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Electromagnetic Duality and its Physical Implications

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Abstract

In this work I will study electromagnetic duality, a symmetry between the electric and the magnetic field in Maxwell equations. I will discuss its properties and implications in the conventional formulation of electromagnetism and in a self-dual and Lorentz invariant extension. Then I will add a massless Majorana partner to the gauge field, and show that the theory is supersymmetric.

Introduction

Throughout the history, symmetries have always played a fundamental role in physics. Symmetries are what allow us to find order in the laws of nature. Finding symmetries in physical systems makes it a lot easier to study them. The importance of symmetry emerged with Noether's theorem, which gave a new perspective on the nature of physical theories from their symmetries.
Dualities are phenomena analogous to symmetries: roughly speaking they are a non-trivial formal equivalence between different theories, two different points of view to look at the same object. They are very powerful in the construction of theories, as they allow physicists to do calculations in regimes that are otherwise inaccessible to conventional tools, and they give new insights into the physics of a problem.
The difference between a symmetry and a duality is that the first is a relation within a single theory, the latter is a similar kind of relation but in general between different theories. But duality may also be a symmetry of one theory alone, as the one to be considered here.
In this work I will study electromagnetic duality, which is an equivalence between electric and magnetic fields in Maxwell equations. It is a particular type of duality, as it is an example of *self-duality*, since it takes place within the same theoretical context. This phenomenon manifests itself as a symmetry of the action. While it is not manifest in the conventional Maxwell action (the "discrete" duality transformation, which interchanges the two fields does not leave the action invariant, it only leaves the equations of motion invariant), it is manifest in the Hamiltonian formulation [DT76] and its manifestly Lorentz invariant extension proposed in [PST95b].
A nice property of electromagnetic duality is that it can be made a continuous symmetry, a rotation between the electric and magnetic fields: this is physically very important, because due to Noether's theorem, this implies that there exists a conserved physical quantity, which is the optical helicity, i.e. the difference between the right and left circularly polarized photons, as derived in [Cal65] (which is the one of the first analyses on electromagnetic duality).
A non-negligible detail of electromagnetic duality, is that it only works in the vacuum, since there is no *one-to-one* correspondence between electric and magnetic charges. This is true, unless the theory includes magnetic charges as well. The existence of magnetically charged particles was assumed by Dirac in [Dir31] as a consequence of electromagnetic duality. From an experimental point of view, magnetic monopoles have not been observed as fundamental particles yet. The search for magnetic monopoles is still very active. It is of great interest to find them, because grand unified theories and superstring theories predict their existence. An obstacle is that magnetic charges could be found at very high energy scales. Nevertheless, monopoles and associated currents were directly measured in experiments and identified as topological quasiparticle excitations in emergent condensed matter systems, both in a classical and a quantum regime [CMS08] [DTV20].

In the first chapter of this work I will define electromagnetic duality and show its implications in the conventional Maxwell theory.
In the second chapter I will introduce the PST action and highlight the different properties with respect to the Maxwell action. An important feature is that, even though the action is different and there are new different fields, the physical quantities remain the same.
In the last chapter I will illustrate the basic features of fermionic fields and supersymmetry, and show that the

PST action with a massless fermionic partner to the gauge field is supersymmetric.

Notation

In this work I will use the Minkowski space-time metric with signature $(+,-,-,-)$. Greek indices will be used for space-time indices $0,1,2,3$, latin lower case indices will be used as three-dimensional space indices $1,2,3$.

Chapter 1

Electromagnetic Duality

Electromagnetic duality is a discrete symmetry ($\mathbb{Z}_2$), that can be turned into a continuous one ($SO(2)$), of Maxwell's equations in the vacuum:

$$\begin{cases} \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{cases} \tag{1.1}$$

Electromagnetic duality consists in the transformation

$$\begin{cases} \mathbf{E} \to \mathbf{B} \\ \mathbf{B} \to -\mathbf{E} \end{cases} \tag{1.2}$$

Which is obviously a symmetry of the theory.
When plugging in the electric matter density and current, this symmetry is broken, since Maxwell's theory does not take into account magnetic sources, that have not been observed (yet).
However, many of the modern extensions of electrodynamics require the existence of magnetic monopoles (and magnetic currents), which until now seems the only way to explain the experimental fact which is the quantization of electric charge, via Dirac quantization condition or similar conditions [Dir31][BS88]. In this way adding to the (vacuum) duality transformation, the transformation of sources $(\rho_{el}, \mathbf{j}_{el}) \to (\rho_m, \mathbf{j}_m)$, $(\rho_m, \mathbf{j}_m) \to (-\rho_{el}, -\mathbf{j}_{el})$, this would be a symmetry of the whole electromagnetic theory.
The symmetry can actually be seen as a continuous symmetry:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \mapsto R(\beta) \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \tag{1.3}$$

Where $R(\beta)$ is the $SO(2)$ rotation

$$R(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \tag{1.4}$$

Which reduces to the discrete transformation for $\beta = \frac{\pi}{2}$.
It is trivial to check that Maxwell's equaitons are indeed invariant under this transformation.[a]

[a]When coupled to sources indeed the transformation will be $(s_{(el)}, s_{(m)}) \mapsto R(\beta)(s_{(el)}, s_{(m)})$, where s are the sources.

1.1 Duality in Maxwell Theory

Studying duality in the covariant formulation of Maxwell theory gives deeper results about this symmetry, in particular studying the Lagrangian (and Hamiltonian) formalism.
The electromagnetic fields are represented by the tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.5}$$

such that:

$$\begin{cases} F_{0i} = E^i \\ F_{ij} = -\epsilon_{ijk} B^k \end{cases} \tag{1.6}$$

Maxwell equations in the covariant formalism can be written as the Bianchi identity

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \tag{1.7}$$

(which corresponds to Maxwell equations that do not depend on sources) and the equations that are commonly referred to as Maxwell equations

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{1.8}$$

Where $j = (\rho, \mathbf{j})$ is the 4-current (electric).[b]
The Lagrangian density of Maxwell theory reads

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\nu j^\nu \tag{1.9}$$

Which for our purposes will just be considered to be the free Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1.10}$$

Then the duality symmetry of the free Maxwell equations can be more elegantly written as the exchange of $F_{\mu\nu}$ with its Hodge dual field strength

$$F_{\mu\nu} \;\mapsto\; \widetilde{F}_{\mu\nu} = (\star F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{1.11}$$

In fact:

$$\begin{cases} E^i = F^{i0} \;\mapsto\; \widetilde{F}^{i0} = \frac{1}{2}\epsilon^{i0jk}F_{jk} = +\frac{1}{2}\epsilon^{ijk}\epsilon^{jk\ell}B^\ell = B^i \\ B^i = \frac{1}{2}\epsilon^{ijk}F_{jk} \;\mapsto\; \frac{1}{2}\epsilon^{ijk}\widetilde{F}_{jk} = \frac{1}{4}\epsilon^{ijk}\left(2\epsilon_{jk0\ell}F^{0\ell}\right) = -F_{0i} = -E^i \end{cases} \tag{1.12}$$

The Hodge operator can easily be checked to satisfy $\star \circ \star = -1$ in dimension 4.
The infinitesimal transformation corresponding to the continuous $SO(2)$ rotation is:

$$\delta F_{\mu\nu} = \beta \widetilde{F}_{\mu\nu} \tag{1.13}$$

Even though the equations of motion are indeed invariant under discrete duality transformations, the Lagrangian is not (note that $\widetilde{F}^{\mu\nu}\widetilde{F}_{\mu\nu} = -F^{\mu\nu}F_{\mu\nu}$). So we will say that Maxwell theory is not manifestly self-dual, though

[b] Note that with the presence of magnetic sources, Bianchi identitiy would have to be written as $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = j^\sigma_{(m)}$, where $j_{(m)}$ is the magnetic 4-current. This indeed leads to apparent problems due to the non existence of a global vector potential A, which can be solved with the so-called *Dirac String* procedure [BS88].

Maxwell Lagrangian is invariant under infinitesimal transformations (1.13), modulo a total derivative. Indeed, let us see how the action transforms under infinitesimal duality:

$$\delta_\beta S = \delta_\beta \left(-\frac{1}{4} \int \mathrm{d}^4 x F^{\mu\nu} F_{\mu\nu} \right) = -\frac{1}{2} \int \mathrm{d}^4 x F^{\mu\nu} \delta_\beta F_{\mu\nu} = -\frac{1}{2}\beta \int \mathrm{d}^4 x F^{\mu\nu} \widetilde{F}_{\mu\nu} \tag{1.14}$$

$$= -\beta \int \mathrm{d}^4 x \left(\partial^\mu A^\nu\right) \widetilde{F}_{\mu\nu} = -\beta \int \mathrm{d}^4 x \partial^\mu \left(\widetilde{F}_{\mu\nu} A^\nu \right) + \overbrace{\cancel{\frac{1}{2}\beta \int \mathrm{d}^4 x A^\nu \epsilon_{\mu\nu\rho\sigma} \partial^\mu F^{\rho\sigma}}}^{\text{Bianchi}} \tag{1.15}$$

$$= -\beta \int \mathrm{d}^4 x \partial_\mu \left(\widetilde{F}^{\mu\nu} A_\nu \right) \tag{1.16}$$

Therefore, since the variation of the action is the integral of a 4-divergence: it can be made 0 with appropriate boundary conditions (i.e. $\widetilde{F}_{\mu\nu} = 0$ on the boundary). Hence, the action is invariant under infinitesimal duality transformations.

