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Accoppiamento tra le soluzioni alle equazioni di Einstein di buco nero e quelle cosmologiche

Coupling between black hole and cosmological solutions to Einstein's equations

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Abstract

Italiano

In relatività generale i buchi neri sono descritti attraverso soluzioni nel vuoto simmetriche rispetto a un'asse alle equazioni di Einstein. Poichè sono soluzioni nel vuoto, esse tendono a uno spaziotempo di Minkowski piatto all'infinito delle coordinate spaziali. Il modello cosmologico attualmente accettato è però una metrica di Robertson-Walker, che può essere solo spazialmente piatto e, in particolare, prevede che l'universo espanda isotropicamente e omogeneamente. Questa tesi investiga la generalizzazione di McVittie del più semplice, non rotante e statico buco nero, descritto attraverso la soluzione di Schwarzschild alle equazioni di Einstein. Questa nuova soluzione tende correttamente alla soluzione cosmologica di Robertson-Walker all'infinito spaziale. La principale differenza tra tale generalizzazione e l'originale buco nero di Schwarzschild è che la sua massa non è più una costante nel tempo ed è difatti accoppiata al fattore di scala cosmologico. In particolare la massa di un buco nero di McVittie cresce con l'espansione dell'universo e questo effetto compensa esattamente la corrispondente rarefazione di una distribuzione omogenea di buchi neri. Il bilanciamento tra questi due effetti può essere visto come una possibile origine della costante cosmologica, interpretata come una componente della densità di energia che non dipende dal fattore di scala dell'universo. Recenti osservazioni mostrano delle consistenze tre la masse misurate dei buchi neri, costante cosmologica e la predizione di McVittie.

English

In general relativity black holes are described as vacuum axisymmetric solutions to Einstein's equations. Since they are vacuum solutions, they tend to flat Minkowski spacetime at spatial infinity. The presently accepted relativistic cosmological model, however, is a Robertson-Walker metric that may only be spatially flat and, in particular, the universe expands, isotropically and homogeneously. This thesis investigates McVittie's generalization of the simplest, non-rotating, static black hole, described by the Schwarzschild solution to Einstein's equation. This new solution correctly tends to the Robertson-Walker cosmological solution at spatial infinity. The main difference between this generalization and the original Schwarzschild black hole is that its mass is not constant in time anymore and is, in fact, coupled with the cosmological scale factor. In particular the mass of a McVittie black hole grows as the universe expands and this effect exactly compensates the corresponding rarefaction of a spatially homogeneous distribution of black holes. The balance between these two effects can be seen as one possible origin of the cosmological constant seen as a component of the energy density that does not depends on the scale factor of the universe. Recent observations show some agreements between the measured mass of black holes, cosmological constant and the McVittie predicition.

CONTENTS

CHAPTER

ONE

INTRODUCTION

General relativity is today the main theory of interest to describe the physical nature of our universe in astronomical scales. In particular, quantum theory is not needed when dealing with a model of the universe here-now, since only for hot, dense matter quantum effects are dominant. One intuitively believes that these effects become dominant when the scalar curvature **S** is of the same magnitude as the inverse of the Planck constant h^{-1} [18]. Such conditions are those of the first instances of the universe. We shall not deal with it in this thesis. All the following results and equations will be treated in natural units such that c=8πG=1. Einstein's field equations of general relativity are tensorial relations that can be written as follows [2]

$$G_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{1.0.1}$$

or equivalently

$$Ric_{\mu\nu} = (T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) + \Lambda g_{\mu\nu} \tag{1.0.2}$$

Where $G_{\mu\nu} = Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress energy tensor.

Two kinds of solutions to Einstein's equations are used in astronomy to describe different phenomena:

i) Local solutions are spacetimes whose coordinates are the ones of an observer near a mass source. The most important example of local solutions is the **Schwarzchild metric** which describes the gravitational field of spherically symmetric vacuum space-times. The Schwarzchild solution is used to describe the motion of the planets around the sun and non-rotating black holes. The latter is not that common in the universe since a non-homogeneous distribution of matter in a collapsing star generates a rotating black hole which is described by the axisymmetric Kerr solutions.

ii) Cosmological solutions describe the whole universe and its history. Most of these solutions start from a reasonable principle- the **Copernican principle** - which implies that the universe, on its largest scale, looks the same everywhere. As a result, spatial isotropy and spatial homogeneity are assumed. These metrics are written in "comoving coordinates" which are used when the system of nebulae is taken as the basis reference. The most general solution for isotropic, homogeneous cosmological metrics is the Robertson-Walker metric.

Despite local black hole solutions have been one of the main topics of general relativity, they have been often treated isolated from the rest of the universe. On the other hand, cosmological black holes (CBHs) are surrounded by the matter of the universe, living in a cosmological spacetime [20]. This is a more realistic concept that correlates local solutions to the cosmological one. They must thus be compatible. In particular, the local solution at spatial infinity must reproduce the cosmological one. However, the curvature of a black hole solution such as the Schwarzschild solution and the Kerr solution is null approaching spatial infinity, while the cosmological solution of the general Robertson-Walker class might only be spatially flat and not globally flat. The Robertson-Walker metric is the one that reproduces a spatially isotropic and homogeneous space which is modulated by a *scale factor* $\mathbf{a(t)}$. The spatial curvature of this metric is modeled by an adjustable parameter k, and cosmologists tend to agree to the fact that a perturbed Robertson-Walker metric is the best description of the actual universe, at least here-now. The case $\mathbf{k}=0$ is the flat one and can be called *simple cosmological spacetime*. Einstein-de Sitter metric belongs to a subclass of Robertson-Walker flat metrics of vacuum solutions, where the cosmological constant Λ is carrying a kind of vacuum energy (dark energy) that allows the vacuum spacetime to expand. Robertson-Walker metrics which describe universes filled with a perfect fluid obey a class of equations called **Friedmann equations**. These equations rule the evolution of the universe, and in particular the evolution of the scale factor. Spacetimes that obey to these equations are called *Friedmann-Robertson-Walker* (FRW) universes. The present day accepted model is a perturbed flat Robertson Walker solution called **Λ-CDM** model, that includes multiple density parameters to describe dark matter, dark energy and radiation. The presence of dark energy is a need to describe an accelerating universe, in accordance with the latest observations. However, the source of dark energy is still a matter of debate and black holes themselves might represent the main source.

Some solutions to the incompatibility issue have been proposed. The Einstein-Strauss model "pasts" a vacuum sphere of arbitrary radius representing the Schwarzschild solution in an expanding Friedmann universe [19]. It has also been called, as the intuitive picture of spacetime suggests, a "Swiss-cheese model". The result is that the condition of asymptotic flatness is relaxed. Another similar attempt by Vaidya was made to represent the model with a superposition of the Kerr solution with the metric of the Einstein universe [21]. The result is the *Vaidya-Einstein-Schwarzschild* (VES) spacetime. Furthermore, a time-dependent solution for the Schwarzschild black hole living in a cosmological spacetime was proposed by Sultana and Dyer [20]. Nevertheless, the solution investigated by this thesis is the one discussed by McVittie, the first of its kind, which represents a "cosmological" analogue of the Schwarzschild metric, describing a particle surrounded by a perfect fluid. McVittie expresses the metric in terms of "*cosmical coordinates*", compatible at infinity with the RW metric, differing only in terms of negligible contributes [12]. The cosmical coordinates reduce to the comoving FRW coordinates, thus they represent a particular reference frame of the galaxies composing the universe. However, a causal structure analysis of the surfaces of interests of the McVittie metric shows that only a restricted subclass of solutions can represent BHs. Moreover, some further results regarding the McVittie Black holes have been treated by Nolan (1993) who constructed a non-singular interior for this solution. Faraoni & Jacques (2007) treated dynamical phenomena of this kind of black holes resulting in effects including a comoving horizon.

Since the Schwarzschild solution is the ideal one of a non-rotating black hole, a generalization of the Kerr black hole of the same kind is needed. However, no analytical solution has been found at the present day. Local Kerr solutions describe with excellent results only black holes in timescales from millisecond to hours, and spatial scales up to milliparsecs. When the scales become cosmological, the absence of a coupling with the global solution leads to inconsistencies. A BH solution that satisfies observational constraints at small and large scales simultaneously has yet to be found. However, a more general relation between relativistic material and the expansion of the universe was found by Croker & Weiner (2019). The consequences of this kind of coupling concerning Kerr BHs are dealt in [7]. The latter article investigates such effect in a new light, giving experimental results and conclusions about the contributions of black holes as a dark energy species, stating that the late-time accelerating expansion of the universe is driven by these kind black holes interiors. This thesis aims to give a first introduction of local and cosmological solutions in general relativity, to investigate the McVittie computations and physical implications, and to discuss the new conclusions about the origin of dark energy made by Farrah et al. (2023).

CHAPTER

TWO

EXACT SOLUTIONS

The following results, if not differently specificated, are taken from [18] and treated and discussed in the related mathematical formalism.

2.1 The Schwarzchild black hole

Assume $(\mathcal{S}^2, h, \xi)$ to be the unit 2-sphere space-time, where h is the metric associated with it and ξ the volume element. Let $\mu \in (0, \infty)$ [1]. Define $\mathcal{A} \subset \mathbb{R}^2$ by $\mathcal{A} = (u^1)^{-1}[(0, 2m) \cup (2m, \infty)]$. $M = \mathcal{S}^2 \times \mathcal{A}$ is then the manifold of interest which is the Cartesian product between a 2-sphere (where the angular coordinates shall live) and a rectangle (where the radial and time coordinates shall live[2]). Let $P : M \to \mathcal{S}^2$ and $Q : M \to \mathcal{A}$ be the natural projections. Now define $r = u^1 \circ Q$ and $t = u^2 \circ Q$.

Definition 2.1. The **Schwarzchild metric** is a Lorentzian metric [3] **g** on M defined by [18, p. 30]

$$g = \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2 P^* h - \left(1 - \frac{2\mu}{r}\right) dt^2 \tag{2.1.1}$$

which can be explicated by

$$g = \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 - \left(1 - \frac{2\mu}{r}\right) dt^2 \tag{2.1.2}$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$.
It is clear how the latter metric reduces to a flat Minkowski metric as $r \to \infty$ i.e. infinitely away from the source. The Schwarzschild space-time is composed of two connected components:

i) $(N, g|_N)$ where $N = r^{-1}(2\mu, \infty)$ is the *normal Schwarzschild space-time of active mass* $\bar{m} = 8\pi\mu$. This space-time is sufficient to describe stars whose radius is smaller than 2μ. In fact, the Schwarzschild solution is a vacuum solution and so it wouldn't describe the interior of the star anyway. $g|_N$ is a Lorentzian metric with $\left.\frac{\partial}{\partial t}\right|_N$ timelike while r is a kind of radius.

ii) $(B, g|_B)$ where $B = r^{-1}(0, 2\mu)$ is the *Schwarzschild black hole of active mass m*. A star of active mass at least twice as big as the one of the sun, which runs out of fuel, starts collapsing

[1] The mass of the black hole is actually not defined a priori. It is a constant of the metric which can be linked to the mass through the weak field limit.

[2] It may be misleading to name the first coordinate "time" since in one of the two connected components (the black hole component) of $\mathcal{A}$ it is not a time at all. In fact, it is neither spacelike nor timelike.

[3] A metric with signature=2.

inexorably because of the gravitation field of its own mass. The result is a black hole, a body with a density beyond the current scope of physics. No signal can escape from it and the only trace that the black hole leaves is its gravitational influence on the rest of the universe. In this region $\left.\frac{\partial}{\partial r}\right|_B$ is timelike so that $-dr|_B$ is a kind of time. Every timelike and null geodesic must go in the direction of the black hole heading to the singularity of infinite curvature.

We stated before that the limit $r \to \infty$ brings the metric to a flat Minkowski one. A formal way to state that the latter limit corresponds to approaching spatial infinity is the following. Define $f : U \to \mathbb{R}$ where $U = N \cup B$ and $f = R^{ijkl} R_{ijkl}$ which measures the overall magnitude of the curvature. An observer $\gamma : (a, b) \to M$ is said to *escape to infinity* iff $lim_{u \to b} f \gamma u = 0$.

Proposition 2.1. $f = 144\frac{\mu^2}{r^6}$

Proof. Omitted. □

The Schwarzschild metric is the unique spherically symmetric vacuum solution (thus *Ricci flat*) as stated by **Birkhoff's theorem** [2]. The result is a *static metric.* Following are reported some definitions to point out what the latter means.

2.2 Reference frames

Definition 2.2. Let (M,g) be a spacetime. An *observer* in M is a future-pointing timelike curve $\gamma : \varepsilon \to M$ such that $\mid \gamma_* \mid = 1$.

Normalization is imposed for convenience.
An *instantaneous observer* is an ordered pair (z,Z), where $z \in M$ and Z is a future-pointing timelike unit vector in M_z

Definition 2.3. A *reference frame* **Q** on a space-time M is a vector field each of whose integral curves is an observer. **Q** is a *geodesics* reference frame iff $\nabla_{\mathbf{Q}}\mathbf{Q} = 0$.

In other words a vector field **Q** is a reference frame iff g(**Q**,**Q**)=-1 and **Q** is future pointing. Now, let ω be the 1-form equivalent to Q (i.e. $\omega(\cdot) = g(\mathbf{Q}, \cdot)$). Let $\gamma : \varepsilon \to M$ be an observer in **Q**. Then $(\gamma^*\omega)(d/du) = \omega(\gamma_*) = g(\mathbf{Q}, \gamma_*) = g(\gamma_*, \gamma_*) = -1$ since γ must be future pointing. Thus, it is immediate that $du = -\gamma^*\omega$. A reference frame **Q** may also have some additional properties that will be useful in the next sections:

i) **Q** is called *locally synchronizable* iff $\omega \wedge d\omega = 0$

ii) **Q** is called *locally proper time synchronizable* iff $d\omega = 0$

iii) **Q** is called *synchronizable* iff there exist $h, t : C^\infty(M) \to \mathbb{R}$ such that $h > 0$ and $\omega = -h\ dt$. The function t is then called *time function* which is not unique. In this case $du = (h \circ \gamma)\gamma^*\ dt$.

iv) **Q** is called *proper time synchronizable* iff h identically equals 1. Then $\omega = -dt$ and t is called the *proper time function* which is also not unique. In this case $du = \gamma^*\ dt$.

Proposition 2.2. *If* **Q** *is an arbitrary reference frame on M with the following properties: i) dt is nowhere zero, and ii) the level hypersurfaces of the time function t are orthogonal to* **Q**.
Then, **Q** *is syncronizable and* $\pm t$ *is a time function for* **Q**.

Proof. Omitted. □

Now let **Q** be a proper time synchronizable reference frame and $\mathcal{N}_a$ be the level hypersurface of the proper time t defined by t=a. Assume that each observer intersects $\mathcal{N}_a$ only once. The observers belonging to the reference frame can agree to set up their "atomic clocks" in order to make it mark

$t = 0$ when they cross $\mathcal{N}_0$. Since $du = -\gamma^* \, dt$, each of them will cross $\mathcal{N}_a$ at $t = a$. When the reference frame is only synchronizable and not proper time synchronizable, it is possible to express $(t \circ \gamma)$ as an explicit function of u dependent by h. Physically, observers in a (proper time) synchronizable reference frame can use light signals to experimentally correlate by "radar" their reference times. In the former time will coincide, while in the latter they'll reach a compromise time.

