ALGANT MASTER THESIS

Resolution of singularities for algebraic surfaces

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1 Introduction

This thesis discusses Lipman's method of the resolution of singularities for algebraic surfaces based on [2].

More precisely, let X be a surface, i.e. a Noetherian, normal, connected, and excellent scheme. Denote $X = X_0$ and define a sequence of surfaces

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow ... \tag{1.1}$$

Here, X_i is the normalization of the blowing-up of X_{i-1} along its singular locus, and the arrows are the corresponding morphisms $f_i : X_i \to X_{i-1}$. According to Lipman, when i is large enough, X_i is nonsingular.

The proof starts by showing that for sufficiently large n in the above sequence, all points of X_n are rational singularities (Section 10). In Section 10.1, it is shown that nonsingular points are rational singularities. At a singular point p, the lengths of the stalks of $R^1 f_* \mathscr{O}_{X'}$ are bounded independently of modifications $f : X' \to X$ (Section 10.2). These facts about the length are due to the theory of dualizing sheaves discussed in Section 8. With these two facts, one can find some sequence of surfaces that blows-up the singular points into rational singularities.

Now the problem is reduced to the case when all points of a surface X_0 are rational singularities. The second step of the proof, discussed in Section 12, is to show that when $m \gg 0$, the dualizing sheaf of X_m in (1.1) is locally free. From this, all rational singularities can be shown to have multiplicity at most 2.

Finally, the schematic-fibre of rational singularities are analysed . For each point with multiplicity 2, its schematic-fibre under the blow-up map is either a nonsingular curve, a reduced cone, or a double line, and all singular points in the fibre can be resolved by a sequence of normalized blow-up into nonsingular points.

Due to time constraints, only the first two steps are discussed in this thesis. Also, the following restriction is made.

Definition 1.1. A surface X is an integral normal separated scheme of finite type and of dimension 2 over an infinite perfect closed field k.

2 Multiplicity

The aim of this section is to prove the following proposition:

Proposition 2.1. Let A be a d-dimensional Cohen-Macaulay local ring with maximal ideal m and infinite residue field k. Then there exists a regular sequence $a_1, \cdots, a_d \in A$ such that the multiplicity of m is the length of the ring $A/(a_1, \cdots, a_d)$.

To do so, this section collects a list of facts about multiplicity and other concepts of commutative algebra from [11].

2.1 Definition of multiplicity

The lemma below motivates the definition of multiplicity.

Lemma 2.2. Let A be a d-dimensional Noetherian local ring and I an m-primary ideal. For every finitely generated A-module M, denote $l(M)$ the length of M as an A-module. Then for sufficienly large n, there is some $e(I, M) \in \mathbb{Z}$ such that

$$l(M/I^{n+1}M) = \frac{e(I, M)}{d!} n^d + \text{(lower order terms)}$$

Definition 2.3. The $e(I, M)$ in the above definition is called the multiplicity of I. If m is the maximal ideal of A, $e(m)$ denotes $e(m, A)$.

2.2 Regular sequences

Definition 2.4. Let A be a ring and M be an A-module. Then a sequence of elements $a_1, \cdots a_n \in A$ is called an M-regular sequence if $M/(a_1, \cdots, a_n)M \neq 0$ and for $1 \leq i \leq n$, a_i is not a zero-divisor of $M/(a_1, \cdots, a_{i-1})M$.

Definition 2.5. Let A be a ring, M be an A-module, and $a_1, \cdots, a_n \in A$.

Define $K_0 = A$, $K_i = \oplus_{1 \leq j_1 < \cdots < j_i \leq n} A e_{j_1 \cdots j_n}$ for $1 \leq i \leq n$, and $K_i = 0$ for $i > n$. For $i \geq 2$, define the following map:

$$\begin{aligned} d : K_i &\to K_{i-1} \\ e_{j_1 \cdots j_i} &\mapsto \sum_{k=1}^{i} (-1)^{k-1} a_{j_k} e_{j_1 \cdots \hat{j_k} \cdots j_i}. \end{aligned}$$

Also define $d(e_j) = x_j$. This d is a differential map that defines a complex $K_\bullet$, called the Koszul complex. The homology groups are denoted $H_i(a, M)$.

Lemma 2.6. Let A be a ring, M be an A-module, and $a_1, \cdots a_n \in A$ be an M-regular sequence. Then the only nonzero homology group of the Koszul complex of A is $H_0(a, M) = M/(a_1, \cdots, a_n)M$.

2.3 System of parameters

Lemma 2.7. Let A be a d-dimensional Noetherian local ring and m the maximal ideal. Then every generating system of an m-primary ideal has at least d elements. Also, there exists an m-primary ideal generated by d elements.

Definition 2.8. Let A be a d-dimensional Noetherian local ring and m the maximal ideal. Any d-element subset of A that generates an m-primary ideal is called a system of parameters.

The following formula relates Koszul complexes and multiplicities.

Proposition 2.9. Let A be a d-dimensional Noetherian local ring and M a finitely generated A-module. Let $\{a_1, \cdots a_d\}$ be a system of parameters and $I = (a_1, \cdots, a_d)$. Then we have

$$e(I, M) = \sum_{i=0}^{d} (-1)^i l(H_i(a, M)).$$

Lemma 2.10. Let A be a d-dimensional Noetherian local ring with maximal ideal m and infinite residue field k. Let I be an m-primary ideal. Then there is a system of parameters $\{a_1, \cdots a_d\}$ such that

$$e(I, A) = e((a_1, \cdots, a_d), A).$$

Lemma 2.11. [11, Theorem 17.4(iii)] Let A be a Noetherian local ring with maximal ideal m. Let $a_1, \cdots, a_r \in m$. Then if $a_1, \cdots, a_r$ is an A-regular sequence, there is a system of parameters of A that contains $a_1, \cdots, a_r$.

2.4 Depth and Cohen-Macaulay rings

Lemma 2.12. Let A be a Noetherian ring, I be an ideal of A, and M a finitely generated A-module such that $M \neq IM$. Then all maximal M-regular sequences in I have the same length.

Definition 2.13. Let A be a Noetherian ring, I be an ideal of A, and M a finitely generated A-module such that $M \neq IM$. The length of a maximal M-regular sequence in I is called the I-depth of M and is denoted as $\mathrm{depth}(I, M)$.

If A is a local ring with maximal ideal m, $\mathrm{depth}(m, M)$ is also called the depth of M and is also denoted $\mathrm{depth}(M)$.

Definition 2.14. A Noetherian local ring A is called a Cohen-Macaulay ring if $\mathrm{depth}(A) = \dim(A)$.

A Noetherian ring A is called a Cohen-Macaulay ring if for every maximal ideal m, A_m is Cohen-Macaulay.

Lemma 2.15. Regular local rings are Cohen-Macaulay.

Local rings of surfaces are Cohen-Macaulay.

Lemma 2.16. Every integral normal ring of dimension 2 is Cohen-Macaulay.

Proof. By the Auslander-Buchsbaum formula, $\mathrm{depth}(A_m) \leq \dim(A_m) = 2$. The reverse inequality follows from the Serre criterion of normality. □

Over local Cohen-Macaulay rings, systems of parameters coincide with regular sequences.

Lemma 2.17. [11, Theorem 17.4(iii)] Let A be a Cohen-Macaulay local ring with maximal ideal m. Let $a_1, \cdots, a_r \in m$. Then $a_1, \cdots, a_r$ is an A-regular sequence if and only if there is a system of parameters of A that contains $a_1, \cdots, a_r$.

The proof of Proposition 2.1 can now be given.

Proof. Let $\{a_1, \cdots a_d\}$ be a system of parameters such that $e(I, A) = e((a_1, \cdots, a_d), A)$. Since this system produces an A-regular sequence, all homology groups of positive degrees are zero for the Koszul complex with the sequence $a_1, \cdots, a_d$. Therefore, $e(I, A) = l(A/(a_1, \cdots, a_d))$. □

This section ends with the definition of multiplicity over a scheme.

Definition 2.18. Let X be a reduced scheme of finite type over a field k. Then the multiplicity of a point $x \in X$ is the multiplicity of the maximal ideal m_x in the local ring $\mathscr{O}_{X,x}$.

3 Blow-ups

3.1 Existence of blow-ups

Blow-ups is an essential tool to resolve singularities. The material of this section is based on [8, Section (13.19)].

Definition 3.1. [8, Definition 13.90] Let X be a scheme and Z be a closed subscheme. A blow-up of X along Z is a scheme X' and a morphism $f : X' \to X$ such that $f^{-1}(Z)$ is an effective Cartier divisor and X' and f satisfies the universal property: if $f' : X'' \to X$ is any morphism such that $f'^{-1}(Z)$ is an effective Cartier divisor, there exists a unique morphism $g : X'' \to X'$ such that $f' = f \circ g$.

Definition 3.2. With notations as above, Z is called the centre of the blow-up and $f^{-1}(Z)$ the exceptional divisor of the blow-up.

Blow-ups exist for every closed subscheme. With the above universal property, a blow-up is unique up to an unique isomorphism. The following two lemma describes blow-ups and their exceptional divisors explicitly.

Lemma 3.3. [8, Proposition 13.92] Let Z be a closed subscheme of a scheme X with associated ideal sheaf $\mathscr{I}$. Then the blow-up of X along Z is $\mathrm{Proj}(\oplus_{d\geq 0}\mathscr{I}^d)$.

Lemma 3.4. [8, Remark 13.94] Let Z be a closed subscheme of a scheme X with associated ideal sheaf $\mathscr{I}$. Then the exceptional divisor of the blow-up of X along Z is $\mathrm{Proj}(\oplus_{d\geq 0}\mathscr{I}^d/\mathscr{I}^{d+1})$.

3.2 Properties of Blow-up

We first notice that blow-up can be studied locally.

Lemma 3.5. [8, Proposition 13.91(2)] Let X be a scheme, Z a closed subscheme of X, and $f : X' \to X$ the blow-up of X along Z. Further let U be an open subset of X and U' be the blow-up of U along $Z \cap U$. Then $f^{-1}(U) = U'$.

Lemma 3.6. [8, Proposition 13.91(3)] Let X be a scheme, Z a closed subscheme of X, and $f : X' \to X$ the blow-up of X along Z. Then $f|_{f^{-1}(X\setminus Z)}$ is an isomorphism.

With these lemmas, one can study the behaviour of the blow-up locally, such as studying the case when X is an affine open scheme.

Lemma 3.7. Let A be a ring and I be an ideal of A generated by n elements. Suppose $X = \mathrm{Spec}(A)$ and $b : X' \to X$ is the blow-up of X along $V(I) \subset X$. Then b is a projective morphism and X' is isomorphic to a closed subscheme of $\mathbb{P}^{n-1}_A$.

Proof. Let $I = (a_0, \cdots, a_{n-1})$. Then there is a surjective A-homomorphism:

$$\begin{aligned} \alpha : A[x_0, x_1, \cdots, x_{n-1}] &\to A \oplus I \oplus I^2 \oplus \cdots, \\ 1 &\mapsto 1 \in A, \\ x_i &\mapsto a_i \in I. \end{aligned}$$

Taking Proj on both sides, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^{n-1}_A & \xleftarrow[i]{} & X' \\ & \searrow^{\bar{b}} & \downarrow b \\ & & X \end{array}$$

with $X' \cong V^+(\ker \alpha)$. □

3.3 Blow-ups over regular surfaces

In this subsection, the proof of the following lemma, as discussed in [1], will be given:

Lemma 3.8. The blow-up of a nonsingular scheme along a nonsingular closed subscheme is nonsingular.

Since nonsingularity of a scheme is a local property, the above lemma can be proved by using an affine open cover. The above lemma restricted to an affine scheme then becomes:

Lemma 3.9. Let A be a nonsingular ring. Let I be an ideal of A such that A/I is nonsingular. Then $\mathrm{Proj}(\oplus_{d \geq 0} I)$ is a nonsingular scheme.

To prove this lemma, the following facts about nonsingular rings and regular sequences are collected:

Lemma 3.10. [12, Tag 00NR] Let A be a nonsingular local ring with an ideal I such that A/I is also a nonsingular local ring. Then there exists a system of parameters $\{a_1, \cdots, a_d\}$ such that

- this system $\{a_1, \cdots, a_d\}$ generates m and,
- there exists $1 \leq c \leq d$ such that $I = (a_1, \cdots, a_d)$.

Lemma 3.11. [12, Tag 00NU] Let A be a Noetherian local ring. Let $a_1, \cdots, a_c$ be a A-regular sequence such that $A/(a_1, \cdots, a_c)$ is a nonsingular local ring. Then A is also a nosingular local ring.

