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Gauge transformations and gauge-invariant cosmological perturbations

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Introduction

The study of cosmological perturbations brought to light misleading interpretations and solutions of questionable physical meaning. In fact the invariance of General Relativity under gauge transformations introduces a redundant degree of freedom that is eliminated by a gauge choice. However the gauge-dependent quantities studied in the literature could not be considered physical, because they could change value and dependence on time under gauge transformations. Furthermore there are some cases in which the perturbation seems to grow when it is not even within the particle horizon.

A first step toward the solution of the gauge problem was made by Hawking [1], who formulated the perturbation equations in a covariant way but did not totally solve the problem of gauge ambiguity, because in order to define a density perturbation it is still necessary a time slicing, which breaks the invariance of the approach. A different attempt was made by Sakai [2], who found preferential coordinate systems in order to exclude fictitious solutions for density perturbations, which are the ones moving with the average density of matter.

The very solution of the problem was the introduction of gauge-invariant variables through which to describe cosmological perturbations and their study. This idea was conceived by Bardeen [6] in 1980, and then was developed by Kodama and Sasaki [7]. By writing the perturbations in terms of invariant quantities the formulation results no more ambiguous, and the corresponding perturbation equations yield results that are physically significant and trustworthy.

In this thesis we will follow the historical path of the gauge problem and its resolution, emphasizing the innovation that the introduction of invariant quantities has led to cosmological perturbation theory. In chapter 1 an introduction to cosmological perturbation theory and gauge transformations is presented. In chapter 2 the gauge problem is studied in deep for the non conformal synchronous gauge and the conformal Newtonian gauge, to highlight the meaningless results that particular choices of gauge could lead to. Chapter 3 follows Bardeen's approach to the subject, defining gauge-invariant quantities separately for scalar, vector and tensor perturbations. We will also derive perturbation equations for these variables and solve them without losing any generality. To make more intuitive the physical meaning of the new variables introduced in chapter 3, in chapter 4 they are examined in the conformal Newtonian gauge, in which they assume a particularly simple form. The solution of Einstein's equations in terms of these invariant quantities shows that the gauge problem is finally solved.

Notation

To lighten the formulation we will adopt the notation

$$c = 8\pi G = 1$$

where c is the velocity of light and G the gravitational constant.

Along this thesis we will need to distinguish between the background universe and the physical one, so the covariant derivative with respect to the three-dimensional background universe will be denoted with a slash, while a semicolon will denote the covariant derivative with respect to the three-dimensional perturbed universe.

Greek indices will range from 0 to 3, Latin indices will mark only spatial coordinates and range from 1 to 3.

Chapter 1

Cosmological perturbations and gauge transformations

Einstein's equations describe the spacetime geometry as a function of the stress-energy tensor $T_{\mu\nu}$. Explicitly they read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein's tensor, while $R_{\mu\nu}$ is the Ricci's tensor, contraction of the curvature tensor, $R \equiv R^{\gamma}_{\gamma}$ and $g_{\mu\nu}$ is the metric tensor.

Einstein's equations can be seen as a system of nonlinear second order differential equations for the metric components $g_{\mu\nu}$, thus no general methods exist through which it is possible to obtain all solutions. There are only a few exact solutions of physical interest, such as the Schwarzschild solution for a spherical simmetric gravitational field and the Friedmann-Lemaitre-Robertson-Walker (FLRW) solution for an homogeneus and isotropic universe. [3] [4]

1.1 Cosmological perturbations

The exact solutions of Einstein's equations describe only particular cases of symmetries. We can extend our knowledge of the physical universe by considering small deviations from these symmetries: this is the so called perturbation theory.

A gravitational perturbation can be written as a small variation of the metric [5]:

$$g_{ij} \rightarrow g_{ij} + \delta g_{ij} \tag{1.2}$$

where the unperturbed metric represents the background universe, which we will consider to be the FLRW metric for homogeneous and isotropic universes

$$ds^2 = a^2(\tau)[-d\tau^2 + {}^3g_{ij} dx^i dx^j] \tag{1.3}$$

where $a(\tau)$ is the scale factor, τ is the conformal time and ${}^3g_{ij}$ is the metric tensor for a 3-dimensional space of uniform curvature K, and the choice of the space coordinates is left arbitrary.

The unperturbed energy-momentum tensor is the one that represents a perfect fluid at rest relative to the comoving coordinates, so that the only non-zero components are

$$T^0_0 = -\rho_0 \tag{1.4a}$$
$$T^i_j = P_0\delta^i_j \tag{1.4b}$$

where $\rho_0(\tau)$ is the background energy density and $P_0(\tau)$ the background pressure. Let us define the quantities

$$w = \frac{P_0}{\rho_0} \tag{1.5}$$

$$c_s^2 = \frac{dP_0}{d\rho_0} \tag{1.6}$$

that will be useful later on.

The homogeneity and isotropy of the background allow a separation of the time dependence and the spatial one, so without losing any generality we can expand an arbitrary perturbation over spatial spherical harmonics $Q^{(n)}$. Through these functions perturbations can be classified in scalar, vector and tensor quantities, according to how they transform under spatial coordinate transformations in the background spacetime [6][7].

A *scalar perturbation* has a spatial dependence derived from scalar harmonics, which are the solutions of the *scalar Helmholtz's equation*:

$$\Delta Q^{(0)} + k^2 Q^{(0)} = 0 \tag{1.7}$$

where $-k^2$ is the eigenvalue of the Laplace-Beltrami operator Δ.

Vectors and tensor quantities associated with scalar perturbations must be constructed from covariant derivatives of $Q^{(0)}$ and the spatial metric tensor; let us define the vector

$$Q^{(0)}_i = -\frac{1}{k}Q^{(0)}_{|i} \tag{1.8}$$

and the traceless symmetric tensor

$$Q^{(0)}_{ij} = \frac{1}{k^2}Q^{(0)}_{|ij} + \frac{1}{3}{}^3g_{ij}Q^{(0)} \; . \tag{1.9}$$

A *vector perturbation* is proportional to $Q^{(0)}_i$, but it has a divergenceless component that cannot be constructed from scalar harmonics; instead it must be proportional to vector harmonic functions, which are solutions of the *vector Helmholtz's equation*

$$\Delta Q^{(1)i} + k^2 Q^{(1)i} = 0 \; . \tag{1.10}$$

The second rank traceless symmetric tensor associated with the vector harmonics is

$$Q^{(1)ij} = -\frac{1}{2k}(Q^{(1)i|j} + Q^{(1)j|i}) \; . \tag{1.11}$$

In the same way a *tensor perturbation* will be proportional to the solutions of the *tensor Helmholtz's equation*

$$\Delta Q^{(2)ij} + k^2 Q^{(2)ij} = 0 \; . \tag{1.12}$$

1.1.1 Scalar perturbations

First of all we will consider scalar perturbations. The metric tensor components g_{00}, g_{0i} and g_{ij} under spatial coordinate transformations transform as a scalar, a vector and a tensor respectively. Hence let us consider the perturbation of the metric

$$g_{00} = -a^2(\tau)[1 + 2A(\tau)Q^{(0)}(\vec{x})] \tag{1.13a}$$

$$g_{0i} = -a^2 B^{(0)}(\tau) Q_i^{(0)}(\vec{x}) \tag{1.13b}$$

$$g_{ij} = a^2[1 + 2H_L(\tau)Q^{(0)}(\vec{x})]^3 g_{ij}(\vec{x}) + 2H_T^{(0)}(\tau)Q_{ij}^{(0)}(\vec{x}) \tag{1.13c}$$

where $\vec{x}$ denotes the dependence on the spatial coordinates. $A(\tau)$ is the amplitude of the perturbation in the lapse function, which represents the ratio between the proper-time distance and the coordinate-time distance between two neighboring constant time hypersurfaces. $B^{(0)}(\tau)$ is interpreted as the amplitude of a perturbation in the shift vector, which represents the rate of deviation of a constant space-coordinate line from a line normal to a constant time hypersurface. H_L is the amplitude of perturbation of a unit spatial volume, and H_T represents the amplitude of anisotropic distortion of each constant time hypersurface.

