty} \lambda_p^{V_p}, \tag{2.1.13}$$

Now, since there are r external legs, the relation (2.1.6) between P and V is modified to:

$$P = \sum_{p=3}^{+\infty} \frac{p}{2} V_p - \frac{r}{2}, \tag{2.1.14}$$

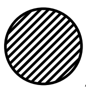
because the r external legs are attached to r vertex lines, which do not have to be counted as propagators in the sum. Therefore, as we did for vacuum diagrams, the contribution of (2.1.13) can be rearranged using (2.1.3) with the generalized Euler characteristic (2.1.12), (2.1.5) and (2.1.14). Thus the number of closed lines becomes:

$$I = 2 - 2g - b + P - V = 2 - 2g - b + \sum_{p=3}^{+\infty} \left(\frac{p}{2} - 1\right) V_p - \frac{r}{2},$$

hence we have:

$$N^{I-\alpha P+\sum_p \left(1-\frac{p}{2}+\frac{p}{2}\alpha\right)V_p} \prod_{p=3}^{+\infty} \lambda_p^{V_p} = N^{2-2g-b+(\alpha-1)\frac{r}{2}} \prod_{p=3}^{+\infty} \lambda_p^{V_p}.$$

The contribution of the diagram is then:



$$_{r;\,\mathrm{VPI}} \sim \frac{1}{N^{2g-2+b+(1-\alpha)\frac{r}{2}}} \prod_{p=3}^{+\infty} \lambda_p^{V_p}, \tag{2.1.15}$$

This shows that the weight factor of a diagram with external legs in a theory written through 't Hooft couplings not only depends on the genus of the diagram, but also on the number of boundaries of the associated surface and on the number of the external legs. Unlike the case of vacuum diagrams, here there is a dependence on the explicit definition of the action, indeed (2.1.15) depends on the explicit value of α. However the ratio of the contributions of diagrams with the same number of external legs is general, indeed, if their genus is respectively g_1 and g_2 and the number of holes is b_1 and b_2, such ratio is $1/N^{2(g_1-g_2)+(b_1-b_2)}$ whatever the explicit form of the action is. This result generalizes the one obtained for vacuum graphs to all amputated diagrams.

Figure 2.1.5. *Connected diagrams associated to surfaces with respectively one and two boundaries (dashed lines)*

Before considering the case of non-amputated connected diagrams, it is important to better understand the connection between the external legs and the boundaries of the associated surface.

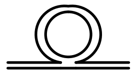
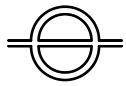
If we consider a diagram with all external legs attached to the same internal line, we have a surface with one boundary because all the external lines can be attached to it (Figure 2.1.5a, where the dashed box represents the boundary), so that the graph has $b = 1$.

If the external legs are attached to two different internal lines, the associated surface must have two boundaries because the external lines cannot be attached to the same boundary without cutting the graph (Figure 2.1.5b). Hence the surface associated to a diagram has as many boundaries as different internal lines containing external legs there are in the graph.

In such a way, both diagrams in Figure 2.1.4 are graphs with null genus, but (a) is associated to a sphere with one boundary, whereas (b) is associated to a sphere with two boundaries. Therefore, using (2.1.15), we have that (b) is subleading with respect to (a) of a factor $1/N$, which agrees with the prediction given above.

Let us consider now the connected correlation functions. They are described by non-amputated connected diagrams, i.e. graphs where the external legs are propagators (henceforth we will refer to them just as connected diagrams). They have the same structure of the associated amputated diagrams, except for the external propagators, each one producing an extra factor $1/N^{\alpha}$. Therefore the weight factor of a connected diagram with r external legs can be computed using (2.1.15) and multiplying it by $1/N^{\alpha r}$, which is the contribution of the external propagators. In such a way the ratio of the contributions of two connected diagrams with the same number of external legs and characterized respectively by (g_1, b_1) and (g_2, b_2) is still $1/N^{2(g_1-g_2)+(b_1-b_2)}$.

For this reason the leading contribution to the r-point function comes from diagrams with sphere topology and one boundary, i.e. with all external legs attached to the same internal line. Hence such graphs are called planar diagrams, whereas all the other ones are suppressed by powers of N and so they are called non-planar (included null genus graphs with more than one boundary). For this reason, in the large N expansion, the correlation functions receive contribution only from planar diagrams. For example the 2-point function can be expanded as:

$\langle M_{i_1 j_1}(x_1) M_{i_2 j_2}(x_2) \rangle_{\text{conn}} \quad \sim$  $+$  $+ \cdots .$

In such a way the whole theory in the large N limit is described only by planar diagrams (both vacuum graphs and diagrams with external legs), thus such regime is also called planar limit.

In this section we have outlined the general features of the large N expansion of matrix theories. For vacuum diagrams we have developed a very elegant proof of the dependence of the weight factor on the genus. Then we have extended such argument to diagrams with external legs.

In the next section we will apply these concepts to a single scalar matrix theory in order to clarify them. First of all we will provide, through a redefinition of the field, a particularly useful form of the action, then, focusing on vacuum diagrams, we will perform explicit calculations in order to confirm the prediction (2.1.9) coming from the topological analysis. In the end, considering diagrams with external legs we will develop a very useful formalism to describe correlation functions and the generating functional for both connected and connected amputated diagrams.

2.2. Scalar matrix model

Consider a theory of a scalar $N \times N$ hermitian matrix $M(x)$. Let us take the action:

$$\widehat{S}[M] = \int d^d x \left(\frac{1}{2} \operatorname{tr} (\partial M)^2 - \sum_{p=3}^{+\infty} g_p \operatorname{tr} M^p \right), \tag{2.2.1}$$

where, for simplicity, we consider only single-trace non-derivative interactions.

Through this definition of the action the propagator does not carry any power of N, thus it corresponds to $\alpha = 0$. Hence 't Hooft couplings are defined as:

$$\lambda_p \coloneqq g_p N^{\frac{p}{2}-1}, \tag{2.2.2}$$

so that $\widehat{S}$ can be written as:

$$\widehat{S}[M] = \int d^d x \left(\frac{1}{2} \mathrm{tr}\,(\partial M)^2 - \sum_{p=3}^{+\infty} \lambda_p \, N^{1-\frac{p}{2}} \, \mathrm{tr}\, M^p \right). \tag{2.2.3}$$

The free energy is:

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}M \, e^{-\widehat{S}[M]}, \tag{2.2.4}$$

where $\mathcal{N}$ is a normalization, which can be fixed requiring that $Z|_{\lambda_p=0} = 1$, so:

$$\mathcal{N} = \left(\int \mathcal{D}M \, e^{-\frac{1}{2}\int d^d x \, \mathrm{tr}\,(\partial M)^2} \right)^{-1} = \left(\pi \det \Box^{-1}\right)^{-\frac{N^2}{2}},$$

because we can write the exponent as $\mathrm{tr}\,(M \Box M)$ and perform a gaussian integral.

This being done, we can provide a very useful redefinition of the field, which actually is the standard way of writing a theory which admits a consistent large N expansion:

$$\Phi(x) \coloneqq \frac{M(x)}{\sqrt{N}}, \tag{2.2.5}$$

so the action becomes:

$$\widehat{S}[\Phi] = N \int d^d x \left(\frac{1}{2} \mathrm{tr}\,(\partial \Phi)^2 - \sum_{p=3}^{+\infty} \lambda_p \, \mathrm{tr}\, \Phi^p \right) \coloneqq N S[\Phi], \tag{2.2.6}$$

where the new action S is defined carrying out of $\widehat{S}$ the factor N. Thus the free energy is now defined as:

$$e^{-N^2 \mathcal{F}} = \mathcal{N} \int \mathcal{D}\Phi \, e^{-N S[\Phi]}, \tag{2.2.7}$$

where the normalization $\mathcal{N}$ becomes:

$$\mathcal{N} = \left(\frac{\pi \det \Box^{-1}}{N} \right)^{-\frac{N^2}{2}}.$$

In such a way, because of the term N^2 ahead $\mathcal{F}$, its topological expansion reads:

$$\mathcal{F} = \sum_g \frac{\mathcal{F}^{(g)}}{N^{2g}}, \tag{2.2.8}$$

thus the contribution to the free energy of a Feynman diagram of genus g now carries a weight factor $1/N^{2g}$.

This definition of the theory involves directly 't Hooft parameters λ_p and makes the action $S[\Phi]$ independent of N. In this picture, given an arbitrary diagram, either vacuum or with external legs, every vertex carries a factor N, no matter what the number of external legs it has is, every propagator carries $1/N$ and, as usual, every closed line produces a factor

N, so this corresponds to $\alpha = 1$. For this reason it is very simple to obtain the weight factor counting its propagators, vertices and closed lines.

Moreover, within this definition, the large N expansion is straightforward because every term in S does not change while sending $N \mapsto +\infty$ with λ_p kept fixed. For this reason (2.2.6) is the standard way of defining a theory which admits the large N limit.

2.2.1. Vacuum diagrams

In the first section of this chapter we have developed the topological analysis for Feynman diagrams. At a certain point, considering some vacuum diagrams in Figure 2.1.2, we have estimated the N-dependence of the ratio of diagrams with different topology and shown that it agrees with the general predictions. Here we explicitly compute the single diagrams and confirm the general analysis.

Consider the free energy $\mathcal{F}$ and focus only on the λ_4 vertex, which produces diagrams in Figure 2.1.2:

$$e^{-N^2\mathcal{F}} = \mathcal{N} \int \mathcal{D}\Phi \, e^{-N \int d^dx \left(\frac{1}{2}\mathrm{tr}\,(\partial\Phi)^2 - \lambda_4 \,\mathrm{tr}\,\Phi^4 + \cdots\right)},$$

where dots represent the other interaction terms we are not interested in. Expanding the exponential in series of the couplings we can evaluate perturbative corrections to $\mathcal{F}$:

$$e^{-N^2\mathcal{F}} = \mathcal{N} \int \mathcal{D}\Phi \left(1 + N\lambda_4 \int d^dx \,\mathrm{tr}\,\Phi^4(x) + \cdots + O(\lambda_4^2)\right) e^{-\frac{N}{2}\int \mathrm{tr}\,(\partial\Phi)^2}, \tag{2.2.9}$$

The correlation functions of the free theory are:

$$\langle \mathcal{O}(x) \rangle_0 = \frac{\int \mathcal{D}\Phi \, \mathcal{O}(x) \, e^{-\frac{N}{2}\int \mathrm{tr}\,(\partial\Phi)^2}}{\int \mathcal{D}\Phi \, e^{-\frac{N}{2}\int \mathrm{tr}\,(\partial\Phi)^2}} = \mathcal{N} \int \mathcal{D}\Phi \, \mathcal{O}(x) \, e^{-\frac{N}{2}\int \mathrm{tr}\,(\partial\Phi)^2}, \tag{2.2.10}$$

where $\mathcal{O}$ is an arbitrary operator, thus (2.2.9) can be written as:

$$e^{-N^2\mathcal{F}} = 1 + N\lambda_4 \int d^dx \left\langle \mathrm{tr}\,\Phi^4(x) \right\rangle_0 + \cdots + O(\lambda_4^2). \tag{2.2.11}$$

Now, recalling the method used in subsection 1.1.1, we have that:

$$\langle \Phi_{ij}(x) \, \Phi_{kl}(y) \rangle_0 = \frac{\int \mathcal{D}\Phi \, \Phi_{ij}(x) \, \Phi_{kl}(y) \, e^{-\frac{N}{2}\int \mathrm{tr}\,(\partial\Phi)^2}}{\int \mathcal{D}\Phi \, e^{-\frac{N}{2}\int \mathrm{tr}\,(\partial\Phi)^2}} = \frac{\delta_{il}\,\delta_{jk}}{N} \, f(x,y), \tag{2.2.12}$$

where f is the ordinary Green function for a scalar field, which does not depend on N. In the following we will only concentrate on the N-dependence and drop all space-time dependent factors, including loop integrals.

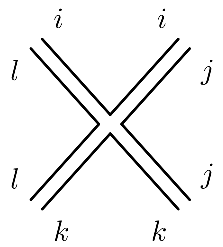
At this point we can compute the first order corrections to $\mathcal{F}$ using Wick's theorem:

$$\begin{aligned}\left\langle \operatorname{tr}\Phi^4 \right\rangle_0 &= \left\langle \Phi_{ij}\Phi_{jk}\Phi_{kl}\Phi_{li} \right\rangle_0 \\ &= \left\langle \Phi_{ij}\Phi_{jk} \right\rangle_0 \left\langle \Phi_{kl}\Phi_{li} \right\rangle_0 + \left\langle \Phi_{ij}\Phi_{kl} \right\rangle_0 \left\langle \Phi_{jk}\Phi_{li} \right\rangle_0 + \left\langle \Phi_{ij}\Phi_{li} \right\rangle_0 \left\langle \Phi_{jk}\Phi_{kl} \right\rangle_0 \\ &= \frac{1}{N^2}\delta_{ik}\,\delta_{jj}\,\delta_{ik}\,\delta_{ll} + \frac{1}{N^2}\delta_{il}\,\delta_{jk}\,\delta_{ji}\,\delta_{kl} + \frac{1}{N^2}\delta_{ii}\,\delta_{jl}\,\delta_{lj}\,\delta_{kk},\end{aligned} \tag{2.2.13}$$

where repeated indices are summed. In such a way, since $\delta_{ii} = N$, the correlator reads:

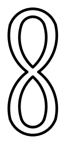
$$N\lambda_4 \left\langle \operatorname{tr}\Phi^4 \right\rangle_0 = \frac{\lambda_4}{N}\Big(\delta_{ik}\delta_{jj}\delta_{ik}\delta_{ll} + \delta_{il}\delta_{jk}\delta_{ji}\delta_{kl} + \delta_{ii}\delta_{jl}\delta_{lj}\delta_{kk}\Big) = \lambda_4(N^2 + 1 + N^2). \tag{2.2.14}$$

Considering the diagrammatic representation, we can associate every term to its corresponding graph. Starting from the vertex, we have:



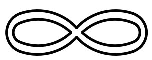
$$= N\lambda_4 \int d^d x\, \operatorname{tr}\Phi^4(x), \tag{2.2.15}$$

hence we can obtain the contribution of every term in (2.2.14) connecting the external lines of the vertex according to the contractions of the indices coming from the deltas. Thus we have:

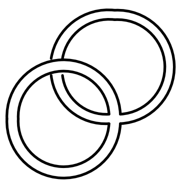


$$\sim \quad N\lambda_4 \left\langle \Phi_{ij}\Phi_{li} \right\rangle_0 \left\langle \Phi_{jk}\Phi_{kl} \right\rangle_0 = \lambda_4 N^2 \tag{2.2.16}$$

and:



$$\sim \quad N\lambda_4 \left\langle \Phi_{ij}\Phi_{jk} \right\rangle_0 \left\langle \Phi_{kl}\Phi_{li} \right\rangle_0 = \lambda_4 N^2 \tag{2.2.17}$$



$$\sim \quad N\lambda_4 \left\langle \Phi_{ij}\Phi_{kl} \right\rangle_0 \left\langle \Phi_{jk}\Phi_{li} \right\rangle_0 = \lambda_4 \tag{2.2.18}$$

which means that the computation of the graphs produces the weight factor $1/N^{2g-2}$ because (2.2.16) and (2.2.17) are planar whereas (2.2.18) has genus 1.

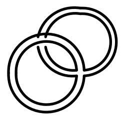
Consider now further orders in the expansion (2.2.11), such as the second order contribution:

$$N^2\lambda_4^2 \int d^d x\, d^d y\, \left\langle \operatorname{tr}\Phi^4(x)\operatorname{tr}\Phi^4(y) \right\rangle_0 .$$

The computation of such terms produces the exponential series of (2.2.14), so that we have:

$$e^{-N^2\mathcal{F}} = \exp\big(N\lambda_4 \int d^dx \left\langle \operatorname{tr}\Phi^4 \right\rangle_0 + \cdots\big). \tag{2.2.19}$$

In such a way, since (2.2.14) is composed by the three diagrams (2.2.16), (2.2.17) and (2.2.18), they directly contribute to $\mathcal{F}$:

$\mathcal{F} = -\frac{\lambda_4}{N^2}\Bigg($  $+$  $+$  $+\cdots\Bigg) \sim \lambda_4\Big(1 + \frac{1}{N^2} + 1 + \cdots\Big).$ (2.2.20)

Hence, thanks to the definition (2.2.7) with the factor N^2 in the exponential, these diagrams actually give contribution to $\mathcal{F}$ with a weight factor $1/N^{2g}$ instead of $1/N^{2g-2}$.

2.2.2. Correlation functions and diagrams with external legs

In the end of the first section of this chapter we have developed a topological analysis for diagrams with external legs, obtaining that the ratio of the contributions of diagrams with the same number of external lines depends only on the genus and the number of boundaries of the associated surface.

In this part we perform explicit calculations within the scalar theory considered above in order to confirm the general result of the topological analysis and, moreover, we develop a very useful formalism to rewrite the generating functional in terms of invariant source operators.

To do so, we study the connected correlation functions. Diagrammatically they are described by diagrams where the external legs are propagators, thus their weight factor can be computed using (2.1.15) and multiplying it by $1/N^r$, where r is the number of external legs. Let us start considering the 2-point function:

$$\langle \Phi_{i_1j_1}(x)\,\Phi_{i_2j_2}(y)\rangle = \frac{\int \mathcal{D}\Phi\,\Phi_{i_1j_1}(x)\,\Phi_{i_2j_2}(y)\,e^{-NS[\Phi]}}{\int \mathcal{D}\Phi\,e^{-NS[\Phi]}}. \tag{2.2.21}$$

In this case, instead of directly computing the connected correlator, we consider the total correlation function defined by (2.2.21), neglecting the disconnected contributions. In such a way we can use the perturbative method expanding in series the exponential, so that:

$$\begin{aligned}\langle \Phi_{i_1j_1}(x)\,\Phi_{i_2j_2}(y)\rangle =& \langle \Phi_{i_1j_1}(x)\,\Phi_{i_2j_2}(y)\rangle_0 \\ &+ N\lambda_4 \int d^dz \left\langle \Phi_{i_1j_1}(x)\,\Phi_{i_2j_2}(y)\operatorname{tr}\Phi^4(z)\right\rangle_0 + \cdots ,\end{aligned} \tag{2.2.22}$$

where, for simplicity, we focus only on the λ_4 vertex.

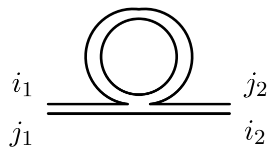
The correlator $\langle \Phi_{i_1j_1}\Phi_{i_2j_2}\operatorname{tr}\Phi^4\rangle_0$ represents the first order correction and can be computed using Wick's theorem. Dropping contributions dependent on space-time degrees of freedom and loop integrals, as we did in (2.2.12), we can focus only on the N-dependence, so that we have:

$$\left\langle \Phi_{i_1 j_1} \Phi_{i_2 j_2} \operatorname{tr} \Phi^4 \right\rangle_0 = \langle \Phi_{i_1 j_1} \Phi_{i_2 j_2} \Phi_{ab} \Phi_{bc} \Phi_{cd} \Phi_{da} \rangle_0$$
$$= \langle \Phi_{i_1 j_1} \Phi_{ab} \rangle_0 \langle \Phi_{i_2 j_2} \Phi_{bc} \rangle_0 \langle \Phi_{cd} \Phi_{da} \rangle_0 + \langle \Phi_{i_1 j_1} \Phi_{ab} \rangle_0 \langle \Phi_{i_2 j_2} \Phi_{cd} \rangle_0 \langle \Phi_{bc} \Phi_{da} \rangle_0 + \cdots , \tag{2.2.23}$$

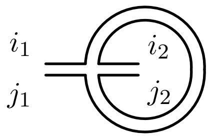
Neglecting contributions coming from disconnected diagrams, the remaining terms can be absorbed in only two contributions:

$$\langle \Phi_{i_1 j_1} \Phi_{ab} \rangle_0 \langle \Phi_{i_2 j_2} \Phi_{bc} \rangle_0 \langle \Phi_{cd} \Phi_{da} \rangle_0 = \frac{1}{N^3} \delta_{i_1 b} \delta_{j_1 a} \delta_{i_2 c} \delta_{j_2 b} \delta_{ca} \delta_{dd} = \frac{\delta_{i_1 j_2} \delta_{i_2 j_1}}{N^2} \tag{2.2.24}$$
$$\langle \Phi_{i_1 j_1} \Phi_{ab} \rangle_0 \langle \Phi_{i_2 j_2} \Phi_{cd} \rangle_0 \langle \Phi_{bc} \Phi_{da} \rangle_0 = \frac{1}{N^3} \delta_{i_1 b} \delta_{j_1 a} \delta_{i_2 d} \delta_{j_2 c} \delta_{ba} \delta_{cd} = \frac{\delta_{i_1 j_1} \delta_{i_2 j_2}}{N^3} . \tag{2.2.25}$$

Considering their diagrammatic representation we have:



$$\sim N\lambda_4 \langle \Phi_{i_1 j_1} \Phi_{ab} \rangle_0 \langle \Phi_{i_2 j_2} \Phi_{bc} \rangle_0 \langle \Phi_{cd} \Phi_{da} \rangle_0 = \lambda_4 \frac{\delta_{i_1 j_2} \delta_{i_2 j_1}}{N} \tag{2.2.26}$$

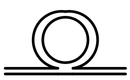
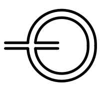


$$\sim N\lambda_4 \langle \Phi_{i_1 j_1} \Phi_{ab} \rangle_0 \langle \Phi_{i_2 j_2} \Phi_{cd} \rangle_0 \langle \Phi_{bc} \Phi_{da} \rangle_0 = \lambda_4 \frac{\delta_{i_1 j_1} \delta_{i_2 j_2}}{N^2} . \tag{2.2.27}$$

Unlike diagrams in Figure 2.1.4, these are non-amputated graphs, thus their contribution takes into account the presence of the external propagators. Anyway, according to the topological analysis, (2.2.27) is subleading with respect to (2.2.26) by a factor $1/N$.

Because of the invariance under cyclic permutations, these terms represent the only two possible inequivalent contractions of the indices of the external lines. Hence, regrouping all equivalent terms, we obtain:

$$N\lambda_4 \int d^d z \left\langle \Phi_{i_1 j_1}(x) \, \Phi_{i_2 j_2}(y) \operatorname{tr} \Phi^4(z) \right\rangle_0 =$$

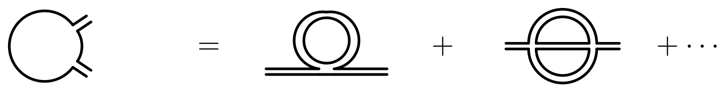
$= 8$  $+ 4$  $\sim 8\lambda_4 \frac{\delta_{i_1 j_2} \delta_{i_2 j_1}}{N} + 4\lambda_4 \frac{\delta_{i_1 j_1} \delta_{i_2 j_2}}{N^2}$, (2.2.28)

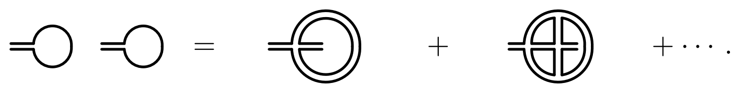
Since (2.2.26) and (2.2.27) represent the only two inequivalent contractions of indices in the 2-point function, all further order terms must be proportional to one of them. Hence we can write the connected 2-point function as:

$$\frac{N^2}{2!} \langle \Phi_{i_1 j_1} \Phi_{i_2 j_2} \rangle_{\text{conn}} = f(0,1) \frac{\delta_{i_1 j_2} \delta_{i_2 j_1}}{N} + f(2,0) \frac{\delta_{i_1 j_1} \delta_{i_2 j_2}}{N^2} , \tag{2.2.29}$$

where $f(0,1)$ is the contribution of all diagrams of all orders corresponding to the contraction $\delta_{i_1 j_2}\delta_{i_2 j_1}$, whereas $f(2,0)$ comes from all diagrams with $\delta_{i_1 j_1}\delta_{i_2 j_2}$. The term $1/2!$ is conventional, whereas the power of N multiplying every term in (2.2.29) is exactly the contribution to the weight factor coming from the boundaries $1/N^b$. Thus, since the contraction of $f(0,1)$ corresponds to graphs with one boundary, it has $1/N$, whereas $f(2,0)$ has $1/N^2$ because it corresponds to diagrams with two boundaries. Moreover, the factor N^2 ahead the 2-point function in the l.h.s. of (2.2.29) is the contribution to the weight factor coming from the external propagators, thus in general it is N^r for the r-point function. Through this definition, being the contribution of the boundaries and the external legs inserted explicitly in the expansion of the correlation function, the amplitudes $f(0,1)$ and $f(2,0)$ contain the contribution to the weight factor coming only from the genus of their diagrams.

The notation $f(\nu_1,\nu_2)$ introduced above develops a useful graphical method to build the diagrams composing it. Indeed the first entry in the argument of f, i.e. ν_1, is the number of internal lines with one external leg attached to them, whereas the second entry, ν_2, is the number of internal lines with two external legs attached to them. Hence $f(\nu_1,\nu_2)$ is the contribution of all diagrams with $\nu_1 + 2\nu_2$ external legs, composed by ν_1 lines with one external leg and ν_2 lines with two external legs. Thus every diagram in $f(\nu_1,\nu_2)$ is associated to a surface with $b = \nu_1 + \nu_2$ boundaries and so such term is multiplied by the factor $1/N^{\nu_1+\nu_2}$. For example in (2.2.29) we have:

$f(0,1) =$  (2.2.30)

$f(2,0) =$  (2.2.31)

This notation is defined so that the contraction structure of every diagram in $f(0,1)$, as well as of each one in $f(2,0)$, is the same. Indeed every graph in (2.2.30) has the same number of internal lines with the same number of external legs, as well as every diagram in (2.2.31).

The generalization of this formalism to the case of the connected r-point function is straightforward. Indeed in such function there are as many different terms as the number of all inequivalent contractions of the external leg indices among themselves. Thus, since the contraction structure depends on how many external legs are attached to a single closed line and how many of these lines are there in the graph, we can generalize $f(0,1)$ and $f(2,0)$ to a function $f(\{\nu_k\})$, where $\{\nu_k\}$ is a sequence of natural numbers such that:

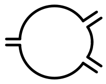
$$\sum_k k\nu_k = r, \tag{2.2.32}$$

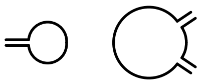
which is the constraint imposing the number of external legs to be equal to r. In such a way ν_k is the number of closed lines with k external legs attached to them, thus $f(\{\nu_k\})$ is the contribution of all connected diagrams with r external legs with contraction structure $\{\nu_k\}$.

