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A rigorous derivation of the Bogoliubov-Gross-Pitaevskii equation for superfluids

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Introduction

In the present Thesis, a rigorous derivation of the Bogoliubov-Gross-Pitaevskii (BGP) equation, which completely describes the dynamics of the condensed phase of a boson fluid at zero temperature [7], is presented. It must be emphasized that this is a very active field of research, catalysed by recent experimental advancements in the analysis of condensation for interacting bosons, and where fundamental results on the mathematical side have been reached quite recently [13]. The problem is dealt within the formalism of second quantization, first introduced by Bogoliubov [1], with an external trapping potential working as a vessel, the well amplitude size ruling the large size limit of the system (at a fixed density).

To begin with, in the **First Chapter**, the minimal necessary scaling hypotheses are discussed and compared with both the theoretical and the experimental ones existing in the literature [8]. This is relevant in a problem where the existence of an effective equation in the thermodynamic limit almost always requires to let some physical parameters characterizing the system (e.g. the range of the two body potential) to depend on the size of the latter.
Once determined the right scaling regime, one is left with a problem in dimensionless form where, essentially, the dynamics of the boson quantum field is proven to be close, up to a small remainder term whose norm is vanishing in the thermodynamic limit, to that of a problem where the two-body potential is delta-like, multiplied by a coupling constant that is explicitly computed in terms of all the parametres of the system (such as number density, two-body interaction, and so on). On the other hand, the fundamental boson commutation rules satisfied by the rescaled quantum field are of the semi-classical form, with a commutator that vanishes in the large size limit, as hypothesised by Bogoliubov [1].

At this stage, by analogy with what has been done for the finite version of the problem, i.e. for Bose-Hubbard models [14], in the **Second Chapter** we take the expectation of the quantum field on a coherent state distributed according to a quantum invariant Gaussian thermal measure. Such an expectation, or Wick symbol, defines the scalar field that satisfies the BGP equation in a suitable infrared limit, as first conceived, to our knowledge, by Langer [3, 4]. The procedure requires the introduction of a suitable ultraviolet cut-off regularisation of the field, in such a way that one first works on a finite model, reducing the confrontation between operator and scalar dynamics to the reconstruction of the scalar BGP equation à la Galerkin.
Finally, the convergence of the time-dependent Wick symbol defined above to the solution of the scalar BGP equation, in measure norm, is proven; specifically, the bounding constant of the distance is found to be depending linearly on time: such a dependence represents an improvement of the existing estimates (displaying instead an exponential dependence [12]). Further, it is shown that such constant goes to zero in the thermodynamic limit, thus ensuring the exact convergence to BGP scalar dynamics at zero temperature, though a condition needs to be satisfied by the trapping potential.

Conventions and notations

Hereby a series of conventions and notations used through all of the present work is presented.

1. The vector space $\mathbb{R}^d$ will be assumed equipped with the standard Lebesgue measure

$$\mathrm{d}^d x := \mathrm{d}\lambda_d(x).$$

2. Given a measure space (X, σ, μ) with sigma-algebra σ and measure μ, we will denote as $L^2(X, \mathrm{d}\mu)$ the space of complex-valued square-summable functions on X, that is

$$L^2(X, \mathrm{d}\mu) := \left\{ f : X \to \mathbb{C} \ : \ \|f\|_\mu^2 := \int_X |f|^2 \, \mathrm{d}\mu < +\infty \right\}.$$

In particular, since by point 1. the measure on $\mathbb{R}^d$ is the standard one, we will denote $L^2(\mathbb{R}^d, \mathrm{d}\lambda_d)$ simply as $L^2(\mathbb{R}^d)$.

3. The above $L^2(\mathbb{R}^d)$ admits as a dense subset the Schwartz space

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{C}) \ : \ \forall \alpha, \beta, \ \|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} \left| x_1^{\alpha_1} \cdots x_d^{\alpha_d} \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}(x) \right| < +\infty \right\}.$$

The semi-norms $\|\cdot\|_{\alpha,\beta}$ induce on $\mathcal{S}(\mathbb{R}^d)$ a stronger topology than the standard norm one given by the immersion in $L^2(\mathbb{R}^d)$, so that its topological dual, the space of tempered distributions, i.e. the space of linear continuous functionals on $\mathcal{S}(\mathbb{R}^d)$, contains $(L^2(\mathbb{R}^d))'$ and is denoted $\mathcal{S}'(\mathbb{R}^d)$.

4. We will use a natural system of units, by imposing the reduced Planck and Boltzmann constants to satisfy

$$\hbar = 1 \qquad\qquad k_b = 1.$$

5. For the QFT Hilbert space $\mathcal{H}$, Dirac notation will be used, denoting vectors (kets), linear functionals (bras) and their coupling (bracket) as

$$\mathcal{H} \ni |\varphi\rangle \qquad \wedge \qquad \langle\chi| \in \mathcal{H}' \simeq \mathcal{H} \qquad \Longrightarrow \qquad \langle\chi|\varphi\rangle \in \mathbb{C}.$$

Furthermore, to distinguish graphically between Wick symbols (intended as phase space functions) and linear operators on $\mathcal{H}$, the latter will be written in a boldfaced sans-serif fashion. For instance the field operator and the order parametre (i.e. its coherent expectation) differ as

$$\mathsf{\Psi} \qquad\qquad \psi := \langle\alpha| \, \mathsf{\Psi} \, |\alpha\rangle \, .$$

6. Given a generic Hilbert space $\mathcal{H}$, we will indicate by

$$\mathsf{L}(\mathcal{H}) := \left\{ \mathsf{A} : D(\mathsf{A}) \to \mathcal{H} \ : \ D(\mathsf{A}) \subseteq \mathcal{H}, \ \overline{D(\mathsf{A})} = \mathcal{H} \right\}$$

the space of densely-defined linear operators over $\mathcal{H}$.

Chapter 1

Fundamental hypotheses and scaling

In this first chapter of the present work, a general class of systems of trapped bosons interacting between themselves via a mean field Hartree-type potential v will be presented through the formalism of second quantisation, instead of the quantum many-body setting employed, for instance, in [9] and in [5].
Secondly, the relevant couplings of the aforementioned bosonic Quantum Field Theory (QFT) will be introduced and their physical significance will be analysed.
Eventually, a scaling of the couplings of the theory will be devised via algebraic adimensionalisation of the field operators and physical insight; this scaling will be employed to prove the convergence of operator Hartree dynamics to the operator Bogoliubov-Gross-Pitaevskii (BGP) equation

$$i\frac{\partial \Phi}{\partial \tau} = \left(\mathsf{k} + \gamma \Phi^\dagger \Phi\right) \Phi,$$

with k the (adimensionalised) single-particle Hamiltonian and γ the so called Gross-Pitaevskii coupling constant, in the thermodynamic limit, amounting to $N \to \infty$, $\rho = N/V = \text{const.}$ given the boson number N and volume V of the system.

1.1 Bosonic Quantum Field theory

In second quantisation, the information about a system of non-relativistic interacting spinless bosons in dimension d is encoded in:

1. an Hilbert space $\mathcal{H}$;

2. a strongly continuous unitary representation U on $\mathcal{H}$ of the Euclidean group[1] $\mathbb{E}(d)$ for space translations and rotations:

$$\mathsf{U} : \mathbb{E}(d) \to \mathsf{L}(\mathcal{H}) \qquad (c, R) \longmapsto \mathsf{U}(c, R) = \exp\left(ic \cdot \mathsf{P}\right)\mathsf{U}(0, R) \qquad \mathsf{U}(c, R)^{-1} = \mathsf{U}(c, R)^\dagger$$

3. a strongly continuous unitary $\mathbb{R}$-representation of time translations, given by a densely defined positive Hamiltonian H

$$\mathsf{V} : \mathbb{R} \to \mathsf{L}(\mathcal{H}) \qquad \mathsf{V}(t) = \exp\left(-it\mathsf{H}\right);$$

4. an algebra of *field operators* indicised over spacetime $\{\Psi(t, x), (t, x) \in \mathbb{R}^{d+1}\}$ satisfying the the *equal time commutation relations* (ETCR) and transforming covariantly under $\mathbb{E}(d)$

$$\left[\Psi(t, x), \Psi^\dagger(t, y)\right] = \delta^d(x - y) \qquad \mathsf{U}(c, R)\Psi(t, x)\mathsf{U}(c, R)^\dagger = \Psi(t, Rx + c)$$

where δ^d is the d-dimensional Dirac delta distribution;

[1]This is the group of space isometries, given by the semi-direct product $\mathbb{R}^d \rtimes SO(d)$.

5. a unique state $|\Omega\rangle \in \mathcal{H}$, called the *vacuum* of the theory, invariant under spacetime traslations:

$$\mathsf{H}\,|\Omega\rangle = 0 \qquad \wedge \qquad \mathsf{U}(c,R)\,|\Omega\rangle = |\Omega\rangle\,.$$

Remark. As suggested by the ETCRs at point 4 of the previous list, $\Psi(x)$ is localised at $x \in \mathbb{R}^d$; more rigorously, Ψ is an operator valued tempered distribution, that is

$$\Psi \in \mathcal{S}'(\mathbb{R}^d) \otimes \mathsf{L}(\mathcal{H}) \qquad\qquad \mathsf{L}(\mathcal{H}) \ni \Psi(f;t) = \int_{\mathbb{R}^d} \bar{f}(x)\Psi(t,x)\,\mathrm{d}^d x, \tag{1.1}$$

for any smearing functions f in Schwartz[2] space $\mathcal{S}(\mathbb{R}^d)$, so that the ETCRs read

$$\left[\Psi(f;t), \Psi^\dagger(g;t)\right] = \langle f, g\rangle_{L^2(\mathbb{R}^d)} \qquad\qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

In the following exposition, if not otherwise specified, in order to simplify the notation, we will commit a slight abuse of notation by treating $\Psi(t,x)$ as an operator on $\mathcal{H}$.

1.2 Hartree Hamiltonian and normal mode decomposition

We will consider the following Hartree Hamiltonian of identical, spinless, non-relativistic, trapped, interacting bosons of mass m:

$$\mathsf{H} := \int_{\mathbb{R}^d} \Psi^\dagger(x)\mathsf{h}(x)\Psi(x)\,\mathrm{d}^d x + \frac{1}{2}\int_{\mathbb{R}^{2d}} \Psi^\dagger(x)\Psi^\dagger(y)v(\|x-y\|)\Psi(y)\Psi(x)\,\mathrm{d}^d x\mathrm{d}^d y, \tag{1.2}$$

where

$$\mathsf{L}(L^2(\mathbb{R}^d)) \ni \mathsf{h}(x) := -\frac{1}{2m}\nabla^2 + u(x)$$

is the single-particle Hamiltonian and u and v are, respectively, the trapping and the spherically-symmetric interaction potential. The functional structure of these potentials will be expressed later in the chapter on the basis of a few generic physical assumptions.
It is easy to check, with the help of the ETCRs, that the field operator satisfies the following Heisenberg equation.

$$i\frac{\partial\Psi}{\partial t} = [\Psi, \mathsf{H}] = \left(\mathsf{h} + v * \Psi^\dagger\Psi\right)\Psi, \tag{1.3}$$

where, as usual, $*$ denotes convolution.

Remark. This equation may be thought as a non-linear deformation of the second-quantised Schrödinger equation: indeed the latter describes non-interacting bosons in an external potential.

Although (1.3) may seem daunting in terms of integrability, it still admits a conserved charge, namely the *number operator*.

Definition 1.1. The number operator N is defined as

$$\mathsf{N} := \int_{\mathbb{R}^d} \Psi^\dagger(x)\Psi(x)\,\mathrm{d}^d x \tag{1.4}$$

Proposition 1.1. *The number operator is preserved by time evolution.*

$$\dot{\mathsf{N}}(t) = -i\,[\mathsf{N}(t), \mathsf{H}] \equiv 0$$

[2]This is the space of rapidly decreasing complex-valued function in $\mathbb{R}^d$; its topological dual is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

Proof. It suffices to notice that ETCRs imply, for all $(t,x) \in \mathbb{R}^{d+1}$,

$$\left[\mathsf{N}(t), \Psi^\dagger(t,x)\right] = \Psi^\dagger(t,x) \qquad \wedge \qquad [\mathsf{N}(t), \Psi(t,x)] = -\Psi(t,x),$$

hence N commutes with any operator containing an equal number of Ψs and $\Psi^\dagger$s, in particular the Hamiltonian H. □

Remark. Indeed, N is the Nöther charge associated to the global $U(1)$ internal symmetry of the Hamiltonian

$$\Psi \longmapsto e^{i\alpha}\Psi.$$

As usual, we may expand the field operators in terms of an orthonormal basis of the single particle Hilbert space $L^2(\mathbb{R}^d)$: this would enable us to express the field algebra representation in terms of another one. In particular we choose the normal mode decomposition, that permits us to diagonalise the quadratic part of the Hamiltonian, and since it is physically meaningful.
Suppose $\mathsf{h} \in \mathsf{L}(L^2(\mathbb{R}^d))$ to have a completely discreet spectrum $\sigma(\mathsf{h}) \simeq \mathbb{N}^d$, meaning there exists a bijective map

$$\omega : \mathbb{N}^d \longrightarrow \sigma(\mathsf{h}) \subset \mathbb{R} \qquad\qquad k := (k_1, \dots, k_d) \longmapsto \omega_k$$

indicising single-particle energy eigenvalues. Let $\{\phi_k\}_{k\in\mathbb{N}^d} \subset L^2(\mathbb{R}^d)$ be the countable set of eigenfunctions of h, i.e.

$$\mathsf{h}\phi_k = \omega_k\phi_k,$$

then the following statement holds.