2.3 The Schwarzschild metric as a static metric

As we are dealing with a time-independent source of gravity, a few more implications must be treated. This peculiarity can be only valued by postulating the existence of a reference frame of observers that don't experience a change in the local geometry.

Definition 2.4. Let **Z** be a reference frame on M. **Z** is defined as *stationary iff* there is a positive function f on M such that $f\mathbf{Z}$ is a Killing vector field. **Z** is then static *iff* it is stationary and irrotational[4]. M is *stationary (respectively, static) iff* there exists on M **Z** above.

Furthermore M is an *absolute* stationary reference frame if it is unique. An observer $(\gamma u, \gamma_* u)$ is at *rest* iff it is in absolute reference frame. Irrotational reference frames are privileged reference frames. In fact, neighbors reference frames can relate their times to each others.

Proposition 2.3. *A reference frame is irrotational iff it is locally synchronizable.*

Proof. Omitted. □

Heuristically we can say that stationary spacetimes are generated by a time-independent source while static spacetimes have the additional property of a non rotating source.

Proposition 2.4. *The normal Schwarzschild space (2.1.2,ii) is a static spacetime.* $a(\partial/\partial t)|_N, \forall a \in \mathbb{R}$, *is a killing future pointing vector.* $\boldsymbol{Z} = (1 - 2\mu/r)^{-1/2}(\partial/\partial_t)$ *is a static, absolute reference frame on N. An observer* γ *is at rest iff* $r \circ \gamma \in (2\mu, \infty)$ *constant and* $P \circ \gamma \in \mathcal{S}^2$ *constant, where* $P : M \to \mathcal{S}^2$ *is the natural projection on the angular coordinates.*

Z is orthogonal to the family of hypersurfaces t=const, and in particular it is synchronizable but not proper time synchronizable. Two observers in this reference frame are able to find a compromise time that won't be the same in general. The distance between two observers $(x_1, \mathbf{Z}x_1)$ and $(x_2, \mathbf{Z}x_2)$ who are at rest is independent of time. In fact, the two can relate their proper times u_1 and u_2 to the coordinate t. At a constant t, the distance between them is given by

$$\Delta s = \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \tag{2.3.1}$$

Supposing without loss of generality that $Px_1 = Px_2$. Then

$$\Delta s = \int \left(1 - \frac{2\mu}{r}\right)^{-1} dr \tag{2.3.2}$$

which is not a function of t.

2.4 Isotropic coordinates

Definition 2.5. A metric expressed in **isotropic coordinates** (r, θ, ϕ, t) is a Lorentzian metric that can be written in the form

$$g = A(r,t)^2(dr^2 + r^2 d\Omega^2) - B(r,t)^2 dt^2 \tag{2.4.1}$$

[4] a specific definition of irrotational is given in Sachs-Wu 2.3. Physically, the observers in a small neighborhood of an observer in $x \in M$ exhibit no overall rotation.

The Shwarzschild metric can be also expressed in isotropic coordinates, which are obtained by a diffeomorphic transformation, resulting in a physically equivalent space-time. As a consequence, the latter is just given without further comments [22, p. 181] by

$$r = r_1 \left(1 + \frac{\mu}{2r_1}\right)^2 \tag{2.4.2}$$

In the new coordinates, the Schwarzschild metric can be indeed written

$$g = \left(1 + \frac{\mu}{2r_1}\right)^4 dr_1^2 + r_1^2 d\Omega^2 - \left(\frac{1-\mu/2r_1}{1+\mu/2r_1}\right)^2 dt^2 \tag{2.4.3}$$

This coordinate system doesn't allow to describe spacetime inside the event horizon. In fact, the condition that emerges from the change of coordinate is

$$r_1 > \frac{\mu}{2}$$

Nevertheless, the latter form will be useful in discussing the McVittie metric.

2.4.1 The Kerr Solution

The Kerr metric, found only in 1963, represents a spinning black hole. Such property is reasonable in every realistic scenario, since a slight non-homogeneity of a collapsing star would produce a rotation. The solution is much more difficult than the Schwarzschild one, and the derivation starts from the assumptions of axial symmetry (around the rotation axis) and stationary solution. The result is [2]

$$g = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2\theta d\phi^2 + \frac{2\mu r}{\rho^2}(a\ \sin^2\theta d\phi - dt)^2 \tag{2.4.4}$$

where

$$\Delta(r) = r^2 - 2\mu r + a^2$$
$$\rho^2(r,\theta) = r^2 + a^2 cos^2\theta$$

The active mass of the black hole is $\bar{m} = 8\pi\mu$, while the variable a measures the spin. Note that for $a \to 0$ the solution reduces to Schwarzschild one.
The metric produces two killing vectors: $\xi^\mu = \partial_t$ that makes the solution stationary, but not static (since it's spinning); and $\eta^\mu = \partial_\phi$ which expresses the axial symmetry. Further properties are resumed by Carrol (1997) [2].

2.5 Cosmological Models

2.5.1 Matter Models

We want to describe the universe and its history through a spacetime (M, g, ∇) where ∇ is the relative Levi-Civita connection. The approximation used for the Schwarzschild solution of a vacuum universe is not valid anymore. One needs to define a stress-energy tensor that describes the energy distribution of the universe and other pre-relativistic concepts such as energy of electromagnetism, momentum per unit volume, energy flux, and momentum flux. In doing so a matter model is needed.

Definition 2.6. A *particle of rest-mass* m is a future-pointing curve $\gamma : \varepsilon \to M$ such that $g(\gamma_*, \gamma_*) = -m^2$ where $m \in [0, \infty)$ is allowed to be 0 for mass-less particles such as photons.

Suppose now that an enormous number of particles of mass m have the same energy-momentum.

Definition 2.7. A particle flow ($\mathbf{P}$,η) of rest-mass m is a function $\eta : M \to [0, \infty)$, called the *world density*, and an *energy-momentum* vector field $\mathbf{P}$: $M \to TM$ such that each integral curve of $\mathbf{P}$ is a particle rest-mass m.

This description is useful to describe the collective behavior of a distribution of particles that are future-pointing and with a similar momentum. The particle world density is conserved *iff* $div(\eta\mathbf{P}) = 0$. The stress-energy tensor is then built from the latter. If the particles of the flow of mass m all have the same charge $\mathbf{e} \in \mathbb{R}$, then (m,e,$\mathbf{P}$,$\eta$) is the *type of the particle flow.* We call $\hat{E}$ the (2,0)- stress-energy tensor field on M and $\mathbf{E}$ the physically equivalent (0,2)-tensor field. Suppose any instantaneous observer (z,Z) measures the energy in any unit 3-volume of his local rest space $Z^{\perp}$. We assume $\mathbf{E}$(Z,Z)=measured energy density $\forall z, Z$.

Definition 2.8. Let ($\mathbf{P}$,η) be a particle flow on spacetime M. The stress-energy tensor of ($\mathbf{P}$,η) is $\hat{T} = \eta\mathbf{P}\otimes\mathbf{P}$.

The stress-energy tensor $\mathbf{E}$ of an electromagnetic field $\mathbf{F}$ on M is then defined as

$$E_{ij} = \frac{1}{4\pi}[F_{im}F_j^{\ m} - \frac{1}{4}g_{ij}F^{mn}F_{mn}] \tag{2.5.1}$$

The basic object of interest in mathematical general relativity is a triple (M,$\mathcal{M}$,$\mathbf{F}$) called a *relativistic model.* Here, M is a Riemannian manifold, $\mathbf{F}$ is an electromagnetic field and $\mathcal{M}$ is a *matter model.* In most of the non-quantum physical situations a matter model can be described by a collection $\mathcal{M}$=(m_A, e_A,$\mathbf{P}_A, \eta_A$) | $A = 1, ..., N$ of N particle flows M Where N is a non-negative integer. By making N sufficiently large one can basically describe any form of matter.
Some useful examples of matter models are reported in the following examples. Each of these is necessary to understand the *perfect fluid* matter model which is the one used to model cosmological spacetimes and to compute the *Friedmann equations.* Let $\mathbf{J}$ be the charge-current density.

Example 2.1 (Dust)**.** Let (m,0,$\mathbf{P}$,η) be a particle flow on M with m,η nowhere zero. Then $Z = m^{-1}\mathbf{P}$ is a reference frame on M and $\rho = m^{-2}\eta$ is a function $\rho : M \to (0, \infty)$.
(M,$\mathcal{M}$,$\mathbf{F}$) is defined as *dust* iff:

i) $\mathcal{M}$ =($\mathbf{Z}$,ρ), where $\mathbf{Z}$ is a reference frame on M and ρ is a C^{∞} function.

ii) $\hat{T} = \rho\mathbf{Z}\otimes\mathbf{Z}$ and $\mathbf{J}$=0.

$\mathbf{Z}$ is then defined as the *comoving reference frame.* $\forall z \in M$, the instantaneous observer (z,$\mathbf{Z}$z) measures energy density $\mathbf{T(Z,Z)}$z=ρz so ρ is defined as the *comoving energy density.* We say (M,$\mathcal{M}$, $\mathbf{F}$) obeys the *dust matter equation iff* div($\rho\mathbf{Z}$)=0=$D_{\mathbf{Z}}\mathbf{Z}$.

If (M,$\mathcal{M}_1$,$\mathbf{F}$) and (M,$\mathcal{M}_2$,$\mathbf{F}$) are relativistic models, then the *superposition* (M,$\mathcal{M}$,$\mathbf{F}$) is defined as:
$\mathcal{M}$ is the pair $\{\mathcal{M}_1, \mathcal{M}_2\}$ and $\hat{T} = \hat{T}_1 + \hat{T}_2$, $J = J_1 + J_2$. Then $\mathbf{M}_1$ and $\mathcal{M}_2$ are the *components* of $\mathcal{M}$. We require that the matter equations of the two components are the same and that the superposition is collision-free i.e. there aren't any interactions between them.

Example 2.2 (Quasi-gas)**.** $\mathcal{M}$ is a *quasi-gas* on M *iff* $\mathcal{M}$ is a finite superposition [5] of particle flows on m obeying

i) $\mathcal{M}$ is a non-empty set;

ii) $\forall$ particle-flow in $\mathcal{M}$, the world density is nowhere zero.

It is reasonable to assume that a stress-energy tensor singles out a reference frame in a natural way. $\hat{T}$ is defined to be *normal* at $x \in M$ *iff* $\tilde{T}$X[6] is timelike $\forall$ causal $X \in M_x$. X is then called an *eigenvector* of $\hat{T}$ at x *iff* $\tilde{T}X = aX$ for some $a \in \mathbb{R}$.

Proposition 2.5. *If a stress-energy tensor $\hat{T}$ is normal at $x \in M$, then $\hat{T}$ has a timelike eigenvector which is unique up to nonzero multiples.*

[5] the "quasi-gas" characterization refers to the fact that gas must be described by an infinite collection of particle flows.

[6] the equivalent (1,1)-tensor of $\hat{T}$.

Proof. omitted. □

Now $\hat{T}$ is called *normal* if it is normal t every $x \in M$. A nowhere zero vector field **X** on M is called an *eigenvector field of* $\hat{T}$ *iff* there is a function (eigenfunction) f on M such that $\hat{T}X = fX$.

Proposition 2.6. *A normal stress-energy tensor T on M posseses a unique future-pointing unit timelike eigenvector field.*

Proof. Omitted. □

The concept of eigenvector fields is of fundamental importance in defining the *comoving reference frame* in a cosmological spacetime. If we assume the Einstein field equations, it is easy to demonstrate that an eigenvector field of $\hat{T}$ is necessary and sufficiently an eigenvector field of $\hat{G}$.

Example 2.3 (Perfect fluid)**.** The perfect fluid matter model is the one used to define the Friedmann-Robertson-Walker universes. A stress-energy tensor $\hat{T}$ on a spacetime M is defined as *spatially isotropic* at $z \in M$ *iff* there exists one instantaneous observer (z,Z) for whom T is spatially isotropic i.e they are left invariant by the unique extension of the transformation $\psi : M_z \to M_z$ such that $\psi \in \mathcal{O}^3$.

Proposition 2.7. *Let* $\mathcal{M}$ *be a quasi-gas on M whose stress-energy tensor* $\hat{T}$ *is spatially isotropic at each* $z \in M$*. Then:*

i) $\hat{T}$ *is normal;*

ii) $\hat{T} = \rho \mathbf{Z} \otimes \mathbf{Z} + p(\hat{g} + \mathbf{Z} \otimes \mathbf{Z})$*, where* **Z** *is the reference frame which is an eigenvector field of T.* ρ, p *are functions on M such that* $\rho > 0$ *and* $\rho \geq 3p \geq 0$*. Furthermore, T is spatially isotropic for (z,Z) iff* **Z**=**Z***z;*

iii) $\rho z = 3pz$ *for one* $z \in M$ *iff* $\rho = 3p$ *iff each component of* $\mathcal{M}$ *has zero rest mass.*

Proof. Omitted. □

We call ρ and p respectively *the energy density and the pressure of the quasi-gas* $\mathcal{M}$. The physical interpretation of the latter is in terms of "random velocities" of the quasi-gas. J is set to zero. The condition in *iii)* is the one of a *radiation-dominated universe.* Furthermore, in a Robertson-Walker universe, $\mathbf{Z} = \partial_4$ is comoving.

Definition 2.9. A relativistic model (M,$\mathcal{M}$, **F**) is a *perfect fluid* iff

i) $\mathcal{M} = (\mathbf{Z}, \rho, p)$ where **Z** is a reference frame on M, and ρ, p are C^∞ functions with $\rho > 0$ and $\rho \geq 3p \geq 0$;

ii) J=0 and $\hat{T} = \rho \mathbf{Z} \otimes \mathbf{Z} + p(\hat{g} + \mathbf{Z} \otimes \mathbf{Z})$

By definition, $\mathcal{M}$ is then a *perfect fluid* on M and **Z** is the *comoving reference frame.*

A perfect fluid can be seen as an enormous number of particle flows with lots of random collisions. These fluids are isotropic in their rest frames. In these frames $T^{\mu\nu} = diag(p, p, p, \rho)$. Here ∂_4 is taken to be the timelike coordinate. The stress-energy tensor with an index raised by the metric is

$$T^\mu_\nu = diag(p, p, p, -\rho) \tag{2.5.2}$$

and the trace is given by

$$T = T^\mu_\mu = -\rho + 3p; \tag{2.5.3}$$

To make progress one has to define an **equation of state** of the type

$$p = \omega\rho \tag{2.5.4}$$

which is respected by essentially every perfect fluid relevant to cosmology.

