UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Corso di Laurea Magistrale in Fisica

Tesi di Laurea

An Effective Field Theory Approach to the Gravitational Two-Body Problem

Relatore

Prof. Pierpaolo Mastrolia

Correlatore

Prof. Riccardo Sturani

Laureando

Andrea Cristofoli

Anno Accademico 2017/2018

ad Alison

Abstract

The aim of this thesis is to introduce the reader on gravitational wave physics, developing modern methods for the study of the two-body problem in General Relativity. Given that no exact solution for binary dynamics is viable using analytical tools, in the present thesis we will adopt a perturbative scheme called PN (post-Newtonian) expansion. Useful for bound systems with weak gravitational fields and slow velocities compared to the speed of light, in the thesis we will use this scheme to show how the conservative dynamics of a slow inspiral binary takes deviations from the Newton's two body potential in terms of n PN contributions, i.e. with factors proportional to $G_N^{n-l} v^{2l}$ where $0 \leq l \leq n-1$, being v a typical three velocity of the system and n, l natural numbers. As a convenient tool for perturbative calculations, we will introduce the so called EFT (Effective Field Theory) approach, so as to model the inspiral phase of a binary similarly as done with the heavy quark field theory in particle physics. Exploiting this approach, we will cast the perturbative nature of PN corrections into Feynman diagrams, making possible to define a clear power counting in v^2/c^2 and organizing specific PN calculations in terms of a well defined subset of diagrams. In order to build them we will derive a finite set of Feynman rules, adopting a non manifest covariant parametrization for the metric tensor. Suitable for the EFT of PN systems, this choice leads to diagrams with a definite scaling in powers of G_N and v^2/c^2. Remarkably, we will see how these classical contributions give rise to integrals similar to multi-loop massless two-point functions, a relation that has been recently recognized by Mastrolia, Foffa, Sturani and Sturm. Taking advantage of this relation, we will introduce the most modern multi-loop techniques so as to perform high precision calculations in gravitational physics. We will introduce Integration by part identities for Feynman integrals within the dimensional regularization scheme, and apply them for the decomposition of multi-loop amplitudes, in terms of MI's (Master Integrals). We will also deal with mathematical techniques for the evaluation of the latter. The results of the relevant Feynman amplitudes (i.e. PN contributions) will be given as a series expansion in ε, around $d = 3 + \varepsilon$ space dimensions. Using all the high precision offered by multi-loop techniques, we will perform several PN calculations concerning a non spinning binary system as the complete Einstein-Infeld-Hoffmann Lagrangian at 1 PN order, two contributions at 2 PN order scaling respectively as G_N^3 and $G_N^2 v^2$ and a 4 PN one

with a $G_N^4 v^2$ dependence. This 4 PN correction was never calculated before with the EFT approach and it represents the original contribution of the thesis.
It has involved a three-loop amplitude which has been reduced into three MI's with the use of a Mathematica package called FIRE. All MI's have been calculated separately, leading to the evaluation of this PN contribution.
The methodologies and the results of this thesis can thus be used for planning new calculations that will be relevant for future gravitational waves detectors as the 5 PN sector of a non spinning binary black-hole, currently unknown, and the so called post-Minkowskian sector given by unbound systems with arbitrarily high velocities and weak gravitational fields, which will be relevant for the future space gravitational wave detector LISA.

Contents

Introduction **11**

1 General Relativity: a derivation **17**
1.1 The Pauli-Fierz Lagrangian for a graviton . . . 18
1.2 Newton's law from gravitons . . . 21
1.2.1 The graviton propagator . . . 22
1.2.2 Fixing the coupling matter-gravity . . . 25
1.3 From gravitons to General Relativity . . . 26
1.3.1 The self-energy of the gravitational field . . . 26
1.3.2 Failure of $\partial_\mu T^{\mu\nu} = 0$ in presence of gravity . . . 28
1.3.3 The Einstein-Hilbert action . . . 30

2 Gravitational Waves **35**
2.1 Gravitational waves from General Relativity . . . 36
2.1.1 The energy-momentum tensor for a gravitational wave . . . 37
2.1.2 The energy and momentum flux of a gravitational wave . . . 38
2.2 Generation of gravitational waves . . . 39
2.2.1 Multipoles expansion . . . 40
2.2.2 The quadrupole radiation . . . 43
2.2.3 Radiation from point particle sources . . . 44
2.3 Compact binaries as a radiating system . . . 47
2.3.1 The coalescence of compact binaries . . . 48
2.3.2 Estimates and numerical values for compact binaries . . . 49
2.4 Beyond linearized General Relativity . . . 50
2.4.1 The phase of a gravitational wave: post-Newtonian corrections 51
2.4.2 The amplitude of a gravitational wave beyond linearized GR . 52

3 The Effective Field Theory approach **55**
3.1 The Effective Field Theory approach described . . . 56
3.1.1 Binary systems as an EFT . . . 57
3.1.2 Calculating observables from the EFT of a binary . . . 59
3.2 The EFT of a binary system in scalar gravity . . . 60

3.2.1 Local EFT via a non relativistic limit 63
3.2.2 A nod on time non-localities 66
3.3 The EFT of a binary system within General Relativity 67
3.4 Feynman rules for the non relativistic EFT of a real binary 69
3.4.1 The ϕ propagator 69
3.4.2 The A_i propagator 70
3.4.3 The σ_{nm} propagator 71
3.4.4 Source-gravity interactions rules 72
3.4.5 Fields interactions rules 73
3.4.6 Feynman rule for σ-ϕ-ϕ vertex 74
3.4.7 Feynman rule for σ-σ-ϕ-ϕ vertex 76
3.5 Building effective Feynman diagrams 81
3.5.1 A relation between effective diagrams and QFT amplitudes . . 82

4 Multi-loop techniques in QFT 83
4.1 Dimensional regularization 84
4.2 Feynman parametrization and loop integrals 85
4.2.1 The scalar integral $I_S(d, a, b)$ 85
4.2.2 The scalar integral $I_F(d, a)$ 87
4.2.3 The tensorial generalization of $I_F(d, a)$ 89
4.3 Integration by parts identities (IBP) 90
4.3.1 IBP for the massive one-loop tadpole 91
4.3.2 Properties of IBP 91
4.4 Massless propagators 92
4.4.1 The massless one-loop propagator 92
4.4.2 The massless two-loop propagator 95
4.4.3 The massless three-loop propagator 98
4.5 The star triangle rule (STR) for an ideal vertex 103
4.5.1 The STR and the massless four-loop propagator 105

5 Post-Newtonian corrections 109
5.1 Organizing post-Newtonian calculations 110
5.2 The Einstein-Infeld-Hoffmann Lagrangian from a 1 PN calculation . . 112
5.3 Topologies from 2 up to 4 PN order 115
5.3.1 2-nd Post Newtonian order 115
5.3.2 3-rd Post Newtonian order 116
5.3.3 4-th Post Newtonian order 117
5.4 Evaluation of two 2-PN amplitudes 118
5.4.1 A non static 2-PN amplitude from a G_N^2 topology 118
5.4.2 A static 2-PN amplitude from a G_N^3 topology 120
5.5 A non static 4-PN amplitude from a G_N^4 topology 125

Conclusions **131**

Appendices **135**

A Degree of divergence for the four-loop massless propagator **137**

B The non static 2-PN amplitude **139**

C The gamma function and the measure of a unit sphere **143**

Introduction

On September 14, 2015, LIGO [1] (*Laser Interferometer Gravitational-wave Observatory*) detected for the first time gravitational waves emitted by the coalescence and merging of two colliding black holes [3]. Since then four binary black-hole coalescences have been observed [2, 8, 7, 6], followed by the inspiral of neutron stars [9], opening a new era of precision in gravitational astrophysics.
The relevance of these measurements is extraordinary because they made possible to test GR (General Relativity) in the so called strong field regime [5], while until recently we were restricted only to solar system observations and binary pulsar ones [75]. These recent tests were viable only because gravitational waves carry fingerprints of a binary dynamics, for this reason a complete understanding of the gravitational two-body problem has become more crucial than ever.
Unfortunately this is an extremely difficult goal to reach in GR and currently it has been partially accomplished using numerical relativity [58] or perturbative schemes as the so called PN (post-Newtonian) expansion [14].
Developed already in 1916 by Einstein himself [32], the PN approach is useful for bound systems having weak gravitational fields and slow velocities compared to the speed of light. Within this dynamical regime, it can be proved that the conservative dynamics of a binary takes deviations from the ordinary Newton's two body potential in terms of n PN contributions, i.e. with factors proportional to $G_N^{n-l}v^{2l}$ where $0 \leq l \leq n-1$ being n, l natural numbers and v a typical velocity of the binary.
The 1 PN correction to a non spinning binary system was calculated already in 1917 by Lorentz and Droste, and it took decades in order to perform higher PN calculations as the 4 PN one. This was calculated for the first time by Damour, Jaranowski and Schäfer in 2014 [29] and currently it represents the state of art.
Given these results on the conservative dynamics, one can include radiation effects on the binary motion adopting the so called Effective One Body approach [16].
Developed by Buonanno and Damour in 1998 (for a review see [23]), this approach combines in a suitably resummed format perturbative PN results on the motion and radiation of compact binaries, with some non-perturbative information extracted from numerical simulations.

This scheme have been used for the development of the most accurate templates for gravitational waves and binary coalescences [17, 21, 27] and it have been crucial for the first detection of LIGO.
Given that planned gravitational wave detectors [64] will work at higher experimental precision compared to the current one, several research groups are attempting to study with increasing precision the details of a binary dynamics, both on the conservative and radiative sector, adopting several approaches and new techniques.
The present work of thesis is inserted within this line of research and will introduce the reader on modern gravitational wave physics showing how to perform PN calculations adopting a modern Effective Field Theory (EFT) approach, commonly used in particle physics [36].
First introduced in gravitational physics by Damour and Farese [25] and later systematized by Goldberger and Rothstein [39], the EFT approach casts the perturbative nature of PN physics into Feynman diagrams, making possible to define a clear power counting in an expansion parameter and connecting physics at separate scales with the flow of the renormalization group.
As we will see, this approach shares many features with QFT's (Quantum Field Theories), but despite the terminology our amplitudes will only provide classical contributions and should therefore not be considered of any quantum nature.
Given this clarification, it is remarkable how PN contributions can be topologically mapped into massless multi-loop two point functions, a relation that has been recently introduced in [34] by Mastrolia, Foffa, Sturani and Sturm.
Exploiting this mapping, we will perform high precision calculations introducing the most modern multi-loop techniques used in particle physics [61, 18].
In this regard, when a direct integration of Feynman integrals is prohibitive, the evaluation of multi-loop amplitude will be addressed in two stages.
In a preliminary one, by exploiting some remarkable properties of dimensional regularized integrals, namely Integration by parts identities (IBP's), Lorentz invariance identities, and further ones due to kinematic symmetry specific of each diagram, one establishes several relations among a whole set of scalar integrals associated to the original Feynman diagram called MI's (Master Integrals)
The second phase consists of the actual evaluation of the MI's. The basis of MI's is not known apriori, and these are identified at the end of the reduction procedure, and only afterwards the problem of their evaluation arises. The most used multi-loop reduction technique is the well-known Laporta algorithm [55, 56], based on the solution of algebraic systems of equations obtained through IBP's [19, 71, 40].
Owing to IBP's, MIs are found to obey systems of difference [55, 56, 70, 57] and differential [52, 44] equations, reviewed in [10, 45].
Supplemented by boundary conditions, often derived from simpler integrals, these methods offer valid alternatives to the evaluation of MI's, which can be found as solution of those systems of equations instead of direct integration. Within the thesis, we will use all this technology to evaluate multi-loop amplitudes linked with several

PN corrections, evaluating a 4 PN contribution proportional to $G_N^4 v^2$ that so far wasn't calculated before within the EFT approach.
In this perspective, this work will also explain in a comprehensive way the methods and techniques that can be borrowed from particle physics so as to perform new and relevant calculations which are urgent for nowadays gravitational physics. As for the material treated, it is divided into five Chapters:

- In the first Chapter it is derived General Relativity, so far the best description of gravity at classical level, via a field theory construction. Starting from the Fierz-Pauli Lagrangian describing a massless spin 2 field, we will see how consistency requirements will bring us to the simplest action of gravity given by the Einstein-Hilbert one describing GR. In doing so, we will point out the non uniqueness of the procedure, showing how this derivation accommodate in a natural way deviations suggested by cosmological considerations (e.g. non local terms [60]).

- In the second Chapter, having derived GR, we will focus on gravitational wave physics studying their behavior far from sources and how they generate from dynamical ones, introducing the so called multipole expansion useful for a non relativistic treatment of the radiated power from a binary.
  As application, we will study gravitational waves emitted by a point-particle binary deriving the so called *chirping* and coalescence of a binary system.
  A detailed analysis will be made on how deviations from Newtonian gravity influence the phase and amplitude of the emitted gravitational waves.

- In the third Chapter, we will introduce the Effective Field Theory approach whereby PN corrections to a binary dynamics are expressed in terms of Feynman diagrams. We will give an example of how the approach works by studying a toy model of binary held together by scalar gravity, passing then to the case of a real binary within GR.
  We will derive several Feynman rules useful for the evaluation of high post-Newtonian corrections showing in the end a remarkable relation that makes possible to map these classical contributions into massless two point functions at n-loop.

- In the fourth Chapter, given that classical PN corrections can be seen as loop amplitudes, we will introduce the most modern multi-loop techniques.
  We will start with dimensional regularization and Feynman parametrization, evaluating a relevant class of scalar loop integrals.
  Alongside these, the methods of Integration by parts identities and the Star triangle rule will be introduced. All these techniques will be used for the evaluation of two point functions at one-two-three loops and for the estimate of the degree of divergence of a four loop amplitude.

- In the fifth Chapter, we will collect everything learned from the previous ones in order to calculate several PN contributions to non spinning binary systems. It will be evaluated the complete Einstein-Infeld-Hoffmann Lagrangian at 1 PN order, a sector of the 2 PN order coming from two diagrams proportional to G_N^3 and $G_N^2 v^2$ and we will calculate for the first time a 4 PN contribution from a diagram scaling as $G_N^4 v^2$.
  In doing so, attention will be paid on the current status of PN calculations and how diagrams can be organized in order to evaluate a given PN order.

In conclusion, we will point out how this thesis can be used for planning new calculations that will be relevant for future gravitational waves detectors.
Among these, we cite the 5 PN sector of a non spinning binary black-hole, currently unknown, and the so called *post-Minkowskian* sector given by unbound system with arbitrarily high velocities and weak gravitational fields, which will be relevant for the future space gravitational wave detector LISA [11].

Notations

Constant and units

We will denote the speed of light with c, the Planck constant with $\hbar$ and Newton's gravitational constant with G_N. Occasionally we will set $c = 1$. As for $\hbar$, we will always work in natural units where $\hbar = 1$, while G_N it will be always written explicitly.

Metric signature

For the flat Minkowskian metric we have chosen the most common convention in General Relativity

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

while for coordinates and the derivative operator

$$x^\mu = (ct, \vec{x}) \quad , \quad \partial_\mu = \left(\frac{1}{c}\partial_t, \partial_i\right)$$

Scalar products and multi-loop integrals

Depending on the type of calculations we will be interested in, we will say that a loop integral is Minkowskian when its integrand can be expressed as a function of Minkowskian contractions (e.g. $p^2 = (p^0)^2 - \vec{p} \cdot \vec{p}$), while Euclidean if dependent on Euclidean ones only (e.g. $k^2 = \vec{k} \cdot \vec{k}$).

Chapter 1

General Relativity: a derivation

General Relativity (GR) is commonly introduced as a geometric theory of space-time in many textbooks [73, 62], with an entire building which could appear different at first sight from the Standard Model one.
Although elegant and intuitive, this geometrical point of view is not a necessary condition in order to develop GR: it is possible to derive the same field equations and underlying concepts, starting from field theoretical considerations, in a bottom-up approach at the end equivalent to Einstein's formulation less than few aspects.
This novel procedure, that had its forerunner in a 1954 paper of Suraj Gupta [43], is based on a series of lectures given by Feynman at Caltech in 1962-1963 on gravitational physics [33] in which gravity is seen on the same footage of every other field theory: in doing so, we will find convenient to talk of *graviton* as a mediator of gravity at quantum level, nevertheless it should keep in mind that our theory has to be trusted only on its classical sector.
Given these considerations, the chapter is so divided:

- It is shown how to derive the Lagrangian describing a free, massless spin 2 particle, by means of field theory assumptions. By taking care that gauge invariance is preserved at classical level, interactions will be added in terms of a free coupling g.

- It is derived the graviton propagator and shown how the non relativistic limit of our theory reduces to Newtonian gravity. The procedure will also fix in a unique way the absolute value of the coupling between gravity and matter.

- To maintain consistency at classical level, it will been shown how the previous theory have to develop non linear interactions among gravitons. Clearness will be made on the reasons of this consistency requirement and it will be demonstrated how it leads naturally to General Relativity.

1.1 The Pauli-Fierz Lagrangian for a graviton

Let's put aside the geometrical foundation of General Relativity (GR) and suppose we are only interested to describe gravity as an ordinary Quantum field theory (QFT) defined on a flat space-time, which for the beginning is a reasonable assumption.
To fulfill our purpose we need to define a functional action of the following form

$$\mathcal{S} \equiv \int d^4x \mathcal{L}(\phi_i, \partial\phi_i) \tag{1.1}$$

where ϕ_i denotes a generic relativistic field, not necessarily scalar, which transform as a finite irreducibile representation of the Poincaré group $\mathcal{P} = SO(1,3) \bowtie R^4$.
A consistent QFT requires 1.1 to be invariant under the action of $\mathcal{P}$. Since the d^4x measure is unaffected by these transformations, one is required to define a scalar Lagrangian $\mathcal{L}$ in terms of relativistic fields.
We are also interested to identify the mediator of the gravitational field, so we will take advantage of the fact that classical relativistic fields, needed to define $\mathcal{L}$, transform as finite irreducible representations of the Poincaré group P, while spin particles arise from the decomposition of the same in terms of finite irreducible representations of the rotation subgroup $SO(3)$.
We can now start our derivation by noticing that classical forces, like gravity, are mediated at quantum level by bosons[1], which means that we can restrict ourselves to those representations of $SO(1,3)$ that contain particles with integer spin.
In particular, since the gravitational field exercises a long range force, it is reasonable to assume that gravity is mediated by a massless boson.[2]
A theorem due to Weinberg [74] states that it is not possible to develop a consistent QFT of interacting massless bosons with spin higher than 2 (see also [66] at p.155 for a modern derivation), which means gravity can only be described as a massless bosons with elicity $\pm 2s$ where s could be $0, 1, 2$.
As additional hint let's notice that gravity couples to masses (i.e energy), whose density is the 00 component of an energy-momentum tensor $T_{\mu\nu}(x)$.
Assuming a local theory[3] the only interactions allowed will be developed in the Lagrangian as scalar contractions with a $T_{\mu\nu}(x)$, function of the space-time point,

[1]The concept of "force" is a classical and in a quantum theory is linked with the Feynman propagator of a tree level amplitude with initial states equal to final: if a fermion would mediate such interactions we would have violation of angular momentum at a vertex, this is the reason why it is usually said that "bosons" are force carrier

[2]A priori we cannot exclude that gravity is mediated by a massive boson leading to a Yukawa potential where the mass is constrained by current experiments: we won't consider this possibility here, but for a detailed analysis see [46], Section 2.3, p.81

[3]In our approach GR is intended as an Effective field theory, so one could also include non local terms in the action. For simplicity we will not consider them, however allowing them has deep phenomenological implications, for a review see [60]

and the corresponding gravitational field in that place. In the later we will just ignore self-interactions of gravity due to a gravitational $T_{\mu\nu}(x)$, considering only an external $T_{\mu\nu}(x)$ such that $\partial_\mu T^{\mu\nu}(x) = 0$.
Given these considerations, we are able to single out which Lagrangian and quantum mediator should correctly describe gravity.
If the mediator of gravity is a $s = 0$ particle, the gravitational field would be described by a scalar field with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + g\phi T \tag{1.2}$$

where g is a free-dimensional coupling and T the trace of the energy-momentum tensor of some external field. This is the only viable interaction which is local and respects Poincaré invariance. Unfortunately this would mean that gravity cannot couple to electromagnetism which has $T = 0$, in contrast with the fact that light bending is a well established experimental evidence in gravitational physics [4].
Therefore $s = 0$ is the wrong choice. We can check if $s = 1$ is a viable option, in that case gravity would be described by a vector field $A^\mu(x)$, the simplest representation containing a spin 1 particle.
Such a theory is constrained by $U(1)$ gauge invariance and any local interaction with an external $T_{\mu\nu}$ would have to be invariant under the replacement $A^\mu(x) \to A^\mu(x) + \partial^\mu\alpha(x)$.
This rules out interactions like $A_\mu A_\nu T^{\mu\nu}$ which are not gauge invariant, but also $\partial_\mu A_\nu T^{\mu\nu}$ that can be rewritten as $\partial_\mu(A_\nu T^{\mu\nu}) - A_\nu\partial_\mu T^{\mu\nu}$ and eliminated as a boundary term plus a null term due to the conserved $T^{\mu\nu}$. These were the only viable local and Poincaré invariant interactions under the constraint of gauge invariance.
This means that also $s = 1$ is wrong and that the only possibility that is left is a massless spin 2 particle, that we will call *"graviton"*.
The simplest finite irreducible representation of $SO(1,3)$, which contains a spin 2 representation for the Lie subgroup $SO(3)$, is a symmetric and traceless tensor with two indices

$$B^{\mu\nu} \in 0 \oplus 1 \oplus 2 \tag{1.3}$$

However if we build our Lagrangian with this field only we will obtain a mismatch of degrees of freedom: 9 for the classical field against the 2 of the graviton.
As in the case of electromagnetism, we can implement a sort of gauge invariance, which will help us in getting riddle of them.
Before to proceed further let's use a symmetric tensor with non zero trace $h_{\mu\nu} \in 0 \oplus (0 \oplus 1 \oplus 2)$: it only increase by one the undesired degrees of freedom.

[4]Indeed, this was the first confirmation of General Relativity. For a geometric derivation of the result see [73] p.136, Section 6.3

For such a symmetric tensor field, the simplest gauge transformation that we could guess is the following

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{1.4}$$

where ξ_μ is a generic 4-vector.
Up to four divergences, which can easily integrated out from the total action, there is a unique Lagrangian for $h_{\mu\nu}$ which respects Poincaré and gauge invariance

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \partial_\mu h_{\alpha\beta}\partial^\beta h^{\mu\alpha} - \partial^\mu h_{\mu\nu}\partial^\nu h \tag{1.5}$$

This is the so called *Fierz-Pauli Lagrangian*, from the names of the first who derived it[5].
As for interactions, the simplest coupling with an external energy momentum tensor is given by the following action

$$\mathcal{S}_{int} = \frac{g}{2}\int d^4x \; h_{\mu\nu}T^{\mu\nu} \tag{1.6}$$

where g is a free parameter with mass dimension [-2].
Beware only that in order to be consistent with previous requirements, it is crucial that the external $T^{\mu\nu}$ is conserved otherwise gauge invariance would be violated

$$\mathcal{S}'_{int} = \frac{g}{2}\int d^4x \; h'_{\mu\nu}T^{\mu\nu} = \frac{g}{2}\int d^4x \; h_{\mu\nu}T^{\mu\nu} + 2\partial_\mu \xi_\nu T^{\mu\nu} = \tag{1.7}$$

$$\frac{g}{2}\int d^4x \; h_{\mu\nu}T^{\mu\nu} - g\int d^4x \xi_\nu \partial_\mu T^{\mu\nu} = \mathcal{S}_{int} \quad \Leftrightarrow \quad \partial_\mu T^{\mu\nu} = 0 \tag{1.8}$$

In this respect, the action describing a spin 2 particle, coupled to an external energy-momentum tensor is

$$\mathcal{S} = \int d^4x \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \partial_\mu h_{\alpha\beta}\partial^\beta h^{\mu\alpha} - \partial^\mu h_{\mu\nu}\partial^\nu h + \frac{g}{2}h_{\mu\nu}T^{\mu\nu} \tag{1.9}$$

What remains undefined is the value of the g-coupling. As we will see in the next section, its absolute value can be fixed by matching the non relativistic limit of the underlying quantum theory with Newtonian gravity.

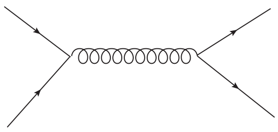
[5]see [33], Section 3.6 for a straightforward derivation

1.2 Newton's law from gravitons

To test the validity of our construction, we will take advantage of the fact that in the non relativistic limit of any Quantum field theory it is possible define a classical potential $V(x)$ by means of the following formula [6]

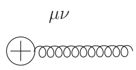
$$V(x) = -i \int \frac{d^3q}{(2\pi)^3} M_{if}(q) e^{iq\cdot x} \tag{1.10}$$

to see if it is possible to reproduce the Newton potential.
Here $M_{if}(q)$ is a three level amplitude of a $2 \to 2$ scattering with initial states equal to the finals and $q = (0, \vec{q})$. A tree level scattering mediated by a graviton would be described by different channels, like the s one



(1.11)

where two dashed lines (not necessarily fermions) exchange a graviton.
The vertex can be read off from the Lagrangian as $h_{\mu\nu}T^{\mu\nu}$ and the fact that $T^{\mu\nu}$ has at least two external fields. However, according to our approach, this is a given classical field and we can simply assume the following Feynman rule for the vertex



$$= -\frac{ig}{2}\tilde{T}^{\mu\nu}(q) \tag{1.12}$$

Equation 1.12 can be considered has the emission of a graviton from a static source and it can be derived from the Lagrangian by taking the field derivatives in $h_{\mu\nu}$, and multiplying by $-i$ after its Fourier Transform.
At this point we are only missing the free graviton propagator. As it is customary in ordinary QFT, this can be defined as the Green function of classical equations of motion, expressed according the $i\epsilon$ prescription. Given this, we can proceed with its evaluation in momentum space[7]

[6] see Maggiore's book [59] Section 6.6 and eqs.(7.56)-(7.59)for a derivation of this formula.
[7] For the example of a scalar field, see Pierre Ramond [65] Section 3.2

1.2.1 The graviton propagator

For a graviton, the kinetic part of the action is

$$\mathcal{S}_{kin} = \int d^4x\, \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \partial_\mu h_{\alpha\beta}\partial^\beta h^{\mu\alpha} - \partial^\mu h_{\mu\nu}\partial^\nu h \tag{1.13}$$

Let's express the $h_{\mu\nu}$ field via Fourier transform as

$$h_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{h}_{\mu\nu}(k)\, e^{ikx} \quad , \quad \tilde{h}^*_{\mu\nu}(k) = \tilde{h}_{\mu\nu}(-k) \tag{1.14}$$

Plugging it in the action gives

$$\begin{aligned}\mathcal{S}_{kin} = \int d^4x\, \frac{d^4k}{(2\pi)^4}\, \frac{d^4p}{(2\pi)^4}\, e^{ix\cdot(k+p)}\, \frac{k\cdot p}{2}\Big[\tilde{h}_{\mu\nu}(k)\tilde{h}^{\mu\nu}(p) - \tilde{h}(k)\tilde{h}(p)\Big] \\ -k^\mu p^\nu \Big[\tilde{h}^\alpha_\nu(k)\tilde{h}_{\mu\alpha}(p) - \tilde{h}_{\mu\nu}(k)\tilde{h}(p)\Big]\end{aligned} \tag{1.15}$$

The integral in x gives a Dirac's delta, which can be further integrated over the p momentum in order to give

$$\mathcal{S}_{kin} = \int \frac{d^4k}{(2\pi)^4}\frac{k^2}{2}\Big[\tilde{h}(k)\tilde{h}(-k) - \tilde{h}_{\mu\nu}(k)\tilde{h}^{\mu\nu}(-k)\Big] + k^\mu k^\nu \Big[\tilde{h}_{\alpha\nu}(k)\tilde{h}^\alpha_\mu(-k) - \tilde{h}_{\mu\nu}(k)\tilde{h}(-k)\Big] \tag{1.16}$$

In a compact form it can be expressed as

$$\mathcal{S}_{kin} = \int \frac{1}{2}\frac{d^4k}{(2\pi)^4}\, A^{\alpha\beta\mu\nu}(k)\tilde{h}_{\mu\nu}(k)\tilde{h}_{\alpha\beta}(-k) \tag{1.17}$$

where

$$A^{\alpha\beta\mu\nu} \equiv k^2\Big(\eta^{\alpha\beta}\eta^{\mu\nu} - \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu} - \frac{1}{2}\eta^{\alpha\nu}\eta^{\beta\mu}\Big) - k^\mu k^\nu \eta^{\alpha\beta} - k^\alpha k^\beta \eta^{\mu\nu} + \frac{1}{2}k^{(\mu}k^\alpha \eta^{\beta\nu)} \tag{1.18}$$

As we will see, the minimum of 1.17 is given (in Fourier space) by the following algebraic equations

$$A^{\alpha\beta\mu\nu}\tilde{h}_{\mu\nu}(k) = 0 \tag{1.19}$$

At this point, one would invert 1.18 in order to define a Feynman propagator for a free graviton. Unfortunately, this is not possible due to the presence of a redundancy in the description of the gravitational field.
The solution to this obstruction applies to all gauge theories: gauge invariance is broken at classical level adding a suitable gauge fixing term to the action, which makes possible to extract a suitable propagator.
In our theory, the gauge fixing term is called *De Donder*

$$\mathcal{S}_{GF} = -\int d^4x\Big(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h\Big)^2 = \int d^4x \partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\rho h_{\mu\nu}\partial^\nu h^{\rho\mu} - \frac{1}{4}\partial_\mu h \partial^\mu h \quad (1.20)$$

As before, the fields are expressed using a Fourier transform

$$\mathcal{S}_{GF} = \int d^4x\, d^4k\, d^4p\, e^{ix\cdot(k+p)}\, k_\mu p_\nu \Big[-\tilde{h}^{\mu\nu}(k)\tilde{h}(p) + \tilde{h}^{\mu\beta}(k)\tilde{h}^\nu_\beta(p)\Big] + \frac{1}{4}k\cdot p\,\tilde{h}(k)\tilde{h}(p)$$
(1.21)

Performing the x and ensuing p integration gives

$$\mathcal{S}_{GF} = \int d^4k\, k_\mu k_\nu \Big[\tilde{h}^{\mu\nu}(k)\tilde{h}(-k) - \tilde{h}^{\mu\beta}(k)\tilde{h}^\nu_\beta(-k)\Big] - \frac{k^2}{4}\,\tilde{h}(k)\tilde{h}(-k) \qquad (1.22)$$

Expressed in an equivalent way

$$\mathcal{S}_{GF} = \frac{1}{2}\int d^4k\, B^{\alpha\beta\mu\nu}\tilde{h}_{\mu\nu}(k)\tilde{h}_{\alpha\beta}(-k) \qquad (1.23)$$

where

$$B^{\alpha\beta\mu\nu} = -\frac{1}{2}k^2\eta^{\alpha\beta}\eta^{\mu\nu} + k^\mu k^\nu \eta^{\alpha\beta} + k^\alpha k^\beta \eta^{\mu\nu} - \frac{1}{2}k^{(\mu}k^\alpha\eta^{\beta\nu)} \qquad (1.24)$$

Summing to the kinetic term the gauge fixing one, results in the total action

$$\mathcal{S} = \mathcal{S}_{kin} + \mathcal{S}_{GF} = \frac{1}{2}\int d^4x\, C^{\alpha\beta\mu\nu}\tilde{h}_{\mu\nu}(k)\tilde{h}_{\alpha\beta}(-k) \qquad (1.25)$$

with

$$C^{\alpha\beta\mu\nu} = \frac{k^2}{2}(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\beta\nu} - \eta^{\mu\beta}\eta^{\nu\alpha}) \qquad (1.26)$$

The inverse of this tensor should be a complete symmetric tensor such that

$$C^{\alpha\beta\mu\nu}D_{\mu\nu\gamma\delta} = \frac{1}{2}\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + \frac{1}{2}\delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} \tag{1.27}$$

Expliciting the first term

$$k^2(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\beta\nu} - \eta^{\mu\beta}\eta^{\nu\alpha})D_{\mu\nu\gamma\delta} = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} \tag{1.28}$$

$$k^2(\eta^{\alpha\beta}D^{\mu}_{\mu\gamma\delta} - D^{\alpha\beta}_{\gamma\delta} - D^{\beta\alpha}_{\gamma\delta}) = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} \tag{1.29}$$

$$k^2(\eta^{\alpha\beta}D^{u}_{u\gamma\delta} - 2D^{\alpha\beta}_{\gamma\delta}) = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} \tag{1.30}$$

Taking the trace in the first pair of indexes, gives

$$k^2 D^{\mu}_{\mu\gamma\delta} = \eta_{\gamma\delta} \tag{1.31}$$

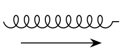
Inserting this trace in the previous equation gives

$$-2k^2 D^{\alpha\beta}_{\gamma\delta} = -\eta^{\alpha\beta}k^2 D^{\mu}_{\mu\gamma\delta} + \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} = -\eta^{\alpha\beta}\eta_{\gamma\delta} + \delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma} \tag{1.32}$$

Dividing by $2k^2$, raising all indexes and multiplying by i gives the desired propagator

$$D^{\alpha\beta\gamma\delta} = \frac{i}{2(k^2 + i\epsilon)}\Big(- \eta^{\alpha\beta}\eta^{\gamma\delta} + \eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma}\Big) \tag{1.33}$$

where it has been introduced the $i\epsilon$ prescription that select the desired Feynman propagator.
Equation 1.33 can be expressed in a diagrammatic way as

$\alpha\beta$  $\gamma\delta$ k $\equiv D^{\alpha\beta\gamma\delta}$ (1.34)

1.2.2 Fixing the coupling matter-gravity

Now we have all the elements to calculate the non relativistic $2 \to 2$ scattering. From 1.12 and 1.34 one can built a unique tree level amplitude

 $= M_{if}(q)$ (1.35)

$$M_{if}(q) = -\frac{g^2}{4}\tilde{T}_1^{\mu\nu}(q)D_{\mu\nu\alpha\beta}(q)\tilde{T}_2^{\alpha\beta}(-q) \tag{1.36}$$

As source of external fields we will use free relativistic particles on a given trajectory, with energy-momentum tensor given by (see Landau and Lifshitz, Vol.2 [54])

$$T^{\mu\nu} = \frac{p^\mu p^\nu}{p^0}\delta^3(x - x_0(t)) \tag{1.37}$$

Here $x_0(t)$ denotes a trajectory that in a non relativistic and static limit can be set equal to zero with $p^\mu = (m, \vec{0})$. After a Fourier transform of 1.37, one has that the amplitude can be approximated to its non relativistic expression using $|\vec{q}| >> |q^0|$

$$-\frac{g^2}{4}T_1^{00}(q)D_{0000}T_2^{00}(-q) = \frac{-ig^2 m_1 m_2}{8(q^2 + i\epsilon)} = \frac{-ig^2 m_1 m_2}{8|\vec{q}|^2} + O({}^{q^0}/_{|\vec{q}|}) \tag{1.38}$$

where we have removed the $i\epsilon$ prescription since no poles are encountered. Inserting this result in 1.10 one has that the induced potential is equal to

$$V(x) = -\frac{g^2 m_1 m_2}{8}\int \frac{d^3q}{(2\pi)^3}\frac{1}{|\vec{q}|^2}e^{iq\cdot x} \tag{1.39}$$

which gives[8]

$$V(x) = -\frac{g^2 m_1 m_2}{32\pi r} \tag{1.40}$$

This potential has to be matched with the well known Newton potential. As a remarkable fact there is a unique absolute value for which one obtains Newton's law of gravitation

$$V(x) = -\frac{G_N m_1 m_2}{r} \quad \Leftrightarrow \quad |g| = \sqrt{32\pi G} \tag{1.41}$$

This proves the consistency of our construction.[9]

[8] See Chapter 5, Section 5.2, for a derivation of the Fourier transform of ${}^{1}/_{|\vec{q}|^2}$

[9] Only powers of the coupling squared enters in physical observables, so sign ambiguities due to the absolute value of g has no physical consequences.

1.3 From gravitons to General Relativity

We have derived the action for a spin 2 particle coupled to an external energy-momentum tensor such that $\partial_\mu T^{\mu\nu} = 0$. The underlying quantum theory leads naturally to Newtonian gravity. However from Noether's theorem applied on 1.5 one should expect that also the gravitational field has an energy-momentum tensor, which means that the assumption $\partial_\mu T^{\mu\nu} = 0$ cannot hold for the external matter only.
As we will see, the resolution of this inconsistency will lead to General Relativity.

1.3.1 The self-energy of the gravitational field

To overcome the previous inconsistency we will remain within the classical limit of our theory.
This can be done by taking the minimum of the functional action, regarded as a function of $h_{\mu\nu}$.
It is a well known fact that minimum solutions are described by the following Euler-Lagrange equations

$$\frac{\delta \mathcal{S}}{\delta h_{\mu\nu}} = 0 \quad \Leftrightarrow \quad \partial_\rho \frac{\partial \mathcal{L}}{\partial_\rho h_{\mu\nu}} - \frac{\partial \mathcal{L}}{\partial h_{\mu\nu}} = 0 \tag{1.42}$$

Applying this procedure to 1.9 gives

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha\partial_\nu \bar{h}_{\mu\alpha} - \partial^\alpha\partial_\mu \bar{h}_{\nu\alpha} = -\frac{g}{2}T_{\mu\nu} \quad , \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha_\alpha \tag{1.43}$$

We can now take advantage of gauge invariance and impose the so called *De Donder gauge*

$$\partial_\mu \bar{h}^{\mu\nu} = 0 \tag{1.44}$$

To prove the validity of this gauge condition, let's assume to start with a $\bar{h}_{\mu\nu}$ field that doesn't satisfies 1.44 and let's see if exist a ξ_μ field that render the gauge transformed $h^{'}_{\mu\nu}$ the desidered one.
If so, we would have

$$\bar{h}^{'}_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho) \quad \Rightarrow \quad \partial_\mu \bar{h}^{\mu\nu} = \Box\xi^\nu \tag{1.45}$$

where partial derivative of 1.4 has been taken, here expressed in terms of $\bar{h}_{\mu\nu}$.
The equation so obtained admits always a solution, since the d' Alembertian operator is invertible, which proves 1.44.

