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Graphs encoding properties of finite groups

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A Mamma e Papà,
perché possiate non dimenticare mai
quanto importante sia stato
quello che avete fatto per me.

Spero che siate orgogliosi
dell'uomo che sono oggi.

Grazie, vi voglio bene...

To all the people struggling,
no matter the reason.

In this period of crisis,
we have the opportunity to learn
that as in science, so in society,
the only way to solve problems
and live better
is doing it together.

Contents

Notation

A_n	The alternating group on n letters;
$\mathfrak{A}$	The class of abelian groups;
C_n	The cyclic group with n elements;
D_n	The dihedral group with $2n$ elements;
$\mathfrak{D}$	The class of groups with nilpotent derived subgroup;
$\mathrm{diam}(\Gamma)$	The diameter of the graph Γ;
$\mathrm{dist}_\Gamma(a, b)$	The distance between the vertices a and b in Γ;
$\mathfrak{F}$	A generic class (or formation) of groups;
$\mathfrak{F}\mathfrak{G}$	The product of the formations $\mathfrak{F}$ and $\mathfrak{G}$;
$\mathrm{F}(G)$	The Fitting subgroup of G;
$\phi(G)$	The Frattini subgroup of G;
$\phi_{\mathfrak{F}}(G)$	The intersection of all the $\mathfrak{F}$-maximal subgroups of G;
G	A generic finite group;
$\|G : H\|$	The index of H in G;
$\widetilde{\Gamma}_{\mathfrak{F}}(G)$	The preliminary non-$\mathfrak{F}$ graph of G;
$\Gamma_{\mathfrak{F}}(G)$	The non-$\mathfrak{F}$ graph of G;
$\mathcal{I}_{\mathfrak{F}}(G)$	The set of isolated vertices of $\widetilde{\Gamma}_{\mathfrak{F}}(G)$;
$\mathfrak{N}$	The class of nilpotent groups;
$\mathfrak{N}^t$	The class of groups with Fitting length less or equal then t;
$\mathrm{R}(G)$	The soluble radical of G;
S_n	The symmetric group on n letters;
$\mathfrak{S}$	The class of soluble groups;
$\mathfrak{S}_p$	The class of p-groups;
$\mathfrak{U}$	The class of supersoluble groups;
$\mathrm{w}\mathfrak{U}$	The class of widely supersoluble groups;
$\mathrm{Z}(G)$	The center of G;
$\mathrm{Z}_\infty(G)$	The hypercenter of G;
$x \sim y$	x and y are connected by an edge;
$x \approx y$	x and y are in the same connected component.

Chapter 1

Introduction

This thesis is about finite groups and some graphs related to them. All the groups, unless otherwise specified, are considered to be finite. The study of groups with graphs was introduced at first by Paul Erdős who defined the non-commuting graph of a group G: a graph whose vertices are elements in $G \setminus \mathrm{Z}(G)$ and two vertices are connected if the subgroup generated by the corresponding elements is not abelian. Following the steps of Erdős, at the end of the 20th Century, it was introduced the generating graph. This graph is built in the following way: take a preliminary graph $\widetilde{V}(G) = (G, E)$, where two vertices are connected if the corresponding elements generate the group G. The generating graph $V(G)$ is obtained by deleting the isolated vertices of $\widetilde{V}(G)$. This graph has been studied in the literature, obtaining many results in the case of finite groups. In the following years many such graphs have shown up in the mathematical community: for instance the non-nilpotent graph, the non-soluble graph and some generalization of the generating graph. Apart from the graphs related to generation, all the other ones are built taking a class $\mathfrak{X}$ of groups (e. g. nilpotent groups, soluble groups) with the following procedure: firstly it is defined a preliminary graph $\widetilde{\Gamma}_{\mathfrak{X}}(G) = (G, E)$, where two vertices are connected if the subgroup generated by the corresponding elements is not in the class $\mathfrak{X}$; then it is studied the graph $\Gamma_{\mathfrak{X}}(G)$ (which we call non-$\mathfrak{X}$ graph) obtained by removing the isolated vertices of $\widetilde{\Gamma}_{\mathfrak{X}}$.

From now on, with an abuse of notation, we use the words “vertex” and “element” meaning the same thing; this should create no confusion.

In the first part of the work we summarize the main results related to these different graphs present in the literature, while in the second part we obtain some original results.

In particular we dedicate a chapter to the study of the non-nilpotent graph. In [AZ10], it was made an attempt to find the best possible bound to the diameter of the non-nilpotent graph of a finite group. It was obtained a bound of 6. We improve this bound to 3 and we show that it is the best possible. Moreover we prove that two vertices outside the Fitting subgroup have distance at most 2.

After that, in the main part of the work, we try to carry out a generalization of these graphs already studied in the literature. In particular we study the graphs obtained when the class $\mathfrak{X}$ is an hereditary saturated formation $\mathfrak{F}$ of soluble groups containing all the abelian groups. We study mainly the set of isolated vertices of the preliminary graph, in particular we are interested in understanding when it is a subgroup. On this line of investigation we define *regular formations*, which are formations in which the set of isolated vertices correspond to the intersection of all the $\mathfrak{F}$-maximal subgroups and we give a characterization of them in function of the soluble strongly critical groups for $\mathfrak{F}$. We prove in fact

Theorem 1.1. *Let $\mathfrak{F}$ be an hereditary saturated formation, with $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$. Then $\mathfrak{F}$ is regular if and only if every finite group G which is soluble and strongly critical for $\mathfrak{F}$ has the property that $G/\operatorname{soc}(G)$ is cyclic.*

We give examples of regular formations, among which there are the formations of nilpotent groups and soluble groups, which actually motivated the definition of regularity.

Since there are examples of non-regular formations in which the isolated vertices form a subgroup, we weaken the definition of regularity to that of *semiregular formations.*

Semiregular formations are all the formations in which the set of isolated vertices form a subgroup for every finite group. Given a formation with some technical properties, we investigate the structure of groups of minimal order with respect to the property that the isolated vertices do not form a subgroup, if it exists.

With this approach we show that some notable formations are semiregular, in fact we prove

Theorem 1.2. *The following formations are semiregular:*

1. *the formation $\mathfrak{U}$ of the finite supersoluble groups.*
2. *the formation $\mathfrak{D} = \mathfrak{N}\mathfrak{A}$ of the finite groups with nilpotent derived subgroup.*
3. *the formation $\mathfrak{N}^t$ of the finite groups with Fitting length less or equal then t, for any $t \in \mathbb{N}$.*
4. *the formation $\mathfrak{S}_p\mathfrak{N}^t$ of the finite groups G with $G/\operatorname{O}_p(G) \in \mathfrak{N}^t$.*

Moreover, we exhibit an example of hereditary saturated formation which is not semiregular: the formation $\mathrm{w}\mathfrak{U}$ of widely supersoluble groups.

We then investigate when the non-$\mathfrak{F}$ graph is connected; we say that $\mathfrak{F}$ is *connected* if the non-$\mathfrak{F}$ graph is connected for every finite group. We prove that the formations of Theorem 1.2 are connected and that

Theorem 1.3. *Let $\mathfrak{F}$ be an hereditary saturated formation, with $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$. If $\mathfrak{F}$ is regular, then $\mathfrak{F}$ is connected.*

Finally, we investigate when the non-$\mathfrak{F}$ graph is planar and we prove:

Theorem 1.4. *Let $\mathfrak{F}$ be a 2-recognizable, hereditary, semiregular formation, with $\mathfrak{N} \subseteq \mathfrak{F}$, and let G be a finite group. Then $\Gamma_{\mathfrak{F}}(G)$ is planar if and only if either $G \in \mathfrak{F}$ or $G \cong S_3$.*

These last two theorems generalize some of the work made by Abdollahi and Zarrin in [AZ10] for the non-nilpotent graph.

The results of this chapter were proven using properties of soluble groups, since in our conditions it is possible to reduce general questions to the universe of soluble groups, thanks to the work in [GKPS06]; however it is important to say that the paper uses the Classification of Finite Simple Groups, so our proofs rely indirectly on it too.

Finally, the research carried out in this part is restricted to formations of groups with certain properties. Although the considered formations contain most of the notable classes, we are not able to give precise statements for a generic class: this leaves many open questions which could be the object of further research.

Chapter 2

Some examples

In this chapter we are going to analyze some examples of the graphs pointed out in the introduction.

1 The non-commuting graph

The non-commuting graph is one of the first ones which has shown up in literature. Define $\widetilde{\Gamma}_{\mathfrak{A}}(G) = (G, E)$, where the vertices are the elements of G and two vertices are joined if the corresponding elements do not commute. Let us define also:

$$\mathcal{I}_{\mathfrak{A}}(G) := \{g \in G : (g, x) \notin E \quad \forall x \in G\},$$

the set of isolated vertices of $\widetilde{\Gamma}_{\mathfrak{A}}(G)$. We have that

$$\mathcal{I}_{\mathfrak{A}}(G) = \{g \in G : [g, x] = 1 \quad \forall x \in G\} = \mathrm{Z}(G).$$

The non-commuting graph is the graph obtained from $\widetilde{\Gamma}_{\mathfrak{A}}(G)$, by removing the isolated vertices:

$$\Gamma_{\mathfrak{A}}(G) = (G \setminus \mathcal{I}_{\mathfrak{A}}(G), E).$$

This graph was studied in several papers, see for instance [AAM06]. We highlight here some of its main properties. Of course we consider G to be a non-abelian group, otherwise the graph is trivial. Firstly, we have

Proposition 2.1. $\Gamma_{\mathfrak{A}}(G)$ *is connected and* $\mathrm{diam}(\Gamma_{\mathfrak{A}}(G)) = 2$*.*

Proof. Let $x \in G$, the neighbors of x are the elements of $G \setminus \mathrm{C}_G(x)$. If we consider $x, y \in G \setminus \mathrm{Z}(G)$, we have that the set of vertices connected to both of them is $X := G \setminus (\mathrm{C}_G(x) \cup \mathrm{C}_G(y))$. If $\mathrm{C}_G(x)$ and $\mathrm{C}_G(y)$ are proper subgroups, their union can't be G (groups can't be union of two proper subgroups), so X is non-empty and this means that there exists z such that $x \sim z \sim y$. Hence $\mathrm{diam}(\Gamma_{\mathfrak{A}}(G)) \leq 2$ and the graph is connected. Suppose now $\mathrm{diam}(\Gamma_{\mathfrak{A}}(G)) = 1$: it means that the graph is complete. Let a be a non-central element. $[a, a^{-1}] = 1$, so $a = a^{-1} \Rightarrow a^2 = 1$, otherwise they should be connected. Moreover let b be a central element; ab is non-central, so $a^2b^2 = (ab)^2 = 1$, hence $b^2 = 1$. Every element of the group has order 2 and it is well known that groups with exponent 2 are abelian, contradiction. The diameter is exactly 2. □

We have also many other graph theoretic properties for $\Gamma_{\mathfrak{A}}(G)$, whose proofs can be found in [AAM06]:

Proposition 2.2. *$\Gamma_{\mathfrak{A}}(G)$ is Hamiltonian.*

Proposition 2.3. *$\Gamma_{\mathfrak{A}}(G)$ is planar if and only if G is isomorphic to one of S_3, D_4 or Q_8.*

Proposition 2.4. *If $\Gamma_{\mathfrak{A}}(G)$ is regular, then G is nilpotent of class at most 3 and $G = P \times A$ where A is an abelian group and P is a p-group with $\Gamma_{\mathfrak{A}}(P)$ regular.*

We end this section by recalling the question posed by Paul Erdös, which gave rise to the study of such graph, more for historical reasons than for the effective use of the result in this work.