2.5.2 Robertson-Walker spacetime

Now that a justified stress-energy tensor is defined it is possible to build a cosmological model.
As we anticipated the Copernican principle states that the universe must be homogeneous and isotropic. These conditions have mathematical implications [2]:

i) **Spatial homogeneity**: $\forall p, q \in M$ there is an isometry ϕ on M such that $\phi p = q$;

ii) **Spatial isotropy** for an observer(z,Z): let ϕ be an isometry on M such that $\phi z = z$ and $\phi_* Z = Z$. Then $\phi_* \in \mathcal{O}^3$. Then (M,g) is called *spatially isotropic* for (z,Z) *iff* given any two vectors $X_1, X_2 \in R$ (local rest space of (z,Z)), there is such ϕ with $\phi_* X_1 = X_2$.

On the other hand, when we look at distant galaxies, they appear to be receding from us. As a consequence, the universe must be not static so homogeneity and isotropy are not assumed in time. Define now $M = \Sigma \times \mathcal{F}$, where $\mathcal{F}$ is an open interval in $\mathbb{R}$ and Σ is a maximally symmetric three-dimensional manifold. We basically foliate spacetime in isotropic, homogeneous, spacelike slices. The metric can thus be taken to be of the form[2]

$$g = -dt^2 + a^2(t)d\sigma^2 \tag{2.5.5}$$

where $d\sigma^2 = \gamma_{ij}dx^i dx^j$ is the 3-D maximally symmetric submanifold metric. $a(t)$ is called the **scale factor**, and tells us how big the spacelike manifold is at a time t. The Riemann tensor of a maximally symmetric space is uniquely specified by a *curvature constant* K and by the number of eigenvalues of the metric [22] in the following way

$$R_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \tag{2.5.6}$$

and

$$R_{jl} = 2K\gamma_{jl} \tag{2.5.7}$$

A space with these properties is called a *space of constant curvature.* The metric for Σ is distinctively determined, up to coordinate transformations from (2.5.6)[13]. A spherical symmetric space is in the form

$$d\sigma^2 = e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{2.5.8}$$

Comparing the Ricci tensor to the one in (2.5.7), one can find the **Robertson-Walker metric**

$$g = a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right] - dt^2 \tag{2.5.9}$$

It is obtained without the use of the field equations but just from the already treated generic assumptions. Since the substitution

$$K \to \frac{K}{\mid K \mid} \quad r \to \sqrt{\mid K \mid} r \quad a(t) \to \frac{a(t)}{\sqrt{\mid K \mid}} \tag{2.5.10}$$

leaves the metric (2.5.9) invariant, the only relevant parameter is $k = \frac{K}{|K|}$, provided that the scale factor a(t) must be renormalized and the coordinate r must be transformed as above. The parameter k defines three cases of interest:

i) k=0 (**flat**): There is no curvature on the spatial sub-manifold $\mathbb{R}^3$;

ii) k=1 (**closed**): positive curvature on $\mathbb{R}^3$;

iii) k=-1(**open**): negative curvature on $\mathbb{R}^3$.

The metric (2.5.9) can be also written [10]

$$g = a(t)^2\left(dr^2 + \chi^2(r)d\Omega^2\right) - dt^2 \tag{2.5.11}$$

where

$$\chi(r) = \begin{cases} sin(r) & \text{if k=+1} \\ r & \text{if k=0} \\ sinh(r) & \text{if k=-1} \end{cases} \tag{2.5.12}$$

These three can be combined notationally as

$$\chi(r) = \frac{sin(\sqrt{k}r)}{\sqrt{k}} \tag{2.5.13}$$

In fact, the case k=0 is interpreted as a limit

$$\lim_{k\to 0} \chi(r) = \lim_{k\to 0} \frac{sin(\sqrt{k}r)}{\sqrt{k}} = \lim_{k\to 0} \frac{\sqrt{k}r}{\sqrt{k}} = r \tag{2.5.14}$$

and the case k=-1 gives

$$\frac{sin(\sqrt{-1}r)}{\sqrt{-1}} = \frac{sin(ir)}{i} = sinh(r) \tag{2.5.15}$$

. The non-trivial results above are thus recovered. Finally, through the coordinate change

$$\overline{r} = 2\frac{tan(\sqrt{k}r/2)}{\sqrt{k}} \tag{2.5.16}$$

and renaming $\overline{r}$ as r, we obtain the *Lamaître coordinate system* of the RW metric

$$g = \frac{a^2(t)}{\left(1 + \frac{kr^2}{4}\right)^2} \left\{dr^2 + r^2 d\Omega^2\right\} - dt^2 \tag{2.5.17}$$

It is easy to verify that starting from the elements of the spacelike metric $d\sigma^2$

$$\gamma_{ij} = (1 + \frac{kr^2}{4})^{-2}\delta_{ij} \tag{2.5.18}$$

that the equation (2.5.6) holds [13].
The spacelike submanifold is modulated by the renormalized scale factor and thus the equation (2.5.6) becomes

$$R_{ijkl} = [k/a^2(t)](\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \tag{2.5.19}$$

The proper distance between two different galaxies, measured at the same cosmic time t is consequentially [22]

$$d_{prop}(t) = \int_0^{r_1} \sqrt{g_{rr}} dr = a(t) \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} \tag{2.5.20}$$

A perfect fluid relativistic model of the kind of (2.9) is now used to describe the universe matter distribution since this type of fluid is isotropic in his rest frame, then the metric expressed in the isotropic frame we are seeking to build is going to identify the same frame, i.e. the *comoving frame*. The comoving observer is intuitively speaking moving with the matter of the universe. Here-now galaxies seem to form the predominant form of matter. The galaxies random velocities are quite small compared to the speed of light. Thus, we can interpret the history of each galaxy with an integral curve of **Z**.

It is seen in proposition 2.7 that $\omega \leq \frac{1}{3}$ and the relation becomes an equality for a radiation-dominated model $\omega = \frac{1}{3}$.

We also see that by computing the conservation of energy equation of the zero component of T^{μ}_{ν} we obtain [2]

$$0 = \nabla_{\mu} T^{\mu}_{0} = -\dot{\rho} - 3\frac{\dot{a}}{a}(\rho + 3p) \tag{2.5.21}$$

where the dot indicates a partial derivation over t. The latter with the equation of state (2.5.4) gives [23]

$$\rho = \rho_0 a^{-3(1+\omega)} \tag{2.5.22}$$

Where ρ_0 is the density at the present time. In other words, the density of the fluid decreases with a weight ω given by the equation of state. If the stress energy tensor is given by the superposition of multiple different perfect fluids, the above equation is valid for each fluid

$$\rho_x = \rho_{(x,0)} a^{-3(1+\omega_x)} \; x = 1, ..., N \tag{2.5.23}$$

where N is the number of fluids taken in consideration.
Empirically we can state that our universe here-now is a matter universe with $\dfrac{\rho_{mat}}{\rho_{rad}} \sim 10^{67}$.
We can write $\hat{T} = \hat{T}_g + \hat{T}_p + \hat{T}'$ where $\hat{T}_g$ is due to the matter in galaxies, $\hat{T}_p$ is due to the microwave photons and $\hat{T}'$ includes the contribution of all other forms of matter such as neutrinos. Near here-now $\hat{T}_g$ must certainly dominate. Thus a dust cosmological model is preferred. Dust is collision-less, non-relativistic matter. The weight $\omega_m = 0$ (p=0) implies $\rho_m = \rho_{m,0} a^{-3}$. Furthermore, ordinary matter doesn't completely describe all the matter composing the universe. Inconsistencies in the observations of spiral galaxies by Rubin [23] showed that a big amount of non-luminous matter must affect observations. The theorizing of *dark matter* followed quickly.

Dealing with radiation, in order to include further corrections, one discovers by relating the two forms of the stress-energy tensor (2.5.2) and (2.5.1) that the weight of radiation is $\omega_r = 1/3$. Thus $\rho_r = \rho_{r,0} a^{-4}$.

These components have been supposed enough to describe an economic model of the universe here-now, at least before the more recent discoveries made by the Supernova Cosmology Project and the High-z Supernova Search Team from observations of standard candles and type Ia supernovae in 1998 [23]. A cosmological model built with only matter and radiation necessary leads to an expanding universe with a decreasing velocity. Experimental data supported this behavior until the latter discovery that suggested an accelerating universe. In order to explain data, the cosmological constant Λ was recovered. Originally, Λ was added by Einstein to the field equations in order to allow the universe to be static. On the other hand, a static universe would have been unstable and cosmological data soon supported the opposite view of a dynamic universe. After the above observations, the cosmological constant was recovered to make the universe accelerate.

Λ adds a vacuum energy, also called *dark energy.* The field equations (3.2.6) in vacuum are of the same form as the equations with no cosmological constant but an energy momentum for the vacuum

$$T^{(vac)}_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{2.5.24}$$

The structure of this tensor is the same of a perfect fluid with

$$\rho = -p = \Lambda \tag{2.5.25}$$

Therefore the cosmological constant can also be treated as a negative density component in the stress-energy tensor.

Using the field equations in the form of (2.5.4), we obtain the **Friedmann equations** [2]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{2.5.26}$$

[7] However, the energy density in radiation dominated at very early times...

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3} \tag{2.5.27}$$

Metrics that follow the latter equations are called *Friedmann-Robertson-Walker* (FRW) universes.
The **Λ-CDM** model is a perturbed flat FRW model that includes dark matter, radiation, and dark energy, given by the cosmological constant. As suggested by recent observations, the Λ-CDM model is the one preferred to describe the universe here-now. In fact, it is a simple and economic model that doesn't give up to describe in good approximation the observations [23].
Note that the Friedmann equations (2.5.26) and (2.5.27) can be also written without the cosmological constant Λ, and by including it in the density. In this way, pressure can also be negative, and in order to get an accelerating universe the following condition must hold:

$$-1 \leq \omega \leq -\frac{1}{3} \tag{2.5.28}$$

where $\omega = -1$ is the only vacuum case.
We then call $H = \frac{\dot{a}}{a}$ the *Hubble parameter* which characterizes the rate of expansion.
The *density parameter* is defined by $\Omega = \frac{\rho}{\rho_{crit}}$
where $\rho_{crit} = 3H^2$ is the density obtained by imposing k=0 (flat universe) in (2.5.26), and Λ included as a energy density.
The present experimental Hubble parameter is $H_0 = 2.18 \times 10^{-18}\ s^{-1}$ [23].
Therefore, the present day critical density is

$$\rho_{(crit,0)} = 3H_0^2 = 3(2.18 \times 10^8\ s^{-1})^2 = 1.43 \times 10^{-35}\ s^{-2}$$

in natural units.
Thus, if the current total energy density of the universe is greater than or less than $\rho_{crit,0}$, the universe will either be spherical or hyperbolic. Ω gives information about the curvature of the universe since the Friedmann equation (2.5.26) can be written

$$\Omega - 1 = \frac{k}{H^2 a} \tag{2.5.29}$$

that selects the three cases: i) $\Omega < 1 \longleftrightarrow$ open; ii) $\Omega = 1 \longleftrightarrow$ flat; i) $\Omega > 1 \longleftrightarrow$ closed.
In a model where more constituents are taken, these definitions must be treated separately for each component

$$\Omega_x = \frac{\rho_x}{\rho_{crit}} \quad ; \quad \Omega_{(x,0)} = \frac{\rho_{(x,0)}}{\rho_{crit}} \tag{2.5.30}$$

Using then Ω as a parameter in the equation (2.5.23) it is possible to predict the evolution of

$$\rho_x = 3H_0^2 \Omega_{(x,0)} a^{-3(1+\omega_x)} \quad ; \quad \Omega_x = \frac{H_0^2}{H^2} \Omega_{(x,0)} a^{-3(1+\omega_x)} \tag{2.5.31}$$

2.5.3 Simple cosmological spacetimes

Robertson-Walker flat spaces are interesting since they describe the universe in a surprisingly good approximation. We shall call them *simple cosmological spacetime.* They can be written in the form

$$g = \left\{ (e^{(\beta \circ t)} \left(dr^2 + r^2 d\Omega^2 \right) \right\} - dt^2 \tag{2.5.32}$$

where $\exp(\beta/2) = a$.
Finally, it is called *original de Sitter universe* a simple cosmological spacetime in the form (2.5.32) and $\beta \circ t = at$ with $a = \pm\sqrt{\frac{\Lambda}{3}}$ is a constant [2].
The latter universe can be obtained from the Friedmann equations for a simple cosmological spacetime in vacuum filled with energy given by a cosmological constant $\Lambda > 0$. Actually, whatever the space curvature is, all the vacuum spacetimes with $\Lambda > 0$ are equivalent and maximally symmetric. The $\Lambda < 0$ solution is also maximally symmetric, and it is known as the *anti-de Sitter universe.*

CHAPTER

THREE

THE MCVITTIE MASS-PARTICLE

3.1 The McVittie mass-particle in a cosmological universe

The McVittie mass-particle is a generalization of the better known Schwarzschild static mass-particle expressed in isotropic coordinates (2.4.3)

$$g = \left(1 + \frac{\mu}{2r_1}\right)^4 dr_1^2 + r_1^2 d\Omega^2 - \left(\frac{1 - \mu/2r_1}{1 + \mu/2r_1}\right)^2 dt^2 \tag{3.1.1}$$

The wanted generalization is needed when an observer is near a strong source of gravity and wants to model it, embedded in the cosmological universe. In fact, this new metric describes a spherically symmetric massive body, such as a black hole or a star, which is compatible at infinity with the RW metric, eventually written in the Lemaître coordinates (2.5.17)

$$g = \frac{a^2(t)}{\left(1 + \frac{kr^2}{4}\right)^2} \left\{dr^2 + r^2 d\Omega^2\right\} - dt^2 \tag{3.1.2}$$

where the constant k gives the curvature of the space as a whole.
The simple cosmological metric already discussed in (2.5.32) is the special case of (3.1.2), in which the curvature of space is set to zero. While the cosmological metrics (2.5.32) and (3.1.2) are expressed in the *cosmic coordinate*[1] r, the local metric uses the *observer's coordinate* r_1. These are related by a change of coordinate made by the observer at a specific time t_1 written in the form $r_1 = re^{\beta(t_1)/2}$. In this way, the observer's coordinate is not independent of the time at a fixed value of r. In particular, the metric (3.1.2) in observer's coordinates becomes

$$\frac{e^{\beta(t)-\beta(t_1)}}{\left(1 + \frac{ke^{-\beta(t_1)}r_1^2}{4}\right)^2} \left\{dr_1^2 + r_1^2 d\Omega^2\right\} - dt^2 \tag{3.1.3}$$

which at the time t_1 becomes a static spacetime, possibly just curved in space with spatial curvature $K = e^{-\beta(t_1)}k$.
In this way, in a flat RW spacetime (2.5.32), the observer lives in a flat Minkowski spacetime at t_1.

[1]The "cosmical coordinates" are called in this way because the "cosmical time" reduces to the ordinary FRW comoving time when the center mass is absent.

3.2 Derivation of the metric

The goal is to build a metric respecting the above conditions, whose coordinates are the ones set up by an observer close to the mass-particle that is taken to be the origin of the spatial coordinates. The assumptions made by the observer are firstly the ones that are always set in any model of relativity [12]:

i) The length of a measuring rod is constant in time and independent of orientation around a given point;

ii) The backwards and forwards velocity of light between any two points is the same;

iii) The velocity of light is the same in every direction around a given point.

