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Dipartimento di Fisica e Astronomia "Galileo Galilei"

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Entanglement Entropy in Black Hole microstates

Relatore	**Laureando**
Prof. Stefano Giusto	**Giulia Fardelli**

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Introduction

Black holes have always been an ideal framework to study gravity: from classical General Relativity (GR) to the more recent quantum theories of gravity, providing a unique probe of their non-perturbative aspects. The discoveries that BHs carry an entropy and Hawking radiate have opened puzzling questions, that a consistent and reliable theory of gravity should be able to solve.
When one tries to associate to a black hole an entropy, found to be proportional to its horizon area, a first problem immediately arises. In fact, according to Statistical Mechanics, we would expect that a macroscopic entropy is associated to a large degeneracy of microscopic states and we would like to find $\mathcal{N} \sim e^{S_{BH}}$ microstates for a black hole. Such an interpretation, however, is inconsistent with the *no-hair* theorem, which states that a black hole is unique once one specifies its mass, charges and angular momentum.
Another puzzle due to uniqueness theorems and thermodynamic properties, in particular to the thermal character of the Hawking radiation, is the *information paradox*, which emerges in the process of formation and evaporation of a black hole. When one considers the formation, information is lost in the sense that, when the BH has formed, it is described only by very few parameters, loosing completely track of the matter producing it. Also during the evaporation process, we assist to a loss of information caused by the fact that the emitted radiation depends only on BH temperature and consequently, could not carry other data. So in both cases, it turns out that a a pure state evolves into a mixed one, determining a violation of unitarity. Many efforts have been done trying to solve the paradox to regain a unitary theory. In an attempt to give a solution to the problem, someone suggests that Planck size corrections to Hawking computations could change the situation. However, in [1], S. Mathur shows that not even these subtle quantum corrections can solve the paradox and thus proposes, as an alternative, a completely new interpretation of black holes beyond General Relativity.

In String Theory, a black hole can be constructed as a bound state of some extended objects, called *D-branes*. Different types of D-branes exist according to their dimension and for each of these types we can introduce a *charge* for the black hole. So, in the following, we will refer to a n-charge BH, as a black hole containing n different types of brane. The D-brane description turns out to be very useful in counting microstates degeneracy, allowing to study the problem at at weak coupling (i.e. without gravity). A problem remains: how these microstates look like at strong coupling? An interesting proposal for their gravitational nature was made by Mathur in [2] and is know as the *Fuzzball proposal*. According to this program, there should be an exponential number of horizon-free solutions associated to each black hole, representing the different microstates and at least a subset of them can be well described by solutions in Supergravity, the low energy limit of String Theory. The microstates one can obtain differ from the *naive* geometry, obtained as a simple Supergravity solution, at the horizon scale. The naive solution is recovered as a description of the ensemble of the system, representing a sort of statistical average of all the quantum microstates. One would expect that the number of such microstates should reproduce the macroscopic entropy.

The Fuzzball proposal represents a very appealing picture. It succeeds in constructing examples of microstates for specific types of extremal black holes and appears to solve some of the puzzles associated with black hole physics and in particular the information paradox.

However, it is a great effort to find and characterise all the microstates solutions. And even if it has been done for the D1-D5 system, for the next simplest case, i.e. 3-charge black hole only few examples have been derived so far and the current results are still incomplete.
It is important to construct microstates for the D1-D5-P system, because it is the simplest BPS black hole one can construct in five dimensions. In fact, as we will see in detail, in classical SUGRA the 1- and 2-charge are "degenerate" black holes, in the sense that they have vanishing horizon area. Ad well, the 2-charge solutions remain important to learn features that are valid also in the black hole case.
In this work we will then concentrate on a special set of microstates carrying the same charges of the D1-D5-P supersymmetric BHs, dubbed *Superstrata* ([3, 4, 5]).

An alternative and complementary way to look at microstates is provided by AdS/CFT correspondence. According to this conjecture, a gravity theory in $(d+1)$-dimensional Anti de Sitter space-time is intimately related to the d-dimensional Conformal Field Theory living in its boundary. This duality can be applied to our case, because in a special limit, known as *decoupling* or *near horizon* limit, the 2 and 3-charge geometries become asympototically $\mathrm{AdS}_3 \times S^3$.
A holographic dictionary can be established between CFT states and geometries and it is useful both for having a better understanding of the known microstate solutions and as a guide to construct new ones. We will apply this dictionary in the study of $\frac{1}{4}$ and $\frac{1}{8}$-BPS states.

An important quantity, admitting both a CFT and a gravity interpretation is *Entanglement Entropy* (EE). Its importance relies on the fact that it provides an interesting way to probe the space-time and, conjecturally, reconstruct it. For this reason we concentrate part of this work to the derivation and computation of EE for the special class of Superstrata solutions mentioned above.
The idea of EE starts in quantum mechanics, where given a system we can divide it into two subsystem A and B and compute the Von Neumann entropy for the density matrix restricted to the subsystem A. The reduced density matrix ρ_A is obtained by tracing over the degrees of freedom of the system B from the total density matrix ρ.
The definition and the computation of EE can be extended in QFT. We will skip the details of these computations, to focus our attention on the definition of EE on the gravity side. In fact, from AdS/CFT duality, we would expect that EE can be defined also in the gravity theory. An expression for this holographic EE has appeared for the first time in a work by Ryu and Takayanagi ([6]), for D dimensional AdS space-time. Following the Ryu-Takayanagi proposal, we consider a dual CFT in $(D-1)$ dimensions, defined on a manifold $\mathbb{R}\times N$, and we divide N into two subregions A and B. According to the prescription by Ryu and Takayanagi, the EE of the region A is:

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N^D}$$

where γ_A is a $(D-2)$ minimal surface in AdS_D whose boundary is given by ∂A and G_N^D is the Newton constant in D dimensions.
When we try to apply the RT formula to our 6D microstates geometries, however, we need to face the problem that these metrics are only asymptotically $\mathrm{AdS}_3 \times S^3$ and, in general, there is no way to decouple the AdS_3 from the compact part in the spacetime interior. As suggested in [7, 8] it is necessary to generalize the RT prescription to adapt it to the 6 dimensional case. In addressing this problem, the Superstrata solutions we consider represent a very special case. For them, indeed, it is possible to define a reduced Einstein 3D metric $\tilde{g}^E_{\mu\nu}$ asymptotically AdS, which does not depend on the coordinates on S^3. One of the major aims of this Thesis has been the attempt to give a general proof that when there exists such an S^3-independent 3D Einstein metric, the 6D minimization problem reduces to the 3D one. This result allows us to restrict to the three dimensional part of the metric to compute the EE of a generic interval, thus it is sufficient to compute geodesics of $\tilde{g}^E$ and apply the RT prescription. In the final part of this work, we compute the EE a particular

3-charge microstate in the limit in which is a small correction around the classical black hole.

This thesis is organized as follows.
In Ch. 1 we give a brief introduction of the laws of BH thermodynamic in GR, then we review how to construct black holes in SUGRA, giving the explicit derivations for the 1-, 2- and 3-charge BH.
In Ch. 2 we introduce the *Fuzzball proposal* and we show how it successfully accounts for the microscopic counting of states. The second part of the chapter, instead, is devoted to AdS/CFT duality, with special attention to the *D1-D5* CFT dual to our gravity solutions.
The particular class of 3-charge microstates, we work with, is presented in Ch. 3. We sketch the details of their derivation and review briefly how these states can be interpreted in CFT.
Ch. 4 and Ch. 5 are about Entanglement Entropy. In Ch. 4 we introduce this concept in a general way on both the CFT and the gravity side. Then we specify to the derivation of how the 6D problem of determining EE reduces to the simpler lower dimensional one.
Ch. 5 contains all the explicit computations of the Entanglement Entropy for the Supestrata solutions introduced in Ch.3.

Chapter 1

Black Holes in Supergravity

Black holes represent an ideal system to study a quantum theory of gravity.
Classical General Relativity (GR) predicts the existence of BHs as solutions of the Einstein-Hilbert equations. Studying them in detail, then one discovers that they carry entropy and admit a sort of thermodynamic description. However a BH Thermodynamics poses some puzzling problems: first of all how can the macroscopic entropy have a statistical interpretation as the logarithm of the number of microstates, if classical BHs have to satisfy *no-hair theorem*. This theorem states, indeed, that stationary, charged, asymptotically flat black holes are fully characterized by their mass, charges and angular momentum. So it is incompatible with the existence of different microstates for the same black hole.
Another challenge of a quantum theory of gravity is proving a solution to the Hawking *information paradox*. The information loss, related to the Hawking radiation, during the process of formation and evaporation of a black hole, implying a violation of unitarity, creates a problem not resolved yet in a satisfactory way by classical GR.

Progress in resolving this issues has been made, at least, for a special class of supersymmetric black holes constructed in String Theory and in its low energy limit, Supergravity (SUGRA). A promising idea of how one can count and possibly construct microstates of these black holes is presented in the *Fuzzball Proposal* [2] by S. Mathur. According to these proposal microstates should be described as horizon-free solutions and at least a class of them admit a Supergravity description.

In this chapter, we start with a brief introduction of BH Thermodynamics and related open questions, such as microscopic interpretation of their entropy and the Hawking information paradox.
Since the study of black holes in Supergravity is an essential point to understand the microstates construction within the Fuzzball program, we then give an insight in the main ideas of SUGRA. In the end we review the explicit construction of some types of these black hole as bound state of some extended objects, known as p-branes.

1.1 BH thermodynamics and the information paradox

The idea that black holes could manifest a thermodynamic behaviour was motivated in part by the discovery that the horizon has a non decreasing area A and thus could be interpreted as an entropy. Between the late 1960's and the early 1970's, works by Hawking and Bekenstein further extend this thermodynamic analogy. In particular, it was discovered that a temperature can be associated to a BH and that it is related to its surface gravity $\hat{\kappa}$ via:

$$T_H = \frac{\hat{\kappa}}{2\pi} \tag{1.1}$$

This relation fixes the exact expression for the Bekenstein-Hawking entropy to be [9]:

$$S_{BH} = \frac{A}{4G_N} \tag{1.2}$$

where G_N is the Newton constant and A is the are of the horizon[1].

It is possible to formulate four laws of BH mechanics by analogy with the ones of Thermodynamics [10]:

0^{th} law: surface gravity $\hat{\kappa}$ for a stationary BH is constant over the horizon;

1^{st} law: for a black hole of mass M, angular momentum J and charge Q

$$dM = \hat{\kappa}\frac{dA}{8\pi} + \omega_H dJ + \Phi_e dQ \tag{1.3}$$

where ω_H is the angular velocity at the horizon and Φ_e is the electrostatic potential

2^{nd} law: in any classical process the horizon area must be non decreasing, $\delta A \geq 0$

3^{rd} law: it is not possible to achieve $\hat{\kappa} = 0$ through a finite number of physical operations.

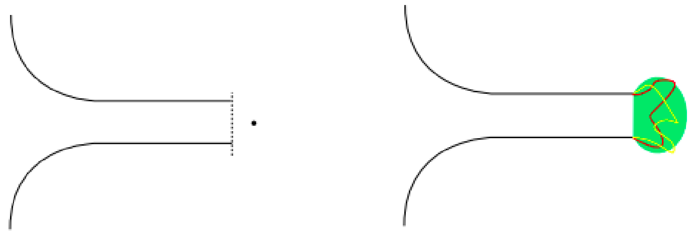
Moreover Hawking has shown that a black hole emits a thermal radiation, whose temperature, as seen from an asymptotically far away observer, is exactly the T_H in Eq. (1.1).

The thermodynamic interpretation as well as the thermal character of the emitted radiation lead to some problems and open unresolved questions. Among them there is the *Hawking information paradox*: using a semi-classical approximation, it can be shown [11] that information is lost in the processes of formation and evaporation of a black hole. For semi-classical approximation we intend that we treat matter quantum mechanically, gravity, instead, is not quantized and we use a classical metric as our background.
Let us start considering the formation of a black hole from a shell of matter in some pure quantum state. According to the "no-hair" conjecture, we expect that after its creation, the black hole will be completely characterised by very few parameters: its mass, charges and angular momentum. However, if that is true, then, we have inevitably lost the information about the initial state. In other words, if the conjecture applies, the black hole is described only by the 3 parameters listed before, which stay the same, independently from the matter collapsing to form the black hole. One might think that information could be recovered later in some way through the emitted radiation. However, because of the thermal character of this last one, this possibility can not realize. So in the process of formation it seems that there is no way to avoid the occurrence of the paradox. Now let us consider particles emission from the black hole. These particles are emitted as entangled pairs: let us call b_i the members of the pairs that stay outside the horizon and can escape to infinity, while the particles inside the horizon are denoted with c_i. If one looks only to the radiation outside the horizon, then the system is in a mixed state. At this point this result, however, does not imply any information lost. The mixed nature of the outgoing radiation is due to the fact that we are looking only at half of the system, but there exist still the quanta c_i, with which the b_is can be entangled, avoiding the paradox. The problem arises when the black hole completely evaporates because of the infalling matter. Since these particles carry negative energy, they cause the mass of the black hole to shrink and consequently they determine an increase in the surface gravity and related temperature, which leads eventually to the completely disappearance of the hole in a finite time. After the complete evaporation we are left only with the b_i particles entangled with nothing, but this means that a pure initial state has evolved to a totally mixed one (the thermal outgoing radiation) and consequently it implies a undesired violation of unitarity.

To evade the paradox and try to restore unitarity, one might think that small corrections, at scale $\frac{\ell_p}{R_H}$[2], could in some ways carry the information and solve the problem. However, in

[1] Here and in the following we use natural units $c = 1 = \hbar$

[2] Where ℓ_p is Planck length and R_H the horizon radius



(a) (b)

Figure 1.1: Comparison between the traditional black hole picture (a) and the Fuzzball proposal (b).

[12] Mathur has shown that small deviations from Hawking calculations can not provide a successfull resolution and that only order one corrections could avoid the information loss. We postpone a more detailed description of Mathur argument to Ch. 4, where we introduce entanglement entropy and related properties, which are essential for Mathur derivation. Anyway, this argument strongly suggests that in attempt to have an unitary theory without losing information, it is necessary a deep reinterpretation of what a black hole is. In particular, Mathur proposal, encoded in his *Fuzzball program* [2], is to overcome the traditional picture of BHs and consider, instead, a black hole as an effective description of an ensemble of geometries. These microstates differ from the traditional BH at the horizon scale: where traditional BHs have a horizon and a singularity, a microstatate is regular ending in a smooth cap, as depicted in Fig. 1.1. This difference is the key to escape from Hawking paradox: we will point it out more clearly and explain better in the following.

In the next chapter we will analyse in detail this Fuzzball proposal, but let us first introduce the Supergravity framework, where this program is realised.

1.2 Supergravity Framework

According to the Fuzzball proposal, black holes microstates can be derived as smooth and regular solutions of classical Supergravity. So a brief introduction about this theory is necessary, even if a complete dissertation is away from the purposes of this work.

A Supergravity theory is a theory with local Supersymmetry. One first introduces Supersymmetry as a global space-time symmetry and then localizes it. As a global symmetry, SUSY interchanges bosons B and fermions F and determines, in this way, that every particle has to be accompanied by a superpatner with opposite statistics and together they form super-multiplets. Supersymmetric generators are a set of fermionic operators Q_α, where α is a spinor index, mapping bosons into fermions and whose infinitesimal action (parametrized by a spinor ϵ) can be written as:

$$\delta B \sim \bar{\epsilon} F, \qquad \delta F \sim \sigma^\mu \epsilon \partial_\mu B$$

In generic D dimensions they satisfy an algebra of the form [13]:

$$[P, Q] = 0\,, \qquad [M, Q] \sim Q \tag{1.4a}$$

$$\{Q_\alpha, Q_\beta\} \sim (\mathcal{C}\Gamma^\mu)_{\alpha\beta}\, P_\mu + a\, (\mathcal{C}\Gamma^{\mu_1 \cdots \mu_p})_{\alpha\beta}\, Z_{[\mu_1 \ldots \mu_p]} \tag{1.4b}$$

where P and M are the generators of the Poincaré algebra, $\mathcal{C}$ is the charge conjugation, Γ's are antisymmetrised products of gamma matrices and finally Z are called the central charges, since it can be shown they commute with all the supersymmetric algebra.

Then, when one allows the parameter ϵ to depend on the spacetime coordinates, then Supersymmetry becomes local. The locality of the Supersymmetry implies invariance under diffeomorphism, but a theory invariant under general coordinate transformation is nothing but a theory of gravity. One can conclude that a theory having local Supersymmetry contains automatically gravity, we call such a theory Supergravity and its gauge mediator is a spin 3/2 particle labelled *gravitino*.

Either the Supergravity theory and supersymmetric algebra depend and take a specific form according to the dimension D of the space-time and to the number of supersymmetries $\mathcal{N}$ (then generators take a further index: Q^I_α with $I = 1, \dots, \mathcal{N}$). The number $\mathcal{N}$ and the dimensionality of the space time are related: it can be shown that in order to have a consistent theory of gravity, in D=4 the maximum allowed number of supersymmetries is $\mathcal{N} = 8$ (in D=4, it corresponds to $4\mathcal{N} = 32$ real supercharges[3]) , while the biggest dimension for a Supergravity theory is D=11 with $\mathcal{N} = 1$.

The dimension of the spacetime D and the number of supersymmetries $\mathcal{N}$ differentiates supergravity theories. It can be demonstrated that in $D = 4$ the maximum number of supersymmetries allowed in order to have a consistent theory of gravity is $\mathcal{N} = 8$ and that that the biggest dimension for a supergravity theory is $D = 11$ ($\mathcal{N} = 1$).

Let us conclude this introduction to Supergravity with a comment on a particular class of multiplets called *BPS states*. Let us specify for the moment to the case $D = 4$, we introduce the fermion generators Q^I_α and its conjugate $\bar{Q}^I_\alpha$ ($\alpha = 1, 2$). The corresponding Supersymmetric algebra contains the commutator:

$$\{Q^I_\alpha, \bar{Q}^J_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu \delta^{IJ}\,, \qquad \{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta} Z^{IJ} \quad (Z^{IJ} = -Z^{JI}) \tag{1.5}$$

where we have defined $\sigma^\mu = (\mathbb{1}, -\sigma_i)$, σ_i are Pauli matrices. Now let us consider for simplicity the case in which there are 2 real generators: $Q_1 = Q_1^\dagger$ and $Q_2 = Q_2^\dagger$ and to be in the rest frame ($P_\mu = m$). Eqs. (1.5) become:

$$Q_1^2 = m = Q_2^2, \qquad \{Q_1, Q_2\} = Z \tag{1.6}$$

Let us call $|\psi\rangle$ a generic state, then exploiting the hermiticity of the supercharges we get:

$$\langle\psi| (Q_1 \pm Q_2)^2 |\psi\rangle \geq 0 \tag{1.7}$$

$$\langle\psi| (Q_1 \pm Q_2)^2 |\psi\rangle = 2m \pm Z \tag{1.8}$$

so

$$m \geq \frac{|Z|}{2} \tag{1.9}$$

States saturating the bound, that is to say states such that $2m = |Z|$, are called BPS (from Bogomol'nyi–Prasad–Sommerfield) and they are invariant under half of the SUSY charges. As an example, consider the state $(Q_1 - Q_2)|\psi\rangle$. If it saturates the condition (1.9), then $(Q_1 - Q_2)|\psi\rangle = 0 \Rightarrow Q_1|\psi\rangle = Q_2|\psi\rangle$. These two states are not independent, so the multiplet is correctly described by the *short multiplet*:

$$\{|\psi\rangle, Q_1|\psi\rangle\} \tag{1.10}$$

where we could have chosen equivalently Q_2 instead of Q_1.
Similar arguments apply in higher dimension. In general, from the positiveness of the Hilbert space one can infer a relation similar to Eq. (1.9) and obtain the Bogomol'nyi–Prasad–Sommerfield bound:

$$E \geq |Z_r|, \qquad r = 1 \ \dots, \frac{\mathcal{N}}{2} \tag{1.11}$$

where Z_r is any central charge eigenvalue. Equality holds for *BPS* states and as a consequence they must be annihilated by at least one SUSY generator Q, in this sense they are supersymmetric.

[3]The number of real supercharges depends on the dimension d_R of the spinorial representation in D dimension:

D	2	4	6	10	11
d_R	2	4	8	16	32

1.2.1 10 dimensional *SUGRA*

In this Thesis we will work only with 10 dimensional Supergravity. It represents the low energy limit of *Type IIA* and *Type IIB* Superstring Theory [13] and carries the highest number of allowed supercharges, 32, with in $D = 10$ corresponds to $\mathcal{N} = 2$ supersymmetries. Type IIB and IIA differ for their field content. So let us look in detail to their massless bosonic spectrum, which can be divided in 2 sectors:

- **NS-NS sector**, containing the string metric $G_{\mu\nu}$, the 2-form Kalb Ramond potential B_2 and the scalar dilaton Φ (such that the string coupling is $g_s = e^{\Phi_\infty}$). The fundamental string, denoted with F1, is electrically charged w.t.r. to the field B_2 while magnetically to the NS5-branes (in Sec.1.2.2 we will explain what these extended objects are and we will introduce the concept of D-branes);
- **R-R sector**, whose fields consist in p-form potentials C_p (p is even in Type IIB and odd in Type IIA), whose associated field strength is defined as $F_{p+1} = dC_p$. The R-R gauge fields $C_{(p+1)}$ couple electrically to Dp-branes and magnetically to D$(6-p)$-branes.

The bosonic part of the low energy 10 D effective action, at the second derivative level, is [14]:

$$S_{10d} = S_{NS\text{-}NS} + S_{R\text{-}R} + S_{CS} \tag{1.12}$$

S_{NS} contains contributions from NS-NS fields and it is the same for both Type IIA-IIB SUGRA:

$$S_{NS\text{-}NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) \tag{1.13}$$

where H is B_2 field strength ($H \equiv dB$). The ten-dimensional gravitational coupling constant κ_{10} is related to G_N^{10} (ten-dimensional Newton constant) and to the string coupling g_s and string length ℓ_s:

$$2\kappa_{10}^2 = 16\pi G_N^{10} = \frac{(2\pi\ell_s)^8 g_s^2}{2\pi} \tag{1.14}$$

This action is given in the *string frame*. It can be anyway rewritten in a different frame, the *Einstein frame*, where the first term in (1.13) is expressed in the standard Einstein-Hilbert form [4] just rescaling the metric as:

$$g_{\mu\nu}^E = e^{-\Phi/2}G_{\mu\nu} \tag{1.17}$$

The second and the third terms in (1.12), respectively dubbed Ramond and Chern-Simons terms, depend instead on the specific Supergravity type we are considering because they contain R-R fields. In the table below, we report the expressions for these actions for Type IIA and IIB:

	Type IIA	Type IIB
$S_{R\text{-}R}$	$-\frac{1}{4\kappa_{10}^2}\int_{10}\sqrt{-G}\left(\lvert F_2\rvert^2 + \left\lvert\tilde{F}_4\right\rvert^2\right)$	$-\frac{1}{4\kappa_{10}^2}\int_{10}\sqrt{-G}\left(\lvert F_1\rvert^2 + \left\lvert\tilde{F}_3\right\rvert^2 + \frac{1}{2}\left\lvert\tilde{F}_5\right\rvert^2\right)$
S_{CS}	$-\frac{1}{4\kappa_{10}^2}B_2 \wedge F_4 \wedge F_4$	$-\frac{1}{4\kappa_{10}^2}C_4 \wedge H_3 \wedge F_3$

[4] Let us consider the action (1.13) in generic D dimension. We want to write the piece $\int \sqrt{-G}e^{-2\Phi}R$ as the standard Einstein Hilbert action $\int d^Dx\sqrt{-g^E}R_E$ through a metric redefinition $G_{\mu\nu} = e^{\Delta}g^E_{\mu\nu}$. In order to fix Δ let us remember that under a rescaling the determinant transform as $G \equiv \det G_{\mu\nu} \to e^{D\Delta}g$ and $R \to e^{-\Delta}R_E$. To write the action in the standard form it is required:

$$e^{\frac{D\Delta}{2}-2\Phi-\Delta} = 1 \qquad \Leftrightarrow \qquad \Delta = \frac{4\Phi}{D-2} \tag{1.15}$$

. So going from the string frame to the Einstein one, the metric changes as:

$$g_{\mu\nu}^E = e^{-\frac{4\phi}{D-2}}G_{\mu\nu} \tag{1.16}$$

where we have defined $\sqrt{-G}|F_i|^2 \equiv F_i \wedge *F_i$ and:

$$\tilde{F}_3 \equiv F_3 - C_0 \wedge H_3, \qquad \tilde{F}_4 = F_4 - C_1 \wedge H_3, \qquad \tilde{F}_5 \equiv F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$$

For Type IIB Supergravity one has to impose an additional self duality constraint on the 5-form strength : $\tilde{F}_5 = *\tilde{F}_5$.

Before describing Dp branes, the building blocks of BH microstate, let us introduce two necessary tools for this construction: dimensional reduction and dualities.

Dimensional reduction

Dimensional reduction allows to pass from a higher dimensional theory to a lower one through the compactification of someone of the original space directions.
As an example, consider the *Kaluza Klein* reduction of the D dimensional action containing graviton and dilaton, by compactifying one direction on a circle of radius R [15]. The D dimensional action, we start with is:

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-G} e^{-2\Phi_D} \left(R + 4\partial_M \Phi_D \partial^D \Phi_D \right) \tag{1.18}$$

where $x^M \equiv (x^\mu, x)$, $\mu = 0, \ldots, D-2$ and x is periodic ($x \simeq x + 2\pi R$).
To get the (D-1) action, one has to:

- decompose the metric in terms of the lower dimensional metric $\bar{g}_{\mu\nu}$, the Kaluza-Klein gauge fields A_μ and scalar σ as

$$G_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} + e^{2\sigma} A_\mu A_\nu & e^{2\sigma} A_\mu \\ e^{2\sigma} A_\nu & e^{2\sigma} \end{pmatrix} \tag{1.19}$$

- consider the (D-1) dimensional dilaton

$$\bar{\Phi}_{D-1} = \Phi_D - \frac{\sigma}{2} \tag{1.20}$$

- consider the lower dimensional Ricci scalar $\bar{R}$ which is related to the the one defined in D dimensions via:

$$R = \bar{R} - \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu}$$

  with $F_{\mu\nu}$ the field strength of A_μ.

If we Fourier expand the action (1.18) and considering only massless, x-independent contributions of the reduced action, we finally obtain for the reduced action:

$$S = \frac{1}{2\kappa_{D-1}^2} \int d^{D-1} x \sqrt{-\bar{g}} e^{-2\bar{\Phi}} \left(\bar{R} + 4\partial_\mu \bar{\Phi} \partial^\mu \bar{\Phi} - \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} \right) \tag{1.21}$$

$$\frac{1}{\kappa_{D-1}^2} = \frac{2\pi R}{\kappa_D^2} \tag{1.22}$$

S duality

S duality is a duality mapping states and vacua of a theory with coupling constant g to the ones of a theory with coupling $\frac{1}{g}$.
In Type IIB Supergravity this duality acts:

- changing the sign of the dilaton Φ and consequently, since its VEV is related to the string coupling constant, it maps $g_s \to g'_s = \frac{1}{g_s}$ (in a way that $(g'_s)^{1/4} \ell'_s = g_s^{1/4} \ell_s$);

- exchanging B_2 and C_2, while $\tilde{F}_5$, together with momentum P, stay unaltered. As a consequence, fields in the action change accordingly as

$$D_1 \leftrightarrow F_1, \qquad D_5 \leftrightarrow NS_5, \qquad D_3 \leftrightarrow D_3 \tag{1.23}$$

In the string frame the relevant set of transformation is:

$$\Phi' = -\Phi \tag{1.24a}$$

$$G'_{\mu\nu} = e^{-\Phi} G_{\mu\nu} \tag{1.24b}$$

$$B'_2 = C_2 \tag{1.24c}$$

$$C'_2 = -B_2 \tag{1.24d}$$

$$\tag{1.24e}$$

In type IIB S duality relates, without changing type, two versions of the same theory: one with strong coupling and the other one with small coupling.
The strong coupling limit of Type IIA, instead, is a completely different 11 dimensional theory, called *M theory*.

T duality

T duality is a symmetry in String Theory. T duality on a circle switches the winding and the momentum of a string and interchanges Type IIA and Type IIB.
Suppose that we want to apply T duality along an isometry direction y, such that y parametrizes a circle of radius R. In the passage from Type IIA to Type IIB:

- the radius of the circle changes as

$$\frac{R'}{\ell'_s} = \frac{\ell_s}{R}, \qquad \ell'_s = \ell_s$$

- the coupling constant gets rescaled, in order to leave invariant the low energy effective action, as

$$\frac{g'_s}{\sqrt{R'/\ell'_s}} = \frac{g_s}{\sqrt{R/\ell_s}}$$

and this affects consequently the dilaton.

To be more precise, let us write the metric as:

$$ds^2 = g_{yy}(dy + A_\mu dx^\mu)^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu \tag{1.25a}$$

$$B_2 = B_{\mu y} dx^\mu \wedge (dy + A_\nu dx^\nu) + \hat{B}_2 \tag{1.25b}$$

$$C_p = C_{p-1} \wedge (dy + A_\mu dx^\mu) + \hat{C}_p \tag{1.25c}$$

where hatted quantities have no component along y, the direction of the duality.
Under T-duality, metric and fields get transformed as:

$$ds'^2 = g_{yy}^{-1}(dy + B_{\mu y} dx^\mu)^2 + \hat{g}_{\mu\nu} dx^\mu dx^\nu \tag{1.26a}$$

$$e^{2\Phi'} = g_{yy}^{-1} e^{2\Phi} \tag{1.26b}$$

$$B'_2 = A_\mu dx^\mu \wedge dy + \hat{B}_2 \tag{1.26c}$$

$$C'_p = \hat{C}_{p-1} \wedge (dy + B_{\mu y} dx^\mu) + C_p \tag{1.26d}$$

The exchange of the fields $g_{\mu y}$ and $B_{\mu y}$ has the physical meaning of swapping winding number (F1) and momentum P. From the transformations of C_p forms we can guess how the dimension of the Dp-branes is affected by T-duality. If y is a direction parallel to the brane worldvolume, then D$p \to$ D$(p-1)$; while if the transformation is performed along a transverse direction the brane acquire an additional Neumann boundary condition[5].

[5] How spatial dimensions of branes change under a T duality depends on the exchange of Neumann and Dirichlet boundary conditions.

1.2.2 Branes

Type II supergravity admits as solutions some extended object known as p-branes. As mentioned before these membranes are charged with the respect to the forms appearing in the Supergravity action. To understand better their behaviour, let us first recall how a gauge field interacts with a point particle in classical electrodynamics. Let us consider a point particle in 4 dimensions with charge q interacting with the 1 form A_1; the interaction Lagrangian is given by:

$$\mathcal{L}_{int} = q \int_\gamma A_1 \tag{1.27}$$

where γ is the world-line of the particle. We know that the electric charge (Q_e) and magnetic one (Q_m) can be read directly from the field strength $F_2 = dA_1$ (and its Hodge dual $\tilde{F} = *F$) as an integral over a 2-sphere S^2:

$$Q_e = \int_{S^2} \tilde{F}, \qquad Q_m = \int_{S^2} F \tag{1.28}$$

We can generalize Eq. (1.27) for a p dimensional spacelike surface, with charge density μ, interacting with a $(p+1)$ form. The lagrangian is:

$$\mathcal{L}_{int} = \mu \int_{\gamma_{p+1}} A_{p+1} \tag{1.29}$$

We have seen that in SUGRA there are C_p forms, whose field strength is defined as $F_{p+1} \equiv dC_p$. In the light of the previous reasonings, one expects that

- a p form couples electrically to a $(p-1)$ brane
- and magnetically to a brane of dimension $D - p - 3$.

Now we can interpret the fields of the Supergravity action as electric and magnetic sources for the B field (respectively, F1 fundamental string and NS 5-brane), while RR fields are such that a p-brane in 10 D is charged electrically for C_{p+1} and magnetically for C_{7-p}.

It was Polchinski in [16] to propose that objects carrying RR charges are Dp-branes. Moreover he showed that D-branes breaks half of the supersymmetries of the original theory and thus they are BPS states. Qualitatively, we can think that Dp-branes and p-branes describe the same BPS states, but in different regime of validity: the first ones in String Theory, while the second ones in the context of Supergravity. The two descriptions are anyway related: the number of Dp-branes is connected to the charges of p-branes, trough quantization.

1.3 Constructing Black Holes

In this section we are going to present how to construct extremal BHs (or equivalently BPS states) in Supergravity, using p-branes (see for example [13, 2]). We have already said that each type of D-brane halves the number of supersymmetries of the vacuum solution, which we assume to be a maximally supersymmetric one (i.e. 32 supercharges). For this reason we

D branes are defined in String theory as the objects where open strings can end. Let us consider an open string $X^\mu(\tau,\sigma)$, where τ and σ parametrize its world-sheet ($\mu = 0, \dots D$). According to String Theory, the end points ($\sigma = 0$ and $\sigma = \sigma_1$) of an open string must satisfy one of the following boundary conditions:

- *Neumann boundary condition*: $\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=0,\sigma_1} = 0$. In this case the endpoints are free in space and they are only constrained to move at the speed of light and orthogonal to the string velocity vector.
- *Dirichlet boundary condition*: $\partial_\tau X^\mu(\tau,\sigma)|_{\sigma=0,\sigma_1} = 0$, fixed endpoint condition.

Now if $X^0, \dots, X^p$ are Neumann, they define a p spatial dimensional hyperplane, a Dp-brane. To satisfy the remaining $D - p$ boundary conditions, the endpoints must lie on these p-spatial dimensional objects.

will refer to the solution with only one type of brane as $\frac{1}{2}$-BPS state (16 supercharges left) or 1-charge solution, to the D1-D5 system as $\frac{1}{4}$-BPS and so on... We are focusing on BPS states because of their importance in computing entropy, as we will clarify in a moment.

Considering BPS states is a crucial point in the research for a statistical description of BHs as an ensemble of different microstates. In Sec. 1.1 we have seen that a BH presents a macroscopic entropy S_{BH} of Eq. (1.2). Taking a statistical point of view, one would interpret S in terms of the number N_{ms} of microscopic states as

$$N_{ms} = e^{S_{BH}} \tag{1.30}$$

As we have already stressed before, such a description causes a problem, since it seems to contradict the *no hair* theorem. How it can be possible to conciliate the existence of N_{ms} different microstates, with the statement that a BH is unique after the definition of its mass, charges and angular momentum?
In the attempt to obtain a consistent picture, it is necessary to go beyond GR and to look at how to construct BH in String Theory and in Supergravity. In these frameworks, one can find and investigate microstates. A consistent microscopical analysis is possible essentially by the use of BPS states. Thanks to their supersymmetric nature, for these states it is meaningful a direct comparison between S_{BH} and the statistical one $S_{stat} = \log N_{ms}$. Let us clarify better this point.

S_{BH} is determined in a strong coupling regime because it represents the entropy of the BH and a black hole exists only when the coupling $g_s \gg 1$. On the other hand, we are able to count the degeneracy of the states for the S_{stat} only in the opposite regime $g_s \to 0$. If the coupling is small, indeed, the theory can be well described by a free field theory and it is easier to determine the number of microstates.
In general, results obtained at different value of g can not be directly compared. However, if states are BPS, then the index related to the difference of bosonic and fermionic states, which accounts for the degeneracy, is protected by Supersymmetry and does not change when g_s changes. This is the reason why from the beginning, we have specialized to BPS solutions.
In particular, in the following, we will show explicitly the accordance between S_{BH} and S_{stat} studying a special class of five dimensional black holes, obtained from the dimensional reduction of p-branes solutions in 10 D Supergravity.