1.2 Noether's Theorem

A very deep result of lagrangian field theory is Noether's Theorem ([Noe18])

Noether's Theorem

For every infinitesimal global symmetry of the action, there exists a 4-current satisfying the continuity equation and a corresponding conserved charge, i.e. given a Lagrangian density $\mathcal{L}\left(\{\varphi_r\}, \{\partial_\mu \varphi_r\}\right)$ depending on a set of fields $\{\varphi_r\}$, for any infinitesimal transformation of the fields $\delta\varphi_r$ such that

$$\delta S \left[\{\varphi_r\}, \{\partial_\mu \varphi_r\} \right] = \int \mathrm{d}^4 x \partial_\mu X^\mu \tag{1.17}$$

Where S is the corresponding action, and X^μ is some 4-vector, then the 4-vector

$$J^\mu = \sum_{\{\varphi_r\}} \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_r} \delta\varphi_r - X^\mu \tag{1.18}$$

Satisfies the continuity equation (on-shell)

$$\partial_\mu J^\mu = 0 \tag{1.19}$$

And the quantity

$$Q = \int \mathrm{d}^3 x J^0(x) \tag{1.20}$$

is conserved in time

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \tag{1.21}$$

This definitely applies to the infinitesimal duality rotation, and we have

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = -\frac{1}{2} F^{\rho\sigma} \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho \right) = -F^{\mu\nu} \tag{1.22}$$

δA_ν has yet to be calculated, since we defined the transformation only on the field strength F.
This can be done in multiple ways. One is noting that Maxwell's equations (in the vacuum) read $\partial_\mu F^{\mu\nu} = 0$, which is equivalent to saying

$$\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \widetilde{F}^{\rho\sigma} = 0 \tag{1.23}$$

Which implies the existence on-shell of a "dual" 4-vector potential A^d_ν, such that:

$$\widetilde{F}_{\mu\nu} = \partial_\mu A^d_\nu - \partial_\nu A^d_\mu \tag{1.24}$$

which is then related by the vector potential by the equation

$$\partial_\mu A^d_\nu - \partial_\nu A^d_\mu = \epsilon_{\mu\nu\rho\sigma}\partial^\rho A^\sigma \tag{1.25}$$

Then the infinitesimal duality rotation on the 4-potential A is simply $\delta A_\mu = \beta A^d_\mu$, so that on-shell

$$\delta F_{\mu\nu} = \delta\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = \beta\left(\partial_\mu A^d_\nu - \partial_\nu A^d_\mu\right) = \beta\widetilde{F}_{\mu\nu} \tag{1.26}$$

which was the original transformation.
Therefore the Noether current associated to duality is

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu}\delta A_\nu - X^\mu = \beta\left(-\widetilde{F}^{\mu\nu}A^d_\nu + \widetilde{F}^{\mu\nu}A_\nu\right) \tag{1.27}$$

which, leaving the irrelevant factor β out, reduces to

$$J^\mu = \widetilde{F}^{\mu\nu}A_\nu - F^{\mu\nu}A^d_\nu \tag{1.28}$$

The corresponding conserved charge is

$$Q = \int \mathrm{d}^3x J^0 = \int \mathrm{d}^3x\left(\widetilde{F}^{0i}A_i - F^{0i}A^d_i\right) = \int \mathrm{d}^3x\left(\langle\mathbf{B},\mathbf{A}\rangle - \left\langle\mathbf{E},\mathbf{A}^d\right\rangle\right) \tag{1.29}$$

Where $\langle\cdot,\cdot\rangle$ denotes the 3-dimensional euclidean scalar product.
Notice that although the current is not gauge invariant, the conserved charge is, since integrating by parts after a gauge transformation would eliminate the "transformed part".
This charge is often called *optical helicity*, and it is proportional to the difference between the right and left circularly polarized photons, i.e. the net helicity, of the electromagnetic radiation.
An interesting peculiarity of this charge is that, when the theory is quantized, in the presence of a curved space-time, it is not conserved! This is an example of a *quantum anomaly*. This phenomenon implies that, under strong gravitational backgrounds, the optical helicity is not conserved, and opens the possibility of extracting information from strong gravitational fields through the observation of the polarization of photons. A deep analysis on this was made in [ARN18].

Chapter 2

PST Action

As it has been mentioned already, Maxwell equations are indeed duality invariant, but the Lagrangian is not (under finite $SO(2)$ duality rotations). It has been a goal for theoretical physicists for decades to find a manifestly dual formulation of electromagnetism, not just for aesthetic reasons, but for connections of this problem with supergravity and string theory, and because it is desirable to have such a formulation to describe electrically and magnetically charged particles on equal footing. Many attempts have been made over the last decades, which implied giving up manifest Lorentz invariance or introducing auxiliary fields.
In this work I will study the Lagrangian proposed by Pasti, Sorokin and Tonin in [PST95b], which is manifestly Lorentz invariant, manifestly self-dual, introducing an auxiliary scalar field.

2.1 Basic properties

The main features of this action are:

- 2 vector fields A^I, $I = 1, 2$, which are meant to be interpreted as the 4-vector potential(s) of electromagnetism, introduced in a dual manner (i.e. A_μ and A^d_μ of the previous section).
- The electromagnetic field strength(s)
$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu \tag{2.1}$$
- A scalar field $a(x)$, which is introduced with the objective of having a manifestly Lorentz invariant Lagrangian. This field will turn out to be an auxiliary field. The duality condition arises from the field equations of motion of A^I_μ, which will result in the vanishing of the tensor:
$$\mathcal{F}^{I\mu\nu} := \epsilon^{IJ} F^{J\mu\nu} - \widetilde{F}^{I\mu\nu} \tag{2.2}$$
where ϵ^{IJ} is the 2 dimensional Levi-Civita symbol, i.e. $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$[a].

The action reads:
$$S = \int \mathrm{d}^4x \left(-\frac{1}{8} F^{I\mu\nu} F^I_{\mu\nu} - \frac{1}{4} \frac{1}{\sqrt{\partial_\tau a \partial^\tau a}} \left(\partial_\mu a\right) \mathcal{F}^{I\mu\nu} \mathcal{F}^I_{\nu\rho} \partial^\rho a \right) \tag{2.3}$$

Note. *For the calculations it is useful to define the following 4-vectors:*
$$\begin{cases} u_\mu := \dfrac{\partial_\mu a}{\sqrt{(\partial a)^2}} \\ f^I_\mu := \mathcal{F}^I_{\mu\nu} u^\nu \end{cases} \tag{2.4}$$

[a] The uppercase latin *duality* indices $I, J, \ldots$ will have the usual summation convention, even though they are all upper indices.

The variation of the action is, modulo 4-divergences[b]:

$$\delta S = \int \mathrm{d}^4x \left[\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(f^I_\nu u_\rho\right)\right]\delta A^I_\sigma + \left[\frac{1}{2}\epsilon^{IJ}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\frac{u_\nu}{\sqrt{(\partial a)^2}}f^I_\rho f^J_\sigma\right)\right]\delta a \tag{2.5}$$

2.2 Symmetries

This action, as mentioned before, has the feature of being manifestly covariant (invariant under Poincaré transformations) and also manifestly self-dual, with the duality transformation that links the 2 vector fields A^I, $I = 1, 2$:

$$A^I \mapsto R^{IJ}A^J \tag{2.6}$$

Where R^{IJ} is an $SO(2)$ matrix.
We will explore this symmetry in a deeper manner later.
This action also possesses 3 local symmetries, one being the usual $U(1)$ gauge symmetry of electromagnetism, and 2 new local symmetries specific of this action:

1. A potential gauge symmetry:

$$\begin{cases}\delta A^I_\mu = \partial_\mu\Lambda^I \\ \delta a = 0\end{cases} \tag{2.7}$$

$$\delta\mathcal{L} \equiv \epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(f^I_\nu u_\rho\right)\partial_\sigma\Lambda^I = \partial_\sigma\left(\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(f^I_\nu u_\rho\right)\Lambda^I\right) - \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\sigma\left(f^I_\nu u_\rho\right)\Lambda^I \tag{2.8}$$

$$\equiv 0 \tag{2.9}$$

2. A local symmetry with parameter $\lambda^I : \mathbb{R}^4 \to \mathbb{R}$, $\chi(x)^I := \frac{\lambda^I(x)}{\sqrt{(\partial a)^2}}$

$$\begin{cases}\delta A^I_\mu = u_\mu\lambda^I = (\partial_\mu a)\chi^I \\ \delta a = 0\end{cases} \tag{2.10}$$

   This symmetry is fundamental in getting self-duality as an equation of motion.
   Let us rewrite the variation of the Lagrangian with respect to A as:

$$\delta_A\mathcal{L} \equiv -f^{I\mu}u^\nu\delta_A\widetilde{F}^I_{\mu\nu} \tag{2.11}$$

   And

$$\delta_A\widetilde{F}^I_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\gamma\lambda}\delta_A F^{I\gamma\lambda} = \epsilon_{\mu\nu\gamma\lambda}\partial^\gamma\left(\partial^\lambda a\chi^I\right) = \epsilon_{\mu\nu\gamma\lambda}\partial^\lambda a\partial^\gamma\chi^I \tag{2.12}$$

   And the contraction of the indices ν and the indices λ in the variation of the Lagrangian, by symmetry, gives 0.