At this point, the dynamics can be cast in this simpler form

$$\Box\bar{h}^{\mu\nu} = -\frac{g}{2}T^{\mu\nu} \tag{1.46}$$

The validity of equations 1.44, 1.46 necessarily implies the conservation of the external energy momentum tensor, which is possible only if $T^{\mu\nu}$ is a classical external source, and not a dynamical one.
Since in a full theory, it should be the total energy momentum tensor to be conserved, and not the matter one only, one could assume that the correct dynamics is

$$\Box\bar{h}^{\mu\nu} = -\frac{g}{2}(T^{\mu\nu}_m + t^{\mu\nu}_2) \tag{1.47}$$

where $t^{\mu\nu}_2$ is the energy momentum tensor coming from Noether's theorem applied on the Fierz-Pauli Lagrangian (the subscript 2 emphasize its quadratic dependence in $h_{\mu\nu}$).
This modified dynamics is compatible with the *De Donder gauge* condition since $\partial_\mu\bar{h}^{\mu\nu} = 0$ implies

$$\partial_\mu(T^{\mu\nu} + t^{\mu\nu}_2) = 0 \tag{1.48}$$

which is the conservation of the total energy momentum tensor, matter plus gravitational field, that we were looking for.
In order to derive 1.47 from a Lagrangian one has to add to the Fierz-Pauli one, a term cubic in $h_{\mu\nu}$ and proportional to g, thus leading to a non linear coupling of gravitons among themselves, typical of non abelian gauge theories.

$$\mathcal{L} = \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \partial_\mu h_{\alpha\beta}\partial^\beta h^{\mu\alpha} - \partial^\mu h_{\mu\nu}\partial^\nu h + \frac{g}{2}h^{\mu\nu}T_{\mu\nu} + \frac{g}{2}h^{\mu\nu}S_{\mu\nu} \tag{1.49}$$

where the cubic term in $h^{\mu\nu}$ involves contractions of at least three $h_{\mu\nu}$ with their derivatives.
The similarity with non abelian gauge theories can be seen also at the level of gauge transformations since the previous linear one is no longer valid. In fact the $h_{\mu\nu}T^{\mu\nu}$ term is no longer invariant since now $\partial_\mu T^{\mu\nu} \neq 0$. To overcome this obstacle one has to promote the previous linear gauge transformation to a non linear one of the form

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + gO(h\partial\xi) \tag{1.50}$$

where the tensorial structure of $O(h\partial\xi)$ can be fixed in order to render the previous Lagrangian gauge invariant.
The problem is that the procedure so defined cannot be stopped at a finite step.
Noether's theorem applied on 1.49 leads to a contribute to the total energy momentum tensor cubic in $h_{\mu\nu}$ and proportional to g, let's call it $gt^{\mu\nu}_3$.

Again, this should be inserted in the dynamics in order to obtain the conservation of the total energy momentum tensor

$$\Box \bar{h}_{\mu\nu} = -\frac{g}{2}(T^{\mu\nu} + t_2^{\mu\nu} + g t_3^{\mu\nu}) \tag{1.51}$$

however this should derive from a Lagrangian with a new term proportional to g^2 and quartic in $h_{\mu\nu}$ and so on.
The main result of this investigation is the following:
In order to define a consistent theory of interacting gravitons, one should include non linear terms with powers of g within a non linear gauge transformation.
What remains undefined is how to fix these non linear terms and the complete gauge transformation.

1.3.2 Failure of $\partial_\mu T^{\mu\nu} = 0$ in presence of gravity

Assuming that $\partial_\mu T^{\mu\nu} = 0$ holds for matter only, means that we are neglecting a mutual influence between the gravitational field and particles. In fact 1.5 states only the dynamics of the gravitational field due to a presence of matter, and not viceversa.
In order to relax this assumption we need to define the action for matter.
In case of free relativistic particles, one has

$$S_{particle} = \frac{m}{2}\int ds\, \dot{x}^\mu(s)\dot{x}^\nu(s)\eta_{\mu\nu} \tag{1.52}$$

where s is the proper time elapsed in the particle's frame and the derivatives are respect to this evolution parameter. As for particle-gravity interactions, this can be read from 1.5 as

$$S_{int} = \frac{g}{2}\int d^4x\, T^{\mu\nu}h_{\mu\nu} \tag{1.53}$$

Expressing 1.37 in the following equivalent way

$$T^{\mu\nu} = m\int ds\, \delta^4(x - x_0(s))\dot{x}_0^\mu \dot{x}_0^\nu \tag{1.54}$$

one has that the interacting term can be written as

$$S_{int} = \frac{mg}{2}\int ds \dot{x}_0^\mu \dot{x}_0^\nu h_{\mu\nu}(x_0(s)) \tag{1.55}$$

Thus, relativistic particles coupled to gravity has the following action

$$S = \frac{m}{2}\int ds\, \dot{x}_0^\mu(s)\dot{x}_0^\nu(s) g_{\mu\nu}(x_0(s)) \quad ; \quad g_{\mu\nu} = \eta_{\mu\nu} + g h_{\mu\nu} \tag{1.56}$$

where we have defined the so called *metric tensor* on the left, including both $\eta_{\mu\nu}$ and $h_{\mu\nu}$.
Particles trajectories are now derived by minimizing 1.56 with respect to the world-lines

$$\delta_x S = 0 \quad \Leftrightarrow \int ds\, \delta_x \left(\dot{x}^\mu \dot{x}^\nu g_{\mu\nu} \right) = 0 \tag{1.57}$$

$$\int ds\, 2 g_{\mu\nu} \dot{x}^\nu \delta \dot{x}^\mu + \dot{x}^\mu \dot{x}^\nu \delta_x g_{\mu\nu} = 0 \tag{1.58}$$

Integrating by parts the first term and explicitng the variation for the metric tensor

$$\int ds\; - 2 g_{\mu\nu} \ddot{x}^\nu \delta x^\mu - 2 \partial_\alpha g_{\mu\nu} \dot{x}^\alpha \dot{x}^\nu \delta x^\mu + \dot{x}^\mu \dot{x}^\nu \partial_\alpha g_{\mu\nu} \delta x^\alpha = 0 \tag{1.59}$$

At this point we symmetrize in α, ν the second term, while exchanging the indices α and μ in the last term

$$\int ds\; - 2 g_{\mu\nu} \ddot{x}^\nu \delta x^\mu - \partial_\alpha g_{\mu\nu} \dot{x}^\alpha \dot{x}^\nu \delta x^\mu - \partial_\nu g_{\alpha\mu} \dot{x}^\nu \dot{x}^\alpha \delta x^\mu + \dot{x}^\alpha \dot{x}^\beta \partial_\mu g_{\alpha\beta} \delta x^\mu = 0 \tag{1.60}$$

We are now able to express the total variation in a close form as

$$\int ds\, \delta x^\mu \left[- 2 g_{\mu\nu} \ddot{x}^\nu - \partial_\alpha g_{\mu\nu} \dot{x}^\alpha \dot{x}^\nu - \partial_\nu g_{\alpha\mu} \dot{x}^\nu \dot{x}^\alpha + \dot{x}^\alpha \dot{x}^\beta \partial_\mu g_{\alpha\beta} \right] = 0 \tag{1.61}$$

The integral is null for a generic δx^μ only if it is null the term in the square brackets. This defines particle's motion influenced by gravity as

$$g_{\mu\nu} \ddot{x}^\nu = - \Gamma_{\mu\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{1.62}$$

$$\Gamma_{\mu\alpha\beta} \equiv \frac{1}{2} \left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta} \right) \tag{1.63}$$

This are the geodesic equations for particles influenced by gravity where 1.63 is usually called *Christoffel symbol of the first kind*.
Now we have all the elements to understand why $\partial_\mu T^{\mu\nu} = 0$ cannot hold for matter only.
To realize it, let's take the partial derivative of $T^{\mu\nu}$ matter as expressed in 1.54

$$\partial_\mu T^{\mu\nu} = \int ds\, \partial_\mu \delta(x - x_0(s)) \dot{x}^\mu(s) \dot{x}^\nu(s) \tag{1.64}$$

Using the fact that

$$\partial_\mu f(x(s))\frac{dx^\mu}{ds} = \frac{df}{ds}(x(s)) \tag{1.65}$$

we have

$$\partial_\mu T^{\mu\nu} = -\int ds\, \delta^4(x - x_0(s))\ddot{x}^\nu(s) \tag{1.66}$$

where we have integrated by parts the derivative of the Dirac's delta so obtained.
Let's now multiply by $g_{\nu\sigma}$, expressing the acceleration of the worldline due to gravity by means of 1.63 we arrive at

$$g_{\sigma\nu}\partial_\mu T^{\mu\nu} + \Gamma_{\sigma\alpha\beta}T^{\alpha\beta} = 0 \tag{1.67}$$

This is the continuity equation for a matter $T^{\mu\nu}$ in presence of gravity. From it is now clear that $\partial_\mu T^{\mu\nu} = 0$ was wrong, since we have neglected the mutual influence of gravity and matter.

1.3.3 The Einstein-Hilbert action

To be able to define a consistent theory of gravity, containing non-linear interactions among gravitons, we must find a functional action, generalizing the *Fierz-Pauli* such that

$$\frac{\delta\mathcal{S}}{\delta h_{\mu\nu}} = -\frac{g}{2}T^{\mu\nu} \tag{1.68}$$

being $T^{\mu\nu}$ an external energy momentum tensor now satisfying 1.67.
This new action will have to reduce to the solely action describing massless spin 2 particle, once self-interactions are negligible.
Given this considerations, let's take the partial derivative ∂_μ on both sides of 1.68, followed by a contraction with the metric tensor

$$g_{\sigma\nu}\partial_\mu\frac{\delta\mathcal{S}}{\delta h_{\mu\nu}} = -\frac{g}{2}g_{\sigma\nu}\partial_\mu T^{\mu\nu} \tag{1.69}$$

This can be expressed in terms of the solely gravitational field using 1.67 and 1.68

$$g_{\sigma\nu}\partial_\mu\frac{\delta\mathcal{S}}{\delta h_{\mu\nu}} + \Gamma_{\sigma\alpha\beta}\frac{\delta\mathcal{S}}{\delta h_{\mu\nu}} = 0 \tag{1.70}$$

Unfortunately, equation 1.70 is a differential functional equation and there is no simple procedure which generate solutions to this vector equation, stated as it is.
An alternative way to express 1.70 is by a contraction with an arbitrary A^σ and integrating over the space-time in order to obtain a scalar equation.
This yields to

$$\int d^4x A^\sigma g_{\sigma\nu}\partial_\mu\frac{\delta\mathcal{S}}{\delta h_{\mu\nu}} + A^\sigma\Gamma_{\sigma\alpha\beta}\frac{\delta\mathcal{S}}{\delta h_{\mu\nu}} = 0 \tag{1.71}$$

If satisfied by an arbitrary A^σ then 1.71 implies 1.70.
We may now integrate by part the first term in the integrand so as to get rid of the derivative in ∂_μ. We deduce

$$\int d^4x \frac{\delta \mathcal{S}}{\delta h_{\mu\nu}}[-\partial_\mu(A^\sigma g_{\sigma\nu}) + A^\sigma \Gamma_{\sigma\alpha\beta}] = 0 \tag{1.72}$$

where the square brackets is intended as symmetric under the exchange of μ and ν, since coupled to the symmetric variation $\delta h_{\mu\nu}$.
We can interpret this equation in another useful way.
Under an infinitesimal shift in $h_{\mu\nu}$, by a generic quantity, $\epsilon_{\mu\nu}$ one has

$$\mathcal{S}(h_{\mu\nu} + \epsilon_{\mu\nu}) = \mathcal{S}(h_{\mu\nu}) + \int d^4x \, \frac{\delta \mathcal{S}}{\delta h_{\mu\nu}} \epsilon_{\mu\nu} + .. \tag{1.73}$$

By matching the infinitesimal $\epsilon_{\mu\nu}$ to the terms in the square bracket of 1.72 we deduce

$$\mathcal{S}(h_{\mu\nu} + \epsilon_{\mu\nu}) = \mathcal{S}(h_{\mu\nu}) \tag{1.74}$$

which means that the desired action has to be invariant under the following infinitesimal transformation

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\partial_\mu(A^\sigma g_{\sigma\nu}) - \frac{1}{2}\partial_\nu(A^\sigma g_{\sigma\mu}) + A^\sigma \Gamma_{\sigma\mu\nu} \tag{1.75}$$

To render explicit the meaning of this requirement, we can express 1.75 in terms of $g_{\mu\nu}$

$$g'_{\mu\nu} = g_{\mu\nu} + \partial_\mu \xi^\sigma g_{\sigma\nu} + \partial_\nu \xi^\sigma g_{\sigma\mu} + \xi^\sigma \partial_\sigma g_{\mu\nu} \tag{1.76}$$

where we have defined $-\frac{g}{2}A^\sigma = \xi^\sigma$.
In equation 1.76 we can recognize the transformation of $g_{\mu\nu}$, regarded as a tensor under infinitesimal change of coordinates $x'^\mu = x^\mu + \xi^\mu$.
Given this symmetry, the task of finding solutions to the previous functional equation is now reduced to the simpler construction of a functional action invariant under infinitesimal change of coordinates.
However, these also affect the measure of integration so one is first required to define an invariant measure, otherwise it is hopeless the definition of invariant involving $g_{\mu\nu}$.
Let's first define the inverse of the metric tensor, such that

$$g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu \tag{1.77}$$

We now examine an invariant measure which may easily be found by looking at the transformation properties of the determinant.

It can be shown that determinant of a matrix satisfies the following relation

$$\det A = e^{Tr \log A} \tag{1.78}$$

we shall not stop here to prove it, but to make it appear reasonable, let's notice that this equality is trivial satisfied for a diagonal matrix since

$$\det A = A_{11}A_{22}...A_{nn} = e^{\log A_{11}+\log A_{22}+...} = e^{TrlogA} \tag{1.79}$$

We now use 1.78 in order to evaluate the determinant of a matrix $A+B$ when B is infinitesimal.

$$\det(A+B) = \det A \ \det(1+A^{-1}B) = \det A \, e^{Tr \log(1+A^{-1}B)} \tag{1.80}$$

$$\log \ \det(A+B) = \log \ \det A + TrA^{-1}B + O(B^2) \tag{1.81}$$

According to the transformation rule 1.76, we can regard the terms proportional to ξ as an infinitesimal matrix. Applying on both sides the determinant the expansion 1.81 leads to

$$\log \ \det(-g^{'}_{\mu\nu}) = \log \ \det(-g_{\mu\nu}) + 2\partial_\alpha\xi^\alpha + \xi^\sigma\partial_\sigma \log \ \det(-g_{\mu\nu}) \tag{1.82}$$

where for the last term we have used the following relation

$$\partial_\sigma g_{\mu\nu} g^{\mu\nu} = \partial_\sigma \log \ \det(-g_{\mu\nu}) \tag{1.83}$$

We set $C = \frac{1}{2}\log \ \det(-g_{\mu\nu})$ and rewrite the resulting equation as

$$C^{'} = C + \partial_\mu\xi^\mu + \partial_\mu C\xi^\mu \tag{1.84}$$

The last terms start to resemble total derivative which can be integrated out in order to obtain our invariant.
This can be noticed by exponentiation, followed by a subsequent expansion at linear order in ξ^μ

$$e^{C'} = e^C + e^C(\partial_\mu\xi^\mu + \partial_\mu C^\mu) = e^C + \partial_\mu(e^C\xi^\mu) \tag{1.85}$$

Integrating over all space-time we get that the following is an invariant measure of integration

$$\int d^4x\sqrt{-\det \ g_{\mu\nu}} \tag{1.86}$$

Unfortunately, even if 1.86 is solution to the functional differential equation previously stated, this cannot be regarded as a solution needed for our theory since there are no derivatives in $g_{\mu\nu}$.
To proceed further, let's first define the so called *Christoffel symbols of the second kind*

$$\Gamma^\tau_{\mu\nu} \equiv g^{\tau\sigma}\Gamma_{\sigma\mu\nu} \tag{1.87}$$

Within $\Gamma^{\tau}_{\mu\nu}$ it is possible to construct a tensor under arbitrary change of coordinates containing second derivatives of $g_{\mu\nu}$. It is usually called *Riemann curvature tensor*

$$R^{\tau}_{\mu\nu\rho} \equiv \partial_\rho \Gamma^{\tau}_{\mu\nu} - \partial_\nu \Gamma^{\tau}_{\mu\rho} + \Gamma^{\tau}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\tau}_{\nu\lambda}\Gamma^{\lambda}_{\mu\rho} \tag{1.88}$$

It's transformation properties can be derived in a straightforward, even if tedious, way from those of $g_{\mu\nu}$ as

$$R^{\tau}_{\mu\nu\rho}{}' = R^{\tau}_{\mu\nu\rho} + \partial_\nu \xi^\lambda R^{\tau}_{\mu\lambda\rho} + \partial_\mu \xi^\lambda R^{\tau}_{\lambda\nu\rho} + \partial_\rho \xi^\lambda R^{\tau}_{\mu\lambda\nu} + \partial_\lambda \xi^\tau R^{\lambda}_{\mu\rho\nu} + \xi^\lambda \partial_\lambda R^{\tau}_{\mu\nu\rho} \tag{1.89}$$

From it, one defines the so called *Ricci tensor* and *Ricci scalar*

$$R^{\tau}_{\mu\nu\tau} \equiv R_{\mu\nu} \quad R \equiv g^{\mu\nu} R_{\mu\nu} \tag{1.90}$$

the last, being the simplest invariant under infinitesimal change of coordinates containing second derivatives of the metric tensor.
Using these tensors, the simplest proposal for a complete theory is given by the following action

$$\mathcal{S}(\eta_{\mu\nu} + g h_{\mu\nu}) = \frac{2}{g^2} \int d^4x \sqrt{-\det\ g_{\mu\nu}}\, R \tag{1.91}$$

where the g^2 coupling is there for dimensional arguments.
Regarding the metric tensor $g_{\mu\nu}$ as the correct Lagrangian variable, we conclude that

$$\mathcal{S}(g_{\mu\nu}) = \frac{1}{16\pi G_N} \int d^4x \sqrt{-\det\ g_{\mu\nu}}\, R \tag{1.92}$$

This is the *Einstein-Hilbert* action for General Relativity.
The minimum of this functional action defines the so called *Einstein's field equations* which nowadays are the best description of gravity at large scales[10]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \bar{T}_{\mu\nu} \quad ; \quad \bar{T}_{\mu\nu} \equiv \frac{T_{\mu\nu}}{\sqrt{-\det\ g_{\mu\nu}}} \tag{1.93}$$

[10] see [73] p.450 Appendix E.1 for a derivation of this result

This dynamics correctly reduces to that coming from 1.43.
It's enough to set $g_{\mu\nu} = \eta_{\mu\nu} + gh_{\mu\nu}$, so that to linear order in $h_{\mu\nu}$ the Riemann tensor becomes

$$R^{\mu}_{\nu\rho\sigma} = \frac{g}{2}(\partial_\nu \partial_\rho h^{\mu}_{\sigma} + \partial^\mu \partial_\sigma h_{\nu\rho} - \partial_\nu \partial_\rho h^{\mu}_{\sigma} - \partial_\nu \partial_\sigma h^{\mu}_{\rho}) \tag{1.94}$$

where the indices are raised and lowered using the Minkowski metric $\eta_{\mu\nu}$.
It is now a direct calculation to show that equations 1.93 reduces to

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_\nu \bar{h}_{\mu\alpha} - \partial^\alpha \partial_\mu \bar{h}_{\nu\alpha} = -\frac{g}{2}T_{\mu\nu} \quad , \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\alpha}_{\alpha} \tag{1.95}$$

thus showing the correctness of our method.
In order to make the physical meaning of this linearization as clear as possible, one can rescale the $h_{\mu\nu}$ field by a g^{-1} factor in the definition of the metric tensor so as to make it adimensional. From this point for view, the linearization in $h_{\mu\nu}$ can be regarded as a weak field expansion around the Minkowski one, being

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad ; \quad |h_{\mu\nu}| << 1 \tag{1.96}$$

The associated theory is usually called linearized General Relativity and its action is

$$\mathcal{S} = \frac{1}{g^2}\int d^4x \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \partial_\mu h_{\alpha\beta}\partial^\beta h^{\mu\alpha} - \partial^\mu h_{\mu\nu}\partial^\nu h \tag{1.97}$$

whose corresponding Lagrangian is nothing else than the rescaled *Fierz-Pauli* one.
As for the complete theory, the *Einstein-Hilbert action* is not the solely one which reproduces this action. Indeed, one could also add to the action powers of the *Ricci* scalar, contractions involving the *Ricci* or *Riemann* tensor, or non local terms: all these corrections goes under the name of *modified gravity*, and are object of current research and we will not address them here.
As for the purpose of this thesis, we will restrict to the *Einstein-Hilbert* action as the correct one describing gravity.

Chapter 2

Gravitational Waves

One of the most important predictions of General Relativity (GR) is the existence of gravitational waves emitted by the dynamics of binary systems.
Although theorized since the formulation of GR, the announcement of their first detection was made only recently, precisely on February 11 2016 by the LIGO and Virgo collaborations (see [1, 3]). This has signed a milestone in the history of modern physics since it has opened the era of gravitational-wave astronomy and cosmology. For this reason it is of extreme importance a complete knowledge on gravitational waves, to which this entire chapter is devoted to. The list of topics that will be treated is the following:

- Gravitational waves will be introduced as solutions to linearized GR. It will be derived a formula for the energy and momentum flux emitted by a gravitational wave exploiting Noether's theorem on the Pauli-Fierz Lagrangian.

- It will be derived the expression for gravitational waves emitted by a source. The process will be studied in a non relativistic fashion by means of multipoles expansion. It will be derived the quadrupole radiation formula and the proportionality relation between the frequency of the orbital motion and that of the emitted gravitational radiation.

- As application, it will be studied the inspiral phase of a binary using linearized GR, studying its radiated power and time of coalescence. Numerical estimates will be treated in detail, useful for providing a phenomenological view on these systems.

- In order to describe more accurately a binary system it will be introduced the *post-Newtonian* approach. It will be shown how physical quantities like the phase and amplitude of a gravitational wave gets modified within the *post-Newtonian* scheme respect to the solely linearized GR.

2.1 Gravitational waves from General Relativity

In the previous chapter we have derived the *Einstein-Hilbert* action showing that its equations of motion reduces to linearized GR once the gravitational field is weak. As we have seen, this linearized dynamics is governed by the rescaled *Fierz-Pauli* one. In empty space $T^{\mu\nu} = 0$, and in the *De Donder* gauge, one has

$$\Box \bar{h}_{\mu\nu} = 0 \quad \Rightarrow \quad \bar{h}_{\mu\nu} = e_{\mu\nu}(k) e^{ik\cdot x} \tag{2.1}$$

Solutions to 2.1 are called *Gravitational waves*, since they are plane waves with $k^\mu = (\omega/c, \vec{k})$ and $\omega/c = |k|$. However, the *De Donder* gauge hasn't fix the gauge completely, since it is still possible to perform a gauge transformation that satisfies

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho) \quad \Leftrightarrow \quad \Box \xi^\mu = 0 \tag{2.2}$$

By looking at the degrees of freedom of $h_{\mu\nu}$ it makes sense: this is a symmetric tensor of 10 components describing a massless spin 2 particle, since the *De Donder* gauge has eliminated 4 of them, we should expect that other 4 can be eliminated in order to match the 2 degrees of freedom of the corresponding graviton.
This can be accomplished fixing the remaining four components of the ξ vector of 2.2 in order to have

$$\bar{h} = 0 \quad , \quad \bar{h}^{0\mu} = 0 \quad \Rightarrow \quad \bar{h}_{\mu\nu} = h_{\mu\nu} \tag{2.3}$$

Unite with the De Donder gauge, the tranverse-traceless gauge (TT), so called, can be expressed as

$$h^i_i = 0 \quad , \quad h^{0\mu} = 0 \quad , \quad \partial_i h^{ij} = 0 \tag{2.4}$$

Using this, and assuming for definiteness that the gravitational wave travels along the z-axis, a solution to 2.1 can be expressed in terms of a 3×3 symmetric tensor with null trace[1]

$$h^{TT}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[w(t-z)] \tag{2.5}$$

The solution is expressed in terms of the two amplitudes h_+ and $h_\times$: these are the remaining degrees of freedom of our gravitational wave and matches exactly those of a graviton, as it must be by construction of the theory.

[1]The upperscript TT stands for the tranverse-traceless gauge: it can be shown (see [46] p.8) that this gauge is valid only in empty space, or far from a source that is producing gravitational waves.

2.1.1 The energy-momentum tensor for a gravitational wave

The dynamics of a gravitational wave in empty space derives from the Fierz-Pauli Lagrangian.
It is a well known fact that a Lagrangian invariant under space-time translations admits, via Noether theorem, a conserved energy momentum tensor given by

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial_\mu \phi_i}\partial^\nu \phi_i + \eta^{\mu\nu}\mathcal{L} \quad ; \quad \partial_\mu T^{\mu\nu} = 0 \tag{2.6}$$

being ϕ_i the proper relativistic field.
For the case of linearized GR one has

$$t_{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu h_{\alpha\beta})}\partial_\nu h_{\alpha\beta} + \eta_{\mu\nu}\mathcal{L} \quad \Rightarrow \quad \partial_\mu t^{\mu\nu} = 0 \tag{2.7}$$

where we have kept the same conventions of the previous chapter to denote the gravitational energy-momentum tensor. Using the expression for the rescaled Fierz-Pauli Lagrangian, valid in linearized General Relativity, the energy-momentum tensor for a gravitational wave in absence of matter is

$$t_{\mu\nu} = \frac{c^4}{32\pi G_N}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} \tag{2.8}$$

where we have restored the velocity of light c.
One of the peculiar aspects of 2.8 is that, unlike electromagnetic waves, it is not possible to use it in order to define a local energy density which is gauge invariant. In electromagnetism, the energy-momentum tensor makes possible to define $\frac{1}{2}(E^2 + B^2)$ as a gauge invariant quantity whose integral gives the total energy of the system; in linearized General Relativity, it can be shown that it doesn't exist a gauge invariant tensor with this property [46].
This is due to the fact that 2.8 can always be set to zero thanks to equivalence principle since, according to it, it is always possible to define an inertial frame so that the effect of gravity are locally absent.
However, the lack of a gauge invariant local energy density it is not a problem for our purpose, since the only measurable quantity is the total energy emitted, which is not a local quantity.

2.1.2 The energy and momentum flux of a gravitational wave

In order to derive the energy flux, we can integrate on a fixed volume V the right part of 2.7 in order to obtain the following continuity equation

$$\int_V d^3x\, \partial_\mu t^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{P}^\nu = -\int_{\partial V} d\vec{A}\, t_{i\nu} \tag{2.9}$$

where we have defined $P^\nu = t^{0\nu}$.
Posing $\nu = 0$ leads to the conservation of the energy

$$\frac{dE}{dt} = -\int_{\partial V} d\vec{A} \cdot t_{i0} \tag{2.10}$$

Assuming that the volume is a sphere, we have $d\vec{A} = r^2 d\Omega \vec{n}$

$$\frac{dE}{dt} = \frac{c^4 r^2}{32\pi G_N} \int d\Omega\, n_i\, \partial_0 h_{\alpha\beta} \partial^i h^{\alpha\beta} \tag{2.11}$$

where the overall sign is positive since we are interested in the received energy at infinity.
In particular at sufficiently large distances r, it can be shown[2] that a gravitational wave has the general form

$$h_{\alpha\beta} = \frac{1}{r} f_{\alpha\beta}(t - {}^r\!/_c) \quad \Rightarrow \quad \partial_i h_{\alpha\beta} = -\partial_0 h_{\alpha\beta} + O({}^1\!/_{r^2}) \tag{2.12}$$

where the i direction has been chosen in order to be parallel to the radial vector $\vec{r}$, while $f_{\alpha\beta}$ stands for a generic tensor function of the retarded time $t - {}^r\!/_c$.
Imposing the TT gauge outside the source, we arrive at the following simple result

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G_N} \int d\Omega \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \tag{2.13}$$

In a similar manner, one can derive from 2.9 the flux of momentum emitted by a gravitational wave

$$\frac{dP^k}{dt} = -\frac{c^3 r^2}{32\pi G_N} \int d\Omega \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \tag{2.14}$$

[2] see Maggiore [46], section 3.1

2.2 Generation of gravitational waves

In this section we are going to study how gravitational waves can be generated from the dynamics of physical sources.
Let's assume that the background is the usual Minkowski one, so that the source is able to produce only a weak gravitational field. Thus the gravitational waves produced from the motion of a source satisfy the following equation in *De Donder* gauge

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G_N}{c^4} T_{\mu\nu} \tag{2.15}$$

where the $T_{\mu\nu}$ is the stress energy tensor of the source.
A solution to 2.15 can be found using the method of the Green's function[3], that is assuming

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G_N}{c^4} \int d^4y \, G(x-y) T_{\mu\nu}(y) \quad \Leftrightarrow \quad \Box G(x-y) = \delta^4(x-y) \tag{2.16}$$

For a radiation problem, a convenient solution for the right part of 2.16 is given by the so called retarded *Green* function

$$G(x-y) = -\frac{1}{4\pi |x-y|} \delta^0(x^0_{ret} - y^0) \quad , \quad x^0_{ret} = ct - |x-y| \tag{2.17}$$

Replacing this solution in 2.16 we obtain the following solution to 2.15

$$\bar{h}_{\mu\nu}(x) = \frac{4G_N}{c^4} \int d^3y \, \frac{1}{|x-y|} T_{\mu\nu}(ct - |x-y|, y) \tag{2.18}$$

In order to reduce redundancy in the degrees of freedom, we can fix the TT gauge outside the source where the vacuum equation holds, thus obtaining

$$h^{TT}_{ij}(x) = \frac{4G_N}{c^4} \, \Lambda_{ijkl}(\vec{n}) \int d^3y \, \frac{1}{|x-y|} T_{kl}(ct - |x-y|, y) \tag{2.19}$$

where we have defined for convenience the following projector

$$\Lambda_{ijkl}(\vec{n}) \equiv P_{ik}P_{jl} - \frac{1}{2} P_{ij}P_{kl} \quad , \quad P_{ij} = \delta_{ij} - n_i n_j \quad , \quad n^i n_i = 1 \tag{2.20}$$

such that

$$\Lambda_{ijkl}\Lambda_{klmn} = \Lambda_{ijmn} \quad , \quad n^i \Lambda_{ijkl} = 0 \quad , \quad \Lambda_{iikl} = \Lambda_{ijkk} = 0 \tag{2.21}$$

[3] see Jackson [48] for a pedagogical introduction on the subject.

2.2.1 Multipoles expansion

Normally, the source of gravitational waves and the point at which they are detected are separated by a distance $r >> d$ where d denotes the typical radius of the source. Within this assumption we can express the retarded time as

$$ct - |x - y| = ct - r - y \cdot n + O(d^2/r^2) \quad , \quad \vec{x} = r\vec{n} \tag{2.22}$$

$$h_{ij}^{TT}(x) = \frac{4G_N}{rc^4}\,\Lambda_{ijkl}(\vec{n}) \int d^3y\, T_{kl}(ct - r - y \cdot n, y) \tag{2.23}$$

were the integration is restricted to $y \leq d$, since outside $T_{\mu\nu}$ is null.
In general it is not possible to evaluate this three-dimensional integral, and it is necessary to adopt other approximations besides the large distances one: one of them is a perturbative approach called *multipoles expansion* which is valid only if the typical velocity of the source is non relativistic.
In order to define the expansion, first one should Fourier transform the expression for the energy-momentum tensor in 2.23 as

$$T_{kl}(ct - r + y \cdot n, y) = \int \frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega, k)e^{-i(\omega t - \frac{r}{c} + \frac{y \cdot n}{c}) + ik \cdot y} \tag{2.24}$$

For a typical non relativist source, one should expect that the energy momentum tensor is peaked around a typical frequency ω_s such that $\omega_s d \sim v << c$ where v denotes a typical velocity of the source.
This means that the following relations holds for a non relativistic source

$$\frac{\omega_s}{c} y \cdot n \leq \frac{\omega_s d}{c} \sim \frac{v}{c} << 1 \tag{2.25}$$

Using 2.25 we can expand the exponential in 2.24 using $\omega y \cdot n$ as an expansion parameter since, as it will be demonstrated, $\omega_s \sim \omega$

$$e^{-i(\omega t - \frac{r}{c} + \frac{y \cdot n}{c})} = e^{-i(\omega t - \frac{r}{c}}\Big(1 - i\frac{\omega}{c}y^i n^i + \frac{1}{2}\Big(-i\frac{\omega}{c}\Big)^2 y^i y^j n^i n^j + ...\Big) \tag{2.26}$$

This gives the so called "multipoles expansion"

$$T_{kl}(ct - r + y \cdot n, y) \simeq T_{kl}(ct - r, y) + \frac{y^i n^i}{c}\partial_0 T_{kl} + \frac{1}{2c^2}y^i y^j n^i n^j \partial_0^2 T_{kl} + ... \tag{2.27}$$

which can be regarded as a Taylor series using $(y \cdot n)/c$ as an expansion parameter. As for the terms in the sum, the first is called "dipole", the second "quadrupole", the third "sixtupole" and so on.

The starting point in the study of multipoles expansion come with the definition of the momenta for the stress-energy tensor T^{ij}

$$S^{ij}(t) \equiv \int d^3x\, T^{ij}(x,t) \tag{2.28}$$

$$S^{ij,k}(t) \equiv \int d^3x\, T^{ij}(x,t)\, x^k \tag{2.29}$$

$$S^{ij,kl}(t) \equiv \int d^3x\, T^{ij}(x,t)\, x^k x^l \tag{2.30}$$

and similarly for higher order momenta.
In terms of them, equation 2.19 becomes

$$h_{ij}^{TT}(x) = \frac{4G_N}{rc^4}\Lambda_{ijkl}(\vec{n})\Big[S^{kl}(ct-r)+\frac{n_m}{c}\dot{S}^{kl,m}(ct-r)+\frac{n_p n_m}{2c^2}\ddot{S}^{kl,mp}(ct-r)+..\Big] \tag{2.31}$$

Equation 2.31 is the basis for the multipoles expansion.
From definitions 2.28,2.29,2.30 we see that, respect to S^{ij}, $S^{ij,k}$ has an additional factor $x^m \sim O(d)$. Since time derivatives bring a factor $O(\omega_s)$ we have that $\dot{S}^{ij,k}$ has an additional factor $O(\omega_s d \sim v)$ with respect to S^{ij}, which means that $\frac{n_m}{c}\dot{S}^{kl,m}$ introduces $O(\frac{v}{c})$ corrections with respect to S^{kl}. Similarly the term $\frac{n_p n_m}{2c^2}\ddot{S}^{kl,mp}$ introduces $O(\frac{v^2}{c^2})$ corrections and so on. Therefore, the multipoles expansion can be regarded as a non relativistic expansion in powers of $\frac{v}{c}$.
At this point, is convenient to define the momenta of the other components of $T^{\mu\nu}$.
For T^{00} one has

$$M \equiv \frac{1}{c^2}\int d^3x T^{00}(t,x) \tag{2.32}$$

$$M^i \equiv \frac{1}{c^2}\int d^3x T^{00}(t,x)\, x^i \tag{2.33}$$

$$M^{ij} \equiv \frac{1}{c^2}\int d^3x T^{00}(t,x)\, x^i x^j \tag{2.34}$$

$$M^{ijk} \equiv \frac{1}{c^2}\int d^3x T^{00}(t,x)\, x^i x^j x^k \tag{2.35}$$

while for the momenta of T^{0i}

$$P^i \equiv \frac{1}{c}\int d^3x T^{0i}(t,x) \tag{2.36}$$

$$P^{i,j} \equiv \frac{1}{c}\int d^3x T^{0i}(t,x)\, x^j \tag{2.37}$$

$$P^{i,jk} \equiv \frac{1}{c} \int d^3x T^{0i}(t,x)\, x^j x^k \tag{2.38}$$

and so on. The link between the various momenta comes from the conservation of the associated energy momentum tensor. As example

$$c\dot{M}^i = \int_V d^3x\, x^i \partial_0 T^{00} = -\int_V d^3x\, x^i \partial_j T^{0j} = \int_V d^3x\, (\partial_j x^i) \partial_0 T^{0j} = cP^i \tag{2.39}$$

Using the same procedure, it is possible to derive relations for all the momenta involved. Here we will report only few of them

$$\dot{M} = 0 \tag{2.40}$$

$$\dot{M}^i = P^i \tag{2.41}$$

$$\dot{M}^{ij} = P^{ij} + P^{ji} \tag{2.42}$$

$$\dot{M}^{ij,k} = P^{i,jk} + P^{j,ki} + P^{k,ij} \tag{2.43}$$

while for the momenta of T^{0i}

$$\dot{P}^i = 0 \tag{2.44}$$

$$\dot{P}^{i,j} = S^{ij} \tag{2.45}$$

$$\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j} \tag{2.46}$$

where equations $\dot{M} = 0$ and $\dot{P}^i = 0$ can be regarded as the conservation of the mass and the total momentum of the source.
Using these relations we are able to express the various momenta S^{ij}, $\dot{S}^{ij,k}$ and so on, in terms of the two set $M, M^{ij}, ..$ and $P^i, P^{i,j}, ...$
For the first term in the multipoles expansion 2.31 we get

$$S^{ij} = \frac{1}{2}\ddot{M}^{ij} \tag{2.47}$$

Combining equation 2.43 with equation 2.46 instead, we have

$$\dddot{M}^{ijk} = 2(\dot{S}^{ij,k} + \dot{S}^{j,ik} + \dot{S}^{k,ij}) \tag{2.48}$$

which can be used in order to define the second term in the multipoles expansion

$$\dot{S}^{ij,k} = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} + \ddot{P}^{k,ij}\right) \tag{2.49}$$

In a similar manner, one proceed if higher multipole corrections are needed.