Before proceeding with the microstate counting, let us fist review how Black Holes can be constructed as solution of classical Supergravity. In SUGRA, there are two ways in which one can derive a BH solution.

- A direct method: we can start directly from a BPS configuration deriving the expressions for the metric and for the fields from supersymmetry constraints.
- An indirect method, in which we can work with non BPS states starting from a generic vacuum solution (satisfying the SUGRA equations) and then, applying a set of boosts and dualities, we can add charges to the original state. Only in the end, we take the extremal limit

In this Thesis, we will work only using the second approach. In particular we are interesting in Type II Supergravity solutions, whose topology reduces to $\mathbb{R}^{1,4} \times S^1 \times T^4$ asymptotically. Let us denote by (t, x^i) the non-compact direction and choose polar coordinates for $\mathbb{R}^4$:

$$\begin{cases} x^1 = r\sin\theta\cos\phi \\ x^2 = r\sin\theta\sin\phi \\ x^3 = r\cos\theta\cos\psi \\ x^4 = r\cos\theta\sin\psi \end{cases} \qquad \theta \in \left[0, \frac{\pi}{2}\right] \qquad \phi, \psi \in [0, 2\pi]$$

Finally y is the coordinate of the S^1 ($y \simeq y + 2\pi R$) and z_a ($a = 1, \ldots, 4$) label T^4 directions. The *solution generating technique* we will follow, is a pure algebraic procedure to obtain new

BHs solutions from known ones. Let us start from a five dimensional version of Schwarzschild BH, direct product with $S^1 \times T^4$:

$$ds_{10}^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + dy^2 + (dz^a)^2 \tag{1.31}$$

sum over a are understood. The power of $\frac{1}{r}$ appearing in the solution, depends on the dimensions of the compact space.
Since we have taken the Schwarzschild BH, which is a vacuum solution of GR, and we have just taken its direct product with $S^1 \times T^4$, the metric (1.31) is a solution of 10 dimensional Supergravity. Since there are no dilaton neither gauge fields, it can be interpreted both in string and Einstein frame. Starting from Eq.(1.31), we can dress it with charges and in some limits obtain the desidered BPS solutions. Once we have derived the metric, then to compute the horizon area we can either[2]:

1. look at the 5D non compact dimensions after reduction on $S^1 \times T^4$ and compute the area A_5 in Einstein metric, obtaining the entropy as $S_{BH} = \frac{A_5^E}{4G_5}$;
2. derive the horizon area directly from the 10 dimensional metric and then use $S_{BH} = \frac{A_{10}^E}{4G_{10}}$

The two results must match up to the identification:

$$G_5 = \frac{G_{10}}{(2\pi R)(2\pi)^4 V_4}, \qquad G_{10} = \frac{(2\pi)^7 g_s^2 \ell_s^8}{16\pi} \tag{1.32}$$

where $(2\pi)^4 V_4$ is the volume of the torus.

1.3.1 1-charge solution

Now suppose to start from Eq. (1.31) and to add a charge. This can be done through a *boost*:

$$\begin{pmatrix} dt \\ dy \end{pmatrix} \to \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} dt \\ dy \end{pmatrix} \equiv \begin{pmatrix} dt' \\ dy' \end{pmatrix} \tag{1.33}$$

The boosting procedure takes solution to solution, in the sense that the transformed metric still satisfies the equations of motion, but the metric we obtain is really a new and different one. The reason is that a boost is not a globally defined transformation. Since it involves a compact direction y, with a definite periodicity, it creates an identification also on the time coordinate, which has no physical meaning. So the right way to proceed is to decompactify first, then apply the boost changing the coordinates as Eq. (1.33) and finally compactify back the S^1 direction. The solution obtained in this way is really a new one.
After the boost, the metric becomes:

$$\begin{aligned} ds^2 = &\underbrace{\left(1 + \frac{2M}{r^2}\mathrm{sh}^2\alpha\right)}_{S_\alpha} dy^2 - \left(1 - \frac{2M}{r^2}\mathrm{ch}^2\alpha\right) dt^2 + \frac{4M}{r^2}\mathrm{ch}\alpha\,\mathrm{sh}\alpha\, dydt \\ &+ \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + (dz^a)^2 \end{aligned} \tag{1.34}$$

This can be interpreted in Type IIA SUGRA as a wave carrying momentum P_y (some other details can be found in Ap.B). To pass to F1, we need to T-dualize along y ($P_y \xrightarrow{\text{T duality}} F1_y$). In order to apply transformation rules (1.25)-(1.26) it is useful to rewrite the metric as:

$$\begin{cases} ds^2 = S_\alpha \left(dy + S_\alpha^{-1}\frac{2M}{r^2}\text{ch}\alpha\,\text{sh}\alpha\,dt\right)^2 - S_\alpha^{-1}\left(1-\frac{2M}{r^2}\right)dt^2 + \left(1-\frac{2M}{r^2}\right)^{-1}dr^2 + \\ \qquad r^2 d\Omega_3^2 + (dz^a)^2 \\ \Phi = 0 \\ B = 0 \\ C = 0 \end{cases} \tag{1.35}$$

After the T-duality:

$$\begin{cases} ds^2 = S_\alpha^{-1}dy^2 - S_\alpha^{-1}\left(1-\frac{2M}{r^2}\right)dt^2 + \left(1-\frac{2M}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2 + (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} \\ B_2 = S_\alpha^{-1}\frac{2M}{r^2}\text{ch}\alpha\,\text{sh}\alpha\,dt\wedge dy \end{cases} \tag{1.36}$$

This solution does not represent yet a BPS state. The extremal case is obtained only performing the *BPS limit*, which consists of:

$$\begin{cases} \alpha \to \infty \\ M \to 0 \end{cases} \qquad \text{but keeping fixed} \qquad Me^{2\alpha} = 2Q \tag{1.37}$$

where Q is the charge of the string. In this limit the function S_α becomes:

$$S_\alpha = 1 + \frac{2}{r^2}\frac{Me^{2\alpha} + Me^{-2\alpha} - M}{4} \to 1 + \frac{Q}{r^2} \equiv Z(r) \tag{1.38}$$

This limit applied to (1.36) give the $\frac{1}{2}$-BPS solution [6]

$$\begin{cases} ds^2 = Z^{-1}\left(-dt^2 + dy^2\right) + dr^2 + r^2 d\Omega_3^2 + (dz^a)^2 \\ e^{2\Phi} = Z^{-1} \\ B_2 = -Z^{-1}dt\wedge dy \end{cases} \tag{1.39}$$

in the end we want to compute the the area of the horizon from the 5 dimensional metric. Reduction on T^4 is trivial, while reducing along y determines:

$$\begin{cases} ds_5^2 = -Z^{-1}dt^2 + dr^2 + r^2 d\Omega_3^2 \\ e^{2\Phi_5} = e^{2\Phi_{10}}e^{-\sigma} = Z^{-1}Z^{1/2} = Z^{1/2} \end{cases} \tag{1.40}$$

To compute S_{BH} we have to express the metric in Einstein frame, so we multiply ds^2 by a factor $e^{-4\Phi_5/3}$ (from (1.16)):

$$ds_5^2|_E = -Z^{2/3}dt^2 + Z^{1/3}\left(dr^2 + r^2 d\Omega_3^2\right) \tag{1.41}$$

The horizon is located at $r = 0$ and its area is:

$$A = 2\pi^2\left(rZ^{1/6}\right)^3 = 2\pi^2 r^3\left(1 + \frac{Q}{r^2}\right)^{1/2} \xrightarrow{r\to 0} A \sim Qr^2 = 0 \tag{1.42}$$

So we have found that the 1-charge solution has $S_{BH} = 0$ and we will see later on, this is consistent with the microscopic count [2].

[6] Notice that we have shifted the B field. If one simply takes the BPS limit on B_2, one would obtain:

$$B_{ty} = \frac{Q}{r^2}\frac{1}{1+Q/r^2} = 1 - \left(1 + \frac{Q}{r^2}\right)^{-1}$$

The constant shift can be neglected since it has no physical meaning.

1.3.2 2-charge solution

To add another charge to the solution (1.36), we can perform another boost along y, introducing a new parameter β:

$$ds^2 = S_\alpha^{-1} S_\beta \left(dy + S_\beta^{-1} \frac{2M}{r^2} \text{ch}\beta\, \text{sh}\beta\, dt \right)^2 - S_\alpha^{-1} S_\beta^{-1} \left(1 - \frac{2M}{r^2} \right) dt^2$$
$$+ \left(1 - \frac{2m}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2 + (dz^a)^2 \tag{1.43}$$

This is a solution of Type IIB supergravity describing a fundamental string F1 carrying momentum P_y.
The final solution we want to describe is a bound state of D1-D5 branes, it can be generated from the F1-P solutions using a set of S and T dualities. We summarize them above, while explicit calculations are collected in Ap.B.

$$\begin{pmatrix} F1_y \\ P_y \end{pmatrix} \xrightarrow[\text{along } y]{\text{S duality}} \begin{pmatrix} D1_y \\ P_y \end{pmatrix} \xrightarrow{T_{z1,2,3,4}} \begin{pmatrix} D_5 \\ P_y \end{pmatrix} \xrightarrow{S} \begin{pmatrix} NS5 \\ P_y \end{pmatrix} \xrightarrow{T_y} \begin{pmatrix} NS5 \\ F1_y \end{pmatrix} \xrightarrow{T_{z1}} \begin{pmatrix} NS5 \\ F1_y \end{pmatrix} \xrightarrow{S} \begin{pmatrix} D5 \\ D1 \end{pmatrix}$$

The solution describing the D1-D5 system is:

$$\begin{cases} ds^2 = S_\beta^{-1/2} S_\alpha^{-1/2} \left[dy^2 - \left(1 - \frac{2M}{r^2}\right) dt^2 \right] + S_\alpha^{1/2} S_\beta^{1/2} \left[\left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \right] \\ \qquad + S_\beta^{1/2} S_\alpha^{-1/2} (dz^a)^2 \\ e^{2\Phi} = S_\beta S_\alpha^{-1} \\ C_2 = -S_\beta^{-1} \frac{2M}{r^2} \text{ch}\beta\, \text{sh}\beta\, dt \wedge dy - F(\theta, \alpha, \beta) d\phi \wedge d\psi \\ B_2 = 0 \end{cases} \tag{1.44}$$

Let us take the BPS limit: $M \to 0$ and $\alpha, \beta \to \infty$, keeping fixed:

$$M e^{2\alpha} = 2Q_5, \qquad M^{2\beta} = 2Q_1$$

The functions $S_{\alpha,\beta}$ becomes:

$$S_\alpha \to 1 + \frac{Q_5}{r^2} \equiv Z_5, \qquad S_\beta = 1 + \frac{Q_1}{r^2} \equiv Z_1 \tag{1.45}$$

In the limit (1.45), the metric (1.44) reduces to:

$$\begin{cases} ds^2 = \frac{1}{\sqrt{Z_1 Z_5}} \left(-dt^2 + dy^2 \right) + \sqrt{Z_1 Z_5} \left(dr^2 + r^2 d\Omega_3^2 \right) + \sqrt{\frac{Z_1}{Z_5}} (dz^a)^2 \\ e^{2\Phi} = \frac{Z_1}{Z_5} \\ C_2 = -\left\{ \left(1 - Z_1^{-1}\right) dt \wedge dy - Q_5 \sin^2\theta d\phi \wedge d\psi \right\} \end{cases} \tag{1.46}$$

Now, we decide to compute the horizon area directly in 10 D, so let us first express the metric in Einstein frame:

$$ds^2|_E = \left(\frac{Z_5}{Z_1} \right)^{1/4} ds^2 \tag{1.47}$$
$$= Z_5^{-1/4} Z_1^{-3/4} \left(-dt^2 + dy^2 \right) + Z_5^{3/4} Z_1^{1/4} \left(dr^2 + r^2 d\Omega_3^2 \right) + Z_5^{-1/4} Z_1^{1/4} (dz^a)^2 \tag{1.48}$$

The area of the horizon is the product of contributions deriving from the length of the S^1 direction, from the 3-sphere and the torus:

$$A = \underbrace{2\pi^2 \left(r^2 Z_5^{3/4} Z_1^{1/4} \right)^{3/2}}_{S^3} \underbrace{2\pi R \left(Z_5^{-1/4} Z_1^{-3/4} \right)^{1/2}}_{S^1} \underbrace{(2\pi)^4 V_4 \left(Z_1^{1/4} Z_5^{-1/4} \right)^{4/2}}_{T^4}$$
$$= 2\pi^2 (2\pi R) (2\pi)^4 V_4 r^3 (Z_1 Z_5)^{1/2} \xrightarrow{r \to 0} r \sqrt{Q_1 Q_5} = 0 \tag{1.49}$$

Again the singularity is located at $r = 0$ and the horizon area together with S_{BH} vanishes. However, as we will see in Ch. 2, the count of microstate for the D1-D5 system is found to be different from zero. This fact highlights some problems concerning the solution (1.46). So let us give some further details about the 2-charge geometry. First of all, if one computes its curvature, he realizes that it blows up as $r \to 0$. Thus, the metric (1.46) can not be a good Supergravity solution and we must include higher order derivatives corrections, which might generate a finite horizon.
Another fact about the naive geometries obtained in SUGRA, as Eq. (1.46) and (1.52) in the next section, is that they do not represents microstates. These classical SUGRA solutions provide only the description of the ensemble of the microstates and for this reason they are expected to have an entropy different from zero. Microstates geometries, instead, must be derived from a physical source in String Theory, which for example could never be localized in a point as the 2-charge naive solution. So, it is a general result, that to construct microstates we need to take a different point of view, the one of String Theory. In particular for the 2-charge case, microstates, as we will show later, are produced by stringy objects, a fundamental string carrying momentum to be precise.

1.3.3 3-charge solution

The last solution we will see is the one with three charges: reducing in 5 dimension to the so called Strominger-Vafa Black hole. Differently from the previous cases, the 3-charge geometry is a BPS solution of classical SUGRA with finite horizon area and thus it represents a real BH.

Following the indirect procedure, presented above. Take a boost with parameter γ to add another charge to (1.44). The resulting metric describes a D1-D5-P$_y$ ststem:

$$\begin{aligned} ds^2 = {} & S_\alpha^{-1/2} S_\beta^{-1/2} S_\gamma \left(dy + S_\gamma^{-1} \frac{2M}{r^2} \mathrm{sh}\gamma\, \mathrm{ch}\gamma\, dt \right)^2 - S_\alpha^{-1/2} S_\beta^{-1/2} S_\gamma^{-1} \left(1 - \frac{2M}{r^2}\right) dt^2 \\ & + S_\alpha^{1/2} S_\beta^{1/2} \left[\left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \right] + S_\beta^{1/2} S_\alpha^{-1/2} \left(dz^a\right)^2 \end{aligned} \tag{1.50}$$

Finally we apply the BPS limit, with the additional constraint $Me^{2\gamma} = 2Q_P$, we define:

$$Z_1 = 1 + \frac{Q_1}{r^2} \qquad Z_5 = 1 + \frac{Q_5}{r^2} \qquad Z_P = 1 + \frac{Q_P}{r^2} \tag{1.51}$$

The resulting metric is:

$$\begin{cases} ds^2 = \frac{1}{\sqrt{Z_1 Z_5}} \left[-dt^2 + dy^2 + \frac{Q_P}{r^2} (dy + dt)^2 \right] + \sqrt{Z_1 Z_5} \left(dr^2 + r^2 d\Omega_3^2 \right) + Z_1^{1/2} Z_5^{-1/2} \left(dz^a\right)^2 \\ e^{2\Phi} = \frac{Z_1}{Z_5} \end{cases} \tag{1.52}$$

To obtain the D1D5P black hole in 5D Einstein frame, we need to:

1. reduce on the T^4. The dilaton changes according to Eq.(1.20) as:

$$e^{2\Phi_6} = \frac{Z_1}{Z_5} \left(Z_1^{-1/4} Z_5^{1/4} \right)^4 = 1$$

2. then on S_1:

$$e^{2\Phi_5} = Z_5^{1/4} Z_1^{1/4} Z_P^{-1/2}$$

3. finally express the metric in the Einstein frame:

$$ds_5^2|_E = \left(Z_5^{-1/6} Z_1^{-1/6} Z_P^{1/3} \right) ds_5^2 \tag{1.53}$$

$$= -(Z_5 Z_1 Z_P)^{-2/3} dt^2 + (Z_5 Z_1 Z_P)^{1/3} \left(dr^2 + r^2 d\Omega_3^2 \right) \tag{1.54}$$

From the last expression we can read off directly the horizon area and compute it at the horizon ($r = 0$):

$$A = 2\pi^2 r^3 (Z_5 Z_1 Z_P)^{1/2} \xrightarrow{r\to 0} = 2\pi^2 \sqrt{Q_5 Q_1 Q_P}$$

which, as announced before, is different from zero. The associated entropy is:

$$S_{BH} = \frac{A}{4G_5} = \frac{2\pi R V_4}{g_s^2(\alpha')^4}\sqrt{Q_1 Q_5 Q_P} \tag{1.55}$$

Now that we have an expression for the entropy in terms of the macroscopic charges, we would like to interpret it from a microscopic point of view as the number of branes constructing the BH. This interpretation is made possible by the quantization of the charges Q_i. Let us look in detail to this quantization.

Let us start from Eq. (1.52) to extract the relation between mass and charge. It is sufficient to look at the expansion of g_{tt} term:

$$g_{tt} \simeq -1 + \frac{2}{3}\frac{Q_1 + Q_5 + Q_P}{r^2}$$

and it has to be compared to $g_{tt} \simeq -1 + \frac{16\pi G_5}{3\Omega_3}\frac{M}{r^2}$. We conclude that (using relations (1.32))

$$Q = \frac{2\pi g_s^2(\alpha')^4}{(2\pi R)V_4} M$$

Remember that for a Dp-brane:

$$M = n \cdot \tau_p \cdot L$$

where n is the number of superimposed D-branes, L is the length (or volume for $p > 1$) and τ_p the tension ($\tau_p = \frac{1}{g_s(2\pi)^p(\alpha')^{(p+1)/2}}$). For the momentum charge, instead, we take $M = \frac{n_p}{R}$.

$$Q_1 = n_1 \frac{g_s(\alpha')^3}{V_4} \tag{1.56a}$$

$$Q_5 = n_5 g_s(\alpha') \tag{1.56b}$$

$$Q_P = n_P \frac{g_s^2(\alpha')^4}{V_4 R^2} \tag{1.56c}$$

What is really surprising is that, if now we substitute the quantized values, all the numerical factors cancel to give:

$$S_{BH} = 2\pi\sqrt{n_1 n_5 n_P} \tag{1.57}$$

which is exactly the same as we will obtain in Ch.2 from the microscopic computations.

1.3.4 3-charge solution: the rotating BH

To conclude this chapter, we present a more general 3-charge solution, now with an additional angular momentum. To have a rotating solution, we start from the five dimensional extension of Kerr metric (Myers-Perry [17]) times $T^4 \times S^1$:

$$\begin{aligned} ds^2 = {}& dy^2 - \left(1 - \frac{M}{f}\right)dt^2 + \frac{f r^2}{(r^2+a_1^2)(r^2+a_2^2) - Mr^2}dr^2 + f d\theta^2 \\ &+ \left\{r^2 + a_1^2\left(1 + \frac{M}{f}\cos^2\theta\right)\right\}\cos^2\theta d\psi^2 + \frac{2Ma_1a_2}{f}\sin^2\theta\cos^2\theta d\psi d\phi \\ &+ \left\{r^2 + a_2^2\left(1 + \frac{M}{f}\sin^2\theta\right)\right\}\sin^2\theta d\phi^2 + \frac{2M}{f}\left(a_1\cos^2\theta d\psi + a_2\sin^2\theta d\phi\right)dt + (dz^a)^2 \end{aligned} \tag{1.58}$$

where

$$f = r^2 + a_1^2 \sin^2\theta + a_2^2 \cos^2\theta$$

and a_1 and a_2 are parameter related to the two angular momenta.
The procedure is analogous to the previous one, so we will not repeat the chain of dualities needed to add the three charges and we will report directly the results obtained in [18].

The non extremal rotating D1-D5-P solution (we slightly change the notation calling δ_i the boost parameters, $i = 1, 5, P$) is found to be [18]:

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{M\cosh^2\delta_p}{f}\right)\frac{dt^2}{\sqrt{Z_1 Z_5}} + \left(1 + \frac{M\sinh^2\delta_p}{f}\right)\frac{dy^2}{\sqrt{Z_1 Z_5}} - \frac{M\sinh 2\delta_p}{f\sqrt{Z_1 Z_5}}dtdy \qquad (1.59)\\
&+ f\sqrt{Z_1 Z_5}\left(\frac{r^2 dr^2}{(r^2 + a_1^2)(r^2 + a_2^2) - Mr^2} + d\theta^2\right)\\
&+ \left[\left(r^2 + a_1^2\right)\sqrt{Z_1 Z_5} + \frac{\left(a_2^2 - a_1^2\right)K_1 K_5 \cos^2\theta}{\sqrt{Z_1 Z_5}}\right]\cos^2\theta d\psi^2\\
&+ \left[\left(r^2 + a_2^2\right)\sqrt{Z_1 Z_5} + \frac{\left(a_1^2 - a_2^2\right)K_1 K_5 \sin^2\theta}{\sqrt{Z_1 Z_5}}\right]\sin^2\theta d\phi^2\\
&+ \frac{M}{f\sqrt{Z_1 Z_5}}\left(a_1\cos^2\theta d\psi + a_2 \sin^2\theta d\phi\right)^2\\
&\frac{2M\cos^2\theta}{f\sqrt{Z_1 Z_5}}[(a_1\cosh\delta_1\cosh\delta_5\cosh\delta_p - a_2\sinh\delta_1\sinh\delta_5\sinh\delta_p)\,dt\\
&+ (a_2\sinh\delta_1\sinh\delta_5\cosh\delta_p - a_1\cosh\delta_1\cosh\delta_5\sinh\delta_p)\,dy]d\psi\\
&\frac{2M\sin^2\theta}{f\sqrt{Z_1 Z_5}}[(a_2\cosh\delta_1\cosh\delta_5\cosh\delta_p - a_1\sinh\delta_1\sinh\delta_5\sinh\delta_p)\,dt\\
&+ (a_1\sinh\delta_1\sinh\delta_5\cosh_p - a_2\cosh\delta_1\cosh\delta_5\sinh\delta_p)\,dy]d\phi + \sqrt{\frac{Z_1}{Z_5}}\left(dz^a\right)^2
\end{aligned}$$

$$e^{2\Phi} = \frac{Z_1}{Z_5} \qquad (1.60)$$

where:

$$Z_i = 1 + K_i = 1 + \frac{M\sinh^2\delta_i}{f} \qquad (1.61)$$

The quantized angular momenta are defined as ($J_{\phi,\psi} \in \mathbb{Z}$):

$$J_\psi = -M(a_1\cosh\delta_1\cosh\delta_5\cosh\delta_p - a_2\sinh\delta_1\sinh\delta_5\sinh\delta_p)\frac{\pi}{4G_5} \qquad (1.62)$$

$$J_\phi = -M(a_2\cosh\delta_1\cosh\delta_5\cosh\delta_p - a_1\sinh\delta_1\sinh\delta_5\sinh\delta_p)\frac{\pi}{4G_5} \qquad (1.63)$$

The BPS limit take the solution to the extremal case, whose 5D reduction reproduces the BMPV [7] Black Hole. The conditions we have to apply in taking the BPS limit are

$$M \to 0, \qquad \delta_i \to \infty \quad \text{with} \quad Me^{2\delta_i} = Q_i \qquad (1.64)$$

There is an additional constraint on the a_i, accounting for the fact that a BPS black hole in 5 dimensions can have only one angular momentum. In particular from computations, one realizes that for general J_ϕ and J_ψ, the extremal limit is singular and that it is necessary to chose $J_\psi = -J_\phi$. In the BPS limit, the parameters a_i admit an expansion for $M \to 0$ as:

$$\begin{aligned}
a_1 &= -\frac{J}{2\sqrt{Q_1 Q_5}}\frac{M}{\sqrt{Q_p}} + O(M^{3/2}) \qquad (1.65)\\
a_2 &= \frac{J}{2\sqrt{Q_1 Q_5}}\frac{M}{\sqrt{Q_p}} + O(M^{3/2})
\end{aligned}$$

[7]BMPV stands for Breckenridge, Myers, Peet and Vafa [19].

where we have introduced the dimensionful angular momentum in terms of the quantized integers $J_{\psi,\phi}$:

$$J \equiv J_\psi \frac{4G_5}{\pi} = -J_\phi \frac{4G_5}{\pi} \tag{1.66}$$

In the limit (1.64) the functions Z_i become :

$$Z_i \to 1 + \frac{Q_i}{r^2}$$

which we will keep calling Z_i for ease of notation.
The BPS metric, accordingly to (1.64), (1.65), becomes:

$$\begin{aligned} ds^2 =& -\frac{1}{\sqrt{Z_1 Z_5}}\left(1 - \frac{Q_p}{r^2}\right)dt^2 + \frac{1}{\sqrt{Z_1 Z_5}}\left(1 + \frac{Q_p}{r^2}\right)dy^2 - \frac{2Q_p}{r^2\sqrt{Z_1 Z_5}}dtdy \\ &+ \sqrt{Z_1 Z_5}\left(dr^2 + r^2 d\theta^2\right) + r^2\sqrt{Z_1 Z_5}(\cos^2\theta d\psi^2 + \sin^2\theta d\phi^2) \\ &+ \frac{2J}{r^2\sqrt{Z_1 Z_5}}(-dt + dy)\left(\cos^2\theta d\psi - \sin^2\theta d\phi\right) + \sqrt{\frac{Z_1}{Z_5}}\left(dz^a\right)^2 \end{aligned} \tag{1.67}$$

$$e^{2\Phi} = \frac{Z_1}{Z_5} \tag{1.68}$$

To find the metric of the 5 dimensional BMPV black hole, we first perform the reduction on the T^4, then the dilaton gets rescaled as:

$$e^{2\Phi_6} = 1 \tag{1.69}$$

Reducing on the S^1 is less easy and it is convenient to rewrite the metric as:

$$\begin{aligned} ds_6^2 =& \frac{Z_p}{\sqrt{Z_1 Z_5}}\left\{dy - \frac{Q_p}{r^2 Z_p}dt + \frac{J}{r^2 Z_p}(\cos^2\theta d\psi - \sin^2\theta d\phi)\right\}^2 - \frac{Z_p^{-1}}{\sqrt{Z_1 Z_5}}dt^2 \\ &+ \sqrt{Z_1 Z_5}(dr^2 + r^2 d\theta^2) + r^2\sqrt{Z_1 Z_5}(\cos^2\theta d\psi^2 + \sin^2\theta d\phi^2) \\ &- \frac{J^2}{r^4 Z_p\sqrt{Z_1 Z_5}}(\cos^2\theta d\psi - \sin^2\theta d\phi)^2 - \frac{2JZ_p^{-1}}{r^2\sqrt{Z_1 Z_5}}(\cos^2\theta d\psi - \sin^2\theta d\phi)dt \end{aligned} \tag{1.70}$$

After the reduction along y the dilaton becomes:

$$e^{2\Phi_5} = Z_p^{-1/2} Z_1^{1/4} Z_5^{1/4} \tag{1.71}$$

Written in Einstein frame, the metric is:

$$\begin{aligned} ds_5^2|_E =& -\left(Z_1 Z_5 Z_p\right)^{-2/3} dt^2 + \left(Z_1 Z_5 Z_p\right)^{1/3}\left[dr^2 + r^2 d\theta^2 + r^2(\cos^2\theta d\psi^2 + \sin^2\theta d\phi^2)\right] \\ &- \left(Z_1 Z_5 Z_p\right)^{-2/3}\left\{\frac{J^2}{r^4}(\cos^2\theta d\psi - \sin^2\theta d\phi)^2 + \frac{2J}{r^2}(\cos^2\theta d\psi - \sin^2\theta d\phi)dt\right\} \end{aligned} \tag{1.72}$$

The horizon area is computed at $r_{hor} = 0$:

$$A = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta}(g_{\psi\psi}g_{\phi\phi} - g_{\phi\psi})^2} \tag{1.73}$$

$$= 2\pi^2\sqrt{(Z_1 Z_5 Z_p)r^6 - J^2} \xrightarrow{r\to 0} 2\pi^2\sqrt{Q_1 Q_5 Q_P - J^2} \tag{1.74}$$

The resulting BH entropy,in terms of the quantized charges and angular momentum is:

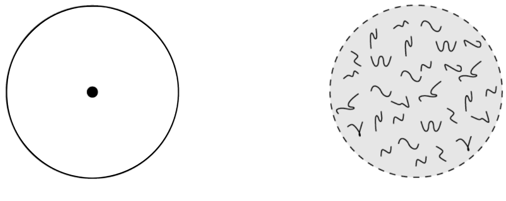
$$S_{BH} = 2\sqrt{\left(\frac{\pi R V_4}{g_s^2(\alpha')^4}\right)^2 Q_1 Q_5 Q_p - \underbrace{\left(\frac{\pi}{4G_5}\right)^2 J^2}_{J_\psi^2}} = 2\pi\sqrt{n_1 n_5 n_P - J_\psi^2}. \tag{1.75}$$

Chapter 2

The Fuzzball Proposal and the D1-D5 CFT

In the previous chapter, we have tried to emphasize the paradoxes and problems arising from a classical description of BHs, focusing our attention on the entropy problem and the information paradox. We have also guessed that a possible way to avoid the information loss is to allow for order one corrections to Hawking computations and think of the information as distributed all throughout an horizon sized region.
An alternative interpretation for the BHs, in the context of a quantum theory of gravity, is represented by the *Fuzzball proposal* [2]. Black holes are no more described as in the traditional way depicted in Fig. (2.1)(a), but is rather described by a "quantum fuzz"[1] of horizon size (Fig.2.1b). Fuzzballs represent microstates of the BH, one would like to count and construct them explicitly in order to make a comparison with the Bekenstein-Hawking entropy. And even if all the microstates for the 2-charge system have been identified, already for the next simple case, the 3-charge one, we still lack a complete characterisation for them. In this work we will work only with BPS states, taking the advantage from their supersymmetric nature. Supersymmetry and the large charge limit we consider, make the degeneracy of the states not to change as we change the coupling g_s. So we are allowed to count microstates at $g_s \to 0$, where there exists a treatable dual description in terms if a free Conformal Field Theory (CFT). The results we obtain will remain valid for large g_s, which is the regime in which the BH is supposed to form.

In this Chapter we will first review the *Fuzzball proposal* and the microscopic count of states for the D1-D5 and D1-D5-P system. Here and in Ch.3, we will focus on 3-charge microstates, since from the classical point of view a real black hole, with non vanishing horizon, must carry at least 3 charges.



(a) (b)

Figure 2.1: View of a traditional black hole (a) and a *fuzzball*(b)

2.1 The Fuzzball Proposal for Black Holes

Given the thermodynamic description of black holes one would expect that there exist $e^{S_{BH}}$ microstates. An attempt to construct them explicitly was started by [20, 2, 21] and, at least for the 2 charge case, all the microscopic geometries are *fuzzballs*: smooth and regular solutions with no horizon.

2.1.1 The microscopic count of state

In Sec.1.3 we have reviewed how we can construct BHs in Supergravity, we call these geometries *naive*. We expect that these naive geometries account for the macroscopic characteristics of the black holes and that they have an entropy different from zero. We have already mentioned that S_{BH} for the 2-charge BH vanishes and we have provided a possible explanation in Sec. 1.3.2.
If the classical BH represents the ensemble of the microscopic degrees of freedom and for this reason should have a non trivial S, on the other hand microstates are expected not to carry an entropy. This explains why we are looking for the D1-D5 and D1-D5-P microstates to be horizonless. It is the number of these microstates, expressed in terms of $S_{stat} = \log N_{ms}$, to be supposed to reproduce S_{BH}.

Let us now specify to 1-, 2-, 3-charge BHs introduced before and try to compute their microscopic entropy starting from their "stringy" representation.
Remember that for the 1-charge case, in SUGRA, we have obtained $S_{BH}^{1-charge} = 0$ and this result is found to be consistent with the microscopic computations. Consider the fact that the 1-charge solution is represented by a fundamental string F1 wrapping n_1 times around the compact direction S^1 in its ground state. As a consequence, its degeneracy comes from the massless zero modes descending in Superstring Theory. The total number of degrees of freedom is 256: 128 are bosonics, accounting for the degrees of freedom of fields and C forms, while the other 128 are fermionic. So $S_{stat} = \log 256$ and since it does not depend on n_1, in the large charge limit $n_1 \to \infty$, it is zero at leading order reproducing the macroscopic computation.

Introducing the 2-charge solution in Sec. 1.3.2, we have mentioned the fact that a singularity, as the one it presents, located at a point is not allowed in String Theory. Thus when considering possible microstates carrying the same charges as the D1-D5 system, we need to change completely our point of view. In particular, these microstates, as suggested in [21, 2], should be constructed from the S-dual system, that is the F1-P bound state. It consists of a long single string closing on itself after n_1 loops around the sphere S^1. The momentum P is bounded to the string, in the sense that it manifests as a travelling wave along y, carrying n_P units of momentum. This momentum can be distributed as transverse vibrations in a lot of different ways, giving rise to a large degeneracy that can be computed in the small coupling regime. To visualize the vibrations of the string we open it into a long string of length $2\pi R n_1 \equiv L_T$. The momentum is carried by transverse vibrations along T^4 and $\mathbb{R}^4$ (8 bosonic vibrations and 8 fermionic). Each harmonic of vibration of the string behaves like a harmonic oscillator and each Fourier mode k has energy equal to momentum, given by:

$$e_k = p_k = \frac{2\pi}{L_T} k \tag{2.1}$$

The total momentum, instead, is:

$$P = \frac{n_P}{R} = \frac{2\pi}{L_T}(n_1 n_P) \tag{2.2}$$

If we assume that there are m_i units of the harmonic k_i, each one carrying momentum (2.1), then to obtain total momentum P (2.2), it is necessary that:

$$\sum_i m_i k_i = n_1 n_P \tag{2.3}$$

So to determine the degeneracy, it is sufficient to count in how many different ways one can distribute $n_1 n_P$ harmonics. The number of partitions of the integer $n_1 n_P$, is well approximated, by the leading term of Hardy-Ramanujan formula:

$$\mathfrak{N} = e^{2\pi\sqrt{\frac{n_1 n_P}{6}}} \tag{2.4}$$

To reproduce the correct degeneracy, we need to remember that our fundamental string vibrates along transverse directions bosonic and fermionic, so we have to consider them and correct Eq. (2.4) by a factor α: $\mathfrak{N} \sim e^{2\pi\sqrt{\alpha\frac{n_1 n_P}{6}}}$.
To determine α, let us start considering n_B bosonic directions. The momentum is distributed among bosons as $\frac{n_1 n_p}{n_B}$ and we have n_B directions, then Eq. (2.4) gets modified as:

$$\left(e^{2\pi\sqrt{\frac{n_1 n_p}{6 n_B}}}\right)^{n_B} \quad \Rightarrow \quad \alpha = n_B \tag{2.5}$$

Since fermions count half of the bosons, the correct factor α is:

$$\alpha = n_B + \frac{n_F}{2} \tag{2.6}$$

In our case:

$$\alpha = 8 + 4 = 12 \qquad \Rightarrow \qquad e^{2\pi\sqrt{2 n_1 n_P}} \tag{2.7}$$

And finally the entropy:

$$S_{stat} = \log \mathfrak{N} = 2\sqrt{2}\pi\sqrt{n_1 n_P} \tag{2.8}$$

As announced before, we obtain a different result from the computation from the naive geometry (1.46) for the reasons clarified before. In works as [21, 22] microstates geometries describing a fundamental string vibrating have been constructed. They are found to be smooth, horizonless, i.e. fuzzballs able to reproduce result (2.8). We will report some explicit example in the next Chapter.