3. A local symmetry with parameter $\varphi : \mathbb{R}^4 \to \mathbb{R}$:

$$\begin{cases}\delta A^I_\mu = \dfrac{\varphi}{\sqrt{(\partial a)^2}}\epsilon^{IJ}f^J_\mu \\ \delta a = \varphi\end{cases} \tag{2.13}$$

[b] The derivation is shown in A.1.

By direct calculation (using the expression in term of $\partial_\mu \delta a$ of the variation of the Lagrangian):

$$\delta\mathcal{L} = -\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(u_\nu f^I_\rho\right)\frac{\varphi}{\sqrt{(\partial a)^2}}\epsilon^{IJ}f^J_\sigma - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{IJ}u_\nu f^I_\rho f^J_\sigma\frac{1}{\sqrt{(\partial a)^2}}\overbrace{\partial_\mu\varphi}^{\delta\partial_\mu a} \tag{2.14}$$

$$= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{IJ}\partial_\mu\left(\frac{1}{\sqrt{(\partial a)^2}}u_\nu f^I_\rho f^J_\sigma\varphi\right) \tag{2.15}$$

which is a 4-divergence!

2.3 Equations of Motion

Setting the variation of the action equal to 0 we have

$$\begin{cases} \dfrac{\delta S}{\delta A^I_\mu} = 0 \implies \epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(f^I_\nu u_\rho\right) = 0 \\ \dfrac{\delta S}{\delta a} = 0 \implies \dfrac{1}{2}\epsilon^{IJ}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\dfrac{u_\nu}{\sqrt{(\partial a)^2}}f^I_\rho f^J_\sigma\right) = 0 \end{cases} \tag{2.16}$$

The second equation is implied by the first one: in fact, denoting $k^I_\mu = \frac{f^I_\mu}{\sqrt{(\partial a)^2}}$ the first equation of motion takes the form:

$$0 = \epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(k^I_\nu\partial_\rho a\right) = \epsilon^{\mu\nu\rho\sigma}(\partial_\nu a)(\partial_\mu k^I_\nu) \tag{2.17}$$

So the second equation is identically satisfied when the first one holds:

$$\epsilon^{IJ}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\partial_\nu a k^I_\rho k^J_\sigma\right) = 2\epsilon^{IJ}\epsilon^{\mu\nu\rho\sigma}\partial_\nu(\partial_\mu k^I_\rho)k^J_\sigma = 0 \tag{2.18}$$

Which highlights the auxiliary nature of the scalar field $a(x)$.
Now, the first equation implies that $f^I_\nu u_\rho$ is a closed form, so, in a topologically trivial space, due to Poincaré's Lemma, there exists a vector field $\phi^I : \mathbb{R}^4 \to \mathbb{R}^4$ (and ψ and $\tilde{\lambda}$[c]) such that

$$u_{[\nu}f^I_{\rho]} = \partial_{[\nu}\phi^I_{\rho]} = (\partial_{[\nu}a)\psi^I_{\rho]} = \partial_{[\nu}a\partial_{\rho]}\tilde{\lambda}^I \tag{2.19}$$

By the first local symmetry we described, we can always fix a gauge such that

$$u_{[\nu}f^I_{\rho]} = 0 \tag{2.20}$$

In fact

$$\delta F^I_{\mu\nu} = 2\partial_{[\mu}\delta A^I_{\nu]} = 2\partial_{[\mu}(\partial_{\nu]}a)\chi^I = 2(\partial_{[\nu}a)(\partial_{\mu]}\chi^I) \tag{2.21}$$

$$\delta\widetilde{F}^I_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}(\partial^\rho a)(\partial^\sigma\chi^I) \tag{2.22}$$

$$\delta f^I_\rho = \left(\epsilon^{IJ}\delta F^J_{\rho\sigma} - \delta\widetilde{F}^I_{\rho\sigma}\right)u^\sigma = 2\epsilon^{IJ}u^\sigma\partial_{[\sigma}a\partial_{\rho]}\chi^J - \overset{\mu,\sigma}{\cancel{\epsilon_{\rho\sigma\mu\nu}(\partial^\mu a)(\partial^\nu\chi^I)u^\sigma}} \tag{2.23}$$

$$= \epsilon^{IJ}\sqrt{(\partial a)^2}\partial_\rho\chi^J - \epsilon^{IJ}u_\rho\partial^\sigma a\partial_\sigma\chi^J \tag{2.24}$$

So we have:

$$\delta\left(u_{[\nu}f^I_{\rho]}\right) = \epsilon^{IJ}\partial_{[\nu}a\partial_{\rho]}\chi^J \tag{2.25}$$

[c] This is a trickier part which will not be proved here.

Thus we can fix $\chi^J = -\epsilon^{IJ}\tilde{\lambda}^I$ such that $u_{[\nu}f^I_{\rho]}$ is equivalent to 0 by the local symmetry. This implies

$$0 = (u_{[\nu}f^I_{\rho]})u^\nu = \frac{1}{2}f^I_\rho - u_\rho u^\nu \mathcal{F}^I_{\nu\mu}u^\mu = \frac{1}{2}f^I_\rho \tag{2.26}$$

By the property of skew-symmetric tensors introduced in A.1, we can write $\mathcal{F}^I$ in terms of f^I (as a homogeneous combination) so we get, as an equation of motion:

$$\mathcal{F}^I_{\mu\nu} = \epsilon^{IJ}F^J_{\mu\nu} - \widetilde{F}^I_{\mu\nu} = 0 \tag{2.27}$$

And this implies Maxwell's equations, because, by the Bianchi identity on the F^I 2-form, we get Maxwell's equations on the F^J, $J \neq I$, 2-form:

$$\partial_\mu F^{I\mu\nu} = 0 \tag{2.28}$$

But since the two Maxwell fields are related by the duality condition (2.27) only one of them is physically independent.

2.4 Electroamagnetic Duality and its Conserved Charge

The discrete duality transformation sends each 4-vector potential to its "dual", i.e.

$$A^I \;\mapsto\; \epsilon^{IJ}A^J \tag{2.29}$$

The corresponding continuous $SO(2)$ infinitesimal rotation is $\delta A^I = \beta\epsilon^{IJ}A^J$, which is the infinitesimal transformation of

$$\begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \tag{2.30}$$

We then have

$$\delta F^{I\mu\nu} = \beta\epsilon^{IJ}F^{J\mu\nu} \overset{eom}{\equiv} \beta\widetilde{F}^{I\mu\nu} \tag{2.31}$$

So we get on-shell

$$\delta_\beta\mathcal{L} = -\frac{1}{4}F^{I\mu\nu}(\beta\widetilde{F}^{I\mu\nu}) + \frac{1}{2}\overbrace{f^{I\mu}}^{0 \text{ on-shell}}\delta_\beta(f^I_\mu) = -\frac{\beta}{4}F^{I\mu\nu}\epsilon_{\mu\nu\rho\sigma}\partial^\rho A^{I\sigma} \tag{2.32}$$

$$\overset{\text{Leibniz+Bianchi}}{\overset{\uparrow}{=}} \beta\partial^\rho\left(-\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}F^{I\mu\nu}A^{I\sigma}\right) = \partial^\rho\left(-\frac{\beta}{2}\widetilde{F}^I_{\rho\sigma}A^{I\sigma}\right) \tag{2.33}$$

Which proves that duality is indeed a global symmetry of the action.
The corresponding Noether current is then given by

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu}\delta A_\nu - X^\mu \tag{2.34}$$

where X^μ is the term coming from transforming the fields in the action, i.e.

$$X_\mu = -\beta\frac{1}{2}\widetilde{F}^I_{\mu\nu}A^{I\nu} \tag{2.35}$$

The first part of the current is, on-shell, (derivation in *Appendix* A.2):

$$\left(\frac{1}{2}\epsilon^{IJ}\widetilde{F}^{J\mu\nu} - \epsilon^{\mu\nu\rho\sigma}\overbrace{f^I_\rho}^{0 \text{ on-shell}}u_\sigma\right)\overbrace{\delta A^I_\nu}^{\beta\epsilon^{IK}A^K_\nu} = \frac{1}{2}\beta\widetilde{F}^{I\mu\nu}A^I_\nu \tag{2.36}$$