Let us define the 4-velocity u^μ as the velocity of the rest frame of matter, i.e. the frame in which the energy flux vanishes. The three-velocity associated to u^μ is

$$\frac{u^i}{u^0} = v^{(0)}(\tau)Q^{(0)i}(\vec{x}) \; , \tag{1.14}$$

and to first order the normalization $u^\mu u_\mu = -1$ gives the perturbation of the zeroth component

$$u^0 = a^{-1}(1 - AQ^{(0)}) \; . \tag{1.15}$$

In this frame the energy density ρ is given by

$$\rho = -T_0^0 = \rho_0(\tau)[1 + \delta(\tau)Q^{(0)}(\vec{x})] \tag{1.16}$$

where T_ν^μ is the stress-energy tensor and $\delta(\tau)$ the amplitude of the perturbation that depends on the conformal time.

The spatial stress-energy tensor T_j^i is represented by an isotropic pressure

$$P = \frac{1}{3}T_i^i = P_0(\tau)[1 + \pi_L(\tau)Q^{(0)}(\vec{x})] = P_0 + \delta P \tag{1.17}$$

where π_L is the amplitude of the isotropic perturbation, and a traceless anisotropic stress

$$T^i_j = P_0[(1 + \pi_L Q^{(0)})\delta^i_j + \pi^{(0)}_T(\tau)Q^{(0)i}_j] \tag{1.18}$$

where π_T is the amplitude of the anisotropic perturbation. Transforming back from this reference frame to the coordinate frame the only components that change at first order are

$$T^0_i = (\rho_0 + P_0)(v^{(0)} - B^{(0)})Q^{(0)}_i \tag{1.19a}$$
$$T^i_0 = -(\rho_0 + P_0)v^{(0)}Q^{(0)i} \;. \tag{1.19b}$$

The difference between the pressure perturbation and what expected from the background pressure-energy density is the so called entropy perturbation, and it is expressed by:

$$\eta(\tau)Q^{(0)} = \frac{1}{w}(w\pi_L - c_s^2\delta)Q^{(0)} \;. \tag{1.20}$$

1.1.2 Vector perturbations

In the case of vector perturbations the scalar quantities considered must be unperturbed. Therefore in the metric tensor the component g_{00} is unchanged, while g_{0i} and g_{ij} vary as

$$g_{0i} = -a^2(\tau)B^{(1)}(\tau)Q^{(1)}_i(\vec{x}) \tag{1.21a}$$
$$g_{ij} = a^2[{}^3g_{ij}(\vec{x}) + 2H^{(1)}_T(\tau)Q^{(1)}_{ij}(\vec{x})] \tag{1.21b}$$

where the quantities implied are interpreted in the same way as the ones described above. Define the three-velocity as

$$\frac{u^i}{u^0} = v^{(1)}Q^{(0)i} \tag{1.22}$$

in the same way we did for scalar perturbations. The stress-energy tensor changes as

$$T^0_i = (\rho_0 - P_0)(v^{(1)} - B^{(1)})Q^{(1)}_i \tag{1.23a}$$
$$T^i_j = P_0[\delta^i_j + \pi^{(1)}_T Q^{(1)i}_j] \tag{1.23b}$$

1.1.3 Tensor perturbations

Tensor perturbations affect only the traceless part of the metric tensor and the stress-energy tensor:

$$g_{ij} = a^2(\tau)[{}^3g_{ij}(\vec{x}) + 2H^{(2)}_T(\tau)Q^{(2)}_{ij}(\vec{x})] \tag{1.24}$$

$$T^i_j = P_0[\delta^i_j + \pi^{(2)}_T(\tau)Q^{(2)i}_j] \;. \tag{1.25}$$

Note that no density or isotropic pressure perturbation is associated with vector or tensor perturbations [6][7].

1.2 Gauge transformations

General Relativity is invariant under diffeomorphisms: it can be easily seen by writing the action of the metric, the Einstein-Hilbert action. Considering diffeomorphisms $\delta g_{\mu\nu} = \mathcal{L}g_{\mu\nu}$, where $\mathcal{L}g_{\mu\nu}$ is the Lie derivative of the metric tensor, it is immediate to observe that this object is invariant [8].

This invariance generates a redundant degree of freedom, that must be suppressed because it has no physical meaning. The traditional way to do it is through a gauge fixing.

A gauge is a one-to-one correspondence between the background spacetime and the physical one, i.e. the perturbed one. Let us think at the two spacetimes like two manifolds: a gauge is a map from the background to the physical universe. Obviously there are infinite possible maps between the two, and they are all equivalent for General Relativity. So we have to choose one of them to obtain physical solutions: the perturbations defined above are clearly gauge dependent, for their dependence on the point of the spacetime where they are calculated.

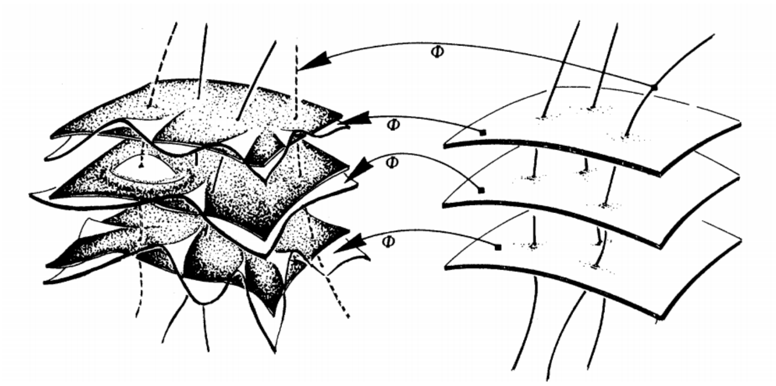

Figure 1.1: Φ is the map from the background spacetime (the one on the right) to the physical one (on the left) at different hypersurfaces with constant perturbation of the energy density.

A change of this correspondence is a gauge transformation. It is very important to distinguish between gauge tranformations and coordinate transformations: the first ones change the point in the background spacetime corresponding to a point in the physical spacetime, while the second ones only change the labeling of the points in the background and physical spacetime together [6][9].

For a more rigorous formulation of this concept let us now define two vector fields X and Y: their integral curves define two flows ϕ_λ and ψ_λ from the background spacetime $\mathcal{M}_0$ to the perturbed one $\mathcal{M}_\lambda$, where λ is a parameter that indicates the model of the chosen physical spacetime. Thus X and Y are everywhere transverse to $\mathcal{M}_\lambda$ and points lying on the same integral curve of either of the two are to be regarded as the same point within the respective gauge: ϕ_λ and ψ_λ are both point identification maps, i.e.

two different gauge choices.

We can define the gauge transformation Φ_λ: $\mathcal{M}_0 \to \mathcal{M}_0$

$$\Phi_\lambda := \phi_{-\lambda} \circ \psi_\lambda \ . \tag{1.26}$$

This mathematical formulation is made clearer by figure 1.2.

