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Preiss' Theorem on the Geometry of Measures in $\mathbb{R}^n$

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Introduction

In this work we study the proof of Preiss' Theorem, which states that a locally finite Borel measure on $\mathbb{R}^n$ with positive and finite density for almost every point in the support of μ is rectifiable. During all this work we consider only Borel measures, then we will omit it in the statements.

Theorem 0.1 (Preiss' Theorem)**.** Let μ be a locally finite measure on $\mathbb{R}^n$ and let $m \in \mathbb{Z}_{>0}$, $m \leq n$. Assume that the limit

$$\lim_{r \downarrow 0} \frac{\mu\left(B_r(x)\right)}{r^m}$$

exists, is finite and positive for μ-a.e. $x \in \operatorname{supp}(\mu)$.

Then there exist a Borel measurable function f, a countable collection $\{\Gamma_i\}_i$ of Lipschitz m-dimensional submanifolds of $\mathbb{R}^n$ and an m-dimensional set E such that

$$H^m\left(E \setminus \bigcup_i \Gamma_i\right) = 0 \text{ and } \mu = f H^m \llcorner E,$$

where $H^m \llcorner E$ is the m-dimensional Hausdorff measure restricted to the set E.

In the original paper [P] Preiss proved a stronger version of this theorem, but the proof of Theorem 0.1 contains most of the deep ideas, then we decided to focus on this weaker, but not so much easier, version. In order to do that we followed the method shown by De Lellis in [DL].

In the next section we give the first definitions and some results without proofs that we will use in the following chapters.

0.1 Preliminary results and notation

First of all we define the m-density of a measure.

Definition 0.1. Let μ be a positive Radon measure on $\mathbb{R}^n$ and $m \in \mathbb{Z}_{>0}$ Then we define the upper m-density of μ at x as

$$\theta^{*m}(\mu, x) := \limsup_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_m r^m},$$

where ω_m is the m-dimensional Hausdorff measure of the unit ball. Analogously, we define the lower m-density of μ at x as

$$\theta_*^m(\mu, x) := \liminf_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_m r^m}.$$

If $\theta_*^m(\mu, x) = \theta^{*m}(\mu, x)$ we define the m-density of μ at x as

$$\theta^m(\mu, x) := \lim_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_m r^m}.$$

We note that we could define the m-densities for $m \in \mathbb{R}_{>0}$ too, but Martstrand proved that if the m-density exists then m is an integer, therefore, given the hypotheses of Preiss' Theorem, we can restrict to this case. The following definitions too can be defined for $m \in \mathbb{R}_{>0}$.

A first property of measures with positive and finite upper density is the following, for which we omit the proof.

Theorem 0.2. Let μ be a measure and $m \in \mathbb{Z}_{>0}$ such that

$$0 < \theta^{*m}(\mu, x) < \infty$$

for μ-a.e. x. Then there exist an m-dimensional set E and a Borel function f such that $\mu = f H^m \llcorner E$.

Now we define m-uniform measures.

Definition 0.2. We say that μ is an m-uniform measure if, for every $r > 0$ and every $x \in \operatorname{supp}(\mu)$,

$$\mu(B_r(x)) = \omega_m r^m.$$

We denote by $U^m(\mathbb{R}^n)$ the set of m-uniform measures with 0 in their support.

A first important observation on uniform measures is that if $\mu \in U^m(\mathbb{R}^n)$ and $\operatorname{supp}(\mu) \subset V$, where V is an m-dimensional linear plane, then

$$\mu = H^m \llcorner V.$$

In order to see that we study $U^m(\mathbb{R}^m)$. We observe that, from the Besicovitch Differentiation Theorem, $\mu = fL_m$, where L_m is the m-dimensional Lebesgue measure and

$$f(x) = \lim_{r\downarrow 0} \frac{\mu(B_r(x))}{\omega_m r^m}$$

for L_m-a.e. x. Since μ is uniform, for every $x \in \operatorname{supp}(\mu)$ the limit is equal to 1, and for $x \notin \operatorname{supp}(\mu)$ it is euqal to 0. Then $f = Id_E$, where $E := \operatorname{supp}(\mu)$, but $0 \in \operatorname{supp}(\mu)$, then for every $r > 0$:

$$\mu(B_r(0) \cap E) = \omega_m r^m = L_m(B_r(0)),$$

therefore, since E is closed, we obtain $B_r(0) \subset E$ for every $r > 0$, hence $E = \mathbb{R}^m$.

For $\mu \in U^m(\mathbb{R}^n)$ it suffices to take an orthonormal basis in $\mathbb{R}^n$ with $V = \langle e_1, ..., e_m \rangle$ and we conclude that $\mu = H^m \llcorner V$.

Now we give the definition of the set of tangent measures.

Definition 0.3. Let μ be a measure, $x \in \mathbb{R}^n$, and $r \in \mathbb{R}_{>0}$. The measure $\mu_{x,r}$ is defined by

$$\mu_{x,r}(A) = \mu(x + rA)$$

for all Borel sets $A \subset \mathbb{R}^n$.

For every $m \in \mathbb{Z}_{>0}$ we define the set of m-tangent measures to μ at x, $\operatorname{Tan}_m(\mu, x)$, as the set of all measures ν for which there exists a sequence of radii $r_i \downarrow 0$ such that

$$\frac{\mu_{x,r_i}}{r_i^m} \overset{*}{\rightharpoonup} \nu.$$

Now we state a relation between tangent measures and uniform measures omitting the proof. The proof can be found, for example, in [DL], Proposition 3.4.

Theorem 0.3. Let μ be a measure with positive and finite m-density. Then

$$\emptyset \neq \operatorname*{Tan}_m(\mu, x) \subset \{\theta^m(\mu, x)\nu : \nu \in U^m(\mathbb{R}^n)\}.$$

We state now a series of known computations which we will use a lot of times in Chapters 3, 4 and 5, when we study the moments of μ. The proofs of these computations can be found in [DL], Appendix B.

Lemma 0.1. The following formulas hold:

$$\int_{\mathbb{R}^m} e^{-|z|^2} dL_m(z) = \pi^{m/2};$$

$$\omega_{2m} := L_{2m}\left(B_1(0)\right) = \frac{\pi^m}{m!};$$

$$\omega_{2m+1} := L_{2m+1}\left(B_1(0)\right) = \frac{2^{m+1}\pi^m}{(2m+1)!!};$$

$$\int_{\mathbb{R}^m} |z|^{2j} e^{-|z|^2} dL_m(z) = \pi^{m/2} \prod_{i=1}^{j} \left(i - 1 + \frac{m}{2}\right);$$

$$\int_{\mathbb{R}^m} |z| e^{-|z|^2} dL_m(z) = \frac{m\omega_m}{(m+1)\omega_{m+1}} \pi^{(m+1)/2};$$

$$\int_{\mathbb{R}^m} |z|^{2j+1} e^{-|z|^2} dL_m(z) = \frac{m\omega_m}{(m+1)\omega_{m+1}} \pi^{(m+1)/2} \prod_{i=1}^{j} \left(i + \frac{m-1}{2}\right).$$

0.2 Plan of the work

In Chapter 1 we study the proof of an important rectifiability criterion for Borel sets with positive and finite measure, due to Martstrand and Mattila. Using that we prove a corollary that links tangent measures to rectifiability: it states that if the upper density of μ is finite, the lower density of μ is positive and every tangent measure to μ at x are of the form $cH^k \llcorner V$ where V is a k-dimensional linear plane and c is a positive constant, then μ is rectifiable.

Martstrand, knowing that the tangent measures to μ at x are of the form $\theta^m(\mu, x)\nu$, where ν is a uniform measure, conjectured that every uniform measure was of the form $H^k \llcorner V$. This is true for $k \leq 2$ and it would conclude the proof of Theorem 0.1, but Preiss found a counterexample for $k = 3$.

In Chapter 2 we outline the proof of Preiss' Theorem: we study the set of tangent measures in order to prove that if the measure has positive and finite density then we can apply the corollary proved in Chapter 1. In order to do that we state three theorems that we prove in Chapters 3, 4 and 5. Using those three theorems we conclude the proof of Preiss' Theorem following two steps: first of all we prove that given those hypotheses for μ-a.e. x there exists a plane V such that $\theta^m(\mu, x)H^k \llcorner V$ is tangent to μ in x, then we prove that the set of tangent measures to μ at x can not contain a measure of the form $\theta^m(\mu, x)H^k \llcorner V$ and a measure which is not of that form. This means that they are all of that form and we conclude.

In Chapter 3 we prove the first theorem that we assumed to prove Preiss' Theorem, which states that if the measure is uniform then the set of its tangent measures at infinity is a singleton, therefore there exists a unique tangent measure at infinity. To do that we introduce the moments of μ, $b^{\mu}_{k,s}$, and we prove that they admit a Taylor expansion.

In Chapter 4 we prove the second theorem that we stated in Chapter 2, which states that if the tangent measure at infinity of a uniform measure is sufficiently near to a flat measure than it is flat. In order to do that we prove that the theorem is true for every conical measure. We prove it studying the form $b_2^{\mu,(1)}$, which is the second term of the Taylor expansion of $b^{\mu}_{2,s}$.

In Chapter 5 we prove the last theorem we used, which states that if a uniform measure is flat at infinity then it is flat. This proof is based on the study of the forms $b_1^{\mu,(1)}$ and $b_2^{\mu,(2)}$.

Chapter 1

The Martstrand-Mattila Rectifiability Criterion

In this chapter we study the proof of the Marstrand-Mattila Rectifiability Criterion and a corollary that links rectifiable measures with tangent measures.

Definition 1.1. An m-dimensional Borel set $E \subset \mathbb{R}^n$ is called rectifiable if there exists a countable family $\{\Gamma_i\}_i$ of m-dimensional Lipschitz graphs such that $H^m\left(E \setminus \bigcup \Gamma_i\right) = 0$.

An m-dimensional set $E \subset \mathbb{R}^n$ is called purely unrectifiable if $H^m(E)$ is finite and if for every m-dimensional Lipschitz graph Γ it holds $H^m(E \cap \Gamma) = 0$.

A measure μ is called rectifiable if there exist an m-dimensional rectifiable set E and a Borel function f such that $\mu = f H^m \llcorner E$.

Definition 1.2. Let $E \subset \mathbb{R}^n$ be an m-dimensional set and fix $x \in \mathbb{R}^n$. E is weakly linearly approximable at x if for every $\eta > 0$ there exist $\lambda > 0$ and $r > 0$ such that for every $\rho \in (0, r)$ there exists an m-dimensional linear plane W for which the following conditions hold:

$$H^m\left(E \cap B_\rho(x) \setminus \{z : \mathrm{dist}(x + W, z) < \eta\rho\}\right) < \eta\rho^m; \tag{1.1}$$

$$H^m\left(E \cap B_{\eta\rho}(z)\right) \geq \lambda\rho^m, \text{ for all } z \in (x + W) \cap B_\rho(x). \tag{1.2}$$

The first condition of this definition means that in a small ball around x most of E is contained in a tubular neighborhood of $x + W$. The second condition means that in every small ball centered at a point of $x + W$ there is a significant portion of E.

We prove now that if E is purely unrectifiable and weakly linearly approximable at H^m-a.e. point then its projection on every m-plane has H^m

measure 0. To do that we need to restrict on a compact subset where the conditions of weak linear approximation are uniform and a geometric lemma.

Lemma 1.1. Let E be a Borel set which is weakly linearly approximable at H^m-a.e. $x \in E$ and let $\varepsilon > 0$. Then there exist a compact set $C \subset E$ and positive numbers r_0, η, δ such that $H^m(E \setminus C) < \varepsilon$ and for every $a \in C$ and every $r \in (0, r_0)$, $H^m(E \cap B_r(a)) \geq \delta r^m$ and there exists a m-dimensional linear plane W such that:

$$C \cap B_r(a) \subset \{z : \operatorname{dist}(z, a + W) \leq \eta r\}. \tag{1.3}$$

Proof. Since E is weakly linearly approximable at H^m-a.e. $x \in E$, then by definition we can select a compact $C' \subset E$ such that $H^m(E \setminus C') < \varepsilon/2$ and there exist $r_1 > 0$ and $\delta > 0$ such that $H^m(E \cap B_r(a)) \geq \delta r^m$ for every $r \in (0, r_1)$.

Now we can select a compact subset $C \subset C'$ and two positive numbers $\eta < \delta\varepsilon$, $r_0 \in (0, r_1)$ such that $H^m(C' \setminus C) < \varepsilon/2$ and for every $a \in C$ and every $r \in (0, r_0)$ there exists an m-dimensional linear plane W such that

$$H^m(E \cap B_{2r}(a) \setminus \{z : \operatorname{dist}(z, a + W) \leq \eta r/2\}) < \delta \left(\frac{\eta r}{2}\right)^m.$$

We prove that this plane satisfies (1.3).

To do that we argue by contradiction: if it were false, then there would exist $z \in C \cap B_r(a)$ with $\operatorname{dist}(z, a + W) > \eta r$. Therefore

$$B_{\eta r/2}(z) \subset B_{2r}(a) \setminus \{z : \operatorname{dist}(z, a + W) \leq \eta r/2\}.$$

Hence:

$$H^m(B_{2r}(a) \cap E \setminus \{z : \operatorname{dist}(z, a + W) \leq \eta r/2\}) \geq H^m\left(B_{\eta r/2}(z)\right) \geq \delta \left(\frac{\eta r}{2}\right)^m.$$

This is a contradiction. □

Let V be an m-plane. We will indicate with P_V the orthogonal projection on V, with Q_V the orthogonal projection on $V^\perp$ and we define the m-cone $C(x, V, \alpha)$ as

$$C(x, V, \alpha) := x + \{y \in \mathbb{R}^n : |Q_V(y)| \leq \alpha |P_V(y)|\},$$

with $\alpha \in (0, +\infty)$ and $x \in \mathbb{R}^n$.

Lemma 1.2 (Geometric Lemma)**.** Let $F \subset \mathbb{R}^n$ and assume that there exists a m-dimensional plane V and a positive number α such that $F \subset C(x, V, \alpha)$ for every $x \in F$. Then there exists a Lipschitz map $f : V \to V^\perp$ such that F is contained in the graph of f.

Lemma 1.3. Let E be a purely unrectifiable set which is weakly linearly approximable at H^m-a.e. $x \in E$. Then $H^m(P_V(E)) = 0$ for every m-dimensional linear plane V.

Proof. We fix $\varepsilon \in (0, 1/2)$ and let C be as in lemma 1.1. We fix a m-dimensional linear plane V and for every $i \in \mathbb{Z}_{>0}$ we define

$$C_i := \{a \in C : C \cap B_{i^{-1}}(a) \setminus C\left(a, V, \eta^{-1}\right) = \emptyset\}.$$

By the Geometric Lemma, the intersection of C_i with a ball of radius $i^{-1}/2$ is contained in a Lipschitz graph, but C is purely unrectifiable, then:

$$H^m\left(\bigcup_i C_i\right) = 0.$$

For H^m-a.e. $a \in C$ there exists $b \in C \cap B_{r_0}(a) \cap B_{i^{-1}}(a)$ such that

$$|Q_V(b-a)| > \frac{|P_V(b-a)|}{\eta} \Rightarrow |P_V(b-a)| < \eta|b-a|.$$

Set $r := |b-a|$, let W be as in Lemma 1.1 and set $c := P_W(b-a) + a$. Lemma 1.1 implies that b verifies $|c-b| \leq \eta r$, since $b \in C \cap B_r(a)$ and $c \in a + W$ with W satisfying (1.3). Furthermore, P_W is a projection, then $|P_W(b-a)| = |c-a| \leq |b-a| = r$, and we have that $\eta < \varepsilon < 1/2$. Then we conclude that $|c-a| \geq |b-a| - |c-b| \geq r - \eta r > r/2$.

Let $w := (c-a)/|c-a|$;

$$|P_V(w)| = \frac{1}{|c-a|}|P_V(c-b) + P_V(b-a)| \leq \frac{2\eta r}{|c-a|} \leq \eta.$$

Now we prove that

$$H^m\left(P_V(\{z : \operatorname{dist}(z, a+W) < \eta r\} \cap B_r(a))\right) \leq 2^{m+2}\eta r^m. \tag{1.4}$$

After translating and rescaling this is equivalent to prove that

$$H^m\left(P_V(\{z : |Q_W(z)| < \eta\} \cap B_1(0))\right) \leq 2^{m+2}\eta.$$

Let W' be the subset of W perpendicular to w and set $V' := P_V(W')$. V' is a linear space with dimension at most $m-1$, then we can choose a unit vector $v \in V$ perpendicular to V'. We know that $|\langle w, v\rangle| \leq |P_V(w)| \leq \eta$, but a generic $\zeta \in W \cap B_1(0)$ can be written as $\zeta = \alpha w + w'$ with $w' \in W'$ and $|\alpha| < 1$, then

$$|\langle \zeta, v\rangle| = |\alpha||\langle w, v\rangle| + |\langle w', v\rangle| \leq \eta$$

for every $\zeta \in W \cap B_1(0)$. Therefore for every $\zeta' \in B_1(0)$:

$$|\langle \zeta', v\rangle| \leq |\langle P_W(\zeta), v\rangle| + |\langle Q_W(\zeta), v\rangle| \leq \eta + |Q_W(\zeta)|.$$

This means that

$$P_V\left(\{z : |Q_W(z)| < \eta\} \cap B_1(0)\right) \subset \{z : |\langle z, v\rangle| \leq 2\eta\} \cap B_1(0) \cap V.$$

Now we can fix an orthonormal basis of V with v as first element, then using the notation $z = (z_1, ..., z_m)$ we obtain:

$$\{z : |\langle z, v\rangle| \leq 2\eta\} \cap B_1(0) \cap V \subset \{z : |z_1| \leq 2\eta, |z_i| \leq 1, \text{ for } i = 2, ..., m\}.$$

Hence:

$$H^m\left(P_V(\{z : |Q_W(z)| < \eta\} \cap B_1(0))\right) \leq$$
$$\leq H^m\left(\{z : |z_1| \leq 2\eta, |z_i| \leq 1, \text{ for } i = 2, ..., m\}\right) = 2^{m+2}\eta.$$

Now with a rescalation and a translation we obtain (1.4).

For definition of W we know that $C \cap B_r(a) \subset \{z : \mathrm{dist}(z, a + W) < \eta r\}$, then

$$H^m\left(P_V(C \cap B_r(a))\right) \leq 2^{m+2}\eta r^m,$$

and obviously

$$H^m\left(P_V(C \cap \bar{B}_{r/2}(a))\right) \leq 2^{m+2}\eta r^m. \tag{1.5}$$

Using the Vitali-Besicovitch Covering Theorem we can choose a countable set of balls $\bar{B}_{r_i}(a_i)$ which are pairwise disjoint, cover H^m-almost all C, are centered at $a_i \in C$ for all i, $r_i \in (0, r_0/2)$ and satisfy (1.5) when we replace $r/2$ and a with r_i and a_i. Hence

$$H^m(P_V(C)) \leq \sum_i H^m\left(P_V(C \cap \bar{B}_{r_i}(a_i))\right) \leq \sum_i 2^{2m+2}\eta r_i^m \leq$$

$$\leq 2^{2m+2}\frac{\eta}{\delta}\sum_i H^m\left(E \cap \bar{B}_{r_i}(a_i)\right) \leq 2^{2m+2}\frac{\eta}{\delta}H^m(E) \leq 2^{2m+2}\varepsilon H^m(E).$$

Moreover P_V is a projection, then $H^m\left(P_V(E \setminus C)\right) \leq H^m\left(E \setminus C\right) \leq \varepsilon$. Hence:

$$H^m\left(P_V(E)\right) \leq H^m\left(P_V\left(E \setminus C\right)\right) + H^m\left(P_V(C)\right) \leq \left(1 + 2^{2m+2}H^m(E)\right)\varepsilon,$$

and by the arbitrariness of ε we can conclude that $H^m\left(P_V(E)\right) = 0$. □

The last tool we need in order to prove the Marstrand-Mattila rectifiability criterion is the following decomposition theorem.

Theorem 1.1. Let E be a Borel set such that $H^m(E) < \infty$. Then there exist two Borel sets $E^r, E^u \subset E$ such that $E^r \cup E^u = E$, with E^r rectifiable and E^u purely unrectifiable. Moreover this decomposition is unique up to H^m-null set.

Proof. Let $R(E) := \{E' \subset E : E' \text{ is a Borel and rectifiable set}\}$ and define $\alpha := \sup_{E' \in R(E)} H^m(E')$.

We take a sequence $\{E_i\} \subset R(E)$ such that $\lim_{i\to\infty} H^m(E_i) = \alpha$, then we set $E^r := \bigcup_i E_i$. E^r is rectifiable because it is a countable union of rectifiable sets, $E^r \subset E$ and $H^m(E^r) = \alpha$. Let $E^c := E \setminus E^r$. If there were a Lipschitz graph Γ such that $H^m(E^c \cap \Gamma) > 0$, then we would have that $E^r \cup (\Gamma \cap E^c)$ is rectifiable and $H^m(E^r \cup (\Gamma \cap E^c)) > \alpha$, that is a contradiction.

It remains to prove uniqueness: the intersection of a rectifiable and a purely unrectifiable set has always H^m measure 0. If we have two decompositions $E^r + E^u = E = F^r + F^u$ with E^r, F^r rectifiable sets and E^u, F^u purely unrectifiable sets, then we know that

$$H^m(E^r \cap E^u) = H^m(E^r \cap F^u) = H^m(F^r \cap E^u) = H^m(F^r \cap F^u) = 0.$$

This means that:

$$H^m(E^r \setminus F^r) = H^m(F^r \setminus E^r) = H^m(E^u \setminus F^u) = H^m(F^u \setminus E^u) = 0.$$

□

Theorem 1.2 (Marstrand-Mattila Rectifiability Criterion)**.** Let E be a Borel set such that $0 < H^m(E) < \infty$ and assume that E is weakly linearly approximable at H^m-a.e. $x \in E$. Then E is rectifiable.

Idea of the proof: we will argue by contradiction; we suppose that there exists a purely unrectifiable set E which is weakly linearly approximable at H^m-a.e. point. Then we fix a point x where the set is weakly linearly approximable and a ball $B_r(x)$. There we can select some pairwise disjoint cylinders and inside each of them we choose N pairwise disjoint balls that give a significant contribution to the measure of $E \cap B_r(x)$. Then we can fix the constants in order to reach a contradiction with the upper density of the set E.

Proof. Step 1. We prove that if the theorem were false then there would exists a purely unrectifiable set E with $H^m(E) > 0$ which is weakly linearly approximable at H^m-a.e. $x \in E$.

Indeed let F be an unrectifiable set which is weakly linearly approximable at H^m-a.e. $x \in F$ and let F^u be its purely unrectifiable part as given in the previous theorem. Using the Besicovitch Differentiation Theorem we see that

$$\lim_{r \to 0^+} \frac{H^m\left(F^u \cap B_r(x)\right)}{H^m\left(F \cap B_r(x)\right)} = 1$$

for H^m-a.e. $x \in F^u$. Then F weakly linearly approximable at x implies that F^u is weakly linearly approximable at x too, therefore F^u is purely unrectifiable and weakly linearly approximable at H^m-a.e. $x \in F^u$.

Then if the theorem were false, than there would exist a Borel set E such that $0 < H^m(E) < \infty$, $H^m(P_V(E)) = 0$ for every m-dimensional plane V and E is weakly linearly approximable at H^m-a.e. $x \in E$.