Lemma 3.12. Let A be a Noetherian ring, and let $I \subset A$ be an ideal generated by a A-regular sequence $a_1, \cdots, a_c$. Then there is the following isomorphism

$$(A/I)[x_1, \cdots, x_c] \to \oplus_{d \geq 0} I^d / I^{d+1},$$
$$x_i^d \mapsto \overline{a_i}^d \in I^d / I^{d+1}.$$

Using these facts, the nonsingularity of the blowup can be proved. We shall first prove that the exceptional divisor is nonsingular. Note that is the exceptional divisor of the blow-up of Spec(A) along the closed subscheme associated to I is $\mathrm{Proj}(\oplus_{d \geq 0} I^d / I^{d+1})$.

Lemma 3.13. Let A be a nonsingular ring. Let I be an ideal of A such that A/I is nonsingular. Then $\mathrm{Proj}(\oplus_{d \geq 0} I^d / I^{d+1})$ is also nonsingular.

Proof. Let $p \in \mathrm{Spec}(A)$. To prove this lemma, we consider the blow-up of $\mathrm{Spec}(A_p)$ along the closed subschmeme associated to the ideal I_p. Note that this is possible since $\mathrm{Spec}(A_p)$ is the intersection of all open subschemes that contain p. Then the exceptional divisor is $\mathrm{Proj}(\oplus_{d \geq 0} I_p^d / I_p^{d+1})$.

Using Lemma 3.10, there is a A_p-regular sequence such that $I_p = (a_1, \cdots, a_c)$. Then by Lemma 3.12, $\mathrm{Proj}(\oplus_{d \geq 0} I_p^d / I_p^{d+1}) = \mathbb{P}^{c-1}_{A_p/I_p}$, which is nonsingular. □

Now the proof of Lemma 3.8 will be given.

Proof. First note that b induces an isomorphism from $X' \setminus b^{-1}(x)$ to $X \setminus \{p\}$. Thus X' is nonsingular outside $b^{-1}(x)$.

Let $x' \in b^{-1}(x)$. Since $b^{-1}(x)$ is an effective Cartier divisor, there is some $a \in \mathscr{O}_{X',x'}$ such that a is not a zero divisor and $\mathscr{O}_{b^{-1}(x),x'} = \mathscr{O}_{X',x'}/(a)$. Since $\mathscr{O}_{b^{-1}(x),x'}$ is nonsingular, $\mathscr{O}_{X',x'}$ is nonsingular by Lemma 3.11. □

4 Grassmannian and line bundles on closed subschemes

4.1 Grassmannian functor

The Grassmannian functor classifies a collection of locally free subsheaves and can be seen as a scheme defined through a functor.

Definition 4.1. [8, Section (8.4)] Let $0 \leq d \leq n$. The Grassmannian functor $\mathrm{Grass}_{d,n} : (\mathrm{Sch})^{\mathrm{op}} \to (\mathrm{Sets})$ is the contravariant functor defined as follows

- For every scheme S, define

$$\begin{aligned}\mathrm{Grass}_{d,n}(S) = \{&\mathscr{U} \subset \mathscr{O}_S^n : \\ &\mathscr{O}_S^n/\mathscr{U} \text{ is a locally free } \mathscr{O}_S\text{-submodule of rank } n-d\}.\end{aligned}$$

- For every morphism $f : T \to S$, define

$$\begin{aligned}\mathrm{Grass}_{d,n}(f) : \mathrm{Grass}_{d,n}(S) &\to \mathrm{Grass}_{d,n}(T), \\ \mathscr{U} &\mapsto f^*\mathscr{U}.\end{aligned}$$

Lemma 4.2. [8, Lemma 8.13 and Corollary 8.15] The functor $\mathrm{Grass}_{d,n}$ is represented by a scheme locally covered by a finite number of copies of $\mathbb{A}_{\mathbb{Z}}^{d(n-d)}$.

The Grassmannian functor can be generalized. In the above definition of the Grassmannian functor, the $\mathscr{O}_S^n$ can be replaced by an arbitrary quasi-coherent sheaf, as follows.

Definition 4.3. [8, Section (8.6)] Let $e \geq 0$, X a scheme, and $\mathscr{E}$ a quasi-coherent $\mathscr{O}_X$-module. The Grassmannian functor $\mathrm{Grass}^e(\mathscr{E}) : (\mathrm{Sch}/X)^{\mathrm{op}} \to (\mathrm{Sets})$ is a contravariant functor defined as follows:

- For every X-scheme $h : S \to X$, define

$$\begin{aligned}\mathrm{Grass}^e(\mathscr{E})(S) = \{&\mathscr{U} \subset h^*(\mathscr{E}) : \\ &h^*(\mathscr{E})/\mathscr{U} \text{ is a locally free } \mathscr{O}_S\text{-submodule of rank } e\}.\end{aligned}$$

- For every morphism $f : T \to S$, define

$$\begin{aligned}\mathrm{Grass}_{d,n}(f) : \mathrm{Grass}_{d,n}(S) &\to \mathrm{Grass}_{d,n}(T), \\ \mathscr{U} &\mapsto f^*\mathscr{U}.\end{aligned}$$

Lemma 4.4. [8, Proposition 8.17(1)] The functor $\mathrm{Grass}^e(\mathscr{E})$ is represented by a X-scheme.

The following simple lemma relates both version of the Grassmannian functors.

Lemma 4.5. The equality $\mathrm{Grass}^e(\mathscr{O}_X^n) = \mathrm{Grass}_{n-e,n} \times_{\mathbb{Z}} X$ holds.

Proof. Consider a X-scheme $h : T \to X$. Then

$$\begin{aligned} &(\mathrm{Grass}_{n-e,n} \times_{\mathbb{Z}} X)(T) \\ =&(\mathrm{Grass}_{n-e,n} \times X)(T) \\ =&\,\mathrm{Grass}_{n-e,n}(T) \times X(T) \\ =&\,\mathrm{Grass}_{n-e,n}(T) \times \{h\}. \end{aligned}$$

Since $h^*(\mathscr{O}_X^n) = \mathscr{O}_T^n$, there is a bijection

$$\mathrm{Grass}^e(\mathscr{O}_X^n)(T) \to (\mathrm{Grass}_{n-e,n} \times_{\mathbb{Z}} X)(T)$$

functorial in T. □

Before continuing, some properties of modules of finite presentation and finite type are recalled.

Lemma 4.6. [8, Propositon 7.28] Let X be a ringed space. Consider an exact sequence of $\mathscr{O}_X$-modules:

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

with $\mathscr{F}$ of finite presentation and $\mathscr{F}''$ of finite type. Then $\mathscr{F}'$ is of finite type.

Lemma 4.7. [8, Proposition 8.10] Let X be a scheme, and $\iota : \mathscr{F}' \to \mathscr{F}$ be a a homomorphism of $\mathscr{O}_X$-modules. Further suppose that $\mathscr{F}'$ is of finite type and $\mathscr{F}$ is locally free of finite type. Then the following statements are euqivalent.

1. The map ι is injective and $\mathscr{F}'' := \mathscr{F}/\iota(\mathscr{F}')$ is locally free of finite type.

2. For every open affine subset U of X, there exists a homomorphism $\pi : \mathscr{F}|_U \to \mathscr{F}'|_U$ such that $\rho \circ \iota|_U = \mathrm{id}$

3. If $f : T \to X$ is a morphism of schemes, $f^*(\iota) : f^*(\mathscr{F}') \to f^*(\mathscr{F})$ is injective.

4. If $x \in X$, the morphism $\iota_0 := \iota \otimes \mathrm{id}_{\kappa(x)} : \mathscr{F}'(x) \to \mathscr{F}(x)$ is injective.

5. The $\mathscr{O}_X$-module $\mathscr{F}'$ is locally free of finite type and the dual $\iota^\vee : \mathscr{F}^\vee \to \mathscr{F}'^\vee$ is surjective.

Proof. (1 $\Rightarrow$ 2) Let U be an affine open subset of X and $U = \mathrm{Spec}(A)$. Since $\mathscr{F}$ and $\mathscr{F}''$ are locally free and hence quasi-coherent, there exists free A-modules M and M'' such that $\tilde{M} = \mathscr{F}|_U$ and $\tilde{M''} = \mathscr{F}|_U$.

Denote $\pi : \mathscr{F} \to \mathscr{F}''$ and $p : M \to M''$. Then $\tilde{p} = \pi|_U$. By [8, Proposiiton 7.14], $\ker(\pi|_U) = \ker(p)^{\sim}$. Let $M' = \ker(p)$. Then $\mathscr{F}'|_U = \tilde{M'}$.

The above discussion yields the following exact sequence:

$$0 \to M' \xrightarrow{i} M \to M'' \to 0.$$

Since M'' is free, M'' is projective. Then the exact sequence splits and i has an left inverse r. One can then take $\rho = \tilde{r}$.

(2 $\Rightarrow$ 3) By applying f^* to both sides of $\rho \circ \iota = \mathrm{id}$, $f^*(\iota)$ has a left inverse $f^*(\rho)$.

(3 $\Rightarrow$ 4) Replace f in statement 3 by the morphism $\mathrm{Spec}(\kappa(x)) \to X$.

(4 $\Rightarrow$ 1) The injectivity of ι will be checked first. Let $x \in X$. By assumption, the map ι_0 is an injective morphism of $\kappa(x)$-vector space. Therefore, there is an left inverse ρ_0 of ι_0.

Now note that the morphism $\mathscr{F}_x \to \mathscr{F}(x) \xrightarrow{\rho_0} \mathscr{F}'(x)$ is surjective. Also, $\mathscr{F}_x$ is locally free of finite type and thus projecive. Therefore, there is a morphism $\rho_1 : \mathscr{F}_x \to \mathscr{F}'_x$ such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathscr{F}'_x & \xrightarrow{\iota_x} & \mathscr{F}_x & \xrightarrow{\rho_1} & \mathscr{F}'_x \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{F}'(x) & \xhookrightarrow{\iota_0} & \mathscr{F}(x) & \xrightarrow{\rho_0} & \mathscr{F}'(x).
\end{array}$$

Note that $\mathscr{F}'_x = \mathrm{im}(\rho_1 \circ \iota_x) + m_x \mathscr{F}'_x$. By Nakayama's lemma, $\rho_1 \circ \iota_x$ is surjective, hence bijective. Then ι has a left inverse $(\rho_1 \circ \iota_x)^{-1} \circ \rho_1$ and is injective.

The above discussion also shows that the following exact sequence:

$$0 \to \mathscr{F}'_x \xrightarrow{\iota_x} \mathscr{F}_x \to \mathscr{F}''_x \to 0$$

splits. As such, $\mathscr{F}''_x$ is a direct summand of $\mathscr{F}_x$ the free $\mathscr{O}_{X,x}$-module. Since $\mathscr{F}'$ and $\mathscr{F}$ is locally free of finite type, $\mathscr{F}''$ is of finite presentation. By [8, Proposition 7.47], $\mathscr{F}''$ is locally free of finite type.

(4 $\Rightarrow$ 5) Up to this point, it is shown that statements 1-4 are equivalent. Therefore, it is possible to continue the discussion in the proof of (1 $\Rightarrow$ 2) as follows. Since the exact sequence $0 \to M' \to M \to M''$ splits, M' is a direct summand of M'' and thus free. Therefore, $\mathscr{F}' \cong \iota(\mathscr{F}')$ is locally free.

Since $\mathscr{F}'$ and $\mathscr{F}$ are locally free of finite type, the equality $(\iota_0)^\vee = \iota^\vee \otimes \mathrm{id}_{\kappa(x)}$ holds for all x. Since $\iota^\vee \otimes \mathrm{id}_{\kappa(x)}$ is surjective, the map $\iota_x^\vee$ is surjective by Nakayama's lemma.

(5 $\Rightarrow$ 4) Since $\mathscr{F}'$ and $\mathscr{F}$ are locally free of finite type, the equality $(\iota^\vee \otimes \mathrm{id}_{\kappa(x)})^\vee = \iota \otimes \mathrm{id}_{\kappa(x)}$ holds for all x. Since $\iota^\vee \otimes \mathrm{id}_{\kappa(x)}$ is surjective, $\iota \otimes \mathrm{id}_{\kappa(x)}$ is injective. □

Corollary 4.8. Let X be a scheme, $n \geq 1$, and $\mathscr{U} \subset \mathscr{O}_X^n$ such that $\mathscr{O}_X^n/\mathscr{U}$ is a locally free $\mathscr{O}_X$-submodule of rank $n-d$. Then $\mathscr{U}$ is locally free module of rank d.