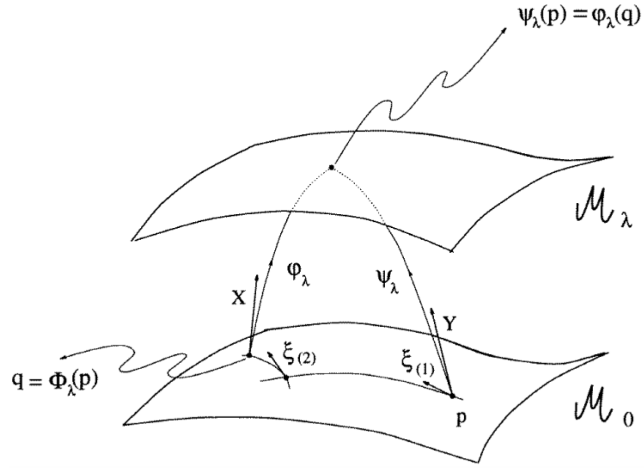

Figure 1.2: Representation of the gauge transformation Φ_λ

The tensor fields T^X_λ and T^Y_λ defined by the gauges ϕ_λ and ψ_λ are connected at first order by:

$$T^Y_\lambda = T^X_\lambda + \lambda \mathcal{L}_\xi T^X_\lambda \tag{1.27}$$

where ξ^μ is a generator of the diffeomorfism Φ_λ, i.e. the gauge tranformation, and $\mathcal{L}_\xi T^X_\lambda$ is the Lie derivative of T^X_λ along the vector ξ^μ [10].

Particularly the Lie derivative along ξ^μ of the metric tensor operates like

$$\mathcal{L}_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} \ . \tag{1.28}$$

1.2.1 Active and passive approach

There are two ways to calculate how perturbations change under small gauge transformations: the *active* and the *passive* approach [11].

In the *active approach* the focus is on how the perturbations change under a mapping, once the generator of the gauge transformation ξ^μ has been specified. The map is then the exponential map, and a generic tensor T varies like

$$\widetilde{T} = e^{\mathcal{L}_\xi} T \ . \tag{1.29}$$

Splitting the tensors to the first order and separating the terms we obtain at zeroth order

$$\widetilde{T}_0 = T_0 \; , \tag{1.30}$$

and at first order

$$\widetilde{\delta T}_1 = \delta T_1 + \mathcal{L}_\xi T_1 \; . \tag{1.31}$$

Applying the map to coordinate functions we get the relation between coordinates of a point q and the ones of another point p:

$$x^\mu(q) = e^{\mathcal{L}_\xi} x^\mu(p) = e^{\xi^\sigma \frac{\partial}{\partial x^\sigma}\big|_p} x^\mu(p) \tag{1.32}$$

that expanded at first order gives

$$x^\mu(q) = x^\mu(p) + \lambda \xi^\mu(p) \; . \tag{1.33}$$

The *passive approach* instead specifies the relation between two coordinate systems directly, and the change of the perturbation is calculated with respect to this coordinate change. In this view the transformation is taken at the same physical point, while in the active view it is taken at the same coordinate point. The transformation at first order from the coordinate system x^μ to $\tilde{x}^\mu$ is

$$\tilde{x}^\mu(q) = x^\mu(q) - \lambda \xi^\mu(q) \; . \tag{1.34}$$

Let us consider the energy density ρ and split it into an homogeneous background quantity and a perturbation at first order as

$$\rho(x^\mu) = \rho_0(x^0) + \lambda \delta\rho(x^\mu) \; . \tag{1.35}$$

The energy density is a scalar and does not change under coordinate transformations, so

$$\rho(x^\mu) = \tilde{\rho}(\widetilde{x}^\mu) \; . \tag{1.36}$$

Expanding both sides of (1.36) using (1.34) and (1.35) we obtain:

$$\begin{aligned} \rho_0(x^0) + \lambda \delta\rho(x^\mu) &= \rho_0(\widetilde{x}^0) + \lambda \widetilde{\delta\rho}(\widetilde{x}^\mu) \\ &= \rho_0(x^0) - \lambda \frac{\partial \rho_0}{\partial x^0}(x^0)\xi^0(x^\mu) + \lambda \widetilde{\delta\rho}(x^\mu) \end{aligned} \tag{1.37}$$

where x^0 and ξ^0 (with upper indices) are the 0-component of x^μ and ξ^μ. The transformation rule is then

$$\widetilde{\delta\rho} = \delta\rho + \frac{\partial \rho_0}{\partial x^0} \xi^0 \; . \tag{1.38}$$

From now on we will use the passive approach to study cosmological perturbations.

Chapter 2

Gauge problem

The perturbations defined in the previous chapter are clearly gauge dependent. In lot of cases this brings to misleading interpretations or even to non physical solutions [12]. There are no physical reasons to choose between gauges that give very different results, each mathematically correct. Furthermore, if the gauge condition imposed leaves a residual gauge freedom, the perturbation equations will have solutions which have no physical reality, and can be annulled by a gauge transformation [6].

In this chapter we will see a few examples of non physical results obtained from particular choices of gauge, considering only the case of scalar perturbations in the dust solution of the Einstein-de Sitter model, i.e. setting $K = 0$ and $w = 0$.

2.1 Non-comoving synchronous gauge

The most evident example of the absurd results that a gauge choice can bring is the matter-dominated limit of the non-comoving synchronous gauge [12].

The condition of the synchronous gauge is the one that allows synchronization of clocks in different space points, and it's expressed by $g_{00} = 1$ and $g_{0i} = 0$.

Matter-dominated limit means that the matter dominates the energy density, so that we can ignore the other components. By "matter" it is meant non-relativistic matter, whose pressure is so small compared to the energy density that we can ignore it and consider $P = w = 0$. According to our present understanding, the universe was radiation-dominated for the first few ten thousand years, after which it became matter-dominated [13].

The major problem, dealing with this gauge, has to do with those perturbations whose spatial wavelengths λ are larger than the particle horizon, such that

$$\lambda = \frac{2\pi a(t)}{q} \gg ct \tag{2.1}$$

where q is the wavenumber in comoving coordinates, and t is the proper time.

Let us define the quantity

$$\phi(t) \equiv \frac{a^2}{\dot{a}} u^j_{|j} \tag{2.2}$$

where the dot denotes the derivative with respect to the time t, and the new variable $\eta = ln\, t$ so that

$$\frac{d}{dt} = \frac{\dot{a}}{a}\frac{d}{d\eta} \ . \tag{2.3}$$

In the following we will denote with a prime the derivative with respect to η [12].

Let's call h_{ij} the component of the perturbation of the metric tensor introduced in (1.2) that represents density perturbations only, rescaled of a factor a^2. The perturbed Einstein's field equations yield to the equations [12][14]:

$$h'' + \frac{1}{2}h' - 3\delta = 0 \tag{2.4}$$

$$\delta' + \phi - \frac{1}{2}h' = 0 \tag{2.5}$$

$$\phi' + \frac{1}{2}\phi = 0 \tag{2.6}$$

where $h \equiv h^i_i$, and δ is the energy perturbation defined in equation (1.16).

There are four different independent solutions of these equations, each varying as $e^{m\eta}$ for some eigenvalue m, and four different eigenvectors $(\delta, \phi, h, h')e^{m\eta}$. The time dependence (with respect to t) of a mode with eigenvalue m is

$$(\delta, \phi, h, h') \propto t^{\frac{2m}{3}} \ . \tag{2.7}$$

Explicitly the four eigenmodes are:

$$m = -\frac{3}{2} \qquad (\delta, \phi, h, h') \propto \left(\frac{1}{2}, 0, 1, -\frac{3}{2}\right)t^{-1} \tag{2.8a}$$

$$m = -\frac{1}{2} \qquad (\delta, \phi, h, h') \propto \left(0, -\frac{1}{2}, 2, -1\right)t^{-\frac{1}{3}} \tag{2.8b}$$

$$m = 0 \qquad (\delta, \phi, h, h') \propto (0, 0, 1, 0) \tag{2.8c}$$

$$m = 1 \qquad (\delta, \phi, h, h') \propto \left(\frac{1}{2}, 0, 1, 1\right)t^{\frac{2}{3}} \ . \tag{2.8d}$$

Every solution is the linear combination of the four modes above. To recognize the physical validity of a solution we have to check whether it is due to the combination of gauge solutions: we have to find the most general solution due to gauge transformations, and to verify if the combination of the four modes above could be seen as a gauge mode. In this case we have to reject the solution, because no possible measurements could distinguish its presence or absence.