The last case we have to consider is the D1-D5-P system. To study its degeneracy it is useful to work in the dual frame, where D1-D5-P system is mapped to the F1-NS5-P. To start with consider only a NS5 brane, i.e. $n_5 = 1$. The system is always described by a fundamental string carrying momentum P as travelling wave, but this time it lies along the NS5 and can vibrate only inside of it. Thus the allowed directions of vibrations are only the 4 ones belonging to the NS5 and transverse to the F1. So α changes as:

$$\alpha = 4 + 2 = 6 \qquad \Rightarrow \qquad S_{stat} = 2\pi\sqrt{n_1 n_P}$$

If now we allow n_5 to be greater than one, considering the fact that the result should be symmetric under the permutation of the charges due to dualities, we have that the entropy for the three charge state is:

$$S_{stat} = 2\pi\sqrt{n_1 n_5 n_P} \tag{2.9}$$

which perfectly reproduces the Bekenstein-Hawking entropy (1.57). This is a remarkable result, first derived by Stroeminger and Vafa in [23] for a slightly different system.

Similar arguments apply also on the CFT side, reproducing the microscopic computations, as we will see in detail in Sec. 2.4

2.1.2 Microstates construction

Now we can return to the problem of how to construct 2-charge microstates. We know that they should be solutions for a fundamental string with transverse vibrations (longitudinal vibrations of the fundamental string make no sense). Classically, these oscillations can be parametrized through a function $\vec{g}(v)$, where $\vec{g}$ has components along directions transverse to the string and we have defined $v \equiv t + y$. The *profile function* $\vec{g}$ can not depend also on $u = t - y$ in order to preserve supersymmetry.

In the most general case, $\vec{g}$ has 8 components g^A, with A taking value on $T^4 \times \mathbb{R}^4$. We will restrict to the case $A = 1 \dots 4$ or $A = 1 \dots 4$ plus a non trivial component along one of the T^4 direction (in Ch.3), because in the duality frame, where the system is described by a D1-D5 bound state, these solutions have rotationally invariance in the T^4.
The F1 string can have different strands and each one can carry a different vibration profile $g^{(s)}$. The solution representing this system is [2]

$$\begin{cases} ds^2_{string} = Z^{-1}\left(-dudv + Kdv^2 + 2A_i dx^i dv\right) + \left(dx^i\right)^2 + (dz^a)^2 \\ B = -\frac{1}{2}(Z^{-1} - 1)du \wedge dv + Z^{-1}A_i dv \wedge dx^i \\ e^{2\Phi} = Z^{-1} \end{cases} \tag{2.10}$$

where

$$Z = 1 + \sum_{s=0}^{n_1} \frac{Q_1^{(s)}}{\left|\vec{x} - \vec{g}^{(s)}(v)\right|^2}, \qquad K = \sum_{s=0}^{n_1} \frac{Q_1^{(s)}\left|\dot{\vec{g}}^{(s)}(v)\right|^2}{\left|\vec{x} - \vec{g}^{(s)}(v)\right|^2}, \qquad A_i = -\sum_{s=0}^{n_1} \frac{Q_1^{(s)}\dot{g}_i^{(s)}(v)}{\left|\vec{x} - \vec{g}^{(s)}(v)\right|^2} \tag{2.11}$$

where dot indicates derivative with the respect to v.
In the "black hole limit", i.e. $n_1, n_P \to \infty$ keeping g, R, V fixed, the sum over s can be approximated by an integral:

$$\sum_{s=0}^{n_1} \to \int_0^{n_1} ds = \int_{y=0}^{2\pi R n_1} \underbrace{\frac{ds}{dy}}_{1/(2\pi R)} dy = \frac{1}{2\pi R}\int_0^{L_T} dy \tag{2.12}$$

Finally, since the integrand functions depend only on v, we can further express the integral as:

$$\frac{1}{2\pi R}\int_0^{L_T} dv \tag{2.13}$$

and the charges:

$$Q_1^{(s)} \to \frac{Q_1}{n_1} \tag{2.14}$$

In this limit, the general form of the solution (2.10) stays unvaried and only the functions get modified as:

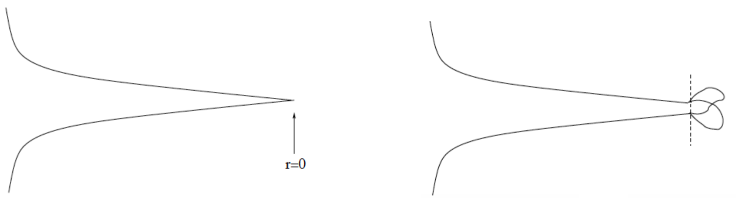
$$Z = 1 + \frac{Q_1}{L_T}\int_0^{L_T} \frac{dv}{|\vec{x} - \vec{g}(v)|^2}, \quad K = \frac{Q_1}{L_T}\int_0^{L_T} dv \frac{\left|\dot{\vec{g}}(v)\right|^2}{|\vec{x} - \vec{g}(v)|^2}, \quad A_i = -\frac{Q_1}{L_T}\int_0^{L_T} dv \frac{\dot{g}_i(v)}{|\vec{x} - \vec{g}(v)|^2} \tag{2.15}$$

In order to obtain the D1-D5 bound state, we perform the chain of dualities in Sec.1.3.2, remembering how charges change under these operations (B.14), thus:

$$Z \to 1 + \frac{Q_5'}{L}\int_0^L \frac{dv}{|\vec{x} - \vec{g}(v)|^2} \equiv Z_5 \tag{2.16}$$

$$1 + K \to 1 + \frac{Q_5'}{L}\int_0^L dv \frac{\left|\dot{\vec{g}}(v)\right|^2}{|\vec{x} - \vec{g}(v)|^2} \equiv Z_1 \tag{2.17}$$

$$A_i \to -\frac{Q_5'}{L_T}\int_0^{L_T} dv \frac{\dot{g}_i(v)}{|\vec{x} - \vec{g}(v)|^2} \equiv A_i \tag{2.18}$$



(a) (b)

Figure 2.2: (a) Naive 2 charge geometry with a singularity at $r = 0$ (b) Fuzzball geometries of 2-charge D1-D5, the dashed line accounts for an area such that $\frac{A}{4G} \sim \sqrt{n_1 n_5}$.

where $L = 2\pi \frac{Q_5'}{R}$. Finally, the D1-D5 solution is:

$$e^{2\Phi} = \frac{Z_1}{Z_5} \tag{2.19}$$

$$ds^2 = \frac{1}{\sqrt{Z_1 Z_5}} \left[(dy + B_i dx_i)^2 - (dt - A_i dx_i)^2 \right] + \sqrt{Z_1 Z_5} \left(dx^i \right)^2 + \sqrt{\frac{Z_1}{Z_5}} \left(dz^a \right)^2 \tag{2.20}$$

where following [22] we have defined $B_i \equiv B_{ti} = B_{iy}$ such that $dB = - *_{\mathbb{R}^4} dA$. We will return to this solution, even if with a slightly different notation in Ch.3, where we will report some examples.
The differences between the solution (2.20) and the naive geometry compared to the naive geometry are shown in Fig. 2.2.
Let us look in detail to the behaviour of the D1-D5 geometry (2.20) for different values of the parameters involved. Let us define $|\vec{x}|^2 = r^2$ and distinguish 3 different regions:

- $r^2 \gg Q_{1,5} \leftrightarrow$ **asymptotically flat regime**, in this limit

$$Z_1 \to 1, \qquad Z_5 \to 1, \qquad A_i \to 0 \tag{2.21}$$

  so the metric (2.20) becomes:

$$ds^2 \simeq -dt^2 + dy^2 + \sum_{i=1}^{4} dx^i dx^i + \sum_{a=1}^{4} dz^a dz^a \tag{2.22}$$

  and we recover Minkowski space-time.

- $r^2 \sim Q_{1,5} \gg |\vec{g}|^2$, so that[1]

$$Z_1 \to 1 + \frac{Q_1}{r^2}, \qquad Z_5 \to 1 + \frac{Q_5}{r^2}, \qquad A_i \to 0 \tag{2.23}$$

  and the metric becomes exactly the same as the naive one (1.46).

- the most interesting region is the **decoupling** or **near-horizon** limit, which occurs for $|\vec{g}|^2 \ll Q_{1,5}$ and $r^2 \ll Q_{1,5}$. It consists essentially in neglecting the asymptotically flat part in $Z_{1,5}$ (i.e. the factor 1). In this region thanks to the remaining dependence on the shape of $|g|$, the presence of difference and diverse microstates appear.
  In the decoupling region, we can further take the asymptotic limit, $r^2 \gg |\vec{g}|^2$, then

$$Z_1 \to \frac{Q_1}{r^2}, \qquad Z_5 \to \frac{Q_5}{r^2}, \qquad A_i \to 0 \tag{2.24}$$

  and the metric:

$$ds^2 \simeq \frac{r^2}{\sqrt{Q_1 Q_5}}(-dt^2 + dy^2) + \frac{\sqrt{Q_1 Q_5}}{r^2} \left(dx^i \right)^2 + \sqrt{\frac{Q_1}{Q_5}} \left(dz^a \right)^2 \tag{2.25}$$

[1]Notice that $Q_1 = \frac{Q_5}{L} \int_0^L dv \left| \dot{\vec{g}} \right|^2$

and rewriting using polar coordinates for $\mathbb{R}^4$ and changing $r \to \tilde{r} \equiv \frac{r}{\sqrt{Q_1 Q_5}}$

$$ds^2 \simeq \underbrace{\sqrt{Q_1 Q_5}\left(-dt^2 + dy^2 + \frac{d\tilde{r}^2}{\tilde{r}^2}\right)}_{\text{AdS}_3 \text{ in Poincaré coordinates}} + \underbrace{\sqrt{Q_1 Q_5}\, d\Omega_3^2}_{S^2} + \underbrace{\sqrt{\frac{Q_1}{Q_5}}\,(dz^a)^2}_{T^4} \tag{2.26}$$

with the radius $R_{\text{AdS}} = (Q_1 Q_5)^{1/4} = R_{S^3}$.
Eq. (2.26) tells us that all fuzzball solutions are asymptotically $\text{AdS}_3 \times S^3 \times T^4$. The existence of this limit motivates their study in the dual CFT.

To conclude we want to make a comment about the regularity of the solutions obtained above. While the F1-P solution contains a physical singularity, it can be shown that in the D1-D5 frame, where we define microstates, solutions are completely regular. So let us concentrate on D1-D5 solutions defined by the fields (2.16)-(2.18). The potential source of singularity are the points $\vec{x} \to \vec{g}(v)$, defining a curve in $\mathbb{R}^4$. We will soon discover that this is only a coordinate singularity.
In the curve $\vec{x} = \vec{g}$ choose a point v_0 and introduce a coordinate z to measure distances along the curve, $z \simeq \left|\dot{\vec{g}}(v_0)\right|(v - v_0)$. In the plane transverse to the curve define the coordinates $x_\perp$, then:

$$Z_5 \simeq \frac{Q_5}{L}\int_{-\infty}^{+\infty} \frac{dv}{|x_\perp|^2 + z^2} = \frac{Q_5}{L\left|\dot{\vec{g}}(v_0)\right|}\int_{-\infty}^{+\infty}\frac{dz}{\rho^2 + z^2} = \frac{Q_5}{L\left|\dot{\vec{g}}(v_0)\right|}\frac{\pi}{\rho} \tag{2.27}$$

$$Z_1 \simeq \frac{Q_5\left|\dot{\vec{g}}(v_0)\right|}{L}\frac{\pi}{\rho} \tag{2.28}$$

$$A_z \simeq -\frac{Q_5}{L}\frac{\pi}{\rho} \tag{2.29}$$

where we have used polar coordinates for the plane. From the expression for A and the from the relation $dB = - *_4 dA$, we get:

$$B = \tilde{Q}(\cos\theta - 1)d\phi$$

where $\tilde{Q} \equiv \frac{Q_5 \pi}{L}$. Now concentrate on the $y - x_\perp$ part of the 2-charge metric (2.20):

$$ds^2|_{y,x_\perp} \simeq \frac{\rho}{\tilde{Q}}\left[dy - \tilde{Q}(1-\cos\theta)d\phi\right]^2 + \frac{\tilde{Q}}{\rho}\left(d\rho^2 + \rho^2 d\theta^2 + \rho^2\sin^2\theta d\phi^2\right) \tag{2.30}$$

which is the metric of a Kaluza-Klein monopole expanded around $\rho \simeq 0$. From KK monopole theory, we know that the potential singularity at $\rho = 0$ is nothing more than a coordinate singularity. We can easily show it looking the relevant part which can develop a singularity:

$$ds^2 \simeq \frac{\rho}{\tilde{Q}}dy^2 + \frac{\tilde{Q}}{\rho}d\rho^2 \overset{\rho = r^2}{=} \frac{r^2}{\tilde{Q}}dy^2 + 4\tilde{Q}dr^2 = 4\tilde{Q}\left(dr^2 + \frac{r^2}{4\tilde{Q}^2}dy^2\right) \tag{2.31}$$

the last one is the metric of a completely regular $2D$ plane after the identification of $\frac{y}{2\tilde{Q}}$ as an angular coordinate, but this happens if and only if y has periodicity $4\pi\tilde{Q}$. This condition is trivially verified if one remembers:

$$\tilde{Q} = \frac{Q_5 \pi}{L} = \frac{Q_5\pi}{2\pi Q_5/R} \quad \Rightarrow \quad 4\pi\tilde{Q} = 2\pi R \tag{2.32}$$

Since the regularity requirement is satisfied, we can conclude that the y, ρ, θ, ϕ part of the metric is regular. For what concern the $t - z$ part we have:

$$ds^2|_{t,z} \simeq -\frac{\rho}{\tilde{Q}}\left(dt + \frac{\tilde{Q}}{\rho}dz\right)^2 + \frac{\tilde{Q}}{\rho}dz^2 = -\frac{\rho}{Q}dt^2 - 2dtdz \tag{2.33}$$

which is regular when $\rho \to 0$ and the same is true for the T^4 part.

So far we have shown that the D1-D5 solutions we have constructed have no singularity, nor horizon (otherwise we will associate to them an entropy and they would be no more microstates). Differently from the naive geometry, as shown in Fig. (2.2), these microstates end in different caps according to the choice of the profile function $\vec{g}$. To summarize different microstates can be derived considering different profiles of oscillations of a F1-P system. For a generic state it can be shown that the wavelength of vibration is [2]:

$$\lambda \sim R\sqrt{\frac{n_1}{n_P}}$$

Due to the oscillations, the transverse coordinates undergo a change:

$$\Delta x \sim \left|\dot{\vec{g}}\right|\lambda = \left|\dot{\vec{g}}\right| R\sqrt{\frac{n_1}{n_p}} \sim \Delta x \sim R\sqrt{\frac{Q_P}{Q_1}} R\sqrt{\frac{n_1}{n_p}} \sim \sqrt{\alpha'}$$

where we have used $Q_P \sim \left|\dot{\vec{g}}\right| Q_1$. The value of Δx gives a roughly approximation of the size of the region in which we are sensitive to the presence of different microstates. The naive geometry is a good approximation as far as $r > \sqrt{\alpha'}$ and for smaller values the effect of oscillations start to manifest themselves. The value $r = \sqrt{\alpha'}$ is represented in Fig.2.2 by the dashed line and delimits a ball shaped region which gives the "size" of the bound state. In order to compute the area of this ball, consider the expression for the area obtained in Eq. (1.49) and evaluate it for $r = \sqrt{\alpha'}$, then:

$$A_E \sim \sqrt{Q_1 Q_P} R V_4 \alpha'^{1/2} \quad \Rightarrow \quad \frac{A_E}{4G_{10}} \sim \sqrt{n_1 n_P} \tag{2.34}$$

This result agrees with the microscopic entropy, up to an undetermined constant, which we can not fix by this rough evaluation.

2.1.3 Summary and conclusion

We have shown that the microstates we have constructed so far for the 2-charge system are horizonless and perfectly regular. The region where they differ from the naive geometry presents a "fuzzball boundary" or stretched horizon satisfying a Bekestein-Hawking entropy relation (2.34). In this way, S_{BH} acquires a statistical connotation as the log of the number of microstates.

In our derivation we have assumed that the quantum states of the string have a large occupation number in each harmonic, and consequently they are well approximated by a classical profile. Moreover, we have seen that our microstates are horizonless and the only thing that can be interpreted as a horizon is the fuzzball boundary, depicted in Fig.2.2, by a dashed line. However it is not a true horizon as in classical GR, providing a solution for the information paradox. The absence of horizon, indeed, invalidates one of the fundamental hypothesis Hawking has done in his derivation.

To conclude, remember that for the moment the *Fuzzball proposal* remains only a conjecture. It succeeds to give a meaning to the microscopical entropy of the 2-charge microstates and tries to give an answer to the information paradox, admitting that the emitted radiation retains in some way the information relative to each microstate. At the same time, however, the proposal is not totally exhaustive: for example, has not yet been able to give a complete description of the 3-charge fuzzball microstates. As we will see in the next chapters only a subclass of geometries has been found for the the $\frac{1}{8}$-BPS state, having a good description in Supergravity. We will review the general expression of this subclass of geometries with 3 charges and try compute their Entanglement Entropy.

At this point it is useful to introduce the AdS/CFT correspondence. It provides a map between geometries and CFT states, turning out to be a useful tool for degeneracy counting and for understanding more deeply the characteristics of microstates. It eventually reveals very important to interpret the holographic EE.

2.2 AdS/CFT conjecture

Starting from the late 90th, it was proposed that there could exist a correspondence between a $d+1$ dimensional AdS space-time, defined as the *bulk*, and a Conformal Field Theory (CFT) in d dimensions living in its *boundary*. This proposal is know as *AdS/CFT correspondence* and it is the most explicit and powerful instance of gravitational holography. This conjecture was motivated mainly by two arguments:

- the number of degrees of freedom of a CFT$_d$, measured by its entropy, goes like its spatial volume, i.e. $S_{CFT} \propto \mathrm{Vol}_{d-1}$. At the same time, we have observed that the Bekestein-Hawking formula relates the entropy of BH to the area of the horizon. So, if we consider a BH in $d+1$ dimension, its entropy is proportional to the area of the boundary of a d dimensional region Σ_d, that is to say $S_{BH} \propto \mathrm{Area}(\partial\Sigma_d) = \mathrm{Vol}(\Sigma_{d-1})$. So we retrieve the same behaviour on the gravity side as well as on the CFT one.

- Another argument suggesting the existence of this correspondence follows from 't Hooft *large N limit*. 't Hooft showed that when one considers a gauge $U(N)$ theory, one can make an expansion of the amplitudes for $N \to \infty$ keeping g^2N fix, where g is the coupling constant of the Yang-Mills theory. It turns out that the different powers of N correspond to the different topologies of Feynman diagrams and consequently only certain graphs are not suppressed as N becomes large. The main contributions are found to derive from planar graphs (those that can be drawn on a paper without self-crossing). This type of expansion is very similar to the perturbative analysis of a closed string theory.

Both these arguments suggest the idea of an intimate correlation between a gravity description and CFT. What one finds, then, is the two theory are related in such a way that they are valid in opposite coupling regimes, i.e. when the gauge theory is strongly coupled, the string one is weakly coupled and viceversa.

The groundbreaking work on AdS/CFT by Maldacena [24] concentrated, in particular, on the equivalence between a four dimensional $SU(N)$ super Yang-Mills theory and Type IIB closed superstring theory defined on AdS$_5 \times S^5$. Let us summarize briefly the main points, just to provide a general idea.
In the original work [24] the correspondence is introduced as an open/closed string duality. Let us consider N coincident D3-branes in ten dimensional type IIB string theory. They can be viewed equivalently:

1. as the submanifold where open-strings can end and with which closed strings can interact;

2. are solutions of closed superstring theory, in the sense that they create a background where closed string can propagate

Now consider the low energy limit. Then the description

1. reduces now to two decoupled systems, closed strings propagating in flat space-time and massless open strings attached to the D3-branes realizing (after quantization) a $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory (SYM) with gauge group $SU(N)$;

2. represents massless closed strings in a Type IIB supergravity background, described

in general by the metric:

$$\begin{aligned} ds^2 &= Z(r)^{-1/2} \underbrace{\sum_{\mu=0}^{3} (dx^\mu)^2}_{\parallel \text{ to D3-branes}} + Z(r)^{1/2} \underbrace{\left(dr^2 + r^2 d\Omega_5^2\right)}_{\perp \text{ to D3-branes}} \\ Z(r) &= 1 + \frac{4\pi g_s N \ell_s^4}{r^4} \equiv 1 + \frac{\mathfrak{R}^4}{r^4} \end{aligned} \tag{2.35}$$

The low energy limit is equivalent to the near horizon limit: so we perform an expansion around $r \to 0$ and introduce the variable $z \equiv \frac{\mathfrak{R}^2}{r}$ then:

$$ds^2|_{nh} \simeq \mathfrak{R}^2 \frac{dz^2 + (dx^\mu)^2}{z^2} + \mathfrak{R}^2 d\Omega_5^2 \tag{2.36}$$

which is nothing than AdS5 $\times$ S^5 (both of radius $\mathfrak{R}$)

These observations lead to the correspondence

$$\begin{array}{ccc} SU(N)\ \text{SYM} & & \text{IIB on AdS}_5 \times S^5 \\ g_{YM}\ \text{s.t.}\ g_{YM}^2 = 4\pi g_s & \Leftrightarrow & (\mathfrak{R}/\ell_s)^4 = g_{YM}^2 N \end{array} \tag{2.37}$$

The two descriptions are tractable in two different regimes of the parameters, in particular:

- Type IIB becomes simple when $\frac{\mathfrak{R}}{\ell_s} \gg 1$ when it can be described as a supergravity theory. However, also the effective coupling $g_{YM}^2 N$ is large and consequently the gauge theory is not tractable in this limit;
- in the opposite limit, i.e. $g_{YM}^2 N \ll 1$, SYM can be successfully treated perturbatevely.

From these observations, it seems that boundary and bulk descriptions can not be in general tested simultaneously. It is generally true, except in certain special situations: for example when we consider observables not depending on the effective coupling constant or preserving some supersymmetries.

The correspondence between the gravity and the gauge theory is evident also when one looks at the symmetries characterising these theories. Before concentrating on these symmetries, let us make a brief comment on the SYM we are dealing with. The Yang-Mills $SU(N)$ theory, realized on the world-volume of D3-branes, has $\mathcal{N} = 4$ SUSY, that is to say it has 16 supercharges in D=4. Since it is also a conformal theory we need to add to them other 16 superconformal charges, obtaining a total of 32 supercharges. The theory admits an additional $\mathcal{R}$-symmetry $SU(4)_{\mathcal{R}}$, where an $\mathcal{R}$ symmetry does non commute with the supercharges. We can summarize the correspondence and the symmetries identifications as:

$\mathcal{N} = 4$**SYM**	$\longleftrightarrow$	**IIB on AdS**$_5 \times S^5$
$O(4,2)$ conformal group		$O(4,2)$ isometries of AdS$_5$
16 SUSY+16 conformal charges		32 supercharges in the near horizon
$SU(4)$ $\mathcal{R}$ symmetry		$SO(6) \simeq SU(4)$ S^5 isometries

Schematically, the relevant aspects of this correspondence we need to retain are that the boundary of AdS$_5$ is $\mathbb{R}^{1,3}$ and here the dual CFT lives. In a CFT states and operators are the same and they are related to fields in AdS, in such a way that the conformal dimension of CFT operators corresponds to energy in AdS.

From its appearance the AdS/CFT correspondence has been a very appealing idea, which has been extended beyond the case AdS$_5 \times S^5$ and which has found lots of possible applications. In the next section, we will investigate the CFT describing our microstates, introducing a "holographic dictionary" able to map gravity geometries to states in the dual CFT.

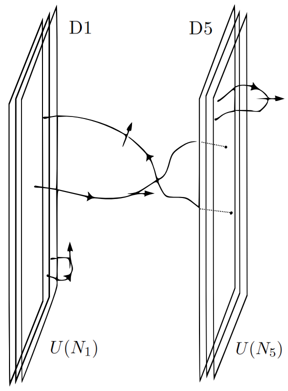


Figure 2.3: Open strings attached to n_1 parallel D1 branes and n_5 parallel D5 branes. Open strings modes give rise respectively to a $U(n_1)$ and $U(n_5)$ gauge theory.

2.3 The D1-D5 CFT

In Sec. 2.1.2 we have seen that for small r, in the decoupling limit, D1-D5 geometries are asymptotically $\mathrm{AdS}_3 \times S^1 \times T^4$. The AdS/CFT correspondence, then, suggests that there should exist a dual Conformal Field Theory, a CFT_2, carrying 8 supercharges and providing an alternative description of the same system. This field theory is called *D1-D5 CFT*. We will introduce it following mainly [25, 7, 20].

2.3.1 A general introduction

Before starting, remember that in Supergravity we have introduced the D1-D5 system as a bound state of n_1 D1 branes wrapped around the sphere S^1 and n_5 D5 branes extended in $T^4 \times S^1$. From this moment on, we will assume that the S^1 radius R is very large compared to the length of the T^4 ($R \gg \sqrt[4]{V_4}$), so that the effective description of our system is in two dimensions.
Considering the D1-D5 system, we can take 2 different prospective: we can consider it as a gauge theory or in a "branes within branes" description. Let us give more details.

Gauge Theory description

We know that open strings can start and end on the D-branes. In the case of two systems of branes (D1 and D5 in our specific case), one has to consider open strings with both endpoints on the same type of branes (let us indicate them has $1-1$ and $5-5$ strings) or one starting and ending on different branes ($1-5$ and $5-1$), as shown in Fig. 2.3.

- $5-5$ case: open strings modes gives rise to a $U(n_5)$ gauge theory with 16 supercharges in the 6 dimensional world-volume. Modes parallel to the brane represent $U(n_5)$ gauge fields, while those transverse give adjoint scalars. A similar reasoning applies to $1-1$ open strings.
- $1-5$ and $5-1$ strings: in this case we are left only with 8 supercharges as we want to describe a bound state of D1-D5 branes. So we expect that this type of string is the one describing the states we are interested in.

Remember that we have assumed that the radius of the S^1 is much larger that the length of the T^4, thus we can dimensionally reduce our theory to 1+1 dimension (time t and S^1 direction y). The potential for this 2D theory presents two class of minima known as

Coulomb and *Higgs branch*. The first one refers to the configuration in which the branes separates in the non compact space breaking the gauge group. When they are coincident, instead, we talk about the Higgs branch. This last one accounts for the bound states of the D1-D5 branes and it is the one we are interested in.

Instantonic description

Another equivalent way to look at the the D1-D5 system is to consider D1-branes as instantons of the D5-branes gauge theory. As we have already said, open string modes on n_5 coincident D5-branes give rise to a $U(n_5)$ gauge theory, defined in the 5+1 worldvolume of the branes and having 16 supercharges. Within this world-volume, the $U(n_5)$ SYM theory admits instantonic solutions localized in T^4 but which can still maintain a dynamics along S^1 and such that break half of the original supersymmetries. These instantons can be identified with the D1-branes wrapping the y direction. So we can look at the brane bound state as n_1 strings of instantons of the $U(n_5)$ gauge theory, localized in T^4 and whose moduli space we denote with $\mathcal{M}_{inst}$. Then we can describe D1-D5 states trough a 1+1 sigma model with target space $\mathcal{M}_{inst}$. This is a $\mathcal{N} = (4,4)$ SCFT whose central charge is related to the dual AdS radius via the relation:

$$c = \frac{3R_{\mathrm{AdS}_3}}{2G_3} = 6n_1n_5 \tag{2.38}$$

The identification of the moduli space is definitely non trivial. Since we have always in mind to work with quantities protected by Supersymmetry, we can restrict ourselves to a special point of the moduli space, the *orbifold point*. Here the target space becomes the symmetrized product of $N \equiv n_1n_5$ copies of the T^4, i.e. $\mathcal{M}_{inst} = {(T^4)^N}/{S_N}$ (S_N is the group of permutation and its presence determines the appearance of *twisted sectors*).
Summarizing, the sigma model at the orbifold point is:

$$\underbrace{(t,y)}_{\mathbb{R}\times S^1} \xrightarrow{\Sigma} \frac{(T^4)^N}{S_N} \tag{2.39}$$

and the theory is just a collection of $4n_1n_5$ free bosons and $4n_1n_5$ doublets of chiral and antichiral fermions.

For the D1-D5 system, the correspondence is established between the near horizon limit of D1-D5 geometries (Type IIB SUGRA compactified on T^4), which is asymptotically $\mathrm{AdS}_3 \times S^3$ and the $\mathcal{N} = (4,4)$ CFT_2, living in the Anti de Sitter boundary. From the duality, we expect that the 2 theories have the same symmetries. We need to say that in the near horizon limit, the 8 supercharges of the D1-D5 bound states are enhanced to 16. A similar situation happens for the CFT, the $\mathcal{N} = (4,4)$ theory has only 8 supercharges, but since the theory is conformal, we need also to consider the commutators of the supercharges with the conformal transformations. These commutators give addition superconformal charges and as a consequence SUSY is doubled. In the table below, we have summarized the main group correspondences.

AdS side	**CFT side**
• $SO(2,2)$ isometry group of AdS_3, this can be written as $SO(2,2) \simeq SL(2,\mathbb{R})_L \otimes SL(2,\mathbb{R})_R$. Where we have distinguished between right and left accordingly to the fact that the AdS Killing vectors can depend either in the combination $t-y$ or $t+y$.	• $SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})$ globally defined Virasoro algebra in 2D. In 2 dimensions, in general, the Virasoro generators L_n and $\bar{L}_n$ are infinite ($n \in \mathbb{Z}$), but the globally well-behaved subalgebras contains only $n = 0, \pm 1$.
• $SO(4)_E$ isometries of S^3	• $SU(2)_L \otimes SU(2)_R \simeq SO(4)$ $\mathcal{R}$-symmetry. In our CFT we have 4 left and 4 right moving supercharges, and the $SU(2)$ group rotates the charges in the same sector (they transform as a doublet).
• $SO(4)_I$ rotations of T^4	• $SU(2)_1 \otimes SU(2)_2$ symmetry.

For what concerns the description of states in the 2 dual theories we refer to the next section. Let us only anticipate that the asymptotically flat microstates are dual to states in the RR sector. Fermions must be in the Ramond sector, i.e. they must be periodic in the S^1 direction, to reproduce flat space-time at infinity. In fact, in the Ramond sector the vacuum energy is zero thanks to the fact that worldsheet susy is realised (i.e. every fermion cancels exactly every boson). On the contrary, if we had antiperiodic fermions in the gravity description, we would have found a non trivial vacuum energy and we would have lost asymptotic flatness required for our geometries.

2.3.2 Orbifold Model for the D1D5 CFT

In the limit of a free field theory, the CFT is well described by a 2 dimensional sigma model defined on $\mathbb{R} \times S^1$, whose $(T^4)^N/S_N$.
Now we can define the field content of the theory. For each copy of the T^4 there are:

- 4 bosonic excitations (X), giving the position of the effective string on the torus;
- 4 left-moving fermions (ψ);
- 4 right-moving fermions ($\tilde{\psi}$)

where fermions can be periodic on S^1 (R sector) or antiperiodic (NS sector). The central charge is then $c = 6$ for each of the $n_1 n_5$ copies.

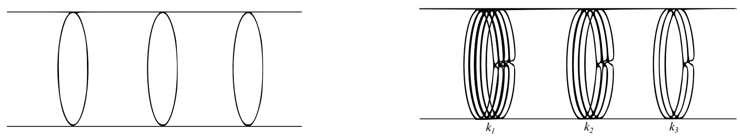
We can classify the fields according to the representations of the symmetry group listed above. Let us define the indices:

- α and $\dot{\alpha}$ referring to doublet of, respectively, $SU(2)_L$ and $SU(2)_R$ ($\alpha, \dot{\alpha} = \pm$);
- A and $\dot{A}$ for $SU(2)_1$ and $SU(2)_2$ doublets;
- an index r to label the copy ($r = 1 \ldots, n_1 n_5$).

According to these classification, the fields content of the theory is:

$$\left\{ X^{A\dot{A}}_{(r)}(t,\sigma), \psi^{\alpha\dot{A}}_{(r)}(t+\sigma), \tilde{\psi}^{\dot{\alpha}\dot{A}}_{(r)}(t-\sigma) \right\} \tag{2.40}$$

where we have defined $\sigma \equiv \frac{y}{R}$ and from this moment on all the tilded quantities we will refer to the right moving sector. Pictorially we can think of the CFT as a collection of $n_1 n_5$ strings or "strands". The base space of the sigma model is a cylinder ($\sigma \sim \sigma + 2\pi$), where strings can be singly wound (*untwisted sector*), as in Fig.2.4(a) or can be attached together in longer "component strings" (b) (*twisted sector*). The existence of untwisted and twisted



(a) (b)

Figure 2.4: Untwisted (a) and twisted sector (b).

sector is due to S_N, which can act mapping some copies of the T^4 into each other. The *untwisted sector* defined by the condition:

$$X^{A\dot{A}}_{(r)}(t, \sigma + 2\pi) = X^{A\dot{A}}_{(r)}(t, \sigma) \tag{2.41}$$

and we are left with $n_1 n_5$ strands. Strings joined together, instead, form the *twisted* sector. It can be defined, for example, by a condition:

$$X^{A\dot{A}}_{r}(t, \sigma + 2\pi) = X^{A\dot{A}}_{(r+1)})(t, \sigma) \tag{2.42}$$

In general a component string can go around the cylinder k_i times before closing to herself, but if we have m_i strands of such winding k_i, they must satisfy the constraint:

$$\sum_i k_i m_i = N \tag{2.43}$$

If we perform a Wick rotation ($t \to \tau = i\frac{t}{R}$), then, the cylinder is mapped to a complex plane spanned by the coordinates

$$z = e^{\tau + i\sigma}, \qquad \bar{z} = e^{\tau - i\sigma} \tag{2.44}$$

Consequently left moving fermions become holomorphic functions $\psi^{\alpha\dot{A}} \equiv \psi^{\alpha\dot{A}}(z)$, while right moving are antiholomorphic ones depending on $\bar{z}$. Bosonic coordinates, in the end, can be factorized as $X^{A\dot{A}}(z)$ and $X^{A\dot{A}}(\bar{z})$.
In addition, we know that each sector [2] realizes a $\mathcal{N} = 4$ superconformal symmetry. At a generic point in the moduli space, the current algebra is composed by the stress energy current $T(z)$, the supercurrents $G^{\alpha A}(z)$ and the $SU(2)_L$ currents $J^a(z)$ (a accounts for $\mathfrak{su}(2)$ generators: $a = 1, 2, 3$ or $a = \pm, 3$). The corresponding modes are L_n, $G^{\alpha A}_n$ J^a_n. In principle they generate an infinite dimensional algebra, but we are interested only on the globally defined subalgebra, which consists of the generators annihilating the NS vacuum:

$$\left\{ L_{\pm 1},\, L_0,\, G^{\alpha A}_{\pm\frac{1}{2}},\, J^a_0 \right\} \tag{2.45}$$

It is worth mentioning that the eigenvalue of L_0 gives the conformal dimension h and we can further classify states according to the eigenvalues j_3 of J^3_0 and j of $(J^a_0)^2$.