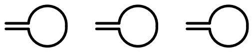
Solving the constraint, we can find the number of inequivalent ways of contracting external indices, namely the number of different values of $f(\{\nu_k\})$. Each one of these values is multiplied by its associated contraction term (the product of deltas) and by the factor $1/N^{b=\sum_k \nu_k}$ corresponding to the number of boundaries of such structure. For example consider the connected 3-point function:

$$\langle \Phi_{i_1 j_1}(x_1)\, \Phi_{i_2 j_2}(x_2)\, \Phi_{i_3 j_3}(x_3) \rangle_{\text{conn}} \,. \tag{2.2.33}$$

In order to expand it in terms of $f(\{\nu_k\})$ we have to solve the constraint (2.2.32) with $r = 3$. There are three solutions: $(\nu_1 = 0, \nu_2 = 0, \nu_3 = 1)$, $(\nu_1 = 1, \nu_2 = 1, \nu_3 = 0)$ and $(\nu_1 = 3, \nu_2 = 0, \nu_3 = 0)$. Thus we have:

$f(0,0,1) =$  $\sim \dfrac{\delta_{i_1 j_2}\delta_{i_2 j_3}\delta_{i_3 j_1}}{N}$ (2.2.34a)

$f(1,1,0) =$  $\sim \dfrac{\delta_{i_1 j_2}\delta_{i_2 j_1}\delta_{i_3 j_3}}{N^2}$ (2.2.34b)

$f(3,0,0) =$  $\sim \dfrac{\delta_{i_1 j_1}\delta_{i_2 j_2}\delta_{i_3 j_3}}{N^3}$ (2.2.34c)

Hence (2.2.34a) is weighted with $1/N$ because its external legs are attached to the same internal line, so $b = 1$, (2.2.34b) has $1/N^2$ because it has $b = 2$ whereas (2.2.34c) is multiplied by $1/N^3$ because it has $b = 3$. Hence, considering only the matrix structure of the theory, the 3-point function can be written as:

$$\begin{aligned} \frac{N^3}{3!} \langle \Phi_{i_1 j_1} \Phi_{i_2 j_2} \Phi_{i_3 j_3} \rangle_{\text{conn}} = f(3,0,0) &\frac{\delta_{i_1 j_1}\delta_{i_2 j_2}\delta_{i_3 j_3}}{N^3} \\ + f(1,1,0)\frac{\delta_{i_1 j_2}\delta_{i_2 j_1}\delta_{i_3 j_3}}{N^2} &+ f(0,0,1)\frac{\delta_{i_1 j_2}\delta_{i_2 j_3}\delta_{i_3 j_1}}{N}. \end{aligned} \tag{2.2.35}$$

This formalism is important because, as already said, it separates the contribution to the weight factor of boundaries and external legs from that of the genus, which is contained in $f(\{\nu_k\})$. Thus, considering the general case of the r-point function, we have:

$$\frac{N^r}{r!} \langle \Phi_{i_1 j_1} \cdots \Phi_{i_r j_r} \rangle_{\text{conn}} = \sum_{\substack{\{\nu_k\} \\ \sum_k k\nu_k = r}} f(\{\nu_k\}) \frac{(\text{contractions of structure } \{\nu_k\})}{N^{\sum_k \nu_k}}. \tag{2.2.36}$$

Now $f(\{\nu_k\})$ contains the contribution of every diagram of structure $\{\nu_k\}$, therefore it is associated to graphs of every genus. The contribution of boundaries to the weight factor of such diagrams being represented by the factor $1/N^{\sum_k \nu_k}$ outside $f(\{\nu_k\})$, we can expand the latter through the genus of its diagrams. Thus we have the expansion:

$$f(\{\nu_k\}) = \sum_g \frac{f^{(g)}(\{\nu_k\})}{N^{2g-2}}, \tag{2.2.37}$$

where $f^{(g)}(\{\nu_k\})$ is the contribution of diagrams with structure $\{\nu_k\}$ and genus g. In such a way, reconstructing the total power of N of every diagram of genus g and structure $\{\nu_k\}$ (i.e. with $b = \sum_k \nu_k$ boundaries) in the r-point function, we have $1/N^{2g-2+b+r}$, which is exactly the result one expects from the topological analysis (we are working with $\alpha = 1$ and r external propagators, thus each one carries a factor $1/N$). From this we can explicitly see that only planar diagrams give a finite contribution to the correlation functions in the large N limit.

This formalism is then very useful to determine the behavior of the generating functional in the planar limit. Thus, consider the generating functional of connected diagrams of the scalar theory:

$$e^{-N^2\mathcal{F}[J]} = \mathcal{N}\int \mathcal{D}\Phi\, e^{-N\left(S[\Phi]-\int d^dx\,\mathrm{tr}\,J(x)\Phi(x)\right)}, \tag{2.2.38}$$

which is defined from the free energy (2.2.7) introducing the external source $J(x)$, which is a $N \times N$ hermitian matrix like Φ. The connected correlation functions can be computed by the relation:

$$\langle \Phi_{i_1 j_1}(x_1) \cdots \Phi_{i_r j_r}(x_r)\rangle_{\text{conn}} = -N^{2-r}\left.\frac{\delta^r \mathcal{F}[J]}{\delta J_{i_1 j_1}(x_1)\cdots \delta J_{i_r j_r}(x_r)}\right|_{J=0}, \tag{2.2.39}$$

where $\delta/\delta J_{ij}$ is the functional derivative with respect to the matrix field J. Now, expanding $\mathcal{F}[J]$ in a formal Taylor series, we have:

$$\mathcal{F}[J] = \mathcal{F}[0] + \sum_{r=1}^{+\infty}\frac{1}{r!}\int d^dx_1\cdots d^dx_r \left.\frac{\delta^r \mathcal{F}[J]}{\delta J_{i_1 j_1}(x_1)\cdots \delta J_{i_r j_r}(x_r)}\right|_{J=0} \times J_{i_1 j_1}(x_1)\cdots J_{i_r j_r}(x_r), \tag{2.2.40}$$

where repeated indices are summed. $\mathcal{F}[0]$ represents the free energy $\mathcal{F}$ defined by (2.2.7), which, as already studied, contains the contribution of all vacuum diagrams of the theory. Since we have already described the expansion of $\mathcal{F}$ through the genus (2.2.8), here we focus on the series of functional derivatives, which describes the contribution of the diagrams with external legs.

Using (2.2.39) we can write:

$$\mathcal{F}[J] = \mathcal{F} - \sum_{r=1}^{+\infty}\frac{N^{r-2}}{r!}\int d^dx_1\cdots d^dx_r\,\langle \Phi_{i_1 j_1}(x_1)\cdots \Phi_{i_r j_r}(x_r)\rangle_{\text{conn}} \times J_{i_1 j_1}(x_1)\cdots J_{i_r j_r}(x_r), \tag{2.2.41}$$

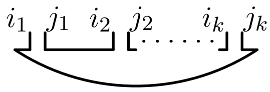
where, dropping again the space-time degrees of freedom (dof's), we can replace the correlator $\langle \Phi_{i_1 j_1} \cdots \Phi_{i_r j_r} \rangle_{\text{conn}}$ with its form in terms of $f(\{\nu_k\})$, so that the generating functional reads:

$$\begin{aligned}\mathcal{F}[J] &= \mathcal{F} + \sum_{r=1}^{+\infty} \frac{1}{N^2} \sum_{\substack{\{\nu_k\} \\ \sum_k k\nu_k = r}} f(\{\nu_k\}) \frac{(\text{contractions of structure } \{\nu_k\})}{N^{\sum_k \nu_k}} J_{i_1 j_1} \cdots J_{i_r j_r} \\ &= \mathcal{F} + \frac{1}{N^2} \sum_{\{\nu_k\}} f(\{\nu_k\}) \frac{(\text{contractions of structure } \{\nu_k\})}{N^{\sum_k \nu_k}} J_{i_1 j_1} \cdots J_{i_r j_r}. \end{aligned} \tag{2.2.42}$$

For simplicity we can absorb the factor $1/N^2$ into the definition of $f(\{\nu_k\})$, so that its expansion becomes:

$$f(\{\nu_k\}) = \sum_g \frac{f^{(g)}(\{\nu_k\})}{N^{2g}}. \tag{2.2.43}$$

At this point, the product of the contraction terms with the source tensor $J_{i_1 j_1} \cdots J_{i_r j_r}$ produces a scalar term depending on J. Its exact form can be understood considering that in a diagram of structure $\{\nu_k\}$ there are ν_k lines with k external legs producing the saturation of the indices attached to them, such as:



$$\sim \quad \delta_{i_1 j_k} \delta_{i_2 j_1} \cdots \delta_{i_k j_k - 1}. \tag{2.2.44}$$

Hence the product of this term with the source tensor gives the trace operator $\text{tr}\, J^k$. Therefore the whole contraction term of structure $\{\nu_k\}$ produces the operator: $\prod_k \left(\text{tr}\, J^k\right)^{\nu_k}$. Moreover, since every contraction term of such structure is weighted with a factor $1/N^{\sum_k \nu_k}$, we have:

$$\frac{1}{N^{\sum_k \nu_k}} \prod_k (\text{tr}\, J^k)^{\nu_k} = \prod_k \left(\frac{\text{tr}\, J^k}{N} \right)^{\nu_k}.$$

Therefore the whole generating functional can be written as:

$$\mathcal{F}[J] = \mathcal{F} + \sum_{\{\nu_k\}} f(\{\nu_k\}) \prod_k \left(\frac{\text{tr}\, J^k}{N} \right)^{\nu_k}. \tag{2.2.45}$$

Reintroducing the space-time dof's we can express it in momentum space:

$$\begin{aligned}\mathcal{F}[J] = \mathcal{F} + \sum_{\{\nu_k\}} \int \prod_{k, i_k, j_k} d^d p_{k,j_k}^{(i_k)}\, \delta^d\Big(\sum_{k, i_k, j_k} p_{k,j_k}^{(i_k)} \Big) f(\{\nu_k\}; p_{k,j_k}^{(i_k)}) \\ \times \prod_k \prod_{i_k=1}^{\nu_k} \frac{\text{tr}\left(J(p_{k,1}^{(i_k)}) \cdots J(p_{k,k}^{(i_k)}) \right)}{N}, \end{aligned} \tag{2.2.46}$$

where $p_{k,j_k}^{(i_k)}$ is the momentum of the j_k-th external leg ($j_k = 1, \cdots, k$) attached to the i_k-th line with k external legs ($i_k = 1, \cdots, \nu_k$) and to the i_k-th boundary, whereas $f(\{\nu_k\}; p_{k,j_k}^{(i_k)})$ is the amplitude of diagrams with structure $\{\nu_k\}$, which depends on their external momenta.

The operators $\mathrm{tr}\, J^k$ are trace operators, hence they are not invariant as N varies. Indeed, since the trace of a $N \times N$ matrix is the sum of the N diagonal entries, we have that $\mathrm{tr}\,(N \times N \text{ matrix}) \sim N$. For this reason it is worth introducing invariant operators defined by:

$$j_k(p_1, \cdots, p_k) := \frac{\mathrm{tr}\,(J(p_1) \cdots J(p_k))}{N}. \tag{2.2.47}$$

which is of order $O(N^0 = 1)$. They are called source operators so that the structure of the generating functional is:

$$\mathcal{F}[J] = \mathcal{F} + \sum_{\{\nu_k\}} f(\{\nu_k\}) \prod_k j_k^{\nu_k}. \tag{2.2.48}$$

Hence (2.2.46) becomes:

$$\mathcal{F}[J] = \mathcal{F} + \sum_{\{\nu_k\}} \int \prod_{k,i_k,j_k} d^d p_{k,j_k}^{(i_k)} \, \delta^d\Big(\sum_{k,i_k,j_k} p_{k,j_k}^{(i_k)}\Big) f(\{\nu_k\}; p_{k,j_k}^{(i_k)}) \times \prod_k \prod_{i_k=1}^{\nu_k} j_k(p_{k,1}^{(i_k)}, \cdots, p_{k,k}^{(i_k)}), \tag{2.2.49}$$

This is the structure of the generating functional in terms of the amplitudes $f(\{\nu_k\}; p_{k,j_k}^{(i_k)})$ and of the invariant source operators. Hence, since these amplitudes can be expanded through the genus as in (2.2.43) and recalling that the free energy has a similar expansion:

$$\mathcal{F} = \sum_g \frac{\mathcal{F}^{(g)}}{N^{2g}}, \tag{2.2.50}$$

we can see that the whole generating functional can be written as:

$$\mathcal{F}[J] = \sum_g \frac{\mathcal{F}^{(g)}[J]}{N^{2g}}. \tag{2.2.51}$$

where $\mathcal{F}^{(g)}[J]$ is the contribution coming from all diagrams of genus g, thus it reads:

$$\mathcal{F}^{(g)}[J] = \mathcal{F}^{(g)} + \sum_{\{\nu_k\}} \int \prod_{k,i_k,j_k} d^d p_{k,j_k}^{(i_k)} \, \delta^d\Big(\sum_{k,i_k,j_k} p_{k,j_k}^{(i_k)}\Big) f^{(g)}(\{\nu_k\}; p_{k,j_k}^{(i_k)}) \times \prod_k \prod_{i_k=1}^{\nu_k} j_k(p_{k,1}^{(i_k)}, \cdots, p_{k,k}^{(i_k)}). \tag{2.2.52}$$

This proves that the generating functional receives contribution from all diagrams with sphere topology in the large N limit. So it contains also non-planar contributions, because in its expansion the only important topological property is the genus. This is due to the fact that the operators j_k, which contain the contribution of boundaries to the weight factor, are invariant in the large N expansion.

Anyway, when taking the derivatives with respect to J_{ij} of $\mathcal{F}[J]$ in order to compute correlation functions, the powers of N contained in the source operators explicitly appear again, so that all the non-planar amplitudes are suppressed with respect to the planar ones.

The application of this formalism to the generator of connected and amputated diagrams is straightforward. This will be important in the next section for the computation of the Wilsonian action of the theory and of its evolution along the RG flow.

2.3. Wilsonian action and RG flow

In this section we apply WRG to the scalar matrix theory considered in the previous section. In particular we study the RG equation in the large N expansion, noting that a closed RG flow still exists in this limit, as shown by Becchi, Giusto and Imbimbo in [16].

Let us consider the action defined in (2.2.6), which is now treated as the bare action. Hence we have to introduce the UV cut-off into the propagator and we do so introducing the scale $t_0 := \Lambda_0^{-2}$, so that the kinetic term becomes:

$$\frac{1}{2}\int d^dx\,\mathrm{tr}\;(\partial\Phi)|^2_{t_0} = -\frac{1}{2}\int d^dx\,\mathrm{tr}\,(\Phi\,\Box|_{t_0}\,\Phi) = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}P^{-1}_{t_0}(k)\,\mathrm{tr}\,\Phi_k\Phi_{-k}, \tag{2.3.1}$$

where $\Box|_{t_0}$ is the d'Alembert operator with a cut-off t_0, explicitly defined in momentum space by:

$$P_{t_0}(k) := \frac{K(t_0k^2)}{k^2}, \tag{2.3.2}$$

which is the propagator with cut-off t_0, where the function K is the same defined for (1.1.3).

Through this definition the bare action becomes:

$$S_0[\Phi] = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}P^{-1}_{t_0}(k)\,\mathrm{tr}\,\Phi_k\Phi_{-k} - S^{(0)}_{\mathrm{int}}[\Phi], \tag{2.3.3}$$

where:

$$S^{(0)}_{\mathrm{int}}[\Phi] = \int d^dx\sum_p \lambda_p\,\mathrm{tr}\,\Phi^p(x), \tag{2.3.4}$$

is the bare interaction term, so that the whole action is single-trace[5].

Proceeding along the RG flow, S_0 is replaced by an effective action S_t defined at the scale $t := \Lambda^{-2} \leq t_0$, whose interaction term is the Wilsonian action:

$$S_t[\Phi] = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}P^{-1}_{t}(k)\,\mathrm{tr}\,\Phi_k\Phi_{-k} - N\mathcal{H}_t[\Phi], \tag{2.3.5}$$

where P_t is the propagator with cut-off t and $\mathcal{H}_t$ is the Wilsonian action. Since $\mathcal{H}_t$ is also the generator of connected amputated diagrams of the bare theory with double cut-off, we

[5] An operator is called single-trace if it is proportional to a trace of some product of fields. Hence a single-trace coupling is the parameter associated to a single-trace operator and a functional is called single-trace if it is made of single-trace operators.

can consider the functional:

$$
\begin{aligned}
e^{-N^2\mathcal{F}_{t,t_0}[J]} =& \mathcal{N}\int\mathcal{D}\Phi\, e^{-N\left(\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}P_t^{t_0\,-1}\operatorname{tr}\Phi_k\Phi_{-k}-S^{(0)}_{\text{int}}[\Phi]-\int\frac{d^dk}{(2\pi)^d}\operatorname{tr}J_k\Phi_{-k}\right)} \\
:=& \mathcal{N}\int\mathcal{D}\Phi\, e^{-S_t^{t_0}[\Phi,J]},
\end{aligned}
\tag{2.3.6}
$$

which is the generator of connected diagrams with double cut-off, where J is the source and $P_t^{t_0}$ is the propagator with double cut-off, defined in the same way as (1.2.16). Moreover the normalization term $\mathcal{N}$ is:

$$
\mathcal{N}=\left(\frac{\pi\det P_t^{t_0}}{N}\right)^{-\frac{N^2}{2}}.
$$

Defining $\varphi_k := P_t^{t_0}(k)J_k$ we can recall the connection between Wilsonian action and generating functional (1.2.24), so that:

$$
\mathcal{H}_t[\varphi]=-\mathcal{F}_{t,t_0}[P_t^{t_0\,-1}\varphi]-\frac{1}{2N}\int\frac{d^dk}{(2\pi)^d}P_t^{t_0\,-1}(k)\operatorname{tr}\varphi_k\varphi_{-k}. \tag{2.3.7}
$$

Taking the derivative with respect to t of this equation we have:

$$
\partial_t\mathcal{H}_t[\varphi]=-\partial_t\mathcal{F}_{t,t_0}[P_t^{t_0\,-1}\varphi]+\frac{1}{2N}\int\frac{d^dk}{(2\pi)^d}\frac{\partial_tP_t^{t_0}(k)}{P_t^{t_0}(k)}\operatorname{tr}\varphi_k\varphi_{-k}. \tag{2.3.8}
$$

Computing the derivatives of $\mathcal{F}_{t,t_0}$ and writing them in terms of $\mathcal{H}_t$ using (2.3.7), one can obtain the RG equation for the Wilsonian action. Therefore we can start doing so considering:

$$
\langle\mathcal{O}(k)\rangle_\varphi := \frac{\int\mathcal{D}\Phi\,\mathcal{O}(k)\,e^{-S_t^{t_0}[\Phi,\varphi]}}{\int\mathcal{D}\Phi\,e^{-S_t^{t_0}[\Phi,\varphi]}}, \tag{2.3.9}
$$

which is a sort of correlator where the source φ is not switched off. In such a way we have:

$$
\begin{aligned}
\partial_t\mathcal{F}_{t,t_0} =& -\frac{1}{N^2}\frac{\partial_t\mathcal{N}}{\mathcal{N}}-\frac{1}{2N}\int\frac{d^dk}{(2\pi)^d}\frac{\partial_tP_t^{t_0}}{P_t^{t_0\,2}}\langle\operatorname{tr}\Phi_k\Phi_{-k}\rangle_\varphi \\
&+\frac{1}{N}\int\frac{d^dk}{(2\pi)^d}\frac{\partial_tP_t^{t_0}}{P_t^{t_0\,2}}\langle\operatorname{tr}\varphi_k\Phi_{-k}\rangle_\varphi .
\end{aligned}
\tag{2.3.10}
$$

Now we have to evaluate every single piece of the second member of the last equation. Let us start from the first one:

$$
\frac{\partial_t\mathcal{N}}{\mathcal{N}}=\mathcal{N}^{-1}\partial_t\left(\frac{\pi\det P_t^{t_0}}{N}\right)^{-\frac{N^2}{2}}=-\frac{N^2}{2}\frac{\partial_t\det P_t^{t_0}}{\det P_t^{t_0}}=-\frac{N^2}{2}\partial_t\log\left(\det P_t^{t_0}\right), \tag{2.3.11}
$$

thus $\det P_t^{t_0}$ is a functional determinant whose derivative can be computed using a property which connects it to the functional trace:

$$
\det P_t^{t_0}=\exp\left(\operatorname{Tr}\log P_t^{t_0}\right),
$$

where the functional trace is represented by the symbol Tr to distinguish it from the ordinary matrix trace tr. In such a way:

$$\partial_t \log\left(\det P_t^{t_0}\right) = \partial_t \mathrm{Tr}\left(\log P_t^{t_0}\right) = \int \frac{d^d k}{(2\pi)^d} \frac{\partial_t P_t^{t_0}(k)}{P_t^{t_0}(k)}, \tag{2.3.12}$$

thus:

$$-\frac{1}{N^2}\frac{\partial_t \mathcal{N}}{\mathcal{N}} = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \frac{\partial_t P_t^{t_0}(k)}{P_t^{t_0}(k)}. \tag{2.3.13}$$

Considering the other two terms in (2.3.8) and taking the derivatives with respect to φ_k of $e^{-N^2 \mathcal{F}_{t,t_0}}$, we can show that:

$$\begin{aligned}
\langle \mathrm{tr}\, \Phi_k \Phi_{-k} \rangle_\varphi &= -{P_t^{t_0}}^2(k) \mathrm{tr}\, \frac{\delta^2 \mathcal{F}_{t,t_0}}{\delta\varphi_k\, \delta\varphi_{-k}} + N^2 {P_t^{t_0}}^2(k) \mathrm{tr}\, \frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_k} \frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_{-k}} \\
\langle \mathrm{tr}\, \varphi_k \Phi_{-k} \rangle_\varphi &= -N P_t^{t_0}(k) \mathrm{tr}\, \varphi_k \frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_k}.
\end{aligned}$$

At this point, inserting these results in (2.3.8), we have:

$$\begin{aligned}
\partial_t \mathcal{H}_t = &-\frac{1}{2}\int \frac{\partial_t P_t^{t_0}}{P_t^{t_0}} - \frac{1}{2N}\int \partial_t P_t^{t_0} \mathrm{tr}\, \frac{\delta^2 \mathcal{F}_{t,t_0}}{\delta\varphi_k\, \delta\varphi_{-k}} \\
&+ \frac{N}{2}\int \partial_t P_t^{t_0} \mathrm{tr}\, \frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_k} \frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_{-k}} + \int \frac{\partial_t P_t^{t_0}}{P_t^{t_0}} \mathrm{tr}\, \varphi_k \frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_k} + \frac{1}{2N}\int \frac{\partial_t P_t^{t_0}}{P_t^{t_0}} \mathrm{tr}\, \varphi_k \varphi_{-k}.
\end{aligned} \tag{2.3.14}$$

Using (2.3.7) we can rewrite the derivatives of $\mathcal{F}_{t,t_0}$ in terms of those of $\mathcal{H}_t$:

$$\frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_k} = -\frac{\delta \mathcal{H}_t}{\delta\varphi_k} - \frac{1}{N}{P_t^{t_0}}^{-1}(k)\varphi_{-k}, \tag{2.3.15}$$

thus:

$$\frac{\delta^2 \mathcal{F}_{t,t_0}}{\delta\varphi_k\, \delta\varphi_{-k}} = -\frac{\delta^2 \mathcal{H}_t}{\delta\varphi_k\, \delta\varphi_{-k}} - \frac{1}{N}\frac{\delta {P_t^{t_0}}^{-1}(k)\varphi_{-k}}{\delta\varphi_{-k}} \tag{2.3.16}$$

$$\frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_k}\frac{\delta \mathcal{F}_{t,t_0}}{\delta\varphi_{-k}} = \frac{\delta \mathcal{H}_t}{\delta\varphi_k}\frac{\delta \mathcal{H}_t}{\delta\varphi_{-k}} + \frac{1}{N}{P_t^{t_0}}^{-1}\left(\frac{\delta \mathcal{H}_t}{\delta\varphi_k}\varphi_k + \varphi_{-k}\frac{\delta \mathcal{H}_t}{\delta\varphi_{-k}}\right) + \frac{1}{N^2}{P_t^{t_0}}^{-2}\varphi_k\varphi_{-k}, \tag{2.3.17}$$

so (2.3.14) becomes:

$$\begin{aligned}
\partial_t \mathcal{H}_t = &-\frac{1}{2}\int \frac{\partial_t P_t^{t_0}}{P_t^{t_0}} + \frac{1}{2N}\int \partial_t P_t^{t_0} \mathrm{tr}\, \frac{\delta^2 \mathcal{H}_t}{\delta\varphi_k\, \delta\varphi_{-k}} \\
&+ \frac{1}{2N^2}\int \partial_t P_t^{t_0} \mathrm{tr}\, \frac{\delta {P_t^{t_0}}^{-1}(k)\varphi_{-k}}{\delta\varphi_{-k}} + \frac{N}{2}\int \partial_t P_t^{t_0} \mathrm{tr}\, \frac{\delta \mathcal{H}_t}{\delta\varphi_k}\frac{\delta \mathcal{H}_t}{\delta\varphi_{-k}} + \int \frac{\partial_t P_t^{t_0}}{P_t^{t_0}} \mathrm{tr}\, \varphi_k \frac{\delta \mathcal{H}_t}{\delta\varphi_k} \\
&+ \frac{1}{2N}\int \frac{\partial_t P_t^{t_0}}{{P_t^{t_0}}^2} \mathrm{tr}\, \varphi_k\varphi_{-k} - \int \frac{\partial_t P_t^{t_0}}{P_t^{t_0}} \mathrm{tr}\, \varphi_k \frac{\delta \mathcal{H}_t}{\delta\varphi_k} - \frac{1}{2N}\int \frac{\partial_t P_t^{t_0}}{{P_t^{t_0}}^2} \mathrm{tr}\, \varphi_k\varphi_{-k}.
\end{aligned}$$

In such a way the terms proportional to $\mathrm{tr}\, \varphi^2$ and $\mathrm{tr}\, \varphi \frac{\delta \mathcal{H}_t}{\delta\varphi}$ cancels out, whereas the field independent terms proportional to $\int \partial_t P_t^{t_0}/P_t^{t_0}$ can be reabsorbed and neglected as we did in (1.2.10). Hence we have:

$$\partial_t \mathcal{H}_t = \int \frac{d^d k}{(2\pi)^d} \partial_t P_t^{t_0}(k) \Big[\frac{1}{2N} \mathrm{tr}\, \frac{\delta^2 \mathcal{H}_t}{\delta\varphi_k\, \delta\varphi_{-k}} + \frac{N}{2} \mathrm{tr}\, \frac{\delta \mathcal{H}_t}{\delta\varphi_k} \frac{\delta \mathcal{H}_t}{\delta\varphi_{-k}} \Big], \tag{2.3.18}$$

which is the WPE for the scalar matrix theory with UV cut-off t_0, indeed it has the same structure of the equation (1.2.14) we have found in chapter 1.