Furthermore, the observer sets up a system of coordinates *orthogonal* and *isotropic* in space, possibly through a change of coordinates.
In addition, the *Copernican principle* is assumed resulting in the following assumptions:

iv) The matter in the universe is distributed with spherical symmetry around the origin;

v) There is no flow of the matter as a whole either towards or away from the origin. Consequently, the pressure at any point is isotropic, i.e. the random velocities of the particles composing the galaxies have no preferential direction. We'll see how this "non-accretion" condition can be eliminated in order to get a slightly different kind of solution.

Nevertheless, the resolution of Einstein's equations shall be treated with the use of cosmical coordinates that satisfy the Copernical principle, and so the conditions (iv) and (v). We shall then show, that after a transformation into the observer's coordinates the same conditions hold. The most general metric respecting orthogonality, isotropy and spherical symmetry around the origin is in the form

$$g = e^{\nu(r,t)}\left\{dr^2 + r^2 d\Omega^2\right\} - e^{\xi(r,t)}dt^2 \tag{3.2.1}$$

where r indicates the cosmical coordinate. The stress-energy tensor, chosen starting from these assumptions, is the one describing a *perfect fluid* in the comoving coordinates (2.5.2)

$$T^{\mu}_{\nu} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & -\rho \end{pmatrix} \tag{3.2.2}$$

having *trace* $T = (3P - \rho)$.
The stress-energy component T^4_0 is set to zero in order to respect the condition of *no-accretion* (v) and thus no radial flow. However, this hypothesis is not well motivated at a physical level. In particular, black holes are highly dynamical objects that live in a universe full of matter and radiation [11]. Furthermore, LIGO and VIRGO gravitational wave detectors showed that black holes can grow feeding other black holes [1]. Nevertheless, such condition is realistic for other kinds of objects.
The analytical procedure to find the functions ν and ξ is based on the resolution of Einstein's equations.
Starting from the general form of the metric (3.2.1) one can compute the Christoffel symbols. Conventionally the coordinates are identified by $\partial_r = \partial_1$, $\partial_\theta = \partial_2$, $\partial_\phi = \partial_3$, and $\partial_t = \partial_4$. Furthermore the dot denotes differentiation with respect to t, and the dash denotes differentiation with respect to r. The non-null Christoffel symbols follow:

$$\Gamma^1_{11} = \frac{1}{2}\nu' \qquad \Gamma^1_{14} = \Gamma^1_{41} = \frac{1}{2}\dot{\nu} \qquad \Gamma^1_{22} = -\frac{1}{2}r\left(r\nu' + 2\right) \qquad \Gamma^1_{33} = -\frac{1}{2}r\sin^2(\theta)\left(r\nu' + 2\right)$$

$$\Gamma^1_{44} = \frac{1}{2}\xi' e^{\xi} - e^{\nu} \qquad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2}\nu' + \frac{1}{r} \qquad \Gamma^2_{24} = \Gamma^2_{42} = \frac{1}{2}\dot{\nu}$$

$$\Gamma^2_{33} = -\sin(\theta)\cos(\theta) \qquad \Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{2}\nu' + \frac{1}{r} \qquad \Gamma^3_{32} = \Gamma^3_{23} = \cot(\theta)$$

$$\Gamma^4_{11} = \frac{1}{2}\dot{\nu}e^{\nu}e^{-\xi} \qquad \Gamma^4_{41} = \Gamma^2_{14} = \frac{1}{2}\xi' \qquad \Gamma^4_{22} = \frac{1}{2}r^2\dot{\nu}e^{\nu}e^{-\xi}$$

$$\Gamma^4_{33} = \frac{1}{2}r^2\sin^2(\theta)\dot{\nu}e^{\nu}e^{-\xi} \qquad \Gamma^4_{44} = \frac{1}{2}\xi' \tag{3.2.3}$$

The Ricci tensor is then built from the latter symbols. The non-null components are

$$Ric_{11} = e^{\nu}e^{-\xi}\left(-\frac{1}{4}\dot{\nu}\dot{\xi} + \frac{3}{4}(\dot{\nu})^2 + \frac{1}{2}\ddot{\nu}\right) + \left(\frac{1}{4}\nu'\xi' - \nu'' - \frac{1}{4}(\xi')^2 - \frac{1}{2}\xi'' - \frac{\nu'}{r}\right)$$

$$Ric_{22} = e^{\nu}e^{-\xi}\left(-\frac{1}{4}r^2\dot{\nu}\dot{\xi} + \frac{3}{4}r^2(\dot{\nu})^2 + \frac{1}{2}r^2\ddot{\nu}\right) + \left(-\frac{1}{4}r^2(\nu')^2 - \frac{3}{2}r\nu' - \frac{1}{4}r^2\nu'\xi' - \frac{r}{2}\xi' - r^2\frac{\nu''}{2}\right)$$

$$Ric_{33} = \sin^2(\theta)\left[e^{\nu}e^{-\xi}\left(-\frac{1}{4}r^2\dot{\nu}\dot{\xi} + \frac{3}{4}r^2(\dot{\nu})^2 + \frac{1}{2}r^2\ddot{\nu}\right) + \left(-\frac{1}{4}r^2\nu'\xi' - \frac{1}{4}r^2(\nu')^2 - \frac{1}{2}r^2\nu'' - \frac{1}{2}r\xi' - \frac{3}{2}r\nu'\right)\right]$$

$$Ric_{44} = \frac{3}{4}\dot{\nu}\dot{\xi} - \frac{3}{4}(\dot{\nu})^2 - \frac{3}{2}\ddot{\nu} + e^{\xi-\nu}\left(\frac{1}{4}(\xi')^2 + \frac{1}{2}\xi'' + \frac{1}{4}\nu'\xi' + \frac{\xi'}{r}\right)$$

$$Ric_{14} = Ric_{41} = \frac{1}{2}\dot{\nu}\xi' - \dot{\nu}' \tag{3.2.4}$$

Plugging the Ricci tensors in the Einstein field equations (1.0.2) gives the four equations

$$-\frac{1}{2}(\rho + 3p) + \Lambda = e^{-\xi}\left\{\frac{3}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{3}{4}\dot{\xi}\dot{\nu}\right\} - e^{-\nu}\left\{\frac{1}{2}\xi'' + \frac{1}{4}(\xi')^2 + \frac{1}{r}\xi' + \frac{1}{4}\xi'\nu'\right\} \tag{3.2.5}$$

$$-\frac{1}{2}(p - \rho) + \Lambda = e^{-\xi}\left\{\frac{1}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{4}\dot{\xi}\dot{\nu}\right\} - e^{-\nu}\left\{\frac{1}{2}\xi'' + \frac{1}{4}(\xi')^2 + \frac{1}{r}\nu' + \nu'' - \frac{1}{4}\xi'\nu'\right\} \tag{3.2.6}$$

$$-\frac{1}{2}(p - \rho) + \Lambda = e^{-\xi}\left\{\frac{1}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{4}\dot{\xi}\dot{\nu}\right\} - e^{-\nu}\left\{\frac{1}{2}\nu'' + \frac{3}{2r}\nu' + \frac{1}{4}(\nu')^2 + \frac{1}{2r}\xi' + \frac{1}{4}\xi'\nu'\right\} \tag{3.2.7}$$

$$\dot{\nu}' - \frac{1}{2}\dot{\nu}\xi' = -T_{14} = 0 \tag{3.2.8}$$

Setting equal the second members of (3.2.6) and (3.2.7), it is possible to obtain the two fundamental equations for determining the coefficients of the metric

$$\xi'' + \nu'' - \frac{1}{r}(\nu' + \xi') - \nu'\xi' - \frac{1}{2}(\nu')^2 + \frac{1}{2}(\xi')^2 = 0 \tag{3.2.9}$$

$$\dot{\nu}' - \frac{1}{2}\dot{\nu}\xi' = 0 \tag{3.2.10}$$

3.3 Solution of the equations

The fundamental equations (3.2.9) and (3.2.10) can be used to determine the function $\nu(r,t)$ and $\xi(r,t)$. The eq. (3.2.9) can be written as follows

$$\frac{\partial\dot{\nu}}{\partial r} = \frac{1}{2}\dot{\nu}\xi'$$

$$\frac{\partial\dot{\nu}}{\dot{\nu}} = \frac{1}{2}\xi'\partial r$$

$$\log(\dot{\nu}) = \frac{1}{2}\xi + a(t)$$

$$\dot{\nu} = \alpha(t)e^{\xi(r,t)/2}$$

where $\alpha(t) = e^{a(t)}$. This can be put in integral form as

$$\nu(r,t) = \int \alpha(t)e^{\xi(r,t)/2}dt + g(r) \tag{3.3.1}$$

The latter equation relates the two functions of interest. Then, we can study restricted cases to arrive at the wanted solution.

3.3.1 ξ is a function of r alone

If we assume $\xi(r,t) = \xi(r)$ the equation (3.3.1) becomes

$$\nu = \beta(t)e^{\xi(r)/2} + g(r) \tag{3.3.2}$$

where

$$\beta = \int \alpha(t)dt$$

Thus the derivatives of ν with respect to r can be written

$$\nu' = \frac{1}{2}\beta(t)\xi'(r)e^{\xi(r)/2} + g'(r)$$
$$\nu'' = \frac{1}{4}\beta(t)\left(\xi'(r)\right)^2 e^{\xi(r)/2} + \frac{1}{2}\beta(t)\xi''(r)e^{\xi(r)/2} + g''(r)$$

Therefore, the equation (3.2.10) can be developed further as

$$\xi'' - \frac{1}{4}\beta(\xi')^2e^{\xi/2} + \frac{1}{4}\beta\xi''e^{\xi/2} + g'' - \frac{1}{2r}\beta\xi'e^{\xi/2} + \tag{3.3.3}$$
$$-\frac{g'}{r} - \frac{\xi'}{r} - g'\xi' - \frac{1}{8}\beta^2(\xi')^2e^{\xi} - \frac{(g')^2}{2} - \frac{1}{2}\beta g'\xi'e^{\xi/2} + \frac{1}{2}(\xi')^2 = 0 \tag{3.3.4}$$

We can treat this equation as a second degree equation for β rewriting it in the canonical form

$$A(r)\beta(t)^2 + B(r)\beta(t) + C(r)$$

where

$$A(r) = -\frac{1}{8}(\xi')^2e^{\xi}$$
$$B(r) = \frac{1}{4r}\left(-r(\xi')^2 + 2r\xi'' - 2\xi' - 2rg'\xi'\right)e^{\xi/2}$$
$$C(r) = \xi'' + g'' - \frac{g'}{r} - \frac{\xi'}{r} - g'\xi' - \frac{(g')^2}{2} + \frac{(\xi')^2}{2}$$

Hence, two sub-cases occur:

i) **A(r)≠0**: The solution of the equation (3.3.4) can be put in the form

$$\beta(t) = \frac{-B(r) \pm \sqrt{B^2(r) - 4A(r)C(r)}}{2A(r)}$$

It is then necessary that $\beta(t) = \beta = constant$ since the left member is a function of t, while the right member is a function of r.
Now, using eq. (3.3.2), it is immediate that both ν and ξ are functions of r alone. Therefore, the result is a static metric, which is not the solution we are seeking for;

ii) **A(r)=0**: Such condition implies $\xi' = 0$. Thus, ξ is a constant. Equation (3.3.4) becomes

$$g'' - \frac{g'}{r} - \frac{(g')^2}{2} = 0 \tag{3.3.5}$$

This differential equation can be treated as a first order differential equation by making the substitution $g' = z^{-1}$. Hence

$$g'' = -\frac{z'}{z^2}$$

and (3.3.5) turns into

$$z' + \frac{z}{r} = -\frac{1}{2}$$

The related homogeneous solution can be find with

$$\frac{dz}{dr} = -\frac{z}{r}$$
$$\frac{dz}{z} = -\frac{dr}{r}$$
$$\log(z) = \log(r^{-1}) + C$$
$$z_o(r) = \frac{K}{r}$$

where $K = e^C$.
A particular solution is obtained with the method of variation of the parameters. Hence

$$z_p(r) = \frac{C(r)}{r}$$

and

$$C'(r) = -\frac{r}{2} \quad \Longrightarrow \quad C(r) = -\frac{r^2}{4}$$

setting the integration constant to 0. Thus

$$z_p(r) = -\frac{r}{4}$$
$$z(r) = \frac{K}{r} - \frac{r}{4}$$
$$g' = \left(\frac{K}{r} - \frac{r}{4}\right)^{-1} = -\frac{4r}{r^2 - 4K}$$

The solution of the differential equation is then

$$g(r) = C - \int \frac{4r}{r^2 - 4K} dr = C - 2\log(r^2 - 4k) =$$
$$= C - 2\log(-4k) - 2\log\left(1 + \frac{r^2}{4K}\right) = -2\log\left(1 + \frac{kr^2}{4}\right)$$

where $k = -\frac{1}{K}$ and the constant of integration cancel the constant term $[-2\log(-4k)]$ for simplicity.
Plugging this result into (3.3.2) we obtain

$$\nu = \beta(t)e^{-\xi/2} - 2\log\left(1 + \frac{kr^2}{4}\right) \tag{3.3.6}$$

Remembering that ξ is a constant, one can see that the latter gives the RW metric, up to a rescaling of t.

Therefore, no generalization in terms of cosmical coordinates of the Schwarzschild metric exists, in which the "mass"[2] of the particle enters as a constant independent of time. In fact, to get a metric with the discussed properties ξ must depend on the cosmical time.