Question 2.5. *Let G be a (not necessarily finite) non-abelian group whose non-commuting graph has no infinite clique. Is it true that $\Gamma_{\mathfrak{A}}(G)$ has a finite clique number?*

The problem was solved by B. H. Neumann in [Neu76] with the following:

Theorem 2.6. *Let G be a (not necessarily finite) non-abelian group, whose non-commuting graph has no infinite clique. Then $|G : \mathrm{Z}(G)|$ is finite and $\omega(\Gamma_{\mathfrak{A}}(G))$ is finite.*

2 The non-soluble graph

In this section we analyze another graph which, in this case, is related to the property of solvability, see [HR13]. We define $\widetilde{\Gamma}_{\mathfrak{S}}(G) := (G, E)$ the graph, built exactly as above, with just a difference: two vertices are connected if the subgroup generated by them is not soluble. Let $\mathcal{I}_{\mathfrak{S}}(G)$ be the set of isolated vertices of $\widetilde{\Gamma}_{\mathfrak{S}}(G)$. The non soluble graph is then

$$\Gamma_{\mathfrak{S}}(G) := (G \setminus \mathcal{I}_{\mathfrak{S}}(G), E).$$

In [GKPS06], the set $\mathcal{I}_{\mathfrak{S}}(G)$ is proved to be actually a subgroup: the soluble radical of G. To study this graph, the notion of *solvabilizer* was introduced, which has an analogous in the other graphs, and it is a useful tool. The solvabilizer of an element x in G is the set

$$\mathrm{sol}_G(x) := \{y \in G : \langle x, y \rangle \text{ is soluble}\},$$

which represents the set of vertices which are not connected to x. We define also the solvabilizer of a subset S of G as the set of elements of G which generate a soluble subgroup with every element of S; in other words:

$$\mathrm{sol}_G(S) := \bigcap_{s \in S} \mathrm{sol}_G(s).$$

With this definition we have

$$\mathrm{sol}_G(G) = \mathcal{I}_{\mathfrak{S}}(G) = \mathrm{R}(G).$$

Finally, we state a very important result, that we are going to use quite often:

Theorem 2.7. [GKPS06, Theorem 6.4] *Let $x, y \notin \mathcal{I}_{\mathfrak{S}}(G)$, then there exists $s \in G$ such that $x \sim s \sim y$. This means, in particular, that $\Gamma_{\mathfrak{S}}(G)$ is connected and* $\operatorname{diam} \Gamma_{\mathfrak{S}}(G) \leq 2$.

3 The generating graph

In this section we talk about one of the most important graphs associated to finite groups, which is different from the other ones investigated in this thesis, but it is widely studied, since it helps to understand the way elements of a group generate the group itself. Moreover, it offers many tools to prove theorems about the graphs of our interest.

Let us define $\widetilde{V}(G) := (G, E)$ to be the graph on the elements of G, with $g, h \in G$ connected if $\langle g, h \rangle = G$; the generating graph of G is the graph $V(G) := (G \setminus \mathcal{I}, E)$, where $\mathcal{I}$ is the set of isolated vertices of $\widetilde{V}(G)$.

Firstly, we give a proposition about the set of isolated vertices $\mathcal{I}$.

Proposition 2.8. *The Frattini subgroup is always a subset of $\mathcal{I}$, which is not always a subgroup.*

Proof. The first part of the statement follows from the fact that if $H \leq G$ and $G = \phi(G)H$, then $H = G$. The second part follows considering the group $S_3 \times S_3$, in which $|\mathcal{I}| = 15$, and since the order of the group is 36, it can't be a subgroup. □

The generating graph is quite a mysterious object for a generic group, but in some cases we have more information, for example when the group is a finite non-abelian simple group, the generating graph is connected, with diameter equal to 2. Moreover we have the following

Theorem 2.9. [CL13, Luc17] *Let G be a finite soluble group, then*

- *$V(G)$ is connected;*
- $\operatorname{diam}(V(G)) \leq 3$.

The problem of whether the graph is connected or not for a generic finite group is still open.

Chapter 3

The non-nilpotent graph

In this chapter we are going to analyze the non-nilpotent graph $\Gamma_{\mathfrak{N}}(G)$ of a group G. One of the main reference about this graph is [AZ10]. We will now recall some of the main known facts. First of all, we consider the set of isolated vertices: it turns out that $\mathcal{I}_{\mathfrak{N}}(G) = \mathrm{Z}_\infty(G)$, the hypercenter of G. The hypercenter of a finite group G is defined to be the greatest element of the upper central series. We define also

$$\mathrm{nil}_G(x) := \{y \in G : \langle x, y\rangle \in \mathfrak{N}\}$$

which is called the *nilpotentizer* of x and

$$\mathrm{nil}_G(S) := \bigcap_{s\in S} \mathrm{nil}_G(s),$$

where we omit the subscript G when there is no possibility of confusion. With this notation we have $\mathrm{nil}(G) = \mathrm{Z}_\infty(G)$. In their paper, Abdollahi and Zarrin proved

Proposition 3.1. $\Gamma_{\mathfrak{N}}(G)$ *is connected with diameter at most* 6.

They also proved that in many cases, for example when the nilpotentizers are subgroups for every element of the group, the diameter is equal to 2. This led them to conjecture that $\mathrm{diam}(\Gamma_{\mathfrak{N}}(G)) = 2$ for every finite group G, but this is false. Andrew Davis, Julie Kent and Emily McGovern, three students of the Missouri State University, investigated the non-nilpotent graph of the semidirect product $\langle a\rangle \rtimes S_4$, where $|a|$ is odd and $a^\sigma = a^{\mathrm{sgn}(\sigma)}$ for every $\sigma \in S_4$. Let $g = a^i\sigma \in G$. If $\langle a, g\rangle$ is not nilpotent, then $\sigma \notin A_4$, while if $\langle (1,2)(3,4), g\rangle$ is not nilpotent then σ is a 3-cycle. This implies that the vertices a and $(1,2)(3,4)$ do not have a common neighbor in the graph $\Gamma_{\mathfrak{N}}(G)$, so $\mathrm{dist}_{\Gamma_{\mathfrak{N}}(G)}(a, (1,2)(3,4)) \geq 3$. However this is the worst possible situation. Indeed our main result is the following.

Theorem 3.2. *If G is a finite group, then* $\mathrm{diam}(\Gamma_{\mathfrak{N}}(G)) \leq 3$.

Our second result says that if $\mathrm{dist}_{\Gamma_{\mathfrak{N}(G)}}(x,y)=3$, then at least one of the two elements x and y belong to the Fitting subgroup $\mathrm{F}(G)$ of G.

Theorem 3.3. *If G is a finite group and $x,y\notin \mathrm{F}(G)$, then $\mathrm{dist}_{\Gamma_{\mathfrak{N}(G)}}(x,y)\leq 2$.*

1 Proofs of Theorems 3.2 and 3.3

Throughout this section, we will say that g is a p-element, where p is a prime, meaning that the order of g is a power of p.

Lemma 3.4. *Let G be a finite group and let $g\in G$. If H is a subgroup of G and $g\notin H$, then there exist a prime p and a positive integer n such that g^n is a p-element and $g^n\notin H$.*

Proof. Let $|g|=p_1^{n_1}\cdots p_r^{n_r}$, with $p_1,\dots,p_r$ distinct primes. For $1\leq i\leq r$, set $m_i=\prod_{j\neq i}p_j^{n_j}$. Since $\langle g^{m_1},\dots,g^{m_r}\rangle=\langle g\rangle\not\leq H$, there exists $i\in\{1,\dots,r\}$ such that the p_i-elements $g_i^{m_i}$ does not belong to H. □

Lemma 3.5. *Let p be a prime and x a p-element of a finite group G. If $x\notin \mathrm{Z}_\infty(G)$, then there exist a prime $q\neq p$ and a q-element y such that $\langle x,y\rangle$ is not nilpotent.*

Proof. Suppose, by contradiction, that $\langle x,y\rangle$ is nilpotent for every q-element y and every prime $q\neq p$. Let $K:=\langle Q\mid Q\in \mathrm{Syl}_q(G), q\neq p\rangle$. Then K is a normal subgroup of G and $K\leq \mathrm{C}_G(x)$. Moreover $|G/K|$ is a p-group, so if P is a Sylow subgroup of G containing x, then $G=KP$. Let g be an arbitrary element of G and write $g=ab$, with $a\in K$ and $b\in P$. Then $\langle x,x^g\rangle=\langle x,x^{ab}\rangle=\langle x,x^b\rangle\leq P$. By a theorem of R. Baer (see for example [Isa08, 2.12]) $x\in \mathrm{O}_p(G)$. In particular, if z is a p-element of G, then $\langle x,z\rangle$ is a p-group. Now let g be an arbitrary element of G and write $g=\alpha\beta$, where α is a p-element, β a p'-element and $[\alpha,\beta]=1$. We have $\langle x,g\rangle=\langle x,\alpha\beta\rangle=\langle x,\alpha,\beta\rangle=\langle x,\alpha\rangle\langle\beta\rangle\cong\langle x,\alpha\rangle\times\langle\beta\rangle$, since $\beta\in K\leq \mathrm{C}_G(x)$. But, as we noticed before, $\langle x,\alpha\rangle$ is a p-group, and so $\langle x,g\rangle\cong\langle x,\alpha\rangle\times\langle\beta\rangle$ is nilpotent. This implies $x\in \mathrm{nil}(G)=\mathrm{Z}_\infty(G)$, against our assumption. □

Proof of Theorem 3.2. Let x_1,x_2 be two distinct elements of $G\setminus \mathrm{Z}_\infty(G)$. By Lemma 3.4, there exist two positive integers m_1, m_2 and two primes p_1,p_2 such that $x_1^{m_1}$ is a p_1-element, $x_2^{m_2}$ is a p_2-element and $x_1^{m_1},x_2^{m_2}\notin \mathrm{Z}_\infty(G)$. By Lemma 2, there exist two primes $q_1\neq p_1$ and $q_2\neq p_2$, a q_1-element z_1 and a q_2-element z_2 such that $\langle x_1^{m_1},z_1\rangle$ and $\langle x_2^{m_2},z_2\rangle$ are not nilpotent. If $\langle z_1,z_2\rangle$ is not nilpotent, then (x_1,z_1,z_2,x_2) is a path in the graph $\Gamma_{\mathfrak{N}}(G)$ joining x_1 and x_2 and $\mathrm{dist}_{\Gamma_{\mathfrak{N}(G)}}(x_1,x_2)\leq 3$. So we may assume that $\langle z_1,z_2\rangle$ is nilpotent. If $q_1\neq q_2$, then $\langle z_1,z_2\rangle=\langle z_1z_2\rangle$. This implies that $\langle x_1,z_1z_2\rangle=\langle x_1,z_1,z_2\rangle$ and $\langle x_2,z_1z_2\rangle=\langle x_2,z_1,z_2\rangle$ are not nilpotent, and (x_1,z_1z_2,x_2) is a path in $\Gamma_{\mathfrak{N}}(G)$. If $q_1=q_2$, then $q_1\neq p_2$. If $\langle x_1,z_2\rangle$ is not nilpotent, then