Hence, arguing by contradiction, we suppose that there exists such a set E.

Step 2. We reduce the set E losing a small quantity of measure and gaining some useful properties.

First of all we choose a compact $F \subset E$ such that $0 < H^m(F) < \infty$ and such that there exist r_0, δ positive numbers such that $H^m(E \cap B_r(a)) \geq \delta r^m$ for every $a \in F$ and $r < r_0$.

Next we fix a positive $\eta \in (0, 1)$. We prove that there exists a compact set $F_1 \subset F$ such that $0 < H^m(F_1) < \infty$ and such that there exist $r_1 \in (0, r_0)$ and $\gamma > 0$ such that for every $r \in (0, r_1)$ and every $a \in F_1$ there exists an m-dimensional plane W with the following properties:

$$F \cap B_{2r}(a) \subset \{z : \operatorname{dist}(z, a + W) < \eta r\}; \tag{1.6}$$

$$H^m\left(E \cap B_{\eta r}(b)\right) \geq \gamma(\eta r)^m \text{ for all } b \in (a + W) \cap B_r(a). \tag{1.7}$$

For definition of weak linear approximability there exist a compact $F_1 \subset F$, $r_1 \in (0, r_0)$ and $\gamma > 0$ such that $0 < H^m(F_1) < \infty$, (1.7) holds and

$$H^m\left(E \cap B_{2r}(a) \setminus \{z : \operatorname{dist}(z, a + W) \leq \eta r/2\}\right) < \gamma \left(\frac{\eta r}{2}\right)^m .$$

We know that $H^m\left(E \cap B_r(a)\right) \geq \delta r^m$ holds too, then we can argue as in the proof of lemma 1.1 and we obtain (1.6).

Now we prove that there exists a compat $G \subset F_1$ with positive measure such that there exists $r_2 \in (0, r_1)$ such that for every $r \in (0, r_2)$ and every $a \in G$ there exists an m- dimensional linear plane W which satisfies (1.6), (1.7) and

$$(a + W) \cap B_r(a) \subset \{z : \operatorname{dist}(z, F) < \eta r\}. \tag{1.8}$$

For every $a \in G$ and $r < r_2$ we select W such that (1.6) and (1.7) hold.

Since $H^m(E) < \infty$, then there exists a constant $\tilde{c}$ such that

$$H^m\left(E \cap B_{2r}(a)\right) \leq \tilde{c} r^m.$$

$$H^m\left(E \cap B_{2r}(a)\right) = H^m\left((E \setminus F) \cap B_{2r}(a)\right) + H^m\left(F \cap B_{2r}(a)\right) \Rightarrow$$

$$\Rightarrow 1 = \frac{H^m\left((E \setminus F) \cap B_{2r}(a)\right)}{H^m\left(E \cap B_{2r}(a)\right)} + \frac{H^m\left(F \cap B_{2r}(a)\right)}{H^m\left(E \cap B_{2r}(a)\right)}.$$

Since $a \in F$, then for the Besicovitch Differentiation Theorem

$$\lim_{r \to 0} \frac{H^m\left(F \cap B_{2r}(a)\right)}{H^m\left(E \cap B_{2r}(a)\right)} = 1 \Rightarrow \lim_{r \to 0} \frac{H^m\left((E \setminus F) \cap B_{2r}(a)\right)}{H^m\left(E \cap B_{2r}(a)\right)} = 0.$$

Hence for every $\varepsilon > 0$ there exists $\bar{r} > 0$ such that for every $r \in (0, \bar{r})$ $\frac{H^m((E \setminus F) \cap B_{2r}(a))}{H^m(E \cap B_{2r}(a))} \leq \varepsilon$. Now applying Egorov's Theorem we can make the convergence of the functions $\frac{H^m((E \setminus F) \cap B_{2r}(a))}{H^m(E \cap B_{2r}(a))}$ uniform on a subset $G \subset F$ with positive measure, then there exists $r_2 \in (0, r_1)$ such that

$$H^m\left((E \setminus F) \cap B_{2r}(a)\right) \leq \varepsilon H^m\left(E \cap B_{2r}(a)\right) \leq \varepsilon \tilde{c} r^m$$

for all $a \in G$ and $r \in (0, r_2)$. We choose $\varepsilon = \gamma \eta^m / \left(2^m \tilde{c}\right)$, then

$$H^m\left((E \setminus F) \cap B_{2r}(a)\right) \leq \gamma \left(\frac{\eta r}{2}\right)^m$$

for every $a \in G$ and $r \in (0, r_2)$. Now, if (1.8) were false then it would exists $b \in (a + W) \cap B_r(a)$ such that $B_{\eta r}(b) \cap F = \emptyset$. Therefore,

$$H^m\left(E \cap B_{\eta r}(b)\right) = H^m\left((E \setminus F) \cap B_{\eta r}(b)\right) \leq H^m\left((E \setminus F) \cap B_{2r}(a)\right) \leq \gamma \left(\frac{\eta r}{2}\right)^m,$$

that contradicts (1.7).

Let $t \in (0, \gamma \eta^m / 2)$ and $a \in G$ such that $\theta^{*m}(G, a) \leq 1$ and

$$\lim_{r \to 0} r^{-m} H^m\left((E \setminus G) \cap B_r(a)\right) = 0.$$

Without loss of generality we assume $a = 0$ and we select $r_3 \in (0, r_2)$ such that for every $r \in (0, r_3)$ the following conditions hold:

$$H^m\left(E \cap B_r(0)\right) < 2\omega_k r^m; \tag{1.9}$$

$$H^m\left((E \setminus G) \cap B_{2r}(0)\right) < t r^m. \tag{1.10}$$

Now we fix $r =: \sigma < r_3$ and select W which satisfies (1.6) and (1.8). From hypotheses

$$H^m(P_W(G)) \leq H^m(P_W(E)) = 0. \tag{1.11}$$

We want to show that for η and t small enough the conditions from (1.6) to (1.11) lead to a contradiction.

Step 3. Now we define the cylinders that will lead us to the conclusion.

For $b \in W$ and $\rho \in \mathbb{R}^+$ we define

$$D_\rho(b) := B_\rho(b) \cap W \text{ and } C_\rho(b) := \{x : P_W(x) \in D_\rho(b)\}.$$

Let $H := D_\rho(0) \setminus P_W\left(G \cap \bar{B}_{2\sigma}(0)\right)$; H is open since G is compact. For every $x \in H$ we define

$$\rho(x) := \operatorname{dist}\left(x, P_W\left(G \cap \bar{B}_{2\sigma}(0)\right)\right),$$

and we note that if $\rho(x) > \eta\sigma$ then we would have $B_{\eta\sigma}(0) \cap G = \emptyset$, therefore

$$H^m\left(E \cap B_{\eta\sigma}(x)\right) = H^m\left((E \setminus G) \cap B_{\eta\sigma}(x)\right) \leq H^m\left((E \setminus G) \cap B_{2\sigma}(x)\right) \leq t\sigma^m,$$

and for t sufficiently small this is in contradiction with (1.7). Then

$$\rho \leq \eta\sigma. \tag{1.12}$$

Using the $5r$-Covering Lemma we find a countable set $\{x_i\}_{i\in I}$ of points in $H \cap D_{\sigma/4}(0)$ such that $\left\{D_{20\rho(x_i)}(x_i)\right\}_{i\in I}$ is a covering of $H \cap D_{\sigma/4}(0)$ and the disks $\{D_{4\rho(x_i)}(x_i)\}_{i\in I}$ are pairwise disjoint; we define $\rho_i := \rho(x_i)$.

Since $H^m\left(H \cap D_{\sigma/4}(0)\right) = H^m\left(D_{\sigma/4}(0)\right) = \omega_m(\sigma/4)^m$, then

$$\sum_{i\in I} \omega_m \rho_i^m = \frac{1}{20^m} \sum_{i\in I} \omega_m \left(20\rho_i\right)^m \geq \frac{H^m\left(H \cap D_{\sigma/4}(0)\right)}{20^m} = \frac{\omega_m \sigma^m}{80^m} \tag{1.13}$$

Now we split the indices in two sets:

$$J := \left\{i \in I : C_{\rho_i/2}(x_i) \cap F \cap B_\sigma(0) \neq \emptyset\right\} \text{ and } K := I \setminus J.$$

Step 4. We study the sum of $\omega_m \rho_i^m$ on J and we reach an estimate of the same sum on K.

For every $i \in J$ let $y_i \in C_{\rho_i/2}(x_i) \cap F \cap B_\sigma(0)$. Since $y_i \in F \cap B_\sigma(0)$, then from (1.6) we have that $|y_i - P_W(y_i)| \leq \eta\sigma$, therefore

$$|y_i| \leq \eta\sigma + \rho_i/2 \Rightarrow B_{\rho_i/2}(y_i) \subset B_{\eta\sigma+\rho}(0),$$

but from (1.12) $B_{\eta\sigma+\rho}(0) \subset B_{2\eta\sigma}(0)$. Taking $\eta < 1/2$ we obtain that $B_{\rho_i/2}(y_i) \subset B_\sigma(0)$. Moreover $E \cap B_{\rho_i/2}(y_i) \subset C_{\rho_i}(x_i) \cap (E \setminus G)$, then:

$$H^m\left(C_{\rho_i}(x_i) \cap (E \setminus G) \cap B_\sigma(0)\right) \geq H^m\left(E \cap B_{\rho_i/2}(y_i)\right) \geq \frac{\delta\rho_i^m}{2^m}.$$

We note that $\{C_{\rho_i}(x_i)\}_{i\in I}$ are pairwise disjoint, then:

$$\sum_{i\in J} \omega_m \rho_i^m \leq \sum_{i\in J} \frac{\omega_m 2^m}{\delta} H^m\left(C_{\rho_i}(x_i) \cap (E \setminus G) \cap B_\sigma(0)\right) \leq$$

$$\leq \frac{\omega_m 2^m}{\delta} H^m\left((E \setminus G) \cap B_\sigma(0)\right) \leq \frac{t\omega_m 2^m \sigma^m}{\delta}.$$

From this estimate and (1.13) we can conclude that if we choose t sufficiently small there exists a constant $\bar{c}$ such that

$$\sum_{i\in K} \omega_m \rho_i^m \geq \bar{c}\sigma^m.$$

Indeed

$$\sum_{i\in K} \omega_m \rho_i^m \geq \frac{\omega_m \sigma^m}{80^m} - \sum_{i\in J} \omega_m \rho_i^m \geq \omega_m \sigma^m \left(\frac{1}{80^m} - \frac{2^m t}{\delta}\right),$$

then if we take $t \leq \delta/(2 \cdot 160^m)$ we obtain

$$\sum_{i\in K} \omega_m \rho_i^m \geq \frac{\omega_m}{2 \cdot 80^m} \sigma^m.$$

Step 5. We focus on the cylinders $C_{\rho_i}(x_i)$ with $i \in K$ and we search some pairwise disjoint balls in order to reach the contradiction with (1.9).

For every $i \in K$ there exists a point $z_i \in \partial C_{\rho_i}(x_i) \cap G \cap \bar{B}_{2\sigma}(0)$, moreover $\rho_i/(8\eta) \leq \rho_i/\eta \leq \sigma < r_3$, then we can fix an m-dimensional plane W_i which meets the conditions (1.6) and (1.8) for $a = z_i$ and $r = \frac{\rho_i}{8\eta}$.

Since $i \in K$, $C_{\rho_i/2}(x_i) \cap F \cap B_\sigma(0) = \emptyset$. Since (1.8) holds and $\eta r = \rho_i/8$, we have that

$$(z_i + W_i) \cap C_{\rho_i/4}(x_i) \cap B_{3\sigma/4} = \emptyset.$$

We prove that $(z_i + W_i) \cap C_{2\rho_i}(x_i) \cap B_{\sigma/2}$ contains a segment S_i of length $\rho_i/(8\eta)$.

Let $A_i := B_{\rho_i/(8\eta)}(z_i) \cap (z_i + W_i)$. From (1.8) $A_i \subset \{z : \operatorname{dist}(z, F) < \rho_i/8\}$, then $x_i \notin P_W(A_i)$, indeed we assume that there exists $x \in A_i$ such that $P_W(x) = b_i$. There would exists $y \in F$ such that $|x - y| \leq \rho_i/8$, then $P_W(y) \in B_{\rho_i/2}(x_i)$, but $C_{\rho_i/2}(x_i) \cap F \cap B_\sigma(0) = \emptyset$.

Let I_i be the segment with end-points x_i and $P_W(z_i)$; since $x_i \in I_i \setminus P_W(A_i)$ and $P_W(z_i) \in I_i \cap P_W(A_i)$, then $I_i \cap \partial_W P_W(A_i) \neq \emptyset$.

Since $\partial_W P_W(A_i) = P_W(\partial_{W_i} A_i)$, then we can choose $a_i \in \partial_{W_i} A_i$ such that $P_W(a_i) \in I_i$ and we define S_i as the segment with end-points a_i and z_i.

We have that

$$S_i \subset A_i \subset (z_i + W_i) \cap C_{2\rho_i}(x_i) \cap B_{\sigma/2},$$

and S_i has length $\rho_i/(8\eta)$ as we wanted.

Therefore we can find N points $z_i^j \in S_i$ for $j = 1, ..., N$ with

$$N > |S_i|/(2\rho_i) = 1/(16\eta)$$

such that the balls $B_{\rho_i/2}(z_i^j)$ are pairwise disjoint. By (1.8) each ball $B_{\rho_i/8}(z_i^j)$ must contain a point $w_i^j \in F$, therefore

$$H^m \left(E \cap B_{\rho_i/8}\left(w_i^j\right)\right) \geq \frac{\delta \rho_i^m}{8^m}.$$

Since $B_{\rho_i/8}(w_i^j) \subset B_{\rho_i/4}(z_i^j)$, then $\{B_{\rho_i/8}(w_i^j)\}_j$ are pairwise disjoint and they are contained in $C_{4\rho_i}(x_i)$, but $\{C_{4\rho_i}(x_i)\}_{i\in K}$ are pairwise disjoint, then the balls $B_{\rho_i/8}(w_i^j)$ with $i \in K$ and $j = 1, ..., N$ are pairwise disjoint.

Then we can conclude:

$$H^m \left(E \cap B_\sigma(0)\right) \geq \sum_{i\in K} \sum_{j=1}^{N} H^m \left(E \cap B_{\rho_i/8}\left(w_i^j\right)\right) \geq \sum_{i\in K} \sum_{j=1}^{N} \frac{\delta \rho_i^m}{8^m} =$$

$$= \frac{N\delta}{8^m \omega_m} \sum_{i\in K} \omega_m \rho_i^m \geq \frac{\delta \bar{c}}{2^{3m+4} \omega_m \eta} \sigma^m.$$

Therefore we can choose η small enough to obtain a contradiction with (1.9), and this complete the proof. □

Now we prove the corollary that we will use to prove Preiss' Theorem.

Theorem 1.3. Let μ be a measure such that for μ-a.e. x the densities $\theta_*^m(\mu, x)$ and $\theta^{*m}(\mu, x)$ are positive and finite, and such that every tanget measure to μ at x is of the form $\alpha H^m \llcorner V$ for some m-dimensional linear plane V. Then μ is a rectifiable measure.

Proof. Since $0 < \theta^{*m}(\mu, x) < \infty$ for μ-a.e. x, there exist a Borel function f and a Borel set E such that $\mu = f H^m \llcorner E$. The thesis is equivalent to prove that $E \cap \{f > 0\}$ is rectifiable. Then it is enough to prove that $E_c := E \cap \{c \leq f \leq c^{-1}\}$ is rectifiable for any $c \in (0, 1)$, because $E = \bigcup_{i\in\mathbb{N}_{>1}} E_{1/i}$, and if $E_{1/i}$

is rectifiable for every $i > 1$ then E is a countable union of countable unions of Lipschitz graphs, hence it is a countable union of Lipschitz graphs.

We fix $c \in (0,1)$ and let $\nu := H^m \llcorner E_c$. Then, by the Besicovich Differentiation Theorem we have that

$$\theta^{*m}(\nu,x) = \frac{\theta^{*m}(\mu,x)}{f(x)}, \text{ and } \theta^m_*(\nu,x) = \frac{\theta^m_*(\mu,x)}{f(x)}$$

for H^m-a.e. $x \in F$, then

$$0 < \theta^m_*(\nu,x) \leq \theta^{*m}(\nu,x) < \infty. \tag{1.14}$$

From the locality of $\mathrm{Tan}_m(\nu,x)$ it follows that $\mathrm{Tan}_m(\nu,x) = \mathrm{Tan}_m(\mu,x)/f(x)$ for H^m-a.e. $x \in F$, therefore

$$\mathrm{Tan}_m(\nu,x) \subset \{aH^m \llcorner V : a \geq 0 \text{ and } V \text{ is an } m\text{-dimensional plane}\}. \tag{1.15}$$

We prove that E_c is weakly linearly approximable at every point x which satisfies (1.14) and (1.15)

We argue by contradiction: we assume that there exists x that satisfies (1.14) and (1.15) but E_c is not weakly linearly approximable at x. Without loss of generality we assume that $x = 0$; then there exist $\eta > 0$ and a decreasing sequence $r_j \downarrow 0$ that for every m-dimensional plane W and every j either

$$H^m(E_c \cap B_{r_j}(0) \setminus \{z : \mathrm{dist}(W,z) \leq \eta r_j\}) \geq \eta r_j^m \tag{1.16}$$

or there exists $z_{j,W} \in W \cap B_{r_j}(0)$ with

$$\frac{H^m(E_c \cap B_{\eta r_j}(z_{j,W}))}{r_j^m} \leq \frac{1}{j}. \tag{1.17}$$

Set $\nu_j := r_j^{-m}\nu \llcorner B_{r_j}(0)$: since $\theta^{*m}(\nu,0) < \infty$ there exists a subsequence $\{\nu_{j_i}\}_i$ that converges to $\nu_\infty \in \mathrm{Tan}_m(\nu,0)$.

From (1.15) it follows that there exist an m-dimensional plane W and a constant $\bar{c} \geq 0$ such that $\nu_\infty = \bar{c}H^m \llcorner W$. Moreover either (1.16) or (1.17) holds for an infinite number of indices i, then we can take a subsequence $\{\nu_{j_{i_l}}\}_l$ such that it holds the same condition for all radii. We indicate that subsequence with $\{\nu_l\}_l$.

In case (1.16) holds for all radii then

$$\nu_l(B_1(0) \setminus \{z : \mathrm{dist}(W,z) \leq \eta\}) \geq \eta.$$

Let Ω be the closure of $B_1(0) \setminus \{z : \mathrm{dist}(W,z) \leq \eta\}$. Then

$$\bar{c}H^m(\Omega \cap W) = \nu_\infty(\Omega) \geq \limsup_{l\to\infty} \nu_l(\Omega) \geq \eta,$$

but $\Omega \cap W = \emptyset$, therefore there is a contradiction.

If (1.17) holds for all radii, then there exists a sequence of points $y_l \in W \cap B_1(0)$ such that

$$\lim_{l\to\infty} \nu_l(B_\eta(y_l)) = 0.$$

We take a converging subsequence $\{y_{l_h}\}_h$: $y_{l_h} \to y \in W$. Then

$$\bar{c}\omega_m \eta^m = \bar{c}H^m\left(W \cap B_\eta(y)\right) = \nu_\infty(B_\eta(y)) \leq \lim_{h\to\infty} \nu_{l_h}\left(B_\eta\left(y_{l_h}\right)\right) = 0,$$

hence $\bar{c} = 0$.

On the other hand for L^1-a.e. $\rho > 0$:

$$\theta_*^m(\nu, y) = \liminf_{r\to 0} \frac{\nu(B_r(y))}{\omega_m r^m} \leq \frac{\nu_\infty(B_\rho(0))}{\omega_m \rho^m} = \frac{\bar{c}}{\omega_m} = 0,$$

but $\theta_*^m(\nu, y) > 0$ for (1.14), hence we reached a contradiction. $\square$

Chapter 2

Preiss' Theorem

In this chapter we give a proof of Preiss' theorem, that is the main result of this work, skipping the proof of three steps that we will discuss in the following chapters.

Theorem 2.1 (Preiss' Theorem)**.** Let m be a positive integer and μ a locally finite measure on $\mathbb{R}^n$ such that

$$0 < \theta_*^m(\mu, x) = \theta^{*m}(\mu, x) < \infty$$

for μ-a.e. x. Then μ is an m-rectifiable measure.

To prove this theorem we follow this strategy: first of all we prove that if μ satisfies those hypotheses then for μ-a.e. x there exists an m-dimensional plane W_x such that $\theta(\mu, x)H^m \llcorner W_x \in \mathrm{Tan}_m(\mu, x)$; then we prove that if $\mathrm{Tan}_m(\mu, x) \subset \theta(\mu, x)U^m(\mathbb{R}^n)$ and it contains a measure of the form $\theta(\mu, x)H^m \llcorner V$ for an m-dimensional plane V, then all the measures in $\mathrm{Tan}_m(\mu, x)$ are of that form. After these two steps we can conclude that at μ-a.e. x the set of tangent measures at x consists of measures of the form $\theta(\mu, x)H^m \llcorner V$, then we can apply Theorem 1.3 and conclude that μ is rectifiable.

2.1 Part A of Preiss' strategy

In this first section we prove the first step of the strategy described before.

The first tool that we need is a corollary of the Marstrand Theorem that we state omitting the proof.

Corollary 2.1. Let m be an integer and $\mu \in U^m(\mathbb{R}^n)$. Then there exist an m-dimensional linear plane $V \subset \mathbb{R}^n$ and two sequences $\{x_i\} \subset \mathrm{supp}(\mu)$ and

$\{r_i\} \subset (0,1]$ such that

$$\frac{\mu_{x_i,r_i}}{r_i^m} \stackrel{*}{\rightharpoonup} H^m \llcorner V$$

in the sense of measures.

For the proof see *Chapter 3* of [DL].

Lemma 2.1. Let μ be as in the Preiss' Theorem, then for μ-a.e. x the following property holds: if $\nu \in \mathrm{Tan}_m(\mu, x)$, then $r^{-m}\nu_{y,r} \in \mathrm{Tan}_m(\mu, x)$ for every $y \in \mathrm{supp}(\nu)$ and $r > 0$.

Proof. The thesis is equivalent to prove that for μ-a.e. a the following property holds: if $\nu \in \mathrm{Tan}_m(\mu, a)$ and $x \in \mathrm{supp}(\nu)$ then $\nu_{x,1} \in \mathrm{Tan}_m(\mu, a)$.

Indeed, let a be a point where this last property holds, let $\xi \in \mathrm{Tan}_m(\mu, a)$ and fix $b \in \mathrm{supp}(\xi)$ and $r > 0$. Let $\nu := r^{-m}\xi_{0,r}$. We see that $\nu \in \mathrm{Tan}_m(\mu, a)$, $b/r \in \mathrm{supp}(\nu)$, and $r^{-m}\xi_{b,r} = \nu_{b/r,1}$. The property we are assuming implies that $\nu_{b/r,1} \in \mathrm{Tan}_m(\mu, a)$, then $r^{-m}\xi_{b,r} \in \mathrm{Tan}_m(\mu, a)$, that is our thesis.

Now we prove that property. For every $j, k \in \mathbb{N}$ we define $A_{k,j}$ as the set

$$\left\{a \in \mathbb{R}^n : \exists \nu \in \mathrm{Tan}_m(\mu, a), x \in \mathrm{supp}(\nu) \text{ with } d(r^{-m}\mu_{a,r}, \nu_{x,1}) \geq \frac{1}{k}\ \forall r < \frac{1}{j}\right\}$$

where d is the metric of the weak* topology.