2.2.2 The quadrupole radiation

Using 2.47, we can express equation 2.31 as

$$h_{ij}^{TT} = \frac{2G_N}{rc^4}\Lambda^{ijkl}(n)\,\ddot{M}_{kl}(ct-r) + O(v/c) \tag{2.50}$$

where from now on we will neglect $O(v/c)$ terms.
Defining the quadrupole momentum[4] as the traceless part of M^{ij}

$$Q^{ij} \equiv \left(M^{ij} - \frac{1}{3}\delta^{ij}M_k^k\right) \tag{2.51}$$

we arrive at the following equation

$$h_{ij}^{TT} = \frac{2G_N}{rc^4}\Lambda^{ijkl}(n)\,\ddot{Q}_{kl}(ct-r) \tag{2.52}$$

where the remaining part of M^{ij}, proportional to a δ^{ij} it has been eliminated due to contraction with the Λ^{ijkl} projector and its symmetries.
The last equation is very important as it is thanks to this that it is possible to calculate an exact formula for the power radiated per unit solid angle in the quadrupole approximation.
This can be derived inserting 2.52 in 2.13, expressed in terms of the power radiated

$$\frac{dP}{d\Omega} = \frac{r^2c^3}{32\pi G_N}\dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT} = \frac{G_N}{8\pi c^5}\Lambda^{ijkl}(\vec{n})\dddot{Q}_{ij}\dddot{Q}_{kl} \tag{2.53}$$

where in the last equality it has been performed a straightforward contraction between two Λ^{ijkl} projectors.
What is left in 2.53 is the integration over the solid angle, which can be easily performed noting that the angular dependence is in the projectors.
Using

$$\int d\Omega\Lambda^{ijkl}(\vec{n}) = \frac{2\pi}{15}(11\delta^{ik}\delta^{jl} - 4\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk}) \tag{2.54}$$

we arrive at the total radiated power in quadrupole approximation

$$P_{quad} = \frac{G_N}{5c^5}\dddot{Q}^{ij}\dddot{Q}_{ij} \tag{2.55}$$

[4] Beware that some authors define the quadropole momentum using a different normalization factor (e.g. Landau and Lifshitz [54]). Here we will follow the definition of Maggiore ([46] Section 3.3.2)

2.2.3 Radiation from point particle sources

Within linearized GR, the simplest source that one can study is represented by a closed system of point particles moving under their mutual gravitational attraction. As we will see this is a good approximation for many physical situations and it offer a first sight on the gravitational dynamics of a real binary system.
For point-particles the total energy-momentum tensor can be expressed in a non covariant way as

$$T^{\mu\nu}_{tot}(t,x) = \sum_a \frac{p^\mu_a p^\nu_a}{\gamma_a m_a}\delta^3(x - x_a(t)) = \sum_a \gamma_a m_a \frac{dx^\mu_a}{dt}\frac{dx^\nu_a}{dt}\delta^3(x - x_a(t)) \tag{2.56}$$

where $\gamma = (1 - {}^{v^2}/{}_{c^2})^{1/2}$ is the usual Lorentz factor and the index a run over the number of constituents, 2 for a binary.
We can start by defining the second mass momentum for the system as

$$M^{ij}(t) = \sum_a m_a \int d^3x\, x^i x^j \delta^3(x - x_a(t)) = \sum_a m_a x^i_a(t) x^j_a(t) \tag{2.57}$$

Under a translation of the reference frame by $x^i \to x^i + a^i$ the second mass momentum transform as

$$M^{ij}(t) \quad \to \quad M^{ij}(t) + a^i\Big(\sum_a m_a x^j_a(t)\Big) + a^j\Big(\sum_a m_a x^i_a(t)\Big) + a^i a^j \sum_a m_a \tag{2.58}$$

which means that its value depend on the origin that we have chosen, in particular for its time derivative one has

$$\dot{M}^{ij}(t) \quad \to \quad \dot{M}^{ij}(t) + a^i P^j_{tot} + a^j P^i_{tot} \quad ; \quad P^i_{tot} = \sum_a m_a \dot{x}^i_a \tag{2.59}$$

Since for a closed system P_{tot} is conserved (i.e. $\dot{P}_{tot} = 0$), one has that the second time derivative of $M^{ij}(t)$ is invariant under coordinate translations. This is a reasonable result as otherwise quadrupole radiation would depend on the choice of coordinates, as it can be seen from

$$S^{ij} = \frac{1}{2}\ddot{M}^{ij} \tag{2.60}$$

We can use this invariance in order to calculate the second mass momentum in the center of mass frame, which for a non relativistic binary system is given by [5]

$$x^i_{CM} = \frac{\sum_a m_a x^i_a}{\sum_a m_a} \quad , \quad x^i_0 = x^i_1 - x^i_2 \tag{2.61}$$

Inserting this change of coordinates into the expression of the second mass momentum gives

$$M^{ij}(t) = m_1 x^i_1 x^j_1 + m_2 x^i_2 x^j_2$$

$$= m x^i_{CM} x^j_{CM} + \mu(x^i_{CM} x^j_0 + x^j_{CM} x^i_0) + \mu x^i_0 x^j_0 \quad ; \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{2.62}$$

where we have defined the so called *reduced mass* μ.
We can set the origin in the center of mass frame such that $x_{CM} = 0$ thus yielding this simpler expression

$$M^{ij}(t) = \mu x^i_0 x^j_0 \tag{2.63}$$

At this point one should determine the dynamics of the relative distance due to gravitational attraction on the system and then use 2.63 in order to calculate the radiated power.
As example, let's assume that the dynamics of the relative distance is given so that the system describes a gravitational oscillator with mass m around a heavier mass M. The relative coordinate is moving along the z-axis with harmonic oscillations

$$z_0(t) = a \ \cos(\omega_s t) \tag{2.64}$$

The second mass momentum becomes

$$M^{ij}(t) = \delta^{i3}\delta^{j3}\,\frac{\mu a_1^2}{2}(\,1 + 2\cos(2\omega_s t)) \quad \Rightarrow \quad \ddot{M}^{ij} = -2\delta^{i3}\delta^{j3}\,\omega_s^2\,\mu a^2 \cos(2\omega_s t) \tag{2.65}$$

where we have expressed $\cos^2(\omega_s t)$ in terms of $\cos(\omega_s t)$ using basic trigonometric relations.
At this point one should note that the gravitational waves produced by the system are, in quadrupole approximation

$$h^{TT}_{ij} = \frac{2G_N}{rc^4}\Lambda^{ijkl}(n)\ddot{M}_{kl}(ct - r) \quad \Rightarrow \quad h^{TT}_{ij} = C_{ij}\cos(2\omega_s t) \tag{2.66}$$

for a time independent tensor C_{ij}.

[5] It can be shown that this definition for the center of mass-frame of a relativistic system no longer applies and requires a generalization. See Blanchet et al.[31], equation (3.8)

A comparison between equation 2.64 and the right part of 2.66 bring us to the following conclusion:

"For a non relativistic, closed gravitational system the gravitational waves produced in quadrupole approximation oscillates with a frequency ω_{gw} proportional to the typical frequency of the system with $\omega_{gw} = 2\omega_s$".

This proportionality relation between the orbital frequency and the radiation frequency is also valid for more complex systems, and retails its validity for higher multipole expansion with the only difference that the proportionality factor could change.
As for the radiated power, one needs the entire expression of h_{ij}^{TT} simply derivable from 2.66 by means of contractions.
Plugging the result in 2.53 we deduce

$$\frac{dP}{d\Omega} = \frac{G_N \mu^2 a^4 \omega_s^6}{2\pi c^5} \sin^4(\theta) \quad \Rightarrow \quad P_{quad} = \frac{16}{15} \frac{G_N \mu^2}{c^5} a^4 \omega_s^6 \tag{2.67}$$

which is the final result of this section.

2.3 Compact binaries as a radiating system

In this section we will apply what we have developed so far in order to study a more realistic system as the inspiral of two compact binary.
We will assume a Minkowskian background, which means it is possible to apply Newton's gravitational law, in particular we will suppose for simplicity that the motion is on a given circular orbit of constant ω_s, which is allowed since it is a Keplerian one.
Denoting the orbital distance with R and the total mass as $m_1 + m_2 = m$ one has that the third Kepler's law holds

$$\omega_s^2 = \frac{mG_N}{R^3} \tag{2.68}$$

where ω_s denotes the orbital frequency of the circular motion.
We choose the frame so that the orbit lies on the x-y plane and the trajectories of the binaries are

$$x(t) = R\cos(\omega_s t) \quad , \quad y(t) = R\sin(\omega_s t) \quad , \quad z(t) = 0 \tag{2.69}$$

Moving in the center of mass-frame, one can proceed with the evaluation of the second mass momenta using 2.63, which yields

$$M_{11} = \mu R^2 \Big(\frac{1-\cos(2\omega_s t)}{2}\Big) \quad , \quad M_{22} = \mu R^2 \Big(\frac{1+\cos(2\omega_s t)}{2}\Big) \tag{2.70}$$

$$M_{12} = -\frac{1}{2}\mu R^2 \sin(2\omega_s t) \tag{2.71}$$

while the other components are null.
Given these, one can derive from 2.66 the expression for the gravitational waves produced in the TT gauge. In terms of their two physical components

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G_N\mu\omega_s^2R^2}{c^4}\Big(\frac{1+\cos^2\theta}{2}\Big)\cos(2\omega_s t_{ret}) \tag{2.72}$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G_N\mu\omega_s^2R^2}{c^4}\cos\theta\ \cos(2\omega_s t_{ret}) \tag{2.73}$$

being θ the angle between the line of sight and the normal to the orbit, and $\phi = 2\omega_s t$ the angle lying on the $x-y$ plane.
Also in this case we recognize that the gravitational waves produced oscillates with a time frequency ω_{gw} such that $\omega_{gw} = 2\omega_s$.
At this point it is a straightforward calculation the evaluation of the total power radiated in quadrupole approximation. Introducing the so called *mass chirp* M_c we have

$$P_{quad} = \frac{32}{5}\frac{c^5}{G_N}\Big(\frac{G_N M_c \omega_s}{c^3}\Big)^{10/3} \quad , \quad M_c \equiv \mu^{3/5} m^{2/5} \tag{2.74}$$

2.3.1 The coalescence of compact binaries

Until now, we have assumed that compact binaries are on a fixed Keplerian orbit, a circular one in our case.
However, this is inconsistent with the conservation of energy since the system emits a non null total energy whose release must take place at the expense of the internal energy of the system. Within linearized GR this can expressed as the sum of the kinetic and potential energy

$$E_{int} = E_{kin} + E_{pot} = -\frac{G_N m_1 m_2}{2R} \tag{2.75}$$

Due to the emission of gravitational waves, E_{int} has to become more and more negative. This means that R must decrease in time, which from third Kepler's law means that ω_s has to increase, the same for the emitted energy from 2.74. The natural conclusion is that at the end there is the coalescence of the binary system.
In order to give an estimate of the process we can express the energy of the binary in terms of ω_s using 2.3

$$E_{int} = -\Big(\frac{G_N^2 M_c^5 \omega_s^2}{8}\Big)^{1/3} \tag{2.76}$$

Due to conservation of the energy one must have that $P = -\frac{dE_{int}}{dt}$ which gives, unite with (2.74), the following differential equation in ω_s

$$\dot{\omega_s} = \frac{192}{5}\Big(\frac{G_N M_c}{c^3}\Big)^{5/3} \omega_s^{11/3} \tag{2.77}$$

The integration of this first order differential equation exhibit a divergence for ω_s in a finite time, that we will call t_{coal}.
In terms of $\tau = t_{coal} - t$ the solution is

$$\omega_s = \Big(\frac{5}{256}\frac{1}{\tau}\Big)^{3/8}\Big(\frac{G_N M_c}{c^3}\Big)^{-5/8} = \frac{\omega_{gw}}{2} \tag{2.78}$$

What we have obtained is the time behavior of the orbital frequency which, it must be remembered, is also proportional at the frequency of the emitted gravitational waves. Equation 2.78 shows that with the passage of time the two binaries become closer and closer with an increasing orbital frequency, leading at the end to an artificial singularity which only reveals that our estimate is no longer valid when the two binary merges. Several factors can explain the birth of this divergence, first of all having treated components of a binary system as point-like when in reality the binary ones have a physical extension. Nevertheless, this does not change the physical meaning of 2.78, which can be used for phenomenological considerations.

2.3.2 Estimates and numerical values for compact binaries

We can now base a series of phenomenological considerations on 2.78 expressd in terms of the frequency $f_{gw} = \frac{\omega_{gw}}{2\pi}$. We find convenient to express numerical values, included Newton's constant and velocity of light, in terms of the mass sun $M_\odot$

$$f_{gw} = \frac{1}{\pi}\Big(\frac{5}{256}\frac{1}{\tau}\Big)^{3/8}\Big(\frac{G_N M_c}{c^3}\Big)^{-5/8} \simeq 143Hz\Big(\frac{1.21M_\odot}{M_c}\Big)^{5/8}\Big(\frac{1s}{\tau}\Big)^{3/8} \tag{2.79}$$

Relation 2.79 can be inverted as

$$\tau \simeq 2s\Big(\frac{1.21M_\odot}{M_c}\Big)^{5/3}\Big(\frac{100Hz}{f_{gw}}\Big)^{3/8} \tag{2.80}$$

which gives an estimate of the time missing at the coalescence in terms of the frequency of the gravitational waves emitted.

In this respect, if a detector would receive a gravitational wave from a binary system with $M_c = 1.21M_\odot$ this would mean that for frequencies detected around $f_{gw} = 10Hz$ the system is at 17 minutes to coalescence, for frequencies at $f_{gw} = 100Hz$ at few second, and so on, until the two merge and our model fail to be valid.

Gravitational wave detectors makes also possible to estimate the number of cycles spent in the accessible detector bandwidth $[f_{min}, f_{max}]$ which can be expressed as

$$N_{cycles} = \int_{t_t}^{t_f} dt\, f_{gw} = \int_{f_{min}}^{f_{max}} df_{gw}\, \frac{f_{gw}}{\dot{f}_{gw}} \tag{2.81}$$

Using 2.77 integration is straightforward giving as result

$$N_{cycles} = \frac{1}{32\pi^{8/3}}\Big(\frac{G_N M_c}{c^3}\Big)^{-5/3}(f_{min}^{-5/3} - f_{max}^{-5/3}) \tag{2.82}$$

Assuming again that $M_c = 1.21M_\odot$ one has

$$N_{cycles} \simeq 1.6\; 10^4\Big(\frac{10Hz}{f_{min}}\Big)^{5/3} \tag{2.83}$$

where we have neglected the contribute coming from f_{max} which for a typical detector is negligible.

Equation 2.83 states that for a detector sensitive to mHz frequencies the number of cycles spent in the detector could be more than a million [6] which means that in principle the knowledge of the incoming gravitational wave is possible with great accuracy. As we will see, this is the basis to test physics beyond linearized GR that we have assumed here.

[6] Indeed, this is the case for the space interferometer LISA which will be designed to have $f_{min} \simeq 10^{-4}Hz$.

2.4 Beyond linearized General Relativity

In dealing with non relativistic compact binaries we have assumed that the background is the usual Minkowski one and we have computed the generation of gravitational waves as a multipoles expansion in $\frac{v}{c}$ being v a typical internal speed of the system.
One could be attempt to apply the same procedure in order to derive higher contribute to the total emitted radiation by means of a higher multipoles expansion around a flat metric. In order to be valid, this procedure has to assume that the background space-time curvature and velocity of the source can be treated as independent variables. This is indeed the case when the system is governed by non gravitational forces like the electromagnetic one. In this case one can still perform a multipoles expansion which maintains its validity also for highly relativistic particles accelerated in an external electric field.
Unfortunately this doesn't happen when the system is governed by gravitational forces.
Indeed, for a self-gravitational system as a binary, one has that the virial theorem holds

$$\frac{v^2}{c^2} \sim \frac{R_s}{R} \quad ; \quad R_s = \frac{2G_N m}{c^2} \tag{2.84}$$

where R is the typical size of the system, m the total mass and R_s the Schwarzschild radius that measures the strength of the gravitational field around the source[7].
Equation 2.84 means that increasing multipoles corrections is consistent only at the price of modifying the background space-time.
A solution to this problem is the so-called *Post Newtonian* approach, developed already in 1916 by Einstein himself. At the base of this scheme there is the possibility to cast deviations from Newton's law due to a curved background by means of corrections to the Newton Potential introducing new terms proportional to powers of the Newton constant G_N and v^2. [8]
It is usually said that 1 PN corrections are considered once the two bodies potential gets corrected by terms proportional to G_N^2 and $G_N v^2$; that 2 PN corrections are considered once G_N^3, $G_N^2 v^2$ and $G_N v^4$ contributes are taken into account and so on for higher contributes.
For the moment we take for granted that these corrections to the Newton's potential exist focusing on their phenomenological implications, dedicating the next chapter of the thesis to their systematic deduction.

[7] see Wald [73] p.124 who entroduce the concept within the Schwarzschild metric.

[8] see Luc Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries [14] for a review on the subject.

2.4.1 The phase of a gravitational wave: post-Newtonian corrections

Let's start by analyzing the consequences of these corrections on a physical measurable quantity as the phase of a gravitational wave

$$\Phi(t) = 2\int_{t_i}^{t} dt'\omega(t') \tag{2.85}$$

being ω the orbital angular frequency.
For simplicity we will assume that the orbits are circular[9].
The first thing to note is that due to deviations from Newton's law, we should expect that the internal energy 2.75 gets modified to an expression of the form

$$E = -\frac{\mu G_N}{2} v^2 (1 + e_{v^2}(\mu/m)v^2 + e_{v^4}(\mu/m)v^4 + ...) \tag{2.86}$$

being $e_{v^{2n}}(\mu/m)$ a generic correction.
In a similar manner, also the radiated power from the binary 2.74 should change as

$$P = \frac{32\mu^2}{5G_N m^2}\frac{1}{G_N c^5} v^{10}(1 + f_{v^2}(\mu/m)v^2 + f_3(\mu/m)v^3 + ...) \tag{2.87}$$

where we have expressed all formulas using v^2 instead of ω by means of

$$v^3 = \omega m G_N \tag{2.88}$$

a simple relation that can be legitimate using the third Kepler's law unit with the virial theorem[10] for the case of circular orbits.
All these quantities can now be used for an estimate of the phase of a gravitational wave since[11]

$$\Phi = 2\int_{t_i}^{t} dt'\omega(t') = 2\int_{v_i}^{v_f} dv\, \omega(v)\, \frac{dE}{dv}\frac{dt}{dE} = \frac{2}{G_N m}\int_{v_i}^{v_f} dv\, \frac{v^3}{P}\,\frac{dE}{dv} \tag{2.89}$$

having used the following relation $P = -\frac{dE}{dt}$ due to the conservation of energy.

[9]Usually binary orbits are eccentric, however they usually circularize long before they have reached the coalescence phase. This fact is proved in Maggiore [46] Section 4.1.3

[10]In this case, the precise numerical factor is $v^2/c^2 = R_s/2d$

[11]This derivation is based on a course held by Riccardo Sturani (see http://www.ictp-saifr.org/wp-content/uploads/2013/07/cursoGW-ICTP-Sturani1.pdf)

Plugging equations 2.86,2.87 into 2.89 one has the following post-Newtonian expression for the phase of a gravitational wave

$$\Phi = \frac{5m}{16\mu}\int_{v_i}^{v_f} dv\, \frac{1}{v^6}(1 + p_{v^2}v^2 + p_{v^3}v^3 + ..) \tag{2.90}$$

In equation 2.90 the first term of the expansion gives in terms of v the Newtonian phase previously calculated, while the remaining gives post-Newtonian corrections parameterized in terms of p_{v^n}.
The presence of these terms modify the prediction on the number of cycles spent by a gravitational wave in the frequency band, which is a physical quantity that a detector can measure with high accuracy 2.83. In this respect post-Newtonian corrections due to General Relativity, or its modifications, are testable.

2.4.2 The amplitude of a gravitational wave beyond linearized GR

Another quantity of interest that gets modified going beyond Newtonian physics is the amplitude of a gravitational wave.
We have already seen that in quadrupole approximation, the Newtonian amplitude h_+ of a gravitational wave can be expressed as

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G_N\mu\omega_s^2R^2}{c^4}\Big(\frac{1+\cos^2\theta}{2}\Big)\cos(\omega_{gw}t_{ret}) \tag{2.91}$$

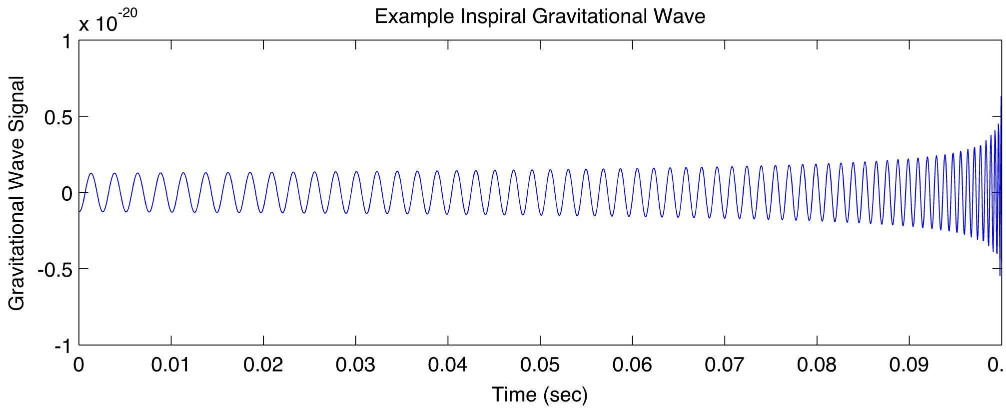
where equation 2.91 was derived assuming a constant orbital frequency ω_s.
In order to take care of the coalescence of the binary, one should beware that:

- $\omega_{gw}t_{ret}$ must be replaced by Φ unite with its temporal dependence.
- in the factors in front of the amplitude, ω_{gw} has to be replaced by $\omega_{gw}(t)$.
- one has to include contributions coming from the derivatives of the orbital distance $R(t)$ and $\omega_{gw}(t)$.

The time dependence of Φ, within Newtonian physics, can be deduced from the first term of 2.90 expressed in terms of $\tau = t_{coal} - t$.
The result is simple and given by

$$\Phi(\tau) = -2\bigg(\frac{5G_NM_c}{c^3}\bigg)^{-5/8}\tau^{5/8} + \Phi_0 \tag{2.92}$$

where $\Phi_0 = \Phi(\tau = 0)$ is equal to the value of Φ at coalescence.
As for the radial velocity $\dot{R}$, we assume it is negligible or equivalently $\dot{\omega}_{gw} << \omega_{gw}^2$.



(2.94)

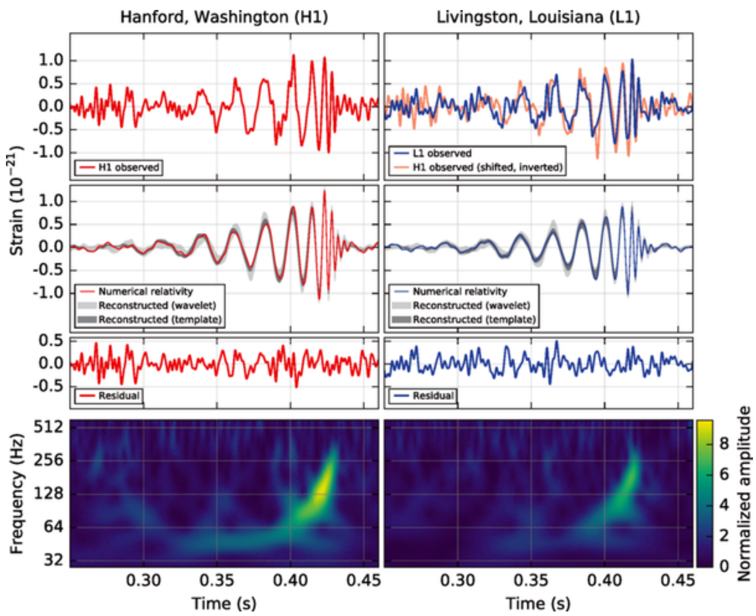
In this respect the resulting amplitude can be expressed in terms of the solely $\tau = t_{coal} - t$ as

$$h_{+}(\tau) = \frac{1}{r}\left(\frac{G_N M_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(\Phi(\tau)) \tag{2.93}$$

If we remain at the level of Newtonian physics, Φ is equal to 2.92 and the corresponding time evolution of a gravitational wave amplitude, valid for the inspiral phase, can be seen from Fig.2.94. Observing the figure[12] it can be noticed that once the coalescence is approached the amplitude increase, a behavior which is usually referred as "chirping".
Of course, the time evolution of a gravitational wave should also take care of post-Newtonian effects, like the modification of the phase seen in equation 2.92.
This effect, along with others like back-reaction in the motion due to radiation and the merging and ring-down phase, have been considered for the first time within the so called *Effective one body* approach developed by Thibault Damour and Alessandra Buonanno (See [16], [26], [22], for a review on the subject see also [24]).
This scheme adopts a perturbative post-Newtonian approach to define the conservative dynamics, but it is also able to combine, in a suitably resummed format, some non perturbative informations extracted from numerical simulations of coalescing binaries.

[12]The image has been taken from https://www.ligo.org/science/GW-Inspiral.php. The amplitude of the gravitational wave lies on the y axis, while on the x axis there is the time t, where it is assumed that the coalescence occurs at $t_{coal} = 0.1$.

This provided the first templates which has been subsequently used during the first detection of a gravitational wave made on 14 September 2015 and then announced by the LIGO and Virgo collaborations on 11 February 2016 [4]. After this many others followed. The second observation of gravitational waves was made on 26 December 2015 [2] and announced on 15 June 2016 while three more observations were made in 2017.
For completeness we report below the temporal evolution of the first signal, which has been called GW150914 (from "Gravitational Wave" and the date of observation 2015-09-14)



(2.95)

From the first boxes on top we can easily recognize the "chirp" due to the emission of gravitational radiation during the inspiral phase. This is followed by a peak and a subsequent decreasing trend due to the merging and ring-down phase of the binary. The agreement between theoretical predictions and measurements, here reported in the residual, can be considered as a test that General Relativity is valid also in its strong-field regime, which means that no modifications are currently needed to explain gravitational phenomena.

Chapter 3

The Effective Field Theory approach

In the previous chapter we have seen how deviations from Newton's law can modify the prediction of physical quantities like the phase and amplitude of a gravitational wave. These corrections fall within the so-called *post-Newtonian* scheme and give rise to additional term to Newton's potential as powers of G_N and v^2, being v a typical three velocity of the system. For their estimate we will adopt a modern method which goes under the name of Effective Field Theory approach [1]. Mainly based on a diagrammatic scheme, it was first introduced in particle physics for the study of a heavy quark field theory (see [36]). Developed in gravitational physics by Damour and Farése [25] and later systematized by Goldberger and Roshtein [39], within this approach the action for a binary can be evaluated by means of Feynman diagrams, with the possibility to introduce modern QFT techniques for high precision calculations.
Regarding the topics that will be dealt with, the Chapter is so divided:

- It is introduced the Effective Field Theory (EFT) approach showing how an effective theory at low energies can derive from a complete one. This will allow us to introduce some fundamental concepts typical of field theories, with the possibility of constructing an effective theory for binaries.

- It is developed a toy model of binary system, in which gravity is modeled as a scalar field. The effective action for binary only will be reduced to a local one via a non relativistic limit and we will show how its action can be derived from a Feynman diagrammatic approach.

- It is developed in detail the EFT of a real binary system kept together by gravity, deriving a finite subset of Feynman rules. In addition, it will be shown the link between classical effective diagrams and QFT amplitudes.

[1] For a review on the subject see Goldberger [38]

3.1 The Effective Field Theory approach described

Consider a field theory of two interacting scalars fields with action $\mathcal{S}(\phi, \sigma)$ where ϕ has light degrees of freedom while σ has a mass M near the UV scale.
Let's assume we want to work at energies $E << M$ where the σ field is not excited. In this energy regime it is impractical to take into account its interactions with ϕ, therefore, rather than studying the mutual interactions of the two fields, in this situations it is easier to solve the dynamics in σ, plugging then the solution back in $\mathcal{S}(\phi, \sigma)$ in order to define an effective action for ϕ only.
The result of this procedure can be expressed in a compact way using the following functional integral

$$e^{i\mathcal{S}_{eff}(\phi)} = \int D\,\sigma e^{iS(\phi,\sigma)} \tag{3.1}$$

The effective action so obtained can be assumed to be a local[2] function of ϕ as

$$\mathcal{S}_{eff}(\phi) = \sum_n c_n \int d^4x\, O_n(x) \tag{3.2}$$

where $O_n(x)$ describes a local operators of ϕ while c_n is an energy dependent term called *Wilson coefficient*. If O_n has a mass dimension Δ_n in natural units $\hbar = c = 1$, then the corresponding Wilson coefficient evaluated at a renormalization point μ of order λ, will scale as a power of λ as

$$c_n(\mu = \lambda) = \frac{\alpha_n}{\lambda^{\Delta_n - 4}} \quad , \quad \alpha_n \sim O(1) \tag{3.3}$$

This observation is crucial since we can start to distinguish two kinds of effects of the UV physics on the lower energy effective one.
Indeed for energies $E << M$ there is an infinite number of irrelevant operators with mass dimensions $\Delta_n > 4$ that can be neglected, which means that only a finite number of terms are needed. This fact is usually called *decoupling* and it states that at a given energy scale the effective behavior of a theory is simply described by a finite number of terms allowed by symmetries. In this sense, being interested at a specific physical scale, it is much easier to work with an effective field theory rather than the complete one, since the last can be constructed from basic assumptions.
This procedure can also be applied if ϕ is a light field while σ is a massless one which interacts with the other via a coupling Λ, which as a negative mass dimension. What it is important here it is not that the field that we are integrating out is massive or not, but that manifests itself at a well-defined scale of energy, which can be that given by its mass or by a coupling.

[2] In reality, this procedure can generate effective actions containing different non-local terms see [60]. In the case of a binary, we will in fact see time non-localities, nevertheless introducing an appropriate non-relativistic limit we will show how it is possible to reduce to a local theory.

3.1.1 Binary systems as an EFT

Effective field theories (EFT) are important in order to deal with problems that involve simultaneously separate physical scales. This is the case for the inspiral phase of a binary, which is characterized by the presence of the size of compact objects r_s, the orbital radius r and the frequency of the emitted radiation ω.
All these are linked by the following relations

$$v^2 \sim \frac{r_s}{r} \quad , \quad v \sim \omega r \tag{3.4}$$

both proved to hold, at least approximately, in the precedent chapter.
According to the EFT approach, given a scale of interest one has to recognize the relevant degrees of freedom and allowed symmetries in order to define the most general effective action.
As for the binary we can assume that the system is composed by black holes[3] interacting with a low frequency gravitational field. During the inspiral phase the components are at a such distance that they can be viewed as point-particles probe of the background space-time.
At this scale, the relevant degrees of freedom are

- The background gravitational field $g_{\mu\nu}(x)$.
- The black hole's worldline coordinate $x^\mu(\lambda)$ being λ an affine parameter.
- An orthonormal frame $e^\mu_a(x)$ localized on the black hole's worldline such that $g_{\mu\nu}e^\mu_a e^\nu_b = \eta_{ab}$. This describes how the binary is spinning relative to the gravitational field.

Once the degrees of freedom have been identified, one must construct an action that respects a given number of symmetries, which in our case are

- General coordinates invariance $x^{\mu'} = x^{\mu'}(x)$
- Worldline reparametrization invariance $\lambda' = \lambda'(\lambda)$
- $SO(3)$ invariance: this guarantees that the compact objects are perfectly spherical with no permanent moments relative to its own rest frame.

For simplicity in the follow we will neglect spin degrees of freedom for black holes, thus assuming they are Schwarzschild ones.

[3]Neutron star cannot be only described as point particles since they contain additional low frequency modes. See [50] for a review on neutron star/black hole spectroscopy.

In this respect, the most general effective action describing the slow inspiral phase of a binary made of black holes is given by

$$\mathcal{S}_{eff}(g_{\mu\nu}, x^{\mu}) = \mathcal{S}_{pp}(g_{\mu\nu}, x^{\mu}) + \mathcal{S}_{E-H}(g_{\mu\nu}) \tag{3.5}$$

In equation 3.5, $\mathcal{S}_{E-H}$ is given by the Einstein-Hilbert action for the gravitational field 1.91

$$\mathcal{S}_{E-H}(g_{\mu\nu}) = -2m_{pl}^2 \int d^4x \sqrt{-\det g_{\mu\nu}} R(g_{\mu\nu}) \tag{3.6}$$

where we have introduced the so called *Planck mass* as $m_{pl} \equiv g^{-1}$. As for S_{pp}, this has to be the most general point-particle action coupled to $g_{\mu\nu}$ respecting coordinate and reparametrization invariance

$$\mathcal{S}_{pp}(g_{\mu\nu}, x_a) = -\sum_{a=1,2} m_a \int d\tau_a + ... \tag{3.7}$$

The first term of 3.7 is the elapsed proper time by a black hole worldline

$$-m \int d\tau = -m \int d\lambda \sqrt{g_{\mu\nu}(x)\dot{x}^{\mu}(\lambda)\dot{x}^{\nu}(\lambda)} \tag{3.8}$$

the simplest allowed scalar action which reduces to 1.56 in a weak field approximation. In addition to this, one can add terms in order to describe the internal of black holes. In principle there is an infinite number of them but the simplest are [4]

$$\mathcal{S}_{pp}(g_{\mu\nu}, x_a) = -\sum_{a=1,2} m_a \int d\tau_a + c_E \int d\tau_a E_{\mu\nu} E^{\mu\nu} + c_B \int d\tau_a B_{\mu\nu} B^{\mu\nu} + ... \tag{3.9}$$

where the E_{μ}, $B_{\mu\nu}$ tensors are the gravitational analog of the decomposition of $F_{\mu\nu}$ into components of electric and magnetic type respectively

$$E_{\mu\nu}(x_a) = R_{\mu\nu\alpha\beta}\dot{x}_a^{\alpha}\dot{x}_a^{\beta} \quad , \quad B_{\mu\nu}(x_a) = \epsilon_{\mu\alpha\beta\rho} R^{\alpha\beta}_{\gamma\nu}\dot{x}_a^{\rho}\dot{x}_a^{\gamma} \tag{3.10}$$

As for the coefficients c_{E-B} it can be shown[5] that these are proportional to $m_{pl}^2 r_s^5$, which means that they vanish rapidly as the size of the black hole goes to zero. Thus, at first approximation, the slow inspiral phase of a binary can be described in terms of point-particles interacting with a gravitational field $g_{\mu\nu}$.

[4]We won't consider terms proportional to the Ricci tensor or scalar since, due to the corresponding Einstein's field equation, one has $R_{\mu\nu} = 0$

[5]See Goldberger [38] Section 2.6

3.1.2 Calculating observables from the EFT of a binary

Given the effective action of a binary system, one can start by integrating out the gravitational field according to 3.1, in order to define a new effective action involving the binary constituents only.
In order to see the relevance of this procedure for the evaluation of physical observables, we should remember that the phase of a gravitational wave emitted from a binary during the slow inspiral phase can be expressed in terms of its internal energy $E(v)$ and its power emitted $P(v)$ as

$$\phi = 2\int dv\,\frac{v^3}{P}\frac{dE}{dv} \tag{3.11}$$

It is a easy task to prove that $E(v)$ and $P(v)$ can be derived from the action of a binary once the gravitational field has been integrated out.
First of all one should derive $\mathcal{S}_{eff}(x)$ by a saddle point approximation on

$$e^{i\mathcal{S}_{eff}(x)} = \int Dh_{\mu\nu}e^{i(\mathcal{S}_{E-H}(h_{\mu\nu})+S_{pp}(h_{\mu\nu},x))} \tag{3.12}$$

where we have assumed that $g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_{pl}}$
As the extrema with respect to $h_{\mu\nu}$ of the exponent in 3.12 are simply gauge-fixed classical solutions $h_{\mu\nu}(x_a)$ of Einstein's field equations at given worldlines , one immediately see that

$$\mathcal{S}_{eff}(x_a) = \mathcal{S}_{eff}(x_a, h_{\mu\nu}(x_a)) + O(\hbar) \tag{3.13}$$

where quantum contributes can be neglected for the phenomenology we are aiming.
Once the action has been derived, one can use the real part of 3.13 to define the classical equations of motion for worldlines only

$$\delta_x[Re(\mathcal{S}_{eff}(x))] = 0 \tag{3.14}$$

which can be further used to define the internal energy of the binary $E(v)$ as a constant of motion, using Noether's theorem.
In the same manner, the imaginary part of 3.13 can be used to define the total radiated power as

$$P = \lim_{T\to\infty}\frac{2}{T}[Im(\mathcal{S}_{eff}(x))] \tag{3.15}$$

the demonstration of which will be entrusted in the following section using a simplified model of binary system with scalar gravity.

3.2 The EFT of a binary system in scalar gravity

In this section we provide a specific application of the Effective field theory approach dealing with a simplified model of binary system with scalar gravity[6].
In this theory binary components interacts with a massless scalar field which should represent gravity. Of course, this is not possible since we have demonstrated that gravity is described by a tensorial field, in any case we think it is instructive to start with this example, before to approach the case of real gravity.
We can start by introducing the analog of 3.5 for the case of a non self-interacting scalar gravity, where the index sum is implied

$$\mathcal{S}(\phi, x^\mu) = \int d^4x\, \partial_\mu\phi\partial^\mu\phi \;-\; m_a \int d\tau_a \Big(1 + \frac{\phi(x_a)}{2\sqrt{2}m_p}\Big) \tag{3.16}$$

Let's evaluate the effective action for the black holes/point particles by means of the following functional integral

$$e^{i\mathcal{S}_{eff}(x^\mu)} = \int D\phi\; e^{i\mathcal{S}(\phi, x^\mu)} \tag{3.17}$$

In the evaluation of 3.17 we will introduce a perturbative approach based on Feynman diagrams.
Let's begin by defining

$$J(x) \equiv \sum_{a=1}^{2} -\frac{m_a}{2\sqrt{2}m_p} \int d\tau_a \delta^4(x - x_a) \tag{3.18}$$

Denoting by $< ... >_x$ a generic integration over space-time coordinates, the previous functional integral can be expressed as

$$e^{i\mathcal{S}_{eff}(x^\mu)} = e^{i\mathcal{S}_{kin}(x_a)}\; Z(J) \tag{3.19}$$

$$Z(J) \equiv \int D\phi\; e^{i<\partial_\mu\phi\partial^\mu\phi>+i<J\phi>} \tag{3.20}$$

where $\mathcal{S}_{kin}(x^\mu)$ is the elapsed proper time on the black hole worldlines and $Z(J)$ the so called *functional generator of disconnected n-points functions*, typical of quantum field theories.