Proposition 1.2. *Consider the normal mode expansion*

$$\Psi(t,x) = \sum_k \mathsf{a}_k(t)\phi_k(x) \qquad\qquad \mathsf{a}_k(t) = \int_{\mathbb{R}^d} \bar{\phi}_k(x)\Psi(t,x)\,\mathrm{d}^d x \equiv \Psi(\phi_k;t); \tag{1.5}$$

then, for any time, the operators $\mathsf{a}_k(t)$ satisfy the canonical commutation relations (CCR) algebra

$$[\mathsf{a}_k(t), \mathsf{a}_l^\dagger(t)] = \delta_{kl} \qquad\qquad [\mathsf{a}_k(t), \mathsf{a}_l(t)] = [\mathsf{a}_k^\dagger(t), \mathsf{a}_l^\dagger(t)] = 0. \tag{1.6}$$

Proof. Multiplying equation 1.5 by $\phi_l^*(x)$ and integrating over space, we obtain

$$\int_{\mathbb{R}^d} \Psi(t,x)\bar{\phi}_l(x)\,\mathrm{d}^d x = \sum_k \mathsf{a}_k(t)\langle\phi_l, \phi_k\rangle = \mathsf{a}_l(t),$$

where the last passage involves the orthonormalcy of the basis. Now, we may compute

$$[\mathsf{a}_k(t), \mathsf{a}_l^\dagger(t)] = \int_{\mathbb{R}^{2d}} [\Psi(t,x), \Psi^\dagger(t,y)]\bar{\phi}_k(x)\phi_l(y)\,\mathrm{d}^d x\mathrm{d}^d y = \langle\phi_k, \phi_l\rangle = \delta_{kl},$$

using the ETCRs. □

Note. The multi-index sum in the above Proposition should be understood as the conventional

$$\sum_k \equiv \sum_{k\in\mathbb{N}^d} := \prod_{i=1}^d \sum_{k_i=0}^\infty;$$

instead, we define, for future use (in particular, in the next chapter), the following norm on multi-indices

$$|k| := \max_{i\in\{1,\dots,d\}} k_i$$

so that the set of multi-indices having norm lesser than a positive integer Λ is an hypercube of side Λ, with volume

$$\sum_k^\Lambda 1 \equiv \sum_{k\,:\,|k|<\Lambda} 1 = \Lambda^d.$$

This is better than the usual norm $|k|_{std} := k_1 + \cdots + k_d$, for which the same set is (isomorphic to) a standard simplex, with a dimension-dependent combinatorial factor appearing in the volume.
Notice that both norms coincide for $d = 1$.

Remark. The physical interpretation of the CCR algebra operators is clear: $\mathsf{a}_k^\dagger$ creates a bosonic excitation in the k-th single-particle energy level, when applied to the QFT vacuum $|\Omega\rangle$, and correspondingly a_k destroys it.
Further, the problem of defining an uncountable number of operators satisfying the ETCRs is reduced to that of giving an Hilbert space representation of the CCR.

The Hamiltonian H, expressed in terms of Dirac operators a, has the following form,

$$\mathsf{H} = \sum_k \omega_k \mathsf{a}_k^\dagger \mathsf{a}_k + \frac{1}{2} \sum_{klmn} v_{klmn} \mathsf{a}_k^\dagger \mathsf{a}_l^\dagger \mathsf{a}_m \mathsf{a}_n, \tag{1.7}$$

where the coefficients v_{klmn} are defined as

$$v_{klmn} := \int_{\mathbb{R}^{2d}} v(\|x-y\|)\bar{\phi}_k(x)\bar{\phi}_l(y)\phi_m(x)\phi_n(y)\,\mathrm{d}^d x \mathrm{d}^d y.$$

Remark. The coefficients v_{klmn} carry information of both the two-body interaction v and the external potential u, since they explicitly depend on the former and on the eigenfunctions of h, and, for H to be self-adjoint, they satisfy the following relation.

$$\bar{v}_{klmn} = v_{mnkl}$$

The number operator N is easily computed to be

$$\mathsf{N} = \sum_k \mathsf{a}_k^\dagger \mathsf{a}_k, \tag{1.8}$$

that is, the sum of the occupation numbers of each single-particle energy level.
Furthermore, Dirac operators satisfy the following Heisenberg equation.

$$i\dot{\mathsf{a}}_k(t) = i\frac{\mathrm{d}\mathsf{a}_k}{\mathrm{d}t} = \omega_k \mathsf{a}_k + \sum_{lmn} v_{(kl)mn} \mathsf{a}_l^\dagger \mathsf{a}_m \mathsf{a}_n, \tag{1.9}$$

with $v_{(kl)mn}$ the symmetrised coefficient

$$v_{(kl)mn} := \frac{v_{klmn} + v_{lkmn}}{2}.$$

1.3 Scaling of the couplings

We would like to analyse the behaviour of the above formulated bosonic Hartree QFT when the *thermodynamic limit*[3] ($N \to \infty$, with fixed density $\rho := N/V$) is taken; in particular, it will be shown that the equations of motion (1.3) converge to the field operator equations for bosonic Bogoliubov-Gross-Pitaevskii QFT.
In order to do so, we first need to explicit the couplings of the system, and then bind them to the number of bosons N through a scaling procedure.

[3]Actually, N will be defined a fortiori, in the second Chapter, as the expectation value of the number operator N over a certain (mixed) state ϱ, i.e. $N := \operatorname{tr} \mathsf{N}\varrho$.

1.3.1 Couplings of the system

Definition 1.2 (Trap)**.** We define the trapping potential u to be of the form

$$u(x) := \varepsilon_0 f(\|x\| / L)$$

with $f \in \mathcal{C}^2(\mathbb{R}^+)$ such that

$$f'(0) = 0 \qquad f''(0) > 0 \qquad f(r) \underset{r\to\infty}{\longrightarrow} +\infty.$$

Then ε_0 is to be interpreted as the *intensity* of the trap, whereas L is the *characteristic well length* of the system, allowing us to define the confinement volume V as

$$V = L^d.$$

Remark. The simplest example of trap to bear in mind is the isotropic harmonic one

$$f(r) = r^2 \qquad \Longrightarrow \qquad u(x) = \frac{1}{2} m\omega^2(L)\|x\|^2 = \frac{1}{2mL^2} f(\|x\| / L)$$

This is also of physical relevance because of the use of optical traps in the study and experiments about ultracold trapped bosons (see [7] for a review of the theory of condensation of confined bosons).

Definition 1.3 (Interaction)**.** The repulsive two-body interaction potential v is assumed to be of the form

$$v(\|x-y\|) := \varepsilon_1 g(\|x-y\| / r_0)$$

with $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$g(s) \underset{s\to\infty}{\longrightarrow} 0 \qquad \wedge \qquad 0 \neq g_0 := \int_{\mathbb{R}^d} g(\|\xi\|) \, \mathrm{d}^d\xi < \infty.$$

Here, ε_1 plays the role of interaction intensity, and r_0 is the characteristic length of interaction.

Remark. A simple example is given by a finite range g, that is

$$g(s) = 0 \qquad \forall s > 1,$$

in which case r_0 is interpreted as the finite range of interaction, and, denoting $\mathcal{B}^d$ the d-dimensional ball of radius 1,

$$g_0 = \int_{\mathcal{B}^d} g(\|\xi\|) \, \mathrm{d}^d\xi.$$

Then, assigning a scaling to these parametres corresponds to giving a set of functions

$$\varepsilon_0(N) \quad \varepsilon_1(N) \quad r_0(N) \qquad\qquad \text{or, equivalently} \quad \varepsilon_0(L) \quad \varepsilon_1(L) \quad r_0(L)$$

obeying some constraints of geometrical or physical nature.

1.3.2 Adimensionalised QFT

In this section, we express all the relevant quantities in terms of dimensionless variables, in order to give some constraints on the scaling of the parametres introduced above.
Recall the single-particle time independent Schrödinger equation

$$\mathsf{h}\phi_k(x) = \left(-\frac{1}{2m}\nabla^2 + u(x)\right)\phi_k(x) = \omega_k\phi_k(x), \tag{1.10}$$

and define the following dimensionless quantities:

$$\xi := \frac{x}{L} \qquad \mathsf{k} := \frac{\mathsf{h}}{\varepsilon_0} \qquad \epsilon_k := \frac{\omega_k}{\varepsilon_0}.$$

Proposition 1.3. *Provided* h *satisfies equation (1.10), the adimensionalised Hamiltonian* k *satisfies the following eigenvalue problem:*

$$\mathsf{k}\varphi_k(\xi) = \left(-\frac{\mu^2}{2}\nabla_\xi^2 + f(\|\xi\|)\right)\varphi_k(\xi) = \epsilon_k\varphi_k(\xi)$$

where

$$\mu^2 := \frac{1}{m\varepsilon_0 L^2} \qquad \wedge \qquad \varphi_k(\xi) := L^{d/2}\phi_k(x).$$

Proof. The thesis is obtained by simply multiplying both sides of equation (1.10) by $L^{d/2}/\varepsilon_0$ and recalling the definition of u in terms of f. □

In order to set the field algebra in dimensionless form, we further rescale time and creation operators as follows,

$$\tau := \varepsilon_0 t \qquad \mathsf{b}_k := \frac{\mathsf{a}_k}{\sqrt{N}}$$

keeping in mind that $\hbar = 1$ and $N = \rho L^d$.

Proposition 1.4. *The rescaled field operator is defined as*

$$\Phi(\tau,\xi) := \sum_k \mathsf{b}_k(\tau)\varphi_k(\xi) = \frac{1}{\sqrt{\rho}}\Psi(t,x)$$

and obeys the following commutation relations

$$[\Phi(\tau,\xi),\Phi^\dagger(\tau,\eta)] = \frac{1}{N}\delta^d(\xi-\eta) \qquad [\mathsf{b}_k,\mathsf{b}_l^\dagger] = \frac{1}{N}\delta_{kl}$$

Proof. From the definitions of φ_k and b_k, it follows that

$$\Phi(\tau,\xi) := \sum_k \mathsf{b}_k(\tau)\varphi_k(\xi) = \sqrt{\frac{L^d}{N}}\sum_k \mathsf{a}_k(t)\phi_k(x) = \frac{1}{\sqrt{\rho}}\Psi(t,x).$$

Furthermore, the commutation relations for b_k are obtained by their definition and by CCRs, while

$$[\Phi(\tau,\xi),\Phi^\dagger(\tau,\eta)] = \frac{L^d}{N}[\Psi(t,x),\Psi^\dagger(t,y)] = \frac{L^d}{N}\delta^d(L(\xi-\eta)) = \frac{1}{N}\delta^d(\xi-\eta)$$

where in the second passage we used ETCRs and in the last one the scaling property of the Dirac distribution was employed. □

Remark. In the large N limit the field algebra is trivialised, i.e. all commutators vanish, a fact already pointed out, and employed, by N. Bogoliubov in [1].

The dimensionless field definition suggests the rescaling for the Hamiltonian written below.

$$\mathsf{K} := \frac{\mathsf{H}}{N\varepsilon_0}$$

Proposition 1.5. *The rescaled Hamiltonian has explicit form*

$$\mathsf{K} = \int_{\mathbb{R}^d}\Phi^\dagger(\xi)\mathsf{k}(\xi)\Phi(\xi)\,\mathrm{d}^d\xi + \frac{N}{2\varepsilon_0}\int_{\mathbb{R}^{2d}} v(L\,\|\xi-\eta\|)\Phi^\dagger(\xi)\Phi^\dagger(\eta)\Phi(\eta)\Phi(\xi)\,\mathrm{d}^d\xi\mathrm{d}^d\eta$$

and the field equations read

$$i\frac{\partial\Phi}{\partial\tau} = N[\Phi,\mathsf{K}]$$

Proof. The first term in the Hamiltonian is computed as

$$\frac{1}{N\varepsilon_0}\int_{\mathbb{R}^d} \Psi^\dagger(x)\mathsf{h}(x)\Psi(x)\, L^d\, \mathrm{d}^d\xi = \int_{\mathbb{R}^d} \Phi^\dagger(\xi)\mathsf{k}(\xi)\Phi(\xi)\,\mathrm{d}^d\xi;$$

the second one is similarly found by noting that

$$\frac{N}{2N^2\varepsilon_0}\Psi^\dagger(x)\Psi^\dagger(y)\Psi(y)\Psi(x)\, L^{2d}\,\mathrm{d}^d\xi\mathrm{d}^d\eta = \frac{N}{2\varepsilon_0}\Phi^\dagger(\xi)\Phi^\dagger(\eta)\Phi(\eta)\Phi(\xi)\,\mathrm{d}^d\xi\mathrm{d}^d\eta.$$

As for time evolution,

$$i\frac{\partial\Phi}{\partial\tau} = \frac{i}{\sqrt{\rho}\varepsilon_0}\frac{\partial\Psi}{\partial t} = \frac{1}{\sqrt{\rho}\varepsilon_0}[\Psi,\mathsf{H}] = N[\Phi,\mathsf{K}],$$

where in the second passage the equations of motion of Ψ were used. □

Using the commutation relations in Proposition 1.4, the equations of motion of Φ are explicitly computed to be

$$i\frac{\partial\Phi}{\partial\tau}(\tau,\xi) = N[\Phi,\mathsf{K}] = \left(\mathsf{k} + \frac{N\varepsilon_1}{\varepsilon_0}\int_{\mathbb{R}^d} g\left(\frac{L}{r_0}\|\xi-\eta\|\right)\Phi^\dagger(\tau,\eta)\Phi(\tau,\eta)\,\mathrm{d}^d\eta\right)\Phi(\tau,\xi) \tag{1.11}$$