Proof. Combine the previous 2 lemmas. □

4.2 Global sections of line bundles and closed subschemes

Lemma 4.9. Let k be an infinite field, X be a k-scheme of finite type, and Z be a n-dimensional closed subscheme. Let $\mathscr{L}$ be a line bundle on X generated by global sections. Then there exist $n+1$ global sections $s_0, \cdots s_n \in \Gamma(X, \mathscr{L})$ such that for every $z \in Z$ the $(s_i)_z$ generate the $\mathscr{O}_{X,z}$-module $\mathscr{L}_z$.

Proof. By collecting all global generators that generate the finite number of elements that generates $\mathscr{L}$ locally, one can show that $\mathscr{L}$ is generated by finitely many global sections. Fix the number of collected global sections to be N. This produces the following exact sequence

$$0 \to \mathscr{K} \xrightarrow{\iota} \mathscr{O}_X^N \to \mathscr{L} \to 0.$$

By Corollary 4.8, $\mathscr{K}$ is a direct summand of $\mathscr{O}_X^N$ and is locally free of rank $N-1$.

Notice that if there are $n+1$ elements $s_0, \cdots, s_n$ in $\Gamma(X, \mathscr{L})$ such that $\sum_{i=0}^n s_i|_Z \mathscr{O}_Z \not\subset \mathscr{K}|_Z$, the image of $\sum_{i=0}^n s_i|_Z \mathscr{O}_Z$ is the whole $\mathscr{L}|_Z$.

Consider the Grassmannian varieties

$$S = \mathrm{Grass}^{N-n-2}(\mathscr{K}|_Z), \qquad R = \mathrm{Grass}^{N-n-1}(\mathscr{O}_Z^N).$$

It will be shown that there is a morphism $u_1 : S \to R$ using the Yoneda's lemma. To do so, it will be first shown that $S(T) \subset R(T)$ for every Z-scheme T with structure morphism $u : T \to Z$.

Let $\mathscr{U} \in S(T)$. Applying statement 3 of Lemma 4.7, the inclusion relations $\mathscr{U} \subset u^*\mathscr{K}|_Z \subset u^*\mathscr{O}_Z^N$ holds.

To show that $u^*\mathscr{O}_Z^N$ is a locally free $\mathscr{O}_T$-module, one first consider the exact sequence:

$$0 \to \mathscr{U} \xrightarrow{\iota'} u^*\mathscr{K}|_Z \to u^*\mathscr{K}|_Z/\mathscr{U} \to 0.$$

Again by Lemma 4.7, the map $\iota'^\vee : (u^*\mathscr{K}|_Z)^\vee \to \mathscr{U}^\vee$ is also surjective. Also, since $\mathscr{K}$ and $\mathscr{O}_X^N$ satisfies statement 3 of Lemma 4.7, so does $u^*\mathscr{K}$ and $u^*\mathscr{O}_Z^N$. Then the morphism $(u^*\mathscr{O}_Z^N)^\vee \to (u^*\mathscr{K}|_Z)^\vee$ is surjective. Therefore, the morphism $(u^*\mathscr{O}_Z^N)^\vee \to (u^*\mathscr{K}|_Z)^\vee \to (\mathscr{U})^\vee$ is surjective. Since $\mathscr{U}$ is locally free, $u^*\mathscr{O}_Z^N/\mathscr{U}$ is locally free of rank $N-n-1$.

Now we have obtain the morphism $u_1 : S \to R$. Since $R = \mathrm{Grass}^{N-n-1}(\mathscr{O}_k^N) \times_{\mathrm{Spec}\, k} Z$ by Lemma 4.5, this produces another natural morphism

$$u_2 : R \to \mathrm{Grass}^{N-n-1}(\mathscr{O}_k^N)$$

Denote $P = \mathrm{Grass}^{N-n-1}(\mathscr{O}_k^N)$ and $u = u_2 \circ u_1 : S \to P$. We now need to show that $P \setminus u(S)$ has a k-valued point. Note u is not surjective, since $\dim S = (N-n-2)(n+1) + n < (N-n-1)(n+1) = \dim P$.

Since Z is a k-scheme of finite type, S is a projective Z-scheme, and P is a projective k-scheme, u is of finite type. Since P is covered by a finite number of copies of $\mathbb{A}_k^{(N-n-1)(n+1)}$, P is Noetherian. Then Chevalley's theorem ([8, Theorem 10.20]) shows that $u(S)$ is a finite union of locally closed subsets of P.

Notice that the closure of an irreducible component of $u(S)$ is an irreducible component of $\overline{u(S)}$. Also, all irreducible components of $\overline{u(S)}$ are obtained in this way. They have the same generic point and thus the same field of rational functions. Therefore, $\dim \overline{u(S)} = \dim u(S) < \dim T$.

Therefore, there exists an open subscheme $V \subset T$ such that $V \cap u(S) \neq \emptyset$ and such that V is isomorphic to an open subscheme of $\mathbb{A}_k^{(N-n-1)(n+1)}$. Let f be a polynomial such that $D(f) \subset V$. Since k is infinite, there is some $x \in k^{(N-n-1)(n+1)}$ such that $f(x) \neq 0$ and there is some k-point in $T \setminus \overline{u(S)}$.

Note that this closed point corresponds to a $(n+1)$-dimensional k-vector space $W \subset \Gamma(X, \mathscr{L})$. Let $s_1, \cdots, s_n$ be a basis of W. Then $\mathscr{U} = \oplus_{i=1}^n s_i|_Z \mathscr{O}_Z \not\subset \mathscr{K}|_Z$. □

Corollary 4.10. Let k be an infinite field, X be a k-scheme of finite type, and Z be a n-dimensional closed subscheme. Let $\mathscr{L}$ be a line bundle on X generated by global sections. Then there exist some open subscheme $U \subset X$ containing Z and $n+1$ global sections $s_0, \cdots s_n \in \Gamma(X, \mathscr{L})$ such that the $s_i|_U$ generate $\mathscr{L}|_U$.

Proof. Let $z \in Z$. By the previous lemma, there exist $n+1$ global sections $s_0, \cdots s_n \in \Gamma(X, \mathscr{L})$ such that for every $z \in Z$ the $(s_i)_z$ generate the $\mathscr{O}_{X,z}$-module $\mathscr{L}_z$. Then there is an open enighbourhood $U_z \in X$ of z such that $s_i|_{U_z}$ generate $\mathscr{L}|_{U_z}$. The proof is completed by taking $U = \cup_{z \in Z} U_z$ □

5 Modification

Definition 5.1. A morphism of schemes $f : X' \to X$ is called a modification if f is birational and projective.

5.1 Examples of modification

Let us give some examples of modifications.

Lemma 5.2. [8, Corollary 13.97] Blow-ups are modifications.

Lemma 5.3. Normalizations of surfaces are modifications.

Proof. [8, Proposition 12.44] implies that normalization is birational. Since X is quasi-excellent, by [8, Theorem 12.50], normalization is finite, hence projective. □

5.2 Properties of modifications

Proposition 5.4. Let $f : X' \to X$ be a modification of surfaces. Then there exist closed points $x_1, \cdots x_n \in X$ such that f induces an isomorphism $X' \setminus f^{-1}(\{x_1, ...x_n\}) \cong X \setminus \{x_1, ...x_n\}$

Proof. Let U be the maximal open subset of X such that $f|_{f^{-1}(U)}$ is an isomorphism. Since X' is integral and X is Noetherian, [12, Tag 0BFP] implies that U contains all $x \in X$ such that

- x is of codimension 0, or
- x is of codimension 1 and its stalk $\mathscr{O}_{X,x}$ is a discrete valuation ring.

Recall that normal rings of dimension 1 are regular and hence are discrete valuation rings. Since X is normal, all points of codimension 1 are in U.

This means that $X \setminus U$ is a closed subset of X and only has points of codimension 2, i.e. closed points. Therefore, $X \setminus U$ is a finite set of closed points of X, and the required isomorphism follows. □

The direct image functors attached to modifications maps structure sheaves to structure sheaves, in the following situation.

Proposition 5.5. [10, Corollary 4.4.3(a)] Let $f : X' \to X$ be a proper birational morphism from an integral scheme X' to a normal locally Noetherian scheme X. Then the canonical homomorphism $\mathscr{O}_{X'} \to f_*\mathscr{O}_X$ is an isomorphism.

5.3 Domination

Definition 5.6. Let $f : X' \to X$ be a modification of surfaces. The morphism f is said to be dominated by another modification $g : X'' \to X$ if the induced birational map $\pi : X'' \to X'$ is defined on the whole X''.

Lemma 5.7. The map π in the above definition is also a modification.

Proof. π is birational because $K(X'') = K(X') = K(X)$.

Now we need to show that π is projective. Since projective morphisms are separated, this is true by [6, (5.5.5)]. □

The following proposition is useful

Proposition 5.8. Let $f : X' \to X$ be a modification of surfaces. Then f is dominated by a sequence of normalized blow-up of closed points. More precisely, there are surfaces $X''_1, ..., X''_n$ such that the following diagram commutes and all maps in the upper row and the right column are normalized blow-ups.

$$\begin{array}{ccccccc} X''_n & \longrightarrow & \cdots & \longrightarrow & X''_2 & \longrightarrow & X''_1 \\ \downarrow & & & & & & \downarrow \\ X' & & & \xrightarrow{\quad f \quad} & & & X \end{array}$$

6 Homological algebra

6.1 Higher direct images

Lemma 6.1. [9, III, Corollary 8.2] Let $f : X' \to X$ be a continuous map of topological spaces and $\mathscr{F}$ be a sheaf of abelian groups on X'. Then for $q > 0$ and open $U \subset X$, $(R^q f_* \mathscr{F})|_U = R^q(f|_{f^{-1}(U)})_*(\mathscr{F}|_{f^{-1}(U)})$.

Lemma 6.2. [9, III, Corollary 8.8] Let $f : X' \to X$ be a projective morphism of Noetherian schemes. Let $\mathscr{F}$ be a coherent $\mathscr{O}_{X'}$-module. Then for $i \geq 0$, $R^i f_*(\mathscr{F})$ is a coherent $\mathscr{O}_{X'}$-module.

Proposition 6.3. [2, (1.5)(a)] Let $f : X' \to X$ be a birational and projective morphism of surfaces and $\mathscr{F}$ a coherent $\mathscr{O}_{X'}$-module. Then for $i \geq 2$, $R^i f_* \mathscr{F} = 0$. Also, $\dim \operatorname{supp} R^1 f_* \mathscr{F} = 0$. If $x \in R^1 f_* \mathscr{F}$, $(R^1 f_* \mathscr{F})_x$ is a finite length $\mathscr{O}_{X,x}$-module.

6.2 Spectral sequence

The material of this subsection is based on [13, Chapter 5]

Definition 6.4. Let $\mathcal{A}$ be an abelian category and $a \in \mathbb{Z}$. A (cohomology) spectral sequence in $\mathcal{A}$ consists of the following data:

- A family $\{E_r^{pq}\}_{p,q\in\mathbb{Z},r\geq a}$ of objects in $\mathcal{A}$.
- Maps $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ that satisfies $d_r^{p+r,q-r+1} \circ d_r^{pq} = 0$.
- An isomoprhism $E_{r+1}^{pq} \xrightarrow{\sim} \ker(d_r^{pq})/\operatorname{im}(d_r^{p-r,q+r-1})$

Note that the last isomorphism means that for $r \geq a$, E_{r+1}^{pq} is a quotient of some subobject of E_r^{pq}.

Definition 6.5. A spectral sequence is bounded if for all $n \in \mathbb{Z}$, there are only finitely many pairs (p, q) such that $p + q = n$ and E_a^{pq} is nonzero.

Definition 6.6. Let $\{E_r^{pq}\}$ be a bounded spectral sequence. Then for each p, q, there is some $r(p, q) \geq a$ such that for $r \geq r(p, q)$ $E_r^{pq} = E_{r(p,q)}^{pq}$. We denote $E_\infty^{pq} := E_{r(p,q)}^{pq}$

Definition 6.7. A bounded spectral sequence $\{E_r^{pq}\}$ is said to converge to H^* if there is a family of objects H^n of $\mathcal{A}$, each having a family of objects F^*H^n such that

- $F^pH^n \supset F^{p+1}H^n$ for all p,
- There exists $s_n < t_n$ such that $F^{s_n}H^n = H^n$ and $F^{t_n}H^n = 0$, and
- there is an isomorphism $E_\infty^{pq} \xrightarrow{\sim} F^pH^{p+q}/F^{p+1}H^{p+q}$.

Symbolically, such convergence is denoted $E_a^{pq} \Rightarrow H^{p+q}$. The family of objects H^* is called the limit of the spectral sequence.