So we have a Noether current

$$J^{\mu} = \widetilde{F}^{I\mu\nu} A^{I}_{\nu} = \widetilde{F}^{1\mu\nu} A^{1}_{\nu} + \widetilde{F}^{2\mu\nu} A^{2}_{\nu} \stackrel{!}{=} \widetilde{F}^{\mu\nu} A_{\nu} - F^{\mu\nu} A^{d}_{\nu} \tag{2.37}$$

Where by A^d we meant the "dual" vector potential A^2 (the *electric* vector potential), satisfying $\mathrm{d}A^d = \star F$, and we implied that A^1 is the Maxwell vector potential.
This 4-current is exactly the one we found in Maxwell theory, and therefore so is its corresponding conserved charge.
It is important to note that, due to the equations of motion, this current does not depend on the scalar field $a(x)$, confirming its auxiliary nature, since it does not contribute to the physical quantities such as the conserved current of duality symmetry.
The corresponding conserved charge is then

$$Q = \int \mathrm{d}^3x J^0 = \int \mathrm{d}^3x \left(\widetilde{F}^{0\nu} A_{\nu} - F^{0\nu} A^{d}_{\nu} \right) = \int \mathrm{d}^3x \left(\langle \mathbf{A}, \mathbf{B} \rangle - \left\langle \mathbf{A}^d, \mathbf{E} \right\rangle \right) \tag{2.38}$$

2.5 Constraints and Hamiltonian Formulation

Some interesting properties of this formulation come out of the Hamiltonian analysis.
The Hamiltonian formulation of a field theory, breaks manifest Lorentz invariance, but it is useful for the quantization of the theory, and the analysis of constraints on phase space has nice physical properties.
The Hamiltonian (density) of a field theory, with a Lagrangian $\mathcal{L}\left(\{\varphi_r\}_r\right)$ is defined as:

$$\mathcal{H} = \sum_{\{\varphi_r\}_r} \pi_r \dot{\varphi}_r \Bigg|_{\dot{\varphi}_r(\pi_r)} - \mathcal{L}|_{\dot{\varphi}_r(\pi_r)} \tag{2.39}$$

Where π_r are the canonical momenta conjugate to the fields φ_r:

$$\pi_r := \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_r} \tag{2.40}$$

The Hamiltonian density is then calculated with $\dot{\phi}_r$ as a function of π_r, inverting the equation above.

2.5.1 Constraints

Some problems may arise here, when there are particular redundancies of the theory. One may have **constraints** $\chi(\phi_r, \pi_r)$, which, simply by applying the definition (2.40), without requiring the equations of motion, satisfy:

$$\chi \equiv 0 \tag{2.41}$$

This kind of problem is present in classical mechanics as well, it was addressed by Dirac, with the theory introduced in [Dir64], which is based on the analysis of the constraints and their classification, and it is important for the analysis of local symmetries and quantization.
An important distinction in this theory is between *First-Class* and *Second-Class* constraints:

- **First-Class** constraints are those which commute with all other constraints, up to other constraints, i.e. a constraint $\tilde{\chi}$ is first-class if, given the set of all other constraints χ_r, the Poisson Bracket :

  $$\{\tilde{\chi}, \chi_r\} = \sum_m f^m \chi_m \approx 0 \quad , \quad \forall r \tag{2.42}$$

  for some functions f^m. The symbol $\approx$ is called *weak* equivalence, so a function vanishes *weakly* if it is 0 when all the constraints are set to 0.
  First-Class constraints form a closed algebra with the Poisson Brackets.

- **Second-Class** constraints are those which are not first-class.

Since these constraints are equivalent to 0, the Hamiltonian is defined up to a sum of these constraints with respective Lagrange multipliers, which have to satisfy some consistency equations[d]. It can be proved that the generic Hamiltonian is written as a first-class Hamiltonian plus some first-class constraints. These different Hamiltonians have the same physical meaning, but lead to different time evolutions for physical quantities, how do we interpret these redundancies in the physical quantities generated by the constraints? They are precisely gauge symmetries! Gauge invariant quantities will not be affected by this (they commute with the generators, i.e. the constraints).

2.5.2 Momentum and Constraints in the PST action

As shown in *Appendix* A.2, the canonical momenta conjugate to the gauge fields A^I_μ and the scalar field a are:

$$\begin{cases} \pi_A^{K\nu} = \dfrac{1}{2}\epsilon^{IJ}\widetilde{F}^{J0\nu} - \epsilon^{0\nu\rho\sigma} f^I_\rho u_\sigma \\ \pi := \pi_a = \dfrac{1}{2}\dfrac{f^I_\nu}{\sqrt{(\partial a)^2}}\left(\mathcal{F}^{I\nu 0} - f^{I\nu}u^0\right) \end{cases} \tag{2.43}$$

There is one obvious constraint, analogous to the Maxwell case:

$$\pi_A^{I0} = 0 \tag{2.44}$$

which expresses the nature of A^I_0 not as a dynamical degree of freedom, but as a Lagrange multiplier in the Hamiltonian formulation.
Another straightforward constraint is:

$$\left(\pi_A^{Ii} - \frac{1}{2}\epsilon^{IJ}\widetilde{F}^{J0i}\right) u_i = 0 \tag{2.45}$$

Two less straightforward constraints, but still direct calculations are:

$$\partial_i \pi_a^{Ii} = 0 \tag{2.46}$$

and

$$\Lambda = \epsilon^{IJ}\epsilon_{ijk}\left(\pi_A^{Ii} - \frac{1}{2}\epsilon^{IK}\widetilde{F}^{K0i}\right)\left(\pi_A^{Ii} - \frac{1}{2}\epsilon^{JL}\widetilde{F}^{L0j}\right)\partial^k a + \pi_a(\partial_i a)^2 = 0 \tag{2.47}$$

These constraints form a closed algebra with the Poisson Brackets, i.e. they are all first-class! This simplifies the quantization of the theory.
What's the meaning of the two constraints, (2.45) and (2.47), which are not present in Maxwell theory?
By applying Poisson brackets, and introducing "parameter" functions, one finds by direct calculation that these constraints generate respectively local symmetries 2. and 3. discussed in this chapter.
In this work the Hamiltonian analysis will not be taken any further.

[d]See [Dir64]

Chapter 3

Adding a Fermion and Supersymmetry

In this chapter I will briefly define and discuss properties of fermions and supersymmetry, then I will show that adding a free Dirac action for a massless Majorana spin $\frac{1}{2}$ fermion to the PST-action, the resulting action is supersymmetric [PST95a].
Fermions arise from a spin $\frac{1}{2}$ representation of the Poincaré group, i.e. the group that ensures Lorentz and space-time translation invariance of physical theories[Min08]. I will introduce fermions as they arise from the mathematical structure of the Poincaré group.
Supersymmetry is a hypothetical extension of Poincaré symmetry, with fermionic generators[GL71]. Therefore, in order to introduce these physical concepts, I will first talk about the Poincaré group and then I will introduce the Super-Poincaré group (for deeper and more rigorous explanations see [Osb19] for the Poincaré group, [Wip16], [Ber20], [WB92] for supersymmetry).

3.1 Poincaré Group

The Lorentz group is the group of transformations $\Lambda \in O(1,3)$, i.e. the group of matrices that preserves the Minkowski scalar product

$$AB = \eta_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu \tag{3.1}$$

for any A, B 4-vectors.
This is equivalent to

$$\Lambda^T \eta \Lambda = \eta \tag{3.2}$$

For physical purposes it is sufficient to focus on the subgroup $SO(1,3)$ of $O(1,3)$, which is the subgroup of transformations with determinant $+1$.
Another useful restriction is focusing on the subgroup of $SO(1,3)$ with ${\Lambda^0}_0 \geq 1$[a], often referred to as the orthocronus Lorentz group. These mathematical restrictions physically represent the removal of the transformations which include time reversal and parity transformation (space inversion).
It can be proved that this subgroup is connected.
Furthermore out of the 16 parameters in the 4×4 Λ matrix, only 6 of them are independent. It can be shown that these 6 degrees of freedom correspond to 3 rotations and 3 Lorentz boosts, which, by composition, generate the whole Lorentz group.
The Poincaré group is the group of Lorentz transformations just described and space time translations. Its elements are represented by $(\Lambda, a) \in SO(1,3) \rtimes \mathbb{R}^4$[b], so the free parameters are now 10.

[a] This can be done since there are 2 disconnected components of this group, one with $\Lambda^0_0 \geq 1$, one with $\Lambda^0_0 \leq -1$.
[b] The semi-direct product is due to the fact that translations don't commute with the Lorentz group.

