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Black hole microstates in string theory and massless probes

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Introduction

In the recent years, the world has become more familiar with the term *black hole*. Although the allure characterizing black holes is much older, in the last few years new results, such as the first observation of gravitational waves from two merging supermassive black holes in 2016 [1], or, even more recently, the first photo of a black hole, published just over three years ago, in 2019 [2], have brought up to the general public the concept of black holes. But if these results allowed the uninitiated to get a little more familiar with the concept of black holes, the same cannot be said regarding the initiated. Indeed, the scientific community, and in particular the physics community, is (and has been) interested in studying these peculiar objects, that are nothing but solutions to the classical Einstein equations from General Relativity whose trademarks are a singularity shielded by event horizon(s), i.e. hypersurfaces separating two regions that are causally disconnected: the inside of the horizon and the outside of it, also because, among other reasons, black holes provide us with a nice proving ground where Quantum Mechanics and General Relativity meet each other, giving us the possibility to test our theories that should enable the two to meet with no drama.

One of such theories (actually, one of the most promising candidates) is String Theory, namely a theory with strings as fundamental objects rather than particles. The theory was initially developed in the sixties to model the force responsible for keeping the quarks bound inside protons and neutrons; i.e., the strong nuclear force. Among the things that inhibited String Theory to reach success in this field was the presence of a massless, spin-two particle in the spectrum of the theory; something that did not belong to the world of hadrons. However, gravitational waves are indeed characterized by helicity equal to two, and the graviton, meaning the quantum of gravitation, is thought to be a massless, spin-two particle. This was enough of a justification to take a closer look at String Theory as a theory able to unify Quantum Mechanics and General Relativity. It was 1974.

Since then, String theory had its successes. For instance, and here we come back to black holes, it is thanks to String Theory if we were able to produce the first counting of black holes microstates. Basically, because of Bekenstein's argument, it is known that black holes should carry some form of entropy because, if this was not the case, the entropy carried by matter falling inside a black hole would disappear behind the event horizon, thus violating the second principle of thermodynamics. Indeed, Bekenstein argued that the entropy characterizing a black hole is proportional to the event horizon's area, which grows as energy falls inside the hole. But, thanks to Boltzmann's formula, we know that entropy is also associated with the number of microstates giving rise to the same macroscopic system properties (that, in the case of black holes, are the mass, charge and angular momentum); unfortunately, since General Relativity never deals with such microstates, but just with the macroscopic charges, nobody knew how to construct them. It was in 1996, thanks to the work of Strominger and Vafa [3], that the correct counting of a black hole's microstates to reproduce the appropriate Bekenstein entropy was performed in the framework of String Theory for a specific black hole solution.

Another issue String Theory has to deal with if it wants to be the definitive theory of quantum gravity is the information paradox, namely a paradox highlighting an incompatibility between General Relativity and Quantum Mechanics. The paradox was proposed by Hawking and stems from the fact that black holes emit radiation that is perfectly thermal; i.e., it is the radiation a black body with temperature equal to the Hawking temperature of the black hole would emit. As the emission process goes on, the hole loses energy, thus shrinking in size. Eventually, the hole evaporates completely and, assuming nothing is left behind in this process, this would lead to an evolution from a pure state to a mixed one; something that is impossible in Quantum Mechanics. The main issue is that the radiation cannot be in casual contact with the microstates of the black hole because of the event horizon; if the event horizon did not prohibit the transfer of information between the microstates and the radiation the process of evaporation would be similar to any burning process, which does not violate Quantum Mechanics. A possible solution to this incompatibility seems to arise from the so-called *fuzzballs*, i.e. peculiar branes and strings configurations in String Theory with non-trivial, horizon-scale structures proposed by Mathur [4] to model black holes microstates. In the two-charge fuzzball composed by a vibrating string, each microstate corresponds to a different vibrational profile of the string, and the metrics produced by two different microstates differ strongly inside a region having the correct surface to produce the appropriate Bekenstein-Hawking entropy.

In this thesis we first aim at reviewing a method used to reproduce the supergravity metric for peculiar two-charge fuzzballs (but that can actually be extended to three-charge ones) known as *supertubes*; in particular, we will deal with the NS5-P and NS5-F1 fuzzballs. The idea is to start from an appropriate WZW model with target space G having signature $(10, 2)$, and reduce down to the $(9, 1)$ signature of the coset G/H by means of two (null) gauge transformations, with H being the gauge group. The action obtained by integrating out the gauge fields is the one of a string on background-metric given by the decoupling limit of the supertube one. The advantage gained in using this procedure is that the String Theory on G is exactly solvable, hence containing corrections otherwise invisible to the supergravity approximation, and, once we know how to construct the correct coset G/H, we could in principle implement such corrections in the supergravity metric. On that note, this thesis moves the first steps towards connecting the geodesics in G/H to the ones in G. However, the principal result of this work is the study of the motion of massless probes in different supertube backgrounds, showing that the radii of trajectories orbiting around the black holes depend on parameters characterizing the specific microstate at study. One of the reasons why the study of geodesics is interesting, is that it gives the opportunity of studying how fuzzballs influence the spacetime around them, thus understanding their absorption properties, and how these relate with the absorption properties of GR black holes, which capture everything falling inside the event horizon. The end goal of this is to hopefully understand how the geometry is influenced once the whole ensemble of microstates is considered. With respect to this, some results from the last chapter seem to suggest the idea that, once the average over the microstate ensemble forming the given black hole is performed, the absorption properties of the different microstates come together to reproduce the result we expect from typical General Relativity black holes: absorption of light getting too close to the hole.

Outline of the thesis

Chapter 1 briefly introduces the concepts of hypersurfaces, killing vectors, killing and event horizons and surface gravity, as well as how the conserved charges and currents are related to a given Killing vector. We then move to the presentation of some standard black holes solutions in General Relativity: the Schwarzschild, Reissner-Nordström and finally Kerr black holes. Following the brief review of the above black hole solutions, is the section introducing the concept of black hole temperature in the special case of a Schwarzschild solution, immediately followed by the introduction of the Bekenstein entropy, still in the case of Schwarzschild black hole. Ending the chapter, after the enunciation of the four laws of black holes mechanics and a simple derivation of the Hawking radiation emitted by a $1 + 1$ dimensional Schwarzschild black hole, is a qualitative discussion on the information paradox.

Chapter 2 is dedicated to the review of the String Theory tools that we will need in the subsequent chapters. In closer detail, the chapter starts off with the discussion of bosonic string, moving from the Nambu-Goto action, Polyakov action with the appropriate gauge fixing rules and boundary conditions, to the introduction of lightcone coordinates and the presentation of the mode expansion for both closed and open strings. Then, the lightcone quantization is carried out, which then enables us to spend a few words on the spectrum of bosonic string theory, again for both closed and open strings; in this context, an argument based on the representation of the Lorentz group for the choice of the spacetime dimensionality in which the bosonic string theory lives is given. The next section is dedicated to Conformal Field Theory, defining what a conformal transformation is, to then concentrate on two-dimensional Conformal Field Theories, summarizing the main results as the stress-energy tensor in a Conformal Field Theory, operator product expansion and how this, together with the stress-energy tensor, is related to the transformation of a given operator under a generic conformal transformation. The section then ends with the introduction of the Virasoro algebra and a concise discussion on the relationship between operators and states. In the next section we then review how the background fields such as a general metric $g_{\mu\nu}$, B field and dilaton are introduced in the string theory. The second-to-last section is dedicated to the Ramond-Neveu-Schwarz strings and, in particular, to the unveiling of their (massless) spectrum, thus introducing the spectrum of Type IIB and IIA theories, and the stable D-branes they contain. Finally, the last section comprises a discussion on T and S duality.

Chapter 3 presents some D-branes configurations that possess the trademarks of black holes. The first of such solutions examined is the four-charge black hole in Type IIA supergravity, made up of three D2 branes and one D6 brane. In particular, we give the metric produced by such an arrangement of branes, and also hint as to how this solution is similar to the extremal Reissner-Nordström from Chapter 1. The next solution considered is the three-charge black hole made up of three M2 branes, the eleven-dimensional counterpart to the D2 branes of the previous hole. On top of giving the metric produced by the chosen branes arrangement, along with the asymptotic behaviours as done for the four-charge solution, in this case we compute the Bekenstein-Hawking entropy associated with the three-charge hole. After a chain of T and S dualities, a microscopic counting of the degrees of freedom of the three-charge hole is carried out, showing that the entropy emerging from this counting precisely reproduces the Bekenstein-Hawking one. Ending the chapter we have a section reviewing the fuzzball proposal. Following the introduction of the general idea in the NS5-P frame, and a chain of T and S dualities, is the explicit computation of a fuzzball metric for a circular profile in the D1-D5 picture.

Chapter 4 begins with a review of Wess-Zumino-Witten models, as these will be needed in the rest of the chapter. The following section is then dedicated to computing the supergravity metric for a circular array of NS5 branes in the so-called *decoupling limit* for the NS5 branes, to then examine how the (transverse) supergravity metric for such a brane configuration can be retrieved from a gauged Wess-Zumino-Witten model, introducing the model and the two gauge groups that are required to achieve such a result. This procedure, meaning the introduction of an appropriate gauge transformation in a Wess-Zumino-Witten model, is then generalized to the case of NS5-P supertube and, through a T duality, to the case of the NS5-F1 supertube. Concluding the chapter is a brief discussion on the decoupling limit and how it modifies the asymptotics for the NS5-F1 supertube.

Chapter 5 deals with the motion of massless probes in some supergravity backgrounds already met in preceding chapters, except for the elliptical NS5-F1 supertube. The main focus is the study of null paths that get captured in closed trajectories around the supertubes, thus mimicking what a physical black hole is mostly known for: trapping light. In this context, the main result is the computation of how such trajectories depend on some microstates-determining parameters, in particular in for case of the elliptical NS5-F1 supertube. Furthermore, we also move the first steps in the direction of connecting the geodesics on G/H with the ones on G.

Chapter 1

Black holes review

In this chapter we present a brief review of the black-hole-solutions to Einstein's equations in four dimensional General Relativity (GR), following quite closely the treatment from [5] and [6].

1.1 Preliminary concepts

Before diving into the review of some black hole solutions, let us review some preliminary concepts.

1.1.1 Killing vectors

In GR, if the metric $g_{\mu\nu}$ admits a vector K^μ satisfying the so-called *Killing's equation*, i.e.

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = \nabla_{(\mu} K_{\nu)} = 0, \tag{1.1}$$

we say that the vector K^μ, known as *Killing vector*, generates a symmetry of the metric. The corresponding conserved charge for a particle moving along a geodesic $X^\mu(\lambda)$, with λ proper, is

$$Q = K_\mu(X(\lambda))p^\mu(\lambda), \tag{1.2}$$

where $p^\mu(\lambda) = \frac{dX^\mu(\lambda)}{d\lambda}$.

If a metric admits a timelike Killing vector near infinity the metric is said to be *stationary*; in this case we can find coordinates such that $K = \partial_t$ and the metric components do not depend on t. Furthermore, if K is also orthogonal to a family of hypersurfaces, the metric is said to be *static*. Finally, a metric (expressed in appropriate coordinates) having Killing vector $K = \partial_\phi$ is said to be *axisymmetric*.

1.1.2 Event and Killing horizons

Given a smooth function $f(x)$ of the spacetime coordinates x^μ, a hypersurface Σ is defined by requiring $f(x) = \text{const}$ on Σ. We then define the tangent vector $v = v^\mu \partial_\mu$ to Σ so that $v^\mu \partial_\mu f = 0$, since f is constant along the hypersurface, and the normal vector $\xi = g^{\mu\nu}\partial_\nu f \partial_\mu$.

Using the above definitions, we can group hypersurfaces into three categories: timelike hypersurfaces, if $\xi \cdot \xi > 0$, spacelike hypersurfaces, if $\xi \cdot \xi < 0$ and null hypersurfaces if $\xi \cdot \xi = 0$.

Event and Killing horizons are null hypersurfaces, so let us focus on the last case; let's start with event horizons. An *event horizon* is formally defined as the boundary of the causal past of future null infinity, i.e. $J^-(\mathfrak{I}^+)$, or, more intuitively, points "inside" the event horizon cannot communicate with points at infinity. Looking at Figure 1.1, the gray region is $J^-(\mathfrak{I}^+)$, while the inside of the horizon is the white region.

If the tangent vector to the null hypersurface $\mathcal{N}$ is a Killing vector, $\mathcal{N}$ is addressed as *Killing horizon* and, given a Killing horizon, we can always associate to it its *surface gravity* κ such that

$$\nabla_\mu(\xi \cdot \xi) = -2\kappa \xi_\mu \quad \text{on } \mathcal{N}. \tag{1.3}$$

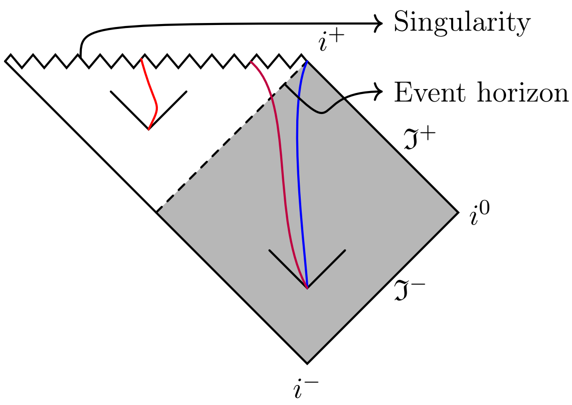


Figure 1.1: Conformal diagram of an asymptotically flat metric containing a black hole. The event horizon is represented by the dashed line, while the zigzag line is the singularity of the black hole. $\mathfrak{I}^{\pm}$ ($i^{\pm}$) represent, respectively, future and past null (timelike) infinity; finally i^0 represent spacelike infinity. The gray region is $J^-(\mathfrak{I}^+)$, the white region is the inside of the horizon.

In terms of a scalar quantity, we can also write

$$\kappa^2 = -\frac{1}{2}\nabla^\mu\xi^\nu\nabla_\mu\xi_\nu \quad \text{on } \mathcal{N}. \tag{1.4}$$

The reason why κ is addressed as "surface gravity" comes from considering a static, asymptotically flat spacetime; here, indeed, κ is the acceleration of a static observer near the horizon, as measured by a static observer at infinity.

We will later see that the surface gravity is also related to the Hawking temperature.

1.1.3 Conserved charges

We now review how to define charge, mass and spin in GR since these quantities characterize completely a stationary black hole.

Electric charge is defined starting from Maxwell's equations

$$\nabla^\nu F_{\nu\mu} = -4\pi j_\mu, \quad \nabla_{[\mu}F_{\nu\rho]} \tag{1.5}$$
$$d{\star}F = -4\pi{\star}j, \quad dF = 0. \tag{1.6}$$

From the first relation of (1.6), we deduce that j is conserved (i.e. $d{\star}j = 0$) hence, thanks to Stokes theorem, we can define the electric charge on a spacelike hypersurface Σ as

$$Q = \int_\Sigma {\star}j = \frac{1}{4\pi}\int_\Sigma d{\star}F = \frac{1}{4\pi}\int_{\partial\Sigma} {\star}F. \tag{1.7}$$

In four dimensions, introducing spherical coordinates on Σ, and indicating with S_r^2 the 2-sphere at fixed radius r, we get

$$Q = \frac{1}{4\pi}\lim_{r\to\infty}\int_{S_r^2} {\star}F. \tag{1.8}$$

Other conserved charges are obtained starting from a Killing vector K, applying Bianchi's identity to said Killing vector and using Killing equation (1.1) as well as the definition of the Riemann tensor $[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma$. See [6] for further details. By this procedure we arrive at

$$\star d{\star}dK = 8\pi G j \tag{1.9}$$

with

$$j_\mu = 2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)K^\nu. \tag{1.10}$$

The current j is conserved (as $d\star j = 0$), hence we can obtain a conserved charge from its integration on a hypersurface Σ as

$$Q_K = c\int_\Sigma \star j = \frac{c}{8\pi G}\int_\Sigma d\star dK = \frac{c}{8\pi G}\int_{\partial\Sigma}\star dK \tag{1.11}$$

for some constant c; this is known as *Komar integral.*

Considering a stationary, asymptotically flat space, meaning a space for which a Killing vector $K = \partial_t$ exists, we can define the Komar mass (or energy) as

$$M_{\text{Komar}} = -\frac{1}{8\pi G}\lim_{r\to\infty}\int_{S_r^2}\star dK. \tag{1.12}$$

If the space is also axisymmetric, meaning if the space admits a Killing vector $\widetilde{K} = \partial_\phi$, with $[\widetilde{K}, K] = 0$, we can define the angular momentum

$$J_{\text{Komar}} = \frac{1}{16\pi G}\lim_{r\to\infty}\int_{S_r^2}\star d\widetilde{K}. \tag{1.13}$$

1.2 Black holes in General Relativity

Now we move to presenting the most well-known black hole solutions to Einstein's equations.