2.3.3 Untwisted sector

The untwisted sector of the CFT is characterised by all $k = 1$. We will review schematically the mode expansion of fields and currents and we will discuss the vacuum states together with other interesting operators. We refer to [25] for a complete description. Only in the end we will briefly make some comments about the twisted sector.

First of all we need to define how field boundary conditions change passing from the theory defined on the cylinder to the one on the complex plane. While bosons keep the same periodicity, fermions undergo a change [3]:

[2] We will treat in details the left sector, but the same is true for the right one. It is sufficient to put a tilde above all operators and functions and substituting z with $\bar{z}$

[3] When one maps to the complex plane, there is a Jacobian factor that switches the sign so that periodic fermions in the z-plane correspond to anti-periodic fermions on the cylinder and vice-versa

	cylinder	$\mathbb{C}$ plane
R sector	periodic	antiperiodic
NS sector	antiperiodic	periodic

These periodicities justify the following mode expansions (we report only the left case, the right one is analogous):

- for bosons

$$\partial X^{A\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \alpha^{A\dot{A}}_{(r)n} z^{-n-1} \tag{2.46}$$

- fermions in the R sector

$$\psi^{\alpha\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}} \psi^{\alpha\dot{A}}_{(r)n} z^{-n-\frac{1}{2}} \tag{2.47}$$

- fermions in the NS sector

$$\psi^{\alpha\dot{A}}_{(r)}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi^{\alpha\dot{A}}_{(r)n} z^{-n-\frac{1}{2}} \tag{2.48}$$

satisfying the following commutation relations:

$$\left[\alpha^{A\dot{A}}_{(r)n}, \alpha^{B\dot{B}}_{(s)m}\right] = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}} n\delta_{n+m,0}\delta_{rs}\,, \qquad \left\{\psi^{1\dot{A}}_{(r)n}, \psi^{2\dot{B}}_{(s)m}\right\} = \epsilon^{\dot{A}\dot{B}}\delta_{n+m,0}\delta_{rs} \tag{2.49}$$

Vacuum states and some important operators

In order to fully characterise a vacuum state, one needs to specify the states of bosons and fermions, and each of these, by itself, is a direct product of the state in the left and right sector.

- the vacuum state for bosons is $|00\rangle_{(r)}$ such that:

$$\begin{cases} \left(\alpha^{A\dot{A}}_{n}\right)_{(r)} |00\rangle_{(r)} = 0 \\ \left(\tilde{\alpha}^{A\dot{A}}_{n}\right)_{(r)} |00\rangle_{(r)} = 0 \end{cases} \qquad \forall n \geq 0,\, \forall A, \dot{A}$$

- vacuum state of fermions in the NS sector $|0\rangle^{NS}_{(r)}$ are such that:

$$\begin{cases} \left(\psi^{\alpha\dot{A}}_{n}\right)_{(r)} |0\rangle^{NS}_{(r)} = 0 \\ \left(\tilde{\psi}^{\dot{\alpha}\dot{A}}_{n}\right)_{(r)} |0\rangle^{NS}_{(r)} = 0 \end{cases} \qquad \forall n > 0,\, \forall \alpha, \dot{\alpha}, \dot{A}$$

- Ramond vacuum state deserves a special mention. Because of the presence of zero modes, there is not only one vacuum, but a set of degenerate vacua. In particular among the zero mode we decide to take as creation operators $\psi^{2\dot{A}}_{0\,(r)}$ and $\tilde{\psi}^{2\dot{A}}_{0\,(r)}$, while we choose 4 annihilation operators to be $\psi^{1\dot{A}}_{0}$, $\tilde{\psi}^{1\dot{A}}_{0}$). In this way we can define, for each copy, one of the Ramond vacua as $|{+}{+}\rangle^{R}_{(r)}$ through:

$$\begin{cases} \psi^{\alpha\dot{A}}_{n} |{+}{+}\rangle^{R} = 0 \\ \tilde{\psi}^{\dot{\alpha}\dot{A}}_{n} |{+}{+}\rangle^{R} = 0 \end{cases} \quad \forall n > 0\,, \forall \alpha, \dot{\alpha}, \dot{A} \quad \vee \quad \begin{cases} \psi^{1\dot{A}}_{0} |{+}{+}\rangle^{R} = 0 \\ \tilde{\psi}^{\dot{1}\dot{A}}_{0} |{+}{+}\rangle^{R} = 0 \end{cases} \quad \text{but} \quad \begin{cases} \psi^{2\dot{A}}_{0} |{+}{+}\rangle^{R} \neq 0 \\ \tilde{\psi}^{\dot{2}\dot{A}}_{0} |{+}{+}\rangle^{R} \neq 0 \end{cases}$$

which carries the maximum possible eigenvalues for $SU(2)_L \otimes SU(2)_R$ symmetry. The other vacua differer for the values of j_3 and $\tilde{j}_3$.

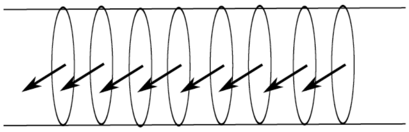


Figure 2.5: Singly wound components strings with spins represented by arrows

To better characterise the degenerate vacua, let us define:

$$J^{+}_{(r)} = \frac{1}{2}\psi^{1\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\psi^{1\dot{B}}_{(r)} \tag{2.50a}$$

$$J^{-}_{(r)} = -\frac{1}{2}\psi^{2\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\psi^{2\dot{B}}_{(r)} \tag{2.50b}$$

$$J^{3}_{(r)} = -\frac{1}{2}\left(\psi^{1\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\psi^{2\dot{B}}_{(r)} - 1\right) \tag{2.50c}$$

$$\tag{2.50d}$$

The mode expansion is:

$$J^{(a)}_{(r)}(z) = \sum_{n\in\mathbb{Z}} J^{(a)}_{(r)n} z^{-(n+1)} \tag{2.50e}$$

and equivalently for $\tilde{J}_{(r)}(\bar{z})$.
Degenerate R-R vacua are obtained by the action of these operators on $|{+}{+}\rangle^{R}_{(r)}$ and are classified according to the respective eigenvalues, i.e. $\left|j^R_3, \tilde{j}^R_3\right\rangle$ (j^R_3 and $\tilde{j}^R_3$ can take values $\pm\frac{1}{2} \equiv \pm$). In detail:

$$J^{-}_{(r)0}\,|{+}{+}\rangle^{R}_{(r)} = |{-}{+}\rangle^{R}_{(r)} \tag{2.51a}$$

$$\tilde{J}^{-}_{(r)0}\,|{+}{+}\rangle^{R}_{(r)} = |{+}{-}\rangle^{R}_{(r)} \tag{2.51b}$$

$$J^{-}_{(r)0}\tilde{J}^{-}_{(r)0}\,|{+}{+}\rangle^{R}_{(r)} = |{-}{-}\rangle^{R}_{(r)} \tag{2.51c}$$

The last vacuum we consider is the one with $j^R_3 = 0 = \tilde{j}^R_3$. Actually, there are other 3 bosonic RR vacua, which we neglect because they are not T^4-invariant. The state $|00\rangle_{(r)}$ can be obtained from $|{+}{+}\rangle$ with the action of another operator:

$$O^{\alpha\dot{\alpha}}_{(r)}(z,\bar{z}) = -\frac{i}{\sqrt{2}}\psi^{\alpha\dot{A}}_{(r)}\epsilon_{\dot{A}\dot{B}}\tilde{\psi}^{\dot{\alpha}\dot{B}}_{(r)} = \sum_{n,m\in\mathbb{Z}} O^{\alpha\dot{\alpha}}_{(r)mn} z^{-(n+1/2)}\bar{z}^{-(m+1/2)} \tag{2.52}$$

In particular:

$$|00\rangle^{R}_{(r)} = O^{2\dot{2}}_{(r)00}\,|{+}{+}\rangle^{R}_{(r)} \tag{2.53}$$

In the pictorial representation of Fig.2.4, we can think about the ground states as singly wound strings to which we assign a spin (Fig. 2.5).

Spectral flow

States carrying D1-D5 charges (in the near horizon limit) are RR ground states. Let us consider a very specific example whose gravity dual will be derived in Sec. 3.2, starting from the circular profile (3.9). At the orbifold point in CFT, this configuration corresponds to the product of $n_1 n_5$ $|{+}{+}\rangle_1$ states. The total state is an eigenstate of $(J^3, \tilde{J}^3)$ with eigenvalues $\left(\frac{N}{2}, \frac{N}{2}\right)$.

Consider the associated metric in the decoupling limit:

$$
\begin{aligned}
ds^2_{dec} \simeq & -\frac{r^2+a^2}{\sqrt{Q_1Q_5}}dt^2 + \frac{r^2}{\sqrt{Q_1Q_5}}dy^2 + \frac{\sqrt{Q_1Q_5}}{r^2+a^2}dr^2 + \\
& \sqrt{Q_1Q_5}\left[d\theta^2 + \sin^2\theta\left(d\phi - \frac{dt}{R}\right)^2 + \cos^2\theta\left(d\psi - \frac{dy}{R}\right)^2\right] + \sqrt{\frac{Q_1}{Q_5}}d\hat{s}_4^2
\end{aligned}
\tag{2.54}
$$

Perform the change of coordinates:

$$
\phi \to \tilde{\phi} \equiv \phi - \frac{t}{R}, \qquad \psi \to \tilde{\psi} \equiv \psi - \frac{y}{R} \tag{2.55}
$$

under which the metric (2.54) becomes global $\mathrm{AdS}_3 \times S^3 \times T^4$. In CFT the global $\mathrm{AdS}_3 \times S^3$ metric corresponds to the NS-NS vacuum state $|0\rangle^{NS}$. This coordinate change (2.55), which notice it is non vanishing at the boundary of AdS, is defined to be, in the dual CFT frame, a *spectral flow*. Spectral flow is a transformation mapping states from the R sector to the NS one and viceversa.

Under spectral flow the modes of the operator change as:

$$
\begin{aligned}
L'_n &= L_n - \alpha J^3_n + \frac{c\alpha^2}{24}\delta_{n0} \\
J^{3\prime}_n &= J^3_n - \frac{c\alpha}{12}\delta_{n0} \\
J^{\pm\prime}_n &= J^{\pm}_{n\mp\alpha} \\
G^{\pm A\prime}_n &= G^{\pm A}_{n\mp\frac{\alpha}{2}}
\end{aligned}
\tag{2.56}
$$

where $\alpha \equiv 2a+1$ with $a \in \mathbb{Z}$.
Consequently conformal dimension and J^3 eigenvalue transform:

$$
h' = h + \alpha j_3 + \alpha^2\frac{c}{24} \tag{2.57}
$$

$$
j'_3 = j_3 + \alpha\frac{c}{12} \tag{2.58}
$$

To pass from R to NS, $\alpha = 1$, while $\alpha = -1$ for the opposite case.

Chiral primaries

Operators as $J^{(a)}$ and $O^{\alpha\dot{\alpha}}$ are *chiral primary operators.* A state $|\psi\rangle$ is a *global primary* if

$$
L_{+1}|\psi\rangle = 0 = G^{\alpha A}_{+\frac{1}{2}}|\psi\rangle \tag{2.59}
$$

while it is *chiral* if, for every A

$$
G^{+A}_{-\frac{1}{2}}|\chi\rangle = 0 \tag{2.60}
$$

and antichiral,

$$
\tilde{G}^{+A}_{-\frac{1}{2}}|\chi\rangle = 0 \tag{2.61}
$$

Chiral primaries are states satisfying both these conditions. They are characterised by having $h = j_3 = j$ (and in the right sector $\tilde{h} = \tilde{j}_3 = \tilde{j}$) if they are chiral, or $h = -j_3$ for antichiral.

In the untwisted sector chiral primaries are:

- $h = j = 0$ the NS vacuum $|0\rangle^{NS}$, which is the lowest dimensional existing primary;
- two with $h = j = \frac{1}{2}$, $\psi^{+\dot{A}}_{-\frac{1}{2}}|0\rangle^{NS}$;

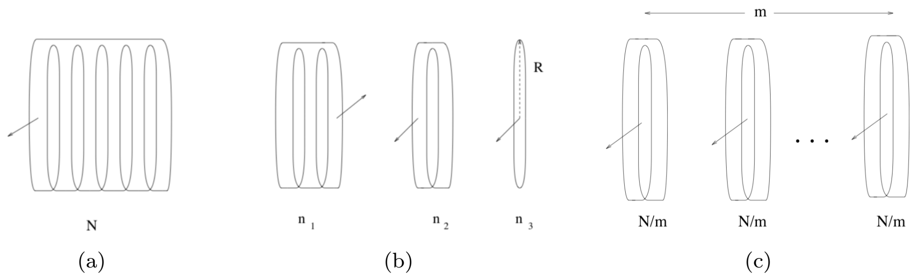


Figure 2.6: Various examples of state with different number of component strings. Arrows indiacate spins. R states correspond to chiral primaries with: (a) $k_i = N$, $n = 1$, (b) generic state, (c) $\left[\Sigma^{s_i \dot{s}_i}_{N/m}\right]^m$ with all the component strings with the same diemnsion.

- $J^+_{-1} \left|0\right\rangle^{NS}$, which has $j = h = 1$.

States obtained starting from these chiral primaries and acting upon them with L_{-1}, J^-_0 and $G^{-A}_{-\frac{1}{2}}$ are called *(super-)descendants.*

To conclude, remember the action of spectral flow on the charge and dimension of a state (Eq. (2.57)-(2.58)). Ramond vacua are one unit of spectral flow from chiral primary ($h = j_3$) states in the NS sector; or equivalently, negative one units of spectral flow from anti-chiral primary ($h = -j_3$) states. Therefore, there is a one-to-one correspondence between the R vacua and the NS chiral primary states.

2.3.4 Twist operators

The orbifolding by the permutation group, generates 'twist' sectors, created by twist operators joining together some strings in a longer "component string". We will not be working with the twist sector, let us only mention the fact that boundary conditions and mode expansions in the twisted sector become slightly more complicated with the respect to the untwisted case.

Twist operators, denoted σ_k for bosons and Σ_k for fermions, take k single strands and sew them together in a string of length k creating identifications between the different copies. As a consequence, vacuum states are the same of the untwisted case, except for an additional k label, while operator definitions receive a further sum over the number of the k strings joined together.
For L and R bosons, NS and R fermions we have different twist operators. For bosons we introduce the operators σ^X_k and $\tilde{\sigma}^X_k$, such that it creates a ground state of length k as;

$$\lim_{z\to 0} \sigma^X_k(z)\tilde{\sigma}^X_k(\bar{z}) \left(\otimes^k_{r=1} \left|00\right\rangle_{(r)}\right) = \left|00\right\rangle_k \tag{2.62}$$

In the NS sector, remembering that these fermions are scalar under $SU(2)_L \otimes SU(2)_R$

$$\lim_{z\to 0} \Sigma_k(z,\bar{z}) \left(\otimes^k_{r=1} \left|0\right\rangle^{NS}_{(r)}\right) = \left|0\right\rangle^{NS}_k \tag{2.63}$$

and finally in the R sector:

$$\lim_{z\to 0} \Sigma^{s_1\dot{s}_2}_k(z,\bar{z}) \left(\otimes^k_{r=1} \left|{+}{+}\right\rangle^R_{(r)}\right) = \left|{+}{+}\right\rangle^{NS}_k \tag{2.64}$$

where s_1 and $\dot{s}_2$ spin values are adjusted in order to conserve spin from the r.h.s. to the l.h.s.

Twist operators are very useful to visualize the D1-D5 system [20]. Indeed let us consider the most generic NS chiral primary:

$$\prod_{i=1}^{n} \left[\Sigma_{k_i}^{s_i \dot{s}_i}\right]^{m_i}, \qquad \sum_{i=1}^{n} k_i m_i = N \equiv n_1 n_5 \tag{2.65}$$

The corresponding R ground states, represented in Fig.2.6, are m_i multiwound string wrapped k_i times around the S^1 for a total of N component strings. s_i and $\dot{s}_i$ ($s_i, \dot{s}_i = \pm$) refer to the the $(j^R, \tilde{j}^R)$ spin carried by the component string. For example in Fig. 2.6(c) it is represented a state with m component strings with all the spin aligned and if we choose $s = \dot{s} = -$, then the CFT state has a total spin $j \equiv j = \tilde{j} = \frac{m}{2}$, which is the maximum value of j.
In the previous chapter, to construct D1-D5 geometries we started from the dual representation as a fundamental string F1 of total length L_T carrying momentum P. This momentum is represented as a quantum of vibration in the transverse direction (let us indicate with $i = 1, 2, 3, 4$ the $\mathbb{R}^4$ coordinates) generated by the creation operator of the corresponding oscillator α^i_{-n}. In general if the string has n_1 units of momentum , the state of the string can be written as:

$$\left[\alpha^{i_1}_{-k_1}\right]^{m_1} \cdots \left[\alpha^{i_1}_{-k_n}\right]^{m_n} |0\rangle\,, \qquad \sum_{l=1}^{n} k_l m_l = N \tag{2.66}$$

Now comparing states (2.65) and (2.66), we can argue that if in the D1-D5 R ground state we have a component string wrapping k_i times, then in the F1-P system it is mapped to a momentum mode α_{-k_l} of wavelength $\frac{L_T}{k_l}$ and the 4 spin polarization $(s_l, \dot{s}_l) = (\pm, \pm)$ correspond to the the different $i = 1$ of the α^i modes[4].

2.4 Holographic dictonary

We have introduced the D1-D5 CFT with the aim to provide an alternative description of microstate geometries. In particular, it is possible to write down a quite systematic identification procedure linking 2 and 3- charge geometries to CFT states. These states are superposition of vacua or of excitated vacua in the Ramond sector. In particular, since on the gravity side we are considering extremal geometries, only one among the left and right sector can be excited. We will always assume that the right movers stay unaltered in the RR ground state, while we can act on the left sector.

To conclude this chapter, we want to show how the CFT description gives an alternative perspective on the counting of microstates of Sec. 2.1.1. For the D1-D5 case, let us start considering $n_5 = 1$, then we can visualize the system as D1 branes living on the other brane. We have seen that the state can be represented in the R sector as a collection of component strings of different lengths. The extremal situations are n_1 singly wound loops or only one string wrapping n_1 times. But also all the other configurations are allowed, they must only respect the constraint that the total windings is n_1. If now n_5 becomes greater than 1, the total number of strands become $n_1 n_5$ and the D1 "fractionates". To understand what we mean with "fractionate" suppose to have $n_1 = 1$ D1 brane bound to n_5 D5 branes. Fractionation means that the D1 brane appear as n_5 "fractional" D1 branes, each one with a tension equal to $\frac{\tau_{D1}}{n_5}$. Thus if we have n_1 units of D1 charge, we have $n_1 n_5$ units of fractional D1 charge.
Also for $n_5 > 1$ the logic stays the same, we need to look at all the possible configurations of these strands to count microstates. This is equivalent to the number of partition of $n_1 n_5$;

[4] One can show [20], [22] that the choice of all spins aligned corresponds to a fundamental string rotating in a plane (as the simplest 2 charge case we have study). If spins are not aligned instead, the circle becomes an ellipse.

remembering that each component string has 16 possible R ground states: 8 bosonic and 8 fermionic, we obtain the same result as in Eq. 2.8, that the number of microstates is:

$$\mathfrak{N} = e^{2\sqrt{2}\pi\sqrt{n_1 n_5}}$$

In the 3-charge case, we have a system of D1 and D5 branes, giving a total winding number of $n_1 n_5$. We can arrange them in different configurations with the usual constraint that:

$$\sum_i m_i k_i = n_1 n_5 \tag{2.67}$$

Then we want to add n_P units of momentum and in principle we can just distribute it in any configurations respecting Eq. (2.67). It can be shown [2], however, that in computing the number of microstates the strand configuration which gives the major contribution to the entropy (at least at leading order) is the one with only one component string with winding $k = n_1 n_5$. In this extremal case each unit of momentum becomes an excitation of the string at level $n_1 n_5$ and if we consider all the n_P units the effective level of excitation comes to be $n_1 n_5 n_P$ giving the entropy (2.9):

$$S = 2\pi\sqrt{n_1 n_5 n_P} \tag{2.68}$$

In the next chapters we will look in detail to some of this $\frac{1}{4}$ and $\frac{1}{8}$-BPS states. We will see how the holographic dictionary one can construct, gives us a powerful tool to relate the profile function and the 1-form appearing in the supergravity solutions to specific CFT states. In particular one can connect the warp factors Z_i to the VEVs of CFT anti-chiral primary operators. In doing so, we will try to resume the main characteristics of this CFT interpretation, but we refer to [7, 3, 26] for a more complete explanation.

Chapter 3

Superstrata solutions

So far, we have introduced in general the idea behind the *Fuzzball proposal* and we have said that the microstates, we are going to consider, are BPS states. On the gravity side they can be described by solutions of ten dimensional Type IIB Supergravity, whose ten dimensional space-time has the topology $\mathbb{R}^{1,4} \times S^1 \times T^4$.

The problem of constructing and classifying fuzzball microstates for the 2 charge extremal D1-D5 system has been started by [20] and completed in later works by I.Kanitscheider, K.Skenderis and M.Taylor. These $\frac{1}{4}$-BPS states are described by smooth classical solutions of supergravity and they reproduce exactly the microscopic counting of microstates.
In this work, instead, we will investigate *Superstrata*, smooth and horizonless geometries preserving 4 supersymmetries on ($\frac{1}{8}$-BPS) and having the same charges as the D1-D5-P black hole. These class of microstates is less understood and only partially classified, however it is of great importance. In fact, in classical SUGRA the D1-D5-P system has a finite horizon, representing a true BH, hence it is an ideal framework in which to examine the fuzzball conjecture. To construct these new geometries, it was supposed, by analogy with the 2.charge case, that at least a subclass of them, can be constructed as classical supergravity solutions. Many works, such as [26, 27, 28, 3, 4], have been devoted to the construction of these new geometries. They are found to have neither horizon nor singularity, to present an AdS throat region and at spatial infinity they reduce to Strominger-Vafa BH.

After a general introduction on the supersymmetry equations these geometries need to satisfy, we will review the construction of D1-D5 microstates as in Eq. (2.20) and starting from them we will construct 1/8 BPS states. After having constructed them as solutions of IIB supergravity, we will interpret these states in the light of the *D1 D5 CFT*. In the dual CFT, the construction procedure [26] starts from a Ramond-Ramond ground state(RR) and we show that new solutions can be obtained through the action of the generators of the Virasoro algebra or/and of the $\mathcal{R}$ symmetry one. Geometrically this procedure is equivalent to very specific change of coordinates, which lead to new 3-charge gravity solutions called *super-descendants*.

3.1 Supergravity equations

Assuming only invariance under T^4 rotations, in [27] the general 3-charge solution of type IIB supergravity compactified on $S^1 \times T^4$ is derived. The metric , in string frame, representing these solutions can be expressed as:

$$ds^2_{(10)} = -\frac{2\alpha}{\sqrt{Z_1 Z_2}}(dv+\beta)\left[du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right]+\sqrt{Z_1 Z_2}ds^2_4+\sqrt{\frac{Z_1}{Z_2}}d\hat{s}^2_4 \tag{3.1}$$

where ds_4^2 and $d\hat{s}_4^2$ are respectively the Euclidean metric of the non compact spacial directions (diffeomorphic to $\mathbb{R}^4$) and the flat metric on the torus; the time t and the S^1 direction y are expressed in terms of the null coordinates:

$$v = \frac{t+y}{\sqrt{2}}, \qquad u = \frac{t-y}{\sqrt{2}}$$

The other functions appearing in the metric are the 0-forms defined on $\mathbb{R}^4$ $\mathcal{F}$ and Z_1, Z_2, Z_4. α is defined as a combination of these functions:

$$\alpha = \frac{Z_1 Z_2}{Z_1 Z_2 - Z_4^2}$$

Finally ω and β appearing in Eq.(3.1) are 1-forms in $\mathbb{R}^4$. For the microstates of interest, β is assumed to be v-independent and to have self dual strenght $d\beta = *d\beta$ (differential and hodge dual are intended w.t.r. to ds_4^2) and ds_4^2 is taken to be the flat Euclidean metric. Reduction on the T^4 and the passage to the Einstein frame, lead Eq. (3.1) to 6D metric:

$$ds_6^2|_E = -\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)\left[du + \omega + \frac{\mathcal{F}}{2}(dv+\beta)\right] + \sqrt{\mathcal{P}} ds_4^2 \tag{3.2}$$

where we have called $\mathcal{P} = Z_1 Z_2 - Z_4^2$.
In order to complete the description of the Type IIB solution, we need to specify:

- the dilaton: $e^{2\phi} = \alpha \frac{Z_1}{Z_2}$
- the Kalb-Ramond field, represented by the 2 form

  $$B = -\alpha \frac{Z_4}{Z_1 Z_2}(du+\omega)\wedge(dv+\beta) + a_4 \wedge (dv+\beta) + \delta_2$$

  where we have introduced the 1-form a_4 and the 2-form δ_2 on $\mathbb{R}^4$
- the RR 0-,2- and 4-forms

  $$\begin{aligned} C_0 &= \frac{Z_4}{Z_1} \\ C_2 &= -\frac{\alpha}{Z_1}(du+\omega)\wedge(dv+\beta) + a_1 \wedge (dv+\beta) + \gamma_2 \\ C_4 &= \frac{Z_4}{Z_2}\hat{\mathrm{vol}}_4 - \alpha\frac{Z_4}{Z_1 Z_2}\gamma_2 \wedge (du+\omega)\wedge(dv+\beta) + x_3 \wedge (dv+\beta) \end{aligned}$$

  where a_1, γ_2 and x_3 are in order a 1,2,3-form and $\hat{\mathrm{vol}}_4$ is the torus volume form. As an alternative to C_2 one can consider its dual form C_6

  $$C_6 = \hat{\mathrm{vol}}_4 \wedge \left\{ -\frac{Z_1}{\mathcal{P}}(du+\omega)\wedge(dv+\beta) + a_2 \wedge (dv+\beta) + \gamma_1 \right\} \tag{3.3}$$

  where a_2 and γ_1 are respectively a 1- and a 2-form.

Two important remarks: except for β and the $\mathbb{R}^4$ metric, all the other forms can depend non trivially on v; to have supersymmetry the metric and all the functions presented above have not to depend on u.

To preserve supersymmetry and to satisfy the equations of motion these geometries should fulfill some constraints encoded in a set of equations derived in [27] and reported below. Let us define $\mathcal{D} \equiv d_{\mathbb{R}^4} - \beta \wedge \frac{d}{dv}$ and indicate with a dot the derivative with the respect to v, then

- Equations for Z_1 and Z_2 together with $\Theta_1 \equiv \mathcal{D}a_1 + \dot{\gamma}_2$ and $\Theta_2 \equiv \mathcal{D}a_2 + \dot{\gamma}_1$

  $$\mathcal{D} *_4 \mathcal{D} Z_1 = -\Theta_2 \wedge d\beta \tag{3.4a}$$
  $$\mathcal{D} *_4 \mathcal{D} Z_2 = -\Theta_1 \wedge d\beta \tag{3.4b}$$

$$\mathcal{D}\Theta_2 = *_4\mathcal{D}\dot{Z}_1, \quad \Theta_2 = *_4\Theta_2 \tag{3.5a}$$
$$\mathcal{D}\Theta_1 = *_4\mathcal{D}\dot{Z}_2, \quad \Theta_1 = *_4\Theta_1 \tag{3.5b}$$

- Equations for Z_4 and $\Theta_4 \equiv \mathcal{D}a_4 + \dot{\delta}_2$:

$$\mathcal{D} *_4 \mathcal{D}Z_4 = -\Theta_4 \wedge d\beta \tag{3.6a}$$
$$\mathcal{D}\Theta_4 = *_4\mathcal{D}\dot{Z}_4, \quad \Theta_4 = *_4\Theta_4 \tag{3.6b}$$

As we can see, equations for Z_i and Θ_i are linear and homogeneous. Analogous expressions exist for $\omega, \mathcal{F}$, they are still linear but not homogeneous. The non homogeneous terms are quadratic in Z_i and Θ_i:

$$\mathcal{D}\omega + *_4\mathcal{D}\omega + \mathcal{F}d\beta = Z_1\Theta_1 + Z_2\Theta_2 - 2Z_4\Theta_4 \tag{3.7}$$
$$*_4\mathcal{D} *_4 \left(\dot{\omega} - \frac{1}{2}\mathcal{D}\mathcal{F}\right) = \partial_v^2(Z_1Z_2 - Z_4^2) - \left(\dot{Z}_1\dot{Z}_2 - \dot{Z}_4^{\,2}\right) - \frac{1}{2} *_4 (\Theta_1 \wedge \Theta_2 - \Theta_4 \wedge \Theta_4) \tag{3.8}$$

To construct $\frac{1}{8}$-BPS solutions, we will follow a *solution generating technique* [4, 3]. This procedure allows to solve the BPS equations, dividing the problem in different steps:

- *zero layer*, i.e. solving equations for ds_4^2 and β. In this work, we will use a simple configuration in which ds_4^2 is flat and β is v independent and with a selfdual strenght.
- *first layer*, finding the expressions for Z_1, Z_2 and Z_4 and the respective Θ. We start from the 2-charge microstate solution and add momentum charge. Exploiting the linearity of the BPS equations, we can find a basis and express the Z_i as a superposition of this basis. Requiring the regularity of the solutions, then further constrains their form.
- *second layer*, once know the expression for Z_i and Θ_i, one can try to solve for ω and $\mathcal{F}$. This is in general, a very non trivial task.

3.2 2-charge solutions

In Sec. 2.1.2 we have already said that 2-charge microstates are constructed as vibrations of a fundamental string, classically parametrized by a profile function $\vec{g}(v)$. Then a chain of S and T duality maps the F1-P state to the D1-D5 one. We restrict to the subclass where A takes values $A = 1, \dots, 5$, where the first 4 are the direction of $\mathbb{R}^4$ and the other one is the direction of the T^4. In the D1-D5 frame this subclass has an invariance under T^4-rotations.

The simplest profile one can consider is the circular one:

$$g_1(v) = a\cos\left(\frac{2\pi v}{L}\right), \qquad g_2(v) = a\sin\left(\frac{2\pi v}{L}\right) \tag{3.9}$$

where $L = 2\pi\frac{Q_5}{R}$ and Q_5 is the D5 charge. A more interesting example occurs when another oscillation is turned on along the T^4 direction (which is singled out in the chain of T and S duality).

$$g_1(v) = a\cos\left(\frac{2\pi v}{L}\right), \qquad g_2(v) = a\sin\left(\frac{2\pi v}{L}\right), \qquad g_5(v) = -\frac{b}{k}\sin\left(\frac{2\pi k v}{L}\right) \tag{3.10}$$

In Appendix C we report the expression for the warp factors Z_1, Z_2, Z_4 and of β and ω along with some explicit derivations. The geometry associated with the profile (3.10) is:

$$ds^2_{(10)} = -2\frac{\sqrt{Z_1 Z_2}}{Z_1 Z_2 - Z_4^2}(dv+\beta)\,(du+\omega) + \sqrt{Z_1 Z_2}ds_4^2 + \sqrt{\frac{Z_1}{Z_2}}d\hat{s}_4^2 \tag{3.11a}$$

$$Z_2 = 1 + \frac{Q_5}{\Sigma}, \qquad Z_4 = Rba^k\frac{\sin^k\theta\cos(k\phi)}{\Sigma(r^2+a^2)^{k/2}} \tag{3.11b}$$

$$Z_1 = 1 + \frac{R^2}{Q_5}\left\{\frac{\left(a^2+\frac{b^2}{2}\right)}{\Sigma} + \frac{b^2}{2}a^{2k}\frac{\sin^{2k}\theta\cos(2k\phi)}{\Sigma(r^2+a^2)^k}\right\} \tag{3.11c}$$

$$\beta = \frac{Ra^2}{\sqrt{2}\Sigma}\left(\sin^2\theta d\phi - \cos^2\theta d\psi\right), \qquad \omega = \frac{Ra^2}{\sqrt{2}\Sigma}\left(\sin^2\theta d\phi + \cos^2\theta d\psi\right) \tag{3.11d}$$

$$ds_4^2 = \frac{\Sigma}{r^2+a^2}dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2 \tag{3.11e}$$

where the parameters a and b are related via:

$$a^2 + \frac{b^2}{2} = \frac{\sqrt{Q_1 Q_5}}{R} \tag{3.12}$$

As expected for fuzzball microstates, for both cases presented:

- asymptotically, for large r, the ten dimensional metric reduces to $\mathbb{R}^{4,1}\times S^1\times T^4$;
- in the asymptotic **decoupling limit** the metric tends to $\mathrm{AdS}_3\times S^3\times T^4$.

The last thing we want to stress is that, as expected, these geometries have no horizon nor singularity.

3.3 Solution generating technique

Among 3-charge microstates, we will focus our attention to *superstrata solutions* [3], which are smooth microstate geometries with have the same charges as D1-D5-P black holes in five dimensions. In the dual CFT, they were first derived in [26] as descendants of chiral primaries .
Let us list the main point of the solution generating technique ([3, 4]) we are going to follow:

1. The starting point is the rotating 2 charge state (3.10) in the decoupling limit;
2. Then we add momentum while maintaining the BPS nature of the states. This is achieved exciting only one sector, that is to say we act only in the left moving-sector, keeping the right one untouched. Some intermediate steps are needed:
   (a) *Spectral flow from R to NS sector.* To apply the solution generating technique of [3], we need to work with NS states, so we have to go from the R ground state to the NS sector, via de coordinate transformation (2.55). The result of spectral flow is a chiral primary state in the NS sector ($h^{NS} = j^{NS}$ and $\tilde{h}^{NS} = \tilde{j}^{NS}$).
   (b) *Chiral rotation.* We can act on the chiral primaries with the generators of the $SL(2,\mathbb{R})_L\times SU(2)_L$ algebra. On the gravity side their action is a diffeomorphism

which is non trivial at the AdS boundary.

$$L_0 = \frac{iR}{2}(\partial_t + \partial_y) \tag{3.13a}$$

$$L_{\pm 1} = ie^{\pm i\frac{\sqrt{2}v}{R}} \left[-\frac{R}{2} \left(\frac{r}{\sqrt{r^2+a^2}}\partial_t + \frac{\sqrt{r^2+a^2}}{r}\partial_y \right) \pm \frac{i}{2}\sqrt{r^2+a^2}\partial_r \right] \tag{3.13b}$$

$$J_0^3 = -\frac{i}{2}\left(\partial_{\tilde{\phi}} + \partial_{\tilde{\psi}}\right) \tag{3.13c}$$

$$J_0^{\pm} = \frac{i}{2}e^{\pm i(\tilde{\phi}+\tilde{\psi})}\left(\mp i\partial_\theta + \cot\theta\partial_{\tilde{\phi}} - \tan\theta\partial_{\tilde{\psi}}\right) \tag{3.13d}$$

They satisfy the usual algebraic relations:

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \qquad [L_{+1}, L_{-1}] = 2L_0$$
$$\left[J_0^3, J_0^{\pm}\right] = \pm J_0^{\pm}, \qquad \left[J_0^+, J_0^-\right] = 2J_0^3$$

The action of J^0 corresponds to S^3 rotations, while L generates conformal transformation in AdS$_3$.