(x_1, z_2, x_2) is a path in $\Gamma_{\mathfrak{N}}(G)$. Otherwise $\langle x_1^{m_1}, z_2\rangle \leq \langle x_1, z_2\rangle$ is nilpotent, hence $\langle x_1^{m_1}, z_2\rangle = \langle x_1^{m_1} z_2\rangle$ and $(x_1, z_1, x_1^{m_1} z_2, x_2)$ is a path in $\Gamma_{\mathfrak{N}}(G)$. □

Lemma 3.6. *Let G be a finite group. If $x, y \notin \mathrm{F}(G)$ and $\gcd(|x|, |y|) = 1$, then $\mathrm{dist}_{\Gamma_{\mathfrak{N}}(G)}(x, y) \leq 2$.*

Proof. Assume, by contradiction, $\mathrm{dist}_{\Gamma_{\mathfrak{N}}(G)}(x, y) > 2$. Since $x, y \notin \mathrm{F}$ by [Isa08, 2.12] there exist g and h in G such that $\langle x, x^g\rangle$ and $\langle y, y^h\rangle$ (and consequently also $\langle x, x^{g^{-1}}\rangle$ and $\langle y, y^{h^{-1}}\rangle$ are not nilpotent). If $\langle x^g, y^{h^{-1}}\rangle$ were nilpotent, then $[x^g, y^{-h}] = 1$ and $(x, x^g y^{h^{-1}}, y)$ would a path in $\Gamma_{\mathfrak{N}}(G)$. So $\langle x^g, y^{h^{-1}}\rangle$ (and consequently also $\langle x, y^{h^{-1}g^{-1}}\rangle$ and $\langle x^{gh}, y\rangle$) is not nilpotent. We prove, by induction on n, that $\langle x^{(gh)^n}, y\rangle$ is not nilpotent, for every $n \in \mathbb{N}$. Indeed, assuming that $\langle x^{(gh)^n}, y\rangle$ is not nilpotent, then $\langle x^{(gh)^n}, y^{(gh)^{-1}}\rangle$ is also non nilpotent, otherwise $[x^{(gh)^n}, y^{(gh)^{-1}}] = 1$ and $(x, x^{(gh)^n} y^{(gh)^{-1}}, y)$ would be a path in $\Gamma_{\mathfrak{N}}(G)$. But then, taking $n = |gh|$, we get that $\langle x, y\rangle$ is not nilpotent and $\mathrm{dist}_{\Gamma_{\mathfrak{N}}(G)}(x, y) = 1$, against our assumption. □

Lemma 3.7. *Let G be a finite soluble group and let p be a prime. If $g_1, g_2 \in G\backslash \mathrm{nil}(G)$ are p-elements such that $\mathrm{dist}_{\Gamma_{\mathfrak{N}}(G)}(g_1, g_2) > 2$, then $g_1, g_2 \in \mathrm{O}_p(G)$.*

Proof. Let $C_1 := \mathrm{C}_G(g_1)$ and $C_2 := \mathrm{C}_G(g_2)$. By Lemma 3.5, there exist a prime $q \neq p$ and a q-element x such that $\langle g_1, x\rangle$ is not nilpotent. Let K be a p-complement in G containing x. It must be $K \subseteq C_1 \cup C_2$ (indeed if $y \in K \setminus (C_1 \cup C_2)$, then (g_1, y, g_2) would be a path in $\Gamma_{\mathfrak{N}}(G)$). Hence either $K \leq C_1$ or $K \leq C_2$. However $x \in K \setminus C_1$, so we must exclude the first possibility and conclude $K \leq C_2$. In particular $|G : C_2|$ is a p-power and therefore $G = C_2 P$, being P a Sylow p-subgroup of G containing g_2. As in the proof of Lemma 3.5, applying Baer's theorem we conclude $g_2 \in \mathrm{O}_p(G)$. With the same argument we can prove $g_1 \in \mathrm{O}_p(G)$. □

Proof of Theorem 3.3. By Lemma 3.4, we may assume that there exists two primes p and q such that x is a p-element and y is a q-element. By Lemma 3.6, we may assume $p = q$. If $x, y \notin \mathrm{R}(G)$ (where $\mathrm{R}(G)$ denotes the soluble radical of G), then, by [GKPS06, Theorem 6.4], there exists $z \in G$ such that $\langle x, z\rangle$ and $\langle y, z\rangle$ are not soluble. Hence (x, z, y) is a path in $\Gamma_{\mathfrak{N}}(G)$ and $\mathrm{dist}_{\Gamma_{\mathfrak{N}}(G)}(x, y) \leq 2$. So it is not restrictive to assume $x \in \mathrm{R}(G)$. In particular $H = \mathrm{R}(G)\langle y\rangle$ is a soluble group containing x, so by Lemma 3.7, either $\mathrm{dist}_{\Gamma_{\mathfrak{N}}(G)}(x, y) \leq \mathrm{dist}_{\Gamma_{\mathfrak{N}}(H)}(x, y) \leq 2$ or $x, y \in \mathrm{F}(H)$. However in the second case, we would have $x \in \mathrm{F}(H) \cap \mathrm{R}(G) \leq \mathrm{F}(\mathrm{R}(G)) \leq \mathrm{F}(G)$. □

Chapter 4

A generalization: non-$\mathfrak{F}$ graphs

1 Summary of the results

In this chapter we are going to build the general framework pointed out in the introduction.

Consider $\mathfrak{F}$ to be a class of groups. A group (resp. subgroup) is called an $\mathfrak{F}$-*group* (resp. $\mathfrak{F}$-*subgroup*) if it belongs to $\mathfrak{F}$. We say that $\mathfrak{F}$ is *hereditary* whenever if $G \in \mathfrak{F}$ and $H \leq G$, then $H \in \mathfrak{F}$. If $\mathfrak{F}$ is hereditary, it is interesting to consider the intersection $\phi_{\mathfrak{F}}(G)$ of all maximal $\mathfrak{F}$-subgroups of G, that is, the subgroups which are maximal with respect to being an $\mathfrak{F}$-group. It turns out that if $\mathfrak{F} \in \{\mathfrak{A}, \mathfrak{N}, \mathfrak{S}\}$, then $\phi_{\mathfrak{F}}(G) = \mathcal{I}_{\mathfrak{F}}(G)$ for any finite group G. Indeed $\mathcal{I}_{\mathfrak{A}}(G) = \mathrm{Z}(G)$, $\mathcal{I}_{\mathfrak{N}}(G) = \mathrm{Z}_{\infty}(G)$ [AZ10, Proposition 2.1], $\mathcal{I}_{\mathfrak{S}}(G) = \mathrm{R}(G)$ [GKPS06, Theorem 1.1]. This motivates the following definition: we say that $\mathfrak{F}$ is *regular* if $\mathfrak{F}$ is hereditary and $\phi_{\mathfrak{F}}(G) = \mathcal{I}_{\mathfrak{F}}(G)$ for every finite group G.

The first question that we address in this chapter is how to characterize the hereditary saturated formations that are regular. Recall that a formation $\mathfrak{F}$ is a class of groups which is closed under taking homomorphic images and subdirect products. The second condition ensures the existence of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of each group G, that is, the smallest normal subgroup of G whose factor group is in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if $G \in \mathfrak{F}$ whenever the Frattini factor $G/\Phi(G)$ is in $\mathfrak{F}$. A group G is *critical* for $\mathfrak{F}$ (or $\mathfrak{F}$-*critical*) if $G \notin \mathfrak{F}$ and every proper subgroup of G lies in $\mathfrak{F}$, while a group G is *strongly critical* for $\mathfrak{F}$ if $G \notin \mathfrak{F}$ and every proper subgroup and proper quotient of G lies in $\mathfrak{F}$.

Theorem 4.1. *Let $\mathfrak{F}$ be an hereditary saturated formation, with $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$. Then $\mathfrak{F}$ is regular if and only if every finite group G which is soluble and strongly critical for $\mathfrak{F}$ has the property that $G/\operatorname{soc}(G)$ is cyclic.*

It follows from Theorem 4.1 that a formation is not in general regular. For example, if $\mathfrak{U}$ is the formation of the finite supersoluble groups, then there exists a strongly critical group G for $\mathfrak{U}$ such that $\mathrm{soc}(G)$ is an elementary abelian group of order 25 and $G/\mathrm{soc}(G)$ is isomorphic to the quaternion group Q_8. It is an interesting question to see if and when $\mathcal{I}_{\mathfrak{F}}(G)$ is a subgroup of G.

Consider the class $\mathfrak{F}$ of finite groups in which normality is transitive. The group $G =: \langle a, b, c \mid a^5 = 1, b^5 = 1, c^4 = 1, [a, b] = 1, a^c = a^2, b^c = b^3 \rangle$ is critical for $\mathfrak{F}$ (see [Rob69]). Then $\langle a, g\rangle$ and $\langle b, g\rangle$ are proper subgroups for every $g \in G$, so they belong to the class, while $\langle ab, y\rangle = G$ does not belong to the class. Thus $a, b \in \mathcal{I}_{\mathfrak{F}}(G)$ but $ab \notin \mathcal{I}_{\mathfrak{F}}(G)$. So in general $\mathcal{I}_{\mathfrak{F}}(G)$ is not a subgroup of G.

We say that a formation $\mathfrak{F}$ is *semiregular* if $\mathcal{I}_{\mathfrak{F}}(G) \leq G$ for any finite group G. In Section 4 we will investigate the structure of a group G which is minimal with respect to the property that $\mathcal{I}_{\mathfrak{F}}(G)$ is not a subgroup. To state our result we need to recall another definition: we say that $\mathfrak{F}$ is *2-recognizable* whenever a group G belongs to $\mathfrak{F}$ if all 2-generated subgroups of G belong to $\mathfrak{F}$.

Theorem 4.2. *Let $\mathfrak{F}$ be an hereditary saturated formation, with $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$. Assume that $\mathfrak{F}$ is 2-recognizable and not semiregular and let G be a finite group of minimal order with respect to the property that $\mathcal{I}_{\mathfrak{F}}(G)$ is not a subgroup of G. Then G is a primitive monolithic soluble group. Moreover, if $N = \mathrm{soc}(G)$ and S is a complement of N in G, then the following hold.*

1. *$N = \mathrm{soc}(G) = G^{\mathfrak{F}}$.*
2. *$N\langle s\rangle \in \mathfrak{F}$ for every $s \in S$; in particular S is not cyclic.*
3. *if $n \in N$ and $s \in S$, then $ns \in \mathcal{I}_{\mathfrak{F}}(G)$ if and only if $N\langle s, t\rangle \in \mathfrak{F}$ for all $t \in S$; in particular $ns \in \mathcal{I}_{\mathfrak{F}}(G)$ if and only if $s \in \mathcal{I}_{\mathfrak{F}}(G)$.*
4. *Suppose that $\mathfrak{F}$ is locally defined by the formation function f and, for every prime p, let $\overline{f(p)}$ be the formation of the finite groups X with the property that $X/\mathrm{O}_p(X) \in f(p)$. If $K \leq S$, we have that $NK \in \mathfrak{F}$ if and only if $K \in \overline{f(p)}$, in particular $\mathcal{I}_{\mathfrak{F}}(G) = N\mathcal{I}_{\overline{f(p)}}(S)$, where p is the unique prime dividing $|N|$.*

As an application of the previous theorem we will prove.