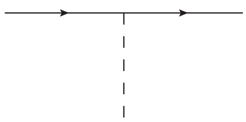
[6] This model has been taken from Goldberger [38]

In equation 3.19 one could recognize the functional generator of a free scalar field interacting with an external source $J(x)$ and conclude[7] that

$$Z(J) = e^{-\frac{1}{2} < J(x) J(y) G_2(x-y) >_{x,y}} \tag{3.21}$$

$$G_2(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} \, e^{ik^\mu (x-y)_\mu} \frac{i}{k^2 + i\epsilon} \quad ; \quad k^2 = k^\mu k^\nu \eta_{\mu\nu} \tag{3.22}$$

where 3.22 is the so called *two point function* of a free scalar field.
However, this analytic derivation is only possible because the functional integral is Gaussian, a fact that ceases to apply when adding to the scalar field generic self-interactions given by a potential $V(\phi)$. Therefore, to deal with these cases, we will use a diagrammatic approach typical of QFT.
First of all, let's define the Feynman rule in momentum space for the interaction between $\phi(x)$ and $J(x)$.
This can be read from 3.16 as



$$= i \int d^4x \; J(x) e^{-ik\cdot x} = \sum_{a=1}^{2} -i \frac{m_a}{2\sqrt{2} m_{pl}} \int d\tau_a e^{-ik_\mu \cdot x_a^\mu} \tag{3.23}$$

which can be described as the emission from the black hole of a scalar particle (the absorption is given by the same rule, with the only difference of the sign at the exponential).
As for the propagator of the scalar field in momentum space we can proceed as we have done for the graviton propagator obtaining

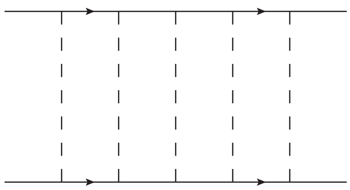
$$= \frac{i}{k^2 + i\epsilon} \tag{3.24}$$

Equations 3.23,3.24 can now be used to build amplitudes in momentum space, whose Fourier transform gives the contribute to the desired functional generator. As example, we will evaluate the most generic amplitude describing the emission and absorption of n particles from the two black holes.

[7]See Ramond [65] Section 3.1

This is given by

$$M(k_i) = \quad (3.25)$$



$$M(k_i) = \frac{1}{2^n\, n!}\prod_{i=1}^{n}\left[i\int d^4x_i J(x_i)e^{-ik_i\cdot x_i}\right]\left[\frac{i}{k_i^2+i\epsilon}\right]\left[i\int d^4y_i J(y_i)e^{ik_i\cdot y_i}\right] \quad (3.26)$$

$$= \frac{1}{n!}\prod_{i=1}^{n}\left[-\frac{1}{2}\int d^4x_i d^4y_i J(x_i)J(y_i)e^{-ik_i\cdot(x-y)_i}\frac{i}{k_i^2+i\epsilon}\right] \quad (3.27)$$

where the index i runs over the intermediate lines, while $2^{-n}n!$ is a symmetry factor associated to the topology of the diagram. After a Fourier transform of each k_i momentum, the contribute to the functional generator becomes

$$Z_n(J) = \frac{1}{n!}\prod_{i=1}^{n}\left[-\frac{1}{2}\int d^4x_i d^4y_i J(x_i)J(y_i)G_2(x_i-y_i)\right] \quad (3.28)$$

$$= \frac{1}{n!}\left[-\frac{1}{2}\int d^4x d^4y J(x)J(y)G_2(x-y)\right]^n \quad (3.29)$$

We can recognize in 3.29 the n-th term of an exponential expansion.
Since the functional generator is given uniquely by the sum over all 3.28 we conclude that

$$Z(J) = \sum_{n=1}^{\infty} Z_n(J) = e^{iW(J)} \quad (3.30)$$

$$W(J) \equiv \frac{i}{2}\int d^4x d^4y J(x)J(y)G_2(x-y) \quad (3.31)$$

where we have defined $W(J)$, the so called *functional generator of connected n-points functions*. The reason for this name can be seen from its expression compared to that of $Z(J)$: while the last takes contribute from all connected and disconnected diagrams, the functional $W(J)$ depends only on the simpler connected ones.

This result is true regardless the specific field theory and it will be useful in dealing with the EFT of real binaries.
Using equation 3.19 we can now express the effective action for the binary system as

$$e^{i\mathcal{S}_{eff}(x)} = e^{i\mathcal{S}_{kin}(x)}e^{iW(J)} \quad \Leftrightarrow \quad S_{eff}(x) = S_{kin}(x) + W(J) \tag{3.32}$$

$$\mathcal{S}_{eff}(x_a) = -m_a \int d\tau_a + \frac{i}{2}\sum_{a,b}^{a\neq b} \frac{m_a m_b}{8m_p^2} \int d\tau_a d\tau_b G_2(x_a - x_b) \tag{3.33}$$

where divergent terms due to $a = b$ have been removed since they cannot affect physical observables. This conclude the evaluation of the effective action for the toy model of binary we have presented. In order to make the model even more realistic, one could also introduce self-interactions for the scalar field as a potential of the type $V(\phi) = \lambda\phi^n$ for some coupling λ. This would have generated new Feynman rules, and therefore additional contributions, in form of coupling powers, to the effective action[8]

3.2.1 Local EFT via a non relativistic limit

The effective action previously derived is non local in proper times of the two black holes.
From a relativistic point of view this is natural: the dynamics of a worldline at a certain time, depends on the motion of the other at a retarded one, which in turn depends on the dynamics of the previous to another delayed time and so on.
However, since we are interested in the slow inspiral phase of a binary, we can apply a proper non relativistic limit to 3.33 in order to neglect these time-non localities, at least up to a given PN order.
To apply this procedure, we should divide the region of the elapsed momenta in two subsets:

- Potential region $\mathcal{V}(k)$: in this subset, the emitted particles have a space-like momentum (i.e. $k^2 > 0$) of the form $\left(k^0 \sim \frac{v}{r}\ ,\ k^i \sim \frac{1}{r}\right)$
- Radiation region $\mathcal{R}(k)$: in this subset, the exchanged particles have a null-like momenta $k^2 = 0$ of the form $\left(\ k^0 \sim \frac{v}{r}\ ,\ k^i \sim \frac{v}{r}\ \right)$.

The presence of non localities is due to the integration on the radiation region in the two point function 3.22. In order to avoid this region we can restrict ourselves to the potential one where $k^0 << |\vec{k}|$.

[8]It should be pointed out that couplings are constrained in order to define a consistent perturbative approach. For the example of an EFT of a binary held together by a self-interacting scalar field, see [63].

Restricting to this region, the two point function becomes

$$G_2^{NR}(x_a - x_b) = i\int_{\mathcal{V}(k)} \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik^\mu(x_a-x_b)_\mu}}{k^2+i\epsilon} = i\int_{\mathcal{V}(k)} \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik^\mu(x_a-x_b)_\mu}}{-(k^0)^2+|\vec{k}|^2} \tag{3.34}$$

$$= i\int_{\mathcal{V}(k)} \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik^\mu(x_a-x_b)_\mu}}{|\vec{k}|^2}\frac{1}{1-\left(\frac{k^0}{|\vec{k}|}\right)^2} \tag{3.35}$$

$$= i\int_{\mathcal{V}(k)} \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik^\mu(x_a-x_b)_\mu}}{|\vec{k}|^2}\sum_{n=0}^{\infty}\left(\frac{k^0}{|\vec{k}|}\right)^{2n} \tag{3.36}$$

Defining $\omega = k^0$ we have that

$$G_2^{NR}(x_a - x_b) = i\sum_{n=0}^{\infty}\int \frac{d^3k}{(2\pi)^3}\,\frac{e^{-i\vec{k}\cdot(\vec{x}_a-\vec{x}_b)}}{|\vec{k}|^{2(n+1)}}\int\frac{d\omega}{2\pi}\,e^{i\omega(t_a-t_b)}\omega^{2n} \tag{3.37}$$

The ω integral is straightforward

$$\int\frac{d\omega}{2\pi}\,e^{i\omega(\tau_a-\tau_b)}\omega^{2n} = (-1)^n\frac{\partial^{2n}}{\partial^{2n}(t_a-t_b)}\int\frac{d\omega}{2\pi}e^{i\omega(\tau_a-\tau_b)} = (-1)^n\frac{\partial^{2n}\delta(\tau_a-\tau_b)}{\partial^{2n}(\tau_a-\tau_b)} \tag{3.38}$$

where in the last passage we have simply used the definition of Dirac's delta.
As for the $\vec{k}$ integral, this is[9]

$$\int \frac{d^3k}{(2\pi)^3}\,\frac{e^{-i\vec{k}\cdot(\vec{x}_a-\vec{x}_b)}}{|\vec{k}|^{2(n+1)}} = \frac{\Gamma(1/2-n)}{\Gamma(n+1)}\frac{1}{(4\pi)^{3/2}}\left(\frac{r^2}{4}\right)^{n-1/2} \tag{3.39}$$

being $r = |\vec{x}_a - \vec{x}_b|$ the Euclidean distance between x_a and x_b.
By combining the two results we have that the non relativistic expansion of the two point function is

$$G_2^{NR}(x_a - x_b) = i\sum_{n=0}^{\infty}(-1)^{1+n}\frac{\partial^{2n}}{\partial^{2n}(\tau_a-\tau_b)}\Big[\delta(\tau_a-\tau_b)\Big]\frac{\Gamma(1/2-n)}{\Gamma(n+1)}\frac{1}{(4\pi)^{3/2}}\left(\frac{r^2}{4}\right)^{n-1/2} \tag{3.40}$$

[9]See Chapter 4, Section 4.2.2 for a derivation of this integral in arbitrary d-dimensions.

As example, let's evaluate the effects of the first term of 3.40 into 3.33. This is given by

$$G_2^{NR}(x_a - x_b) = -\frac{i\delta(\tau_a - \tau_b)}{4\pi r} \tag{3.41}$$

By inserting this expression into 3.33 and evaluating the non relativistic expression for the elapsed proper time we arrive at

$$\mathcal{S}_{eff}(x_a) = -m_a \int d\tau_a + \frac{1}{2}\sum_{a,b}^{a\neq b} \frac{m_a m_b}{8m_p^2} \int d\tau_a d\tau_b \frac{\delta(\tau_a - \tau_b)}{4\pi r} \tag{3.42}$$

Due to our non relativistic limit, we can now define a unique time $\tau_a = t$ so that

$$\mathcal{S}_{eff}(x_a) = \int dt \bigg(\frac{m_a v_a^2}{2} + \frac{m_1 m_2 G_N}{r} \bigg) \tag{3.43}$$

which is the well known Lagrangian for the classical two body problem.
In addition to this, one should also consider terms coming from the total non relativistic expansion of 3.40. Since these are proportional to derivatives of Dirac's delta, one can always eliminate time non localities after an integration by parts, with the effect of adding to 3.43 time derivatives of the two worldlines.
In this respect, one is always able to define a local Lagrangian for a binary, which can be easily used to define the energy of the system E as a conserved quantity via Noether's theorem.
As for the evalutation of P, we should evaluate the immaginary part of 3.33 which is possible by restricting the two point function in the radiation region $\mathcal{R}(k)$.
This is equivalent to integrate over the region with $k^2 = 0$

$$G_2^{Rad}(x_a - x_b) = i \int_{\mathcal{R}(k)} \frac{d^4k}{(2\pi)^4} \frac{e^{-ik^\mu (x_a - x_b)_\mu}}{k^2 + i\epsilon} = \pi \int \frac{d^4k}{(2\pi)^4}\, e^{-ik^\mu (x_a - x_b)_\mu} \delta(k^2) \tag{3.44}$$

where Dirac's Delta has the effect to ensure that the emitted particles are on-shell[10].
Inserting 3.44 into 3.33 gives

$$Im\mathcal{S}_{eff}(x_a) = \sum_{a,b}^{a\neq b} \frac{\pi m_a m_b}{16 m_{pl}^2} \int d\tau_a d\tau_b \int \frac{d^4k}{(2\pi)^4}\, e^{-ik^\mu (x_a - x_b)_\mu} \delta(k^2) \tag{3.45}$$

[10]It can be proved that a propagator is real only when particles are on-shell. Thus the condition $k^2 = 0$ is equivalent to evaluate the imaginary part of the propagator which is equal to $-\pi\delta(k^2)$. See Schwartz [66] Section 24.1.2 for a derivation.

After this we can now use the following relation

$$\delta(k^2) = \frac{\delta(\omega - |\vec{k}|)}{2|\vec{k}|} + \frac{\delta(\omega + |\vec{k}|)}{2|\vec{k}|} \tag{3.46}$$

Performing a integration with respect to ω we arrive at

$$Im\mathcal{S}_{eff}(x_a) = \frac{1}{16m_{pl}^2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2|\vec{k}|}\left|\sum_a m_a \int d\tau_a e^{-ik\cdot x_a}\right|^2_{\omega=|\vec{k}|} \tag{3.47}$$

From 3.47 one can recover the differential emitted power as

$$\frac{dP}{d\Omega d|\vec{k}|} = \frac{1}{T}\frac{G_N}{4\pi^2}|\vec{k}|^2\left|\sum_a m_a \int d\tau_a e^{-ik\cdot x_a}\right|^2_{\omega=|\vec{k}|} \tag{3.48}$$

3.2.2 A nod on time non-localities

The non relativistic procedure here derived makes possible to eliminate perturbatively time non localities due to the expansion of the propagator denoted as G_2.
However, as seen by Damour and Blanchet in [15], the formal expansion of this propagator in flat space time breaks down at the so called 4 post-Newtonian (PN) approximation to the conservative dynamics of a binary. Indeed, at this level of accuracy it is crucial to take account of the fact that the gravitational propagator in the curved space-time, generated by the binary system, contains a significant tail contribution whose support is not limited to lightlike intervals, but extends to strongly time-nonlocal intervals.
Exploiting this fact, in [30] Damour, Jaranowski and Schäfer were able to solve infrared ambiguities which had blocked a previous PN calculation.

3.3 The EFT of a binary system within General Relativity

Arrived at this point, we have all the ingredients to approach the effective field theory of a realistic binary. For concreteness, let's assume we are interested in the description of a spin-less binary-black hole in its slow inspiral phase.
At this scale, we have already seen that the effective action is that of point particles coupled to an external gravitational field.
In order to derive an action for binaries only one should be aware that in $3+1$ space-time dimensions there are divergent contributes to the effective action. These can easily handled in dimensional regularization, i.e. by working in $d+1$ dimensions and taking only at the end the physical limit $d \to 3$. We will explain this procedure in detail in the next chapter, for the moment, neglecting finite size effects, we will just assume that our effective theory is given by

$$\mathcal{S}_{eff}(g_{\mu\nu}, x^\mu) = -2\Lambda^2 \int d^d x dt \sqrt{-\det g_{\mu\nu}} R(g_{\mu\nu}) - m_a \int d\tau_a \quad ; \quad \Lambda \equiv \frac{m_{pl}}{l^{d-3}} \tag{3.49}$$

where we have introduced an arbitrary physical length scale l in order to render the action adimensional in natural units.
Given 3.49, we can start by taking advantage of the diffeorphism invariance of General Relativity so as to impose a Kaluza-Klein parametrization for the metric tensor based on the use of the so called *Kol-Smolkin* variables [51].
According to these, the metric tensor is decomposed in terms of a scalar field ϕ, a d-dimensional vector field A_i and a $d \times d$ symmetric tensor field σ_{nm} as

$$g_{\mu\nu} = e^{\frac{2\phi}{\Lambda}} \begin{pmatrix} -1 & \frac{A_i}{\Lambda} \\ \frac{A_i}{\Lambda} & e^{-c_d \frac{\phi}{\Lambda}} \gamma_{ij} - \frac{A_i A_j}{\Lambda^2} \end{pmatrix} \quad , \quad \gamma_{ij} = \delta_{ij} + \frac{\sigma_{ij}}{\Lambda} \tag{3.50}$$

where $c_d = 2\left(\frac{d-1}{d-2}\right)$.
As shown by Gilmore and Ross in [37], the use of this parametrization is useful since it is suitable for the non relativistic corrections we want to calculate. Inserting 3.50 into the Einstein-Hilbert action we obtain

$$\mathcal{S}_{E-H}(\phi, A^i, \sigma_{nm}) = \int d^d x dt - 2\Lambda^2 \sqrt{\gamma} R_d - c_d \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} F_{ij} F^{ij} e^{-c_d \frac{\phi}{\Lambda}} \tag{3.51}$$

$$\gamma \equiv \det \gamma_{ij} \quad , \quad F_{ij} \equiv \partial_i A_j - \partial_j A_i \tag{3.52}$$

where R_d is the Ricci scalar of a d-dimensional manifold with metric tensor γ_{ij}, while F_{ij} is the Euclidean analog of the curvature tensor used in electromagnetism.

As for the point-particle action in 3.49 we can take advantage of reparametrization invariance, in order to parametrize a black hole worldline with t, the time of an external inertial observer. With this choice the corresponding action becomes

$$\mathcal{S}_{pp} = -m_a \int dt \sqrt{-\dot{x}_a^u \dot{x}_b^v g_{uv}(x_a)} \quad , \quad \dot{x}^\mu = (1, {}^{v^i}\!/_c) \tag{3.53}$$

which in terms of the Kol-Smolkin variables read

$$\mathcal{S}_{pp} = -m_a \int dt\, e^{\frac{\phi}{\Lambda}} \sqrt{1 - \frac{2v_i A_i}{\Lambda} - \gamma_{ij} v^i v^j e^{-\frac{c_d \phi}{\Lambda}} + \frac{(A_i v^i)^2}{m_{pl}^2}} \tag{3.54}$$

Given 3.51,3.54, the effective action for the binary only can be obtained by integrating out the fields ϕ, A_i, σ_{nm} as we have done for the scalar case

$$e^{i\mathcal{S}_{eff}(x_a)} = \int D\phi D A_i D\sigma_{nm}\, e^{i\mathcal{S}_g + i\mathcal{S}_{pp} + i\mathcal{S}_{GF}} \tag{3.55}$$

$$\mathcal{S}_{GF} \equiv -\Lambda^2 \int d^d x dt \sqrt{-g}\, \Gamma^\mu_{\alpha\beta} \Gamma^\nu_{\gamma\delta} g_{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} \tag{3.56}$$

The only difference here is the presence of a gauge-fixing action $\mathcal{S}_{GF}$.
The introduction of this term can be justified within the Faddev-Popov procedure, which makes possible to physically define a consistent functional integral by first fixing a proper gauge. In this case, the gauge chosen is the De-Donder one, while the absence of ghosts is due to the fact that we are only interested in the evaluation of classical contributes to the effective action[11] (i.e proportional to $\hbar^0$).
Therefore, assuming that 3.55 is our starting point, we can start by deriving the Feynman rules in momentum space which defines our effective field theory.
What we need are the rules for the emission and absorption from external sources and interactions ones between ϕ, A^i and σ_{nm}, unite with their propagators.
All these can be found by expanding in Λ^{-1} the Einstein-Hilbert action plus the gauge fixing one

$$\begin{aligned}\mathcal{S}_{E-H} + \mathcal{S}_{GF} = \int d^d x dt \sqrt{\gamma} \Bigg[\frac{1}{4} \Big((\nabla\sigma)^2 - 2(\nabla\sigma_{ij})^2 - (\dot{\sigma}^2 - 2\dot{\sigma}^{ij}\dot{\sigma}_{ij}) \Big) \Bigg] \\ - c_d \Big[(\nabla\phi)^2 - \dot{\phi}^2 \Big] + \Big[\frac{1}{2} F_{ij} F^{ij} + (\vec{\nabla} \cdot \vec{A})^2 - \dot{\vec{A}} \cdot \dot{\vec{A}} \Big] + ...\end{aligned} \tag{3.57}$$

similarly for the point-particle action.

[11]For an accurate description of how to define a consistent functional integral, particularly in the case of internal symmetry groups such as gauge theories, consult Ramond [65] Section 7.2-7.3

3.4 Feynman rules for the non relativistic EFT of a real binary

Having imposed the Kaluza-Klein parametrization for the metric tensor has lead to a theory that can be seen as that of point-particles interacting with three fields. Of course, these fields are nothing else than components of the metric tensor, so every time we will talk about the emission of a particle it should be kept in mind that this is just a pictorial point of view useful in the perturbative evaluation of the functional integral. Baring this fact in mind, we can start by evaluating the Feynman rules that we need starting from fields propagators. As for these, we will derive their relativistic expression, taking at the end the limit for $\omega << |\vec{k}|$.

3.4.1 The ϕ propagator

Let's start from the scalar field, by taking care of the fact that the action is defined in an arbitrary space-time dimensions $d+1$.
The kinetic term is

$$\mathcal{S}_{kin}(\phi) = -c_d \int d^d x dt \Big[(\nabla\phi)^2 - \dot{\phi}^2 \Big] = -c_d \int d^d x dt \, \partial_\mu \phi \partial^\mu \phi \tag{3.58}$$

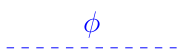
Let's Fourier transform the fields involved

$$\mathcal{S}_{kin}(\phi) = c_d \int d^d x dt \, \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}p}{(2\pi)^{1+d}} \, e^{ix^\mu (k+p)_\mu} \Big[k^\mu p_\mu \, \tilde{\phi}(k) \tilde{\phi}(p) \Big] \tag{3.59}$$

$$= -c_d \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}p}{(2\pi)^{d+1}} \, \delta^{d+1}(k+p) \, k^2 \tilde{\phi}(k) \tilde{\phi}(p) \tag{3.60}$$

$$= \frac{1}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} C_\phi(k) \tilde{\phi}(k) \tilde{\phi}(-k) \quad , \quad C_\phi(k) = -2c_d k^2 \tag{3.61}$$

Remembering the procedure used for the derivation of the graviton propagator seen in Chapter 1, we define the Feynman propagator as i times the inverse of 3.61



$$= P_\phi \equiv -\frac{i}{2c_d(k^2 + i\epsilon)} \tag{3.62}$$

where the $i\epsilon$ prescription select the desired propagator. As convention, a blue line will represent the ϕ field. As for the corresponding non relativistic limit of this rules, one is simply required to replace k^2 with $|\vec{k}|^2$ and remove the $i\epsilon$ since no poles are present in the potential region where the non relativistic limit applies

3.4.2 The A_i propagator

Despite the scalar case, in order to invert the kinetic term for the vector field A_i one should add to it a gauge fixing term, which in our case comes from 3.56.
The proper action can be read from 3.57 as

$$\mathcal{S}_{kin+GF}(A_i) = \int d^dxdt\, \frac{1}{2}F_{ij}F^{ij} + (\vec{\nabla}\cdot\vec{A})^2 - \dot{\vec{A}}\cdot\dot{\vec{A}} \tag{3.63}$$

By expressing all contractions we have

$$\mathcal{S}_{kin+GF}(A_i) = \int d^dxdt\, \partial_i A_j \partial^i A^j - \partial_i A_j \partial^j A^i + \partial_i A^i \partial_j A^j - \dot{A}_i \dot{A}_j \delta^{ij} \tag{3.64}$$

$$= \int d^dxdt\, \partial_i A_j \partial_n A_m (\delta^{in}\delta^{jm} - \delta^{im}\delta^{jn} + \delta^{ij}\delta^{nm}) - \dot{A}_j \dot{A}_m \delta^{jm} \tag{3.65}$$

At this point we Fourier transform all the fields involved in the action getting

$$\mathcal{S}_{kin+GF}(A_i) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}p}{(2\pi)^{d+1}} \delta^{d+1}(k+p) \Big[|\vec{k}|^2\delta^{jm} - \omega_k^2\delta^{jm}\Big] \tilde{A}_j(k)\tilde{A}_m \tag{3.66}$$

$$= \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}p}{(2\pi)^{d+1}} \delta^{d+1}(k+p) \Big[k^2\delta^{jm}\Big] \tilde{A}_j(k)\tilde{A}_m \tag{3.67}$$

$$= \frac{1}{2}\int \frac{d^{d+1}k}{(2\pi)^{d+1}} C^{jm}(k)\tilde{A}_j(k)\tilde{A}_m \quad , \quad C^{jm} = 2k^2\delta^{jm} \tag{3.68}$$

It is now straightforward to define the propagator of A^i in Fourier space as

$$r \overset{A_i}{\sim\!\sim\!\sim\!\sim\!\sim} t \quad = P^{rt} \equiv \frac{i\delta^{rt}}{2(k^2 + i\epsilon)} \tag{3.69}$$

As convention, a red line will represent a A_i field.
As for its non relativistic expression, this is given by removing $i\epsilon$ prescription and replacing k^2 with $|\vec{k}|^2$.

3.4.3 The σ_{nm} propagator

The procedure to derive the propagator for σ_{nm} is similar to that of A^i. Due to a lack of invertibility of the kinetic term of σ_{nm}, one should add to it a quadratic term coming from the gauge fixing action.
This is given by

$$\mathcal{S}_{kin+GF}(\sigma_{nm}) = \int d^dx dt\, \frac{1}{4}\Big((\nabla\sigma)^2 - 2(\nabla\sigma_{ij})^2 - \dot{\sigma}^2 + 2\dot{\sigma}^{ij}\dot{\sigma}_{ij}\Big) \tag{3.70}$$

Expliciting all contractions

$$\begin{aligned}\mathcal{S}_{kin+GF}(\sigma_{nm}) = \int d^dx dt\ & \frac{1}{4}\partial_\alpha\sigma_{ij}\partial^\alpha\sigma_{nm}(\delta^{ij}\delta^{nm}) - \frac{1}{4}\partial_\alpha\sigma_{ij}\partial^\alpha\sigma_{nm}(\delta^{in}\delta^{jm} + \delta^{im}\delta^{jn}) \\ & -\frac{1}{4}\dot{\sigma}_{ij}\dot{\sigma}_{nm}(\delta^{ij}\delta^{nm}) + \frac{1}{4}\dot{\sigma}_{ij}\dot{\sigma}_{nm}(\delta^{in}\delta^{jm} + \delta^{im}\delta^{jn})\end{aligned} \tag{3.71}$$

$$= \int d^dx dt\ \Big[\frac{1}{4}\partial_\alpha\sigma_{ij}\partial^\alpha\sigma_{nm} - \frac{1}{4}\dot{\sigma}_{ij}\dot{\sigma}_{nm}\Big](\delta^{ij}\delta^{nm} - \delta^{in}\delta^{jm} - \delta^{im}\delta^{jn}) \tag{3.72}$$

Fourier transforming the fields involved gives

$$\begin{aligned}\mathcal{S}_{kin+GF}(\sigma_{nm}) = \frac{1}{4}\int & \frac{d^{d+1}k}{(2\pi)^{d+1}}\frac{d^{d+1}p}{(2\pi)^{d+1}}\,\delta^{d+1}(k+p)\,[-\vec{k}\cdot\vec{p} + \omega_k\omega_p] \\ & \tilde{\sigma}_{ij}(k)\tilde{\sigma}_{nm}(p)(\delta^{ij}\delta^{nm} - \delta^{in}\delta^{jm} - \delta^{im}\delta^{jn})\end{aligned} \tag{3.73}$$

$$= \frac{1}{2}\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\, C^{ijnm}\tilde{\sigma}_{ij}(k)\tilde{\sigma}_{nm}(-k) \quad , \quad C^{ijnm} = \frac{k^2}{8}(\delta^{ij}\delta^{nm} - \delta^{in}\delta^{jm} - \delta^{im}\delta^{jn}) \tag{3.74}$$

At this point, all the calculations are equal to those that we have performed for the graviton case. The desired propagator in momentum space can thus be expressed as

$$ij \ \overset{\sigma_{rs}}{\rule{4cm}{0.4pt}}\ nm \quad = \quad P_\sigma^{ijnm} \tag{3.75}$$

where

$$P_\sigma^{ijnm} = -\frac{i}{2(k^2 + i\epsilon)}\Big(\frac{2}{2-d}\,\delta^{ij}\delta^{nm} + \delta^{in}\delta^{jm} + \delta^{im}\delta^{jn}\Big) \tag{3.76}$$

As convention, a green line will represent a generic σ_{rs} tensor. As for its non relativistic expression, this is given by removing $i\epsilon$ and replacing k^2 with $|\vec{k}|^2$.

3.4.4 Source-gravity interactions rules

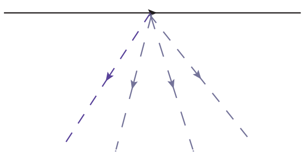
We can now proceed to derive the Feynman rules for the source gravity interactions, which can visualized as the emission (or absorption) of particles from the external classical sources (e.g. black holes).
The advantage of the Kaluza-Klein parametrization is now evident, since as we will show ϕ couples to the mass of the objects only, A_i to its velocity, while σ_{ij} to the symmetric tensor $v^i v^j$ constructed from a single three velocity. To see it, let's proceed from the usual point-particle action for binaries. Neglecting finite size effects, we have already seen that this is given by

$$\mathcal{S}_{pp} = -m_a \int dt\, e^{\frac{\phi}{\Lambda}} \sqrt{1 - \frac{2v_i A_i}{\Lambda} - \gamma_{ij} v^i v^j e^{-\frac{c_d \phi}{\Lambda}} + \frac{(A_i v^i)^2}{m_{pl}^2}} \tag{3.77}$$

Looking carefully at this equation, it should be clear that one has to deal with an infinite number of source-gravity Feynman rules, given by the expansion of 3.77 in terms of the velocities and Λ^{-1}.
For the static case (i.e. $v_a^i = 0$) one has that 3.77 reduces to

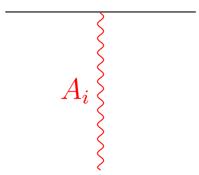
$$\mathcal{S}_{pp} = -m_a \int dt\, e^{\frac{\phi(x_a)}{\Lambda}} \tag{3.78}$$

This term can be expanded as a power series in ϕ^n, where each of them can be interpreted as the emission of n scalar fields from an external source with Feynman rule



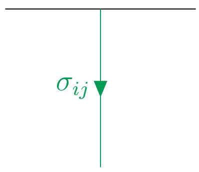
$$= -\frac{i m_a}{\Lambda^n n!} \tag{3.79}$$

At the same time, restoring v_a dependencies and neglecting ϕ fields, one can derive similar rules for the emission of vector particle or tensorial one. In equation 3.77, at first order in velocities and Λ^{-1} one can derive the following Feynman rule for the emission of a vector particle



$$= \frac{i m_a v_i}{\Lambda} \tag{3.80}$$

Similarly, at second order in the velocities and first order in Λ^{-1} one gets the rule for the emission of a sigma tensor



$$= \frac{i m_a v^i v^j}{2\Lambda} \tag{3.81}$$

By a further expansion of the point-particle action, one could get several rules whit the source emitting or absorbing simultaneously particles of different type.

3.4.5 Fields interactions rules

We are now going to evaluate Feynman rules for the interactions between ϕ, A^i and σ_{nm}. As we have demonstrated in the first Chapter, by a proper expansion of the Einstein-Hilbert action one gets a theory of a self interacting field with an infinite number of self-interactions: the same happens here with the expansion of 3.51 in powers of Λ^{-1}. For this reason, we will restrict ourselves only to a finite subset of these Feynman rules showing the general procedure behind the calculations.
The procedure that we will adopt can be applied to all kind of field theories: we will isolate the term in the action which correspond to the desired interaction, we Fourier transform all the fields and we take as many field derivatives as many fields are in our term. At this point the result is multiplied by i and this give us the corresponding Feynman rules where all the momenta are incoming. For future porpoises, we define the functional derivative of a scalar field in Fourier space as a derivative operator which satisfy the following identities

$$\frac{\delta\tilde{\phi}(q)}{\delta\tilde{\phi}(p)} = \delta^{d+1}(q+p) \quad \Rightarrow \quad \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \frac{\delta\tilde{\phi}(q)}{\delta\tilde{\phi}(p)} = 1 \tag{3.82}$$

$$\frac{\delta[\tilde{\phi}(q)\tilde{\phi}(k)]}{\delta\tilde{\phi}(p)} = \frac{\delta\tilde{\phi}(q)}{\delta\tilde{\phi}(p)}\tilde{\phi}(k) + \frac{\tilde{\phi}(k)}{\delta\tilde{\phi}(p)}\delta\tilde{\phi}(q) \tag{3.83}$$

and similarly for A^i and σ_{nm}.
For simplicity we will only consider static interactions, i.e. only those terms in the action that do not depend on time derivatives[12].
in particular we will evaluate two static interactions: the three vertex $\sigma \to \phi\phi$ and the four vertex $\sigma\sigma \to \phi\phi$.

[12]The case of non static interaction is straightforward and similar to the static one, involving only the presence of additional time derivatives in the action.

3.4.6 Feynman rule for σ-ϕ-ϕ vertex

The first static interaction that we will describe is the $\sigma \to \phi\phi$ one, which is represented in 3.57 by the term with a unique σ_{nm} tensor and two scalar fields ϕ.
The corresponding action is given by

$$\mathcal{S}_{\phi,\sigma_{nm}} = -c_d \int d^d x dt\ \partial_i \phi\, \partial_j \phi\, \gamma^{ij} \sqrt{\det \gamma} \tag{3.84}$$

In order to extract from 3.84 the desired interaction we have to expand the determinant and the γ-metric in terms of Λ.
The following identities hold

$$\sqrt{\det \gamma} = e^{\frac{1}{2}\, Tr \log\ \gamma_{ij}} = e^{\frac{1}{2}\, Tr \log\ \left[\delta_{ij} + \frac{\sigma_{ij}}{\Lambda}\right]} = e^{\frac{\sigma^i_i}{2\Lambda} + O(\Lambda^{-2})} = 1 + \frac{\sigma^i_i}{2\Lambda} + O(\Lambda^{-2}) \tag{3.85}$$

$$\gamma^{ij} = \delta^{ij} - \frac{\sigma^{ij}}{\Lambda} + O(\Lambda^{-2}) \tag{3.86}$$

Inserting these in 3.84 gives

$$\mathcal{S}_{\phi,\sigma_{nm}} = -c_d \int d^d x dt\ \partial_i \phi\, \partial_j \phi\ \Big(\delta^{ij} - \frac{\sigma^{ij}}{\Lambda}\Big)\Big(1 + \frac{\sigma^i_i}{2\Lambda}\Big) + O(\Lambda^{-2}) \tag{3.87}$$

$$= c_d \int d^d x dt\ \ - \partial_i \phi\, \partial^i \phi - \frac{1}{2\Lambda} \partial_i \phi\, \partial^i \phi\, \sigma^j_j\ + \frac{1}{\Lambda} \partial_i \phi \partial_j \phi\, \sigma^{ij} + O(\Lambda^{-2}) \tag{3.88}$$

$$= \mathcal{S}_{kin}(\phi) + \mathcal{S}^{\Lambda^{-1}}_{int} + O(\Lambda^{-2}) \tag{3.89}$$

where we have defined the desired interacting action as

$$\mathcal{S}^{\Lambda^{-1}}_{int} \equiv \frac{c_d}{\Lambda} \int d^d x dt \Big(- \frac{1}{2} \partial_i \phi\, \partial^i \phi\, \sigma^j_j\ + \partial_i \phi \partial_j \phi\, \sigma^{ij} \Big) \tag{3.90}$$

We can now proceed by expressing all the fields involved by means of a $d+1$-dimensional Fourier transform as

$$\phi(x) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ik^\mu x_\mu} \tilde{\phi}(k) \quad , \quad \sigma_{ij}(x) = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} e^{ik^\mu x_\mu} \tilde{\sigma_{ij}}(k) \tag{3.91}$$

Inserting these in 3.90 we get

$$\mathcal{S}_{int}^{\Lambda^{-1}} = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{d^{d+1}q}{(2\pi)^{d+1}} \delta^{d+1}(k+q+p)\Big(-\frac{c_d}{\Lambda}\Big)\Big(k^i p^j - k\cdot p\frac{\delta^{ij}}{2}\Big)\tilde{\sigma}_{ij}(q)\tilde{\phi}(k)\tilde{\phi}(p) \tag{3.92}$$

Using the following identity

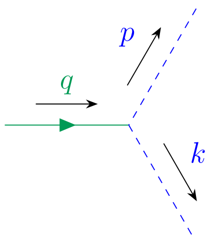
$$\frac{\delta\tilde{\sigma}_{ij}(q)}{\delta\tilde{\sigma}_{nm}(p)} = \frac{1}{2}\big(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn}\big)\delta^{d+1}(q+p) \tag{3.93}$$

we can take the field derivatives in $\tilde{\phi}$ and $\tilde{\sigma}$

$$\frac{\delta^3\mathcal{S}_{int}^{\Lambda^{-1}}}{\delta\tilde{\phi}(k)\delta\tilde{\phi}(p)\delta\tilde{\sigma}_{nm}(q)} = \int d^d q\, \delta^{d+1}(k+q+p)\Big(-\frac{2c_d}{\Lambda}\Big)\Big(k^i p^j - k\cdot p\frac{\delta^{ij}}{2}\Big)\frac{\delta\sigma_{ij}}{\delta\sigma_{nm}} \tag{3.94}$$

$$\frac{\delta^3\mathcal{S}_{int}^{\Lambda^{-1}}}{\delta\tilde{\phi}(k)\delta\tilde{\phi}(p)\delta\tilde{\sigma}_{nm}(q)} = \delta^{d+1}(k+q+p)\Big(-\frac{c_d}{\Lambda}\Big)\Big(k_n p_m + k_m p_n - k\cdot p\,\delta_{nm}\Big) \tag{3.95}$$

where no new momenta symbols has been introduced after functional derivation, since the procedure is implied. Multiplying by $-i$ we select our Feynman rule where



$$= \delta^{d+1}(k+q+p)\Big(\frac{ic_d}{\lambda}\Big)\Big(k\cdot p\,\delta_{nm} - k_n p_m - k_m p_n\Big) \tag{3.96}$$

the Dirac's delta has been left for the conservation of the total momentum and energy.[13]

[13] In the diagrammatic representation of the interaction, momenta are always incoming vertices: in this particular case we have exchanged the directions of $\vec{p}$, $\vec{k}$ vectors without affecting the Feynman rule since it is symmetric under this change.