NS states obtained by spectral flow have maximum angular momentum, so they are annihilated by J_0^- and L_{+1}. The action of J_0^+ and L_{-1}, instead, is non trivial and generates new states labelled *superdescendants*. To understand better, we can look at this construction from the CFT point of view. As it will clarified later, in the NS-NS sector the state related to Z_4 is represented by an (anti)chiral primary state, defined $|00\rangle_k^{NS}$. We can act on this m times with J_0^+ ($m \leq k$, $m = k$ being the maximum value of the angular momentum) and n times with L_{-1}(we can act as many times as we want). Then the corresponding super-descendants are:

$$\left(J_0^+\right)^m (L_{-1})^n |00\rangle_k^{NS} \xrightarrow[\text{flow}]{\text{spectral}} \left(J_{-1}^+\right)^m \left(L_{-1} - J_{-1}^3\right)^n |00\rangle_k^R$$

(c) *Inverse spectral flow.* Using the inverse of (2.55) we bring the solution back to the R sector in order to read again the metric in the original ansatz presented before.

3. At the linearized level (in b) we can act with the infinitesimal transformations (3.13) to get perturbatevely new solutions [29] and take a linear combinations of them to make the superstratum. However it is possible using the linear structure of the equations expleined above to promote the result to a non perturbative level.

   In the next section we will see in more detail how this can be done through an explicit example, following the procedure of [3] and [4]. In particular we will see that the infinitesimal solutions for Z_4 can be used directly for the finite case (arbitrary large b) and how to solve the BPS equations to get the new smooth 6D metric removing all the possible sources of singularities.

3.4 Constructing the solutions

Following the point (3) above, we can generate new solutions acting with L_{-1} or J_0^3 on NS states. At the linear order in b $ds_4^2, \beta, \omega, \mathcal{F}, Z_{1,2}$ do not receive corrections and they remain the same as (C.3), (C.9), (C.11) and only Z_4 and Θ_4 changes. According to the technique of [30], we can read new geometries, acting with the infinitesimal transformations (3.13) on the scalar function $C_0 = \frac{Z_4}{Z_2} = \hat{Y}$, where

$$Y \equiv e^{-ik\tilde{\phi}} \sin^k\theta \frac{e^{-\frac{ikt}{R}}}{\left(a^2+r^2\right)^{k/2}} \tag{3.14}$$

Since at the linearised level Z_2 does not change, then a change in $\hat{Y}$ is equivalent to a change in Z_4.
We will concentrate only on descendants generated by the action of the $SL(2,\mathbb{R})$ generator [29] [1], they are obtained as:

$$(L_{-1})\,\hat{Y} = \frac{kr}{(a^2+r^2)^{\frac{k}{2}+\frac{1}{2}}}\sin^k\theta\left(-e^{-\frac{ikt}{R}-ik\tilde{\phi}-\frac{i(t+y)}{R}}\right) \tag{3.15}$$

Iterating n times, remembering that we have defined $\sqrt{2}v = t+y$ and expressing the result in the RR sector (so that $ik\left(\frac{t}{R}+\tilde{\phi}\right) = ik\phi$): k and n generic

$$Z_4 = bR\frac{\Delta_{k,n}}{\Sigma}e^{-i\hat{v}_{k,n}} \equiv bz_{k,n} \tag{3.16}$$

$$\begin{aligned}\Theta_4 = &-\sqrt{2}b\Delta_{k,n}e^{-i\hat{v}_{k,n}}\left[i\left(nr\sin\theta - n\frac{\Sigma}{r\sin\theta}\right)\left(\frac{dr\wedge d\theta}{(r^2+a^2)}\cos\theta + \frac{r\sin\theta}{\Sigma}d\phi\wedge d\psi\right)\right.\\ &\left.-n\left(\frac{dr\wedge d\phi}{r} - \cot\theta d\theta\wedge d\psi\right)\right] \equiv b\vartheta_{k,n}\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}\Delta_{k,n} &= \left(\frac{a}{\sqrt{r^2+a^2}}\right)^k\left(\frac{r}{\sqrt{r^2+a^2}}\right)^n\sin^k\theta\\ \hat{v}_{k,n} &= n\frac{\sqrt{2}v}{R} + k\phi\end{aligned}$$

We have verified that these field satisfy the BPS equation, in particular that Θ_4 is self dual with the respect to ds_4^2 metric and that:

$$*_4\mathcal{D}\dot{Z}_4 = \mathcal{D}\Theta_4$$

3.4.1 First layer of BPS equations

Exploiting the linearity of the first layer of BPS differential equations (3.4)-(3.6), i.e. the conditions for Z_i, we can take as Z_4 and Θ_4 a linear combination of the modes (3.16) and (3.17):

$$Z_4 = \sum_{k,n} b_4^{k,n}z_{k,n}, \qquad \Theta_4 = \sum_{k,n} b_4^{k,n}\vartheta_{k,n} \tag{3.18}$$

Since the equations for the other Z are the same as the ones for Z_4, we can think to expand also these fields using the same modes:

$$Z_1 = \frac{Q_1}{\Sigma} + \sum_{k,n} b_1^{k,n}z_{k,n}, \qquad Z_2 = \frac{Q_5}{\Sigma} + \sum_{k,n} b_2^{k,n}z_{k,n} \tag{3.19}$$

and analogously for the Θs.

Now, in order to extend these solutions not only to the infinitesimal case, but to the full finite, non linear problem, some important remarks are needed.

- First of all, since the BPS equations are linear, the Z_i and Θ_i obtained as superposition of modes $z_{k,n}$ and $\vartheta_{k,n}$ remain solutions even when the b_i coefficients are taken to be finite.

[1] In the conventions of [3] and [4] we have put $m = 0$

- We require that the expressions for ds_4^2 and β do not change:

$$
\begin{aligned}
ds_4^2 &= \Sigma\left(\frac{dr^2}{r^2+a^2}+d\theta^2\right)+(r^2+a^2)\sin^2\theta d\phi^2+r^2\cos^2\theta d\psi^2 \\
\beta &= \frac{Ra^2}{\sqrt{2}\Sigma}(\sin^2\theta d\phi-\cos^2\theta d\psi)
\end{aligned}
$$

- The last step needed to construct the metric is solving the *second layer* of BPS equation the one for $\mathcal{F}$ and ω and requiring that the solution is regular and free from singularities. It is found in [3], [4] that these requirements strongly constrain the form of the b_i coefficients and consequently of the Z_i.

3.4.2 Second layer of BPS equations

Solutions for $\mathcal{F}$ and ω are not known in general and must be constructed case by case. It can be shown ([3, 4, 26]) that exact smooth solutions of the form (3.18), (3.19), constructed from the 2-charge configuration (3.10), exist if:

- $b_2^{k,n}=0$, i.e. all the Z_2 modes are trivial;
- one adjusts the coefficients $b_1^{k,n}$ and $b_4^{k,n}$ in such a way that the ω and $\mathcal{F}$ depend only on the difference of modes ($\hat{v}_{k,n}-\hat{v}_{k',n'}$). In fact solutions to Eq.s (3.7)-(3.8) will depend in principle on both the sum and the difference of modes. However, solutions containing the sum of modes become in general singular, thus it is necessary to rearrange the b_1-coefficients to cancel the terms containing these sums. This tuning of the coefficients is known as *coiffuring* and it is a fundamental tool to guarantee the regularity of the solution.

We now restrict to the one single mode case, that is to say we take only a single term in the sums appearing in Eq. (3.18), 3.19. For this particular case, the result of *coiffuring* is that Z_1 depends on $\hat{v}_{2k,2n5}$ and $b_1^{k,n}=\left(b_4^{k,n}\right)^2\equiv b^2$ and, in this way, it is possible to solve completely the second layer.
The complete ansatz for the single-mode superstratum is:

$$
Z_2=\frac{Q_5}{\Sigma},\qquad Z_1=\frac{Q_1}{\Sigma}+\frac{b^2R^2}{2Q_5}\frac{\Delta_{2k,2n}}{\Sigma}\cos\hat{v}_{2k,2n},\qquad Z_4=Rb\frac{\Delta_{k,n}}{\Sigma}\cos\hat{v}_{2k,2n} \tag{3.20a}
$$

$$
\Theta_1=0,\qquad \Theta_2=\frac{b^2}{2}\frac{R}{Q_5}\vartheta_{2k,2n},\qquad \Theta_4=b\vartheta_{k,n} \tag{3.20b}
$$

$$
\mathcal{F}_{1,n}=-\frac{b^2}{a^2}\left(1-\frac{r^{2n}}{(r^2+a^2)^n}\right) \tag{3.21}
$$

$$
\omega_{1,n}=\frac{R}{\sqrt{2}\Sigma}\left[\sin^2\theta\left(a^2+b^2\left(1-\frac{r^{2n}}{(r^2+a^2)^n}\right)\right)d\phi+a^2\cos^2\theta d\psi\right] \tag{3.22}
$$

where we have reported the solution for ω and $\mathcal{F}$ only for $k=1$ and n arbitrary [2]. We have concentrated on this case because metric derived from this anstatz presents a remarkable and very interesting property. Its reduced 3D metric in Einstein frame does not depend on S^3 coordinates. This characteristic is quite incredible and no other cases are known at the moment. We will use this independence when we will compute the EE of this class of solution in Sec. 4.2.

[2] A complete dissertation for generic k, m and n can be found in [5]

3.4.3 Metric for $k = 1$ and arbitrary n

Let us recall the generic expression for the 6 dimensional metric after the reduction on T^4:

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv+\beta)\left(du+\omega+\frac{\mathcal{F}}{2}(dv+\beta)\right)+\sqrt{\mathcal{P}}ds_4^2$$

where

$$\mathcal{P} \equiv Z_1 Z_2 - Z_4^2$$

Let us substitute the expression (3.21), (3.22) and Eq.(3.20a) for $k = 1$ and keep in mind Eq. (3.12). Then we obtain:

$$\mathcal{P} = \frac{R^2}{2\Sigma^2}\left(2a^2+b^2-(1-F_0)\frac{b^2a^2}{r^2+a^2}\sin^2\theta\right) \tag{3.23}$$

$$ds_6^2|_E = -\frac{dt^2}{\sqrt{Q_1Q_5}}\frac{\Sigma}{\Lambda}\left(1-\frac{b^2}{2a^2}F_0\right)+\frac{dy^2}{\sqrt{Q_1Q_5}}\frac{\Sigma}{\Lambda}\left(1+\frac{b^2}{2a^2}F_0\right)+\frac{\Sigma}{\sqrt{Q_1Q_5}}\frac{b^2}{a^2\Lambda}F_0dtdy \tag{3.24}$$

$$+\sqrt{Q_1Q_5}\Lambda\left(\frac{dr^2}{r^2+a^2}+d\theta^2\right)+\frac{\sqrt{Q_1Q_5}}{\Lambda}\sin^2\theta\left(d\phi^2-\frac{2a^2}{R(2a^2+b^2)}2dtd\phi\right) \tag{3.25}$$

$$+\frac{\sqrt{Q_1Q_5}}{\Lambda}\cos^2\theta F_1\left(d\psi^2-\frac{b^2F_02dtd\psi+(2a^2+b^2F_0)2dyd\psi}{R(2a^2+b^2)F_1}\right) \tag{3.26}$$

We have introduced the useful functions:

$$\Lambda \equiv \frac{\sqrt{\mathcal{P}}\Sigma}{\sqrt{Q_1Q_5}}$$

$$F_1(r) \equiv 1-\frac{a^2b^2}{2a^2+b^2}\frac{r^{2n}}{(r^2+a^2)^{n+1}}$$

$$F_0(r) \equiv 1-\frac{r^{2n}}{(r^2+a^2)^n}$$

Now we want to rewrite Eq. (3.24) in a convenient way, i.e. we want that the Einstein metric in six dimensions takes the form[3]

$$ds_6^2|_E = g_{\mu\nu}dx^\mu dx^\nu + G_{\alpha\beta}(dy^\alpha + A_\mu^\alpha dx^\mu)(dy^\beta + A_\nu^\beta dx^\nu) \tag{3.27}$$

where μ and ν are indices such that at the boundary become coordinates of AdS_3 ($x^\mu = \{r,t,y\}$) while α and β of S^3 ($y^\alpha = \{\theta,\phi,\psi\}$).

$$\begin{aligned}ds_6^2|_E = {} & \Lambda\sqrt{Q_1Q_5}\frac{dr^2}{r^2+a^2}+\Lambda\sqrt{Q_1Q_5}d\theta^2+\frac{\sqrt{Q_1Q_5}}{\Lambda}\sin^2\theta\left(d\phi-\frac{2a^2}{R(2a^2+b^2)}dt\right)^2+ \\ & \frac{\sqrt{Q_1Q_5}}{\Lambda}\cos^2\theta F_1\left(d\psi-\frac{b^2F_0dt+(2a^2+b^2F_0)dy}{R(2a^2+b^2)F_1}\right)^2+ \\ & \frac{\sqrt{Q_1Q_5}}{\Lambda R^2(2a^2+b^2)}\Bigg\{-dt^2\left[\Sigma\frac{2a^2-b^2F_0}{a^2}+\frac{4a^4\sin^2\theta}{2a^2+b^2}+\frac{b^4F_0^2\cos^2\theta}{F_1(2a^2+b^2)}\right]+ \\ & dy^2\left[\Sigma\frac{2a^2+b^2F_0}{a^2}-\cos^2\theta\frac{(2a^2+b^2F_0)^2}{(2a^2+b^2)F_1}\right]+2dtdy(b^2F_0)\left[\frac{\Sigma}{a^2}-\cos^2\theta\frac{2a^2+b^2F_0}{(2a^2+b^2)F_1}\right]\Bigg\}\end{aligned} \tag{3.28}$$

From this expression we can read directly:

$$G_{\alpha\beta} = \begin{pmatrix}\Lambda\sqrt{Q_1Q_5} & 0 & 0\\ 0 & \frac{\sqrt{Q_1Q_5}}{\Lambda}\sin^2\theta & 0\\ 0 & 0 & \frac{\sqrt{Q_1Q_5}}{\Lambda}F_1\cos^2\theta\end{pmatrix}, \qquad \det G_{\alpha\beta} = (Q_1Q_5)^{3/2}\frac{F_1}{\Lambda}\sin^2\theta\cos^2\theta$$

[3]This form of the metric will be useful to Entanglement Entropy computation of Ch. 5.

$$A_t^\phi = \frac{2a^2}{R(2a^2+b^2)}; \qquad A_t^\psi = \frac{b^2 F_0}{R(2a^2+b^2)F_1}; \qquad A_y^\psi = \frac{2a^2+b^2F_0}{R(2a^2+b^2)F_1}$$

Finally we reduce on the S_3 finding the 3D Einstein metric, which will be crucial to the Entanglement Entropy computations of Ch.4 :

$$\tilde{g}^E_{\mu\nu} = g_{\mu\nu}\frac{\det G_{\alpha\beta}}{\det G_{\alpha\beta}|_{r\to\infty}} \tag{3.29}$$

$$\begin{aligned}\tilde{g}^E_{\mu\nu}dx^\mu dx^\nu = &\frac{\sqrt{Q_1Q_5}}{(r^2+a^2)^2}\left[r^2+\frac{R^2a^4}{Q_1Q_5}\left(1+\frac{b^2}{2a^2}F_0\right)\right]dr^2 - \frac{r^2\left(1-\frac{b^2}{2a^2}F_0\right)+\frac{R^2a^4}{Q_1Q_5}}{\sqrt{Q_1Q_5}}dt^2 \\ &+\frac{r^2\left(1+\frac{b^2}{2a^2}F_0\right)}{\sqrt{Q_1Q_5}}dy^2+\frac{b^2}{a^2}\frac{r^2F_0}{\sqrt{Q_1Q_5}}dtdy\end{aligned} \tag{3.30}$$

3.5 CFT dual states

In order to understand better what the geometries we have constructed are, it can be useful to give their description in the dual *D1-D5 CFT*.

3.5.1 $\frac{1}{4}$-BPS states

According to the holographic dictionary, 2-charge microstate geometries are represented at the orbifold point in the R sector by a collection of N_k strands of winding k, we will denote with $|s\rangle_k$ (where $s = (j_3, \tilde{j}_3) = (0,0)\,(\pm,\pm)$). Hence a generic ground state $\psi_{\{N_k^{(s)}\}}$ is specified by the partition of $\{N_k^{(s)}\}$ satisfying the usual constraint:

$$\sum_{k,s} kN_k^{(s)} = N \tag{3.31}$$

So

$$\psi_{\left\{N_k^{(s)}\right\}} \equiv \prod_{k,s}(|s\rangle_k)^{N_k^{(s)}} \tag{3.32}$$

These states are, by construction, eigenstates of the zero modes of $J^3 = \sum_{r=1}^N J^3_{(r)}$ and $\tilde{J}^3 = \sum_{r=1}^N \tilde{J}^3_{(r)}$ with eigenvalues $\sum_{k,s} sN_j^{(s)}$.
A generic geometry is dual to a superposition of such R ground states and from the gravity side, the D1-D5 state is specified by the profile functions $g_i(v)$, $i = 1, \cdots, 5$. This suggests the existence of a correspondence between RR states and profile functions. The holographic recipe fixes the nature of this correspondence, after Fourier expanded the $g^i(v)$:

- the five i components of g^i are related to the 5 possible spin configuration $|s\rangle = |00\rangle, |{+}{+}\rangle, |{-}{+}\rangle, |{+}{-}\rangle, |{-}{-}\rangle$;
- the length of the strands is determined by the harmonic number in the Fourier expansion;
- finally the number of each type of strands depends on the magnitude of the harmonic mode

In particular,

$$g_1(v) + ig_2(v) = \sum_{k=1}^{\infty} \frac{1}{R}\sqrt{\frac{Q_1Q_5}{N}} \left(\frac{A_k^{(++)}}{k} e^{\frac{2\pi ik}{L}v} + \frac{A_k^{(--)}}{k} e^{-\frac{2\pi ik}{L}v} \right) \tag{3.33}$$

$$g_3(v) + ig_4(v) = \sum_{k=1}^{\infty} \frac{1}{R}\sqrt{\frac{Q_1Q_5}{N}} \left(\frac{A_k^{(+-)}}{k} e^{\frac{2\pi ik}{L}v} + \frac{A_k^{(-+)}}{k} e^{-\frac{2\pi ik}{L}v} \right) \tag{3.34}$$

$$g_5(v) = -\operatorname{Im}\left(\sum_{k=1}^{\infty} \frac{1}{R}\sqrt{\frac{2Q_1Q_5}{N}} \frac{A_k^{(00)}}{k} e^{\frac{2\pi ik}{L}v} \right) \tag{3.35}$$

where A_k is a dimensionless parameter satisfying the condition:

$$\sum_{k,s} \left| A_k^{(s)} \right|^2 = N \tag{3.36}$$

and where we have express explicitly the $(J^3, \tilde{J}^3)$ values for each $A^{(J^3,\tilde{J}^3)}$.
From the expressions (3.33)-(3.35), we understand that, given set of Fourier coefficient $\{A_k^{(s)}\}$, it specifies a profile, the profile determines the geometry according to Eq.(C.1a) and finally the CFT state dual to the geometry is given by the superposition:

$$\psi\left(\left\{A_k^{(s)}\right\}\right) = \sum_{\left\{N_k^{(s)}\right\}}{}^{'} \left(\prod_{k,s} A_k^{(s)} \right)^{N_k^{(s)}} \cdot \psi_{\left\{N_k^{(s)}\right\}} = \sum_{\left\{N_k^{(s)}\right\}}{}^{'} \prod_{k,s} \left(A_k^{(s)} \left|s\right\rangle_k \right)^{N_k^{(s)}} \tag{3.37}$$

where the prime over the sum reminds us the constraint (3.31).
However the only superposition giving a state well described in Supergravity, is the one with average numbers of strands $\bar{N}_k^{(s)} \gg 1$. The average number, then, is related to the Fourier coefficient via:

$$\bar{N}_k^{(s)} = \frac{\left| A_k^{(s)} \right|^2}{k} \tag{3.38}$$

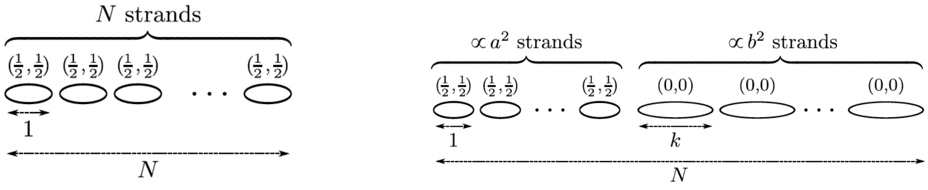
In the limit $\bar{N}_k^{(s)} \gg 1$, the sum in Eq. (3.37) is peaked around this average value, thus we can neglect the sum and consider the state as the simple product of the angular momentum eigenstates.

According to this picture in terms of strands, we can visualize the $\frac{1}{4}$ BPS states, constructed explicitly in Sec. 3.2, as depicted in Fig. 3.1. In particular the profile (3.9) corresponds to the state with N single strands (Fig. 3.1a), each one with the maximum angular momentum $(+,+)$. The total system is a Ramond ground state with $j = \tilde{j} = \frac{n_1n_5}{2}$ and $h = \frac{n_1n_5}{4} = \tilde{h}$.
Profile (3.10) has, in addition to strands of length one, another type of strands with quantum number $(0,0)$ and length k. The number of strands $(+,+)$ is proportional to a^2, while $\frac{b^2}{2k}$ counts the number of second type strands[4]. Explicitly the geometry described by Eq. (3.11a) can be associated to a CFT state, given by the coherent superposition of $N_k^{(00)} \equiv p\left|00\right\rangle_k$ and $N_1^{(++)}\left|{++}\right\rangle_1$ (where in order to satisfy Eq.(3.31) $N_1^{(++)} = N - kp$)

$$\left|\Psi\right\rangle = \sum_{p=1}^{N/k} \left(A_1^{(++)} \left|{++}\right\rangle_1 \right)^{N-kp} \left(A_k^{(00)} \left|00\right\rangle_k \right)^p \tag{3.39}$$

The correspondence goes further: through an expansion in spherical harmonics of the Z_i and computing the expectation values of some specific chiral primary operators in CFT,

[4]To be precise, one should consider coherent states, summing over all possible configurations. However, as previously said, since we are working in the limit of $N_k^{(s)}$ large, the sum is peaked over the average number which is determined by the amplitude of the Fourier coefficients.



(a) Eq. (3.9) (b) Eq. (3.10)

Figure 3.1: Types of strands in the CFT correspondind to the analysed profiles

it is possible to relate directly the gravity parameters appering in the Z_i to the Fourier coefficients A_i. We will not treat this topic here and we refer for example to [31] for details. We report only the result for our example, in this case we have:

$$\begin{cases} A_1^{(++)} = R\sqrt{\frac{N}{Q_1 Q_5}}a \\ A_k^{(00)} = R\sqrt{\frac{N}{2Q_1 Q_5}}b_k \end{cases} \tag{3.40}$$

In the limit $N_k^{(s)}$ large $\left|A_1^{(++)}\right|^2$ and $\left|A_k^{(00)}\right|^2/k$ count, respectively, the number of $|{++}\rangle$ and $|00\rangle_k$ states.

3.5.2 $\frac{1}{8}$-BPS states

The holographic dictionary for $\frac{1}{8}$-BPS states is less understood than the previous one. As we have done in the gravity framework, we want to construct these 3 charge geometries starting from $\frac{1}{4}$ states. The ground state we start with (in the approximation of $N_k^{(s)}$ large) is the product of $|{++}\rangle_1$ and $|00\rangle_k$ states. To increase by one the number of charges we can act n times, in the NS sector, with the operator $(L_{-1} - J_{-1}^3)$ on $|00\rangle_k$. The result is adding n_P units of momentum to the state but no angular momentum. The momentum n_P is generally defined in the R sector to be $n_P = h^R - \tilde{h}^R$.

A generic $\frac{1}{8}$-BPS will be a superposition:

$$\psi(A_1, B_{k,n}) = \sum_{N_1, N_{k,n} \text{s.t.}(3.43)} A_1^{N_1} (B_{k,n})^{N_{k,n}} \psi_{N_1, N_{k,n}} \tag{3.41}$$

$$\psi_{N_1, N_{k,n}} = (|{++}\rangle_1)^{N_1} \left(\frac{\left(L_{-1} - J_{-1}^3\right)^n}{n!} |00\rangle_k \right)^{N_{k,n}} \tag{3.42}$$

with the constraint:

$$N_1 + k N_{k,n} = N \tag{3.43}$$

However, since we work in the N large limit, we can neglect the sum and considering only the saddle points values for $N_1, N_{k,n}$ [4]:

$$\bar{N}_1 = |A_1|^2, \qquad k\bar{N}_{k,n} = \binom{n+k-1}{n} |B_{k,n}|^2 \tag{3.44}$$

where the CFT dimensionless coefficients A and B are related to the Supergravity variables a and $b_4^{k,n}$ through:

$$|A_1| \equiv= R\sqrt{\frac{N}{Q_1 Q_5}}a, \qquad |B_{k,n}| \equiv= R\sqrt{\frac{N}{Q_1 Q_5}} \binom{n+k-1}{n}^{-1} b_4^{k,n} \tag{3.45}$$

Knowing the average number $\bar{N}_{k,n}$ we can read directly the total momentum charge: since each $|00\rangle_k$ strand carries one unit of momentum, it is given by n times the average number:

$$n_p = n\bar{N}_{k,n} = \frac{R^2 N}{Q_1 Q_5} \frac{n}{2k} \binom{n+k-1}{n}^{-1} \left(b_4^{k,n}\right)^2 \tag{3.46}$$

Chapter 4

Entanglement Entropy

4.1 A general introduction

Entanglement Entropy (EE) is a very useful quantity when we want to investigate the properties of a given quantum field theory. In this chapter we want to review the basic ideas of what EE is and give a description from different view points, quantum mechanics, QFT and in the end we will give a holographic interpretation.

4.1.1 Basic definitions and properties

Let us start from an example: consider a quantum mechanical system in a *pure* state $|\Psi\rangle$, then its density matrix is:

$$\rho_{tot} = |\Psi\rangle \langle\Psi|$$

For the total system one can define the von Neumann entropy as:

$$S_{tot} = -\mathrm{tr}\rho_{tot} \log \rho_{tot}$$

This quantity is trivially zero for a pure state, but imagine, now, to divide the system into two subsystems A and B. We make this split in such a way that the total Hilbert space can be written as a direct product: $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then, we can introduce a *reduced* density matrix for the subsystem A, tracing over B degrees of freedom:

$$\rho_A \equiv \mathrm{tr}_B \rho_{tot}$$

and equivalently for B. With these few definitions one is already able to introduce the notion of *entanglement entropy*(EE): the EE of the subsystem A is given by the von Neumann entropy of the reduced density matrix :

$$S_A = -\mathrm{tr}_A \log \rho_A \tag{4.1}$$

Note that even if the initial state is a pure one, in general the reduced density is not, so S_A is in principle different from zero.

In order to clarify better these ideas let concentrate on a pure state $|\Psi\rangle$ of the form:

$$|\Psi\rangle = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

where $|i\rangle_A$ and $|j\rangle_B$ are respectively a base for $\mathcal{H}_A$ (of dimension d_A) and $\mathcal{H}_B$ (dimension d_B). According to the form of the coefficients matrix $c_{i,j}$, two different types of states can exist:

- **separable state** $\Leftrightarrow |\Psi\rangle = |\Psi\rangle_A \otimes |\Psi\rangle_B$, i.e. c_{ij} factorize as $c_{ij} = c_i^A c_j^B$. In this case also the reduced density matrix is that of a pure state and EE vanishes ($S_A = 0$);
- **entangled state** $\Leftrightarrow c_{ij} \neq c_i^A c_j^B$, then the state can not be written as a product. In this case when we trace out the degrees of freedom of the subsystem B we are left with a reduced density matrix ρ_A of a mixed state, so its EE is now non trivial.

From this simple example, it is clear that EE measures how much a given state differs from a separable one, or equivalently how much is entangled with another. It goes from zero (separable state) to its maximum $S_A^{max} = \log(\min(d_A, d_B))$, when a given state is a superposition of all possible quantum states with an equal weight [32].
Consider for instance a system of two particles A and B of spin 1/2, their Hilbert space being spanned by $\mathcal{H}_{A,B} = \left\{ |+\rangle_{A,B}, |-\rangle_{A,B} \right\}$. Suppose the state is an entangled one:

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{2}} \left(|+\rangle_A |-\rangle_B - |-\rangle_A |+\rangle_B \right) \Leftrightarrow \rho_{tot} = |\Psi\rangle \langle\Psi| \\ \rho_A &= {}_B\langle +| \rho_{tot} |+\rangle_B + {}_B\langle -| \rho_{tot} |-\rangle_B \\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{aligned}$$

From the *reduced* density matrix we can finally compute the EE:

$$\begin{aligned} S_A &= -\mathrm{tr} \left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\log 2 & 0 \\ 0 & -\log 2 \end{pmatrix} \right] \\ &= \log 2 \end{aligned}$$

This result tells us that the state $|\Psi\rangle$ is maximally entangled, since it saturates the upper bound for $d_A = d_B = 2$.

The Entanglement Entropy is characterised by some properties:

- for a pure state the EE of a subsystem A and its complement B is the same $S_A = S_B$. It is not true if the state is mixed;
- *subadditivity*, given two disjoint systems A and B

$$S_A + S_B \geq S_{A\cup B}$$

- *strong subadditivity* for any three disjoint systems A, B and C

$$\begin{aligned} S_{A\cup B\cup C} + S_B &\leq S_{A\cup B} + S_{B\cup C} \\ S_A + S_C &\leq S_{A\cup B} + S_{B\cup C} \end{aligned}$$

Before going on with the derivation of EE in QFT, we want to use the concepts and the properties just introduced, to review the Mathur argument, cited in Ch. 1, concerning the failure of small corrections as a solution of the information paradox. In particular in [33] he shows, using strong subadditivity, that small corrections to the leading order Hawking's computations can not solve the problem and only order unity modifications could fix the paradox.

Let us consider the creation of a black hole from matter collapsing in a semiclassical approximation, then gravity background is Schwarzschild BH:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

while the general state of this system is a quantum one:

$$|\Psi\rangle \approx |\psi\rangle_M \otimes |\psi\rangle_{pair} \tag{4.2}$$

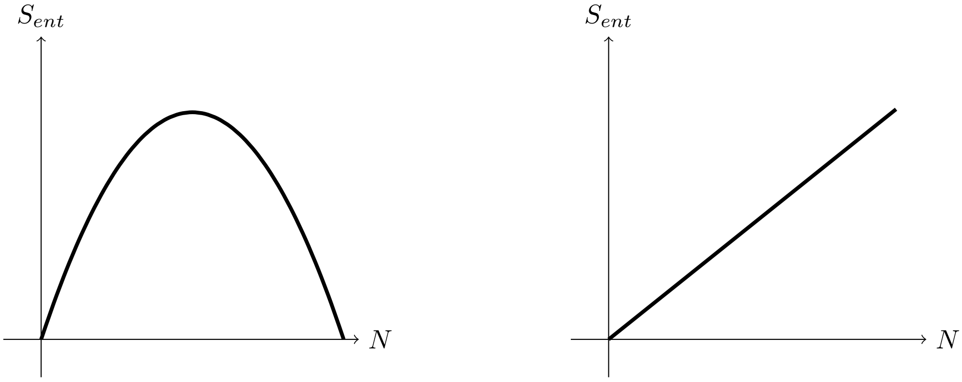


Figure 4.1: Entanglement entropy: for a body with its radiation rises and then falls, for a BH with Unruh vacuum rises monotically

Particle pairs created from the vacuum are in an entangled state. The pair is composed by a particle outside the horizon, which can escape to infinity and we will call b and the another one, we denote with c, which is inside the horizon and can fall in it. $|\psi\rangle_M$ represents the matter, which is far away from the created quanta.
For each time t, a pair is created. The form of the state is assumed to be:

$$|\psi\rangle_{pair} = \frac{1}{\sqrt{2}}\Big(|0\rangle_{c_{N+1}}|0\rangle_{b_{N+1}} + |1\rangle_{c_{N+1}}|1\rangle_{b_{N+1}}\Big) \tag{4.3}$$

Let us denote $|\Psi_{M,c}, \psi_b(t_N)\rangle$ the generic state at the time t_N, when N pairs have been generated. At the following step another pair is added and if we assume that all the created couples have the same form (4.3), then:

$$|\Psi_{M,c}, \psi_b(t_{N+1})\rangle = |\Psi_{M,c}, \psi_b(t_N)\rangle \, \frac{|0\rangle_{c_{N+1}}|0\rangle_{b_{N+1}} + |1\rangle_{c_{N+1}}|1\rangle_{b_{N+1}}}{\sqrt{2}} \tag{4.4}$$

Since particles b_i are entangled with the c_i's, at any step we can define the Entanglement entropy of the state as:

$$S_{EE} \equiv -\mathrm{tr}_b\left(\rho_b \log \rho_b\right) \tag{4.5}$$

We have seen that for an entangled state $S = \log 2$. At each step a new pair appears, the particle created earlier move away from each other and from the region of production. Then the EE for N pairs is:

$$S_{EE}(t_N) = N \log 2 \tag{4.6}$$

Notice that the entropy of our system keeps rising monotonically with N. This is a characteristic of BH radiation, which is completely different from the common behaviour under evaporation of a normal body. The latter, as shown in Fig.4.1, has an S_{ent}, which rises and finally returns to zero.

Now we want to deform the leading state (4.4) by a small perturbation. In the simplest case, this can be done simply considering a slight modification of (4.3) and (4.4) and specifying what we mean for a small correction. So first of all, let us introduce two orthonormal basis ψ_i for the M and c quanta and χ_i for the b:

$$|\Psi_{M,c}, \psi_b(t_n)\rangle = \sum_i C_i \psi_i \chi_i \tag{4.7}$$

The expression of the entropy thus changes as:

$$S_{EE}(t_N) = -\sum_i |C_i|^2 \log |C_i|^2 \equiv S_N \tag{4.8}$$

Now assume that time evolution does not affect the b_i created at earlier times. It is reasonable since they are far away from the hole, but changes the form of ψ_i. So at time t_{N+1}, we have:

$$\chi_i(t_{N+1}) = \chi(t_N) \tag{4.9}$$
$$\psi_i(t_{N+1}) = \psi_i^{(1)} S^{(1)} + \psi_i^{(2)} S^{(2)} \tag{4.10}$$

where we have introduced the orthonormal vectors:

$$S^{(1)} = \frac{|0\rangle_{c_{N+1}} |0\rangle_{b_{N+1}} + |1\rangle_{c_{N+1}} |1\rangle_{b_{N+1}}}{\sqrt{2}}$$
$$S^{(2)} = \frac{|0\rangle_{c_{N+1}} |0\rangle_{b_{N+1}} - |1\rangle_{c_{N+1}} |1\rangle_{b_{N+1}}}{\sqrt{2}}$$

Consequently the state becomes:

$$|\Psi_{M,c}, \psi_b(t_{N+1})\rangle = S^{(1)} \underbrace{\sum_i C_i \psi_i^{(1)} \chi_i}_{\Lambda^{(1)}} + S^{(2)} \underbrace{\sum_i C_i \psi_i^{(2)} \chi_i}_{\Lambda^{(2)}} \tag{4.11}$$

Assuming normalization of $|\Psi\rangle$, then:

$$||\Lambda^{(1)}||^2 + ||\Lambda^{(2)}||^2 = 1 \tag{4.12}$$

A direct comparison between the state (4.4) and (4.11) shows the identification at leading order:

$$\psi_i^{(1)} = \psi_i, \qquad \psi_i^{(2)} = 0$$

Now we can define a small deformation: a correction is small if:

$$||\Lambda^{(2)}|| < \varepsilon, \qquad \varepsilon \ll 1 \tag{4.13}$$

The interpretation of this requirement is clear: $||\Lambda^{(2)}||$ small means that the probability to find the state of created pair in $S^{(2)}$ is much less than unity and the most probable state remains the original one $S^{(1)}$. If a deformation does not satisfies (4.13), instead, we will call it an "order unity" correction.