Changing variable $\lambda(\eta) := L(\eta-\xi)/r_0$, and omitting time for simplicity, the second term in the parenthesis becomes

$$\frac{N\varepsilon_1}{\varepsilon_0}\int_{\mathbb{R}^d} g\left(\frac{L}{r_0}\|\xi-\eta\|\right)\Phi^\dagger(\eta)\Phi(\eta)\,\mathrm{d}^d\eta = \left(\frac{\varepsilon_1}{\varepsilon_0}\rho g_0 r_0^d\right)\Phi^\dagger\Phi(\xi) + \tag{1.12}$$

$$+\left(\frac{\varepsilon_1}{\varepsilon_0}\rho g_0 r_0^d\right)\int_{\mathbb{R}^d} g(\|\lambda\|)\left[\Phi^\dagger(\xi+r_0\lambda/L)\Phi(\xi+r_0\lambda/L) - \Phi^\dagger(\xi)\Phi(\xi)\right]\mathrm{d}^d\lambda \tag{1.13}$$

1.3.3 Scaling of the couplings and convergence to BGP

Definition 1.4. The dimensionless Gross-Pitaevskii constant is defined as

$$\gamma := \frac{\varepsilon_1}{\varepsilon_0}\rho g_0 r_0^d.$$

Proposition 1.6. *Suppose γ to be constant, r_0 to be of sublinear growth in L, and g to be of range 1; then (1.13) vanishes in the thermodynamic limit $L\to\infty$.*

Proof. From hypotheses

$$r_0(L)/L \underset{L\to\infty}{\longrightarrow} 0,$$

hence the integrand behaves as

$$\Phi^\dagger\Phi(\xi+r_0\lambda/L) - \Phi^\dagger\Phi(\xi) \underset{L\to\infty}{\sim} \frac{r_0}{L}\mathrm{d}(\Phi^\dagger\Phi)_\xi(\lambda) = \frac{r_0}{L}\lambda\cdot\nabla(\Phi^\dagger\Phi)(\xi).$$

Overestimating g by its sup over $\mathcal{B}^d$, we obtain the estimate below.

$$\left\|\int_{\mathbb{R}^d} g(\|\lambda\|)\left[\Phi^\dagger\Phi(\xi+r_0\lambda/L) - \Phi^\dagger\Phi(\xi)\right]\mathrm{d}^d\lambda\right\|_{op} \le \tag{1.14}$$

$$\le \int_{\mathbb{R}^d} g(\|\lambda\|)\left\|\Phi^\dagger\Phi(\xi+r_0\lambda/L) - \Phi^\dagger\Phi(\xi)\right\|_{op}\mathrm{d}^d\lambda \sim \tag{1.15}$$

$$\sim \frac{r_0}{L}\sup_{\mathcal{B}^d} g\int_{\mathbb{R}^d}\left\|\lambda\cdot\nabla(\Phi^\dagger\Phi)(\xi)\right\|_{op}\mathrm{d}^d\lambda \underset{L\to\infty}{\longrightarrow} 0. \tag{1.16}$$

In the previous calculation, $\|\cdot\|_{op}$ denotes the Frobenius norm on the space $\mathsf{L}(\mathcal{H})$ of operators defined on the QFT Hilbert space $\mathcal{H}$, and more rigorously the entire estimate should be done with a suitable smearing of $\Phi^\dagger\Phi$ with a function f in $\mathcal{S}(\mathbb{R}^d)$ or in $\mathcal{C}_c^\infty(\mathbb{R}^d)$ as illustrated in Equation (1.1). □

We now have obtained two relevant constraints by which we will enact the scaling of the couplings ε_0, ε_1, and r_0 of our theory. Indeed, for the previous Proposition to be valid and to ensure the stability of the single-particle Schrödinger equation, we require the following quantities to be constants.

$$\mu^2 = \frac{1}{m\varepsilon_0(L)L^2} \qquad \gamma = \frac{\varepsilon_1(L)}{\varepsilon_0(L)}\rho g_0 r_0^d(L) \tag{1.17}$$

The first one immediately implies that

$$\varepsilon_0(L) = \frac{1}{m\mu^2 L^2} \propto L^{-2} \propto N^{-2/d}. \tag{1.18}$$

Remark. Equation (1.18) implies that in the $N \to \infty$ limit the trap vanishes; secondly, it shows that the dependence $\varepsilon_0 \propto L^{-2}$ is a general dependence of the trapping potential intensity, not a peculiarity of the harmonic one.

Physically, we expect the intensity of the interaction potential to be independent of N, hence by requiring

$$\varepsilon_1(L) \equiv \varepsilon_1 \tag{1.19}$$

the second constraint gives the following scaling for the interaction range,

$$r_0(L) = \left(\frac{\gamma}{m\rho\mu^2\varepsilon_1 g_0}L^{-2}\right)^{1/d} \propto L^{-2/d} \propto N^{-2/d^2} \tag{1.20}$$

and, for fixed ε_1, the behavior of the scaling of the couple (r_0,ε_0) for different dimensions is graphically represented in Figure 1.1.

Remark (I)**.** Notice that the requirements of Proposition 1.6 are met by this scaling choice, r_0 being of sublinear growth in L.

Remark (II)**.** Although it might seem a bit unsettling that the range of interaction vanishes in the $L \to \infty$ limit, there is a nice intuitive explanation: any Fluid Dynamics model should in principle be obtained by carrying out a statistical average of the underlying microscopic theory on a mesoscopic scale, a procedure that leaves only a few relevant physical variables (viscosity, pressure, etc.); in our case, instead, we obtain an hydrodynamic equation, namely the BGP one, via a scaling procedure, hence, since there is no mesoscopic scale and all the information about the boson-boson interaction must be encoded in the intensity γ, the only way to integrate out r_0 is to make it vanish.

Remark (III)**.** For a rather thorough discussion of the various possible scaling hypotheses and their relevance in the obtainment of BGP as an effective equation in the thermodynamic limit, we refer the interested reader to the systematic work by A. Michelangeli in [8] and [9].

A direct consequence of Proposition 1.6 is the following Theorem.

Theorem 1.1. *The Hartree equation of motion for the dimensionless field operator* Φ *reduces, in the thermodynamic limit, to the operator Bogoliubov-Gross-Pitaevskii (BGP) equation.*

$$i\frac{\partial\Phi}{\partial\tau} = \left(\mathsf{k} + \gamma\Phi^\dagger\Phi\right)\Phi \tag{1.21}$$

Proof. It is sufficient to enforce the above mentioned scaling of the couplings in compliance to the hypotheses of Proposition 1.6. □

Remark. Although we have obtained an important result, namely that a large class of systems enters the BGP regime in the thermodynamic limit, we would like to show that, following Bogoliubov's approach in [1], actually the dynamics may be encoded in a yet-to-be-determined scalar field ψ obeying the scalar BGP equation

$$i\frac{\partial\psi}{\partial t} = \left(\mathsf{k} + \gamma|\psi|^2\right)\psi = \left(-\frac{\mu^2}{2m}\nabla^2 + f + \gamma|\psi|^2\right)\psi.$$

The determination of ψ and proof of the above claim is the subject of the next Chapter.

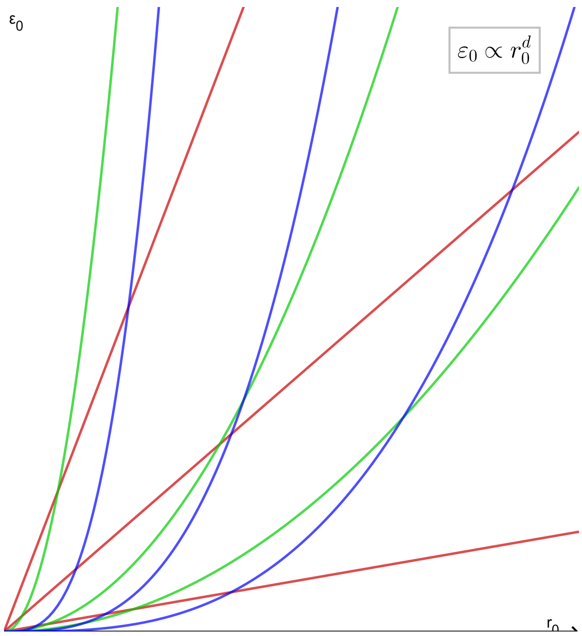

Figure 1.1: Couplings' qualitative flow $\varepsilon_0 = r_0^d$ (from top-right to bottom-left) in the thermodynamic limit for dimensions d of physical interest: 1 (red), 2 (green) and 3 (blue). The origin (quantum BGP) is an infrared fixed point.

Chapter 2

Convergence to scalar BGP

2.1 UV regularisation

Let us recall from the previous Chapter the time evolution equations obeyed by the normal modes a_k of the field operator Ψ,

$$\dot{\mathsf{a}}_k = -i\omega_k \mathsf{a}_k - i \sum_{lmn} v_{(kl)mn} \mathsf{a}_l^\dagger \mathsf{a}_m \mathsf{a}_n; \tag{2.1}$$

this is an infinite system of countably many strongly coupled operator differential equations.

We regularise these relations by inserting an ultra-violet (UV) cut-off $\Lambda \in \mathbb{N}$, requiring the sum to run only on multi-indices of norm lesser than Λ; then, employing the short-hand already introduced in a Note on Proposition 1.2, namely

$$\sum_k^\Lambda \equiv \sum_{k\,:\,|k|<\Lambda}$$

we obtain the following finite system of equations

$$\dot{\mathsf{a}}_{k,\Lambda} = -i\omega_k \mathsf{a}_{k,\Lambda} - i \sum_{lmn}^\Lambda v_{(kl)mn} \mathsf{a}_{l,\Lambda}^\dagger \mathsf{a}_{m,\Lambda} \mathsf{a}_{n,\Lambda} \qquad \forall k \;:\; |k| < \Lambda. \tag{2.2}$$

This is equivalent to UV truncating the normal modes expansion of the bosonic fields as

$$\Psi(t,x) = \sum_k \mathsf{a}_k(t)\phi_k(x) \qquad \Longrightarrow \qquad \Psi_\Lambda(t,x) = \sum_k^\Lambda \mathsf{a}_{k,\Lambda}(t)\phi_k(x)$$

with corresponding number operator

$$\mathsf{N}_\Lambda = \sum_k^\Lambda \mathsf{a}_{k,\Lambda}^\dagger \mathsf{a}_{k,\Lambda}$$

and Hamiltonian

$$\mathsf{H}_\Lambda = \sum_k^\Lambda \omega_k \mathsf{a}_{k,\Lambda}^\dagger \mathsf{a}_{k,\Lambda} + \frac{1}{2}\sum_{klmn}^\Lambda v_{klmn} \mathsf{a}_{k,\Lambda}^\dagger \mathsf{a}_{l,\Lambda}^\dagger \mathsf{a}_{m,\Lambda} \mathsf{a}_{n,\Lambda}.$$

Remark. Since the notation becomes heavy, the subscript Λ on the operators $\mathsf{a}_{k,\Lambda}$, marking these latter operators as satisfying the finite system of equations (2.2), will not be carried in the following Sections. Still, to remind us of the presence of the cut-off, all other relevant quantities will maintain the subscript.

The objective of regularising the system being achieved, we would like to reduce it to a set of ordinary differential equations: this will be done by the use of *coherent states*.

Definition 2.1. A QFT coherent state $|\alpha\rangle \in \mathcal{H}$ is a (possibly generalised) state that is eigenvector of all of the creation operators.

$$\mathsf{a}_k |\alpha\rangle = \alpha_k |\alpha\rangle \qquad \forall k \qquad \langle\alpha| \mathsf{a}_k^\dagger = \langle\alpha| \bar{\alpha}_k \tag{2.3}$$

Remark. This is simply a generalisation of the harmonic oscillator coherent states employed in Quantum Mechanics.

Remark. To our knowledge, Langer in [3] and [4] was the first to utilise these states in the topic of Bose-Einstein condensation and superfluids.

Consequently, following Langer, by taking the quantum expectation over $|\alpha\rangle$ of equations 2.2 and denoting by

$$a_k(t;\alpha) := \langle\alpha|\, \mathsf{a}_k(t)\, |\alpha\rangle$$

the expectation value of $\mathsf{a}_k(t)$ (recall that we are in Heisenberg picture), we obtain the following Cauchy problem

$$\dot{a}_k(t) = -i\omega_k a_k(t) - i\sum_{lmn}^{\Lambda} v_{(kl)mn}\, \langle\alpha|\, \mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)\, |\alpha\rangle \qquad \wedge \qquad a_k(0;\alpha) = \alpha_k$$

which would correspond to the scalar version of the Hartree equation, if not for the failure of the operator product to map onto the pointwise one, the latter being commutative.
Hence, in the next Section, we proceed to formalise Langer's intuition by using Wick Deformation Quantisation.

2.2 Wick star product

Definition 2.2 (Coherent phase space)**.** The vector space $\mathbb{C}^{\Lambda^d}$, equipped with linear coordinates

$$\alpha = (\alpha_k)_{|k|<\Lambda},$$

will be called *coherent phase space*, since its points constitute coherent state eigenvalues of the UV-regularised QFT.