The thesis is equivalent to prove that $\mu(A_{k,j}) = 0$ for all $k, j \in \mathbb{R}^n$. We argue by contradiction: assume then that $\mu(A_{k,j}) > 0$ for some k and j. Then there exists $R > 0$ such that the set

$$A_{k,j} \cap \{a : R^{-1} < \theta^m(\mu, a) \leq R\}$$

has positive measure. Let B be that set for that choice of R and let

$$S := \{\nu_{x,1} : \nu \in \mathrm{Tan}_m(\mu, a) \text{ for some } a \in B, x \in \mathrm{supp}(\nu)\}.$$

We note that

$$\nu_{x,1}(B_r(0)) = \nu(B_r(x_a)) \leq \liminf_{i\to\infty} \frac{\mu_{a,\rho_i}(B_r(x_a))}{\rho_i^m} =$$

$$= \liminf_{i\to\infty} \frac{\mu(a + B_{r\rho_i}(\rho_i x_a))}{\omega_m (r\rho_i)^m}\omega_m r^m = \theta(\mu, a)\omega_m r^m \leq R\omega_m r^m,$$

therefore $S \subset \{\nu : \nu(B_r(0)) \leq R\omega_m r^m\ \forall r > 0\} =: C$.

The set C is compact with respect to the metric d, then we can cover it with a finite family of sets G_i of type

$$G_i = \left\{\zeta : d(\zeta, \zeta_i) < \frac{1}{4k}\right\}.$$

Consider the sets D_i of points $a \in B$ for which there exists at least a measure $\nu^a \in \mathrm{Tan}_m(\mu, a)$ and $x_a \in \mathrm{supp}(\nu^a)$ such that $\nu^a_{x_a,1} \in G_i$ and $d(r^{-m}\mu_{a,r}, \nu^a_{x_a,1}) \geq 1/k$ for every $r \in (0, 1/j)$. The family sets $\{D_i\}$ is a finite covering of B, hence there exists a set D in that family such that $\mu(D) > 0$, and let G be the corresponding G_i.

For any $a \in D$ we fix a measure ν^a and a point x_a which satisfy the previous inequality and $\nu^a_{x_a,1} \in G$. If $a, b \in D$ then

$$d(\nu^a_{x_a,1}, \nu^b_{x_b,1}) < \frac{1}{2k},$$

since $\nu^a_{x_a,1}, \nu^b_{x_b,1} \in G$.

Since D is μ-measurable we can choose $a \in D$ such that

$$\lim_{i \to 0} \frac{\mu(D \cap B_r(a))}{\mu(B_r(a))} = 1. \tag{2.1}$$

Then we choose $r_i \downarrow 0$ and $\{a_i\} \subset D$ such that

$$\frac{\mu_{a,r_i}}{r_i^m} \stackrel{*}{\rightharpoonup} \nu^a;$$

$$|a_i - (a + r_i x_a)| < \mathrm{dist}(a + r_i x_a, D) + \frac{r_i}{i}.$$

Now we prove that

$$\lim_{i \to \infty} \frac{\mathrm{dist}(a + r_i x_a, D)}{r_i} = 0.$$

Arguing by contradiction, we suppose that there exists a positive constant c such that

$$\limsup_{i \to \infty} \frac{\mathrm{dist}(a + r_i x_a, D)}{r_i} > c.$$

We prove now that $\nu^a(B_c(x_a)) = 0$, that is in contradiction with the condition $x_a \in \mathrm{supp}(\nu^a)$:

$$\nu^a(B_c(x_a)) \leq \liminf_{i \to \infty} \frac{\mu_{a,r_i}(B_c(x_a))}{r_i^m} = \liminf_{i \to \infty} \frac{\mu(B_{r_i c}(a + r_i x_a))}{r_i^m}.$$

Since (2.1) holds, for every $\varepsilon > 0$ there exists $R > 0$ such that for every $r < R$ it holds

$$\frac{\mu(B_r(a))}{r^m} \leq \frac{\mu(D \cap B_r(a))}{r^m} + \varepsilon.$$

Now we can take a decreasing subsequence $\{r_{i_j}\}_j$ such that $\mathrm{dist}(a + r_{i_j} x_a, D) > r_{i_j} c$ for every j and such $r_{i_2} < c r_{i_1}/(c + |x_a|)$, with $r_{i_1} < R$.

For every $j > 1$ we have that

$$B_{cr_{i_j}}(a + r_{i_j} x_a) \subset B_{r_{i_j}(|x_a|+c)}(a) \subset B_{cr_{i_1}}(a + r_{i_1} x_a).$$

Then we can conclude

$$\nu^a(B_c(x_a)) \leq \liminf_{i\to\infty} \frac{\mu(B_{r_i c}(a + r_i x_a))}{r_i^m} \leq \liminf_{j\to\infty} \frac{\mu(B_{r_{i_j} c}(a + r_{i_j} x_a))}{r_{i_j}^m} \leq$$

$$\leq \liminf_{j\to\infty} \frac{\mu(B_{r_{i_j}(c+|x_a|)}(a))}{r_{i_j}^m} \leq \liminf_{j\to\infty} \frac{\mu(B_{r_{i_j}(c+|x_a|)}(a) \cap D)}{r_{i_j}^m} + \varepsilon \leq$$

$$\leq \liminf_{j\to\infty} \frac{\mu(B_{r_{i_1} c}(a + r_{i_1} x_a) \cap D)}{r_{i_j}^m} + \varepsilon = \varepsilon.$$

By the arbitrariness of ε it follows that $\nu^a(B_c(x_a)) = 0$, and then we reached the contradiction.

Then it follows that

$$\left| \frac{a_i - a}{r_i} - x_a \right| \leq \frac{\mathrm{dist}(a + r_i x_a, D)}{r_i} + \frac{1}{i} \xrightarrow[i\to\infty]{} 0.$$

Now we can note that

$$\frac{\mu_{a_i,r_i}}{r_i^m} = \left(\frac{\mu_{a,r_i}}{r_i^m} \right)_{\frac{a_i - a}{r_i}} \overset{*}{\rightharpoonup} \nu^a_{x_a,1}.$$

Therefore, we can choose $r_i < 1/j$ sufficiently small such that

$$d(\nu^a_{x_a,1}, r_i^{-m} \mu_{a_i,r_i}) < \frac{1}{2k}.$$

Since $a_i \in D$, then we conclude:

$$\frac{1}{k} < d(\nu^{a_i}_{x_{a_i},1}, r_i^{-m} \mu_{a_i,r_i}) \leq d(\nu^{a_i}_{x_{a_i},1}, \nu^a_{x_a,1}) + d(\nu^a_{x_a,1}, r_i^{-m} \mu_{a_i,r_i}) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}.$$

Then we reached a contradiction and this conclude the proof. □

Now we prove the first part of the Preiss' strategy.

Theorem 2.2. Let μ be as in the Preiss' Theorem, then for μ-a.e. $x \in \mathbb{R}^n$ there exists a plane W_x such that $\theta(\mu, x) H^m \llcorner W_x \in \mathrm{Tan}_m(\mu, x)$.

Proof. Let x be a point where Corollary 2.1 and Lemma 2.1 hold and the density $\theta(\mu, x)$ exists.

If $\nu \in \mathrm{Tan}_m(\mu, x)$, then for Lemma 2.1 $r^{-m}\nu_{y,r} \in \mathrm{Tan}_m(\mu, x)$ for every $y \in \mathrm{supp}(\nu)$ and $r > 0$. Furthermore ν is of the form $\theta(\mu, x)\zeta$ with ζ uniform, hence for Corollary 2.1 there exist an m-dimensional plane V and two sequences $\{x_i\} \subset \mathrm{supp}(\nu)$ and $\{r_i\} \subset (0, 1]$ such that

$$\frac{\zeta_{x_i,r_i}}{r_i^m} \overset{*}{\rightharpoonup} H^m \llcorner V.$$

Multiplying for the density we have that $\theta(\mu, x)r_i^{-m}\zeta_{x_i,r_i} = r_i^{-m}\nu_{x_i,r_i}$, and $r_i^{-m}\nu_{x_i,r_i} \in \mathrm{Tan}_m(\mu, x)$, then

$$\frac{\nu_{x_i,r_i}}{r_i^m} \overset{*}{\rightharpoonup} \theta(\mu, x)H^m \llcorner V.$$

We conclude that the weak* closure of $\mathrm{Tan}_m(\mu, x)$ contains a measure of the form $\theta(\mu, x)H^m \llcorner V$, where V is an m-dimensional plane.

We prove now that $\mathrm{Tan}_m(\mu, x)$ is closed, and this concludes the proof.

For every $\rho \in (0, 1]$ consider the set

$$C_\rho := \{\sigma^{-m}\mu_{x,\sigma} : 0 < \sigma \leq \rho\}.$$

Let $\xi_\sigma := \sigma^{-m}\mu_{x,\sigma}$. We prove that $\xi_\sigma(B_r(0))$ is bounded from above from a constant depending only on μ and r. For $\sigma \geq \varepsilon > 0$ we have that $\xi_\sigma(B_r(0))$ is bounded because μ is locally bounded; near 0 instead, we have that the density of μ exists, then there exists the limit

$$\lim_{\sigma\to 0} \xi_\sigma(B_r(0)) = \lim_{\sigma\to 0} \frac{\mu(B_{r\sigma}(x))}{\omega_m(r\sigma)^m}\omega_m r^m = \theta(\mu, x)\omega_m r^m.$$

Hence there exists a constant $c(r)$ depending only on r and μ such that

$$C_\rho \subset \{\xi : \xi(B_r(0)) \leq c(r) \ \ \forall r > 0\}$$

for every $\rho \in (0, 1]$.

As we can see in *Theorem 2.6* of [DL], on this set the weak* topology is metrized by a metric d. Let $\bar{C}_\rho$ be the closure of C_ρ in the metric d: since

$$\mathrm{Tan}_m(\mu, x) = \bigcap_{0<\rho<1} \bar{C}_\rho,$$

then $\mathrm{Tan}_m(\mu, x)$ is weakly* closed. $\square$

2.2 Part B of Preiss' strategy

In this section we outline a proof of the second step of Preiss' strategy, stating three theorems that we will prove in the next chapters and showing how our goal follows from them.

Theorem 2.3. Let μ be as in the Preiss' Theorem and let x be a point such that $\mathrm{Tan}_m(\mu, x) \subset \theta(\mu, x)U^m(\mathbb{R}^n)$ and such that $\mathrm{Tan}_m(\mu, x)$ contains a measure of the form $\theta(\mu, x)H^m \llcorner V$ for some m-dimensional plane V. Then $\mathrm{Tan}_m(\mu, x) \subset \theta(\mu, x)G_m(\mathbb{R}^n)$, where $G_m(\mathbb{R}^n)$ is the set of flat measures.

First of all we define the set of tanget measures at infinity; the first result is the uniqueness theorem of tangent measures at infinity for uniform measures.

Definition 2.1. Let $\alpha \in \mathbb{R}^+$ and μ be a locally finite measure. Then we define the set $\mathrm{Tan}_\alpha(\mu, \infty)$ as the set of measures ν such that there exists a sequence of radii $r_i \uparrow \infty$ with

$$\frac{\mu_{0,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu.$$

Theorem 2.4 (Uniqueness Theorem)**.** If $\nu \in U^m(\mathbb{R}^n)$, then there exists $\zeta \in U^m(\mathbb{R}^n)$ such that $\mathrm{Tan}_m(\nu, \infty) = \{\zeta\}$.

From this theorem it follows that the whole family $\{r^{-m}\nu\}_{r>0}$ converges to ζ as $r \to \infty$, then we can define ζ as the tangent measure at infinity of ν. We will give a proof of Theorem 2.4 in Chapter 3.

Definition 2.2. We say that $\nu \in U^m(\mathbb{R}^n)$ is flat at infinity if its tangent measure at infinity is flat.

The following theorem states that if ν is uniform and its tangent measure at infinity is sufficiently close to a flat measure, then ν is flat at infinity. We will prove it in Chapter 4.

Theorem 2.5. There exists a constant $\varepsilon > 0$ depending only on m and n such that if $\nu \in U^m(\mathbb{R}^n)$, ζ is its tangent measure at infinity and

$$\min_{V \in G(m,n)} \int_{B_1(0)} [\mathrm{dist}(x, V)]^2 d\zeta(x) \leq \varepsilon,$$

then ζ is flat, where $G(m, n)$ is the set of m-dimensional linear planes in $\mathbb{R}^n$.

The third theorem we state will be proved in the last chapter, Chapter 5, and it gives a relation between flatness at infinity and flatness.

Theorem 2.6. If $\nu \in U^m(\mathbb{R}^n)$ is flat at infinity, then ν is flat.

The last result we need to prove Theorem 2.3 is the following lemma.
We indicate with $M(\mathbb{R}^n)$ the set of nonnegative locally finite measures.

Lemma 2.2. Let $\varphi \in C_c(\mathbb{R}^n)$ be a nonnegative function and consider the functional $F : M(\mathbb{R}^n) \to \mathbb{R}$ given by

$$F(\mu) := \min_{V \in G(m,n)} \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, V)]^2 d\mu(z).$$

If $\mu_i \overset{*}{\rightharpoonup} \mu$ then $F(\mu_i) \to F(\mu)$.

Proof. Let V_i be such that

$$F(\mu_i) = \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, V_i)]^2 d\mu_i(z).$$

Up to a subsequence we can assume that $\{V_i\}$ converges to an m-dimensional plane V_∞; then the sequence of functions $\varphi(\cdot)[\mathrm{dist}(\cdot, V_i)]^2$ converges uniformly to $\varphi(\cdot)[\mathrm{dist}(\cdot, V_\infty)]^2$. This implies that

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, V_i)]^2 d\mu_i = \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, V_\infty)]^2 d\mu,$$

then

$$\liminf_{i\to\infty} F(\mu_i) = \lim_{i\to\infty} \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, V_i)]^2 d\mu_i = \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, V_\infty)]^2 d\mu \geq$$

$$\geq \min_{V \in G(m,n)} \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, V)]^2 d\mu(z) = F(\mu).$$

Let $\bar{V}$ be an m-dimensional plane such that

$$F(\mu) = \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, \bar{V})]^2 d\mu(z).$$

Then it holds that

$$F(\mu) = \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, \bar{V})]^2 d\mu(z) = \lim_{i\to\infty} \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, \bar{V})]^2 d\mu_i(z) \geq$$

$$\geq \limsup_{i\to\infty} \int_{\mathbb{R}^n} \varphi(z)[\mathrm{dist}(z, \bar{V}_i)]^2 d\mu_i(z) = \limsup_{i\to\infty} F(\mu_i).$$

This concludes the proof. □

Furthermore, with this definition of F we can note that if $\nu \in U^m(\mathbb{R}^n)$ than ν is flat if and only if $F(\nu) = 0$.

Now we give a proof of Theorem 2.3, which concludes the proof of Theorem 2.1.

Proof. We argue by contradiction: let x be a point such that $\mathrm{Tan}_m(\mu, x)$ is contained in $\theta(\mu, x)U^m(\mathbb{R}^n)$ and such that there exist $\nu, \zeta \in \mathrm{Tan}_m(\mu, x)$ with $\nu/\theta(\mu, x)$ flat and $\zeta/\theta(\mu, x)$ not flat. Without loss of generality we assume $\theta(\mu, x) = 1$.

Let χ be the tangent measure at infinity to ζ and fix $\varphi \in C_c(B_2(0))$ such that $\varphi = 1$ on $B_1(0)$ and $\varphi(x) \geq 0$ for every x.

Since ζ is not flat, for Theorem 2.6 χ cannot be flat, then, for Theorem 2.5, we have that $F(\chi) > \varepsilon$.

Moreover we can note that $\chi \in \mathrm{Tan}_m(\mu, x)$, indeed we know that there exist two sequence $\rho_i \downarrow 0$ and $\sigma_j \uparrow \infty$ such that

$$\frac{\mu_{x,\rho_i}}{\rho_i^m} \overset{*}{\rightharpoonup} \zeta \quad \text{and} \quad \frac{\zeta_{0,\sigma_j}}{\sigma_j^m} \overset{*}{\rightharpoonup} \chi,$$

then

$$\frac{\mu_{x,\rho_i\sigma_j}}{(\rho_i\sigma_j)^m} \xrightarrow[i\to\infty]{*} \frac{\zeta_{0,\sigma_j}}{\sigma_j^m}.$$

It follows that $\sigma_j^{-m}\zeta_{0,\sigma_j} \in \mathrm{Tan}_m(\mu, x)$ for every j, but in the proof of Theorem 2.2 we proved that $\mathrm{Tan}_m(\mu, x)$ is weakly* closed, therefore $\chi \in \mathrm{Tan}_m(\mu, x)$. Then we can fix two sequences of radii, $r_k \downarrow 0$ and $s_k \downarrow 0$, such that

$$\frac{\mu_{x,r_k}}{r_k^m} \overset{*}{\rightharpoonup} \nu \quad \text{and} \quad \frac{\mu_{x,s_k}}{s_k^m} \overset{*}{\rightharpoonup} \chi$$

and such that $s_k < r_k$ for every k.

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $f(r) := F(r^{-m}\mu_{0,r})$.

Since ν is flat, we have that

$$\lim_{k\to\infty} f(r_k) = F(\nu) = 0;$$

then, for r_k sufficiently small, $f(r_k) < \varepsilon$.

Focusing on χ instead of ν:

$$\lim_{k\to\infty} f(s_k) = F(\chi) > \varepsilon;$$

then, for s_k sufficiently small, $f(s_k) > \varepsilon$.

Note that for Lemma 2.2 f is continuous, then for every k we can fix $\sigma_k \in [s_k, r_k]$ such that $f(\sigma_k) = \varepsilon$ and $f(r) \leq \varepsilon$ for $r \in [\sigma_k, r_k]$.

We know that if a sequence of measures is locally uniformly bounded, then there exists a subsequence which converges in the weakly* topology. Since $\theta^{*m}(\mu, x) < \infty$, then we have that for every $\rho > 0$ the set of numbers

$$\frac{\mu(B_{\rho r}(x))}{r^{-m}} = \frac{\mu_{x,r}(B_\rho(0))}{r^{-m}}$$

is uniformly bounded, therefore the family of measures $\{r^{-m}\mu_{x,r} : r \in (0,1]\}$ is locally uniformly bounded, then for any sequence of radii there exists a subsequence that converges in the weakly* topology.

Therefore we can assume that, up to a subsequence, $\sigma_k^{-m}\mu_{x,\sigma_k}$ converges in the weak* topology to a measure $\xi \in U^m(\mathbb{R}^n)$. We have that

$$F(\xi) = \lim_{k\to\infty} f(\sigma_k) = \varepsilon,$$

then ξ is not flat.

Now we prove that $r_k/\sigma_k \to \infty$. If it existed a sequence of indices $\{k_i\}$ and a constant $C \in [1, \infty)$ such that $r_{k_i}/\sigma_{k_i} \to C < \infty$, then we would have that

$$\nu \overset{*}{\leftharpoondown} \frac{\mu_{x,r_k}}{r_k^m} = \left(\frac{\sigma_k}{r_k}\right)^m \left(\frac{\mu_{x,\sigma_k}}{\sigma_k^m}\right)_{0,r_k/\sigma_k} \overset{*}{\rightharpoonup} C^{-m}\xi_{0,C},$$

hence, ξ would be flat.

Now we note that for every $R > 0$ we have $(R\sigma_k)^{-m}\mu_{0,R\sigma_k} \overset{*}{\rightharpoonup} R^{-m}\xi_{0,R}$, then

$$F(\xi_{0,R}) = \lim_{k\to\infty} f(R\sigma_k).$$

Let $R \geq 1$. Since $r_k/\sigma_k \to \infty$, there exists $\bar{k}(R)$ such that for any $k > \bar{k}(R)$ we have $R\sigma_k \in [\sigma_k, r_k]$, then

$$F(R^{-m}\xi_{0,R}) \leq \varepsilon$$

for every $R \geq 1$.

Let ψ be the tangent measure at infinity to ξ:

$$F(\psi) = \lim_{R\to\infty} F(R^{-m}\xi_{0,R}) \leq \varepsilon,$$

then, for Theorem 2.5, ψ is flat, hence ξ is flat too for Theorem 2.6. Here we reached a contradiction. □

Chapter 3

Uniqueness Theorem for tangent measures at infinity

In this chapter we prove Theorem 2.4, that is the uniqueness of tangent measures at infinity for uniform measures.

The first lemma states that the integral of a radial function on an uniform measure μ does not depend on μ. We will use this fact several times in this chapter and in the next ones.

Lemma 3.1. Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a Borel function, μ an m-uniform measure and $y \in \operatorname{supp}(\mu)$. Then

$$\int_{\mathbb{R}^n} \varphi(|x|) d\mu(x) = \int_{\mathbb{R}^n} \varphi(|x-y|) d\mu(x) = \int_{\mathbb{R}^m} \varphi(|z|) dL_m(z),$$

where L_m is the Lebesgue measure.

Proof. Since $\mu(B_r^n(0)) = \mu(B_r^n(y)) = \omega_m r^m = L_m(B_r^m(z))$, the identity is true if φ is piecewise constant; then we can argue by density to conclude that it is true for every Borel function φ. □

We indicate with μ_r the measure $r^{-m} e^{-|\cdot|^2} \mu_{0,r}$. Then, for every Borel function φ we have that

$$\int_{\mathbb{R}^n} \varphi(x) d\mu_r(x) = r^{-m} \int_{\mathbb{R}^n} e^{-|x|^2/r^2} \varphi\left(\frac{x}{r}\right) d\mu(x).$$

Let $\nu \in \operatorname{Tan}_m(\mu, \infty)$ and $r_i \uparrow \infty$ a sequence such that $r_i^{-m} \mu_{0,r_i} \stackrel{*}{\rightharpoonup} \nu$, then $\mu_{r_i} \stackrel{*}{\rightharpoonup} e^{-|\cdot|^2} \nu$. The uniqueness of the tangent measure at infinity is then equivalent to the existence of a unique limit of μ_r for $r \uparrow \infty$.

Let P be a polynomial and let $F_P(r) := \int_{\mathbb{R}^n} P(z) d\mu_r(z)$. With a density argument we will prove that the existence of a unique limit of μ_r for $r \uparrow \infty$ is equivalent to the existence of the limit

$$\lim_{r \to \infty} F_P(r). \tag{3.1}$$

For $\mu \in U^m(\mathbb{R}^n)$ and $s > 0$ we indicate with $I(s)$ the integral

$$I(s) := \int_{\mathbb{R}^n} e^{s|z|^2} d\mu(z).$$

Definition 3.1. Let $\mu \in U^m(\mathbb{R}^n)$, $k \in \mathbb{Z}_{>0}$, $u_1, ..., u_k \in \mathbb{R}^n$ and $s \in \mathbb{R}^+$. Then we define the moments $b^\mu_{k,s}(u_1, ..., u_k)$ as

$$b^\mu_{k,s}(u_1, ..., u_k) := \frac{(2s)^k}{k!} I(s)^{-1} \int_{\mathbb{R}^n} \langle z, u_1 \rangle ... \langle z, u_k \rangle e^{-s|z|^2} d\mu(z).$$

We will prove then that the existence of the limit (3.1) is equivalent to the existence of the limit

$$\lim_{s \downarrow 0} \frac{b^\mu_{N,s}}{s^{N/2}}. \tag{3.2}$$

We study now the existence of the limit (3.2). In order to do that we need a Taylor expansion for $b^\mu_{k,s}$, that we will reach using the estimates in the following lemmas.