Since we are interested in the large N limit of the theory, we realize that the term proportional to the second derivative of $\mathcal{H}_t$ in (2.3.18) is subleading with respect to the other one. Thus one may expect that in such limit only the term proportional to $(\frac{\delta\mathcal{H}_t}{\delta\varphi})^2$ survives. This is not true because the second derivative term actually produces some contribution of the same order as the leading term. This is due to the fact that the equation is written in terms of the matrices φ, which are not invariant when N increases. Thus, in order to analyze what happens to (2.3.18) in the large N expansion, we have to use invariant operators defined as:

$$Y_m(p_1, \cdots, p_m) \coloneqq \frac{\mathrm{tr}\,(\varphi_{p_1} \cdots \varphi_{p_m})}{N}. \tag{2.3.19}$$

Hence we can rewrite the functional derivatives of $\mathcal{H}_t$ using the chain rule:

$$\frac{\delta \mathcal{H}_t}{\delta \varphi_k} = \sum_m \int \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_m}{(2\pi)^d} \frac{\delta \mathcal{H}_t}{\delta Y_m(p_1, \cdots, p_m)} \frac{\delta Y_m(p_1, \cdots, p_m)}{\delta \varphi_k}. \tag{2.3.20}$$

In such a way the matrix structure is in the factor $\frac{\delta Y_m}{\delta\varphi}$, which reads:

$$\begin{aligned} \frac{\delta Y_m(p_1, \cdots, p_m)}{\delta \varphi_k} &= \frac{1}{N} \frac{\delta \mathrm{tr}\,(\varphi_{p_1} \cdots \varphi_{p_m})}{\delta \varphi_k} \\ &= \frac{1}{N} \sum_{i=1}^m (2\pi)^d \delta^d(k - p_i) \varphi_{p_{i+1}} \cdots \varphi_{p_m} \varphi_{p_1} \cdots \varphi_{p_{i-1}}. \end{aligned} \tag{2.3.21}$$

Thus, redefining integration momenta in a suitable way, we can perform the sum over i so that we have:

$$\frac{\delta \mathcal{H}_t}{\delta \varphi_k} = \sum_m m \int \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_{m-1}}{(2\pi)^d} \frac{\varphi_{p_1} \cdots \varphi_{p_{m-1}}}{N} \frac{\delta \mathcal{H}_t}{\delta Y_m(k, p_1, \cdots, p_{m-1})}. \tag{2.3.22}$$

Using this relation we can rewrite the first and the second terms in (2.3.18):

$$\begin{aligned} \mathrm{tr}\, \frac{\delta \mathcal{H}_t}{\delta \varphi_k} \frac{\delta \mathcal{H}_t}{\delta \varphi_{-k}} = \sum_{m,n} mn \int \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_{m-1}}{(2\pi)^d} \frac{d^d q_1}{(2\pi)^d} \cdots \frac{d^d q_{n-1}}{(2\pi)^d} \\ \times \frac{\mathrm{tr}\,(\varphi_{p_1} \cdots \varphi_{p_{m-1}} \varphi_{q_1} \cdots \varphi_{q_{n-1}})}{N^2} \frac{\delta \mathcal{H}_t}{\delta Y_m(k, p_\alpha)} \frac{\delta \mathcal{H}_t}{\delta Y_n(-k, q_\beta)}, \end{aligned}$$

and:

$$\operatorname{tr}\frac{\delta^2\mathcal{H}_t}{\delta\varphi_k\,\delta\varphi_{-k}} = \sum_{m,n} mn\int \frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_{m-1}}{(2\pi)^d}\frac{d^dq_1}{(2\pi)^d}\cdots\frac{d^dq_{n-1}}{(2\pi)^d} \times \frac{\operatorname{tr}\left(\varphi_{p_1}\cdots\varphi_{p_{m-1}}\varphi_{q_1}\cdots\varphi_{q_{n-1}}\right)}{N^2}\frac{\delta^2\mathcal{H}_t}{\delta Y_m(k,p_\alpha)\,\delta Y_n(-k,q_\beta)} + \sum_m m\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_{m-1}}{(2\pi)^d}\frac{1}{N}\operatorname{tr}\left(\frac{\delta\varphi_{p_1}\cdots\varphi_{p_{m-1}}}{\delta\varphi_{-k}}\right)\frac{\delta\mathcal{H}_t}{\delta Y_m(k,p_\alpha)},$$

where, in the argument of Y_m in the derivatives of $\mathcal{H}_t$, p_α and q_β represent respectively the dependence on $p_1,\cdots,p_{m-1}$ and $q_1,\cdots,q_{n-1}$. Moreover:

$$\frac{\delta\varphi_{p_1}\cdots\varphi_{p_{m-1}}}{\delta\varphi_{-k}} = \sum_{i=1}^{m-1}(2\pi)^d\delta^d(k+p_i)\operatorname{tr}\left(\varphi_{p_1}\cdots\varphi_{p_{i-1}}\right)\varphi_{p_{i+1}}\cdots\varphi_{p_{m-1}},$$

which allows us to replace the traces with the operators Y_m:

$$\frac{N}{2}\operatorname{tr}\frac{\delta\mathcal{H}_t}{\delta\varphi_k}\frac{\delta\mathcal{H}_t}{\delta\varphi_{-k}} = \frac{1}{2}\sum_{m,n}mn\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_{m-1}}{(2\pi)^d}\frac{d^dq_1}{(2\pi)^d}\cdots\frac{d^dq_{n-1}}{(2\pi)^d} \times Y_{m+n-2}(p_\alpha,q_\beta)\frac{\delta\mathcal{H}_t}{\delta Y_m(k,p_\alpha)}\frac{\delta\mathcal{H}_t}{\delta Y_n(-k,q_\beta)}, \tag{2.3.23}$$

and:

$$\frac{1}{2N}\operatorname{tr}\frac{\delta^2\mathcal{H}_t}{\delta\varphi_k\,\delta\varphi_{-k}} = \frac{1}{2N^2}\sum_{m,n}mn\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_{m-1}}{(2\pi)^d}\frac{d^dq_1}{(2\pi)^d}\cdots\frac{d^dq_{n-1}}{(2\pi)^d} \times Y_{m+n-2}(p_\alpha,q_\beta)\frac{\delta^2\mathcal{H}_t}{\delta Y_m(k,p_\alpha)\,\delta Y_n(-k,q_\beta)} + \frac{1}{2}\sum_m\sum_{i=1}^{m-1}m\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_{m-1}}{(2\pi)^d} \times (2\pi)^d\delta^d(k+p_i)Y_{i-1}(p_1,\cdots,p_{i-1})Y_{m-i-1}(p_{i+1},\cdots,p_{m-1})\frac{\delta\mathcal{H}_t}{\delta Y_m(k,p_\alpha)}. \tag{2.3.24}$$

This form makes clear that the term proportional to $\frac{\delta^2\mathcal{H}_t}{\delta Y_m\,\delta Y_n}$ is suppressed with respect to (2.3.23), whereas the last term in (2.3.24) is of the same order. For this reason, in the large N expansion, the RG equation (2.3.18) becomes:

$$\partial_t\mathcal{H}_t = \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\partial_t P_t^{t_0}(k)\Bigg[\sum_{m,n}mn\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_{m-1}}{(2\pi)^d}\frac{d^dq_1}{(2\pi)^d}\cdots\frac{d^dq_{n-1}}{(2\pi)^d} \times Y_{m+n-2}(p_\alpha,q_\beta)\frac{\delta\mathcal{H}_t}{\delta Y_m(k,p_\alpha)}\frac{\delta\mathcal{H}_t}{\delta Y_n(-k,q_\beta)} + \sum_m\sum_{i=1}^{m-1}m\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_{m-1}}{(2\pi)^d} \times (2\pi)^d\delta^d(k+p_i)Y_{i-1}(p_1,\cdots,p_{i-1})Y_{m-i-1}(p_{i+1},\cdots,p_{m-1})\frac{\delta\mathcal{H}_t}{\delta Y_m(k,p_\alpha)}\Bigg]. \tag{2.3.25}$$

This is the RG equation of $\mathcal{H}_t$ in the planar limit. It is a Hamilton-Jacobi type equation which describes a system with a t-dependent "hamiltonian" of the sources Y_m.

This equation produces a closed RG flow in the large N limit which differs from the ordinary one. Indeed, though the first term in (2.3.25) is exactly the classical term proportional to $\left(\frac{\delta\mathcal{H}_t}{\delta\varphi}\right)^2$ already present in the ordinary equation (2.3.18), the second term, proportional to $\frac{\delta\mathcal{H}_t}{\delta Y_m}$, is a quantum loop term completely different from that, proportional to $\frac{\delta^2\mathcal{H}_t}{\delta\varphi^2}$, in the ordinary WPE. The presence of the latter term is the crucial point, because it leads to the creation of multi-trace terms[6] in $\mathcal{H}_t$ associated to non-planar amplitudes with sphere topology.

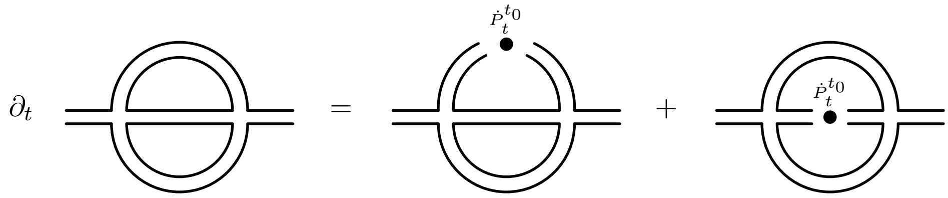


Figure 2.3.1. *The planar RG equation applied to the contribution to $\mathcal{H}_t$ coming from a planar diagram*

Considering a diagrammatic argument, we can show that (2.3.25) actually contains contributions coming from non-planar diagrams. Since $\mathcal{H}_t$ is the generator of connected amputated diagrams with double cut-off, we can describe its evolution considering the action of the derivative with respect to t on graphs composing it. The effect of the application of ∂_t on these graphs is that one internal propagator $P_t^{t_0}$ is cut forming two new external legs, later connected by its derivative $\partial_t P_t^{t_0} = \dot{P}_t^{t_0}$. Therefore, according to the positions of the cut propagator inside a diagram, there are different possible situations. Let us show this fact through the example of the planar diagram in Figure 2.3.1. Here the derivative ∂_t can cut one of the three internal propagators, producing two different diagrams. The first graph in the r.h.s. of such figure represents the situation where ∂_t cuts the upper or, equivalently, the lower propagator. In such a way all the external legs, both the new ones and the original ones, are attached to the same line, thus the diagram is planar. The second graph in the r.h.s. of Figure 2.3.1 represents the situation where ∂_t cuts the middle propagator, therefore the new external legs are attached to a line which is different from that of the original one and so the diagram is non-planar. This shows that the evolution of $\mathcal{H}_t$ in the planar limit is influenced also by non-planar diagrams. Moreover, since every line inside a graph produces the trace of as many sources as external legs attached to it, these non-planar amplitudes are proportional to multi-trace terms and the presence of them in the planar RG equation means that multi-trace interactions are produced in $\mathcal{H}_t$.

This fact is possible only because we have written the theory in a very suitable way, using invariant composite operators Y_m. By this definition the weight $1/N$ of every internal line

[6] A multi-trace term is an operator proportional to a product of different traces of products of fields. Thus a term like $\mathcal{O} \sim \mathrm{tr}\,\phi^p \mathrm{tr}\,\phi^q$ is called double-trace operator because it is the product of two traces. The couplings associated to such terms are called multi-trace couplings.

containing external legs, i.e. of every trace, is absorbed into Y_m, so that the contribution of every diagram depends only on the genus. For this reason both planar and non-planar diagrams with the same genus produce contributions of the same order to (2.3.25). Therefore the evolution of $\mathcal{H}_t$ along the RG flow in the large N limit is influenced by all diagrams with sphere topology, both planar and non-planar. In particular, non-planar amplitudes are responsible for the creation of multi-trace interactions, even though starting only from single-trace terms.

These facts can be remarked also using functional methods. Indeed, since $\mathcal{H}_t$ is connected to the generator of connected diagrams $\mathcal{F}_{t,t_0}$ by (2.3.7), we can expand $\mathcal{H}_t$ through the amplitudes of its diagrams:

$$\mathcal{H}_t[\varphi] = \sum_{\{\nu_k\}} \int \prod_{k,i_k,j_k} d^d p_{k,j_k}^{(i_k)} \, \delta^d\Big(\sum_{k,i_k,j_k} p_{k,j_k}^{(i_k)}\Big) h_t(\{\nu_k\}; p_{k,j_k}^{(i_k)}) \times \prod_k \prod_{i_k=1}^{\nu_k} Y_k(p_{k,1}^{(i_k)}, \cdots, p_{k,k}^{(i_k)}), \quad (2.3.26)$$

where $h_t(\{\nu_k\}; p_{k,j_k}^{(i_k)})$ is the amplitude of all connected amputated diagrams with structure $\{\nu_k\}$ and momenta $p_{k,j_k}^{(i_k)}$. Therefore, such amplitudes are defined in the same way as the amplitudes $f(\{\nu_k\}; p_i)$ of the functional $\mathcal{F}_{t,t_0}[J]$, thus they contain only the contribution coming from the genus of their diagram and so they can be expanded as:

$$h_t(\{\nu_k\}; p_i) = \sum_g \frac{h_t^{(g)}(\{\nu_k\}; p_i)}{N^{2g}}, \quad (2.3.27)$$

so that:

$$\mathcal{H}_t[\varphi] = \sum_g \frac{\mathcal{H}_t^{(g)}[\varphi]}{N^{2g}}, \quad (2.3.28)$$

where $\mathcal{H}_t^{(g)}$ is the contribution of all diagrams of genus g, thus it reads:

$$\mathcal{H}_t^{(g)}[\varphi] = \sum_{\{\nu_k\}} \int \prod_{k,i_k,j_k} d^d p_{k,j_k}^{(i_k)} \, \delta^d\Big(\sum_{k,i_k,j_k} p_{k,j_k}^{(i_k)}\Big) h_t^{(g)}(\{\nu_k\}; p_{k,j_k}^{(i_k)}) \times \prod_k \prod_{i_k=1}^{\nu_k} Y_k(p_{k,1}^{(i_k)}, \cdots, p_{k,k}^{(i_k)}). \quad (2.3.29)$$

This shows that the planar generator $\mathcal{H}_t^{(0)}$ receives contribution from all diagrams with sphere topology, both planar and non-planar, exactly as the generator of connected diagrams $\mathcal{F}_{t,t_0}$.

In conclusion we have seen that the RG flow of a d-dimensional matrix theory with UV cut-off is described in the large N limit by a Hamilton-Jacobi equation (2.3.25), which differs from the ordinary WPE in the quantum term. This equation generates a closed RG flow, which involves both planar and non-planar diagrams with sphere topology and along which multi-trace interactions are produced in the Wilsonian action, even though starting from a single-trace bare action.

This result is important because it can be connected to the RG equations of the gravity dual in the AdS_{d+1} space obtained from holography. Therefore in the next chapter we will develop the AdS/CFT correspondence, arising from the holographic conjecture, in order to describe the holographic Wilsonian renormalization. The latter enables us to obtain RG equations in AdS space and to translate them into the QFT-side of the correspondence.

CHAPTER 3

The AdS/CFT Correspondence

Gauge/Gravity duality is one of the most important results coming from sting theory in the last twenty years and it describes a duality between theories with gravity and theories without gravity. The AdS/CFT correspondence represents a specific case of such duality, which concerns gravity theories in Anti-de Sitter space (AdS) and field theories with conformal invariance.

The first explicit realization of AdS/CFT correspondence is the equivalence between type IIB string theory (and also supergravity) defined on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, proposed by Maldacena in [17]. Other works [18, 19] have developed a general prescription which determines the duality between a gravity theory in a $d+1$-dimensional AdS space and a local conformal field theory (CFT) in d dimensions.

The AdS/CFT correspondence is related to an important physical idea on quantum gravity: holography [20, 21]. This idea comes from the thermodynamics of black holes. Indeed it has been shown that a black hole is a thermodynamic system where the temperature is related to the black body emission, whereas the entropy depends on the area A of the horizon of the black hole through $S = A/4G$, where G is the gravitational constant. Since entropy determines the number of degrees of freedom (d.o.f.'s), the latter equation implies that in a $d+1$-dimensional gravity the number of d.o.f.'s contained in a box depends on the area of such box, i.e. on a d-dimensional object. Instead, in a local field theory the number of degrees of freedom in a box depends on the volume. Therefore, since an area in $d+1$ dimensions is the same as a volume in d dimensions, this means that a gravity theory has the same number of degrees of freedom of a local field theory in one less dimension. This situation is analogous to an hologram, which provides a 3D picture from the information stored in a 2D image, thus this phenomenon is referred to as holography. The AdS/CFT correspondence is a realization of holography since it equalize a theory of gravity in $d+1$-dimensions to a local field theory in d-dimensions.

3.1. The Anti-de Sitter space

Before staring with the proper development of the AdS/CFT correspondence, let us focus on the Anti-de Sitter space in general $d+1$ dimensions (AdS_{d+1}).

From the physical point of view AdS_{d+1} is a maximally symmetric solution of the Einstein equations in $d+1$-dimensions with negative cosmological constant $\Lambda_c < 0$:

$$\mathcal{R}_{mn} - \frac{1}{2} g_{mn} \mathcal{R} = -\frac{\Lambda_c}{2} g_{mn}, \tag{3.1.1}$$

where $m, n = 0, 1, \cdots, d$. From this equation we have:

$$\mathcal{R} = \frac{d+1}{d-1} \Lambda_c \qquad \mathcal{R}_{mn} = \frac{\Lambda_c}{d-1} g_{mn} \ ,$$

hence the maximally symmetric solution is represented by the Riemann tensor:

$$\mathcal{R}_{mnpq} = \frac{\Lambda_c}{d(d-1)} \left(g_{mp} g_{nq} - g_{mq} g_{np} \right). \tag{3.1.2}$$

This solution can be parametrized as the subspace of $\mathbb{R}^{2,d}$ with signature $(-, -, +, + \cdots, +)$, determined by the quadratic equation:

$$-X_{-1}^2 - X_0^2 + \sum_{i=0}^{d} X_i^2 = -R^2, \tag{3.1.3}$$

with $X_{-1} > 0$ and:

$$R^{-2} = -\frac{\Lambda_c}{d(d-1)},$$

where R is the so-called AdS radius.

The equivalent maximally symmetric solution in Euclidean signature is the subspace of $\mathbb{R}^{1,d+1}$ solving the hyperboloid equation:

$$-X_{-1}^2 + X_0^2 + \sum_{i=0}^{d} X_i^2 = -R^2. \tag{3.1.4}$$

Therefore we can easily pass from the Minkowskian to the Euclidean solution with a Wick rotation: $X_0 \mapsto -iX_0$. This operation is very useful since one typically prefers to work in Euclidean signature. Thus, henceforth we will work on the Euclidean version of AdS_{d+1} and we will set $R = 1$ for simplicity.

The coordinates $X_0, X_1, \cdots, X_{d-1}$ are usually represented as a vector X_μ, so that the Minsowskian metric of $\mathbb{R}^{1,d+1}$ induces a metric on AdS_{d+1} of the form:

$$ds^2 = -dX_{-1}^2 + dX_d^2 + dX_\mu dX^\mu, \tag{3.1.5}$$

where $dX_\mu dX^\mu = \delta^{\mu\nu} dX_\mu dX_\nu$. Being X_{-1} related to X_d and X_μ through (3.1.4), we can parametrize the space by $(X_\mu, X_d) \in \mathbb{R}^{d+1}$. This space has a boundary which can be reached sending X_I (with $I = -1, \mu, d$) to infinity with the same rate, i.e. keeping fixed the ratios X_I/X_J, and identifying points according to the equivalence relation $X_I \sim sX_I$, with $s \in \mathbb{R}_+$. In such a way from (3.1.4) for large X_I we have:

$$X_{-1}^2 = X_d^2 + X_\mu X^\mu.$$

Rescaling X_I with $s = 1/X_{-1}$, so that $X_{-1} \sim 1$, we have:

$$X_d^2 + X_\mu X^\mu = 1. \tag{3.1.6}$$

This equation defines the boundary of AdS_{d+1} and shows that such boundary has the topology of a d-dimensional sphere S^d.

A suitable system of coordinates on AdS_{d+1} can be defined by:

$$z := \frac{1}{X_{-1} + X_d} \qquad x_\mu := \frac{X_\mu}{X_{-1} + X_d}\,, \tag{3.1.7}$$

where $z > 0$ is the so-called radial coordinate. Thus:

$$X_{-1} X_d = \frac{1}{z} \qquad X_d - X_{-1} = -\frac{z^2 + x^2}{z} \qquad X_\mu = \frac{x_\mu}{z}\,,$$

where $x^2 = x_\mu x^\mu = \delta_{\mu\nu} x^\mu x^\nu$. Therefore the metric becomes:

$$ds^2 = g_{mn} dx^m dx^n = \frac{(dz^2 + dx_\mu dx^\mu)}{z^2}, \tag{3.1.8}$$

where $x^m = (z, x^\mu)$ and $g_{mn} = z^{-2}\delta_{mn}$. This is a very useful definition of coordinates because the metric g_{mn} is diagonal and it depends only on the radial coordinate z. Moreover, here it is immediate to realize that AdS_{d+1} admits d-dimensional subspaces $z = r \in \mathbb{R}_+$ which are Euclidean slices orthogonal to the radial axis z with induced metric:

$$h_{\mu\nu} = \frac{\delta_{\mu\nu}}{r^2}. \tag{3.1.9}$$

In these coordinates the boundary is parametrized by the points $(z = 0, x)$, reached for X_{-1}, X_μ and X_d going to infinity with the same rate and $X_{-1} \neq X_d$, plus $(z = +\infty, x)$, which corresponds to the case with $X_{-1} = X_d$. Therefore it corresponds to the d-dimensional Euclidean space plus the "point at infinity" $z = +\infty$[1], which means that it has the topology of the sphere S^d.

[1]The proper "point at infinity" of this space is $(z = 0, x = \infty)$, but the isometry which maps $\|x\|_{d+1} \mapsto \|x\|_{d+1}^{-1}$ shows that both $(z = 0, x = \infty)$ and $(z = +\infty, x)$ are mapped to the same point $(z = 0, x = 0)$, thus they can be identified.

The metric (3.1.8) is singular at $z = 0$, thus it cannot induce a metric on the boundary. For this reason one has to consider a positive function f on AdS_{d+1} such that $f \underset{z\mapsto 0}{\sim} z\, g(x)$, i.e. that has a first order zero on the boundary, and redefine the metric as:

$$d\tilde{s}^2 \coloneqq f^2 ds^2. \tag{3.1.10}$$

This new metric for AdS_{d+1} is finite at $z = 0$, thus it can induce a metric on the boundary. The choice of the function f is non-unique because we can always take this transformation:

$$f \mapsto e^w f,$$

with a smooth function w, that changes the metric:

$$d\tilde{s}^2 \mapsto e^{2w} d\tilde{s}^2.$$

This means that the metric $d\tilde{s}^2$ is defined up to conformal transformations parametrized by the function w. Therefore the metric on AdS_{d+1} defines a conformal structure on the boundary.

Since AdS_{d+1} is a subspace of $\mathbb{R}^{1,d+1}$ defined through (3.1.4), its isometries are the isometries of $\mathbb{R}^{1,d+1}$, which form the group $SO(1, d+1)$. Moreover the latter is also the conformal group of the boundary since it has the topology of the sphere S^d. The action of $SO(1, d+1)$ on AdS_{d+1} can be expressed in the basis (z, x_μ) through:

$$z \mapsto \lambda z \qquad x_\mu \mapsto \lambda x_\mu \qquad \lambda \in \mathbb{R}_+, \tag{3.1.11}$$

which is a dilatation. Furthermore, the translations are given by:

$$z \mapsto z \qquad x_\mu \mapsto x_\mu + a_\mu \qquad a_\mu \in \mathbb{R}^d. \tag{3.1.12}$$

The subgroup $SO(d)$ generates rotations:

$$z \mapsto z \qquad x_\mu \mapsto \Lambda_{\mu\nu} x_\nu\ , \tag{3.1.13}$$

with $\Lambda_{\mu\nu} \in SO(d)$, whereas the special conformal transformations are given by:

$$z \mapsto \frac{z}{1 - 2b\cdot x + b^2(z^2 + x^2)} \qquad x_\mu \mapsto \frac{x_\mu - b_\mu(z^2 + x^2)}{1 - 2b\cdot x + b^2(z^2 + x^2)} \tag{3.1.14}$$

with $b_\mu \in \mathbb{R}^d$.

In addition to these transformations we can extend the isometry group $SO(1, d+1)$ including a transformation non-connected with the identity, called inversion:

$$z \mapsto \frac{z}{z^2 + x^2} \qquad x_\mu \mapsto \frac{x_\mu}{z^2 + x^2}\ . \tag{3.1.15}$$

In such a way, including the inversion, the isometry group becomes $O(1, d+1)$. In this form we can easily see that for $z \mapsto 0$ these transformations reduce to the conformal transformations on the boundary.

3.2. The definition of the correspondence

In the previous section we have studied the properties of the AdS space, now we explicitly define the AdS/CFT correspondence following the method proposed by Witten in [19].

The duality determines the equivalence between a gravity theory (supergravity/string theory) in $d+1$-dimensional AdS space[2] (bulk theory) and a local CFT in d-dimensions. Its definition requires a map which relates quantities of the two theories and a prescription which specifies the equivalence between them.

Therefore consider the action of the bulk theory $S_{AdS}[\Phi]$, where the fields are collectively denoted by $\Phi = (\phi, A_m, g_{mn}, \ldots)$. To have an AdS_{d+1} vacuum for S_{AdS} let us assume that the scalar potential has a negative minimum so that it creates a negative cosmological constant. Now focus on the behavior of the theory near the AdS boundary at $z = 0$. Define a set of boundary conditions for the fields which determines their behavior near the boundary:

$$\Phi(z, x) \underset{z \mapsto 0}{\sim} f(z)\Phi_0(x), \tag{3.2.1}$$

where f is a function of z which represents the profile of the field along the radial axis of AdS_{d+1} whose form is determined by the solutions of the equation of motion for small z, whereas Φ_0 is an arbitrary function of $\mathbb{R}^d$ which represents the boundary value of Φ.

The partition function of the theory is the functional integral defined with boundary conditions (3.2.1), thus it reads:

$$\mathcal{Z}_{AdS} = \int_{\Phi \sim f\Phi_0} \mathcal{D}\Phi \, e^{-S_{AdS}[\Phi]}. \tag{3.2.2}$$

At this point, we define the map between the two theories stating that any field Φ in the bulk is associated to a conformal operator $\mathcal{O}$ of the dual CFT, such that it couples to Φ_0 through the term $\int \Phi_0 \mathcal{O}$. Therefore the prescription which defines the equivalence between the two theories is:

$$\left\langle e^{\int \Phi_0 \mathcal{O}} \right\rangle_{\text{CFT}} = \mathcal{Z}_{AdS} = \int_{\Phi \sim f\Phi_0} \mathcal{D}\Phi \, e^{-S_{AdS}[\Phi]}, \tag{3.2.3}$$

which represents the identification of the bulk partition function with boundary state specified by Φ_0 and the generating functional of the d-dimensional CFT, where the boundary value Φ_0 represents the source of the dual operator $\mathcal{O}$.

[2] To be precise the vacuum of such theory should be $AdS_{d+1} \times M$, with M some compact manifold. For simplicity we will always neglect M and refer only to AdS_{d+1}.