Once defined what we mean for small corrections, we want to prove that under this assumption, the entanglement of the b quanta with those in the black hole increases at each stage of time evolution. For ease of notation let us denote $\{b\} \equiv \{b_1, \ldots, b_N\}$ and $(M, \{c\} \equiv \{c_1, \ldots, c_N\})$. Differently from the leading order, now the pair created at time t_{N+1}, i.e. (c_{N+1}, b_{N+1}), can interact weakly with $(M, \{c\})$. The statement we want to prove is that: supposed that the $\{b\}$ quanta has an entropy S_N at time t_N and that the new pair generated at t_{N+1} differers from Hawking state (4.4) only by a small correction (4.13), the entropy of emitted quantity $S(\{b\} + b_{N+1})$ is non decreasing. In order to prove it 3 intermediate passages are necessary, we summarize them below and collect their proofs in Ap. (A):

1. the entanglement of (c_{N+1}, b_{N+1}) with the rest of the system $S(c_{N+1}, b_{N+1})$ is bounded by ε
$$S(c_{N+1}, b_{N+1}) < \varepsilon$$
2. $$S(\{b\}, c_{N+1}, b_{N+1})) \geq S_N - \varepsilon$$
3. $$S(c_{N+1}) > \log 2 - \varepsilon$$

To prove our statement, we need only one version of the *strong subadditivity* theorem:

$$S(A+B)+S(B+C)\geq S(A)+S(C) \tag{4.14}$$

So we set $A=\{b\}$, $B=b_{N+1}$ and $C=c_{N+1}$, then, exploiting the relations obtained above, (4.14) reads:

$$S(\{b\}+b_{N+1})\geq S_N+\underbrace{S(c_{N+1})}_{<\log 2-\varepsilon}-\underbrace{S(c_{N+1},b_{N+1})}_{<\varepsilon} \tag{4.15}$$

$$S(\{b\}+b_{N+1})>S_N+\log 2-2\varepsilon \tag{4.16}$$

We have proven that if corrections are small in the sense of (4.13) then, at each step of the evolution, the entropy has to increase. Consequently, it fails the belief that small departures from the leading state (4.4) can change Hawking's conclusions and solving the information paradox. Thus, if we wish the entanglement to be zero the created pairs should change to a state quasi-orthogonal to the semiclasically expected one, that is to say that we need some "order unity" corrections.

Let us now return to the description of Entanglement Entropy in QFT and in gravity via the AdS/CFT.

4.1.2 Entanglement Entropy in QFT

We can define EE also for Quantum Field Theory. In $d+1$ dimension [6], we consider as a subsystem a d dimensional submanifold A, with boundary ∂A. Since the EE is always divergent in a continuum theory, we need to introduce an UV cut-off a. Then, it can be proved that the EE takes the general form:

$$S_A\propto\frac{\text{Area}(\partial A)}{a^{d-1}}+\text{subleading term}$$

This simple area law does not always describe efficiently EE in generic situations and suffers from different corrections according to the system one wants to describe. However, written in this form, it shows a strong resemblance with Bekenstein-Hawking entropy S_{BH} and it was exactly this similarity that originally motivated research on entanglement entropy and its possible relation with S_{BH}, even though this analogy soon after turns out not to be completely correct.

In QFT Entanglement Entropy is usually computed through a technique known as *replica trick* [34] (in the following we will assume to work with Euclidean time). The main idea of this method is to consider n fictitious copies of our original theory, then it is possible to compute the n^{th} power of ρ_A (ρ_A^n) and express the usual EE as:

$$S_A=\lim_{n\to 1}\frac{\log\text{Tr}_A\left(\rho_A^n\right)}{1-n} \tag{4.17}$$

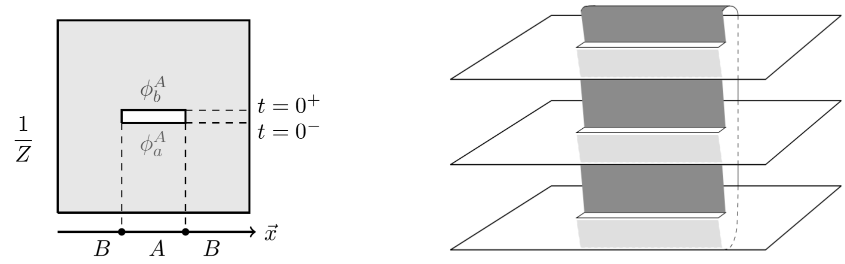
$$=-\frac{\partial}{\partial n}\log\text{Tr}_A\left(\rho_A^n\right)\big|_{n=1} \tag{4.18}$$

where in the last passage we have used the de l'Hôpital theorem and the normalization $\text{Tr}_A\rho_A=1$.[1]

In analogy with QM, in QFT we start considering a complete set of operator $\hat\phi(t,\vec x)$, whose eigenvectors and eigenvalues are determined via $\hat\phi(0,\vec x)\left|\phi_0(\vec x)\right\rangle=\phi_0(\vec x)\left|\phi_0(\vec x)\right\rangle$. Then given

[1] This expression is equivalent to eq.(4.1): let us call λ_i the eigenvalues of the reduced density matrix. By definition $\log\text{Tr}_A\left(\rho_A^n\right)=\log\left(\sum_i\lambda_i^n\right)$, so

$$\frac{\partial}{\partial n}\log\left(\sum_i\lambda_i^n\right)\Bigg|_{n=1}=\frac{1}{\sum_i\lambda_i^n}\Bigg|_{n=1}\sum_i\lambda_i^n\log\lambda_i\big|_{n=1}=\frac{1}{\underbrace{\sum_i\lambda_i}_{1}}\sum_i\lambda_i\log\lambda_i=\text{Tr}\left(\rho_A\log\rho_A\right)$$



(a) (b)

Figure 4.2: (a) Reduced density matric ρ_A [32]. (b)Replica trick for a single interval region. The separated sheets represnt different copies.

a state $|\Psi\rangle$, the functional $\Psi[\phi_0(\vec{x})] = \langle\phi_0(\vec{x})|\Psi\rangle$ corresponds to the usual wave function in QM. We can define the partition function via path integral as:

$$Z = \int \mathcal{D}\phi_0 \, \langle\Psi|\phi_0\rangle \, \langle\phi_0|\Psi\rangle \tag{4.19}$$

Now suppose to divide the system into two subsystems A and B, such that we will call ϕ^B the field with support only on the region B and ϕ^A the field with $\vec{x} \in A$. The reduced density ρ_A is obtained tracing out the B degrees of freedom:

$$\rho_A = \frac{1}{Z} \int \mathcal{D}\phi \, \langle\phi^B|\Psi\rangle \, \langle\Psi|\phi^B\rangle$$

The values ρ_A^{ab} of the reduced density matrix, once specified the boundary conditions for $\phi^A(t = 0^-, \vec{x}) = \phi_a^A$ and $\phi^A(t = 0^+, \vec{x} = \phi_b^A)$,

$$\rho_A^{ab} = \langle\phi_a^A|\,\rho_A\,|\phi_b^A\rangle = \frac{1}{Z}\int_{t=-\infty}^{t=+\infty} \mathcal{D}\phi e^{-S_E[\phi]} \prod_{\vec{x}\in A} \delta\left(\phi(0^+,\vec{x}) - \phi_b^A(\vec{x})\right) \delta\left(\phi(0^-,\vec{x}) - \phi_a^A(\vec{x})\right) \tag{4.20}$$

S_E being the Euclidean action. In Fig. 4.2(a) we report a pictorial representation of the reduced density matrix: if we think of our space-time as a plane of coordinates t and $\vec{x}$, then integrating out the B degrees of freedom can be viewed as sewing together all the $\vec{x} \notin A$, while in correspondence of $\vec{x}_A$ there are cuts (we have assumed that the region A is a single interval).
Replica trick consists in making n identical copies, labelled by an index j, $1 \leq j \leq n$ and then compute the trace sewing them together along the cuts in such a way that $\phi_j^b = \phi_{j+1}^a$ and identifying ϕ_n at $t = 0^+$ with ϕ_1 at $t = 0^-$. This procedure is depicted schematically in Fig. 4.2(b). We define Z_n the partition function of the surface constructed by the n copies, then the trace is:

$$\mathrm{Tr}\rho_A^n = \frac{Z_n}{Z^n} \tag{4.21}$$

where Z^n is the n^{th} power of the original partition function.

In general it remains difficult to compute $\mathrm{Tr}\rho_A^n$, but in the case of 2 dimensional CFT there is a way to rewrite eq.(4.21), making computation easier. In particular, suppose that we want to compute the EE for a single interval A of length l in a state $|s\rangle$ using the replica trick. We have learnt that we need to consider n copies of the CFT and it is necessary to glue them together imposing that the fields ϕ_k $(k = 1, 2, \cdots, n)$, defined on every sheet, satisfy specific boundary conditions. For CFT_2 it can be shown [35] that the surface constituted by the n copies can be mapped to the complex plane and here we can define some local operators, known as *twist fields*. These fields are related to the invariance of the theory under exchange of the copies. So we can introduce two twist fields associated to the opposite cyclic permutation:

$$\mathcal{T}_n, \quad \text{for the permutation} \quad i \to i+1 \bmod n \tag{4.22}$$

$$\mathcal{T}_{-n}, \quad \text{for the permutation} \quad i+1 \to i \bmod n \tag{4.23}$$

It can be proved that

$$\mathrm{Tr}\rho_A^n = \langle s|\, \mathcal{T}_n(z,\bar{z})\mathcal{T}_{-n}(w,\bar{w})\,|s\rangle \tag{4.24}$$

where $z, \bar{z}, w, \bar{w}$ are coordinates on the complex plane. As an example, consider a 1D infinite long system and identify A with the interval $\ell = |u - v|$. This theory is assumed to have central charge c and UV cut-off a (or lattice spacing). Then it can be shown [32] that for this system:

$$\mathrm{Tr}\rho_A^n = c_n \left(\frac{v-u}{a}\right)^{-2d_n} \tag{4.25}$$

where d_n is the conformal dimension of either $\mathcal{T}_n$ and $\mathcal{T}_{-n}$,

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right) \tag{4.26}$$

and c_n is a constant to be fixed in such a way in the limit $n \to 1$ $c_1 = 1$. Now it is sufficient to apply Eq. (4.18) to obtaining:

$$S_A \simeq \frac{c}{3}\log\frac{l}{a} \tag{4.27}$$

4.1.3 Holographic Entanglement Entropy

Determining EE in CFT using (4.24) or (4.20) is not simple in most cases. However, we know that AdS/CFT correspondence relates a $d+1$ dimensional conformal field theory with a bulk theory on the AdS_{d+2} spacetime. In the previous section we have seen that one can define and compute EE in a CFT, so it is natural to ask if it is possible to derive this entropy as a pure geometrical quantity in AdS, exploiting the power of this duality.

In [6] (and its generalization in [36]) Ryu and Takayanagi propose a way to compute EE through AdS/CFT correspondence.
Their conjecture states that the entanglement entropy S_A in CFT_{d+1} can be computed as:

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \tag{4.28}$$

where γ_A is a d-dimensional minimal surface in AdS_{d+2}, which has ∂A as a boundary, while G_N refers to the Newton constant in $(d+2)$ dimensions. The original conjecture in [6] was valid for the static AdS spacetime, but it has been generalized in a covariant way in [36].
To motivate the conjecture, start considering $\mathrm{AdS}_3/\mathrm{CFT}_2$, with the spatial direction of CFT compactified on a circle S^1 of radius R, and focus on EE for a single interval. It can be shown that, looking at Eq. (4.25) in AdS/CFT, the trace can be rewritten as:

$$\mathrm{Tr}\rho_A^n \sim e^{-2d_n \frac{L_\gamma}{R}} \Rightarrow S_A = 2\frac{L_\gamma}{R}\frac{\partial\,(d_n)}{\partial n}\bigg|_{n=1} = \frac{c}{6}\frac{L_\gamma}{R}$$

where L_γ is the length of the geodesic whose boundary is l. Then, remembering

$$c = \frac{3R}{2G_N^{(3)}} \tag{4.29}$$

so

$$S_A = \frac{L_\gamma}{4G_N^{(3)}}$$

which perfectly agrees with the prescription (4.28). The same reasoning can be extended to higher dimensional cases.

4.2 Minimal surface in 6D spacetime

We have seen that for theories, which admit a gravity dual described by classical Einstein gravity and whose states are dual to AdS_{d+2} spacetimes, EE can be computed via RT formula (4.28) or its generalization in [36].
One of our goal is to study the EE for a single interval of length l in the $\frac{1}{8}$-BPS states. These geometries, however, have a product stricture $\mathrm{AdS}_3 \times S^3$ only asymptotically. In general, there is no canonical way to decouple the 3D part asymptotically AdS from the rest of the spacetime. Thus in these cases we can not apply the usual RT formula and we need to extend it. A way to extend the prescription for calculating EE was presented in [8], it applies to stationary geometry asymptotically $\mathrm{AdS}\times S$ and generalizes the covariant EE of [36]. According to [8], given a 1D spacial region $A = [\,0, l\,]$, its EE should be defined as:

$$S_A = \frac{\mathrm{area}\mathcal{A}}{4G_N} \tag{4.30}$$

where, this time, $\mathcal{A}$ is a co-dimensional 2 extremal surface in the six dimension spacetime (such that at the boundary it reduces to $\partial A \times S^3$) and G_N is the 6D Newton constant.

In this work, we want to prove that for a special class of geometries, it is possible to reduce exactly the 6D problem to a 3D one, to which the original RT prescription applies. Let us be more specific.
In [8, 7] it was suggested that it is always possible to rewrite our 6D metric with an almost product structure, as:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + G_{\alpha\beta}\left(dx^\alpha + A^\alpha_\mu dx^\mu\right)\left(dx^\beta + A^\beta_\nu dx^\nu\right) \tag{4.31}$$

where coordinates are such that at the boundary x^μ and x^α are respectively coordinates of AdS_3 and S^3. From the metric written in this way we can define:

$$\tilde{g}^E_{\mu\nu} \equiv g_{\mu\nu}\frac{\det G}{\det G_0} \tag{4.32}$$

which is the reduced 3D Einstein metric, where $\det G = \det G_{\alpha\beta}$ and G_0 is the round metric of S^3.
For some 3-charge solution, in particular for the Superstrata $(k = 1, n)$ in Ch.3, $\tilde{g}^E_{\mu\nu}$ is independent from the coordinates on S^3. For microstate metrics satisfying this characteristic, it is possible to prove that the extremal surface in 6D is equivalent to $x^\mu \times S^3$, where $x^\mu(\lambda)$ is a geodesic of $\tilde{g}^E_{\mu\nu}$. This propriety determines a great simplification in computing EE. In fact it allows, in this case, not to consider the full 6D problem and to restrict the three dimensional part of the metric, which is asymptotically AdS and whose EE can be computed via RT prescription:

$$S_A = \frac{L_\gamma}{4G_N} = \frac{c}{6}\frac{L_\gamma}{R_{\mathrm{AdS}}} = n_1 n_5 \frac{L_\gamma}{R_{\mathrm{AdS}}} \tag{4.33}$$

where we have used the fact that for microstate geometries:

$$G_N = \frac{3}{2}\frac{R_{\mathrm{AdS}}}{c}, \qquad c = 6n_1n_5$$

with c the central charge of the dual CFT, n_1, n_5 the number of D1, D5 branes and R_{AdS} the AdS radius.

4.2.1 Derivation

Let us go through the derivation and verify our statement.
According to Eq. (4.30), in 6D to compute EE we need to find an extremal four dimensional surface, which we parametrize as $x^I(\lambda, x^\alpha)$, $I = \{\mu, \alpha\}$ [2]. The induced metric on the submanifold is:

$$ds_*^2 = g_{\mu\nu}dx_*^\mu dx_*^\nu + G_{\alpha\beta}\left(dx^\alpha + A_\mu^\alpha dx_*^\mu\right)\left(dx^\beta + A_\nu^\beta dx_*^\nu\right) \equiv g_{IJ}^* dx^I dx^J \tag{4.34}$$

whit:

$$dx_*^\mu = \dot{x}^\mu d\lambda + \partial_\alpha x^\mu dx^\alpha$$

where for short we have denoted $\dot{x}^\mu \equiv \frac{\partial x^\mu}{\partial \lambda}$. Following the idea of Ryu and Takayanagi, x^I should extremize the area functional

$$I\left[x^\mu(\lambda, x^\alpha)\right] = \int dx^\alpha d\lambda \sqrt{\det g_*} \tag{4.35}$$

We want to find under which assumptions the minimization problem in higher dimensions is equivalent to the one in 3D. In terms of extremal surfaces, we want to show which conditions the original metric must satisfy in such a way that $\bar{x}^I = (x^\mu(\lambda), x^\alpha)$, where $x^\mu(\lambda)$ is a geodesic of $\tilde{g}^E$, is a solution of the minimization problem (4.35). Extremizing the function I is equivalent to solve Euler-Lagrange equations:

$$\frac{\partial\sqrt{\det g_*}}{\partial x^\rho} - \frac{\partial}{\partial\lambda}\frac{\partial\sqrt{\det g_*}}{\partial\dot{x}^\rho} - \frac{\partial}{\partial x^\xi}\frac{\partial\sqrt{\det g_*}}{\partial\partial_\xi x^\rho} = 0 \tag{4.36}$$

In order to prove that the 3D geodesic is a solution, we will compute Eq. (4.36) in $\bar{x}$ or equivalently consider $\partial_\alpha x^\mu = 0$.

Let us look in detail at the induced metric (4.34):

$$\begin{aligned}
g_{\lambda\lambda}^* &= \left(g_{\mu\nu} + A_\mu^\alpha A_\nu^\beta G_{\alpha\beta}\right)\dot{x}^\mu\dot{x}^\nu \\
g_{\lambda a}^* &= \left(g_{\mu\nu} + G_{\gamma\delta}A_\mu^\gamma A_\nu^\delta\right)\partial_\alpha x^\mu\dot{x}^\nu + G_{\alpha\gamma}A_\mu^\gamma\dot{x}^\mu \\
g_{\alpha\beta}^* &= G_{\alpha\beta} + \left(g_{\mu\nu} + G_{\gamma\delta}A_\mu^\gamma A_\nu^\delta\right)\partial_\alpha x^\mu\partial_\beta x^\nu + A_\mu^\gamma\left(G_{\alpha\gamma}\partial_\beta x^\mu + G_{\beta\gamma}\partial_\alpha x^\mu\right)
\end{aligned}$$

For later use, we report the induced metric evaluated at the solution:

$$g^*\big|_{\bar{x}^I} \equiv \bar{g} = \begin{pmatrix} \left(g_{\mu\nu} + A_\mu^\alpha A_\nu^\beta G_{\alpha\beta}\right)\dot{x}^\mu\dot{x}^\nu & G_{\gamma\alpha}A_\mu^\gamma\dot{x}^\mu \\ G_{\gamma\alpha}A_\mu^\gamma\dot{x}^\mu & G_{\alpha\beta} \end{pmatrix} \tag{4.37}$$

and its inverse:

$$\bar{g}^{\lambda\lambda} = g^{\lambda\lambda}, \qquad \bar{g}^{\lambda\alpha} = -g^{\lambda\lambda}A_\mu^\alpha\dot{x}^\mu, \qquad \bar{g}^{\alpha\beta} = G^{\alpha\beta} + g^{\lambda\lambda}A_\mu^\alpha A_\nu^\beta\dot{x}^\mu\dot{x}^\nu \tag{4.38}$$

where we have denoted $g^{\lambda\lambda}$ the inverse of:

$$g_{\lambda\lambda} \equiv g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

Moreover,

$$\det\bar{g} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\det\left(G_{\alpha\beta}\right) = \tilde{g}_{\mu\nu}^E\dot{x}^\mu\dot{x}^\nu\det\left(G_0\right) \tag{4.39}$$

Looking at the components of g_{IJ}^* we immediately realise that when we consider derivatives w.t.r. to x^ρ and $\dot{x}^\rho$ in (4.36), there is no difference between differentiating the full induced

[2] A generic parametrization is $x^I(\lambda, \xi^\alpha)$. Since the area functional contains an integrand over ξ^α, and is thus invariant under reparametrization of ξ^α, we can, without loss of generality, identify ξ^α with the space-time coordinate x^α $(\xi^\alpha = x^\alpha)$

metric or directly $\bar{g}$ (terms proportional to $\partial_\alpha x^\mu$ are not involved in differentiation and can be simply put to zero). In other words:

$$\left.\frac{\partial\sqrt{\det g_*}}{\partial x^\rho}\right|_{\bar{x}^I} \equiv \frac{\partial\sqrt{\det \bar{g}}}{\partial x^\rho} \tag{4.40}$$

$$\left.\frac{\partial}{\partial\lambda}\frac{\partial\sqrt{\det g_*}}{\partial \dot{x}^\rho}\right|_{\bar{x}^I} \equiv \frac{\partial}{\partial\lambda}\frac{\partial\sqrt{\det \bar{g}}}{\partial \dot{x}^\rho} \tag{4.41}$$

The only non trivial term in the Euler-Lagrange equations is the last one, but even in this case there is a simplification:

$$\left.\left(\frac{\partial}{\partial x^\xi}\frac{\partial\sqrt{\det g_*}}{\partial\partial_\xi x^\rho}\right)\right|_{\bar{x}^I} = \frac{\partial}{\partial x^\xi}\left(\left.\frac{\partial\sqrt{\det g_*}}{\partial\partial_\xi x^\rho}\right|_{\bar{x}^I}\right)$$

So we can firstly concentrate on[3]

$$\left.\frac{\partial\sqrt{\det g_*}}{\partial\partial_\xi x^\rho}\right|_{\bar{x}^I} = \frac{\sqrt{\det \bar{g}}}{2}\bar{g}^{IJ}\left.\frac{\partial g^*_{IJ}}{\partial\partial_\xi x^\rho}\right|_{\bar{x}^I} \tag{4.42}$$

In this case the only components that contribute are $(I,J) = (\lambda,\alpha)$ or (α,β), computed in the solution they are:

$$\begin{cases} \frac{\partial g^*_{\lambda\alpha}}{\partial\partial_\xi x^\rho} = \left(g_{\mu\rho} + G_{\gamma\delta}A^\gamma_\mu A^\delta_\rho\right)\dot{x}^\mu\delta^\xi_\alpha \\ \frac{\partial g^*_{\alpha\beta}}{\partial\partial_\xi x^\rho} = A^\gamma_\rho\left(G_{\alpha\gamma}\delta^\xi_\beta + G_{\beta\gamma}\delta^\xi_\alpha\right) \end{cases}$$

So:

$$\left.\frac{\partial\sqrt{\det g_*}}{\partial\partial_\xi x^\rho}\right|_{\bar{x}^I} = \sqrt{\det \bar{g}}\left(A^\xi_\rho - g^{\lambda\lambda}g_{\mu\rho}A^\xi_\nu\dot{x}^\mu\dot{x}^\nu\right) \tag{4.43}$$

Finally performing the last derivative, one obtains substituting $g_{\mu\nu}$ with $\tilde{g}^E_{\mu\nu}$ and defining $g^{\lambda\lambda} = \frac{1}{g_{\lambda\lambda}} = \frac{1}{\tilde{g}^E_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\cdot\frac{\det(G_0)}{\det G} \equiv \tilde{g}^{\lambda\lambda}\frac{\det(G_0)}{\det G}$:

$$\left.\frac{\partial}{\partial x^\xi}\frac{\partial\sqrt{\det g_*}}{\partial\partial_\xi x^\rho}\right|_{\bar{x}^I} = \frac{\partial}{\partial x^\xi}\left\{\sqrt{\tilde{g}^E_{\pi\sigma}\dot{x}^\pi\dot{x}^\sigma}\cdot\sqrt{\det(G_0)}\left[A^\xi_\rho - \tilde{g}^{\lambda\lambda}\tilde{g}^E_{\mu\rho}A^\xi_\nu\dot{x}^\mu\dot{x}^\nu\right]\right\} \tag{4.44}$$

$$\begin{aligned} &= \frac{\partial\sqrt{\tilde{g}^E_{\pi\sigma}\dot{x}^\pi\dot{x}^\sigma}}{\partial x^\xi}\sqrt{\det G_0}\left[A^\xi_\rho - \tilde{g}^{\lambda\lambda}\tilde{g}^E_{\mu\rho}A^\xi_\nu\dot{x}^\mu\dot{x}^\nu\right] + \\ &+\sqrt{\tilde{g}^E_{\pi\sigma}\dot{x}^\pi\dot{x}^\sigma}\sqrt{\det G_0}\left\{\nabla^0_\xi A^\xi_\rho - \dot{x}^\mu\dot{x}^\nu\left(\frac{\partial\left(\tilde{g}^{\lambda\lambda}\tilde{g}^E_{\mu\rho}\right)}{\partial x^\xi}A^\xi_\nu + \tilde{g}^{\lambda\lambda}\tilde{g}^E_{\mu\rho}\nabla^0_\xi A^\xi_\nu\right)\right\} \end{aligned} \tag{4.45}$$

where ∇^0_ξ is the covariant derivative w.t.r. to the round metric of the 3-sphere[4].
The 3 dimensional geodesic x^μ is a solution if and only if this term vanishes. From eq.(4.45) we see that this happens if two conditions are satisfied:

1. $\partial_\xi\left(\tilde{g}^E_{\mu\nu}\right) = 0$, in this way the first and third term in (4.45) are zero. But this requirement is exactly the definition of a *factorizable* metric, a metric whose 3D Einstein metric reduced on S^3 does not depend on angular coordinates.
2. $\nabla^0_\xi A^\xi_\rho = 0$ *de Donder* gauge condition

[3]Given a generic invertible square matrix A,

$$\det A = e^{\mathrm{Tr}\log A}$$

[4]We have used also the relation: given a generic vector $J^\xi \Rightarrow \nabla_\xi J^\xi = \frac{1}{\sqrt{\det G}}\partial_\xi\left(\sqrt{\det G}J^\xi\right)$

Summarizing our results, we have proven that when a metric is *factorizable* and the gauge fields satisfies the *de Donder* gauge, then (4.36) reduces to:

$$\frac{\partial\sqrt{\tilde{g}^E_{\pi\sigma}\dot{x}^\pi\dot{x}^\sigma}}{\partial x^\rho} - \frac{\partial}{\partial\lambda}\frac{\partial\sqrt{\tilde{g}^E_{\pi\sigma}\dot{x}^\pi\dot{x}^\sigma}}{\partial\dot{x}^\rho} = 0 \tag{4.46}$$

which is the same minimization problem we impose to find the geodesics of $\tilde{g}^E_{\mu\nu}$. Property (1) is non trivial to be realised and might depend on a clever coordinate choice in the original 6D metric. Surprisingly, the geometries of Sec. (3.4.3) are factorizable in the coordinates used to derive the geometry, as can be seen from Eq. (3.30).

Chapter 5

Deriving EE for $\tilde{g}^E_{\mu\nu}$

We have seen that for the geometry of Sec. (3.4.3), the problem of computing the EE can be reduced to the RT recipe applied to an asymptotically AdS$_3$ metric. So let us review the key passages needed to compute the Entanglement Entropy in this case.

5.1 Stationary case

Since we have proven that, for *factorizable* metrics satisfying the de Donder gauge condition, the higher dimensional problem is equivalent to the 3D one, we can start directly with the stationary three dimensional metric($\tilde{g}^E_{\mu\nu} = \tilde{g}^E_{\mu\nu}(r)$):

$$ds^2 = (g_{rr}dr^2 + g_{tt}dt^2 + g_{yy}dy^2 + 2g_{ty}dtdy)$$

where we have assumed $\tilde{g}^E_{ry} = 0 = \tilde{g}^E_{rt} = 0$, which is always true for our geometries.
It is convenient to parametrize the geodesic $x^\mu(\lambda)$ in terms of *proper time*, i.e.:

$$\tilde{g}^E_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1 \tag{5.1}$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$. We know that geodesics are defined by the minimization of the functional:

$$L = \int d\lambda \left(\sqrt{g_{rr}\dot{r}^2 + g_{tt}\dot{t}^2 + g_{yy}\dot{y}^2 + 2g_{ty}\dot{t}\dot{y}}\right) \equiv \int \mathcal{L}d\lambda \Leftrightarrow \frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{\partial}{\partial\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = 0$$

Since we are assuming that the metric does not depend explicitly on t and y, Euler-Lagrange equations with the respect to these coordinates greatly simplify:

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = C_1 \tag{5.2}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{y}} = C_2 \tag{5.3}$$

where C_1 and C_2 are two constants of motion to be fixed later.
Solving (5.2),(5.3) taking into account the condition (5.1), one finds:

$$\dot{t} = \frac{C_1 g_{yy} - C_2 g_{ty}}{g_{tt}g_{yy} - g_{ty}^2} \tag{5.4}$$

$$\dot{y} = \frac{C_2 g_{tt} - C_1 g_{ty}}{g_{tt}g_{yy} - g_{ty}^2} \tag{5.5}$$

$$\dot{r}^2 = \frac{1}{g_{rr}}\left[1 - \left(\frac{g_{yy}C_1^2 + g_{tt}C_2^2 - 2C_1C_2g_{ty}}{g_{tt}g_{yy} - g_{ty}^2}\right)\right] \tag{5.6}$$

The two constants C_1 and C_2 are determined by the choice of boundary conditions. We are interested in a spatial region A at fixed time ($t = \bar{t}$), made of an interval of length l. The endpoints of the geodesic have to lie at the boundary of AdS, but for $r \to \infty$ the area diverges, as expected for the EE in the dual CFT. So, as usual, we introduce an IR cut-off r_0, which we will consider as the AdS boundary for computations, and at the end we will take the results for large r_0. Thus boundary conditions are:

$$0 = \int_{\bar{t}}^{\bar{t}} dt = \int_{\lambda_1}^{\lambda_2} \dot{t} d\lambda = 2 \int_{r*}^{r_0 \gg 1} dr \frac{\dot{t}}{\dot{r}} \tag{5.7}$$

$$l = \int_0^l dy = \int_{\lambda_1}^{\lambda_2} \dot{y} d\lambda = 2 \int_{r*}^{r_0 \gg 1} dr \frac{\dot{y}}{\dot{r}} \tag{5.8}$$

$$L_\gamma = \int_{\lambda_1}^{\lambda_2} d\lambda = 2 \int_{r*}^{r_0 \gg 1} dr \frac{1}{\dot{r}} \tag{5.9}$$

where r^* is the geodesic turning point, such that $\dot{r}\big|_{r=r^*} = 0$.
In the end we compute EE via Eq. (4.33).

5.2 Static case

When the metric is *static* (i.e. there are not terms g_{ty}), we can take a submanifold at constant t and the only relevant components of the metric remain $g_{yy}(r)$ and $g_{rr}(r)$. Eq.(5.4)-(5.6) simply reduce to:

$$\frac{d}{d\lambda}\left(g_{yy}\dot{y}\right) = 0 \quad \Rightarrow \quad \dot{y} = \frac{C}{g_{yy}} \tag{5.10}$$

$$g_{rr}\dot{r}^2 + g_{yy}\dot{y}^2 = 1 \quad \Rightarrow \quad \dot{r} = \sqrt{\frac{g_{yy-C^2}}{g_{rr}g_{yy}}} \tag{5.11}$$

The turning point is now defined through

$$g_{yy}(r^*) - C^2 = 0 \tag{5.12}$$

Boundary conditions are the same as the stationary case, apart the fact that condition (5.7) is automatically satisfied and (5.8) and (5.9) simplify in:

$$l = 2C \int_{r^*}^{r_0} dr \sqrt{\frac{g_{rr}}{g_{yy}\left(g_{yy} - C^2\right)}} \tag{5.13}$$

$$L_\gamma = 2 \int_{r^*}^{r_0} dr \sqrt{\frac{g_{rr}g_{yy}}{g_{yy} - C^2}} \tag{5.14}$$

5.3 Some examples

Now we have all the necessary tools to compute the EE for the superstrata solutions described in Ch. 3. We will first present the case $n = 0, k = 1$ and then pass to the *generic* n metric. In the latter case computations can not be handled analytically, therefore we first consider the $a \to 0$ limit, where the metric reduces to BTZ BH and then we will perform an expansion around this solution. In particular, in order to present explicit results, we will concentrate on the case $n = 1$.

For the sake of clarity, all computations are collected in Appendix D, while only the main relevant results are presented below.

5.3.1 Case $n = 0, k = 1$

When $n = 0$ the reduced metric $\tilde{g}^E_{\mu\nu}$ describing the D1-D5 geometry (3.28) is:

$$ds^2 = \sqrt{Q_1 Q_5}\frac{r^2+\tilde{a}^2}{(r^2+a^2)^2}dr^2 - \frac{r^2+\tilde{a}^2}{\sqrt{Q_1Q_5}}dt^2 + \frac{r^2}{\sqrt{Q_1Q_5}}dy^2$$

where, for short, we have defined $\tilde{a}^2 \equiv \frac{a^4}{a^2+\frac{b^2}{2}}$. Remember that the radius R of the compact direction y is related to a, b and to the $R_{\text{AdS}} \equiv (Q_1Q_5)^{1/4}$ by the relation

$$R = \sqrt{\frac{Q_1Q_5}{a^2+\frac{b^2}{2}}} \equiv \frac{\sqrt{Q_1Q_5}}{a_0} \tag{5.15}$$

The metric is static, so we can use the prescriptions in Sec.5.2. Let us start from determining the turning point r^*, this one is given by the solution of

$$\frac{r^2}{\sqrt{Q_1Q_5}} - C^2 = 0 \quad \Rightarrow \quad r^* = \left|C\,(Q_1Q_5)^{1/4}\right| \equiv |\omega| \tag{5.16}$$

ω is determined by the constraint:

$$\frac{l}{R} = \arccos\left(\frac{\omega^2-\tilde{a}^2}{\omega^2+\tilde{a}^2}\right) - \frac{\omega}{\tilde{a}}\sqrt{\frac{a^2-\tilde{a}^2}{a^2+\omega^2}}\log\left(\frac{2a^2+\omega^2-\tilde{a}^2-2\sqrt{(a^2+\omega^2)(a^2-\tilde{a}^2)}}{\tilde{a}^2+\omega^2}\right) \tag{5.17}$$

For convenience we will indicate with α the ratio $\frac{l}{R}$, $\alpha \in [\,0, 2\pi\,]$.
The Entanglement Entropy of a region A, made of an interval of length l is

$$S_A = n_1 n_5\left\{\log\left(\frac{4r_0^2}{a^2+\omega^2}\right) + \sqrt{\frac{a^2-\tilde{a}^2}{a^2+\omega^2}}\log\left(\frac{2a^2+\omega^2-\tilde{a}^2-2\sqrt{(a^2+\omega^2)(a^2-\tilde{a}^2)}}{\tilde{a}^2+\omega^2}\right)\right\} \tag{5.18}$$

Unfortunately we are not able to invert analytically Eq.(5.17) and express S_A as a function of l. However, in order to have an intuitive idea, we can invert it in some interesting regimes. We decide to keep Q_1, Q_5 and R fixed, so that $\tilde{a}^2 = \frac{a^4}{a_0^2}$ and to look at two opposite limits: when $\frac{a^2}{a_0^2} \ll 1$ and when $\frac{a^2}{a_0^2} \sim 1$ (which is equivalent to consider an expansion in small b).