Starting from the action

$$S_T = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_M, \tag{1.14}$$

where the first term is the so-called Einstein-Hilbert action, with g being the determinant of the metric $g_{\mu\nu}$, and R being the Ricci scalar, we get, by the usual stationary-action principle, the famous Einstein's equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{1.15}$$

Here, $G_{\mu\nu}$ is Einstein's tensor defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and $T_{\mu\nu}$ is the energy-momentum tensor of the matter fields defined by

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}.$$

We now present the main black hole solutions to Einstein's equations in $4d$ General Relativity starting from the most famous of all: Schwarzschild's solution.

1.2.1 Schwarzschild black hole

The spherically symmetric, vacuum solution of Einstein's equations is the famous Schwarzschild metric; in spherical coordinates $\{t, r, \theta, \phi\}$ it is given by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2. \tag{1.16}$$

The metric from (1.16) seems to possess two singularities: one at $r = 0$, and one at $r = 2GM \equiv r_s$ where r_s is known as *Schwarzschild radius.* Of the two, solely the former is a true singularity given that the latter is just a coordinate one, as can be readily checked by computing the scalar curvature

$$R = \frac{12r_s^2}{r^6}.$$

Actually, the surface at $r = r_s$ is the event horizon of the Schwarzschild black hole hence why, looking again at Figure 1.1, which represent part of the conformal diagram for a Schwarzschild black hole, once the purple line crosses the dashed line it cannot turn around and exit the horizon. Additionally, given that the Schwarzschild metric, being stationary, has a Killing vector $K = \partial_t$, and given that its norm (equal to $K^\mu g_{\mu\nu}K^\nu = g_{tt}$) vanishes at $r = r_s$, the event horizon is a Killing horizon as well. In particular, the surface gravity is $\kappa = \frac{1}{4GM}$.

Before moving on, we give a rule of thumb to determine the position of event horizons in stationary metrics. The rule is this: just look at the values r_* such that $g^{rr}(r_*) = 0$.

1.2.2 Reissner-Nordström black hole

In the case where the energy momentum tensor from (1.15) is equal to the electromagnetic one, i.e.

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma},$$

with $F_{\mu\nu}$ being the electromagnetic field strength, but we still have spherical symmetry, the most general metric solving Einstein's equations, now given by

$$G_{\mu\nu} = 8\pi G\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right),$$

is the Reissner-Nordström metric

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2 \quad \text{with} \quad \Delta = 1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}. \tag{1.17}$$

In this case, we have that Q is the total electric charge of the black hole, while P is the total magnetic one, which we will set to zero.

Using the rule of thumb from above, we see that the metric in (1.17) has event horizons where $\Delta(r) = 0$, i.e. at

$$r = r_\pm = MG \pm \sqrt{M^2G^2 - GQ^2} \tag{1.18}$$

The condition for the existence of $r_\pm$ $M^2G \geq Q^2$ generates three cases: *non-extremal* Reissner-Nordström black hole, when $M^2G > Q^2$; *extremal* Reissner-Nordström black hole, when $M^2G = Q^2$ and finally the case $M^2G < Q^2$. We will focus on the extremal case; for further details on all three cases see, for instance, [5].

Setting $M^2G = Q^2$, $r_\pm$ end up coinciding, leading to a single event horizon and to the metric

$$ds^2 = -\left(1 - \frac{GM}{r}\right)^2 dt^2 + \left(1 - \frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega^2. \tag{1.19}$$

Now, changing coordinate from r to $\rho = r - GM$, we get

$$ds^2 = -H^{-2}(\rho)dt^2 + H^2(\rho)d\rho^2 + r^2 d\Omega^2, \tag{1.20}$$

where

$$H = 1 + \frac{GM}{|\boldsymbol{x}|}. \tag{1.21}$$

It is possible to show that the only requirement on the function H to generate a metric of the form (1.20) is to be such that $\Delta H = 0$ with $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$; i.e. it should satisfy Laplace's equation. This means that the $H(\rho)$ in (1.20) could also be given by superposing many solutions of the form (1.21) resulting in

$$H = 1 + \sum_{i=1}^{N} \frac{GM_i}{|\boldsymbol{x} - \boldsymbol{x_i}|} \tag{1.22}$$

where at each point $\boldsymbol{x_i}$ we have the insertion of Reissner-Nordström solution with mass M_i.

These kinds of solutions are only achievable in the extremal black hole case because the condition $GM^2 = Q^2$ implies the balancing, in appropriate units, of gravitational and electric force so that the black holes can form a stable configuration.

1.2.3 Kerr black hole

For this solution we are giving up spherical symmetry in favor of axial symmetry since we suppose the black hole to be rotating with angular momentum J around a given axis. The metric takes the form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{4GMar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 \\ &+ \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\big((r^2+a^2)^2 - a^2\Delta\sin^2\theta\big)d\phi^2, \end{aligned} \tag{1.23}$$

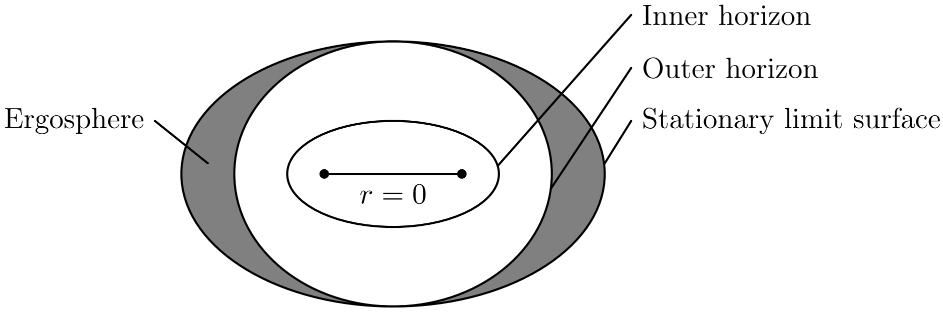


Figure 1.2: Near-horizon structure of Kerr's metric. The gray region is the ergosphere. Inside it, no stationary observer is allowed since t is spacelike there; specifically, you must move in the black hole's rotation direction, that is, ϕ's direction. We then see both horizons and notice that the outer horizon coincides with the stationary limit surface at the top and bottom of the ellipses (i.e. at $\theta = 0, \pi$).

where $\Delta(r) = r^2 - 2GMr + a^2$ and $\rho^2(r, \theta) = r^2 + a^2 \cos^2\theta$ while $a = \frac{J}{M}$ is the angular momentum per unit of mass. The Kerr metric is both stationary, with $K = \partial_t$, and axisymmetric, with $\widetilde{K} = \partial_\phi$, but is not static, as K is not orthogonal to constant-t hypersurface.

The event horizons are again found via the rule given at the end of Section 1.2.1; this leads to (restricting to $GM > a$)

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2}. \tag{1.24}$$

Now, as the norm of $K = \partial_t$ is different from zero at both $r_\pm$, the two are not Killing horizons for K. Actually, the radius at which K's norm vanishes is given by the solution of the following equation

$$(r - GM)^2 = G^2M^2 - a^2\cos^2\theta; \tag{1.25}$$

One solution gives a surface sitting inside the inner horizon; we discard it. The other sits outside the outer horizon (except at angles $\theta = 0, \pi$ where the two coincide) and goes under the name of *stationary limit surface* because, inside it, no stationary observer is allowed since K becomes spacelike there. In between the outer horizon and the stationary limit surface lays a region called *ergosphere*, see Figure 1.2. Finally, using (1.2) with Killing vector $\widetilde{K}$ for a freely falling photon, we get the angular velocity $\frac{d\phi}{dt}$ of the photon, as measured by a stationary observer; evaluating it then at $r = r_+$, we find

$$\Omega_H = \frac{a}{r_+^2 + a^2}. \tag{1.26}$$

1.3 Black holes thermodynamics

1.3.1 Temperature

To define black hole's temperature we start from the path integral

$$\langle f| e^{-\frac{iH\Delta t}{\hbar}} |i\rangle = \int_{\phi(t_i)=\varphi_i}^{\phi(t_f)=\varphi_f} \mathcal{D}\phi \; e^{\frac{i}{\hbar}S[\phi]},$$

where $\Delta t = t_f - t_i$ and $\varphi_{i(f)}$ is the initial (final) field configuration. After performing Wick's rotation $t \to -i\tau$, we take the trace to obtain the partition function for a system at temperature $T = \frac{\hbar}{\Delta\tau}$

$$\mathrm{Tr}\Big(e^{-\frac{\Delta\tau H}{\hbar}}\Big) = \int d\varphi \, \langle\varphi| e^{-\frac{\Delta\tau H}{\hbar}} |\varphi\rangle = \int d\varphi \int_{\phi(t_i)=\varphi}^{\phi(t_f)=\varphi} \mathcal{D}\phi \; e^{-\frac{S_{\mathrm{E}}[\phi]}{\hbar}} = \int \mathcal{D}\phi \; e^{-\frac{S_{\mathrm{E}}[\phi]}{\hbar}}, \tag{1.27}$$

with $\phi(\tau) = \phi(\tau + \Delta\tau)$ (i.e. bosonic field), and S_{E} being the euclidean action.

From this construction we learn that a system with a periodic (imaginary) time coordinate has an equilibrium temperature equal to T; with this knowledge we look at Schwarzschild's metric (1.16) with $t = -i\tau$ and $\frac{\rho^2}{4r_s} = r - r_s$

$$ds_{\mathrm{E}}^2 = \frac{\rho^2}{4rr_s}d\tau^2 + \frac{r}{r_s}d\rho^2 + r^2 d\Omega^2 \approx \frac{\rho^2}{4r_s^2}d\tau^2 + d\rho^2 + r_s^2 d\Omega^2 = d\rho^2 + \rho^2 d(\tau\kappa)^2 + r_s^2 d\Omega^2, \tag{1.28}$$

where in the second-to-last step we expanded for $r \approx r_s$ and $\kappa = \frac{1}{2r_s}$ is Schwarzschild's metric surface gravity. Assigning periodic conditions to $\tau\kappa$, so as to reproduce $\mathbb{R}^2 = d\rho^2 + \rho^2 d(\tau\kappa)^2$, τ acquires periodic conditions $\tau \sim \tau + \frac{2\pi}{\kappa}$, and thus, by the above argument, the black hole has equilibrium temperature

$$T = \frac{\hbar}{\Delta\tau} = \frac{\hbar\kappa}{2\pi}, \tag{1.29}$$

where $\frac{\hbar\kappa}{2\pi}$ is known as *Hawking temperature*.

Although the argument given above was specialized to Schwarzschild's metric, and does indeed seem "hand waving", it is possible to reach the same conclusion by considering the radiation emitted by black holes due particle-antiparticle pairs creation at the horizon. Indeed, Hawking showed in [7] that the emitted radiation makes the black hole behave as if it was a hot body at temperature $T = \frac{\hbar\kappa}{2\pi}$ (we will see this in a simplified scenario in Section 1.4).

1.3.2 Entropy

The entropy of black hole can be derived from the thermodynamic relation $dE = TdS$ with $E = M$, the black hole's mass. Again, considering Schwarzschild's black hole, we have

$$\frac{dS}{dM} = \frac{1}{T} = \frac{2\pi}{\hbar\kappa} = \frac{8\pi GM}{\hbar} \Rightarrow S = \frac{\pi r_s^2}{G\hbar} = \frac{A}{4G\hbar}, \tag{1.30}$$

where A is the horizon's area.

The idea that black holes should carry an entropy was first introduced by Bekenstein [8]; he argued that since once matter has fallen inside a black hole it is inaccessible to an external observer, the entropy the matter carried has either disappeared inside the black hole, thus violating the second law of thermodynamics, or else it increased the black hole's own entropy. Furthermore, he argued that the black hole's entropy should be proportional to the area of hole's horizon, given that this cannot decrease by Hawking's area theorem [9], just as the entropy in thermodynamic's second law.

1.3.3 Black holes mechanics

The analogies between thermodynamics and black holes do not end here, and indeed we can formulate four laws of black hole's mechanics [10].

Generalized second law. *The sum of ordinary entropy outside a black hole and the total black hole entropy never decreases.*

Zeroth law. *The surface gravity is constant over a Killing horizon.* In this definition the surface gravity plays the same role the temperature plays in the zeroth thermodynamic's law, where the temperature of a body at thermal equilibrium is constant in time.

First law. *Two stationary black hole's states, one with mass M, angular momentum J and charge Q, and the other with mass $M + dM$, angular momentum $J + dJ$ and charge $Q + dQ$ are related by*

$$dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ + \Phi_H dQ \tag{1.31}$$

where A is the horizon's area and Φ_H is the electric potential.

Third law. *It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.* This law mimics thermodynamic's third law which states that for a thermal system it is impossible to reach zero temperature in a finite number of steps.

1.4 Hawking radiation

In this treatment we follow [11].

Consider once again Schwarzschild's metric (1.16) and introduce what are known as null Kruskal coordinates U and V such that

$$ds^2 = -\frac{4r_s}{r}e^{-\frac{r}{r_s}}dUdV + r^2 d\Omega_2^2 \quad \text{with} \quad UV = r_s(r_s - r)e^{\frac{r}{r_s}}, \quad \frac{U}{V} = -e^{\frac{r}{r_s}}. \tag{1.32}$$

It is well known that the asymptotic observer, using the coordinates in (1.16) to describe the metric, will not see an infalling one crossing the horizon, while the infalling observer crosses the horizon in a finite proper time τ. Actually, the relation between the time t used by the asymptotic observer and the proper time of the infalling one, is

$$d\tau \propto e^{-\frac{t}{r_s}}dt.$$

Hence, a small interval in τ is perceived as a much longer time interval in t.

The discrepancy between the two time coordinates translates in different field expansions for the asymptotic and infalling observer: the asymptotic one will expand any field in the t-frequency ω, whereas the infalling in the τ-frequency ν. Furthermore, given the relation between t and τ, a given frequency ω, detected by the asymptotic observer, will be much higher when measured using the ν frequency; just a manifestation of red-shift. Because of this, given that the metric changes with characteristic time r_s^{-1}, and that the energy modes in ν are very energetic, thanks to the adiabatic principle[1], we can say to high accuracy $e^{-\mathcal{O}(\nu r_s)}$ that these modes will not be excited; i.e. the infalling observer will not see them.

To make this reasoning more quantitative, consider reducing to a $1+1$ dimensional Schwarzschild metric by neglecting the $d\Omega_2^2$ factor in (1.16) thus obtaining

$$\begin{aligned} ds^2 &= -\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 \\ &= - -\left(1 - \frac{r_s}{r}\right)dudv \\ &= -\frac{4r_s}{r}e^{-\frac{r}{r_s}}dUdV. \end{aligned} \tag{1.33}$$

Where

$$u = t - r_* = -2r_s \ln\left(-\frac{U}{r_s}\right), \quad v = t + r_* = 2r_s \ln\left(\frac{V}{r_s}\right) \quad \text{with} \quad r_* = r + r_s \ln(r - r_s) \tag{1.34}$$

are the null coordinates for the asymptotic observer.

Having solved the correct Klein-Gordon equation for a field ϕ, we can split it into a right-moving (left-moving) field $\phi_{R(L)}$, where the right-moving component will be outgoing with respect to the black hole and be a function of either u or U, while the left-moving one will be ingoing and be a function of either v or V. We can then Fourier-expand ϕ_R as

$$\phi_R = \int_0^\infty \frac{d\nu}{2\pi\sqrt{2\nu}}\left(a_\nu e^{-i\nu U} + a_\nu^\dagger e^{i\nu U}\right) = \int_0^\infty \frac{d\omega}{2\pi\sqrt{2\omega}}\left(b_\omega e^{-i\omega u} + b_\omega^\dagger e^{i\omega u}\right), \tag{1.35}$$

where $[a_\nu, a_{\nu'}^\dagger] = 2\pi\delta(\nu - \nu')$, $[a_\nu, a_{\nu'}] = 0$, $[a_\nu^\dagger, a_{\nu'}^\dagger] = 0$ and similarly for b_ω and $b_\omega^\dagger$.