Since the isometry group of AdS_{d+1} coincides with the conformal group of its d-dimensional boundary, we can think of the dual CFT as "living" on the boundary of AdS_{d+1}, so it will be referred to as the boundary CFT.

Even though the precise map between bulk fields and boundary operators depends on the explicit realizations of the duality, some general aspects can be understood using symmetries. Indeed the source term $\int \Phi_0 \mathcal{O}$ must be invariant under conformal transformations, therefore the $O(1, d+1)$ quantum number of $\mathcal{O}$, such as the scaling dimension, have to be dependent on the transformation properties of Φ_0. In particular, if Φ is a scalar field, then the dual operator $\mathcal{O}$ has to be a scalar operator, so that the source term is invariant. For the same reason if Φ is a vector or tensor field, then $\mathcal{O}$ is a vector or tensor operator. Moreover, let us mention another important aspect related to symmetries: gauge symmetries in the bulk corresponds to global symmetries in the boundary theory. This implies that bulk gauge fields A^a are dual to the conserved currents J_a of the corresponding global symmetry. Particularly important is the case of the conserved energy-momentum tensor in the boundary theory. Indeed, being a symmetric rank 2 tensor, it corresponds to the metric tensor in the bulk theory, implying that the latter is characterized by invariance under diffeomorphism. For this reason the bulk theory is expected to be a gravity theory.

The equation (3.2.3) represents a general prescription for the duality, defined without specifying the complete form of bulk theory and local boundary CFT. However the development of most properties of the correspondence is explicitly obtained in string theory, considering the original duality between type IIB string theory compactified on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions in [17]. Indeed, the identification of the partition functions of the two theories implies that their parameters are related by:

$$\left(\frac{R}{l_s}\right)^4 = g_{YM}^2 N, \tag{3.2.4}$$

where, on the bulk-side, R is the AdS radius and l_s is the string scale length which determines the size of the fluctuations of the string world-sheet, whereas, on the CFT-side, g_{YM} is the gauge coupling and N is the size of the gauge group, so that $\lambda := g_{YM}^2 N$ represents the 't Hooft coupling of Yang-Mills theory. This relation shows some important aspects of the duality. First of all the perturbative string regime, where the supergravity approximation is consistent, is reached when the string fluctuations are small compared to the radius R of AdS_{d+1}, thus for:

$$\left(\frac{R}{l_s}\right)^4 = g_{YM}^2 N \gg 1. \tag{3.2.5}$$

In such regime the 't Hooft coupling λ is strong (large 't Hooft limit), so the boundary CFT is not in perturbative regime. On the other hand, the request of $\lambda \ll 1$ to reach the parturbative regime of the field theory implies that:

$$g_{YM}^2 N = \left(\frac{R}{l_s}\right)^4 \ll 1, \tag{3.2.6}$$

which means that the string fluctuation size is greater than R and so string theory is in non-perturbative regime, where the supergravity approximation breaks down. These facts make clear that the perturbative field theory regime and the perturbative gravity regime are incompatible. This does not corresponds to some contradiction in the correspondence, but represents the fact that the AdS/CFT is a strong-weak duality, which means that, when one side is weakly coupled, the other one is strongly coupled and viceversa.

Moreover, another crucial aspect concerns the classical gravity limit. Indeed another property of the duality is that the string coupling constant g_s, which controls the loop expansion in the bulk theory, is related to the gauge coupling of the boundary field theory through $g_s \sim g_{YM}^2$. Therefore in the supergravity approximation regime (3.2.5) we have:

$$\lambda = g_{YM}^2 N \sim g_s N \gg 1. \tag{3.2.7}$$

This means that, being the 't Hooft coupling λ fixed by (3.2.5), taking the classical gravity limit $g_s \mapsto 0$ necessarily implies that $N \mapsto +\infty$. Therefore the classical gravity regime of the bulk theory is dual to the large N limit of the boundary field theory.

In such limit the bulk action is evaluated on the solutions of the supergravity equations of motion (on-shell). Owing to the boundary conditions (3.2.1), these solutions $\overline{\Phi}$ behave as $\overline{\Phi} \sim f\Phi_0$ for $z \mapsto 0$. However, the e.o.m. is a second order equation, so it requires two conditions to be solved. In addition to (3.2.1), usually one imposes an additional regularity condition at $z = +\infty$. This is useful because it has been shown that such condition makes sure that, given the boundary condition specified by Φ_0, the solution is unique and extends over all the AdS space. This means that $\overline{\Phi}$ can be completely written in terms of Φ_0 and so the on-shell bulk action is a functional of Φ_0. In such a way the gravity partition function in the classical limit reduces to:

$$\mathcal{Z}_{AdS}|_{\bar{\Phi}} = e^{-S_{AdS}[\Phi_0]}. \tag{3.2.8}$$

Therefore the prescription (3.2.3) reduces to a relation between the classical bulk theory and the large N dual CFT:

$$\left\langle e^{\int \Phi_0 \mathcal{O}} \right\rangle_{\text{CFT}} = e^{-S_{AdS}[\Phi_0]}, \tag{3.2.9}$$

where the l.h.s. represents the large N generating functional of the correlation functions of the operator $\mathcal{O}$ of the boundary CFT. This equation implies that the tree level bulk action is identified with the generator of the dual field theory. Owing to this fact, we can use directly S_{AdS} to compute the correlation functions of the operator $\mathcal{O}$ dual to the field Φ. Indeed from (3.2.9) we have that:

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{\text{CFT}} = -\left.\frac{\delta^n S_{AdS}[\Phi_0]}{\delta\Phi_0(x_1)\cdots\delta\Phi_0(x_n)}\right|_{\Phi_0=0}, \tag{3.2.10}$$

which means that we can evaluate holographically the large N CFT correlation functions through the derivatives of the tree level bulk action with respect to the boundary value Φ_0, eventually setting $\Phi_0 = 0$ (holographic method).

The conformal invariance of the boundary field theory fixes the form of the 2- and 3-point function of a conformal operator $\mathcal{O}$ of dimension Δ to:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{\text{CFT}} = \frac{c}{|x_1-x_2|^{2\Delta}} \tag{3.2.11a}$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle_{\text{CFT}} = \frac{d}{|x_1-x_2|^{\Delta}|x_1-x_3|^{\Delta}|x_2-x_3|^{\Delta}}. \tag{3.2.11b}$$

Therefore, computing the correlation functions with the holographic relation (3.2.10), one must ultimately yield exactly this result.

3.3. Dynamics of a scalar field in the bulk

In this part we develop more concrete aspects of the AdS/CFT correspondence through the study of the dynamics of fields in AdS_{d+1}.

Let us consider the simple example of a real massive scalar field $\phi(z,x)$ in AdS_{d+1} with action:

$$S_{AdS}[\phi] = \int dz d^d x\sqrt{g}\left[\frac{1}{2}g^{mn}\partial_m\phi\partial_n\phi + \frac{1}{2}m^2\phi^2 + V(\phi)\right], \tag{3.3.1}$$

where we neglect the back-reaction of the field on the metric and we fix g_{mn} to be the AdS_{d+1} metric (3.1.8), so that $\sqrt{g} = \sqrt{\det g_{mn}} = z^{-1-d}$.

The equation of motion is:

$$(-\Box + m^2)\phi + V'(\phi) = 0, \tag{3.3.2}$$

where:

$$\Box\phi = \frac{1}{\sqrt{g}}\partial_m(\sqrt{g}g^{mn}\partial_n\phi), \tag{3.3.3}$$

which is the $d+1$-dimensional d'Alembert operator (in Euclidean coordinates it reduces to a Laplacian operator), whereas g^{mn} is the inverse of the metric, thus it reads $g^{mn} = z^2\delta_{mn}$. In such a way, the e.o.m. can be written as:

$$z^2\partial_z^2\phi + (1-d)z\partial_z\phi + z^2\partial_\mu^2\phi - m^2\phi - V'(\phi) = 0. \tag{3.3.4}$$

In order to determine the boundary value of ϕ we may restrict to the linearized (free) equation of motion and treat the contributions coming from the interaction term $V(\phi)$ as perturbations which do not affect the leading term of the solution for $z \mapsto 0$. Thus we have to solve:

$$z^2\partial_z^2\phi + (1-d)z\partial_z\phi + z^2\partial_\mu^2\phi - m^2\phi = 0, \tag{3.3.5}$$

This equation can be solved using Bessel functions. However, when $z \mapsto 0$ the x-dependence becomes negligible because the term $z^2\partial_\mu^2\phi$ is of order z^2 whereas all the other terms are of order $z^0 = 1$. Hence near the boundary the solution behaves as $\overline{\phi} \sim z^\Delta$, where Δ satisfies the quadratic equation:

$$\Delta(\Delta - d) - m^2 = 0. \tag{3.3.6}$$

The roots of this equation are:

$$\Delta_\pm = \frac{d}{2} \pm \nu\ , \qquad\qquad \nu = \sqrt{\frac{d^2}{4} + m^2}, \tag{3.3.7}$$

so that the boundary behavior of the solution is of the type:

$$\overline{\phi} \sim Az^{\Delta_-} + Bz^{\Delta_+}, \tag{3.3.8}$$

with A and B imposed by initial conditions. Now, restricting to values of the mass such that ν is real we have that $\Delta_- \leq \Delta_+$, with $\Delta_- = \Delta_+ = d/2$ only for $m^2 = -d^2/4$. Therefore the boundary behavior of $\overline{\phi}$ is dominated by the term scaling as z^{Δ_-}, so the boundary conditions are specified by:

$$\overline{\phi}(z,x) \underset{z\mapsto 0}{\sim} z^{\Delta_-}\phi_0(x), \tag{3.3.9}$$

where the arbitrary function $\phi_0(x)$ is the boundary value representing the source of the dual operator $\mathcal{O}$ in the boundary CFT.

Imposing the regularity conditions at $z = +\infty$ the solution can be uniquely specified on the whole AdS_{d+1} by the boundary value. In such a way we can expand it as:

$$\overline{\phi}(z,x) = z^{\Delta_-}\Big(\phi_0(x) + \sum_{k=1}^{+\infty} z^{2k}\phi_{2k}(x)\Big) + z^{\Delta_+}\Big(A(x) + \sum_{k=0}^{+\infty} z^{2k}A_{2k}(x)\Big), \tag{3.3.10}$$

where the functions ϕ_{2k} can be recursively determined in terms of ϕ_0 by the relation:

$$\phi_{2k} = \frac{\partial_\mu^2\phi_{2(k-1)}}{2k(d - 2\Delta_- - 2k)}, \tag{3.3.11}$$

whereas A and A_{2k} depend on the regularity conditions and can be computed only solving exactly the equation of motion.

Turning back to the restriction of the mass values only on that which provide real solutions of (3.3.6), we note that such condition admits the possibility to have $m^2 < 0$. This implies the possibility to have tachyons in the *AdS* theory. At first sight this may be puzzling, in actual fact a theory in *AdS* space has more "freedom" on the definition of the mass. More precisely, one can choose the mass such that $m^2 < 0$ without generating instability because, as proved by [24], the boundary conditions force the kinetic term not to be vanishing at infinity. This implies more loose stability conditions, which require that m^2 is greater than some negative bound, instead of strictly non-negative. This is called the Breitenlohner-Freedman bound, and reads:

$$R^2 m^2 \geq -\frac{d^2}{4}, \tag{3.3.12}$$

where, for once, we have restored the *AdS* radius R. If m satisfies this condition the energy is positive definite, hence the presence of tachyons has not to be worrisome.

This implies some important facts connected to whether the mass is positive, null or negative. Indeed, if $-d^2/4 \leq m^2 < 0$ we have $\Delta_- > 0$, therefore ϕ vanishes at the boundary, whereas, if $m^2 = 0$ we have the massless case, where $\Delta_- = 0$ and so ϕ approaches to the finite value ϕ_0 at the boundary. In both cases the definition of the boundary value through the linearized e.o.m. is justified and self-consistent, therefore the generating functional in the l.h.s. of (3.2.9) is well-defined.

If, however, $m^2 > 0$ we have $\Delta_- < 0$, so ϕ diverges at the boundary. This means that the boundary behavior of ϕ determined through the linearized e.o.m. is not self-consistent and so we cannot build a well-defined generating functional unless we treat the boundary value ϕ_0 as infinitesimal.

Avoiding for now these issues, let us suppose the generating functional of connected correlation functions of $\mathcal{O}$ to be determined in a consistent way. Thus the duality states that it is equivalent to the partition function of the bulk theory:

$$\left\langle e^{N^2 \int \phi_0 \mathcal{O}} \right\rangle_{\text{CFT}} = \int_{\phi \sim z^{\Delta_-} \phi_0} \mathcal{D}\phi \, e^{-\kappa^{-2} S_{AdS}[\phi]}. \tag{3.3.13}$$

In this example we explicitly keep track of the standard normalization of the dual CFT carrying out a factor N^2, where N represents the matrix size of the boundary fields. The operator $\mathcal{O}$ is dual to the bulk scalar field ϕ, thus it is a scalar operator which have to be of order $O(N^0 = 1)$ so that the source term is of order $O(N^2)$. This means that $\mathcal{O}$ can be written as the trace of a product of fields and derivatives of the fields in the boundary theory with normalization:

$$\mathcal{O} = \frac{\operatorname{tr}(\cdots)}{N}. \tag{3.3.14}$$

On the other side of the duality we explicitly express the gravity coupling $\kappa^{-2} \sim 1/G_N$, where G_N is the Newton's gravitational constant. In string theory G_N is related to the string coupling constant by $G_N \sim g_s^2$. Therefore, since through the duality $g_s \sim 1/N$, we have that $G_N \sim N^{-2}$. In such a way $\kappa^{-2} \sim N^2$, which means that the bulk action is of the same order $O(N^2)$ of the action of the dual CFT. For this reason the classical gravity limit-large N limit connection in the duality is reached by $\kappa^2 \mapsto 0$.

To avoid any confusion, henceforth we will use N^2 for quantities of the boundary field theory and κ^{-2} for quantities in the bulk.

Now the classical gravity duality implies that the on-shell bulk action is the generator of connected correlation functions of $\mathcal{O}$ in the large N limit of the dual CFT, therefore we have that:

$$\left\langle e^{N^2 \int \phi_0 \mathcal{O}} \right\rangle_{\text{CFT}} = e^{-\kappa^{-2} S_{AdS}[\phi_0]}. \tag{3.3.15}$$

The definition of ϕ_0 is arbitrary because it depends on the choice of the coordinate z. Therefore, performing an isometry one could have a change of coordinates which modifies ϕ_0. Doing so, since the isometries of AdS_{d+1} are conformal transformations on the boundary, we can obtain the transformation properties of ϕ_0 under the conformal group. Hence considering, for example, a dilatation (3.1.11), the bulk scalar field ϕ must be invariant under such transformation, hence $\overline{\phi}'(z', x') = \overline{\phi}(z, x)$. Therefore, using (3.3.9), the boundary value transforms as:

$$\phi_0(x) \mapsto \phi_0'(x') = \lambda^{-\Delta_-} \phi_0(\lambda x), \tag{3.3.16}$$

which means that ϕ_0 has scaling dimension $-\Delta_-$, or mass dimension Δ_- (henceforth, the mass dimension will be simply referred to as dimension). Therefore, if we consider the source term $\int \phi_0 \mathcal{O}$ of the CFT, we have that the dimension of the dual operator $\mathcal{O}$ is:

$$\Delta_+ = d - \Delta_- = \frac{d}{2} + \nu. \tag{3.3.17}$$

This means that the mass of the bulk field determines the dimension of the dual operator in the boundary CFT. Thus, adjusting the latter one can obtain an operator in the dual CFT of the desired dimension.

3.3.1. Holographic renormalization

In the previous part of the chapter we have seen that a classical gravity theory (supergravity) in AdS_{d+1} is equivalent to a local large N field theory on the d-dimensional boundary S^d of AdS_{d+1}. In particular the fundamental statement (3.2.9) asserts that the tree level bulk action is the generator of connected correlation functions of the boundary CFT in the planar limit. However, the duality so defined is affected by divergences. Indeed there are IR divergences in the on-shell bulk action because, due to the boundary behavior of the fields, the z-integral diverges as $z \mapsto 0$. This fact makes the prescription (3.3.13) a formal identification between the two theories, unable to yield finite results.

Anyway, we can see that these divergences in the bulk can be related to UV divergences in the dual field theory. This is due to the so-called IR/UV connection [23], which states that the long distance (IR) regime of the bulk theory is connected to the short distance

(UV) regime of the dual field theory. In particular, it has been shown in [23] that a IR regulator paramenter $\epsilon \ll 1$ in the bulk, which shifts the boundary from $z = 0$ to $z = \epsilon$, acts through the duality as a UV cut-off for the correlation functions of the boundary theory for distances much greater than ϵ.

Due to this fact we can identify the radial axis in AdS_{d+1} with the length scale in the boundary theory. This is supported by the fact that, taking a dilatation (3.1.11) in AdS_{d+1} , we have that $z \mapsto \lambda z$, i.e. the radial coordinate is rescaled by a factor λ, whereas, since (3.1.11) represents also a conformal transformation on the boundary theory, in the QFT-side the energy scale Λ is rescaled by $\Lambda \mapsto \Lambda/\lambda$. Therefore we shall make the identification of the radial coordinate z with the inverse of energy scale Λ of the boundary theory.

This is a very important aspect of the AdS/CFT correspondence because it implies that the near-to-boundary region ($z \ll 1$) of the bulk theory corresponds to the UV regime of the dual QFT, whereas the interior region ($z \gg 1$) of the bulk is connected to the IR regime of the field theory.

In this subsection we will outline the procedure of holographic renormalization for the massive scalar field in the bulk introduced above. Thus, first of all let us consider the on-shell bulk action (3.3.1). Using integration by parts and the fact that ϕ satisfies the e.o.m., we can write it as:

$$S[\phi] = \int dz d^d x \Big[\frac{1}{2}\partial_m(\sqrt{g} g^{mn}\phi\partial_n\phi) + \sqrt{g}\Big(V(\phi) - \frac{1}{2}\phi V'(\phi)\Big)\Big]. \tag{3.3.18}$$

Now, the first term can be rewritten as:

$$\begin{aligned} \frac{1}{2}\int dz d^d x\, \partial_m(\sqrt{g} g^{mn}\phi\partial_n\phi) &= \frac{1}{2}\int dz d^d x\, \partial_m(z^{1-d}\delta^{mn}\phi\partial_n\phi) \\ &= \frac{1}{2}\int dz d^d x \Big[\partial_z(z^{1-d}\phi\partial_z\phi) + \partial_\mu(z^{1-d}\delta^{\mu\nu}\phi\partial_\nu\phi)\Big], \end{aligned} \tag{3.3.19}$$

where, since ϕ is set to vanish as x goes to infinity, the only surviving term is the derivative with respect to z, which produces a boundary term:

$$\frac{1}{2}\int dz d^d x\, \partial_z(z^{1-d}\phi\partial_z\phi) = \frac{1}{2}\Big[\int d^d x (z^{1-d}\phi\partial_z\phi)\Big|_{z=0}^{z=+\infty} = -\frac{1}{2}\int d^d x\, \Big(z^{1-d}\phi\partial_z\phi\Big|_{z=0}, \tag{3.3.20}$$

because, thanks to the regularity condition imposed at $z = +\infty$ for the solution of the e.o.m., only the term at $z = 0$ survive and, in particular, is divergent.

This fact is due to the behavior of the on-shell field approaching the boundary. Therefore, we can think of introducing a regularization scheme defining a IR cut-off $\epsilon_0 \ll 1$ on the radial axis, which shifts the boundary from $z = 0$ to $z = \epsilon_0$. In such a way we have a regulated boundary where, as long as we keep $\epsilon_0 \neq 0$, there are no divergences. Like in the standard renormalization method for ordinary QFT, we can proceed evaluating quantities on the boundary at $z = \epsilon_0$ and eventually send $\epsilon_0 \mapsto 0$.

Owing to the regularization procedure, the boundary conditions of ϕ are now specified by:

$$\phi(\epsilon_0, x) = \epsilon_0^{\Delta_-}\phi_0(x), \tag{3.3.21}$$

and the action S is regularized to:

$$S_{\epsilon_0} = -\frac{1}{2}\int d^d x\,\Big(z^{1-d}\phi\partial_z\phi\Big|_{z=\epsilon_0} + \int_{z>\epsilon_0} dz d^d x\sqrt{g}\Big[V(\phi) - \frac{1}{2}\phi V'(\phi)\Big], \tag{3.3.22}$$

The study of S_{ϵ_0} enables us to determine the divergent contributions and so to develop a way to remove them. In order to do so we should insert in the first term of (3.3.22) the solution of the total e.o.m. (3.3.2). Treating such solution in a perturbative way, we can expand it as: $\overline{\phi} = \overline{\phi}^{(0)} + \overline{\phi}^{(1)} + \cdots$, where $\overline{\phi}^{(0)}$ is the free solution, whereas the other terms are perturbations coming from the potential. Such corrections acts modifying the terms of the expansion (3.3.10) of the free solution leaving the structure of the expansion unchanged. Hence we can express $\overline{\phi}$ through the same expansion (3.3.10), where the coefficients contains the free contribution plus other perturbative corrections (for example $A = A^{(0)} + A^{(1)} + \cdots$). Doing so the boundary term of S_{ϵ_0} can be expanded as:

$$\begin{aligned}\Big(z^{1-d}\phi\partial_z\phi\Big|_{z=\epsilon_0} &= \epsilon_0^{2\Delta_- - d}\big(\phi_0 + \sum_k \epsilon_0^{2k}\phi_{2k}\big)\big(\Delta_-\phi_0 + \sum_{k'}(\Delta_- + 2k')\epsilon_0^{2k'}\phi_{2k'}\big) + d\phi_0 A + O(\epsilon_0)\\ &= \epsilon_0^{2\Delta_- - d}\sum_{k,k'=0}^{+\infty}(\Delta_- + 2k')\epsilon_0^{2k+2k'}\phi_{2k}\phi_{2k'} + d\phi_0 A + O(\epsilon_0),\end{aligned} \tag{3.3.23}$$

where $O(\epsilon_0)$ includes non-pathological terms proportional to positive powers of ϵ_0, thus vanishing as $\epsilon_0 \mapsto 0$. The divergent terms are those proportional to $\phi_{2k}\phi_{2k'}$ which satisfies $k + k' < (d - 2\Delta_-)/2 = \nu$, hence we have:

$$S_{\text{div}} = -\frac{1}{2}\epsilon_0^{2\Delta_- - d}\int d^d x \sum_{k+k'<\nu}(\Delta_- + 2k')\epsilon_0^{2k+2k'}\phi_{2k}\phi_{2k'}, \tag{3.3.24}$$

which is the divergent part of the on-shell bulk action and is responsible for the "bad" definition of the correspondence (3.2.9).

Let us remark that S_{div} is a local term. Thus, like standard renormalization in QFT, which involves a procedure of regularization followed by the removal of infinities through the definition of counterterms, we can define a local boundary term $S_{\text{c.t.}}$ to be added to S_{ϵ_0}, which acts like a countertem, removing the divergent part S_{div} of the bulk action, so that the renormalized action is defined as:

$$S_{\text{ren}} \coloneqq \lim_{\epsilon_0 \mapsto 0}\Big(S_{\epsilon_0} + S_{\text{c.t.}}\Big). \tag{3.3.25}$$

To have a finite S_{ren} through this definition, the counterterm action $S_{\text{c.t.}}$ must cancels out the divergent terms, therefore it has to be a boundary term which for $\epsilon \mapsto 0$ reduces on-shell to $-S_{\text{div}}$:

$$S_{\text{c.t.}} \underset{\epsilon_0 \mapsto 0}{\longmapsto} -S_{\text{div}}. \tag{3.3.26}$$

This condition of course leave some freedom in the choice of the boundary term, because one can subtract some arbitrary finite term besides the divergent ones. This means that, as in the standard renormalization case, there are different renormalization schemes which are based on the finite contributions subtracted by the counterterm action. All that said, we will not further investigate this argument and we will just give one choice for $S_{\text{c.t.}}$ without dwelling on other possible subtraction schemes.

Let us now consider the case where $\nu < 1$, i.e. $2\Delta_- - d < 2$. Here the only divergent term is the one corresponding to $k = k' = 0$, thus:

$$S_{\text{div}} = -\frac{1}{2}\Delta_- \epsilon_0^{2\Delta_- - d} \int d^d x\, \phi_0^2(x). \tag{3.3.27}$$

To remove such divergent term the counterterm action must be such that:

$$S_{\text{c.t.}} \underset{\epsilon_0 \mapsto 0}{\longmapsto} \frac{1}{2}\Delta_- \epsilon_0^{2\Delta_- - d} \int d^d x\, \phi_0^2(x). \tag{3.3.28}$$

There are many ways to define $S_{\text{c.t.}}$ satisfying such condition. Here we choose:

$$S_{\text{c.t.}} = \frac{1}{2}\Delta_- \int d^d x \left(\sqrt{h}\phi^2(x)\right|_{z=\epsilon_0}, \tag{3.3.29}$$

where $h_{\mu\nu}$ is the induced metric (3.1.9) on the flat d-dimensional slice of AdS_{d+1} at $z = \epsilon_0$. Evaluating (3.3.29) on the solution $\overline{\phi}$ we have:

$$S_{\text{c.t.}} = \frac{1}{2}\Delta_- \int d^d x \Big(\epsilon_0^{2\Delta_- - d}\phi_0^2(x) + 2\phi_0(x)A(x) + O(\epsilon)\Big), \tag{3.3.30}$$

thus it removes S_{div} plus a finite term proportional to $\phi_0 A$.

If $\nu > 1$, so $2\Delta_- - d > 2$, there are other divergent terms to be removed. To do so we have to include in $S_{\text{c.t.}}$ other boundary terms. For example, consider the second divergent term ($k = 1$, $k' = 0$ and $k = 0$, $k' = 1$) in the series (3.3.24):

$$-(\Delta_- + 1)\epsilon_0^{2\Delta_- - d + 2} \int d^d x\, \phi_0 \phi_2. \tag{3.3.31}$$

The corresponding counterterm can be defined as:

$$S^{(1)}_{\text{c.t.}} = \frac{\Delta_- + 1}{2(d - 2\Delta_- - 2)} \int d^d x \left(\sqrt{h}\phi \Box_h \phi\right|_{z=\epsilon_0}, \tag{3.3.32}$$

where $\Box_h = h^{\mu\nu}\partial_\mu\partial_\nu$. In such a way the counterterm action is $S_{\text{c.t.}} = S^{(0)}_{\text{c.t.}} + S^{(1)}_{\text{c.t.}}$, where $S^{(0)}_{\text{c.t.}}$ is the term in (3.3.29) defined for the previous divergent term.

At this point, defined a counterterm action which subtracts the divergent part of the on-shell bulk action we can send $\epsilon_0 \mapsto 0$ and obtain the renormalized action S_{ren}.