Lemma 3.2. Let $\mu \in U^m(\mathbb{R}^n)$. Then there exists a constant $C(m)$ such that

$$|b^\mu_{k,s}(u_1, ..., u_k)| \le C(m) \frac{2^k k^{k/2}}{k!} s^{k/2} |u_1| ... |u_k|.$$

Proof. Since

$$b^\mu_{k,s}(u_1, ..., u_k) = \frac{(2s)^k}{k!} I(s)^{-1} \int_{\mathbb{R}^n} \langle z, u_1 \rangle ... \langle z, u_k \rangle e^{-s|z|^2} d\mu(z),$$

then

$$|b^\mu_{k,s}(u_1, ..., u_k)| \le |u_1| ... |u_k| \frac{(2s)^k}{k!} I(s)^{-1} \int_{\mathbb{R}^n} |z|^k e^{-s|z|^2} d\mu(z).$$

From Lemma 3.1 it follows that

$$I(s) := \int_{\mathbb{R}^n} e^{-s|z|^2} d\mu(x) = \int_{\mathbb{R}^m} e^{-s|z|^2} dL_m(x) =$$

$$= s^{-m/2} \int_{\mathbb{R}^m} e^{-|y|^2} dL_m(y) = \left(\frac{\pi}{s}\right)^{m/2},$$

then

$$|b^\mu_{k,s}(u_1, ..., u_k)| \le |u_1|...|u_k| \frac{2^k s^{k/2}}{\pi^m k!} s^{k/2+m/2} \int_{\mathbb{R}^n} |z|^k e^{-s|z|^2} d\mu(z). \tag{3.3}$$

Using Lemma 3.1 we conclude that

$$s^{k/2+m/2} \int_{\mathbb{R}^n} |z|^k e^{-s|z|^2} d\mu(z) = \int_{\mathbb{R}^n} |s^{1/2} z|^k e^{-|s^{1/2} z|^2} d[s^{m/2} \mu(z)] =$$

$$= \int_{\mathbb{R}^m} |s^{1/2} z|^k e^{-|s^{1/2} z|^2} d[s^{m/2} L_m(z)] = \int_{\mathbb{R}^m} |y|^k e^{-|y|^2} dL_m(y).$$

We know that there exists a dimensional constant $C_1(m)$ such that

$$\int_{\mathbb{R}^m} |y|^k e^{-|y|^2} dL_m(y) \le C_1(m) k^{k/2},$$

then combining this with (3.3) we conclude the proof. □

We indicate with $b^\mu_{k,s}(x^k)$ the number $b^\mu_{k,s}(x, x, ..., x)$.

Lemma 3.3. Let $\mu \in U^m(\mathbb{R}^n)$. Then there exists a constant $\bar{C}(m)$ such that for every $q \in \mathbb{N}$

$$\left| \sum_{k=1}^{2q} b^\mu_{k,s}(x^k) - \sum_{k=1}^{q} \frac{(s|x|^2)^k}{k!} \right| \le \bar{C}(m)(s|x|^2)^{q+1/2}$$

for every $x \in \operatorname{supp}(\mu)$.

Proof. If $|x| = 0$ the lemma is true.

Let $s|x|^2 \ge 1$, then

$$\left| \sum_{k=1}^{2q} b^\mu_{k,s}(x^k) - \sum_{k=1}^{q} \frac{(s|x|^2)^k}{k!} \right| \le \left| \sum_{k=1}^{2q} b^\mu_{k,s}(x^k) \right| + \left| \sum_{k=1}^{q} \frac{(s|x|^2)^k}{k!} \right| \le$$

$$\le C(m) \sum_{k=1}^{2q} \frac{2^k k^{k/2}}{k!} (s|x|^2)^{k/2} + (s|x|^2)^q \sum_{k=1}^{\infty} \frac{1}{k!} \le$$

$$\le C(m)(s|x|^2)^q \sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{k!} + e(s|x|^2)^q.$$

Since $k! \geq Ck^k e^{-k}$ from Stirling's Formula, we have that the series

$$\sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{k!}$$

converges, then

$$\left| \sum_{k=1}^{2q} b_{k,s}^{\mu}(x^k) - \sum_{k=1}^{q} \frac{(s|x|^2)^k}{k!} \right| \leq C_1(m)(s|x|^2)^q \leq C_1(m)(s|x|^2)^{q+1/2}.$$

Consider now the case $s|x|^2 \in (0,1)$.

Let us fix the convention that $b_{0,s}^{\mu}(x^0) := 1$. We prove that for every $s > 0$ and $x \in \operatorname{supp}(\mu)$ such that $s|x|^2 < 1$ we have that

$$\sum_{k=0}^{\infty} b_{k,s}^{\mu}(x^k) = e^{s|x|^2}. \tag{3.4}$$

From Lemma 3.2 it follows that

$$\sum_{k=1}^{\infty} |b_{k,s}^{\mu}(x^k)| \leq \sum_{k=1}^{\infty} C \frac{2^k k^{k/2}}{k!} (s|x^2|)^{k/2} \leq C \sum_{k=1}^{\infty} \frac{2^k k^{k/2}}{k!} \leq C_2,$$

since, as we saw in the previous case, that series converges. Then the series

$$\sum_{k=0}^{\infty} b_{k,s}^{\mu}(x^k)$$

is summable for $s|x|^2 < 1$, therefore

$$\sum_{k=0}^{\infty} b_{k,s}^{\mu}(x^k) = \lim_{q \to \infty} \sum_{k=0}^{q} I(s)^{-1} \int_{\mathbb{R}^n} \frac{(2s\langle z, x \rangle)^k}{k!} e^{-s|z|^2} d\mu(z).$$

Since $e^{-s(|\cdot|^2 + 2|\cdot||x|)} \in L^1(\mu)$ and

$$\left| \sum_{k=0}^{q} \frac{(2s\langle z, x \rangle)^k}{k!} e^{-s|z|^2} \right| \leq e^{-s|z|^2} \sum_{k=0}^{q} \frac{(2|z||x|)^k}{k!} \leq e^{-s(|z|^2 + 2|z||x|)},$$

then by the Dominated Convergence Theorem we conclude

$$\sum_{k=0}^{\infty} b_{k,s}^{\mu}(x^k) = I(s)^{-1} \int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \frac{(2s\langle z, x \rangle)^k}{k!} \right] e^{-s|z|^2} d\mu(z) =$$

$$= I(s)^{-1}\int_{\mathbb{R}^n} e^{2s\langle z,x\rangle - s|z|^2}d\mu(z) = I(s)^{-1}e^{s|x^2|}\int_{\mathbb{R}^n} e^{-s|z-x|^2}d\mu(z).$$

From Lemma 3.1 we obtain

$$\int_{\mathbb{R}^n} e^{-s|z-x|^2}d\mu(z) = \int_{\mathbb{R}^n} e^{-s|z|^2}d\mu(z) = I(s),$$

and this concludes the proof of (3.4).

Now we can compute the wanted estimate:

$$\left|\sum_{k=1}^{2q} b^{\mu}_{k,s}(x^k) - \sum_{k=1}^{q}\frac{(s|x|^2)^k}{k!}\right| \le$$

$$\le \left|\sum_{k=1}^{\infty} b^{\mu}_{k,s}(x^k) - \sum_{k=1}^{q}\frac{(s|x|^2)^k}{k!}\right| + \left|\sum_{k=1}^{\infty} b^{\mu}_{k,s}(x^k) - \sum_{k=1}^{2q} b^{\mu}_{k,s}(x^k)\right|.$$

We study the first addend:

$$\left|\sum_{k=1}^{\infty} b^{\mu}_{k,s}(x^k) - \sum_{k=1}^{q}\frac{(s|x|^2)^k}{k!}\right| = \left|e^{s|x|^2} - \sum_{k=1}^{q}\frac{(s|x|^2)^k}{k!}\right| = \sum_{k=q+1}^{\infty}\frac{(s|x|^2)^k}{k!} \le$$

$$\le (s|x|^2)^{q+1}\sum_{k=0}^{\infty}\frac{1}{k!} = e(s|x|^2)^{q+1} \le e(s|x|^2)^{q+1/2}.$$

We study the second addend:

$$\left|\sum_{k=1}^{\infty} b^{\mu}_{k,s}(x^k) - \sum_{k=1}^{2q} b^{\mu}_{k,s}(x^k)\right| \le$$

$$\le \sum_{k=2q+1}^{\infty} |b^{\mu}_{k,s}(x^k)| \le C(m)\sum_{k=2q+1}^{\infty}\frac{2^k k^{k/2}}{k!}s^{k/2}|x|^k \le$$

$$\le C(m)(s|x|^2)^{q+1/2}\sum_{k=1}^{\infty}\frac{2^k k^{k/2}}{k!} \le C_2(m)(s|x|^2)^{q+1/2}.$$

Then we conclude:

$$\left|\sum_{k=1}^{2q} b^{\mu}_{k,s}(x^k) - \sum_{k=1}^{q}\frac{(s|x|^2)^k}{k!}\right| \le (e + C_2(m))(s|x|^2)^{q+1/2}.$$

□

Theorem 3.1 (Taylor expansion)**.** Let $\mu \in U^m(\mathbb{R}^n)$ and $k \in \mathbb{Z}_{>0}$. Then there exist symmetric k-linear forms $b_k^{(j)}$, with $j \in \mathbb{Z}_{>0}$, such that for all $q \in \mathbb{Z}_{>0}$ and $x \in \operatorname{supp}(\mu)$ the following three conditions hold:

$$b_{k,s}^{\mu} = \sum_{j=1}^{q} \frac{s^j b_k^{(j)}}{j!} + o(s^q); \tag{3.5}$$

$$b_k^{(j)} = 0 \text{ whenever } j < k/2; \tag{3.6}$$

$$\sum_{k=1}^{2q} b_k^{(q)}(x^k) = |x|^{2q}. \tag{3.7}$$

In order to prove this theorem we need to introduce some notation.

We indicate with $\bigodot^k \mathbb{R}^n$ the vector space of symmetric k-tensor on $\mathbb{R}^n$.

Then $b_{k,s}^{\mu} \in \operatorname{Hom}(\bigodot^k \mathbb{R}^n, \mathbb{R})$, and the function $s \to b_{k,s}^{\mu}$ is a curve in $\operatorname{Hom}(\bigodot^k \mathbb{R}^n, \mathbb{R})$.

We define $X^{k,n} := \mathbb{R}^n \oplus \bigodot^2 \mathbb{R}^n \oplus ... \oplus \bigodot^k \mathbb{R}^n$, and P_j is the canonical projection of $X^{k,x}$ on $\bigodot^j \mathbb{R}^n$.

We indicate with $\langle \cdot, \cdot \rangle_k$ the unique scalar product on $\bigodot^k \mathbb{R}^n$ such that

$$\langle u_1 \odot ... \odot u_k, v_1 \odot ... \odot v_k \rangle_k = \frac{1}{k!} \sum_{\sigma \in G_k} \langle u_1, v_{\sigma(1)} \rangle ... \langle u_k, v_{\sigma(k)} \rangle ,$$

where G_k is the set of permutations of $\{1, 2, ..., k\}$.

Definition 3.2. Let $k, n \in \mathbb{Z}_{>0}$. Then we define on $X^{k,n}$ the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ as

$$\langle\!\langle u, v \rangle\!\rangle := \sum_{j=1}^{k} \frac{2^j \langle P_j(u), P_j(v) \rangle_j}{j!},$$

and we set $\|u\| := \langle\!\langle u, u \rangle\!\rangle^{1/2}$.

We indicate with $V^{\perp}$ the orthogonal subspace with respect of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Proof. Step 0. An overview on the proof of (3.5), that is the core of this proof.

We fix a q. Using the tensor notation, we can see the map

$$s \to b_s := \sum_{k=1}^{2q} b_{k,s}^{\mu} \in \operatorname{Hom}(X^{2q,n}, \mathbb{R})$$

as a curve of linear operators. Then, if $x \in \operatorname{supp}(\mu)$, Lemma 3.3 gives us the expansion

$$b_s(x + x^2 + ... + x^{2q}) = \sum_{k=1}^{q} \frac{s^k |x|^{2k}}{k!} + \|x\|^{2q+1} o(s^q),$$

which defines the function on a vector space V by linearity.

Our goal is then to find an analytic curve ω_s and a certain projection Q_s such that

$$b_s = \omega_s \circ Q_s + o(s^q).$$

Proving the analyticity of an extension of Q_s to $s = 0$ will conclude the proof.

Step 1. We prove now (3.5).

Let $q \in \mathbb{Z}_{>0}$, $X := X^{2q,n}$, and consider the curve

$$s \to b_s := \sum_{k=1}^{2q} b^{\mu}_{k,s} \in \operatorname{Hom}(X, \mathbb{R}), \quad \text{for } s > 0.$$

For every $k \in \mathbb{Z}_{>0}$ let $\hat{\omega}_{2k} \in \operatorname{Hom}(X, \mathbb{R})$ be such that $\hat{\omega}_{2k}(y) = 0$ for every $y \in \bigodot^j \mathbb{R}^n$ with $j \neq 2k$ and

$$\hat{\omega}_{2k}(x^{2k}) = \frac{|x|^{2k}}{k!}.$$

We observe that $\hat{\omega}_{2k} \in \bigodot^{2k} \mathbb{R}^n$ and it is given by

$$\hat{\omega}_{2k}(x_1, ..., x_{2k}) = \frac{1}{k!(2k)!} \sum_{\sigma \in G_{2k}} \langle x_{\sigma(1)}, x_{\sigma(2)} \rangle ... \langle x_{\sigma(2k-1)}, x_{\sigma(2k)} \rangle.$$

Now we define, for $s > 0$, $\omega_s \in \operatorname{Hom}(X, \mathbb{R})$ as

$$\omega_s := \sum_{k=1}^{q} s^k \hat{\omega}_{2k},$$

and V as the linear subspace of X generated by the elements of the form $x + x^2 + ... + x^{2q}$ for $x \in \operatorname{supp}(\mu)$.

Let $a_s(\cdot, \cdot)$ be the bilinear form on X defined by

$$a_s(u, v) := \left\langle\!\!\left\langle \sum_{k=1}^{2q} s^k P_k(u), v \right\rangle\!\!\right\rangle.$$

We note that it is a scalar product on X.

Define F_s as the subspace of X orthogonal to V with respect of the scalar product $a_s(\cdot,\cdot)$, that is

$$F_s := \left\{ u \in X : \left\langle\!\!\left\langle \sum_{k=1}^{2q} s^k P_k(u), v \right\rangle\!\!\right\rangle = 0 \ \ \forall v \in V \right\}.$$

Then we have that $V \oplus F_s = X$.

We define Q_s as the orthogonal projection on V with respect to the scalar product $a_s(\cdot,\cdot)$, then $Q_s : X \to X$ is the linear map such that it is the identity on V and it is 0 on F_s.

We note that

$$\|x + x^2 + ... + x^{2q}\|^2 = \sum_{j=1}^{2q} \frac{2^j}{j!} |x|^{2j},$$

then we have that, for $y \in V$, Lemma 3.3 can be written as

$$b_s(y) = \omega_s(y) + \|y\|^{1+c} o(s^q), \tag{3.8}$$

where $c = 2q$ if $|x| < 1$ and $c = 1/(2q)$ if $|x| \geq 1$.

Moreover we note that

$$b_s(u) = \sum_{k=1}^{2q} I(s)^{-1} \frac{2^k}{k!} s^k \int_{\mathbb{R}^n} \left\langle P_k u, v^k \right\rangle_k e^{-s|v|^2} d\mu(v) =$$

$$= I(s)^{-1} \int_{\mathbb{R}^n} \sum_{k=1}^{2q} \frac{2^k}{k!} \left\langle s^k P_k u, P_k(v + v^2 + ... + v^{2q}) \right\rangle_k e^{-s|v|^2} d\mu(v) =$$

$$= I(s)^{-1} \int_{\mathbb{R}^n} \left\langle\!\!\left\langle \sum_{k=1}^{2q} s^k P_k u, v + v^2 + ... + v^{2q} \right\rangle\!\!\right\rangle e^{-s|v|^2} d\mu(v) =$$

$$= I(s)^{-1} \int_{\mathbb{R}^n} a_s(u, v + v^2 + ... + v^{2q}) e^{-s|v|^2} d\mu(v).$$

Then we conclude that for $u \in F_s$:

$$b_s(u) = 0 = \omega_s(0) = \omega_s(Q_s(u)).$$

Therefore we can write (3.8) as

$$b_s = \omega_s \circ Q_s + \|Q_s\|^{1+c} o(s^q),$$

and since Q_s is a projection, $\|Q_s\| \leq 1$, then

$$b_s = \omega_s \circ Q_s + o(s^q).$$

We note that we can define b_s for $s = 0$ too, and since ω_s can be defined also for $s = 0$ and the curve $s \to \omega_s$ is analytic, we have that b_s is analytic at $s = 0$ if Q_s can be extended analytically to $s = 0$.

If b_s is analytic at $s = 0$ then its components, that are $b^{\mu}_{k,s}$, are analytic, and this concludes the proof.

We prove now that Q_s has an analytic extension at $s = 0$.

We know that

$$\left(\sum_{k=1}^{2q} s^k P_k\right) \circ \left(\sum_{j=1}^{2q} s^{-j} P_j\right) = \sum_{k,j=1}^{2q} s^{k-j} P_k \circ P_j = \sum_{k=1}^{2q} P_k = Id,$$

therefore the map $\sum_{j=1}^{2q} s^{-j} P_j$ is the inverse of the map $\sum_{k=1}^{2q} s^k P_k$. Then $x \in F_s$ if and only if

$$x \in \left[\sum_{j=1}^{2q} s^{-j} P_j\right] (V^{\perp}).$$

Moreover, we can decompose the linear space $V^{\perp}$ in $2q$ linear spaces which are pairwise orthogonal:

$$V_1 := V^{\perp} \cap \bigodot{}^1 \mathbb{R}^n,$$

$$V_2 := \left\{V^{\perp} \cap \left[\bigodot{}^1 \mathbb{R}^n \oplus \bigodot{}^2 \mathbb{R}^n\right]\right\} \cap V_1^{\perp},$$

$$V_k := \left\{V^{\perp} \cap \left[\bigoplus_{j \leq k} \left(\bigodot{}^j \mathbb{R}^n\right)\right]\right\} \cap \bigcap_{j<k} V_j^{\perp}.$$

Let $A_s : X \to X$ be a linear map such that A_s is the identity on V, and on V_k is given by $P_k + sP_{k-1} + ... + s^{k-1} P_1$.

We note that A_s maps V into V and $V^{\perp}$ into F_s, and that the curve $s \to A_s$ is analytic.

We prove that A_0 is invertible, and then we conclude proving that the map $\tilde{Q}_s := P_V \circ A_s^{-1}$ is an analytic extension of Q_s.

Let $w \in X$ such that $A_0(w) = 0$ and decompose it as $w = -v_0 + v_1 + ... + v_{2q}$, where $v_0 \in V$ and $v_1 \in V_i$ for $i = 1, 2, ..., 2q$. Assume by contradiction that there exists $k > 0$ such that $v_k \neq 0$ and choose the smallest k with that property.

Since $v_k \in \bigoplus_{j\leq k}\left(\bigodot^j \mathbb{R}^n\right)$, then

$$v_k = P_k(v_k) + \sum_{j=1}^{k-1} P_j(v_k).$$

On the other hand, we know that

$$0 = A_0(w) = -v_0 + \sum_{j=1}^{2q} P_j(v_i) \Rightarrow v_0 = P_k(v_k) \sum_{j=k+1}^{2q} P_j(v_j).$$

Then we obtain

$$\langle\!\langle v_k, v_0 \rangle\!\rangle = |P_k(v_k)|^2,$$

but $v_k \in V_k \subset V^\perp$, then $P_k(v_k) = 0$. Since $v_k \in \bigoplus_{j\leq k}(\bigodot^j \mathbb{R}^n)$ and we have that $P_k(v_k) = 0$, then $v_k \in \bigoplus_{j\leq k-1}\left(\bigodot^j \mathbb{R}^n\right)$, therefore

$$v_k \in \left\{ V^\perp \cap \left[\bigoplus_{j\leq k-1} \left(\bigodot{}^j \mathbb{R}^n \right) \right] \right\} \cap \bigcap_{j<k} V_j^\perp \subset$$

$$\subset \left\{ V^\perp \cap \left[\bigoplus_{j\leq k-1} \left(\bigodot{}^j \mathbb{R}^n \right) \right] \right\} \cap \bigcap_{j<k-1} V_j^\perp = V_{k-1}.$$

We know that $V_{k-1} \perp V_k$, then $v_k = 0$. This concludes the proof of invertibility of A_0.

Then A_s is analytic and invertible at 0; this implies that A_s is invertible in a neighborhood of 0 and the map $s \to A_s^{-1}$ is analytic.

Let $\tilde{Q}_s := P_V \circ A_s^{-1}$, where P_V is the orthogonal projection on V with respect of the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

We know that $\tilde{Q}_s$ is analytic in a neighborhood of 0, $\tilde{Q}_s$ is the identity on V, and, since for every $s > 0$ A_s^{-1} maps F_s into $V^\perp$, $\tilde{Q}_s$ is 0 on F_s.

Then we proved that $Q_s = \tilde{Q}_s$ for $s > 0$, therefore Q_s has an analytic extension at 0, and this concludes the proof of (3.5).

Step 2. We note that (3.6) is an immediate consequence of (3.5) and Lemma 3.2, since the lemma states that $b^\mu_{k,s}$, for $s \to 0$, goes to 0 faster than $s^{k/2}$, then its Taylor expansion can not go to 0 slower than $s^{k/2}$, then $b_k^{(j)} = 0$ whenever $j < k/2$.

Step 3. Now we prove (3.7), which follows from Lemma 3.3, (3.5), and (3.6).

Let $q \in \mathbb{Z}_{>0}$. From Lemma 3.3 we have that for every $x \in \operatorname{supp}(\mu)$ it holds

$$\left| \sum_{k=1}^{2q} b^{\mu}_{k,s}\left(x^k\right) - \sum_{k=1}^{q} \frac{\left(s|x|^2\right)^k}{k!} \right| \leq \bar{C}(m)\left(s|x|^2\right)^{q+1/2},$$

and from (3.5) we have that for every $x \in \mathbb{R}^n$ it holds

$$b^{\mu}_{k,s}(x^k) = \sum_{j=1}^{q} \frac{s^j b^{(j)}_k(x^k)}{j!} + o\left(s^q\right).$$

Then, since $b^{(j)}_k = 0$ if $j < k/2$,

$$o\left(s^q\right) = \left| \sum_{k=1}^{2q} \sum_{j=1}^{q} \frac{s^j b^{(j)}_k(x^k)}{j!} - \sum_{k=1}^{q} \frac{(s|x|^2)^k}{k!} \right| = \left| \sum_{j=1}^{q} \frac{s^j}{j!} \left(\sum_{k=1}^{2j} b^{(j)}_k(x^k) - |x|^{2j} \right) \right|.$$

If we fix $q = 1$ we find $\left| b^{(1)}_1(x) + b^{(1)}_2(x^2) - |x|^2 \right| = o(s)/s$, then

$$b^{(1)}_1(x) + b^{(1)}_2(x^2) = |x|^2.$$

By induction we have that for $j = 1, 2, ..., 2q$ and $x \in \operatorname{supp}(\mu)$

$$\sum_{k=1}^{2j} b^{(j)}_k\left(x^k\right) = |x|^{2j},$$

and this concludes the proof. □

We note that the existence of the limit (3.2) follows from (3.5) and (3.6); we do not use (3.7) now, but it will be useful in the next chapters.