By chasing the diagram around E_2^{pq}, the following result can be obtained:

Lemma 6.8. Let $\{E_r^{pq}\}_{r\geq 2}$ be a spectral sequence such that $E_r^{pq} = 0$ when p or q is negative and that converges to $\{H^n, F^pH^n\}_{n,p\in\mathbb{Z}}$. Then we have the exact sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2$$

Here $E_2^{0,1} \to E_2^{2,0}$ is the differential map. This exact sequence is called the exact sequence of low degree terms.

Theorem 6.9. (Grothendieck spectral sequence) Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two left exact functors between abelian categories with enough injective objects. If F maps injective objects of $\mathcal{A}$ to G-acyclic objects of $\mathcal{B}$, then for each $A \in \mathcal{A}$ there is a convergent spectral sequence:

$$E_2^{pq} = R^p G(R^q F(A)) \Rightarrow R^{p+q}(G \circ F)(A)$$

Substituting direct image functors to the above theorem and using the proof of [12, Tag 01F5], we obtain the following consequence:

Proposition 6.10. (Relative Leray spectral sequence) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then there is a spectral sequence $\{E_r^{pq}\}_{r\geq 2}$ of $\mathscr{O}_Z$-modules such that $E_2^{pq} = R^p g_*(R^q f_* \mathscr{F}) \Rightarrow R^{p+q}(g \circ f)_* \mathscr{F}$

Corollary 6.11. The exact sequence of low degree terms for Proposition 6.10 is

$$0 \to R^1 g_*(f_* \mathscr{F}) \to R^1(g \circ f)_* \mathscr{F} \to g_*(R^1 f_* \mathscr{F}) \to R^2 g_*(f_* \mathscr{F}) \to R^2(g \circ f)_* \mathscr{F}$$

Lemma 6.12. Let $f : X \to Y$ be an affine morphism of schemes and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then $R^q f_* \mathscr{F} = 0$ for $q > 0$

Proof. Let $V \subset Y$ be an affine open set. Then $f^{-1}(V)$ is affine. By [9, III, Theorem 3.7] of Hartshorne and Lemma 6.1, $R^q f_* \mathscr{F}(V) = H^i(f^{-1}(V), \mathscr{F}) = 0$. □

Proposition 6.13. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Further assume f to be affine. Then for $p \geq 0$, $R^p(g \circ f)_* \mathscr{F} \cong R^p g_*(f_* \mathscr{F})$

Proof. By Proposition 6.10, there is a spectral sequence $\{E_r^{pq}\}_{r\geq 2}$ of $\mathscr{O}_Z$-modules such that $E_2^{pq} = R^p g_*(R^q f_* \mathscr{F}) \Rightarrow R^{p+q}(g \circ f)_* \mathscr{F}$. Denote $H^n = R^n(g \circ f)_* \mathscr{F}$.

If $q > 0$, then $R^q f_* \mathscr{F} = 0$ and thus $E_2^{pq} = 0$. Then for $r \geq 2$, we have $E_r^{pq} = 0$ if $q > 0$ and $E_r^{pq} \cong E_2^{pq}$ if $q = 0$. Therefore, $E_\infty^{pq} = E_{pq}^2$ for all $p, q \in \mathbb{Z}$.

Using the definition of convergence of bounded spectral sequences, we have $E_\infty^{p-k,k} \cong F^{p-k} H^p / F^{p-k+1} H^p$. If $k > 0$, we have $E_\infty^{p-k,k} = 0$ and

$$F^p H^p \cong F^{p-1} H^p \cong F^{p-2} H^p \cong \cdots$$

Similarly, by considering the case of $k < 0$, we have

$$F^{p+1} H^p \cong F^{p+2} H^p \cong F^{p+3} H^p \cong \cdots$$

Therefore, $F^p H^p = H^p$ and $F^{p+1} H^p = 0$. Then $R^p g_*(f_* \mathscr{F}) = E_\infty^{p0} = F^p H^p / F^{p+1} H^p = H^p = R^p(g \circ f)_* \mathscr{F}$. □

7 Reflexive sheaves

The material here is based on [12, Tag 0AVT].

7.1 Definition

Definition 7.1. Let X be a scheme and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then the dual of $\mathscr{F}$ is the $\mathscr{O}_X$-module

$$\mathscr{F}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X).$$

Definition 7.2. Let X be an integral locally Noetherian scheme and $\mathscr{F}$ be a $\mathscr{O}_X$-coherent sheaf. The reflexive hull of $\mathscr{F}$ is

$$\mathscr{F}^{\vee\vee} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X), \mathscr{O}_X).$$

The sheaf $\mathscr{F}$ is called reflexive if the natural map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism.

Example 7.3. Let X be an integral locally Noetherian scheme and $\mathscr{F}$ be a locally free $\mathscr{O}_X$-sheaf. Then $\mathscr{F}$ is reflexive.

In the other direction, a reflexive sheaf is locally free in some open subspace of the base scheme.

Lemma 7.4. [12, Tag 0AY6] Let X be an integral locally Noetherian normal scheme. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then the following statements are equivalent:

- The sheaf $\mathscr{F}$ is reflexive.
- There is an open subscheme $j : U \to X$ such that
  - Every irreducible component of $X \setminus U$ has codimension ≥ 2 in X;
  - $j^*\mathscr{F}$ is finite locally free; and
  - $\mathscr{F} = j_* j^* \mathscr{F}$.

As such, this gives the following definition, generalizing the rank of locally free sheaves:

Definition 7.5. Let X be an integral scheme with generic point η. The rank of a reflexive sheaf $\mathscr{F}$ is $\dim_{K(X)} \mathscr{F}_\eta$.

For surfaces, one can take the above U to be the regular locus:

Lemma 7.6. [12, Tag 0B3N] Let X be a regular scheme of dimension ≤ 2 and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then $\mathscr{F}$ is reflexive if and only if $\mathscr{F}$ is locally free of finite rank.

Here, some basic property of reflexive modules are listed.

Lemma 7.7. [12, Tag 0AY4] Let X be an integral locally Noetherian scheme and $\mathscr{F}$ and $\mathscr{G}$ be coherent $\mathscr{O}_X$-modules. Further assume $\mathscr{G}$ to be reflexive. Then $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is reflexive. In particular, $\mathscr{F}^\vee$ and $\mathscr{F}^{\vee\vee}$ are reflexive.

Theorem 7.8. [12, Tag 0EBJ] Let X be an integral locally Noetherian scheme and $j : U \to X$ be an open immersion with complement Z. Further assume that for $z \in Z$, $\mathscr{O}_{X,z}$ has depth ≥ 2. Then j^* and j_* define an equivalence of category between the category of reflexive $\mathscr{O}_X$-modules and the category of reflexive $\mathscr{O}_U$-modules.

Corollary 7.9. Let X be a surface and $j : U \to X$ be an open immersion such that $X \setminus U$ is a codimension 2 subspace. If $\mathscr{F}$ and $\mathscr{G}$ are two reflexive $\mathscr{O}_X$-module such that $\mathscr{F}|_U = \mathscr{G}|_U$, then $\mathscr{F} = \mathscr{G}$.

Proof. Since X is normal, by Serre's criterion the stalks of all points in $X \setminus U$ are of depth ≥ 2. From the above theorem, we have

$$\mathscr{F} = j_*j^*\mathscr{F} = j_*\mathscr{F}|_U = j_*\mathscr{G}|_U = j_*j^*\mathscr{G} = \mathscr{G}$$

□

7.2 Torsion-free sheaves

It is also known that reflexive sheaves are torsion-free sheaves.

Definition 7.10. [12, Tag 0AVR] Let X be an integral scheme and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then a section s of $\mathscr{F}$ is a torsion element if at the generic point η, $s_\eta = 0$.

Definition 7.11. [12, Tag 0AVR] Let X be an integral scheme, and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then $\mathscr{F}$ is torsion-free if every torsion section of $\mathscr{F}$ is 0.

In other words, an $\mathscr{O}_X$-module $\mathscr{F}$ on an integral scheme X is torsion-free if and only if for all $U \subset X$ open, if η is the generic point of X, then the natural map $\mathscr{F}(U) \to \mathscr{F}_\eta$ is injective.

Lemma 7.12. [12, Tag 0AY2] Let X be an integral locally Noetherian scheme. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then we have

- If $\mathscr{F}$ is reflexive, then $\mathscr{F}$ is torsion free.
- The map $j : \mathscr{F} \to \mathscr{F}^{\vee\vee}$ is injective if and only if $\mathscr{F}$ is torsion free.

7.3 Weil divisors

In this section, the theory of Weil divisors will be generalized to normal schemes, based on [12, Tag 0EBK]. Recall that over integral Noetherian locally factorial schemes, the group of Weil divisors is isomorphic to that of Cartier divisors. Therefore, after passing to linear equivalent classes, both groups are isomorphic to the group of isomorphism classes of line bundles. For more general scheme, reflexive sheaves would be a good substitution. As a motivation, we have the following lemma:

Lemma 7.13. [12, Tag 0EBL] Let X be an integral locally Noetherian normal scheme. Then the map

$$(\mathscr{F}, \mathscr{G}) \mapsto (\mathscr{F} \otimes \mathscr{G})^{\vee\vee}$$

defines an abelian group law on the set of isomorphism classes of rank 1 reflexive $\mathscr{O}_X$-modules.

Lemma 7.14. [12, Tag 0EBM] Let X be an integral locally Noetherian normal scheme. There is an isomorphism between the group of isomorphism classes of rank 1 reflexive $\mathscr{O}_X$-modules and the group of linear equivalent classes of Weil divisors.

Given a Weil divisor Z, we can use the above lemma to construct a corresponding reflexive sheaf, which can be denoted as $\mathscr{O}_X(Z)$ or $\mathscr{O}(Z)$. Furthermore, given a reflexive sheaf $\mathscr{F}$, denote $\mathscr{F}(Z) := (\mathscr{F} \otimes \mathscr{O}_X(Z))^{\vee\vee}$.

Lemma 7.15. Let X be an integral locally Noetherian scheme, Z a principal Weil divisor with corresponding closed immersion i, and $\mathscr{F}$ be a reflexive sheaf. Then we have the exact sequence

$$0 \to \mathscr{F}(-Z) \to \mathscr{F} \to \mathscr{F} \otimes i_*\mathscr{O}_Z \to 0.$$

Proof. Recall that we have the natural exact sequence

$$0 \to \mathscr{O}_X(-Z) \to \mathscr{O}_X \to i_*\mathscr{O}_X \to 0.$$

Since tensor product is right exact, we only need to prove that $\mathscr{F}(-Z) \to \mathscr{F}$ is injective.

Take an open subset $j : U \to X$ such that $X \setminus U$ is of codimension 2 and $\mathscr{F}|_U$ and $\mathscr{O}(Z)|_U$ are locally free. Then we have the following exact sequence:

$$0 \to j_*(\mathscr{F} \otimes \mathscr{O}_X(-Z))|_U \to j_*(\mathscr{F} \otimes \mathscr{O}_X)|_U \to j_*(\mathscr{F} \otimes j_*\mathscr{O}_X)|_U.$$

Note

$$
\begin{aligned}
&(\mathscr{F} \otimes \mathscr{O}_X(-Z))^{\vee\vee} \\
=&j_*j^*((\mathscr{F} \otimes \mathscr{O}_X(-Z))^{\vee\vee}) \\
=&j_*(((\mathscr{F} \otimes \mathscr{O}_X(-Z))|_U)^{\vee\vee}) \\
=&j_*((\mathscr{F} \otimes \mathscr{O}_X(-Z))|_U).
\end{aligned}
$$

Thus we have the required exact sequence. □

8 Duality

Dualizing sheaves is an important ingredient in proving the resolution of singularities. This section aims to introduce concepts used to define dualizing sheaves, starting from concepts in homological algebra and then proceeding to dualizing complexes in commutative algebra. This section then ends with the definition and the properties of dualizing sheaves.

8.1 $R\operatorname{Hom}$ functor

The aim of this section is to define the $R\operatorname{Hom}$ functor, the derived functor of the Hom functor. In this subsection, A is a Noetherian ring. $C^\bullet$ and $D^\bullet$ are complexes of A-modules with differential map d_C and d_D. This subsection begins with the definition of the Hom functor.

Definition 8.1. [12, Tag 0A8H] The Hom complex $\operatorname{Hom}^\bullet(C^\bullet, D^\bullet)$ is defined as follow:

$$\operatorname{Hom}^n(C^\bullet, D^\bullet) = \prod_{i\in\mathbb{Z}} \operatorname{Hom}(C^i, D^{i+n})$$

with the differential map

$$d(f) = d_D \circ f - (-1)^n f \circ d_C$$

for $f \in \operatorname{Hom}^n(C^\bullet, D^\bullet)$.