Theorem 4.3. *The following formations are semiregular:*

1. *the formation $\mathfrak{U}$ of the finite supersoluble groups.*
2. *the formation $\mathfrak{D} = \mathfrak{N}\mathfrak{A}$ of the finite groups with nilpotent derived subgroup.*

3. *the formation* $\mathfrak{N}^t$ *of the finite groups with Fitting length less or equal then* t, *for any* $t \in \mathbb{N}$.

4. *the formation* $\mathfrak{S}_p\mathfrak{N}^t$ *of the finite groups* G *with* $G/\operatorname{O}_p(G) \in \mathfrak{N}^t$.

Moreover, we show that the hereditary saturated formation of widely supersoluble groups is not semiregular.

We will say that a formation $\mathfrak{F}$ is *connected* if the graph $\Gamma_{\mathfrak{F}}(G)$ is connected for any finite group G. In Section 7 we consider the case when $\mathfrak{F}$ is a 2-recognizable hereditary saturated semiregular formation with $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$. In particular we investigate the structure of a group G of minimal order with the property that $\Gamma_{\mathfrak{F}}(G)$ is not connected (when $\mathfrak{F}$ is not connected) and we use this information to prove the following result.

Theorem 4.4. *Let* $\mathfrak{F}$ *be an hereditary saturated formation, with* $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$. *If* $\mathfrak{F}$ *is regular, then* $\mathfrak{F}$ *is connected.*

A corollary of this result is [AZ10, Theorem 5.1], stating that the non-nilpotent graph $\Gamma_{\mathfrak{N}}(G)$ is connected for any finite group G. Moreover our approach allows to prove:

Theorem 4.5. *If* $\mathfrak{F} \in \{\mathfrak{U}, \mathfrak{D}, \mathfrak{S}_p\mathfrak{N}^t, \mathfrak{N}^t\}$, *then* $\mathfrak{F}$ *is connected.*

Recall that a graph is said to be embeddable in the plane, or *planar*, if it can be drawn in the plane so that its edges intersect only at their ends. Abdollahi and Zarrin proved that if G is a finite non-nilpotent group, then the non-nilpotent graph $\Gamma_{\mathfrak{N}}(G)$ is planar if and only if $G \cong S_3$ (see [AZ10, Theorem 6.1]). We generalize this result proving:

Theorem 4.6. *Let* $\mathfrak{F}$ *be a 2-recognizable, hereditary, semiregular formation, with* $\mathfrak{N} \subseteq \mathfrak{F}$, *and let* G *be a finite group. Then* $\Gamma_{\mathfrak{F}}(G)$ *is planar if and only if either* $G \in \mathfrak{F}$ *or* $G \cong S_3$.

2 Some preliminary results

This section contains some auxiliary results, that will be needed in our proofs.

Definition 4.7. *Let* G *be a finite group. We denote by* $V(G)$ *the subset of* G *consisting of the elements* x *with the property that* $G = \langle x, y\rangle$ *for some* y.

Proposition 4.8. *Let* G *be a primitive monolithic soluble group. Let* $N = \operatorname{soc}(G)$ *and* H *a core-free maximal subgroup of* G. *Given* $1 \neq h \in H$ *and* $n \in N$, $hn \in V(G)$ *if and only if* $h \in V(H)$.

Proof. Clearly if $hn \in V(G)$, then $h \in V(H)$. Conversely assume that $h \in V(H)$ and let $n \in N$. There exists $k \in H$ such that $\langle h, k\rangle = H$. For any $m \in N$, let $H_m := \langle hn, km\rangle$. Since $H_mN = \langle h, k\rangle N = G$, either $H_m = G$ or

H_m is a complement of N in G. In particular, if we assume, by contradiction, $hn \notin V(G)$, then H_m is a complement of N in G for any $m \in G$, and consequently $H_m = H^{g_m}$ for some $g_m \in G$. If $H_{m_1} = H_{m_2}$ then $m_1^{-1}m_2 = (km_1)^{-1}(km_2) \in H_{m_1} \cap N = 1$ so $m_2 = m_1$. Since $N_G(H) = H$, H has precisely $|G : H| = |N|$ conjugates in G and therefore $\{H_m \mid m \in N\}$ is the set of all the conjugates in G. This implies $1 \neq hn \in \bigcap_{g\in G} H^g = \mathrm{Core}_G(H) = 1$, a contradiction. □

Lemma 4.9. *Let $\mathfrak{F}$ be a saturated formation with $\mathfrak{F} \subseteq \mathfrak{S}$ and let G be a finite group. Suppose $G \notin \mathfrak{F}$ but every proper quotient is in $\mathfrak{F}$. Then either $\mathrm{R}(G) = 1$ or G is a primitive monolithic soluble group and $\mathrm{soc}(G) = G^{\mathfrak{F}}$.*

Proof. If $\mathrm{R}(G) \neq 1$, we have $G/\mathrm{R}(G) \in \mathfrak{F}$, hence $G/\mathrm{R}(G)$ is soluble, which implies that G is soluble. If G contains two different minimal normal subgroups, N_1 and N_2, then $G = G/(N_1 \cap N_2) \leq G/N_1 \times G/N_2 \in \mathfrak{F}$, against our assumption. So $\mathrm{soc}(G)$ is the unique minimal normal subgroup of G. Moreover $G/\mathrm{soc}(G) \in \mathfrak{F}$, hence $\mathrm{soc}(G) = G^{\mathfrak{F}}$. Finally, since $\mathfrak{F}$ is a saturated formation and $G \notin \mathfrak{F}$, it must be $\phi(G) = 1$, so G is a primitive monolithic soluble group. □

The following is immediate.

Lemma 4.10. *Let $g, h \in G$ and $N \trianglelefteq G$.*

(a) If gN and hN are adjacent vertices of $\Gamma_{\mathfrak{F}}(G/N)$, then g and h are adjacent vertices of $\Gamma_{\mathfrak{F}}(G)$.

(b) If $g \in \mathcal{I}_{\mathfrak{F}}(G)$, then $gN \in \mathcal{I}_{\mathfrak{F}}(G/N)$.

(c) $\mathcal{I}_{\mathfrak{F}}(G)^{\sigma} = \mathcal{I}_{\mathfrak{F}}(G)$ for every $\sigma \in \mathrm{Aut}(G)$.

Proposition 4.11. [Ski11, Theorem A] *Let $\mathfrak{F}$ be a saturated formation. Let $H \leq G$ and $N \trianglelefteq G$. Then*

(a) If $H \in \mathfrak{F}$, then $H\phi_{\mathfrak{F}}(G) \in \mathfrak{F}$;

(b) If $N \trianglelefteq \phi_{\mathfrak{F}}(G)$, then $\phi_{\mathfrak{F}}(G)/N = \phi_{\mathfrak{F}}(G/N)$.

3 Proof of Theorem 4.1

Let $\mathfrak{F}$ be an hereditary saturated formation, with $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$.

First we claim that $\phi_{\mathfrak{F}}(G) \subseteq \mathcal{I}_{\mathfrak{F}}(G)$. Since $\mathfrak{F}$ contains all the cyclic groups, by Proposition 4.11 (a), $\langle x\rangle\, \phi_{\mathfrak{F}}(G) \in \mathfrak{F}$ for any $x \in G$. The conclusion follows from the fact that $\mathfrak{F}$ is hereditary.

Suppose that $\mathfrak{F}$ is regular and let G be a soluble strongly critical group for $\mathfrak{F}$. By Lemma 4.9, G is a primitive monolithic soluble group. Moreover, since G is critical for $\mathfrak{F}$, all the maximal subgroups of G are in $\mathfrak{F}$, and therefore $\mathcal{I}_{\mathfrak{F}}(G) = \phi_{\mathfrak{F}}(G) = \phi(G) = 1$. Let $N = \operatorname{soc}(G)$ and S a complement of N in G. Fix $1 \neq n \in N$. Since $n \notin \mathcal{I}_{\mathfrak{F}}(G)$, $\langle n, g\rangle \notin \mathfrak{F}$ for some $g \in G$. Since G is $\mathfrak{F}$-critical, it must be $\langle n, g\rangle = G$ and therefore G/N is cyclic.

Conversely, suppose that $\mathfrak{F}$ is not regular and every soluble strongly critical group G for $\mathfrak{F}$ is such that $G/\operatorname{soc}(G)$ is cyclic. Let G be a smallest finite group such that $\phi_{\mathfrak{F}}(G) \subset \mathcal{I}_{\mathfrak{F}}(G)$. Of course $G \notin \mathfrak{F}$, otherwise $G = \phi_{\mathfrak{F}}(G) = \mathcal{I}_{\mathfrak{F}}(G)$. Let $x \in \mathcal{I}_{\mathfrak{F}}(G) \setminus \phi_{\mathfrak{F}}(G)$ and let H be an $\mathfrak{F}$-maximal subgroup of G which does not contain x.

Step 1. $G = \langle x, H\rangle$.

Proof. Suppose, by contradiction, $\langle x, H\rangle < G$. Then $x \in \mathcal{I}_{\mathfrak{F}}(\langle x, H\rangle) = \phi_{\mathfrak{F}}(\langle x, H\rangle)$, hence, by Proposition 4.11 (a), $\langle x, H\rangle = \phi_{\mathfrak{F}}(\langle x, H\rangle)H \in \mathfrak{F}$, against the fact that H is an $\mathfrak{F}$-maximal subgroup of G. □

Step 2. *If* $1 \neq M \trianglelefteq G$*, then* $G/M \in \mathfrak{F}$.

Proof. By Lemma 4.10 and the minimality of G, $xM \in \mathcal{I}_{\mathfrak{F}}(G/M) = \phi_{\mathfrak{F}}(G/M)$, hence $G/M = \langle xM, HM/M\rangle = \phi_{\mathfrak{F}}(G/M)HM/M \in \mathfrak{F}$, since $HM/M \cong H/(M \cap H) \in \mathfrak{F}$. □

Step 3. *G is a primitive monolithic soluble group and* $\operatorname{soc}(G) = G^{\mathfrak{F}}$.

Proof. From Step 2 we are in the hypotheses of Lemma 4.9. If $\mathrm{R}(G) = 1$, by [GKPS06, Theorem 6.4], for every $1 \neq g_1 \in G$ there exist $g_2 \in G$ such that $\langle g_1, g_2\rangle$ is not soluble, and then $\langle g_1, g_2\rangle \notin \mathfrak{F}$, since $\mathfrak{F}$ contains only soluble groups. So, $\mathcal{I}_{\mathfrak{F}}(G) = 1$, hence $\phi_{\mathfrak{F}}(G) = 1$, which means $\mathcal{I}_{\mathfrak{F}}(G) = \phi_{\mathfrak{F}}(G)$, against the assumptions on G. □

Let $N = \operatorname{soc}(G)$, S a complement of N in G and write $x = \bar{n}\bar{s}$ with $\bar{n} \in N, \bar{s} \in S$.