Now we prove Theorem 2.4, which is the main result of this chapter.

Proof. Let $N \in \mathbb{Z}_{>0}$. We fix $q > N/2$, and we note that from (3.5) it follows that

$$\lim_{s \downarrow 0} \frac{b^{\mu}_{N,s}(x_1, ..., x_N)}{s^{N/2}} = \lim_{s \downarrow 0} \left[\sum_{j=1}^{q} \frac{s^{j-N/2} b^{(j)}_N(x_1, ..., x_N)}{j!} + o(s^q) \right] =$$

$$= \sum_{j=1}^{q} \lim_{s \downarrow 0} \frac{s^{j-N/2} b^{(j)}_N(x_1, ..., x_N)}{j!}.$$

For (3.6) the limit is 0 for $j < N/2$, and for $j > N/2$ it is 0 since $s^{j-N/2} \to 0$. Therefore we have that the limit (3.2) exists for every N: indeed it is 0 whenever N is odd and it is $b^{(N/2)}_N(x_1, ..., x_N)/(N/2)!$ whenever N is even.

Now we prove that since limit (3.2) exists, then limit (3.1) exists for every polynomial P.

Since $I(s) = (\pi/s)^{m/2}$, we have that

$$\frac{b^{\mu}_{N,s}(u_1, ..., u_N)}{s^{N/2}} = \frac{2^N}{N!\pi^{m/2}} s^{N/2+m/2} \int_{\mathbb{R}^n} \langle z, u_1\rangle ... \langle z, u_N\rangle e^{-s|z|^2} d\mu(z).$$

Let $r := s^{-1/2}$:

$$\frac{b^{\mu}_{N,s}(u_1, ..., u_N)}{s^{N/2}} = \frac{2^N}{N!\pi^{m/2}} r^{-m} \int_{\mathbb{R}^n} \langle r^{-1}z, u_1\rangle ... \langle r^{-1}z, u_N\rangle e^{-|z|^2/r^2} d\mu(z) =$$

$$= \frac{2^N}{N!\pi^{m/2}} \int_{\mathbb{R}^n} \langle z, u_1\rangle ... \langle z, u_N\rangle d\mu_r(z).$$

Then we conclude that the limit (3.1) exists for every polynomial of the form $\langle z, u_1\rangle ... \langle z, u_N\rangle$.

Let $\nu^1, \nu^2 \in \mathrm{Tan}_m(\mu, \infty)$ and $\{r_k\}_k, \{s_k\}_k$ two sequences such that $r_k \uparrow \infty$, $s_k \uparrow \infty$ and

$$\frac{\mu_{0,r_k}}{r_k^m} \overset{*}{\rightharpoonup} \nu^1, \quad \frac{\mu_{0,s_k}}{s_k^m} \overset{*}{\rightharpoonup} \nu^2.$$

We indicate with $\tilde{\nu}^1$ and $\tilde{\nu}^2$ the measures $e^{-|\cdot|^2}\nu^1$ and $e^{-|\cdot|^2}\nu^2$ respectively, then we have

$$\mu_{r_k} \overset{*}{\rightharpoonup} \tilde{\nu}^1, \quad \mu_{s_k} \overset{*}{\rightharpoonup} \tilde{\nu}^2.$$

From the definitions we gave we note that for every $j \in \mathbb{N}$ and $\varepsilon > 0$ there exists $M > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_M(0)} |z|^j d\mu_r(z) \leq \varepsilon,$$

then we can conclude that

$$\lim_{k\uparrow\infty} \int_{\mathbb{R}^n} \langle z, u\rangle^j d\mu_{r_k}(z) = \int_{\mathbb{R}^n} \langle z, u\rangle^j d\tilde{\nu}^1(z),$$

$$\lim_{k\uparrow\infty} \int_{\mathbb{R}^n} \langle z, u\rangle^j d\mu_{s_k}(z) = \int_{\mathbb{R}^n} \langle z, u\rangle^j d\tilde{\nu}^2(z).$$

Therefore, since we proved that the limit exists for $r \uparrow \infty$, we have that for every j

$$\int_{\mathbb{R}^n} \langle z, u\rangle^j d\tilde{\nu}^1(z) = \int_{\mathbb{R}^n} \langle z, u\rangle^j d\tilde{\nu}^2(z).$$

Then for every polynomial P in n variables we conclude

$$\int_{\mathbb{R}^n} e^{-|z|^2} P(z) d\nu^1(z) = \int_{\mathbb{R}^n} e^{-|z|^2} P(z) d\nu^2(z),$$

and using the Taylor expansion for $e^{-a|z|^2}$ we obtain the following equation for every $a \geq 0$:

$$\int_{\mathbb{R}^n} e^{-(1+a)|z|^2} P(z) d\nu^1(z) = \int_{\mathbb{R}^n} e^{-(1+a)|z|^2} P(z) d\nu^2(z).$$

We prove now, with a density argument, that

$$\int_{\mathbb{R}^n} \varphi(z) d\nu^1(z) = \int_{\mathbb{R}^n} \varphi(z) d\nu^2(z) \tag{3.9}$$

for every $\varphi \in C_c(\mathbb{R}^n)$, which concludes the proof.

Let B be the vector space generated by functions of the form

$$b + e^{-(1+a)|z|^2} P(z)$$

where $a \geq 0$, $b \in \mathbb{R}$ and P is a polynomial.

In order to prove (3.9) we show that for every $\psi \in C_c(\mathbb{R}^n)$ there exists a sequence $\{\psi_i\}_i \subset B$ which converges uniformly to ψ.

We fix $\psi \in C_c(\mathbb{R}^n)$ and let $\mathbb{S}^n$ be the usual one-point compactification of $\mathbb{R}^n$. We denote with $\tilde{\psi} \in C_c(\mathbb{S}^n)$ the unique continuous extension of ψ, and we note that for every $\chi \in B$ there exists a unique continuous extension $\tilde{\chi} \in C(\mathbb{S}^n)$, then we indicate with $\tilde{B}$ the vector space of such extensions. $\tilde{B}$ is an algebra of continuous functions on a compact set, it separates the points and it vanishes at no point, then we conclude, using the Stone-Weierstrass Theorem, that there exists a sequence $\left\{\tilde{\psi}_i\right\} \subset \tilde{B}$ which converges uniformly to $\tilde{\psi}$. Now, the corresponding sequence $\{\psi_i\} \subset B$ converges uniformly to ψ.

We conclude now the proof of the theorem using this property: let $\varphi \in C_c(\mathbb{R}^n)$ and choose a sequence $\{\psi_i\} \in B$ which converges uniformly to $\psi := e^{|\cdot|^2}\varphi$. Moreover we note that if $\chi \in B$ then $e^{-|\cdot|^2}\chi$ is a sum of functions of the form $e^{-(1+a)|\cdot|^2}P(\cdot)$, then we can conclude that

$$\int_{\mathbb{R}^n} e^{-|z|^2} \psi_i(z) d\nu^1(z) = \int_{\mathbb{R}^n} e^{-|z|^2} \psi_i(z) d\nu^2(z)$$

for our previous computation. Since $\{\psi_i\}$ is uniformly bounded, we let $i \uparrow \infty$ and we obtain (3.9), which concludes the proof. $\square$

Chapter 4

Flatness Criterion for conical measures

In this chapter we prove Theorem 2.5. In order to do that we introduce conical measures and we prove that if μ is uniform and λ is its tangent at infinity then λ is a conical measure. After that we prove that a stronger version of Theorem 2.5 holds for every conical and uniform measure.

Definition 4.1. A measure λ is called a conical measure if for every $\rho > 0$ it holds

$$\lambda_{0,\rho} = \rho^m \lambda.$$

We see that the conical property of the tangent measure at infinity is an immediate consequence of the uniqueness of tangent measure at infinity.

Corollary 4.1. Let $\mu \in U^m(\mathbb{R}^n)$ and $\lambda \in U^m(\mathbb{R}^n)$ be its tangent measure at infinity. Then λ is a conical measure and it holds that if $x \in \operatorname{supp}(\lambda)$ then $\rho x \in \operatorname{supp}(\lambda)$ for every $\rho > 0$.

Proof. Let $r_i \uparrow \infty$ be a sequence of radii such that $r_i^{-m}\mu_{0,r_i} \overset{*}{\rightharpoonup} \lambda$ and let $\rho > 0$. Then

$$\frac{\mu_{0,\rho r_i}}{(\rho r_i)^m} \overset{*}{\rightharpoonup} \frac{\lambda_{0,\rho}}{\rho^m},$$

therefore $\rho^{-m}\lambda_{0,\rho} \in \operatorname{Tan}_m(\mu, \infty) = \{\lambda\}$. This means that the conical property holds for the tangent measure at infinity of a uniform measure.

Now let $x \in \operatorname{supp}(\lambda)$. Since $\lambda_{0,\rho} = \rho^m\lambda$, then

$$\lambda(B_r(\rho x)) = \rho^m \lambda(B_{r/\rho}(x)) > 0$$

for every $r > 0$. This means that $\rho x \in \operatorname{supp}(\lambda)$, and this concludes the proof. $\square$

The aim of this chapter is then to prove the following theorem, and Theorem 2.5 follows trivially from that and Corollary 4.1.

Theorem 4.1. Let $\lambda \in U^m(\mathbb{R}^n)$ be a conical measure. Then:

- if $m \leq 2$ then λ is flat;
- if $m \geq 3$ then there exists a constant $\varepsilon > 0$ depending only on m and n such that if

$$\min_{V \in G(m,n)} \int_{B_1(0)} [\mathrm{dist}(x,V)]^2 \, d\lambda(x) \leq \varepsilon$$

  then λ is flat.

In order to prove this theorem we need to study the behaviour of the moments $b^\lambda_{k,s}$ when λ is uniform and conical.

Lemma 4.1. Let $\lambda \in U^m(\mathbb{R}^n)$ be conical. Then:

1. $b^\lambda_{2k-1,s} = 0$ and $b^\lambda_{2k,s} = [(k)!]^{-1} s^k b^{\lambda,(k)}_{2k}$, then only one term of the Taylor expansion of $b^\lambda_{2k,s}$ is different from 0;
2. $\mathrm{supp}(\lambda) \subset \{x \in \mathbb{R}^n : b^{\lambda,(k)}_{2k}(x^{2k}) = |x|^{2k}\}$;
3. for every $u \in \mathrm{supp}(\lambda)$, every $w \in \mathbb{R}^m$ such that $|w| = |u|$ and every function $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ with $\varphi(|z|, \langle z,u\rangle) \in L^1(\mathbb{R}^n, \lambda)$ and $\varphi(|x|, \langle x,w\rangle) \in L^1(\mathbb{R}^m)$ it holds

$$\int_{\mathbb{R}^n} \varphi(|z|, \langle z,u\rangle) \, d\lambda(z) = \int_{\mathbb{R}^m} \varphi(|x|, \langle x,w\rangle) \, dL_m(x).$$

Proof. Step 1. We prove the first statement.

Let $x \in \mathrm{supp}(\lambda)$. From a change of variables $w = s^{1/2} z$ and the conical property $\lambda_{0,s^{1/2}} = s^{m/2}\lambda$ follows that

$$b^\lambda_{j,s}(x^j) = \frac{(2s)^j}{j!} I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle x,z\rangle^j d\lambda(z) =$$

$$= \frac{(2s)^j}{j!} I(s)^{-1} s^{-j/2-m/2} \int_{\mathbb{R}^n} e^{-|w|^2} \langle x,w\rangle^j d\lambda(w) =$$

$$= \frac{2^j s^{j/2}}{\pi^{m/2} j!} \int_{\mathbb{R}^n} e^{-|w|^2} \langle x,w\rangle^j d\lambda(w).$$

Then, from the Taylor expansion (3.5) follows that for every $x \in \operatorname{supp}(\lambda)$ if j is odd then we have $b^{\lambda}_{j,s}(x^j) = 0$, if j is even then we have

$$b^{\lambda}_{j,s}\left(x^j\right) = \frac{s^{j/2}}{(j/2)!} b_j^{\lambda,(j/2)}(x^j),$$

with

$$b_j^{\lambda,(j/2)}\left(x^j\right) = \frac{2^j (j/2)!}{\pi^{m/2} j!} \int_{\mathbb{R}^n} e^{-|w|^2} \langle x, w\rangle^j d\lambda(w).$$

Since we can determine a symmetric j-linear form from its values on the elements of the form x^j, we conclude that if j is odd then $b^{\lambda}_{j,s} = 0$, and if j is even then $b^{\lambda}_{j,s} = [(j/2)!]^{-1} s^{j/2} b_j^{\lambda,(j/2)}$.

Step 2. Now we prove the second statement.

From the first statement and the Taylor expansion of $b^{\lambda}_{j,s}$ we have that $b_j^{\lambda,(k)} = 0$ if $j \neq 2k$, and from (3.7) follows that for every $x \in \operatorname{supp}(\lambda)$ we obtain

$$b_{2k}^{\lambda,(k)}\left(x^{2k}\right) = |x|^{2k},$$

which concludes the proof of the second statement.

Step 3. In this step we prove the third statement.

From the first and the second statements follows that for every $s > 0$, for every $u \in \operatorname{supp}(\lambda)$ and for every $k \in \mathbb{Z}_{>0}$ we can compute:

$$\int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, u\rangle^{2k-1} d\lambda(z) = \frac{(2k-1)!}{(2s)^{2k-1}} I(s) b^{\lambda}_{2k-1,s}\left(u^{2k-1}\right) = 0,$$

and

$$\int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, u\rangle^{2k} d\lambda(z) = \left(\frac{\pi}{s}\right)^{m/2} \frac{(2k)!}{2^{2k} s^{2k}} b^{\lambda}_{2k,s}\left(u^{2k}\right) =$$
$$= \left(\frac{\pi}{s}\right)^{m/2} \frac{(2k)!}{k! 2^{2k} s^k} b_{2k}^{\lambda,(k)}\left(u^{2k}\right) = \left(\frac{\pi}{s}\right)^{m/2} \frac{(2k)!}{k! 2^{2k} s^k} |u|^{2k}.$$

Let $e_1, \ldots, e_m$ be an orthonormal base of $\mathbb{R}^m$ and let $w := |u| e_1$. Then:

$$\int_{\mathbb{R}^m} e^{-s|x|^2} \langle x, w\rangle^{2k} dL_m(x) = |u|^{2k} \int_{\mathbb{R}^{m-1}} e^{-s|\xi|^2} dL_{m-1}(\xi) \int_{\mathbb{R}} e^{-s|t|^2} t^{2k} dL_1(t).$$

Integrating by parts the last integral we reach

$$\int_{\mathbb{R}^m} e^{-s|x|^2} \langle x, w\rangle^{2k} dL_m(x) = \left(\frac{\pi}{s}\right)^{m/2} \frac{(2k)!}{k! 2^{2k} s^k} |u|^{2k},$$

then

$$\int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, u\rangle^{j} d\lambda(z) = \int_{\mathbb{R}^m} e^{-s|x|^2} \langle x, w\rangle^{j} dL_m(x). \tag{4.1}$$

By the arbitrariness of the choice of the base $e_1, ..., e_m$ we conclude that (4.1) holds for every $w \in \mathbb{R}^m$ such that $|w| = |u|$.

Let B be the set of Borel functions $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ such that for every $u \in \text{supp}(\lambda)$ and $f \in \mathbb{R}^m$ such that $|w| = |u|$ the following holds:

$$\varphi(|z|, \langle z, u \rangle) \in L^1(\mathbb{R}^n, \lambda)$$

and

$$\int_{\mathbb{R}^n} \varphi(|z|, \langle z, u \rangle) d\lambda(z) = \int_{\mathbb{R}^m} \varphi(|x|, \langle x, w \rangle) dL_m(x).$$

We prove that B contains the set of functions that are continue and with compact support from $\mathbb{R}_{\geq 0} \times \mathbb{R}$ to $\mathbb{R}$.

From (4.1) we know that $B \supset \{e^{-sy_1^2} y_2^j : s > 0,\ j \in \mathbb{N}\}$, and by taking the derivatives in s of (4.1) we obtain that $B \supset \{e^{-sy_1^2} y_1^{2k} y_2^j : s > 0,\ k, j \in \mathbb{N}\}$.

Since B is a vector space, we have that B contains all the fuctions of the form

$$e^{-sy_1^2} y_1^{2k} y_2^j \left(\sum_{i=1}^{N} (-1)^i \frac{s^i y_2^{2i}}{i!} \right) \tag{4.2}$$

with $s > 0$, $k, j \in \mathbb{N}$ and $N \in \mathbb{Z}_{\geq} 0$.

Let $|w| = |u| < 1$. Then we have that for every $N \in \mathbb{Z}_{>0}$

$$\left| e^{-s|x|^2} |x|^{2k} \langle w, x \rangle \left(\sum_{i=1}^{N} (-1)^i \frac{s^i \langle w, x \rangle^{2i}}{i!} \right) \right| \leq$$

$$\leq e^{-s|x|^2} |x|^{2k+j} |w|^j e^{s|w|^2|x|^2} = e^{-s|x|^2(1-|w|^2)} |x|^{2k+j} |w|^j, \tag{4.3}$$

which is an integrable function, then we can apply the Dominated Convergence Theorem:

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^m} e^{-s|x|^2} |x|^{2k} \langle w, x \rangle \left(\sum_{i=1}^{N} (-1)^i \frac{s^i \langle w, x \rangle^{2i}}{i!} \right) dL_m(x) =$$

$$= \int_{\mathbb{R}^m} e^{-s\left(|x|^2 + \langle w, x \rangle^2\right)} |x|^{2k} \langle w, x \rangle^j dL_m(x). \tag{4.4}$$

We note that the function in (4.3) is radial, then it is integrable with respect to the measure $\lambda \in U^m(\mathbb{R}^n)$, and if we replace w with u we can apply the Dominated Convergence Theorem to obtain

$$\lim_{N \uparrow \infty} \int_{\mathbb{R}^n} e^{-s|z|^2} |z|^{2k} \langle u, z \rangle \left(\sum_{i=1}^{N} (-1)^i \frac{s^i \langle u, z \rangle^{2i}}{i!} \right) d\lambda(z) =$$

$$= \int_{\mathbb{R}^n} e^{-s\left(|z|^2+\langle u,z\rangle^2\right)}|z|^{2k}\,\langle u, z\rangle^j\, d\lambda(z). \tag{4.5}$$

Then the two limits (4.4) and (4.5) exist, and from (4.2) they are equal.

Now let $\hat{w} := cw$ and $\hat{u} := cu$ with $|w| = |u| < 1$. We prove that for every $c \in \mathbb{R}$ the integrals in (4.4) and (4.5) are equal with $\hat{w}$ and $\hat{u}$ in place of w and u, and this means that the functions of the form

$$e^{-s|y|^2}y_1^{2k}y_2^j$$

with positive s belong to B.

Indeed, using the conical property of λ and L_m we conclude that

$$\int_{\mathbb{R}^n} e^{-s\left(|z|^2+\langle \hat{u},z\rangle^2\right)}|z|^{2k}\,\langle \hat{u}, z\rangle^j\, d\lambda(z) =$$

$$= \int_{\mathbb{R}^n} e^{-s\left(|cz|^2/c^2+\langle u,cz\rangle^2\right)}\frac{|cz|^{2k}}{c^2k}\,\langle u, cz\rangle^j\, d\lambda(z) =$$

$$= \frac{1}{c^{2k+m}}\int_{\mathbb{R}^n} e^{-\tilde{s}\left(|z'|^2+\langle u,w\rangle^2\right)}|z'|^{2k}\,\langle u, z'\rangle^j\, d\lambda(z'),$$

where $\tilde{s} := s/c^2$. We can do the same computation for L_m to obtain that

$$\int_{\mathbb{R}^m} e^{-s\left(|x|^2+\langle \hat{w},x\rangle^2\right)}|x|^{2k}\,\langle \hat{w}, x\rangle^j\, dL_m(x) =$$

$$= \frac{1}{c^{2k+m}}\int_{\mathbb{R}^m} e^{-\tilde{s}\left(|x'|^2+\langle w,x'\rangle^2\right)}|x'|^{2k}\,\langle w, x'\rangle^j\, dL_m(x'),$$

and we conclude that the two integrals are equal because we just proved it for $|w| = |u| < 1$.

Then B contains any linear combination of functions of the form

$$e^{-s|y|^2}y_1^{2k}y_2^j$$

with $s > 0$.

Let $\varphi \in C_c(\mathbb{R}_{\geq 0} \times \mathbb{R})$ be a nonnegative function.

We defince C as the vector space generated by the functions of the form

$$a + e^{-s|y|^2}Q\left(y_1^2, y_2\right),$$

where $a \in \mathbb{R}$, $s > 0$ and Q are polynomials. Moreover we define X as the one-point compactification of $\mathbb{R}_{\geq 0} \times \mathbb{R}$.

Let $\psi(y_1, y_2) := e^{|y|^2}\varphi(y_1, y_2)$ and let $\tilde{\psi}$ be its extension in $C(X)$, that is $\tilde{\psi} = \psi$ on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ and $\tilde{\psi}(\infty) = 0$. Every function $f \in C$ has a unique continuous extension $\tilde{f} \in C(X)$: let $\tilde{C}$ be the set of such extensions.

The set $\tilde{C}$ is an algebra that separates the points and vanishes at no point, then we can apply the Stone-Weiertstrass Theorem, which gives us a sequence $\left\{\tilde{f}_i\right\} \subset \tilde{C}$ which converges uniformly to $\tilde{\psi}$. Let $\{f_i\} \subset C$ be its corresponding sequence: then the sequence $g_i(y_1, y_2) := e^{-|y|^2} f_i(y_1, y_2)$ converges uniformly to φ and $|g_i(y)| \leq Ce^{-|y|^2}$.

Since $\lambda \in U^m(\mathbb{R}^n)$ and $|g_i(y)| \leq Ce^{-|y|^2}$ we can apply the Dominated Convergence Theorem with respect to both measures L_m and λ and we obtain

$$\int_{\mathbb{R}^m} \varphi(|x|, \langle x, w\rangle) dL_m(x) = \lim_{i\uparrow\infty} \int_{\mathbb{R}^m} g_i(|x|, \langle x, w\rangle) dL_m(x) =$$

$$= \lim_{i\uparrow\infty} \int_{\mathbb{R}^n} g_i(|z|, \langle z, u\rangle) d\lambda(z) = \int_{\mathbb{R}^n} \varphi(|z|, \langle z, u\rangle) d\lambda(z).$$

Therefore $\varphi \in B$, and this concludes the proof. □

Now we focus on $b_2^{\lambda,(1)}$. The second statement of the previous lemma states that for every $x \in \text{supp}(\lambda)$ we have $b_2^{\lambda,(1)}(x^2) = |x|^2$. Then the symmetric bilinear form $b_2^{\lambda,(1)}$ is positive semidefinite, therefore we can fix an orthonormal base $e_1, ..., e_n$ which diagonalizes $b_2^{\lambda,(1)}$ and we can write

$$b_2^{\lambda,(1)}(x \odot y) = \alpha_1 \langle x, e_1\rangle + ... + \alpha_n \langle x, e_n\rangle$$

with $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n \geq 0$.