To define $R\operatorname{Hom}$ functor, some properties of complexes are defined.

Definition 8.2. [12, Tag 010Z] A complex $C^\bullet$ is called acyclic if $H^i(C^\bullet) = 0$ for all i

Definition 8.3. [12, Tag 070H] A complex $I^\bullet$ is called K-injective if for every acyclic complex $C^\bullet$, $\operatorname{Hom}(C^\bullet, I^\bullet) = 0$.

Lemma 8.4. [12, Tag 0914] Let $C^\bullet$ be a complex. Then there exists a K-injective complex $I^\bullet$ and a morphism of complexes $f : C^\bullet \to I^\bullet$ such that f is a quasi-isomorphism.

In the following, $D(A)$ denote the derived category of the category of A-modules. Two complexes $C^\bullet$ and $D^\bullet$ are isomorphic in $D(A)$ if there is a morphism of complexes $f : C^\bullet \to D^\bullet$.

Definition 8.5. [12, Tag 0A5W] Let $C^\bullet$ and $D^\bullet$ be two complexes in $D(A)$. Let $I^\bullet$ be a K-injective complex isomorphic to $D^\bullet$ in $D(A)$. Define a functor $R\operatorname{Hom}$, called the derived Hom functor, as follows:

$$\begin{aligned} R\operatorname{Hom}_A : D(A)^{\mathrm{op}} \times D(A) &\to D(A), \\ (C^\bullet, D^\bullet) &\mapsto Hom^\bullet(C^\bullet, I^\bullet). \end{aligned}$$

Recall that Ext is the derived functor of the left exact Hom functor at any one of the two argument. This subsection ends with a relation between Ext functor and $R\operatorname{Hom}$ functor.

Lemma 8.6. Let $C^\bullet$ and $D^\bullet$ be two complexes in the derived category of modules over a ring A. Then the following equalities hold

$$\operatorname{Ext}^i(C^\bullet, D^\bullet) = \operatorname{Hom}_{D(A)}(C^\bullet, D^\bullet[i]) = H^i(R\operatorname{Hom}(C^\bullet, D^\bullet)).$$

Proof. This is due to [12, Tag 06XQ] and [12, Tag 0A64]. □

8.2 Dualizing complexes

This subsection discusses dualizing complexes and their properties. Before giving the definition of dualizing complexes, the following notations on subcategories of $D(A)$ is defined.

- The subcategory $D^b(A)$ contains all complexes $C^\bullet$ isomorphic to a bounded complex in $D(A)$.
- The subcategory $D_{\mathrm{Coh}}(A)$ contains all complexes $C^\bullet$ whose cohomology modules are finitely generated.
- The subcategory $D^b_{\mathrm{Coh}}(A)$ is the subcategory $D^b(A) \cap D_{\mathrm{Coh}}(A)$.

Definition 8.7. [12, Tag 0A7B] Let A be a Noetherian ring. Denote the complex with only zero terms except an A at the zeroth position as $A[0]$. A complex $\omega_A^\bullet$ is called a dualizing complex if $\omega_A^\bullet$ satisfies the following:

- the sheaf $\omega_A^\bullet$ is isomorphic to a finite complex of injective A-modules in $D(A)$,
- the sheaf $\omega_A^\bullet$ is in $D_{\mathrm{Coh}}(A)$, and
- there is a quasi-isomorphism $A[0] \to R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$.

The following lemma explains the name "dualizing" in dualizing complex.

Lemma 8.8. [7, Proposition 25.75] Let A be a Noetherian ring and $\omega^\bullet$ be a complex satisfying the first two condition of Definition 8.7. Then $\omega^\bullet$ is a dualizing complex if and only if for every $C^\bullet \in D^b_{\mathrm{Coh}}(A)$ such that the natural morphism

$$C^\bullet \to R\operatorname{Hom}(R\operatorname{Hom}(C^\bullet, \omega^\bullet), \omega^\bullet)$$

is an isomorphism.

Not all Noetherian rings have a dualizing complex. However, for the rings that are in consideration, i.e. Noetherian integral normal ring of dimension 2, a dualizing sheaf exists.

Lemma 8.9. [12, Tag 0A7K] Let k be a field. Then every k-algebra of finite type has a dualizing complex.

This subsection ends with a discussion of normalized dualizing complexes, which is motivated by the following lemma.

Lemma 8.10. [12, Tag 0A7L] Let $\omega_A^\bullet$ be a dualizing complex of a local ring A. Let m be the maximal ideal of A. Then there is $n \in \mathbb{Z}$ such that $R\operatorname{Hom}(A/m, \omega_A^\bullet) \cong (A/m)[-n]$.

Since tensor product of $A[-n]$ commutes with the second argument of $R\operatorname{Hom}$ ([12, Tag 0ATK]), the following definition can be made:

Definition 8.11. Let $\omega_A^\bullet$ be a dualizing complex of a local ring A with maximal ideal m. The complex $\omega_A^\bullet$ is normalized if $R\operatorname{Hom}(A/m, \omega_A^\bullet) \cong (A/m)[0]$.

8.3 Dualizing modules

Definition 8.12. [12, Tag 0DW3] Let A be a Noetherian local ring of dimension d and $\omega_A^\bullet$ be a normalized dualizing sheaf of A. Then $H^{-\dim A}(\omega_A^\bullet)$ is the dualizing module of A.

Dualizing modules exist for Cohen-Macaulay local ring.

Lemma 8.13. [12, Tag 0AWS] Let A be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$ and dualizing module ω_A. Then the following are equivalent

- The ring A is Cohen-Macaulay.
- The complex $\omega_A^\bullet$ is concentrated at a single degree.
- The equality $\omega_A^\bullet = \omega_A[\dim(A)]$ holds.

The following lemma shows an association between dualizing modules and the Ext functor. This is the definition of dualizing module used in [4, Definition 1 §9] and is mentioned in [5, Definition 3.3.1].

Lemma 8.14. Let A be a Cohen-Macaulay local ring with residue field k and dualizing module ω_A. Then

$$\dim_k \operatorname{Ext}^i(k, \omega_A) = \begin{cases} 1 & \text{if } i = \dim A \\ 0 & \text{if } i \neq \dim A \end{cases}.$$

Proof. Using Lemma 8.6,

$$\begin{aligned} & \operatorname{Ext}^i_A(k, \omega_A) \\ = & \operatorname{Hom}_{D(A)}(k, \omega_A[\dim A][i - \dim A]) \\ = & H^{i - \dim A}(R\operatorname{Hom}(k, \omega_A[\dim A])) \\ = & \begin{cases} k & \text{if } i = \dim A \\ 0 & \text{if } i \neq \dim A \end{cases} \end{aligned}$$

The last equality is due to the definition of normalized dualizing complex. □

This section ends with a method of computing some dualizing modules.

Lemma 8.15. [5, Theorem 3.3.5(a)] Let A be a Cohen-Macaulay local ring with the dualizing module ω_A. Consider a regular sequence $a_1, \cdots, a_n$ in A. Then $\omega_{A/(a_1, \cdots a_n)} \cong \omega_A/(a_1, \cdots a_n)\omega_A$.

8.4 Dualizing sheaves

This subsection discusses duality for schemes. Note that given a scheme X, we denote $D(\mathscr{O}_X)$ the derived category of the category of A-modules.

Definition 8.16. [12, Tag 0A87] Let X be a scheme. We call $\omega_X^\bullet$ a dualizing complex of X if for every affine open subset $U = \operatorname{Spec} A$ there is a dualizing complex $\omega_A^\bullet$ for the ring A such that $\omega_X|_U$ is isomorphic to $\widetilde{\omega_A^\bullet}$ in $D(\mathscr{O}_X)$.

Definition 8.17. [12, Tag 0AWH] Let X be a Noetherian scheme with a dualizing complex $\omega_X^\bullet$. Let $n \in \mathbb{Z}$ be the smallest integer such that $H^n(\omega_X^\bullet)$ is nonzero. Then the sheaf $H^n(\omega_X^\bullet)$ is called the dualizing sheaf.

Here is some properties of dualizing sheaves.

Lemma 8.18. [12, Tag 0AWH] Let X be a Noetherian equidimensional scheme, U an open subset of X, and ω_X a dualizing sheaf of X. Then $\omega_X|_U = \omega_U$.

Lemma 8.19. [7, Proposition 25.138] Let X be a normal connected Noetherian scheme and ω_X be a dualizing sheaf. Then ω_X is reflexive.

Below is some explicit description of the dualizing sheaves for some schemes.

Lemma 8.20. [7, Corollary 25.130] Let X be a connected separated smooth scheme of dimension d over a field k. Then $\omega_X = \wedge^d \Omega^1_{X/k}$, where $\Omega^1_{X/k}$ is the sheaf of differentials.

Lemma 8.21. [7, Proposition 25.139] If X is an integral normal separated scheme of finite type over a perfect field k, then the dualizing sheaf ω_X of X is the double dual of the canonical sheaf, as below:

$$\omega_X \cong (\wedge^{\dim X} \Omega^1_{X/k})^{\vee\vee}$$

Proof. From the above lemma, we have the following isomorphisms over the smooth locus X_{sm} of X:

$$\omega_X|_{X_{\mathrm{sm}}} = \omega_{X_{\mathrm{sm}}} = \wedge^{\dim X} \Omega^1_{X_{\mathrm{sm}}/k} = (\wedge^{\dim X} \Omega^1_{X/k})|_{X_{\mathrm{sm}}} = (\wedge^{\dim X} \Omega^1_{X/k})^{\vee\vee}|_{X_{\mathrm{sm}}}.$$

The last equality follows from the fact that the canonical bundle over regluar By [12, Tag 0B8X], the nonsingular locus and the smooth locus X_{sm} coincide in X, and this locus is a dense open subscheme of X. Furthermore, since X is normal, by Serre's criterion $X \setminus X_{\mathrm{sm}}$ has codimension ≥ 2.

Together with the fact that ω_X and $(\wedge^{\dim X} \Omega^1_{X/k})^{\vee\vee}$ are reflexive, one can apply Corollary 7.9 and show that the required isomoprhism over X holds □

8.5 Properties of dualizing sheaves

Lemma 8.22. [2, (1.5)(c)] Let X be a surface and $\mathscr{F}$ be a finite length $\mathscr{O}_X$-module. Then $\mathscr{F}$ and $\mathscr{E}xt^2_{\mathscr{O}_X}(\mathscr{F}, \omega_X)$ have the same length.

Lemma 8.23. [2, (1.6)(c)] Let $f : X' \to X$ be a modification of surfaces and $\mathscr{F}'$ be a reflexive $\mathscr{O}'_X$-sheaf. Define $\mathscr{F}'^D = \mathscr{H}om_{\mathscr{O}'_X}(\mathscr{F}', w_{X'})$. Then the following sequence is exact:

$$0 \to f_*(\mathscr{F}'^D) \to (f_*\mathscr{F}')^D \to \mathscr{E}xt^2_{\mathscr{O}_X}(R^1 f_*\mathscr{F}', \omega_X) \to R^1 f_*(\mathscr{F}'^D) \to 0.$$

Lemma 8.24. [2, (3.4)(i)] Let ω be a dualizing sheaf of a surface X and $f : X' \to X$ be a modification of surfaces. Then the finite length $\mathscr{O}_X$-module $R^1 f_*\mathscr{O}_X$ and $\omega_X / f_*\omega_{X'}$ are dual via $\mathscr{E}xt(\cdot, \omega)$ and have the same length.

Proof. Substitute $\mathscr{F} = \mathscr{O}_{X'}$ to Lemma 8.23, we have

$$0 \to f_*(\mathscr{O}^D_{X'}) \to (f_*\mathscr{O}_{X'})^D \to \mathscr{E}xt^2(R^1 f_*\mathscr{O}_{X'}, \omega_X) \to R^1 f_*(\mathscr{O}^D_{X'}) \to 0,$$

Note $\mathscr{O}^D_{X'} = \mathscr{H}om_{\mathscr{O}_{X'}}(\mathscr{O}_{X'}, \omega_{X'}) = \omega_{X'}$ and $(f_*\mathscr{O}_X)^D = \mathscr{H}om_{\mathscr{O}_X}(f_*\mathscr{O}_{X'}, \omega_X) = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \omega_X) = \omega_X$. Also, by the Grauert-Riemenschneider's vanishing theorem [2, Theorem 2.9(i)], $R^1 f_*\omega_{X'} = 0$. Then we have

$$0 \to f_*\omega_{X'} \to \omega_X \to \mathscr{E}xt^2(R^1 f_*\mathscr{O}_{X'}, \omega_X) \to 0$$

Therefore $\omega_X / f_*\omega_{X'} \cong \mathscr{E}xt^2(R^1 f_*\mathscr{O}_{X'}, \omega_X)$ and the duality follows. The equality in length is guaranteed by Lemma 8.22 □

9 Rational singularities

9.1 Definition

Definition 9.1. Let X be a surface. A closed point $p \in X$ is called a rational singularity if for every surface X' and for every modification $f : X' \to X$, $(R^1 f_*\mathscr{O}_{X'})_p = 0$

Remark 9.2. Suppose $f : X' \to X$ is a modification from a regular surface X' to a surface X with a rational singularity x. Using [3, Proposition 1] the arithmetic genus of every irreducible component of $f^{-1}(x)$ is 0. As such, these components are isomorphic to $\mathbb{P}^1_k$, i.e. are rational curves ([9, Example IV.1.3.5]).