This is the holographic renormalization procedure, which provides a recipe to obtain a finite on-shell bulk action. This method is very similar to standard renormalization of UV divergences in ordinary QFT since it requires a regularization of the action and the introduction of counterterms to subtract the divergences, which can be realized through different renormalization schemes. The main difference is that the counterterm action $S_{\text{c.t.}}$ introduced in the holographic renormalization is a boundary term which "modifies" the theory, whereas in ordinary QFT counterterms are introduced through redefinition of the fields. Moreover, in standard renormalization divergences appear at every loop order and so in a infinite number of diagrams, thus renormalization is performed perturbatively order by order, whereas in holographic renormalization divergences depends on the value of ν, i.e. on the mass of the bulk field, thus we can have just few divergent terms. For example, in our case, if $\nu = 0$, so $m^2 = -d^2/4$, there is no divergence coming from the boundary term (3.3.20). Instead, if $\nu < 1$ there is only one divergent term, which corresponds to (3.3.29), whereas if $\nu > 1$ there are many other ones, such as (3.3.31). Hence, since $\nu = \sqrt{\frac{d^2}{4} + m^2}$, adjusting the mass of ϕ in the bulk we can manage at will the structure of the divergent part S_{div}. This means that also the structure of the boundary term $S_{\text{c.t.}}$ depends on the mass m of the bulk fields.

At this point, the holographic renormalization method provides a consistent prescription which identifies the renormalized bulk action S_{ren} to the generating functional of the connected correlators of $\mathcal{O}$ of the large N dual field theory:

$$\left\langle e^{N^2 \int \phi_0 \mathcal{O}} \right\rangle_{\text{CFT}} = e^{-\kappa^{-2} S_{\text{ren}}[\phi_0]} = \lim_{\epsilon_0 \mapsto 0} e^{-\kappa^{-2} S_{\epsilon_0}[\phi_0] - \kappa^{-2} S_{\text{c.t.}}[\phi_0]}. \tag{3.3.33}$$

In such a way the derivatives of S_{ren} with respect to ϕ_0 reproduces holographically the correct correlation functions of $\mathcal{O}$, which have the form given in (3.2.11).

In particular, considering the first derivative of S_{ren}, we can show an important fact: the mode A going as z^{Δ_+} in the solution $\overline{\phi}$ represents the expectation value of the operator $\mathcal{O}$ in the boundary theory.

Therefore, from (3.3.25) we have:

$$\frac{\delta S_{\text{ren}}}{\delta \phi_0(x)} = \lim_{\epsilon_0 \mapsto 0} \left(\frac{\delta S_{\epsilon_0}}{\delta \phi_0(x)} + \frac{\delta S_{\text{c.t.}}}{\delta \phi_0(x)} \right). \tag{3.3.34}$$

Let us start the computation with the derivative of S_{ϵ_0}:

$$\begin{aligned}
\frac{\delta S_{\epsilon_0}}{\delta \phi_0(x)} &= \int dz' d^d x' \left(\frac{\delta S_{\epsilon_0}}{\delta \overline{\phi}(z',x')} \frac{\delta \overline{\phi}(z',x')}{\delta \phi_0(x)} + \frac{\delta S_{\epsilon_0}}{\delta \partial_m \overline{\phi}(z',x')} \frac{\delta \partial_m \overline{\phi}(z',x')}{\delta \phi_0(x)} \right) \\
&= \int dz' d^d x' \left[\left(\frac{\delta S_{\epsilon_0}}{\delta \overline{\phi}(z',x')} - \partial_m \frac{\delta S_{\epsilon_0}}{\delta \partial_m \overline{\phi}(z',x')} \right) \frac{\delta \overline{\phi}(z',x')}{\delta \phi_0(x)} \right. \\
&\quad \left. + \partial_m \left(\frac{\delta S_{\epsilon_0}}{\delta \partial_m \overline{\phi}(z',x')} \frac{\delta \overline{\phi}(z',x')}{\delta \phi_0(x)} \right) \right] = - \int d^d x' \left(\frac{\delta S_{\epsilon_0}}{\delta \partial_z \overline{\phi}(z,x')} \frac{\delta \overline{\phi}(z,x')}{\delta \phi_0(x)} \right|_{z=\epsilon_0},
\end{aligned} \tag{3.3.35}$$

thus:

$$\left.\frac{\delta S_{\epsilon_0}}{\delta\partial_z\overline{\phi}(z,x')}\right|_{z=\epsilon_0} = \frac{1}{2}\int dz'' d^dx''\,(z'')^{1-d}\left.\frac{\delta(\partial_{z''}\overline{\phi})^2}{\delta\partial_z\overline{\phi}(\epsilon_0,x')}\right|_{z=\epsilon_0}$$
$$= \epsilon_0^{1-d}\,\partial_z\overline{\phi}(z,x')\Big|_{z=\epsilon_0} = \epsilon_0^{1-d}\Big(\epsilon_0^{\Delta_--1}(\Delta_-\phi_0+O(\epsilon_0))+\epsilon_0^{\Delta_+-1}(\Delta_+A+O(\epsilon_0))\Big), \quad (3.3.36)$$

whereas, using (3.3.10) we have:

$$\frac{\delta\overline{\phi}(\epsilon_0,x')}{\delta\phi_0(x)} = \epsilon_0^{\Delta_-}\Big(\delta^d(x-x')+O(\epsilon_0)\Big)+\epsilon_0^{\Delta_+}\Big(\frac{\delta A(x')}{\delta\phi_0(x)}+O(\epsilon_0)\Big). \quad (3.3.37)$$

Therefore:

$$\frac{\delta S_{\epsilon_0}}{\delta\phi_0(x)} = -\Delta_+A(x)-\Delta_-\int d^dx'\,\phi_0(x')\frac{\delta A(x')}{\delta\phi_0(x)}+(\text{divergent terms}), \quad (3.3.38)$$

where we have dropped terms proportional to positive powers of ϵ_0 because they vanish as $\epsilon_0 \mapsto 0$.

Now consider the derivative of $S_{\text{c.t.}}$. We compute it with $S_{\text{c.t.}}$ defined in (3.3.29) plus all the further terms necessary to remove the divergences in S, like $S^{(1)}_{\text{c.t.}}$ in (3.3.32). From the computation we can see that the only non-divergent term in the derivative of $S_{\text{c.t.}}$ comes from $S^{(0)}_{\text{c.t.}}$, thus we have:

$$\frac{\delta S_{\text{c.t.}}}{\delta\phi_0(x)} = \frac{\Delta_-}{2}\int d^dx'\left(\sqrt{h}\frac{\delta}{\delta\phi_0(x)}\overline{\phi}^2(z,x')\right|_{z=\epsilon_0}+\frac{\delta S^{(1)}_{\text{c.t.}}}{\delta\phi_0(x)}+\cdots$$
$$= \Delta_-\int d^dx'\,\epsilon_0^{-d}\overline{\phi}(\epsilon_0,x')\frac{\delta\overline{\phi}(\epsilon_0,x')}{\delta\phi_0(x)}+\frac{\delta S^{(1)}_{\text{c.t.}}}{\delta\phi_0(x)}+\cdots. \quad (3.3.39)$$

Expressing $\overline{\phi}$ through the expanded solution (3.3.10) we obtain:

$$\frac{\delta S_{\text{c.t.}}}{\delta\phi_0(x)} = \Delta_-A(x)+\Delta_-\int d^dx'\,\phi_0(x')\frac{\delta A(x')}{\delta\phi_0(x)}+(\text{divergent terms}), \quad (3.3.40)$$

where the divergent terms are exactly the ones required to cancel the divergences of (3.3.38). In such a way, we have that the derivative of S_{ren} is finite and reads:

$$\frac{\delta S_{\text{ren}}}{\delta\phi_0(x)} = -\Delta_+A(x)-\Delta_-\int d^dx'\,\phi_0(x')\frac{\delta A(x')}{\delta\phi_0(x)}$$
$$+\Delta_-A(x)+\Delta_-\int d^dx'\,\phi_0(x')\frac{\delta A(x')}{\delta\phi_0(x)} = -(\Delta_+-\Delta_-)A(x), \quad (3.3.41)$$

so that, since $\Delta_+-\Delta_- = 2\Delta_+-d$, we have:

$$\langle\mathcal{O}(x)\rangle_{\phi_0} = (2\Delta_+-d)A(x), \quad (3.3.42)$$

which is the 1-point function of the operator $\mathcal{O}$ with ϕ_0 switched on. This stresses a very important fact: the expectation value of the operator $\mathcal{O}$ is determined by $A(x)$.

In this subsection we have outlined the holographic renormalization method, which enables us to obtain the correct correlation functions of the boundary CFT (see appendix B) through the renormalized bulk action.

Anticipating the content of the next chapter, holographic renormalization is primarily important for the study of the non-conformal regime of the dual field theory. Indeed, as introduced in the beginning of the subsection, through the IR/UV connection it is possible to interpret the IR infinities of the bulk theory as UV divergences of the dual field theory. Hence the renormalization of the bulk action corresponds to the renormalization of the dual field theory, where the boundary CFT represents a fixed point.

CHAPTER 4

Asymptotic AdS/deformed CFT correspondence

4.1. Introducing a deformation

In the previous chapter we have studied the standard AdS/CFT correspondence, considering the duality (3.2.3) between a bulk theory in AdS_{d+1}, where fields are forced to reach some boundary value specified by the boundary conditions, and a d-dimensional CFT on the boundary of AdS_{d+1}. Every bulk field is associated to a boundary operator in the CFT such that its boundary value is interpreted as the source of such operator. Moreover, the classical limit of the tree level bulk theory is equivalent to the large N limit of the boundary CFT. However, the correspondence so defined is affected by IR divergences in the on-shell bulk action due to the behavior of the bulk fields in the boundary, i.e. for $z \mapsto 0$. Therefore we have presented a renormalization method (holographic renormalization) to remove such divergences, so that the renormalized bulk action is able to yield finite results. In particular, through the holographic method, the derivatives of the renormalized bulk action with respect to the boundary values of the fields give the exact form of the correlation functions of the dual operators imposed by the conformal invariance of the boundary theory.

This technology is applied to field theories with conformal invariance. The latter are not realistic theories of particles because they do not show a running behavior of the couplings. Indeed the RG equations of the couplings are trivial ($\beta_\lambda \equiv 0$ for the generic coupling λ) and so there is no RG flow. Therefore, to be more realistic, we would develop the correspondence for the case of non-conformal field theories. This is possible if we consider theories which, in the UV, behave as deformations of some CFT. In such a way their action can be written as:

$$I = I_{\text{CFT}} + \int d^d x \, g\mathcal{O}, \tag{4.1.1}$$

where I_{CFT} is the action[1] of the original CFT and $\mathcal{O}$ is a conformal operator of dimension Δ which produces a deformation term with coupling g. Such term breaks the conformal invariance inducing a running with the energy scale of the parameter g. Depending on the dimension of $\mathcal{O}$ the behavior of the deformation changes. Indeed, if $\Delta < d$, the deformation is *relevant* and it grows stronger while going to the IR along the RG flow. On the contrary, if $\Delta > d$, the deformation is called *irrelevant* and it grows as the energy increases. Instead, if $\Delta = d$ the deformation is called *marginal* and the action I is still conformal at classical level. In the latter case the symmetry breakdown can occur only considering loop contributions of order g^2 or higher. Depending on the anomalous dimension $\gamma_{\mathcal{O}}$ of $\mathcal{O}$ introduced by the loop corrections, the marginal deformation could have very different behaviors. If $\gamma_{\mathcal{O}} < 0$ it is *marginally relevant* and behaves as a relevant perturbation, if $\gamma_{\mathcal{O}} > 0$ it is *marginally irrelevant* and behaves as an irrelevant one, instead if $\gamma_{\mathcal{O}} \equiv 0$ at every order it is *exactly marginal* and the conformal invariance is preserved also at quantum level ($\beta_g \equiv 0$).

Considering the holographic description of the bulk theory dual to the CFT described by I_{CFT}, we can use the prescription (3.2.3). In the case of a scalar field ϕ we have:

$$\left\langle e^{N^2 \int \phi_0 \mathcal{O}} \right\rangle_{\text{CFT}} = \mathcal{Z}_{AdS} = \int_{\phi \underset{z \mapsto 0}{\sim} z^{\Delta_-} \phi_0} \mathcal{D}\phi \, e^{-\kappa^{-2} S_{AdS}[\phi]}. \tag{4.1.2}$$

The boundary condition in the partition function $\mathcal{Z}_{AdS}$ of the AdS theory determine a deformation on the dual CFT of the form:

$$I_{\text{CFT}} \mapsto I_{\text{CFT}} + N^2 \int d^d x \, \phi_0 \mathcal{O}, \tag{4.1.3}$$

so that the boundary value ϕ_0 is naturally interpreted as the coupling of the deformation term proportional to $\mathcal{O}$.

This holds true also in the classical gravity limit of the duality, where the tree level bulk theory is dual to the large N limit of the CFT described by I_{CFT}. In this case the form of the field in the bulk can be determined by the solutions of the free e.o.m. and reads:

$$\phi = z^{\Delta_-} (\phi_0 + O(z)) + z^{\Delta_+} (A + O(z)), \tag{4.1.4}$$

where ϕ_0 is the boundary value which determines the deformation of the boundary CFT, whereas, as seen in (3.3.42), A represents the expectation value of the operator $\mathcal{O}$. This means that A specifies a choice of the vacuum of the field theory, determining the VEV of $\mathcal{O}$.

Thanks to the IR/UV connection we can identify the radial coordinate z of AdS with the inverse of the energy scale of the dual QFT. In this picture the boundary CFT described by I_{CFT} is supposed to represent a UV fixed point of the deformed QFT, which is the

[1] In this chapter, to avoid any confusion, we will denote the action of the bulk theory with S, whereas the action of the dual field theory with I.

starting point of the RG flow of the deformation term. Indeed, going down to the IR, the perturbation is switched on and it runs along the RG flow. Since the coupling of this deformation is ϕ_0, its evolution along the energy scale has to be associated to the evolution of ϕ along the radial axis of the bulk metric, so that the solution of the bulk e.o.m. with boundary conditions specified by ϕ_0 encodes the informations about the running behavior of the associated deformation of the dual QFT.

From the analysis of the dimension of the operator $\mathcal{O}$, we note that this makes sense only if the deformation is relevant (or marginally relevant). Indeed, recalling that the dimension of $\mathcal{O}$, as given in (3.3.17), is $\Delta_+ = d/2 + \nu$, where ν depends directly on the mass of the dual scalar field ϕ in the bulk, we have that $\mathcal{O}$ is relevant if $-d^2/4 \leq m^2 < 0$, marginal for $m^2 = 0$ and irrelevant for $m^2 > 0$. Therefore, if the deformation is irrelevant the dual scalar field has positive squared mass and is characterized by $\Delta_- < 0$, which implies, through (4.1.4), that ϕ diverges on the boundary. This fact means that the use of the free e.o.m. to determine the boundary behavior of ϕ is not self-consistent, because the diverging behavior requires a non-linearized description of the near-to-boundary regime of the bulk theory. This means that in the QFT-side the partition function of the field theory in the l.h.s. of (4.1.2) is not well-defined and so the description in terms of a flow starting in the UV from a fixed point breaks down. This is consistent with the fact that an irrelevant perturbation grows stronger in the UV and invalidates the description of the theory through a deformation of a CFT which we started with.

On the other hand, considering a relevant perturbation, the dual field ϕ in the bulk is characterized by $\Delta_- > 0$, thus it vanishes on the boundary. This means that the use of linearized solutions of the e.o.m. is consistent in the UV and so the partition function of the dual QFT with the deformation generated by ϕ_0 is well-defined through the duality. Therefore the latter describes the RG flow of a QFT starting from a UV fixed point and evolving to the IR through a deformation generated by the boundary conditions of the fields in the bulk. This works also for marginally relevant deformations. For this reason we will explicitly focus only on relevant and marginally relevant deformations, dropping the case of irrelevant ones.

Since the RG flow generates a running behavior of the couplings, the conformal invariance is broken in the dual QFT. This fact implies that also the isometries of AdS_{d+1} are broken and so the bulk metric is no longer that of AdS_{d+1}, but is deformed. However, the flow has a UV fixed point characterized by I_{CFT}, where the conformal invariance is not broken. This means that the bulk geometry has to be asymptotic to AdS_{d+1} in the IR region, so that it has an AdS boundary where bulk fields behave as (4.1.4) and the duality (4.1.2) can be defined. Assuming that the deformation breaks the conformal invariance but preserves the d-dimensional Poincaré invariance, so that the dual theory is a well-defined relativistic

QFT, we can parametrize the bulk metric by:

$$\begin{aligned} ds^2 &= u^2(z)dz^2 + h_{\mu\nu}(z)dx^\mu dx^\nu \\ h_{\mu\nu}(z) &= v^2(z)\delta_{\mu\nu}, \end{aligned} \tag{4.1.5}$$

where z is the usual radial coordinate and the boundary corresponds to $z = 0$. Approaching the region where z is small, the metric must take an AdS form, thus:

$$u, v \underset{z\mapsto 0}{\sim} z^{-1}. \tag{4.1.6}$$

This is an asymptotic AdS_{d+1} metric ($aAdS_{d+1}$) parametrized by $(z, x) \in \mathbb{R}^{d+1}$ and depending on u and v, which are functions[2] of z.

Considering the bulk theory on this metric does not change the prescription (4.1.2) because the presence of the AdS boundary ensures that the boundary behavior of the bulk theory is the same. For this reason we refer to this duality as the asymptotic AdS/deformed CFT correspondence ($aAdS$/dCFT).

When dealing with a bulk theory in an asymptotic AdS space, the application of almost all the AdS/CFT technology is straightforward, but there is an important aspect which is different from the pure AdS case. That is the relation between a radial cut-off $z = \epsilon$ in the bulk and the energy scale Λ of the dual QFT. In the case of pure AdS_{d+1} this relation is $\Lambda \equiv \epsilon^{-1}$, which means that the radial cut-off in the bulk is the inverse of the energy scale of the dual QFT. The generalization to $aAdS_{d+1}$ is much more subtle and a precise connection has not yet been determined. Some attempts to define an approximate relation between ϵ and Λ have been made, such as in [23, 30], but these are euristic arguments and the topic is still an important open problem. Nevertheless, it is clear that near the boundary, where the metric reduces to AdS_{d+1}, the connection between the radial cut-off and the energy scale reduces to the pure AdS relation.

Considering the technology described in the previous chapter, we can apply the holographic method to the $aAdS$/dCFT case for the computation of the correlation functions for the deformed CFT. Indeed, the correlators of the operator $\mathcal{O}$ when the deformation specified by ϕ_0 is switched on are defined as:

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{\phi_0} := \frac{\left\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)e^{N^2\int\phi_0\mathcal{O}}\right\rangle_{\text{CFT}}}{\left\langle e^{N^2\int\phi_0\mathcal{O}}\right\rangle_{\text{CFT}}}. \tag{4.1.7}$$

Therefore, using the classical gravity limit prescription (3.2.9), we can compute the large N connected correlation functions of $\mathcal{O}$ through the derivatives of the on-shell renormalized

[2] In general u and v are functions of both z and x, but the d-dimensional Poincaré invariance requires them to be independent of the coordinates x.

bulk action $\kappa^{-2}S_{\text{ren}}$:

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{\phi_0,\text{conn}} = -\frac{1}{N^{2n}}\left.\frac{\delta^n\kappa^{-2}S_{\text{ren}}}{\delta\phi_0(x_1)\cdots\delta\phi_0(x_n)}\right|_{\phi_0}, \tag{4.1.8}$$

where ϕ_0 is set to be equal to the value of the effective coupling at the scale we are interested in. Setting $\phi_0 = 0$, one recovers the correlation functions of the boundary CFT.

This method exploits the holographic renormalization, which provides a finite renormalized action S_{ren} through the introduction of a local boundary counterterm $S_{\text{c.t.}}$. The removal of IR divergences of the on-shell bulk action corresponds to removing the UV divergences in the dual QFT, so that the renormalized bulk theory described by S_{ren} is dual to the renormalized QFT on the boundary. In such a way this correspondence is given by the prescription:

$$\left\langle e^{N^2\int\phi_0\mathcal{O}}\right\rangle_{\text{CFT}} = \lim_{\epsilon_0\mapsto 0}\int_{\phi(\epsilon_0)\sim\epsilon_0^{\Delta_-}\phi_0}\mathcal{D}\phi\, e^{-\kappa^{-2}S[\phi]-\kappa^{-2}S_{\text{c.t.}}[\phi]}, \tag{4.1.9}$$

where the r.h.s. represents the partition function of the renormalized bulk theory and ϵ_0 is some IR cut-off on the radial axis.

Many works have developed this aspect, such as [26–28]. In particular it has been shown that the correlation functions computed using (4.1.9), i.e. after the holographic renormalization, satisfy the standard Callan-Symazik equations of the dual QFT. Moreover the RG flow of the deformation couplings can be described in terms of the evolution of the bulk along the radial coordinate. In this picture the bulk fields are interpreted as the running coupling of the deformation terms and their e.o.m.'s represent the RG equations of the deformation couplings. For example, the Hamilton equations for a scalar bulk field ϕ are the RG equations for the deformation coupling ϕ_0:

$$z\frac{d}{dz}\phi = \left.\frac{\delta\mathcal{H}}{\delta\pi}\right|_{\pi=\frac{\delta S}{\delta\phi}} \qquad \longleftrightarrow \qquad \Lambda\partial_\Lambda\phi = \beta_\phi(\Lambda).$$

In the next section we will develop in detail a Wilsonian approach to the holographic RG, in particular, working on a scalar bulk theory.

4.2. *Wilsonian approach to holographic RG*

The development of the $aAdS$/dCFT correspondence improved the use of renormalization methods based on holography. Besides the results mentioned above, a Wilsonian approach has been recently proposed in [30, 31] and later developed in other works, such as [32–34].

In this section we will develop this approach following the original works and focus on the holographic computation of the Wilsonian action of the boundary QFT.

The fact that the radial coordinate z of the bulk is connected to the energy scale of the dual QFT suggests the existence of a relation between the radial evolution of the bulk theory and the Wilsonian RG flow of the dual field theory. This idea is developed reproducing for the radial coordinate in the bulk theory the Wilsonian procedure of separating the functional integration into a UV and a IR part at a certain cut-off and integrating out the UV modes in order to obtain a non-local functional which modify the bare action into the scale-dependent effective action.

Therefore consider a bulk theory in $aAdS_{d+1}$ with action S_0. For simplicity let us restrict only on scalar fields ϕ^i, thus the partition function defined with appropriate boundary conditions reads:

$$\mathcal{Z} = \int_{\phi^i \underset{z\mapsto 0}{\sim} z^{\Delta_-}\phi_0^i} \mathcal{D}\phi\, e^{-\kappa^{-2}S_0[\phi]}, \tag{4.2.1}$$

Let us introduce in the z-axis of $aAdS_{d+1}$ a IR cut-off $\epsilon_0 \ll 1$ which determines a regulated boundary. The action S_0 is regularized by this parameter and treated as the bare action of the bulk theory. On the QFT-side, ϵ_0 is identified with the UV cut-off $\Lambda_0 = \epsilon_0^{-1}$ of the boundary theory. Now define a new radial cut-off $\epsilon \geq \epsilon_0$ which is meant to be the floating cut-off running along the z-axis. This is dual to the floating energy scale $\Lambda(\epsilon)$ of the boundary QFT but the precise relation which connects them depends on the form of the bulk metric (4.1.5). We take ϵ as the reference scale on which we want to develop the duality. Therefore we can split the functional measure of the partition function $\mathcal{Z}$ into three factors:

$$\mathcal{Z} = \int_{\phi^i(\epsilon_0)\sim\epsilon_0^{\Delta_-}\phi_0^i} \mathcal{D}\phi_{(z<\epsilon)}\mathcal{D}\phi_\epsilon\mathcal{D}\phi_{(z>\epsilon)}\, e^{-\kappa^{-2}S_0[\phi]_{z<\epsilon}-\kappa^{-2}S_0[\phi]_{z>\epsilon}}, \tag{4.2.2}$$

where $\phi_\epsilon^i \coloneqq \phi^i(z=\epsilon)$ are the bulk fields evaluated at $z=\epsilon$. Now we define the amplitudes:

$$\Psi_{\mathrm{UV}}(\phi_\epsilon) \coloneqq \int_{\phi^i(\epsilon_0)\sim\epsilon_0^{\Delta_-}\phi_0^i}^{\phi^i(\epsilon)=\phi_\epsilon^i} \mathcal{D}\phi_{(z<\epsilon)}\, e^{-\kappa^{-2}S_0[\phi]_{z<\epsilon}} \tag{4.2.3}$$

$$\Psi_{\mathrm{IR}}(\phi_\epsilon) \coloneqq \int_{\phi^i(\epsilon)=\phi_\epsilon^i} \mathcal{D}\phi_{(z>\epsilon)}\, e^{-\kappa^{-2}S_0[\phi]_{z>\epsilon}}, \tag{4.2.4}$$

where Ψ_{IR} is called IR amplitude because it contains the information about the behavior of the theory in the interior of the bulk, which is dual to the IR regime of the boundary QFT. Instead Ψ_{UV} is called UV amplitude because it contains the information about the near-to-boundary behavior of the bulk theory, which corresponds to the UV regime of the dual QFT. Looking at these definitions, Ψ_{UV} is a functional defined with two boundary conditions. The one imposed at $z=\epsilon_0$ is the proper boundary condition which specifies the behavior of the bulk fields on the boundary ($z=0$), whereas the other one, defined at $z=\epsilon$ and specified by ϕ_ϵ^i, imposes the value of the fields at the floating cut-off scale. For this reason the information about the boundary behavior of the theory (UV data) is contained in Ψ_{UV}. On the other side Ψ_{IR} is defined with the same condition imposed for ϕ^i at $z = \epsilon$ as Ψ_{UV}.

Through these definitions the total partition function of the bulk theory becomes:

$$\mathcal{Z} = \int \mathcal{D}\phi_\epsilon \, \Psi_{\mathrm{UV}}(\phi_\epsilon)\Psi_{\mathrm{IR}}(\phi_\epsilon). \tag{4.2.5}$$

The separation of modes at the floating cut-off can be seen as reflecting the Wilsonian treatment of the dual QFT. Therefore, following [30], we assume that the IR amplitude Ψ_{IR} is equal to the partition function of the dual theory with cut-off at $\Lambda(\epsilon)$, thus:

$$\Psi_{\mathrm{IR}}(\phi_\epsilon) = \int \mathcal{D}M \, e^{-I^{(\Lambda)}_{\mathrm{CFT}}[M] + N^2 \int \epsilon^{-\Delta_-} \phi^i_\epsilon \mathcal{O}_i}, \tag{4.2.6}$$

where M represents the generic matrix field of the dual QFT and $I^{(\Lambda)}_{\mathrm{CFT}}$ is the undeformed action of the boundary theory with cut-off $\Lambda(\epsilon)$. The operators $\mathcal{O}_i$ associated to the bulk fields ϕ^i are single-trace operators with normalization such that the deformation term is of order $O(N^2)$, thus:

$$\mathcal{O}_i = \frac{\mathrm{tr}\,(\cdots)}{N},$$

where the dots represent products of the field M and its derivatives.