Lemma 4.2. Let $\lambda \in U^m(\mathbb{R}^n)$ be conical. Then

$$\text{tr}\left(b_2^{\lambda,(1)}\right) = \text{tr}\left(b_{2,1}^{\lambda}\right) = m.$$

Proof. From the first statement of the previous lemma it follows the first equality: $\text{tr}\left(b_2^{\lambda,(1)}\right) = \text{tr}\left(b_{2,1}^{\lambda}\right)$. Now we compute:

$$\text{tr}\left(b_{2,1}^{\lambda}\right) = \sum_{i=1}^{n} b_{2,1}^{\lambda}(e_i^2) = 2I(1)^{-1} \int_{\mathbb{R}^n} e^{-|z|^2} \sum_{i=1}^{n} \langle e_i, z\rangle^2 d\lambda(z) =$$

$$= 2I(1)^{-1} \int_{\mathbb{R}^n} e^{-|z|^2} |z|^2 d\lambda(z) = 2I(1)^{-1} \int_{\mathbb{R}^m} e^{-|x|^2} |x|^2 dL_m(x).$$

Integrating by parts we know that

$$\int_{\mathbb{R}^m} e^{-|x|^2} |x|^2 dL_m(x) = \frac{m}{2} I(1),$$

then we conclude

$$\text{tr}\left(b_{2,1}^{\lambda}\right) = m.$$

□

In order to prove Theorem 4.1 we need a last lemma.

Lemma 4.3. For every $\delta > 0$ there exists $\tilde{\varepsilon} > 0$ such that for every $\mu \in U^m(\mathbb{R}^n)$, if W is an m-dimensional linear plane such that

$$\int_{\mathbb{R}^n} e^{-|z|^2} [\mathrm{dist}(z, W)]^2 \, d\mu(z) \le \tilde{\varepsilon},$$

then for every $v \in W \cap \bar{B}_1(0)$ there exists $x \in \mathrm{supp}(\mu)$ such that $|x - v| \le \delta$.

Proof. We argue by contradiction. We negate the theorem: there exists $\delta > 0$ such that for every $\varepsilon > 0$ there exists a measure $\mu \in U^m(\mathbb{R}^n)$, an m-dimensional linear plane W and $x \in W \cap \bar{B}_1(0)$ such that

$$\int_{\mathbb{R}^n} e^{-|z|^2} [\mathrm{dist}(z, W)]^2 \, d\mu(z) \le \varepsilon,$$

and $B_\delta(x) \cap \mathrm{supp}(\mu) = \emptyset$.

Let $\varepsilon_k = 1/k$ for every $k \in \mathbb{Z}_{>0}$ and let μ_k, W_k, x_k be the corresponding measure, plane and point that satisfy those two conditions. We can fix an m-dimensional linear plane W and rotate all the measures μ_k in order to have $W_k = W$ for every k.

Then the following three conditions hold:

$$\lim_{k \uparrow \infty} \int_{\mathbb{R}^n} e^{-|z|^2} [\mathrm{dist}(z, W)]^2 \, d\mu_k(z) = 0;$$

$$x_k \in W \cap \bar{B}_1(0) \quad \text{for every } k \in \mathbb{Z}_{>0};$$

$$\mu_k \left(B_\delta(x_k)\right) = 0 \quad \text{for every } k \in \mathbb{Z}_{>0}.$$

Since $W \cap \bar{B}_1(0)$ is compact, there exists a subsequence $\{x_{k_j}\}_j$ which converges to $x \in W \cap \bar{B}_1(0)$. Moreover, since $\mu_k \in U^m(\mathbb{R}^n)$ for every k, the sequence $\{\mu_k\}$ is uniformly locally bounded, then we can assume that up to a subsequence $\mu_{k_j} \overset{*}{\rightharpoonup} \mu$. Then we have that $\mu \in U^m(\mathbb{R}^n)$, $\mathrm{supp}(\mu) \subset W$ and $x_k \notin \mathrm{supp}(\mu)$, but it is impossible, since the first two conditions imply that $\mu = H^m \llcorner V$, which is in contradiction whit the third one. □

Now we prove Theorem 4.1, which concludes the proof of Theorem 2.5.

Proof. Step 1. Trivial case and idea of the proof.

For $m = 0$ we have that $U^0(\mathbb{R}^n) = \{\delta_0\}$, where δ_0 is the Dirac mass concentrated at the origin, then in this case the proposition is true.

For $m \ge 1$ we consider the form $b_2^{(1)}$ and we fix a base $e_1, ..., e_n$ that diagonalizes it as before. We claim that $\alpha_m \ge 1$, then for Lemma 4.2 we have $\alpha_1 = ... = \alpha_m = 1$ and $\alpha_{m+1} = ... = \alpha_n = 0$. Let V be the vector space generated by $e_1, ...e_m$, then $b_2^{(1)}(x^2) = |P_V(x)|^2$, and for Lemma 4.1:

$$\text{supp}(\lambda) \subset \left\{x : |x|^2 = |P_V(x)|^2\right\} = V,$$

then we would conclude $\lambda = H^m \llcorner V$, which is the thesis.

Then to conclude the proof it is enough to prove that $a_m \geq 1$.

Step 2. Case $m = 1, 2$.

Since $\lambda(B_1(0)) = \omega_m > 0$ and

$$\lambda(\{0\}) = \lim_{r\downarrow 0} \lambda(B_r(0)) = \lim_{r\downarrow 0} \omega_m r^m = 0,$$

we have that $\text{supp}(\lambda) \setminus \{0\} \neq \emptyset$. Let $x \in \text{supp}(\lambda) \setminus \{0\}$ and $z := x/|x|$. Since λ is conical then $z \in \text{supp}(\lambda)$, therefore we can apply Lemma 4.1 to reach $b_2^{(1)}(z^2) = |z|^2 = 1$. Then we have

$$\alpha_1 \geq \sup_{|z|=1} b_2^{(1)}(z^2) \geq 1.$$

We proved the case $m = 1$.

Let $m = 2$ and $w \in \mathbb{R}^m$ such that $|w| = |z| = 1$. We consider the function $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}$ given by

$$\varphi(y_1, y_2) := \chi_{\{|y_2|\leq 1\}}.$$

From Lemma 4.1 follows that

$$\lambda(\{y \in \mathbb{R}^n : |\langle y, z\rangle| \leq 1\}) = \int_{\mathbb{R}^n} \chi_{|\langle y,z\rangle|\leq 1} d\lambda(z) =$$

$$= \int_{\mathbb{R}^2} \chi_{|\langle x,w\rangle|\leq 1} dL_2(x) = \infty.$$

Then there exists a sequence $\{z_j'\} \subset \text{supp}(\lambda)$ such that for every j

$$|\langle z_j', z\rangle| \leq 1$$

and

$$\lim_{j\uparrow\infty} |z_j'| = \infty.$$

Let $y_j := z_j'/|z_j'|$. Up to a subsequence we have that $\{y_j\}$ converges to a $y \in \mathbb{R}^n$ with $|y| = 1$, moreover

$$|\langle y, z\rangle| = \lim_{j\uparrow\infty} \frac{|\langle z_j', z\rangle}{|z_j'|} = 0.$$

Since $z'_j \in \text{supp}(\lambda)$, from the conical property we know that $y_j \in \text{supp}(\lambda)$, then we can apply the second statement of Lemma 4.1 and we have

$$b_2^{(1)}(y_j^2) = |y_j|^2 = 1,$$

and passing to the limit in j we have $b_2^{(1)}(y^2) = 1$.

Then we have found a vector y that has norm 1, is orthogonal to z and is such that $b_2^{(1)}(y^2) = b_2^{(1)}(z^2) = 1$. This implies that $\alpha_2 \geq 1$, which is the desired conclusion.

Step 3. Case $m \geq 3$.

Let W be an m-dimensional linear plane. We fix an orthonormal base $\bar{e}_1, ..., \bar{e}_{n-m}$ of $W^\perp$ and we compute:

$$\text{tr}\left(b_2^{(1)} \llcorner W^\perp\right) = \text{tr}\left(b_{2,1} \llcorner W^\perp\right) = \sum_{i=1}^{n-m} b_{2,1}(\bar{e}_i^2) =$$

$$= 2I(1)^{-1} \int_{\mathbb{R}^n} e^{-|z|^2} \sum_{i=1}^{n-m} \langle z, \bar{e}_i \rangle^2 \, d\lambda(z) = 2\pi^{-m/2} \int_{\mathbb{R}^n} e^{-|z|^2} \left[\text{dist}(z, W)\right]^2 d\lambda(z).$$

Let V be the m-dimensional linear plane generated by $e_1, ..., e_m$. Since $V^\perp$ is the $n-m$-dimensional linear plane spanned by the eigenvectors corresponding to the smallest eigenvalues of $b_2^{(1)}$, we have that

$$\text{tr}\left(b_2^{(1)} \llcorner V^\perp\right) = \min_{W \in G(m,n)} \text{tr}(b_2^{(1)} \llcorner W^\perp).$$

Then we conclude

$$\int_{\mathbb{R}^n} e^{-|z|^2} [\text{dist}(z, V)]^2 d\lambda(z) = \min_{W \in G(m,n)} \int_{\mathbb{R}^n} e^{-|z|^2} [\text{dist}(z, W)]^2 d\lambda(z).$$

Now we take $\delta > 0$, that we will fix later, and we apply Lemma 4.3 to $\mu = \lambda$, $W = V$ and $v = e_m$: then there exists $\tilde{\varepsilon} := \tilde{\varepsilon}(\delta)$ such that if

$$\tilde{\varepsilon} \geq \int_{\mathbb{R}^n} e^{-|z|^2} [\text{dist}(z, V)]^2 d\lambda(z) = \min_{W \in G(m,n)} \int_{\mathbb{R}^n} e^{-|z|^2} [\text{dist}(z, W)]^2 d\lambda(z),$$

then there exists $x \in \text{supp}(\lambda)$ such that $|x - e_m| \leq \delta$. Since $\text{tr}\left(b_2^{(1)}\right) = m$ we have that $\alpha_m \leq 1$ and for every $i \leq m-1$ we have

$$\alpha_i + (m-1)\alpha_m \leq \text{tr}\left(b_2^{(1)}\right) = m \Rightarrow \alpha_i - 1 \leq (m-1)(1 - \alpha_m).$$

Since $x \in \mathrm{supp}(\lambda)$ then

$$\sum_{i=1}^{n} \alpha_i \langle x, e_i \rangle^2 = b_2^{(1)}(x^2) = |x|^2 = \sum_{i=1}^{n} \langle x, e_i \rangle^2 ,$$

therefore we can compute

$$0 = \sum_{i=1}^{n} (\alpha_i - 1) \langle x, e_i \rangle^2 \leq \sum_{i=1}^{m} (\alpha_i - 1) \langle x, e_i \rangle^2 \leq$$

$$\leq (m-1)(1-\alpha_m) \sum_{i=1}^{m-1} \langle x, e_i \rangle^2 + (\alpha_m - 1) \langle x, e_m \rangle^2 =$$

$$= (m-1)(1-\alpha_m) \sum_{i=1}^{m-1} \langle x - e_m, e_i \rangle^2 - (1-\alpha_m) \left(\langle e_m, e_m \rangle + \langle x - e_m, e_m \rangle \right)^2 \leq$$

$$\leq (1-\alpha_m) \left((m-1) \sum_{i=1}^{m-1} |x - e_m|^2 - (1 - |x - e_m|)^2 \right) \leq$$

$$\leq (1-\alpha_m) \left((m-1)^2 \delta^2 - (1-\delta)^2 \right) = (1-\alpha_m) \left(\left(m^2 - 2m \right) \delta^2 + 2\delta - 1 \right) =$$

$$= (1-\alpha_m) \left(m^2 - 2m \right) \left(\delta - \frac{1}{2m} \right) \left(\delta + \frac{1}{m-2} \right) .$$

Then we can choose $\delta \in (0, 1/(2m))$ and we obtain that there exists $\tilde{\varepsilon} > 0$ such that if

$$\min_{W \in G(m,n)} \int_{\mathbb{R}^n} e^{-|z|^2} [\mathrm{dist}(z, W)]^2 d\lambda(z) \leq \tilde{\varepsilon},$$

then $\alpha_m \geq 1$, therefore $\alpha_m = 1$, that means that the measure λ is flat.

We prove now that there exists a constant $\tilde{c} > 0$ such that

$$\int_{\mathbb{R}^n} e^{-|z|^2} [\mathrm{dist}(z, W)]^2 d\lambda(z) \leq \tilde{c} \int_{B_1(0)} [\mathrm{dist}(z, W)]^2 d\lambda(z). \tag{4.6}$$

This concludes the proof, because if we choose $\varepsilon = \tilde{\varepsilon}/\tilde{c}$, we have that

$$\int_{B_1(0)} [\mathrm{dist}(z, W)]^2 d\lambda(z) \leq \varepsilon \Rightarrow$$

$$\Rightarrow \int_{\mathbb{R}^n} e^{-|z|^2} [\mathrm{dist}(z, W)]^2 d\lambda(z) \leq \tilde{c} \int_{B_1(0)} [\mathrm{dist}(z, W)]^2 d\lambda(z) \leq \tilde{\varepsilon},$$

and for what we just proved, this means that λ is flat.

The inequality (4.6) follows from the conical property of λ: let

$$J(r) := \int_{B_r(0)} [\text{dist}(z, W)]^2 d\lambda(z).$$

We compute $J(r)$:

$$J(r) = \int_{B_r(0)} [\text{dist}(z, W)]^2 d\lambda(z) = \int_{B_r(0)} r^2 [\text{dist}(z/r, W)]^2 d\lambda(z) =$$

$$= r^{m+2} \int_{B_1(0)} [\text{dist}(y, W)]^2 d\lambda(y) = r^{m+2} J(1).$$

Now we search the constant $\tilde{c}$, knowing that e^{-t^2} is decreasing:

$$\int_{\mathbb{R}^n} e^{-|z|^2} [\text{dist}(z, W)]^2 d\lambda(z) \leq \sum_{j=0}^{\infty} \int_{B_{j+1}(0) \setminus B_j(0)} e^{-j^2} [\text{dist}(z, W)]^2 d\lambda(z) =$$

$$= \sum_{j=0}^{\infty} e^{-j^2} (J(j+1) - J(j)) = \sum_{j=0}^{\infty} e^{-j^2} \left((j+1)^{m+2} - j^{m+2} \right) J(1).$$

Using the ratio test we note that the last series converges. Let $\tilde{c}$ be its limit, then we found

$$\int_{\mathbb{R}^n} e^{-|z|^2} [\text{dist}(z, W)]^2 d\lambda(z) \leq \tilde{c} J(1),$$

and this concludes the proof. $\square$

Chapter 5

Relation between flatness at infinity and flatness

In this chapter we study the proof of Theorem 2.6, that is the relation between flatness at infinity and flatness of a measure. More precisely we prove the following theorem, which concludes the proof of Preiss' theorem.

Theorem 5.1. Let $\mu \in U^m(\mathbb{R}^n)$ and V be an m-dimensional linear plane. If $H^m \llcorner V$ is the tangent measure at infinity to μ, then $\mu = H^m \llcorner V$.

As in Chapter 1, we indicate with P_V the orthogonal projection on the m-dimensional linear plane V and with Q_V the orthogonal projection on $V^\perp$.

In order to prove Theorem 5.1, we prove that under those hypotheses

$$b_1^{\mu,(1)}(x) = |Q_V(x)|^2 \text{ for every } x \in \operatorname{supp}(\mu),$$

and that

$$b_1^{\mu,(1)} = 0,$$

then the support of μ is contained in the plane V; this, together with $\mu \in U^m(\mathbb{R}^n)$ implies that $\mu = H^m \llcorner V$.

In all this chapter we will omit μ in $b_k^{\mu,(j)}$ and $b_{k,s}^{\mu}$, and we will specify when they are about another measure.

Moreover for all this chapter we take the measure μ and the plane V as in Theorem 5.1.

If μ is uniform and flat at infinity, we can compute the moments $b_{2k}^{(k)}$ as follows.

Lemma 5.1. For every $x \in \mathbb{R}^n$ it holds

$$b_{2k}^{(k)}\left(x^{2k}\right) = k! b_{2,1}^{H^m \llcorner V}\left(x^{2k}\right) = b_{2k}^{H^m \llcorner V,(k)}\left(x^{2k}\right) = |P_V(x)|^{2k}.$$

Proof. In the proof of Theorem 2.4 we saw that

$$\frac{b_{2k}^{(k)}(u_1, ..., u_{2k})}{k!} = \lim_{s\downarrow 0} \frac{b_{2k,s}(u_1, ..., u_{2k})}{s^k}.$$

Moreover, with a change of variable $r := s^{-1/2}$ and the fact that μ is flat at infinity, we can compute this limit:

$$\lim_{s\downarrow 0} \frac{b_{2k,s}(u_1, ..., u_{2k})}{s^k} =$$

$$= \lim_{s\downarrow 0} \frac{(2s)^{2k}}{s^k(2k)!} I(s)^{-1} \int_{\mathbb{R}^n} \langle z, u_1\rangle ... \langle z, u_{2k}\rangle\, e^{-s|z|^2} d\mu(z) =$$

$$= \frac{2^{2k}}{(2k)!} I(1)^{-1} \lim_{s\downarrow 0} \int_{\mathbb{R}^n} \left\langle s^{1/2} z, u_1\right\rangle ... \left\langle s^{1/2} z, u_{2k}\right\rangle e^{-|s^{1/2}z|^2} s^{m/2} d\mu(z) =$$

$$= \frac{2^{2k}}{(2k)!} I(1)^{-1} \lim_{r\uparrow\infty} \int_{\mathbb{R}^n} \langle w, u_1\rangle ... \langle w, u_{2k}\rangle\, e^{-|w|^2} d\left[\frac{\mu_{0,r}}{r^m}\right](w) =$$

$$= \frac{2^{2k}}{(2k)!} I(1)^{-1} \int_{\mathbb{R}^n} \langle w, u_1\rangle ... \langle w, u_{2k}\rangle\, e^{-|w|^2} d\left[H^m \llcorner V\right](w) =$$

$$= b_{2k,1}^{H^m \llcorner V}(u_1, ..., u_{2k}).$$

Then we conclude

$$b_{2k}^{(k)}\left(x^{2k}\right) = k! b_{2k,1}^{H^m \llcorner V}\left(x^{2k}\right). \tag{5.1}$$

This concludes the proof of the first and the second equation, because if we took $\mu = H^m \llcorner V$ it would satisfy the hypotheses and we just proved that for every μ which satisfy the hypotheses (5.1) holds. In order to prove the last equation we must compute the second term of (5.1):

$$b_{2k}^{(k)}\left(x^{2k}\right) = k! b_{2k,1}^{H^m \llcorner V}\left(x^{2k}\right) =$$

$$= \frac{2^{2k} k!}{(2k)!} I(1)^{-1} \int_{\mathbb{R}^n} \langle z, x\rangle^{2k} e^{-|z|^2} d\left[H^m \llcorner V\right](z) =$$

$$= \frac{2^{2k} k! |P_V(x)|^{2k}}{(2k)!} I(1)^{-1} \int_{\mathbb{R}^n} \left\langle z, \frac{P_V(x)}{|P_V(x)|}\right\rangle^{2k} e^{-|z|^2} d\left[H^m \llcorner V\right](z).$$

We fix an orthonormal basis of $\mathbb{R}^n$ such that $e_1 := P_V(x)/|P_V(x)|$ and $\{e_1, ..., e_m\}$ is a basis of V:

$$b_{2k}^{(k)}\left(x^{2k}\right) = k! b_{2k,1}^{H^m \llcorner V}\left(x^{2k}\right) =$$

$$= \frac{2^{2k}k!|P_V(x)|^{2k}}{(2k)!} I(1)^{-1} \int_{\mathbb{R}^n} \langle z, e_1 \rangle^{2k} e^{-|z|^2} d\left[H^m \llcorner V\right](z) =$$

$$= \frac{2^{2k}k!|P_V(x)|^{2k}}{(2k)!} I(1)^{-1} \int_{\mathbb{R}^m} z_1^{2k} e^{-z_1^2} e^{-z_2^2 - \ldots - z_m^2} dL_m(z) =$$

$$= \frac{2^{2k}k!|P_V(x)|^{2k}}{(2k)!} I(1)^{-1} \int_{\mathbb{R}^{m-1}} e^{-|w|^2} dL_{m-1}(w) \int_{\mathbb{R}} t^{2k} e^{-t^2} dL_1(t) =$$

$$= \frac{2^{2k}k!|P_V(x)|^{2k}}{\pi^{m/2}(2k)!} \pi^{(m-1)/2} \left(2k - \frac{1}{2}\right) \cdots \frac{3}{2} \cdot \frac{1}{2} \pi^{1/2} = |P_V(x)|^{2k}.$$

Then we just proved the last equation. □

Now we study the moments of the form $b_{2k-1}^{(k)}$.

Lemma 5.2. For every $k \in \mathbb{Z}_{>0}$ it holds

$$b_{2k-1}^{(k)} \llcorner V = 0.$$

Proof. The form $b_{2k-1}^{(k)}$ is symmetric, then to prove that it is 0 on V it suffices to show that $b_{2k-1}^{(k)}\left(y^{2k-1}\right) = 0$ for every $y \in V$.

Let $y \in V \setminus \{0\}$. Since $r^{-m}\mu_{0,r} \overset{*}{\rightharpoonup} H^m \llcorner V$, there exists a sequence $\{x_j\} \subset \mathrm{supp}(\mu)$ such that $|x_j| \to \infty$ and

$$\frac{x_j}{|x_j|} \to \frac{y}{|y|}.$$

Then

$$b_{2k-1}^{(k)}\left(y^{2k-1}\right) = |y|^{2k-1} \lim_{j \uparrow \infty} \frac{b_{2k-1}^{(k)}\left(x_j^{2k-1}\right)}{|x_j|^{2k-1}}.$$

Fixing $q = k$ in (3.7) we have that

$$b_{2k-1}^{(k)}\left(x_j^{2k-1}\right) = |x|^{2k} - b_{2k}^{(k)}\left(x_j^{2k}\right) - \sum_{i=1}^{2k-2} b_i^{(k)}\left(x_j^i\right) =$$

$$= |x|^{2k} - |P_V(x)|^{2k} - \sum_{i=1}^{2k-2} b_i^{(k)}\left(x_j^i\right) \geq - \sum_{i=1}^{2k-2} b_i^{(k)}\left(x_j^i\right),$$

where the last sum is 0 for $k = 1$. Then we have that

$$b_{2k-1}^{(k)}\left(y^{2k-1}\right) \geq -|y|^{2k-1} \lim_{j \uparrow \infty} \frac{1}{|x_j|^{2k-1}} \sum_{i=1}^{2k-2} b_i^{(k)}\left(x_j^i\right).$$

Since $b_i^{(j)}$ are symmetric i-linear forms, there exist two constants C_1, C_2 such that

$$b_i^{(j)}\left(x_j^i\right) \leq C_1|x_j|^i \leq C_2\left(1+|x_j|^{2k-2}\right),$$

and using this we find

$$b_{2k-1}^{(k)}\left(y^{2k-1}\right) \geq -C_2 \lim_{j\uparrow\infty} \frac{1+|x_j|^{2k-2}}{|x_j|^{2k-1}} = 0.$$

Since $y \in V$, then $-y \in V$, and in the same way we find

$$-b_{2k-1}^{(k)}\left(y^{2k-1}\right) = b_{2k-1}^{(k)}\left((-y)^{2k-1}\right) \geq 0,$$

then $b_{2k-1}^{(k)}\left(y^{2k-1}\right) = 0$ for every $y \in V$. □

Let $k = 1$. We proved that $b_1^{(1)}(y) = 0$ for every $y \in V$. This means that there exists $w \in V^\perp$ such that

$$b_1^{(1)}(v) = \langle v, w\rangle$$

for every $v \in \mathbb{R}^n$.