3.4.7 Feynman rule for σ-σ-ϕ-ϕ vertex

We will show the derivation of another Feynman rule, in this case for the four vertex associated with the $\sigma\sigma \to \phi\phi$ interaction.
This rule comes from 3.84 by expanding in Λ^{-1} at a higher order with respect to the previous one, in order to get an interacting action involving two σ_{nm} tensors and two ϕ fields.
Lets' start by expanding the determinant term at the second order in Λ^{-1}

$$\sqrt{\det\gamma} = e^{\frac{1}{2}\,Tr\,\log\,\gamma_{ij}} = e^{\frac{1}{2}\,Tr\,\log\,\left[\delta_{ij}+\frac{\sigma_{ij}}{\Lambda}\right]} = e^{\frac{1}{2\Lambda}\,\sigma^i_i-\frac{1}{4\Lambda^2}\sigma_{ij}\sigma^{ij}+O(\Lambda^{-3})} \tag{3.97}$$

$$= 1+\frac{1}{2\Lambda}\sigma^i_i-\frac{1}{4\Lambda^2}\sigma_{ij}\sigma^{ij}+\frac{1}{2}\Big(\frac{1}{2\Lambda}\sigma^i_i-\frac{1}{4\Lambda^2}\sigma_{ij}\sigma^{ij}\Big)^2+O(\Lambda^{-3}) \tag{3.98}$$

$$= 1+\frac{1}{2\Lambda}\sigma^i_i-\frac{1}{4\Lambda^2}\sigma_{ij}\sigma^{ij}+\frac{1}{8\Lambda^2}(\sigma^i_i)^2+O(\Lambda^{-3}) \tag{3.99}$$

Given 3.99 we proceed by expanding the inverse of γ_{ij}.
First of all, we should notice that the inverse metric is such that

$$\gamma\gamma^{-1} = \mathbf{1}_n \tag{3.100}$$

and by expliciting γ_{ij} in terms of σ_{ij} one has

$$\Big(\mathbf{1}_n+\frac{\sigma}{\Lambda}\Big)\gamma^{-1} = \mathbf{1}_n \quad\Rightarrow\quad \gamma^{-1}+\frac{\sigma}{\Lambda}\gamma^{-1} = \mathbf{1}_n \quad\Rightarrow\quad \gamma^{-1} = \mathbf{1}_n-\frac{\sigma}{\Lambda}\gamma^{-1} \tag{3.101}$$

The last equation give us a recurrence relation which can be used to define the inverse metric via a sort of geometric series

$$\gamma^{-1} = \mathbf{1}_n-\frac{\sigma}{\Lambda}+\Big(\frac{\sigma}{\Lambda}\Big)^2+O(\Lambda^{-2}) \tag{3.102}$$

Inserting 3.99,3.102 into 3.84 we can now isolate the term in the action which contributes to the four vertex.
This is given by

$$\mathcal{S}^{\Lambda^{-2}}_{int} = -\frac{c_d}{\Lambda^2}\int d^dxdt\partial_i\phi\partial^i\phi\Big(\frac{\sigma^2}{8}-\frac{1}{4}\sigma^{nm}\sigma_{nm}\Big)+\partial_i\phi\partial_j\phi\,\sigma^{ij}\frac{\sigma}{2}+\partial_i\phi\partial_j\phi\,\sigma^{il}\sigma^j_l \tag{3.103}$$

At this point, for a simpler derivation, we will separate 3.103 into three contributions by calculating the rules of Feynman separately for each of them, then adding them so as to obtain the rule for the desired vertex.

The division will be the following

$$\mathcal{S}_A = -\frac{c_d}{\Lambda^2}\int d^dxdt\ \partial_i\phi\partial^i\phi\left(\frac{\sigma^2}{8} - \frac{1}{4}\sigma^{nm}\sigma_{nm}\right) \tag{3.104}$$

$$\mathcal{S}_B = -\frac{c_d}{\Lambda^2}\int d^dxdt\ \partial_i\phi\partial_j\phi\,\sigma^{ij}\frac{\sigma}{2} \tag{3.105}$$

$$\mathcal{S}_C = -\frac{c_d}{\Lambda^2}\int d^dxdt\ \partial_i\phi\partial_j\phi\,\sigma^{il}\sigma_l^j \tag{3.106}$$

We start with 3.104 by expressing all the fields involved via a Fourier transform

$$\begin{aligned}\mathcal{S}_A = \frac{c_d}{\Lambda^2}\int d^dxdt\,\frac{d^{d+1}k}{(2\pi)^{d+1}}\,\frac{d^dp}{(2\pi)^{d+1}}\,\frac{d^{d+1}q_1}{(2\pi)^{d+1}}\,\frac{d^{d+1}q_2}{(2\pi)^{d+1}}\,e^{ix^\mu(k+p+q_1+q_2)_\mu}\,k\cdot p\,\tilde{\phi}(k)\tilde{\phi}(p)\\ \left(\frac{1}{8}\delta^{\alpha\beta}\delta^{nm} - \frac{1}{8}P^{\alpha\beta nm}\right)\tilde{\sigma}_{\alpha\beta}(q_1)\tilde{\sigma}_{nm}(q_2)\end{aligned} \tag{3.107}$$

$$\begin{aligned}= -\frac{c_d}{\Lambda^2}\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\frac{d^dp}{(2\pi)^{d+1}}\frac{d^{d+1}q_1}{(2\pi)^{d+1}}\frac{d^{d+1}q_2}{(2\pi)^{d+1}}\delta^{d+1}(k+p+q_1+q_2)\frac{1}{8}k\cdot p\\ Q^{\alpha\beta nm}\tilde{\sigma}_{\alpha\beta}(q_1)\tilde{\sigma}_{nm}(q_2)\tilde{\phi}(k)\tilde{\phi}(p)\end{aligned} \tag{3.108}$$

where we have introduced the following useful operators

$$P^{\alpha\beta nm} \equiv \delta^{\alpha n}\delta^{\beta m} + \delta^{\alpha m}\delta^{\beta n} \quad , \quad Q^{\alpha\beta nm} \equiv P^{\alpha\beta nm} - \delta^{\alpha\beta}\delta^{nm} \tag{3.109}$$

Now we proceed by taking the functional derivatives of the fields. First of all, let's notice that the following identities hold

$$\frac{\delta[\tilde{\sigma}_{\alpha\beta}(k_1)\tilde{\sigma}_{nm}(k_2)]}{\delta\tilde{\sigma}_{ij}(q_1)} = \frac{1}{2}P_{\alpha\beta ij}\tilde{\sigma}_{nm}\delta^{d+1}(q_1+k_1) + \frac{1}{2}P_{nmij}\tilde{\sigma}_{\alpha\beta}\delta^{d+1}(q_1+k_2) \tag{3.110}$$

$$\begin{aligned}\frac{\delta[\tilde{\sigma}_{\alpha\beta}(k_1)\tilde{\sigma}_{nm}(k_2)]}{\delta\tilde{\sigma}_{ij}(q_1)\delta\tilde{\sigma}_{rs}(q_2)} = \frac{1}{4}\big[P_{\alpha\beta ij}P_{nmrs}\delta^{d+1}(q_1+k_1)\delta^{d+1}(q_2+k_2)+\\ P_{nmij}P_{\alpha\beta rs}\delta^{d+1}(q_1+k_2)\delta^{d+1}(q_2+k_1)\big]\end{aligned} \tag{3.111}$$

After a redefinition of the variable of integration, we can perform the following field derivatives so as to maintain the initial momentum variables in Fourier space

$$\frac{\delta^4 \mathcal{S}_A}{\delta\tilde{\phi}(k)\delta\tilde{\phi}(p)\delta\tilde{\sigma}_{ij}(q_1)\delta\tilde{\sigma_{rs}}(q_2)} = -\frac{4c_d}{\Lambda^2}\frac{1}{8}k\cdot p\, Q_{ijrs}\delta^d(k+p+q_1+q_2) \tag{3.112}$$

Multiplying by i, we obtain the first part of the Feynman rule for the process $\sigma\sigma \to \phi\phi$

$$V^A_{ijrs} = -i\frac{4c_d}{\Lambda^2}\quad\frac{1}{8}k\cdot p\, Q_{ijrs}\quad \delta^d(k+p+q_1+q_2) \tag{3.113}$$

Now we focus on the second term of our interacting action

$$\mathcal{S}_B = -\frac{c_d}{2\Lambda^2}\int d^dxdt\;\, \partial_i\phi\partial_j\phi\,\sigma^{ij}\sigma \tag{3.114}$$

As before we proceed by Fourier transforming all the fields involved

$$\begin{aligned}\mathcal{S}_B = -\frac{c_d}{2\Lambda^2}\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{d^{d+1}q_1}{(2\pi)^{d+1}}\frac{d^{d+1}q_2}{(2\pi)^{d+1}}\delta^{d+1}(k+p+q_1+q_2)\\ \left(k_ip_j\delta_{\alpha\beta}\right)\,\tilde{\phi}(k)\tilde{\phi}(p)\,\tilde{\sigma}^{ij}(q_1)\tilde{\sigma}^{\alpha\beta}(q_2)\end{aligned} \tag{3.115}$$

By taking the corresponding fields derivatives we have

$$\begin{aligned}\frac{\delta^4 \mathcal{S}_B}{\delta\tilde{\phi}(k)\delta\tilde{\phi}(p)\delta\tilde{\sigma}_{rs}(q_1)\delta\tilde{\sigma}_{nm}(q_2)} = -\frac{c_d}{4\Lambda^2}\,\delta^d(k+p+q_1+q_2)\left(k_ip_j\delta_{\alpha\beta}\right)\\ \left(P^{ijrs}P^{\alpha\beta nm}+P^{ijnm}P^{\alpha\beta rs}\right)\end{aligned} \tag{3.116}$$

$$\begin{aligned}\frac{\delta \mathcal{S}_B}{\delta^4\tilde{\phi}(k)\delta\tilde{\phi}(p)\delta\tilde{\sigma}_{rs}(q_1)\delta\tilde{\sigma}_{nm}(q_2)} = -\frac{c_d}{2\Lambda^2}\delta^d(k+p+q_1+q_2)\,k_ip_j\\ \left(P^{ijrs}\delta^{nm}+P^{ijnm}\delta^{rs}\right)\end{aligned} \tag{3.117}$$

Multiplying by i we get another term for the Feynman rule we were looking for

$$V^{rsnm,B} = -i\frac{c_d}{2\Lambda^2}\left(k_ip_j\right)\left(P^{ijrs}\delta^{nm}+P^{ijnm}\delta^{rs}\right) \tag{3.118}$$

Which can be explicited to

$$V^{rsnm,B} = -i2\frac{c_d}{\Lambda^2}k^{(r}p^s\delta^{nm)} \tag{3.119}$$

As for the remaining term for the four vertex, this is given by

$$\mathcal{S}_C = -\frac{c_d}{\Lambda^2}\int d^dxdt\ \partial_i\phi\partial_j\phi\,\sigma^{il}\sigma_l^j \tag{3.120}$$

We first Fourier transform the fields involved

$$\begin{aligned}\mathcal{S}_C = \frac{c_d}{\Lambda^2}\int \frac{d^{d+1}k}{(2\pi)^{d+1}}\frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{d^{d+1}q_1}{(2\pi)^{d+1}}\frac{d^{d+1}q_2}{(2\pi)^{d+1}}\delta^{d+1}(k+p+q_1+q_2)\\ k_\alpha p_i\delta_{\beta j}\tilde{\phi}(k)\tilde{\phi}(p)\tilde{\sigma}^{\alpha\beta}(q_1)\tilde{\sigma}^{ij}(q_2)\end{aligned} \tag{3.121}$$

As usual we take the functional derivatives of the fields

$$\begin{aligned}\frac{\delta^4\mathcal{S}_B}{\delta\tilde{\phi}(k)\delta\tilde{\phi}(p)\delta\tilde{\sigma}_{rs}(q_1)\delta\tilde{\sigma}_{nm}(q_2)} = \frac{c_d}{2\Lambda^2}\delta^{d+1}(k+p+q_1+q_2)\Big(k_\alpha p_i\delta_{\beta j}\Big)\\ \Big(P^{\alpha\beta rs}P^{ijnm}+P^{\alpha\beta nm}P^{ijrs}\Big)\end{aligned} \tag{3.122}$$

In 3.122 appears several contractions which deserve to be calculated separately. One is given by

$$P^{\alpha\beta rs}P^{ijnm}\delta_{\beta j} \tag{3.123}$$

$$= \Big(\delta^{\alpha r}\delta^{\beta s}+\delta^{\alpha s}\delta^{\beta r}\Big)\delta_{\beta j}\Big(\delta^{in}\delta^{jm}+\delta^{im}\delta^{jn}\Big) \tag{3.124}$$

$$= \Big(\delta^{\alpha r}\delta^{sj}+\delta^{\alpha s}\delta^{rj}\Big)\Big(\delta^{in}\delta^{jm}+\delta^{im}\delta^{jn}\Big)\Big(\delta^{\alpha r}\delta^{sj}+\delta^{\alpha s}\delta^{rj}\Big)\Big(\delta^{in}\delta^{jm}+\delta^{im}\delta^{jn}\Big) \tag{3.125}$$

$$= \delta^{\alpha r}\delta^{in}\delta^{sm}+\delta^{\alpha r}\delta^{im}\delta^{sn}+\delta^{\alpha s}\delta^{in}\delta^{rm}+\delta^{\alpha s}\delta^{im}\delta^{rn} \tag{3.126}$$

$$= \delta^{\alpha r}P^{ismn}+\delta^{\alpha s}P^{irmn} \tag{3.127}$$

A further contraction, which can be solved using 3.127, is given by

$$k_\alpha p_i\Big(\delta^{\alpha r}P^{ismn}+\delta^{\alpha s}P^{irmn}+\delta^{\alpha n}P^{imsr}+\delta^{\alpha m}P^{inrs}\Big) \tag{3.128}$$

$$= p_i\Big(k^rP^{ismn}+k^sP^{irmn}+k^nP^{imsr}+k^mP^{inrs}\Big) \tag{3.129}$$

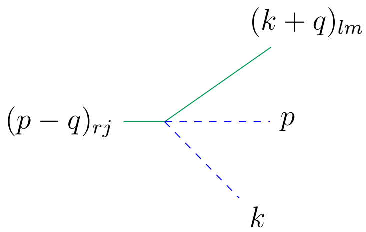
Since equation 3.129 is part of 3.122, we can use it to express the final term for the four vertex as

$$V^{rsmn,C} = i\frac{4c_d}{\Lambda^2}k^{(r}p^m\delta^{sn)} \tag{3.130}$$

At this point, the complete four vertex for the process $\sigma\sigma \rightarrow \phi\phi$ is given by summing up equations 3.113,3.119,3.130 with their proper indexes.
The result is

$$V^{rlmn} = V^{rlmn,A} + V^{rlmn,B} + V^{rlmn,C} \tag{3.131}$$

$$V^{rlmn} = i\frac{4c_d}{\Lambda^2}\Big[k^r p^l \delta^{mn} - \frac{1}{2}k^l p^m \delta^{rj} - \frac{1}{8}p \cdot k Q^{rjlm} + \big(r \longleftrightarrow j, l \longleftrightarrow m\big)\Big] \tag{3.132}$$



$$= V^{rlmn} \tag{3.133}$$

This complete the evaluation of the Feynman rules for the effective field theory of a binary black hole. If needed, further rules can be found by expanding 3.49 in powers of Λ^{-1} and velocities of the binaries.

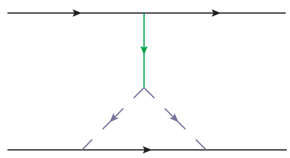
3.5 Building effective Feynman diagrams

We continue the development of our EFT for binary black holes by defining the rules to be followed to calculate post-Newtonian corrections to the effective action.
First of all we define *effective diagram*, an amplitude in momentum space that can be built from the previous Feynman rules which scales with definite powers of the Newton constant G_N and v^2, being v a typical velocity of the system.
We also say that it gives a n post-Newtonian correction if it scales as $G_N^{n-l}v^{2l}$ where l satisfy $0 \leq l \leq n-1$.
At this point, given an effective diagram, one should be aware to follow the subsequent rules:

- Only connected diagrams contributes to the effective action, i.e only those diagrams that cannot be expressed as products of simpler one. This fact was proved to hold for scalar gravity and it maintain its validity regardless the particular EFT one is studying.[14]

- Given a connected diagram, one is required to draw all the possible arrangements which are topological inequivalent after having identified the external worldlines. The number of ways a given diagram can be drawn is the topological weight of the diagram: this has to divide the chosen diagram in order to obtain the correct contribute to the effective action.

- Once a proper effective diagram has been built, one has to integrate over momentum variables by taking care that external momenta (i.e those coming from worldlines) has to be Fourier transformed has seen in scalar gravity, while internal one has to be integrated. This last fact is merely connected with the presence of self-interactions of gravity and it leads to loop integrals.
  However, besides the name of these, one should be aware that there are two kind of loop integrals that one can encounter. The first integrations involves at least an external leg of a worldline, as example

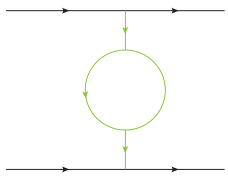
  

  (3.134)

  Since there is no propagator associated to the worldlines, this integral cannot give quantum contributes, therefore, besides they formally resemble loop integral, these are just classical contributes (i.e. there are no $\hbar$ dependencies).

[14] For a demonstration, see Ramond Section 3.2 [65]

As for the remaining, these are real loop integrals in a QFT sense involving no classical worldline as

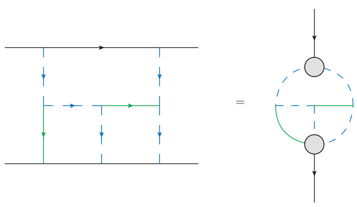


(3.135)

This correction is of quantum nature and brings non null powers of $\hbar$.
A rough estimate shows that similar corrections scale as the ratio of the Planck constant and that of total angular momentum of the system: for a binary black hole this would introduce corrections which are roughly of the order of 10^{-77} respect to classical ones. For this reason, we will neglect these effects. Baring these facts in mind, one can proceed to calculate any relevant corrections to the effective action of a binary.

3.5.1 A relation between effective diagrams and QFT amplitudes

Given that no quantum corrections enter in the evaluation of the EFT of a binary, it is notable that QFT methods can still be applied in order to describe classical contributes. One can realize it recognizing that effective diagrams resembles the topology of two point functions. A simpler prove of this fact can be derived by a procedure of "pinching", where the external lines are stretched to form a single external line which takes care of source-gravity interactions.
Applying this procedure gives where the blob denotes the interactions source-gravity,



(3.136)

while the remaining is the loop-like integral.
Thanks to this observation, it becomes relevant to develop an efficient QFT machinery in order to evaluate high post-Newtonian corrections, which is what the next chapter is devoted to.

Chapter 4

Multi-loop techniques in QFT

In dealing with the Effective Field Theory (EFT) of a binary system, we have seen how post-Newtonian corrections can be cast in form of effective diagrams, showing a deep relation among these and massless two point functions with n loops.
Thanks to this topological relation, it seems natural to borrow QFT techniques into our EFT approach in order to evaluate high post-Newtonian corrections.
This possibility has been exploited in the work of *Mastrolia, Sturani, Foffa* and *Sturm* [34] leading to the complete evaluation of the static sector (i.e. no v^2 dependence) for the effective action of a non spinning binary.
Aim of this chapter is precisely to develop in detail this QFT methods that one can apply in order to evaluate high post-Newtonian corrections, in particular non static ones, that so far haven't been evaluated within these methods.
As for the Chapter it is so divided:

- In order to deal with divergences typical of loop integrals, we will introduce the dimensional regularization technique that allows us to work in $d+1$ space-time dimensions and then take the physical limit for $d \to 3$. In this way, spurious divergences can be isolated and removed so as to obtain a physical and finite contribution for the effective action of a binary.

- In order to calculate loop integrals in arbitrary dimensions, we will introduce the Feynman parameterization technique by applying it to the calculation of integral scalars and tensor types. We will also evaluate their Fourier transform.

- A modern amplitude technique will be presented, based on the so-called *Integration by parts identities* (IBP) together with the concept of *Master Integral*. As an example, we will apply them to reduce a massive one-loop tadpole.

- We will calculate in detail two point massless functions with one-two loops, together with a three loop coming from a topology called Bug type. In addition, we will introduce the *Star triangle rule* (STR) and use it to estimate the degree of divergence of a massless two point function with four loops.

4.1 Dimensional regularization

In dealing with effective diagrams we will be required to calculate Euclidean loop-like integrals of the following type

$$I = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + \Delta^2} = \frac{1}{2\pi^2} \int_0^{+\infty} dp \frac{p^2}{p^2 + \Delta^2} \sim +\infty \quad , \quad \Delta \in \mathbb{R} \tag{4.1}$$

which gives divergent contributes to the effective action.
In general these divergences can be divided in two types: *UV divergences* coming from the region $p^2 \sim +\infty$ and *IR divergences* from the region $p^2 \sim 0^+$.
To overcome UV divergences one can introduce the procedure of dimensional regularization, which was proposed in 1972 by t'Hooft and Veltman (see [68]), here applied for effective field theories.
The basic idea behind this technique is to regard the number of dimensions as an additional complex variable, in our case the d space dimensions, so that UV divergences trivially disappear. Redefining our integrals as function of d, via analytic continuation one can prove that they reduces to the starting divergent ones for a proper choice of d.
For instance, if we regard 4.1 as a d-dimensional integral then we have

$$I(d) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 + \Delta^2} = \frac{2^{1-d}\pi^{-3d/2}}{\Gamma(d/2)} \int_0^{+\infty} dp \frac{p^{d-1}}{p^2 + \Delta^2} \tag{4.2}$$

which converges for $d = 1$ dimension, while for $d = 3$ it reduces to the previous UV divergent one.[1]
The value of this method is that by Laurent expanding around $d = 3 + \varepsilon$, where our physical theory is defined, one can extract divergences as simple poles in ε: to be consistent, our EFT is required to be, at a given post-Newtonian order, free of UV and IR divergences after having summed up every d-dimensional regularized effective diagram that contributes to the effective action.
This explain the choice to work with a $d + 1$ dimensional EFT, as seen introducing the effective action of a binary in 3.49. As for the presence of the arbitrary length scale l, which was introduced in order to render adimensional the effective action, the reader should be aware that it is just an artifact of the procedure of dimensional regularization since it can be proved that it leaves unaffected the EFT at $3 + 1$ dimensions, regardless the specific choice of l one can made.[2]

[1] See Appendix C for the evaluation of a d-dimensional unite sphere, here implied.
[2] For a detailed introduction on the subject see Itzykson-Zuber [47]

4.2 Feynman parametrization and loop integrals

Since many effective diagrams can be mapped into loop integrals, we will introduce a useful parametrization used for their evaluation based on the so called *Feynman parameters*.
Given a generic pair of complex number A and B we have

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A+(B-A)x]^2} = \int_0^1 dxdy\, \delta(x+y-1) \frac{1}{[xA+yB]^2} \tag{4.3}$$

This simple identity is relevant since it can be used to complete squares at the denominator, as in the case of the following Euclidean loop integral

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2(k-q)^2} = \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{[k^2+((k-q)^2-k^2)x]^2} \tag{4.4}$$

$$= \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{[(k-qx)^2+\Delta]^2} \quad , \quad \Delta(q,x) \equiv q^2(1-x)x \tag{4.5}$$

$$= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+\Delta)^2} \tag{4.6}$$

In the last integral we have shifted by qx, obtaining an integrand which is odd as a function of k, depending solely on its norm.
This is as an example of a so called *scalar integral*. Since many effective diagrams will be expressed as their function, we will proceed to evaluate a generic class of scalar integral, unite with their Fourier transform.

4.2.1 The scalar integral $I_S(d,a,b)$

Scalar integrals usually appear in dealing with loop integrals, for this reason we will evaluate the following Euclidean one that regards 4.4 as its subcase

$$I_S(d,a,b) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^{2a}(k^2+\Delta)^b} \tag{4.7}$$

Here a, b and Δ are arbitrary real numbers, for which the integral could converge or not, while the k terms in the integrand stands for the Euclidean norm of $\vec{k}$.[3].

[3] Beware that we are integrating on an Euclidean region of Fourier space. Usually this kind of integrals are presented in a QFT context assuming a Minkowskian measure. To express our result in terms of them one is simple required to perform a Wick rotation. See Schwartz Appendix B.2 [66]

We first split the integration region $\mathbb{R}^d$ using spherical coordinates in d dimensions

$$I_S(d,a,b) = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^{+\infty} dk \frac{k^{d-1}}{k^{2a}(k^2+\Delta)^b} = \frac{2^{1-d}\pi^{-d/2}}{\Gamma(d/2)} \int_0^{+\infty} dk \frac{k^{d-1-2a}}{(k^2+\Delta)^b} \quad (4.8)$$

$$I_S(d,a,b) = \frac{2^{1-d}\pi^{-d/2}}{\Gamma(d/2)} I_{|k|}(d,a,b) \quad (4.9)$$

$$I_{|k|}(d,a,b) = \int_0^{+\infty} dk \, \frac{k^{d-1-2a}}{(k^2+\Delta)^b} \quad (4.10)$$

We can collect the Δ term in 4.10, and perform a rescaling in the variable of integration in order to obtain

$$I_{|k|}(d,a,b) = \int_0^{+\infty} dk \, \frac{k^{d-1-2a}}{(k^2+\Delta)^b} = \Delta^{\frac{d}{2}-a-b} \int_0^{+\infty} dk \, \frac{k^{d-1-2a}}{(k^2+1)^b} \quad (4.11)$$

Introducing the change of coordinates $k = \sqrt{x}$, the following identities hold

$$\int_0^{+\infty} dk \, \frac{k^{d-1-2a}}{(k^2+1)^b} = \frac{1}{2} \int_0^{+\infty} dx \, \frac{x^{\frac{d}{2}-a-1}}{(x+1)^b} = \frac{1}{2}\beta\left(\frac{d}{2}-a, a+b-\frac{d}{2}\right) \quad (4.12)$$

where in the last equivalence we have introduced the so called β-function

$$\beta(x,y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^{+\infty} dt \, \frac{t^{x-1}}{(t+1)^{x+y}} \quad (4.13)$$

Inserting 4.12 into 4.11, we arrive at the final expression for 4.7

$$I_{|k|}(d,a,b) = \Delta^{\frac{d}{2}-a-b} \frac{\Gamma(\frac{d}{2}-a)\Gamma(a+b-\frac{d}{2})}{2\Gamma(b)} \quad (4.14)$$

This gives for 4.7 the following result

$$I_S(d,a,b) = \frac{\Delta^{\frac{d}{2}-a-b}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2}-a)\Gamma(a+b-\frac{d}{2})}{\Gamma(b)\Gamma(d/2)} \quad (4.15)$$

4.2.2 The scalar integral $I_F(d, a)$

Since all our calculations will be made in Fourier space, in order to give back a contribute to the effective action, we will have to perform a Fourier transform.
It is thus necessary to evaluate another class of scalar integral regarded as a Fourier transform. The simplest, is the following Euclidean d-dimensional integral

$$I_F(d,a) = \int \frac{d^d q}{(2\pi)^d} \frac{e^{iq\cdot r}}{q^{2a}} \tag{4.16}$$

Let's assume that $\vec{r}$ represents a d-dimensional z-axis, we then decompose the measure of integration as

$$d^d q = dq_z d^{d-1} Q_\perp = dq_z dQ Q^{d-2} d\Omega_{d-1} \tag{4.17}$$

where $Q_\perp$ is ortogonal to q_z, which is parallel to $\vec{r}$, while Q stand for its Euclidean norm.
Using 4.17, the previous integral becomes

$$I_F(d,a) = \frac{\Omega_{d-1}}{(2\pi)^d} \int_{-\infty}^{+\infty} dq_z \int_0^{+\infty} dQ Q^{d-2} \frac{e^{iq_z r}}{(q_z^2 + Q^2)^a} \tag{4.18}$$

A this point it is useful to introduce the following change of coordinates on the (q_z, Q) upper-half plane

$$\begin{cases} q_z = xQ \\ Q = \frac{y}{xr} \end{cases} \longrightarrow \begin{cases} q_z = \frac{y}{r} \\ Q = \frac{y}{xr} \end{cases} \tag{4.19}$$

which is a map from $\mathbb{R}_{q_z} \times \mathbb{R}^+_Q$ to $\mathbb{R}^2_{x,y}$ with determinant $\frac{y}{(xr)^2}$.
Applying this change to our integral gives

$$I_F(d,a) = \frac{\Omega_{d-1}}{(2\pi)^d} r^{2a-d} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{e^{iy}}{(y^2)^{\frac{1}{2}-\frac{d}{2}+a}} \frac{x^{2a-d}}{(1+x^2)^a} \tag{4.20}$$

which can be reduced to the product of two scalar integrals as

$$I_F(d,a) = \frac{\Omega_{d-1}}{(2\pi)^d} r^{2a-d} \int_{-\infty}^{+\infty} dx \frac{x^{2a-d}}{(1+x^2)^a} \int_{-\infty}^{+\infty} dy \frac{e^{iy}}{(y^2)^{\frac{1}{2}-\frac{d}{2}+a}} \tag{4.21}$$

The integral in the x variable can be easily performed by noticing that it comes from 4.7 choosing $d = \Delta = 1$

$$\int_{-\infty}^{+\infty} dx \frac{x^{2a-d}}{(1+x^2)^a} = \frac{\Gamma(a - \frac{d}{2} + \frac{1}{2})\Gamma(\frac{d-1}{2})}{2\Gamma(a)} \tag{4.22}$$

We are only left with the integral on the second bracket which comes from the more general one

$$L(\alpha) = \int_{-\infty}^{+\infty} dy \frac{e^{iy}}{y^\alpha} \tag{4.23}$$

for a specific choice of α.
In order to evaluate 4.23, we can use integration by parts, assuming $\alpha \in \mathcal{N}$, in order to derive a useful recurrence relation

$$L(\alpha+1) = \frac{iL(\alpha)}{\alpha} \quad \Rightarrow \quad L(\alpha) = \frac{i^{\alpha-1}}{\Gamma(\alpha)} L(1) \tag{4.24}$$

where $L(1)$ is given by the principal value of a contour integral in the complex plane. To show this, let's first define the line of integration by means of a counterclockwise closed curve avoiding $z = 0$ given by $\Omega = \{\Omega_R \cup (\mathbb{R} \setminus [-r, +r]) \cup \Omega_r\}$ where Ω_r=$\{z \in \mathbb{C} \mid z = re^{i\phi}\ ,\ 0 < \phi < \pi\}$, the same for Ω_R.
Since the integrand of $L(1)$ is a meromorphic function inside Ω, one has

$$\oint_\Omega \frac{e^{iz}}{z} = \int_{\Omega_R} \frac{e^{iz}}{z} dz + \int_{\mathbb{R}\setminus[-r,r]} \frac{e^{iz}}{z} dz + \int_{\Omega_r} \frac{e^{iz}}{z} dz = 0 \tag{4.25}$$

Now, sending $R \to +\infty$ the first integral is zero by means of Jordan's lemma, while the second is equal to $L(1)$ given that $r \to 0$. This means that

$$L(1) = -\lim_{r\to 0} \int_{\Omega_r} \frac{e^{iz}}{z} dz \tag{4.26}$$

Integral 4.26 is easily evaluated using the fact that in the integration region r is fixed, $0 < \phi < \pi$, and $dz = izd\phi$

$$L(1) = -i \lim_{r\to 0} \int_\pi^0 e^{ir[\cos(\phi)+isin(\phi)]} d\phi = -\int_\pi^0 d\phi = i\pi \tag{4.27}$$

The result is

$$\int_{-\infty}^{+\infty} dy\,\frac{e^{iy}}{y^\alpha} = \frac{\pi\, i^\alpha}{\Gamma(\alpha)} \quad , \quad \alpha \in \mathcal{N} \tag{4.28}$$

In order to relate 4.28 to the integral in 4.21, we restrict ourselves to even α since the integrand we are looking for depend on y^2, not y only.
With this choice, we can set $\alpha = 2n$ and $n = a + \frac{1}{2} - \frac{d}{2}$ into 4.28, which gives via analytic continuation into a and d the following result

$$\int_{-\infty}^{+\infty} dy \frac{e^{iy}}{(y^2)^{a+\frac{1}{2}-\frac{d}{2}}} = \frac{\pi(-1)^{a+\frac{1}{2}-\frac{d}{2}}}{\Gamma(2a+1-d)} \tag{4.29}$$

Inserting 4.29,4.22 into 4.21, after few manipulations with the Gamma function involved, one can prove that our scalar integral is equal to

$$I_F(d,a) = \frac{\Gamma(\frac{d}{2}-a)}{(4\pi)^{\frac{d}{2}}\Gamma(a)}\Big(\frac{r}{2}\Big)^{2a-d} \tag{4.30}$$

4.2.3 The tensorial generalization of $I_F(d,a)$

Besides the scalar case observed, we will also encounter another integral which can be regarded as the tensorial generalization of 4.16

$$I_F^{ij}(d,a) = \int \frac{d^dq}{(2\pi)^d} e^{iq\cdot r}\,\frac{q^iq^j}{q^{2a}} \tag{4.31}$$

Equation 4.31 can be related to 4.16 using the fact that

$$\partial^i\partial^j I_F(d,a) = -I_F^{ij}(d,a) \quad \Leftrightarrow \quad d \neq 2a \tag{4.32}$$

which is valid only when $I(d,a)$ has a x^i dependence (i.e. when $2a \neq d$).
By extending the function so obtained via analytic continuation also in the region $d = 2a$, one has that the following relation holds

$$I_F^{ij}(d,a) = \frac{\Gamma(\frac{d}{2}-a+1)}{(4\pi)^{\frac{d}{2}}\Gamma(a)}\Big(\frac{r}{2}\Big)^{2a-d-2}\Big[\frac{1}{2}\delta^{ij} - \Big(\frac{d}{2}-a+1\Big)\frac{r^ir^j}{r^2}\Big] \tag{4.33}$$

4.3 Integration by parts identities (IBP)

In this section we will present a modern multi-loop techniques, that were first introduced by *Chetyrkin* and *Tkachov* in [19, 71, 40].
Let's suppose we are interested in the evaluation of a Feynman diagram with n-loop momenta $k_1, ..k_n$ and $p_1, ...p_m$ external momenta

$$\mathcal{M}(p_1, ..., p_m) = \int \frac{d^d k_1}{(2\pi)^d} .... \frac{d^d k_n}{(2\pi)^d} \frac{N_1^{n_1}..N_l^{n_l}}{D_1^{m_1}..D_q^{m_q}} \tag{4.34}$$

where D and N represent respectively a generic term in the denominator (numerator) which is made of scalar products with external and loop momenta.
There exist a way to generate relations that involve 4.34 and diagrams sharing the same topology, based on the simple observation that the integral of the full derivative of a vector field v^i is null, assuming it vanish in a proper way while approaching the border[4] of $\mathbb{R}^d$

$$\int_{R^d} d^d k \, \partial_{k_i} v^i = 0 \quad \Leftrightarrow \quad |v(k)| \sim_{+\infty} \frac{1}{|k|^n} \, , \, n > 1 \tag{4.35}$$

Choosing as a vector field an arbitrary linear combination of momenta multiplied with the integrand of 4.34, one derives the following relation

$$\int \frac{d^d k_1}{(2\pi)^d} .... \frac{d^d k_n}{(2\pi)^d} \frac{\partial}{\partial k_j} \cdot \left([a^i k^i + b^i p^i] \frac{N_1^{n_1}..N_l^{n_l}}{D_1^{m_1}..D_q^{m_q}} \right) = 0 \tag{4.36}$$

Expressing derivatives via Leibniz rule, one sees that the derivative on the linear combination gives back the desired integral multiplied by a proper factor while the action on the integrand produce new scalar products. These fall into two categories: the reducible ones, i.e. those that can be expressed as a linear combination of the denominators and the irreducible that cannot; as a remarkable facts all these operations gives new amplitude sharing the same topology or sub-topology of 4.34. In this respect, equation 4.36 states that $\mathcal{M}$ could be expressed as linear combination of different Feynman amplitudes sharing its same topology. In particular varying the coefficients a_i and b_i of 4.36 one could generate a set of relations called *IBP identities* that could be solved for the initial amplitude in terms of simpler ones: these can be viewed as a basis and are usually called *Master Integral.* The procedure has been implemented by Laporta developing a method for high precision calculations of multi-loop (see [55]) and has been used in several works as [61, 18].

[4] It is nothing else than the Gauss theorem in arbitrary d dimensions.

4.3.1 IBP for the massive one-loop tadpole

As example, let's consider the following one-loop massive vacuum diagram typical of Quantum field theory

$$I_n = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)^n} \tag{4.37}$$

In this case k is the only loop momentum, thus we can generate only one IBP

$$\int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2+m^2)^n} = 0 \tag{4.38}$$

$$(d-2n) \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)^n} = -2nm^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)^{n+1}} \tag{4.39}$$

$$(d-2n) I_n = -2nm^2 I_{n+1} \quad \Rightarrow \quad I_n = \frac{\Gamma(n - d/2)}{(m^2)^{n-1} \Gamma(n) \Gamma(1 - d/2)} I_1 \tag{4.40}$$

where in the last line we can recognize I_1 as the proper Master Integral for the one-loop massive tadpole, which has to be calculated separately.

4.3.2 Properties of IBP

We end this section describing the properties of IBP that are useful:

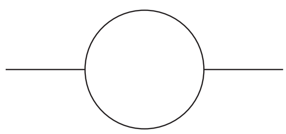
- Given a Feynman amplitude with n-loop momenta and m external legs, one can generate $n(n+m)$ IBP identities. Currently there is no general procedure capable to close the system of IBP identities for arbitrary amplitudes.
- Once individuated a set of IBP identities, all the Feynman diagrams in there will share the same topology. The reason is due to the fact that operations involved for their generation leads only to irreducible scalar products at the numerator, which do not affect the topology; in particular IBP cannot lower the number of loop integrations.
- IBP identities can only express a given diagram in terms of simpler ones, also called Master Integrals. As for their evaluation one has to proceed with alternative methods, as we will see in the next sections.

4.4 Massless propagators

At the end of the previous Chapter we have seen how effective diagrams can be written in terms of QFT amplitudes, involving loop-like integrals. Due to the fact that ϕ, A^i and σ_{nm} can be viewed as massless fields, one can deduce that also the two point functions that one can build are massless.
According to this topological relation, one is required to evaluate the behavior of n-loop amplitudes in order to understand the corresponding post-Newtonian corrections. Given this mapping, we will evaluate in detail the massless two point function with one and two loops, focusing on a specific topology with three loop that can be solved using IBP identities.
All these will be introduced with their topology only and evaluated as Euclidean loop integral.