Let us consider the change of coordinates

$$\widetilde{x}^\mu = x^\mu + \xi^\mu(x) \ . \tag{2.9}$$

The most general form allowed for ξ^μ is

$$\xi^0 = \psi(\vec{x}) \tag{2.10a}$$

$$\xi^i = \int dt \frac{\psi_{|i}}{a^2} + \chi^i(\vec{x}) \tag{2.10b}$$

where ψ and χ are functions of the spatial coordinates alone. The perturbation of the metric tensor varies as

$$h_{ij} = 2\frac{\dot{a}}{a}\psi\delta_{ij} + 2\psi_{|ij}\int\frac{dt}{a^2} + \chi^i_{|j} + \chi^j_{|i} \tag{2.11}$$

that yields

$$h = 6\frac{\dot{a}}{a}\psi + 2\nabla^2\psi\int\frac{dt}{a^2} + 2\chi^j_{|j} \ . \tag{2.12}$$

The quantities ϕ and δ change as:

$$\phi = \frac{\nabla^2\psi}{\dot{a}a} \tag{2.13}$$

$$\delta = 3\frac{\dot{a}}{a}\psi \ . \tag{2.14}$$

It can be verified that h, ϕ and δ in this form satisfy equations (2.4), (2.6) and (2.5).

If $\chi^i = 0$ and $\psi \propto e^{iq_j x^j}$ equations (2.12), (2.13) and (2.14) can be seen as the linear combination of the solutions (2.8a) and (2.8b):

$$(\delta, \phi, h, h') \propto \left(\frac{1}{2}, 0, 1, -\frac{3}{2}\right)\left(\frac{t}{t_0}\right)^{-1} + \left(0, -\frac{1}{2}, 2, -1\right)\left(\frac{t}{t_0}\right)^{-\frac{1}{3}} \tag{2.15}$$

where t_0 is the time such that

$$t_0 = \left(\frac{4}{3}\right)^{\frac{1}{2}}\frac{a(t_0)}{q} \tag{2.16}$$

which is, apart from a factor of order unity, the horizon crossing time of the mode q. At times much before t_0 the solution (2.15) is dominated by the t^{-1} mode, at times much after t_0 by the $t^{-\frac{1}{3}}$ mode.

We have seen that the synchronous gauge does not fix entirely the gauge degrees of freedom, in fact it brings solutions that are constructed from a gauge transformation. There is still a residual degree of freedom that must be eliminated in order to get physical solutions.

One way to remove this gauge freedom is to impose another gauge condition: the *uniform Hubble constant* gauge, that fixes the condition that any physical measurements of ϕ and δ at any epoch are reported in the coordinate system which has, at that epoch, $h = h' = 0$.

Introducing the new variable

$$s \equiv \frac{t}{t_0} \ , \tag{2.17}$$

we can obtain two linearly independent solutions of δ:
the *decaying solution*

$$\delta = \frac{2s^{-\frac{1}{3}}}{2s^{\frac{2}{3}} + 3} \tag{2.18}$$

and the *growing solution*

$$\delta = \frac{2s^{\frac{4}{3}} + s^{\frac{2}{3}}}{2s^{\frac{2}{3}} + 3} \tag{2.19}$$

Consider the limits of these solutions in the cases in which the perturbation is well within the horizon $s \gg 1$ and very far from it $s \ll 1$. For the decaying solution we obtain:

$$\delta \to s^{-1} \qquad s \gg 1 \tag{2.20a}$$

$$\delta \to s^{-\frac{1}{3}} \qquad s \ll 1. \tag{2.20b}$$

For the growing mode we get:

$$\delta \to s^{\frac{2}{3}} \qquad s \gg 1 \tag{2.21a}$$

$$\delta \to s^{\frac{2}{3}} \qquad s \ll 1. \tag{2.21b}$$

We can see that, crossing the horizon, in the decaying mode the trend of the perturbation changes, i.e. in this gauge a discontinuity at the horizon is present. But there is a more substantial problem: the perturbation increases with time. In particular, it goes to infinity when it is not even within the particle horizon. This solution, obtained with no mathematical errors, is physically absurd. This is due to the gauge choice we made at the very beginning of this section; the choice of a gauge condition that may simplify the calculation could lead to non-sense physical results.

2.2 Conformal Newtonian gauge

Let's now consider the conformal Newtonian gauge for scalar perturbations, that is expressed, in terms of the spherical harmonics that describe the perturbation, by the condition:

$$\left(B^{(0)} Q_i^{(0)}\right)^{|i} = 0 \tag{2.22a}$$

$$\left(H_T^{(0)} Q_{ij}^{(0)}\right)^{|j} = 0 \ . \tag{2.22b}$$

This time we will adopt a different approach: starting from the perturbation in the conformal synchronous gauge we will make a gauge transformation to get the solution in the Newtonian gauge [15].

The synchronous gauge, in comoving coordinates, is described by $g_{00} = -a^2(\tau)$ and $g_{0i} = 0$. Set the residual gauge freedom, the energy density perturbation in the growing mode depends on the peculiar gravitational potential φ through the cosmological Poisson equation. Some calculations give

$$\delta_S = \frac{\tau^2}{6}\nabla^2\varphi \tag{2.23}$$

where the subscript S denotes that the quantity is the one characteristic of the synchronous gauge. We can see that the coordinate frame change did not resolve the growing problem, and the perturbations grows as infinite as the time increase.

Let us consider the gauge transformation described in equation (1.34) from the synchronous and the Newtonian gauge, and write the spatial component of ξ^μ as

$$\xi^i = \partial^i b + d^i \tag{2.24}$$

with $\partial_i d^i = 0$. The gauge transformation brings

$$A_N(\tau,\vec{x}) = \dot{\xi}^0 + \frac{\dot{a}}{a}\xi^0 \tag{2.25a}$$

$$H_{L,N}(\tau,\vec{x}) = H_{L,S}(\tau,\vec{x}) + \frac{1}{3}\nabla^2 b + \frac{\dot{a}}{a}\xi^0 \tag{2.25b}$$

$$H^{(0)}_{T,N}(\tau,\vec{x}) = H^{(0)}_{T,S}(\tau,\vec{x}) \tag{2.25c}$$

where the subscripts S and N indicate the synchronous and the Newtonian gauge respectively, the dot denotes the derivative with respect to τ and the spherical harmonics associated with the perturbation amplitudes are included in the spatial dependence of the amplitudes themselves, for simplicity of notation. The parameters ξ^0 and b can be fixed using the fact that $\xi^0 = \dot{b}$, and the property of the synchronous gauge:

$$-\frac{\tau^2}{3}(\varphi_{|ij} - \frac{1}{3}\nabla^2\varphi) + 2b|ij - \frac{2}{3}\nabla^2 b = 0 \tag{2.26}$$

that brings easily

$$b = \frac{\tau^2}{6}\varphi \tag{2.27a}$$

$$\xi^0 = \frac{\tau}{3}\varphi\ . \tag{2.27b}$$

In particular, the energy density perturbation can be calculated from:

$$\delta_N = \delta_S + \dot{\rho}_0\xi^0 \tag{2.28}$$

where ρ_0 is the energy density of the background universe, i.e. the Einstein-de Sitter model. In the Newtonian gauge the energy density perturbation is:

$$\delta_N = -2\varphi + \frac{\tau^2}{6}\nabla^2\varphi\ . \tag{2.29}$$

We can note that the quantity is changed by a factor that depends on the gravitational potential, but not on the conformal time. The problem of the growing mode that we have seen in the synchronous gauge is persistent also in the Newtonian gauge. In this example we have also practically seen that the quantity has changed with the change of the gauge.