It is possible to express b_ω in terms of a_ν and $a_\nu^\dagger$ by means of the *Bogoliubov coefficients*

$$\begin{aligned} \alpha_{\omega\nu} &= \sqrt{\frac{\omega}{\nu}}\int du\ e^{i\omega u - i\nu U} = 2r_s\sqrt{\frac{\omega}{\nu}}(r_s\nu)^{2ir_s\omega}e^{\pi r_s\omega}\Gamma(-2ir_s\omega) \\ \beta_{\omega\nu} &= \sqrt{\frac{\omega}{\nu}}\int du\ e^{i\omega u + i\nu U} = 2r_s\sqrt{\frac{\omega}{\nu}}(r_s\nu)^{2ir_s\omega}e^{-\pi r_s\omega}\Gamma(-2ir_s\omega), \end{aligned} \tag{1.36}$$

[1]The adiabatic principle tells us that if the Hamiltonian for a quantum system is changing slowly compared to the spacing between levels, then the probability for the system to become excited is exponentially small.

as

$$b_\omega = \int_0^\infty \frac{d\nu}{2\pi}(\alpha_{\omega\nu}a_\nu + \beta_{\omega\nu}a_\nu^\dagger). \tag{1.37}$$

From here, exploiting the adiabatic principle to deduce $a_\nu |\psi\rangle = 0$, we get

$$\begin{aligned}\langle\psi|b_\omega^\dagger b_{\omega'}|\psi\rangle &= \int \frac{d\nu}{2\pi}\int\frac{d\nu'}{2\pi}\beta^*_{\omega\nu}\beta_{\omega'\nu'}\langle\psi|a_\nu a^\dagger_{\nu'}|\psi\rangle = \int\frac{d\nu}{2\pi}\beta^*_{\omega\nu}\beta_{\omega'\nu}\\ &\overset{(1.36)}{=} \sqrt{\omega\omega'}4r_s^2(r_s\nu)^{2ir_s(\omega'-\omega)}e^{-\pi r_s(\omega'+\omega)}\Gamma^*(-2ir_s\omega)\Gamma(-2ir_s\omega')I,\end{aligned} \tag{1.38}$$

where

$$I = \int_0^\infty \frac{d\nu}{2\pi\nu}\nu^{2ir_s(\omega'-\omega)} = \frac{\delta(\omega'-\omega)}{2r_s}. \tag{1.39}$$

Substituting (1.39) inside (1.38), and using

$$|\Gamma(iy)|^2 = \frac{\pi}{y\sin(\pi y)} \quad \text{with} \quad y\in\mathbb{R},$$

we finally find

$$\langle\psi|b_\omega^\dagger b_{\omega'}|\psi\rangle = \frac{2\pi\delta(\omega'-\omega)}{e^{\frac{\hbar\omega}{T_H}}-1} \tag{1.40}$$

with T_H being Hawking temperature (1.29).

Hence, even though the state $|\psi\rangle$ is the vacuum for the infalling observer ($a_\nu|\psi\rangle = 0$), and so he sees empty space around him, the asymptotic observer sees the black hole producing radiation as a hot body at temperature T_H, the Hawking temperature of the black hole. This radiation is known as *Hawking radiation* [9].

1.5 Information paradox

In this section we give a brief and qualitative description of the information paradox; for a more in-depth look at [12].

The information paradox was introduced by Hawking in [13]; the idea at its basis is that when a pair of particles is created by vacuum fluctuations near the horizon one of the two particles is generated inside the horizon while the other escapes to infinity. Looking again at the discussion from the previous section, it is possible to show that the particle inside the black hole has negative energy[2] (see [11] and references therein for the details), while the one escaping to infinity has positive energy, thus preserving energy conservation: the black hole "eats" a particle of energy $-E$ and "spits out" a particle of energy E. The two particles are entangled, so as more and more pairs are created, the entanglement entropy increases; e.g. consider having n EPR pairs and throwing one particle from each pair inside a black hole. Since we started from a pure state (the sum of n EPR pairs) $|\phi\rangle$ with density matrix $\rho_i = |\phi\rangle\langle\phi|$, the entropy is initially $S_i = 0$ and with *entanglement* entropy[3] $S_E = n\log 2$; however, once we have thrown in half of the particles, and the black hole has evaporated, what we are left with is a mixed state with density matrix ρ_f given by the partial trace over the in fallen particles of ρ_i and entropy $S_f = -\operatorname{Tr}(\rho_f\log(\rho_f))$. But this is exactly the definition of the entanglement entropy, and so $S_f \equiv S_E = n\log 2$. This means that we started from a *pure* state ρ_i and ended up, after the black hole has completely evaporated, with a *mixed* state ρ_f; a clear violation of unitarity.

What makes this process different from any other burning (or evaporation) process is the presence of the horizon and the hypothesis that beyond it only empty space resides, with all the matter being concentrated at the singularity. Indeed, if you consider a burning coal, the radiation it emits is causally

[2]We do not need to worry about the negative energy: energy is the conserved charge of the appropriate Killing vector that, outside the horizon, is timelike, while inside is spacelike; hence what was energy outside the horizon is momentum inside it.

[3]Recall that the definition of entanglement entropy for a system made by two subsystems A and B with density matrix ρ_{AB}, is $S_E = -\operatorname{Tr}(\rho_A\log(\rho_A))$, where $\rho_A = \operatorname{Tr}_B\rho_{AB}$ and Tr_B is the partial trace over subsystem B. In particular, the entanglement entropy of a single EPR pair is $\log 2$.

connected with the surface of the coal, thus "transporting information" away from the coal. This is impossible in the case of the black hole, because everything inside the horizon is causally disconnected from anything outside.

Chapter 2

String theory review

In this chapter we introduce the concepts that are needed in the subsequent material. Further details on what follows can be found, for instance, in [11], [14] and [15], which are also the references we follow in this chapter.

From here on $\hbar = c = 1$.

2.1 Bosonic string

Our aim is to study the motion of a string in a flat, D-dimensional manifold with metric given by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. Moving in time, the string sweeps a two-dimensional surface in the D-dimensional Minkowski space we are considering; this surface, parametrized by two parameters, a timelike one, τ, and a spacelike one, σ, is known as *worldsheet* and is given by functions $X^\mu(\tau, \sigma)$. For brevity's sake, in the following, we are going to refer to both parameters by the condensed notation $\sigma^\alpha = (\tau, \sigma)$ with $\alpha = 0, 1$.

In the brief review that follows, we will meet two kinds of strings: *closed* strings, for which $\sigma \in [0, 2\pi)$ and X^μ satisfy $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$, and *open* strings, for which $\sigma \in [0, \pi]$ with no periodic boundary conditions on $X^\mu(\tau, \sigma)$.

2.1.1 The Nambu-Goto action

The appropriate action to describe the motion of the string is *Nambu-Goto*'s one, a generalization of Special Relativity's single particle action. There, the single-particle action is proportional to the worldline's length, so, in the case of the string, we expect the appropriate action to be proportional to the worldsheet's area, and indeed so is the Nambu-Goto action. It is given by

$$S_{\mathrm{NG}} = -T \int d^2\sigma \sqrt{-\det \gamma}, \tag{2.1}$$

where

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \tag{2.2}$$

is the metric induced on the worldsheet by the pull-back of Minkowski's one. In the above (2.1) T represents the string tension and is usually written as

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2} \tag{2.3}$$

where α' is known as *universal Regge slope* that, having dimension of $(\mathrm{length})^2$, is also renamed l_s^2, with l_s being the *string length*.

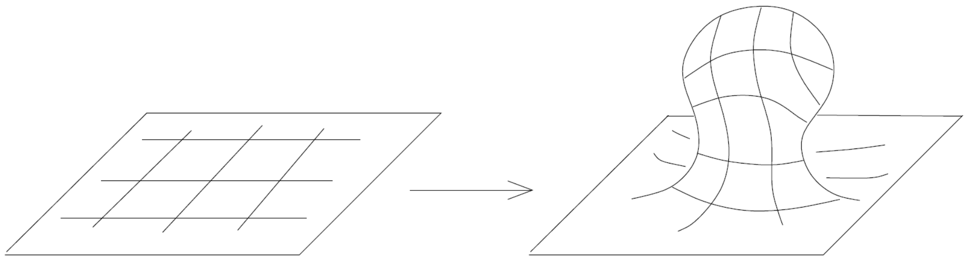


Figure 2.1: Action of a Weyl transformation. It changes the lengths but not the angles. Image from [15].

2.1.2 The Polyakov action

Due to the presence of the square root in the Nambu-Goto action, it is difficult to quantize it. Fortunately, there is an equivalent action, the *Polyakov action*, that can be quantized with path integral techniques; this is given by

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}, \tag{2.4}$$

where $g_{\alpha\beta}$ is a new field. Its equations of motion are polynomial, hence their solutions are easy to find and are given by

$$g_{\alpha\beta} = 2f(\tau,\sigma)\partial_\alpha X\cdot\partial_\beta X \quad \text{with} \quad f(\tau,\sigma) = (g^{\rho\sigma}\partial_\rho X\cdot\partial_\sigma X)^{-1}; \tag{2.5}$$

that is, $g_{\alpha\beta}$ differs from $\gamma_{\alpha\beta}$ by the f factor. Even though this may seem to hinder the equivalence between Nambu-Goto's and Polyakov's action, inserting this solution in (2.4) we have a factor f coming from $\sqrt{-g}$ and a factor f^{-1} from $g^{\alpha\beta}$, hence the two simplify and the f drops out completely from the Polyakov action, that thus results equivalent to the Nambu-Goto one. This is not accidental because Polyakov's action enjoys, on top of the usual Poincaré and reparametrization symmetry, a new kind of symmetry that goes under the name of *Weyl symmetry*. Its action is to send

$$g_{\alpha\beta}(\sigma)\to\Omega^2(\sigma)g_{\alpha\beta}(\sigma),$$

for any given[1] $\Omega^2(\sigma)$. The action of such (gauge, since it depends on σ) symmetry is to leave the angles unchanged while, at the same time, locally changing the distances between points; see Figure 2.1 for a cartoon.

2.1.3 Gauge fixing

Using the two gauge symmetries possessed by Polyakov's action (namely reparametrization and Weyl symmetry), we can turn $g_{\alpha\beta}$ into the flat metric[2] $\eta_{\alpha\beta} = \mathrm{diag}(-1,1)$ and, inserting this metric in (2.4), and computing the equations of motion for X^μ, we get

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta X^\mu)\Big|_{g_{\alpha\beta}=\eta_{\alpha\beta}} = 0 \Rightarrow \partial^\alpha\partial_\alpha X^\mu = 0. \tag{2.6}$$

The equations of motion for the $g_{\alpha\beta}$ in (2.5), now that we have fixed $g_{\alpha\beta}$ through gauge fixing, become constraints to be satisfied. Actually, since they are obtained from the variation of the action with respect to the metric, they also give (something proportional to) the stress-energy tensor $T_{\alpha\beta}$ that, in the flat-metric gauge, is given by

$$T_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}X^\mu X_\mu, \tag{2.7}$$

and that, in light of (2.5), must satisfy $T_{\alpha\beta}=0$. These constraints translate into (here and in the following $\dot{X}\equiv\partial_\tau X$ and $X'\equiv\partial_\sigma X$)

$$\begin{aligned} T_{01} &= \dot{X}\cdot X' = 0\\ T_{00} = T_{11} &= \frac{1}{2}(\dot{X}^2 + X'^2) = 0. \end{aligned} \tag{2.8}$$

[1]Here (and in the following) we write $f(\sigma)$ to indicate $f(\tau,\sigma)$.

[2]This is only possible if there is no topological obstruction.

2.1.4 Boundary conditions

When we vary Polyakov's action in the flat metric gauge, i.e. when we vary the action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X \cdot \partial^\alpha X, \tag{2.9}$$

with respect to X^μ to obtain their equation of motion, we get the boundary term

$$-\frac{1}{2\pi\alpha'}\int d\tau \big(X'_\mu \delta X^\mu\big|_{\sigma=\pi} - X'_\mu \delta X^\mu\big|_{\sigma=0}\big).$$

(Note that we specialized to the case of open strings because, in the case of closed strings, thanks to the periodic boundary conditions[3] $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$, that must be satisfied by the δX^μs as well, the boundary term vanishes automatically). This term vanishes in two cases

1. *Neumann boundary conditions.* In this case we require that the momentum's component normal to the worldsheet's boundary vanishes, namely

$$\partial_\sigma X^\mu = 0 \quad \text{at} \quad \sigma = 0, \pi. \tag{2.10}$$

2. *Dirichlet boundary conditions.* In this case we require that the string's ends stay fixed at some point, meaning

$$X^\mu(\tau, 0) = X_0^\mu \quad \text{and} \quad X^\mu(\tau,\pi) = X_\pi^\mu \tag{2.11}$$

Let's now consider Dirichlet boundary conditions for some coordinates and Neumann for the others, i.e.[4]

$$\begin{aligned} \partial_\sigma X^a = 0 \quad &\text{for} \quad a = 0,\ldots,p \\ X^I = c^I \quad &\text{for} \quad I = p+1,\ldots,D \end{aligned} \tag{2.12}$$

These conditions break the Lorentz group $SO(1, D-1) \to SO(1,p)\times SO(D-p-1)$ (and indeed, because of this, Dirichlet boundary conditions have not been so popular among physicist for a period of time), but at the same time generate a special hypersurface: the one on which the endpoints of the string are fixed. This hypersurface is usually referred to as Dp-brane (or simply D-brane), where D stands for Dirichlet, while p is the number of spatial dimensions of the brane.

As a final remark, we will not consider the case where Dirichlet boundary conditions are applied to X^0 as this would give birth to something called *instantons*.

2.1.5 Lightcone coordinates and modes expansion

We can now solve the equations of motion for X^μ (2.6) by introducing the so-called *lightcone* coordinates $\sigma^\pm = \tau \pm \sigma$ in which the worldsheet metric is $ds^2 = -d\sigma^+ d\sigma^-$. In these coordinates X's equations of motion read $\partial_+\partial_- X^\mu = 0$ and admit the general solution

$$X^\mu(\tau,\sigma) = X^\mu_{\mathrm{L}}(\sigma^+) + X^\mu_{\mathrm{R}}(\sigma^-)$$

where R stands for right-moving while L stands for left-moving. Using this form for the X^μ fields, plus the newly introduced lightcone coordinates, we can express the constraints (2.8) as

$$(\partial_+ X^\mu)^2 = (\partial_- X^\mu)^2 = 0, \tag{2.13}$$

where

$$\partial_- X^\mu = \partial_- X^\mu_{\mathrm{R}} \quad \text{and} \quad \partial_+ X^\mu = \partial_+ X^\mu_{\mathrm{L}}.$$

Furthermore, we can express the left- and right- component fields in terms of Fourier modes. Let us split the discussion on Fourier modes into the closed and open string cases; we first focus on the case of closed strings.

[3]In the case of closed strings the boundary term would be $-\frac{1}{2\pi\alpha'}\int d\tau\big(X'_\mu\delta X^\mu\big|_{\sigma=2\pi} - X'_\mu \delta X^\mu\big|_{\sigma=0}\big)$.
[4]In comparison to (2.11), here we have $X_0^I = X_\pi^I$

Closed strings

In this case, the two components admit the mode expansion given by

$$\begin{aligned} X^\mu_{\mathrm{R}}(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}, \\ X^\mu_{\mathrm{L}}(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}, \end{aligned} \tag{2.14}$$

where x^μ is the center-of-mass position and p^μ is the string's total momentum. Because of the term linear in sigma in both X^μ_{R} and X^μ_{L}, only the sum of them satisfies the closed-string periodic boundary conditions. Moreover, the reality condition for X^μ translates into

$$\alpha^\mu_n = (\alpha^\mu_{-n})^* \quad \text{and} \quad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^*. \tag{2.15}$$

In terms of Fourier modes, the constraints (2.13) become

$$(\partial_- X)^2 = \frac{\alpha'}{2}\sum_{m,p}\alpha_m\cdot\alpha_p e^{-i(m+p)\sigma^-} = \frac{\alpha'}{2}\sum_{m,n}\alpha_m\cdot\alpha_{n-m}e^{-in\sigma^-} \equiv \alpha'\sum_n L_n e^{-in\sigma^-}, \tag{2.16}$$

where we introduced the mode $\alpha^\mu_0 \equiv \sqrt{\frac{\alpha'}{2}}p^\mu$ and the *Virasoro generators*

$$L_n \equiv \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m. \tag{2.17}$$

The same discussion applies to the modes $\tilde{\alpha}^\mu_n$ for which we have $\tilde{\alpha}^\mu_0 \equiv \alpha^\mu_0$ and

$$\tilde{L}_n = \frac{1}{2}\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m.$$

Open strings

In the case of open strings, the mode expansion of the right- and left- components is given by

$$\begin{aligned} X^\mu_{\mathrm{R}}(\sigma^-) &= \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}, \\ X^\mu_{\mathrm{L}}(\sigma^+) &= \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}, \end{aligned} \tag{2.18}$$

where the missing factor of $\frac{1}{2}$ in front of p^μ compared to (2.14) is to retain the definition of p^μ as the string's spacetime momentum when $\sigma \in [0,\pi]$.