Mathematically speaking, the Poincaré group is a non-compact[c] 10-dimensional Lie group[d]

3.1.1 Poincaré Algebra

It is mathematically and physically important to study the Lie Algebra of the group. The algebra describes how the group acts locally in a neighbourhood of the identity.
Mathematically it can be defined as the tangent space at the identity of the Lie group, with a Lie bracket structure (for a matrix group it is the commutator). One great result is that by studying the Lie Algebra you can get many properties of the Lie group. The part of the Lie group connected to the identity is described by the exponential map of elements of the Lie algebra.
Physically, the generators of the Lie algebra of a group, which is a symmetry of the physical theory, represent conserved physical quantities.
The algebraic structure of the algebra is described by the Lie brackets of the basis of the vector space, i.e. the commutators between the generators of the matrix group. The Poincaré algebra has 10 generators: 6 are associated to the Lorentz group, described either by a skew-symmetric matrix $M_{\mu\nu} \in M_4(\mathbb{R})$ or by two 3-vectors, $K^i = M^{0i}$ for Lorentz boosts, $J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}$ for rotations; the remaining 4 generators represent space-time translations, P_μ. The algebra is:

$$\begin{cases} [M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho}\right) \\ [P_\mu, P_\nu] = 0 \\ [M_{\mu\nu}, P_\rho] = i\left(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu\right) \end{cases} \quad \text{or} \quad \begin{cases} [J_i, J_j] = i\epsilon_{ijk}J_k \\ [K_i, K_j] = -i\epsilon_{ijk}K_k \\ [J_i, K_j] = i\epsilon_{ijk}K_k \\ [P_\mu, P_\nu] = 0 \\ [J_i, P_j] = i\epsilon_{ijk}P_k \\ [K_i, P_j] = i\delta_{ij}P_0 \\ [J_i, P_0] = 0 \\ [K_i, P_0] = iP_i \end{cases} \tag{3.3}$$

Now, for the "Lorentz part" of the algebra we see that, defining the following vectors:

$$J_i^\pm = \frac{1}{2}\left(J_i \pm iK_i\right) \tag{3.4}$$

These 2 vectors follow the usual angular momentum commutation relations (the $\mathfrak{su}(2)$ algebra), and they commute with each other:

$$[J_i^\pm, J_j^\pm] = i\epsilon_{ijk}J_k^\pm \quad , \quad [J_i^\pm, J_j^\mp] = 0 \tag{3.5}$$

This shows[e] that the algebra of $SO(1,3)$ is isomorphic to 2 copies of the algebra of the rotation group $SO(3)$ (which is $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$), i.e. $\mathfrak{so}(1,3) \simeq \mathfrak{su}(2) \otimes \mathfrak{su}(2)$.
This argument can be made more formal by finding the universal cover of $SO(1,3)$ (which is not simply connected), which is $SL(2,\mathbb{C})$. One then can focus on the representations of the universal cover. We will not get deep into the mathematics of the problem, but there is one important result, that highlights the difference between $SO(1,3)$ and the "non-relativistic" rotation group $SO(3)$: the fundamental representation of $SL(2,\mathbb{C})$ and its conjugate are not equivalent (not related by a unitary transformation)! So it is like having 2 fundamental representations, that lead to different physics.

[c] The subgroup of rotations is compact, but it is straightforward and intuitive to see that boosts and translations make it non-compact.
[d] A group $(G, \circ, ^{-1})$ is a Lie group if it is a differentiable manifold, the composition $\circ : G \times G \to G$ and the inversion $^{-1} : G \to G$ are differentiable function between manifolds.
[e] It can be formally proved.

3.2 Spinors

3.2.1 2-component spinor notation

The fundamental representation of the Lorentz Group is given by 2-component vectors (**spinors**), which transform by the multiplication with $SL(2,\mathbb{C})$ matrices:

$$\psi_\alpha \xrightarrow{A} A_\alpha^{\ \beta}\psi_\beta, \ \chi^\alpha \xrightarrow{A} \chi^\beta \left(A^{-1}\right)_\beta^{\ \alpha} \tag{3.6}$$

We may raise or lower indices with the 2 dimensional Levi-Civita symbol, so that the representations introduced in the previous equation $\psi_\alpha, \ \chi^\alpha$ are equivalent:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \ , \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta \tag{3.7}$$

as one can see that:

$$\left(A^{-1}\right)_\beta^{\ \alpha} = \epsilon^{\alpha\gamma}A_\gamma^{\ \delta}\epsilon_{\delta\beta} \tag{3.8}$$

Here comes the crucial difference between $SO(1,3)$ spinors and $SO(3)$ spinors: we have an inequivalent representation given by conjugation. So it is convenient to introduce the following notation: the conjugate spinors obtained from ψ_α, χ^β have dotted indices $\dot\alpha = 1,2$. Conjugation interchanges dotted and undotted indices. We then have:

$$\bar\psi_{\dot\alpha} = (\psi_\alpha)^\star \quad , \quad \bar\chi^{\dot\alpha} = (\chi^\alpha)^\star \tag{3.9}$$

Which have the following transformation rules:

$$\bar\psi_{\dot\alpha} \xrightarrow{A} \bar\psi_{\dot\beta}\left(\bar A^{-1}\right)^{\dot\beta}_{\ \dot\alpha} \ , \ \bar\chi^{\dot\alpha} \xrightarrow{A} \bar A^{\dot\alpha}_{\ \dot\beta}\bar\chi^{\dot\beta} \tag{3.10}$$

for

$$\left(\bar A^{-1}\right)^{\dot\alpha}_{\ \dot\beta} = \left(A_\beta^{\ \alpha}\right)^\star \quad \left(\bar A^{-1} = A^\dagger\right) \tag{3.11}$$

Both $A, \bar A \in SL(2,\mathbb{C})$ and obey the same multiplication rules, since $\bar A_1 \bar A_2 = \overline{A_1 A_2}$. So the corresponding dotted Levi-Civita symbols give:

$$\bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta} \quad , \quad \bar\chi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta} \tag{3.12}$$

Any matrix $A \in SL(2,\mathbb{C})$ can be written as

$$A = \exp\left((\beta_j + i\omega_j)\sigma_j\right) \quad , \quad A^\star = \exp\left((\beta_j - i\omega_j)\sigma_j\right) \tag{3.13}$$

Where σ_j ($j = 1,2,3$) are the Pauli matrices.
This shows how, as previously mentioned, $SL(2,\mathbb{C})$ matrices are expressed in terms of the generators of the spin $\frac{1}{2}$ representation of the $SU(2)\times SU(2)^\star$ algebra. A is constructed by exponentiating the J^+ generators, and $A^\star$ by exponentiating the J^- generators. So, denoting a representation by (j_+, j_-), undotted spinors are a $\left(\frac{1}{2}, 0\right)$ representation, dotted spinors are a $\left(0, \frac{1}{2}\right)$ representation.
It is convenient to extend the σ matrices with

$$\sigma_\mu = (\mathbb{I}, \boldsymbol{\sigma}) \ , \ \bar\sigma_\mu = (\mathbb{I}, -\boldsymbol{\sigma}) \tag{3.14}$$

So that the internal Lorentz generator is[f] $\sigma_{\mu\nu}$:

$$\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} = \frac{1}{4}\left[(\sigma^\mu)_{\alpha\dot\gamma}\left(\bar\sigma^\nu\right)^{\dot\gamma\beta} - (\mu \leftrightarrow \nu)\right] \tag{3.15}$$

[f] It can be checked directly that these generators have the correct algebra.

3.2.2 Dirac Spinors

The independent fundamental spinors ψ, χ, and their conjugates $\bar{\psi}, \bar{\chi}$, can be combined as a single 4-component **Dirac spinor** and its conjugate

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \ , \ \bar{\Psi} = \begin{pmatrix} \chi^\alpha \ \bar{\psi}_{\dot{\alpha}} \end{pmatrix} \tag{3.16}$$

Where

$$\bar{\Psi} = \Psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.17}$$

Correspondingly, there are 4×4 Dirac matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \tag{3.18}$$

The internal Lorentz generator is:

$$\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu} \quad , \quad \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \bar{\sigma}^\mu\sigma^\nu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^n u - \sigma^\nu\bar{\sigma}^\mu \end{pmatrix} \tag{3.19}$$

A **Majorana fermion** is a Dirac fermion with $\psi = \chi$.

3.2.3 Dirac Action

With the notation just introduced, it can be proved that there's a natural way to define a Lorentz scalar: for 2 component spinors $\psi\chi := \psi_\alpha\chi^\alpha$. For Dirac spinors $\bar{\Psi}\Psi$. There is also a natural way to define Lorentz 4-vectors, $\chi\sigma^\mu\bar{\psi}$ and $\bar{\Psi}\gamma^\mu\Psi$.
The Dirac action describes the dynamics of free spin $\frac{1}{2}$ fields. With the definitions previously given, one can build a scalar action, that gives first order equations of motion in the correct way defining:

$$S = \int \mathrm{d}^4x\bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi \tag{3.20}$$

treating Ψ and $\bar{\Psi}$ independently. Here $m \in \mathbb{R}$, $m > 0$ is the mass.
The equations of motion (**Dirac equation**) are straightforward to derive:

$$\left(i\gamma^\mu\partial_\mu - m\right)\Psi = 0 \tag{3.21}$$

This is a wave equation (it has wave-like solutions, with both positive and negative energy) of first order and it implies the Klein-Gordon equation.