Let $b = w/2$, then we have that $b \in V^\perp$ and

$$b_1^{(1)}(v) = 2\langle b, v\rangle .$$

The next lemma is an immediate consequence of the Lemma 5.3 and it gives us the first property of $b_1^{(1)}$ that we need in order to prove the theorem.

Lemma 5.3. For every $x \in \operatorname{supp}(\mu)$ the following two relations hold:

- $b_1^{(1)}(x) = |Q_V(x)|^2$;
- $|Q_V(x)| \leq \left\|b_1^{(1)}\right\|$.

Proof. Fixing $q = 1$ in (3.7) and using Lemma 5.1, we have that, for every $x \in \operatorname{supp}(\mu)$,

$$2\langle b, x\rangle + |P_V(x)|^2 = |x|^2 \Rightarrow 2\langle b, x\rangle = |Q_V(x)|^2.$$

The second statment follows from the first and the fact that $b \in V^\perp$:

$$2|b||Q_V(x)| \geq 2\langle b, Q_V(x)\rangle = 2\langle b, x\rangle = |Q_V(x)|^2.$$

□

This lemma means that the distance of $x \in \mathrm{supp}(\mu)$ from V is uniformly bounded by a constant, $\|b_1^{(1)}\|$. Then, if we prove that $b_1^{(1)} = 0$ then we have that $\mathrm{supp}(\mu) \subset V$, which concludes the proof.

We prove now that the distance of $v \in V$ from $\mathrm{supp}(\mu)$ is bounded by a constant r_0.

Lemma 5.4. There exists $r_0 > 0$ such that $\mathrm{dist}(v, \mathrm{supp}(\mu)) < r_0$ for every $v \in V$.

Proof. We argue by contradiction: assume that there exists $\{x_k\}_k \subset V$ such that

$$r_k := \mathrm{dist}\left(x_k, \mathrm{supp}(\mu)\right) \to \infty,$$

and for every k let $y_k \in \mathrm{supp}(\mu)$ such that $|y_k - x_k| = r_k$.

Let $z_k \in V$ be such that $|y_k - z_k| = \mathrm{dist}(y_k, V)$: for Lemma 5.3,

$$|y_k - z_k| = \mathrm{dist}(y_k, V) \le \left\| b_1^{(1)} \right\|.$$

Consider the sequence of measures $\{\mu^k\}_k$ where $\mu^k := r_k^{-m} \mu_{z_k, r_k}$. Those measures are uniformly locally bounded, since

$$\mu^k \left(B_r(x)\right) = \omega_m r^m,$$

then, up to a subsequence, $\mu^k \overset{*}{\rightharpoonup} \mu^\infty$. We note that $x \in \mathrm{supp}(\mu^k)$ if and only if $z_k + r_k x \in \mathrm{supp}(\mu)$.

We verify that $0 \in \mathrm{supp}(\mu^\infty)$:

$$\mathrm{dist}\left(0, \mathrm{supp}(\mu^k)\right) = \frac{1}{r^k} \mathrm{dist}\left(z_k, \mathrm{supp}(\mu)\right) \le$$

$$\le \frac{|z_k - y_k|}{r^k} \le \frac{\left\| b_1^{(1)} \right\|}{r^k} \to 0.$$

Therefore $0 \in \mathrm{supp}(\mu^\infty)$ and $\mu^\infty \left(B_r(x)\right) = \omega_m r^m$, then $\mu^\infty \in U^m\left(\mathbb{R}^n\right)$.

Moreover, if we fix $x \in \mathrm{supp}(\mu^k)$, we have that

$$|Q_V(x)| = \mathrm{dist}(x, V) = \frac{1}{r_k} \mathrm{dist}(z_k + r_k x, V) \le \frac{1}{r_k} \left| b_1^{(1)} \right|,$$

since $z_k + r_k x \in \mathrm{supp}(\mu)$. Then:

$$\mathrm{supp}(\mu^k) \subset \left\{ x \in \mathbb{R}^n \mid |Q_V(x)| \le \frac{1}{r_k} \left| b_1^{(1)} \right| \right\}.$$

Therefore we conclude that $\mathrm{supp}(\mu^\infty) \subset V$ and $\mu^\infty \in U^m(\mathbb{R}^n)$, then $\mu^\infty = H^m \llcorner V$.

Let $w_k := x_k - z_k$. We have that $w_k \in V$ and

$$\lim_{k\uparrow\infty} \frac{|w_k|}{r_k} \le \lim_{k\uparrow\infty} \frac{|x_k - y_k| + |y_k - z_k|}{r_k} = 1,$$

therefore, up to a subsequence, the sequence w_k/r_k converges to a limit $u \in V$.

Since $r_k = \mathrm{dist}(x_k, \mathrm{supp}(\mu))$, we have that $\mu(B_{r_k}(x_k)) = 0$, then:

$$0 = \mu^k(B_1(w_k/r_k)) \to \mu^\infty(B_1(u)),$$

and this contradicts $\mu^\infty = H^m \llcorner V$. □

The next step is to study the trace of $b_2^{(2)}$ proving that

$$\mathrm{tr}\left(b_2^{(2)}\right) \ge \frac{4}{m+2}\left|b_1^{(1)}\right|^2, \tag{5.2}$$

that is the longest part of the proof of Theorem 5.1.

After that, it suffices to prove that $\mathrm{tr}(b_2^{(2)}) = 0$ to conclude.

In order to prove (5.2) we split the trace in two parts:

$$\mathrm{tr}\left(b_2^{(2)}\right) = \mathrm{tr}\left(b_2^{(2)} \llcorner V^\perp\right) + \mathrm{tr}\left(b_2^{(2)} \llcorner V\right). \tag{5.3}$$

In the following lemma we compute the first addend of (5.3).

Lemma 5.5. The following formula holds:

$$\mathrm{tr}\left(b_2^{(2)} \llcorner V^\perp\right) = 2\left|b_1^{(1)}\right|^2. \tag{5.4}$$

Proof. From the Taylor expansion of $b_{2,s}$ we have that

$$b_{2,s} = s b_2^{(1)} + s^2 b_2^{(2)} + o(s^2),$$

then

$$\mathrm{tr}\left(b_2^{(2)} \llcorner V^\perp\right) = 2\lim_{s\downarrow 0} \frac{\mathrm{tr}\left(b_{2,s} \llcorner V^\perp\right) - s\,\mathrm{tr}\left(b_2^{(1)} \llcorner V^\perp\right)}{s^2}.$$

From Lemma 5.1 we have that $\mathrm{tr}\left(b_2^{(1)}\right) = 0$, therefore

$$\mathrm{tr}\left(b_2^{(2)} \llcorner V^\perp\right) = 2\lim_{s\downarrow 0} \frac{\mathrm{tr}\left(b_{2,s} \llcorner V^\perp\right)}{s^2}.$$

We study now $\operatorname{tr}\left(b_{2,s} \llcorner V^{\perp}\right)$. Let $e_1, ..., e_{n-m}$ be an orthonormal basis of $V^{\perp}$. Then:

$$\operatorname{tr}\left(b_{2,s} \llcorner V^{\perp}\right) = \sum_{i=1}^{n-m} 2s^2 I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, e_i \rangle^2 \, d\mu(z) =$$

$$= 2s^2 I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} \sum_{i=1}^{n-m} \langle z, e_i \rangle^2 \, d\mu(z) =$$

$$= 2s^2 I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} |Q_V(z)|^2 d\mu(z).$$

From Lemma 5.3 we have that for every $z \in \operatorname{supp}(\mu)$, $|Q_V(z)|^2 = b_1^{(1)}(z)$, and from the Taylor expansion of $b_{1,s}$ we know that

$$b_{1,s}(z) = s b_1^{(1)}(z) + o(s) = 2 \langle b, z \rangle + o(s).$$

Using these equations we find:

$$\operatorname{tr}\left(b_2^{(2)} \llcorner V^{\perp}\right) = 2 \lim_{s \downarrow 0} \frac{\operatorname{tr}\left(b_{2,s} \llcorner V^{\perp}\right)}{s^2} =$$

$$= 4 \lim_{s \downarrow 0} I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} |Q_V(z)|^2 d\mu(z) =$$

$$= 4 \lim_{s \downarrow 0} I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} b_1^{(1)}(z) d\mu(z) =$$

$$= 4 \lim_{s \downarrow 0} 2I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle b, z \rangle \, d\mu(z) = 4 \lim_{s \downarrow 0} \frac{b_{1,s}(b)}{s} =$$

$$= 8|b|^2 = 2 \left| b_1^{(1)} \right|^2.$$

□

Now we study the second addend, in order to reach the wanted estimate.

We introduce some notation. Let $\gamma := (2\pi)^{-m/2} e^{-|z|^2/2} H^m \llcorner V$, and let $\omega : \odot^2 V \to \mathbb{R}^n$ and $\hat{b} \in \operatorname{Hom}(\odot^2 V, \mathbb{R}^n)$ be two linear maps defined as follow:

- $\langle \omega(u_1 \odot u_2), w \rangle := 3 b_3^{(2)}(u_1 \odot u_2 \odot w) - 4 \langle u_1, u_2 \rangle \langle b, w \rangle$;
- $\hat{b}(u_1, u_2) := b_2^{(2)}(u_1, u_2) + \langle \omega(u_1 \odot u_2), b \rangle$.

In this work we will always use this notation with $u_1 = u_2 =: u$, that is

$$\left\langle \omega(u^2), w \right\rangle = 3b_3^{(2)}(u^2 \odot w) - 4|u|^2 \langle b, w \rangle ,$$

and

$$\hat{b}(u^2) = b_2^{(2)}(u^2) + \left\langle \omega(u^2), b \right\rangle .$$

We note that for every $u, w \in V$ it holds

$$\left\langle \omega(u^2), w \right\rangle = 0,$$

since, from Lemma 5.2, $b_3^{(2)} \llcorner V = 0$ and $\langle b, w \rangle = 0$ because $b \in V^\perp$, then $\omega(u^2) \in V^\perp$ for every $u \in V$.

With this notation we can find an integral formula for $\mathrm{tr}(b_2^{(2)} \llcorner V)$.

Lemma 5.6. Using the notation just introduced, the following formula holds:

$$\mathrm{tr}\left(b_2^{(2)} \llcorner V\right) = \int_{\mathbb{R}^n} \hat{b}(v^2) d\gamma(v).$$

Proof. Since $b_2^{(2)} \llcorner V$ is symmetric, we can fix a system of orthonormal coordinates on V, $v_1, ..., v_m$, where the corresponding orthonormal vector $e_1, ...e_m$ are eigenvectors of $b_2^{(2)}$, and let $\beta_1, ..., \beta_m$ be their corresponding eigenvalues. Then:

$$\int_{\mathbb{R}^n} b_2^{(2)}\left(v^2\right) d\gamma(v) = \int_{\mathbb{R}^n} \left(\beta_1 v_1^2 + ... + \beta_m v_m^2\right) d\gamma(v) =$$

$$= \sum_{i=1}^{m} \beta_i (2\pi)^{-m/2} \int_{\mathbb{R}^n} e^{-|v|^2/2} v_i^2 d\left(H^m \llcorner V\right) =$$

$$= \sum_{i=1}^{m} \frac{\beta_i}{(2\pi)^{m/2}} \int_{\mathbb{R}^{m-1}} e^{-|x|^2/2} dL_{m-1}(x) \int_{\mathbb{R}} t^2 e^{-t^2/2} dt =$$

$$= \sum_{i=1}^{m} \frac{\beta_i}{(2\pi)^{m/2}} (2\pi)^{(m-1)/2} \cdot 2^{3/2} \frac{\pi^{1/2}}{2} = \beta_1 + .. + \beta_m = \mathrm{tr}\left(b_2^{(2)} \llcorner V\right).$$

Therefore we have to prove that

$$\int_{\mathbb{R}^n} \left\langle \omega(v^2), b \right\rangle d\gamma(v) = 0.$$

Let $z \in \mathbb{R}^n$. Then, using the same argument that we just used:

$$\int_{\mathbb{R}^n} \langle z, v \rangle^2 d\gamma(v) = \int_V \langle P_V(z), v \rangle^2 d\gamma(v) =$$

$$= \sum_{i=1}^{m} \int_V [P_V(z)]_i^2 \, v_i^2 d\gamma(v) = \sum_{i=1}^{m} [P_V(z)]_i^2 = |P_V(z)|^2 . \tag{5.5}$$

Now we want to write $\langle \omega(v^2), w \rangle$ as a limit of an integral, when $v \in V$ and $w \in V^{\perp}$.

Let $v, w \in \mathbb{R}^n$; we write the Taylor expansion of $b_{3,s}$ and $b_{2,s}$:

$$b_{3,s} = s b_3^{(1)} + \frac{s^2}{2} b_3^{(2)} + o(s^2) = \frac{s^2}{2} b_3^{(2)} + o(s^2);$$

$$b_{2,s} = s b_2^{(1)} + o(s).$$

Then:

$$b_3^{(2)} \left(v^2 \odot w\right) = \lim_{s \downarrow 0} \frac{2}{s^2} b_{3,s} \left(v^2 \odot w\right) =$$

$$= \lim_{s \downarrow 0} \frac{8s}{3I(s)} \int_{\mathbb{R}^n} \langle z, v \rangle^2 \langle z, w \rangle \, e^{-s|z|^2} d\mu(z);$$

$$b_2^{(1)} \left(v^2\right) = \lim_{s \downarrow 0} \frac{1}{s} b_{2,s} \left(v^2\right) = \lim_{s \downarrow 0} \frac{2s}{I(s)} \int_{\mathbb{R}^n} \langle z, v \rangle^2 \, e^{-s|z|^2} d\mu(z).$$

Therefore we have that for every $v, w \in \mathbb{R}^n$:

$$3 b_3^{(2)} \left(v^2 \odot w\right) - 4 \langle b, w \rangle \, b_2^{(1)} \left(v^2\right) = \lim_{s \downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, v \rangle^2 \langle z - b, w \rangle \, d\mu(z). \tag{5.6}$$

Letting $v \in V$ and $w \in V^{\perp}$, we have that, for Lemma 5.1,

$$b_2^{(1)} \left(v^2\right) = |P_V(v)|^2 = |v|^2,$$

then

$$\lim_{s \downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, v \rangle^2 \langle z - b, w \rangle \, d\mu(z) =$$

$$= 3 b_3^{(2)} \left(v^2 \odot w\right) - 4 \langle b, w \rangle \, b_2^{(1)} \left(v^2\right) =$$

$$= 3 b_3^{(2)} \left(v^2 \odot w\right) - 4|v|^2 \langle b, w \rangle = \left\langle \omega(v^2), w \right\rangle . \tag{5.7}$$

We need to compute the integral of this limit, then we verify that we can apply the Dominated Convergence Theorem in order to switch the limit with the integral:

$$\left| \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, v \rangle^2 \langle z - b, w \rangle \, d\mu(z) \right| \leq$$

$$\leq \frac{8}{\pi^{m/2}} s^{1+m/2} \int_{\mathbb{R}^n} e^{-s|z|^2} |z|^2 |v|^2 \left(|\langle z, w \rangle| + |b||w| \right) d\mu(z) \leq$$

$$\leq \frac{8|v|^2|w|\left(\left\|b_1^{(1)}\right\| + |b|\right)}{\pi^{m/2}} s^{1+m/2} \int_{\mathbb{R}^n} e^{-s|z|^2}|z|^2 d\mu(z) =$$

$$= \frac{8|v|^2|w|\,(3|b|)}{\pi^{m/2}} \frac{m}{2} \pi^{m/2} = 12|b||w||v|^2,$$

which is integrable with respect to the measure γ.

Then, we compute:

$$\int_V \langle \omega(v^2), b \rangle \, d\gamma(v) = \int_V \lim_{s\downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, v \rangle^2 \langle z-b, b \rangle \, d\mu(z) d\gamma(v) =$$

$$= \lim_{s\downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z-b, b \rangle \int_V \langle z, v \rangle^2 \, d\gamma(v) d\mu(z) =$$

$$= \lim_{s\downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \left|P_V(z)\right|^2 \langle z-b, b \rangle \, d\mu(z). \tag{5.8}$$

Our goal is to prove that the last limit is equal to 0. We study the limit with $Q_V(z)$ instead of $P_V(z)$ and a general $w \in V^\perp$ instead of b in the second factor of the last product; after that we can sum the two limits and study the limit with z instead of $P_V(z)$:

$$\left| \int_{\mathbb{R}^n} e^{-s|z|^2} \left|Q_V(z)\right|^2 \langle z-b, w \rangle \, d\mu(z) \right| \leq$$

$$\leq 4|b|^2|w| \int_{\mathbb{R}^n} e^{-s|z|^2} \left(|z| + |b|\right) d\mu(z) = 4|b|^2|w| \left[s^{-(m+1)/2}\hat{c} + s^{-m/2}\pi^{m/2}|b| \right];$$

then the limit is

$$\lim_{s\downarrow 0} \left| \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \left|Q_V(z)\right|^2 \langle z-b, w \rangle \, d\mu(z) \right| \leq$$

$$\leq \lim_{s\downarrow 0} \frac{32|b|^2|w|}{\pi^{m/2}} s^{1+m/2} \left[s^{-(m+1)/2}\hat{c} + s^{-m/2}\pi^{m/2}|b| \right] = 0,$$

hence

$$\lim_{s\downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \left|Q_V(z)\right|^2 \langle z-b, w \rangle \, d\mu(z) = 0. \tag{5.9}$$

Summing (5.8) and (5.9) with $w = b$, we find that our goal is equivalent to prove that

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |z|^2 \langle z-b, b \rangle \, d\mu(z) = 0.$$

We prove that, for every $w \in V^{\perp}$,

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) = 0.$$

That limit exists: indeed, if we fix an orthonormal basis of $\mathbb{R}^n$, we can apply (5.6) to reach

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) =$$

$$= \sum_{i=1}^{n} \lim_{s\downarrow 0} \frac{s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, e_i\rangle^2 \langle z-b, w\rangle \, d\mu(z) =$$

$$= \sum_{i=1}^{n} \frac{3}{8} b_3^{(2)} \left(e_i^2 \odot w\right) - \frac{1}{2} b_2^{(1)} \left(e_i^2\right) \langle b, w\rangle \, .$$

We note that

$$\frac{s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) =$$

$$= \pi^{-m/2} \frac{\int_{\mathbb{R}^n} e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z)}{s^{-1-m/2}} =$$

$$= \pi^{-m/2} \frac{-\frac{d}{ds} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z-b, w\rangle \, d\mu(z)}{-\frac{2}{m} \frac{d}{ds} s^{-m/2}},$$

then, in order to compute the limit for $s \downarrow 0$, we can apply the De L'Hôpital rule:

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |z|^2 \langle z-b, w\rangle \, d\mu(z) =$$

$$= \frac{m}{2\pi^{m/2}} \lim_{s\downarrow 0} \frac{\int_{\mathbb{R}^n} e^{-s|z|^2} \langle z-b, w\rangle \, d\mu(z)}{s^{-m/2}} =$$

$$= \frac{m}{2} \lim_{s\downarrow 0} I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z-b, w\rangle \, d\mu(z) =$$

$$= \frac{m}{2} \lim_{s\downarrow 0} \left[I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, w\rangle \, d\mu(z) - I(s)^{-1} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle b, w\rangle \, d\mu(z) \right] =$$

$$= \frac{m}{2} \lim_{s\downarrow 0} \left[\frac{1}{2} s^{-1} b_{1,s}(w) - \langle b, w\rangle \right] = \frac{m}{2} \left[\frac{1}{2} b_1^{(1)}(w) - \langle b, w\rangle \right] =$$

$$= \frac{m}{2} \left[\langle b, w\rangle - \langle b, w\rangle \right] = 0.$$

Subtracting (5.9) from this we obtain that, for every $w \in V^{\perp}$,

$$\lim_{s\downarrow 0} \frac{s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |P_V(z)|^2 \langle z-b, w\rangle \, d\mu(z) = 0. \tag{5.10}$$

Then, taking $w = b$ in (5.10), the lemma is proved. □

We note that the whole argument with a generic $w \in V^{\perp}$ was not essential for this proof, here we could do all the computations with b instead of w, but (5.10) will be useful in the next proofs, and that is the reason we proved it in a more general case.

The next lemma is an useful equation which follows from (3.7).

Lemma 5.7. For every $z \in \mathrm{supp}(\mu)$ it holds

$$\begin{aligned} b_1^{(2)}(z) + b_2^{(2)}\left(z^2\right) + 3b_3^{(2)}\left((P_V(z))^2 \odot Q_V(z)\right) = \\ = |Q_V(z)|^2\left(|Q_V(z)|^2 + 2|P_V(z)|^2\right). \end{aligned} \tag{5.11}$$

Proof. From (3.7) with $q = 2$, we have that, for every $z \in \mathrm{supp}(\mu)$,

$$b_1^{(2)}(z) + b_2^{(2)}\left(z^2\right) + b_3^{(2)}\left(z^3\right) + b_4^{(2)}\left(z^4\right) = |z|^4. \tag{5.12}$$

We know, from Lemma 5.1, that $b_4^{(2)}\left(z^4\right) = |P_V(z)|^4$. Moreover,

$$\begin{aligned} |z|^4 = \left(|z|^2\right)^2 = \left(|P_V(z)|^2 + |Q_V(z)|^2\right)^2 = \\ = |P_V(z)|^4 + 2|P_V(z)|^2|Q_V(z)|^2 + |Q_V(z)|^4. \end{aligned}$$

Substituting these informations in (5.12), we obtain

$$b_1^{(2)}(z) + b_2^{(2)}\left(z^2\right) + b_3^{(2)}\left(z^3\right) = |Q_V(z)|^2\left(|Q_V(z)|^2 + 2|P_V(z)|^2\right). \tag{5.13}$$

We study now the term $b_3^{(2)}\left(z^3\right)$: our goal is to prove that

$$b_3^{(2)}\left(z^3\right) = 3b_3^{(2)}\left((P_V(z))^2 \odot Q_V(z)\right). \tag{5.14}$$

Since $b_3^{(2)}$ is linear we can write

$$\begin{aligned} b_3^{(2)}\left(z^3\right) = b_3^{(2)}\left((P_V(z))^3\right) + 3b_3^{(2)}\left((P_V(z))^2 \odot Q_V(z)\right) + \\ + 3b_3^{(2)}\left(P_V(z) \odot (Q_V(z))^2\right) + b_3^{(2)}\left((Q_V(z))^3\right). \end{aligned}$$

We prove that for every $v \in V$ and for every $w \in V^{\perp}$,

$$b_3^{(2)}\left(v \odot w^2\right) = b_3^{(2)}\left(v^3\right) = b_3^{(2)}\left(w^3\right) = 0,$$

and this concludes the proof of this lemma.

From Lemma 5.2 follows that $b_3^{(2)}(v^3) = 0$ for every $v \in V$.