Example. For $d = 2$ and $\Lambda = 2$, we obtain the coherent phase space $\mathbb{C}^4$ with complex coordinates

$$\alpha = (\alpha_{(0,0)}, \alpha_{(0,1)}, \alpha_{(1,0)}, \alpha_{(1,1)}).$$

Remark. Coherent phase space is actually just the UV-truncated Fock space of the QFT; indeed, in the UV limit $\Lambda \to \infty$ we obtain

$$\mathbb{C}^{\mathbb{N}^d} \qquad\qquad \alpha = (\alpha_k)_{k\in\mathbb{N}^d}$$

which is exactly the Fock space expressed in terms of the coherent state basis instead of the usual occupation number one.

Proposition 2.1. *Coherent phase space has a natural Kahler structure, given by the canonical form*

$$\Omega = i\sum_{k}^{\Lambda} \mathrm{d}\alpha_k \wedge \mathrm{d}\bar{\alpha}_k$$

and corresponding Poisson structure specified by the Poisson bivector

$$\Pi = -i\sum_{k}^{\Lambda} \frac{\partial}{\partial\alpha_k} \wedge \frac{\partial}{\partial\bar{\alpha}_k} = -i\sum_{k}^{\Lambda}\left(\frac{\partial}{\partial\alpha_k}\otimes\frac{\partial}{\partial\bar{\alpha}_k} - \frac{\partial}{\partial\bar{\alpha}_k}\otimes\frac{\partial}{\partial\alpha_k}\right)$$

Proof. It is a standard computation to show that Π satisfies Schouten's identity, so it is a genuine Poisson tensor. Then, since Π has maximal rank, Ω is the symplectic form obtained by lowering the indices of Π. □

Definition 2.3. We will denote the space of formal power series in α, $\bar{\alpha}$ as

$$\mathcal{P}_\Lambda(\alpha, \bar{\alpha}) := \left\{ c(\alpha, \bar{\alpha}) = \sum_{ij}^{\Lambda} \sum_{nm} c_{ij,nm} \bar{\alpha}_i^n \alpha_j^m \right\}.$$

Definition 2.4. The space of formal power series in terms of a, $\mathsf{a}^\dagger$ will be indicated with

$$\mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger) := \left\{ \mathsf{c}(\mathsf{a}, \mathsf{a}^\dagger) = \sum_{ij}^{\Lambda} \sum_{nm} c_{ij,nm} \mathsf{a}_i^{\dagger n} \mathsf{a}_j^m \right\}.$$

Remark. These definitions should be read in terms of multi-indices; for instance

$$\alpha_i^n = \prod_{k=0}^{d} \alpha_{i_k}^{n_k}.$$

Then, we are able to set a correspondence between these two spaces through *Wick's quantisation map.*

Proposition 2.2. *The map*

$$\mathsf{W} : \mathcal{P}_\Lambda(\alpha, \bar{\alpha}) \to \mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger) \qquad\qquad f \longmapsto \mathsf{W}[f]$$

characterised by the following properties:

1. $\mathsf{W}[1] = 1_{\mathcal{H}}$;
2. *linearity, that is* $\mathsf{W}[af + bg] = a\mathsf{W}[f] + b\mathsf{W}[g]$ *for all* $f, g \in \mathcal{P}_\Lambda(\alpha, \bar{\alpha})$ *and for all* $a, b \in \mathbb{C}$;
3. *normal form compatibility, i.e.* $\mathsf{W}[\bar{\alpha}_i^n \alpha_j^m] = \mathsf{W}[\alpha_j^m \bar{\alpha}_i^n] = \mathsf{a}_i^{\dagger n} \mathsf{a}_j^m$;

is a linear isomorphism and is called (in the deformation quantisation jargon) Wick's quantisation prescription.
Its inverse is given by the quantum expectation over coherent states, that is

$$\mathsf{W}^{-1} = D = \langle \ | \cdot | \ \rangle : \mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger) \to \mathcal{P}_\Lambda(\alpha, \bar{\alpha}) \qquad\qquad \mathsf{F} \longmapsto D[\mathsf{F}] = \langle \ | \mathsf{F} | \ \rangle$$

such that

$$D[\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)](\alpha, \bar{\alpha}) = \langle \alpha | \, \mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger) \, | \alpha \rangle \,,$$

and it is named dequantisation map.

Proof. The well-posedness and invertibility of W are manifest by the requirements 1-2-3 of the Proposition.
Furthermore,

$$\langle \alpha | \, \mathsf{a}_i^{\dagger n} \mathsf{a}_j^m \, | \alpha \rangle = \bar{\alpha}_i^n \alpha_j^m$$

hence

$$D \equiv \mathsf{W}^{-1} = \langle \ | \cdot | \ \rangle$$

by linearity of the quantum expectation and uniqueness of the inverse. □

Remark (I)**.** Essentially, W sends polynomials into polynomial operators expressed in normal form via the prescription $\alpha \to \mathsf{a}$, $\bar{\alpha} \to \mathsf{a}^\dagger$.

Remark (II)**.** Although in the present work polynomials will mostly suffice to achieve the desired results, for an analysis of more general algebras of observables over which to define the Wick map, in the sense of strict deformation quantisation, we invite the interested reader to refer to [10].

Example. The normal form specification is important, since, using the CCRs,

$$\mathsf{W}[|\alpha_k|^2] = \mathsf{a}_k^\dagger \mathsf{a}_k \qquad\qquad \mathsf{W}[|\alpha_k|^2 + 1] = \mathsf{a}_k^\dagger \mathsf{a}_k + 1 = \mathsf{a}_k \mathsf{a}_k^\dagger.$$

Definition 2.5. $\mathsf{W}[f]$ is called quantisation of the function $f \in \mathcal{P}_\Lambda(\alpha, \bar{\alpha})$ and will be denoted by the *sans-serif*

$$\mathsf{f} := \mathsf{W}[f].$$

On the other hand, $\mathsf{W}^{-1}[\mathsf{F}]$ is named Wick symbol of the operator F and it will be expressed with the corresponding *serif*-ed letter

$$F := \mathsf{W}^{-1}[\mathsf{F}].$$

We can now list some properties of the Wick mapping.

Lemma. *For any $x \in \mathbb{R}$, for any $\mathsf{F} \in \mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger)$*

$$e^{x\mathsf{a}_k}\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)e^{-x\mathsf{a}_k} = \mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger + x\mathrm{e}_k)$$

$$e^{-x\mathsf{a}_k^\dagger}\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)e^{x\mathsf{a}_k^\dagger} = \mathsf{F}(\mathsf{a} + x\mathrm{e}_k, \mathsf{a}^\dagger)$$

where e_k represent the unit vector in the k-th direction.

Proof. Since the transformation is linear and F can be developed in terms of $\mathsf{a}_i^{\dagger n}\mathsf{a}_j^m$, we ought to prove the thesis for these.
Then, the problem may be reduced to

$$e^{x\mathsf{a}_k}\mathsf{a}_i^{\dagger n}\mathsf{a}_j^m e^{-x\mathsf{a}_k} = \delta_{ik}\left(e^{x\mathsf{a}_k}\mathsf{a}_i^\dagger e^{-x\mathsf{a}_k}\right)^n \mathsf{a}_j^m.$$

Deriving with respect to x, and using CCRs, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{x\mathsf{a}_k}\mathsf{a}_k^\dagger e^{-x\mathsf{a}_k} = e^{x\mathsf{a}_k}[a_k, a_k^\dagger]e^{-x\mathsf{a}_k} \equiv 1.$$

Direct integration provides

$$e^{x\mathsf{a}_k}\mathsf{a}_k^\dagger e^{-x\mathsf{a}_k} = \mathsf{a}_k + x$$

whence

$$e^{x\mathsf{a}_k}\mathsf{a}_i^{\dagger n}\mathsf{a}_j^m e^{-x\mathsf{a}_k} = \delta_{ik}\left(\mathsf{a}_k^\dagger + x\right)^n \mathsf{a}_j^m.$$

As for the second identity, it is obtained from the first by the substitution $\mathsf{a}_k^\dagger \to \mathsf{a}_k$ and $x \to -x$. $\square$

Remark. We have just proved a special case of the more general Baker-Campbell-Hausdorff (BCH) formula.

Combining the previous Lemma with the Wick map, we arrive to the following Corollary.

Corollary. $\mathsf{F} \in \mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger)$ *and its Wick symbol $F \in \mathcal{P}_\Lambda(\alpha, \bar{\alpha})$ are subject to the equalities below.*

$$\langle\alpha|\, e^{x\mathsf{a}_k}\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)\,|\alpha\rangle = e^{x\alpha_k}F(\alpha, \bar{\alpha} + x\mathrm{e}_k)$$

$$\langle\alpha|\, \mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)e^{x\mathsf{a}_k^\dagger}\,|\alpha\rangle = e^{x\bar{\alpha}_k}F(\alpha + x\mathrm{e}_k, \bar{\alpha})$$

Proof. It is sufficient to apply the previous Lemma, taking into account that

$$e^{-x\mathsf{a}_k}\,|\alpha\rangle = e^{-x\alpha_k}\,|\alpha\rangle \qquad \wedge \qquad \langle\alpha|\, e^{-x\mathsf{a}_j^\dagger} = \langle\alpha|\, e^{-x\bar{\alpha}_j}$$

by coherent state definition. $\square$

Finally, deriving with respect to x and putting $x = 0$, a series of relations between phase space derivation and operator multiplication are obtained:

$$\langle\alpha|\, \mathsf{a}_k\mathsf{F}(a, \mathsf{a}^\dagger)\,|\alpha\rangle = \left(\alpha_k + \frac{\partial}{\partial\bar{\alpha}_k}\right)F\,(\alpha, \bar{\alpha}) \tag{2.4}$$

$$\langle\alpha|\, \mathsf{F}(a, \mathsf{a}^\dagger)\mathsf{a}_k^\dagger\,|\alpha\rangle = \left(\bar{\alpha}_k + \frac{\partial}{\partial\alpha_k}\right)F\,(\alpha, \bar{\alpha}) \tag{2.5}$$

$$\frac{\partial f}{\partial\alpha_k}(\alpha, \bar{\alpha}) = \langle\alpha|\, [\mathsf{a}_k, \mathsf{f}(\mathsf{a}, \mathsf{a}^\dagger)]\,|\alpha\rangle \tag{2.6}$$

$$\frac{\partial f}{\partial\bar{\alpha}_k}(\alpha, \bar{\alpha}) = \langle\alpha|\, [\mathsf{f}(\mathsf{a}, \mathsf{a}^\dagger), \mathsf{a}_k^\dagger]\,|\alpha\rangle \tag{2.7}$$

These latest identities permit us to define a non-commutative product on coherent phase space, and to confront it with the usual pointwise commutative one.

Proposition 2.3. *Let us define the following star product:*

$$\star : \mathcal{P}_\Lambda(\alpha,\bar\alpha) \times \mathcal{P}_\Lambda(\alpha,\bar\alpha) \to \mathcal{P}_\Lambda(\alpha,\bar\alpha) \qquad (f,g) \longmapsto f\star g$$

$$f \star g := \mathsf{W}^{-1}[\mathsf{W}[f]\mathsf{W}[g]];$$

then: $\forall a,b \in \mathbb{C}$, $\forall f,g,h \in \mathcal{P}_\Lambda(\alpha,\bar\alpha)$

1. $\star$ *is linear:* $(af+bg)\star h = af\star h + bg\star h$;
2. $\star$ *is associative:* $f\star(g\star h) = (f\star g)\star h = f\star g\star h$;
3. $\star$ *is in general non-commutative;*

Furthermore, defining the Wick parenthesis as the phase space counterpart of the commutator divided by the imaginary unit i,

$$\{\!\{\cdot,\cdot\}\!\} : \mathcal{P}_\Lambda(\alpha,\bar\alpha) \to \mathcal{P}_\Lambda(\alpha,\bar\alpha) \qquad (f,g) \longmapsto \{\!\{f,g\}\!\} := -i(f\star g - g\star f)$$

the following properties can be stated: $\forall a,b \in \mathbb{C}$, $\forall f,g,h \in \mathcal{P}_\Lambda(\alpha,\bar\alpha)$

1. *linearity:* $\{\!\{af+bg,h\}\!\} = a\{\!\{f,h\}\!\} + b\{\!\{g,h\}\!\}$;
2. *skew-symmetry:* $\{\!\{f,g\}\!\} = -\{\!\{g,f\}\!\}$;
3. $\star$*-Leibniz property:* $\{\!\{f,g\star h\}\!\} = \{\!\{f,g\}\!\}\star h + g\star\{\!\{f,h\}\!\}$;
4. *Jacobi identity:* $\{\!\{f,\{\!\{g,h\}\!\}\}\!\} + \{\!\{h,\{\!\{f,g\}\!\}\}\!\} + \{\!\{g,\{\!\{h,f\}\!\}\}\!\} = 0$;

Proof. These properties descend from the ones enjoyed by the operator product and the commutator through the Wick quantisation map. □

Remark. The equipment of coherent phase space $\mathbb{C}^{\Lambda^d}$ with $(\star, \{\!\{\cdot,\cdot\}\!\})$ turns $\mathcal{P}_\Lambda(\alpha,\bar\alpha)$ into a $\star$-Poisson algebra; therefore, it would be interesting and insightful to confront it with the standard Poisson structure given by $(\cdot,\{\cdot,\cdot\})$ and illustrated in Proposition 2.1.