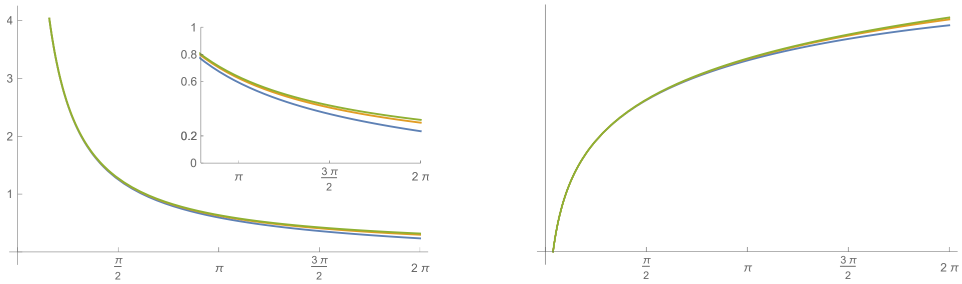
- $\frac{a^2}{a_0^2} \ll 1$

$$\omega = a_0\left(\frac{2}{\alpha} - \frac{a^2}{a_0^2}\frac{\alpha}{3}\right) + \mathcal{O}(a^4)$$

and consequently at this order of approximation:

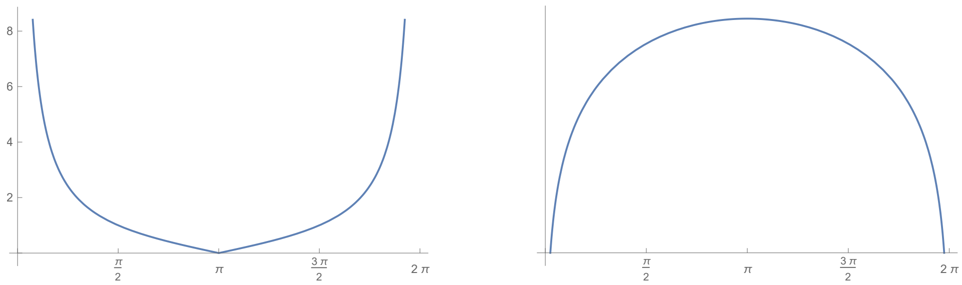
$$S_A \simeq n_1 n_5\left\{\log\left(\frac{\alpha^2 r_0^2}{a_0^2}\right) - \frac{a^2}{a_0^2}\frac{\alpha^2}{6}\right\}$$

In Fig.(5.1) it is represented the behaviour of the turning point r^* and of the EE as functions of the opening angle α. We can notice that S_A is not symmetric. We might expect this result, in fact in the limit $a \to 0$, the geometry reduces to massless BTZ black hole, which is dual to the mixed state representing the statistical ensemble of all the 2-charge states. Since the geometry in this limit represent a BH, which is not a pure state, the EE should be increasing, as we obtain in Fig.5.1(b). The EE of an interval l is equal to that of its complement $\pi - l$ if and only if the state is pure.
Moreover notice. in Fig.5.1(a), that r^* never reaches zero. This characteristic reveals the emergence of a phenomenon known as *Entanglement shadow*. An *Entanglement shadow* appears when the minimal surfaces anchored on the boundary, in our case our geodesics, will not penetrate a region and thus the EE can not probe this portion of space-time.



(a) $\frac{r^*}{a_0}$ as a function of α (b) $\frac{S_A}{n_1 n_5}$ as a function of α

Figure 5.1: r^* and EE in small a approximation. We plot the results for different values of $\epsilon^2 = \left(\frac{a}{a_0}\right)^2$: blue for $\epsilon = 0.2$, orange $\epsilon = 0.1$ and green $\epsilon = 0.01$. In the zoomed graph the effect of entanglement shadow is highlighted.



(a) $\frac{r^*}{a_0}$ as a function of α (b) $\frac{S_A}{n_1 n_5}$ as a function of α

Figure 5.2: r^* and EE in small b approximation. Since at our order of approximation there is no difference between our geometry and AdS, the represented r^* and S_A would be the same for the pure state.

- $\frac{a^2}{a_0^2} \sim 1 \leftrightarrow b \ll 1$, keeping only the order b^2 the problem is exactly the same as *pure AdS* [1]

$$\omega = a_0 \cot\frac{\alpha}{2} + \mathcal{O}(b^4)$$

$$S_A = 2 n_1 n_5 \log\left(\frac{2r_0}{a_0}\sin\left(\frac{\alpha}{2}\right)\right) + \mathcal{O}(b^4)$$

As shown in Fig.(5.2), the EE of a pure state is symmetric, as we expected, and there is no place for entanglement shadows.

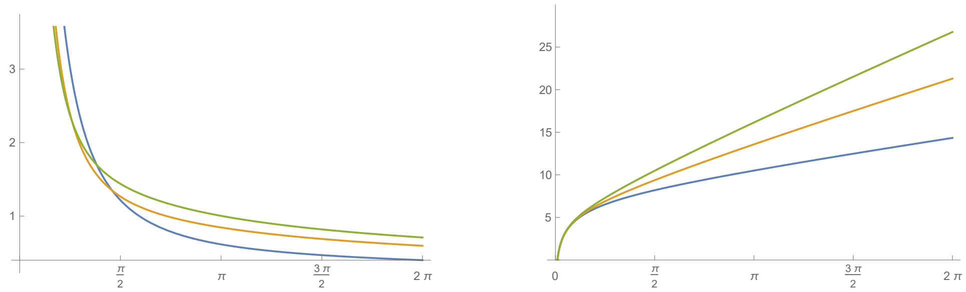
5.3.2 Generic n metric: $a = 0$ limit

As announced before, we now want to compute the EE for the geometries presented in Ch.3. Since for generic n, we are not able to perform the calculation analytically, we start looking at the case $a = 0$. In this limit the metric is that one of an extremal, rotating BTZ

[1] In this approximation ($o(b^2)$) the metric is:

$$ds^2 \simeq \sqrt{Q_1 Q_5}\frac{1}{r^2 + a_0^2}dr^2 - \frac{r^2 + a_0^2 - b^2}{\sqrt{Q_1 Q_5}}dt^2 + \frac{r^2}{\sqrt{Q_1 Q_5}}dy^2$$

There is no difference between the computation in this metric and that one for *pure AdS* since at order b^2 the only term which receive corrections is g_{tt}, which never enters the calculations.



(a) $\frac{r^*}{b}$ as a function of α (b) $\frac{S_A}{n_1 n_5}$ as a function of α

Figure 5.3: r^* and EE for a metric with a generci n in the limit $a = 0$.As explicit examples we plot the results for $n = 1$ (blue), $n = 5$ (orange) and $n = 10$ (green).

black hole. The Bañados-Teitelboim-Zanelli BH is a black hole solution of 2+1 dimensional gravity, characterised by a negative cosmological constant, i.e. it is asymptotically Anti-de Sitter. Written in our coordinates

$$ds^2 = \frac{\sqrt{Q_1 Q_5}}{r^2} dr^2 - \frac{1}{\sqrt{Q_1 Q_5}} \left(r^2 - \frac{b^2 n}{2} \right) dt^2 + \frac{1}{\sqrt{Q_1 Q_5}} \left(r^2 + \frac{b^2 n}{2} \right) dy^2 + \frac{b^2 n}{\sqrt{Q_1 Q_5}} dt dy \tag{5.19}$$

and it has a horizon at $r = 0$.
Following the procedure explained in Sec.5.1, it is possible to determine the turning point, which is the indicator of the presence of *Entanglement shadow* and to compute the EE for this geometry. We skip the calculations, reported in Appendix D.2, to present the two important results:

$$r^* = (2nQ_1Q_5)^{1/4} \sqrt{\frac{b}{l}} \sqrt{\coth\left(\frac{l}{2} \frac{b\sqrt{n}}{\sqrt{2}\sqrt{Q_1 Q_5}} \right)} \tag{5.20}$$

$$= n^{1/4} \sqrt{\frac{2Q_1Q_5}{lR}} \coth\left(\frac{l\sqrt{n}}{2R} \right) \tag{5.21}$$

where the S^1 radius $R^2 = \frac{2Q_1Q_5}{b^2}$.
In Fig.(5.3 a) we plot $\frac{r^*}{b} = \frac{n^{1/4}}{\sqrt{\alpha}} \coth\left(\frac{\alpha\sqrt{n}}{2} \right)$ as a function of $\alpha = \frac{l}{R}$. Since $\coth(x)$ is always equal or greater than one, the turning point

$$r^* \geq \frac{b n^{1/4}}{\sqrt{\alpha}} \tag{5.22}$$

This is a sign of Entanglement shadows, because it indicates that the geodesic is not able to probe spacetime for $r < r^*$.
To conclude the Entanglement Entropy is

$$S_A = n_1 n_5 \log \left(\frac{\sqrt{2} l r_0^2 \sinh\left(\frac{bl\sqrt{n}}{\sqrt{2}\sqrt{Q_1Q_5}} \right)}{b\sqrt{n}\sqrt{Q_1Q_5}} \right) \tag{5.23}$$

In order to match this result with similar computations known in the literature, that in [37] for example [2], we need to introduce the dimensionless UV cut off in the dual conformal field

[2] In order to obtain the metric (2.3) of [37] we need to perform the following coordinate change: $\frac{y}{R_{AdS}} \to \phi; r \to \tilde{r} \equiv \sqrt{r^2 + \bar{r}^2} \Rightarrow ds^2 = \frac{R_{AdS}^2 \tilde{r}^2}{(\tilde{r}^2 - \bar{r}^2)^2} d\tilde{r}^2 - \frac{(\tilde{r}^2 - \bar{r}^2)^2}{R_{AdS}^2 \tilde{r}^2} dt^2 + \tilde{r}^2 \left(d\phi + \frac{\bar{r}^2}{R_{AdS}^2 \tilde{r}^2} dt \right)^2$

theory as $\epsilon = \frac{(Q_1Q_5)^{1/4}}{r_0} \ll 1$ and rearrange the terms in:

$$
\begin{aligned}
S_A = n_1 n_5 \Bigg\{ &\log \left(\underbrace{\frac{l}{(Q_1Q_5)^{1/4}}}_{L} \underbrace{\frac{r_0}{(Q_1Q_5)^{1/4}}}_{1/\epsilon} \right) + \\
&+ \log \left(\frac{(Q_1Q_5)^{1/4}}{b\sqrt{\frac{n}{2}}} \frac{r_0}{(Q_1Q_5)^{1/4}} \sinh \left(\frac{b\sqrt{n}}{\sqrt{2}(Q_1Q_5)^{1/4}} \frac{l}{(Q_1Q_5)^{1/4}} \right) \right) \Bigg\} \\
&= n_1 n_5 \left\{ \log \left(\frac{L}{\epsilon} \right) + \log \left(\frac{R_{\text{AdS}}}{\bar{r}\epsilon} \sinh \left(\frac{\bar{r} L}{R_{\text{AdS}}} \right) \right) \right\}
\end{aligned}
$$

where we have defined[3] $\bar{r} \equiv b\sqrt{\frac{n}{2}}$ and $R_{\text{AdS}} = (Q_1Q_5)^{1/4}$. In this form the expression for S_A agrees with the result found in [37].

5.3.3 EE for $n = 1$

We want to study EE for the generic microstates ($a \neq 0$) in Ch. 3, whose reduced metric in Einstein frame is:

$$
\begin{aligned}
ds_3^2 = &\frac{\sqrt{Q_1Q_5}}{(r^2+a^2)^2} \left(r^2 + \frac{a^4}{a_0^2} \left(1 + \frac{b^2}{2a^2} F_n \right) \right) dr^2 - \frac{1}{\sqrt{Q_1Q_5}} \left(r^2 \left(1 - \frac{b^2}{2a^2} F_n \right) + \frac{a^4}{a_0^2} \right) dt^2 \\
&+ \frac{r^2}{\sqrt{Q_1Q_5}} \left(1 + \frac{b^2}{2a^2} F_n \right) dy^2 + \frac{b^2}{a^2} \frac{r^2 F_n}{\sqrt{Q_1Q_5}} dt dy
\end{aligned}
$$

where $F_n \equiv 1 - \left(\frac{r^2}{r^2+a^2} \right)^n$. As announced before, we will concentrate only in the case $n = 1$ (key results are still valid also for a generic n, but computations are longer and not very enlightening).

With this simplification, the expressions of $\dot{r}, \dot{y}$ and $\dot{t}$ are easier (see (D.4)-(D.6)), but it is not enough to make analytic computations. For this reason, we decide to solve the problem perturbatively in a, keeping Q_1, Q_5 and R fixed, so that

$$
\frac{b^2}{2} = a_0^2 - a^2, \qquad a_0^2 = \frac{Q_1Q_5}{R_y^2}
$$

Rememeber that, as in the case $n = 0$, the ratio $\frac{a}{a_0}$ tell us if our microstate tend to the BTZ black hole, or, in the opposite limit, it is a small deformation of $\text{AdS}_3 \times \text{S}^3$:

$$
0 < \frac{a}{a_0} < 1 \qquad\qquad \begin{cases} \frac{a}{a_0} \to 1 & \Leftrightarrow \quad \text{AdS}_3 \times \text{S}^3 \\ \frac{a}{a_0} \to 0 & \Leftrightarrow \quad \text{BH} \end{cases}
$$

In particular we are interested in the small a expansion. We expand the turning point and the constants of motion k_1 and k_1 (obtained as the sum and the difference of the previous C_1, C_2, see Eq. (D.7)) around the solution obtained for $a = 0$ (labelled with a subscript 0):

$$
\begin{aligned}
r^* &\simeq r_0^* + a^2 \delta r, \qquad & r_0^* &= (Q_1Q_5)^{1/4} \sqrt{\left(\frac{2a_0}{l} \right) \coth \left(\frac{a_0 l}{2\sqrt{Q_1Q_5}} \right)} \qquad (5.24) \\
\text{k}_1 &\simeq \text{k}_1^0 + a^2 \delta_1, \qquad & \text{k}_1^0 &= \frac{2a_0 \coth \left(\frac{a_0 l}{\sqrt{Q_1Q_5}} \right)}{\sqrt[4]{Q_1Q_5}} \\
\text{k}_2 &\simeq \text{k}_2^0 + a^2 \delta_2, \qquad & \text{k}_2^0 &= \frac{2\sqrt[4]{Q_1Q_5}}{l}
\end{aligned}
$$

Explicit integrations, that can be found in D.3, lead to the following results:

[3]In [37] $\bar{r} = r_0$ which is different from our r_0 indicating the cut off

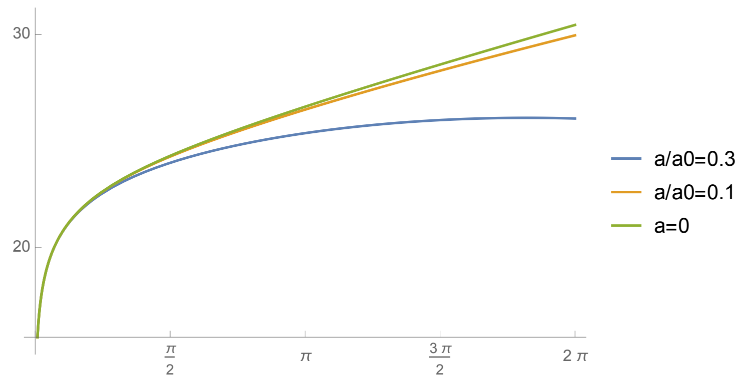


Figure 5.4: $\frac{S_A}{n_1 n_5}$ as a function of α

- the two constants we have to fix, change at order a^2 as shown in eq. (D.18) and (D.19);
- δr^* does not receive corrections at order a^2 and that is true for every n. So the expression for the turning point is the same as (5.20): $r^* = \sqrt[4]{nQ_1Q_5}\sqrt{\frac{2a_0}{l}\coth\left(\frac{a_0 l\sqrt{n}}{2\sqrt{Q_1Q_5}}\right)}$. In particular the microstates have Entanglement shadows.
- Entanglement entropy changes at order a^2 (if we introduce the parameter $\alpha \equiv \frac{l}{R_y}, \alpha \in (0, 2\pi)$):

$$\begin{aligned} S_A \simeq\, & n_1 n_5 \log\left(\frac{r_0^2}{a_0^2}\cdot \alpha \sinh\alpha\right) \\ & - \frac{a^2}{a_0^2}(n_1 n_5)\left\{\frac{\alpha^2}{2}\left(\frac{1}{2}\cosh(\alpha)\text{sech}^2\left(\frac{\alpha}{2}\right)+1\right)+\frac{3}{4}\alpha\sinh(\alpha)\text{sech}^2\left(\frac{\alpha}{2}\right)\right\} \end{aligned} \tag{5.25}$$

In Fig.(5.4), it depicted how the EE changes with the variation of the ratio $\frac{a}{a_0}$, which controls the perturbative expansion. In addition it shows that in microstates there are small deviations from the black hole EE.
We have seen that for a microstate, treated as a small deformation of the case $a = 0$, its EE gets corrections with the respect to the BH. An interpretation of this result and a more deepened study of the Entanglement shadows in the context of the dual CFT, could be interesting and could help to have a clearer understanding of these phenomena. We leave such analysis for future work.

Conclusion

In this work, we have devoted our attention to the study of the Entanglement Entropy of a subclass of $\frac{1}{8}$-BPS microstate solutions. Motivated by the surprising discovery that already in their original coordinates, these configurations present a reduced 3D Einstein metric $\tilde{g}_{\mu\nu}$ independent from x^α coordinates, we study under which assumptions the 6 dimensional minimization problem for EE reduces to a lower dimensional one. We have proved that for *factorizable* metrics satisfying a *de Donder* gauge condition, computing the EE for a single interval in the full 6 dimensional metric is equivalent to determine the length of $\tilde{g}^E$ geodesics.
We then have applied this result to compute explicitly the EE for the $\frac{1}{8}$-BPS microstates with $k=1$ and $n=0$ and the generic case with $n \geq 1$.

We start this work discussing the thermodynamic behaviour of black holes, highlighting how this characteristic together with the *no hair* theorem, causes the appearance of some standing puzzles such as the information loss paradox and the entropy problem. These issues make evident the need of a new perspective of BH Physics beyond classical GR, which must be able to give a description of a black hole in terms of microstates.

We investigate the *Fuzzball proposal* as be a possible solution to these problems. What is promising about the Fuzzball program is that, besides trying to solve the information paradox, it provides us with an effective procedure to construct microstate geometries. These last ones are found to be regular and horizonless solutions of Supergravity. For this reason, a part of this work has been devoted to an introduction to Supergravity, with a special attention to the ten dimensional case. Within this context we have investigated the construction of extremal black holes as bound states of branes. In particular, since the final goal of the Thesis is to study the EE for the $\frac{1}{8}$-BPS states, we have focused our attention on the three-charge system. In five dimensions it reduces to the Stroeminger-Vafa black hole and in 10 D it is constructed as a D1-D5-P system.

The prescription of Ryu and Takayanagi to compute EE is a interesting example of how the Holographic principle applies. To contextualize their proposal we have reviewed the general ideas of the AdS/CFT conjecture. The correspondence reveals to be very useful also in the study of microscopic description of BHs . It provides a deeper insight on the nature of the microstates and establishes a dictionary between geometries ad CFT states in further support of the Fuzzball proposal. With this aim, we have introduced the Conformal Field Theory relevant for the D1-D5 states, the *D1D5 CFT* at the orbifold point. In this particular regime, the theory is distant from being a good description of a black hole. However, some quantities, such as the index related to states degeneracy, are protected by Supersymmetry and do not change if we computed them in the region where the CFT is a free field theory.

In the second part of this work, we have concentrated on Superstrata solutions. These horizonless Supergravity solutions, having the same charges as general supersymmetric D1-D5-P black holes, can be obtained adding momentum to the 2-charge geometries. In the dual CFT description, in the NS sector, the role of adding a momentum charge, is played by the generator of the $SL(2,\mathbb{R})_L \times SU(2)_L$ symmetry group of $\mathrm{AdS}_3 \times S^3$. To obtain the desired states, we can act on the NS (anti-)chiral primary state, we have denoted $|00\rangle_k^{NS}$, m times with J_0^+ and n times with the Virasoro generator L_{-1}.

The resulting state is $\left(J_0^+\right)^m (L_{-1})^n |00\rangle_k^{NS}$.

To derive the corresponding metric, we have restricted to the case $m = 0$ and $k = 1$.
To further investigate these solutions, we devoted the last part of this Thesis to the computation of their Entanglement Entropy. We introduce in a very general way the concept of Entanglement Entropy, assuming the perspectives firstly of Quantum Mechanics and then of QFT (in particular the conformal case). Finally we report the Ryu-Takayanagi prescription for the holographic computation of EE, which applies to AdS space-time (or at most AdS times a compact space).

The 6D metric of our microstates configuration are only asymptotically $\mathrm{AdS}_3 \times S^3$, thus it is impossible to apply the RT formula directly as it is. As suggested in [8, 7], we have to generalize the prescription and adapt it to our case.
In this work we have shown that, at least for the special class of *factorizable* geometries, for what concerns EE computation, it is equivalent to solve the full 6D extremization problem or, instead, finding minimal curve in the 3D asymptotically AdS metric. What is surprising is that the metrics we deal with are factorizable in the same coordinates used in the derivation. We do not rule out the possibility, even if hard to prove, that for every metrics there exists a suitable change of coordinates making $\tilde{g}^E_{\mu\nu}$ x^α-independent.
Once we have proved the equivalence between the 6D problem and the lower dimensional one, we have restricted our attention to the three dimensional part of our solution. In this 3D asymptotically AdS space-time, the RT prescription applies and we can compute geodesics of $\tilde{g}^E$ to find the EE. In the concluding chapter, we have computed explicitly the EE for a single interval of length l in the Superstrata metrics introduced before. In studying their EE we have notice the appearance of a phenomenon known as *Entanglement shadow*, which consists in a region of space-time not probed by the minimal surfaces. It has been conjectured (suggestively) that the whole space-time could be reconstructed from the knowledge of the EE of the boundary CFT. Consequently, the presence of these shadows, might invalidate this conjecture and it might indicate that more refined CFT quantities are needed to reconstruct the space-time. For example, in [38] a new quantity, called *entwinement* is proposed as a good quantity to compute entanglement in the presence of shadows. In [38], it is investigate the static BTZ black hole, where entanglement shadows appear. These shadows appear because in computing the canonical EE one consider only the minimal geodesics and they only penetrate to a certain maximum depth in the spacetime. It is proved, however, that the longer geodesics, discarded in EE, do penetrate the shadows and it is guessed that they must be related in some way to physical quantities in the CFT. In particular, it can be shown that they contribute to *entwinement*, which would be eventually the right picture capturing the entanglement of this system.
The result of our computations, indeed, have to be reinterpreted in the light of the dual CFT, if we want to understand better the meaning and the possible consequences of a presence of entanglement shadows in microstate geometries.

Moreover, when we have considered the generic n metric, in order to carry out the computations, we have performed an expansion in small a. In this approximation, microstates represents a small deformation of the classical BH and the results we have obtained are indeed in agreement with this interpretation. A useful analysis to carry in future, would be to repeat numerically (keeping finite values for a and b) the same computations we have done analytically. In this way, we could improve our comprehension of EE for the microstate geometries and probably clarify the effects of Entanglement shadows.

To conclude let us summarise the main and useful implications of our Entanglement Entropy computation. It reveals the presence of entanglement shadows in microstate geometries, which have to be further analysed. In addition, we have shown that the BH behaviour is corrected in a microstate when we compute the EE. It is only the first step for a deeper CFT interpretation, that we will leave for a future work.

Appendix A

Mathur Theorem

To prove (1), we need to start constructing the density matrix for the system (c_{N+1}, b_{N+1}):

$$\rho_{(c_{N+1},b_{N+1})} = \begin{pmatrix} \left\langle \Lambda^{(1)} \middle| \Lambda^{(1)} \right\rangle & \left\langle \Lambda^{(1)} \middle| \Lambda^{(2)} \right\rangle \\ \left\langle \Lambda^{(2)} \middle| \Lambda^{(1)} \right\rangle & \left\langle \Lambda^{(2)} \middle| \Lambda^{(2)} \right\rangle \end{pmatrix} = \begin{pmatrix} 1-\varepsilon_1^2 & \varepsilon_2 \\ \varepsilon_2 & \varepsilon_1^2 \end{pmatrix} \tag{A.1}$$

where we have defined $\left\langle \Lambda^{(1)} \middle| \Lambda^{(2)} \right\rangle \equiv \varepsilon_2$, $||\Lambda^{(2)}||^2 \equiv \varepsilon_1^2$ and used the relation (4.12). Notice that both ε_i are small, since by hypothesis (4.13) $\varepsilon_1^2 < \epsilon^2$ and by mean of Cauchy-Schwartz inequality[1]

$$\left\langle \Lambda^{(1)} \middle| \Lambda^{(2)} \right\rangle \leq \underbrace{||\Lambda^{(1)}||}_{<1}\,\underbrace{||\Lambda^{(2)}||}_{<\varepsilon} < \varepsilon$$

Since we want to compute the Entanglement Entropy we diagonalize (A.1), its eigenvalues being:

$$\lambda_1 = \frac{1}{2}\left(1+\sqrt{1+4(\varepsilon_1^4-\varepsilon_1^2+\varepsilon_2^2)}\right) = 1-(\varepsilon_1^2-\varepsilon_2^2)+o(\varepsilon^3) \tag{A.2}$$

$$\lambda_2 = \frac{1}{2}\left(1-\sqrt{1+4(\varepsilon_1^4-\varepsilon_1^2+\varepsilon_2^2)}\right) = (\varepsilon_1^2-\varepsilon_2^2)+o(\varepsilon^3) \tag{A.3}$$

and consequently:

$$S_{EE} \simeq -\left[1-(\varepsilon_1^2-\varepsilon_2^2)\right] \underbrace{\log\left[1-(\varepsilon_1^2-\varepsilon_2^2)\right]}_{\simeq -(\varepsilon_1^2-\varepsilon_2^2)} -(\varepsilon_1^2-\varepsilon_2^2)\log\left(\varepsilon_1^2-\varepsilon_2^2\right) \tag{A.4}$$

$$= (\varepsilon_1^2-\varepsilon_2^2)\log\frac{e}{(\varepsilon_1^2-\varepsilon_2^2)} + o(\varepsilon^3) < \varepsilon \tag{A.5}$$

To motivate (2), we need to recall the subadditivity relation: given 2 systems their entropy satisfy

$$S(A+B) \geq |S(A)-S(B)| \tag{A.6}$$

and the inequality in (2) is trivially proven just identifying $A = \{b\}$ $(S(\{b\}) = S_N)$ and $B = (c_{N+1}, b_{N+1})$, so:

$$S(\{b\}, c_{N+1}, b_{N+1})) \geq S(\{b\}) - S(c_{N+1}, b_{N+1})) \geq S_N - \varepsilon \tag{A.7}$$

where in the last passage we have used the result in (A.5).

[1] We do not care about the sign of $\left\langle \Lambda^{(1)} \middle| \Lambda^{(2)} \right\rangle$ since this term will appear always squared, so for simplicity we consider it positive from the beginning

To conclude we consider problem (3), so let us construct the density matrix for c_{N+1} from (4.11):

$$\rho_{c_{N+1}} = \frac{1}{2}\begin{pmatrix} \left\langle \Lambda^{(1)} + \Lambda^{(2)} \middle| \Lambda^{(1)} + \Lambda^{(2)} \right\rangle & 0 \\ 0 & \left\langle \Lambda^{(1)} - \Lambda^{(2)} \middle| \Lambda^{(1)} - \Lambda^{(2)} \right\rangle \end{pmatrix} \tag{A.8}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + 2\varepsilon_2 & 0 \\ 0 & 1 - 2\varepsilon_2 \end{pmatrix} \tag{A.9}$$

the corresponding entropy

$$S(c_{N+1}) = \log 2 - 4\varepsilon_2^2 + o(\varepsilon^3) > \log 2 - \varepsilon \tag{A.10}$$

Appendix B

Further remarks on solution generating technique

B.1 1-charge solution

First of all we want to understand how a boost can add a momentum charge. This becomes clear when we reduce the metric following the Kaluza-Klein procedure. For simplicity let us look only to the 6D part of (1.35), keeping in mind that we want to reduce on S^1:

$$ds^2 = S_\alpha \left(dy + S_\alpha^{-1} \frac{2M}{r^2} \mathrm{ch}\alpha \, \mathrm{sh}\alpha \, dt \right)^2 - S_\alpha^{-1} \left(1 - \frac{2M}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2 \tag{B.1}$$

Written in this way it is very simple to identify the KK gauge field:

$$A_t = S_\alpha^{-1} \frac{2M}{r^2} \mathrm{ch}\alpha \, \mathrm{sh}\alpha$$

and derive the charge[1]

$$Q = R \frac{M \Omega_3}{16 \pi G_5} \mathrm{sh}(2\alpha)$$

The lower dimensional charge is the y component of the six dimensional momentum.

B.2 2-charge solution

The starting point is the F1-P$_y$ system described in Type IIB:

$$\begin{cases} ds^2 = S_\alpha^{-1} S_\beta \left(dy + S_\beta^{-1} \frac{2M}{r^2} \mathrm{ch}\beta \, \mathrm{sh}\beta \, dt \right)^2 - S_\alpha^{-1} S_\beta^{-1} \left(1 - \frac{2M}{r^2} \right) dt^2 \\ \qquad + \left(1 - \frac{2M}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3^2 + (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} \\ B_2 = S_\alpha^{-1} \frac{2M}{r^2} \mathrm{ch}\alpha \, \mathrm{sh}\alpha \, dt \wedge dy \end{cases} \tag{B.2}$$

[1] As it follows from ADM formalism

Thank to S-duality (1.24) we can transform F1-P$_y$ $\to$ D1$_y$-P$_y$ (Type IIB):

$$\begin{cases} ds'^2 = S_\alpha^{1/2} ds^2 = S_\alpha^{-1/2} S_\beta \left(dy + S_\beta^{-1} \frac{2M}{r^2} \mathrm{ch}\beta\, \mathrm{sh}\beta\, dt\right)^2 - S_\alpha^{-1/2} S_\beta^{-1} \left(1 - \frac{2M}{r^2}\right) dt^2 \\ \qquad\qquad + S_\alpha^{1/2} \left\{ \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + (dz^a)^2 \right\} \\ e^{2\Phi} = S_\alpha \\ C_2 = -S_\alpha^{-1} \frac{2M}{r^2} \mathrm{sh}\alpha\, \mathrm{ch}\alpha\, dt \wedge dy \end{cases} \tag{B.3}$$

Now T-duality along the four torus directions take the D1 brane to a D5 one carrying momentum. The resulting solution is again of Type IIB Supergravity . We perform explicitly the first T duality, for example along z^1 (in the convention of (1.25) our C_2 is equal to $\hat{C}_2$ and $A = 0$):

$$\begin{cases} ds'^2 = S_\alpha^{1/2} ds^2 = S_\alpha^{-1/2} S_\beta \left(dy + S_\beta^{-1} \frac{2M}{r^2} \mathrm{ch}\beta\, \mathrm{sh}\beta\, dt\right)^2 - S_\alpha^{-1/2} S_\beta^{-1} \left(1 - \frac{2M}{r^2}\right) dt^2 \\ \qquad\qquad + S_\alpha^{1/2} \left\{ \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + \sum_{a=2}^4 (dz^a)^2 \right\} + S_\alpha^{-1/2} \left(dz^1\right)^2 \\ e^{2\Phi} = S_\alpha^{1/2} \\ C_3 = -S_\alpha^{-1} \frac{2M}{r^2} \mathrm{sh}\alpha\, \mathrm{ch}\alpha\, dz^1 \wedge dt \wedge dy \end{cases} \tag{B.4}$$

Doing the same on the other z^a directions, we get:

$$\begin{cases} ds'^2 = S_\alpha^{1/2} ds^2 = S_\alpha^{-1/2} S_\beta \left(dy + S_\beta^{-1} \frac{2M}{r^2} \mathrm{ch}\beta\, \mathrm{sh}\beta\, dt\right)^2 - S_\alpha^{-1/2} S_\beta^{-1} \left(1 - \frac{2M}{r^2}\right) dt^2 \\ \qquad\qquad + S_\alpha^{1/2} \left\{ \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \right\} + S_\alpha^{-1/2} (dz^a)^2 \\ e^{2\Phi} = S_\alpha^{-1} \\ C_6 = -S_\alpha^{-1} \frac{2M}{r^2} \mathrm{sh}\alpha\, \mathrm{ch}\alpha\, dt \wedge dy \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \end{cases} \tag{B.5}$$

It can be useful to substitute to the 6-form its dual C_2, defined via the relation:

$$* (dC_6) = dC_2 \tag{B.6}$$

First of all the only non trivial component of the differential of C_6 is:

$$(dC_6)_{rty1234} = -2M \mathrm{ch}\alpha\, \mathrm{sh}\alpha \frac{\partial}{\partial r} \left(\frac{S_\alpha^{-1}}{r^2} \right) \tag{B.7}$$

Its hodge dual has only component along the coordinates of the 3-sphere and it is reasonable to assume that it would depend only on θ (and eventually on the boost parameters α and β). So we guess that:

$$dC_2 = * (dC_6) = f(\theta, \alpha, \beta) d\theta \wedge d\phi \wedge d\psi \qquad \Rightarrow \qquad C_2 = F(\theta, \alpha, \beta) d\phi \wedge d\psi \tag{B.8}$$

where we have called F the primitive of f with the respect to θ.
Now we can apply S duality, the NS5-P$_y$ Type IIB solution is represented by:

$$\begin{cases} ds^2 = S_\beta \left(dy + S_\beta^{-1} \frac{2M}{r^2} \mathrm{ch}\beta\, \mathrm{sh}\beta\, dt\right)^2 - S_\beta^{-1} \left(1 - \frac{2M}{r^2}\right) dt^2 \\ \qquad + S_\alpha \left[\left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \right] + (dz^a)^2 \\ e^{2\Phi} = S_\alpha \\ B_2 = F(\theta, \alpha, \beta) d\phi \wedge d\psi \\ C_2 = 0 \end{cases} \tag{B.9}$$

Then a T duality along S^1 takes the system to NS5$_{y1234}$-F1$_y$ (Type IIA):

$$\begin{cases} ds^2 = S_\beta^{-1} dy^2 - S_\beta^{-1} \left(1 - \frac{2M}{r^2}\right) dt^2 + S_\alpha \left[\left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \right] + (dz^a)^2 \\ e^{2\Phi} = S_\beta^{-1} S_\alpha \\ B_2 = S_\beta^{-1} \frac{2M}{r^2} \mathrm{ch}\beta\, \mathrm{sh}\beta\, dt \wedge dy + F(\theta, \alpha, \beta) d\phi \wedge d\psi \\ C_2 = 0 \end{cases} \tag{B.10}$$

Our final goal is to find the solution for a D1-D5 system. However the last solution we have obtained is in IIA, so we first apply a T duality along one of the direction of the torus, for example z^1, and only after that we do the last S-duality transformation. Notice that, since the coefficient behind $(dz^1)^2$ is 1 and the fields have no component along this direction, the action of the T duality acts trivially on the ansatz (B.10) and its only effect is to take the solution from Type IIA to Type IIB. S-duality, instead, changes the solution as:

$$\begin{cases} ds^2 = S_\beta^{-1/2} S_\alpha^{-1/2} \left[dy^2 - \left(1 - \frac{2M}{r^2}\right) dt^2\right] + S_\alpha^{1/2} S_\beta^{1/2} \left[\left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right] \\ \quad + S_\beta^{1/2} S_\alpha^{-1/2} (dz^a)^2 \\ e^{2\Phi} = S_\beta S_\alpha^{-1} \\ C_2 = -S_\beta^{-1} \frac{2M}{r^2} \mathrm{ch}\beta\, \mathrm{sh}\beta\, dt \wedge dy - F(\theta, \alpha, \beta) d\phi \wedge d\psi \\ B_2 = 0 \end{cases} \tag{B.11}$$

In the limit (1.45), the metric becomes:

$$ds^2 = \frac{1}{\sqrt{Z_1 Z_5}} \left(-dt^2 + dy^2\right) + \sqrt{Z_1 Z_5} \left(dr^2 + r^2 d\Omega_3^2\right) + \sqrt{\frac{Z_1}{Z_5}} (dz^a)^2 \tag{B.12}$$

In the BPS limit we can derive the exact form of F, which is found to be, up to an unphysical constant shift:

$$F(\theta) = -Q_5 \sin^2\theta \tag{B.13}$$

It is worth looking at the changes in the parameters of the theory under the chain of dualities from F1-P$_y$ $\to$ D1-D5 (for simplicity we redefine $g_s = g$, $R_{z^1} = R_1$ and the torus volume $V_4 = V$):

$$\begin{pmatrix} g \\ Q_1 \\ R \\ R_1 \\ V \end{pmatrix} \xrightarrow{S} \begin{pmatrix} 1/g \\ Q_1/g \\ R/\sqrt{g} \\ R_1/\sqrt{g} \\ V/g^2 \end{pmatrix} \xrightarrow{T_{1234}} \begin{pmatrix} g/V \\ Q_1/g \\ R/\sqrt{g} \\ \sqrt{g}/R_1 \\ g^2/V \end{pmatrix} \xrightarrow{S} \begin{pmatrix} V/g \\ Q_1 V/g^2 \\ R\sqrt{V}/g \\ \sqrt{V}/R_1 \\ V \end{pmatrix} \xrightarrow{T_y} \begin{pmatrix} \sqrt{V}/R \\ Q_1 V/g^2 \\ g/(R\sqrt{V}) \\ \sqrt{V}/R_1 \\ V \end{pmatrix}$$
$$\xrightarrow{T_1} \begin{pmatrix} R_1/R \\ Q_1 V/g^2 \\ g/(R\sqrt{V}) \\ R_1/\sqrt{V} \\ R_1^2 \end{pmatrix} \xrightarrow{S} \begin{pmatrix} R/R_1 \\ Q_1 V R/(g^2 R_1) \\ g/(\sqrt{R R_1 V}) \\ \sqrt{R R_1/V} \\ R^2 \end{pmatrix} \equiv \begin{pmatrix} g' \\ Q_5' \\ R' \\ R_1' \\ V' \end{pmatrix} \tag{B.14}$$

where we have used the conventions of [22] for:

$$\text{S duality :} \qquad g' = \frac{1}{g}, \qquad R' = \frac{R}{\sqrt{g}}$$
$$\text{T duality on y :} \qquad g' = \frac{g}{R_y}, \quad R_y' = \frac{1}{R_y}$$

Appendix C

2 charges solutions

As explained in [22] it is possible to derive the solution representing the D1-D5 bound states starting from the description of a fundamental string F1 carrying the momentum P, applying a chain of T and S duality. The 2 charges solution is then defined from the profile of the string along $\mathbb{R}^4$, given by the functions $g_i(v')\, i = 1, 2...4$ and along one of the direction of the T^4 (the one used in T duality) denoted by $g(v)$.