Thanks to either Neumann's or Dirichlet's boundary conditions we only have one set of oscillator modes. Indeed, to meet Neumann's boundary conditions $\partial_\sigma X^a = 0$ at $\sigma = 0, \pi$, the Fourier modes must be such that

$$\alpha^a_n = \tilde{\alpha}^a_n$$

while, for Dirichlet's boundary conditions $X^I = c^I$, we must require that

$$x^I = c^I, \quad p^I = 0, \quad \alpha^I_n = -\tilde{\alpha}^I_n.$$

For open strings the zero mode is $\alpha^\mu_0 = \sqrt{2\alpha'}p^\mu$

2.1.6 Lightcone gauge quantization

Having a gauge theory in our hands, and wishing to quantize it, we could follow different strategies: either quantize the theory with all its gauge freedom kept intact, to then impose the constraints dictated by gauge fixing directly on the physical states of the theory (this is what is done in Gubta-Bleuler quantization of QED); or determine the physical states at the classical level by imposing all constraints to then quantize just the physical degrees of freedom. This latter option is the one we will choose for the time being, and it goes under the name of *lightcone quantization.* There is also a third possibility: BRST quantization. We initially focus on the closed-string case to later quote the results for open strings as well.

In Section 2.1.3 we fixed the gauge freedom by setting the worldsheet metric equal to $\eta_{\alpha\beta}$; however this is not the end of the story. Indeed, since we have two gauge transformation in our gauge-shed, we could always design a coordinate change $\sigma \to \tilde{\sigma}(\sigma)$ such that

$$\eta_{\alpha\beta} \to \Omega^2(\sigma)\eta_{\alpha\beta}, \tag{2.19}$$

and then reabsorb the $\Omega^2(\sigma)$ with a Weyl transformation. In terms of lightcone coordinates, wherein the worldsheet metric takes the form $ds^2 = -d\sigma^+ d\sigma^-$, the transformations realizing (2.19) are simply those for which

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \qquad \text{and} \qquad \sigma^- \to \tilde{\sigma}^-(\sigma^-); \tag{2.20}$$

hence these represent the residual gauge freedom we have yet to fix. It turns out that the best way to fix this residual gauge freedom is to choose spacetime coordinates "adapted" to the lightcone ones used in the worldsheet. These are given by

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right) \tag{2.21}$$

while leaving the other $D-2$ fields untouched; in these, the spacetime Minkowski metric then transforms into

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} dX^i dX^i. \tag{2.22}$$

We can now use the freedom from (2.20), and thus at the same time fixing once and for all the gauge freedom, by choosing coordinates in which the solution to the equations of motions for $X^+ = X^+_{\mathrm{L}}(\sigma^+) + X^+_{\mathrm{R}}(\sigma^-)$ such that

$$X^+_{\mathrm{L}} = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^+ \quad \text{and} \quad X^+_{\mathrm{R}} = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^- \quad \Rightarrow \quad X^+ = x^+ + \alpha' p^+ \tau. \tag{2.23}$$

Solving for X^- is particularly easy in the lightcone gauge and using the ansatz $X^- = X^-_{\mathrm{L}}(\sigma^+) + X^-_{\mathrm{R}}(\sigma^+)$: it suffices to employ the constraints (2.13) and the solution for X^+ (2.23) to get

$$\partial_+ X^-_{\mathrm{L}} = \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \quad \text{and} \quad \partial_- X^-_{\mathrm{R}} = \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i, \tag{2.24}$$

which, in terms of the mode expansion (2.14), implies

$$\alpha^-_n = \frac{1}{\sqrt{2\alpha'}p^+}\sum_{m=-\infty}^{+\infty}\sum_{i=2}^{D-2} \alpha^i_{n-m}\alpha^i_m \quad \text{and} \quad \frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0}\alpha^i_n \alpha^i_{-n}\right) \tag{2.25}$$

and similarly for the $\tilde{\alpha}^-_n$ modes. This means that, up to an integration constant represented by x^- in the mode expansion of X^-, we can fully determine X^- starting from the fields in the $i = 1, \ldots, D-2$ *transverse* directions. Employing that the momentum p^- can be determined both in terms of the modes α and $\tilde{\alpha}$, the classical mass of the string is

$$M^2 = -p_\mu p^\mu = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n. \tag{2.26}$$

To summarize, the physical solution is parametrized in terms of $2(D-2)$ transverse oscillators modes α_n^i and $\tilde{\alpha}_n^i$, along with the other parameters x^i, p^i, p^+ and x^-.

Now, we quantize the physical degrees of freedom; to do this we must impose the commutation relations

$$[x^i, p^i] = i\delta^{ij}, \quad [x^-, p^+] = -i, \quad [\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0}. \tag{2.27}$$

From these relations, we see that the oscillator modes α_n^i and α_{-n}^i have (up to a renormalization such as $\alpha_n^i \to n^{-1/2}\alpha_n^i$) the same commutation relations a pair of creation and annihilation operators would have if we took α_n^i to be the annihilation operator for $n > 0$ and the creation one for $n < 0$.

Obtaining the on-shell condition (2.26) starting from the definition of p^- in terms of the oscillator modes in (2.25) (and similarly in terms of $\tilde{\alpha}$), is a tad more challenging since now the modes are operators satisfying non-trivial commutation relation. The usual definition for a product of operators is to use the normal ordering that, in this case, produces a term that was not there at the classical level (we indicate the normal ordering by : :)

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0} :\alpha_{-n}^i\alpha_n^i: + \frac{(D-2)}{2}\sum_{n>0} n\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0} :\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i: + \frac{(D-2)}{2}\sum_{n>0} n\right), \tag{2.28}$$

or, in terms of the number operators of the harmonic oscillator (to be precise, these are not the *number* operators because the oscillators modes are not normalized to be creation and annihilation operators)

$$N = \sum_{i=1}^{D-2}\sum_{n>0} \alpha_{-n}^i\alpha_n^i \qquad \text{and} \qquad \widetilde{N} = \sum_{i=1}^{D-2}\sum_{n>0} \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i, \tag{2.29}$$

as

$$M^2 = \frac{4}{\alpha'}\left(N + \frac{D-2}{2}\sum_{n>0} n\right) = \frac{4}{\alpha'}\left(\widetilde{N} + \frac{D-2}{2}\sum_{n>0} n\right). \tag{2.30}$$

The new term, namely the divergent sum $\sum_{n>0} n$, can be seen to be equal to $-1/12$, hence we have

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\widetilde{N} - \frac{D-2}{24}\right). \tag{2.31}$$

This last condition, along with all the other forms of it, is also known as *level matching* condition, the reason being that it requires the same excitations number in both the right- and left- moving oscillators. Through this condition we can determine the spectrum of the theory; but before doing this a few results about the open string case.

Suppose the ends of the open string are confined on a Dp brane, hence we will have Neumann boundary conditions for X^a with $a = 0, \dots, p$ and Dirichlet ones for the remaining components X^I with $I = p+1, \dots, D$. As we already saw in Section 2.1.5, this implies that x^I and p^I are already fixed; so the non-trivial quantization is necessary just for x^a, p^a and α_n^μ. In this case the lightcone coordinates are chosen as

$$X^\pm = \frac{1}{\sqrt{2}}\big(X^0 \pm X^p\big). \tag{2.32}$$

The quantization now follows the same procedure as for closed strings. Another difference emerges once we arrive at the mass-shell condition which reads

$$M^2 = \frac{1}{\alpha'}\left(\sum_{i=1}^{p-1}\sum_{n>0} \alpha_{-n}^i\alpha_n^i + \sum_{i=p+1}^{D-1}\sum_{n>0} \alpha_{-n}^i\alpha_n^i - a\right) \tag{2.33}$$

with $a = \frac{D-2}{2}$.

Now, back to the spectrum of closed strings.

2.1.7 String spectrum

The foundation of the Hilbert space is the vacuum state $|0; p\rangle$, where 0 refers to the lack of stringy excitations, and p is the momentum operator's eigenvalue.

The Tachyon

The first excited state is the one in which no oscillator is excited, resulting in $N = 0 = \widetilde{N}$ and with a negative mass that can be read off of (2.31). This state is known as *tachyon*. It is a problem for bosonic string theory that will actually get resolved once fermions are introduced.

The first excited state

The next level is for $N = \widetilde{N} = 1$, which means acting with one creation operator α^j_{-1} and, because of the level-matching condition, with an $\tilde{\alpha}^i_{-1}$ operator as well. The $(D-2)^2$ states we obtain are

$$\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0;p\rangle \tag{2.34}$$

having mass

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right). \tag{2.35}$$

The problematic aspect of these states is how to precisely fit them into some representation of the full Lorentz group $\mathrm{SO}(1, D-1)$, because the operators α^i and $\tilde{\alpha}^i$ both sit in the vector representation of $\mathrm{SO}(D-2)$ and, from Wigner's classification of representations of the Poincaré group, we know that massive particles, whose momentum can be set to be $p^\mu = (p, 0, \dots, 0)$, transform under the group of spatial rotations $\mathrm{SO}(D-1)$ while massless particles, whose momentum can be set to be $p^\mu = (p, 0, \dots, 0, p)$, sit in a representation of $\mathrm{SO}(D-2)$. There is no way of fitting the $(D-2)^2$ states into a representation of $\mathrm{SO}(D-1)$, so our only hope if we want to fit these states in representation of the Lorentz group is for them to be massless which then, looking at (2.35), occurs only if

$$D = 26; \tag{2.36}$$

this is the *critical dimension*. Only a spacetime with this dimension will preserve Lorentz symmetry. The states (2.34) then fill out a $\mathbf{24} \otimes \mathbf{24}$ representation of $\mathrm{SO}(24)$ that can be decomposed into three irreducible representations

$$\text{traceless symmetric} \oplus \text{antisymmetric} \oplus \text{singlet=(trace)} \tag{2.37}$$

The tensors associated with each of these three representations are then a symmetric tensor $G_{\mu\nu}(X)$, being the field of the graviton, an antisymmetric tensor $B_{\mu\nu}(X)$, known as *Kalb-Ramon field* and a scalar field $\Phi(X)$ known as *dilaton*.

Higher excited states

We just mention that once $D = 26$ has been selected, it is actually possible to show that all higher excited states fit into $\mathrm{SO}(D-1)$ representations; i.e. they represent massive particles.

Open strings' spectrum

For open strings as well, the choice of $D = 26$ suffices to maintain Lorentz invariance intact; and the similarities do not end here. Indeed, the lowest-lying state, i.e. the one for which

$$N = \sum_{i=1}^{p-1}\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{i=p+1}^{D-1}\sum_{n>0}\alpha^i_{-n}\alpha^i_n \tag{2.38}$$

is equal to zero, is again a tachyon, meaning a state with negative mass. Instead, the first excited states for $N = 1$ are massless and can either

- be generated by the application of α^a_{-1} for $a = 1, \dots, p-1$ to $|0;p\rangle$. These states are the quanta of a spin-1, massless gauge-field A_a living on the brane;
- be generated by the application of α^I_{-1} for $I = p+1, \dots, D-1$ to $|0;p\rangle$. For each I these are the states of a scalar field ϕ^I living on the brane and can be interpreted as representing the oscillations of the brane in the I directions.

2.2 Conformal Field Theory

In this section we continue studying the worldsheet theory using (and introducing) techniques from two-dimensional Conformal Field Theory (CFT for short). As usual, we will give a quite concise treatment of the matter, referring the reader to the references mentioned at the beginning of the chapter for a more in-depth discussion on what is about to follow.

2.2.1 Conformal transformations

For starters, with conformal transformation it is intended a transformation leaving the angles unchanged while, at the same, altering the length of things. For this to be the case, the transformation $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ must act on the metric as

$$g_{\mu\nu}(\sigma) \to \Omega^2(\sigma) g_{\mu\nu}(\sigma). \tag{2.39}$$

A CFT then is nothing more than a field theory that is invariant under conformal transformations.

Since we will deal with the worldsheet theory, this means that we will restrict to the case of two-dimensional CFT. Furthermore, in the context of worldsheet theory, we will restrict ourselves to the case in which the worldsheet metric has euclidean signature, corresponding to choosing worldsheet coordinates $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$, that are later used to define the complex coordinates

$$z = \sigma^1 + i\sigma^2 \quad \text{and} \quad \bar{z} = \sigma^1 - i\sigma^2. \tag{2.40}$$

In terms of z and $\bar{z}$, we define the holomorphic derivatives

$$\partial_z \equiv \partial = \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \partial_{\bar{z}} \equiv \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \tag{2.41}$$

and the components of vectors

$$V^z = V^1 + iV^2, \quad V^{\bar{z}} = V^1 - iV^2, \quad V_z = \frac{1}{2}(V^1 - iV^2), \quad V_{\bar{z}} = \frac{1}{2}(V^1 + iV^2) \tag{2.42}$$

where the indices are raised and lowered with the metric

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0; \qquad g^{z\bar{z}} = g^{\bar{z}z} = 2, \quad g^{zz} = g^{\bar{z}\bar{z}} = 0 \tag{2.43}$$

meaning that the line element consequently being equal to

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz d\bar{z}. \tag{2.44}$$

Using these definitions the volume element is $d^2z = 2d\sigma^1 d\sigma^2$, implying that the delta function in the complex coordinates, giving $\int d^2\delta(z,\bar{z}) = 1$, and the one in the σ coordinates, giving $\int d\sigma^1 d\sigma^2 \delta(\sigma) = 1$ are related to one another by a factor 2.

Looking again at (2.44), and comparing it to (2.39), i.e. the general change of the metric under a conformal transformation, we notice that such transformation is realized by coordinate changes such that $z \to f(z)$ and equally for $\bar{z}$; meaning that, in terms of complex coordinates, the conformal transformations are those generated by holomorphic (and antiholomorphic) functions. Looking at infinitesimal (anti)holomorphic coordinate transformations, it is possible to show that their generators are

$$\ell_n = -z^{n+1}\partial \quad \text{and} \quad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial} \tag{2.45}$$

and that they satisfy the algebra

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} \quad \text{and} \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}, \tag{2.46}$$

while $[\ell_m, \bar{\ell}_n] = 0$.

2.2.2 Stress-energy tensor in classical CFT

In a CFT, the stress-energy tensor $T_{\alpha\beta}$, that we define in the following way,

$$T_{\alpha\beta} = -\frac{4\pi\alpha'}{\sqrt{g}}\frac{\delta S}{\delta g^{\alpha\beta}} \tag{2.47}$$

has the important property of being traceless as a consequence of the action's invariance under a scale transformation. In complex coordinates, the tracelessness condition reads $T_{z\bar{z}} = T_{\bar{z}z} = 0$; and combining this with the conservation equation $\partial_\alpha T^{\alpha\beta} = 0$ gives

$$\partial T_{\bar{z}\bar{z}} = 0 \Rightarrow T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}) \quad \text{and} \quad \bar{\partial} T_{zz} = 0 \Rightarrow T_{zz} \equiv T(z) \tag{2.48}$$

meaning that $T(z)$ ($\bar{T}(\bar{z})$) is a holomorphic (antiholomorphic) function.

Actually, the stress-energy tensors enters the conserved currents generated by general holomorphic and antiholomorphic coordinate transformations. Indeed, considering a given infinitesimal coordinate change such as $z' = z + \epsilon(z)$ and $\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})$ it is possible to show that the conserved currents correlated with these transformations are

$$\begin{aligned} \delta z =& \epsilon(z) \quad \text{and} \quad \delta\bar{z} = 0: \qquad J^z = 0, \quad J^{\bar{z}} = T(z)\epsilon(z);\\ \delta\bar{z} =& \bar{\epsilon}(\bar{z}) \quad \text{and} \quad \delta z = 0: \qquad \bar{J}^z = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \quad \bar{J}^{\bar{z}} = 0, \end{aligned} \tag{2.49}$$

thus J is holomorphic and $\bar{J}$ is antiholomorphic.