A surface whose closed points are rational singularities simplifies the situation, since the blow-up of the surface along rational singularities is normal, see [2, Theorem 4.9]. This means that instead of considering normalized blow-ups, only blow-ups need to be consider.

9.2 Basic property

Lemma 9.3. [2, Proposition (3.2)(i)] Let X, X', X'' be surfaces and $f : X' \to X, g : X'' \to X, \pi : X'' \to X'$ be modifications such that the following diagram is commutative:

$$\begin{array}{ccc} X'' & \xrightarrow{\quad \pi \quad} & X' \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & X & \end{array}$$

There is a natural exact sequence

$$0 \to R^1 f_* \mathscr{O}_{X'} \to R^1 g_* \mathscr{O}_{X''} \to f_* R^1 \pi_* \mathscr{O}_{X''} \to 0$$

Proof. Using Corollary 6.11, we have an exact sequence

$$\begin{aligned} 0 \to & R^1 f_*(\pi_* \mathscr{O}_{X''}) \to R^1 (f \circ \pi)_* \mathscr{O}_{X''} \\ & \to f_*(R^1 \pi_* \mathscr{O}_{X''}) \to R^2 f_*(\pi_* \mathscr{O}_{X''}) \to R^2 (f \circ \pi)_* \mathscr{O}_{X''} \end{aligned}$$

Recall that X, X', and X'' are normal and modifications are birational and projective, hence proper. By Proposition 5.5,

$$\pi_* \mathscr{O}_{X''} = \mathscr{O}_{X'}.$$

Also, by Proposition 6.3,

$$R^2 f_*(\pi_* \mathscr{O}_{X''}) = R^2 g_* \mathscr{O}_{X''} = 0.$$

The proof is completed by substituting the above two equalities into the exact sequence. □

Lemma 9.4. [2, Proposition (3.2)(ii)] Let X and X' be surfaces and $f : X' \to X$ be a modification of surfaces. If X has rational singularities, so does X'.

More precisely, let $x \in X$ be a rational singularity. Then if x' is a closed point of X' such that $f(x') = x$, x is also a rational singularity.

Proof. Let $\pi : X'' \to X'$ be a modification of surfaces and $g = \pi \circ f$. Then $(R^1 g_* \mathscr{O}_{X''})_x = (R^1 f_* \mathscr{O}_{X'})_x = 0$. By Lemma 6.2, both $R^1 g_* \mathscr{O}_{X''}$ and $R^1 f_* \mathscr{O}_{X'}$ are coherent and thus are of finite presentation. Then there is an open neighbourhood U of x such that $(R^1 g_* \mathscr{O}_{X''})|_U$ and $(R^1 f_* \mathscr{O}_{X'})|_U$ are zero sheaves. Therefore, by Lemma 9.3, $(f_* R^1 \pi_* \mathscr{O}_{X''})|_U$ is a zero sheaf.

By Proposition 6.3, $\text{supp}(R^1 \pi_* \mathscr{O}_{X''})|_U = 0$. Then the conclusion follows from the following lemma. □

Lemma 9.5. Let X be a surface, $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module with $\dim \operatorname{supp} \mathscr{F} = 0$. If $\Gamma(X, \mathscr{F}) = 0$, then $\mathscr{F} = 0$.

Proof. Suppose to the contrary $\operatorname{supp} \mathscr{F}$ is nonempty. Let $\operatorname{supp} \mathscr{F} = \{x_1, ..., x_n\}$ and $s_{x_1} \in \mathscr{F}_{x_1}$. Let U_1 be an open neighbourhood of x_1 such that s_{x_1} lifts to some $s_1 \in \Gamma(U_1, \mathscr{F})$ and $x_2, ..., x_n \notin U_1$. Let $U_2 = X \setminus \{x_1\}$ and $s_2 = 0 \in \Gamma(U_2, \mathscr{F})$.

Note that X is covered by open subsets U_1 and U_2. Also, since $(U_1 \cap U_2) \cap \operatorname{supp} \mathscr{F} = \emptyset$, $s_1|_{U_1 \cap U_2} = 0 = s_2|_{U_1 \cap U_2}$. Then there is a $s \in \Gamma(X, \mathscr{F})$ such that $s_1 = s|_{U_1}$ and $s_2 = s|_{U_1}$.

However, it is assumed that $\Gamma(X, \mathscr{F}) = 0$. Therefore $s = 0$ and $s_1 = s|_{U_1} = 0$. This shows that $\mathscr{F}_{x_1} = \{0\}$. □

Lemma 9.6. [2, Proposition (3.4)(ii)] Let X be a surface. Then $x \in X$ is a rational singularity if and only if for every modification of surfaces $f : X' \to X$, the equality $(\omega_{X'}/f_*\omega_X)_x = 0$ holds.

To prove this lemma, we recall the following result about stalks and the functors Ext and $\mathscr{E}xt$:

Lemma 9.7. [9, Proposition 6.8] Let X be a Noetherian scheme, $x \in X$, and $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules. If $\mathscr{F}$ is coherent, for $i \geq 0$ we have

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \cong \operatorname{Ext}^i_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

Now we prove Lemma 9.6.

Proof. ($\Rightarrow$) Recall from Lemma 6.2 that $R^1 f_* \mathscr{O}_{X'}$ is coherent. Using Lemma 8.24, one have

$$\begin{aligned} &(\omega_{X'}/f_*\omega_X)_x \\ =&\mathscr{E}xt^2(R^1 f_*\mathscr{O}_{X'}, \omega_X)_x \\ =&\operatorname{Ext}^2((R^1 f_*\mathscr{O}_{X'})_x, \omega_{X,x}) \\ =&\operatorname{Ext}^2(0, \omega_{X,x}) \\ =&0. \end{aligned}$$

($\Leftarrow$) Since $\omega'_X/f_*\omega$ is coherent annd is dual to $R^1 f_*\mathscr{O}_{X'}$ via $\mathscr{E}xt^2(R^1 f_*\mathscr{O}_{X'}, \omega_X)$ by Lemma 8.24, reversing the role of $\omega'_X/f_*\omega$ and $R^1 f_*\mathscr{O}_{X'}$ in the above chain of equalities proves what is required. □

10 Resolution of points into rational singularities

In this section, we aim to show the following proposition.

Proposition 10.1. Let $x \in X$ be a closed point. Then for some large n, every closed point in the preimage of x becomes a rational singularity of X_n in (1.1).

The proof here is based on [2, Section 3]. It will be divided into two cases: nonsingular closed points (Section 10.1) and singular closed points (Section 10.2).

10.1 Nonsingular closed points

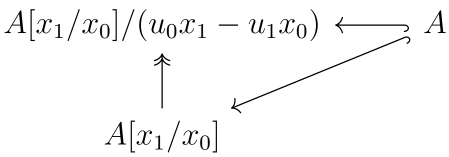
Nonsingular closed points on a surface are already rational singularities. Before showing this, we first discuss higher direct images and blow-ups.

Lemma 10.2. Let $f : X' \to X = \mathrm{Spec}(A)$ be the morphism of blow-up X at a nonsingular closed point p. Then $R^1 f_* \mathscr{O}_{X'} = 0$

Proof. We first define the following list of symbols.

- Denote $\mathbb{P}$ by $\mathbb{P}^1_A$.
- Write $\mathbb{P}^1_A$ as $\mathrm{Proj}(A[x_0, x_1])$.
- Denote by m the maximal ideal corresponding to the closed point p is m.
- Since p is a nonsingular point, let m be generated by two elements u_1, u_2.
- Denote by $\mathscr{I}$ the ideal sheaf $(u_0x_1 - u_1x_0)\mathscr{O}_{\mathbb{P}}$. Note that $X' = V_+(u_0x_1 - u_1x_0)$.
- Let $i : X' \to \mathbb{P}$ be the closed immersion.
- Denote by $g : \mathbb{P} \to X$ the natural map when viewing $\mathbb{P}$ as a A-scheme.

Let $\{U_0, U_1\}$ be the standard open cover of $\mathbb{P}$ and $U_0 = \mathrm{Spec}\, A[x_1/x_0]$. Then the morphisms f, g, i induces the following commutative diagram of ring homomorphisms:



Similar diagram can also be constructed for U_1. Therefore, $f = g \circ i$.

Now consider the following exact sequence of sheaves:

$$0 \to \mathscr{I} \to \mathscr{O}_{\mathbb{P}} \to i_*\mathscr{O}_{X'} \to 0.$$

We then extract the following part of the long exact sequence of higher direct image functor:

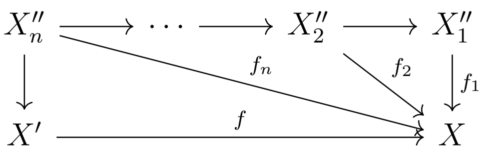
$$R^1g_*\mathscr{O}_{\mathbb{P}} \to R^1g_*(i_*\mathscr{O}_{X'}) \to R^2g_*\mathscr{I}$$

By Proposition 6.3, $R^2g_*\mathscr{I} = 0$. Also, by [12, Tag 01XW], $R^1g_*\mathscr{O}_{\mathbb{P}} = 0$. Therefore by Proposition 6.13, $R^1f_*\mathscr{O}_{X'} = R^1g_*(i_*\mathscr{O}_{X'}) = 0$ □

Proposition 10.3. [2, Proposition (3.2)(iii)] Nonsingular points $p \in X$ are rational singularities.

Proof. By Lemma 6.1, one only needs to show that it is true for some affine neighbourhood of p in X. Also, since the nonsingular locus of X is open, we can further assume that X is nonsingular. Let $X = \operatorname{Spec}(A)$ for some regular ring A.

Let $f : X' \to X$ be a modification. Using Proposition 5.8, there is a sequence of normalized blowing-ups $X''_1, ..., X''_n$ over X that dominates f. Then we have the commutative diagram:



with all arrows of the upper row and the right column normalized blow-ups at a point, which are modifications. The arrow on the left is a birational morphism. Also note that by Lemma 5.7, all maps between the two rows are modifications.

It is now needed to show that $(R^1f_*\mathscr{O}_{X'})_p = 0$. Since blowing-up is an isomorphism outside the centre, we can assume that for every $1 \le m \le n$, $f_m^{-1}(p)$ is within the centre of the blow-up of X''_m.

Since Lemma 3.8 implies that the blow-up at p is nonsingular, normalization is unnecessary and f_1 is the blow-up at p. Hence $R^1(f_1)_*\mathscr{O}_{X'} = 0$ by Lemma 10.2.

Now we show that $(R^1(f_n)_*\mathscr{O}_{X'})_p = 0$ for all n by induction. Consider the commutative diagram below with $R^1(f_{n-1})_*\mathscr{O}_{X'} = 0$.

$$\begin{array}{ccc} X''_n & \xrightarrow{\pi} & X''_{n-1} \\ & \searrow^{f_n} & \downarrow f_{n-1} \\ & & X \end{array}$$

Similar to the discussion on f_1, $R^1\pi_*\mathscr{O}_{X''_{n-1}} = 0$.

Applying Lemma 9.3 we have a short exact sequence

$$0 \to R^1 f_{n-1*}\mathscr{O}_{X''_{n-1}} \to R^1 f_{n*}\mathscr{O}_{X''_n} \to f_{n-1*}R^1\pi_*\mathscr{O}_{X''_n} \to 0.$$

Since all but the middle term is zero, $R^1 f_{n*}\mathscr{O}_{X''_n} = 0$.

Now we show that $R^1 f_*\mathscr{O}_X = 0$. Indeed, applying Lemma 9.3 again, we have an exact sequence

$$0 \to R^1 f_*\mathscr{O}_{X'} \to R^1 f_{n*}\mathscr{O}_{X''_n} = 0.$$

□

10.2 Singular points

We first show a sufficient condition for singular closed points to be blown-up to rational singularities.