Is becoming clearer and clearer that the quantities here defined are not of physical interest, because they change not only their value but also their dependence on time with a gauge transformation. How could such quantities be physically reliable? Moreover we have seen repeatedly that a gauge choice could lead to correct mathematical solutions which have no physical meaning, for example in the cases which we examined the energy density perturbation increases to infinity. A new approach to study cosmological perturbations is clearly required.

Chapter 3

Gauge-invariant variables

In the previous chapter we have seen how the choice of a particular gauge could lead to unphysical results. The gauge problem was overcome for the fist time by James Bardeen, who proposed the introduction of gauge-invariant variables [6]. He understood the need to express cosmological perturbation only in terms of invariant quantities, indeed they are the only quantities that have a real physical meaning: quantities whose course and meaning depend on the gauge choice could not have any physical meaning. Nevertheless these new invariant variables, to have genuine physical significance, have to be constructed from the natural variables of the problem that is matter of consideration, in this case the perturbation considered. Therefore he proposed a new approach to studying cosmological perturbations in a total covariant and unambiguous way.

In this chapter we will follow Bardeen's paper, to understand the innovation that his idea brought to the subject.

3.1 Definition of the variables

3.1.1 Scalar perturbations

Let us consider the perturbation of the metric tensor: two independent gauge-independent quantities can be constructed from the amplitudes of the perturbation, Φ_A and Φ_H:

$$\Phi_A \equiv A + \frac{1}{k}\dot{B}^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}B^{(0)} - \frac{1}{k^2}\left(\ddot{H}_T^{(0)} + \frac{\dot{a}}{a}\dot{H}_T^{(0)}\right) \tag{3.1}$$

$$\Phi_H \equiv H_L + \frac{1}{3}H_T^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}B^{(0)} - \frac{1}{k^2}\frac{\dot{a}}{a}\dot{H}_T^{(0)} \ . \tag{3.2}$$

In fact let us consider the general gauge transformation:

$$\widetilde{\tau} = \tau + T(\tau)Q^{(0)} \tag{3.3a}$$

$$\widetilde{x}^i = x^i + L^{(0)}(\tau)Q^{(0)i} \ . \tag{3.3b}$$

Therefore the amplitudes of the perturbation change as

$$\widetilde{A} = A - \dot{T} - \frac{\dot{a}}{a}T \tag{3.4a}$$
$$\widetilde{B}^{(0)} = B^{(0)} + \dot{L}^{(0)} + kT \tag{3.4b}$$
$$\widetilde{H}_L = H_L - \frac{1}{3}kL^{(0)} - \frac{\dot{a}}{a}T \tag{3.4c}$$
$$\widetilde{H}_T^{(0)} = H_T^{(0)} + kL^{(0)} \tag{3.4d}$$

Let us see if under this transformation Φ_A and Φ_H are actually invariant:

$$\begin{aligned}
\widetilde{\Phi}_A &= \widetilde{A} + \frac{1}{k}\dot{\widetilde{B}}^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}\widetilde{B}^{(0)} - \frac{1}{k^2}\left(\ddot{\widetilde{H}}_T^{(0)} + \frac{\dot{a}}{a}\dot{\widetilde{H}}_T^{(0)}\right) \\
&= A - \dot{T} - \frac{\dot{a}}{a}T + \frac{1}{k}\dot{B}^{(0)} + \frac{1}{k}\ddot{L}^{(0)} + \dot{T} + \frac{1}{k}\frac{\dot{a}}{a}B^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}\dot{L}^{(0)} + \frac{\dot{a}}{a}T \\
&\quad - \frac{1}{k^2}\ddot{H}_T^{(0)} - \frac{1}{k}\ddot{L}^{(0)} - \frac{1}{k^2}\frac{\dot{a}}{a}\dot{H}_T^{(0)} - \frac{1}{k}\frac{\dot{a}}{a}\dot{L}^{(0)} \\
&= A + \frac{1}{k}\dot{B}^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}B^{(0)} - \frac{1}{k^2}\left(\ddot{H}_T^{(0)} + \frac{\dot{a}}{a}\dot{H}_T^{(0)}\right) \\
&= \Phi_A
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\widetilde{\Phi}_H &= \widetilde{H}_L + \frac{1}{3}\widetilde{H}_T^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}\widetilde{B}^{(0)} - \frac{1}{k^2}\frac{\dot{a}}{a}\dot{\widetilde{H}}_T^{(0)} \\
&= H_L - \frac{1}{3}kL^{(0)} - \frac{\dot{a}}{a}T + \frac{1}{3}H_T^{(0)} + \frac{1}{3}kL^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}B^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}\dot{L}^{(0)} \\
&\quad + \frac{\dot{a}}{a}T - \frac{1}{k^2}\frac{\dot{a}}{a}\dot{H}_T^{(0)} - \frac{1}{k}\frac{\dot{a}}{a}\dot{L}^{(0)} \\
&= H_L + \frac{1}{3}H_T^{(0)} + \frac{1}{k}\frac{\dot{a}}{a}B^{(0)} - \frac{1}{k^2}\frac{\dot{a}}{a}\dot{H}_T^{(0)} \\
&= \Phi_H \ .
\end{aligned} \tag{3.6}$$

Neither Φ_A nor Φ_H vary. Note that there are four gauge dependent variables and two gauge functions T and $L^{(0)}$, so the independent quantities must be two.

Under the transformation (3.3a) the velocity $v^{(0)}$ changes as $\widetilde{v}^{(0)} = v^{(0)} + \dot{L}^{(0)}$, so we can define an invariant velocity as it follows:

$$v_s^{(0)} \equiv v^{(0)} - \frac{1}{k}\dot{H}_T^{(0)} \ . \tag{3.7}$$

The energy density perturbation δ is not gauge-independent, thus it must be combined with other quantities in order to be invariant. One restriction for the new variable is that it has to reduce to δ as soon as the perturbation comes inside the horizon. A first possibility is

$$\epsilon_m \equiv \delta + 3(1+w)\frac{1}{k}\frac{\dot{a}}{a}(v^{(0)} - B^{(0)}) \ , \tag{3.8}$$

that is equal to δ in every gauge in which $v^{(0)} = B^{(0)}$, so we can interpret it as the density perturbation from the reference frame of the matter.

Another invariant combination is

$$\epsilon_g \equiv \delta - 3(1+w)\frac{1}{k}\frac{\dot{a}}{a}\left(B^{(0)} - \frac{1}{k}\dot{H}_T^{(0)}\right) , \tag{3.9}$$

that is equal to δ when $B^{(0)} = \frac{1}{k}\dot{H}_T^{(0)}$, that corresponds to the Newtonian gauge.

3.1.2 Vector perturbations

Considering vector perturbations, the only gauge-invariant quantity that we can construct from the amplitudes of the perturbation is

$$\Psi \equiv B^{(1)} - \frac{1}{k}\dot{H}_T^{(1)} . \tag{3.10}$$

To test it, like we did for scalar perturbations, let us consider the general gauge transformation

$$\widetilde{x}^i = x^i + L^{(1)}(\tau)Q^{(1)i} \tag{3.11}$$

and remember that there is no ambiguity for the time coordinate. The perturbation amplitudes change then as

$$\widetilde{B}^{(1)} = B^{(1)} + \dot{L}^{(1)} \tag{3.12a}$$

$$\widetilde{H}_T^{(1)} = H_T^{(1)} + kL^{(1)} \tag{3.12b}$$

and consequently

$$\begin{aligned}\widetilde{\Psi} &= \widetilde{B}^{(1)} - \frac{1}{k}\dot{\widetilde{H}}_T^{(1)} \\ &= B^{(1)} + \dot{L}^{(1)} - \frac{1}{k}\dot{H}_T^{(1)} - \dot{L}^{(1)} \\ &= B^{(1)} - \frac{1}{k}\dot{H}_T^{(1)} \\ &= \Psi.\end{aligned} \tag{3.13}$$

One possible choice of a gauge-invariant velocity for the matter is, in analogy with the one defined in (3.7), the shear velocity

$$v_s^{(1)} \equiv v^{(1)} - \frac{1}{k}\dot{H}_T^{(1)} \tag{3.14}$$

from which we can define

$$v_c \equiv v^{(1)} - B^{(1)} = v_s^{(1)} - \Psi \tag{3.15}$$

that is the source of Ψ in Einstein's equations, and represents the velocity of the normal to time-constant hypersurfaces.