2.2.3 Operator Product Expansion

In CFT the word *field* (or local operator) refers to any local combination we can write down; so if a theory contains a scalar field $\phi(z,\bar{z})$, then ϕ itself will be a field, along with all of its derivatives, and combinations of them. Even e^ϕ will be a field. Furthermore, fields in CFTs are characterized by their *weights* h and $\tilde{h}$, meaning numbers determining the transformation properties of a given field ϕ under $z \to w(z)$ and its antiholomorphic counterpart as

$$\phi(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\tilde{h}} \phi(w,\bar{w}). \tag{2.50}$$

But how does an operator transform under a generic conformal transformation? To answer this question, we must look at *operator product expansion*, or OPE for short. Denoting all the operators of a CFT with $\mathcal{O}_i$, where i labels all said operators, the OPE tells us that we can expand the time-ordered product of fields $\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})\dots\rangle$ as

$$\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})\dots\rangle = \sum_k C^k_{i,j}(z-w,\bar{z}-\bar{w})\langle \mathcal{O}_k(w,\bar{w})\dots\rangle, \tag{2.51}$$

where the ... represent any other operator insertion with the only requirement that these operators must not be as close to $\mathcal{O}_i$ and $\mathcal{O}_j$ as these two are to each other. As $z \to w$, singular terms emerge from the OPE; these will be the interesting feature of OPEs because they dictate how a given field transforms under a conformal transformation. To see this, we first recall the result of Ward identities, which can be obtained starting from the path integral formulation of the time-ordered correlation function. Inside the path integral we can perform the field redefinition $\phi \to \phi' = \phi + \epsilon(\sigma)\phi$ through a symmetry wherein we promote the parameter of the symmetry, here indicated by ϵ, to a function of the spacetime coordinates, here indicated by σ. Since this is just a renaming of dummy integration variable, the variation of the path integral under such renaming must vanish. Then, keeping just the linear terms in ϵ, where ϵ has support just in the vicinity of the insertion point of the operator $\mathcal{O}_1(\sigma_1)$ among all others in the correlation function, allows us to obtain

$$-\frac{1}{2\pi}\int_\epsilon \partial_\alpha \langle J^\alpha(\sigma)\mathcal{O}_1(\sigma_1)\dots\rangle = \langle \delta\mathcal{O}_1(\sigma_1)\dots\rangle, \tag{2.52}$$

where $\int_\epsilon$ stands for an integral over the region where ϵ is non-zero. From this, selecting just the class of conformal transformations, the conserved current in (2.52) is either holomorphic or antiholomorphic, hence the integral on the non-zero ϵ region picks up just the residue; meaning

$$\frac{i}{2\pi}\oint dz\, J_z(z)\mathcal{O}_1(\sigma_1) = -\operatorname*{Res}_{z=z_0}[J_z\mathcal{O}_1]. \tag{2.53}$$

In terms of OPE, this means that the OPE between J_z and $\mathcal{O}_1$ will contain the singular term producing such a residue. Furthermore, using the Ward identities, we see that the variation of the operator $\mathcal{O}_1$ under the change $\delta z = \epsilon(z)$ is

$$\delta\mathcal{O}_1(\sigma_1) = -\operatorname{Res}[J_z(z)\mathcal{O}_1(\sigma_1)] = -\operatorname{Res}[\epsilon(z)T(z)\mathcal{O}_1(\sigma_1)] \tag{2.54}$$

and similarly for the change $\delta\bar{z} = \bar{\epsilon}(\bar{z})$

$$\delta\mathcal{O}_1(\sigma_1) = -\operatorname{Res}[\bar{J}_{\bar{z}}(\bar{z})\mathcal{O}_1(\sigma_1)] = -\operatorname{Res}[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\mathcal{O}_1(\sigma_1)]. \tag{2.55}$$

In the end, all of this means that knowing the OPE between the stress-energy tensor and a given operator equals being capable of computing the transformation properties of said operator under conformal symmetry. In particular, a field having conformal weights $(h, \tilde{h})$ has the following OPE with T and $\bar{T}$

$$\begin{aligned} T(z)\mathcal{O}(w,\bar{w}) &= \dots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \dots \\ \bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) &= \dots + \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \dots \end{aligned} \tag{2.56}$$

For a *primary operator* $\mathcal{O}$ the dots in the above formulas would only contain regular terms.

2.2.4 CFT and Polyakov action

Using the definition (2.47) on the Polyakov action from (2.9) with no minus sign in front of it due to the Euclidean signature of the metric and for a single scalar field X, we find the complex-coordinates stress-energy tensor to be

$$T = -\frac{1}{\alpha'} :\partial X\partial X: \equiv -\frac{1}{\alpha'}\lim_{z\to w}\big(\partial X(z)\partial X(w) - \langle\partial X(z)\partial X(w)\rangle\big), \tag{2.57}$$

where the normal order removes the singular terms, and a similar definition holds for $\bar{T}$ where $\partial \to \bar{\partial}$. Knowing that the OPE between $\partial X(z)$ and $\partial X(w)$ is (from now on we just write the non-singular pieces of the OPEs)

$$\partial X(z)\partial X(w) = -\frac{\alpha'}{2}\frac{1}{(z-w)^2}, \tag{2.58}$$

it is possible to show, by means of the definition of primary operator and Wick theorem, that

- The field ∂X is a primary operator of weights $h = 1$ and $\tilde{h} = 0$;
- The field $:e^{ikX}:$ has weights $h = \tilde{h} = \frac{\alpha' k^2}{4}$.

Computing the OPE of the stress-energy tensor with itself reveals T is not a primary field, the reason being that the OPE has a singular term $\sim (z-w)^{-4}$; indeed,

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{2.59}$$

The issue goes deeper, as this is not just a property of the stress-energy tensor coming from the Polyakov action, but is a general property. In fact, by dimensional reasoning, it is possible to show that in any CFT

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{2.60}$$

where c is known as *central charge* (the central charge for the antiholomorphic component $\bar{T}$ is $\tilde{c}$). Comparing (2.59) and (2.60), one concludes that for a single scalar field $c = \tilde{c} = 1$; for D scalar fields X^μ instead $c = \tilde{c} = D$.

2.2.5 Virasoro algebra

The complex plane can be parametrized by $z = e^{-iw}$ where $w = \sigma^1 + i\sigma^2$ parametrizes a cylinder for $\sigma^1 \in [0, 2\pi]$ and $\sigma^2 \in \mathbb{R}$. In the complex plane, since T and $\bar{T}$ are holomorphic and antiholomorphic respectively, they can be expanded in Laurent series as

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \quad \text{and} \quad \bar{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}} \tag{2.61}$$

with the modes L_m and $\tilde{L}_m$ that can be computed, starting from the T and $\bar{T}$ respectively, by contour integrals encircling the origin as

$$L_m = \frac{1}{2\pi i} \oint \frac{dz}{z} z^{m+2} T(z) \quad \text{and} \quad \tilde{L}_m = \frac{1}{2\pi i} \oint \frac{d\bar{z}}{\bar{z}} \bar{z}^{m+2} \bar{T}(\bar{z}). \tag{2.62}$$

Actually, starting from these definitions, the commutator between L_m and L_n may be computed from

$$[L_m, L_n] = \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{m+1} w^{n+1} T(z) T(w) \tag{2.63}$$

and adequately modifying the contours to find

$$[L_m, L_n] = \oint \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+1} T(z) T(w) \tag{2.64}$$

that, employing the use of the TT OPE, yields

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}. \tag{2.65}$$

This is known as *Virasoro algebra.*

To build a representation of the Virasoro algebra we start from a state $|\psi\rangle$ eigenvector of both L_0 and $\tilde{L}_0$ with eigenvalue h and $\tilde{h}$ respectively. The state $L_n |\psi\rangle$ will then be an L_0-eigenvector with eigenvalue $(h-n)$; thus L_n acts as a raising operator for $n < 0$ and a lowering one for $n > 0$. For the spectrum to be bounded from below, there must be some states that are annihilated by all L_n and $\tilde{L}_n$ for n positive. States satisfying this requirement are known as *primary* states or *highest weight* states. Acting on primary states with raising operators will a tower of so-called *descendants* states; the whole set of such states forms a *Verma* module.

Furthermore, requiring the Hamiltonian to be Hermitian, reflects on the stress-energy tensor modes as $L_n = L^\dagger_{-n}$; this in turns enables us to deduce that the eigenvalue h must be non-negative i.e., $h \geq 0$, while the central charge c must be positive.

2.2.6 Operators and states

The map between the cylinder parametrized by the coordinate w where now $\sigma^2 \leq 0$, and the unit disk given by e^{-iw}, permits the setting up of an isomorphism between operators and states. In essence, specifying an initial state on the cylinder for $\sigma^2 \to -\infty$, maps to specifying the fields' configuration at the point $z = 0$, effectively defining an operator at the origin; namely, the so-called *vertex operator.* We are not going to go into the details about this, so we just cite two of the most relevant examples of such isomorphism which are the identity operator to vacuum correspondence i.e., $\mathbb{1} \leftrightarrow |0;0\rangle$, and the $:e^{ik\cdot X(0,0)}: \leftrightarrow |0;k\rangle$ correspondence. Using this correspondence, it can be showed that the action of L_0 ($\tilde{L}_0$) on the state $|\mathcal{O}\rangle$ corresponding to a given primary operator $\mathcal{O}$ of weight $(h, \tilde{h})$ is such that

$$L_0 |\mathcal{O}\rangle = h |\mathcal{O}\rangle \quad \text{and} \quad \tilde{L}_0 |\mathcal{O}\rangle = \tilde{h} |\mathcal{O}\rangle \tag{2.66}$$

$$L_m |\mathcal{O}\rangle = \tilde{L}_m |\mathcal{O}\rangle = 0 \quad \text{for} \quad m > 0; \tag{2.67}$$

hence the state $|\mathcal{O}\rangle$ is a highest weight state.

2.3 Background fields

Up to this point the metric in the D-dimensional manifold where the motion of the string takes place has been flat; i.e., it has been $\eta_{\mu\nu}$. The generalization to the string theory on non-flat metric passes by the action

$$S_g = \frac{1}{4\pi\alpha'}\int_M d^2z\,\sqrt{h}h^{\alpha\beta}g_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu \tag{2.68}$$

where now $h_{\alpha\beta}$ is the worldsheet metric, while $g_{\mu\nu}$ is the metric in the D-dimensional manifold. In this context, the metric $g_{\mu\nu}$ is said to be a *background field*; and it is not the only such field we can introduce. Indeed, we can also introduce the two-form gauge field $B_{\mu\nu}$, known as B field, by means of the action

$$S_B = \frac{1}{4\pi\alpha'}\int_M d^2z\,\epsilon^{\alpha\beta}B_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu; \tag{2.69}$$

keep in mind though that a B field can only be introduced in theories of oriented, bosonic strings. Finally, the last background field we mention is the dilaton Φ, introduced through the action

$$S_\Phi = \frac{1}{4\pi}\int_M d^2z\,\sqrt{h}\Phi(X)R^{(2)}(h) \tag{2.70}$$

with $R^{(2)}(h)$ is the scalar curvature of the two-dimensional string worldvolume computed from the worldsheet metric $h_{\alpha\beta}$. This last field plays a crucial role in defining the string perturbation expansion, as can be more easily seen in the case of constant dilaton where the action (2.70) turns out to be proportional to the Euler characteristic $\chi(M)$ (the proportionality factor being the constant value of the dilaton).

2.4 Ramond-Neveu-Schwarz strings

In this section we will briefly review how fermions are added to the action of bosonic string theories in the Ramond-Neveu-Schwarz (RNS) formalism and their effect on the spectrum of said theories. We will skip most of the technical discussion on supersymmetry because, at the end of the day, we will not need any of it in what follows. The interested reader may want to read the references cited at the beginning of the chapter.

2.4.1 Adding fermions

In this formalism the addition of fermions simply amounts to the introduction of the fields $\psi^\mu(\sigma)$ which are two-components spinors on the worldsheet and vectors under the D-dimensional Lorentz group, while at same time satisfying (classically) Grassmann statistics $\{\psi^\mu, \psi^\nu\} = 0$. Introducing these fields in the Polyakov action (2.9) yields

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_\alpha X_\mu\partial^\alpha X^\mu + \bar{\psi}^\mu\gamma^\alpha\partial_\alpha\psi_\mu\right) \tag{2.71}$$

where γ^α for $\alpha = 0, 1$ are the two-dimensional Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{satisfying} \quad \{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}. \tag{2.72}$$

The two components ψ^μ_A for $A = \mp$, being real, form the Majorana spinor ψ^μ given by

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}; \tag{2.73}$$

and in terms of them the fermionic part of the action (2.71) is given by (we use lightcone coordinates, so $\partial_+ \equiv \partial_{\sigma^+}$ and similarly for the $-$ and note as well that $\bar{\psi} = \psi^\dagger i\gamma^0$)

$$S_\mathrm{f} = \frac{i}{2\pi\alpha'}\int d^2\sigma\left(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+\right). \tag{2.74}$$

The action (2.71) enjoys a global symmetry exchanging the X^μ bosonic field and the ψ^μ fermionic field; this symmetry is known as *supersymmetry.* A nice way to implement supersymmetry in the action for X and ψ is to move to *superspace*, namely a space where two additional Grassmann coordinates θ_A are added on top of the two worldsheet coordinates σ. In this framework, the conformal symmetry gets generalized to what is known as superconformal symmetry and using the generalization of the Virasoro constraints, it is possible to show that for superstring theories, meaning string theories with supersymmetry added to them, the critical dimension changes from $D = 26$ to $D = 10$.

2.4.2 Boundary conditions and spectrum

Once we start solving the equations of motion for the ψ^μ field, we find that, similarly to what happened for the X^μ field, we must impose some boundary conditions. We will split the discussion for open and closed strings.

Open strings

Setting $\psi^\mu_+|_{\sigma=0} = \psi^\mu_-|_{\sigma=0}$, we find that at the other end of string, for $\sigma = \pi$, we have either

- $\psi^\mu_+|_{\sigma=\pi} = \psi^\mu_-|_{\sigma=\pi}$. These are known as *Ramond* (or R) *boundary conditions* and give rise, through the mode expansion of the field, to spacetime fermions;
- $\psi^\mu_+|_{\sigma=\pi} = -\psi^\mu_-|_{\sigma=\pi}$. These are known as *Neveu-Schwarz* (or NS) *boundary conditions* and give rise, again through the mode expansion of the field, to spacetime bosons.

Once the theory has been quantized with the help of the lightcone gauge, the first few states in the spectrum of the NS sector are a tachyonic spacetime scalar as ground state and an eight-dimensional, massless spacetime vector of SO(8), where 8 is the number of transverse directions. Higher excited states correspond to massive vector bosons. Instead, in the R sector, the ground state is made up of either a positive or negative parity Majorana-Weyl spinor, while higher excited states are massive spacetime spinors.

As of now, the spectrum is not supersymmetric, and it contains a tachyon. To fix these issues there exist a procedure known as GSO projection which, as the name suggests, *projects* out certain states. Even though we are not going to get into the specifics of the GSO projection, allow us to mention that thorough this procedure the tachyon from the NS sector gets eliminated and that, once either the positive or negative chirality ground state from the R sector has been chosen, the resulting spectrum of the open RNS string is supersymmetric, i.e. it exhibits the same number of spacetime fermionic and bosonic degrees of freedom.

Closed strings

In this case the field ψ^μ splits into right- and left-moving modes. The boundary conditions read $\psi^\mu_\pm(\sigma) = \pm\psi^\mu_\pm(\sigma + 2\pi)$, hence amounting to imposing either periodic (R) or antiperiodic (NS) boundary conditions, independently on both right and left moving components. This again divides the discussion into two subclasses (we indicate with A-B the case in which A boundary conditions have been imposed on the right-moving modes, while B on the left-moving ones)

- R-R or NS-NS. In this case we end up with spacetime bosons;
- R-NS or NS-R. In this case we end up with spacetime fermions.

For the analysis of the spectrum we must consider four sectors: R-R, NS-R, R-NS and NS-NS. Depending on whether the chirality of the R sector's ground state is the same for the left- and right-moving R sector, two different theories may be obtained: Type IIB theory if the chiralities are the same, and Type IIA theory if they are not. The massless spectrum of the two theories are as such

- Type IIB theory

- NS-NS sector: it contains a scalar (the dilaton), an antisymmetric two-form gauge field (the B field) and a symmetric, traceless rank-two tensor, the graviton;
- NS-R and R-NS sectors: each of these contains a spin $\frac{3}{2}$ gravitino and a spin $\frac{1}{2}$ fermion, the dilatino. Being this type IIB theory, both the NS-R and R-NS gravitinos and dilatinos have the same chiralities;
- R-R sector: it contains a scalar gauge field, a two-form gauge field and a four-form gauge field with self-dual field strength.

• Type IIA theory

- NS-NS sector: same as before;
- NS-R and R-NS sectors: same as before but with opposite chiralities between the gravitinos and dilatinos coming from the two sectors;
- R-R sector: it contains a one-form gauge field and a three-form gauge field.