Step 4. *There exists* $1 \neq n^* \in N \cap \mathcal{I}_{\mathfrak{F}}(G)$.

Proof. We may assume $\bar{s} \neq 1$ (otherwise $x = \bar{n} \in N \cap \mathcal{I}_{\mathfrak{F}}(G)$) and $\bar{s} \notin V(S)$ (otherwise, by Proposition 4.8, $\langle x, g\rangle = G \notin \mathfrak{F}$ for some $g \in G$ and $x = \bar{n}\bar{s} \notin \mathcal{I}_{\mathfrak{F}}(G)$). Since $\mathrm{C}_G(N) = N$, there exists $m \in N$ such that $x^m \neq x$. We claim that $n^* = [m, x^{-1}] \in N \cap \mathcal{I}_{\mathfrak{F}}(G)$. Indeed let $g \in G$. Since $\bar{s} \notin V(S)$, $K := \langle x, x^m, g\rangle = \langle x, n^*x, g\rangle = \langle \bar{n}\bar{s}, n^*\bar{n}\bar{s}, g\rangle \leq N\langle \bar{s}, g\rangle < G$. In particular, again by the minimality of G, $x, x^m \in \mathcal{I}_{\mathfrak{F}}(K) = \phi_{\mathfrak{F}}(K)$, hence $K = \phi_{\mathfrak{F}}(K)\langle g\rangle$ and, since $\langle g\rangle \in \mathfrak{F}$, $K \in \mathfrak{F}$. Since $\langle n^*, g\rangle \leq K$, we conclude $\langle n^*, g\rangle \in \mathfrak{F}$. □

Step 5. *S is not cyclic.*

Proof. Suppose, by contradiction, $S = \langle s \rangle$. Since N is an irreducible S-module and $n^* \neq 1$, we have $\langle n^*, s \rangle = G$. However $n^* \in \mathcal{I}_{\mathfrak{F}}(G)$, so this would imply $G \in \mathfrak{F}$. □

Step 6. *$N \subseteq \mathcal{I}_{\mathfrak{F}}(G)$.*

Proof. Suppose, by contradiction, that there exist $m \in N$ and $g \in G$ such that $\langle g, m \rangle \notin \mathfrak{F}$. This implies $K := N \langle g \rangle \notin \mathfrak{F}$. By the previous step, $K < G$. By Lemma 4.10, $(n^*)^s \in \mathcal{I}_{\mathfrak{F}}(G)$ for any $s \in S$. So in particular $X = \{(n^*)^s \mid s \in S\} \subseteq \mathcal{I}_{\mathfrak{F}}(K)$. However, by the minimality of G, $\mathcal{I}_{\mathfrak{F}}(K) = \phi_{\mathfrak{F}}(K)$ is a subgroup of G, so $\langle X \rangle = N \leq \phi_{\mathfrak{F}}(K)$ and consequently $K = \phi_{\mathfrak{F}}(K)\langle g \rangle \in \mathfrak{F}$. □

Step 7. *G is a strongly critical group for $\mathfrak{F}$.*

Proof. By Step 2, we just need to prove that every maximal subgroup of G is in $\mathfrak{F}$. Notice that $S \cong G/N \in \mathfrak{F}$, and so does every conjugate of S. The other maximal subgroups of G are of the form $K := NM$, with M maximal in S. In particular, by the minimality of G, $\mathcal{I}_{\mathfrak{F}}(K) = \phi_{\mathfrak{F}}(K)$, and, by the previous step, $N \leq \phi_{\mathfrak{F}}(K)$. Hence $K = \phi_{\mathfrak{F}}(K)M \in \mathfrak{F}$, since $M \in \mathfrak{F}$. □

Finally, G is a soluble strongly critical group for $\mathfrak{F}$, so $G/N \cong S$ is cyclic, but we excluded this possibility in Step 5. We have a contradiction, so $\mathfrak{F}$ must be regular.

4 Proof of Theorem 4.2

To prove the theorem we need the following lemma.

Lemma 4.12. *Suppose that $\mathfrak{F}$ is a 2-recognizable formation. If $\mathcal{I}_{\mathfrak{F}}(G)$ is a subgroup of G and $G = \mathcal{I}_{\mathfrak{F}}(G)\langle g \rangle$ for some $g \in G$, then $G \in \mathfrak{F}$.*

Proof. Let x be an arbitrary element of G. We have $x = ig^\alpha$ for some $i \in \mathcal{I}_{\mathfrak{F}}(G)$ and $\alpha \in \mathbb{N}$. Moreover $\langle g, ig^\alpha \rangle = \langle g, i \rangle \in \mathfrak{F}$, since $i \in \mathcal{I}_{\mathfrak{F}}(G)$. Hence $g \in \mathcal{I}_{\mathfrak{F}}(G)$, so $G = \mathcal{I}_{\mathfrak{F}}(G)$ and, because $\mathfrak{F}$ is 2-recognizable, $G \in \mathfrak{F}$. □

Proof of the Theorem 4.2. Let $x, y \in \mathcal{I}_{\mathfrak{F}}(G)$ such that $xy \notin \mathcal{I}_{\mathfrak{F}}(G)$. There exists $g \in G$ such that $\langle xy, g \rangle \notin \mathfrak{F}$. Notice that the minimality property of G implies $G = \langle x, y, g \rangle$. Let M be a non-trivial normal subgroup of G and set $I/M := \mathcal{I}_{\mathfrak{F}}(G/M) \trianglelefteq G/M$. By Lemma 4.10, $xM, yM \in \mathcal{I}_{\mathfrak{F}}(G/M)$. Since $G = \langle x, y, g \rangle$, we have $\langle gM \rangle I/M = G/M$. By Lemma 4.12, $G/M \in \mathfrak{F}$. So we are in the hypotheses of Lemma 4.9. If $\mathrm{R}(G) = 1$, then, as in the proof of Theorem 4.1, $\mathcal{I}_{\mathfrak{F}}(G) = 1$, in contradiction with the assumption that $\mathcal{I}_{\mathfrak{F}}(G)$ is not a subgroup of G. So G is a primitive monolithic soluble group and $N = \mathrm{soc}(G) = G^{\mathfrak{F}}$.

We will show now that there is an element $1 \neq n^* \in N \cap \mathcal{I}_{\mathfrak{F}}(G)$. We write x in the form $x = \bar{n}\bar{s}$, with $\bar{n} \in N$ and $\bar{s} \in S$. If $\bar{s} = 1$, then

$x \in N \cap \mathcal{I}_{\mathfrak{F}}(G)$ and we are done (notice that $xy \notin \mathcal{I}_{\mathfrak{F}}(G)$ implies $x \neq 1$). Suppose $\bar{s} \neq 1$. Since $G \notin \mathfrak{F}$, $x \notin V(G)$, hence $\bar{s} \notin V(S)$ by Proposition 4.8. Since $\mathrm{C}_N(x) \neq N$, there exist $m \in N$ such that $x^m \neq x$. We claim that $n^* := [m, x^{-1}] \in N \cap \mathcal{I}_{\mathfrak{F}}(G)$. Indeed let $g \in G$. Since $\bar{s} \notin V(S)$, $K := \langle x, x^m, g\rangle = \langle x, n^*x, g\rangle = \langle \bar{n}\bar{s}, n^*\bar{n}\bar{s}, g\rangle \leq N\langle \bar{s}, g\rangle < G$. In particular $x, x^m \in \mathcal{I}_{\mathfrak{F}}(K)$ and $K = \mathcal{I}_{\mathfrak{F}}(K)\langle g\rangle$ and therefore $K \in \mathfrak{F}$ by Lemma 4.12.

We prove now that $N \subseteq \mathcal{I}_{\mathfrak{F}}(G)$. As in Step 6 of the proof of Theorem 4.1, assume by contradiction that $\langle g, m\rangle \notin \mathfrak{F}$, for some $m \in N$ and $g \in G$. Setting $K := N\langle g\rangle$, it follows, with the same argument, that $N \leq \mathcal{I}_{\mathfrak{F}}(K)$ and consequently $K = \mathcal{I}_{\mathfrak{F}}(K)\langle g\rangle \in \mathfrak{F}$ by Lemma 4.12, a contradiction.

Let s be an arbitrary element of S and let $H := N\langle s\rangle$. Since $N \subseteq \mathcal{I}_{\mathfrak{F}}(G) \cap H \subseteq \mathcal{I}_{\mathfrak{F}}(H)$, we deduce that $H \in \mathfrak{F}$ from Lemma 4.12. This proves (2).

Let now $n \in N$ and $s \in S$. If $ns \in \mathcal{I}_{\mathfrak{F}}(G)$, then $ns \notin V(G)$ and therefore $s \notin V(S)$ by Proposition 4.8. Let t be an arbitrary element of S and set $H := N\langle s, t\rangle < G$. Since $H < G$, by the minimality of G, $\mathcal{I}_{\mathfrak{F}}(H)$ is a subgroup of G, and therefore $N\langle s\rangle \leq \mathcal{I}_{\mathfrak{F}}(H)$, and consequently $H = \mathcal{I}_{\mathfrak{F}}(H)\langle t\rangle$ and $H \in \mathfrak{F}$ by Lemma 4.12. If, on the contrary, $ns \notin \mathcal{I}_{\mathfrak{F}}(G)$, then there exist $n^* \in N$ and $s^* \in S$ such that $\langle n^*s^*, ns\rangle \notin \mathfrak{F}$, hence $N\langle s, s^*\rangle \notin \mathfrak{F}$. This proves (3).

Finally, we prove (4). Let $K \leq S$. Suppose $H := NK \in \mathfrak{F}$. Let U/V be a p-chief factor of H with $U \leq N$. Since $H \in \mathfrak{F}$, we have $\mathrm{Aut}_H(U/V) = H/\mathrm{C}_H(U/V) \in f(p)$; moreover, since N is abelian, $N \leq \mathrm{C}_H(U/V)$, so $\mathrm{C}_H(U/V) = N\,\mathrm{C}_K(U/V)$ and hence $\mathrm{Aut}_K(U/V) \cong \mathrm{Aut}_H(U/V) \in f(p)$. Let $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = N$ with N_i/N_{i-1} a chief factor of H for every i. Since N is a p-group, $\mathrm{Aut}_K(N_i/N_{i-1}) \in f(p)$ for every i, so $K/T \in f(p)$ with $T := \bigcap_{i=1}^{t} \mathrm{C}_K(N_i/N_{i-1})$. Since $\mathrm{C}_T(N) \leq \mathrm{C}_K(N) = 1$, T is a p-group, hence $K^{f(p)} \leq T \leq \mathrm{O}_p(K)$ and $K \in \overline{f(p)}$. Conversely, suppose $K \in \overline{f(p)}$. Let $1 = N_0 \trianglelefteq \cdots \trianglelefteq N_t = N = NK_0 \trianglelefteq \cdots \trianglelefteq NK_s = NK = H$ be a chief series of H and denote by $\mathrm{F}(H)$ the Fitting subgroup of H. If $1 \leq i \leq t$, then $\mathrm{Aut}_H(N_i/N_{i-1})$ is an epimorphic image of $H/\mathrm{F}(H)$, since $\mathrm{F}(H) \leq \mathrm{C}_H(N_i/N_{i-1})$. On the other hand, $\mathrm{F}(H) = N\,\mathrm{O}_p(K)$, hence $H/\mathrm{F}(H) \cong K/\mathrm{O}_p(K) \in f(p)$, and so $\mathrm{Aut}_H(N_i/N_{i-1}) \in f(p)$. Consider now $\mathrm{Aut}_H(NK_j/NK_{j-1})$ for $1 \leq j \leq a$ and let q be the prime dividing $|NK_j/NK_{j-1}|$. Then we have $H/\mathrm{C}_H(NK_j/NK_{j-1}) \cong K/\mathrm{C}_K(K_j/K_{j-1}) = \mathrm{Aut}_K(K_j/K_{j-1}) \in f(q)$, since $NK_j/NK_{j-1} \cong K_j/K_{j-1}$ is a chief factor of K and $K \in \mathfrak{F}$. So H satisfies all the local conditions, and then it is in $\mathfrak{F}$. □