4.4.1 The massless one-loop propagator

We start this section evaluating the amplitude of a one-loop massless propagator with arbitrary powers at denominator.
Since we are interested on Euclidean loop integrals, the reader should remember that all these are defined on an Euclidean space, where if needed the QFT counterpart, it is sufficient to perform a Wick Rotation

$I(p) =$  $= \int \frac{d^d k}{D_1^{n_1} D_2^{n_2}}$ (4.41)

$$D_1 = (k+p)^2 \quad , \quad D_2 = k^2 \tag{4.42}$$

In the evaluation of 4.41 one can rewrite the amplitude as

$$I(p) = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int d^d k \int_0^{+\infty} d\alpha_1 \int_0^{+\infty} d\alpha_2 \, \alpha^{n_1 - 1} \alpha^{n_2 - 1} e^{-\alpha_1 D_1 - \alpha_2 D_2} \tag{4.43}$$

In equation 4.43 we have introduced the so called *Schwinger parametrization*, based on the following identity

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^{+\infty} d\alpha\, \alpha^{n-1} e^{-\alpha a} \tag{4.44}$$

To proceed let's perform a shift in the loop variable of 4.43 given by $k \to k - p\frac{\alpha_1}{\alpha_1+\alpha_2}$. After few algebraic manipulations one can write

$$I = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int d^d k \int_0^{+\infty} d\alpha_1 d\alpha_2\, \alpha_1^{n_1-1} \alpha_2^{n_2-1} e^{-(\alpha_1+\alpha_2)k^2} e^{-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2} \tag{4.45}$$

The integration on the loop variable is now straightforward being a Gaussian one

$$I = \frac{\pi^{d/2}}{\Gamma(n_1)\Gamma(n_2)} \int_0^{+\infty} d\alpha_1 \int_0^{+\infty} d\alpha_2\, \alpha_1^{n_1-1} \alpha_2^{n_2-1} (\alpha_1+\alpha_2)^{-\frac{d}{2}} e^{-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2} \tag{4.46}$$

Let's now rescale both α_i variables by $\alpha_i \to {}^{\alpha_i}\!/_{p^2}$, defining a new adimensional function called G-function

$$\begin{gathered} I = \pi^{d/2} p^{d-2n_1-2n_2}\, G(n_1,n_2) \\ G(n_1,n_2) \equiv \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int_0^{+\infty} d\alpha_1 \int_0^{+\infty} d\alpha_2\, \alpha_1^{n_1-1} \alpha_2^{n_2-1} (\alpha_1+\alpha_2)^{-\frac{d}{2}} e^{-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}} \end{gathered} \tag{4.47}$$

In order to arrive at a close form for 4.47 let's make this final change of coordinates $\alpha_1 = \eta x$ and $\alpha_2 = \eta(1-x)$ which gives

$$\begin{aligned} G(n_1,n_2) &= \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx x^{n_1-1}(1-x)^{n_2-1} \int_0^{+\infty} d\eta e^{-\eta x(1-x)} \eta^{-\frac{d}{2}+n_1+n_2-1} \\ &= \frac{\Gamma(-\frac{d}{2}+n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\, x^{\frac{d}{2}-n_2-1}(1-x)^{\frac{d}{2}-n_1-1} \\ &= \frac{\Gamma(-\frac{d}{2}+n_1+n_2)\Gamma(\frac{d}{2}-n_1)\Gamma(\frac{d}{2}-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)} \end{aligned} \tag{4.48}$$

We find convenient to introduce the so called ν function such that

$$\nu(x) = \frac{\Gamma(d/2 - x)}{\Gamma(x)} \tag{4.49}$$

Inserting the expression so found for $G(n_1, n_2)$ into 4.47 and using 4.49 gives

$$I(p) = \pi^{\frac{d}{2}} \frac{\nu(n_1)\nu(n_2)\nu(d - n_1 - n_2)}{p^{2n_1+2n_2-d}} \tag{4.50}$$

As example that we will be used in the follow, for $n_1 = n_2 = 1$ one has

$$I(p) = \pi^{\frac{d}{2}} p^{d-4} G(1,1) \tag{4.51}$$

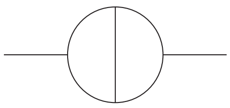
$$G(1,1) \equiv G_1 = \frac{\Gamma(2 - \frac{d}{2})\Gamma(\frac{d}{2} - 1)\Gamma(\frac{d}{2} - 1)}{\Gamma(d-2)} \tag{4.52}$$

$$G_1 = -\frac{2g_1}{(d-3)(d-4)} \quad , \quad g_1 \equiv \frac{\Gamma(3 - \frac{d}{2})\Gamma(\frac{d}{2} - 1)^2}{\Gamma(d-3)} \tag{4.53}$$

In equations 4.52,4.53 we have defined the so called G_n *functions*.
Introduced by Grozin in [42], they have the advantage to express in a clear form the topology relations among certain loop integrals, once their external momentum dependence has been factorized out by dimensional analysis.

4.4.2 The massless two-loop propagator

In this subsection we will evaluate the amplitude associated with a massless two point function with two-loops. The chosen one has represented for many years an important point of access to perturbative aspects in Quantum field theory and it is remarkable, as we will see, that many effective diagrams can be expressed in its term.
Assuming arbitrary powers at the denominator, this is given by

$$I(p) =$$  $$= \int \frac{d^d k d^d q}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} \tag{4.54}$$

$$D_1 = k^2 \quad D_2 = q^2 \quad D_3 = (k-p)^2 \quad D_4 = (q-p)^2 \quad D_5 = (k-q)^2 \tag{4.55}$$

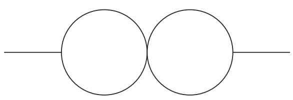
By dimensional analysis we can introduce the corresponding G-function as

$$I(p) = \pi^d p^{2d-2\sum_i n_i} G(n_1, n_2, n_3, n_4, n_5) \tag{4.56}$$

where $G(n_1, n_2, n_3, n_4, n_5)$ is symmetric under $(1 \leftrightarrow 2\,,\, 3 \leftrightarrow 4)$ and $(1 \leftrightarrow 3,\, , 2 \leftrightarrow 4)$, due to the symmetries of 4.54.
In the following, some particular values of the G-function for the two-loop topology.

The case $G(n_1, n_2, n_3, n_4, 0)$

For this choice, the amplitude is given by closing in itself the intermediate line of 4.54 leading to

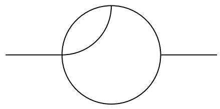
$$I(p) =$$  $$= \pi^d p^{2d-2(n_1+n_2+n_3+n_4)} G(n_1, n_2, n_3, n_4, 0) \tag{4.57}$$

Since 4.57 can be regarded as the square of a one-loop two point function, one can state that $G(n_1, n_2, n_3, n_4, 0) = G_1(n_1, n_3) G_1(n_2, n_4)$, where G_1 is the G-function for the one-loop case.

The case $G(0, n_2, n_3, n_4, n_5)$

Assuming $n_1 = 0$, is equivalent to evaluate the following amplitude, which is a sub-topology of 4.54

$$I(p) =$$



$$= \int \frac{d^d k d^d q}{D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} \tag{4.58}$$

Given that the only k dependence is in D_3 and D_5, one can recognize that the integration on the loop variable k reduces to a one-loop integral leading to

$$I(p) = G(n_3, n_5) \int \frac{d^d q}{D_2^{n_2} D_4^{n_3+n_4+n_5-d/2}} \tag{4.59}$$

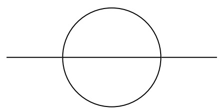
Again, the residual is still a one-loop integral, which yields

$$G(0, n_2, n_3, n_4, n_5) = G(n_3, n_5) G(n_2, n_4 + n_5 + n_3 - \frac{d}{2}) \tag{4.60}$$

A similar expression holds for $n_5 \neq 0$ and other choice of $n_i = 0$ due to the symmetries of the G-function.

The case $G(0, n_2, n_3, 0, n_5)$

The amplitude in this case is given by



$$= \pi^d p^{2d-n_2-n_3-n_5}\, G_2 \tag{4.61}$$

where we have defined $G(0, n_2, n_3, 0, n_5) \equiv G_2$ for future use.
Due to the elimination of two denominators, the evaluation of 4.61 is straightforward, given by a chain of one-loop integrations.
As example, for $n_2 = n_3 = n_5 = 1$ we have

$$G_2 = G(1,1) G(1, 2 - d/2) = \frac{4 g_2}{(d-3)(d-4)(3d-10)(3d-8)} \tag{4.62}$$

$$g_2 \equiv \frac{\Gamma(5-d)\Gamma(d/2-1)^3}{\Gamma(3d/2-5)} \tag{4.63}$$

The general case $G(n_1, n_2, n_3, n_4, n_5)$

If the amplitude has all n_i integer and non null, one can evaluate the corresponding G-function using IBP identities.
According to the procedure described in Section 4.3, we can choose v^i as $k - q$ times the integrand of 4.54, taking a divergence on q

$$\int d^d k d^d q \, \frac{\partial}{\partial q^i} \left[\frac{q^i - k^i}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} \right] = 0 \tag{4.64}$$

Acting with the derivative on the integral and using these relations

$$\begin{gathered} D_5 = (q-k)^2 \quad , \quad D_3 - D_4 - D_5 = 2q \cdot (q-k) \\ D_1 - D_2 - D_5 = 2(q+p)(q-k) \end{gathered} \tag{4.65}$$

one can derive the following identity

$$G(n_1, n_2, n_3, n_4, n_5) = \frac{n_2 G_{(2^+, 5^-)} - n_2 G_{(2^+, 1^-)} + n_4 G_{(4^+, 5^-)} - n_4 G_{(4^+, 3^-)}}{d - n_2 - n_4 - 2n_5} \tag{4.66}$$

where the symbol $l^\pm$ means that the argument of the G-function has to be replaced by $n_l \pm 1$.
As example, when all $n_i = 1$, the identity 4.66 gives

$$G(1,1,1,1,1) = \frac{2}{d-4} \Big[G(1,2,1,1,0) - G(0,2,1,1,1) \Big] \tag{4.67}$$

This can be further simplified in terms of G_1 and G_2 as

$$G(1,1,1,1,1) = \frac{2}{d-4} \Big[G(1,1)G(2,1) - G(1,1)G(2,3 - {}^{d}\!/\!_{2}) \Big] \tag{4.68}$$

$$= \frac{2}{d-4} \left[G_1^2 \, \frac{G(2,1)}{G(1,1)} - G_2 \, \frac{G(2,3 - {}^{d}\!/\!_{2})}{G(1,2 - {}^{d}\!/\!_{2})} \right] \tag{4.69}$$

Using

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = - \frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2 (d - 2 - 2n_2)} \tag{4.70}$$

we derive that the coefficients in front of G_1 and G_2 are

$$\frac{G(2,3 - {}^{d}\!/\!_{2})}{G(1,2 - {}^{d}\!/\!_{2})} = - \frac{(3d-8)(3d-10)}{d-4} \quad , \quad \frac{G(1,2)}{G(1,1)} = -(d-3) \tag{4.71}$$

Inserting 4.71 into 4.69 we arrive at the final expression

$$G(1,1,1,1,1) = \frac{2}{d-4}\left[-(d-3)G_1^2 + \frac{(3d-8)(3d-10)}{d-4}G_2 \right] \tag{4.72}$$

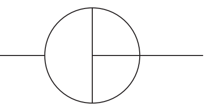
$$= \frac{8(g_2 - g_1^2)}{(d-3)(d-4)^3} \tag{4.73}$$

We end this subsection by discussing the interesting case when one of the n_i coefficient in the G-function is rational.
This case has been object of interest by many authors (see [53] or [41]) and it needs special techniques in order to be solved, dependent on how many rational coefficients are present and in which entry they are.
For the estimate of the degree of divergence of a massless four loop propagator, we will evaluate one of this amplitude with rational coefficients reducing it to a vacuum amplitude with non-integer powers at denominator. Anyway, the reader should be aware that for these cases no general procedure for their evaluation currently exist.

4.4.3 The massless three-loop propagator

There are many topologies describing massless two point functions with three loops, for this reason we will restrict to a single one whose topology is called Bug type

$$I =$$



$$= \int \frac{d^dq_1 d^dq_2 d^dq_3}{D_1D_2D_3D_4D_5D_6D_7} \tag{4.74}$$

where, as before, we have defined the D_i terms as

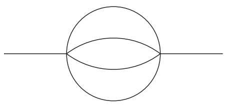
$$\begin{array}{c} D_1 = (q_1 - p)^2 \quad , \quad D_2 = q_1^2 \quad , \quad D_3 = q_2^2 \quad , \quad D_4 = q_3^2 \\ D_5 = (q_2 + q_3)^2 \quad , \quad D_6 = (q_1 + q_2)^2 \quad , \quad D_7 = (q_3 - q_1 + p)^2 \end{array} \tag{4.75}$$

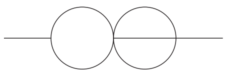
This amplitude can be reduced by means of IBP identities as a linear combination of three Master Integrals: since the amount of algebra involved is cumbersome, the reduction was made using the program Reduze (see [72]) which has been able to generate a series of IBP identities thanks to which I has been expressed in terms of three Master Integral $M_{i=1,2,3}$ and known coefficients.

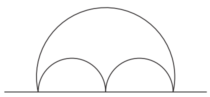
In this respect, equation 4.74 can be written as

$$I = c_1\mathcal{M}_1 + c_2\mathcal{M}_2 + c_3\mathcal{M}_3 \tag{4.76}$$

where the Master Integral are given by

$\mathcal{M}_1 =$  $= \displaystyle\int \frac{d^dq_1 d^dq_2 d^dq_3}{(q_1-p)^2 q_3^2 (q_2+q_3)^2 (q_1+q_2)^2}$ (4.77)

$\mathcal{M}_2 =$  $= \displaystyle\int \frac{d^dq_1 d^dq_2 d^dq_3}{(q_1-p)^2 q_1^2 (q_2+q_3)^2 (q_1+q_2)^2 (q_3-q_1+p)^2}$ (4.78)

$\mathcal{M}_3 =$  $= \displaystyle\int \frac{d^dq_1 d^dq_2 d^dq_3}{(q_1-p)^2 q_2^2 (q_2+q_3)^2 (q_1+q_2)^2 (q_3-q_1+p)^2}$ (4.79)

As for the coefficient in the linear combination 5.38 the output of Reduze gave the following result

$$c_1 = 80\,\frac{(-2800 + d\,3810 - d^4\,36 + d^3\,432 - d^2 1931)}{(d-4)^4 s^3} \tag{4.80}$$

$$c_2 = -4\,\frac{(-240 + d\,242 + d^3\,9 - d^2\,81)}{(d-4)^3 s^2} \tag{4.81}$$

$$c_3 = 64\,\frac{(-63 + d\,60 + d^3\,2 - d^2\,19)}{(d-4)^3 s^2} \tag{4.82}$$

At this point we are only left with the evaluation of the Master Integrals in order to have a complete knowledge on 4.74.
We will deal with each of them separately evaluating them as Euclidean loop integral remembering that this is due to the fact that we are interested on their effective corresponding.

Evaluation of $\mathcal{M}_1$

The first Master Integral is the only with four denominators and it is given by

$$\mathcal{M}_1 = \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1 - p)^2 q_3^2 (q_2 + q_3)^2 (q_1 + q_2)^2} \tag{4.83}$$

We can start by isolating the integration on the q_3 loop momentum

$$\mathcal{M}_1 = \int \frac{d^d q_1 d^d q_2}{(q_1 - p)^2 (q_1 + q_2)^2} \left[\int \frac{d^d q_3}{q_3^2 (q_2 + q_3)^2} \right] \tag{4.84}$$

In the square brackets we can recognize the massless two point function at one-loop that we have calculated in 4.50.
Using this result, equation 4.83 becomes

$$\mathcal{M}_1 = \pi^{d/2} \nu(1)^2 \nu(d-2) \int \frac{d^d q_1 d^d q_2}{(q_1 - p)^2 (q_1 + q_2)^2 q_2^{4-d}} \tag{4.85}$$

The evaluation now is straightforward, since we can integrate first in q_1 variable, followed by a q_2 integration, dealing with a chain of one-loop integrals.
The final result for 4.83 is given by

$$\mathcal{M}_1 = \frac{\pi^{3d/2}}{p^{8-3d}} \nu(1)^4 \nu(d-2) \nu(2 - d/2)^2 \nu(2d-4) \tag{4.86}$$

This expression can be simplified by expressing all the ν functions in terms of Γ ones, leading to the following analytic expression valid for arbitrary d dimensions

$$\mathcal{M}_1 = \frac{\pi^{\frac{3d}{2}}}{p^{8-3d}} \frac{\Gamma(d/2 - 1)^4 \Gamma(4 - 3d/2)}{\Gamma(2d-4)} \tag{4.87}$$

Evaluation of $\mathcal{M}_2$

As for the second Master Integral this is given by

$$\mathcal{M}_2 = \int \frac{d^d q_1\, d^d q_2\, d^d q_3}{(q_1 - p)^2 q_1^2 (q_2 + q_3)^2 (q_1 + q_2)^2 (q_3 - q_1 + p)^2} \tag{4.88}$$

where again we are assuming an Euclidean integral.
We can proceed in the same way as we have done for the evaluation of M_1, by first performing the q_3 integration associated with a one-loop two point function.
The result is

$$\mathcal{M}_2 = \pi^{d/2} \nu(1)^2 \nu(d-2) \int \frac{d^d q_1\, d^d q_2}{(q_1 - p)^2 q_1^2 (q_1 + q_2)^2 (q_2 + q_1 - p)^{4-d}} \tag{4.89}$$

At this point, in order to further reduce the integral, we can make the following change of variables defining $q_1 + q_2 = k$ and $q_1 = q$, such that

$$\mathcal{M}_2 = \pi^{d/2} \nu(1)^2 \nu(d-2) \int \frac{d^d k\, d^d q}{(q-p)^2 q^2 k^2 (k-p)^{4-d}} \tag{4.90}$$

We can now integrate in q which is a straightforward calculation using the previous technique

$$\mathcal{M}_2 = \frac{\pi^d \nu(1)^4 \nu(d-2)^2}{p^{4-d}} \int \frac{d^d k}{k^2 (k-p)^{4-d}} \tag{4.91}$$

Performing the remaining loop integral and expressing all the ν functions in terms of Γ ones we arrive at

$$\mathcal{M}_2 = \frac{\pi^{\frac{3d}{2}}}{p^{10-3d}} \frac{\Gamma(d/2 - 1)^5 \Gamma(2 - d/2) \Gamma(3-d)}{\Gamma(d-2) \Gamma(3d/2 - 3)} \tag{4.92}$$

Evaluation of $\mathcal{M}_3$

The evaluation of the remaining Master Integral is very similar to the previous ones

$$\mathcal{M}_3 = \int \frac{d^d q_1 \, d^d q_2 \, d^d q_3}{(q_1 - p)^2 q_2^2 (q_2 + q_3)^2 (q_1 + q_2)^2 (q_3 - q_1 + p)^2} \tag{4.93}$$

We start by integrating on the q_3 loop variable, which gives

$$\mathcal{M}_3 = \pi^{d/2} \nu(1)^2 \nu(d-2) \int \frac{d^d q_1 \, d^d q_2}{(q_1 - p)^2 q_2^2 (q_1 + q_2)^2 (q_2 + q_1 - p)^{4-d}} \tag{4.94}$$

Also in this case, we can make the following change of variables given by $q_1 + q_2 = k$ and $q_1 = q$

$$\mathcal{M}_3 = \pi^{d/2} \nu(1)^2 \nu(d-2) \int \frac{d^d k \, d^d q}{(k-q)^2 (q-p)^2 k^2 (k-p)^{4-d}} \tag{4.95}$$

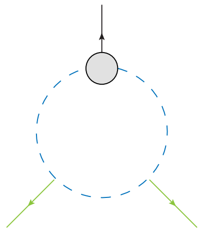
The remaining integrations are straightforward and lead to

$$\mathcal{M}_3 = \frac{\pi^{\frac{3d}{2}}}{p^{10-3d}} \frac{\Gamma(d/2 - 1)^5 \Gamma(2 - d/2)^2 \Gamma(3d/2 - 4) \Gamma(5 - 3d/2) \Gamma(3-d)}{\Gamma(d-2)^2 \Gamma(4-d) \Gamma(2d-5)} \tag{4.96}$$

4.5 The star triangle rule (STR) for an ideal vertex

In this section we introduce the Star triangle rule (STR) for an ideal vertex (for an introduction see Kleinert [49], here presented in Euclidean space due to the features of the loop-like integrals into our EFT.
Let's assume we are required to evaluate the following loop integral coming from an effective diagram

$$\mathcal{M}(q_1, q_2, q_3) = \tag{4.97}$$



$$\mathcal{M}(q_1, q_2, q_3) = \int \frac{d^d p}{(p-q_1)^{2\alpha_1}(p-q_2)^{2\alpha_2}(p-q_3)^{2\alpha_3}} \quad , \quad \sum_{i=1}^{3} \alpha_i = d \tag{4.98}$$

being the left part of 4.98 a constraint, hence the name ideal for this vertex rule. We will prove that for these particular values of α_i, one can evaluate 4.97 expressing it in terms of propagators depending on the differences of external momenta only. In order to prove it, let's perform the following translation in Fourier space $\vec{p} \to \vec{p} + \vec{q}_1$

$$\mathcal{M}(q_1, q_2, q_3) = \int \frac{d^d p}{p^{2\alpha_1}(p+q_1-q_2)^{2\alpha_2}(p+q_1-q_3)^{2\alpha_3}} \tag{4.99}$$

After this, we rescale the loop variable by $\vec{p} \to {}^{\vec{p}}/_{p^2}$, $d^d p \to {}^{d^d p}/_{p^{2d}}$.
Under this rescaling, a generic term in the denominator of 4.98 will transform as

$$(p-q)^{2\alpha} \to \left(\frac{\vec{p}}{p^2} - \vec{q}\right)^{2\alpha} = \frac{(1 - 2p\cdot q + p^2 q^2)^{\alpha}}{p^{2\alpha}} = \frac{(\vec{p} - \frac{\vec{q}}{q^2})^{2\alpha} q^{2\alpha}}{p^{2\alpha}} \tag{4.100}$$

Inserting this into 4.98 gives

$$\int \frac{d^d p}{p^{2(d-\alpha_1)}} \left[\frac{(p - \frac{q_1-q_2}{(q_1-q_2)^2})^2 (q_1-q_2)^2}{p^2}\right]^{-\alpha_2} \left[\frac{(p - \frac{q_1-q_3}{(q_1-q_3)^2})^2 (q_1-q_2)^2}{p^2}\right]^{-\alpha_3} \tag{4.101}$$

$$= \int d^d p \left[\left(p - \frac{q_1 - q_2}{(q_1 - q_2)^2} \right)^{2\alpha_2} \left(p - \frac{q_1 - q_3}{(q_1 - q_2)^2} \right)^{2\alpha_3} (q_1 - q_2)^{2\alpha_2} (q_1 - q_2)^{2\alpha_3} \right]^{-1} \tag{4.102}$$

where p^2 terms at denominator have been eliminated using the constraint 4.98.
At this point we are left with the evaluation of a two point function at one loop, since the other terms can be factorized out.
The result is straightforward and equal to

$$\mathcal{M}(q_1, q_2, q_3) = \frac{\pi^{d/2} \nu(\alpha_1) \nu(\alpha_2) \nu(\alpha_3)}{\left(\frac{q_1 - q_2}{(q_1 - q_2)^2} - \frac{q_1 - q_3}{(q_1 - q_2)^2} \right)^{2\alpha_2 + 2\alpha_3 - d} (q_1 - q_2)^{2\alpha_2} (q_1 - q_2)^{2\alpha_3}} \tag{4.103}$$

Equation 4.103 can be cast in a simpler form by rewriting the various exponents at denominator as

$$\begin{gathered} \alpha_2 = \left(\frac{d}{2} - \alpha_1 \right) + \left(\frac{d}{2} - \alpha_3 \right) \quad , \quad \alpha_3 = \left(\frac{d}{2} - \alpha_2 \right) + \left(\frac{d}{2} - \alpha_3 \right) \\ \alpha_2 + \alpha_3 - \frac{d}{2} = \frac{d}{2} - \alpha_1 \end{gathered} \tag{4.104}$$

Inserting these in 4.103

$$\mathcal{M}(q_1, q_2, q_3) = \frac{\pi^{d/2} \nu(\alpha_1) \nu(\alpha_2) \nu(\alpha_3)}{\left(\frac{q_1 - q_2}{(q_1 - q_2)^2} - \frac{q_1 - q_3}{(q_1 - q_3)^2} \right)^{d - 2\alpha_1} (q_1 - q_2)^{2(d - 2\alpha_1) + (d - 2\alpha_2) + (d - 2\alpha_3)}} \tag{4.105}$$

within the following identity

$$(q_2 - q_3)^2 = (q_1 - q_2)^2 + (q_1 - q_2)^2 - 2(q_1 - q_3) \cdot (q_1 - q_2) \tag{4.106}$$

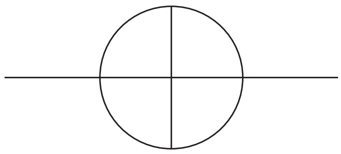
one gets the final result

$$\mathcal{M}(q_1, q_2, q_3) = \frac{\pi^{d/2} \nu(\alpha_1) \nu(\alpha_2) \nu(\alpha_3)}{(q_1 - q_2)^{d - 2\alpha_3} (q_1 - q_3)^{d - 2\alpha_2} (q_2 - q_3)^{d - 2\alpha_1}} \tag{4.107}$$

4.5.1 The STR and the massless four-loop propagator

So far we have studied in detail massless two point functions at one and two loops, solving a specific topology involved at three-loop called Bug type. At higher number of loops, there is no general procedure to adopt for their evaluation and one is forced to face case by case.
A possibility is always given by IBP identities, which makes possible to express a given amplitude as a linear combination of other sharing the same topology, which could be more manageable compared to the initial one.
Unfortunately, some amplitudes cannot be reduced in a useful way, as it happens with the following four-loop massless propagator [5]

$$\mathcal{M}_{3,6}(p) = \tag{4.108}$$



$$= \int \frac{d^d q_1 d^d q_2 d^d q_3 d^d q_4}{(p-q_1)^2 (q_1-q_2)^2 q_2^2 q_3^2 q_4^2 (p-q_1-q_3)^2 (q_2+q_4)^2 (q_1-q_2+q_3-q_4)^2} \tag{4.109}$$

This amplitude is associated with a static post-Newtonian correction at fourth order and it has been first calculated around $d = 3+\varepsilon$ via a semi-analytic method in [34] and later via an analytic method in [28]: the two calculations agree and the result is given by

$$\mathcal{M}_{3,6} = \frac{c(\varepsilon)}{p^4}\Big[\frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon} - 4 + \frac{\pi^2}{24} + O(\varepsilon)\Big] \quad , \quad c(\varepsilon) = e^{2\varepsilon\gamma_E p^{4\varepsilon}}/(4\pi)^{4+2\varepsilon} \tag{4.110}$$

being γ_E the Euler-Mascheroni constant.
Due to the intricacy involved in the evaluation of 4.108, we will focus only on the estimate of its degree of divergence around $d = 3+\varepsilon$, extracting a double pole up to $O(1/\varepsilon)$ factors. In doing so we will take advantage of the Star Triangle rule previously described, showing that 4.108 can be expressed as a vacuum diagram around $d = 3+\varepsilon$ up to factors $1/\varepsilon$.

[5] The reason for the numbers 3 and 6 in the definition comes from the fact that this diagram originally appeared in a list of Master Integrals in particle physics.

Estimate of the divergence for $\mathcal{M}_{3,6}$

We are going to present a procedure thanks to which is possible to estimate the degree of divergence of a given loop integral.
At first glance, some assumptions may appear to be dangerous, however the correctness of our procedure is given by the reproduction of the exact double pole of $\mathcal{M}_{3,6}$ around $d = 3 + \varepsilon$.
First of all, let's isolate in 4.108, the integration in the loop variable q_4 defining

$$Q_4(d) \equiv \int \frac{d^d q_4}{q_4^2 (q_2 + q_4)^2 (q_1 - q_2 + q_3 - q_4)^2} \tag{4.111}$$

If we choose $d = 3$ (the physical dimension of interest) the α_i coefficients involved are all equal to one, rendering possible to calculate $Q_4(3)$ by means of the Star triangle rule.
As a function of the space dimension d, 4.111 it is thus defined at $d = 3$, and assuming continuity in a nearby of d, one has that the following relation holds

$$Q_4(3 + \varepsilon) = \frac{\pi^{3/2} \nu(1)^3}{q_2 (q_2 - q_1 - q_3)(q_1 + q_3)} + O(\varepsilon) \tag{4.112}$$

By substituting the expression so obtained in $\mathcal{M}_{3,6}$ we have

$$\mathcal{M}_{3,6} = \int \frac{d^d q_1 d^d q_2 d^d q_3}{(p - q_1)^2 (q_1 - q_2)^2 \, q_2^3 \, q_3^2 \, (q_1 + q_3)(q_2 - q_1 - q_3)(p - q_1 - q_3)^2} \tag{4.113}$$

$$+ O(\varepsilon) \int \frac{d^d q_1 d^d q_2 d^d q_3}{(p - q_1)^2 (q_1 - q_2)^2 q_2^2 q_3^2 (p - q_1 - q_3)^2} \quad , \quad d = 3 + \varepsilon \tag{4.114}$$

The integral that multiplies $O(\varepsilon)$ in 4.114 has a pole that scale as[6] $1/\varepsilon$ which means that by neglecting 4.114, one is neglecting finite terms around $d = 3 + \varepsilon$.
Proceeding in this direction, let's shift the region of momenta by $q_1 \to q_1 - q_3$, which gives

$$\mathcal{M}_{3,6} = \int \frac{d^d q_1 d^d q_2 d^d q_3}{(p - q_1 + q_3)^2 (q_1 - q_2 - q_3)^2 \, q_2^3 \, q_3^2 \, q_1 (q_2 - q_1)(p - q_1)^2} + O(K) \tag{4.115}$$

being K a generic constant.

[6]See Appendix A for a prove

We now apply the same procedure used for the loop variable q_4, in order to isolate and evaluate, around $d = 3 + \varepsilon$, the integration over the loop momentum q_3.
First, let's define

$$Q_3(d) \equiv \int \frac{d^d q_3}{q_3^2 (q_3 - q_1 + p)^2 (q_2 - q_1 + q_3)^2} \tag{4.116}$$

By applying on 4.116 the STR for $d = 3$ one proves its finiteness. Assuming it is continuous around $d = 3 + \varepsilon$ one has

$$Q_3(d) = \frac{\pi^{3/2} \nu(1)^3}{(p - q_1)(p - q_2)(q_2 - q_1)} + O(\varepsilon) \quad , \quad d = 3 + \varepsilon \tag{4.117}$$

Substituting 4.117 in 4.115 we obtain a result similar to that of 4.114.
Also in this case one can demonstrate that the integral multiplying $O(\varepsilon)$ is proportional to finite term only.
Given these considerations, one has

$$\mathcal{M}_{3,6} = \pi^6 \nu(1)^6 \int \frac{d^d q_1 d^d q_2}{(p - q_1)^3 \, q_1 \, q_2^3 \, (q_2 - q_1)^2 (p - q_2)} + O(K) \tag{4.118}$$

which we recognize as proportional to a G-function with rational coefficient in its entries, more precisely

$$\mathcal{M}_{3,6} = \frac{\pi^{6+d}}{p^{10-2d}} \, G(1/2, 3/2, 1/2, 3/2, 1) + O(K) \quad , \quad d = 3 + \varepsilon \tag{4.119}$$

We can redefine the G-function without affecting its result using the following identity that descends from the invariance of the integral under inversion of integration variables

$$\mathcal{M}_{3,6} = \frac{\pi^{6+d}}{p^{10-2d}} \, G(d-3, d-3, 1/2, 3/2, 1) + O(K) \tag{4.120}$$

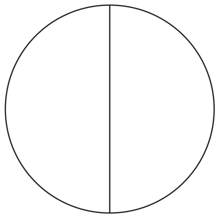
At this point we make the following assumption

$$G(d-3, d-3, 1/2, 3/2, 1) = G(0, 0, 1/2, 3/2, 1) + O(\varepsilon) \tag{4.121}$$

Equation 4.121 is formal and the residual G function can be trusted only in the limit $\varepsilon \to 0$. Thus we will only consider its highest pole where associated to it there is a vacuum diagram with rational powers at denominator.

Its value is known (it can be found in [41]) and it is equal to

$$G(0, 0, n_3, n_4, n_5) = \tag{4.122}$$



$$= \frac{\Gamma(d/2 - n_3)\Gamma(n_1 + n_3 - d/2)\Gamma(n_2 + n_3 - d/2)\Gamma(n_1 + n_2 + n_3 - d)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)\Gamma(n_1 + n_2 + n_3)} \tag{4.123}$$

For $n_3 = 3/2$, $n_4 = 1/2$, $n_5 = 1$ and with $d = 3 + \varepsilon$ we have

$$G(0, 0, 3/2, 1/2, 1) = \frac{4\Gamma(-\varepsilon/2)\Gamma(-\varepsilon)}{\pi^2} = \frac{8}{\pi\varepsilon^2} + .. \tag{4.124}$$

Substituting equation 4.124 into 4.121 we arrive at our final result

$$M_{3,6} = \frac{1}{(4\pi)^4 p^4} \frac{1}{2\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right) \tag{4.125}$$

This matches exactly the double pole with its coefficient as was first individuate by Mastrolia, Foffa, Sturani, Sturm in [34].

Chapter 5

Post-Newtonian corrections

At this point, we have all the tools to perform high-precision calculations concerning post-Newtonian (PN) corrections to binary dynamics, as we will see with their evaluation up to the 4 PN sector. Due to the *Effacement principle* [1], the internal structure of the binary enters from the 5 post-Newtonian order, therefore we will just assume that our system is defined by the same effective action introduced in 3.49, where binary components were described as point particles.
As for the material treated in the Chapter, it is so divided:

- Hinting at the covariant formulation of our Effective field theory (EFT) we will introduce a convenient procedure to be able to identify which effective diagrams contribute at a given post-Newtonian order to the binary effective action.

- It is derived the complete *Einstein-Infeld-Hoffmann* Lagrangian by means of effective diagrams at 1 post-Newtonian order. It is also shown in detail the various topologies involved with the increasing of post-Newtonian calculations.

- Exploiting Quantum field theory (QFT) methods we will calculate two contributes at 2 post-Newtonian order, a static one from a G_N^2 topology and v-dependent one linked with a G_N^3 topology. Both diagrams will be calculated in momentum space within dimensional regularization, unite with their Fourier transform around $d = 3 + \epsilon$.

- It is calculated a non static contribute to the effective action coming from a diagram at 4-PN order and associated with a G_N^4 topology. This represent an original contribute of this thesis, since this calculation has never been calculated before within EFT methods.

[1] For details, see Maggiore [46] Section 5.5

5.1 Organizing post-Newtonian calculations

In order to evaluate the effective action for a binary at a given PN order, one is required to device a general procedure so that it is possible to understand which diagrams enter at the desired PN precision.
In sketching a scheme, we will follow a series of steps first introduced by Gilmore and Ross in [37], and based on a manifest covariant formulation of our EFT.
First of all, we define the effective action of our EFT in $d+1$ space-time dimensions as

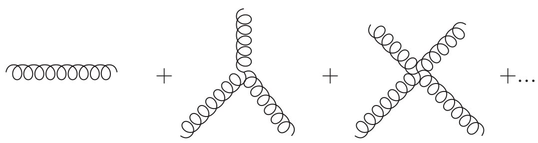
$$\mathcal{S}_{eff}(g_{\mu\nu}, x^\mu) = -2\Lambda^2 \int d^dx dt \sqrt{-\det g_{\mu\nu}} R - m_a \int d\tau_a \sqrt{-g_{\mu\nu}(x_a)\dot{x}_a^\mu \dot{x}_a^\nu} \tag{5.1}$$

At this point, we don't introduce the Kol-Smolin variables, choosing an alternative manifest covariant parametrization for the metric tensor given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{\Lambda} \tag{5.2}$$

Inserting equation 5.2 in 5.1, by expanding in powers of Λ^{-1} we are able to extract a set of Feynman rules in $h_{\mu\nu}$, similar to those previously derived for ϕ, A_i, σ_{nm} in Chapter 3.
The expansion of the Einstein-Hilbert action gives vertexes which scales as Λ^{2-n}, being n the numbers of $h_{\mu\nu}$ fields involved

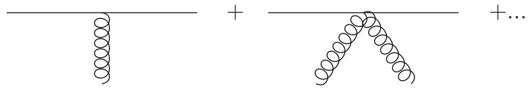
$$-\Lambda^2 \int d^dx dt \sqrt{-\det g_{\mu\nu}} R \quad \Rightarrow \quad \int d^dx dt \left[(\partial h)^2 + \frac{h(\partial h)^2}{\Lambda} + \frac{h^2(\partial h)^2}{\Lambda^2} + .. \right] \tag{5.3}$$



(5.4)

where the required contractions and indexes are implied.
On the other side, the expansion of the point-particle actions gives source-gravity interactions proportional to Λ^{-n}, schematically

$$-m_a \int d\tau_a \sqrt{-g_{\mu\nu}(x_a)\dot{x}_a^\mu \dot{x}_a^\nu} \quad \Rightarrow \quad -m_a \int dt \sqrt{-\dot{x}_a^2} - \frac{m_a}{2\Lambda} \int dt h_{\mu\nu} \frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{-\dot{x}_a^2}} + ... \tag{5.5}$$



(5.6)

Starting from Feynman rules 5.3,5.5, we can develop effective diagrams that scale with a definite power of Λ^{-1} only, not velocities due to the square roots of $\dot{x}_a^2$ in them: this manifest covariant formulation is also called *post-Minkowskian* (PM) and it is useful when no $^v/_c << 1$ assumption is made.
Although not useful for post-Newtonian calculations, its use is still practical given that the topology of a PM diagram can be deployed in terms of PN diagrams.
To do so, we should notice that the $h_{\mu\nu}$ parametrization can be related to the Kol-Smolkin one, by matching their expansion in Λ^{-1} of the metric tensor.
As example, at linear order one has

$$h_{\mu\nu} = \begin{pmatrix} -2\phi & A_i \\ A_i & \sigma_{ij} - c_d \phi \delta_{ij} \end{pmatrix} \tag{5.7}$$

At this point, we can systematically derive which effective diagrams enters at a given PN order:

- Within the manifest covariant parametrization of our EFT, let's fix a power of G_N, writing down all possible diagrams that scale with this power law. No calculations are needed, since it is worth mentioning that the topology of a Feynman rule has a definite scaling in G_N, regardless its specific form.