From the Taylor expansion of $b_{3,s}$, we know that

$$b_{3,s}\left(v \odot w^2\right) = sb_3^{(1)}\left(v \odot w^2\right) + \frac{s^2}{2} b_3^{(1)}\left(v \odot w^2\right) + o(s^2),$$

but we proved that $b_3^{(1)} = 0$, then:

$$b_3^{(2)}\left(v \odot w^2\right) = \lim_{s \downarrow 0} \frac{2}{s^2} b_{3,s}\left(v \odot w^2\right) =$$

$$= \lim_{s \downarrow 0} \frac{8s}{3I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, v \rangle \langle z, w \rangle^2 \, d\mu(z).$$

Since $w \in V^\perp$, $\langle z, w \rangle^2 = \langle Q_V(z), w \rangle^2 \le |w|^2 |Q_V(z)|^2 \le |w|^2 \left|b_1^{(1)}\right|^2$, then

$$\left|b_3^{(2)}\left(v \odot w^2\right)\right| \le \lim_{s \downarrow 0} \frac{8s}{3I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |\langle z, v \rangle| \, |\langle z, w \rangle|^2 \, d\mu(z) \le$$

$$\le \lim_{s \downarrow 0} \frac{8s|v||w|^2 \left|b_1^{(1)}\right|^2}{3I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |z| d\mu(z) =$$

$$= \lim_{s \downarrow 0} \frac{8s|v||w|^2 \left|b_1^{(1)}\right|^2}{3I(s)} \int_{\mathbb{R}^m} e^{-s|x|^2} |x| dL_m(x) =$$

$$\frac{8|v||w|^2 \left|b_1^{(1)}\right|^2}{3\pi^{m/2}} \int_{\mathbb{R}^m} e^{-|y|^2} |y| dL_m(y) \lim_{s \downarrow 0} \frac{s^{1+m/2}}{s^{(m+1)/2}} = 0.$$

We study now $b_3^{(2)}(w^3)$. The computation is similar to the one we just did for $b_3^{(2)}(v \odot w^2)$:

$$b_3^{(2)}\left(w^3\right) = \lim_{s \downarrow 0} \frac{8s}{3I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, w \rangle^3 \, d\mu(z),$$

then

$$\left|b_3^{(2)}\left(w^3\right)\right| \le \lim_{s \downarrow 0} \frac{8s}{3I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |\langle z, w \rangle|^3 \, d\mu(z) \le$$

$$\le \lim_{s \downarrow 0} \frac{8s|w|^3 \left|b_1^{(1)}\right|^3}{3I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |d\mu(z) = \frac{8|w|^3 \left|b_1^{(1)}\right|^3}{3} \lim_{s \downarrow 0} \frac{s}{I(s)} I(s) = 0.$$

□

Then, we can focus on $\hat{b}(v^2)$, with $v \in V$.

Lemma 5.8. For every $v \in V$ it holds the following inequality:

$$\left(\hat{b}(v^2)\right)^2 \leq \left|\omega(v^2)\right|^2 |b|^2.$$

Proof. Let $v \in V$ and $\{t_i\}_i \subset \mathbb{R}$ an increasing divergent sequence: for Lemma 5.4, for every i there exist z_i such that $t_i v + z_i \in \mathrm{supp}(\mu)$ and $|z_i| \leq r_0$. Let $v_i := P_V(z_i)$ and $w_i := Q_V(z_i)$.

Since $w_i \in B_{r_0}(0) \cap V^\perp$, that is compact, we have that up to a subsequence $w_i \to w \in V^\perp$. We apply Lemma 5.7 to $t_i v + v_i + w_i$:

$$b_1^{(2)}(t_i v + v_i + w_i) + b_2^{(2)}\left((t_i v + v_i + w_i)^2\right) + 3b_3^{(2)}\left((t_i v + v_i)^2 \odot w_i\right) =$$
$$= 2|t_i v + v_i|^2 |w_i|^2 + |w_i|^4. \tag{5.15}$$

We divide the equation (5.15) by t_i^2 and we take the limit for $i \to \infty$:

$$b_2^{(2)}\left(v^2\right) + 3b_3^{(2)}\left(v^2 \odot w\right) = 2|v|^2 |w|^2. \tag{5.16}$$

We chose v_i and w_i such that $t_i v + v_i + w_i \in \mathrm{supp}(\mu)$, then from Lemma 5.3 we have

$$b_1^{(1)}(t_i v + v_i + w_i) = |w_i|^2,$$

but $t_i v + v_i \in V$, then, from Lemma 5.2, $b_1^{(1)}(t_i v + v_i) = 0$; hence we conclude that

$$b_1^{(1)}(w_i) = |w_i|^2,$$

and letting $i \to \infty$,

$$b_1^{(1)}(w) = |w|^2.$$

Then we conclude that $|w|^2 = 2\langle b, w\rangle$.

Substituting this in (5.16) we find

$$0 = b_2^{(2)}\left(v^2\right) + 3b_3^{(2)}\left(v^2 \odot w\right) - 4|v|^2 \langle b, w\rangle = b_2^{(2)}\left(v^2\right) + \left\langle \omega\left(v^2\right), w\right\rangle.$$

Summing and subtracting $\langle \omega(v^2), b\rangle$ from this last term we obtain

$$\hat{b}\left(v^2\right) + \left\langle \omega(v^2), w - b\right\rangle = 0.$$

Then we conclude:

$$\left(\hat{b}\left(v^2\right)\right)^2 \leq \left|\omega\left(v^2\right)\right|^2 |w - b|^2 =$$
$$= \left|\omega\left(v^2\right)\right|^2 \left(|w|^2 - 2\langle w, b\rangle + |b|^2\right) = \left|\omega\left(v^2\right)\right|^2 |b|^2.$$

□

Now we are ready to study the second addend of (5.3).

Lemma 5.9. The following inequality holds:

$$\operatorname{tr}\left(b_2^{(2)} \llcorner V\right) \geq -\frac{2m}{m+2}\left|b_1^{(1)}\right|^2. \tag{5.17}$$

Proof. To prove this lemma we need to do some computations: the computation in the first step will be used in the second step to link $\int_V \hat{b}\left(v^2\right) d\gamma$ and $\int_V \left(\hat{b}\left(v^2\right)\right)^2 d\gamma$ and reach the first inequality that will lead to the end, that is

$$\int_V \left(\hat{b}\left(v^2\right)\right)^2 d\gamma(v) \leq -8|b|^2 \int_V \hat{b}\left(v^2\right) d\gamma(v);$$

in the third step we use some of the estimates done in the second step to reach the second and last inequality that will conclude the proof, that is

$$\int_V \left(\hat{b}\left(v^2\right)\right)^2 d\gamma(v) \geq \left(1+\frac{2}{m}\right)\left[\int_V \hat{b}\left(v^2\right) d\gamma(v)\right]^2.$$

Step 1. We know that, for every $z \in \operatorname{supp}(\mu)$,

$$|Q_V(z)|^2 = b_1^{(1)}(z) = b_1^{(1)}\left(Q_V(z)\right) = 2\left\langle b, Q_V(z)\right\rangle,$$

then we substitute it in (5.11), we sum and subtract $\langle \omega((P_V(z))^2), b\rangle$, and we obtain that for every $z \in \operatorname{supp}(\mu)$:

$$\begin{aligned}
0 &= b_1^{(2)}(z) + b_2^{(2)}\left(z^2\right) + 3b_3^{(2)}\left((P_V(z))^2 \odot Q_V(z)\right) + \\
&\quad -|Q_V(z)|^4 - 2|P_V(z)|^2|Q_V(z)|^2 = \\
&= b_1^{(2)}(z) + b_2^{(2)}\left((Q_V(z))^2\right) + 2b_2^{(2)}\left(P_V(z)\odot Q_V(z)\right) + b_2^{(2)}\left((P_V(z))^2\right) + \\
&\quad + \left\langle \omega((P_V(z))^2), b\right\rangle + 3b_3^{(2)}\left((P_V(z))^2 \odot Q_V(z)\right) + \\
&\quad -4\left|P_V(z)\right|^2 \left\langle b, Q_V(z)\right\rangle^2 - \left\langle \omega((P_V(z))^2), b\right\rangle - \left(b_1^{(1)}\left(Q_V(z)\right)\right)^2 = \\
&= \hat{b}\left((P_V(z))^2\right) + \left\langle \omega\left((P_V(z))^2\right), Q_V(z) - b\right\rangle + b_1^{(2)}(z) + \\
&\quad + 2b_2^{(2)}\left(P_V(z)\odot Q_V(z)\right) + b_2^{(2)}\left((Q_V(z))^2\right) - \left(b_1^{(1)}\left(Q_V(z)\right)\right)^2.
\end{aligned}$$

Therefore:

$$\left|\hat{b}\left((P_V(z))^2\right) + \left\langle \omega\left((P_V(z))^2\right), Q_V(z) - b\right\rangle\right| \leq \left|b_1^{(2)}(z)\right| +$$

$$+2\left|b_2^{(2)}\left(P_V(z)\odot Q_V(z)\right)\right|+\left|b_2^{(2)}\left((Q_V(z))^2\right)\right|+\left|\left(b_1^{(1)}\left(Q_V(z)\right)\right)^2\right|.$$

We note that, from Lemma 5.3, the last two terms are bounded, while the first is linear. Moreover:

$$\left|b_2^{(2)}\left(P_V(z)\odot Q_V(z)\right)\right|\leq \sup_{v\in V\cap B_{|z|}(0),w\in V^\perp\cap B_{2|b|}(0)}\left|b_2^{(2)}\left(v\odot w\right)\right|\leq$$

$$\leq 2|z||b|\sup_{v\in V\cap B_1(0),w\in V^\perp\cap B_1(0)}\left|b_2^{(2)}\left(v\odot w\right)\right| =: K_1|z|.$$

Then the second term too has at most a linear growth, therefore there exists a constant $K>0$ such that

$$\left|\hat{b}\left((P_V(z))^2\right)+\left\langle\omega\left((P_V(z))^2\right),Q_V(z)-b\right\rangle\right|\leq K\left(|z|+1\right)$$

for every $z\in\operatorname{supp}(\mu)$.

Using this estimate we compute the following limit:

$$\limsup_{s\downarrow 0}\left|\frac{s}{I(s)}\int_{\mathbb{R}^n}e^{-s|z|^2}\left[\hat{b}\left(P_V(z)^2\right)+\left\langle\omega\left(P_V(z)^2\right),Q_V(z)-b\right\rangle\right]d\mu(z)\right|\leq$$

$$\leq K\lim_{s\downarrow 0}\frac{s}{I(s)}\int_{\mathbb{R}^n}e^{-s|z|^2}\left(|z|^2+1\right)d\mu(z)=0,$$

where the last limit is 0 as we already saw in the computation of (5.9).

Then we reached that

$$\lim_{s\downarrow 0}\frac{s}{I(s)}\int_{\mathbb{R}^n}e^{-s|z|^2}\hat{b}\left((P_V(z))^2\right)d\mu(z)=$$

$$=-\lim_{s\downarrow 0}\frac{s}{I(s)}\int_{\mathbb{R}^n}e^{-s|z|^2}\left\langle\omega\left((P_V(z))^2\right),Q_V(z)-b\right\rangle d\mu(z). \tag{5.18}$$

Step 2. First of all we compute $\int_V\langle\zeta,v\rangle^4\,d\gamma(v)$ with $\zeta\in V$: we fix an orthonormal system of coordinates $x_1,...,x_m$ on V such that $\zeta=(|\zeta|,0,...,0)$. Then

$$\int_V\langle\zeta,v\rangle^4\,d\gamma(v)=(2\pi)^{-m/2}\int_{\mathbb{R}^m}|\zeta|^4x_1^4e^{-|x|^2/2}dx=$$

$$=|\zeta|^4\pi^{-m/2}\int_{\mathbb{R}^m}4y_1^4e^{-|y|^2}dy=4|\zeta|^4\pi^{-m/2}\int_{\mathbb{R}^{m-1}}e^{-|y'|^2}dy'\int_{\mathbb{R}}y_1^4e^{-|y_1|^2}dy_1=$$

$$=4|\zeta|^4\pi^{-m/2}\left(\pi^{(m-1)/2}\right)\left(\frac{3}{4}\pi^{-1/2}\right)=3|\zeta|^4. \tag{5.19}$$

Using an analogue argument we compute the following two integrals. Let $y, z \in V$ be orthogonal and let $x_1, ..., x_m$ be an orthonormal system of coordinates such that $y = (|y|, 0, ..., 0)$ and $z = (0, |z|, 0, ..., 0)$. Then we can compute:

$$\int_V \langle y, v\rangle^2 \langle z, v\rangle^2 \, d\gamma(v) =$$

$$= 4|y|^2|z|^2\pi^{-m/2} \int_{\mathbb{R}^{m-2}} e^{-|x'|^2} dx' \int_{\mathbb{R}} x_1^2 e^{-x_1^2} dx_1 \int_{\mathbb{R}} x_2^2 e^{-x_2^2} dx_2 =$$

$$= 4|y|^2|z|^2\pi^{-m/2}\pi^{(m-2)/2} \left(\frac{1}{2}\pi^{1/2}\right)^2 = |y|^2|z|^2; \tag{5.20}$$

$$\int_V \langle y, v\rangle \langle z, v\rangle^3 \, d\gamma(v) =$$

$$= 4|y|^2|z|^2\pi^{-m/2} \int_{\mathbb{R}^{m-2}} e^{-|x'|^2} dx' \int_{\mathbb{R}} x_1^2 e^{-x_1^2} dx_1 \int_{\mathbb{R}} x_2^2 e^{-x_2^2} dx_2 = 0. \tag{5.21}$$

For general $y, z \in V$, we can write $y = \xi + az$ with ξ and z orthogonal, therefore:

$$\int_V \langle y, v\rangle^2 \langle z, v\rangle^2 \, d\gamma(v) =$$

$$= \int_V \langle \xi, v\rangle^2 \langle z, v\rangle^2 \, d\gamma(v) + 2a \int_V \langle y, v\rangle \langle z, v\rangle^3 \, d\gamma(v) + a^2 \int_V \langle z, v\rangle^4 \, d\gamma(v) =$$

$$= |\xi|^2|z|^2 + 3a^2|z|^4 = \left(|\xi|^2 + a^2|z|^2\right)|z|^2 + 2\left(a|z|^2\right)^2 = |y|^2|z|^2 + 2\langle y, z\rangle^2. \tag{5.22}$$

Now let $y \in V$ and $w \in V^\perp$. Using (5.7), (5.10) and (5.22) we compute

$$\int_V \langle y, v\rangle^2 \left\langle \omega\left(v^2\right), w\right\rangle d\gamma(v) =$$

$$= \int_V \langle y, v\rangle^2 \left[\lim_{s\downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z, v\rangle^2 \langle z - b, w\rangle \, d\mu(z)\right];$$

using the same estimate done for (5.8), we can apply the Dominate Convergence Theorem and the Fubini's Theorem:

$$\int_V \langle y, v\rangle^2 \left\langle \omega\left(v^2\right), w\right\rangle d\gamma(v) =$$

$$= \lim_{s\downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \langle z - b, w\rangle \, d\mu(z) \left[\int_V \langle y, v\rangle^2 \langle z, v\rangle^2 \, d\gamma(v)\right] d\mu(z) =$$

$$= \lim_{s\downarrow 0} \frac{8s}{I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} |P_V(z)|^2 \langle z - b, w\rangle \, d\mu(z) +$$

$$+\lim_{s\downarrow 0}\frac{16s}{I(s)}\int_{\mathbb{R}^n} e^{-s|z|^2}\langle z-b,w\rangle\langle z,y\rangle^2\,d\mu(z)=2\left\langle \omega\left(y^2\right),w\right\rangle.$$

Now we study $\int_V\left(\hat{b}\left(v^2\right)\right)^2 d\gamma(v)$. We know that $\omega\left(v^2\right)\in V^\perp$ for every $v\in V$, then we have that $\langle z-b,\omega(v^2)\rangle=\langle Q_V(z)-b,\omega(v^2)\rangle$, and we can consider (5.7) with $w=\omega\left(v^2\right)$:

$$\int_V\left(\hat{b}\left(v^2\right)\right)^2 d\gamma(v)\le |b|^2\int_V\left|\omega\left(v^2\right)\right|^2 d\gamma(v)=$$

$$=|b|^2\lim_{s\downarrow 0}\int_V\frac{8s}{I(s)}\int_{\mathbb{R}^n} e^{-s|z|^2}\langle z,v\rangle^2\left\langle z-b,\omega\left(v^2\right)\right\rangle d\mu(z)d\gamma(v)=$$

$$=|b|^2\lim_{s\downarrow 0}\frac{8s}{I(s)}\int_{\mathbb{R}^n} e^{-s|z|^2}\int_V\langle z,v\rangle^2\left\langle z-b,\omega\left(v^2\right)\right\rangle d\gamma(v)d\mu(z)=$$

$$=|b|^2\lim_{s\downarrow 0}\frac{8s}{I(s)}\int_{\mathbb{R}^n} e^{-s|z|^2}\int_V\langle P_V(z),v\rangle^2\left\langle Q_V(z)-b,\omega\left(v^2\right)\right\rangle d\gamma(v)d\mu(z)=$$

$$=|b|^2\lim_{s\downarrow 0}\frac{16s}{I(s)}\int_{\mathbb{R}^n} e^{-s|z|^2}\left\langle\omega\left((P_V(z))^2\right),Q_V(z)-b\right\rangle d\mu(z)=$$

$$=-16|b|^2\lim_{s\downarrow 0}\frac{s}{I(s)}\int_{\mathbb{R}^n} e^{-s|z|^2}\hat{b}\left((P_V(z))^2\right)d\mu(z)=$$

$$=-16\pi^{-m/2}|b|^2\int_{\mathbb{R}^n} e^{-|z|^2}\hat{b}\left((P_V(z))^2\right)dH^m\llcorner V=$$

$$=-8|b|^2\int_V\hat{b}\left(v^2\right)d\gamma(v). \tag{5.23}$$

This is the first estimate we need.

Step 3. We can fix coordinates $v_1,...,v_m$ on V such that the corresponding unit vectors $e_1,...,e_m$ are orthonormal and they are the eigenvectors of $\hat{b}$. Let $\beta_1,...,\beta_m$ be the corresponding eigenvalues of $\hat{b}$. Using (5.5), we compute

$$\int_V\hat{b}\left(v^2\right)d\gamma(v)=\int_V\sum_{i=1}^m\beta_i\langle v,e_i\rangle^2 d\gamma(v)=\sum_{i=1}^m\beta_i=\operatorname{tr}\left(\hat{b}\right), \tag{5.24}$$

and

$$\int_V\left(\hat{b}\left(v^2\right)\right)^2 d\gamma(v)=\int_V\left(\sum_{i=1}^m\beta_i\langle v,e_i\rangle^2\right)^2 d\gamma(v)=$$

$$=\sum_{i=1}^m\beta_i^2\int_V\langle v,e_i\rangle^4 d\gamma(v)+\sum_{i\neq j}\beta_i\beta_j\int_V\langle v,e_i\rangle^2\langle v,e_j\rangle^2 d\gamma(v)=$$

$$= 3\sum_{i=1}^{m} \beta_i^2 + \sum_{i\neq j} \beta_i\beta_j = \left(\sum_{i=1}^{m} \beta_i\right)^2 + 2\sum_{i=1}^{m} \beta_i^2.$$

Using the inequality between arithmetic mean and quadratic mean, we conclude:

$$\left(\sum_{i=1}^{m} \beta_i\right)^2 + 2\sum_{i=1}^{m} \beta_i^2 \geq \left(1 + \frac{2}{m}\right)\left(\sum_{i=1}^{m} \beta_i\right)^2 =$$

$$= \left(1 + \frac{2}{m}\right)\left[\int_V \hat{b}\left(v^2\right) d\gamma(v)\right]^2,$$

then

$$\int_V \left(\hat{b}\left(v^2\right)\right)^2 d\gamma(v) \geq \left(1 + \frac{2}{m}\right)\left[\int_V \hat{b}\left(v^2\right) d\gamma(v)\right]^2. \tag{5.25}$$

Combinig (5.23) with (5.25) we obtain

$$\left(1 + \frac{2}{m}\right)\left[\int_V \hat{b}\left(v^2\right) d\gamma(v)\right]^2 \leq -8|b|^2 \int_V \hat{b}\left(v^2\right) d\gamma(v);$$

if $\int_V \hat{b}\left(v^2\right) d\gamma(v) = 0$ the lemma is proved, and it can not be positive, because $\left(1 + \frac{2}{m}\right)\left[\int_V \hat{b}\left(v^2\right) d\gamma(v)\right]^2 > 0$, then we take $-8|b|^2 \int_V \hat{b}\left(v^2\right) d\gamma(v) < 0$ and we divide:

$$\int_V \hat{b}\left(v^2\right) d\gamma(v) \geq -\left(1 + \frac{2}{m}\right)^{-1} 8|b|^2 = -\frac{2m}{m+2}\left|b_1^{(1)}\right|^2. \tag{5.26}$$

From (5.26) and Lemma 5.6 we obtain the thesis. □

Summing (5.4) and (5.17) we reach (5.2), that is the wanted estimate. Now we compute $\mathrm{tr}\left(b_2^{(2)}\right)$.

Lemma 5.10. The trace of the linear form $b_2^{(2)}$ is 0.

Proof. From the Taylor expansion of $b_{2,s}$, we have that

$$b_{2,s} = s b_2^{(1)} + \frac{s^2}{2} b_2^{(2)} + o(s^2),$$

then

$$\mathrm{tr}\left(b_2^{(2)}\right) = 2\lim_{s\downarrow 0} \frac{\mathrm{tr}\left(s^{-1} b_{2,s}\right) - \mathrm{tr}\left(b_2^{(1)}\right)}{s}.$$

Lemma 5.1 with $k = 1$ gives us $b_2^{(1)}(u, v) = \langle P_V(u), P_V(v) \rangle$, therefore

$$\operatorname{tr}\left(b_2^{(1)}\right) = m. \tag{5.27}$$

Now we study $\operatorname{tr}(s^{-1} b_{2,s})$. Let $e_1, ..., e_n$ be an orthonormal basis of $\mathbb{R}^n$:

$$\operatorname{tr}\left(s^{-1} b_{2,s}\right) = s^{-1} \sum_{i=1}^{n} b_{2,s}\left(e_i^2\right) =$$

$$= s^{-1} \frac{(2s)^2}{2I(s)} \int_{\mathbb{R}^n} e^{-s|z|^2} \sum_{i=1}^{n} \langle z, e_i \rangle^2 \, d\mu(z) =$$

$$= \frac{2s^{1+m/2}}{\pi^{m/2}} \int_{\mathbb{R}^n} |z|^2 e^{-s|z|^2} d\mu(z) = \frac{2s^{1+m/2}}{\pi^{m/2}} \int_{\mathbb{R}^m} |x|^2 e^{-s|x|^2} dL_m(x) =$$

$$= \frac{2}{\pi^{m/2}} \int_{\mathbb{R}^m} \left|s^{1/2} x\right|^2 e^{-|s^{1/2} x|^2} s^{m/2} dL_m(x) =$$

$$= \frac{2}{\pi^{m/2}} \int_{\mathbb{R}^m} |y|^2 e^{-|y|^2} dL_m(y) = m.$$

Summing this with (5.27) we obtain

$$\operatorname{tr}\left(b_2^{(2)}\right) = 0.$$

□

Then, from Lemma 5.10 and (5.2), we conclude that $b_1^{(1)} = 0$, which concludes the proof of Theorem 5.1.

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