Proposition 10.4. Let $x \in X$ be a singular closed point such that the length of $(R_1 f_*\mathscr{O}_{X'})_x$ is bounded independently of the modification f. Then for some large n, the singular point becomes a rational singularity of X_n in (1.1).

With this proposition, the proof proceeds as follows:

- Show that the length of $\omega_{X'}/f^*\omega_X$ at x is independently of the modification f.
- Pass this information of bounded length to that of $(R_1 f_*\mathscr{O}_{X'})_x$ using duality.

We now prove this proposition.

Proof. Let $x \in X$ be a singular point. Let $f : X' \to X$ be a modification such that $(R^1 f_*\mathscr{O}_{X'})_x$ has the maximum length among all modification of X.

Step 1: There is some open neighbourhood U of x and a composition $U'' \to U$ of normalized blow-ups such that all closed points of U'' are rational singularities.

Let $\pi : X'' \to X'$ be a modification. By the maximality of $(R^1 f_*\mathscr{O}'_X)_x$, $(R^1(\pi\circ f)_*\mathscr{O}_{X'})_x$ also has the maximum length. This forces $l((f_*R^1\pi_*\mathscr{O}_{X''})_x) =$

0 and $(f_*R^1\pi_*\mathscr{O}_{X''})_x = 0$. Let U be an open neighbourhood of x such that the only singular point in U is x and $(f_*R^1\pi_*\mathscr{O}_{X''})|_U = 0$. Note that U remains a surface. By Proposition 6.3 , the support of $R^1\pi_*\mathscr{O}_{X''}$ is of dimension 0. By Lemma 9.5, $(R^1\pi_*\mathscr{O}_{X''})|_{f^{-1}(U)} = 0$. Hence, all closed point of $f^{-1}(U)$ are rational singularities.

Denote $U' = f^{-1}(U)$. Apply Proposition 5.8 to dominate $U' \to U$ by a sequence of normalized blow-ups.

$$U'' = U''_m \xrightarrow{b_m} U''_{m-1} \xrightarrow{b_m} \cdots \xrightarrow{b_m} U''_1 \xrightarrow{b_m} U''_0 = U \tag{10.1}$$

Denote the composition of the normalized blow-ups as $g : U'' \to U$. This induces a modification $U'' \to U'$. Since all closed points of U' are rational singularities, all closed points of U'' are rational singularities by Lemma 9.4.

Step 2: Complete the proof by comparing the above sequences of normalized blow-ups (1.1) and (10.1).

For $1 \leq l \leq k$, let the blow-up $b_l : X''_l \to X''_{l-1}$ be the blow-up of X''_{l-1} at the point x_l. Suppose the map $\alpha_l : X''_{l-1} \to \cdots \to X''_1 \to X$ does not map x_l to x. Then $b_l|_{b_l^{-1}(X''_{l-1}\setminus\{x_l\})}$ is an isomorphism. Therefore, we can ignore all those blow-ups. Now replace the remaining blow up with the blow-up that blows up all the singular points, and we obtained what is required.

□

What is left to show is that the singular points on the surface have the aforementioned bound.

Lemma 10.5. Let X be a surface. Then for any modification $f : X' \to X$ and closed point $x \in X$, the length of $(\omega_X/f_*\omega_{X'})_p$ is bounded independently of f

Proof. Recall from Lemma 8.21 that the dualizing sheaf of X is $(\wedge^2\Omega^1_{X/k})^{\vee\vee}$. This produces a canonical morphism

$$\phi_1 : \wedge^2\Omega^1_{X/k} \to \omega_X. \tag{10.2}$$

Also, one can construct the following composition of maps:

$$\phi_2 : \wedge^2\Omega^1_X \to f_*(\wedge^2\Omega^1_{X'}) \to f_*\omega_{X'} \to \omega_X. \tag{10.3}$$

The first map is a canonical map from [12, Tag 08RU]. Recall that by definition there is a natural isomorphism $\mathscr{H}om(\Omega^1_{X/k}, f_*\Omega^1_{X'/k}) \to \mathrm{Der}(\mathscr{O}_X, f_*\Omega^1_{X'/k})$, the set of all derivations $\mathscr{O}_X \to f_*\Omega^1_{X'/k}$. Using the natural maps $\mathscr{O}_X \to f_*\mathscr{O}_{X'}$ and $\mathscr{O}_{X'} \to \Omega^1_{X'/k}$, one obtains the first map. The second map is the map from taking double dual, similar to the map (10.2) but applied to X'. The third map is from Lemma 8.24.

Restricting to the smooth locus, it can be checked that the homomorphisms ϕ_1 and ϕ_2 are the same map. Details are omitted. Therefore, $\operatorname{im}\phi_1 \subset \operatorname{im}(f_*\omega_{X'} \to \omega_X) \cong f_*\omega_{X'}$ and the length $\omega_X/f_*\omega_{X'}$ is bounded by $\omega_X/\operatorname{im}\bigwedge^2\Omega^1_X$ independently of f. $\square$

Now we can conclude that all singular points can be blown-up to rational singularities.

Proposition 10.6. For every singular point p, the length of $R_1f_*\mathscr{O}_{X'}$ at p is bounded independently of the modification $f : X' \to X$.

Proof. Using Lemma 8.22, $(R_1f_*\mathscr{O}_{X'})_p$ has the same length as $(\omega_X/f_*\omega_{X'})_p$, which is bounded independently of the modification by Lemma 10.5. $\square$

11 Blowing-up rational singularities

11.1 Modifications and reflexive sheaves

In this section, let $f : X' \to X$ be a modification of surfaces that is an isomorphism outside a rational singularity $x \in X$.

Lemma 11.1. [2, Proposition 4.1] Let $\mathscr{F}'$ be an $\mathscr{O}_{X'}$-module generated by global sections. Then for $n \geq 1$, $R^nf_*\mathscr{F}' = 0$.

Lemma 11.2. [2, Proposition 4.2] Let $\mathscr{L}$ be a reflexive $\mathscr{O}_X$-module. Then $f^*\mathscr{L}/(f^*\mathscr{L})_{\mathrm{tor}}$ is reflexive.

11.2 Euler characteristic

In this section, let $f : X' \to X$ be the blow-up of a surface X along x, and let $Z' = f^{-1}(x)$. Further assume that there are some global sections s_1, s_2 of $\mathscr{O}_{X'}$ such that $\mathscr{I}_{Z'} = (s_1, s_2)\mathscr{O}_{X'}$, which is the case by Corollary 4.10.

Lemma 11.3. [2, Proposition 4.6] Suppose $x \in X$ has multiplicity μ, i.e. the multiplicity of the maximal ideal m_x as a $\mathscr{O}_{X,x}$-module is μ. Then the maximal ideal m_x of the stalk $\mathscr{O}_{X,x}$ satisfies $\dim_k m_x^r/m_x^{r+1} = r\mu + 1$ for all $r \geq 0$.

Lemma 11.4. [2, Proposition 4.8] Let $f : X' \to X$ be the blow-up of the maximal ideal m at a rational singularity with exceptional divisor $Z' = f^{-1}(x)$ and corresponding ideal sheaf $\mathscr{I}_{Z'}$ spanned by two elements. Then $\chi(\mathscr{O}_{Z'}) = 1$ and $\deg(\mathscr{O}'_X(-Z')|_{Z'}) = \mu$.

12 Bounding multiplicities

The next step of the proof is to show that after some composition of blowing-up a surface whose closed points are rational singularities, the singular rational singularities have multiplicity of 2. Such points are called rational double points. Recall from Lemma 11.4 that we only need to consider blow-ups rather than normalized blow-ups from now on.

The multiplicities of nonsingular closed points are always 1. Therefore, we can restrict our attention to an open affine subscheme that contains a single rational singularity.

Let $f : X' \to X$ be the blow-up of a rational singularity $x \in X$. The proof will proceed as follow:

- Show that $\omega_{X'} = f^*\omega_X/(f^*\omega_X)_{\text{tor}}$, the quotient of the inverse image of ω_X by the torsion elements.
- Show that for $n \gg 0$ and the X_n as in (1.1), ω_{X_n} is locally free.
- Show that if the dualizing sheaf of a surface is locally free, all closed points on the surface have multiplicity ≤ 2.

Let $f : X' \to X$ be the blow-up of X along the ideal sheaf $\mathscr{I}_{\{x\}}$ of a rational singularity x, and let $Z' = f^{-1}(x)$. Recalling that multiplicity is a local property and blowing-up can be considered in a local manner, the following assumption on the surface X can be made:

- X is the affine scheme of some ring, say A. Then by viewing $\mathscr{I}_{\{x\}}$ as an ideal of A, we have $\mathscr{I}_{\{x\}} = m$, where m is the maximal ideal of A corresponding to $\{x\}$.
- There are global sections s_1, s_2 of $\mathscr{O}_{X'}$ such that $\mathscr{I}_{Z'} = (s_1, s_2)\mathscr{O}_{X'}$. This is true by Corollary 4.10.
- x is the only singular point in X, by passing to a suitable neighbourhood of x.

12.1 Bound on multiplicities

The last part of this step will be shown first. This is because the following lemma, a generalization of what is required, will be useful for an earlier step.

Lemma 12.1. Let x be a rational singularity of the surface X with multiplicity $\mu > 1$. Let the maximal ideal and the dualizing module of $\mathscr{O}_{X,x}$ be m_x and ω respectively. Then $\dim_{\kappa(x)}(\omega/m_x\omega) = \mu - 1$.

Proof. By Proposition 2.1, let $a_1, a_2 \in \mathscr{O}_{X,x}$ be a regular sequence such that the multiplicity at x is the length of $\mathscr{O}_{X,x}/(a_1, a_2)$.

Denote $\overline{\mathscr{O}} = \mathscr{O}_{X,x}/(a_1, a_2)$ and $\overline{m} = m_x/(a_1, a_2)$. Then the map

$$m_x \twoheadrightarrow \overline{m}$$

induces a surjective $\kappa(x)$-vector space homomorphism

$$m_x/m_x^2 \to \overline{m}/\overline{m}^2$$

with kernel $\kappa(x)a_1 + \kappa(x)a_2$.

Using Lemma 11.3, $\dim_{\kappa(x)} \overline{m}/\overline{m}^2 \geq \mu - 1$. Therefore, $\overline{m}$ cannot be generated by fewer than $\mu - 1$ elements, and the length of $\overline{m}$ as a $\overline{\mathscr{O}}$-module is at least $\mu - 1$.

Since the length of $\overline{\mathscr{O}}$ is μ by the choice of a_1, a_2 and $\overline{m} \neq \overline{\mathscr{O}}$ by Nakayama's lemma, $l_{\overline{\mathscr{O}}}(\overline{m}/\overline{m}^2) = \mu - 1$. Therefore, $l_{\overline{\mathscr{O}}}(\overline{m}^2) = 0$ and $\overline{m}^2 = 0$.

Let

$$0 = M_0 \subsetneq \cdots \subsetneq M_{\mu-1} = \overline{m}$$

be a maximal composition series of $\overline{\mathscr{O}}$-modules. Since $\overline{m}^2 = 0$, this series doubles as a sequence of inclusions of $\kappa(x)$-modules. Let $1 \leq c \leq \mu - 1$ and $b_1, b_2 \in M_c/M_{c-1}$ such that $M_c/M_{c-1} = \kappa(x)b_1 + \kappa(x)b_2$. By Nakayama's lemma, $M_c/M_{c-1} = \overline{\mathscr{O}}b_1 + \overline{\mathscr{O}}b_2$. Note M_c/M_{c-1} is a simple $\overline{\mathscr{O}}$-module. Without loss of generality, let $u \in \overline{\mathscr{O}}$ such that $ub_1 = b_2$. Then this equality can be reduced to $\kappa(x) = \overline{\mathscr{O}}/\overline{m}$. Thus M_c/M_{c-1} is a simple $\kappa(x)$-module. Therefore, $\dim_{\kappa(x)} \overline{m} = \mu - 1$.

Applying Lemma 8.15 to $\overline{\mathscr{O}}$ and the regular sequence a_1, a_2, the dualizing module $\overline{\omega}$ of $\overline{\mathscr{O}}$ is $\omega/(a_1, a_2)$.