Explicitly computing the action of $\star$ with the aid of equations (2.4-2.5), it is easy to check that

$$\alpha_k^n \star g(\alpha,\bar\alpha) = \left(\alpha_k + \frac{\partial}{\partial\bar\alpha_k}\right)^n g(\alpha,\bar\alpha)$$

$$f(\alpha,\bar\alpha)\star\bar\alpha_k^m = \left(\bar\alpha_k + \frac{\partial}{\partial\alpha_k}\right)^m f(\alpha,\bar\alpha)$$

so that

$$f\star g\;(\alpha,\bar\alpha) = f\left(\alpha + \frac{\partial}{\partial\bar\alpha},\bar\alpha\right) g\;(\alpha,\bar\alpha) = g\left(\alpha,\bar\alpha + \frac{\partial}{\partial\alpha}\right) f\;(\alpha,\bar\alpha) \tag{2.8}$$

for all f,g in $\mathcal{P}_\Lambda(\alpha,\bar\alpha)$.
Expanding equation (2.8), we obtain, denoting by $\mathcal{O}(\partial^2)$ terms containing derivatives of at least order 2,

$$f\star g = \sum_{n=0}^{\infty}\sum_{k_1,\ldots,k_n}^{\Lambda}\frac{1}{n!}\frac{\partial^n f}{\partial\alpha_{k_1}\cdots\partial\alpha_{k_n}}\frac{\partial^n g}{\partial\bar\alpha_{k_1}\cdots\partial\bar\alpha_{k_n}} = fg + \sum_k^{\Lambda}\frac{\partial f}{\partial\alpha_k}\frac{\partial g}{\partial\bar\alpha_k} + \mathcal{O}(\partial^2)$$

which shows that $\star$ may be seen as an algebraic deformation of the point-wise product between coherent phase space functions.
Subsequently, Wick's parenthesis shall be understood as an algebraic deformation (see [10]) of the Poisson parenthesis: indeed,

$$\{\!\{f,g\}\!\} := \frac{f\star g - g\star f}{i} = -i\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1,\ldots,k_n}^{\Lambda}\frac{\partial^n f}{\partial\alpha_{k_1}\cdots\partial\alpha_{k_n}}\frac{\partial^n g}{\partial\bar\alpha_{k_1}\cdots\partial\bar\alpha_{k_n}} - \frac{\partial^n f}{\partial\bar\alpha_{k_1}\cdots\partial\bar\alpha_{k_n}}\frac{\partial^n g}{\partial\alpha_{k_1}\cdots\partial\alpha_{k_n}}, \tag{2.9}$$

so that

$$\{\!\{f,g\}\!\} = \{f,g\} + \mathcal{O}(\partial^2) \qquad \{f,g\} := \Pi(\mathrm{d}f,\mathrm{d}g) = -i\sum_k^\Lambda \frac{\partial f}{\partial \alpha_k}\frac{\partial g}{\partial \bar\alpha_k} - \frac{\partial f}{\partial \bar\alpha_k}\frac{\partial g}{\partial \alpha_k}$$

with the canonical Poisson tensor Π as defined in Proposition 2.1.
We now possess the principal tools needed to transport the operator dynamics onto coherent phase space.

2.3 Gaussian thermal measure on coherent phase space

Bogoliubov, in [1], assumed a field expansion of the type

$$\Psi = \psi 1_{\mathcal{H}} + \Theta$$

with ψ representing the superfluid order parametre, i.e the condensate wave-function, and Θ the normal fluid excitation field. Surely, since in the thermodynamical limit the only relevant state ought to be the ground one, it is reasonable to expect our yet-to-be-defined order parametre ψ_Λ to be the vacuum expectation value of Ψ for $N \to \infty$; instead, for finite N, we do not have any definite clue to what should the microscopic state be.
One physically reasonable way to approach the problem is to introduce a heat reservoir to fix the system at a temperature $T = \beta^{-1}$ much smaller than the critical one T_c, and then devise a scaling of β as a function of N so that

$$\lim_{N\to\infty} \beta(N) = +\infty.$$

This is physically significant, since all condensation experiments are performed at *very* small but finite temperatures.
Then, a reasonable candidate for the state over which to average Ψ_Λ is the Gibbs equilibrium mixed state

$$e^{-\beta \mathsf{H}_\Lambda} \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}e^{-\beta\mathsf{H}_\Lambda} = -i[\mathsf{H}_\Lambda, e^{-\beta\mathsf{H}_\Lambda}] \equiv 0.$$

However, this state is not easily transportable over coherent phase space, hence we should search for an approximation; the easiest way to find one is to truncate the Hamiltonian to its quadratic (i.e. non self-interacting) part,

$$\mathsf{H}_{0,\Lambda} = \sum_k^\Lambda \omega_k \mathsf{a}_k^\dagger \mathsf{a}_k \qquad\qquad e^{-\beta\mathsf{H}_{0,\Lambda}},$$

but this results in a non-stationary state, since $[\mathsf{H}_{0,\Lambda}, \mathsf{H}_\Lambda] \neq 0$.
Recalling Proposition 1.1, we have that N_Λ is a Nöther charge, so we can build a Gibbsian equilibrium state as

$$\varrho_\Lambda := \exp(-\beta\omega_0\mathsf{N}_\Lambda)/\operatorname{tr}(\exp(-\beta\omega_0\mathsf{N}_\Lambda)) \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\varrho_\Lambda = -i[\mathsf{H}_\Lambda, \varrho_\Lambda] \equiv 0, \tag{2.10}$$

with ω_0 the single-particle ground state energy.

Remark (I)**.** Physically, this mixed state is obtained by filling with the same fraction of bosons each single-particle energy level ϕ_k satisfying

$$\phi_k \qquad : \qquad |k| < \Lambda.$$

Remark (II)**.** In principle, we could use any other quantum invariant measure, and, although we will stick to ϱ_Λ, an analysis of these other possibilities is contained in [16].

Lemma. *Consider a one dimensional harmonic oscillator with number operator* $\mathsf{N} = \mathsf{a}^\dagger\mathsf{a}$*, then for any* λ *in* $\mathbb{R}$*, and any operator* $\mathsf{F}(\mathsf{a}^\dagger,\mathsf{a})$ *we have that*

$$\frac{\operatorname{tr}(\mathsf{F}e^{-\lambda\mathsf{N}})}{\operatorname{tr}(e^{-\lambda\mathsf{N}})} = \frac{e^\lambda - 1}{\pi}\int_{\mathbb{C}} \langle\alpha|\,\mathsf{F}\,|\alpha\rangle\, e^{-(e^\lambda-1)|\alpha|^2}\,\mathrm{d}\alpha\mathrm{d}\bar\alpha.$$

Proof. Computing the trace over number eigenstates and inserting a Dirac completeness in terms of coherent states

$$\mathsf{1}_{\mathcal{H}} = \int_{\mathbb{C}} \frac{\mathrm{d}\beta \mathrm{d}\bar{\beta}}{\pi} \; |\beta\rangle\langle\beta| \qquad \wedge \qquad \langle n|\beta\rangle = e^{-|\beta|^2/2}\frac{\beta^n}{\sqrt{n!}}$$

we obtain

$$\sum_{n=0}^{\infty} \langle n| \, \mathsf{F} e^{-\lambda \mathsf{N}} \, |n\rangle = \int_{\mathbb{C}} \frac{\mathrm{d}\beta \mathrm{d}\bar{\beta}}{\pi} \sum_{n=0}^{\infty} \langle n| \, \mathsf{F} \, |\beta\rangle \, \langle\beta|n\rangle e^{-\lambda n} = \int_{\mathbb{C}} \frac{\mathrm{d}\beta \mathrm{d}\bar{\beta}}{\pi} \sum_{n=0}^{\infty} \langle n| \, \mathsf{F} \, |\beta\rangle \, e^{-|\beta|^2/2} \frac{(\bar{\beta} e^{-\lambda})^n}{\sqrt{n!}} \tag{2.11}$$

Now, it can be recognised that

$$\sum_{n=0}^{\infty} \langle n| \, e^{e^{-2\lambda}|\beta|^2/2} e^{-e^{-2\lambda}|\beta|^2/2} \frac{(\bar{\beta} e^{-\lambda})^n}{\sqrt{n!}} = \left\langle e^{-\lambda}\beta \right|$$

so that

$$\mathrm{tr}(\mathsf{F} e^{-\lambda \mathsf{N}}) = \int_{\mathbb{C}} \frac{\mathrm{d}\beta \mathrm{d}\bar{\beta}}{\pi} e^{-(1-e^{-2\lambda})|\beta|^2/2} \langle e^{-\lambda}\beta| \mathsf{F} \, |\beta\rangle = e^{\lambda} \int_{\mathbb{C}} \frac{\mathrm{d}\alpha \mathrm{d}\bar{\alpha}}{\pi} e^{-\sinh(\lambda)|\alpha^2|} \langle e^{-\lambda/2}\alpha | \mathsf{F} | e^{\lambda/2}\alpha\rangle \tag{2.12}$$

where in the last passage we changed integration variable to $\alpha = e^{-\lambda/2}\beta$. Coherent states are not orthogonal, they satisfy the following normalisation:

$$\langle\gamma|\eta\rangle = e^{-(|\gamma|^2+|\eta|^2-2\bar{\gamma}\eta)/2} \qquad \Longrightarrow \qquad \langle\gamma|\gamma\rangle = 1.$$

Consequently, we may write

$$\langle e^{-\lambda/2}\alpha | \mathsf{F} | e^{\lambda/2}\alpha\rangle = e^{-(e^{\lambda}+e^{-\lambda}-2)|\alpha|^2/2} F(e^{-\lambda/2}\bar{\alpha}, e^{\lambda/2}\alpha)$$

where F should be understood as the non-diagonal Wick symbol of F. But $F(e^{-\lambda/2}\bar{\alpha}, e^{\lambda/2}\alpha)$ contains terms such as

$$(e^{-\lambda/2}\bar{\alpha})^n (e^{\lambda/2}\alpha)^m = e^{(m-n)\lambda/2}|\alpha|^{n+m} e^{i\arg(\alpha)(n-m)}$$

and only the last factor contributes to the angle integration contained in $|\alpha| \, \mathrm{d}|\alpha| \mathrm{d}(\arg(\alpha))$: specifically this leads to δ_{mn}, which collapses the non-diagonal Wick symbol onto the ordinary one. Hence we obtain

$$\mathrm{tr}(\mathsf{F} e^{-\lambda \mathsf{N}}) = e^{\lambda} \int_{\mathbb{C}} \frac{\mathrm{d}\alpha \mathrm{d}\bar{\alpha}}{\pi} e^{(e^{\lambda}-1)|\alpha|^2} \, \langle\alpha| \, \mathsf{F} \, |\alpha\rangle \, . \tag{2.13}$$

The thesis is achieved by directly computing

$$\mathrm{tr}(e^{-\lambda \mathsf{N}}) = e^{\lambda} \int_{\mathbb{C}} \frac{\mathrm{d}\alpha \mathrm{d}\bar{\alpha}}{\pi} e^{(e^{\lambda}-1)|\alpha|^2} = \frac{e^{\lambda}}{e^{\lambda}-1}$$

and taking the fraction. □

Remark. An combinatorically insightful explanation of the transformation $\lambda \to e^{\lambda} - 1$ in the context of normal ordering $: \cdot :$ of the exponential

$$e^{-\lambda \mathsf{a}^\dagger \mathsf{a}} = : e^{-(e^{\lambda}-1)\mathsf{a}^\dagger \mathsf{a}} :$$

in terms of Stirling numbers and Bell polynomials, and in the spirit of Wick's theorem, is contained in [11].

Proposition 2.4. *Tracing an operator* $\mathsf{F} \in \mathsf{L}(\mathcal{H})$ *against* ϱ_Λ *is equivalent to averaging its Wick symbol* F *over coherent space with respect to a normalised gaussian measure* μ_Λ*, that is*

$$\langle \mathsf{F} \rangle_{\varrho_\Lambda} = \frac{\mathrm{tr}(\mathsf{F} e^{-\beta\omega_0 \mathsf{N}_\Lambda})}{\mathrm{tr}(e^{-\beta\omega_0 \mathsf{N}_\Lambda})} = \int_{\mathbb{C}^{\Lambda^d}} F(\alpha, \bar{\alpha}) \, \mathrm{d}\mu_\Lambda(\alpha) \tag{2.14}$$

with

$$\mathrm{d}\mu_\Lambda(\alpha) = \frac{1}{Z} \prod_{k}^{\Lambda} \exp\left[-B(\beta)|\alpha_k|^2\right] \mathrm{d}\alpha_k \mathrm{d}\bar{\alpha}_k \qquad\qquad Z = \int_{\mathbb{C}^{\Lambda^d}} \prod_{k}^{\Lambda} e^{-B(\beta)|\alpha_k|^2} \, \mathrm{d}\alpha_k \mathrm{d}\bar{\alpha}_k = \left(\frac{\pi}{B(\beta)}\right)^{\Lambda^d}$$

where B *is the Planckian factor* $B(\beta) = e^{\beta\omega_0} - 1$.