3.3 Supersymmetry

Supersymmetry is a hypothetical external, or space-time, symmetry that extends the Poincaré group, that would solve some fundamental problems which the Standard Model cannot solve.
Roughly speaking it consists in enlarging the Poincaré group with fermionic generators (symmetry generators that transform as spin $\frac{1}{2}$ particles), that transform bosons into fermions and vice versa. In a theory in which supersymmetry is realized, each particle has a super-partner. So together they form a super-multiplet.

3.3.1 Why supersymmetry?

There are several reasons that make supersymmetry interesting.

Theoretical reasons:

- Under very reasonable assumptions, the Coleman-Mandula theorem states that the only possible symmetries of the S-matrix (which is related to what we observed in laboratories) are Poincaré symmetries and internal symmetries. One implicit assumption, though, is that it assumes a Lie algebra of commutators for the symmetry. If one extends this theorem with a graded Lie Algebra with anti-commutators as well, and fermionic generators, one gets a larger Super-Poincaré group plus internal symmetries.
- Unification: supersymmetry (and supergravity) is one of the most natural candidates for achieving the long time quest which is the unification of fundamental interactions.
- Supersymmetry is an intrinsic property of String Theory.

Elementary particle theory:

- Hierarchy problem: the experimental value of the Higgs mass is unnaturally smaller than its natural theoretical value, unless one makes a huge fine-tuning. But if one assumes Supersymmetry, the radiative perturbations cancel out exactly, without any unnatural fine-tuning.
- Gauge coupling unification: in the standard model, according to the renormalization group, the running coupling constants approximately meet at an energy scale of $\sim 10^{15}GeV$. At this scale, many problems arise: phenomenological problems, new hierarchy problems and proton decay. Theoretically, the deviation from the exact meeting of these couplings is unnaturally small. Small deviations are unnatural unless there are specific reasons. If one adds SUSY all of these problems are solved in a natural way.

Cosmology:

- Supersymmetry adds to the standard model a candidate for dark matter: the neutralino.

One more theoretical reason is that supersymmetry has very nice properties, such as renormalization properties, that allows to use it as a toy model for more complicated theories.

3.3.2 Supersymmetry Algebra

One can show that (up to conventional multiplicative factors), calling the fermionic generators Q^I_α, the supersymmetry algebra needs to have the form:

$$\text{Poincaré Algebra} + \begin{cases} [P_\mu, Q^I_\alpha] = 0 \\ [P_\mu, \bar{Q}^I_{\dot\alpha}] = 0 \\ [M_{\mu\nu}, Q^I_\alpha] = i\left(\sigma_{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta \\ [M_{\mu\nu}, \bar{Q}^{I\dot\alpha}] = i\left(\bar\sigma_{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta}\, \bar{Q}^{I\dot\beta} \\ \{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ} \\ \{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \quad , \quad Z^{IJ} = -Z^{JI} \\ \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\left(Z^{IJ}\right)^\star \end{cases} \tag{3.22}$$

Where there is an additional index, which is the number of fermionic generators $I = 1, \dots, \mathcal{N}$. I will focus on $\mathcal{N} = 1$ supersymmetry, so that a boson and a fermion are superpartners. For $\mathcal{N} = 1$ there are no *central charges* Z^{IJ}, since they're anti-symmetric in the indices I and J, but $I = J = 1$.
One feature of supersymmetry representations is that for a given multiplet there is a mass degeneracy, i.e. all the

particles in the multiplet have the same mass. Since we're working with electromagnetism, which is mediated by a massless gauge boson, the supersymmetric gauge theory will have a massless boson and a massless fermion.

3.3.3 $\mathcal{N}$=1 Supersymmetric abelian gauge theory.

The simplest form for a supersymmetric Lagrangian for a gauge boson and a majorana spinor super-partner is the following:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + \frac{1}{2}D^2 \tag{3.23}$$

Where D is an auxiliary scalar field which brings no additional degrees of freedom and with trivial equations of motion, which solely serves the purpose of supersymmetry algebra closure.
The supersymmetry transformation is, with a constant Majorana fermion α:

$$\begin{cases} \delta_{\alpha}A_{\mu} = i\bar{\alpha}\gamma_{\mu}\Psi \\ \delta_{\alpha}\Psi = iF^{\mu\nu}\Sigma_{\mu\nu}\alpha + iD\gamma_5\alpha \\ \delta_{\alpha}D = \bar{\alpha}\gamma_5\gamma^{\mu}\partial_{\mu}\Psi \end{cases} \tag{3.24}$$

with $\Sigma_{\mu\nu}$ is the spin component of angular momentum for Dirac fermions, $\Sigma_{\mu\nu} = \frac{i}{2}\gamma_{\mu\nu}$, $\gamma_{\mu\nu} = \frac{1}{2}[\gamma_{\mu}, \gamma_{\nu}]$ and $\gamma_5 := -\frac{i}{24}\epsilon^{\mu\nu\rho\sigma}\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}$.
Note that if one computes the algebra of these transformations, we get an additional term in $\{Q, \bar{Q}\}$: this is due to a redundancy of the system, which is gauge symmetry; calculating $(\delta_{\alpha}\delta_{\beta} - \delta_{\beta}\delta_{\alpha})$ on the fields gives the expected translation (momentum) term, and for the gauge potential A_{μ} one gets an additional term which is a 4 divergence of a scalar field, so it is just a gauge transformation of A_{μ}.

3.4 PST supersymmetric action

Even though in the PST action there are 2 gauge vector fields, there is only one on-shell vector field degree of freedom, since A^1 and A^2 are related by duality.
The supersymmetric action is then the PST action with a massless Majorana fermion Ψ:

$$S = \int \mathrm{d}^4x \,(\mathcal{L}_{PST} + \mathcal{L}_{fermion}) \tag{3.25}$$

Where $\mathcal{L}_{fermion}$ is:

$$\mathcal{L}_{fermion} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi \tag{3.26}$$

Of course, since this is not the usual Maxwell action, the supersymmetry transformations have to be modified slightly:

$$\begin{cases} \delta A^I_{\mu} = i\bar{\Psi}\gamma_{\mu}\epsilon^I \\ \delta\Psi = \frac{1}{8}F^{I\mu\nu}\gamma_{\mu}\gamma_{\nu}\epsilon^I - \frac{1}{4}\epsilon^{IJ}u_{\rho}\mathcal{F}^{J\rho[\mu}u^{\nu]}\gamma_{\mu}\gamma_{\nu}\epsilon^I \\ \delta a = 0 \end{cases} \tag{3.27}$$

With ϵ^I being constant Majorana spinors, $I = 1, 2$, satisfying $\epsilon^I = i\gamma^5\epsilon^{IJ}\epsilon^J$.
On-shell these transformations reduce to the conventional $\mathcal{N} = 1$ supersymmetry transformations of the vector super-multiplet (A^1, Ψ).
Note that the action is invariant under this transformation, but the supersymmetry algebra only closes on-shell.
Due to the presence of the additional local symmetries (not only the A_{μ} $U(1)$ gauge symmetry), in particular the gauge symmetry for the $a(x)$ field that highlights its auxiliary nature, one gets an additional term in the anti-commutator of two supersymmetry generators acting on the field $a(x)$, which generates its gauge transformation (and the corresponding transformation for A) (2.13). This is not a surprise because it ensures that the action of the anticommutator of the supercharges Q on $a(x)$ gives 0, since it is not part of the supersymmetry vector multiplet.

Conclusion

In this work I have introduced the concept of electromagnetic duality as a symmetry of Maxwell equations, in both a non-covariant and a covariant formalism. Since duality can be made continuous ($SO(2)$), I have derived the charge conserved due to Noether's Theorem, optical helicity.
Then I have analyzed the manifestly self-dual extension of Maxwell's action (PST), presenting its peculiarities such as its equations of motion and its local symmetries, and shown that the scalar field $a(x)$ plays an auxiliary role. In particular, it does not appear in the physical relevant quantities, such as the conserved Noether charge.
In the last chapter, after giving a brief summary of the Poincaré group, spinors and supersymmetry, I have shown that the PST action with a massless Majorana fermion is supersymmetric under appropriate supersymmetry transformations.

Future developments directly related to this work may be to complete the Hamiltonian analysis by constructing the total Hamiltonian of the theory à la Dirac, to study quantum anomalies of the duality symmetry and its effect on the auxiliary nature of the scalar field $a(x)$, to introduce into the theory electrically and magnetically charged sources (dyons) and to study the physical meaning of the generalized conserved charge associated with the duality symmetry.