Now consider the exact sequence

$$0 \to \overline{m} \to \overline{\mathscr{O}} \to \kappa(x) \to 0$$

Since the dimension of $\overline{\mathscr{O}}$ is 0, applying the functor $\mathrm{Hom}(-, \overline{\omega})$ by Lemma 8.14 and the definition of dualizing sheaves yields a short exact sequence

$$0 \to \kappa(x) \to \overline{\omega} \to \kappa(x)^{\mu-1} \to 0$$

which in turns yields a surjective map $\overline{\omega}/\overline{m\omega} \to \kappa(x)^{\mu-1}$. Such a surjective map is possible since $\kappa(x) = \mathscr{O}_{X,x}/m_x$, and thus $\overline{m\omega}$ is in the kernel of $\overline{\omega} \to \kappa(x)^{\mu-1}$ in the exact sequence.

Since the length of $\kappa(x)$ as a $\mathscr{O}$-module is 1, the length of $\overline{\omega}$ is μ. By Nakayama's lemma, $\overline{\omega} \neq \overline{m\omega}$. Recall that we have assumed that $\mu > 1$. Hence, $\mathscr{O}_{X,x}$ is not regular and $m_x \neq (a_1, a_2)$. Then $\overline{m} \neq 0$ and $\overline{\omega}/\overline{m\omega}$ has length at most $\mu - 1$. Combining what we have above, $\kappa(x)^{\mu-1} \cong \overline{\omega}/\overline{m\omega} \cong \omega/m_x\omega$. □

Proposition 12.2. Let x be a rational singularity of a surface X with multiplicity μ. Further suppose $\omega_{X,x}$ is a free $\mathscr{O}_{X,x}$-module. Then $\mu \leq 2$.

Proof. By [7, Proposition 25.80], $\omega_{X,x}$ is the dualizing module ω of $\mathscr{O}_{X,x}$ and ω_X is of rank 1 by Lemma 8.21. If $\mu > 1$, $1 = \dim_{\kappa(x)}(\omega/m_x\omega) = \mu - 1$ with m_x the maximal ideal of $\mathscr{O}_{X,x}$. Hence $\mu = 2$. □

12.2 Dualizing sheaves of the blow-up

Proposition 12.3. Let $x \in X$ be a rational singularity of multiplicity $\mu > 1$, and let $f : X' \to X$ be the blowing-up of x. Then $\omega_1 := f^*\omega_X/(f^*\omega_X)_{\text{tor}} = \omega_{X'}$.

One direction of this proposition is easy.

Lemma 12.4. There is an injective map $\omega_1 \to \omega_{X'}$ such that ω_1 is isomorphic to its image in $\omega_{X'}$.

Proof. Taking the adjoint of the isomorphism in Lemma 9.6 produces the map $g : f^*\omega_X \to \omega_{X'}$. Since g induces a nontrivial linear map between two one-dimension $K(X)$-vector spaces by taking stalk at the generic point, $\ker g = (f^*\omega_X)_{\text{tor}}$. □

Now the proof will be done by contradiction. Note that we can assume that $\omega_1 \subset \omega_{X'}$. Suppose $\omega_1 \subset\neq \omega_{X'}$. To show that there is a contradiction, we compare the following Euler characteristics:

$$\chi_1 := \chi(\omega_1|_{Z'}),$$
$$\chi_2 := \chi(\omega_{X'}|_{Z'}).$$

Here $Z' = f^{-1}(x)$ as in the start of the section.

Lemma 12.5. We have $H^1(Z', \omega_1|_{Z'}) = 0$.

Proof. Note that we have the following exact sequence:

$$0 \to \mathscr{I}_{Z'}\omega_1 \to \omega_1 \to \omega_1 \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{Z'} \to 0$$

By the fact that $\mathscr{I}_{Z'}$ is locally principal and Lemma 11.2, the first and second terms are reflexive. Using Lemma 11.1 and the long exact sequence of the derived functor Rf_*, $R^1f_*(\omega_1 \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{Z'}) = 0$. Now [9, Proposition III.8.5] can be applied to show that $H^1(X', \omega_1 \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{Z'}) = 0$. Since $Z' = \operatorname{Supp} \mathscr{I}_{Z'}$, we find $H^1(Z', \omega_1|_{Z'}) = 0$. □

Lemma 12.6. We have $m\omega_X \subset f_*(f^*(m\omega_X)/(f^*(m\omega_X))_{\text{tor}})$.

Proof. Let $U \subset X$ be open. By Proposition 5.5, $\mathscr{O}_{X'}(f^{-1}(U)) = \mathscr{O}_X(U)$. Note that there is an equality

$$\begin{aligned} &\mathscr{O}_{X'}(f^{-1}(U)) \otimes_{\mathscr{O}_X(f(f^{-1}(U)))} m\omega_X(f(f^{-1}(U)))) \\ =& m\omega_X(f(f^{-1}(U)))) \end{aligned}$$

Since $m\omega_X$ is torsion free, the above equality becomes an injective map after taking the quotient modulo the torsion elements. □

Lemma 12.7. We have $\dim_k H^0(Z', \omega_1|_{Z'}) \leq \mu - 1$.

Proof. Note that $f^*m = \mathscr{I}_{Z'}$. Then we have

$$m\omega_X \subset f_*(\mathscr{I}_{Z'}\omega_1) \subset f_*\omega_1 \subset f_*\omega_{X'} = \omega_X.$$

Now recall that $\operatorname{Supp} m_x = \{x\}$ and $\operatorname{Supp} \mathscr{I}_Z' = Z'$. Then

$$\begin{aligned} &H^0(Z', \omega_1|_{Z'}) \\ =&H^0(X', \omega_1/\mathscr{I}\omega_1) \\ =&H^0(X, f_*(\omega_1/\mathscr{I}\omega_1)) \\ \subset&H^0(X, \omega_X/m\omega_X) \\ =&\omega_{X,x}/m_x\omega_{X,x}. \end{aligned}$$

By Lemma 12.1, $\dim_k H^0(Z', \omega_1|_{Z'}) \leq \mu - 1$. □

Lemma 12.8. The equality $\chi_2 = \mu - 1$ holds.

Proof. Now we compute χ_2. Using [7, Corollary 25.130(2) and Remark 19.24], one can see that $\omega_{Z'} = (\omega_{X'} \otimes \mathscr{O}_{X'}(Z'))|_Z$.

Since $f|_{Z'}$ gives us a projective morphism with target $\operatorname{Spec} \kappa(x)$, the spectrum of some finite extension of k, the Riemann-Roch theorem can be applied to Z' and gives

$$\begin{aligned} \chi_2 &= \chi((\omega_{X'} \otimes \mathscr{O}_{X'}(Z'))|_{Z'}) + \deg(\mathscr{O}_{X'}(-Z')|_Z) \\ &= \chi(\omega_{Z'}) + \deg(\mathscr{O}_{X'}(-Z')|_Z). \end{aligned}$$

By Lemma 11.3, $\deg(\mathscr{O}_{X'}(-Z')|_Z) = \mu$. By Lemma 11.3 and Serre duality, $\chi(\omega_{Z'}) = 1$. Therefore, $\chi_2 = \mu - 1$. □

Combining the lemmas above,

Lemma 12.9. we have the inequality $\chi_1 \leq \chi_2$.

Recall from the start of this proof of contradiction that it is assumed that $\omega_1 \subsetneq \omega_{X'}$. Since both ω_1 and $\omega_{X'}$ are rank 1 reflexive modules over a Noetherian integral normal scheme, there is some positive divisor D such that $\omega_{X'} = (\omega_1 \otimes \mathscr{O}_{X'}(D))^{\vee\vee}$.

The following lemma is a general one for arbitrary reflexive sheaves of rank 1. Combining this lemma with the above discussion of strict inclusion, we will show that $\chi_1 > \chi_2$ in Lemma 12.11.

Lemma 12.10. Let $\mathscr{L}$ be any reflexive sheaf of rank 1 on X', and let C' be a connected component of Z'. Then

$$\chi(\mathscr{L}|_{Z'}) > \chi(\mathscr{L}(C')|_{Z'})$$

Proof. Using Lemma 7.15, the following diagram of $\mathscr{O}_X$-modules with exact rows and columns can be constructed:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{L}(-Z') & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{L} \otimes_{\mathscr{O}'_X} \mathscr{O}'_X/\mathscr{I}_{Z'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{L}(C'-Z') & \longrightarrow & \mathscr{L}(C') & \longrightarrow & \mathscr{L}(C') \otimes_{\mathscr{O}_{X'}} \mathscr{O}'_X/\mathscr{I}_{Z'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{L}(C'-Z') \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{C'} & & \mathscr{L}(C') \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{C'} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

There is $\mathscr{I}_{C'}$ the ideal sheaf corresponding the closed subscheme C' of X'.

Since the Euler characteristic is additive in short exact sequences, there is the following system of equations:

$$\begin{cases}
\chi(\mathscr{L}(-Z')) + \chi(\mathscr{L} \otimes_{\mathscr{O}'_X} \mathscr{O}'_X/\mathscr{I}_{Z'}) & = \chi(\mathscr{L}) \\
\chi(\mathscr{L}(C'-Z')) + \chi(\mathscr{L}(C') \otimes_{\mathscr{O}'_X} \mathscr{O}'_X/\mathscr{I}_{Z'}) & = \chi(\mathscr{L}(C')) \\
\chi(\mathscr{L}(-Z')) + \chi(\mathscr{L}(C'-Z') \otimes_{\mathscr{O}'_X} \mathscr{O}'_X/\mathscr{I}_{C'}) & = \chi(\mathscr{L}(C'-Z')) \\
\chi(\mathscr{L}) + \chi(\mathscr{L}(C') \otimes_{\mathscr{O}'_X} \mathscr{O}'_X/\mathscr{I}_{C'}) & = \chi(\mathscr{L}(C'))
\end{cases}.$$

The above system reduces to

$$\begin{aligned}
&\chi(\mathscr{L} \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{Z'}) - \chi(\mathscr{L}(C') \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{Z'}) \\
&= \chi((\mathscr{L}(C'-Z') \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{C'})) - \chi((\mathscr{L}(C') \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{C'})).
\end{aligned}$$

Note that all sheaves above have support within C'. Hence, this quantity also equals

$$= \chi((\mathscr{L}(C'-Z') \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{C'})|_{C'}) - \chi((\mathscr{L}(C') \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}/\mathscr{I}_{C'})|_{C'})$$

Also, they are reflexive sheaves over a 1-dimensional normal scheme. Applying Riemann-Roch to C', this becomes

$$= \deg(\mathscr{O}_X(-Z')|_{C'})$$

Since Z' is the blow-up of x, $\mathscr{I}_{Z'}$ is a very ample line bundle for f ([8, Proposition 13.96]). By [8, Remark 13.53], $\mathscr{I}_{Z'}|_{Z'}$ is very ample for $f|_{Z'} : Z' \to \operatorname{Spec} \kappa(x)$. Applying [8, Theorem 13.59(a)], $\mathscr{I}_{Z'}|_{Z'}$ is ample. Therefore, [8, Proposition 13.51] implies that $\mathscr{I}_{Z'}|_{C'}$ is ample, and thus has positive degree, in particular by [12, Tag 0BEV]. The required inequality follows. □

Lemma 12.11. We have the inequality $\chi_1 > \chi_2$.

Proof. Since D is a finite sum of connected components of Z', we obtain the result by applying Lemma 12.10 multiple times. □

Therefore, we have produce the required contradiction, and conclude that

$$\omega_{X'} = f^*\omega_X/(f^*\omega_X)_{\text{tor}}.$$

12.3 Locally free dualizing sheaves

The proof can now be completed.

Proposition 12.12. Consider the sequence (1.1). For sufficiently large n, for the composition of blow-up of singular points $f : X_n \to X$ we have $\omega_{X_n} = f^*\omega_X/(f^*\omega_X)_{\text{tor}}$ is locally free.

Proof. Since blow-ups can be studied locally, let $X = \operatorname{Spec}(A)$. Also by definition ω_X is coherent. At the generic point η of X, $\omega(X) \subset \omega_\eta \cong K(X)$. Therefore, $\omega(X)$ is isomorphic to a finitely generated A-submodule of $K(X)$, which in turn is isomorphic to an ideal I of A. Let $g : X' \to X$ be the normalized blow-up of X along the closed subscheme $Z = \operatorname{Spec} A/I$. Then the ideal I and thus ω blow-ups to a locally free module $g^*(I\mathscr{O}_X) = g^*\omega_X$.

Now apply Proposition 5.8 to dominate g with a sequence of blow-ups $h : X_n \to X$ with $\pi : X_n \to X'$. Then $h^*\omega_X = \pi^*g^*\omega_X$ is locally free.

While Proposition 5.8 may blow up nonsingular points, by Lemma 7.6, ω_X is locally free at nonsingular points. Therefore, one may modify the blow-up such that the centre of the blow-up has no nonsingular points. Therefore, for sufficiently large n in the sequence (1.1), ω_{X_n} is locally free. □

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