5 Proof of Theorem 4.3

Proposition 4.13. *The formation $\mathfrak{U}$ of finite supersoluble groups is semiregular.*

Proof. The formation $\mathfrak{U}$ is 2-recognizable since every $\mathfrak{U}$-critical group is 2-

generated (see for instance [BH09, Example 1]). Assume by contradiction that $\mathfrak{U}$ is not semiregular and let G be a group of minimal order with respect to the property that $\mathcal{I}_{\mathfrak{U}}(G)$ is not a subgroup. We can apply Theorem 4.2. Let $N = \mathrm{soc}(G)$: we have $|N| = p^k$ for a prime p and some k. Let $q \neq p$ be another prime divisor of the order of a complement S of N in G and choose $s \in S$ with $|s| = q$. By Theorem 4.2, $N\langle s\rangle \in \mathfrak{U}$. Applying Maschke's Theorem, N can be decomposed into a direct sum of irreducible submodules and, since $N\langle s\rangle$ is supersoluble, these submodules must have order p. So s acts faithfully on a cyclic group of order p, hence q divides $p-1$ and in particular $q < p$. If $p \mid |S|$, then p would be the greatest prime divisor of $|S|$. Since $S \in \mathfrak{U}$, the Sylow p-subgroup of S is normal in S. However, since S acts faithfully and irreducibly on the finite p-group N, $\mathrm{O}_p(S) = 1$. This implies $\gcd(|N|, |S|) = 1$ and since $N\langle s\rangle \in \mathfrak{U}$ for every $s \in S$, the exponent of S divides $p-1$. The local definition $f(p)$ of $\mathfrak{U}$ is the formation of abelian group with exponent dividing $p-1$, therefore, since p does not divide $|S|$, $NK \in \mathfrak{U}$ if and only if K is abelian, hence $\mathcal{I}_{\mathfrak{U}}(G) = N\,\mathrm{Z}(S)$ is a subgroup of G, so we reached a contradiction. □

Proposition 4.14. *The formation $\mathfrak{D}$ of the finite groups with nilpotent derived subgroup is semiregular.*

Proof. The $\mathfrak{D}$-critical groups are 2-generated (see for instance [BH09, Example 2]), so $\mathfrak{D}$ is 2-recognizable. Suppose by contradiction it is not semiregular and let G be a minimal example of group such that $\mathcal{I}_{\mathfrak{D}}(G)$ is not a subgroup. We can apply Theorem 4.2. Let $N = \mathrm{soc}(G)$ and S a complement of N. We will prove that if $H \leq S$, then $NH \in \mathfrak{D}$ if and only if H is abelian. Since $\mathfrak{D}$ has local screen f with $f(q)$ the formation of the abelian groups for every prime q, if H is abelian, then $NH \in \mathfrak{D}$. On the other hand, suppose $NH \in \mathfrak{D}$. Let $1 = N_0 \trianglelefteq \cdots \trianglelefteq N_l = N$ be a composition series of N as H-module. Let $V_i := N_i/N_{i-1}$ and $C_i := \mathrm{C}_H(V_i)$. For every $1 \leq i \leq l$, we have that $H/C_i \cong \mathrm{Aut}_{NH}(V_i)$ is abelian, since V_i is a chief factor of a group in $\mathfrak{D}$. Then we have that H/T is abelian, with $T := \bigcap_{i=1}^{l} C_i$. Therefore $H' \leq T$. Since $\mathrm{C}_T(N) \leq \mathrm{C}_H(N) = 1$, T is a p-group, but $|S'|$ is not divisible by p (otherwise, since S' is nilpotent, we would have $\mathrm{O}_p(S) \neq 1$), so $H' \leq T \cap S' = 1$ and H is abelian. Hence $\mathcal{I}_{\mathfrak{D}}(G) = N\,\mathrm{Z}(S)$, a contradiction. □

Let $\mathfrak{N}^t$ the formations of finite groups with Fitting length less or equal then t. It is a 2-recognizable, saturated formation [BH09, Example 3]. As an immediate application of Theorem 4.2, we prove its semiregularity by proving that the formation $\overline{f(p)} = \mathfrak{S}_p\mathfrak{N}^{t-1}$ is semiregular for every prime p. We will need two preliminary lemmas.

Lemma 4.15. *$\mathfrak{S}_p\mathfrak{N}$ is regular for every prime p.*

Proof. Let $G = N \rtimes S$ be a strongly-critical group for $\mathfrak{S}_p\mathfrak{N}$. The socle N of G is a q-group. If $q = p$, then, since $S \cong G/N \in \mathfrak{S}_p\mathfrak{N}$ and $\mathrm{O}_p(S) = 1$, if follows $S \in \mathfrak{N}$ and $G \in \mathfrak{S}_p\mathfrak{N}$, so it must be $q \neq p$. If $K < S$, then $NK \in \mathfrak{S}_p\mathfrak{N}$. Since $\mathrm{C}_S(N) = 1$, we deduce $\mathrm{O}_p(NK) = 1$, hence $NK \in \mathfrak{N}$, which implies that NK is a q-group (otherwise $\mathrm{C}_K(N) \neq 1$). We have then that all proper subgroups of S are q-groups, but S itself is not a q-group, so S must be cyclic of order a prime $r \neq q$. We deduce from Theorem 4.1 that $\mathfrak{S}_p\mathfrak{N}$ is regular. □

Lemma 4.16. *$\mathfrak{S}_p\mathfrak{N}^t$ is a 2-recognizable saturated formation for every t and every prime p.*

Proof. The formation $\mathfrak{S}_p\mathfrak{N}^t$ is saturated (see [DH11, IV, 3.13 and 4.8]). We prove by induction on t that $\mathfrak{S}_p\mathfrak{N}^t$ is a 2-recognizable. We have seen in Lemma 4.15 that $\mathfrak{S}_p\mathfrak{N}$ is 2-recognizable for every prime p. Let $t \neq 1$ and let G be a group of minimal order with respect to the property that every 2-generated subgroup of G is in $\mathfrak{S}_p\mathfrak{N}^t$ but G is not. Clearly G is strongly critical for $\mathfrak{S}_p\mathfrak{N}^t$, so, by Lemma 4.9, $G = N \rtimes S$, where $N = \mathrm{soc}(G)$ is an elementary abelian group of prime power order and $S \in \mathfrak{S}_p\mathfrak{N}^t$. If N is a p-group, then $G \in \mathfrak{S}_p\mathfrak{N}^t$, hence N is a q-group with $q \neq p$. If $K < S$, then $NK \in \mathfrak{S}_p\mathfrak{N}^t$. Since $\mathrm{C}_K(N) = 1$, it must be $\mathrm{O}_p(NK) = 1$ so $NK \in \mathfrak{N}^t$. Moreover the Fitting subgroup $\mathrm{F}(NK)$ of NK coincides with $\mathrm{O}_q(NK) = N\,\mathrm{O}_q(K)$ and therefore $K \in \mathfrak{S}_q\mathfrak{N}^{t-1}$, so S is critical for $\mathfrak{S}_q\mathfrak{N}^{t-1}$. Since, by induction, $\mathfrak{S}_q\mathfrak{N}^{t-1}$ is 2-recognizable, the group S is 2-generated. By Proposition 4.8, G itself is 2-generated and hence $G \in \mathfrak{S}_p\mathfrak{N}^t$, a contradiction. □

Proposition 4.17. *$\mathfrak{S}_p\mathfrak{N}^t$ is semiregular for every t and every prime p.*

Proof. We prove by induction on t that $\mathfrak{S}_p\mathfrak{N}^t$ is semiregular for every t. By Lemma 4.15 we may assume $t > 1$. Suppose by contradiction that $\mathfrak{S}_p\mathfrak{N}^t$ is not semiregular and let G be a minimal example of group such that $\mathcal{I}_{\mathfrak{S}_p\mathfrak{N}^t}(G)$ is not a subgroup. We can apply Theorem 4.2. Let $N = \mathrm{soc}(G)$ and S a complement of N. Since $S \in \mathfrak{S}_p\mathfrak{N}^t$, if N were a p-group, then G would be in $\mathfrak{S}_p\mathfrak{N}^t$, hence N is a q-group with $q \neq p$. Let now $s, t \in S$ and $K := \langle s, t\rangle$: since $\mathrm{F}(NK) = N\,\mathrm{O}_q(K)$, we have $NK \in \mathfrak{S}_p\mathfrak{N}^t$ if and only if $NK \in \mathfrak{N}^t$, if and only if $K \in \mathfrak{S}_q\mathfrak{N}^{t-1}$. Hence by induction we conclude that $\mathcal{I}_{\mathfrak{S}_p\mathfrak{N}^t}(G) = N\mathcal{I}_{\mathfrak{S}_q\mathfrak{N}^{t-1}}(S)$ is a subgroup, a contradiction. □

Proposition 4.18. *$\mathfrak{N}^t$ is semiregular for every t.*

Proof. Since $\overline{f(p)} = \mathfrak{S}_p\mathfrak{N}^{t-1}$, the statement follows from Theorem 4.2 and Proposition 4.15. □

6 w$\mathfrak{U}$ is not semiregular

In this section we exhibit an example of an hereditary saturated formation which is not semiregular.

Definition 4.19. *We say that $H \leq G$ is $\mathbb{P}$-subnormal in G if either $H = G$ or there is a chain of subgroups*

$$H = H_0 < H_1 < \cdots < H_n = G$$

such that $|H_r : H_{r-1}|$ is a prime number for every $r = 1, \ldots, n$.