Appendix A

Calculations

Property of skew-symmetric tensors

$K_{\mu\nu} = -K_{\nu\mu}$, u_μ unit vector ($u_\mu u^\mu = 1$).

$$K_{\mu\nu} = u_{[\mu} Z_{\nu]} + G_{\mu\nu} \tag{A.1}$$

such that:

$$G_{\mu\nu} u^\mu = 0 \ , \ \ Z_\mu u^\mu = 0 \tag{A.2}$$

Let us define $k^\mu = K^{\mu\nu} u_\nu$ and $\tilde{k}^\mu = \widetilde{K}^{\mu\nu} u_\nu$.
It's easy to see that Z is determined by:

$$Z_\mu = -2k_\mu \tag{A.3}$$

So we have:

$$G^{\mu\nu} = K^{\mu\nu} + 2u^{[\mu} K^{\nu]\tau} u_\tau \tag{A.4}$$

Taking the dual

$$\begin{aligned}
\widetilde{G}^{\mu\nu} &= \widetilde{K}^{\mu\nu} + \epsilon^{\mu\nu\rho\sigma} u_\rho K_{\sigma\tau} u^\tau = \widetilde{K}^{\mu\nu} - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\sigma\tau\gamma\lambda} u_\rho \widetilde{K}^{\gamma\lambda} u^\tau && \text{(A.5)}\\
&= \widetilde{K}^{\mu\nu} - \frac{1}{2} u_\rho u^\tau \widetilde{K}^{\gamma\lambda} \left(\delta^\mu_\lambda \delta^\nu_\tau \delta^\rho_\gamma - \delta^\mu_\lambda \delta^\nu_\gamma \delta^\rho_\tau + \delta^\mu_\gamma \delta^\nu_\lambda \delta^\rho_\tau - \delta^\mu_\gamma \delta^\nu_\tau \delta^\rho_\lambda + \delta^\mu_\tau \delta^\nu_\gamma \delta^\rho_\lambda - \delta^\mu_\tau \delta^\nu_\lambda \delta^\rho_\gamma\right) && \text{(A.6)}\\
&= \widetilde{K}^{\mu\nu} - \frac{1}{2}\left(2u_\rho u^\nu \widetilde{K}^{\rho\mu} + 2\widetilde{K}^{\mu\nu} + 2u_\rho u^\mu \widetilde{K}^{\nu\rho}\right) && \text{(A.7)}\\
&= -u_\rho \left(u^\mu \widetilde{K}^{\nu\rho} - u^\nu \widetilde{K}^{\mu\rho}\right) = -2u^{[\mu} \tilde{k}^{\nu]} && \text{(A.8)}
\end{aligned}$$

So:

$$G^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} u_\rho \tilde{k}_\sigma \tag{A.9}$$

To which follows the decomposition:

$$K_{\mu\nu} = -2u_{[\mu} k_{\nu]} + \epsilon_{\mu\nu\rho\sigma} u^\rho \tilde{k}^\sigma \tag{A.10}$$

Applying it to the $\mathcal{F}^I_{\mu\nu}$ tensor, with the vector u, we get:

$$\mathcal{F}^I_{\mu\nu} = -2u_{[\mu} f^I_{\nu]} + \epsilon_{\mu\nu\rho\sigma} u^\rho \tilde{f}^{I\sigma} = -2u_{[\mu} f^I_{\nu]} - \epsilon^{IJ} \epsilon_{\mu\nu\rho\sigma} u^\rho f^{J\sigma}$$

It's straightforward to see that the vector field $f^{I\nu}$ is orthogonal to u_ν:

$$f^{I\nu} u_\nu = \mathcal{F}^{I\nu\rho} u_\rho u_\nu = \mathcal{F}^{I[\nu\rho]} u_{\{\rho} u_{\nu\}} = 0 \tag{A.11}$$

A.1 Action Variation

$$\mathcal{L}_1 = -\frac{1}{8} F^I_{\mu\nu} F^{I\mu\nu} \ , \quad \mathcal{L}_2 = \frac{1}{4} f^{I\mu} f^I_\mu \ , \quad \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \tag{A.12}$$

$$\delta_A \mathcal{L}_1 = -\frac{1}{4} F^I_{\mu\nu} \delta_A F^{I\mu\nu} \tag{A.13}$$

$$\delta_A \mathcal{L}_2 = \frac{1}{2} f^{I\mu} \delta_A f^I_\mu = \frac{1}{2} f^{I\mu} u^\nu \delta_A \mathcal{F}^I_{\mu\nu} = \frac{1}{2} \epsilon^{IJ} \mathcal{F}^{I\mu\rho} u_\rho u^\nu \delta_A F^J_{\mu\nu} - \frac{1}{2} f^{I\mu} u^\nu \delta \widetilde{F}^I_{\mu\nu} \tag{A.14}$$

$$(\star \circ \star = -\mathrm{id}) \tag{A.15}$$

$$= \boxed{\epsilon^{IJ}} \frac{1}{2} \left(-\frac{1}{2} \epsilon^{\mu\rho\gamma\lambda} \boxed{\widetilde{\mathcal{F}}^I_{\gamma\lambda}} \right) u_\rho u^\nu \left(-\frac{1}{2} \epsilon_{\mu\nu\tau\sigma} \delta_A \widetilde{F}^{J\tau\sigma} \right) - \frac{1}{2} f^{I\mu} u^\nu \delta_A \widetilde{F}^I_{\mu\nu} \tag{A.16}$$

$$\left(\epsilon^{IJ} \widetilde{\mathcal{F}}^I_{\gamma\lambda} = -\mathcal{F}^J_{\gamma\lambda} \ , \ J \to I \ \ , \ \epsilon^{\mu\rho\gamma\lambda} \epsilon_{\mu\nu\tau\sigma} = -3! \delta^\rho_{[\nu} \delta^\gamma_\tau \delta^\lambda_{\sigma]} \right) \tag{A.17}$$

$$= \frac{u_\rho u^\nu}{8} \mathcal{F}^I_{\gamma\lambda} \delta_A \widetilde{F}^{I\tau\sigma} \left(\boxed{\delta^\rho_\nu \delta^\gamma_\tau \delta^\lambda_\sigma - \delta^\rho_\nu \delta^\gamma_\sigma \delta^\lambda_\tau} + \boxed{\delta^\rho_\sigma \delta^\gamma_\nu \delta^\lambda_\tau - \delta^\rho_\sigma \delta^\gamma_\tau \delta^\lambda_\nu} + \boxed{\delta^\rho_\tau \delta^\gamma_\sigma \delta^\lambda_\nu - \delta^\rho_\tau \delta^\gamma_\nu \delta^\lambda_\sigma} \right) \tag{A.18}$$

$$- \frac{1}{2} f^{I\mu} u^\nu \delta_A \widetilde{F}^I_{\mu\nu} \tag{A.19}$$

$$(u_\rho u^\rho = 1) \tag{A.20}$$

$$= \frac{1}{4} \left(\boxed{\mathcal{F}^I_{\tau\sigma} \delta_A \widetilde{F}^{I\tau\sigma}} \boxed{-u_\rho u^\nu \mathcal{F}^I_{\tau\nu} \delta_A \widetilde{F}^{I\tau\rho}} \boxed{-u_\rho u^\nu \mathcal{F}^I_{\nu\sigma} \delta_A \widetilde{F}^{I\rho\sigma}} \right) - \frac{1}{2} f^{I\mu} u^\nu \delta_A \widetilde{F}^I_{\mu\nu} \tag{A.21}$$

$$\left(\boxed{\cdot} = -f^I_\tau u_\rho \delta_A \widetilde{F}^{I\tau\rho} = -f^{I\mu} u^\nu \delta_A \widetilde{F}^I_{\mu\nu} \ , \quad \boxed{\cdot} = -u_\rho f^I_\sigma \delta_A \widetilde{F}^{I\sigma\rho} = -f^{I\mu} u^\nu \delta_A \widetilde{F}^I_{\mu\nu} \right) \tag{A.22}$$

$$\left(A_{\mu\nu} \widetilde{B}^{\mu\nu} = \widetilde{A}_{\mu\nu} B^{\mu\nu} \right) \tag{A.23}$$

$$= \frac{1}{4} \mathcal{F}^I_{\mu\nu} \delta_A \widetilde{F}^{I\mu\nu} - f^{I\mu} u^\nu \delta_A \widetilde{F}^I_{\mu\nu} = \frac{1}{4} \widetilde{\mathcal{F}}^I_{\mu\nu} \delta_A F^{I\mu\nu} - f^{I\mu} u^\nu \delta_A \widetilde{F}^I_{\mu\nu} \tag{A.24}$$

$$= \frac{1}{4} \left(\epsilon^{IJ} \widetilde{F}^J_{\mu\nu} \delta_A F^{I\mu\nu} \right) - \frac{1}{4} ((\overbrace{\star \circ \star}^{-1} F^I)_{\mu\nu} \delta_A F^{I\mu\nu}) - f^{I\mu} u^\nu \delta_A \widetilde{F}^I_{\mu\nu} \tag{A.25}$$