These are just the massless spectra of the two Type II theories obtained by ignoring the higher energy excitations. When one limits to just the massless spectrum he is working in the *supergravity* limit where just the interaction between massless degrees of freedom are present. Because of this, the theories possessing the above spectrum are known as Type IIB and Type IIA *supergravity*. Similarly to what occurs in electromagnetism, where the gauge field A_μ couples to the electron, i.e. a particle with a one-dimensional worldvolume and charge equal to e, in the case of the Type II supergravity theories, the gauge fields from the R-R sector couple to extended objects having worldvolumes of appropriate dimension. These extended objects are nothing but Dp-branes, that will in turn couple to a $p+1$ form A_{p+1}. The branes that couple to such gauge fields via actions such as

$$S_{\text{int}} = \mu_p \int A_{p+1} = \frac{\mu_p}{(p+1)!} \int d^{p+1}\sigma \, A_{\mu_1 \dots \mu_{p+1}} \frac{\partial x^{\mu_1}}{\partial \sigma^0} \cdots \frac{\partial x^{\mu_{p+1}}}{\partial \sigma^p}, \tag{2.75}$$

are the only ones, among the many with different dimension that in principle we could implement in the theory, to be stable, thanks to the associated charge (μ_p in the above action) regulating the coupling to the gauge field. The (electric) charge μ_p of the Dp-brane is computed as the flux of $\star F$, where $F = dA$, through S^{8-p} via the surface integral

$$\mu_p = \int_{S^{8-p}} \star F_{p+2}, \tag{2.76}$$

where S^{8-p} is the dimensionally-correct sphere to surround a p brane in a 10-dimensional Lorentzian spacetime; namely the spacetime where both Type IIB and IIA supergravity theories live. Actually, thanks to the electromagnetic duality, the field strength F_{p+2} will generate a magnetic charge associated to the magnetic-dual of the previous Dp-brane via a similar surface integral to the one in (2.76) (modulo the Hodge star and the sphere that will now be S^{p+2}, as this is the correct dimension to surround a D$(6-p)$-brane, that is indeed the magnetic dual to a Dp-brane for $D = 10$).

Furthermore, since the two supergravity theories contain another gauge field, the antisymmetric rank-2 tensor from the NS-NS sector, they will both contain a one-dimensional object that couples electrically to it, and which we will indicate with F1 and is simply a fundamental string, and a five-dimensional object that couples magnetically to it, which will be addressed as NS5-brane.

Summing up, in IIB supergravity we have Dp-branes with $p = -1, 1, 3, 5, 7$ while in IIA we have $p = 0, 2, 4, 6$ plus the F1 and NS5 that belong to both. The D(-1)-brane in IIB is D-instanton, an object localized in time which makes sense in the Euclideanized theory.

2.5 Dualities

We now give a brief definition of both T and S duality since they will be employed in later chapters; let us start with T duality.

2.5.1 T duality

This duality is the one relating a theory with one dimension compactified on a circle of radius R to the one compactified on a circle of radius $\widetilde{R} = \frac{\alpha'}{R}$. The first consequence of acting with T duality on a compact direction is that it interchanges Type IIA and IIB theories, as it changes the chirality of the states, that is indeed the differentiating factor between the two theories. This immediately generalizes to the fact that an even number of T dualities will keep you in the same theory you started from, be it IIA of IIB, while an odd number will cause you to jump from one to the other. T duality also influences momentum charges and winding numbers, since taking the reciprocal of the radius R will transform a momentum charge along the circle of radius R into a winding number on the circle of radius $\widetilde{R}$ and vice versa. Moreover, T duality interchanges Neumann and Dirichlet boundary conditions, meaning that if an open string had Neumann boundary conditions along the compact direction along which T duality acts, it will have Dirichlet boundary conditions along the T-dual compact direction, and vice versa. Conversely, boundary conditions along directions unaffected by the T duality stay the same. Given the correlation between boundary conditions and D-branes, the exchange between Neumann and Dirichlet boundary conditions will change the dimensionality of the Dp-brane on which the open string ends. In fact, if the T duality is performed along one of the p directions, the brane in the T-dual world will be a D$(p-1)$-brane; on the other hand, if the direction along which the T duality acts is not among the p along which the Dp-brane extends, the brane in the T-dual world will be a D$(p+1)$-brane. In the midst of this, we need to keep in mind that we can only perform T duality along directions that are compact. Finally, T duality also affects the dilaton, that gets shifted in such a way that the string coupling constant becomes

$$\tilde{g}_{\mathrm{s}} = \frac{\sqrt{\alpha'}g_{\mathrm{s}}}{R}. \tag{2.77}$$

2.5.2 S duality

While the former T duality was a duality affecting both IIA and IIB theories, S duality acts just on Type IIB supergravity. Its action is to exchange B field from the NS-NS sector with the two form gauge field from the R-R sector; usually called C_2 field in the literature. In doing this, branes that were charged electrically under the B field will become charged electrically under the C_2 field, meaning that the F1 and D1 branes gets exchanged: what was before a D1 brane becomes a F1 brane and vice versa. A similar thing occurs naturally to the D5 and NS5 branes, that gets exchanged as well under S duality. Conversely, the C_4 field, namely the four-form gauge field from the NS-NS sector, remains unchanged under S duality, so the same applies to the D3 brane; finally, we will not mention what goes on with the D7 brane under S duality as it would require a much more involved discussion. The last crucial effect that S duality has is to change the sign of the dilaton, mapping Φ to $-\Phi$, thereby changing the string coupling constant from g_{s} to $\frac{1}{g_{\mathrm{s}}}$, and hence relating theories with strong and weak couplings.

Chapter 3

D-branes and black holes

As we have seen in the preceding chapter, Supergravity is a regime of string theory in which we focus just on the massless excitations of the strings. In this chapter we will focus on black holes metrics emerging from selected D-branes arrangement. We will also review the fuzzball proposal, putting forward a hypothesis regarding the structure of the microstates. In particular, we will concentrate on the NS1-P picture, and the D1-D5 picture, the two being connected by a chain of T and S dualities, as we will later see.

3.1 D-branes solutions

Consider the case of a D2-brane in type IIA supergravity along the 0, 1, and 2 directions. The metric and dilaton are given by [16]

$$ds^2 = H^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2) + H^{1/2}(dx_3^2 + \cdots + dx_9^2), \quad e^{\phi} = H^{1/4}, \quad C_{012} = H^{-1}, \tag{3.1}$$

where H can be thought as a Maxwell potential in the transverse $\mathbb{R}^7 = dx_3^2 + \cdots + dx_9^2$ where, according to supergravity's equations of motion, it must obey Laplace equation

$$\Delta_7 H = \rho_{D2} \tag{3.2}$$

with ρ_{D2} being the source that, for a stack of N_{D2} branes at the origin of the transverse space, is given by $\rho_{D2} = N_{D2}\delta(\boldsymbol{r}_7)$. In this case, the solution to (3.2) is given by

$$H = 1 + \frac{N_{D2}}{r^5} \tag{3.3}$$

and, with this solution, the metric (3.1) presents a naked singularity at $r = 0$ (as can be checked by computing the C-field's energy $\frac{1}{4!}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}$).

3.2 Black holes from D-branes

As we will now see, by properly choosing the D-branes' configuration we can produce solutions that exhibit the trademark of GR's black holes: a singularity shielded by a horizon. To do this, we will focus on supersymmetric black holes, i.e. solutions whose mass M is equal to their charge Q, as in this case multicenter solution are possible thanks to the perfect balancing of gravitational attraction and electrostatic repulsion. This is the same discussion we already met in Section 1.2.2 and, as in that case, for $M = Q$ the solution is said to be *extremal*.

3.2.1 Four charge black hole

We begin the analysis by considering four branes in IIA supergravity: three D2 branes and one D6 brane positioned as indicated in Table 3.1, where a number signals that the given brane extends along that direction, while a "$-$" signals that the brane is smeared along that particular direction (so, the

Table 3.1: Placement of branes in the four-charges black hole. The "$-$" indicates a direction along which the brane is smeared while a number indicates a direction along which the brane extends.

$D2_1$	0	1	2	$-$	$-$	$-$	$-$
$D2_2$	0	$-$	$-$	3	4	$-$	$-$
$D2_3$	0	$-$	$-$	$-$	$-$	5	6
$D6_4$	0	1	2	3	4	5	6

first D2 brane extends along x^1 and x^2 while it is smeared along x^3, x^4, x^5 and x^6). Thanks to the smearing, all branes behave as points in the x^7, x^8 and x^9 directions, hence the warp factors H_i (with $i = 1, \dots, 4$ labelling the branes) are given by

$$\Delta_3 H_i = 0 \Rightarrow H_i = 1 + \frac{Q_i}{r} \tag{3.4}$$

with r being the radius in the space spanned by x^7, x^8 and x^9. Now, writing down the metric for this system amounts to inserting the correct powers of the functions H_i in front of each coefficient. The rule to do this is simply to multiply the coefficient dx_j^2 by $H_i^{-1/2}$ if the ith-brane's worldvolume extends along the x^j-direction, and to multiply by $H_i^{1/2}$ if it does not. By means of this rule, we find

$$\begin{aligned} ds^2 = &-\frac{1}{\sqrt{H_1H_2H_3H_4}}dt^2 + \sqrt{H_1H_2H_3H_4}(dx_7^2 + dx_8^2 + dx_9^2) + \\ &+ \sqrt{\frac{H_2H_3}{H_1H_4}}(dx_1^2 + dx_2^2) + \sqrt{\frac{H_1H_3}{H_2H_4}}(dx_3^2 + dx_4^2) + \sqrt{\frac{H_1H_2}{H_3H_4}}(dx_5^2 + dx_6^2). \end{aligned} \tag{3.5}$$

As $r \to \infty$, the metric above approaches four dimensional, flat Minkowski spacetime plus a six-torus coming from the compactification of $x^1, \dots x^6$, i.e.

$$ds^2 = -dt^2 + dx_7^2 + dx_8^2 + dx_9^2 + ds_{T^6}^2.$$

Instead, as $r \to 0$, the metric approaches

$$\begin{aligned} ds^2 &= -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}(dx_7^2 + dx_8^2 + dx_9^2) + \sqrt{\frac{Q_2Q_3}{Q_1Q_4}}(dx_1^2 + dx_2^2) \\ &+ \sqrt{\frac{Q_1Q_3}{Q_3Q_4}}(dx_3^2 + dx_4^2) + \sqrt{\frac{Q_1Q_2}{Q_3Q_4}}(dx_5^2 + dx_6^2) \\ &= \underbrace{-\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2}_{AdS_2} + \underbrace{R^2 d\Omega_2^2}_{S^2} + \underbrace{ds_{T^6}^2}_{T^6} \end{aligned} \tag{3.6}$$

where we introduced the length scale $R^2 = \sqrt{Q_1Q_2Q_3Q_4}$ for dimensional reasons and, in the last step, we employed spherical coordinates for $\mathbb{R}^3$. As highlighted in (3.6), the metric has an $\mathrm{AdS}_2 \times S^2 \times T^6$ structure[1] and, furthermore, possesses a horizon at $r = 0$ as $g_{tt} = 0$ there. From the two asymptotic behaviors it is possible to infer that this solution, known as *four-charge black hole*, really is a black hole in four dimensions.

The last statement is reinforced by comparing the extremal Reissner-Nordström metric (1.20), i.e.

$$ds^2 = -\left(1 - \frac{Q}{\rho}\right)^2 dt^2 + \left(1 - \frac{Q}{\rho}\right)^{-2} d\rho^2 + \rho^2 d\Omega^2,$$

[1] To better see the AdS structure, substitute $z = \frac{L^2}{r}$ in the AdS_2 metric $ds^2 = \frac{L^2}{z^2}(-dt^2 + dz^2)$ and set $L \equiv R$.

Table 3.2: Configuration of the M2 branes for the three charge black hole. As before, the "$-$" indicates a direction along which the brane is smeared while a number indicates a direction along which the brane extends.

$M2_1$	0	1	2	$-$	$-$	$-$	$-$
$M2_2$	0	$-$	$-$	3	4	$-$	$-$
$M2_3$	0	$-$	$-$	$-$	$-$	5	6

and the (3.5) with $Q_i = Q\ \forall i$, r substituted by $\rho - Q$ and neglecting the T^6 metric. With these substitutions indeed we obtain

$$\begin{aligned} ds^2 &= -\left(1+\frac{Q}{r}\right)^{-2} dt^2 + \left(1+\frac{Q}{r}\right)^{2} dr^2 + \left(1+\frac{Q}{r}\right)^{2} r^2 d\Omega_2^2 \\ &= -\left(1-\frac{Q}{\rho}\right)^{2} dt^2 + \left(1-\frac{Q}{\rho}\right)^{-2} d\rho^2 + \rho^2 d\Omega_2^2. \end{aligned} \tag{3.7}$$

3.2.2 Three charge black hole

In this subsection we move to eleven dimensions, instead of the ten used up to this point, to analyze the system formed by three M2 branes, the eleven dimensional respective of D2 branes. The branes' configuration considered here is presented in Table 3.2; as before, a number indicates the direction along which the brane's worldvolume extends, while a "$-$" indicates a direction along which the brane is smeared.

The convention used to write the metric generated by this configuration of branes is again similar to the one used for the four-charge black hole: multiply the metric components by a factor $H^{-2/3}$ if the brane's worldvolume extends along that direction; multiply instead by a factor $H^{1/3}$ if the brane's worldvolume is orthogonal to that direction. Furthermore, we compactify the $x^1, \dots x^6$ directions into a T^6. Consequently, the metric for the three-charge black hole is

$$\begin{aligned} ds^2 =& -(H_1H_2H_3)^{-2/3}dt^2 + (H_1H_2H_3)^{1/3}(dx_7^2+dx_8^2+dx_9^2+dx_{10}^2) \\ &+ \frac{(H_2H_3)^{1/3}}{H_1^{2/3}}(dx_1^2+dx_2^2) + \frac{(H_1H_3)^{1/3}}{H_2^{2/3}}(dx_3^2+dx_4^2) + \frac{(H_1H_2)^{1/3}}{H_3^{2/3}}(dx_5^2+dx_6^2) \end{aligned} \tag{3.8}$$

with $H_i = 1 + \frac{Q_i}{r^2}$ for $i = 1, 2, 3$ and r being the radius in the space spanned by $x^7, \dots, x^{10}$.

As $r \to \infty$, (3.8) approaches a five dimensional Minkowski space times a constant-radii six-torus. As $r \to 0$ instead, (3.8) approaches

$$ds^2 = -\frac{r^4}{R^4}dt^2 + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_3^2 + ds^2_{T^6}, \tag{3.9}$$

where we introduced spherical coordinates such that $dx_7^2 + \dots + dx_{10}^2 = dr^2 + r^2 d\Omega_3^2$ and $R^2 = (Q_1Q_2Q_3)^{1/3}$. Again, this has a $\mathrm{AdS}_2 \times S^3 \times T^6$ structure[2]and, from the two asymptotic limits, we can infer that the metric describes a black hole in five dimensions; this black hole is known as *three-charge* black hole.

3.3 Entropy: from macro to micro

3.3.1 The macro

As already stated in Section 1.3.2, the entropy of a black hole is given by ($c = \hbar = k_{\mathrm{B}} = 1$) $S = \frac{A_{\mathrm{H}}}{4G_{\mathrm{N}}}$ where A_{H} is the area of the horizon and $G_{\mathrm{N}} = (2\pi)^{D-3}(l_{\mathrm{P}})^{D-2}/(16\pi)$ is Newton's constant in D

[2]This time, to see the AdS structure, start from $ds^2 = \frac{L^2}{z^2}(-dt^2 + dz^2)$ and substitute $z = \frac{R^2}{2r^2}$ setting $L \equiv \frac{R}{2}$.

Table 3.3: Map via T and S dualities from the D2-D2-F1 system (obtained from the M2-M2-M2 system by compactifying the x^6 direction) to the F1-NS5-P system. T_i represents here a T duality along the x^i direction, while S an S duality. F1 represent the fundamental string, NS5 its magnetic dual and P the momentum charge.