Consider the formation $\mathrm{w}\mathfrak{U}$, which is the class of groups with every Sylow subgroups $\mathbb{P}$-subnormal in the group. For further reference see [VVT10]. In the paper it is proven that

Theorem 4.20. $\mathrm{w}\mathfrak{U}$ *is an hereditary saturated formation with $\mathfrak{A} \subseteq \mathrm{w}\mathfrak{U} \subseteq \mathfrak{S}$, with local screen $f(p) = \{G \in \mathfrak{S} : \mathrm{Syl}(G) \subseteq \mathfrak{A}(p-1)\}$, where $\mathfrak{A}(p-1)$ is the formation of the abelian groups with exponent dividing $p-1$.*

We prove that

Proposition 4.21. $\mathrm{w}\mathfrak{U}$ *is 2-recognizable.*

Proof. It is immediate from the fact that $\mathfrak{U}$ is 2-recognizable and that every $\mathrm{w}\mathfrak{U}$-critical group is also a $\mathfrak{U}$-critical group, see [VVT10, Theorem 2.9]. □

Theorem 4.22. $\mathrm{w}\mathfrak{U}$ *is not semiregular.*

Proof. Consider the group $S := P \rtimes \langle \iota \rangle$, where $P = \langle x, y \rangle$ is the unique non-abelian group of order 27 and exponent 3 and ι is an involution such that $x^\iota = x^{-1}$ and $y^\iota = y^{-1}$. The group S acts irreducibly in a unique way on $N := \mathbb{F}_7^3$ and we will show that $\mathcal{I}_{\mathrm{w}\mathfrak{U}}(G)$, with $G := N \rtimes S$, is not a subgroup. If g is an element of order 2, the order of $K_h := \langle g, h \rangle$ divides $7^3 3^2 2$ for every $h \in G$, so K_h has abelian Sylow subgroups. Therefore, if $K_h = NH$, with $H \leq S$, we have that $\mathrm{Syl}(H) \subseteq \mathfrak{A}(6)$. Since $S \in \mathrm{w}\mathfrak{U}$, we conclude $K_h \in \mathrm{w}\mathfrak{U}$ with the same reasoning of Theorem 4.2 (4). So, all the elements of order 2 are isolated, then they are in $\mathcal{I}_{\mathrm{w}\mathfrak{U}}(G)$, but the subgroup generated by all these elements is the whole group, which is not in $\mathrm{w}\mathfrak{U}$ and if $\mathcal{I}_{\mathrm{w}\mathfrak{U}}(G)$ was a subgroup, we would have a contradiction since $\mathrm{w}\mathfrak{U}$ is 2-recognizable. □

7 Connectedness of $\Gamma_{\mathfrak{F}}$

In this section we study for which formations the graph $\Gamma_{\mathfrak{F}}(G)$ is connected for every finite group G. In the spirit of the previous sections we will build, under the additional assumption that $\mathfrak{F}$ is semiregular, a smallest group G such that $\Gamma_{\mathfrak{F}}(G)$ is not connected. First we need a preliminary lemma.

Lemma 4.23. *Let G be a 2-generated finite soluble group, with $G \notin \mathfrak{F}$. If $x, y \in V(G)$, then x and y belong to the same connected component of $\Gamma_{\mathfrak{F}}(G)$.*

Proof. Consider the graph $\Delta(G)$ whose vertices are the elements of $V(G)$ and in which g_1, g_2 are adjacent if and only if $\langle g_1, g_2\rangle = G$. If G is soluble then $\Delta(G)$ is a connected graph (see [BBE06, Theorem 1]). The conclusion follows from the fact that $\Delta(G)$ is a subgraph of $\Gamma_{\mathfrak{F}}(G)$. □

Theorem 4.24. *Let* $\mathfrak{F}$ *be a 2-recognizable, hereditary, saturated formation, with* $\mathfrak{A} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$. *Assume that* $\mathfrak{F}$ *is semiregular and suppose that there exists a finite group* G *such that* $\Gamma_{\mathfrak{F}}(G)$ *is not connected. If* G *has minimal order with respect to this property, then* G *is a primitive monolithic soluble group,* $N = \operatorname{soc}(G) = G^{\mathfrak{F}}$ *and* $N \subseteq \mathcal{I}_{\mathfrak{F}}(G)$. *Moreover, the same statements of point (2-4) of Theorem 4.2 hold. With the same notation, we have also that* $\Gamma_{\overline{f(p)}}(S)$ *is not connected.*

Given a finite group X, we will write $x_1 \sim x_2$ to denote that x_1 and x_2 are two adjacent vertices of $\Gamma_{\mathfrak{F}}(X)$ and $x_1 \approx x_2$ if x_1 and x_2 belong to the same connected component of $\Gamma_{\mathfrak{F}}(X)$. We divide the proof in the following steps.

Step 1. *G is a primitive monolithic soluble group and* $N = \operatorname{soc}(G) = G^{\mathfrak{F}}$.

Proof. Suppose there exists $1 \neq M \trianglelefteq G$ such that $G/M \notin \mathfrak{F}$. Set $I/M := \mathcal{I}_{\mathfrak{F}}(G/M) \trianglelefteq G/M$ and let $a_1M, a_2M \notin I/M$. We have $a_1M \approx a_2M$ by minimality of G. Since, by Lemma 4.10 (a), $g_1M \sim g_2M$ implies $g_1 \sim g_2$, we can "lift" a path from a_1M to a_2M in $\Gamma_{\mathfrak{F}}(G/M)$ to a path from a_1 to a_2 in $\Gamma_{\mathfrak{F}}(G)$, so $a_1 \approx a_2$. So there exists a unique connected component of $\Gamma_{\mathfrak{F}}(G)$, say Ω, containing $G \setminus I$. If $I \in \mathfrak{F}$, then every element of $I \setminus \mathcal{I}_{\mathfrak{F}}(G)$ must be adjacent to an element of $G \setminus I$, so $I \setminus \mathcal{I}_{\mathfrak{F}}(G) \subseteq \Omega$. But this implies $\Omega = G \setminus \mathcal{I}_{\mathfrak{F}}(G)$, and consequently $\Gamma_{\mathfrak{F}}(G)$ is connected. Therefore $I \notin \mathfrak{F}$. Since $\mathfrak{F}$ is 2-recognizable, this implies $\mathcal{I}_{\mathfrak{F}}(I) < I$. Let H be a maximal subgroup of G containing I. Since $\Gamma_{\mathfrak{F}}(H)$ is connected, there exists a unique connected component of $\Gamma_{\mathfrak{F}}(G)$, say Δ, containing $H \setminus \mathcal{I}_{\mathfrak{F}}(H)$. Of course $I \setminus \mathcal{I}_{\mathfrak{F}}(I) \subseteq H \setminus \mathcal{I}_{\mathfrak{F}}(H)$, so $I \setminus \mathcal{I}_{\mathfrak{F}}(I) \subseteq \Delta$. Recall that $G \setminus I \subseteq \Omega$. Moreover if $x \in \mathcal{I}_{\mathfrak{F}}(I) \setminus \mathcal{I}_{\mathfrak{F}}(G)$, then $x \sim y$ for some $y \in G \setminus I$, so $\mathcal{I}_{\mathfrak{F}}(I) \setminus \mathcal{I}_{\mathfrak{F}}(G) \subseteq \Omega$. If $\Delta \cap \Omega \neq \emptyset$, then $\Delta = \Omega = G \setminus \mathcal{I}_{\mathfrak{F}}(G)$ and $\Gamma_{\mathfrak{F}}(G)$ is connected. So we may assume $\Delta \cap \Omega = \emptyset$, and consequently $(H \setminus \mathcal{I}_{\mathfrak{F}}(H)) \cap (H \setminus I) = \emptyset$, i.e. $H = I \cup \mathcal{I}_{\mathfrak{F}}(H)$. Since $H \notin \mathfrak{F}$ and $\mathfrak{F}$ is 2-recognizable, $\mathcal{I}_{\mathfrak{F}}(H) \neq H$, and consequently $H = I$. If $g \in G \setminus I$, then $G = \langle g\rangle I$, so $G/M = \langle gM\rangle I/M = \langle gM\rangle \mathcal{I}_{\mathfrak{F}}(G/M)$ and, by Lemma 4.12, $G/M \in \mathfrak{F}$, a contradiction. So all the proper factors of G are in $\mathfrak{F}$ and we may use Lemma 4.9. If $\mathrm{R}(G) = 1$, then $\mathcal{I}_{\mathfrak{F}}(G) = 1$. Let $a, b \in G$, both different from 1. By [GKPS06, Theorem 6.4] there is a path in $\Gamma_{\mathfrak{S}}(G)$ from a to b. This path is also a path in $\Gamma_{\mathfrak{F}}(G)$ since $H \notin \mathfrak{S}$ implies $H \notin \mathfrak{F}$ for every group H. So if $\mathrm{R}(G) = 1$, then $\Gamma_{\mathfrak{F}}(G)$ is connected. Hence we conclude that G is a primitive monolithic soluble group and $N = \operatorname{soc}(G) = G^{\mathfrak{F}}$. □

Step 2. $N \subseteq \mathcal{I}_{\mathfrak{F}}(G)$.

Proof. Since $\mathcal{I}_{\mathfrak{F}}(G) \trianglelefteq G$ and N is the unique minimal normal subgroup, if $\mathcal{I}_{\mathfrak{F}}(G) \neq 1$, then $N \subseteq \mathcal{I}_{\mathfrak{F}}(G)$. Hence we may assume by contradiction that $\mathcal{I}_{\mathfrak{F}}(G) = 1$. Let S be a complement of N in G. Suppose that $S = \langle s \rangle$ is cyclic. Since S is a maximal subgroup of G, $\langle g, s \rangle = G$ for any $g \notin \langle s \rangle$, hence there exists a connected component Λ of $\Gamma_{\mathfrak{F}}(G)$ containing s and $G \backslash \langle s \rangle$. Moreover, every nontrivial element of S, being non-isolated in $\Gamma_{\mathfrak{F}}(G)$, is adjacent to some element of $G \backslash S$, so $\Lambda = G \backslash \{1\}$ and $\Gamma_{\mathfrak{F}}(G)$ is connected, a contradiction. So we may assume that S is not cyclic. Take now $n_1, n_2 \in N \setminus \{1\}$ and for $i \in \{1, 2\}$ let $M_i < S$ such that $n_i \notin \mathcal{I}_{\mathfrak{F}}(NM_i)$ (this is possible since S is not cyclic). We have $N_i := N \cap \mathcal{I}_{\mathfrak{F}}(NM_i) < N$, so $N_1 \cup N_2 \neq N$ and there exists $n \in N \setminus (N_1 \cup N_2)$. We have then $n_1 \approx n$ in $\Gamma_{\mathfrak{F}}(NM_1)$ and $n_2 \approx n$ in $\Gamma_{\mathfrak{F}}(NM_2)$, therefore $n_1 \approx n_2$ in $\Gamma_{\mathfrak{F}}(G)$. Hence there exists a connected component Π of $\Gamma_{\mathfrak{F}}(G)$ containing $N \setminus \{1\}$. Let now $g = ns$ be an arbitrary element of $G \setminus N$. First assume $g \notin V(G)$. Since $N \not\leq \mathrm{C}_G(g)$, there exists $n^* \in N \setminus \{n\}$ with the property that $g = (n^* s)^x$ for some $x \in G$. We claim that $g \in \Pi$. Since $n^* n^{-1} \neq 1$, there exists $\bar{g} = \bar{n}\bar{s}$ such that $\bar{g} \sim n^* n^{-1}$. Set $H := N\langle s, \bar{s} \rangle$ (it is a proper subgroup of G, since for Proposition 4.8, $s \notin V(S)$). If $g \notin \mathcal{I}_{\mathfrak{F}}(H)$, then $ns \approx n^* n^{-1}$ (since $\Gamma_{\mathfrak{F}}(H)$ is connected) and then $g \in \Pi$. Assume $g \in \mathcal{I}_{\mathfrak{F}}(H)$. We have $n^* s \notin \mathcal{I}_{\mathfrak{F}}(H)$, (otherwise, since $\mathcal{I}_{\mathfrak{F}}(H)$ is a subgroup, $(n^* s)(ns)^{-1} = n^* n^{-1} \in \mathcal{I}_{\mathfrak{F}}(H)$), but then $n^* s \approx n^* n^{-1}$ in $\Gamma_{\mathfrak{F}}(H)$ and consequently $n^* s \in \Pi$. This implies $g = (n^* s)^x \in \Pi^x = \Pi$ (notice that $\Pi^x = \Pi$ since $N \setminus \{1\} \in \Pi \cap \Pi^x$). Suppose now $g \in V(G)$. Choose $n_1, n_2 \in N$ and $t \in S$ such that $n_2 \sim n_1 t$ and set $H := N \langle s, t \rangle$. If $H = G$, then $t \in V(S)$ and consequently $n_1 t \in V(G)$. Since G is soluble, it follows from Lemma 4.23 that $g \approx n_1 t \approx n_2$ and $g \in \Pi$. If $H < G$, then $ms \notin \mathcal{I}_{\mathfrak{F}}(H)$ for some $m \in N$ (otherwise $N \leq \mathcal{I}_{\mathfrak{F}}(H)$). By Proposition 4.8, $ms \in V(G)$ and, again by Lemma 4.23, $g \approx ms$. Moreover, since $\Gamma_{\mathfrak{F}}(H)$ is connected, $ms \approx n_2$. So $g \approx n_2$ and therefore $g \in \Pi$. We reached in this way the conclusion that $\Gamma_{\mathfrak{F}}(G)$ is connected, against the assumptions on G. □