- Always within the previous parametrization, we define G_N^n topology the equivalence relation of all connected diagrams that scales as G_N^n. Given all these diagrams, we can proceed by filling each $h_{\mu\nu}$ line with ϕ, A_i and σ_{nm} in order to switch to the Kaluza-Klein parametrization.

- At last, in order to restore the complete v-dependencies of effective diagrams, one can expand every Feynman rule in a non relativistic way as described in Section 5.3.1.

The result is that starting from the lowest powers of G_N, one can systematize which diagrams enter at a given PN order and which not. As a remarkable fact, one should notice that a given G_N^n topology enters at different PN order due to the non relativistic expansion of fields propagators.

5.2 The Einstein-Infeld-Hoffmann Lagrangian from a 1 PN calculation

Exploiting the previous method we will derive the complete effective action for a non spinning binary at 1 PN order, i.e. taking care of corrections that scales as $G_N v^2$ and G_N^2.
Let's start our calculation by defining the effective action as

$$\mathcal{S}_{eff}(x_a) = -m_a \int dt \sqrt{1 - \frac{v_a^2}{c^2}} - iW(x_a, v_a) \tag{5.8}$$

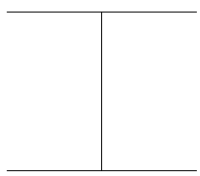
where W is evaluated in terms of connected effective diagrams as

$$W(x_{a,b}, v_{a,b}) = \int d\tau_a d\tau_b A(x_{a,b}, v_{a,b}) \tag{5.9}$$

$$A(x_{a,b}, v_{a,b}) = \int \frac{d^d k}{(2\pi)^d} \frac{d\omega}{2\pi} e^{ik\cdot(x_a - x_b)} e^{-i\omega(\tau_a - \tau_b)} \mathcal{M}(k, \omega) \tag{5.10}$$

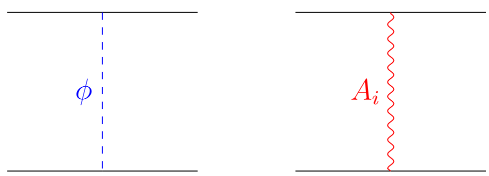
$$\mathcal{M}(k, \omega) \sim G_N v^2, G_N^2 \tag{5.11}$$

In order to find which PN effective diagrams participate in 5.11 we start by looking at the G_N and G_N^2 topologies. For semplicity, from now on we will adopt a thick black line for a graviton, instead of the gluon one.
As for the G_N topology, it is characterized by a unique PM diagram given by



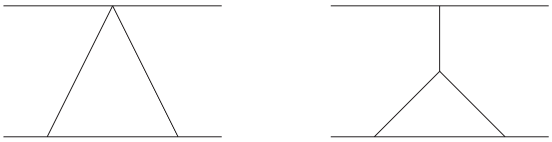
(5.12)

We can now switch to the Kol-Smolkin variables by filling the diagram with ϕ, A_i, and neglecting σ_{nm} since its emission and absorption would be proportional to v^4.
So far the connected diagrams are



(5.13)

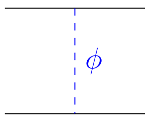
As for the G_N^2 topology, there are two PM diagrams



(5.14)

Introducing the Kol-Smolkin variables, it can be proved that the diagram on the right give corrections which are at least 2 PN, while the diagram on the left gives a unique contribute at 1 PN given by expressing the graviton lines in terms of two static ϕ fields. Having identify all the effective diagrams we can proceed to evaluate their amplitude in momentum space. As for 5.13, we adopt source-gravity interactions at v^2 evaluating the propagator at ω^2

$$\mathcal{M}_\phi(k,\omega) \quad = \tag{5.15}$$



$$= \frac{im_1 m_2}{2c_d\Lambda^2}\frac{1}{|\vec{k}|^2}\left[1+\frac{3}{2}(v_1^2+v_2^2)+\frac{\omega^2}{|\vec{k}|^2}+O\left(\frac{\omega}{|k|}\right)^4\right] \tag{5.16}$$

$$A_\phi = \frac{im_1 m_2}{2c_d\Lambda^2}\int \frac{d^d k}{(2\pi)^d}\frac{d\omega}{2\pi}e^{ik\cdot(x_a-x_b)}e^{-i\omega(\tau_a-\tau_b)}\frac{1}{|\vec{k}|^2}\left[1+\frac{3}{2}(v_1^2+v_2^2)+\frac{\omega^2}{|\vec{k}|^2}\right] \tag{5.17}$$

$$A_\phi = \delta(\tau_a-\tau_b)\frac{im_1 m_2}{2c_d\Lambda^2}\int \frac{d^d k}{(2\pi)^d}e^{ik\cdot(x_a-x_b)}\frac{1}{|\vec{k}|^2}\left[1+\frac{3}{2}(v_1^2+v_2^2)+v_1^i v_2^j\frac{k_i k_j}{|\vec{k}|^2}\right] \tag{5.18}$$

As for the remaining k integral, the calculation follow what has been seen in Sections 4.2.2-4.2.3; the amplitude is finite at $d=3$ and it gives

$$A_\phi = \delta(\tau_a-\tau_b)\frac{m_1 m_2 G_N}{r}\left[1+\frac{3}{2}(v_1^2+v_2^2)+\frac{1}{2}(v_1\cdot v_2 - v_1\cdot\hat{r}\, v_2\cdot\hat{r})\right] \tag{5.19}$$

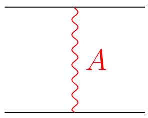
being $\hat{r}$ the unit vector parallel to $\vec{r}$.

Inserting 5.19 in 5.9 one gets the following contribute to the effective action 5.8

$$-iW_\phi = \int dt\, \frac{m_1 m_2 G_N}{r}\left[1 + \frac{3}{2}(v_1^2 + v_2^2) + \frac{1}{2}(v_1 \cdot v_2 - v_1 \cdot \hat{r}\, v_2 \cdot \hat{r})\right] \tag{5.20}$$

We now proceed evaluating the effective diagram involving A_i, which we will calculate within static source-gravity interactions and propagators

$$\mathcal{M}_A(k) =$$



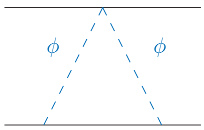
$$= -\frac{i m_1 m_2}{2\Lambda^2}\frac{v_1 \cdot v_2}{|k|^2} \tag{5.21}$$

A straightforward Fourier transform gives the following finite contribute for $d = 3$

$$-iW_A = \int dt\, -\frac{4 m_1 m_2 G_N}{r} v_1 \cdot v_2 \tag{5.22}$$

It only remains the effective diagram involving two ϕ, which will be evaluated using static source-gravity interactions and propagators

$$\mathcal{M}_{\phi\phi}(k_1, k_2) =$$



$$= \frac{i m_1^2 m_2}{8 c_d^2 \Lambda^4 |k_1|^2 |k_2|^2} \tag{5.23}$$

Also in this case, the contribute is finite for $d = 3$.
Taking care that there is also a symmetric contribute given by $1 \leftrightarrow 2$ interchanged one has

$$-iW_{\phi\phi} = \int dt\, \frac{G_N^2 (m_1 + m_2) m_1 m_2}{2r^2} \tag{5.24}$$

Inserting 5.20,5.22,5.24 into 5.8 one gets the action of a local theory defined by the following Lagrangian

$$\mathcal{L}_{EIH} = \frac{m_a v_a^2}{2} + \frac{m_a v_a^4}{8} - \frac{G_N m_1 m_2}{2r}\left[2 + 3(v_1^2 + v_2^2) - 7 v_1 \cdot v_2 - v_1 \cdot \hat{r} v_2 \cdot \hat{r} + \frac{G_N(m_1 + m_2)}{r}\right] \tag{5.25}$$

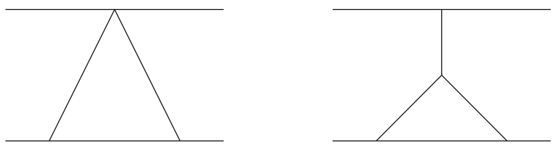
Equation 5.25 was first derived by Albert Einstein, Leopold Infeld and Banesh Hoffmann in 1938 (see [32]) and both results agrees.

5.3 Topologies from 2 up to 4 PN order

Before moving to amplitude calculations we will examine the topologies and number of effective diagrams involved from 2 up to 4 PN order for a spin-less binary system.

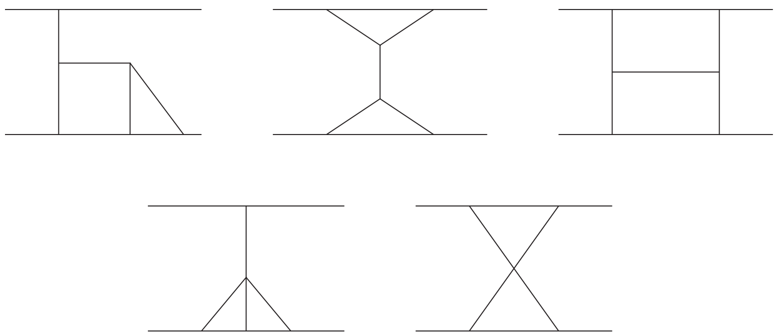
5.3.1 2-nd Post Newtonian order

Being interested in 2 PN contributes, one has to find those diagrams that scale as G_N^3, $G_N^2 v^2$ and $G_N v^4$.
As customary, one has to proceed by looking at the G_N topologies involved, which in this case are given by the previous encountered plus the G_N^3 one. Given this, one has to consider PN diagrams from the G_N topology expanded at v^4, while from the G_N^2 topology one has to consider the following PM diagrams at v^2



(5.26)

As for the G_N^3 topology, this is composed by five PM diagrams

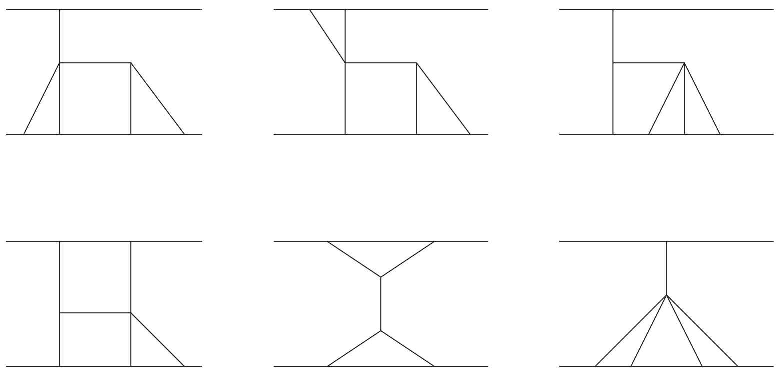


(5.27)

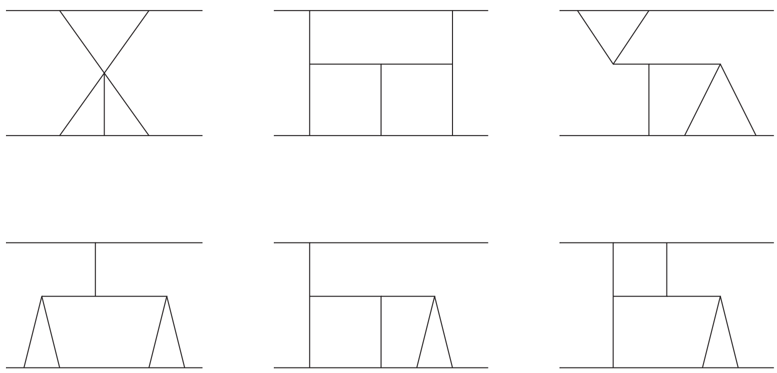
Among these, only the upper three gives 2 PN contributes, while the other enters only at 3 PN. It can be proved that this procedure give rise to 14 PN diagrams participating to the complete effective action of a non spinning binary at 2 PN. These evaluation, within Effective field methods, has been first performed by Gilmore and Ross in [37] and Chu in [20]

5.3.2 3-rd Post Newtonian order

At 3-rd Post Newtonian order, we are looking for corrections to the effective action of a binary which are proportional to G_N^4, $G_N^3 v^2$, $G_N^2 v^4$ and $G_N v^6$. We can start the examination of the G_N topologies involved, starting from the highest one, the G_N^4. This is composed by 12 PM diagrams



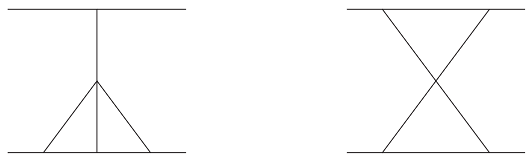
(5.28)



(5.29)

As one can prove, none of them contribute at 3-PN. In order to contribute at this PN order they have to be evaluated as static, which means that the external sources are forced to emit scalar fields only, otherwise it would increase the PN order of the associate diagram. In doing so, one can easily recognize that for every diagram we are forced to introduce vertexes that are not present in our theory, which proves the previous statement.

As for the G_N^3 topology, all its PM diagrams participate at 3 PN order, in particular the following which previous was discarded



(5.30)

As for the remaining topologies, both the G_N^2 and G_N partecipate. Filling all these classified PM diagrams within the Kol-Smolkin variables lead to a total number of 80 effective diagrams.
Within the Effective field theory method, *Foffa* and *Sturani* have been able to evaluate the complete effective action at 3 PN order within a semi-automated algorithm (see [35]).
As remarkable fact, at this PN order every effective diagrams can be calculated within IBP identities and one-loop integrations.

5.3.3 4-th Post Newtonian order

At this post-Newtonian order the contributes we are looking for are expressed by effective diagrams which scales as G_N^5, $G_N^4 v^2$, $G_N^3 v^4$, $G_N^2 v^6$ and $G_N v^8$.
As for the G_N^5 topology, there are 50 topologies associated which we cannot draw due for practical reasons.
As for the $G_N^4 v^2$ contributes, these comes from the G_N^4 topology: the same happen for the other contributes at 4 PN and thus don't constitute any difference within the previous cases.
Currently, within the Effective field theory methods none has succeed to calculate the complete action of a non spinning binary at 4 PN, however the static sector has been evaluatede by *Mastrolia*, *Foffa*, *Sturani*, *Sturm* in [34]. The reason why the evaluation of complete action within EFT methods is missing are multiples. First of all the number of effective diagrams involved, 595, makes apparently impossible to derive their contribute via analytic tools only. Secondly the technology required for the evaluation of some effective diagrams has revealed an intriguing challenge, as seen in [34] with the Master Integral $M_{3,6}$.

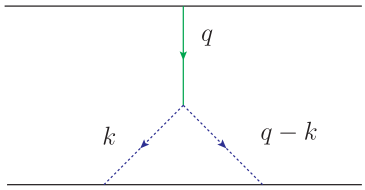
5.4 Evaluation of two 2-PN amplitudes

In this section we present the calculation of two of the 14 effective diagrams that contribute to the effective action of a non spinning binary at 2 PN order.

5.4.1 A non static 2-PN amplitude from a G_N^2 topology

The chosen diagram is the following, which gives a contribute that scales as $G_N^2 v^2$

$\mathcal{M}(q) =$



(5.31)

According to the Feynman rules developed in Chapter 3, the amplitude 5.31 in dimensional regularization is given by

$$\begin{aligned}\mathcal{M}(q) = \int \frac{d^dk}{(2\pi)^d}\Big(-\frac{im_1}{2\Lambda}v^iv^j\Big)\Big(P_\sigma^{ijrs}\Big)\Big(\frac{ic_d}{\Lambda}[\delta_{rs}k\cdot(q-k)-\\ k_r(q-k)_s - k_s(q-k)_r]\Big)\Big(-\frac{i}{2c_d}\frac{1}{k^2}\Big)\Big(-\frac{i}{2c_d}\frac{1}{(q-k)^2}\Big)\Big(-i\frac{m_2}{\Lambda}\Big)^2\end{aligned} \tag{5.32}$$

where the round brackets have been used in order to enlighten the Feynman rules that enters in the evaluation of 5.31.
The expression so obtained can be written in a simpler way performing all the contractions involved

$$\mathcal{M}(q) = -\int \frac{d^dk}{(2\pi)^d}\frac{m_1m_2^2}{8\Lambda^4c_d}\frac{v_iv_jP_\sigma^{ijrs}}{(q-k)^2k^2}[\delta_{rs}k\cdot p - k_r(q-k)_s - k_s(q-k)_r] \tag{5.33}$$

$$= -\frac{im_1m_2^2}{16\Lambda^4c_3}\int\frac{d^3k}{(2\pi)^3}\frac{2v_iv_j}{(q-k)^2k^2q^2}\Big(k^i(q-k)^j + k^j(q-k)^i\Big) + O(\varepsilon) \tag{5.34}$$

$$\mathcal{M}(q) = -im_1m_2^2\pi^2G_N^2 64\int\frac{d^3k}{(2\pi)^3}\frac{v\cdot k\, v\cdot(q-k)}{(q-k)^2q^2k^2} \tag{5.35}$$

where for simplicity the amplitude have been calculated at $d = 3$, since as we will see, no divergences appear for this effective diagram.

In order to estimate the amplitude, we first concentrate on the loop integral given by the integration in the k variable defining

$$\mathcal{M}(q) = -im_1 m_2^2 \pi^2 64 \frac{G_N^2}{q^2} v^i v^j B^{ij}(q) \tag{5.36}$$

$$B^{ij}(q) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{k^i}{k^2} \frac{(q-k)^j}{(q-k)^2} \tag{5.37}$$

where the tensor 5.37 can be decomposed as

$$B^{ij}(q,p) = q^j \int \frac{d^3k}{(2\pi)^3} \frac{k^i}{k^2(q-k)^2} - \int \frac{d^3k}{(2\pi)^3} \frac{k^i k^j}{k^2(q-k)^2} = q^j I^i - S^{ij} \tag{5.38}$$

Here we just report the value for the two integrals in 5.38, leaving to the Appendix (part B) their complete derivation

$$I^i = \frac{q^i}{16q} \quad , \quad S^{ij} = -\frac{q}{64}\delta^{ij} + \frac{3}{64}\frac{q^i q^j}{q} \tag{5.39}$$

Thus, the tensor 5.38 can be expressed as

$$B^{ij} = \frac{q}{64}\delta^{ij} + \frac{1}{64}\frac{q^i q^j}{q} \tag{5.40}$$

Inserting 5.40 into 5.36 we arrive at

$$\mathcal{M}(q) = -im_1 m_2^2 \pi^2 G_N^2 v^i v^j \left(\frac{\delta^{ij}}{q} + \frac{q^i q^j}{q^3} \right) \tag{5.41}$$

Finally, the contribute to the effective action is given by

$$\mathcal{S}_{eff}^{2PN} = -iW = \int d\tau_1 d\tau_2 A(x_1, x_2) \tag{5.42}$$

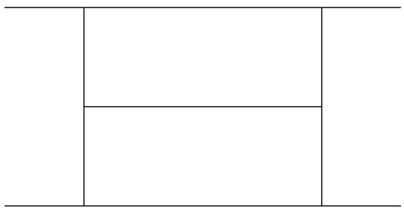
$$A(x_1, x_2) = \int \frac{d^3q}{(2\pi)^3} \frac{d\omega}{2\pi} e^{-i\omega(\tau_1-\tau_2)+ik\cdot(x_1(\tau_1)-x_2(\tau_2))} \mathcal{M}(q) \tag{5.43}$$

The Fourier transform of $\mathcal{M}$ is straightforward giving as a final contribute

$$\mathcal{S}_{eff}^{2PN} = \frac{m_1 m_2^2 G^2}{2r^2} \left[\frac{(v\cdot r)^2}{r^2} - v^2 \right] \tag{5.44}$$

5.4.2 A static 2-PN amplitude from a G_N^3 topology

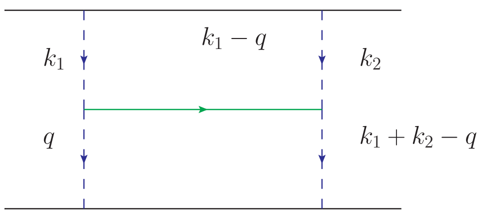
In this subsection we will evaluate an amplitude at 2-PN order that comes from the following H-shape topology at G_N^3 order



(5.45)

We have chosen this diagram since it has divergences for $d = 3$ that can be consistently handled by means of QFT techniques (e.g. with dimensional regularization), despite to the previous case.
Associated to 5.45 there is a unique static PN diagram involving ϕ and σ_{mn} given by

$\mathcal{M}_H(k_1, k_2) =$



$, \quad p = k_1 + k_2$ (5.46)

The amplitude is equal to

$$\mathcal{M}_H(k_1, k_2) = \int \frac{d^d q}{(2\pi)^d} \Big(-\frac{im_1}{\Lambda} \Big)^2 \Big(-\frac{im_2}{\Lambda} \Big)^2 \Big(-\frac{i}{2c_d} \Big)^4 \frac{P_\sigma^{ijnm} v_{ij} v_{nm}}{k_1^2 \, k_2^2 \, q^2 \, (p-q)^2} \tag{5.47}$$

where the contractions involves the following terms

$$P_\sigma^{ijnm} = -\frac{i}{2(k_1 - q)^2} \left(\frac{2}{2-d} \, \delta^{ij} \delta^{nm} + L^{ijnm} \right) \tag{5.48}$$

$$v_{ij} = -\frac{c_d}{\Lambda} \big[-\delta_{ij} k_1 \cdot q + (k_1, q)_{ij} big \big] \tag{5.49}$$

$$v_{nm} = -\frac{c_d}{\Lambda}\big[-\delta_{nm} k_2 \cdot (p-q) + (k_2, p-q)_{nm}\big] \tag{5.50}$$

$$(v,q)_{ij} = v_i q_j + v_j q_i \quad , \quad L^{ijmn} = \delta^{in}\delta^{jm} + \delta^{im}\delta^{jn} \tag{5.51}$$

The first contraction we perform is given by

$$P^{ijnm} v_{ij} = \frac{ic_d}{2\Lambda(k_1-q)^2}\bigg(\frac{2}{2-d}\,\delta^{ij}\delta^{nm} + L^{ijnm}\bigg)\bigg(-\delta_{ij} k_1 \cdot q + (k_1,q)_{ij}\bigg) \tag{5.52}$$

$$= \frac{ic_d}{2\Lambda(k_1-q)^2}\bigg[k_1 \cdot q\Big(\frac{4}{2-d}\delta^{nm} - \frac{2d}{2-d}\delta^{nm} + 2\delta^{nm}\Big) + 2(k_1,q)^{nm}\bigg] \tag{5.53}$$

$$= \frac{ic_d}{\Lambda(k_1-q)^2}(k_1,q)^{nm} \tag{5.54}$$

The similar one involving v_{nm} is straightforward and it can be used to shown that the amplitude 5.46 is equal to

$$\mathcal{M}_H(k_1,k_2) = -\frac{im_1^2 m_2^2}{8\Lambda^6 c_d^2}\int \frac{d^d q}{(2\pi)^d}\frac{N(k_1,k_2,q)}{D_1 D_2 D_3 D_4 D_5} \tag{5.55}$$

$$N(k_1,k_2,q) = [k_1 \cdot k_2]\,[q\cdot(p-q)] + [k_1\cdot(p-q)]\,[k_2\cdot q] - [k_2\cdot(p-q)]\,[k_1\cdot q] \tag{5.56}$$

$$D_1 = k_1^2 \quad D_2 = q^2 \quad D_3 = (k_2)^2 \quad D_4 = (q-p)^2 \quad D_5 = (q-k_1)^2 \tag{5.57}$$

At this point we can proceed with a Fourier transform in the k_1, k_2 variables in order to obtain the corresponding contribute to the effective action

$$\mathcal{S}_{eff}^{2PN} = -i \lim_{d\to 3}\int \frac{d^d k_1}{(2\pi)^d}\frac{d^d k_2}{(2\pi)^d} e^{i(k_1+k_2)\cdot r}\mathcal{M}_H(k_1,k_2) \tag{5.58}$$

where time integrals have not been reported given that they merely simplify.
In order to deal with 5.58 we find convenient to define a new integration region from (k_1,k_2) to (k_1,p) where from now on the subscript 1 for the external momentum will be omitted.
According to this prescription, the desired contribute can be written as the Fourier transform of a function depending solely on p, in accordance with the relation among effective diagrams and QFT amplitudes shown in Chapter 3.

The contribute now becomes

$$\mathcal{S}_{eff}^{2PN} = -i \lim_{d\to 3} \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot r} I(p) \tag{5.59}$$

$$I(p) = -\frac{i m_1^2 m_2^2}{32\Lambda^6 c_d^2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{N}{D_1 D_2 D_3 D_4 D_5} \tag{5.60}$$

where indeed, in equation 5.60 we can recognize the amplitude of a two point function with three loops.
Given this consideration, which is natural given the relations among effective diagrams and QFT amplitudes, we can evaluate it according to the techniques developed in Chapter 4.
We can start by expressing the numerator in term of the denominators by expressing the scalar products involved as a linear combination of their square, and that of their sum or difference[2]

$$k\cdot(p-k) = \frac{1}{2}\big(p^2 - D_1 - D_3\big) \quad , \quad q\cdot(p-q) = \frac{1}{2}\big(p^2 - D_2 - D_4\big) \tag{5.61}$$

$$k\cdot(p-q) = \frac{1}{2}\big(p^2 - D_2 - D_3 + D_5\big) \quad , \quad q\cdot(p-k) = \frac{1}{2}\big(p^2 - D_1 - D_4 + D_5\big) \tag{5.62}$$

$$(p-k)\cdot(p-q) = \frac{1}{2}\big(D_3 + D_4 - D_5\big) \quad , \quad q\cdot k = \frac{1}{2}\big(D_1 + D_2 - D_5\big) \tag{5.63}$$

According to these, the amplitude $I(p)$ becomes

$$I(p) = -\frac{i m_1^2 m_2^2}{32\Lambda^6 c_d^2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left[\frac{(p^2 - D_1 - D_3)(p^2 - D_2 - D_4)}{D_1 D_2 D_3 D_4 D_5}\right] + \tag{5.64}$$

$$\left[\frac{(p^2 - D_2 - D_3 + D_5)(p^2 - D_1 - D_4 + D_5)}{D_1 D_2 D_3 D_4 D_5}\right] + \left[\frac{(D_4 + D_3 - D_5)(D_2 + D_1 - D_5)}{D_1 D_2 D_3 D_4 D_5}\right]$$

[2]Beware that this is not always possible: as we will see in a 4-PN calculation, some scalar products are *irreducibile*, in the sense that they cannot be expressed as linear combination of terms in the denominator

$$I(p) = -\frac{im_1^2 m_2^2}{16\Lambda^6 c_d^2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} p^4 \left[\frac{1}{D_1 D_2 D_3 D_4 D_5}\right] + p^2 \left[\frac{D_5 - D_1 - D_2 - D_3 - D_4}{D_1 D_2 D_3 D_4 D_5}\right]$$

$$+\frac{1}{2}\left[\frac{(D_1 + D_3)(D_2 + D_4) + (D_1 + D_4)(D_3 + D_2) - (D_4 + D_3)(D_1 + D_2)}{D_1 D_2 D_3 D_4 D_5}\right] \quad (5.65)$$

In a more compact way, the amplitude $I(p)$ can be expressed as

$$I(p) = -\frac{im_1^2 m_2^2}{16\Lambda^6 c_d^2} \quad \left(p^4 A + p^2 B + C\right) \quad (5.66)$$

where we have introduced the following integrals

$$A = \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left[\frac{1}{D_1 D_2 D_3 D_4 D_5}\right] \quad (5.67)$$

$$B = \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left[\frac{D_5 - D_1 - D_2 - D_3 - D_4}{D_1 D_2 D_3 D_4 D_5}\right] \quad (5.68)$$

$$C = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left[\frac{2D_1 D_2 + 2D_3 D_4}{D_1 D_2 D_3 D_4 D_5}\right] \quad (5.69)$$

What is relevant, is that all these integrals comes from the same Bug topology studied in Chapter 4, as one can easily see since we have maintained the same notations.

In this respect, their analytic evaluation can be performed with the use of IBP identities giving

$$A = -\frac{p^{2d-10}}{[2\pi]^{2d}} \frac{8(g_2 - g_1^2)}{(d-3)(d-4)^3} \pi^d \quad (5.70)$$

$$B = -\frac{p^{2d-8}}{[2\pi]^{2d}} \left[\frac{4g_1^2}{(d-3)^2(d-4)^2} + \frac{8g_1 g_3}{(d-3)(d-4)}\right] \pi^d \quad , \quad C = 0 \quad (5.71)$$

$$g_1 = \frac{\Gamma(3-\frac{d}{2})\Gamma(\frac{d}{2}-1)^2}{\Gamma(d-3)} \quad g_2 = \frac{\Gamma(5-d)\Gamma(\frac{d}{2}-1)^3}{\Gamma(\frac{3d}{2}-5)} \tag{5.72}$$

$$g_3 = \frac{\Gamma(4-d)\Gamma(d-3)\Gamma(\frac{d}{2}-1)}{\Gamma(3-\frac{d}{2})\Gamma(\frac{3d}{2}-4)} \tag{5.73}$$

After resummation, the Fourier transform of $I(p)$ gives the following final contribute to the effective action in d-dimensions

$$\mathcal{S}^{2PN}_{eff}(d) = -\frac{m_1^2 m_2^2\, r^{6-3d}}{\Lambda^6 c_d^2\, 2^{10}\, [\pi]^{\frac{3d}{2}}} \left[\frac{8(g_2-g_1^2)}{(d-4)^3} + \frac{4g_1^2}{(d-3)(d-4)^2} + \frac{8g_1 g_3}{(d-4)}\right]\frac{\Gamma(\frac{3d}{2}-3)}{\Gamma(4-d)} \tag{5.74}$$

In order to recover the contribute in $d=3$ we will expand all the d-dependent terms around $d = 3+\varepsilon$ by then taking the physical limit for $\varepsilon \to 0$

$$g_1 = \pi^{\frac{3}{2}}\varepsilon + O(\varepsilon^2) \quad , \quad g_2 = -\frac{\pi}{2} + O(\varepsilon) \quad , \quad g_3 = \frac{\varepsilon}{\sqrt{\pi}} + O(\varepsilon^2) \tag{5.75}$$

Using these expansions, the second and third term of the sum in the square bracket of 5.74 goes to zero, leading to the final result

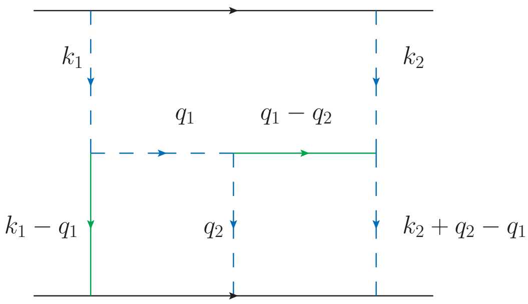
$$\mathcal{S}^{2PN}_{eff} = -\frac{m_1^2 m_2^2}{\Lambda^6 2^{13}\pi^3}\frac{1}{r^3} + O(\epsilon) \quad \Rightarrow \quad \mathcal{S}^{2PN}_{eff} = -2\frac{G_N^3 m_1^2 m_2^2}{r^3} \tag{5.76}$$

where in the final passage we have expressed Λ in terms of G_N and divided by two since the same result could be derived interchanging 1 with 2.

5.5 A non static 4-PN amplitude from a G_N^4 topology

In this final section we present for the first time the evaluation of a 4-PN non static contribute to the Newton Potential which scales as $G_N^4 v^2$.
The amplitude is given by the following diagram

$$\mathcal{M}_B(k_1, k_2) = \tag{5.77}$$



The amplitude in momentum space is given in d space dimensions by

$$\mathcal{M}_B = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \Big(- \frac{i m_1}{\Lambda} \Big)^2 \Big(- \frac{i m_2}{\Lambda} \Big)^2 \Big(- \frac{i}{2 c_d k_1^2} \Big) \Big(- \frac{i}{2 c_d q_1^2} \Big) \Big(- \frac{i}{2 c_d k_2^2} \Big)$$
$$\Big(- \frac{i}{2 c_d q_2^2} \Big) \Big(\frac{-i}{2 c_d (k_2 + q_1 - q_2)^2} \Big) \Big(\frac{i m_a v^i v^j}{2\Lambda} \Big) \Big[P_\sigma^{ijrs} V_{rs} \Big] \Big[V_{nm} P_\sigma^{nmlt} V_{lt} \Big] \tag{5.78}$$

We first evaluate separately, the tensor contractions involved in the square brackets of 5.78

$$P_\sigma^{ijrs} V_{rs} = \frac{i c_d}{\Lambda} \frac{(k_1, q_1)^{ij}}{(k_1 - q_1)^2} \tag{5.79}$$

$$V_{nm} P_\sigma^{nmlt} V_{lt} = \frac{-i 2 c_d^2}{\Lambda^2 (q_1 - q_2)^2} \Big(2 q_1 \cdot k_2 \, q_2 \cdot k_2 + q_1^2 q_2 \cdot k_2 - q_2^2 q_1 \cdot k_2 - k_2^2 q_1 \cdot q_2 \Big) \tag{5.80}$$

Substituting 5.79,5.80 into the expression of the initial amplitude 5.78 we obtain

$$\mathcal{M}_B(k_1, k_2) = \frac{m_1^3 m_2^2}{16 \Lambda^8 c_d^2} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \tilde{\mathcal{M}}_B(k_1, k_2, q_1, q_2) \tag{5.81}$$

$$\tilde{\mathcal{M}}_B(k_1, k_2, q_1, q_2) = \frac{\Big[k_1 \cdot v \, q_1 \cdot v \Big] \Big[2 q_1 \cdot k_2 \, q_2 \cdot k_2 + q_1^2 q_2 \cdot k_2 - q_2^2 q_1 \cdot k_2 - k_2^2 q_1 \cdot q_2 \Big]}{(k_1 - q_1)^2 (q_1 - q_2)^2 k_1^2 k_2^2 q_1^2 q_2^2 (k_2 + q_1 - q_2)^2}$$
(5.82)

The contribute to the effective action can be derived according to the same Fourier transform done in the previous calculations.
The result describes a 4 PN correction to the effective action of a non spinning binary system given by

$$\mathcal{S}_{eff}^{4PN} = -i \lim_{d\to 3} \frac{m_1^3 m_2^2}{16\Lambda^8 c_d^2} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} e^{i(k_1+k_2)\cdot r} \tilde{\mathcal{M}}_B(q_1, q_2, k_1, k_2) \quad (5.83)$$

In order to evaluate 5.83, let's change variables of integration defining $k_1 + k_2 = p$ and $k_2 = q_3$

$$\mathcal{S}_{eff}^{4PN} = -i \lim_{d\to 3} \frac{m_1^3 m_2^2}{16\Lambda^8 c_d^2} \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot r} \Big[\int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{d^d q_3}{(2\pi)^d} \tilde{\mathcal{M}}_B(q_1, q_2, q_3, p) \Big] \quad (5.84)$$

As expected, by isolating the integral over the total momentum exchanged p, we get a typical loop amplitude, in this case a 3-loop one.
In order to explicit its calculation, we first factorize the velocity dependent terms defining

$$\tilde{\mathcal{M}}_B(q_1, q_2, q_3, p) = v^i v^j \tilde{\mathcal{M}}_B^{ij}(q_1, q_2, q_3, p)$$

$$\tilde{\mathcal{M}}_B^{ij} \equiv \frac{\Big[(p-q_3)^i \, q_1^j\Big] \Big[2q_1\cdot q_3 \, q_2\cdot q_3 + q_1^2 q_2\cdot q_3 - q_2^2 q_1\cdot q_3 - q_3^2 q_1\cdot q_2\Big]}{(p-q_3-q_1)^2 (q_1-q_2)^2 (p-q_3)^2 \, q_1^2 \, q_2^2 \, q_3^2 (q_3+q_1-q_2)^2} \quad (5.85)$$

We can now assume that the following ansatz holds

$$\tilde{\mathcal{M}}_B^{ij}(q_1, q_2, q_3, p) = a(q_1, q_2, q_3, p)\, \delta^{ij} + b(q_1, q_2, q_3, p)\, p^i p^j \quad (5.86)$$

which makes possible to evaluate in a convenient way the three loop amplitude associated with 5.85.
By taking separately a trace and a contraction with the $p_i p_j$ tensor, the a and b factors in 5.86 can be expressed as

$$a(q_{i=1,2,3}, p) = \frac{Tr\tilde{\mathcal{M}}_B}{(d-1)} - \frac{\tilde{\mathcal{M}}_B^{ij} p_i p_j}{p^2(d-1)} \quad , \quad b(q_{i=1,2,3}, p) = \frac{d\tilde{\mathcal{M}}_B^{ij} p_i p_j}{p^4(d-1)} - \frac{Tr\tilde{\mathcal{M}}_B}{p^2(d-1)} \quad (5.87)$$

$$Tr\tilde{\mathcal{M}}_B = \frac{\Big[(p-q_3)\cdot q_1\Big] \Big[2q_1\cdot q_3 \, q_2\cdot q_3 + q_1^2 q_2\cdot q_3 - q_2^2 q_1\cdot q_3 - q_3^2 q_1\cdot q_2\Big]}{(p-q_3-q_1)^2 (q_1-q_2)^2 (p-q_3)^2 \, q_1^2 \, q_2^2 \, q_3^2 (q_3+q_1-q_2)^2} \quad (5.88)$$

$$\tilde{\mathcal{M}}_B^{ij} p_i p_j = \frac{\Big[(p-q_3)\cdot p \; p\cdot q_1\Big] \Big[2q_1\cdot q_3 \, q_2\cdot q_3 + q_1^2 q_2\cdot q_3 - q_2^2 q_1\cdot q_3 - q_3^2 q_1\cdot q_2\Big]}{(p-q_3-q_1)^2 (q_1-q_2)^2 (p-q_3)^2 \, q_1^2 \, q_2^2 \, q_3^2 (q_3+q_1-q_2)^2} \quad (5.89)$$