3.3.2 ξ is a function of both r and t, g(r)=0

In the latter section it was shown that the function g(r) influences the curvature of space. Thus, we expect that by imposing g(r)=0, the wanted generalization will be compatible only with a flat RW metric (2.5.32). Starting from the reasonable requirement that the solution must have a singularity

[2]The concept of mass must be taken with caution since here $\xi(r,t)$ is a function dependent on the coordinate system.

at the origin, just like the normal Schwarzschild solution, it is possible to expand ν, ξ as power series in u=1/r by setting

$$\gamma = e^{\xi/2} = 1 + \eta_1(t)u^{m_1} + \eta_2(t)u^{m_2} + ... = 1 + \sum_{i=1}^{\infty} \eta_i(t)u^{m_i} \tag{3.3.7}$$

In this way the limit $r \to \infty$ implies $\gamma \to 1$, which leads to a flat Minkowski space as expected.
On the other hand, if $r \to 0$ it is immediate that $\gamma \to \infty$ and thus there is a singularity.
By substituting (3.3.7) into (3.3.1) we obtain

$$\nu = \int \alpha(t)\gamma(t)dt = \int \alpha(t)dt + \int \alpha(t) \left(\sum_{i=1}^{\infty} \eta_i(t)u^{m_i} \right) dt = \beta(t) + \sum_{i=1}^{\infty} \beta_i(t)u^{m_i} \tag{3.3.8}$$

with

$$\beta(t) = \int \alpha(t)dt \tag{3.3.9}$$

and

$$\beta_i(t) = \int \alpha(t)\eta_i(t)dt \tag{3.3.10}$$

These definitions can also be put in differential form as

$$\dot{\beta}(t) = \alpha(t) \tag{3.3.11}$$

$$\dot{\beta}_i(t) = \alpha(t)\eta_i(t) = \dot{\beta}(t)\eta_i(t) \tag{3.3.12}$$

Equation (3.2.10) can be rewritten by substituting the independent variable r into $u = 1/r$, and by substituting the function ξ int $\gamma = e^{\xi/2}$. In doing so, the new derivatives can be computed to find

$$\frac{\partial}{\partial r} = -\frac{1}{r^2}\frac{\partial}{\partial u} \quad ; \quad \frac{\partial^2}{\partial r^2} = \frac{1}{r^4}\left(2r\frac{\partial}{\partial u} + \frac{\partial^2}{\partial u^2}\right) \quad ;$$

and by using the chain rule, we get

$$\frac{\partial \xi}{\partial u} = \frac{\partial(2\log\gamma)}{\partial u} = \frac{2}{\gamma}\frac{\partial \gamma}{\partial u} \quad ; \quad \frac{\partial^2 \xi}{\partial u^2} = \frac{\partial^2(2\log\gamma)}{\partial u^2} = \frac{2}{\gamma^2}\left[\gamma\frac{\partial^2\gamma}{\partial u^2} - \left(\frac{\partial\gamma}{\partial u}\right)^2\right]$$

Eq. (3.2.10) then becomes

$$\gamma\left(3\frac{\partial\nu}{\partial u} + u\frac{\partial^2\nu}{\partial u^2}\right) + 2u\frac{\partial^2\gamma}{\partial u^2} + \left(6 - 2u\frac{\partial\nu}{\partial u}\right)\frac{\partial\gamma}{\partial u} - \frac{1}{2}u\gamma\left(\frac{\partial\nu}{\partial u}\right)^2 = 0 \tag{3.3.13}$$

where

$$\frac{\partial\nu}{\partial u} = \sum_{i=1}^{\infty} m_i\beta_i(t)u^{m_i-1} \quad ; \quad \frac{\partial^2\nu}{\partial u^2} = \sum_{i=1}^{\infty} m_i(m_i-1)\beta_i(t)u^{m_i-2}$$

$$\frac{\partial\gamma}{\partial u} = \sum_{i=1}^{\infty} m_i\eta_i(t)u^{m_i-1} \quad ; \quad \frac{\partial^2\gamma}{\partial u^2} = \sum_{i=1}^{\infty} m_i(m_i-1)\eta_i(t)u^{m_i-2}$$

Now, by substituting the series forms of ν and γ given by (3.3.7) and (3.3.8), the latter equation leads to

$$\begin{aligned} &\left(\sum_{i=1}^{\infty} m_i(m_i+2)\beta_i(t)u^{m_i-1}\right)\left(1 + \sum_{j=1}^{\infty}\eta_j(t)u^{m_j}\right) + \left(\sum_{i=1}^{\infty} 2m_i(m_i+2)\eta_i(t)u^{m_i-1}\right) + \\ &- \left(\sum_{i=1}^{\infty} 2m_i\beta_i(t)u^{m_i}\right)\left(\sum_{j=1}^{\infty} m_j\eta_j(t)u^{m_j-1}\right) + \\ &- \left(\sum_{i=1}^{\infty}(1/2)m_i\beta_i(t)u^{m_i}\right)\left(\sum_{j=1}^{\infty} m_j\beta_j(t)u^{m_j-1}\right)\left(1 + \sum_{k=1}^{\infty}\eta_k(t)u^{m_k}\right) = 0 \end{aligned} \tag{3.3.14}$$

$m_i - 1$	$m_1 - 1$	$m_2 - 1$	$m_3 - 1$	$m_4 - 1$	$m_5 - 1$	...
$m_i + m_j - 1$	$2m_1 - 1$	$m_1 + m_2 - 1$	$2m_2 - 1$	$m_2 + m_3 - 1$	$2m_3 - 1$	...
$m_i + m_j + m_k - 1$	$3m_1 - 1$	$2m_1 + m_2 - 1$	$2m_2 + m_1 - 1$	$3m_2 - 1$	$2m_2 + m_3 - 1$	...

Table 3.1: Table of the possible indices produced by equation (3.3.14).

The result is an infinite number of indicial equations, obtained by equating every power of u to zero. In particular, we see that the lowest power of u turns out to be u^{m_1-1}.
Equating its coefficients to zero, we get

$$m_1(m_1+2)(2\eta_1(t)+\beta_1(t)) = 0 \quad \Longrightarrow \quad 2\eta_1(t) = -\beta_1(t) \quad \Longrightarrow \quad 2\dot{\eta}_1(t) = -\dot{\beta}_1(t) \tag{3.3.15}$$

Note that only the first and the second term contribute to the latter. This, combined with (3.3.12), result in the useful equation

$$\frac{\dot{\eta}_1(t)}{\eta_1(t)} = -\frac{1}{2}\dot{\beta}(t) \tag{3.3.16}$$

Hence

$$\frac{\dot{\eta}_1(t)}{\eta_1(t)} = -\frac{1}{2}\frac{\dot{\beta}_i(t)}{\eta_i(t)} \tag{3.3.17}$$

In general, the possible powers are of the type

$$m_i - 1 \quad ; \quad m_i + m_j - 1 \quad ; \quad m_i + m_j + m_k - 1$$

We put these indices in a table, as shown in tab. (3.1).
By equating the coefficients on the diagonals, starting from the left side, it becomes soon clear that we can state $m_i = im_1$. Hence, if the wanted solution needs to include the first power of u, the necessary condition is

$$m_i = i \tag{3.3.18}$$

Let's now impose the coefficients of u^1 equal to zero:

$$m_2(m_2+2)(2\eta_2(t)+\beta_2(t)) - 4m_1\eta_1^2 = 0 \quad \Longrightarrow \quad 2\eta_2(t)+\beta_2(t) = k_2\eta_1^2(t) \tag{3.3.19}$$

where $k_2 = 1/2$.
After differentiating this relation with respect to t, it follows that

$$2\dot{\eta}_2 + \dot{\beta}_2 = \eta_1\dot{\eta}_1 \tag{3.3.20}$$

Using now eq. (3.3.17) to substitute $\dot{\beta}_2$, we get

$$2\dot{\eta}_2 - 2\frac{\eta_2}{\eta_1}\dot{\eta}_1 = \eta_1\dot{\eta}_1$$

$$2\dot{\eta}_2 = \dot{\eta}_1\left(\eta_1 + 2\frac{\eta_2}{\eta_1}\right)$$

$$\frac{d\eta_2}{d\eta_1} = \eta_1 + 2\frac{\eta_2}{\eta_1}$$

We can treat the latter through the associated differential equation

$$2y'(x) = x + 2\frac{y}{x} \tag{3.3.21}$$

than can also be written

$$2\left(\frac{y'}{x} - \frac{y}{x^2}\right) = 1 \tag{3.3.22}$$

Then, we proceed with the substitution

$$z(x) = \frac{y}{x} \quad ; \quad z'(x) = \frac{y'}{x} - \frac{y}{x^2}$$

Thus, eq. (3.3.21)

$$z'(x) = \frac{1}{2} \quad \Longrightarrow \quad z(x) = \frac{x}{2} \quad \Longrightarrow \quad y(x) = \frac{x^2}{2}$$

Where the constant of integration has been set to zero.
Therefore it is possible to write

$$\eta_2(t) = \frac{1}{2}\eta_1^2(t) \tag{3.3.23}$$

which plugged in eq. (3.3.19) results in

$$\beta_2(t) = -\eta_1^2(t) \tag{3.3.24}$$

Proceeding in this manner, we see that in general, for any power of u, the above equations can be written

$$\begin{aligned} 2\eta_n + \beta_n &= k_n \eta_1^n \\ 2\dot{\eta}_n + \dot{\beta}_n &= n k_n \eta_1^{n-1} \dot{\eta}_1 \\ 2\frac{d\eta_n}{\eta_1} &= n k_n \eta_1^{n-1} + 2\frac{\eta_n}{\eta_1} \end{aligned}$$

and in conclusion

$$\eta_n(t) = c_n \eta_1^n(t) \tag{3.3.25}$$

$$\beta_n(t) = -\frac{2c_n}{n}\eta_1^n(t) \tag{3.3.26}$$

where $c_n = \frac{n}{n-1}\frac{k_n}{2}$ are constants.
Therefore, if the solution we want exists must be of the form

$$\gamma = 1 + \sum_{n=1}^{\infty} c_n \left(\frac{\mu(t)}{r}\right)^n \tag{3.3.27}$$

$$\nu = \beta(t) - 2\sum_{n=1}^{\infty} \frac{c_n}{n} \left(\frac{\mu(t)}{r}\right)^n \tag{3.3.28}$$

after renaming $\eta_1(t) \equiv \mu(t)$ and with the condition

$$\frac{1}{2}\dot{\beta} = -\frac{\dot{\mu}}{\mu} \tag{3.3.29}$$

Such condition can be seen as a constraint of no-accretion, since it is a direct consequence of the field equation (3.2.9).
Next, we want to show that these equations can be expressed in finite form. In order to do so, let's compute some derivatives to find out that

$$\frac{\partial \nu}{\partial r} = \frac{2}{r}\sum_{n=1}^{\infty} c_n \left(\frac{\mu}{r}\right)^n = \frac{2(\gamma-1)}{r} \tag{3.3.30}$$

$$\frac{\partial \nu}{\partial t} = -2\frac{\dot{\mu}}{\mu} - 2\sum_{n=1}^{\infty} c_n \frac{\dot{\mu}}{\mu}\left(\frac{\mu}{r}\right)^n = -2\frac{\dot{\mu}}{\mu}\gamma \tag{3.3.31}$$

and

$$\frac{\partial^2 \nu}{\partial r^2} = \frac{2}{r^2}\left(r\frac{\partial \gamma}{\partial r} - \gamma + 1\right) \quad ; \quad \frac{\partial \xi}{\partial r} = \frac{2}{\gamma}\frac{\partial \gamma}{\partial r} \quad ; \quad \frac{\partial^2 \xi}{\partial r} = \frac{2}{\gamma^2}\left[\gamma\frac{\partial^2 \gamma}{\partial r^2} - \left(\frac{\partial \gamma}{\partial r}\right)\right]$$

Thus, the second fundamental (3.2.10) equation becomes

$$r^2\frac{\partial^2\gamma}{\partial r^2} - r(\gamma-1)\frac{\partial\gamma}{\partial r} - \gamma(\gamma^2-1) = 0 \tag{3.3.32}$$

Through the substitution

$$r = e^x \quad ; \quad \frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial x} \quad ; \quad \frac{\partial^2}{\partial r^2} = \frac{1}{r^2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) \tag{3.3.33}$$

it is possible to rewrite (3.3.32) in the other form

$$\frac{\partial^2\gamma}{\partial x^2} - \gamma\frac{\partial\gamma}{\partial x} - \gamma(\gamma^2-1) = 0 \tag{3.3.34}$$

which is an autonomous O.D.E. of the second order. The general integral of the latter is found by doing another substitution $z = \gamma^2 - 1$ and by defining $\frac{\partial\gamma}{\partial x} = z(1+\omega)$.
In this way, the following derivatives can be computed:

$$\frac{\partial^2\gamma}{\partial x^2} = \frac{\partial z}{\partial x}(1+\omega) + z\frac{\partial\omega}{\partial x}$$

$$\frac{\partial z}{\partial x} = 2\gamma\frac{\partial\gamma}{\partial x} = 2\gamma z(1+\omega) \quad \Longrightarrow \quad \gamma = \frac{\partial z}{\partial x}\frac{1}{2z(1+\omega)}$$

These can be then insert into the differential equation

$$(1+\omega)\frac{\partial z}{\partial x} + z\frac{\partial\omega}{\partial x} - \frac{z(1+\omega)+z}{2z(1+\omega)}\frac{\partial z}{\partial x}$$

Multiplying every term for dx, reduces the latter to the O.D.E. of the first order, with separable variables

$$\frac{\partial z}{z} = -2\frac{1+\omega}{\omega(2\omega+3)}\partial\omega \tag{3.3.35}$$

that can be put in its integral form

$$\log(z) = -(2/3)\int\left(\frac{1}{\omega} + \frac{1}{2\omega+3}\right)d\omega \quad \Longrightarrow \quad \log(z) = \log\left[\omega^2(2\omega+3)\right]^{-3} + K(t) \quad \Longrightarrow$$

$$\Longrightarrow \quad z^3\omega^2(2\omega+3) = A^3(t) \tag{3.3.36}$$

Where $A(t) = e^{K(t)}$.
Then, we express ω and z in terms of γ to get

$$\left(\frac{\partial\gamma}{\partial x} - \gamma^2 + 1\right)^2\left(2\frac{\partial\gamma}{\partial x} + \gamma^2 - 1\right) = A^3(t) \tag{3.3.37}$$

The series form of γ includes one "arbitrary constant" $\mu(t)$, with respect to integrations by r. Thus, the final solution must be obtained starting from a particular first integral as above. Therefore, we decide to set A=0.
Since, the particular solution

$$\frac{\partial\gamma}{\partial x} - \gamma^2 + 1 = 0 \tag{3.3.38}$$

is singular, the particular first integral we require is the alternative one provided by (3.3.37):

$$2\frac{\partial\gamma}{\partial x} + \gamma^2 - 1 = 0 \tag{3.3.39}$$

which can be solved by separating variables in the following way:

$$-2\frac{d\gamma}{\gamma^2-1} = dx$$

$$\int\left(\frac{1}{\gamma+1} - \frac{1}{\gamma-1}\right)d\gamma = x + K(t)$$

$$\frac{\gamma+1}{\gamma-1} = K'(t)e^x$$

where $K'(t) = e^K(t)$.
Hence, solving the equation for γ gives the finite form

$$\gamma(t) = \frac{1 - \frac{\mu(t)}{2r}}{1 + \frac{\mu(t)}{2r}} \tag{3.3.40}$$

where $\mu(t) = -1/K'(t)$ is chosen to match the expansion in series form (3.3.27), and $r = e^x$ is recovered.
We then use the finite form of γ to compute the spatial derivative (3.3.30)

$$\frac{\partial \nu}{\partial r} = 4\frac{\mu(t)}{((2r + \mu(t))r} \tag{3.3.41}$$

which can be solved by separating the variables

$$\begin{aligned} \frac{1}{4}d\nu &= \left(\frac{2}{2r + \mu(t)} - \frac{1}{r}\right) dr \\ \frac{\nu}{4} &= \log\left(\frac{2r + \mu(t)}{r}\right) + K(t) \\ \nu(t) &= \beta(t) + 4\log\left(1 + \frac{\mu(t)}{2r}\right) \end{aligned} \tag{3.3.42}$$

where $4K(t) + \log 2 = \beta(t)$. So, the finite form of ν is found, too.
The time derivative

$$\frac{\partial \nu}{\partial t} = -2\frac{\dot{\mu}}{\mu}\gamma$$

gives the condition

$$\dot{\beta} + 4\left(\frac{(\dot{\mu}/2r)}{1 + (\mu(t)/2r}\right) = -2\frac{\dot{\mu}}{\mu}\left(\frac{1 - (\mu(t)/2r)}{1 + (\mu(t)/2r)}\right) \tag{3.3.43}$$

that leads to

$$\dot{\beta} = -2\frac{\dot{\mu}}{\mu} \tag{3.3.44}$$

which is the wanted condition.
Now that we found the functions of interest, we can plug them into the general metric (3.2.1). The result is the **McVittie generalization of the Schwarzschild metric in a flat universe**, expressed in cosmical coordinates

$$g = \left(1 + \frac{\mu(t)}{2r}\right)^4 e^{\beta(t)}\left\{dr^2 + r^2 d\Omega^2\right\} - \left(\frac{1 - (\mu(t)/2r)}{1 + (\mu(t)/2r)}\right)^2 dt^2 \tag{3.3.45}$$

where the condition (3.3.61) must hold.
It's easy to show that the flat McVittie metric reduces to the Schwarzschild metric (2.4.3) when $e^{\beta(t)/2} = a \equiv 1$.