Proof. The thesis is a direct consequence of the previous Lemma, provided that $\lambda = \beta\omega_0$, since

$$e^{-\beta\omega_0 \mathsf{N}_\Lambda} = \prod_k^\Lambda e^{-\beta\omega_0 \mathsf{a}_k^\dagger \mathsf{a}_k}$$

and the k-th exponential acts on the corresponding term of the coherent state

$$|\alpha\rangle = |\alpha_0\rangle \otimes \cdots \otimes |\alpha_k\rangle \otimes \cdots |\alpha_{k_{\max}}\rangle$$

where $k_{\max} := \max_{|k|<\Lambda} k$. $\square$

Proposition 2.5. *The following identities hold: for any k*

$$\langle \mathsf{a}_k^\dagger \mathsf{a}_k \rangle_{\varrho_\Lambda} = \int_{\mathbb{C}^{\Lambda^d}} |\alpha_k|^2 \, \mathrm{d}\mu_\Lambda(\alpha) = \frac{1}{B} \qquad \langle (\mathsf{a}_k^\dagger)^2 \mathsf{a}_k^2 \rangle_{\varrho_\Lambda} = \int_{\mathbb{C}^{\Lambda^d}} |\alpha_k|^4 \, \mathrm{d}\mu_\Lambda(\alpha) = \frac{2}{B^2}.$$

More in general, for any $\sigma \geq 0$

$$\int_{\mathbb{C}^{\Lambda^d}} |\alpha_k|^\sigma \, \mathrm{d}\mu_\Lambda(\alpha) = \frac{1}{B^{\sigma/2}} \Gamma\left(\frac{\sigma}{2} + 1\right),$$

where $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is the (unextended) Euler Gamma function, defined as

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t.$$

In particular, all these averages are k-indipendent.

Proof. Since mixed terms are not present in the measure, the integrals over coherent phase space reduce to the k-th coordinate subspace $\mathbb{C}$; then, by expressing in polar coordinates $\alpha_k(u, \vartheta) = ue^{i\vartheta}/\sqrt{B}$, we find

$$\int_{\mathbb{C}^{\Lambda^d}} |\alpha_k|^\sigma \, \mathrm{d}\mu_\Lambda(\alpha) = B^{-\sigma/2} \int_0^\infty 2u^{\sigma+1} e^{-u^2} \, \mathrm{d}u = B^{-\sigma/2} \Gamma\left(\frac{\sigma}{2} + 1\right).$$

The first two identities are a consequence of the above one, since $\Gamma(2) = 1$ and $\Gamma(3) = 2$. $\square$

Definition 2.6. The above Proposition permits to define *a fortiori* the number N of bosons in the system as the average of the number operator.

$$N := \langle \mathsf{N}_\Lambda \rangle_{\varrho_\Lambda} = \sum_k^\Lambda \int_{\mathbb{C}^{\Lambda^d}} |\alpha_k|^2 \, \mathrm{d}\mu(\alpha) = \frac{\Lambda^d}{B}. \tag{2.15}$$

Remark (I)**.** This definition should be assumed as an implicit constraint on the relevant scaling behaviours of Λ and B

$$N \longmapsto (\Lambda(N), B(N)) \qquad : \qquad \frac{\Lambda^d(N)}{B(N)} \overset{!}{=} N$$

It should be noted that Λ^d needs to be of super-linear growth in N, since we are interested in the low temperature and high number of bosons ($1 \ll B, N$) regime.

Remark (II)**.** Given the previous results and observations, and recalling that the weak (i.e. distributional) zero-variance limit of a normalised gaussian is a Dirac distribution, it is reckoned that in the thermodynamic limit

$$\mathcal{S}' - \lim_{N\to\infty} \mathrm{d}\mu_{\Lambda(N)}(\alpha_{(\Lambda)}) = \text{“}\delta[\alpha]\mathcal{D}\alpha\text{”}$$

with $\mathcal{D}\alpha$ some functional measure on coherent field space and δ a functional generalisation of the Dirac delta distribution. Subsequently, as hypothesised by Bogoliubov in [1], the finite temperature average "converges" to the (Fock) vacuum expectation value.

$$\text{“} \lim_{\beta(N)\to\infty} \mathrm{tr}\left(\mathsf{F}_{\Lambda(N)} e^{-\beta(N)\omega_0 \mathsf{N}_{\Lambda(N)}}\right) = \int F[\alpha, \bar{\alpha}] \delta[\alpha] \, \mathcal{D}\alpha = F[0] = \mathrm{tr}\left(\mathsf{F}\,|0\rangle\langle 0|\right) = \langle 0| \, \mathsf{F} \, |0\rangle \text{”}$$

2.4 Identification and dynamics of the order parametre

Definition 2.7. The superfluid order parametre ψ_Λ is defined to be the Wick symbol of the field operator

$$\psi_\Lambda(t,x) := \mathsf{W}^{-1}[\Psi_\Lambda(t,x)].$$

Since $\Psi_\Lambda(t,x) = \sum_k^\Lambda \mathsf{a}_k(t)\phi_k(x)$, we find that the condensate wavefunction admits a normal modes expansion in terms of the Wick symbols of the annihilation operators $\mathsf{a}_k(t)$.

$$\psi_\Lambda(t,x) = \sum_k^\Lambda a_k(t)\phi_k(x) \qquad a_k(t) := \mathsf{W}^{-1}[\mathsf{a}_k(t)] \qquad a_k(0;\alpha) = \alpha_k$$

We ought to confront ψ_Λ with a (regularised) scalar field σ_Λ built out of the low energy normal modes of the scalar Hartree equation, that is

$$\sigma_\Lambda = \sum_k^\Lambda c_k(t)\phi_k(x)$$

$$\dot{c}_k = -i\frac{\partial H_\Lambda}{\partial c_k} = \{c_k, H_\Lambda\} = -i\omega_k c_k - i\sum_{lmn}^\Lambda v_{(kl)mn}\bar{c}_l c_m c_n \qquad \wedge \qquad c_k(0;\alpha) = \alpha_k$$

with

$$H_\Lambda(\alpha,\bar\alpha) = \mathsf{W}^{-1}[\mathsf{H}_\Lambda](\alpha,\bar\alpha) = \sum_k^\Lambda \omega_k|\alpha_k|^2 + \frac{1}{2}\sum_{klmn}^\Lambda v_{klmn}\bar\alpha_k\bar\alpha_l\alpha_m\alpha_n.$$

Our goal is to show that there is a time-uniform estimate in a suitable norm topology

$$\left\|\psi_{\Lambda(N)}(t,\cdot) - \sigma_{\Lambda(N)}(t,\cdot)\right\| \le C_N|t|,$$

with convergence for all times in the thermodynamic limit, that is $C_N \underset{N\to\infty}{\longrightarrow} 0$.

We will consider the norm $\|\cdot\|_{d,\mu}$ defined on $L^2(\mathbb{R}^d \times \mathbb{C}^{\Lambda^d}, \mathrm{d}^d x\mathrm{d}\mu_\Lambda)$, the latter space being isomorphic to the tensor product of standard single-particle Hilbert space and (a generalisation of) Bargmann (see [2]) space $L^2(\mathbb{R}^d, \mathrm{d}^d x) \otimes L^2(\mathbb{C}^{\Lambda^d}, \mathrm{d}\mu)$; then

$$\|\psi_\Lambda(t) - \sigma_\Lambda(t)\|^2_{d,\mu} = \sum_{k,k'}^\Lambda \int_{\mathbb{R}^d} \bar\phi_k\phi_{k'}(x)\,\mathrm{d}^d x \int_{\mathbb{C}^{\Lambda^d}} (\bar a_k(t) - \bar c_k(t))(a_{k'}(t) - c_{k'}(t))\,\mathrm{d}\mu_\Lambda \tag{2.16}$$

Since $\langle\phi_k,\phi_{k'}\rangle_d = \delta_{kk'}$

$$\|\psi_\Lambda(t) - \sigma_\Lambda(t)\|^2_{d,\mu} = \sum_k^\Lambda \|a_k(t) - c_k(t)\|^2_\mu \tag{2.17}$$

Hence, for finite N, in order to estimate the deviation of operatorial dynamics from the effective scalar one, it is sufficient to estimate the deviation of the corresponding normal modes. In order to do this, we first need to write explicitly the equations of motion for the Wick symbols a_k.

Proposition 2.6. *a_k satisfies the following Cauchy problem*

$$\dot a_k = (\mathcal{L}_0 + \mathcal{L}_1)a_k \qquad \wedge \qquad a_k(0) = \alpha_k \tag{2.18}$$

where $\mathcal{L}_0 := \{\cdot, H_\Lambda\}$ is the Lie derivative along the Hamiltonian flow associated to H_Λ (which is exactly the scalar dynamics) and

$$\mathcal{L}_1 = \frac{1}{2i}\sum_{ij}^\Lambda \frac{\partial^2 H}{\partial\bar\alpha_i\partial\bar\alpha_j}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j} - \frac{\partial^2 H}{\partial\alpha_i\partial\alpha_j}\frac{\partial^2}{\partial\bar\alpha_i\partial\bar\alpha_j}$$

Proof. By definition of Wick symbol, the operator Heisenberg equation

$$\dot{\mathsf{a}}_k(t) = -i[\mathsf{a}_k(t), \mathsf{H}_\Lambda(t)] \qquad \mathsf{a}_k(0) = \mathsf{a}_k$$

is mapped to

$$\dot{a}_k(t) = \{\!\{a_k(t), H_\Lambda(t)\}\!\} \qquad a_k(0) = \alpha_k.$$

Since H_Λ is a constant of motion, i.e. $\{\!\{H_\Lambda, H_\Lambda\}\!\} = 0$, we can evaluate it at $t = 0$; then, recalling the explicit form of the Wick parenthesis as expressed in equation (2.9), we obtain

$$\dot{a}_k(t) = \{\!\{a_k(t), H_\Lambda\}\!\} = \{a_k(t), H_\Lambda\} + \frac{1}{2i}\sum_{ij}^{\Lambda} \frac{\partial^2 H}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}\frac{\partial^2 a_k(t)}{\partial\alpha_i\partial\alpha_j} - \frac{\partial^2 H}{\partial\alpha_i\partial\alpha_j}\frac{\partial^2 a_k(t)}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}$$

exactly, since $\{\!\{\cdot, H_\Lambda\}\!\}$ does not contain $\mathcal{O}(\partial^3)$ terms because H_Λ is a polynomial of degree 2 in α and in $\bar{\alpha}$. The identification of $\mathcal{L}_0$ and $\mathcal{L}_1$ is now straight-forward. □

Remark. $\mathcal{L}_1$ is precisely the deviation term from the scalar dynamics, and we notice that it is due to the self-interaction part of the Hamiltonian. Secondly, it should be noted that, to permit such a decomposition, the Hamiltonian structure of the equations of motion is crucial.

Definition 2.8. Let us denote the normal mode deviation function as

$$\delta_k(t) := a_k(t) - c_k(t).$$

Proposition 2.7. *The deviation term satisfies the equations of motion below:*

$$\dot{\delta}_k = \mathcal{L}_0\delta_k + \mathcal{L}_1 a_k \qquad \wedge \qquad \delta_k(0) = a_k(0) - c_k(0) = 0;$$

furthermore, it can be implicitly computed to be

$$\delta_k(t) = \int_0^t e^{(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)\,\mathrm{d}s$$

Proof. Since $\dot{c}_k = \{c_k, H_\Lambda\} = \mathcal{L}_0 c_k$ and since the Lie derivative $\mathcal{L}_0$ is linear, it is clear that

$$\dot{\delta}_k = \dot{a}_k - \dot{c}_k = \mathcal{L}_0(a_k - c_k) + \mathcal{L}_1 a_k = \mathcal{L}_0\delta_k + \mathcal{L}_1 a_k.$$

Let us now introduce an auxiliary function

$$\eta_k(t) := e^{-t\mathcal{L}_0}\delta_k(t) \qquad \Longrightarrow \qquad \dot{\eta}_k(t) = e^{-t\mathcal{L}_0}\mathcal{L}_1 a_k\,(t) \qquad \eta_k(0) = 0,$$

then

$$\eta_k(t) = \int_0^t e^{-s\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)\,\mathrm{d}s \qquad \Longrightarrow \qquad \delta_k(t) = e^{t\mathcal{L}_0}\eta_k(t)$$

whence the thesis. □

Note. $e^{s\mathcal{L}_0}$ is just a useful way of denoting the pull-back operation through the Hamiltonian flow Φ_{H_Λ}. Indeed, define

$$\Phi_{H_\Lambda} := \mathbb{R}\times\mathbb{C}^{\Lambda^d} \qquad (t,\alpha) \longmapsto \Phi^t_{H_\Lambda}(\alpha)$$

as the ODE flow associated to the set of equations

$$\dot{c}_k = \{c_k, H_\Lambda\} \quad \wedge \quad c_k(0) = \alpha_k \qquad \Longrightarrow \qquad c(t) = \Phi^t_{H_\Lambda}(\alpha)$$

then

$$e^{s\mathcal{L}_0}f = (\Phi^s_{H_\Lambda})^* f := f \circ \Phi^s_{H_\Lambda}$$

Remark. The above integration is implicit, since we do not know the function $t \mapsto a_k(t)$.

We can now compute an estimate for the $L^2(\mathbb{C}^{\Lambda^d}, \mathrm{d}\mu_\Lambda)$-norm of δ_k, but first we need to bear in mind the following Proposition.