The Lunin-Mathur solution can be written in the form of (3.1) [1]: it is sufficient to choose, after having defined the length of the F1 as $L = 2\pi \frac{Q_5}{R}$:

$$Z_1 = 1 + \frac{Q_5}{L} \int_0^L dv' \frac{|\dot{g}_i(v')|^2 + |\dot{g}(v')|^2}{|x_i - g_i(v')|^2}, \qquad Z_2 = 1 + \frac{Q_5}{L} \int_0^L dv' \frac{1}{|x_i - g_i(v')|^2} \tag{C.1a}$$

$$Z_4 = -\frac{Q_5}{L} \int_0^L dv' \frac{\dot{g}(v')}{|x_i - g_i(v')|^2} \tag{C.1b}$$

$$a_1 = a_4 = x_3 = 0, \qquad \mathcal{F} = 0 \tag{C.1c}$$

$$\beta = \frac{B - A}{\sqrt{2}}, \quad \omega = -\frac{A + B}{\sqrt{2}} \quad \text{where} \quad A = -\frac{Q_5}{L} \int_0^L dv' \frac{\dot{g}_j dx^j}{|x_i - g_i(v')|^2}, \quad dB = - *_4 dA \tag{C.1d}$$

We review the construction of two examples of 2 charges solution: the first one being the simplest possible case, while the second geometry will be the starting point upon which constructing the 3 charges microstates.

Let us begin with the circular profile, which has non trivial oscillation in the plane defined by the first two coordinates of $\mathbb{R}^4$

$$g_1(v') = a \cos\xi, \qquad g_2(v') = a \sin\xi \tag{C.2}$$

where we have define $\xi \equiv \frac{2\pi}{L} v'$. On $\mathbb{R}^4$ it is useful to introduce a set of coordinates

$$x_1 = \tilde{r} \sin\tilde{\theta} \cos\tilde{\phi}, \qquad x_3 = \tilde{r} \cos\tilde{\theta} \cos\tilde{\psi}$$
$$x_2 = \tilde{r} \sin\tilde{\theta} \sin\tilde{\phi}, \qquad x_4 = \tilde{r} \cos\tilde{\theta} \sin\tilde{\psi}$$

so that $|x_i - g_i(v')|^2 = \tilde{r}^2 + a^2 - 2a\tilde{r} \sin\tilde{\theta} \cos\Big(\xi - \tilde{\phi}\Big)$ Then:

$$Z_2 = 1 + \frac{Q_5}{2\pi} \int_0^{2\pi} d\xi \frac{1}{\tilde{r}^2 + a^2 - 2a\tilde{r} \sin\tilde{\theta} \cos\Big(\xi - \tilde{\phi}\Big)} = 1 + \frac{Q_5}{\sqrt{a^4 + \tilde{r}^4 + 2a^2\tilde{r}^2 \cos 2\tilde{\theta}}}$$
$$= 1 + \frac{Q_5}{r^2 + a^2 \cos^2\theta} \tag{C.3}$$

[1]We report only the most relevant points, an extended version can be find in the original work [22] or in [26].

after having performed another change of coordinates

$$\tilde{r}^2 = r^2 + a^2 \sin^2\theta, \qquad \cos^2\tilde{\theta} = \frac{r^2\cos^2\theta}{r^2 + a^2\sin^2\theta}$$

In the very same way we can derive:

$$Z_1 = 1 + Q_5 a^2 \left(\frac{2\pi}{L}\right)^2 \frac{1}{r^2 + a^2\cos^2\theta} = 1 + \frac{Q_1}{r^2 + a^2\cos^2\theta} \tag{C.4}$$

where it is clear the identification $Q_5 a^2 \left(\frac{R^2}{Q_5^2}\right) = Q_1$. It defines the relation between the charges, the S^1 radius R and the parameter a

$$R = \frac{\sqrt{Q_1 Q_5}}{a} \tag{C.5}$$

$$\begin{aligned} A = \frac{Ra}{2\pi}\int_0^{2\pi} d\xi \Bigg\{ & \frac{\sin\left(\xi - \tilde{\phi}\right)}{\tilde{r}^2 + a^2 - 2a\tilde{r}\sin\tilde{\theta}\cos\left(\xi - \tilde{\phi}\right)} \left(d\tilde{r}\sin\tilde{\theta} + \tilde{r}\cos\tilde{\theta}d\tilde{\theta}\right) \\ & - \frac{\tilde{r}\sin\tilde{\theta}\cos\left(\xi - \tilde{\phi}\right)}{\tilde{r}^2 + a^2 - 2a\tilde{r}\sin\tilde{\theta}\cos\left(\xi - \tilde{\phi}\right)} d\tilde{\phi} \Bigg\} \end{aligned} \tag{C.6}$$

$$= -a\sqrt{Q_1 Q_5}\frac{\sin^2\theta}{r^2 + a^2\cos^2\theta} d\phi \tag{C.7}$$

We know that $F \equiv dB = - *_4 dA$, so first we need to compute[2]

$$dA = -a\sqrt{Q_1 Q_5}\frac{\left(a^2 + r^2\right)\sin(2\theta)}{\left(a^2\cos^2\theta) + r^2\right)^2} d\theta \wedge d\phi + a\sqrt{Q_1 Q_5}\frac{2r\sin^(\theta)}{\left(a^2\cos^2\theta) + r^2\right)^2} dr \wedge d\phi \tag{C.8a}$$

$$F_{r\psi} = - *_4 (dA)_{\theta\phi} = a\sqrt{Q_1 Q_5}\frac{2r\cos^2(\theta)}{\left(a^2\cos^2\theta + r^2\right)^2} \equiv \partial_r B_\psi \tag{C.8b}$$

$$F_{\theta\psi} = - *_4 (dA)_{r\phi} = a\sqrt{Q_1 Q_5}\frac{r^2\sin(2\theta)}{\left(a^2\cos^2\theta + r^2\right)^2} \equiv \partial_\theta B_\psi \tag{C.8c}$$

So:

$$B = -a\sqrt{Q_1 Q_5}\frac{\cos^2\theta}{r^2 + a^2\cos^2\theta} d\psi \tag{C.8d}$$

Finally, if we define $\Sigma = r^2 + a^2\cos^2\theta$ and using (C.5)

$$\beta = \frac{Ra^2}{\sqrt{2}\Sigma}\left(\sin^2\theta d\phi - \cos^2\theta d\psi\right), \qquad \omega = \frac{Ra^2}{\sqrt{2}\Sigma}\left(\sin^2\theta d\phi + \cos^2\theta d\psi\right) \tag{C.9}$$

The second, more interesting for future purposes, 2 charges solution we want to analyse is the one with the same g_i of (C.2), but with an additional

$$g(v') = -\frac{b}{k}\sin(k\xi) \tag{C.10}$$

where k is a natural number. Computations are very similar to the ones we have done for the previous example and indeed β, ω and Z_2 are exactly the same (we choose the same coordinates for ds_4^2), the only difference is that there is an additional piece in Z_1 due to the integration of g and this time $Z_4 \neq 0$. Even though there is a non trivial component of the

[2] In the new coordinate $ds_4^2 = \frac{(r^2 + a^2\cos^2\theta)}{r^2 + a^2} dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2$

profile along one of the direction of the torus, the geometry is still invariant under rotations of the T^4. The expression for Z_1 and Z_4 are:

$$Z_1 = 1 + \frac{R^2}{Q_5}\left\{\frac{\left(a^2 + \frac{b^2}{2}\right)}{\Sigma} + \frac{b^2}{2} a^{2k} \frac{\sin^{2k}\theta \cos(2k\phi)}{\Sigma(r^2 + a^2)^k}\right\} \tag{C.11}$$

$$Z_4 = Rba^k \frac{\sin^k \theta \cos(k\phi)}{\Sigma(r^2 + a^2)^{k/2}} \tag{C.12}$$

$$\Theta_1 = \Theta_2 = \Theta_4 = 0 \tag{C.13}$$

and this time

$$R^2 = \frac{Q_1 Q_5}{a^2 + \frac{b^2}{2}} \tag{C.14}$$

Notice that even if the appearance in Z_4 of an explicit dependence on ϕ the 6 dimensional metric, defined as (3.2) is still ϕ-independent, since this warp factor appear only in $\mathcal{P} = Z_1 Z_2 - Z_4^2$ and $\mathcal{P} = \frac{R^2}{2\Sigma}\left[\frac{2a^2+b^2}{\Sigma} - \frac{b^2 a^{2k} \sin^{2k}\theta}{(r^2+a^2)\Sigma}\right]$, which depends only on r and θ.

Appendix D

Explicit computations for EE

D.1 Case $n = 0$

In order to find $\omega = r^*$ as a function of the interval length l, we need to solve the constraint:

$$
\begin{aligned}
l &= 2\omega\sqrt{Q_1 Q_5}\int_{\omega}^{r_0 \gg 1} dr \frac{1}{r(r^2+a^2)}\sqrt{\frac{r^2+\tilde{a}^2}{r^2-\omega^2}} \\
&= \frac{\sqrt{Q_1 Q_5}}{a^2\sqrt{a^2+\omega^2}}\Bigg\{2\omega\sqrt{a^2-\tilde{a}^2}\,\mathrm{arctanh}\left(\sqrt{\frac{(a^2-\tilde{a}^2)(r^2-\omega^2)}{(r^2+\tilde{a}^2)(a^2+\omega^2)}}\right) \\
&- i\tilde{a}\sqrt{a^2+\omega^2}\log\left[\frac{i\tilde{a}}{\omega}-\frac{i\omega}{\tilde{a}}-\frac{2i\tilde{a}\omega}{r^2}+2\frac{\sqrt{(\tilde{a}^2+r^2)(r^2-\omega^2)}}{r^2}\right]\Bigg\}\Bigg|_{\omega}^{r_0\gg 1} \\
&= \underbrace{\tilde{a}^2\frac{\sqrt{Q_1 Q_5}}{a^2}}_{R}\left\{\arccos\left(\frac{\omega^2-\tilde{a}^2}{\omega^2+\tilde{a}^2}\right)-\frac{\omega}{\tilde{a}}\sqrt{\frac{a^2-\tilde{a}^2}{a^2+\omega^2}}\log\left(\frac{2a^2+\omega^2-\tilde{a}^2-2\sqrt{(a^2+\omega^2)(a^2-\tilde{a}^2)}}{\tilde{a}^2+\omega^2}\right)\right\}
\end{aligned}
$$

Note that for evaluating the integral in the extreme r_0, which is the IR cut-off, we perform an expansion around $r_0 \to \infty$.
Using eq.(5.14) we can determine the geodesic length in terms of ω and consequently of l.

$$
\begin{aligned}
L_\gamma &= 2(Q_1 Q_5)^{1/4}\int_{\omega}^{r_0}\frac{r}{r^2+a^2}\sqrt{\frac{r^2+\tilde{a}^2}{r^2-\omega^2}} \\
&= 2(Q_1 Q_5)^{1/4}\left\{-\sqrt{\frac{a^2-\tilde{a}^2}{a^2+\omega^2}}\mathrm{arctanh}\left(\sqrt{\frac{(a^2-\tilde{a}^2)(r^2-\omega^2)}{(\tilde{a}^2+r^2)(a^2+\omega^2)}}\right)+\log\left(\sqrt{r^2+\tilde{a}^2}+\sqrt{r^2-\omega^2}\right)\right\}\Bigg|_{\omega}^{r_0} \\
&= (Q_1 Q_5)^{1/4}\left\{\log\left(\frac{4r_0^2}{a^2+\omega^2}\right)+\sqrt{\frac{a^2-\tilde{a}^2}{a^2+\omega^2}}\log\left(\frac{2a^2+\omega^2-\tilde{a}^2-2\sqrt{(a^2+\omega^2)(a^2-\tilde{a}^2)}}{\tilde{a}^2+\omega^2}\right)\right\}
\end{aligned}
$$

D.2 Generic n: $a = 0$ limit

Let us start from expressing (5.4), (5.5) and (5.6) in terms of the components of the metric in (5.19)

$$\begin{aligned}
\dot{t} &= \sqrt{Q_1Q_5}\left(\frac{b^2n}{2}\frac{C_2-C_1}{r^4}-\frac{C_1}{r^2}\right)\\
\dot{y} &= \sqrt{Q_1Q_5}\left(\frac{b^2n}{2}\frac{C_1-C_2}{r^4}+\frac{C_2}{r^2}\right)\\
\dot{r}^2 &= \frac{r^2}{\sqrt{Q_1Q_5}}\left\{1-\sqrt{Q_1Q_5}\frac{C_2-C_1}{r^4}\left[(C_1+C_2)r^2-\frac{b^2n}{2}(C_2-C_1)\right]\right\}\Rightarrow\\
\dot{r} &= \frac{1}{r\sqrt{Q_1Q_5}}\sqrt{r^4-\sqrt{Q_1Q_5}\left[(C_2^2-C_1^2)r^2-\frac{b^2n}{2}(C_2-C_1)^2\right]}
\end{aligned}$$

The turning point r^* is given by the largest root of:

$$r^4-\sqrt{Q_1Q_5}\left[(C_2^2-C_1^2)r^2-\frac{b^2n}{2}(C_2-C_1)^2\right]=0$$

$$\begin{aligned}
r_- &= \pm\frac{(Q_1Q_5)^{1/4}}{\sqrt{2}}\sqrt{C_2^2-C_1^2-\sqrt{(C_1-C_2)^2\left((C_1+C_2)^2-2\frac{b^2n}{\sqrt{Q_1Q_5}}\right)}}\\
r_+ &= \pm\frac{(Q_1Q_5)^{1/4}}{\sqrt{2}}\sqrt{C_2^2-C_1^2+\sqrt{(C_1-C_2)^2\left((C_1+C_2)^2-2\frac{b^2n}{\sqrt{Q_1Q_5}}\right)}}
\end{aligned}$$

Since $r_+ > r_-$, we take $r^* = r_+$.
C_1 and C_2 are fixed by the boundary conditions on y and t. For easier calculations it is convenient to consider, instead of (5.7) and (5.8) separately, their sum:

$$\begin{aligned}
\frac{l}{2} &= \int_{r^*}^{r_0} dr\frac{\dot{t}+\dot{y}}{\dot{r}} = (Q_1Q_5)^{3/4}(C_2-C_1)\int_{r^*}^{r_0} dr\frac{1}{r\sqrt{r^4-\sqrt{Q_1Q_5}\left[(C_2^2-C_1^2)r^2-\frac{b^2n}{2}(C_2-C_1)^2\right]}}\\
&= -\frac{\sqrt{Q_1Q_5}}{\sqrt{2}b\sqrt{n}}\log\left(\frac{r^2}{\Delta}\right)\Bigg|_{r^*}^{r_0\gg 1}
\end{aligned}$$

where:

$$\begin{aligned}
\Delta =& b\sqrt{n}\sqrt[4]{Q_1Q_5}\sqrt{b^2n\sqrt{Q_1Q_5}(C_1-C_2)^2+2r^2\left(\sqrt{Q_1Q_5}(C_1^2-C_2^2)+r^2\right)}+\\
&+\sqrt{Q_1Q_5}\left(b^2n(C_1-C_2)+r^2(C_1+C_2)\right)
\end{aligned}$$

And finally evaluating it in the extremes:

$$\frac{l}{2} = \frac{\sqrt{Q_1Q_5}}{\sqrt{2}b\sqrt{n}}\log\left(\frac{\sqrt{2}b\sqrt{n}+\sqrt[4]{Q_1Q_5}(C_1+C_2)}{\sqrt{\sqrt{Q_1Q_5}(C_1+C_2)^2-2b^2n}}\right)$$

Solving for C_2

$$C_2 = \frac{\sqrt{2}b\sqrt{n}}{\sqrt[4]{Q_1Q_5}}\coth\left(l\frac{b\sqrt{n}}{\sqrt{2}\sqrt{Q_1Q_5}}\right)-C_1 \tag{D.1}$$

The other constraint follows from:

$$0 = \int_{r^*}^{r_0} dr \frac{\dot{t}}{\dot{r}} = (Q_1 Q_5)^{3/4} \int_{r^*}^{r_0} dr \frac{\left(\frac{b^2 n}{2} \frac{C_2 - C_1}{r^2} - C_1\right)}{r\sqrt{r^4 - \sqrt{Q_1 Q_5}\left[(C_2^2 - C_1^2) r^2 - \frac{b^2 n}{2}(C_2 - C_1)^2\right]}}$$

$$= \frac{\sqrt{b^2 n \sqrt{Q_1 Q_5}(C_1 - C_2)^2 + 2r^2\left(\sqrt{Q_1 Q_5}(C_1 - C_2)(C_1 + C_2) + r^2\right)}}{r^2 (C_1 - C_2)} + \frac{\sqrt[4]{Q_1 Q_5}}{b\sqrt{n}} \log\left(\frac{\Delta}{r^2}\right)\Bigg|_{r^*}^{r_0 \gg 1}$$

$$= \frac{\sqrt{2}}{C_1 - C_2} + \frac{\sqrt[4]{Q_1 Q_5}}{b\sqrt{n}} \log\left(\frac{\left(\sqrt{2} b\sqrt{n} + \sqrt[4]{Q_1 Q_5}(C_1 + C_2)\right)}{\sqrt{\sqrt{Q_1 Q_5}(C_1 + C_2)^2 - 2b^2 n}}\right)$$

This result can be substituted in eq.(D.1) to obtain:

$$C_1 = \frac{b\sqrt{n} \coth\left(l \frac{b\sqrt{n}}{\sqrt{2}\sqrt{Q_1 Q_5}}\right)}{\sqrt{2}(Q_1 Q_5)^{1/4}} - \frac{(Q_1 Q_5)^{1/4}}{l} \tag{D.2}$$

$$C_2 = \frac{b\sqrt{n} \coth\left(l \frac{b\sqrt{n}}{\sqrt{2}\sqrt{Q_1 Q_5}}\right)}{\sqrt{2}(Q_1 Q_5)^{1/4}} + \frac{(Q_1 Q_5)^{1/4}}{l} \tag{D.3}$$

With these results the expressions for r^* and $\dot{r}$ become:

$$r^* = (2nQ_1 Q_5)^{1/4} \sqrt{\frac{b}{l}} \sqrt{\coth\left(\frac{l}{2} \frac{b\sqrt{n}}{\sqrt{2}\sqrt{Q_1 Q_5}}\right)}$$

$$\dot{r} = \frac{1}{r\sqrt[4]{Q_1 Q_5}} \sqrt{\frac{2b^2 n Q_1 Q_5}{l^2} - \frac{2\sqrt{2} b\sqrt{n}\sqrt{Q_1 Q_5} r^2 \coth\left(l \frac{b\sqrt{n}}{\sqrt{2}\sqrt{Q_1 Q_5}}\right)}{l} + r^4}$$

Integration of $\frac{1}{\dot{r}}$ leads immediately to the geodesic length

$$L_\gamma = \sqrt[4]{Q_1 Q_5} \log\left(\frac{2 l r_0^2 \sinh\left(\frac{b l \sqrt{n}}{\sqrt{2}\sqrt{Q_1 Q_5}}\right)}{\sqrt{2} b\sqrt{n}\sqrt{Q_1 Q_5}}\right)$$

D.3 $n = 1$

Let us start reporting the explicit expressions of (5.6) and of the combination (5.4) with (5.5) when we choose $n = 1$ in the metric of the superstratum: In this case the expression for:

$$\dot{r} = \frac{a_0\left(a^2 + r^2\right)^{3/2}}{\sqrt[4]{Q_1 Q_5}} \left\{ \frac{1}{a^4\left(a_0^2 + r^2\right) + a^2 a_0^2 r^2 + a_0^2 r^4} \right. \tag{D.4}$$
$$\left. - \frac{\sqrt{Q_1 Q_5}\left(a^6 (\mathrm{k}_1 + \mathrm{k}_2)^2 + a^4 r^2 (\mathrm{k}_1 + \mathrm{k}_2)^2 + 4a^2 a_0^2 \mathrm{k}_2 r^2 (\mathrm{k}_1 + \mathrm{k}_2) + 4a_0^2 \mathrm{k}_2 r^2\left(\mathrm{k}_1 r^2 - a_0^2 \mathrm{k}_2\right)\right)}{4\left(a^4 r\left(a_0^2 + r^2\right) + a^2 a_0^2 r^3 + a_0^2 r^5\right)^2} \right\}^{1/2}$$

$$\frac{\dot{y} + \dot{t}}{\dot{r}} = \frac{(Q_1 Q_5)^{3/4}\left(a^4 (\mathrm{k}_1 + \mathrm{k}_2) + 2a_0^2 \mathrm{k}_2 r^2\right)}{a_0 r\sqrt{a^2 + r^2}} \cdot \Delta^{-1/2} \tag{D.5}$$

$$\frac{\dot{y} - \dot{t}}{\dot{r}} = \frac{(Q_1 Q_5)^{3/4}\left((a^6 + a^4 r^2)(\mathrm{k}_1 + \mathrm{k}_2) + 2a^2 a_0^2 r^2 (\mathrm{k}_1 + 2\mathrm{k}_2) - 4a_0^4 \mathrm{k}_2 r^2 + 2a_0^2 \mathrm{k}_1 r^4\right)}{a_0 r\left(a^2 + r^2\right)^{3/2}} \cdot \Delta^{-1/2} \tag{D.6}$$

Where we have defined

$$\begin{aligned}\Delta \equiv &- a^6\sqrt{Q_1Q_5}(\mathrm{k}_1+\mathrm{k}_2)^2 + 4r^4\left(a^4+a^2a_0^2-a_0^2\mathrm{k}_1\mathrm{k}_2\sqrt{Q_1Q_5}\right)+\\ &-r^2\left(\sqrt{Q_1Q_5}\left(a^4(\mathrm{k}_1+\mathrm{k}_2)^2+4a^2a_0^2\mathrm{k}_2(\mathrm{k}_1+\mathrm{k}_2)-4a_0^4\mathrm{k}_2^2\right)-4a^4a_0^2\right)+4a_0^2r^6\end{aligned}$$

Note that, w.t.r to the previous case, we have introduced two new constants:

$$C_2 + C_1 = k_1 \tag{D.7}$$
$$C_2 - C_1 = k_2 \tag{D.8}$$
$$\tag{D.9}$$

First of all we need to find the geodesic turning point δ_{r^*}

$$\begin{aligned}&\dot r(r_0^*+a^2\delta_r,\mathrm{k}_1^0+a^2\delta_1,\mathrm{k}_2^0+a^2\delta_2)=\underbrace{\dot r(r_0^*,\mathrm{k}_1^0,\mathrm{k}_2^0)}_{0}\\ &+a\frac{\tanh\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)}{l\sqrt[4]{Q_1Q_5}\sqrt{a_0l}}\Bigg[-\frac{1}{2}a_0^2\delta_2l^3\sqrt[4]{Q_1Q_5}\sinh\left(\frac{a_0l}{\sqrt{Q_1Q_5}}\right)\operatorname{csch}^4\left(\frac{a_0l}{2\sqrt{Q}}\right)\\ &-2a_0\delta_1l^2Q^{3/4}\coth^2\left(\frac{a_0l}{2\sqrt{Q}}\right)+2\sqrt{2}a_0\delta_rl^2\sqrt[4]{Q_1Q_5}\sqrt{a_0l\coth\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)}\operatorname{csch}^2\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)\\ &-l^2\left(2\sqrt{Q_1Q_5}\coth\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)+a_0l\right)\Bigg]^{1/2}+o(a^3)\end{aligned}$$

So equating to zero we get the expression for the variation of the turning point in terms of the other variations:

$$\begin{aligned}\delta_{r^*}=&\frac{\delta_1}{2}\sqrt{\frac{Q_1Q_5\sinh\left(\frac{a_0l}{\sqrt{Q_1Q_5}}\right)}{a_0l}}\cosh\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)+\frac{\delta_2\sqrt{a_0l\coth\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)}}{2\sqrt{2}}\\ &+\frac{\sinh^2\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)\left(4\sqrt{Q_1Q_5}\coth\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)+2a_0l\right)}{4\sqrt{2}a_0\sqrt[4]{Q_1Q_5}\sqrt{a_0l\coth\left(\frac{a_0l}{2\sqrt{Q_1Q_5}}\right)}}\end{aligned}\tag{D.10}$$

Using the same expansions as in (5.24) we get the following approximate forms for:

$$\begin{aligned}\frac{\dot t+\dot y}{\dot r}=&\frac{2Q_1Q_5}{r\sqrt{f(r)}}+a^2\frac{(Q_1Q_5)^{3/4}}{r^3f(r)^{3/2}}\Bigg(-2\sqrt[4]{Q_1Q_5}\left(2a_0^2Q_1Q_5+l^2r^4-2Q_1Q_5r^2\right)+\\ &-2a_0l\sqrt{Q_1Q_5}r^2\coth\left(\frac{a_0l}{\sqrt{Q_1Q_5}}\right)\left(\delta_2lr^2-4\sqrt[4]{Q_1Q_5}\right)+\delta_2l^3r^6+2\delta_1lQ_1Q_5r^4\Bigg)+o(a^4)\end{aligned}\tag{D.11}$$

$$\frac{\dot y-\dot t}{\dot r}=-\frac{2a_0\left(2a_0Q_1Q_5-l\sqrt{Q_1Q_5}r^2\coth\left(\frac{a_0l}{\sqrt{Q_1Q_5}}\right)\right)}{r^3\sqrt{f(r)}}\tag{D.12}$$

$$\begin{aligned}&+a^2\frac{\sqrt{Q_1Q_5}}{r^5f(r)^{3/2}}\Bigg\{\sqrt[4]{Q_1Q_5}\Bigg[24a_0^4(Q_1Q_5)^{5/4}+2a_0^2r^2\left(4\sqrt[4]{Q_1Q_5}\left(l^2r^2+Q_1Q_5\right)-\delta_2l^3r^4\right)\\ &+l^2r^6\left(\delta_1lr^2+4\sqrt[4]{Q_1Q_5}\right)\Bigg]+2a_0lr^2\coth\left(\frac{a_0l}{\sqrt{Q_1Q_5}}\right)\Bigg[-18a_0^2Q_1Q_5+\\ &+a_0l\sqrt[4]{Q_1Q_5}r^2\coth\left(\frac{a_0l}{\sqrt{Q_1Q_5}}\right)\left(\delta_2lr^2+4\sqrt[4]{Q_1Q_5}\right)-r^2\left(l^2r^2+\delta_1l(Q_1Q_5)^{3/4}r^2+6Q_1Q_5\right)\Bigg]\Bigg\}+o(a^4)\end{aligned}$$

where we have defined

$$f(r)\equiv 4a_0^2Q_1Q_5-4a_0l\sqrt{Q_1Q_5}r^2\coth\left(\frac{a_0l}{\sqrt{Q_1Q_5}}\right)+l^2r^4$$

Now in order to find the condition for δ_1 and δ_2 we have to integrate the expression in (D.11) and (D.17) from $r^* + a^2\delta_{r^*}$ and the cut-off r_0 and we must recover the result $= \left(\frac{l}{2}\right)$ at the order $a = 0$.

$$\int_{r^*+a^2\delta_{r^*}}^{r_0} dr\frac{\dot{t}+\dot{y}}{\dot{r}} = \frac{l}{2} - a\left(\delta_{sum_1} + \delta_{sum_2}\right) + o(a^2) \tag{D.13}$$

$$\delta_{sum_1} = -\frac{l(Q_1Q_5)^{3/8}\sqrt{\frac{\delta_r \sinh\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right)}{a_0}}}{\sqrt[4]{2}\left(a_0 l\coth\left(\frac{a_0 l}{2\sqrt{(Q_1Q_5)}}\right)\right)^{3/4}} \tag{D.14}$$

$$\delta_{sum_2} = \frac{\delta_1(Q_1Q_5)^{7/8}}{4\;2^{3/4}\sqrt{\delta_r}\sqrt[4]{a_0^7 l\coth\left(\frac{a_0 l}{2\sqrt{(Q_1Q_5)}}\right)}\operatorname{csch}^{\frac{3}{2}}\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right)} + \tag{D.15}$$

$$\frac{\delta_2(Q_1Q_5)^{3/8}\sqrt[4]{\frac{l^3\tanh\left(\frac{a_0 l}{2\sqrt{(Q_1Q_5)}}\right)}{a_0^3}}}{4\;2^{3/4}\sqrt{\delta_r \operatorname{csch}\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right)}} +$$

$$-\frac{-2(Q_1Q_5)\cosh\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right) - a_0 l\sqrt{(Q_1Q_5)}\left(\sinh\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right) - 2\tanh\left(\frac{a_0 l}{2\sqrt{(Q_1Q_5)}}\right)\right) + 2(Q_1Q_5)}{8\;2^{3/4}a_0^2\sqrt{a_0\delta_r(Q_1Q_5)^{3/4}\sqrt{a_0 l\coth\left(\frac{a_0 l}{2\sqrt{(Q_1Q_5)}}\right)}\operatorname{csch}\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right)}}$$

$$\propto \sqrt{\delta_r}$$

$$\int_{r^*+a^2\delta_{r^*}}^{r_0} dr\frac{\dot{y}-\dot{t}}{\dot{r}} = \frac{l}{2} - a\delta_{diff} + o(a^2) \tag{D.16}$$

$$\delta_{diff} = \frac{3\sqrt{\delta_r}\operatorname{sech}\left(\frac{a_0 l}{2\sqrt{Q_1Q_5}}\right)}{2\;2^{3/4}\sqrt[8]{Q_1Q_5}\sqrt[4]{\frac{a_0\coth\left(\frac{a_0 l}{2\sqrt{Q_1Q_5}}\right)}{l^5}}} \tag{D.17}$$

So at the first order in a the condition imposed by (D.17) is simply

$$\delta_r = 0$$

Then taking into account this condition, we consider also the contribution at order a^2 in (D.13),(D.16) obtaining:

$$\delta_1 = \frac{\coth\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right)\left(a_0^2 l^2\operatorname{csch}^2\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right) - 2(Q_1Q_5)\right) + a_0 l\sqrt{(Q_1Q_5)}\left(\operatorname{csch}^2\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right) - 1\right)}{2a_0(Q_1Q_5)^{5/4}} \tag{D.18}$$

$$\begin{aligned}\delta_2 = \frac{1}{2a_0^2 l(Q_1Q_5)^{3/4}}&\left\{2\left(a_0 l\sqrt{(Q_1Q_5)}\coth\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right) + (Q_1Q_5)\right)\right.\\ &\left.- a_0 l\left(a_0 l\coth\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right) + 3\sqrt{(Q_1Q_5)}\right)\operatorname{csch}\left(\frac{a_0 l}{\sqrt{(Q_1Q_5)}}\right)\right\}\end{aligned} \tag{D.19}$$

Finally we can compute the geodesic length L_γ

$$\frac{L_\gamma}{2} = \int_{r^*+a^2\delta_{r^*}}^{r_0} dr \frac{1}{\dot{r}} \tag{D.20}$$

$$= \frac{L_\gamma^0}{2} - a^2 \frac{l\left(3\sqrt{Q_1Q_5}\sinh\left(\frac{a_0 l}{\sqrt{Q_1Q_5}}\right) + 2a_0 l\cosh\left(\frac{a_0 l}{\sqrt{Q_1Q_5}}\right) + a_0 l\right)}{8a_0(Q_1Q_5)^{3/4}\left(\cosh\left(\frac{a_0 l}{\sqrt{Q_1Q_5}}\right)+1\right)} + o(a^4) \tag{D.21}$$

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