Step 3. *Statements (2-4) of Theorem 4.2 hold.*

Proof. We can use the same argument of the proof of Theorem 4.2. □

Step 4. $\Gamma_{\overline{f(p)}}(S)$ *is not connected.*

Proof. Suppose that $\Gamma_{\overline{f(p)}}(S)$ is connected. Let $s, t \in S$ such that $s \sim t$ in $\Gamma_{\overline{f(p)}}(S)$. We claim that $ns \approx mt$ for every $n, m \in N$. Suppose $\langle s, t \rangle = S$. By Proposition 4.8 $ns, mt \in V(G)$ so, by Lemma 4.23, they are in the same connected component of $\Gamma_{\mathfrak{F}}(G)$. Suppose instead that $\langle s, t \rangle < S$. We have that $H := N \langle s, t \rangle < G$ is not in $\mathfrak{F}$ since $\langle s, t \rangle \notin \overline{f(p)}$. Therefore ns and mt are not isolated in H and, for minimality, $\Gamma_{\mathfrak{F}}(H)$ is connected, so $ns \approx mt$ in $\Gamma_{\mathfrak{F}}(G)$ too. Choose now two non-isolated vertices $n_1 s_1, n_2 s_2 \in \Gamma_{\mathfrak{F}}(G)$ with $n_1, n_2 \in N$ and $s_1, s_2 \in S$. Since they are not isolated, $s_1, s_2 \notin \mathcal{I}_{\overline{f(p)}}(S)$, hence there is a path $s_1 = z_0 \sim \cdots \sim z_l = s_2$ in $\Gamma_{\overline{f(p)}}(S)$ and since $z_i \sim z_{i+1}$, we have, for every $m, h \in N$ and every i, that $mz_i \approx hz_{i+1}$ in $\Gamma_{\mathfrak{F}}(G)$ and so $n_1s_1 \approx n_2s_2$, a contradiction. □

Proof of Theorem 4.4. Suppose G has minimal order with respect to the property that $\Gamma_{\mathfrak{F}}(G)$ is not connected. By Theorem 4.24, G is a primitive monolithic group and $N \trianglelefteq \mathcal{I}_{\mathfrak{F}}(G) = \phi_{\mathfrak{F}}(G)$. By Proposition 4.11, $\phi_{\mathfrak{F}}(G)/N = \phi_{\mathfrak{F}}(G/N) = G/N$, hence $\phi_{\mathfrak{F}}(G) = G$, a contradiction. □

Proof of Theorem 4.5. It follows applying Theorem 4.24, noticing that:

- If $\mathfrak{F} \in \{\mathfrak{U}, \mathfrak{D}\}$, then $\Gamma_{\overline{f(p)}}(S) = \Gamma_{\mathfrak{A}}(S)$ is connected.
- If $\mathfrak{F} = \mathfrak{S}_p\mathfrak{N}^t$ for some prime p and some t, then $\Gamma_{\overline{f(p)}}(S) = \Gamma_{\mathfrak{S}_q\mathfrak{N}^{t-1}}(S)$ for some other prime q. Therefore we can use induction on t, considering that $\mathfrak{S}_p\mathfrak{N}$ is regular for every p and that Theorem 4.4 holds.
- If $\mathfrak{F} = \mathfrak{N}^t$ for some t, then $\Gamma_{\overline{f(p)}}(S) = \Gamma_{\mathfrak{S}_p\mathfrak{N}^{t-1}}(S)$ for some prime p and we can use the point above. □

8 Planarity of $\Gamma_{\mathfrak{F}}$

The generating graph $\tilde{\Delta}(G)$ of a finite group G is the graph whose vertices are the elements of G and in which two vertices g_1 and g_2 are adjacent if and only if $\langle g_1, g_2\rangle = G$. Moreover $\Delta(G)$ is the subgraph of $\tilde{\Delta}(G)$ induced by the subset of its non isolated vertices. Notice that if G is a 2-generated $\mathfrak{F}$-critical group, then $\Gamma_{\mathfrak{F}}(G) \cong \Delta(G)$.

Proof of Theorem 4.6. One implication is easy: if $G \in \mathfrak{F}$ then $\Gamma_{\mathfrak{F}}(G)$ is a null graph, while if $G \cong S_3$ and $S_3 \notin \mathfrak{F}$, then $\Gamma_{\mathfrak{F}}(G) \cong \Delta(G)$ is planar, as it is noticed in [Luc20]. Conversely, suppose $G \notin \mathfrak{F}$ and $\Gamma_{\mathfrak{F}}(G)$ is planar. Since $\mathfrak{F}$ is 2-recognizable, there exist $a, b \in G$ such that $\langle a, b\rangle \notin \mathfrak{F}$. Since $\Delta(\langle a, b\rangle)$ is a subgraph of $\Gamma_{\mathfrak{F}}(G)$, it must be planar. Finite groups with planar generating graph have been completely classified in [Luc20]. In particular, if $\Delta(X)$ is planar, then either X is nilpotent or $X \in \{S_3, D_6\}$. Since $\mathfrak{N} \subseteq \mathfrak{F}$, $\langle a, b\rangle$ is not nilpotent, so either $\langle a, b\rangle \cong S_3$ or $\langle a, b\rangle \cong D_6$. Since $D_6 \cong S_3 \times C_2$ and $C_2 \in \mathfrak{F}$, $D_6 \notin \mathfrak{F}$ implies $S_3 \notin \mathfrak{F}$. Let A be the set of the non-central involutions of D_6 and let B the set of the elements of D_6 of order divisible by 3: then $\Gamma_{\mathfrak{F}}(D_6)$ contains the complete bipartite graph whose partition has the parts A and B, so it is not planar. Hence $\langle a, b\rangle$ can only be isomorphic to S_3. We show that all the elements of G have order less or equal to 3. Suppose in fact that there is $g \in G$ such that $|g| \geq 4$. Since $\Gamma_{\mathfrak{F}}(G)$ is planar, $g \notin \mathcal{I}_{\mathfrak{F}}(G)$ would imply that it generates a copy of S_3 with another element, but this is impossible since $|g| \geq 4$. We have then that $g \in \mathcal{I}_{\mathfrak{F}}(G)$ and therefore $|\mathcal{I}_{\mathfrak{F}}(G)| \geq 4$. We claim that this is not possible. Indeed G

contains $X = \langle a, b \rangle \cong S_3 \notin \mathfrak{F}$. Since $\mathfrak{F}$ is semiregular, $I := \mathcal{I}_{\mathfrak{F}}(G)$ is a normal subgroup of G. Since $I \cap X = 1$, for every $x, y \in I$ we have

$$\frac{\langle ax, by \rangle}{I \cap \langle ax, by \rangle} \cong \frac{\langle ax, by \rangle I}{I} \cong \frac{\langle a, b \rangle I}{I} \cong \langle a, b \rangle \cong S_3 \notin \mathfrak{F},$$

hence $\langle ax, by \rangle \notin \mathfrak{F}$. But then $\Gamma_{\mathfrak{F}}(G)$ contains the complete bipartite graph on the two parts aI and bI and then it is not planar. We have so proved that all the elements of G have order order less or equal than 3. Groups with this property have been classified in [Neu37]. Since G is not nilpotent and contains a subgroup isomorphic to S_3, $G \cong A \rtimes \langle x \rangle$, with $A \cong C_3^t$ and x acting on A sending every element into its inverse. In particular the subgraph of $\Gamma_{\mathfrak{F}}(G)$ induced by the 3^t involutions is complete, so it is planar only if $t = 1$, i.e. $G \cong S_3$. □

9 Some open questions

The material contained in this chapter leaves a lot of open questions which could be object for further research; in this section we propose some of them.

Firstly, given a class of groups $\mathfrak{F}$, we give the following definition:

Definition 4.25. *The $\mathfrak{F}$-izer of the element x in a subset S of a group G is the set*

$$\mathcal{I}_{\mathfrak{F},S}(x) := \{y \in S : \langle x, y \rangle \in \mathfrak{F}\}.$$

This is the set of elements in S which generate with x an element in $\mathfrak{F}$ or, in other words, the elements in S which are not connected to x in $\widetilde{\Gamma}_{\mathfrak{F}}(G)$. It is a notion which generalizes the one of centralizer, nilpotentizer and solvabilizer when $\mathfrak{F}$ is respectively the class of abelian, nilpotent and soluble groups. Apart from centralizer which are subgroups, the other ones (and hence $\mathfrak{F}$-izers) are in general only subsets. Both in [AZ10] and in [HR13] it was posed the problem of studying groups in which nilpotentizers and solvabilizers are subgroups for every element. This could be extended to $\mathfrak{F}$-izers and it could be interesting to understand which properties $\mathfrak{F}$ should have in order to be such that every $\mathfrak{F}$-izer is a subgroup for every group, as in the case of $\mathfrak{F} = \mathfrak{A}$.

Another interesting problem is to study the subgroup of isolated vertices in semiregular formations. We have that for $\mathfrak{F} \in \{\mathfrak{A}, \mathfrak{N}, \mathfrak{S}\}$, $\mathcal{I}_{\mathfrak{F}}(G)$ is a well known characteristic subgroup of G, but for $\mathfrak{F} \in \{\mathfrak{U}, \mathfrak{D}, \mathfrak{N}^t\}$, which are notable formations, we have no description apart from the graph theoretic one.

We were not able to find a non-connected formation. It would be interesting to find a semiregular formation which is not connected, if it exists.

For the graphs studied in the literature, we find many problems which have not been analyzed in this thesis and could be seen in this general framework, such as: estimation of other graph parameters (e.g. clique number, genus, connectivity), existence of hamiltonian cycles, estimation on the numbers of edges, properties shared by groups with isomorphic non-$\mathfrak{F}$ graphs and many others.

These are just examples, but plenty of questions are available to the interested reader.

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This part is left to the reader. □