3.1.3 Tensor perturbations

No gauge transformation can be constructed from the harmonic tensor $Q^{(2)ij}$, and the direct consequence of this fact is that all the quantities considered in this case are already gauge-invariant. We can then consider $H_T^{(2)}$ and $\pi_T^{(2)}$ as the natural gauge invariant variables regarding tensor perturbations.

3.2 Perturbation equations

Now that we have defined a set of invariant variables we can study cosmological perturbations using a totally covariant form, that makes the survey more physically direct. In general, the perturbed Einstein's equations are

$$\delta G^{\mu}_{\nu} = \delta R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R = \delta T^{\mu}_{\nu} \ . \tag{3.16}$$

3.2.1 Scalar perturbations

Starting from the variation of the Ricci's tensor under scalar perturbation is possible to construct two invariant combinations of the perturbed Einstein's tensor, and through equation (3.16) relate them to the perturbed stress-energy tensor. The equations obtained are

$$2\frac{k^2 - 3K}{a^2}\Phi_H = \rho_0 \epsilon_m \tag{3.17}$$

and

$$-\frac{k^2}{a^2}(\Phi_A + \Phi_H) = P_0 \pi_T^{(0)} \tag{3.18}$$

that are express only in terms of invariant quantities. Note that for a perfect fluid equation (3.18) gives $\Phi_A = -\Phi_H$.

The dynamics of scalar perturbations is derived from the continuity equation

$$T^{\mu}_{\nu;\mu} = 0. \tag{3.19}$$

The momentum equation $T^{\mu}_{i;\mu} = 0$, elaborated in terms of invariant quantities, gives:

$$\dot{v}_s^{(0)} + \frac{\dot{a}}{a} v_s^{(0)} = k\Phi_A + \frac{k}{1+w}(c_s^2 \epsilon_m + w\eta) - \frac{2}{3}\frac{k}{1+w}\left(1 - \frac{3K}{k^2}\right) w \pi_T^{(0)} \ . \tag{3.20}$$

We can interpret $\rho_0(1+w)$ as the inertial mass per unit volume, so that the second term of the right hand side of the equation is related to the acceleration of the rest frame of matter due to the pressure-gradient force, and the third term is proportional to the acceleration due to the divergence of the anisotropic part of the stress tensor. Both terms are the corresponding acceleration times the scale factor a. Besides the first term is the gravitational acceleration in the Newtonian gauge associated with the perturbation.

The energy equation, starting from $T^{\mu}_{0;\mu} = 0$, can be obtained firstly in the conformal synchronous gauge, in which the calculation is easier. Then, having obtained an invariant formula, it can be generalized to every gauge. After some computation we get the equation

$$\dot{\Phi}_H + \frac{\dot{a}}{a}\Phi_H = -\frac{1}{2}\frac{(\rho_0+P_0)a^2}{k}v_s^{(0)} - \frac{P_0a^2}{k^2}\frac{\dot{a}}{a}\pi_T^{(0)} \tag{3.21}$$

that can be expressed also like

$$[\rho_0 \dot{a^3} \epsilon_m] = -\left(1-\frac{3K}{k^2}\right)(\rho_0+P_0)a^3kv_s^{(0)} - 2\left(1-\frac{3K}{k^2}\right)\rho_0a^2\dot{a}\pi_T^{(0)} \tag{3.22}$$

where the dot is applied over the whole quantity $[\rho_0 a^3 \epsilon_m]$. This form is more familiar, in the sense that equation (3.22) has some similarity with the special relativistic energy equation, even though it is not exactly the same.

3.2.2 Vector perturbations

Using the same approach of the previous section, we can derive the equations that describe vector perturbations. Taking the $0-0$ component of Einstein's equations we get

$$\frac{1}{2}\frac{k^2-2K}{a^2}\Psi Q_i^{(1)} = (\rho_0+P_0)v_c Q_i^{(1)} \ . \tag{3.23}$$

The equation of motion is obtained starting from the continuity equation of matter, that gives

$$\dot{v}_c = \frac{\dot{a}}{a}(3c_s^2-1)v_c + \frac{kw}{1+w}\pi_T^{(1)} \ . \tag{3.24}$$

3.2.3 Tensor perturbations

In the case of tensor perturbations there is only one equation obtainable, that is

$$\frac{1}{a^2}\left(\ddot{H}_T^{(2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(2)} + (k^2+2K)H_T^{(2)}\right) = P_0\pi_T^{(2)} \ . \tag{3.25}$$

3.3 Solutions of the perturbation equations

We are going to search a general solution for the perturbation equations. We will consider the approximation $w = c_s^2 = const$ and $K = 0$, that is well justified for perturbation on the scale of galaxies.

3.3.1 Scalar perturbations

Define the new variables:

$$\beta \equiv \frac{2}{3w+1} \tag{3.26}$$

$$x \equiv k\tau \tag{3.27}$$

$$f \equiv x^{\beta-2}\epsilon_m = \frac{2}{3}\beta^{-2}x^{\beta}\Phi_H \ . \tag{3.28}$$

Note that β ranges from 2 ($w = 0$) to $\frac{1}{2}$ ($w = 1$), and let us denote with a prime the derivative with respect to x. Bringing together equations (3.17), (3.20) and (3.21), and expressing them in terms of the new variables, we can get the equation

$$f'' + 2x^{-1}f' + \left(c_s^2 - \beta(\beta+1)x^{-2}\right)f = -x^{\beta-2}\left[w\eta - \frac{2}{3}w\pi_T^{(0)} + 2\beta x\left(x^{-2}w\pi_T^{(0)}\right)'\right] \quad (3.29)$$

The solution of this equation for an arbitrary source can be derived by constructing the Green's function. Applying the condition that the perturbation vanishes at $x = 0$, and considering the approximations for the limit $c_s x \ll 1$, i.e. outside the sound horizon, we get

$$f(x) \simeq (2\beta+1)^{-1}\left[x^{\beta}\int_0^x dy\, y^{-1}\left(\frac{2}{3}\frac{\beta+1}{2\beta-2}w\pi_T^{(0)} - w\eta\right)\right.$$
$$\left. + x^{-\beta-1}\int_0^x dy\, y^{2\beta}(-2\beta(2\beta+1)y^{-2}w\pi_T^{(0)} + w\eta)\right] . \quad (3.30)$$

Let us briefly have some comments about this result: it is composed of a growing mode and a decaying mode, in a way similar to what we have seen in section 2.1. The entropy perturbation contributes in amounts of the same order in growing and decaying mode to f, and consequently to ϵ_m and Φ_H.

If the perturbation is turned off for $x \ll 1$, by the time it reaches the particle horizon $x = 1$ the contribution to the decaying mode is negligible compared to the one of the growing mode.

The anisotropic stress contribution is of the same order of the entropy one in the growing mode, while in the decaying mode is of order x^{-2} relative to the entropy perturbation term. If the anisotropic perturbation turns off at a certain time $x_1 \ll 1$, then by the time in which it reaches the particle horizon $x = 1$ the contribute to the energy density perturbation amplitude in the growing mode is $w\pi_T^{(0)}$, while the contribute in the decaying mode is $x_1^{2\beta-1}w\pi_T^{(0)}$, very small compared to the growing one. Note that the contribution to ϵ_m is always small if $w\pi_T^{(0)} \ll 1$, while the anisotropic perturbation term can be larger than one in $\Phi_H = x^{-2}\epsilon_m$ if the perturbation is applied very early in time, i.e. $x^2 < w\pi_T^{(0)}$.