3.3.3 ξ is a function of both r and t, g(r)$\neq$0

The curved generalization of the McVittie metric can be found by setting the functions of interests in a form that can be compatible with the flat case (3.4.1) in small regions near the origin. On the other hand, the wanted generalization must reduce to the Lamaître metric in the limit $r \to \infty$. By analogy with the flat case, we try to start from the forms

$$\gamma = e^{\xi/2} = \frac{1 - y(r,t)}{1 + y(r,t)} \qquad \nu = \beta(t) + 4\log\left[1 + y(r,t)\right] + g(r) \tag{3.3.46}$$

where

$$g(r) = -2\log\left(1 + \frac{kr^2}{4}\right) \tag{3.3.47}$$

as we found in subsection (3.3.1) for the case $\xi(r,t) \equiv \xi(r)$. The relative derivatives are

$$\frac{\partial \xi}{\partial r} = -\frac{4}{1-y^2}\frac{\partial y}{\partial r} \quad ; \quad \frac{\partial^2 \xi}{\partial r^2} = -\frac{4}{(1-y^2)}\frac{\partial^2 y}{\partial r^2} - \frac{8y}{(1-y^2)^2}\left(\frac{\partial y}{\partial r}\right)$$

and

$$\frac{\partial \nu}{\partial r} = \frac{4}{1+y}\frac{\partial y}{\partial r} + \frac{\partial g}{\partial r} \quad ; \quad \frac{\partial^2 \nu}{\partial r^2} = \frac{4}{(1+y)}\frac{\partial^2 y}{\partial r^2} - \frac{4}{(1+y^2)^2}\left(\frac{\partial y}{\partial r}\right) + \frac{\partial^2 g}{\partial r^2}$$

Furthermore, the derivatives of g are

$$\frac{\partial g}{\partial r} = \frac{kr}{1+\frac{kr^2}{4}} \quad ; \quad \frac{\partial^2 g}{\partial r^2} = \frac{\frac{kr^2}{4} - k}{\left(1+\frac{kr^2}{4}\right)}$$

where we can state from a direct computation that

$$\frac{\partial^2 g}{\partial r^2} - \frac{1}{r}\frac{\partial g}{\partial r} - \frac{1}{2}\left(\frac{\partial g}{\partial r}\right)^2 = 0$$

By defining the functions of interests in this way, the second fundamental equation (3.2.10) becomes

$$-\frac{4y}{1-y^2}\frac{\partial^2 y}{\partial r^2} + 12\frac{1}{1-y^2}\left(\frac{\partial y}{\partial r}\right)^2 + \frac{4y}{r(1-y^2)}\left(1 + r\frac{\partial g}{\partial r}\right)\frac{\partial y}{\partial r} + \frac{\partial^2 g}{\partial r^2} - \frac{1}{r}\frac{\partial g}{\partial r} - \frac{1}{2}\left(\frac{\partial g}{\partial r}\right)^2 = 0$$

which, combined with the latter equality gives

$$\frac{\partial^2 y}{\partial r^2} - \frac{3}{y}\left(\frac{\partial y}{\partial r}\right)^2 - \frac{\partial y}{\partial r}\left(\frac{1}{r} - \frac{kr}{1+\frac{kr^2}{4}}\right) \tag{3.3.48}$$

We can now treat the differential equation as y is a function of r alone, and then renaming the arbitrary constants of integration as functions of t. The equation can be written

$$yy'' - 3(y')^2 - yy'\left(\frac{1}{r} - \frac{kr}{1+\frac{kr^2}{4}}\right) = 0 \tag{3.3.49}$$

The method of resolution consists in doing the following substitution:

$$z = \frac{y}{y'} \quad ; \quad z' = \frac{(y')^2 - y''}{(y')^2} \quad \Longrightarrow \quad yy'' = (y')^2 - z'(y')^2 \tag{3.3.50}$$

That transforms the differential equation in a first order O.D.E.

$$z' + \left(\frac{1}{r} - \frac{kr}{1+\frac{kr^2}{4}}\right)z = -2 \tag{3.3.51}$$

The associated homogeneous equation is

$$z' + \left(\frac{1}{r} - \frac{kr}{1+\frac{kr^2}{4}}\right)z = 0 \tag{3.3.52}$$

The solution to (3.3.52) can be found by separating the variables

$$\log z = -\int \left(\frac{1}{r} - 2\frac{\frac{kr}{2}}{1+\frac{kr^2}{4}}\right)dr$$

$$z_o(r) = \frac{A}{k}\frac{\left(1+\frac{kr^2}{4}\right)^2}{r} \tag{3.3.53}$$

while the particular solution is of the form

$$z_p(r) = C(r) z_o(r)$$

with

$$C(r) = -\frac{4}{k}\int \frac{\frac{kr}{2}}{\left(1+\frac{kr^2}{4}\right)^2} dr$$

Hence,

$$z(r) = \frac{A}{k}\frac{\left(1+\frac{kr^2}{4}\right)^2}{r} + \frac{4}{k}\frac{1+\frac{kr^2}{4}}{r} \tag{3.3.54}$$

From the substitution (3.3.50) we then obtain

$$\frac{dy}{y} = \frac{dr}{z} \tag{3.3.55}$$

Hence, the related integral form is

$$\log(y) = 2\int \frac{1}{(1+u)(A+4+Au)} du =$$
$$= (1/2)\int \left(\frac{1}{1+u} - \frac{A}{A+4+Au}\right) du = \log \frac{1+\frac{kr^2}{4}}{A+4+Au} + B \tag{3.3.56}$$

where $u = \frac{kr^2}{4}$. Then

$$y^2 = \frac{B}{A+4+Ak\frac{r^2}{4}}\left(1+\frac{kr^2}{4}\right) \tag{3.3.57}$$

Now, the only way to obtain a value of y that reduces to $\mu/2r$ in regions where kr^2 is negligible is to impose

$$A = -4 \quad ; \quad B = -\frac{k\mu^2(t)}{4}$$

that is

$$y = \frac{\mu(t)}{2r}\left(1+\frac{kr^2}{4}\right)^{1/2} \tag{3.3.58}$$

Substituting y in (3.3.1) we obtain the final forms

$$\nu(r,t) = \beta(t) + 4\log\left\{1+\frac{\mu(t)}{2r}\left(1+\frac{kr^2}{4}\right)^{1/2}\right\} - 2\log\left(1+\frac{kr^2}{4}\right) \tag{3.3.59}$$

$$\xi(r,t) = 2\log\left\{\frac{1-\frac{\mu(t)}{2r}\left(1+\frac{kr^2}{4}\right)^{1/2}}{1+\frac{\mu(t)}{2r}\left(1+\frac{kr^2}{4}\right)^{1/2}}\right\} \tag{3.3.60}$$

Direct substitutions in (3.2.9) show that

$$\frac{\dot{\mu}}{\mu} = -\frac{1}{2}\dot{\beta} \tag{3.3.61}$$

holds in the curved case too.
The final form, expressed in cosmical coordinates, of the **generalized McVittie metric in an arbitrary curved space** is finally revealed:

$$g = \frac{\left\{1+\frac{\mu(t)}{2r}\left(1+\frac{kr^2}{4}\right)^{1/2}\right\}^4}{\left(1+\frac{kr^2}{4}\right)^2} e^{\beta(t)}\left\{dr^2 + r^2 d\Omega^2\right\} - \left\{\frac{1-\frac{\mu(t)}{2r}\left(1+\frac{kr^2}{4}\right)^{1/2}}{1+\frac{\mu(t)}{2r}\left(1+\frac{kr^2}{4}\right)^{1/2}}\right\}^2 dt^2 \tag{3.3.62}$$

which has to be coupled with the condition (3.3.61), that relies on the "non-accretion" hypothesis. Moreover, the function $\mu(t)$ must not be taken to represent the mass of the particle since it's just a metric coefficient in a particular coordinates system.

Then, it is now clear that, as $r \to \infty$, the metric (3.3.62) tends to the form

$$g = \frac{\left(1 + \frac{\sqrt{k}\mu(t)}{4}\right)^4}{\left(1 + \frac{kr^2}{4}\right)^2} e^{\beta(t)} \left\{dr^2 + r^2 d\Omega^2\right\} - \left(\frac{1 - \frac{\sqrt{k}\mu}{4}}{1 + \frac{\sqrt{k}\mu}{4}}\right)^2 dt^2 \tag{3.3.63}$$

which differs from the FRW solution only in terms of $\mu/\sqrt{k}$ that are negligibly small when μ is the mass of a star and $1/\sqrt{k}$ is of the order of the radius of space. Such limit can also be obtained by setting the "mass" $\mu(t) = 0$.

3.4 The observer's system

The Λ-CDM model assumes a flat RW metric, since it seems to be a really good fit for the observations. Moreover, we expect that the spatial curvature of the FRW geometry doesn't influence the behavior of the metric near a mass source, provided that its radius is smaller than the radius of the curvature [11]. Therefore, a good approximation of the McVittie metric in cosmical coordinates is

$$g = \left(1 + \frac{\mu(t)}{2r}\right)^4 e^{\beta(t)} \left\{dr^2 + r^2 d\Omega^2\right\} - \left(\frac{1 - (\mu(t)/2r)}{1 + (\mu(t)/2r)}\right)^2 dt^2 \tag{3.4.1}$$

The latter can be also linearized in the limit $\mu(t)/r << 1$, in which case it reduces to

$$g = e^{\beta(t)} \left(1 + 2\frac{\mu(t)}{r}\right) \left\{dr^2 + r^2 d\Omega^2\right\} - \left(1 - 2\frac{\mu(t)}{r}\right) dt^2 \tag{3.4.2}$$

that is a perturbed FRW cosmology. Then, one might be tempted to build an arbirtray "multicentered" solution by using linear superposition.
Let's consider a generic observer situated at a cosmical distance r from the mass-particle, and at any instant t_1. Then, he is entitled to make a coordinate transformation

$$r_1 = e^{\beta(t_1)/2} r$$

In fact, if we compute the stress-energy tensor components $(T_{ij})^*$ in the observer's coordinates, it is easy to find that

$$(T^1_1)* = T^1_1 = -\rho \quad ; (T^2_2)* = (T^3_3)* = (T^4_4)* = T^2_2 = T^3_3 = T^4_4 = p \tag{3.4.3}$$

Thus, the conditions of section (3.2, iv & v) are satisfied in the observer's system.
Let's define

$$\mu_H = \mu(t_1) e^{\beta(t_1)/2} = \mu(t_1) a(t_1) \tag{3.4.4}$$

Then, the *Hawking-Haywards* mass of the McVittie mass-particle is $m_H = 8\pi\mu_H$ [5].
This is the real physically relevant quantity representing the mass of the particle.
In fact, it is easy to show that such mass respects the non-accretion condition

$$\dot{\mu_H}(t) = \left(\dot{\mu}(t) + \frac{\mu(t)}{2}\dot{\beta}(t)\right) e^{\beta(t)/2} = 0 \tag{3.4.5}$$

where we used relation (3.3.61).
The metric (3.4.1) can be thus written

$$\left(1 + \frac{\mu_H}{2r_1}\right)^4 \left\{dr_1^2 + r_1^2 d\Omega^2\right\} - \left(\frac{1 - \mu_H/2r_1}{1 + \mu_H/2r_1}\right)^2 dt^2 \tag{3.4.6}$$

Hence, the observer's metric is exactly the Swarzschild static field in the form (2.4.3), independently of the instant at which the transformation is made. The observer will always feel to live near a static field.
The operational procedure is actually the opposite: The observer sets a coordinate system, and in particular he chooses a "time", that has no correlation with the time of other reference frames. Then, he is allowed by (3.4.3) to make a transformation to cosmical coordinates, as a mathematical tool. Stationary observers are then able to find a compromised time to build a relation between their clocks. Furthermore, if we take m_H as the mass of the Sun, the terms m_H/r_1 are already negligible at the distance of the earth, approaching a flat RW metric (2.5.32). Therefore, a human observer living on the surface of the earth is able to account for the "recession" of a distant galaxy by assigning a fixed value r to the galaxy, and measuring the red-shift as an effect of the change in the coordinate r_1 due to the non-null time travel of light. Since we defined

$$a(t) = e^{\beta(t)/2}$$

it is easy to show that

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{\beta}(t)e^{\beta(t)}}{e^{\beta(t)}} = \frac{\dot{\beta}}{2} \tag{3.4.7}$$

that is the Hubble parameter, which can be measured as above.
The numerical number of the present Hubble parameter H_0 is reported in subsection (2.5.2), and in particular it is positive.
Hence, by using (3.3.61) and the point of view of cosmical coordinates, the (non-covariant) "mass" of the particle $\mu(t)$ decreases.