D2	0	1	2	–	–	–		D1	0	–	–	–	–	5		F1	0	–	–	–	–	5
D2	0	–	–	3	4	–	$\xrightarrow{T_1 T_2 T_5}$	D5	0	1	2	3	4	5	$\xrightarrow{S}$	NS5	0	1	2	3	4	5
F1	0	–	–	–	–	5		P	0	–	–	–	–	5		P	0	–	–	–	–	5

dimensions. Starting from this definition, the Bekenstein-Hawking entropy for the three charge black hole (N_i is the number of the i-th M2 brane) is

$$S = 2\pi\sqrt{N_1 N_2 N_3} \tag{3.10}$$

since

$$A_{\rm H} = \int_{S^3} \sqrt{g_{S^3}} \int_{T^6} \sqrt{g_{T^6}} = 2\pi^2\sqrt{Q_1Q_2Q_3}\int_{T^6} dx^1 \dots dx^6 = 2\pi^2\sqrt{Q_1Q_2Q_3}\prod_{i=1}^{6}(2\pi L_i), \tag{3.11}$$

with L_i being the radius of the circle into which x^i is compactified, and the charges Q_i being related to the i-th brane's numbers N_i by

$$Q_1 = \frac{N_1 (l_{\rm P})^6}{L_3L_4L_5L_6}, \quad Q_2 = \frac{N_2 (l_{\rm P})^6}{L_1L_2L_5L_6}, \quad Q_3 = \frac{N_3 (l_{\rm P})^6}{L_1L_2L_3L_4}. \tag{3.12}$$

The last relations are obtained from (we specialize to the case N_1) the integration over $\Sigma = S^3 \times T^4_{3\dots6}$, i.e. the surface surrounding the $\mathrm{M2}_1$ brane, of the magnetic field strength $\tilde{F}_7 = \star_{11} F$ having non-zero component equal to $(\tilde{F}_7)_{34560\phi\psi} = \sqrt{-g}\epsilon_{34560\phi\psi 012r} g^{00}g^{11}g^{22}g^{rr}F_{012r}$, with $F_{012r} = \partial_r H_1^{-1}$ and θ, ϕ and ψ being the angles parametrizing S^3. Altogether, this means that (we use the metric (3.8) with $r \to 0$)

$$(2\pi l_{\rm P})^6 N_1 = \int_\Sigma \tilde{F}_7 = \int\limits_{T^4_{3\dots6}\times S^3} d\Omega_3 dx^3 \dots dx^6 2Q_1 = (2\pi^2)(2\pi)^4 L_3L_4L_5L_6 2Q_1 \Rightarrow Q_1 = \frac{N_1(l_{\rm P})^6}{L_3L_4L_5L_6}. \tag{3.13}$$

Since the end result (3.10) depends solely on the number of branes, and not on their nature, the application to the system under consideration (i.e. the three-charge black hole) of any combination of T and S duality does not change the value of the entropy; even if the branes responsible for the entropy are not the same. Taking advantage of this feature, which essentially boils down to the solution being supersymmetric, it is possible to map the three charge black hole formed by three M2 branes with metric (3.8) into a three-charge black hole in IIB supergravity by compactifying the M2-branes-black-hole's x^6 direction, applying three T dualities (one along x^1 direction, one along x^2 and one along x^5), and, finally, acting with an S duality; see Table 3.3. The result is the F1-NS5-P system, that is formed by N_1 fundamental strings (indicated by F1), N_5 NS5 branes and $N_{\rm p}$ momentum charges (indicated by P). The metric for this solution is given by [4]

$$ds^2 = H_1^{-1}(-dt^2 + dy^2 + K(dt+dy)^2) + H_5\sum_{i=1}^{4} dx^i dx^i + \sum_{a=1}^{4} dz^a dz^a \tag{3.14}$$

$$H_1 = 1 + \frac{Q_1}{r^2}, \qquad H_5 = 1 + \frac{Q_5}{r^2}, \qquad K = \frac{Q_{\rm P}}{r^2}, \qquad e^{2\phi} = \frac{H_5}{H_1} \tag{3.15}$$

where Q_1 is the charge of the F1 branes, Q_5 the one of the NS5 branes and $Q_{\rm p}$ is the momentum charge; ϕ is the dilaton. As a consequence of the arguments given in the preceding paragraph, the Bekenstein-Hawking entropy of this black hole is given by $S = 2\pi\sqrt{N_1N_5N_{\rm p}}$.

Now, it is possible to obtain the same entropy from the count of the string's microscopic degrees of freedom; this is the focus of the next section.

3.3.2 The micro

From classical Thermodynamics we know that the entropy of a system is related to the microscopic degrees of freedom by the famous Boltzmann formula (we reinstate k_{B} just in this formula)

$$S = k_{\mathrm{B}} \log \mathcal{N} \tag{3.16}$$

where $\mathcal{N}$ is the number of different microscopic configurations resulting in the same macroscopic state. So, given the different-from-zero entropy that the above D-branes configurations posses, we would like to interpret this entropy in terms of configurations of microscopic degrees of freedom. Thanks to String Theory this is possible, and we will show this by firstly setting $N_5 = 0$, i.e. by focussing on the two-charge case F1-P; we will later extend the result to the three-charge case.

We begin by joining together all the N_1 strings into a single string wounding N_1 times around the S^1 circle along the y-direction, with the N_{p} momentum charges generating (transverse) waves travelling along the multiply-wound string. In this picture the different microscopic degrees of freedom are due to different partitioning of the momentum charges between the numerous harmonics. Indeed, indicating with R the y-coordinate circle's radius, the total momentum on the string is given by

$$P = \frac{N_{\mathrm{p}}}{R} \tag{3.17}$$

because of the string's wave function's periodicity in the y coordinate. At the same time, a harmonic with wave number k travelling along the string, and responsible for the momentum of said string, will carry momentum equal to

$$p_k = \frac{2\pi k}{L_{\mathrm{T}}} = \frac{2\pi k}{2\pi N_1 R}; \tag{3.18}$$

summing the momentum charges due to all the harmonics should give the total momentum (3.17), i.e.

$$\sum_m \frac{n_m k_m}{N_1 R} = \frac{N_{\mathrm{p}}}{R} \quad \Leftrightarrow \quad \sum_m n_m k_m = N_{\mathrm{p}} N_1 \tag{3.19}$$

where n_m is the number of harmonics k_m present on the string. The number of different combinations of k_m and n_m leading to $N_1 N_{\mathrm{p}}$ is our $\mathcal{N}$.

To compute S, we are going to treat the harmonics as a 1-dimensional gas of massless quanta confined on the string of length L_{T} and, taking into account the fact that for a mode k the bosonic and fermionic partition functions are given, respectively, by

$$Z_k^{\mathrm{B}} = \sum_{m_k=0}^{\infty} e^{-\beta m_k e_k} = \frac{1}{1 - e^{\beta e_k}} \quad \text{and} \quad Z_k^{\mathrm{F}} = \sum_{m_k=0}^{1} e^{-\beta m_k e_k} = 1 + e^{\beta e_k}, \tag{3.20}$$

with $e_k = p_k$ being the k's mode energy, we can write the logarithm of the total partition function Z as the sum of $\log Z^{\mathrm{B}}$ and $\log Z^{\mathrm{F}}$ both obtained by approximating the sum over k by an integral over e_k leading to

$$\begin{aligned} \log Z = \log Z^{\mathrm{B}} + \log Z^{\mathrm{F}} &= \frac{L_{\mathrm{T}}}{2\pi}\left(-\int_0^{\infty} de_k \, \log(1 - e^{\beta e_k}) + \int_0^{\infty} de_k \, \log(1 + e^{\beta e_k}) \right) \\ &= \frac{L_{\mathrm{T}}}{2\pi}\left(\frac{\pi^2}{6\beta} + \frac{\pi^2}{12\beta} \right) = \frac{\pi L_{\mathrm{T}}}{4\beta}. \end{aligned} \tag{3.21}$$

The last result is valid for one bosonic and one fermionic degree of freedom; having 8 of each we just need to multiply by 8. The entropy is now given by ($E = -\partial_\beta \log Z$)

$$S = \log Z + \beta E = 2\pi\sqrt{2 N_1 N_{\mathrm{p}}}. \tag{3.22}$$

Extending the result (3.22) to the three charge case is a matter of convincing oneself, via the aid of dualities, that the winding number N_1 transforms into the "effective winding number" $N_1 N_5$. The dualities we are alluding to are those connecting the F1(N_1)-P(N_{p}) system to the NS5(N_1)-F1(N_{p})

one, where we indicated the number of each charge between the braces; for the details concerning the dualities see [4]. Indeed, through this chain of dualities, the discussion on the partitioning of the N_p momentum charges on the harmonics translates into the division of each N_p F1 string into N_1 pieces, for a grand total of $N_1 N_\mathrm{p}$ pieces of strings that have to be merged together to construct the full string. The different ways in which the pieces can be glued together give rise to the entropy in the F1-NS5 picture. Putting this into mathematical language gives us the right-hand-side of (3.19) where now k_m is the winding number of the n_m pieces of strings. It turns out that, once we add to the F1-NS5 system (now with charges N_1 and N_5 respectively) N_p P charges, the leading order contribution to the entropy comes from the partitioning of these N_p units of momentum onto the $N_1 N_5$-times wound string. The number of such partitioning is known to be equal to $\mathcal{N} \approx \exp\left(2\pi\sqrt{\frac{c}{6} N_1 N_5}\right)$, where c is the sum of bosonic and fermionic degrees of freedom, now equal to 6. Plugging these numbers in we get

$$S = \log \mathcal{N} = 2\pi\sqrt{N_1 N_5 N_\mathrm{p}} \tag{3.23}$$

that is exactly equal to (3.10) with $N_2 = N_5$ and $N_3 = N_\mathrm{p}$.

3.4 The structure of the microstates: fuzzballs

In the previous section we have seen that String Theory enables us to count the microstates of a black hole, and that this counting perfectly reproduces the Bekenstein-Hawking entropy. What we are still missing at the moment is the knowledge about how these microstates look like; i.e., what is their structure. A possible solution to this problem comes from the *fuzzball* proposal that we are now going to briefly review. This will also shed new light onto a puzzling fact: taking $N_5 = 0$ in (3.23) yields zero, meaning that the Bekenstein-Hawking entropy of a two-charge black hole is zero despite having seen that the microscopic entropy is not zero, see (3.22). So, to look into the structure of the microstates of the two-charge black hole, we are going to critically analyze the metric produced by a representative of such class of solutions.

The metric for the F1-P black hole could be obtained by setting Q_5 to zero in (3.14). That would give us ($u = t + y$, $v = t - y$)

$$\begin{gathered} ds^2 = H(-du dv + K dv^2) + \sum_{i=1}^{4} dx^i dx^i + \sum_{a=1}^{4} dz^a dz^a, \\ H^{-1} = 1 + \frac{Q_1}{r^2}, \quad K = \frac{Q_\mathrm{p}}{r^2}, \quad e^{2\phi} = H; \end{gathered} \tag{3.24}$$

however this is not the metric produced by typical configurations of the F1-P system. The reason for this is that this metric assumes that the F1 string sits at the position $r = 0$, while instead we know that the momentum charges cause the string to vibrate in the transverse directions, which then moves away from the position $r = 0$. In the oscillating process the N_1 strands that constitute the string separate in the transverse directions $\boldsymbol{x}$ following the transverse displacement profile $\boldsymbol{F}(v)$, so, intuitively, we will need to replace the r in (3.24) with $|\boldsymbol{x} - \boldsymbol{F}(v)|$, since this is now the new location of the string. The metric for a single strand is known (see [17, 18]) and from it, by superposing the single-strand harmonic functions, we retrieve the multiple strand's metric given by

$$ds^2 = H(-du dv + K dv^2 + 2A_i dx_i dv) + \sum_{i=1}^{4} dx^i dx^i + \sum_{a=1}^{4} dz^a dz^a, \tag{3.25}$$

with H, K and A_i equal to

$$\begin{gathered} (H(\boldsymbol{x}, v))^{-1} = 1 + \sum_s \frac{Q_1^{(s)}}{|\boldsymbol{x} - \boldsymbol{F}(v)|^2}, \quad K(\boldsymbol{x}, v) = \sum_s \frac{Q_1^{(s)} |\dot{\boldsymbol{F}}^{(s)}(v)|^2}{|\boldsymbol{x} - \boldsymbol{F}(v)|^2}, \\ A_i(\boldsymbol{x}, v) = -\sum_s \frac{Q_1^{(s)} \dot{F}_i^{(s)}(v)}{|\boldsymbol{x} - \boldsymbol{F}(v)|^2}, \end{gathered} \tag{3.26}$$

with the sum on s being the sum on the different strands. Specializing to our case, we have N_1 strands. To simplify the treatment we focus on a configuration with wave length $\lambda \gg 2\pi R$ that, although being far from the typical wave length that can be obtained from the analysis of the previous section, enables us to convert the sum on the strands into an integral since the string's typical displacement in the transverse space and the distance between neighboring strands are given, respectively, by

$$\Delta x \sim |\dot{\boldsymbol{F}}|\lambda, \qquad \delta x \sim |\dot{\boldsymbol{F}}| 2\pi R\lambda \quad \Rightarrow \quad \frac{\delta x}{\Delta x} \sim \frac{2\pi R}{\lambda} \ll 1, \tag{3.27}$$

and so all strands give a similar contribution to the harmonic functions. We then have

$$\sum_{s=1}^{N_1} \to \int_0^{N_1} ds = \int_0^{L_{\mathrm{T}}} dv \tag{3.28}$$

so the metric (3.25) becomes

$$ds^2 = H(-dudv + Kdv^2 + 2A_i dx_i dv) + \sum_{i=1}^{4} dx^i dx^i + \sum_{a=1}^{4} dz^a dz^a, \tag{3.29}$$

with H, K and A_i equal to

$$\begin{gathered}(H(\boldsymbol{x}))^{-1} = 1 + \frac{Q_1}{L_{\mathrm{T}}}\int_0^{L_{\mathrm{T}}} \frac{dv}{|\boldsymbol{x}-\boldsymbol{F}(v)|^2}, \quad K(\boldsymbol{x}) = \frac{Q_1}{L_{\mathrm{T}}}\int_0^{L_{\mathrm{T}}} \frac{dv|\dot{\boldsymbol{F}}(v)|^2}{|\boldsymbol{x}-\boldsymbol{F}(v)|^2}, \\ A_i(\boldsymbol{x}) = -\frac{Q_1}{L_{\mathrm{T}}}\int_0^{L_{\mathrm{T}}} \frac{dv\dot{F}_i(v)}{|\boldsymbol{x}-\boldsymbol{F}(v)|^2},\end{gathered} \tag{3.30}$$

Before moving on, a hint as to how to solve the discrepancy we mentioned at the beginning of the section. It is possible to show that the area of the region typically occupied by the oscillating string is such to satisfy a Bekenstein type relation, even though this region is not a proper horizon. Indeed, this region separates the portion of space where the geometry has to be described by the metric (3.29) (the inside) from the region where we could get away with using (3.24) (the outside). For a nice discussion on the absence of horizon for the microstates we redirect the reader again to [4].

Moving on then, (3.29) is the metric for a F1-P brane configuration; it is possible, through a chain of dualities, to get the metric for a D1-D5 brane configuration. For this, recall that an S duality, on top of sending the dilaton ϕ into $-\phi$, acts on the string-frame metric $g_{\mu\nu}$ in such a way that the Einstein-frame metric $g^{\mathrm{E}}_{\mu\nu} = \sqrt{g_{\mathrm{s}}}e^{-\phi/2}g_{\mu\nu}$ remains invariant under it. On top of that, from requiring the metric to approach Minkowski's one at infinity, we will need to rescale the coordinates by powers of the string coupling constant g_{s} at each step. Finally, recall that under T duality along a compact direction with radius R, the dilaton shifts so that the new string coupling constant is given by $\tilde{g}_{\mathrm{s}} = \frac{g_{\mathrm{s}}}{R}$. The chain of T and S dualities action on some parameters of the theory is the following

$$\begin{pmatrix} g_{\mathrm{s}} \\ Q_1 \\ R \\ R_6 \\ V \end{pmatrix} \xrightarrow{S} \begin{pmatrix} 1/g_{\mathrm{s}} \\ Q_1/g_{\mathrm{s}} \\ R/\sqrt{g_{\mathrm{s}}} \\ R_6/\sqrt{g_{\mathrm{s}}} \\ V/g_{\mathrm{s}}^2 \end{pmatrix} \xrightarrow{T_6\,T_7\,T_8\,T_9} \begin{pmatrix} g_{\mathrm{s}}/V \\ Q_1/g_{\mathrm{s}} \\ R/\sqrt{g_{\mathrm{s}}} \\ \sqrt{g_{\mathrm{s}}}/R_6 \\ g_{\mathrm{s}}^2/V \end{pmatrix} \xrightarrow{S} \begin{pmatrix} V/g_{\mathrm{s}} \\ (Q_1 V)/g_{\mathrm{s}}^2 \\ (R\sqrt{V})/g_{\mathrm{s}} \\ \sqrt{V}/R_6 \\ V \end{pmatrix} \xrightarrow{T_5\,T_6} \begin{pmatrix} R_6/R \\ (Q_1 V)/g_{\mathrm{s}}^2 \\ g_{\mathrm{s}}/(R\sqrt{V}) \\ R_6\sqrt{V} \\ R_6^2 \end{pmatrix} \tag{3.31}$$

At this point we obtained a NS5-F1 configuration; to get to the D1-D5 configuration we need to perform a final S duality

$$\begin{pmatrix} R_6/R \\ (Q_1 V)/g_{\mathrm{s}}^2 \\ g_{\mathrm{s}}/(R\sqrt{V}) \\ R_6\sqrt{V} \\ R_6^2 \end{pmatrix} \xrightarrow{S} \begin{pmatrix} R/R_6 \\ (Q_1 R V)/(g_{\mathrm{s}}^2 R_6) \\ g_{\mathrm{s}}/\sqrt{R R_6 V} \\ \sqrt{R_6 R}/\sqrt{V} \\ R^2 \end{pmatrix} \equiv \begin{pmatrix} g'_{\mathrm{s}} \\ Q'_5 \\ R' \\ R'_6 \\ V' \end{pmatrix} \tag{3.32}$$

Due to the dualities, all lengths are scaled by a factor

$$\mu = \sqrt{\frac{VR}{g_s^2 R_6}} \tag{3.33}$$

and the final metric for the D1-D5 system is given by

$$ds^2 = \sqrt{\frac{H}{1+K}}\left[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2\right] + \sqrt{\frac{1+K}{H}} + dx^i dx^i + \sqrt{H(1+K)}dz^a dz^a \tag{3.34}$$

where now H, K, and A_i are given by

$$(H(\boldsymbol{x}))^{-1} = 1 + \frac{\mu Q_1}{L_\mathrm{T}}\int_0^{\mu L_\mathrm{T}} \frac{dv}{\left|\boldsymbol{x} - \mu \boldsymbol{F}(v)\right|^2}, \quad K(\boldsymbol{x}) = \frac{\mu Q_1}{L_\mathrm{T}}\int_0^{\mu L_\mathrm{T}} \frac{dv\left|\mu^2 \dot{\boldsymbol{F}}(v)\right|^2}{\left|\boldsymbol{x} - \mu \boldsymbol{F}(v)\right|^2} \tag{3.35}$$

$$A_i(\boldsymbol{x}) = -\frac{\mu Q_1}{L_\mathrm{T}}\int_0^{\mu L_\mathrm{T}} \frac{dv \mu \dot{F}_i(v)}{\left|\boldsymbol{x} - \mu \boldsymbol{F}(v)\right|^2}, \quad dB = -\star_4 dA \tag{3.36}$$

with $\star_4$ being the duality operator in the space spanned by $x_1, \ldots, x_4$ using the flat metric.