3.3.2 Vector perturbations

The solution of equation (3.24), assuming no perturbation initially and after the perturbation has been on for a sufficient amount of time, is

$$v_c \approx \frac{x}{1-2\beta}\frac{w}{1+w}\pi_T^{(1)} , \quad (3.31)$$

and after the perturbation turns off

$$v_c \propto \frac{1}{a^4(\rho_0 + P_0)} \approx x^{-2(\beta-1)} . \quad (3.32)$$

Using equation (3.23) to gain the form of Ψ we get:

$$\Psi = \frac{2}{k^2 - 2K} a^2 (\rho_0 + P_0) v_c \approx 4\beta(\beta+1) x^{-2} v_c \ . \tag{3.33}$$

Notice that even if $w\pi_T^{(1)} \ll 1$, Ψ could be larger than one: it means that it does not exist a gauge in which all the perturbations are small, hence a time-like observer cannot be at rest relative to a coordinate system in which $\dot{H}_T^{(1)} = 0$, i.e. in which the shear velocity v_s is equal to $v^{(1)}$.

3.3.3 Tensor perturbations

The solution of the tensor perturbation equation is the one that corresponds to gravitational waves of amplitude $H_T^{(2)}$, and it is not any different from the solutions found in the literature for particular gauges, because the variables in question are already invariant.

Chapter 4

Conformal Newtonian gauge

In this chapter we will briefly see how the study of cosmological perturbations changes using the variables introduced by Bardeen [13]. The most direct and intuitive example is the conformal Newtonian gauge, because in this particular gauge the potentials Φ_A and Φ_H assume a very simple and naturally physical form.

In this gauge, by definition,

$$B^{(0)} = H_T^{(0)} = 0 \ , \tag{4.1}$$

so the two potentials Φ_A and Φ_H introduced by Bardeen become:

$$\Phi_A = A \tag{4.2}$$

$$\Phi_H = H_L \ . \tag{4.3}$$

The amplitude Φ_A measures the fractional perturbation in the lapse function and represents the Newtonian spatial curvature, while Φ_H is the amplitude of the spatial metric perturbations, and plays the role of the gravitational potential in the Newtonian approximation [6][16].

The metric for scalar perturbations is then explicitly described by

$$g_{00} = -a^2(\tau)(1 + 2\Phi_A) \tag{4.4a}$$

$$g_{0i} = 0 \tag{4.4b}$$

$$g_{ij} = a^2(\tau)[1 + 2\Phi_H Q^{(0)} \delta_{ij}] \tag{4.4c}$$

where we have made explicit the spatial metric as the Minkowskian one, expressed by a Kronecker delta. Note that the metric is diagonal; this simplifies the calculations and leads to simple geodesic equations. In this gauge no gauge modes are present, so the description is not obscured in the meaning by unphysical modes [17].

Furthermore the invariant velocity $v_s^{(0)}$ introduced in (3.7) and the gauge-invariant energy density ϵ_g defined in (3.9) assume the trivial form:

$$v_s^{(0)} = v^{(0)} \tag{4.5}$$

$$\epsilon_g = \delta \ . \tag{4.6}$$

4.1 Einstein's equations

Let us now derive the perturbed Einstein's equations using the invariant approach. To construct Einstein's equations we need to find the form of the Ricci's tensor for the perturbed metric in the gauge we are considering. The Ricci's tensor is by definition:

$$R_{\mu\nu} \equiv \Gamma^{\alpha}_{\nu\mu|\alpha} - \Gamma^{\alpha}_{\alpha\mu|\nu} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\nu\mu} - \Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\alpha\mu} \tag{4.7}$$

where $\Gamma^{\alpha}_{\nu\mu}$ is the Christoffel's symbol, that is defined by

$$\Gamma^{\alpha}_{\nu\mu} \equiv \frac{1}{2} g^{\alpha\sigma}(g_{\nu\sigma|\mu} + g_{\mu\sigma|\nu} - g_{\nu\mu|\sigma}) \tag{4.8}$$

so we need to know firstly how the Christoffel's symbols vary under scalar perturbations of the metric [18]. Using the definition of the metric in this gauge we get:

$$\Gamma^0_{00} = \mathcal{H} + \dot{\Phi}_A \tag{4.9a}$$
$$\Gamma^0_{0k} = \Phi_{A|k} \tag{4.9b}$$
$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \left(2\mathcal{H}(\Phi_A - \Phi_H) - \dot{\Phi}_H\right)\delta_{ij} \tag{4.9c}$$
$$\Gamma^i_{00} = \Phi_{A|i} \tag{4.9d}$$
$$\Gamma^i_{0j} = \mathcal{H}\delta^i_j + \dot{\Phi}_H\delta^i_j \tag{4.9e}$$
$$\Gamma^i_{jk} = \Phi_{H|k}\delta^i_j + \Phi_{H|j}\delta^i_k - \Phi_{H|i}\delta_{jk} \tag{4.9f}$$

where we considered only the terms at first order in Φ_A and Φ_H and defined $\mathcal{H} \equiv \frac{\dot{a}}{a}$, that is the Hubble parameter. In the form of the perturbed connection coefficients the unperturbed quantities and the perturbations are separated: the latter are represented by the terms in which Bardeen's potentials appear, usually the second terms. Notice that in some cases the unperturbed value is zero and only the perturbation is present, i.e. in Γ^0_{0k} and Γ^i_{00}.

Using the definition of the Ricci's tensor (4.7) it is possible to calculate its form; the components of the perturbed Ricci's tensor are:

$$R_{00} = -3\dot{\mathcal{H}} - 3\ddot{\Phi}_H + \nabla^2\Phi_A + 3\mathcal{H}(\dot{\Phi}_A + \dot{\Phi}_H) \tag{4.10a}$$
$$R_{0i} = -2(\dot{\Phi}_H - \mathcal{H}\Phi_A)_{|i} \tag{4.10b}$$
$$\begin{aligned} R_{ij} = {} & (\dot{\mathcal{H}} + 2\mathcal{H}^2)\delta_{ij} + [\ddot{\Phi}_H - \nabla^2\Phi_H - \mathcal{H}(\dot{\Phi}_A - 3\dot{\Phi}_H) \\ & - (2\dot{\mathcal{H}} + 4\mathcal{H}^2)(\Phi_A - \Phi_H)]\delta_{ij} - (\Phi_H + \Phi_A)_{|ij} \, . \end{aligned} \tag{4.10c}$$

Raising an index and contracting it we get the Ricci's scalar R as:

$$\begin{aligned} R = {} & \frac{6}{a^2}(\dot{\mathcal{H}} + \mathcal{H}^2) + \frac{1}{a^2}[6\ddot{\Phi}_H - 2\nabla^2(2\Phi_H + \Phi_A) \\ & - 6\mathcal{H}(\dot{\Phi}_A - \dot{\Phi}_H) - 12(\dot{\mathcal{H}} + \mathcal{H}^2)\Phi_A] \, . \end{aligned} \tag{4.11}$$

Finally, we can construct the Einstein's tensor, whose components are:

$$G_0^0 = R_0^0 - \frac{1}{2}R = -\frac{3}{a^2}\mathcal{H}^2 + \frac{1}{a^2}(2\nabla^2\Phi_H - 6\mathcal{H}\dot{\Phi}_H + 6\mathcal{H}^2\Phi_A) \tag{4.12a}$$

$$G_i^0 = R_i^0 = \frac{2}{a^2}(\dot{\Phi}_H - \mathcal{H}\Phi_A)_{|i} = -R_0^i = -G_0^i \tag{4.12b}$$

$$\begin{aligned} G_j^i = R_j^i - \frac{1}{2}\delta_j^i R = \frac{1}{a^2}(-2\dot{\mathcal{H}} - \mathcal{H}^2)\delta_j^i + \frac{1}{a^2}[-2\ddot{\Phi}_H + \nabla^2(\Phi_A + \Phi_H) \\ + 2\mathcal{H}\dot{\Phi}_A + (4\dot{\mathcal{H}} + 2\mathcal{H}^2)\Phi_A]\delta_j^i - \frac{1}{a^2}(\Phi_H + \Phi_A)_{|ij} \ . \end{aligned} \tag{4.12c}$$

Let us observe that, like in equations (4.9), we can distinguish the unperturbed terms from the perturbation terms, in which the potentials Φ_A and Φ_H are present.