The equation (4.2.6) is taken in [30] as a postulate which represents the key point to develop the holographic Wilsonian approach. Starting from this and inserting (4.2.6) in the total bulk partition function (4.2.5) we obtain:

$$\mathcal{Z} = \int \mathcal{D}M \, e^{-I^{(\Lambda)}_{\mathrm{CFT}}[M]} \int \mathcal{D}\phi_\epsilon \, \Psi_{\mathrm{UV}}(\phi_\epsilon) \, e^{N^2 \int \epsilon^{-\Delta_-} \phi^i_\epsilon \mathcal{O}_i}. \tag{4.2.7}$$

At this point, the integral over ϕ_ϵ is interpreted as the Wilsonian action I_Λ at the scale $\Lambda(\epsilon)$ of the dual QFT:

$$e^{-I_\Lambda[\mathcal{O}]} = \int \mathcal{D}\phi_\epsilon \, \Psi_{\mathrm{UV}}(\phi_\epsilon) \, e^{N^2 \int \epsilon^{-\Delta_-} \phi^i_\epsilon \mathcal{O}_i}, \tag{4.2.8}$$

which means that:

$$\mathcal{Z} = \int \mathcal{D}M \, e^{-I^{(\Lambda)}_{\mathrm{CFT}}[M] - I_\Lambda[\mathcal{O}]} := \int \mathcal{D}M \, e^{-I_{\mathrm{eff}}[M,\mathcal{O}]}. \tag{4.2.9}$$

In such a way the total bulk partition function represents to the partition function of the dual renormalized QFT described in Wilsonian approach by the effective action $I_{\mathrm{eff}} = I^{(\Lambda)}_{\mathrm{CFT}} + I_\Lambda$.

The Wilsonian action I_Λ at the scale $\Lambda(\epsilon)$ is then connected to the UV amplitude Ψ_{UV} of the bulk theory defined at the separation scale ϵ through the integral transform (4.2.8), which means that moving ϵ along the z-axis corresponds to considering the dual theory at the scale Λ.

Analyzing this relation, Ψ_{UV} is a non-local functional due to the integration of propagating bulk modes confined in the region $\epsilon_0 \leq z < \epsilon$. This extends to the Wilsonian action through the integral transform (4.2.8), so that I_Λ can be seen as containing non-local contributions coming from integrated modes above $\Lambda(\epsilon)$. As we have seen in subsection 1.1.1, I_Λ is then expected to be a functional localized on the scale $\Lambda(\epsilon)$ and containing general multi-trace terms coming from the expansion of the non-local contributions.

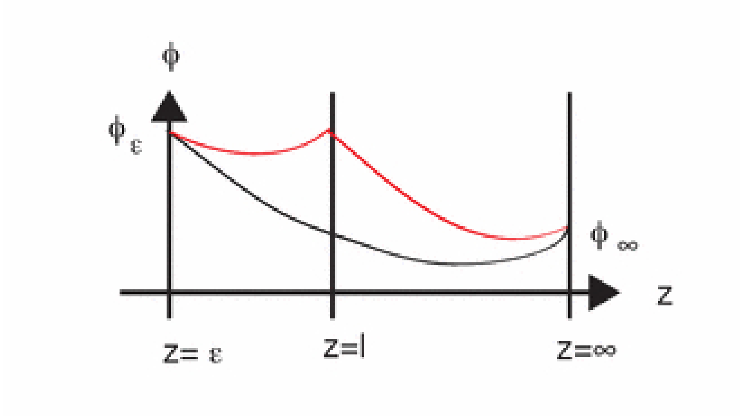


Figure 4.2.1. *Saddle point evaluation of the total partition function* (4.2.5)*. The red line is the result of the separate computation of* $\Psi_{\rm UV}$ *and* $\Psi_{\rm IR}$*. The black line is the correct procedure obtained minimizing the amplitudes with respect to* ϕ^i_ϵ*.*

Considering the classical gravity approximation, one would be led to compute $\Psi_{\rm IR}$ and $\Psi_{\rm UV}$ on their saddle point and evaluating them on the solutions of the bulk e.o.m. with boundary conditions specified by the limits of the functional integrals in (4.2.4) and (4.2.3). Therefore $\Psi_{\rm IR}$ would be determined by a solution satisfying a regularity condition at $z = +\infty$ and the boundary condition $\phi^i(\epsilon) = \phi^i_\epsilon$, whereas $\Psi_{\rm UV}$ would be the solution that satisfies the same condition as $\Psi_{\rm IR}$ at $z = \epsilon$ and the boundary condition $\phi^i(\epsilon_0) \sim \epsilon_0^{\Delta_-}\phi^i_0$ on the *AdS* boundary. However, this way of proceeding is not correct because, for arbitrary values of ϕ^i_ϵ, the union of the two solutions is in general not smooth at $z = \epsilon$ (red line in Figure 4.2.1).

Instead of evaluating separate saddle points for $\Psi_{\rm IR}$ and $\Psi_{\rm UV}$, the smooth solution (black line in Figure 4.2.1) is obtained taking the saddle point directly in (4.2.5), i.e. varying ϕ^i_ϵ to have a stationarized functional. In this case the solution is smooth and depends only on the boundary conditions specified by ϕ^i_0 and the regularity conditions at $z = +\infty$. This means that the Wilsonian action of the large N dual QFT is determined by the saddle point of the integral transform of (4.2.8). As we will see explicitly in the next subsection, also the large N Wilsonian action contains multi-trace terms at the leading order in the $1/N$ expansion. Thus the planar RG flow of the dual QFT is described also by non-planar multi-trace amplitudes.

The evolution equation of the Wilsonian action I_Λ can be obtained using the fact that the total bulk partition function is independent of the floating cut-off scale ϵ. Therefore, taking the derivative of $\mathcal{Z}$ with respect to ϵ and setting it to zero, we have:

$$0 = \partial_\epsilon \mathcal{Z}|_{\phi_0} = \partial_\epsilon \left\langle e^{-I_\Lambda[\mathcal{O}]} \right\rangle_{\Lambda(\epsilon)}, \tag{4.2.10}$$

where the brackets $\langle\cdot\rangle_{\Lambda(\epsilon)}$ represent the path integral of the undeformed theory with cut-off $\Lambda(\epsilon)$. Working with this equation is unwieldy since the derivative ∂_ϵ acts both on the exponential and on the cut-off of the partition function. However there is an alternative and more useful method to derive it, based on a hamiltonian formalism. Considering the radial coordinate z as a "time" coordinate, we can define a radial hamiltonian for the bulk theory:

$$H(\phi,\pi) := \int d^dx\,\Big(-\pi^i\partial_z\phi^i + \sqrt{g}\mathcal{L}(\phi^i,\partial\phi^i)\Big), \tag{4.2.11}$$

where $\mathcal{L}$ is the lagrangian density and π^i is the conjugate momentum associated to ϕ^i:

$$\pi^i := \frac{\partial\sqrt{g}\mathcal{L}}{\partial\partial_z\phi^i}. \tag{4.2.12}$$

Now, in a quantum mechanical formalism the path integral of the theory represents the transition amplitude between two boundary states:

$$\mathcal{Z} = \int_{\phi^i(\epsilon_0)\sim\epsilon_0^{\Delta-}\,\phi_0^i} \mathcal{D}\phi\, e^{-\kappa^{-2}S_0[\phi]} = \langle\mathrm{IR}|\,U(\infty,\epsilon_0)\,|\phi_0\rangle\,, \tag{4.2.13}$$

where $|\phi_0\rangle$ is the boundary state at $z=\epsilon_0$, specified by the boundary conditions, and $|\mathrm{IR}\rangle$ is the initial state, specified by the regularity conditions at $z=+\infty$. The operator $U(\infty,\epsilon_0)$ is the radial evolution operator:

$$U(\epsilon_2,\epsilon_1) = \mathrm{T}\exp\Big(-\kappa^{-2}\int_{\epsilon_1}^{\epsilon_2} dz\,H\Big), \tag{4.2.14}$$

which is the T-ordered exponential of the radial hamiltonian, as in canonical quantization. In this formalism the separation of the path integral at $z=\epsilon$ corresponds to insert the completeness of an intermediate state $|\phi_\epsilon\rangle$ in the amplitude:

$$\langle\mathrm{IR}|\,U(\infty,\epsilon_0)\,|\phi_0\rangle = \int\mathcal{D}\phi_\epsilon\,\langle\mathrm{IR}|\,U(\infty,\epsilon)\,|\phi_\epsilon\rangle\,\langle\phi_\epsilon|\,U(\epsilon,\epsilon_0)\,|\phi_0\rangle\,, \tag{4.2.15}$$

so that Ψ_{IR} and Ψ_{UV} represent the intermediate transition amplitudes:

$$\Psi_{\mathrm{IR}} = \langle\mathrm{IR}|\,U(\infty,\epsilon)\,|\phi_\epsilon\rangle \qquad\qquad \Psi_{\mathrm{UV}} = \langle\phi_\epsilon|\,U(\epsilon,\epsilon_0)\,|\phi_0\rangle\,. \tag{4.2.16}$$

In such a way their evolution along the scale ϵ is determined by radial Schrödinger equations:

$$\partial_\epsilon\Psi_{\mathrm{IR}}(\phi_\epsilon) = \kappa^{-2}H(\phi_\epsilon,\pi_\epsilon)\Psi_{\mathrm{IR}}(\phi_\epsilon) \tag{4.2.17}$$

$$\partial_\epsilon\Psi_{\mathrm{UV}}(\phi_\epsilon) = -\kappa^{-2}H(\phi_\epsilon,\pi_\epsilon)\Psi_{\mathrm{UV}}(\phi_\epsilon), \tag{4.2.18}$$

where $\pi_\epsilon^i = -i\kappa^2\frac{\delta}{\delta\phi_\epsilon^i}$ is the conjugate momentum associated to ϕ_ϵ^i.

These equations determine the evolution along the scale ϵ of the amplitudes of the bulk theory. Therefore taking the derivative ∂_ϵ in (4.2.8) and using (4.2.18), we can obtain the

RG flow equation of the Wilsonian action. By doing so, we have:

$$\begin{aligned}\mathcal{D}_\epsilon e^{-I_\Lambda[\mathcal{O}]} &= \int \mathcal{D}\phi_\epsilon \, \partial_\epsilon \Psi_{\mathrm{UV}} \, e^{N^2 \int \epsilon^{-\Delta_-} \phi_\epsilon^i \mathcal{O}_i} \\ &= \int \mathcal{D}\phi_\epsilon \Big[-\kappa^{-2} H(\phi_\epsilon, \pi_\epsilon) \Psi_{\mathrm{UV}}\Big] e^{N^2 \int \epsilon^{-\Delta_-} \phi_\epsilon^i \mathcal{O}_i},\end{aligned} \tag{4.2.19}$$

where:

$$\mathcal{D}_\epsilon := \partial_\epsilon + \frac{\Delta_-}{\epsilon} \int d^d x \, \mathcal{O}_i \frac{\delta}{\delta \mathcal{O}_i}. \tag{4.2.20}$$

Now, thanks to the properties of the functional derivatives, we can perform some substitution in the integrand of the r.h.s. of (4.2.19):

$$\begin{aligned}\phi_\epsilon^i \quad &\longrightarrow \quad \kappa^2 \epsilon^{\Delta_-} \frac{\delta}{\delta \mathcal{O}_i} \\ \pi_\epsilon^i = -i\kappa^2 \frac{\delta}{\delta \phi_\epsilon^i} \quad &\longrightarrow \quad i\epsilon^{-\Delta_-} \mathcal{O}_i,\end{aligned}$$

In such a way the equation becomes:

$$\mathcal{D}_\epsilon e^{-I_\Lambda[\mathcal{O}]} = -\kappa^{-2} H\left(\kappa^2 \epsilon^{\Delta_-} \frac{\delta}{\delta \mathcal{O}}, i\epsilon^{-\Delta_-} \mathcal{O}\right) e^{-I_\Lambda[\mathcal{O}]}. \tag{4.2.21}$$

This is RG flow equation for the Wilsonian action I_Λ of the dual QFT. It represents the main result derived in [30] because it describes the evolution of I_Λ along the RG flow, so that it can be compared to the ordinary equations obtained in the QFT-side.

The classical gravity limit of the bulk theory is reached by $\kappa^2 \mapsto 0$. In this regime the on-shell bulk theory is dual to the large N limit of the boundary QFT. This means that, applying this limit to (4.2.21), we obtain the RG equation in the planar limit:

$$\mathcal{D}_\epsilon \widetilde{I}_\Lambda = H\left(-\epsilon^{\Delta_-} \frac{\delta \widetilde{I}_\Lambda}{\delta \mathcal{O}}, i\epsilon^{-\Delta_-} \mathcal{O}\right), \tag{4.2.22}$$

where $I_\Lambda[\mathcal{O}] := \kappa^{-2} \widetilde{I}_\Lambda[\mathcal{O}]$. This is a Hamilton-Jacobi-like equation which represents the RG equation of I_Λ in the planar limit. We would like to have a matching between this equation and the planar WPE obtained in chapter 2 for a general scalar matrix theory in d dimensions. As we will see in the next chapter, the identification is not clear and requires a careful analysis.

In this section we have seen that the procedure of separating the functional integral of the bulk theory at a certain cut-off and integrating out the near-to-boundary part produces an holographic description of the Wilsonian treatment for the dual QFT. In particular it provides the Wilsonian action and its RG equation, so that we can determine the RG flow of the boundary theory. For this reason we call this method holographic Wilsonian renormalization group (hWRG).

Briefly summarizing the meaning of this procedure we can see the connection with the holographic renormalization. Indeed the starting point is that the standard AdS/CFT prescription is affected by divergences and so it has to be renormalized. Holographic renormalization represents the standard renormalization for the dual QFT, so it provides a recipe (4.1.9) which connects the renormalized theories and yields consistent results.

Like the ordinary Wilsonian treatment in QFT, the hWRG method outlined here develops a different point of view. The bulk action S_0, regularized at the IR cut-off ϵ_0, is treated as bare action. Thus, moving from ϵ_0 to a floating scale ϵ along the z-axis, we integrate out the field-modes in the interval $[\epsilon_0, \epsilon]$, so that the partition function is split into a factor containing the bare action S_0 with cut-off lowered from ϵ_0 to ϵ (determined by Ψ_{IR}) and a factor containing the integrated part of the theory (Ψ_{UV}). The latter term corresponds holographically to the contribution of the integrated modes of the dual QFT with momentum above $\Lambda(\epsilon)$ and provides, through (4.2.8), the Wilsonian action of the dual theory defined at the scale $\Lambda(\epsilon)$. One of the crucial aspects of this procedure is that the scale ϵ, where the separation of the path integral is taken, corresponds to the cut-off scale $\Lambda(\epsilon)$ at which the dual QFT is considered.

Following this point of view, the prescription (4.1.9), provided by holographic renormalization, can be seen as determining the duality where the separation is taken at $\epsilon_0 \mapsto 0$, thus where the boundary QFT has a fixed point. Therefore the boundary term $S_{\text{c.t.}}$ added to S_0 to remove its divergences represents in hWRG formalism the UV amplitude evaluated at ϵ_0:

$$\Psi_{\text{UV}}(\epsilon_0) = e^{-\kappa^{-2} S_{\text{c.t.}}[\phi]}. \tag{4.2.23}$$

This is the connecting point between hWRG and holographic renormalization. In particular it is clear that Wilsonian treatment extends the duality at every scale of the bulk metric, including the other procedure in a more general method.

4.2.1. Free massive scalar field in the bulk

In this subsection we perform hWRG in the explicit example of a free real scalar field ϕ in the bulk with negligible backreaction and metric (4.1.5), taking for simplicity $u = z^{-1}$, [30, 31]. The action is defined as:

$$S_0 = \frac{1}{2} \int dz d^d x \sqrt{g} \Big(g^{mn} \partial_m \phi \partial_n \phi + m^2 \phi^2 \Big), \tag{4.2.24}$$

so that the radial hamiltonian at $z = \epsilon$ reads:

$$H = \frac{1}{2} \int d^d x \, \epsilon^{-1} \Big[v^{-d} \pi_\epsilon^2 + v^d \big(v^{-2} (\partial_\mu \phi_\epsilon)^2 + m^2 \phi_\epsilon^2 \big) \Big]. \tag{4.2.25}$$

The $aAdS$/dCFT correspondence states that ϕ is dual to a single-trace operator $\mathcal{O}$ of the boundary QFT with dimension $\Delta_+ = d/2 + \nu$. The boundary conditions (4.1.4) specify that the boundary value ϕ_0 is the coupling of the single-trace deformation induced in the dual field theory.

Now, applying the holographic Wilsonian procedure at the scale ϵ, we can assume that the UV amplitude can be written as:

$$\Psi_{\mathrm{UV}}(\phi_\epsilon) = e^{-\kappa^{-2} S_{\mathrm{UV}}[\phi_\epsilon]}, \tag{4.2.26}$$

where S_{UV} is a functional of ϕ_ϵ which contains the integrated modes of S_0 in the region $\epsilon_0 \leq z < \epsilon$. Since the bulk action is quadratic in the field ϕ, we can reasonably consider the expansion of S_{UV} to contain terms at most quadratic in ϕ_ϵ, thus we can write:

$$S_{\mathrm{UV}} = C(\epsilon) - \int \frac{d^d k}{(2\pi)^d} \sqrt{h} J(-k,\epsilon)\phi_{\epsilon k} + \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \sqrt{h} F(k,\epsilon)\phi_{\epsilon k}\phi_{\epsilon -k}, \tag{4.2.27}$$

where the parameters C, J and F determine the evolution of S_{UV} along the scale ϵ.

Now, taking the classical gravity limit of the duality $\kappa^2 \mapsto 0$, we obtain the large N limit of the dual QFT and the evolution equation (4.2.18) for the UV amplitude becomes:

$$\partial_\epsilon S_{\mathrm{UV}} = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \frac{R}{\epsilon}\left[-v^{-d}\frac{\delta S_{\mathrm{UV}}}{\delta \phi_{\epsilon k}}\frac{\delta S_{\mathrm{UV}}}{\delta \phi_{\epsilon -k}} + v^d (v^{-2}k^2 + m^2)\phi_{\epsilon k}\phi_{\epsilon -k}\right]. \tag{4.2.28}$$

From this we can obtain the evolution equations of the parameters of S_{UV}. Focusing on J and F, we have:

$$\frac{1}{\sqrt{g}}\partial_\epsilon\big(\sqrt{h}J(k,\epsilon)\big) = -J(k,\epsilon)F(k,\epsilon) \tag{4.2.29}$$

$$\frac{1}{\sqrt{g}}\partial_\epsilon\big(\sqrt{h}F(k,\epsilon)\big) = -F^2(k,\epsilon) + (v^{-2}k^2 + m^2). \tag{4.2.30}$$

These equations describe the evolution of J and F with the scale ϵ (see appendix C for the solutions).

Now, once specified the form of the UV amplitude Ψ_{UV} of the free bulk scalar theory and the evolution for its parameters, we consider the Wilsonian action of the dual QFT. In this analysis let us restrict to pure AdS_{d+1} for simplicity. Inserting the explicit expression of Ψ_{UV} in (4.2.8) we can obtain I_Λ:

$$e^{-I_\Lambda[\mathcal{O}]} = \int \mathcal{D}\phi_\epsilon \, e^{-\kappa^{-2} S_{\mathrm{UV}}[\phi_\epsilon] + N^2 \int \epsilon^{-\Delta_-}\phi_\epsilon \mathcal{O}}. \tag{4.2.31}$$

Due to the form (4.2.27) of S_{UV}, we can reorganize the integral over ϕ_ϵ into a Gaussian integral and then explicitly perform it. In such a way, neglecting the field-independent terms arising from the integration, the Wilsonian action is:

$$I_\Lambda[\mathcal{O}] = N^2 \int \frac{d^d k}{(2\pi)^d}\Big(\epsilon^{-\Delta_-}\lambda(-k,\epsilon)\mathcal{O}_k + \frac{1}{2}\epsilon^{2\nu} f(k,\epsilon)\mathcal{O}_k\mathcal{O}_{-k}\Big), \tag{4.2.32}$$

where λ_k and f are the effective couplings of the dual QFT, defined by:

$$\lambda(k,\epsilon) := -\frac{J(k,\epsilon)}{F(k,\epsilon)} \tag{4.2.33}$$

$$f(k,\epsilon) := -\frac{1}{F(k,\epsilon)}. \tag{4.2.34}$$

Therefore λ is a single-trace dimensionless coupling which represents the evolution of the initial deformation. Instead f is a double-trace dimensionless coupling generated along the flow. The RG equations of λ and f can be obtained from the evolution equations of J and F. Indeed, using (4.2.29) and (4.2.30), we have that:

$$\epsilon\partial_\epsilon\lambda(k,\epsilon) = (\epsilon^2k^2+m^2)\lambda(k,\epsilon)f(k,\epsilon) \tag{4.2.35}$$

$$\epsilon\partial_\epsilon f(k,\epsilon) = (\epsilon^2k^2+m^2)f^2(k,\epsilon) - df(k,\epsilon) - 1. \tag{4.2.36}$$

These equations determine the evolution along the energy scale ($\Lambda = \epsilon^{-1}$ in pure AdS_{d+1}) of the effective couplings in the large N limit. Solving them we can determine the planar RG flow of the dual QFT.

First of all, if we focus on the small momentum regime ($\epsilon k \ll 1$), we can note that f has two fixed points ($\epsilon\partial_\epsilon f = 0$):

$$f_\pm := \frac{\Delta_\pm}{m^2} = -\frac{1}{\Delta_\mp}, \tag{4.2.37}$$

where the QFT gains conformal invariance. On these points the single-trace coupling λ behaves as:

$$\lambda \sim \epsilon^{\Delta_\pm}. \tag{4.2.38}$$

This means that on f_- the operator $\mathcal{O}$ has dimension Δ_+. Thus this fixed point corresponds to the UV fixed point of the boundary field theory determined by the AdS/CFT prescription specified by the boundary value ϕ_0 (standard quantization). On the other hand, on f_+ the operator $\mathcal{O}$ has dimension Δ_-. This corresponds to a UV fixed point of the boundary QFT in a different definition of the correspondence, called alternative quantization [29, 31]. The latter is defined choosing as boundary value of ϕ the mode A going as z^{Δ_+} instead ϕ_0 as in the standard case. In such a way the dual field theory is different from that of the standard quantization because the dual operators $\mathcal{O}$ have different dimension. This definition is consistent only for $\nu \in (0,1)$, because for $\nu \geq 1$ the unitarity of the dual theory is not guaranteed everywhere in the AdS space. After mentioning this alternative definition, henceforth we will keep developing only the standard quantization.

Imposing the latter quantization, the initial value of f must represent the UV fixed point f_-, thus we set $f(\epsilon_0) = f_-$. In such a way we can define:

$$f := \bar{f} + f_-, \tag{4.2.39}$$

where $\bar{f}$ is the actual double-trace deformation generated along the RG flow. Indeed, since f_- is the fixed point which determines the boundary CFT, its contribution is included in the definition of I_{CFT}. This means that the RG flow of the CFT described by I_{CFT} with the single-trace deformation specified by ϕ_0 contains also a double-trace deformation $\bar{f}$ generated along the flow. The RG equation of $\bar{f}$ can be obtained inserting (4.2.39) in the equation (4.2.36) for $\epsilon k \ll 1$, so that:

$$\epsilon \partial_\epsilon \bar{f} = m^2 \bar{f}^2 - 2\nu \bar{f}. \tag{4.2.40}$$

This result is important because, as we will see in the next chapter, it is exactly the double-trace β-function obtained with field theory computations for a large N CFT deformed by a single-trace term [36, 37].

Until now we have considered the RG flow of f only in the small momentum regime ($\epsilon k \ll 1$). However, since the general solution (see appendix C) guarantees that f is analytic also at $k = 0$ (localized function), we can make the expansion:

$$f(k, \epsilon) = \sum_{n=0}^{+\infty} f_n(\epsilon) k^{2n}, \tag{4.2.41}$$

where f_n is the effective coupling of the derivative operator $\mathcal{O}\partial^{2n}\mathcal{O}$ in I_Λ. In such a way the RG equation of f can be expanded in:

$$\epsilon \partial_\epsilon f_n = \epsilon^2 \sum_{m=0}^{n-1} f_m f_{n-m-1} + m^2 \sum_{m=0}^{n} f_m f_{n-m} - d f_n - \delta_{n,0}. \tag{4.2.42}$$

This equation describes the RG flow of the coupling f_n, which is influenced the other couplings f_m with $m \neq n$. In particular the r.h.s. shows that only parameters f_m with $m \leq n$ enters the equation for f_n, so an iterative solution is possible. In such a way the first term in the series (4.2.41), which is the effective coupling of $\mathcal{O}^2$, does not depends on the other parameters. Moreover, since its equation is exactly that of f for $\epsilon k \ll 1$, it describes the total deformation in such regime. Therefore we can actually define $\bar{f}$ as the actual coupling of $\mathcal{O}^2$ with RG equation (4.2.40).

In conclusion, this example shows that the dual QFT of a free single scalar theory in $aAdS_{d+1}$ for the standard quantization is a deformed CFT where the Wilsonian action contains a single- and a double-trace deformation. The latter is generated along the flow even though we started only from a single-trace deformation and it produces an infinite series of double-trace interactions proportional to the derivatives of $\mathcal{O}^2$. The RG flow is then closed on these deformations and no other multi-trace term is generated.

The main results of this part are the RG equations for the actual double-trace coupling $\bar{f}$ and for the n-th term f_n of the expansion (4.2.41), which represents the coupling of the n-th derivative term, proportional to $\mathcal{O}\partial^{2n}\mathcal{O}$.

In the next chapter we will analyze the results obtained from field theory methods in order to compare them with the hWRG ones. In particular we will see that, considering a deformed CFT defined through composite single-trace operators, like $\mathcal{O}$, the RG flow equations agree with the holographic result, whereas, using the formalism of [16] described in chapter 2 for matrix theories defined through elementary fields, the RG equations seem not to be consistent with the holographic ones.

CHAPTER 5

The holographic Wilsonian treatment and field theory methods

In the previous chapter we have describe a Wilsonian approach to holographic RG (hWRG), proposed by [30, 31]. This method enables us to compute the Wilsonian action of the dual QFT and its RG equation (4.2.21) holographically. In particular, considering the classical limit of the bulk theory, one can obtain the planar RG equation (4.2.22), which describes the RG flow of the dual QFT in the large N limit.

In chapter 2 we have studied the RG flow of d-dimensional matrix theories in Wilsonian approach. In particular we have described the Wilson-Polchinski RG equation (WPE) in the planar limit for the case of scalar matrices, first obtained in [16]. Like the finite N ordinary WPE, the planar equation in independent on the coupling regime of the theory, i.e. it holds true at both weak and strong coupling. Therefore it is expected to be valid for a general planar Wilsonian RG flow.

For this reason we are led to think that the planar hWRG equation obtained in chapter 4 has in general the same structure of the planar WPE. However the matching between the two equations seems not to be clear and our aim is to investigate this issue.