Proposition 2.8. *The gaussian measure μ_Λ is invariant under the scalar flow, that is*

$$\mathrm{d}\mu(\Phi^t_{H_\Lambda}(\alpha)) = \mathrm{d}\mu(\alpha)$$

for all $\alpha \in \mathbb{C}^{\Lambda^d}$, for all times t.
Furthermore, averages with respect to μ are invariant under the full quantum evolution: for all $t \in \mathbb{R}$, for all $\mathsf{F} \in \mathsf{P}_\Lambda(\mathsf{a}, \mathsf{a}^\dagger)$

$$\int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\, \mathsf{F}(\mathsf{a}(t), \mathsf{a}^\dagger(t))\, |\alpha\rangle\ \mathrm{d}\mu(\alpha) = \int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\, \mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)\, |\alpha\rangle\ \mathrm{d}\mu(\alpha)$$

Proof. Denoting by N_Λ the Wick symbol of N_Λ, the measure, interpreted as a volume form on coherent phase space, can be written as

$$\mathrm{d}\mu(\alpha) = Z^{-1} e^{-BN_\Lambda(\alpha,\bar\alpha)} \sum_k^\Lambda \mathrm{d}\alpha_k \wedge \mathrm{d}\bar\alpha_k$$

Since $\{N_\Lambda, H_\Lambda\} \equiv 0$, we obtain

$$\mathrm{d}\mu(\Phi^t_{H_\Lambda}(\alpha)) = Z^{-1} e^{-BN_\Lambda(\alpha,\bar\alpha)} \sum_k^\Lambda \mathrm{d}(\Phi^t_{H_\Lambda}(\alpha))_k \wedge \mathrm{d}(\bar\Phi^t_{H_\Lambda}(\alpha))_k = \det(\mathrm{d}(\Phi^t_{H_\Lambda})_\alpha)\, \mathrm{d}\mu(\alpha) = \mathrm{d}\mu(\alpha)$$

because $\det(\mathrm{d}(\Phi^t_{H_\Lambda})_\alpha) = 1$, since Φ_{H_Λ} is a one-parametre group of symplectomorphisms (i.e. canonical transformations).
The second result follows by recalling the definition of μ with respect to the Wick map.

$$\int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\, \mathsf{F}(\mathsf{a}(t), \mathsf{a}^\dagger(t))\, |\alpha\rangle\ \mathrm{d}\mu(\alpha) = \mathrm{tr}(\mathsf{F}(\mathsf{a}(t), \mathsf{a}^\dagger(t))\varrho_\Lambda) = \mathrm{tr}(e^{it\mathsf{H}_\Lambda}\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)e^{-it\mathsf{H}_\Lambda}\varrho_\Lambda) = \mathrm{tr}(\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)\varrho_\Lambda)$$

where in the last passage cyclicity of the trace and $[\mathsf{H}_\Lambda, \varrho_\Lambda] = 0$ were employed. □

Proposition 2.9. *The norm of the deviation δ_k satisfies the following upper bound.*

$$\|\delta_k(t)\|^2_\mu \le \left(\int_0^t \sqrt{\int_{\mathbb{C}^{\Lambda^d}} |\mathcal{L}_1 a_k(s)|^2\, \mathrm{d}\mu\, \mathrm{d}s}\right)^2 \tag{2.19}$$

Proof. By Proposition 2.7,

$$\|\delta_k(t)\|^2_\mu = \int_{\mathbb{C}^{\Lambda^d}} \int_{[0,t]^2} \overline{e^{(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)}\, e^{(t-u)\mathcal{L}_0}\mathcal{L}_1 a_k\,(u)\, \mathrm{d}s\mathrm{d}u\, \mathrm{d}\mu_\Lambda$$

For fixed times s, u, employing the Cauchy-Schwarz inequality in $L^2(\mathbb{C}^{\Lambda^d}, \mathrm{d}\mu_\Lambda)$ the previous expression becomes

$$\|\delta_k(t)\|^2_\mu \le \int_{[0,t]^2} \left\|e^{(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)\right\|_\mu \left\|e^{(t-u)\mathcal{L}_0}\mathcal{L}_1 a_k\,(u)\right\|_\mu \mathrm{d}s\mathrm{d}u.$$

However, since by Proposition 2.8 the measure μ_Λ is invariant under the scalar flow, we have that

$$\left\|e^{(t-s)\mathcal{L}_0}\mathcal{L}_1 a_k\,(s)\right\|_\mu = \|\mathcal{L}_1 a_k\,(s)\|_\mu$$

hence

$$\|\delta_k(t)\|^2_\mu \le \int_{[0,t]^2} \|\mathcal{L}_1 a_k\,(s)\|_\mu\, \|\mathcal{L}_1 a_k\,(u)\|_\mu\ \mathrm{d}s\mathrm{d}u = \left(\int_0^t \|\mathcal{L}_1 a_k\,(s)\|_\mu\ \mathrm{d}s\right)^2$$

because each norm is t-independent. □

Remark. This is the most general result we can make out without an explicit computation. The next sub-section will be dedicated to the computation of this norm as a function of the parametres and variables of the system.

2.4.1 Computation of $\|\mathcal{L}_1 a_q(s)\|_\mu$

In Proposition 2.6 we determined

$$\mathcal{L}_1 a_q(s) = \frac{1}{2i}\sum_{ij}^{\Lambda} \frac{\partial^2 H}{\partial\bar\alpha_i\partial\bar\alpha_j}\frac{\partial^2 a_q(s)}{\partial\alpha_i\partial\alpha_j} - \frac{\partial^2 H}{\partial\alpha_i\partial\alpha_j}\frac{\partial^2 a_q(s)}{\partial\bar\alpha_i\partial\bar\alpha_j};$$

therefore taking into account the explicit expression of H_Λ and employing equations (2.6-2.7) to transform the second derivatives of $a_q(s)$ into coherent expectations of commutators, through simple algebraic manipulations we may write

$$\mathcal{L}_1 a_q(s) = -\frac{i}{2}\sum_{klmn}^{\Lambda} v_{(kl)(mn)}(\alpha_m\alpha_n \langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\,|\alpha\rangle - 2\bar\alpha_k\alpha_m\alpha_n\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\,|\alpha\rangle - \\ - \bar\alpha_k\bar\alpha_l\langle\alpha|\,\mathsf{a}_m\mathsf{a}_n\mathsf{a}_q(s)\,|\alpha\rangle + 2\bar\alpha_k\bar\alpha_l\alpha_m\langle\alpha|\,\mathsf{a}_n\mathsf{a}_q(s)\,|\alpha\rangle) \tag{2.20}$$

Lemma. *The following inequalities are true.*

$$\sqrt{\langle\alpha|\,\mathsf{a}_i\mathsf{a}_j\mathsf{a}_i^\dagger\mathsf{a}_j^\dagger\,|\alpha\rangle} \le 2 + 2|\alpha_i| + 2|\alpha_j| + |\alpha_i\alpha_j|$$

$$\sqrt{\langle\alpha|\,\mathsf{a}_i\mathsf{a}_i^\dagger\,|\alpha\rangle} \le 1 + |\alpha_i|$$

Proof. The inequalities are obtained by using the CCRs to bring the as to the right and by sublinearity of the square root function,

$$\sqrt{\langle\alpha|\,\mathsf{a}_i\mathsf{a}_j\mathsf{a}_i^\dagger\mathsf{a}_j^\dagger\,|\alpha\rangle} = \sqrt{\langle\alpha|\,1+\delta_{ij}+\mathsf{a}_i^\dagger\mathsf{a}_i+\mathsf{a}_j^\dagger\mathsf{a}_j+2\delta_{ij}\mathsf{a}_i^\dagger\mathsf{a}_i+\mathsf{a}_i^\dagger\mathsf{a}_j^\dagger\mathsf{a}_i\mathsf{a}_j\,|\alpha\rangle} \le 2+2|\alpha_i|+2|\alpha_j|+|\alpha_i\alpha_j|$$

$$\sqrt{\langle\alpha|\,\mathsf{a}_i\mathsf{a}_i^\dagger\,|\alpha\rangle} = \sqrt{\langle\alpha|\,\mathsf{a}_i^\dagger\mathsf{a}_i+1\,|\alpha\rangle} \le 1+|\alpha_i|$$

bearing in mind that $\mathsf{a}_i = \mathsf{a}_i(0)$. □

Proposition 2.10. *The squared-module of $\mathcal{L}_1 a_q(s)$ may be estimated from above by the product of a time-dependent Wick symbol and a time-independent polynomial.*

$$|\mathcal{L}_1 a_q(s)|^2 \le \langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle\,(p(\alpha,\bar\alpha))^2,$$

with $p : \mathbb{C}^{\Lambda^d} \to \mathbb{R}^+$ defined as

$$p(\alpha,\bar\alpha) := 3\sum_{klmn}^{\Lambda} |v_{(kl)(mn)}|(|\alpha_k\alpha_l\alpha_m\alpha_n| + |\alpha_k\alpha_l\alpha_m| + |\alpha_k\alpha_m\alpha_n| + |\alpha_k\alpha_l| + |\alpha_m\alpha_n|).$$

Proof. Employing the following form of the triangular inequality,

$$|z+w|^2 \le (|z|+|w|)^2 \qquad \forall z,w\in\mathbb{C},$$

and taking into account equation 2.20, we obtain

$$|\mathcal{L}_1 a_q(s)|^2 \le \left\{\sum_{klmn}^{\Lambda}\frac{|v_{(kl)(mn)}|}{2}\Big[|\alpha_m\alpha_n||\,\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\,|\alpha\rangle\,| + 2|\alpha_k\alpha_m\alpha_n||\,\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\,|\alpha\rangle\,| + \\ + |\alpha_k\alpha_l|\,|\langle\alpha|\,\mathsf{a}_m\mathsf{a}_n\mathsf{a}_q(s)\,|\alpha\rangle| + 2|\alpha_k\alpha_l\alpha_m|\,|\langle\alpha|\,\mathsf{a}_n\mathsf{a}_q(s)\,|\alpha\rangle|\Big]\right\}^2$$

Using the Cauchy-Schwarz inequality and the previous Lemma for the first two summands,

$$\left|\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\,|\alpha\rangle\right| \le \sqrt{\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle}\sqrt{\langle\alpha|\,\mathsf{a}_k\mathsf{a}_l\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\,|\alpha\rangle} \le \sqrt{\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle}(2+2|\alpha_k|+2|\alpha_l|+|\alpha_k\alpha_l|)$$

$$\left|\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\,|\alpha\rangle\right| \le \sqrt{\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle}\sqrt{\langle\alpha|\,\mathsf{a}_l\mathsf{a}_l^\dagger\,|\alpha\rangle} \le \sqrt{\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle}\,(1+|\alpha_l|)$$

obtaining analogous expressions for the third and fourth one, and taking into account index symmetrisation, eventually we are lead to

$$|\mathcal{L}_1 a_q(s)|^2 \le \langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle \left\{\sum_{klmn}^{\Lambda} |v_{(kl)(mn)}|\Big[|\alpha_k\alpha_l\alpha_m\alpha_n| + 3|\alpha_k\alpha_l\alpha_m| + \right.$$
$$\left. +3|\alpha_k\alpha_m\alpha_n| + 2|\alpha_k\alpha_l| + 2|\alpha_m\alpha_n|\Big]\right\}^2.$$

Overestimating each of the constant factors multiplying the polynomial summands with 3, the statement is proven. ☐

Proposition 2.11. *The norm $\|\mathcal{L}_1 a_q(s)\|_\mu$ is actually estimated by a q and s independent quantity.*

$$\|\mathcal{L}_1 a_q(s)\|_\mu \le \left(\sum_{klmn}^{\Lambda} \frac{3\pi}{2}|v_{(kl)(mn)}|\right)\left(B^{-1} + \mathcal{O}(B^{-2})\right)$$

Proof. Due to the previous Proposition, it is clear that

$$\begin{aligned}
\|\mathcal{L}_1 a_q(s)\|_\mu^2 &\le \int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle\, p^2(\alpha)\ \mathrm{d}\mu(\alpha) \le \\
&\le \sqrt{\int_{\mathbb{C}^{\Lambda^d}} \left|\langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle\right|^2 \mathrm{d}\mu(\alpha)}\ \left\|p^2\right\|_\mu \le \\
&\le \sqrt{\int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\,\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\,|\alpha\rangle\,\mathrm{d}\mu(\alpha)}\ \left\|p^2\right\|_\mu = \\
&= \sqrt{\int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\,\mathsf{a}_q\mathsf{a}_q^\dagger\mathsf{a}_q\mathsf{a}_q^\dagger\,|\alpha\rangle\,\mathrm{d}\mu(\alpha)}\ \left\|p^2\right\|_\mu
\end{aligned}$$

where in the second passage Cauchy-Schwarz inequality with respect to μ_Λ, and in the last one the invariance of the measure under the full quantum evolution, were used. Notice that time-independence is already manifest.
Let us compute separately the two factors.

$$\int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\,\mathsf{a}_q\mathsf{a}_q^\dagger\mathsf{a}_q\mathsf{a}_q^\dagger\,|\alpha\rangle\,\mathrm{d}\mu(\alpha) = \int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\,1 + 3\mathsf{a}_q^\dagger\mathsf{a}_q + \mathsf{a}_q^{\dagger 2}\mathsf{a}_q^2\,|\alpha\rangle\,\mathrm{d}\mu(\alpha) = \int_{\mathbb{C}^{\Lambda^d}} 1 + 3|\alpha_q|^2 + |\alpha_q|^4\,\mathrm{d}\mu(\alpha)$$