Let us consider from now on, for simplicity of calculation, only the case of a perfect fluid. The perturbation in the stress-energy tensor is then

$$\delta T_0^0 = -\epsilon_g \tag{4.13a}$$

$$\delta T_i^0 = (\rho_0 + P_0)v_{|i} = -\delta T_0^i \tag{4.13b}$$

$$\delta T_j^i = \delta P \delta_j^i \ . \tag{4.13c}$$

where δP is the perturbation of the pressure.

We can finally write the perturbed Einstein's equations:

$$\nabla^2\Phi_H - 3\mathcal{H}(\dot{\Phi}_H - \mathcal{H}\Phi_A) = -\frac{1}{2}a^2\epsilon_g \tag{4.14a}$$

$$(\dot{\Phi}_H - \mathcal{H}\Phi_A)_{|i} = \frac{1}{2}a^2(\rho_0 + P_0)v_{|i} \tag{4.14b}$$

$$-\ddot{\Phi}_H + \mathcal{H}\dot{\Phi}_A + (2\dot{\mathcal{H}} + \mathcal{H}^2)\Phi_A + \frac{1}{3}\nabla^2(\Phi_A + \Phi_H) = \frac{1}{2}a^2\delta P \delta_j^i \tag{4.14c}$$

$$-(\partial_i\partial_j + \frac{1}{3}\delta_j^i\nabla^2)(\Phi_H + \Phi_A) = 0 \tag{4.14d}$$

where we separated the spatial component δG_j^i into its trace and traceless part. Equation (4.14b) can be simplified considering that the spatial average of a perturbation is always zero. As a consequence, the equality of gradients of the two perturbations implies the equality of the perturbations themselves. Hence equation (4.14b) becomes

$$\dot{\Phi}_H - \mathcal{H}\Phi_A = \frac{1}{2}a^2(\rho_0 + P_0)v \ . \tag{4.15}$$

Equation (4.14d) gives

$$\Phi_H = -\Phi_A \tag{4.16}$$

which means that there is only one degree of freedom left, and we can identify Φ_H with the opposite of Φ_A in all the other Einstein equations, and call it simply Φ.

4.2 Matter-dominated limit

Let us see the results of Einstein's equations in the matter dominated limit. Let us recall that in this limit we can ignore the pressure, so we can assume $P_0 = w = \delta P = 0$. Equations (4.14a), (4.15) and (4.14c) assume the simpler form:

$$\nabla^2\Phi = \frac{1}{2}a^2\rho_0(\epsilon_g - 3\mathcal{H}v) \tag{4.17a}$$

$$\dot{\Phi} + \mathcal{H}\Phi = -\frac{1}{2}a^2\rho_0 v \tag{4.17b}$$

$$\ddot{\Phi} + \mathcal{H}\dot{\Phi} + (2\dot{\mathcal{H}} + \mathcal{H}^2)\Phi = 0\ . \tag{4.17c}$$

From Friedmann's equations, that describe the FLRW background universe, we can obtain the relation

$$2\dot{\mathcal{H}} + \mathcal{H}^2 = 0\ , \tag{4.18}$$

that inserted in equation (4.17c) gives the second order equation

$$\ddot{\Phi} + \mathcal{H}\dot{\Phi} = 0 \tag{4.19}$$

whose solution is

$$\Phi(\tau, \vec{x}) = C_1(\vec{x}) + C_2(\vec{x})\tau^{-5} \tag{4.20}$$

where the coefficient C_1 and C_2 depend only on the spatial coordinates. The second term is clearly the decaying mode and is proportional to τ^{-5}, while the first term is not growing any more, yet it is constant in time. The coefficients C_1 and C_2 can be determined from the initial value of Φ. Unless there are very special initial conditions, that make the value of C_1 vanishing, the decaying term soon becomes negligible compared to the constant one, and we can then write

$$\Phi(\tau, \vec{x}) = \Phi(\vec{x})\ . \tag{4.21}$$

For the matter-dominated limit of the conformal Newtonian gauge the Bardeen's potential is constant in time for scalar perturbations.

From the form of Φ and using equation (4.17a) and (4.17b) we can derive the energy density perturbation $\delta = \epsilon_g$, that is

$$\epsilon_g = -2\Phi + \frac{2}{3\mathcal{H}^2}\nabla^2\Phi\ . \tag{4.22}$$

Passing to the Fourier space we can easily obtain the behaviour of ϵ_g for superhorizon and subhorizon scales. As of superhorizon scales $k \ll \mathcal{H}$, then the energy density perturbation stays constant

$$\epsilon_g = -2\Phi\ , \tag{4.23}$$

whereas for subhorizon scales $k \gg \mathcal{H}$, so

$$\epsilon_g = -\frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi \propto \tau^2 \propto a \propto t^{\frac{2}{3}} \tag{4.24}$$

i.e. it grows proportional to the scale factor. We can see that the perturbation begins to grow when it comes inside the particle horizon. Let us also notice that at subhorizon scales general relativistic effects become negligible and we can consider a Newtonian description, in which the energy density and the velocity perturbations of the conformal Newtonian gauge become the corresponding quantities of the Newtonian description. The Bardeen's potential then can be interpreted as the Newtonian gravitational potential due to scalar density perturbations, like we have mentioned at the beginning of this chapter.

What we have reached in this formulation is an important result, as we can see that we have finally solved the gauge problem: Φ and ϵ_g are physical variables that do not change under gauge transformations, and their behaviours are of physical understanding.

Conclusions

In the study of cosmological perturbations a problem of coherence and physical meaning emerged, the so called gauge problem. General Relativity invariance under gauge transformation leads to redundant degrees of freedom, that are classically eliminated through a gauge choice. However we have seen that choosing a specific gauge instead of another one could lead to different results. In the study of the non conformal synchronous gauge and the comoving Newtonian gauge we have observed that the energy density perturbation assumes an infinite value on superhorizon scales.

To overcome this problem of non-physical meaning of the results we have followed the approach of Bardeen, who proposed the use of gauge-invariant variables. We have defined the two invariant Bardeen's potentials Φ_A and Φ_H for scalar perturbations, the potential Ψ for vector perturbations and we ascertained that the natural quantities suggested by tensor perturbations are already invariant. We have defined also gauge-invariant velocities and energy density perturbations. All these quantities are built from the natural variables that the physical perturbation suggested, so they can be referred to the corresponding non-invariant quantities, yet the study of these new variables is not misleading nor contradictory for changes among gauges.

To make more intuitive the physical meaning of the invariant variables, we have studied them specifically in the conformal Newtonian gauge, in which they assume a particularly simple form. We have consider scalar perturbations of the metric, and we have derived the perturbed Einstein's equations of a perfect fluid in terms of Φ_A and Φ_H. The solution we found for the matter dominated limit has made clear that the gauge problem has finally been solved. In fact the behaviour of the potential is neither meaningless nor unphysical any more; on the contrary it is plausible both on superhorizon and subhorizon scales.

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