5.1. Comparing hWRG and WPE

Considering first the ordinary Wilsonian treatment in a matrix theory, we can sketch the structure of the planar WPE dropping for simplicity space-time labels and integrals:

$$\partial_t \mathcal{H}_t \sim -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\dot{P}_t\Big(\sum_{mn} mnY_{m+n-2}\frac{\delta\mathcal{H}_t}{\delta Y_m}\frac{\delta\mathcal{H}_t}{\delta Y_n} + \sum_m\sum_{i=1}^{m-1} mY_{i-1}Y_{m-i-1}\frac{\delta\mathcal{H}_t}{\delta Y_m}\Big), \qquad (5.1.1)$$

where $\mathcal{H}_t$ is the Wilsonian action of the theory at the cut-off scale $t = \Lambda^{-2}$, with Λ the actual energy scale of the QFT. Moreover, $\dot{P}_t$ is the derivative with respect to t of the propagator with cut-off, whereas Y_m are single-trace operators defined as the trace of the product of m elementary fields Φ normalized with the factor $1/N$. Therefore (5.1.1) is a Hamilton-Jacobi type equation where Y_m are treated as sources and the hamiltonian has the form:

$$H \sim Y\left(\frac{\delta \mathcal{H}_t}{\delta Y}\right)^2 + Y^2\left(\frac{\delta \mathcal{H}_t}{\delta Y}\right). \tag{5.1.2}$$

On the other hand, within hWRG, the dynamics of the bulk theory determines the evolution of the Wilsonian action $\widetilde{I}_\Lambda$ of the dual QFT. In such a way the planar RG equation is a Hamilton-Jacobi type equation:

$$\mathcal{D}_\epsilon \widetilde{I}_\Lambda = H\left(-\epsilon^{\Delta_-}\frac{\delta \widetilde{I}_\Lambda}{\delta \mathcal{O}_i}, i\epsilon^{-\Delta_-}\mathcal{O}_i\right), \tag{5.1.3}$$

where H is the hamiltonian of the bulk theory. The planar limit of the dual QFT corresponds, through the duality, to the classical gravity limit in the bulk. Therefore, in such regime, the bulk theory can be taken in supergravity approximation. This means that H is a supergravity hamiltonian.

The operators $\mathcal{O}_i$ are the single-trace operators associated to the bulk fields. Since they are defined as the trace of products of elementary fields and normalized with a factor $1/N$, we can think to identify them with the operators Y_m of the matrix theory so as to establish a connection between the two formulations. Due to this fact, comparing the structures of (5.1.1) and (5.1.3), we can see that the hamiltonian (5.1.2) of WPE, since it is a cubic functional, does not have the form of a supergravity hamiltonian, which instead is at least quadratic, as already pointed out in [30]. For this reason the two equations actually seem to have a different structure.

This fact is quite puzzling and a clear understanding of this mismatch is yet to be obtained. However, one could speculate, like in [30], that the hamiltonian (5.1.2) of WPE should be interpreted as a string field theory hamiltonian, where Y is a string creation operator and $\delta/\delta Y$ is a string annihilation operator. This would suggests that the ordinary WRG could correspond through the AdS/CFT duality to the full string theory in the bulk. In this case a supergravity description could be obtained from (5.1.2) by integrating out super-massive excited string modes, which are associated to extra-high dimension operators in the dual QFT. Therefore hWRG could be interpreted as a "supergravity" truncation of the ordinary WRG, where such operators do not appear because their flow is extremely convergent.

At a more basic level, one would like to have a better understanding of the nature of the mismatch between the field-theoretic and holographic RG. We think this is due to a fundamental difference in the introduction of the cut-off in the two approaches. Indeed in

WPE formulation, even though the planar RG equation is specified in terms of the composite operators Y_m, the matrix theory is explicitly defined through the elementary field Φ. This implies that the floating cut-off t is introduced in the theory inside the propagator P_t of Φ, defined in (2.3.2). In such a way the RG equation explicitly depends on the derivative $\dot{P}_t$ of the cut-off propagator. Moreover, the description in terms of elementary fields enables us to develop a diagrammatic interpretation of the planar WPE, where it is clear that the derivative with respect to the cut-off acts through $\dot{P}_t$ cutting the internal lines of the diagrams composing $\mathcal{H}_t$ and so produces new single and multi-trace contributions.

Instead, in holographic Wilsonian treatment the QFT is defined through the AdS/CFT correspondence as dual to a bulk theory in $aAdS_{d+1}$. For this reason the structure of the field theory is determined only by the composite operators $\mathcal{O}_i$ dual to the bulk fields. In particular the explicit form of the boundary action I_{CFT}, which represents the UV fixed point of the flow, is unknown and its description in terms of elementary fields is not given. This means that we have no understanding of the implementation of the cut-off scale Λ inside the QFT at the level of elementary fields. Indeed, as seen above, in (5.1.3) the evolution of the Wilsonian action is determined by the radial evolution in the bulk, so the scale Λ enters the equation through the radial cut-off ϵ, which represents the separation point of the path integral (4.2.5). This implies that the derivative with respect to the scale does not acts explicitly on the propagator of the elementary fields, which in this framework is not defined, and so we cannot give a diagrammatic interpretation of the RG equation because we cannot build Feynman diagrams.

The difference in the definition of the theory appears then to be the striking problem. To get some insight into this analysis, we can think to apply the ordinary WRG approach of WPE to the simple example of the field theory dual to a free scalar field in the bulk, already studied in the previous chapter. By doing so we can explicitly compare the RG equations obtained from the two formulations.

In Wilsonian holographic treatment we have obtained a large N Wilsonian action composed by a single- and a double-trace term in the operator $\mathcal{O}$, dual to the scalar bulk field:

$$\widetilde{I}_\Lambda = \int \frac{d^d k}{(2\pi)^d} \Big(\lambda(-k,\epsilon)\mathcal{O}_k + \frac{1}{2} f(k,\epsilon)\mathcal{O}_k\mathcal{O}_{-k} \Big). \tag{5.1.4}$$

The RG equations of λ and f are:

$$\begin{aligned} \epsilon\partial_\epsilon \lambda(k,\epsilon) &= (\epsilon^2 k^2 + m^2)\lambda(k,\epsilon) f(k,\epsilon) \\ \epsilon\partial_\epsilon f(k,\epsilon) &= (\epsilon^2 k^2 + m^2) f^2(k,\epsilon) - d f(k,\epsilon) - 1, \end{aligned} \tag{5.1.5}$$

so that the planar RG flow of the theory is closed on λ and f, and no other multi-trace term appear in $\widetilde{I}_\Lambda$.

In order to apply WPE to the same case we can identify the operator $\mathcal{O}$ with a single-trace operator $Y_n = \operatorname{tr}\Phi^n/N$. The dimension Δ_+ of $\mathcal{O}$ is then related to the power n of Y_n

by:

$$\Delta_+ = n\Big(\frac{d-2}{2}\Big),$$

because $(d-2)/2$ is the dimension of the elementary field Φ in d dimensions. Therefore, since $\Delta_+ = d/2+\nu$ and ν depends on the mass of the bulk field ϕ, we have that the value of n is specified adjusting the mass of the dual bulk field.

Now we can impose $\mathcal{H}_t$ to have the form of $\widetilde{I}_\Lambda$, so we can restrict its expansion (2.3.26) to:

$$\mathcal{H}_t = \int \frac{d^dk}{(2\pi)^d}\left(\lambda(-k,t)Y_n(k) + \frac{1}{2}f(k,t)Y_n(k)Y_n(-k)\right). \tag{5.1.6}$$

Inserting this equation in (5.1.1) we obtain:

$$\begin{aligned}\partial_t\lambda(-k)Y_n(k) + \frac{1}{2}\partial_t f(k)Y_n(k)Y_n(-k) &= -\frac{n^2}{2}\dot{P}_t(k)\lambda(k)\lambda(-k)Y_{2n-2}(k)\\ &- n^2\dot{P}_t(k)\lambda(-k)f(k)Y_{2n-2}(k)Y_n(k) - \frac{n^2}{2}\dot{P}_t(k)f^2(k)Y_{2n-2}(k)Y_n(k)Y_n(-k)\\ &- \frac{n}{2}\dot{P}_t(k)\sum_{i=1}^{n-1}\Big(\lambda(-k)Y_{i-1}(k)Y_{n-i-1}(k) + f(k)Y_{i-1}(k)Y_{n-i-1}(k)Y_n(k)\Big).\end{aligned} \tag{5.1.7}$$

It is immediately clear that this equation is not consistent as it stands. This is due to the presence of terms proportional to Y_{2n-2}, which is different from Y_n and so determines a new deformation term, and of extra multi-trace terms proportional to Y^3, which are not present in $\mathcal{H}_t$. This means that the planar RG flow obtained from WPE is not closed on the couplings λ and f of Y_n and Y_n^2, but new single- and double-trace terms are produced along the flow. Therefore, unlike the hWRG result (5.1.5), the truncation to the single- and double-trace sector of the operator $\mathcal{O}$ is not well-defined in the WPE formulation.

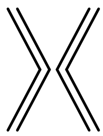
Consider for instance the specific case of $n=2$, which represents the first non-trivial case that can be considered for $\mathcal{H}_t$. The equation (5.1.7) reduces to:

$$\begin{aligned}\partial_t\lambda(-k)Y_2(k) + \frac{1}{2}\partial_t f(k)Y_2(k)Y_2(-k) &= -\dot{P}_t(k)\lambda(-k) - \dot{P}_t(k)\Big(2\lambda(k)\lambda(-k) - f(k)\Big)Y_2(k)\\ &- 4\dot{P}_t(k)\lambda(-k)f(k)Y_2(k)Y_2(-k) - 2\dot{P}_t(k)f^2(k)Y_2^2(k)Y_2(-k).\end{aligned} \tag{5.1.8}$$

Equating the terms proportional to Y_2 and Y_2^2 of the two sides of the equation above, one obtains the RG equations for the effective couplings:

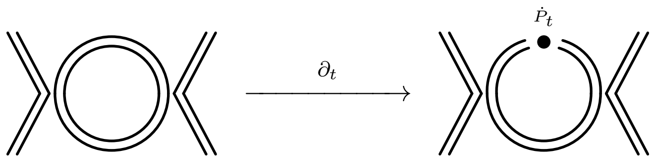
$$\begin{aligned}\partial_t\lambda(k,t) &= -\dot{P}_t(k)\Big(2\lambda(k,t)\lambda(-k,t) - f(k,t)\Big)\\ \partial_t f(k,t) &= -8\dot{P}_t(k)\lambda(-k,t)f(k,t),\end{aligned} \tag{5.1.9}$$

which do not match at all with the corresponding RG equations (5.1.5) on the holographic side. Moreover, the r.h.s. of (5.1.8) contains a multi-trace contribution proportional to Y_2^3, which leads to the generation of new terms in the $\mathcal{H}_t$. Considering the diagrammatic interpretation of the planar WPE, we can see that such contributions are naturally produced by the action of the derivative of the cut-off scale ∂_t on the terms of the Wilsonian action $\mathcal{H}_t$. Indeed, the double-trace term fY_2^2 in $\mathcal{H}_t$ encodes the contributions of the diagrams characterized by two traces, i.e. two internal lines with external legs attached to them. Therefore, since $Y_2 \sim \operatorname{tr}\Phi^2$, such term can be represented by the vertex:



$$\sim \left(\operatorname{tr}\Phi^2\right)^2, \tag{5.1.10}$$

where every branch line represents a trace of two fields, in contrast with the vertex (2.2.15) of $\operatorname{tr}\Phi^4$, where all the lines are attached to the same trace. In such a way, at order f^2, $\mathcal{H}_t$ has a contribution given by the diagram:



(5.1.11)

When ∂_t acts on such a diagram, it cuts one internal line replacing it with $\dot{P}_t$. Therefore, when the central circle line is cut, a new diagram containing three traces is produced, which then gives a contribution proportional to Y_2^3.

From this analysis some facts appear clear:

- The production of new multi-trace terms in the RG equation (5.1.7) obtained in ordinary WRG treatment is due to the action of the cut-off in the contributions present in the initial form of the Wilsonian action $\mathcal{H}_t$. Therefore we generally expect that in such formulation the RG flow cannot be consistently truncated.
- The RG flow obtained in (5.1.3) with the holographic Wilsonian treatment reflects the dynamics of the bulk theory, which determines the dual QFT only in terms of the composite operators associated to the bulk fields. Therefore, since the cut-off is not explicitly defined inside the boundary theory, we cannot keep track of its action on the structure of the Wilsonian action as in WPE.

This lead us to think of the different definition of the field theory in the two approaches as the main cause of the inconsistency between the RG equations. In particular we believe that the role of the elementary and composite operators, and the subsequent definition of the cut-off scale, represent the key points on which to investigate.

5.2. RG flow for a deformed CFT

As an attempt to clarify the difference between the role of elementary and composite operators in Wilsonian RG, we can consider the case of a deformed CFT defined through composite operators and compute its RG flow with field theory methods [35–37].

Let us take the action of the theory equal to:

$$I = I_{\text{CFT}} + N\lambda \int d^d x\, \mathcal{O}(x) + \frac{f}{2} \int d^d x\, \mathcal{O}^2(x), \tag{5.2.1}$$

where I_{CFT} is the undeformed action which is conformal invariant up to terms of order $O(1/N)$. We assume that the operator $\mathcal{O}$ is a single-trace operator with normalization such that $\mathcal{O} = \text{tr}\,(\cdots)$, where the dots represent some product of elementary fields and their derivatives, so that λ and f are proper 't Hooft couplings. Being Δ the dimension of $\mathcal{O}$ in absence of deformation, the 2-point function in such regime reads:

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle_{\text{CFT}} = \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{v}{|x-y|^{2\Delta}}\,, \qquad v > 0. \tag{5.2.2}$$

Now we can compute the RG equations of the deformation couplings λ and f employing a Wilsonian approach. Indeed, considering (5.2.1) as the bare action defined at the UV cut-off Λ_0, we can obtain the effective theory at the scale $\Lambda < \Lambda_0$ integrating out the UV modes of the bare action in the range between Λ and Λ_0. By doing so we encounter the expansion:

$$\begin{aligned} 1 - N\lambda \int d^d x\, \mathcal{O}(x) - \frac{f}{2} \int d^d x\, \mathcal{O}^2(x) + \frac{N^2}{2}\lambda^2 \int d^d x d^d y\, \mathcal{O}(x)\mathcal{O}(y) \\ + \frac{N}{2}\lambda f \int d^d x d^d y\, \mathcal{O}(x)\mathcal{O}^2(y) + \frac{f^2}{8} \int d^d x d^d y\, \mathcal{O}^2(x)\mathcal{O}^2(y) + \cdots . \end{aligned} \tag{5.2.3}$$

To evaluate the higher order corrections to λ and f we should know the correct operator product expansion of $\mathcal{O}(x)\mathcal{O}(y)$ for $x \sim y$. In the large N limit the structure simplifies very much and the products of operators can be expanded in terms of the correlation functions, so that we have:

$$\begin{aligned} 1 - N\lambda \int d^d x\, \mathcal{O}(x) + N\lambda f \int d^d x d^d y\, \mathcal{O}(x) \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_{\text{CFT}} \\ - \frac{f}{2} \int d^d x\, \mathcal{O}^2(x) + \frac{f^2}{2} \int d^d x d^d y\, \mathcal{O}(x)\mathcal{O}(y) \langle \mathcal{O}(x)\mathcal{O}(y)\rangle_{\text{CFT}} + \cdots , \end{aligned} \tag{5.2.4}$$

where the term proportional to λ^2 has been neglected because it only produces a field-independent correction. Now, we can compute the corrections in (5.2.4) setting $w = y - x$, so that we have:

$$\begin{aligned} & N\lambda f \int d^d x d^d w\, \mathcal{O}(x) \langle \mathcal{O}(0)\mathcal{O}(w)\rangle_{\text{CFT}} = N\lambda f \int d^d x\, \mathcal{O}(x) \int d^d w\, \langle \mathcal{O}(0)\mathcal{O}(w)\rangle_{\text{CFT}} \\ & \frac{f^2}{2} \int d^d x d^d w\, \mathcal{O}(x)\mathcal{O}(w+x) \langle \mathcal{O}(0)\mathcal{O}(w)\rangle_{\text{CFT}} \sim \frac{f^2}{2} \int d^d x\, \mathcal{O}^2(x) \int d^d w\, \langle \mathcal{O}(0)\mathcal{O}(w)\rangle_{\text{CFT}}\,, \end{aligned}$$

where we have used the property of invariance under translations of the 2-point function and expanded $\mathcal{O}(w+x)$ around x stopping at order zero. Since we are computing the Wilsonian effective action at the scale Λ, the integral over w is evaluated in the range $1/\Lambda_0 < |w| < 1/\Lambda$, thus it reads:

$$\int d^d w \, \langle \mathcal{O}(0)\mathcal{O}(w)\rangle_{\text{CFT}} = -\frac{v}{2\Delta - d}\Big(\Lambda^{2\Delta-d} - \Lambda_0^{2\Delta-d}\Big). \tag{5.2.5}$$

The contribution of this integral introduces the dependence on the energy scale in the deformation couplings, so this represents the way of introducing the cut-off in the theory emerging in this approach. Therefore, let us remark that, being the elementary structure of I_{CFT} unknown, such scale is not explicitly defined inside the propagator of the elementary fields, but enters the theory through the integral of the 2-point function of the composite operator $\mathcal{O}$.

In such a way the large N renormalized couplings at the scale Λ at the second order in the expansion (5.2.4) are:

$$\lambda(\Lambda) = \lambda(\Lambda_0) + \frac{v}{2\Delta - d}\lambda(\Lambda_0) f(\Lambda_0)\Big(\Lambda^{2\Delta-d} - \Lambda_0^{2\Delta-d}\Big) \tag{5.2.6}$$

$$f(\Lambda) = f(\Lambda_0) + \frac{v}{2\Delta - d} f^2(\Lambda_0)\Big(\Lambda^{2\Delta-d} - \Lambda_0^{2\Delta-d}\Big). \tag{5.2.7}$$

Higher order terms of the expansion (5.2.4) do not give further contributions to the running of λ and f thanks to the factorization property of the correlation functions in the large N limit. Indeed, considering for example the third order contribution proportional to f^3, we have that the only term which gives contribution is:

$$f^3 \int d^d x d^d y d^d z \, \mathcal{O}^2(x)\mathcal{O}(y)\mathcal{O}(z) \, \langle \mathcal{O}(y)\mathcal{O}(z)\rangle_{\text{CFT}} \sim f^3 \int d^d w \, \langle \mathcal{O}(0)\mathcal{O}(w)\rangle_{\text{CFT}} \times \int d^d x d^d y \, \mathcal{O}^2(x)\mathcal{O}^2(y), \tag{5.2.8}$$

whereas all the other terms are suppressed in the large N expansion. This term does not represent a new correction to the running of f, but it is a contribution to the second order term in the exponential series of the double-trace deformation. Therefore we have that (5.2.6) and (5.2.7) are the exact running couplings of the theory in the large N limit and the higher order terms reconstruct their exponential series in the path integral.

Now, defining the dimensionless couplings $\overline{\lambda} := \lambda\Lambda^{\Delta-d}$ and $\bar{f} := f\Lambda^{2\Delta-d}$, we can compute their β-functions by differentiating with respect to Λ:

$$\Lambda\partial_\Lambda \overline{\lambda} = v\overline{\lambda}\Big(\bar{f} + \frac{\Delta - d}{v}\Big) \tag{5.2.9}$$

$$\Lambda\partial_\Lambda \bar{f} = v\bar{f}^2 + (2\Delta - d)\bar{f}. \tag{5.2.10}$$

These are the RG equations of the single-trace coupling $\overline{\lambda}$ and the double-trace coupling $\bar{f}$. Setting the dimension of the operator $\mathcal{O}$ to Δ_+ the equation of $\bar{f}$ recovers the RG equation (4.2.40) of the actual double-trace coupling obtained through holographic Wilsonian treatment in subsection 4.2.1. Moreover, since the term $\bar{f} + (\Delta - d)/v$ in the equation of $\overline{\lambda}$ can be seen as reconstructing the double-trace deformation in (4.2.32), the equation (5.2.9) recovers the RG equation of the single-trace coupling (4.2.35) obtained in hWRG formulation. This is an important result because it has been obtained studying the RG flow of a theory defined as a deformed CFT in terms of composite operators only. Indeed this framework reflects that of hWRG, where the dual QFT is defined in the same way. Therefore this result shows that, considering a CFT deformed by some composite operator, the RG flow of the deformation terms has the same structure if obtained holographically through hWRG or directly with field theory computations.

5.3. Conclusions

Summarizing what we have described in this chapter, we can see that, for a theory defined only through composite operators and where it is not clear how the cut-off scale is introduced in the action, the planar RG flow computed with field theory methods has the same structure of the holographic Wilsonian RG in the large N limit, described in chapter 4. On the other hand the planar RG flow of a matrix theory obtained through ordinary Wilsonian treatment has a very different structure. This is due to the fact that in this case the theory has an explicit structure in terms of elementary fields and, in particular, that the cut-off is explicitly defined inside the propagator, so that we can keep track of how its derivative acts on the Wilsonian action and generates new interaction terms.

Therefore our proposal is that the underlying reason why the holographic Wilsonian treatment seems not to be consistent with ordinary Wilsonian approach to large N QFT stands in the fact that the structure of the theory in the two approaches is determined respectively through composite operators and elementary fields. This leads to different descriptions of the RG flow, due, in particular, to the fact that the cut-off scale enters the theory in very different ways.

APPENDIX **A**

Calculus of loop integrals

In this appendix we perform the explicit calculations of loop integrals encountered in subsection 1.1.1 considering the UV corrections to the ϕ^4 theory in Wilsonian approach.

The first integral we have considered is:

$$= \int \frac{d^4k}{(2\pi)^4} \frac{\widehat{\Theta}(k)}{k^2 + m^2}, \tag{A.1}$$

where $\widehat{\Theta}(k)$ is the function which selects momenta in the range $[\Lambda, \Lambda_0]$. We can evaluate it writing the measure in polar coordinates, thus:

$$d^4k = d\Omega_4 dk k^3,$$

where $k = (k^2)^{\frac{1}{2}} = (k^\mu k_\mu)^{\frac{1}{2}}$. So we have:

$$\frac{1}{(2\pi)^4} \int d\Omega_4 \int_\Lambda^{\Lambda_0} dk \frac{k^3}{k^2 + m^2}. \tag{A.2}$$

Now the integral over the solid angle is:

$$\int d\Omega_4 = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2,$$

where Γ is the Euler's gamma function, so, since $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, we have that $\Gamma(2) = 1$.

The integral over k can be computed using a change of variables:

$$y^2 \coloneqq k^2 + m^2. \tag{A.3}$$

In such a way we have:

$$dk = \frac{y}{\sqrt{y^2 - m^2}} dy,$$

whereas the integration range becomes $\left[\sqrt{\Lambda^2 + m^2}, \sqrt{\Lambda_0^2 + m^2}\right]$. Therefore the integral in (A.1) can be written as:

$$\frac{1}{8\pi^2} \int_{\sqrt{\Lambda^2+m^2}}^{\sqrt{\Lambda_0^2+m^2}} dy \left(y - \frac{m^2}{y} \right) = \frac{1}{8\pi^2} \left(\frac{1}{2} y^2 - m^2 \log y \right|_{\sqrt{\Lambda^2+m^2}}^{\sqrt{\Lambda_0^2+m^2}}.$$

Hence the result is:

$$\int \frac{d^4k}{(2\pi)^4} \frac{\widehat{\Theta}(k)}{k^2 + m^2} = \frac{1}{16\pi^2} \left[(\Lambda_0^2 - \Lambda^2) - m^2 \log \left(\frac{\Lambda_0^2 + m^2}{\Lambda^2 + m^2} \right) \right]. \tag{A.4}$$

The next loop integral that we are going to perform comes from the UV correction to the ϕ^4 vertex:

$$\zeta = -\frac{3}{2} \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{\widehat{\Theta}(k)}{(k^2 + m^2)^2}. \tag{A.5}$$

Such integral can be computed in the same way as the previous one. Indeed we can write it as:

$$\int \frac{d^4k}{(2\pi)^4} \frac{\widehat{\Theta}(k)}{(k^2 + m^2)^2} = \frac{1}{8\pi^2} \int_{\Lambda}^{\Lambda_0} dk \frac{k^3}{(k^2 + m^2)^2}.$$

Now, considering the change of variables (A.3), we have:

$$\int_{\Lambda}^{\Lambda_0} dk \frac{k^3}{(k^2 + m^2)^2} = \int_{\sqrt{\Lambda^2+m^2}}^{\sqrt{\Lambda_0^2+m^2}} dy \left(\frac{1}{y} - \frac{m^2}{y^3} \right) = \left(\log y + \frac{m^2}{2y^2} \right|_{\sqrt{\Lambda^2+m^2}}^{\sqrt{\Lambda_0^2+m^2}}.$$

Hence the result is:

$$\int \frac{d^4k}{(2\pi)^4} \frac{\widehat{\Theta}(k)}{(k^2 + m^2)^2} = \frac{1}{16\pi^2} \left[\log \left(\frac{\Lambda_0^2 + m^2}{\Lambda^2 + m^2} \right) + m^2 \left(\frac{1}{\Lambda_0^2 + m^2} - \frac{1}{\Lambda^2 + m^2} \right) \right]. \tag{A.6}$$

The last loop integral we explicitly perform comes from the UV correction which produces a ϕ^6 vertex:

$$\xi = -15\lambda^3 \int \frac{d^4k}{(2\pi)^4} \frac{\widehat{\Theta}(k)}{(k^2 + m^2)^3}. \tag{A.7}$$

Following the usual method, the integral reads:

$$\int \frac{d^4k}{(2\pi)^4} \frac{\widehat{\Theta}(k)}{(k^2 + m^2)^3} = \frac{1}{8\pi^2} \int_{\Lambda}^{\Lambda_0} dk \frac{k^3}{(k^2 + m^2)^3},$$

so, performing the change of variables (A.3), we have:

$$\int_{\Lambda}^{\Lambda_0} dk \frac{k^3}{(k^2 + m^2)^3} = \int_{\sqrt{\Lambda^2+m^2}}^{\sqrt{\Lambda_0^2+m^2}} dy \left(\frac{1}{y^3} - \frac{m^2}{y^5} \right) = \left(-\frac{1}{2y^2} + \frac{m^2}{4y^4} \right|_{\sqrt{\Lambda^2+m^2}}^{\sqrt{\Lambda_0^2+m^2}}.$$

Hence the result is:

$$\int \frac{d^4k}{(2\pi)^4} \frac{\widehat{\Theta}(k)}{(k^2+m^2)^3} = -\frac{1}{16\pi^2}\left[\frac{1}{\Lambda_0^2+m^2} - \frac{1}{\Lambda^2+m^2} - \frac{m^2}{2}\left(\frac{1}{(\Lambda_0^2+m^2)^2} - \frac{1}{(\Lambda^2+m^2)^2}\right)\right]. \tag{A.8}$$

A particular mention is required for derivative terms in (1.1.37), arising from further orders in the Taylor expansion of $\phi_{\rm L}^2(y)$ around $y = x$. In subsection 1.1.1 we have noted, through symmetry properties, that A^μ is vanishing. Here, for completeness, we explicitly derive this fact.