Thanks to these relations we can express the three loop amplitude 5.84 using 5.86 as

$$\mathcal{S}_{eff}^{4PN} = -i \lim_{d\to 3} \frac{m_1^3 m_2^2}{16\Lambda^8 c_d^2} \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot r} \Big[I_a(p^2) v^2 + I_b(p^2)(v\cdot p)^2 \Big] \tag{5.90}$$

where for convenience we have defined

$$I_a(p^2) \equiv \int \frac{dq_1}{(2\pi)^d} \frac{dq_2}{(2\pi)^d} \frac{dq_3}{(2\pi)^d} \, a(q_1, q_2, q_3, p) \tag{5.91}$$

$$I_b(p^2) \equiv \int \frac{dq_1}{(2\pi)^d} \frac{dq_2}{(2\pi)^d} \frac{dq_3}{(2\pi)^d} \, b(q_1, q_2, q_3, p) \tag{5.92}$$

Finally, using equations 5.87,5.88,5.89 we can express the previous integrals as

$$I_a(p^2) = \frac{I_{Tr}}{(d-1)} - \frac{I_{p^i p^j}}{p^2(d-1)} \tag{5.93}$$

$$I_b(p^2) = d\, \frac{I_{p^i p^j}}{p^4(d-1)} - \frac{I_{Tr}}{p^2(d-1)} \tag{5.94}$$

where I_{Tr} and $I_{p^i p^j}$ stands for the integration in $q_{1,2,3}$ of 5.88 and 5.89, thus yielding physical three-loop amplitudes.
Let's start from their computation.
This has been accomplished by means of an automated reduction package called "FIRE" [67] which is able to express a generic Feynman amplitude in terms of its Master Integrals and relative coefficients.
As input-Feynman amplitudes we have used I_{Tr} and $I_{p^i p^j}$ while the output gave for both an equivalent expression in terms of three Master Integrals that are exactly those encountered in Chapter 4 while studying the Bug topology.
The output of FIRE is here reported, where the Master integral are those defined in the previous Chapter for the Bug topology

$$I_{Tr} = \frac{1}{(2\pi)^{3d}} \Big[\mathcal{M}_1 c_1 + p^2\, \mathcal{M}_2 c_2 + p^2\, \mathcal{M}_3 c_3 \Big] \tag{5.95}$$

$$c_1 = \frac{(d-2)(11d^2 - 64d + 92)}{6(d-3)(d-4)^2} \tag{5.96}$$

$$c_2 = \frac{(2-d)}{12(d-4)} \quad , \quad c_3 = \frac{(2-d)}{8(2d-5)} \tag{5.97}$$

$$I_{p^ip^j} = \frac{1}{(2\pi)^{3d}}\left[\mathcal{M}_1 g_1 + p^4\,\mathcal{M}_2 g_2 + p^4\,\mathcal{M}_3 g_3\right] \tag{5.98}$$

$$g_1 = \frac{(d-2)(111d^3 - 985d^2 + 2924d - 2896)}{48(d-4)^3(d-3)} \tag{5.99}$$

$$g_2 = \frac{d(d-2)}{24(d-4)^2} \quad , \quad g_3 = \frac{-30d^3 + 225d^2 - 556d + 448}{64(d-4)^2(2d-5)} \tag{5.100}$$

Having solved 5.95,5.98, we can insert their expression into 5.93 which makes possible to express it as

$$I_a(p^2) = \frac{1}{(2\pi)^{3d}}\left[\mathcal{M}_1\alpha_1 + \mathcal{M}_2\alpha_2 p^2 + \mathcal{M}_3\alpha_3 p^2\right] \tag{5.101}$$

$$\alpha_1 = \frac{c_1 - g_1}{(d-1)} = -\frac{(d-2)(23d^2 - 52d - 16)}{48(d-1)(d-4)^3} \tag{5.102}$$

$$\alpha_2 = \frac{c_2 - g_2}{(d-1)} = -\frac{(d-2)(3d-8)}{24(d-1)(d-4)^2} \tag{5.103}$$

$$\alpha_3 = \frac{c_3 - g_3}{(d-1)} = \frac{22d^3 - 145d^2 + 300d - 192}{64(d-1)(2d-5)(d-4)^3} \tag{5.104}$$

The same procedure can be applied for 5.94 which gives

$$I_b(p^2) = \frac{1}{(2\pi)^{3d}}\left[\frac{\mathcal{M}_1}{p^2}\beta_1 + \mathcal{M}_2\beta_2 + \mathcal{M}_3\beta_3\right] \tag{5.105}$$

$$\beta_1 = \frac{dg_1 - c_1}{d-1} = \frac{(d-2)(3d-8)(37d^3 - 259d^2 + 572d - 368)}{48(d-1)(d-3)(d-4)^3} \tag{5.106}$$

$$\beta_2 = \frac{dg_2 - c_2}{d-1} = \frac{(d-2)^2(d+4)}{24(d-1)(d-4)^2} \tag{5.107}$$

$$\beta_3 = \frac{dg_3 - c_3}{d-1} = -\frac{(3d-8)(5d-8)(2d^2 - 7d + 4)}{64(d-4)^2(2d-5)(d-1)} \tag{5.108}$$

For simplicity we express the Master Integrals involved separating their p^2 dependence from their adimensional one

$$\mathcal{M}_1 = \frac{\tilde{\mathcal{M}}_1}{(p^2)^{4-\frac{3d}{2}}} \quad , \quad \tilde{\mathcal{M}}_1 = -i\pi^{\frac{3d}{2}} \frac{\Gamma(d/2-1)^4\Gamma(4-3d/2)}{\Gamma(2d-4)} \tag{5.109}$$

$$\mathcal{M}_2 = \frac{\tilde{\mathcal{M}}_2}{(p^2)^{5-\frac{3d}{2}}} \quad , \quad \tilde{\mathcal{M}}_2 = -i\pi^{\frac{3d}{2}} \frac{\Gamma(d/2-1)^5\Gamma(3-d)\Gamma(2-d/2)}{\Gamma(d-2)\Gamma(3d/2-3)} \tag{5.110}$$

$$\mathcal{M}_3 = \frac{\tilde{\mathcal{M}}_3}{(p^2)^{5-\frac{3d}{2}}} \quad , \quad \tilde{\mathcal{M}}_3 = -i\pi^{\frac{3d}{2}} \frac{\Gamma(d/2-1)^5\Gamma(3d/2-4)\Gamma(5-3d/2)\Gamma(2-d/2)^2}{\Gamma(d-2)^2\Gamma(4-d)\Gamma(2d-5)} \tag{5.111}$$

Thanks to these definitions, the following compact expressions for I_a and I_b is possible

$$I_a = \frac{1}{(2\pi)^{3d}}\left[\frac{\tilde{\mathcal{M}}_i\alpha_i}{(p^2)^{4-\frac{3d}{2}}}\right] \quad , \quad I_b = \frac{1}{(2\pi)^{3d}}\left[\frac{\tilde{\mathcal{M}}_l\beta_l}{(p^2)^{5-\frac{3d}{2}}}\right] \tag{5.112}$$

By inserting 5.112 into 5.90, we are a step closer to the complete evaluation of the desired post-Newtonian contribute

$$\mathcal{S}_{eff}^{4PN} = -i\lim_{d\to 3}\frac{m_1^3m_2^2}{16\Lambda^8c_d^2(2\pi)^{3d}}\left(\tilde{\mathcal{M}}_i\alpha_i v^2\left[\int\frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot r}}{(p^2)^{4-\frac{3d}{2}}}\right]+\tilde{\mathcal{M}}_l\beta_l v_i v_j\left[\int\frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot r}p^ip^j}{(p^2)^{5-\frac{3d}{2}}}\right]\right) \tag{5.113}$$

The Fourier transform involved in the square brackets can be easily done using the results of the fourth Chapter, which are here recalled

$$\int\frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot r}}{(p^2)^{4-\frac{3d}{2}}} = \frac{\Gamma(2d-4)}{\Gamma(4-3d/2)(4\pi)^{d/2}}\left(\frac{r}{2}\right)^{8-4d} \tag{5.114}$$

$$\int\frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot r}p_ip_j}{(p^2)^{5-\frac{3d}{2}}} = \left[\delta_{ij}+(4-2d)\frac{r_ir_j}{r^2}\right]\frac{\Gamma(2d-4)}{\Gamma(5-3d/2)(4\pi)^{d/2}}\left(\frac{r}{2}\right)^{8-4d} \tag{5.115}$$

The final contribute, in arbitrary d dimension is given by

$$\mathcal{S}_{eff}^{4PN} = -i \lim_{d\to 3} \frac{m_1^3 m_2^2}{16\Lambda^8 c_d^2} \frac{\Gamma(2d-4)}{\Gamma(4-{}^{3d}/_{2})(2\pi)^{3d}(4\pi)^{d/2}} \left(\frac{r}{2}\right)^{8-4d} \left[\tilde{\mathcal{M}}_i \gamma_i v^2 + \tilde{\mathcal{M}}_i \Omega_i \left(\frac{v \cdot r}{r}\right)^2\right] \tag{5.116}$$

$$\gamma_1 = \alpha_1 - \frac{\beta_1}{3d-8} = -\frac{(d-2)(15d^3 - 95d^2 + 178d - 80)}{12(d-1)(d-3)(d-4)^3} \tag{5.117}$$

$$\gamma_2 = \alpha_2 - \frac{\beta_2}{3d-8} = -\frac{(d-2)(5d^2 - 23d + 28)}{12(d-1)(d-4)^2(3d-8)} \tag{5.118}$$

$$\gamma_3 = \alpha_3 - \frac{\beta_3}{3d-8} = \frac{8d^3 - 49d^2 + 94d - 56}{16(d-1)(d-4)^2(2d-5)} \tag{5.119}$$

$$\Omega_1 = \frac{4(d-2)\beta_1}{3d-8} = \frac{(d-2)^2(37d^3 - 259d^2 + 572d - 368)}{12(d-1)(d-3)(d-4)^3} \tag{5.120}$$

$$\Omega_2 = \frac{4(d-2)\beta_2}{3d-8} = \frac{(d-2)^3(d+4)}{6(d-1)(d-4)^2(3d-8)} \tag{5.121}$$

$$\Omega_3 = \frac{4(d-2)\beta_3}{3d-8} = -\frac{(d-2)(5d-8)(2d^2 - 7d + 4)}{16(d-1)(d-4)^2(2d-5)} \tag{5.122}$$

The result 5.116 is not defined at $d = 3$, thus it has then been expanded within Mathematica as a Laurent series around $d = 3$, leading to

$$\begin{aligned}\mathcal{S}_{eff}^{4PN} = &-\frac{4}{3}\frac{(v\cdot r)^2 m_1^3 m_2^2 G_N^4}{(d-3)r^6} + \frac{G_N^4 m_1^3 m_2^2 (v\cdot r)^2}{r^6}\left(\frac{16}{3}\log({}^{r}/_{l}) + \frac{7\pi^2}{4} - \frac{80}{3}\right) \\ &+ \frac{G_N^4 m_1^3 m_2^2 v^2}{r^4}\left(\frac{28}{3} - \frac{\pi^2}{4}\right) + O(d-3)\end{aligned} \tag{5.123}$$

From 5.123 we see that the contribute to the Newton Potential at $d = 3$ is composed by a divergent term proportional to $\frac{v \cdot r}{r}$ and finite terms containing irrational π^2 terms.
Given that the static 4 PN Lagrangian for a spin-less binary has been calculated leading to no π^2 terms, after resummation, it would be interesting to understand if the same eliminations hold for the non static part of this PN order, which currently is unknown.

Conclusions

Gravitational physics has currently entered in a new and exciting era of precision due to the first detection on 14 September 2015 of gravitational waves by two colliding black holes [3]. Given that these signals carry fingerprints of binary systems, recently a lot of attention has been paid on the gravitational two body problem, which have been studied in the present thesis adopting an analytical and perturbative scheme called PN (post-Newtonian) expansion [14].
Useful for bound system with weak gravitational fields and slow velocities compared to the speed of light, in the thesis we have proved how the conservative dynamics of a slow inspiral binary takes deviations from the Newton's two body potential in terms of n PN contributions, i.e. with factors proportional to $G_N^{n-l} v^{2l}$ where $0 \leq l \leq n-1$, v is a typical three velocity of the system and n, l are natural numbers.
As a convenient tool to deal with PN physics, we have introduced the so called EFT (Effective Field Theory) approach [38], so as to model the inspiral phase of a binary similarly as done with the heavy quark field theory [36] in particle physics.
Exploiting this approach, we have been able to cast the perturbative nature of PN corrections into Feynman diagrams, making possible to define a clear power counting in v^2/c^2 and organizing specific PN calculations in terms of a well defined subset of Feynman diagrams. In order to build them, we have derived a finite number of Feynman rules adopting a non manifest covariant parametrization for the metric tensor [51]. Suitable for the EFT of PN systems, this choice leads to diagrams with a definite scaling in powers of G_N and v^2/c^2. Remarkably, these classical contributions can be topologically mapped into massless multi-loop two point functions, a relation that has been recently introduced in [34] by Mastrolia, Foffa, Sturani and Sturm. Taking advantage of this mapping, we have performed high precision calculations adopting the most modern multi-loop techniques.
Among them we have used Integration by parts identities [19] and the concept of MI (Master integral), with the use of which multi-loop amplitudes have been reduced in terms of simpler ones, whose analytic calculation was viable.
Several PN calculations have been performed, concerning a non spinning binary system, as the complete Einstein-Infeld-Hoffmann Lagrangian at 1 PN order, two contributions at 2 PN order scaling respectively as G_N^3 and $G_N^2 v^2$ and a 4 PN one with a $G_N^4 v^2$ dependence that was never calculated before with the EFT approach.

This 4 PN calculation is the original contribution of the thesis.
It has involved a three-loop amplitude which has been reduced into three MI's with the use of a Mathematica package called FIRE [67]. All MI's have been calculated, leading to the evaluation of this PN contribution which so far was unknown within the EFT approach.
Even if the complete 4 PN sector was already calculated with other approaches [28],[13], the EFT one is far from being a re-derivative scheme and it is currently creating a fruitful convergence among gravitational and multi-loop physics environments. On the one hand, the desire for greater precision in the description of gravitational processes has lead to the interest in novel techniques, on the other the possibility of extending multi-loop techniques in branches that until now have never been hypothesized has revealed an intriguing challenge.
As an example of these cross interests, we mention the possibility of evaluating, through EFT techniques, the 5 PN sector of non-spinning binary black holes.
The evaluation of this sector would be important for both communities: from one side it would provide a precision in the description of binary dynamics that on theoretical ground is not yet possible, on the other it would mean to evaluate contributions which can be mapped into 5-loop amplitudes that currently represent the state of art for the amplitude community.
Thanks to these convergences, there is also the possibility of explaining notable π^2 erase deletions in the 4 PN sector [34, 28], cancellations which could be enlighten inside the EFT approach with a recent technique called *Double-copy* [12], able to map tree level diagrams of gravity in analogous contributions coming from a gauge theory: in this perspective, it could be possible to explain such cancellations in terms of a kinematic-color symmetry as in QCD.
In addition to this, the EFT approach can be used to perform other calculations which are required by the gravitational community in the so called PM (post-Minkowskian) regime where a binary system is unbound having arbitrarily velocities and a weak gravitational field. This dynamics can be encountered in the first moments of the inspiral, where circularization have not yet occurred and both binary components lie on an elliptic orbit with highly variable velocities.
Given that LISA will be able to detect gravitational waves from the inspiral of PM systems, several research groups are trying to understand in detail this dynamics and the behavior of the gravitational waves emitted.
From a diagrammatic point of view, a PM contribute to the conservative dynamics can be evaluated by summing up at fixed power of G_N, all contributions from powers of ${}^{v^2}\!/_{c^2}$. This is possible assuming no expansion in the effective action for small ${}^{v}\!/_{c}$, thus deriving Feynman rules involving powers of G_N only.
It would therefore be natural to apply modern multi-loop techniques for PM calculations: as example we cite the unknown and required evaluation of 3 PM contributions (i.e. G_N^2 scaling) to the conservative sector of a binary black holes dynamics.
In light of all these considerations, it is reasonable to expect that the cross interests

between gravitational and amplitude communities will lead to interesting results in the coming years.

Appendices

Appendix A

Degree of divergence for the four-loop massless propagator

In the estimate of the degree of divergence for the four-loop massless, the calculation was performed looking only at the highest pole in $1/\varepsilon^2$.
In this section we are going to demonstrate why we have lost finite terms in our estimate.
The starting point is the following integral

$$B_1 = O(\varepsilon) \int \frac{d^d q_1 d^d q_2 d^d q_3}{(p-q_1)^2 (q_1-q_2)^2 q_2^2 q_3^2 (p-q_1-q_3)^2} \tag{A.1}$$

This integral can be solved by first integrating in q_3, which is a simple one-loop integral

$$B_1 = O(\varepsilon) \int \frac{d^d q_1 d^d q_2}{[(p-q_1)^2]^{3-d/2} \, (q_1-q_2)^2 \, q_2^2} \tag{A.2}$$

where finite terms were collected in $O(\varepsilon)$.
Again, the integral in q_2 is a straightforward one-loop calculation giving

$$B_1 = O(\varepsilon) \int \frac{d^d q_1}{[(p-q_1)^2]^{3-d/2} \, [(q_1)^2]^{2-d/2}} \tag{A.3}$$

As before, by collecting in $O(\varepsilon)$ those terms that are finite around $d = 3 + \varepsilon$, what remain is

$$B_1 = O(\varepsilon)\Gamma(\varepsilon) = O(K) \tag{A.4}$$

where by $O(K)$ we means a negligible term compared to a generic constant.

The reason of this result is now clear: since the Gamma function has a simple pole, multiplying it by an arbitrary function negligible compared to ε, what is left is an arbitrary constant that can only affect the finite part of $M_{3,6}$.
The same happens for the other integral multiplied by $O(\varepsilon)$ that we have discarded in our estimate

$$B_2 = O(\varepsilon) \int \frac{d^d q_1 d^d q_2}{(p - q_1)^2 \left[(q_1 - q_2)^2\right]^{1/2} \left[q_2^2\right]^{3/2} \left[q_1^2\right]^{1/2}} \tag{A.5}$$

The integration over q_2 gives

$$B_2 = O(\varepsilon)\nu(3/2) \int \frac{d^d q_1}{(p - q_1)^2 \left[q_1^2\right]^{5 - d/2}} \tag{A.6}$$

The last integration gives only finite terms for $d = 3$, thus we are left with

$$B_2 = O(\varepsilon)\nu(3/2) = O(\varepsilon)\Gamma\left(\frac{\varepsilon}{2}\right) = O(K) \tag{A.7}$$

Appendix B

The non static 2-PN amplitude

The three-vector integral I^i

In this section we will evaluate the following three-vector integral

$$I^i = \int \frac{d^3k}{(2\pi)^3} \frac{k^i}{k^2(q-k)^2} \tag{B.1}$$

This integral can be solved using Feynman parameters

$$I^i = \int \frac{d^3k}{(2\pi)^3} \frac{k^i}{k^2(q-k)^2} = \int_0^1 dx \int \frac{d^3k}{(2\pi)^3} \frac{k^i}{[(k-qx)^2+\Delta]^2} \tag{B.2}$$

$$\Delta = q^2 x(1-x) \tag{B.3}$$

We can change variables of integration by a shift of qx obtaining

$$I^i = \int_0^1 dx \int \frac{d^3k}{(2\pi)^3} \frac{k^i + q^i x}{[k^2+\Delta]^2} = q^i \int_0^1 dx\, x \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2+\Delta]^2} \tag{B.4}$$

where in the last equality we have removed the k^i term since it is integrated on a symmetric region.
At this point we are left with a scalar integral in k which can be easily solved

$$I^i = q^i \int_0^1 dx\, x \int \frac{d^3k}{(2\pi)^3} \frac{1}{[k^2+\Delta]^2} = q^i \int_0^1 dx\, x \left[\frac{1}{(4\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{\Delta}} \Gamma\Big(\frac{1}{2}\Big) \right] \tag{B.5}$$

Expliciting Δ as a function of x we arrive at the final result

$$I^i = \frac{q^i}{q}\Gamma\left(\frac{1}{2}\right)\frac{1}{(4\pi)^{\frac{3}{2}}}\int_0^1 dx\,\frac{x}{\sqrt{x(1-x)}} = \frac{q^i}{q}\sqrt{\pi}\frac{1}{(4\pi)^{\frac{3}{2}}}\left(\frac{\pi}{2}\right) = \frac{q^i}{16q} \tag{B.6}$$

The tensorial integral S^{ij}

In this section we will evaluate the following tensorial integral

$$S^{ij} = \int \frac{d^3k}{(2\pi)^3}\frac{k^i k^j}{k^2(k-q)^2} = a\delta^{ij} + bq^iq^j \tag{B.7}$$

By taking the trace once time and separately a contraction in q^iq^j we get a system of two equations

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2} = ad + bq^2 \tag{B.8}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{(k\cdot q)^2}{k^2(k-q)^2} = aq^2 + bq^4 \tag{B.9}$$

The first scalar integral is equal to zero since there is no physical scale associated with it, while for the second we should use again the Feynman parameters

$$\int \frac{d^3k}{(2\pi)^3}\frac{(k\cdot q)^2}{k^2(k-q)^2} = \int_0^1 dx \int \frac{d^3k}{(2\pi)^3}\frac{(k\cdot q)^2}{[(k-qx)^2+\Delta]^2} \tag{B.10}$$

By a shift of qx and eliminating the terms that gives a null integral we obtain

$$\int_0^1 dx \int \frac{d^3k}{(2\pi)^3}\frac{(k\cdot q)^2}{[(k-qx)^2+\Delta]^2} = \int_0^1 dx \int \frac{d^3k}{(2\pi)^3}\frac{(k\cdot q)^2 + q^4x^2}{[k^2+\Delta]^2} \tag{B.11}$$

This is made of two integrals, a scalar and sill a tensorial one

$$\int_0^1 dx\left[\int \frac{d^3k}{(2\pi)^3}\frac{(k\cdot q)^2}{[k^2+\Delta]^2} + \int \frac{d^3k}{(2\pi)^3}\frac{x^2q^4}{[k^2+\Delta]^2}\right] \tag{B.12}$$

The tensorial integral can be reduced to a scalar one by noticing that

$$\int \frac{d^3k}{(2\pi)^3}\frac{k^i k^j}{[k^2+\Delta]^2} = \frac{\delta^{ij}}{3}\int \frac{d^3k}{(2\pi)^3}\frac{k^2}{[k^2+\Delta]^2} \tag{B.13}$$

Using this identity we have

$$\int_0^1 dx\left[\frac{q^2}{3}\int \frac{d^3k}{(2\pi)^3}\frac{k^2}{[k^2+\Delta]^2} + \int \frac{d^3k}{(2\pi)^3}\frac{x^2q^4}{[k^2+\Delta]^2}\right] \tag{B.14}$$

$$= \frac{q^2}{3}\int_0^1 dx\left[\int \frac{d^3k}{(2\pi)^3}\frac{k^2}{[k^2+\Delta]^2}\right] + q^4\int_0^1 x^2\left[\int \frac{d^3k}{(2\pi)^3}\frac{1}{[k^2+\Delta]^2}\right] \tag{B.15}$$

Both integrals converge and gives

$$\frac{q^2}{3}\int_0^1 dx\left[\frac{1}{(4\pi)^{\frac{3}{2}}}\sqrt{\Delta}\,\frac{\Gamma\Big(-\frac{1}{2}\Big)\Gamma\Big(\frac{5}{2}\Big)}{\Gamma\Big(\frac{3}{2}\Big)}\right] + q^4\int_0^1 x^2\left[\frac{1}{(4\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{\Delta}}\,\Gamma\Big(\frac{1}{2}\Big)\right] \tag{B.16}$$

As before we will explicit Δ as a function of x obtaining

$$-q^3\frac{\sqrt{\pi}}{(4\pi)^{\frac{3}{2}}}\int_0^1 dx\sqrt{x-x^2} \;+\; q^3\,\Gamma\Big(\frac{1}{2}\Big)\frac{1}{(4\pi)^{\frac{3}{2}}}\int_0^1 \frac{x^2}{\sqrt{x-x^2}} \tag{B.17}$$

$$= -q^3\frac{\sqrt{\pi}}{(4\pi)^{\frac{3}{2}}}\Big(\frac{\pi}{8}\Big) \;+\; q^3\Big(\sqrt{\pi}\Big)\frac{1}{(4\pi)^{\frac{3}{2}}}\Big(\frac{3\pi}{8}\Big) = \frac{q^3}{32} \tag{B.18}$$

Remember from B.9, that this is the value of $S^{ij}q_iq_j$.
Using the last result, the system B.9,B.8 can be finally solved leading to

$$\begin{cases} 0 = 3a + bq^2 \\ \frac{q^3}{32} = aq^2 + bq^4 \end{cases} \quad\Longrightarrow\quad a = -\frac{q}{64} \quad b = \frac{3}{q\,64} \tag{B.19}$$

This means that the tensor S^{ij} can be expressed as

$$S^{ij} = a\delta^{ij} + br^ir^j = -\frac{q}{64}\delta^{ij} + \frac{3}{64}\frac{q^iq^j}{q} \tag{B.20}$$

Appendix C

The gamma function and the measure of a unit sphere

Definition and properties of gamma function

The gamma function defined as $\Gamma(n)$ is an extension of the factorial to complex and real number arguments.
It is related to the factorial by

$$\Gamma(n) = (n-1)! \quad , \quad n \in \mathcal{N} \tag{C.1}$$

For an arbitrary $n \in \mathcal{C}$, the gamma function can be defined as a definite integral

$$\Gamma(n) = \int_0^{+\infty} dt t^{n-1} e^{-t} \quad , \quad n \in Re(n) \tag{C.2}$$

which reduces to C.1 for natural n, after integrations by parts.
As for C.2, it is analytic eveywhere expect for $n = 0, -1, -2, \ldots.$
Among its remarkable properties we list the following identities

$$\Gamma(1+n) = n\Gamma(n) \tag{C.3}$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{sin(\pi n)} \tag{C.4}$$

$$\Gamma(2n) = (2\pi)^{-1/2} 2^{2n-1/2} \Gamma(n)\Gamma(n+1/2) \tag{C.5}$$

where the last identity goes under the name of Legendre duplication formula.

The measure of a unit d-dimensional sphere

Let's assume we are interested in the evaluation of $\Omega_d = \{\vec{x} \in \mathcal{R}^{d+1} \, , \, ||\vec{x}|| = 1\}$.
Let's start from the following simple relation

$$(\sqrt{\pi})^d = \left(\int_{-\infty}^{+\infty} dx \, e^{-x^2} \right)^d \tag{C.6}$$

which holds for arbitrary d and comes from the known Gaussian integral.
We rewrite C.6 as d dimensional integral

$$\int d^d x e^{-||x||^2} = \int d\Omega_d \int_0^{+\infty} dr \, r^{d-1} e^{-r^2} \tag{C.7}$$

$$\Omega_d \int_0^{+\infty} dr \, r^{d-1} e^{-r^2} = \pi^{d/2} \tag{C.8}$$

With a simple algebraic operation, in equation (C.8) we can recognize the definition of a Gamma function, from which the final result

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{C.9}$$

Bibliography

[1] B. P. Abbott et al. LIGO: The Laser interferometer gravitational-wave observatory. *Rept. Prog. Phys.*, 72:076901, 2009.

[2] B. P. Abbott et al. GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence. *Phys. Rev. Lett.*, 116(24):241103, 2016.

[3] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.*, 116(6):061102, 2016.

[4] B. P. Abbott et al. Observing gravitational-wave transient GW150914 with minimal assumptions. *Phys. Rev.*, D93(12):122004, 2016. [Addendum: Phys. Rev.D94,no.6,069903(2016)].

[5] B. P. Abbott et al. Tests of general relativity with GW150914. *Phys. Rev. Lett.*, 116(22):221101, 2016.

[6] B.. P.. Abbott et al. GW170608: Observation of a 19-solar-mass Binary Black Hole Coalescence. *Astrophys. J.*, 851(2):L35, 2017.

[7] B. P. Abbott et al. GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence. *Phys. Rev. Lett.*, 119(14):141101, 2017.

[8] Benjamin P. Abbott et al. GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2. *Phys. Rev. Lett.*, 118(22):221101, 2017.

[9] B.P. Abbott et al. GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral. *Phys. Rev. Lett.*, 119(16):161101, 2017.

[10] Mario Argeri and Pierpaolo Mastrolia. Feynman Diagrams and Differential Equations. *Int. J. Mod. Phys.*, A22:4375–4436, 2007.

[11] Stanislav Babak, Jonathan R. Gair, Antoine Petiteau, and Alberto Sesana. Fundamental physics and cosmology with LISA. *Class. Quant. Grav.*, 28:114001, 2011.

[12] Zvi Bern, John Joseph M. Carrasco, and Henrik Johansson. Perturbative Quantum Gravity as a Double Copy of Gauge Theory. *Phys. Rev. Lett.*, 105:061602, 2010.

[13] Laura Bernard, Luc Blanchet, Guillaume Faye, and Tanguy Marchand. Center-of-Mass Equations of Motion and Conserved Integrals of Compact Binary Systems at the Fourth Post-Newtonian Order. *Phys. Rev.*, D97(4):044037, 2018.

[14] Luc Blanchet. Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries. *Living Rev. Rel.*, 17:2, 2014.

[15] Luc Blanchet and Thibault Damour. Tail Transported Temporal Correlations in the Dynamics of a Gravitating System. *Phys. Rev.*, D37:1410, 1988.

[16] A. Buonanno and T. Damour. Effective one-body approach to general relativistic two-body dynamics. *Phys. Rev.*, D59:084006, 1999.

[17] Alessandra Buonanno and Thibault Damour. Transition from inspiral to plunge in binary black hole coalescences. *Phys. Rev.*, D62:064015, 2000.

[18] K. G. Chetyrkin, Johann H. Kuhn, P. Mastrolia, and Christian Sturm. Heavy-quark vacuum polarization: First two moments of the O(alpha**3(s) n**2(f)) contribution. *Eur. Phys. J.*, C40:361–366, 2005.

[19] K. G. Chetyrkin and F. V. Tkachov. Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops. *Nucl. Phys.*, B192:159–204, 1981.

[20] Yi-Zen Chu. The n-body problem in General Relativity up to the second post-Newtonian order from perturbative field theory. *Phys. Rev.*, D79:044031, 2009.

[21] Thibault Damour. Coalescence of two spinning black holes: an effective one-body approach. *Phys. Rev.*, D64:124013, 2001.

[22] Thibault Damour. Coalescence of two spinning black holes: an effective one-body approach. *Phys. Rev.*, D64:124013, 2001.

[23] Thibault Damour. Introductory lectures on the Effective One Body formalism. *Int. J. Mod. Phys.*, A23:1130–1148, 2008.

[24] Thibault Damour. Introductory lectures on the Effective One Body formalism. *Int. J. Mod. Phys.*, A23:1130–1148, 2008.

[25] Thibault Damour and Gilles Esposito-Farese. Testing gravity to second post-Newtonian order: A Field theory approach. *Phys. Rev.*, D53:5541–5578, 1996.

[26] Thibault Damour, Bala R. Iyer, Piotr Jaranowski, and B. S. Sathyaprakash. Gravitational waves from black hole binary inspiral and merger: The Span of third postNewtonian effective one-body templates. *Phys. Rev.*, D67:064028, 2003.

[27] Thibault Damour, Bala R. Iyer, and Alessandro Nagar. Improved resummation of post-Newtonian multipolar waveforms from circularized compact binaries. *Phys. Rev.*, D79:064004, 2009.

[28] Thibault Damour and Piotr Jaranowski. Four-loop static contribution to the gravitational interaction potential of two point masses. *Phys. Rev.*, D95(8):084005, 2017.

[29] Thibault Damour, Piotr Jaranowski, and Gerhard Schaefer. On the determination of the last stable orbit for circular general relativistic binaries at the third postNewtonian approximation. *Phys. Rev.*, D62:084011, 2000.

[30] Thibault Damour, Piotr Jaranowski, and Gerhard Schfer. Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems. *Phys. Rev.*, D89(6):064058, 2014.

[31] Vanessa C. de Andrade, Luc Blanchet, and Guillaume Faye. Third postNewtonian dynamics of compact binaries: Noetherian conserved quantities and equivalence between the harmonic coordinate and ADM Hamiltonian formalisms. *Class. Quant. Grav.*, 18:753–778, 2001.

[32] Albert Einstein, L. Infeld, and B. Hoffmann. The Gravitational equations and the problem of motion. *Annals Math.*, 39:65–100, 1938.

[33] R.P. Feynman, F.B. Morinigo, W.G. Wagner, and B. Hatfield. *Feynman Lectures on Gravitation*. Advanced book program. Addison-Wesley, 1995.

[34] Stefano Foffa, Pierpaolo Mastrolia, Riccardo Sturani, and Christian Sturm. Effective field theory approach to the gravitational two-body dynamics, at fourth post-Newtonian order and quintic in the Newton constant. *Phys. Rev.*, D95(10):104009, 2017.

[35] Stefano Foffa and Riccardo Sturani. Effective field theory calculation of conservative binary dynamics at third post-Newtonian order. *Phys. Rev.*, D84:044031, 2011.

[36] Howard Georgi. An Effective Field Theory for Heavy Quarks at Low-energies. *Phys. Lett.*, B240:447–450, 1990.

[37] James B. Gilmore and Andreas Ross. Effective field theory calculation of second post-Newtonian binary dynamics. *Phys. Rev.*, D78:124021, 2008.

[38] Walter D. Goldberger. Les Houches lectures on effective field theories and gravitational radiation. In *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime Les Houches, France, July 31-August 25, 2006*, 2007.

[39] Walter D. Goldberger and Ira Z. Rothstein. An Effective field theory of gravity for extended objects. *Phys. Rev.*, D73:104029, 2006.

[40] A. G. Grozin. Integration by parts: An Introduction. *Int. J. Mod. Phys.*, A26:2807–2854, 2011.

[41] A. G. Grozin. Massless two-loop self-energy diagram: Historical review. *Int. J. Mod. Phys.*, A27:1230018, 2012.

[42] Andrey Grozin. Lectures on QED and QCD. In *3rd Dubna International Advanced School of Theoretical Physics Dubna, Russia, January 29-February 6, 2005*, pages 1–156, 2005.

[43] Suraj N. Gupta. Gravitation and electromagnetism. *Phys. Rev.*, 96:1683–1685, Dec 1954.

[44] Johannes M. Henn. Multiloop integrals in dimensional regularization made simple. *Phys. Rev. Lett.*, 110:251601, 2013.

[45] Johannes M. Henn. Lectures on differential equations for Feynman integrals. *J. Phys.*, A48:153001, 2015.

[46] Sascha Husa. Michele Maggiore: Gravitational waves. Volume 1: Theory and experiments. *Gen. Rel. Grav.*, 41:1667–1669, 2009.

[47] C. Itzykson and J. B. Zuber. *Quantum Field Theory*. International Series In Pure and Applied Physics. McGraw-Hill, New York, 1980.

[48] John David Jackson. *Classical Electrodynamics*. Wiley, 1998.

[49] H. Kleinert and V. Schulte-Frohlinde. *Critical properties of phi**4-theories*. 2001.

[50] Kostas D. Kokkotas and Bernd G. Schmidt. Quasinormal modes of stars and black holes. *Living Rev. Rel.*, 2:2, 1999.

[51] Barak Kol and Michael Smolkin. Classical Effective Field Theory and Caged Black Holes. *Phys. Rev.*, D77:064033, 2008.

[52] A. V. Kotikov. Differential equation method: The Calculation of N point Feynman diagrams. *Phys. Lett.*, B267:123–127, 1991. [Erratum: Phys. Lett.B295,409(1992)].

[53] A. V. Kotikov and S. Teber. New results for a two-loop massless propagator-type Feynman diagram. 2016.

[54] L.D. Landau and .. . *The Classical Theory of Fields*. Course of theoretical physics. Butterworth-Heinemann, 1975.

[55] S. Laporta. High precision calculation of multiloop Feynman integrals by difference equations. *Int. J. Mod. Phys.*, A15:5087–5159, 2000.

[56] S. Laporta. Calculation of master integrals by difference equations. *Phys. Lett.*, B504:188–194, 2001.

[57] R. N. Lee. Space-time dimensionality D as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to D. *Nucl. Phys.*, B830:474–492, 2010.

[58] Luis Lehner. Numerical relativity: A Review. *Class. Quant. Grav.*, 18:R25–R86, 2001.

[59] M. Maggiore. *A Modern Introduction to Quantum Field Theory*. Oxford Master Series in Physics. OUP Oxford, 2004.

[60] Michele Maggiore. Nonlocal Infrared Modifications of Gravity. A Review. *Fundam. Theor. Phys.*, 187:221–281, 2017.

[61] P. Mastrolia and E. Remiddi. Precise evaluation of the electron (g-2) at four loops: The algebraic way. *Nucl. Phys. Proc. Suppl.*, 89:76–81, 2000. [,76(2000)].

[62] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. Number pt. 3 in Gravitation. W. H. Freeman, 1973.

[63] Rafael A. Porto and Riccardo Sturani. Scalar gravity: Post-Newtonian corrections via an effective field theory approach. In *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime Les Houches, France, July 31-August 25, 2006*, 2007.

[64] M. Punturo et al. The third generation of gravitational wave observatories and their science reach. *Class. Quant. Grav.*, 27:084007, 2010.

[65] Pierre Ramond. *Field Theory : A Modern Primer (Frontiers in Physics Series, Vol 74)*. Westview Press, 2 edition, 2001.

[66] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014.

[67] A. V. Smirnov. Algorithm FIRE – Feynman Integral REduction. *JHEP*, 10:107, 2008.

[68] Gerard 't Hooft and M. J. G. Veltman. Regularization and Renormalization of Gauge Fields. *Nucl. Phys.*, B44:189–213, 1972.

[69] Andrea Taracchini et al. Effective-one-body model for black-hole binaries with generic mass ratios and spins. *Phys. Rev.*, D89(6):061502, 2014.

[70] O. V. Tarasov. Connection between Feynman integrals having different values of the space-time dimension. *Phys. Rev.*, D54:6479–6490, 1996.

[71] F. V. Tkachov. A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions. *Phys. Lett.*, 100B:65–68, 1981.

[72] A. von Manteuffel and C. Studerus. Reduze 2 - Distributed Feynman Integral Reduction. 2012.

[73] Robert M. Wald. *General Relativity*. 1984.

[74] Steven Weinberg. Photons and gravitons in s-matrix theory: Derivation of charge conservation and equality of gravitational and inertial mass. *Phys. Rev.*, 135:B1049–B1056, Aug 1964.

[75] Clifford M. Will. The Confrontation between General Relativity and Experiment. *Living Rev. Rel.*, 17:4, 2014.