3.5 Pressure, density, Λ

The need of a cosmological constant has been widely discussed over the last century. The experimental evidence of an accelerating universe emphasizes the necessity of it, directly from the Friedmann equations. We shall show that the result is the same, by using the McVittie metric. We start from the field equations (3.2.5), (3.2.6) in cosmical coordinates and by solving for the density ρ and the pressure p, in order to express them in a more immediate form.
Firstly, we solve for the density:

$$4\rho + 4\Lambda = 3e^{-\xi}(\dot{\nu})^2 - e^{-\nu}\left\{2\xi'' + (\xi')^2 + 6\nu'' + \frac{6}{r}\nu' - \frac{2}{r}\xi' - 2\xi'\nu'\right\}$$

we then use the second fundamental equation (3.2.10), and get

$$\rho = \Lambda + \frac{3}{4}e^{-\xi}(\dot{\nu})^2 - e^{-\nu}\left\{\nu'' + \frac{2}{r}\nu' + \frac{1}{4}(\nu')^2\right\} \tag{3.5.1}$$

Secondly, we solve for the pressure:

$$4P - 4\Lambda = -e^{-\xi}\left\{4\ddot{\nu} + 3(\dot{\nu})^2 - 2\dot{\nu}\dot{\xi}\right\} + e^{-\nu}\left\{2\xi'' + (\xi')^2 + \frac{2}{r}\xi' + 2\nu'' + \frac{2}{r}\nu'\right\}$$

that becomes

$$4P - 4\Lambda = -e^{-\xi}\left\{4\ddot{\nu} + 3(\dot{\nu})^2 - 2\dot{\nu}\dot{\xi}\right\} +$$
$$+e^{-\nu}\left\{\left(\xi'' + \nu'' - \frac{1}{r}(\nu' + \xi') - \nu'\xi' - \frac{1}{2}(\nu')^2 + \frac{1}{2}(\xi')^2\right) + \left(\xi'' + \nu'' + \frac{3}{r}(\xi' + \nu') + \frac{1}{2}(\nu')^2 + \frac{1}{2}(\xi')^2 + \nu'\xi'\right)\right\}$$

Now, for (3.2.10), the first part of the second parenthesis is zero. Then

$$p = \Lambda - e^{-\xi}\left\{\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{2}\dot{\xi}\dot{\nu}\right\} + \frac{e^{-\nu}}{4}\left\{\xi'' + \nu'' + \frac{3}{r}(\xi' + \nu') + \frac{1}{2}(\xi' + \nu')^2\right\} \tag{3.5.2}$$

Now, consider an observer's system at the instant t_1. Denote ρ_1, p_1 the values of density and pressure in this system at the instant t_1. Then, denote by $\dot{H}_1$ and $\ddot{H}_1$ the Hubble parameter $\dot{H}$ and the acceleration parameter $\ddot{H}$ at the instant t_1. Since the transformation $r_1 = e^{\beta(t_1)/2}r$ leaves the stress-energy tensor invariant as demonstrated in the previous section, it is possible to get the correspondents equations in the observer's system directly by substituting the new variable and by introducing the constant mass $m_H = e^{\beta(t_1)/2}$.
Explicitly, on a fixed t_1-slice, we obtain

$$\rho_1 = -\Lambda + 3\dot{H}_1^2 + 3k\left\{1 + \frac{\mu_H}{2r_1}\left(1 + \frac{k_1 r_1^2}{4}\right)^{1/2}\right\}^{-4} - 3k\frac{\mu_H}{r_1}\left(1 + \frac{k_1 r_1^2}{4}\right)^{1/2}\left\{1 + \frac{\mu_H}{2r_1}\left(1 + \frac{k_1 r_1^2}{4}\right)^{1/2}\right\}^{-5}$$

$$p_1 = \Lambda - \frac{1 + \frac{\mu_H}{2r_1}\left(1 + \frac{k_1 r_1^2}{4}\right)^{1/2}}{1 - \frac{\mu_H}{2r_1}\left(1 + \frac{k_1 r_1^2}{4}\right)^{1/2}}\frac{1}{2}\ddot{H}_1 - 3\dot{H}_1^2 - k\left\{1 + \frac{\mu_H}{2r_1}\left(1 + \frac{k_1 r_1^2}{4}\right)^{1/2}\right\}^{-4} \tag{3.5.3}$$

The latter reduces to the better known FRW equations in the limit $r_1 \to \infty$ (neglecting terms $\mu_H/\sqrt{k}$). In the flat case (3.4.1), which is the main case of interest, they further simplify to

$$\rho_1 = 3H_1^2 \tag{3.5.4}$$

$$p_1 = -2\frac{1 + \frac{m_H}{2r_1}}{1 + \frac{m_H}{2r_1}}\dot{H}_1 - 3H_1^2 \tag{3.5.5}$$

where we used relation (3.4.7).
The first equation is the ordinary FRW equation (2.5.26), and therefore it is homogeneous. The second equation is different from (2.5.27), and in particular, it contains a non-homogeneous term $\propto H^2$, except in the case of a pure cosmological constant ($\dot{H} = 0$). That makes the pressure perceived by an observer inhomogeneous in space. This can be explained as follows: the presence of a mass-particle must break the homogeneity of the stress-energy on spatial slices, expecially in the case of a massive particle, such as a black hole. However, the density is a function of t alone. Therefore, the gradient of the pressure must counterbalance this effect in order to get a homogeneous stress-energy [11].

3.6 McVittie BHs

The McVittie solution is interesting in its own right, just to be an exact non-linear solution of the second order to Einstein's field equations. Furthermore, it is the first real attempt to describe a dynamical black hole living in a FRW universe filled with a perfect fluid. However, the solution presents some physical pathologies when we want to deal with objects that originate from the collapsing of a star, i.e. *cosmological black holes* (CBHs).

3.6.1 Apparent horizons

The conventional definition of black holes includes asymptotic flatness and a global definition of the event horizon. In CBHs the first condition is obviously relaxed. Therefore, local definitions of the structures and their horizons are needed.
The *event horizon* of a stationary black hole is a co-dimension one null hypersurface defined as the *boundary of the region which is not in the causal past of the future null-infinity* [1]. Thus, the event horizon is a global property of the spacetime, and one cannot locate the event horizon with local experiments in a finite interval of time [1]. On the other hand, an *apparent horizon* is a surface where at least one congruence of null geodesics changes its focusing properties. In other words, a specific family of geodesics flips from converging to diverging, by passing through the apparent horizon [11]. Apparent horizons depend on the embedding of the surface in spacetime. Therefore, they don't carry geometric invariant data. In any case, the existence of an apparent horizon implies the appearance of a future event horizon outside of it [1]. More formal definitions of *apparent horizons* and *trapped regions* can be found in [8], [9], and they are not going to be further treated in this thesis.

3.6.2 The Schwarzschild-de Sitter metric

The flat McVittie solution can be written in "cosmical coordinates" in a more modern notation:

$$g = \left(1 + \frac{\mu_H}{2a(t)r}\right)^4 a^2(t)\left\{dr^2 + r^2 d\Omega^2\right\} - \left(\frac{1 - \frac{\mu_H}{2a(t)r}}{1 + \frac{\mu_H}{2a(t)r}}\right)^2 dt^2 \tag{3.6.1}$$

where we emphasize the fact that the real physical quantity is $m_H = 8\pi\mu_H$.
A remarkable property is that the latter metric for a de Sitter background ($a(t) = a_0 e^{H_0 t}$ with $H_0 = \sqrt{\Lambda/3}$) reduces to a *Schwarzschild-de Sitter* black hole solution in the static form (after a proper change of coordinate) [5]:

$$g = \left(1 - \frac{2\mu_H}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2 - \left(1 - \frac{2\mu_H}{r} - \frac{\Lambda r^2}{3}\right) dt^2 \tag{3.6.2}$$

This solution has been deeply discussed in literature because of its interesting thermodynamical features in its dynamical horizons.

3.6.3 McVittie singularities

Returning to the general flat McVittie solution, a curvature singularity at $r = \frac{\mu_H}{2a(t)}$ is present, which can be seen thanks to the Ricci scalar

$$R = 12H^2 + 6\frac{1 + \frac{\mu_H}{2a(t)}}{1 - \frac{\mu_H}{2a(t)}}\dot{H} \tag{3.6.3}$$

where the pressure approaches infinity. This singularity is spacelike, and extends all the way to spatial infinity in the sense of proposition (2.1). Therefore, it should be seen as a *big bang* singularity. However, this singularity is absent when $\dot{H} = 0$, making the hyperfsurface regular. In contrast, the Schwarzschild solution has a coordinate singularity in the horizon, which is disposable thanks to a change of coordinates.
Moreover, Nolan argued that the null black hole horizon of the McVittie metric is at infinite distance, representing a null boundary. Hence, the metric outside this surface is geodesically complete and thus, in order to respect the cosmic censorship hypothesis, it doesn't describe a black hole [15]. However, it was shown by the paper of Kaloper, Kleban, and Martin (2010), that this assertion is wrong and such null surface is at a finite distance [11]. Or rather, the assertion is wrong just for the subclass of solutions represented by these which scale factor asymptotes to de Sitter spacetime

$$a(t) \to e^{H_0 t} \quad \Longrightarrow \quad H \to H_0 = constant$$

with $H_0 > 0$, obtained thanks to the addition of a cosmological constant.
In such case, the exponential nature of the scale factor allows the stress-energy density to decrease rapidly enough that all curvature invariants reduce to the ones of a pure Schwarzschild-de Sitter black hole. On the other hand, a polynomial form of the scale factor

$$a(t) \sim t^p \quad \Longrightarrow \lim_{t\to\infty} H(t) \to H_0 = 0$$

allows naked singularities [11].
Therefore, at least in the case of an approaching de Sitter spacetime, the McVittie solution can represent a CBH, with regular horizon. Note that the isotropic Schwarzschild metric, approached by a McVittie solution, has the unwanted feature that the coordinate r covers the exterior of the black hole twice. Thus, it's possible to express the McVittie solution in a form that gives a more intuitive behavior in the surfaces of interests.
Thanks to the transformation

$$\vec{r} \to (1 + \frac{\mu_H}{2a(t)r})^2 a(t)\ \vec{r} \tag{3.6.4}$$

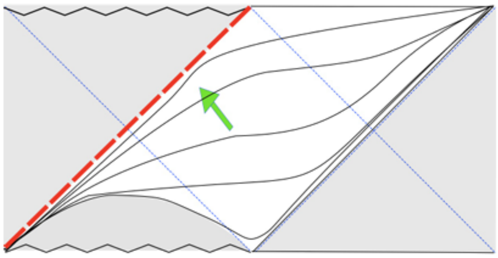

Figure 3.1: Schwarzschild-de Sitter conformal diagram. Black solid lines are the surfaces of constant t, the green arrow represents ingoing observers, and the red dashed line is the null apparent horizon at $r = r_-$ and $t = \infty$ [11].

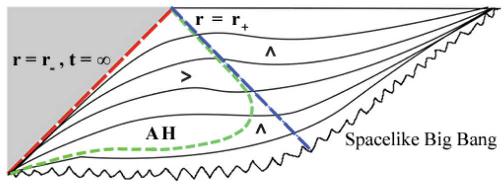

Figure 3.2: Conformal diagram of the McVittie solutions, restricted to the ones asymptotic to a de Sitter space. In addition, the blue dashed line is the cosmological horizon at $r = r_+$, the broken curve at the bottom is the spacelike big bang at $r = 2m$, and the thin green dashed line is the union of the two branches of apparent horizons at a finite t [11].

it is possible to write the McVittie metric in a form that, for $H = const$, reduces to the Scwharzschild-de Sitter metric, analogous to the outgoing Eddington-Finkelstein coordinate system for a flat Schwarzschild black hole, that is

$$g = -\frac{2Hr}{\sqrt{1-(2\mu_H/r)}}drdt + \frac{dr^2}{1-(2\mu_H/r)} + r^2 d\Omega^2 - f(r,t)dt^2 \tag{3.6.5}$$

where $f(r,t) = 1 - (2\mu_H/r) - H^2(t)r^2$.

3.6.4 Causal structure

Paper [11] demonstrates that the apparent horizons can be located at the two roots of $f(r,t)$, distinguishing the case of t finite and $t \to \infty$. Assuming the **dominant energy condition** ($\rho \geq | p |$) and an asymptotic de Sitter behavior ($\lim_{t\to\infty} H \to H_0$), we summarize the five surfaces of interest [11]:

i) A surface at $r = 2\mu_H$ [3] and $t =$ finite. This is the **big bang** singularity. It is a spacelike 3-surface that lies in the causal past of all spacetime points in the patch represented by the metric (3.6.5);

ii) A null apparent horizon at $r = r_-$ and $t = \infty$. r_- is identified with the smaller root of $f(r,t)$. If $H_0 > 0$ this is a **regular black hole event horizon**. If $H_0 = 0$ it is a null singularity. In both cases it is at a finite spatial distance from the interior. For the case $H_0 \neq 0$ this surface can be accessed in finite affine parameter by ingoing null geodesics, and it is traversable. Hence, the surface is not a boundary and the spacetime is geodesically incomplete being compatible with the censorship hypothesis.

iii) A null surface ending at the point $r = r_+$ and $t = \infty$. r_+ is identified with the greater root of $f(r,t)$. If $H_0 > 0$ this is a **cosmological event horizon**. If $H_0 = 0$ this is a null FRW infinity at $r_+ = \infty$. Furthermore, a null geodesic gets to the cosmological horizon at $r = r_+$ in finite affine parameter. Thus, this surface belongs to the spacetime.

iv) Two spacelike apparent horizons that live at the two roots $r = r_-$ and $r = r_+$, but at a finite time t. The first one evolves inward in time, along spacelike directions. On the other hand, the second branch evolves outward becoming timelike, and eventually asymptoting to null at $t = \infty$. Such branch always remains *inside* the Schwarschild-de Sitter horizon at $r = r_+$.
They link up thanks to a **bifurcation point** and delimit a normal region of spacetime from an antitrapped region[4]. The coordinate r is spacelike only in the normal region. It is clear from the form of the function $f(r,t)$ that, at a fixed time t_1, the areas of the apparent horizons are the same of the ones of a Schwazschild-de Sitter black hole with $H = H(t_1)$.

[3]Such surface seems different from the on found in equation (3.6.3) , but note that we changed the radial coordinate to a new one, which is now physical.

[4]both expansions of the ingoing and outgoing geodesics are positive.

Therefore, the McVittie solutions in spacetimes which are asymptotically dominated by a positive cosmological constant are CBHs [11].

3.7 Cosmological Black Holes (CBHs)

3.7.1 The non-accretion condition

In the last section, we discussed the McVittie solution for a generic mass-particle, which is coupled with the scale factor in a FRW universe. We've seen how such model struggles in describing cosmological black holes because of some intrinsic singularities, which are not compatible with a strict definition of CBHs, violating cosmological censorship in some cases. However, in spacetimes which are asymptotically dominated by a positive cosmological constant, the mas-particle can represent a CBH. This special case is actually a possible physical scenario. There is another problem though: the non-accretion condition is highly unrealistic and struggles to describe massive gravitational objects like CBHs. One could think that the $r = \mu_H/2$ surface behaves like a wall that stops all the cosmic fluid from the external universe. Related to this, it was shown that the *accretion rate* of a test fluid for a spherical symmetric Schwarschild solution follows the law

$$\dot{\mu_H} = 4\pi D\mu_H^2(p_\infty + \rho_\infty) \tag{3.7.1}$$

where p_∞ and ρ_∞ are the pressure and density at spatial infinity, and D is a constant first integral of motion [5]. Therefore, in a Schwarzschild-de Sitter solution, where the universe is dominated by a vacuum constant ($p = -\rho = -\Lambda$), the non-accretion condition seems to hold. However, since the cosmological spacetime described by the Λ-CDM model considers other kinds of fluid, this result is not completely satisfying.

3.7.2 Further models of CBHs

Despite McVittie solution still has to be completely defined and understood, being object of confused literature, further models of CBHs have been introduced. Studies by Nolan (1993) developed a new interior metric for the McVittie solution, by replacing it with a different geometry at small radius. Such metric can be used to describe external fields of finite size objects or exterior bubbles separating different spacetime regions [15]. Earlier studies by Einstein and Straus introduced the Swiss-cheese model, a spacetime where the vacuum Schwazschild black hole is glued in a homogeneous FRW cosmology. However, the model can't describe the solar system and hardly describes non-spherical solutions [20]. Vaidya (1977) followed a a similar road, representing a superposition of the Schwarschild metric with the one of an Einstein static universe [21]. Such solution is called *Vaidya-Einstein-Schwarzschild* (VES) spacetime. Moreover, it describes a perfect fluid with equation of state $\rho + 3p = 0$. Nayak (2000) later matched the VES spacetime with Schwarzschild vaccum solution representing the interior of the BH [14]. In these two models, the condition of asymptotic flatness is relaxed. However, they are time independent solutions that unlikely describe a real CBH. In fact, both of them admit a horizon defined as a Killing horizon, but a time dependent cosmological spacetime cannot admit a a global timelike killing vector [20].