Recalling the results in Proposition 2.5, coherent phase space average gives

$$\int_{\mathbb{C}^{\Lambda^d}} \langle\alpha|\,\mathsf{a}_q\mathsf{a}_q^\dagger\mathsf{a}_q\mathsf{a}_q^\dagger\,|\alpha\rangle\,\mathrm{d}\mu(\alpha) = 1 + \frac{3}{B} + \frac{2}{B^2},$$

making q independence also manifest.
Meanwhile,

$$\left\|p^2\right\|_\mu^2 = \int_{\mathbb{C}^{\Lambda^d}} \left\{\sum_{klmn}^{\Lambda} 3|v_{(kl)(mn)}|\Big[|\alpha_k\alpha_l\alpha_m\alpha_n| + |\alpha_k\alpha_l\alpha_m| + |\alpha_k\alpha_m\alpha_n| + |\alpha_k\alpha_l| + |\alpha_m\alpha_n|\Big]\right\}^4 \mathrm{d}\mu(\alpha)$$

Since the averaging over coherent phase space cancels any information about the averaged index, and each integrated index contributes with a weight

$$\int_{\mathbb{C}^{\Lambda^d}} |\alpha_k|\,\mathrm{d}\mu(\alpha) = \sqrt{\frac{\pi}{4B}},$$

the above norm results in

$$\left\|p^2\right\|_\mu^2 = \left(\sum_{klmn}^{\Lambda} 3|v_{(kl)(mn)}|\right)^4 g\left(\frac{1}{B}\right)$$

with g some polynomial of degree 8 in B^{-1}. We are interested in the high B, i.e. low temperature, regime, hence we may approximate g with its lowest order monomial,

$$g(B^{-1}) = 16\left(\frac{\pi}{4B}\right)^{8/2} + \mathcal{O}(B^{-5}) = \left(\frac{\pi}{2B}\right)^4 + \mathcal{O}(B^{-5}).$$

Then, putting together all the separate calculations

$$\left\|\mathcal{L}_1 a_q(s)\right\|_\mu \leq \left(\sum_{klmn}^{\Lambda} \frac{3\pi}{2}|v_{(kl)(mn)}|\right)(B^{-1} + \mathcal{O}(B^{-2}))$$

which is the thesis. □

2.5 Assessment of the convergence to scalar BGP

Now that we have established an estimate for $\left\|\mathcal{L}_1 a\right\|_\mu$ (multi-index and time are omitted due to their irrelevance), we may compute the distance in norm between the full (operatorial) quantum dynamics, and the effective (scalar) one in terms of normal modes.
In particular, by combining the results of Propositions 2.9 and 2.11, we find $\left\|a_k(t) - c_k(t)\right\|_\mu$ to satisfy the following relation:

$$\left\|a_k(t) - c_k(t)\right\|_\mu =: \left\|\delta_k(t)\right\|_\mu \leq \left\|\mathcal{L}_1 a\right\|_\mu |t| \leq b_{N,\Lambda,B}|t| \tag{2.21}$$

with the time-independent constant $b_{N,\Lambda,B}$ (reinstating all dependences) equal to

$$b_{N,\Lambda,B} = \left(\sum_{klmn}^{\Lambda} \frac{3\pi}{2}|v_{(kl)(mn)}(N)|\right)\left(\frac{1}{B} + \mathcal{O}(B^{-2})\right)$$

However, N, Λ and $B := e^{\beta\omega_0} - 1$ are not independent, since they must satisfy the constraints

$$B(N) \underset{N\to\infty}{\longrightarrow} \infty \qquad\qquad \frac{\Lambda^d}{B(N)} = N.$$

A nice way to fulfill these requirements is to devise a scaling scheme for Λ as a function of N. In particular, for both these conditions to be contemporarily verified, any map such that

$$N \longmapsto \Lambda(N) \qquad\qquad \lim_{N\to\infty} \frac{N}{\Lambda^d(N)} = 0$$

is a viable candidate. Therefore, we may write the control constant as

$$b_N = \left(\sum_{klmn}^{\Lambda(N)} \frac{3\pi}{2}|v_{(kl)(mn)}(N)|\right)\left(\frac{N}{\Lambda^d(N)} + \mathcal{O}\left(\frac{N^2}{\Lambda^{2d}}\right)\right);$$

we observe that the N depedence of b_N is rather complex, and, in order to analyse its asymptotic behaviour, we need to simplify this expression.

Proposition 2.12. *The summands in the expression of b_N may be written by explicitating the N dependence as*

$$v_{klmn} = \varepsilon_1 g_0 \left(\frac{r_0}{L}\right)^d v'_{klmn} + \mathcal{O}\left(\left(\frac{r_0}{L}\right)^{d+1}\right)$$

where v'_{klmn} are the interaction-independent dimensionless coefficients

$$v'_{klmn} = \int_{\mathbb{R}^d} \bar{\varphi}_k \bar{\varphi}_l \varphi_m \varphi_n (\xi)\, \mathrm{d}^d\xi \qquad\qquad \varphi_k(\xi) = L^{d/2}\phi_k(x/L)$$

Proof. Let us recall the definition of v_{klmn}:

$$v_{klmn} = \int_{\mathbb{R}^{2d}} v(\|x-y\|)\bar{\phi}_k(x)\bar{\phi}_l(y)\phi_m(x)\phi_n(y)\,\mathrm{d}^d x\,\mathrm{d}^d y \qquad v(\|x\|) = \varepsilon_1 g(\|x\|/r_0),$$

with ε_1 constant and g a range one positive function with integral g_0. Adimensionalising, we find that

$$v_{klmn} = \varepsilon_1 \int_{\mathbb{R}^{2d}} g(L\|\xi-\eta\|/r_0)\bar{\varphi}_k(\xi)\bar{\varphi}_l(\eta)\varphi_m(\xi)\varphi_n(\eta)\,\mathrm{d}^d\xi\,\mathrm{d}^d\eta =$$
$$= \varepsilon_1\left(\frac{r_0}{L}\right)^d \int_{\mathbb{R}^d\times\mathcal{B}^d} g(\|\lambda\|)\bar{\varphi}_k\varphi_m(\xi)\,\bar{\varphi}_l\varphi_n(\xi + r_0\lambda/L)\,\mathrm{d}^d\xi\,\mathrm{d}^d\lambda$$

where in the last passage the change of variable

$$\eta(\lambda) = \xi + \frac{r_0}{L}\lambda$$

was performed. Since $r_0(L) \propto L^{-2/d}$, we can expand the integrand in powers of r_0/L and integrate in λ the zero order term to obtain the thesis. □

Consequently, the superfluid order parametre $\psi_N \equiv \psi_{\Lambda(N)}$ deviates in $L^2(\mathbb{R}^d \times \mathbb{C}^{\Lambda^d(N)})$-norm from the regularised Gross-Pitaevskii field σ_N uniformly in time, since, from equation 2.17,

$$\|\psi_N(t) - \sigma_N(t)\|_{d,\mu} = \sqrt{\sum_k^{\Lambda(N)} \|a_k(t) - c_k(t)\|_\mu^2} \leq C_N|t| \tag{2.22}$$

with the control constant C_N expressed in terms of N as

$$C_N = \Lambda^{d/2}(N) b_N = \frac{3\pi}{2}\varepsilon_1 g_0 \frac{N}{\Lambda^{d/2}(N)}\left(\frac{r_0}{L}\right)^d \sum_{klmn}^{\Lambda(N)} v'_{klmn}.$$

However, we can arrange C_N in a nicer form by recalling that

$$\varepsilon_1 g_0 r_0^d \frac{N}{L^d} = \varepsilon_0\gamma \qquad \varepsilon_0^{-1} = m\mu^2 L^2 \qquad \rho = \frac{N}{L^d}$$

where γ is the adimensional Gross-Pitaevskii constant as defined in Definition 1.4 and μ is the parametre appearing in the dimensionless form of the single-particle Schrödinger equation discussed in Proposition 1.3.
Hence, C_N simplifies to

$$C_N = \frac{3\pi}{2m}\frac{\rho^{2/d}\gamma}{\mu^2}\frac{1}{\Lambda^{d/2}(N)N^{2/d}}\sum_{klmn}^{\Lambda(N)} v'_{klmn}. \tag{2.23}$$

We would like to give some sufficient condition to obtain the convergence

$$\psi_N(t) - \sigma_N(t) \xrightarrow[N\to\infty]{L^2} 0 \qquad \forall t \in \mathbb{R}.$$

This is equivalent to an asymptotic analysis of C_N; since N appears in the sum only through Λ, let us suppose that

$$\sum_{klmn}^{\Lambda(N)} v'_{klmn} = \mathcal{O}(\Lambda^{d\nu}(N)) \qquad \nu \in \mathbb{R},$$

and impose super-linear growth on $N \mapsto \Lambda^d(N)$ through the ansatz

$$\Lambda^d(N) = N^{1+\epsilon} \qquad \epsilon > 0,$$

then the following proposition holds.

Proposition 2.13. *Assuming the notation assumed above, consider the following three mutually exclusive cases:*

a. mild divergence: $\nu \leq \frac{1}{2}$*;*

b. intermediate divergence: $\frac{1}{2} < \nu < \frac{1}{2} + \frac{2}{d}$*;*

c. strong divergence: $\nu \geq \frac{1}{2} + \frac{2}{d}$*;*

then

1. if a. is true,

$$C_N \underset{N\to\infty}{\longrightarrow} 0 \qquad\qquad \forall\epsilon > 0;$$

2. if b. holds,

$$C_N \underset{N\to\infty}{\longrightarrow} 0 \qquad\qquad \forall\epsilon \ : \ 0 < \epsilon < \frac{4 - d(2\nu - 1)}{d(2\nu - 1)};$$

3. if c. is true,

$$C_N \underset{N\to\infty}{\longrightarrow} +\infty \qquad\qquad \forall\epsilon > 0.$$

Proof. For $\nu \leq 1/2$, Λ is ininfluent in the asymptotic limit, hence the result is evident.
For the second case, let us assume $\nu > 1/2$; then

$$C_N \sim N^{[(2\nu-1)(1+\epsilon)d-4]/2d} \longrightarrow 0 \qquad \Longleftrightarrow \qquad \epsilon < \frac{4 - d(2\nu - 1)}{d(2\nu - 1)};$$

for such a condition to be compatible with $\epsilon > 0$, we need to require

$$\frac{4 - d(2\nu - 1)}{d(2\nu - 1)} > 0 \qquad \Longrightarrow \qquad \nu < \frac{1}{2} + \frac{2}{d}.$$

From this computation, the third statement also follows. □

Remark (I)**.** Since ϵ, i.e. the scaling of Λ, is a degree of freedom we did not fix yet, if the trapping potential satisfies either a. or b. of the previous Proposition, then we can always devise $\Lambda(N)$ such that we have convergence to the Gross-Pitaevskii regime.

Remark (II)**.** This Proposition ought to be interpreted as a working hypothesis, due to the fact that, in general, it is difficult to extract, from a qualitative knowledge of the single-particle Hamiltonian, the eigenfunction map

$$k \longmapsto \varphi_k,$$

much harder than obtaining qualitative informations on the eigenvalue map

$$k \longmapsto \omega_k.$$

Thus, a further investigation of the validity of the hypotheses for typical traps needs to be conducted.

Finally, consider the full Bogoliubov-Gross-Pitaevskii field σ, such that its dimensionless form, in analogy to the procedure to prove Theorem 1.1,

$$\varsigma(\tau, \xi) := \frac{1}{\sqrt{\rho}}\,\sigma(t, x) \qquad\qquad \tau = \varepsilon_0(L)t = \frac{t}{m\mu^2 L^2} \qquad \xi = \frac{x}{L}$$

satisfies the scalar BGP equation

$$i\frac{\partial\varsigma}{\partial\tau} = \left(\mathsf{k} + \gamma|\varsigma|^2\right)\varsigma = \left(-\frac{\mu^2}{2}\nabla^2_\xi + f(\xi) + \gamma|\varsigma|^2\right)\varsigma, \tag{2.24}$$

and denote by $P_{<\Lambda}$ the orthogonal projector on the space spanned by the normal modes carrying energy ω_k, with $|k| < \Lambda$; then, the really needed norm estimate is, for any N,

$$\left\|\psi_{\Lambda(N)}(t) - P_{<\Lambda(N)}\sigma(t)\right\|_{d,\mu}.$$

Although this may seem difficult at first, we can actually use the triangular inequality to calculate

$$\begin{aligned}\|\psi_\Lambda(t) - P_{<\Lambda}\sigma(t)\|_{d,\mu} &= \|\psi_\Lambda(t) - \sigma_\Lambda(t) + \sigma_\Lambda(t) - P_{<\Lambda}\sigma(t)\|_{d,\mu} \leq \\ &\leq \|\psi_\Lambda(t) - \sigma_\Lambda(t)\|_{d,\mu} + \|\sigma_\Lambda(t) - P_{<\Lambda}\sigma(t)\|_{d,\mu} \leq C_N|t| + \|\sigma_\Lambda(t) - P_{<\Lambda}\sigma(t)\|_{d,\mu}\end{aligned} \tag{2.25}$$

thus reducing the problem of convergence, in the infra-red (IR) thermodynamic limit, of operator dynamics to scalar BGP as an effective equation, to a case of *à la Galerkin* reconstruction of a partial differential equation (PDE), since

$$\|\sigma_\Lambda(t) - P_{<\Lambda}\sigma(t)\|_{d,\mu}$$

does not carry any information on the quantum system (namely, the order parametre); rather it estimates the error committed by regularising through the Galerkin UV cut-off Λ the full scalar BGP equation.

This peculiarly PDE-oriented problem of reconstruction bounds has been extensively studied in the mathematical analysis literature: for instance, we refer to [6] for the study of the analogous case for Korteweg-de Vries (KdV) equation.

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