The parameter A^μ is the coupling of a $\phi^3\partial_\mu\phi$ vertex, thus it reads:

$$\begin{aligned} A^\mu &= -3\lambda^2 \int d^4w\, w^\mu \,\langle\phi_{\rm H}(0)\phi_{\rm H}(w)\rangle_0^2 \\ &= -3\lambda^2 \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{\widehat{\Theta}(k)}{(k^2+m^2)}\frac{\widehat{\Theta}(p)}{(p^2+m^2)}\int d^4w\, w^\mu\, e^{-iw\cdot(k+p)}. \end{aligned} \tag{A.9}$$

Now we have:

$$w^\mu\, e^{-iw\cdot(k+p)} = i\frac{\partial}{\partial k_\mu} e^{-iw\cdot(k+p)}, \tag{A.10}$$

thus:

$$\int d^4w\, w^\mu\, e^{-iw\cdot(k+p)} = i\frac{\partial}{\partial k_\mu}(2\pi)^4\delta^4(k+p). \tag{A.11}$$

In such a way, using the properties of the delta function, A^μ becomes:

$$A^\mu = 3i\lambda^2 \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{\partial}{\partial k_\mu}\left(\frac{\widehat{\Theta}(k)}{(k^2+m^2)}\right)\frac{\widehat{\Theta}(p)}{(p^2+m^2)}(2\pi)^4\delta^4(k+p), \tag{A.12}$$

where:

$$\frac{\partial}{\partial k_\mu}\widehat{\Theta}(k) = \frac{\partial k}{\partial k_\mu}\partial_k\big(\theta(\Lambda_0-k)-\theta(\Lambda-k)\big) = \frac{k^\mu}{k}\big(\delta(\Lambda-k)-\delta(\Lambda_0-k)\big) \tag{A.13}$$

$$\frac{\partial}{\partial k_\mu}\frac{1}{k^2+m^2} = -\frac{2k^\mu}{k^2+m^2}, \tag{A.14}$$

so that the whole derivative reads:

$$\frac{\partial}{\partial k_\mu}\left(\frac{\widehat{\Theta}(k)}{(k^2+m^2)}\right) = \frac{k^\mu}{k^2+m^2}\left[\frac{1}{k}\big(\delta(\Lambda-k)-\delta(\Lambda_0-k)\big) - 2\frac{\widehat{\Theta}(k)}{k^2+m^2}\right]. \tag{A.15}$$

Since we can split the measure in:

$$d^4k = d\Omega_4 dk\, k^3, \tag{A.16}$$

and redefine the four-momentum k as:

$$k^\mu = k\hat{k}^\mu \qquad \hat{k}^\mu\hat{k}_\mu = 1, \tag{A.17}$$

we have:

$$A^\mu = \frac{3i\lambda^2}{(2\pi)^4} \int d\Omega_4 \, \widehat{k}^\mu \int dk \left(\frac{k^3 \widehat{\Theta}(k)}{(k^2+m^2)^2} \big(\delta(\Lambda - k) - \delta(\Lambda_0 - k)\big) - 2 \frac{k^4 \widehat{\Theta}(k)}{(k^2+m^2)^3} \right) \tag{A.18}$$

Now let us focus on the first integral:

$$\int d\Omega_4 \, \widehat{k}^\mu . \tag{A.19}$$

This integral is equal to zero because the measure is invariant under the redefinition such that $\widehat{k}^\mu \mapsto -\widehat{k}^\mu$. Therefore the derivative coupling A^μ is vanishing.

APPENDIX B

Holographic computation of the correlation functions

The renormalized on-shell action S_{ren} is a finite functional of the boundary value ϕ_0 which yields the correlation functions of the dual operator $\mathcal{O}$ in the boundary field theory through its derivatives with respect to ϕ_0:

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{\text{CFT}} = -\left.\frac{\delta^n S_{\text{ren}}[\phi_0]}{\delta\phi_0(x_1)\cdots\delta\phi_0(x_n)}\right|_{\phi_0=0}. \tag{B.1}$$

In such a way we are capable to compute them evaluating S_{ren} on the complete solution $\overline{\phi}$ of the e.o.m. (3.3.2) written in terms of ϕ_0. However, since the n-point function depends on the n-th derivative of S_{ren} with respect to ϕ_0 calculated in $\phi_0 = 0$, every term in S_{ren} proportional to some power of ϕ_0 greater than n does not give contribution and can be neglected.

Therefore, the 1-point function would need just single-field terms which are not present in S_{ren}, thus we can say that it is vanishing. Instead, for the 2-point function we need the action up to quadratic terms in ϕ_0, i.e. just the free part. For this reason we can consider the solution of the free e.o.m., which is the Klein-Gordon equation (3.3.5). Such solution can be expressed in terms of a function $K(z; x - x')$ which satisfies:

$$\begin{aligned} &(-\Box + m^2)K(z; x - x') = 0 \\ &K(z; x - x') \underset{z\mapsto 0}{\sim} z^{\Delta_-}\delta^d(x - x'), \end{aligned} \tag{B.2}$$

so that:

$$\overline{\phi}(z, x) = \int d^d x' K(z; x - x')\phi_0(x'). \tag{B.3}$$

Since $K(z; x-x')$ connects the solution $\overline{\phi}$ evaluated in the bulk to its boundary value ϕ_0, it is called bulk-to-boundary propagator. Moreover it can be computed considering the Green's function G of the free e.o.m. operator in AdS_{d+1}:

$$(-\Box + m^2)G(z, x; z', x') = \frac{1}{\sqrt{g}}\delta(z - z')\delta^d(x - x'), \tag{B.4}$$

so that, through Green's theorem, we have that:

$$\begin{aligned}\overline{\phi}(z', x') = &\int d^d x\, z^{1-d} \left(\overline{\phi}(z, x)\partial_z G(z, x; z', x') - G(z, x; z', x')\partial_z \overline{\phi}(z, x)\right|_{z=0} \\ &- \int d^d x\, z^{1-d} \left(\overline{\phi}(z, x)\partial_z G(z, x; z', x') - G(z, x; z', x')\partial_z \overline{\phi}(z, x)\right|_{z=+\infty}, \end{aligned}\tag{B.5}$$

because the boundary of AdS_{d+1} is the union of $(z = 0, x)$ with $(z = +\infty, x)$. The imposition of regularity conditions at $z = +\infty$ ensures that the second term in the previous equation does not give contribution. Unlike K, the Green's function G connects two different bulk points (z, x) and (z', x'), thus it is can be called bulk-to-bulk propagator to be distinguished from K. Moreover G satisfies (B.4), thus it is a solution of the free e.o.m., and so its behavior for $z \mapsto 0$ is:

$$G(z, x; z', x') \underset{z\mapsto 0}{\sim} z^{\Delta_-} G_-(x; z', x') + z^{\Delta_+} G_+(x; z', x'), \tag{B.6}$$

where G_- is a non-normalizable amplitude. Thus, in order to have a normalizable Green's function, we set $G_- \equiv 0$ and so, using the boundary conditions for $\overline{\phi}$ in (B.5), we have:

$$\overline{\phi}(z', x') = (\Delta_+ - \Delta_-) \int d^d x\, G_+(x; z', x')\phi_0(x), \tag{B.7}$$

which allows us to connect G_+ to the bulk-to-boundary propagator:

$$K(z'; x' - x) = (\Delta_+ - \Delta_-)G_+(x; z', x'), \tag{B.8}$$

and then:

$$G(z, x; z', x') \underset{z\mapsto 0}{\sim} z^{\Delta_+} \frac{K(z'; x' - x)}{\Delta_+ - \Delta_-}. \tag{B.9}$$

Therefore, computing the Green's function we can obtain the explicit form of the bulk-to-boundary propagator. To do so we can note that G is invariant under the isometry group $O(1, d + 1)$ of AdS_{d+1}, hence it must be a function of the AdS invariant distance:

$$w^2 = -\eta^{IJ}(X - X')_I(X - X')_J = 2 - 2\eta^{IJ} X_i X'_J = \frac{(z - z')^2 + (x - x')^2}{zz'} := \frac{1}{u}, \tag{B.10}$$

where $I = -1, 0, 1, \cdots, d$ and $\eta^{IJ} = \mathrm{diag}(-1, +1, \cdots, +1)$.

Solving the equation (B.4) of G in terms of u we can obtain its form for $z \mapsto 0$, i.e. $u \mapsto 0$:

$$G \underset{z\mapsto 0}{\sim} cu^{\Delta_+} \underset{z\mapsto 0}{\longmapsto} cz^{\Delta_+} \left(\frac{z'}{z'^2 + |x - x'|^2}\right)^{\Delta_+} \tag{B.11}$$

From this we obtain that:

$$K(z;x-x') = c_+ \frac{z^{\Delta_+}}{\left(z^2+|x-x'|^2\right)^{\Delta_+}}, \tag{B.12}$$

where $c_+ = c(\Delta_+ - \Delta_-)$ is a normalization, which is fixed by boundary conditions, thus it has to satisfy:

$$\lim_{z\mapsto 0} c_+ \frac{z^{\Delta_+-\Delta_-}}{\left(z^2+|x-x'|^2\right)^{\Delta_+}} = \delta^d(x-x'), \tag{B.13}$$

or equivalently:

$$\lim_{z\mapsto 0} c_+ \int d^dx \frac{z^{\Delta_+-\Delta_-}}{\left(z^2+|x-x'|^2\right)^{\Delta_+}} = 1. \tag{B.14}$$

Performing a change of variables $y := |x-x'|/z$ we can solve the integral. This enables us to determine the explicit form of the normalization, which is:

$$c_+ = \frac{\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\Delta_+ - \frac{d}{2})}, \tag{B.15}$$

where Γ is the Euler's gamma function. Therefore, since $\Delta_+ \geq d/2$ and $\Gamma(x) > 0$ for $x \geq 0$, the normalization c_+ is always non-negative[1].

Now, knowing the correct form of the bulk-to-boundary propagator, we can give a general solution of the free e.o.m. with boundary value ϕ_0:

$$\overline{\phi}(z,x) = c_+ \int d^dx' \frac{z^{\Delta_+}}{\left(z^2+|x-x'|^2\right)^{\Delta_+}} \phi_0(x'). \tag{B.16}$$

Considering the expanded solution (3.3.10), we can obtain the form of $A(x)$ in terms of ϕ_0 looking at the contribution proportional to z^{Δ_+} for $z \mapsto 0$:

$$A(x) = c_+ \int d^dx' \frac{\phi_0(x')}{|x-x'|^{2\Delta_+}}. \tag{B.17}$$

Recalling from (3.3.42) that the derivative of S_{ren} with respect to ϕ_0 has the form:

$$\langle \mathcal{O}(x) \rangle_{\phi_0} = -\frac{\delta S_{\text{ren}}}{\delta \phi_0(x)} = (2\Delta_+ - d)A(x), \tag{B.18}$$

we have that the 1-point function of $\mathcal{O}$ in the boundary CFT is obtained setting $\phi_0 = 0$. So, using (B.17), we have:

$$\langle \mathcal{O}(x) \rangle_{\text{CFT}} = 0, \tag{B.19}$$

which reflects our initial prediction and satisfies the conformal invariance.

[1] In fact, $c_+ > 0$ except for $\Delta_+ = d/2$, where the Gamma function $\Gamma(\Delta_+ - d/2) = \infty$ and so $c_+ \mapsto 0$.

To compute the 2-point function we need the second derivative of S_{ren}, thus we can apply another derivative to (B.18):

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{\phi_0} = \frac{\delta \langle \mathcal{O}(x_1)\rangle_{\phi_0}}{\delta\phi_0(x_2)} = (2\Delta_+ - d)\frac{\delta A(x_1)}{\delta\phi_0(x_2)}. \tag{B.20}$$

Using (B.17), we have:

$$\frac{\delta A(x_1)}{\delta\phi_0(x_2)} = \frac{c_+}{|x_1 - x_2|^{2\Delta_+}}. \tag{B.21}$$

Therefore the 2-point function of the operator $\mathcal{O}$ in the boundary CFT is:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{\text{CFT}} = (2\Delta_+ - d)\frac{c_+}{|x_1 - x_2|^{2\Delta_+}}. \tag{B.22}$$

This agrees with the expected result for the correlation function in a conformal field theory.

The computation of the 3-point function requires the introduction of terms proportional to ϕ_0^3 in S_{ren}, so we have to include in the computation the part of the potential $V(\phi)$ proportional to $\lambda\phi^3$. In such a way the e.o.m. is modified to:

$$(-\Box + m^2)\phi + \lambda\phi^2 = 0. \tag{B.23}$$

The solution can be calculated perturbatively in λ. Thus we can expand $\overline{\phi} = \overline{\phi}^{(0)} + \overline{\phi}^{(1)} + \cdots$, where $\overline{\phi}^{(0)}$ is the free solution (B.16), whereas $\overline{\phi}^{(1)}$ is the contribution of order λ, which solves the perturbative equation:

$$(-\Box + m^2)\overline{\phi}^{(1)} + \lambda(\overline{\phi}^{(0)})^2 = 0. \tag{B.24}$$

Being the latter an inhomogeneous Klein-Gordon equation, its solution can be written using the Green's function G which satisfies (B.4):

$$\overline{\phi}^{(1)}(z,x) = -\lambda \int dz' d^d x' \sqrt{g} G(z,x;z',x')\Big(\overline{\phi}^{(0)}(z',x')\Big)^2. \tag{B.25}$$

Recalling the boundary behavior of the bulk-to-bulk propagator (B.9), we have:

$$\begin{aligned}\overline{\phi}^{(1)}(z,x) \underset{z\mapsto 0}{\sim} -\frac{\lambda z^{\Delta_+}}{2\Delta_+ - d}\int dz' d^d x_1 d^d x_2 d^d x_3 \sqrt{g} K(z';x-x_1)K(z';x_1-x_2) \\ \times K(z';x_1-x_2)\phi_0(x_2)\phi_0(x_3).\end{aligned} \tag{B.26}$$

This is a term going as z^{Δ_+}, so it represents a correction of order λ to $A(x)$:

$$\begin{aligned}A^{(1)}(x) = -\frac{\lambda}{2\Delta_+ - d}\int dz d^d x_1 d^d x_2 d^d x_3 \sqrt{g} K(z;x-x_1)K(z;x_1-x_2) \\ \times K(z;x_1-x_3)\phi_0(x_2)\phi_0(x_3),\end{aligned} \tag{B.27}$$

so that the term A in the expansion of $\overline{\phi}$ can be written as $A = A^{(0)} + A^{(1)}$, where $A^{(0)}$ is the free contribution (B.17), whereas $A^{(1)}$ is the latter correction.

Now, like for the 2-point function, we can write the 3-point function as:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle_{\phi_0} = \frac{\delta^2 \langle \mathcal{O}(x_1)\rangle_{\phi_0}}{\delta\phi_0(x_2)\delta\phi_0(x_3)} = (2\Delta_+ - d)\frac{\delta^2 A(x_1)}{\delta\phi_0(x_2)\delta\phi_0(x_3)}. \tag{B.28}$$

Since the free part $A^{(0)}$ is proportional to ϕ_0, its second derivative vanishes, thus only $A^{(1)}$ gives a finite contribution:

$$\frac{\delta^2 A^{(1)}(x_1)}{\delta\phi_0(x_2)\delta\phi_0(x_3)} = -\frac{2\lambda}{2\Delta_+ - d}\int dz d^d x\sqrt{g}K(z;x-x_1)K(z;x-x_2)K(z;x-x_3), \tag{B.29}$$

which means that the 3-point function of $\mathcal{O}$ in the boundary CFT is:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle_{\text{CFT}} = -2\lambda\int dz d^d x\sqrt{g}K(z;x-x_1)K(z;x-x_2)K(z;x-x_3). \tag{B.30}$$

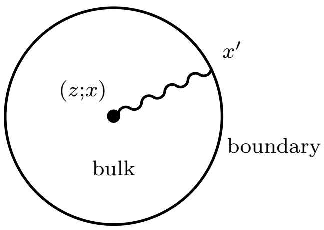
Exploiting the transformation properties of the bulk-to-boundary propagator one can perform the integral and show that:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle_{\text{CFT}} \sim \frac{1}{|x_1 - x_2|^{\Delta_+}|x_1 - x_3|^{\Delta_+}|x_2 - x_3|^{\Delta_+}}, \tag{B.31}$$

which is the expected result for the 3-point function in a conformal field theory.

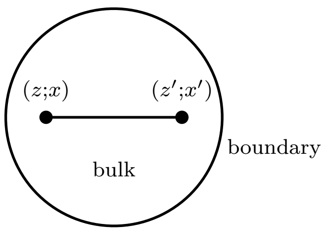
Looking at the form of (B.30) we can give a diagrammatic interpretation of the contributions to the correlation functions. Indeed we have defined K as the bulk-to-boundary propagator because it connects a point in the bulk to a point on the boundary, thus we can imagine to represent it as a line which starts in the bulk and ends on the boundary:

$K(z;x-x') \quad =$

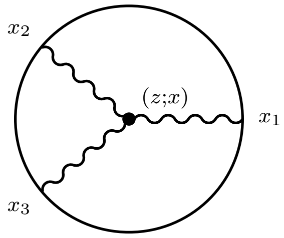


On the contrary, the bulk-to-bulk propagator connects two different points in the bulk. Therefore we can represent $G(z,x;z',x')$ as a line in the interior of the bulk connecting the points (z,x) and (z',x'):

$G(z,x;z',x') \quad =$

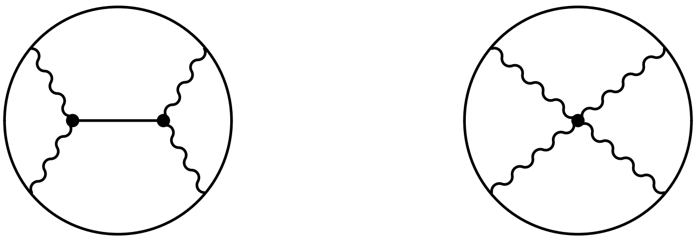


Through this method we can represent the contributions of the correlation functions as diagrams with as many bulk-to-boundary lines as functions K there are in the integral and as many bulk-to-bulk lines as Green's functions G. For example, the 3-point function is represented by:



where the integral $\lambda \int K(z; x - x_1)K(z; x - x_2)K(z; x - x_3)$ is represented by an interaction of three bulk-to-boundary propagators happening in the bulk at (z, x).

Using this representation we can argue the contributions of every correlation function. For example, the 4-point function is determined by two different terms:



The first diagram is of order λ because it comes from the term $\lambda\phi^3$ in $V(\phi)$ and involves two vertices connected by a bulk-to-bulk propagator G. The second diagram is the contribution coming from the term $\mu\phi^4$ in $V(\phi)$ so it is of order μ and involves the vertex of four bulk-to-boundary propagators.

This is the standard technology for the computation of the correlation functions of the operator $\mathcal{O}$ in the boundary CFT. We have shown that this method provides the expected results for the 1-, 2- and 3-point functions. This provides a consistency verification of the AdS/CFT correspondence statement (3.2.9).

Let us remark that this method yields finite and well-defined results only after the holographic renormalization has removed the divergences on the bulk action.

APPENDIX C

Solution of the RG equations

In this appendix we explicitly solve the evolution equations (4.2.29) and (4.2.30) of the parameters J and F of the UV amplitude (4.2.27) for the free scalar field in the bulk considered in subsection 4.2.1:

$$\frac{1}{\sqrt{g}}\partial_\epsilon(\sqrt{h}J(k,\epsilon)) = -J(k,\epsilon)F(k,\epsilon) \tag{C.1}$$

$$\frac{1}{\sqrt{g}}\partial_\epsilon(\sqrt{h}F(k,\epsilon)) = -F^2(k,\epsilon) + (v^{-2}k^2 + m^2). \tag{C.2}$$

The general solutions can be specified in terms of initial conditions $J_0 \coloneqq J(\epsilon_0)$ and $F_0 \coloneqq F(\epsilon_0)$ defined on the regulated boundary. Looking at the first order form of the bulk e.o.m.:

$$\partial_z\phi_k = \frac{z^{-2}}{\sqrt{g}}\pi_k, \qquad \partial_z\pi_k = \sqrt{g}(v^{-2}k^2 + m^2)\phi_k, \tag{C.3}$$

where π is the conjugate momentum of ϕ, we can see that J and F can be parametrized in terms of the general solution $(\widehat{\phi}, \widehat{\pi})$ of such equation:

$$J(k,\epsilon) = \frac{1}{\sqrt{h}\widehat{\phi}_k(\epsilon)}, \qquad F(k,\epsilon) = \frac{\widehat{\pi}_k(\epsilon)}{\sqrt{h}\widehat{\phi}_k(\epsilon)}. \tag{C.4}$$

This means that the initial conditions (J_0, F_0) can be translated into the boundary conditions of $(\widehat{\phi}, \widehat{\pi})$. Since the metric is asymptotic to AdS_{d+1} near the boundary, for $\epsilon_0 \mapsto 0$ we have:

$$\widehat{\phi}_k(\epsilon_0) \sim \alpha_k\epsilon_0^{\Delta_-} + \beta_k\epsilon_0^{\Delta_+}. \tag{C.5}$$

In such a way the functions α and β specify the solution $(\widehat{\phi}, \widehat{\pi})$. Thus if we pick two independent particular solutions $(\widehat{\phi}_1, \widehat{\pi}_1)$ and $(\widehat{\phi}_2, \widehat{\pi}_2)$ of (C.3), such that:

$$\widehat{\phi}_1 \sim z^{\Delta_-}, \quad \widehat{\phi}_2 \sim z^{\Delta_+} \quad \text{for } z \mapsto 0,$$

the general solution can be expressed as:

$$\widehat{\phi}_k = \alpha_k \widehat{\phi}_1(\epsilon, k) + \beta_k \widehat{\phi}_2(\epsilon, k) \qquad \widehat{\pi}_k = \alpha_k \widehat{\pi}_1(\epsilon, k) + \beta_k \widehat{\pi}_2(\epsilon, k). \tag{C.6}$$

For this reason, using (C.4) evaluated at ϵ_0, we can determine the relation between (J_0, F_0) and (α, β):

$$\alpha = \frac{\Delta_+ - F_0}{2\nu J_0} \epsilon_0^{\Delta_+} \qquad \beta = \frac{F_0 - \Delta_-}{2\nu J_0} \epsilon_0^{\Delta_-}. \tag{C.7}$$

Instead of using β, it is more convenient to define $\chi := \beta/\alpha$, so that the initial conditions of J and F can be expressed in terms of (α, χ). Thus we have that:

$$\chi = \frac{F_0 - \Delta_-}{\Delta_+ - F_0} \epsilon_0^{-2\nu}. \tag{C.8}$$

At this point we can write the solutions of (4.2.29) and (4.2.30) as:

$$\sqrt{h} J(k, \epsilon) = \frac{1}{\alpha_k} \frac{1}{\widehat{\phi}_1(\epsilon, k) + \chi_k \widehat{\phi}_2(\epsilon, k)} \tag{C.9}$$

$$\sqrt{h} F(\epsilon, k) = \frac{\widehat{\pi}_1(\epsilon, k) + \chi_k \widehat{\pi}_2(\epsilon, k)}{\widehat{\phi}_1(\epsilon, k) + \chi_k \widehat{\phi}_2(\epsilon, k)}. \tag{C.10}$$

In the case of pure AdS_{d+1} metric (3.1.8) we can obtain an explicit solution in the regime of small momenta ($\epsilon k \ll 1$). Indeed, in such case we can consider:

$$\widehat{\phi}_1 = z^{\Delta_-}, \qquad \widehat{\phi}_2 = z^{\Delta_+}, \tag{C.11}$$

so that from (C.9) and (C.10) we obtain:

$$J(\epsilon) = \frac{1}{\alpha} \frac{\epsilon^{\Delta_+}}{1 + \chi \epsilon^{2\nu}} \tag{C.12}$$

$$F(\epsilon) = \frac{\Delta_- + \Delta_+ \chi \epsilon^{2\nu}}{1 + \chi \epsilon^{2\nu}}. \tag{C.13}$$

Let us remark that the solution $(\widehat{\phi}, \widehat{\pi})$ of (C.3) does not correspond to the classical solution $\overline{\phi}$ where the on-shell bulk action is evaluated. This is due to the fact that the boundary conditions specified by (α, χ) of $\widehat{\phi}$ depends on the initial conditions (J_0, F_0) of the flow of the parameters J and F of S_{UV}, whereas $\overline{\phi}$ is defined through (4.1.4) plus some regularity condition at $z = +\infty$ in order to determine a single deformation of the boundary CFT.

Using hWRG we have seen that the Wilsonian action (4.2.32) of the dual QFT contains a single- and a double-trace deformation in the operator $\mathcal{O}$, respectively determined by the couplings λ and f. Since they are related to J and F through:

$$\lambda(k, \epsilon) = -\frac{J(k, \epsilon)}{F(k, \epsilon)} \qquad f(k, \epsilon) = -\frac{1}{F(k, \epsilon)}, \tag{C.14}$$

the solutions of their RG equations (4.2.35) and (4.2.36) can be found from the solutions of the evolution equations of J and F. Therefore in pure AdS_{d+1} and for $\epsilon k \ll 1$ we have:

$$\lambda(k,\epsilon) = -\frac{1}{\alpha_k}\frac{\epsilon^{\Delta_+}}{\Delta_- + \Delta_+\chi_k\epsilon^{2\nu}} \qquad f(k,\epsilon) = -\frac{1+\chi_k\epsilon^{2\nu}}{\Delta_- + \Delta_+\chi_k\epsilon^{2\nu}}. \tag{C.15}$$

If we want the solutions for the more general metric (4.1.5) we can use (C.9), (C.10) and write it in terms of $(\widehat{\phi}_1, \widehat{\pi}_1)$ and $(\widehat{\phi}_2, \widehat{\pi}_2)$:

$$\lambda(k,\epsilon) = -\frac{1}{\alpha_k}\frac{1}{\widehat{\pi}_1(\epsilon,k) + \chi_k\widehat{\pi}_2(\epsilon,k)} \qquad f(k,\epsilon) = -\sqrt{h}\frac{\widehat{\phi}_1(\epsilon,k) + \chi_k\widehat{\phi}_2(\epsilon,k)}{\widehat{\pi}_1(\epsilon,k) + \chi_k\widehat{\pi}_2(\epsilon,k)}. \tag{C.16}$$

The imposition of the standard quantization requires that the initial value $f(\epsilon_0) = f_-$. This implies that $F_0 = \Delta_+$ and so, from (C.8), we have that $\chi = \infty$. Hence:

$$f(k,\epsilon) = -\sqrt{h}\frac{\widehat{\phi}_2(\epsilon,k)}{\widehat{\pi}_2(\epsilon,k)}. \tag{C.17}$$

Indeed at $\epsilon = \epsilon_0$ we have $\widehat{\phi}_2 \sim \epsilon_0^{\Delta_+}$ and $\widehat{\pi}_2 \sim \Delta_+\epsilon_0^{-\Delta_-}$, thus $f(\epsilon_0) = -1/\Delta_+ = f_-$. This is the general solution for the double-trace coupling of the Wilsonian action.

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