3.7.3 Sultana & Dyer solution

Later model by Sultana and Dyer (2005) described a primordial dynamical black hole, existing *ab initio*, interacting with a k=0 (flat) FRW universe with $a(t) \sim t^{2/3}$. The authors computed a conformal transformation of the Schwarzschild metric

$$g_{ab}^{(S)} \to a^2(t) g_{ab}^{(S)}$$

and admitted a conformal Killing vector field for $\xi^c \nabla_c a(t) \neq 0$, generating a conformal Killing horizon. The chosen fluid is a superposition of an ordinary massive dust, and a null dust

$$T_{ab} = T_{ab}^{(massive)} + T_{ab}^{(null)} = \rho u_a u_b + \rho_n k_a k_b = \rho u_a u_b$$

The result is the metric

$$g = \left(1 + \frac{\mu_0}{2r}\right)^4 a^2(t)\left\{dr^2 + r^2 d\Omega^2\right\} - \left(\frac{1 - \frac{\mu_0}{2r}}{1 + \frac{\mu_0}{2r}}\right)^2 dt^2 \tag{3.7.2}$$

which is similar to the McVittie metric, with the difference that the mass parameter is now a constant μ_0. This implies that the no-accretion condition is relaxed and we get $\mu_H(t) = \mu_0 a(t)$. This solution is non-singular at the surface $\mu_H/2$, but has the problem that the fluid becomes tachyonic (negative energy density) at late times near the horizon, and it is a special case of dust dominated universe, which is not compatible with the Λ-CDM model [5].

3.7.4 Faraoni & Jacques solution

Marginal evidence of *phantom energy*, i.e. an energy that violates the dominant energy condition, and thus $\omega < -1$, led theorists to take in consideration the possibility of a *Big Rip* at late times of the universe[5]. Dealing with this scenario, Faraoni & Jacques (2018) wondered if highly bounded cosmological objects such as CBHs would be so highly deformed causing an expansion of the event horizon, and thus risking to violate cosmic censorship hypothesis. In this sense, they investigated the magnitude of the coupling of CBHs with the scale factor a(t) [5]. Earlier work by Price and Romano [17] showed a "all or nothing" behavior, suggesting that weekly coupling in atoms makes them to comove with the universe, while strongly coupled atoms are only slightly perturbed by a transient and don't expand [5]. However, large structures in an arbitrary FRW universe behave differently. For example, a black hole generated by a Schwarzschild-de Sitter solution[6] (3.6.2) admits a inner horizon at a radius $r = \frac{\mu(t)}{2} = \frac{\mu_H}{a(t)2}$. Therefore, the area of the correspondent 2-dimensional sphere is

$$\mathcal{A} = \int\int d\theta d\phi \sqrt{g_\Sigma} = \int\int d\Omega\, a^2(t)\left(1 + \frac{\mu(t)}{2r}\right)^4 r^2 = 16\pi\mu_H^2 \tag{3.7.3}$$

and thus, the Schwarzschild radial coordinate corresponding to the horizon is the physical curvature coordinate

$$r_{phys} = \sqrt{\frac{\mathcal{A}}{4\pi}} = 2\mu_H \tag{3.7.4}$$

which is not coupled. The latter suggests that in a de Sitter universe only weekly coupled objects participate in the cosmic expansion. However, this is not true in an arbitrary FRW universe. In fact, by taking a dust-dominated Sultana-Dyer solution (3.7.2), the latter reduces to

$$r_{phys} = 2\mu_H(t) = 2\mu_0 a(t) \tag{3.7.5}$$

The radius is thus comoving with the expansion of the universe.
Faraoni & Jacques built a perfectly comoving solution of the McVittie form

$$g = a^2(t)A^4(r,t)\left\{dr^2 + r^2 d\Omega^2\right\} - \frac{B^2(r,t)}{A^2(r,t)}dt^2 \tag{3.7.6}$$

where

$$A(r,t) = 1 + \frac{\mu(t)}{2r} \quad B(r,t) = 1 - \frac{\mu(t)}{2r}$$

which describes a CBH embedded in a universe with generic factor a(t) and filled with an imperfect fluid of the form

$$T_{ab} = (p+\rho)u_a u_b + p g_a b + q_a u_b + q_b u_a \tag{3.7.7}$$

where the spatial vector q^c describes a radial energy flow,

$$u^a = (A/B, 0, 0, 0)\,, \quad q^b = (0, q, 0, 0), \quad q^c u_c = 0, \quad u^c u_c = -1$$

Such solution makes the surface $r = \mu(t)/2$ non singular, and makes the black hole embedded in a hypothetical phantom-dominated universe disappear at late times, protecting cosmic censorship. However, it has the problem that the accretion flow becomes superluminal.

[5]A cosmological scenario where the expansion is so fast that cosmological object are most likely teared apart.

[6]and thus, a McVittie black hole solution, since we had to impose the constraint of a late times de Sitter asymptotic behavior.

3.7.5 Crooker & Weiner coupling

Crooker & Weiner (2019) found a way to couple all the relativistic material, including the interior of compact objects, with the expansion of the universe, by deriving Friedmann equations through the Einstein-Hilbert action in a perturbed FRW universe. By starting from a few basic reasonable assumptions [3], the result is

$$\frac{d^2 a}{d\eta^2} = \frac{a^3}{6}\langle \rho(\eta, x) - \sum_{i=1}^{3} p_i(\eta, x)\rangle_{\mathcal{V}} \tag{3.7.8}$$

where η is the conformal time, and the spatial average is made inside a volume $\mathcal{V}$. Such volume corresponds to a 3-ball centered in a point P of space, such that moving the center of the ball to another point U doesn't change the average estimation. Because the universe is homogeneous in its largest scale, there must exists a radius $b_{\mathcal{V}}$, great enough so that this is possible approximately. Observations suggest that, at the present epoch such radius is

$$b_{\mathcal{V}} \sim 180Mpc$$

Therefore, this coupling is a consequence of an averaging process of all pressures composing the universe. From the conservation law of the stress-energy tensor, they showed that the material contributing in this way to the expansion must shift locally in energy [3]. The paper further argued that the vacuum Kerr solution (and thus the Schwarzschild solution) respects the assumptions in a specific domain, but it doesn't affect Friedmann's equation. Furthermore, asymptotic flat solutions of any kind are valid only for short intervals. On the other hand, Schwarzschild's constant density sphere (interior solution) and the isolated de-Sitter sphere (representing a GEODE[7]) both couple with the expansion by affecting the Friedmann equations. GEODEs are explicit GR solutions maximally relativistic: they saturate the dominant energy condition. Furthermore, the article states that any source that contributes to the cosmologically averaged pressure must itself evolve cosmologically, but the effect is non-negligible only for relativistic objects[8] [3]. The energy density can be written

$$\rho(a) = E\mathcal{N} \tag{3.7.9}$$

where $\mathcal{N}$ is the physical number density of the object population. Moreover, in an expanding universe

$$\mathcal{N} \sim a^{-3} \tag{3.7.10}$$

and (2.5.22) becomes

$$E \sim a^{-3\omega} \tag{3.7.11}$$

In other words, this is a generalized photon red shift for timelike trajectories, which successfully reduces to the better known photon case for $\omega = 1/3$.
For GEODEs the effect is maximal with

$$E \sim a^3 \tag{3.7.12}$$

The cosmological energy shift is completely unaffected by the spatial distribution of material in the universe. This model justifies a single-parameter model of cosmological coupling [4]:

$$m(a) = m_0 \left(\frac{a}{a_i}\right)^k \tag{3.7.13}$$

with $k = -3\omega$, and a_i is the scale factor at which the object becomes cosmologically coupled.

[7]A simple model of a Generic Object of Dark Energy.
[8]composed by materials such that $\mid \omega \mid = \mid \frac{p}{\rho} \mid$ is not negligible.

3.8 Have we found the source of dark energy?

The Kerr solution (1963) showed excellent consistency with observations of gravitational waves from binary BH mergers, on timescales from millisecond to hours, and spatial scales of up to milliparsecs [7]. Croker (2019) showed that such consistency can hold in a restricted dominion of spacetime, since a Kerr BH is a local solution asymptotic to flat Minkowski spacetime.
The need of CBHs solution is obvious, but since the discovery of the McVittie metric, not many crucial results have been found. Generalizations of the Schwarzschild solutions well behave only for specific asymptotic spacetimes, while we are far from finding a Kerr CBH. However, a paper made by Farrah et al. (2023) [7] tried to use the simple model of cosmological energy shift made by Crooker (3.7.13) to validate observations from five different high-redshift samples[9], and one local sample ($z \sim 0$), of elliptical galaxies. The experimental results were presented in an earlier paper by Farrah et al. (2023) [6]. The goal of the article is to perform a direct test of BH mass growth due to cosmological coupling. The test relies on the measurement of the increasing ratio of a supermassive black hole (SMBH) to host stellar mass

$$R = \frac{M_{BH}}{M_*} \tag{3.8.1}$$

which can be compared with the mass of the local sample. In fact, a translational offset was found comparing the two samples in the M_{BH}-M_* plane: a small translation offset τ_* in stellar mass M_* was detected, while the magnitude of the M_{BH} offset τ_{BH} is much larger.
Then, a parameterization of the energy shift can be made thanks to (3.7.13)[10]. SMBH growth via accretion is expected to be negligible, and galaxy-galaxy mergers should not in average increase the SMBH mass. One would think that the only component of the analyzed system that couples is the highly relativistic supermassive black hole. The result is

$$k = 3.11^{+1.19}_{-1.33} \quad \text{(90\% confidence level)} \tag{3.8.2}$$

which excludes the uncoupled case (k=0) at 99.98% confidence. The related value of $k \sim 3$ is consistent with a vacuum interior, just like the one of GEODEs.
Equation (3.7.13) suggests that vacuum interior black holes with $k \sim 3$ will gain mass proportional to a^3, while the number density of such objects will decrease like a^{-3} because of cosmological expansion in a FRW universe. When accretion is negligible, this population of BHs will participate to expansion as a dark energy density ρ_Λ. From the conservation of the stress-energy tensor this is possible only if they also contribute cosmological pressure $P_\Lambda = -\rho_\Lambda$. $k \sim 3$ BHs can be thus treated as cosmological dark energy species. The natural implication is that a cosmologically realistic BH solution with a non-singular vacuum interior for GR must exist. Morevre, the paper goes behind this conclusion, stating that BHs contribute as the unique cosmological dark energy species, driving the late time accelerating expansion with the Planck measured value $\Omega_\Lambda = 0.68$. In order to do so, the authors built a mathematical model of the cosmic star formation rate density (SFRD). Then, they showed that an estimate of the mass of the population of BHs made by (3.7.13), distributed as a population of Massive Compact Halo Objects (MACHO), is consistent with the mathematical model. In conclusion, stellar remnant BHs must produce the totality of dark energy.

On the other hand, a more recent paper by Parnovsky [16] introduced some serious issues in the latter model. These are briefly summarized as follows

i) Firstly, a well behaving solution for Kerr cosmological black holes has yet to be found and eq. (3.7.13) must be taken carefully. Another issue follows directly: while in an asymptotic flat BH solution the mass of the object can be detected from the asymptotical form of the metric, in cosmological black holes this is not true anymore and ambiguity in defining the invariant mass is present, especially because of the absence of an exact solution.

[9] two from WISE survey ($z = 0.75$) and ($z = 0.85$), two from the SDSS ($z = 0.75$) and ($z = 0.85$), and one from COSMOS field ($z = 1.6$).
[10] further technical details can be found in [7] and [6], since they are not the object of this thesis.

ii) In a process of galaxies merging, it is true that the mass of the stellar population of the formed galaxy can be approximately considered equal, while the final formed SMBH doesn't exceed the sum of the masses of the two original SMBHs[11]. An increase in the BH mass at accretion is compensated by a decrease in the matter mass outside of it. Thus, the BH mass can increase, while the total mass cannot. However it is difficult to imagine that the rate of accretion is related to the scale factor a(t). An argument in contrast with this objection is that, by assuming that a(t) is a monotonic function, the scale factor function can be inverted in order to obtain t(a).

iii) Let's assume that the black hole interior is actually dark energy density. The density increase due to cosmological coupling is perfectly balanced by the expansion as stated before. This feature is consistent with the constancy of the dark energy density. In fact it cannot be transformed in something else or viceversa. The equation of state $p = -\rho$ provides a negative constant pressure and an antigravity effect (since $\rho + 3p < 0$). However, the black hole doesn't have a negative pressure and thus it cannot provide antigravity. The reason is simple. Black holes are highly compact objects that occupy only a little fraction of space. Thus, the antigravitational effects would be seen firstly in the region around the black hole. Moreover, the small fraction of BHs matter cannot produce the present day dark energy.

iv) Assume that these vacuum energy interiors (ρ_v, p_v) exist, taking the role of dark energy. Let there be only matter in the universe, which is cold ($p_m \sim 0$), while the radiation can be considered negligible in this analysis. Then, to obtain an accelerating universe as a whole, the condition $\rho + 3P = \rho_m + \rho_v + 3P_v < 0$ must hold. However since for vacuum $\rho_v = -p_v$, the latter becomes $\rho_v > \rho_m/2$.
It is unlikely that this kind of BHs provide more than a third of the total mass in the universe.

3.9 Conclusions

Conclusions by Farrah et al. (2023) [7] concerning the origin of dark energy are yet to be completely discussed. However, a paper by Parnovsky [16] shows some persuasive physical objections to the latter coupling model, suggesting that black holes can't be the cause of the late time accelerating universe. Therefore, the model built in [7] may have an internal consistency, but it's important to don't jump on cosmological conclusions over the dark energy origin. In fact, the used coupling models are far from describing a realistic solution of cosmological black holes. Despite this, the McVittie solution, which is the first attempt to couple a mass-particle with an expanding FRW universe, is interesting in its own right, since it is an exact solution of the second order to Einstein's equations. However, McVittie solution can't always describe a black hole embedded in a FRW universe because of its singularities. Kaloper [11] showed the the only subclass of McVittie solutions that can describe a CBH is the one represented by metrics that are asymptotic to a de Sitter spacetime (and thus driven by a cosmological constant). Other coupling solutions have been found: some examples are the Nolan interior [15], the Sultana-Dyer accretion solution [20], and the solution given by Faraoni and Jaques that produces a comoving horizon [5]. Crooker [3] showed that BH solutions that are asymptotic to flat Minkowski universe can describe a physical solution only in a small subdomain of spacetime. Moreover, in the same paper a redshift generalization for timelike trajectories was presented. Recent observations, showed excellent consistency with the spinning Kerr solution (1963), only in short time/space-scales. The need of an exact cosmological solution of the Kerr metric is obvious, in particular if new relations between black holes and the expanding universe want to be found.

[11]part of the mass may decrease because of the emission of gravitational waves during the process.

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