Now, the first term is a 4-divergence:

$$\widetilde{F}^J_{\mu\nu} \delta_A F^{I\mu\nu} = 2 \widetilde{F}^J_{\mu\nu} \partial^\mu \delta A^{I\nu} = 2 \partial^\mu \left(\widetilde{F}^J_{\mu\nu} A^{I\nu} \right) - \underbrace{\cancel{\left(\epsilon_{\mu\nu\rho\sigma} \partial^\mu F^{J\rho\sigma} \right)}}_{\text{Bianchi}} A^{I\nu} \tag{A.26}$$

So the variation (with respect to the potential A) is:

$$\delta_A \mathcal{L} = \frac{\epsilon^{IJ}}{2} \partial^\mu \left(\widetilde{F}^J_{\mu\nu} \delta A^{I\nu} \right) - \partial^\rho \left(\epsilon_{\mu\nu\rho\sigma} f^{I\mu} u^\nu \delta A^{I\sigma} \right) + \epsilon_{\mu\nu\rho\sigma} \partial^\rho \left(f^{I\mu} u^\nu \right) \delta A^{I\rho} \tag{A.27}$$

$$\equiv \epsilon^{\mu\nu\rho\sigma} \partial_\rho \left(f^I_\mu u_\nu \right) \delta A^I_\sigma \quad (\mathrm{mod}\ 4 - \mathrm{div}) \tag{A.28}$$

Now we vary the action with respect to the a field, using the property of skew-symmetric tensors proved above.

$$\delta_a\mathcal{L} = \frac{1}{2}f^{I\mu}\mathcal{F}^I_{\mu\nu}\delta_a u^\nu = -f^{I\mu}u_{[\mu}f^I_{\nu]}\delta_a u^\nu + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}u^\rho f^{I\mu}(\widetilde{\mathcal{F}}^{I\sigma\tau}u_\tau)\delta_a u^\nu \tag{A.29}$$

$$= \frac{1}{2}f^{I\mu}\left(f^I_\mu u_\nu - f^I_\nu u_\mu\right)\delta_a u^\nu - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}\overbrace{\mathcal{F}^{J\sigma\tau}u_\tau}^{f^{J\sigma}}\delta_a u^\nu \tag{A.30}$$

$$\left(f^{I\mu} = \mathcal{F}^{I\mu\rho}u_\rho = \frac{1}{2}\epsilon^{IJ}\epsilon^{\mu\rho\gamma\lambda}\mathcal{F}^J_{\gamma\lambda}u_\rho\right) \tag{A.31}$$

$$= \frac{1}{2}f^{I\mu}f^I_\mu \overbrace{u_\nu\delta_a u^\nu}^{\frac{1}{2}\delta_a(u_\nu u^\nu)=0} - \overbrace{\cancel{\frac{1}{4}\epsilon^{IJ}\epsilon^{\mu\rho\gamma\lambda}\mathcal{F}^J_{\gamma\lambda}u_\rho u_\mu f^I_\nu\delta_a u^\nu}}^{\mu,\rho} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}f^{J\sigma}\delta_a u^\nu \tag{A.32}$$

$$= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}f^{J\sigma}\delta_a\left(\frac{\partial^\nu a}{\sqrt{(\partial a)^2}}\right) \tag{A.33}$$

$$= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}f^{J\sigma}\left(\frac{\delta_a\partial^\nu a}{\sqrt{(\partial a)^2}}\right) + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}f^{J\sigma}\left(\frac{\partial^\nu a}{\sqrt{(\partial a)^2}}\frac{\partial_\gamma a\delta_a\partial^\gamma a}{(\partial a)^2}\right) \tag{A.34}$$

$$= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}f^{J\sigma}\left(\frac{\delta_a\partial^\nu a}{\sqrt{(\partial a)^2}}\right) + \overbrace{\cancel{\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}f^{J\sigma}\left(\frac{u^\nu u_\gamma\delta_a\partial^\gamma a}{\sqrt{(\partial a)^2}}\right)}}^{\nu,\rho} \tag{A.35}$$

$$= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}f^{J\sigma}\left(\frac{\delta_a\partial^\nu a}{\sqrt{(\partial a)^2}}\right) \tag{A.36}$$

$$= -\partial^\nu\left(\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\epsilon^{IJ}u^\rho f^{I\mu}f^{J\sigma}\frac{\delta a}{\sqrt{(\partial a)^2}}\right) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{IJ}\partial^\nu\left(\frac{u_\rho}{\sqrt{(\partial a)^2}}f^I_\mu f^J_\nu\right)\delta a \tag{A.37}$$

$$\equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{IJ}\partial_\nu\left(\frac{u_\rho}{\sqrt{(\partial a)^2}}f^I_\mu f^J_\sigma\right)\delta a \quad \text{(mod 4-div)} \tag{A.38}$$

So, up to 4-divergences, we can rewrite the total variation as

$$\delta\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(f^I_\nu u_\rho\right)\delta A^I_\sigma + \frac{1}{2}\epsilon^{IJ}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\frac{u_\nu}{\sqrt{(\partial a)^2}}f^I_\rho f^J_\sigma\right)\delta a \tag{A.39}$$

A.2 Calculation

$$\Pi^{K\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu A^K_\nu} = \overbrace{-\frac{1}{4}F^{I\rho\sigma}\frac{\partial F^I_{\rho\sigma}}{\partial\partial_\mu A^K_\nu}}^{\Pi^{K\mu\nu}_M} + \overbrace{\frac{1}{2}f^I_\rho\frac{\partial f^{I\rho}}{\partial\partial_\mu A^K_\nu}}^{\Pi^{K\mu\nu}_D} \tag{A.40}$$

The first part is (half) the usual Maxwell term to A:

$$\Pi^{K\mu\nu}_M = -\frac{1}{4}\delta^{IK}F^{I\rho\sigma}(2\delta^\mu_\rho\delta^\nu_\sigma) = -\frac{1}{2}F^{K\mu\nu} \tag{A.41}$$

For the second part we have

$$\Pi_D^{K\mu\nu} = \frac{1}{2} f_\rho^I \frac{\partial}{\partial\partial_\mu A_\nu^K} \left(\epsilon^{IJ} F^{J\rho\sigma} - \widetilde{F}^{I\rho\sigma} \right) u_\sigma \tag{A.42}$$

$$= \frac{1}{2} f^{I\rho} u^\sigma \epsilon^{IK} 2(\delta^\mu_{[\rho}\delta^\nu_{\sigma]}) - \frac{1}{4}\epsilon^{\rho\sigma\alpha\beta} f_\rho^K u_\sigma (2\delta^\mu_\alpha \delta^\nu_\beta) \tag{A.43}$$

$$= f^{I[\mu} u^{\nu]} \epsilon^{IK} - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} f_\rho^K u_\sigma = \frac{\epsilon^{IK}}{2}\left(-2u^{[\mu} f^{I\nu]} - \epsilon^{IJ}\epsilon^{\mu\nu\rho\sigma} f_\rho^J u_\sigma \right) \tag{A.44}$$

$$= \frac{\epsilon^{IK}}{2} \mathcal{F}^{I\mu\nu} - \epsilon^{\mu\nu\rho\sigma} f_\rho^K u_\sigma \tag{A.45}$$

Where in the last step we used the orthogonal decomposition of a skew-symmetric tensor (which differed from the expression in the brackets just for a sign).
So adding these terms:

$$\Pi^{K\mu\nu} = -\frac{1}{2} F^{K\mu\nu} + \frac{F^{K\mu\nu}}{2} - \frac{1}{2}\epsilon^{IK}\widetilde{F}^{I\mu\nu} - \epsilon^{\mu\nu\rho\sigma} f_\rho^K u_\sigma = \frac{1}{2}\epsilon^{KJ}\widetilde{F}^{J\mu\nu} - \epsilon^{\mu\nu\rho\sigma} f_\rho^K u_\sigma \tag{A.46}$$

We then calculate the "derivative" term relative to the scalar field a:

$$\pi^\mu = \frac{1}{2} f_\nu^I \frac{\partial f^{I\nu}}{\partial\partial_\mu a} = \frac{1}{2} f_\nu^I \mathcal{F}^{I\nu\rho} \frac{\partial u_\rho}{\partial\partial_\mu a} = \frac{1}{2} f_\nu^I \mathcal{F}^{I\nu\rho} \left(\frac{\delta^\mu_\rho}{\sqrt{(\partial a)^2}} - \frac{\partial_\rho a \partial^\tau a \delta^\mu_\tau}{\sqrt{(\partial a)^2}(\partial a)^2} \right) \tag{A.47}$$

$$= \frac{1}{2} \frac{f_\nu^I}{\sqrt{(\partial a)^2}} \left(\mathcal{F}^{I\nu\mu} - f^{I\nu} u^\mu \right) \tag{A.48}$$

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