Even though it is difficult to appreciate at this level, the metric (3.34) is perfectly regular at all points. One could be concerned, looking at the expressions of H and the other fields in (3.35), that the point $\boldsymbol{x} = \mu \boldsymbol{F}(v)$ might create some problems; however it is possible to show (see [19]), through a coordinate change, that the metric is indeed regular at the location of the string, where it ends in a so-called cap whose shape depends on the specific profile $\boldsymbol{F}(v)$.

3.4.1 Circular profile function

The simplest example in which the integrals from above can be carried out explicitly is the one where the displacement profile $\boldsymbol{F}(v)$ describes a circular profile. Starting from the F1-P case, suppose that

$$\boldsymbol{F}(v) = (\hat{a}\cos(\omega v), \hat{a}\sin(\omega v), 0, 0) \tag{3.37}$$

with $\hat{a}$ =const and $\omega = \frac{1}{N_1 R}$ so that the string swings a single turn of helix in the $x_1 - x_2$ plane. Then, introducing the coordinates

$$x_1 = \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi}, \quad x_2 = \tilde{r}\sin\tilde{\theta}\sin\tilde{\phi}, \quad x_3 = \tilde{r}\cos\tilde{\theta}\cos\tilde{\psi}, \quad x_4 = \tilde{r}\cos\tilde{\theta}\sin\tilde{\psi}, \tag{3.38}$$

the integral for H in (3.30) can be promptly computed ($\xi = \omega v$)

$$\begin{aligned} H &= 1 + \frac{Q_1}{2\pi}\int_0^{2\pi} \frac{d\xi}{\tilde{r}^2 + \hat{a}^2 - 2\tilde{r}\hat{a}\sin\tilde{\theta}\cos(\tilde{\phi}+\xi)} = 1 + \frac{Q_1}{\sqrt{(\tilde{r}^2+\hat{a}^2)^2 - 4\hat{a}^2\tilde{r}^2\sin^2\tilde{\theta}}} \\ &= 1 + \frac{Q_1}{r^2 + \hat{a}^2\cos^2\theta}; \end{aligned} \tag{3.39}$$

where, in the last step, we introduced coordinates r and θ such that

$$\tilde{r} = \sqrt{r^2 + \hat{a}^2\sin^2\theta}, \quad \cos\tilde{\theta} = \frac{r\cos\theta}{\sqrt{r^2 + \hat{a}^2\sin^2\theta}} \tag{3.40}$$

The remaining integrals are computed using the useful rule

$$\int_0^{2\pi} d\alpha \, \frac{\cos^n\alpha}{1 + \beta\cos\alpha} = \frac{2\pi}{\sqrt{1-\beta^2}}\left(\frac{\sqrt{1-\beta^2} - 1}{\beta}\right)^n. \tag{3.41}$$

The results are

$$\begin{aligned} K &= \frac{\hat{a}^2 Q_1}{N_1^2 R^2 (r^2 + \hat{a}^2\cos^2\theta)}, \quad A_{x_1} = \frac{Q_1 \hat{a}^2 \sin\tilde{\phi}\sin\theta}{RN_1\sqrt{r^2+\hat{a}^2}(r^2 + \hat{a}^2\cos^2\theta)}, \\ A_{x_2} &= -\frac{Q_1 \hat{a}^2 \cos\tilde{\phi}\sin\theta}{RN_1\sqrt{r^2+\hat{a}^2}(r^2 + \hat{a}^2\cos^2\theta)}, \quad A_{x_3} = A_{x_4} = 0. \end{aligned} \tag{3.42}$$

(a) Typical BH picture *(b) Fuzzball*

Figure 3.1: Fuzzball cartoon. On the RHS we see the "typical" picture of a black hole: flat spacetime at large distances from the singularity with in-between a throat like region. On the LHS instead, we see the cartoon for a fuzzball; the main difference is in the singularity region where the fuzzball ends "smoothly" in a cap whose shape depends on the specific branes configurations.

Expressing A in coordinates r, θ, $\phi \equiv \tilde{\phi}$ and $\psi \equiv \tilde{\psi}$, results in

$$A_\phi = A_{x_1}\frac{\partial x_1}{\partial \phi}A_{x_2}\frac{\partial x_2}{\partial \phi} = -\frac{Q_1 \hat{a}^2 \sin^2\theta}{RN_1(r^2+\hat{a}^2\cos^2\theta)}. \tag{3.43}$$

We now turn our attention to the D1-D5 picture. Thanks to what we saw in Section 3.4, the D1-D5's displacement profile $\boldsymbol{F}(v)$ is simply given by $\mu\boldsymbol{F}_{(3.37)}(v)$, with $a \equiv \mu\hat{a}$ and μ from (3.33). H, K, A_ϕ and B_ψ are given by

$$H^{-1} = 1 + \frac{\mu^2 Q_1}{f} \equiv 1 + \frac{Q_5'}{f}, \qquad K = \mu^2\frac{Q_\mathrm{p}}{f} \equiv \frac{Q_1'}{f} \qquad A_\phi = -a\sqrt{Q_1'Q_5'}\frac{\sin^2\theta}{f}, \tag{3.44}$$
$$B_\psi = -a\sqrt{Q_1'Q_5'}\frac{\cos^2\theta}{f}$$

with $f \equiv r^2 + a^2\cos^2\theta$ and a connected to the charges and new y's radius $R' = g_\mathrm{s}/\sqrt{RR_6V}$ by

$$a = \frac{\sqrt{Q_1'Q_5'}}{R'}. \tag{3.45}$$

Inserting the (3.44) in (3.34) results in

$$\begin{aligned} ds^2 = &-\frac{1}{h}(dt^2 - dy^2) + hf\left(d\theta^2 + \frac{dr^2}{r^2+a^2}\right) - \frac{2a\sqrt{Q_1'Q_5'}}{hf}(\cos^2\theta dy d\psi + \sin^2\theta dt d\phi) \\ &+ h\left[\left(r^2 + \frac{a^2 Q_1'Q_5'\cos^2\theta}{h^2f^2}\right)\cos^2\theta d\psi^2 + \left(r^2 + a^2 - \frac{a^2Q_1'Q_5'\sin^2\theta}{h^2f^2}\right)\sin^2\theta d\phi^2\right] \\ &+ \sqrt{\frac{Q_1'+f}{Q_5'+f}}dz_a dz_a \end{aligned} \tag{3.46}$$

where

$$h = \left[\left(1+\frac{Q_1'}{f}\right)\left(1+\frac{Q_5'}{f}\right)\right]^{\frac{1}{2}}. \tag{3.47}$$

At large r we have that $f \to r^2$ and $h \to 1$, implying that the metric approaches flat space. For $r \ll (Q_1'Q_5')^{1/4}$ (with $r' = r/a$) the metric becomes

$$\begin{aligned} ds^2 = &-(r'^2)\frac{a^2dt^2}{\sqrt{Q_1'Q_5'}} + r'^2\frac{a^2dy^2}{\sqrt{Q_1'Q_5'}} + \sqrt{Q_1'Q_5'}\frac{dr'^2}{r'^2+1} \\ &+ \sqrt{Q_1'Q_5'}[d\theta^2 + \cos^2\theta d\psi'^2 + \sin^2\theta d\phi'^2] + \sqrt{\frac{Q_1'}{Q_5'}}dz_a dz_a \end{aligned} \tag{3.48}$$

with the new angles ψ' and ϕ' equal to

$$\psi = \psi - \frac{a}{\sqrt{Q_1' Q_5'}} y, \quad \phi = \phi - \frac{a}{\sqrt{Q_1' Q_5'}} t \tag{3.49}$$

that are well-defined that to (3.45). Actually, using (3.45) to simplify (3.48) we find

$$\begin{aligned} ds^2 = &\sqrt{Q_1' Q_5'} \left[-(r'^2 + 1) \frac{dt^2}{R^2} + r'^2 \frac{dy^2}{R^2} + \frac{dr'^2}{r'^2 + 1} \right] \\ &+ \sqrt{Q_1' Q_5'} \left[d\theta^2 + \cos^2\theta d\psi'^2 + \sin^2\theta d\phi'^2 \right] + \sqrt{\frac{Q_1'}{Q_5'}} dz_a dz_a \end{aligned} \tag{3.50}$$

i.e. an $\mathrm{AdS}_3 \times S^3 \times T^4$ structure. As stressed before for the general metric in Section 3.4, this metric is perfectly regular, ending in a sort of cap, at $r = 0$. It is also flat at infinity and has a throat for small r. See Figure 3.1 for a cartoon.

Chapter 4

WZW models and black hole microstates

In this chapter we will examine the relationship between some supergravity systems (NS5, NS5-P and NS5-F1) and the gauged Wess-Zumino-Witten models (WZW models for short). The connection between the two arises from the fact that, as we will show, by introducing appropriate gauge fields in a WZW model, and integrating them out by means of their equations of motion, the action that results is exactly the sum of action (2.68) and (2.69), meaning the actions for strings coupled respectively to a non-trivial metric and B field, for metric and B field given by the supergravity solutions for the aforementioned systems in the so-called *decoupling limit*. The main advantage in using this technique is that the theory in the "upstairs" world (meaning the theory in G) can be quantized exactly, thus preserving all the UV corrections that are otherwise lost in the supergravity limit. Then, once one moves to the "downstairs" world (i.e., to the coset G/H and after integrating out the gauge fields), it is possible to implement such corrections in the supergravity computations.

We start with a brief review of Wess-Zumino-Witten models.

4.1 Wess-Zumino-Witten model

In this section we follow [20] and [21]. The Wess-Zumino-Witten model can be obtained starting from the *principal chiral model* whose action is given by

$$S_0[g] = \frac{1}{4\lambda^2} \int_{S^2} d^2x \, \mathrm{Tr}\left[g^{-1}\partial_\mu g g^{-1}\partial^\mu g\right] \tag{4.1}$$

where

$$g \equiv g(z, \bar{z}) : S^2 \to G \tag{4.2}$$

are matrices generating a faithful representation of the Lie group G, and S^2 is the Riemann sphere.

This model enjoys a $\mathrm{G} \times \mathrm{G}$ symmetry that corresponds to sending the g into $g_\mathrm{L} g(z, \bar{z}) g_\mathrm{R}$ for some $g_\mathrm{L}, g_\mathrm{R} \in \mathrm{G}$. The conserved currents related to this symmetry can be computed from the equation of motion that read

$$\partial_\mu(g^{-1}\partial^\mu g) = 0 \quad \Longleftrightarrow \quad g\partial_\mu(g^{-1}\partial^\mu g)g^{-1} = \partial_\mu[(\partial^\mu g)g^{-1}] = 0; \tag{4.3}$$

the right-hand side comes directly from the variation of S_0, and gives the conserved current $J^\mu = g^{-1}\partial^\mu g$ for the multiplication by g_R, while the left-hand side gives the conserved current $\tilde{J}^\mu = (\partial^\mu g)g^{-1}$ for the multiplication by g_L.

Changing coordinates for complex ones, the current conservation condition becomes

$$\partial J_{\bar{z}} + \bar{\partial} J_z = 0 \tag{4.4}$$

which would remind of CFT if the currents had holomorphic and antiholomorphic components. Unfortunately, this is incompatible with a non-Abelian Lie group. This issue though is promptly solved by the introduction of the so-called *Wess-Zumino term* (WZ term for short)

$$\begin{aligned} \Gamma[g] &= \frac{i}{12\pi} \int_B d^3y \, \epsilon_{\alpha\beta\gamma} \, \mathrm{Tr}\left[g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\right] \\ &= \frac{i}{2\pi} \int_B \mathrm{Tr}\left[g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right] \end{aligned} \tag{4.5}$$

where B is the ball you get by considering the interior of the Riemann sphere, and g is now a field on B that coincides with the g in S_0 on $\partial B = S^2$. A somewhat detailed discussion on why such an analytic continuation is possible can be read in [20]; here we give a summary of it. Essentially, because of properties of every Lie group, every map g from (4.2) can be continuously deformed into the constant map; once this is done, it is possible to extend the map g from S^2 to its interior B. Bearing in mind that the continuation is not unique, it can be showed that, thanks to fact that the third homotopy group of every simple Lie group G is $\pi_3(\mathrm{G}) \cong \mathbb{Z}$, the difference between two Γ coming from topologically different gs is just proportional to $2\pi i$. Consequently, the path integral is well-defined as is the case for the Quantum theory.

The action for the *Wess-Zumino-Witten model* (WZW model for short), is then given by

$$S = S_0[g] + k\Gamma[g] \tag{4.6}$$

with $k \in \mathbb{Z}$ being the *level* of the model. From the equations of motion (see Appendix A.1), together with the conditions $k \in \mathbb{Z}_{>0}$ and $\lambda^2 = 2\pi/k$, the conserved currents associated to the $\mathrm{G} \times \mathrm{G}$ symmetry are now antiholomorphic and holomorphic

$$\bar{J} = kg^{-1}\bar{\partial}g \qquad \text{and} \qquad J = -k(\partial g)g^{-1}. \tag{4.7}$$

To end this section, we mention that the WZW model is a CFT also at the quantum level. To prove this it suffices to show that the quantum theory admits a stress-energy tensor with OPE as in (2.60); and indeed, thanks to a procedure known as *Sugawara construction*, this is feasible. Briefly, it is possible to construct the OPE between the conserved currents J^a, where a is an index in the adjoint representation of the algebra of G, on the basis of general dimensional and unitary principles to be

$$J^a(z)J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{(z-w)}, \tag{4.8}$$

where κ^{ab} is the Killing form and $f^{ab}{}_c$ are the structure constants. Starting from this OPE, one can show that the modes of the conserved currents satisfy the so-called *affine Kac-Moody algebra*

$$\left[J^a_m, J^b_n\right] = km\delta^{ab}\delta_{n+m,0} + if^{ab}{}_c J^c_{n+m}. \tag{4.9}$$

Once the currents are in place, the correct stress-energy tensor turns out to be (again, through the Sugawara construction)

$$T(z) = \gamma :J^a J^a: (z) \tag{4.10}$$

with γ being fixed by computing the TJ OPE. Built in this way, the stress-energy tensor satisfies the Virasoro algebra, and the spectrum of the theory can be built from the action of the Kac-Moody algebra on highest weight states.

4.2 NS5 branes and gauged WZW model

In this section we are firstly going to examine how to build the solution for the NS5 branes system in the decoupling limit, and secondly we will show how the same metric can be obtained by null gauging an appropriate WZW model. To know more than what will be here presented about these topics we redirect the reader to [22, 23] and references therein; these will also be the main references of the following sections.

The notation used is similar to that of the previous chapter: we will indicate with $\tilde{y}$ a direction compactified into a circle of radius $R_{\tilde{y}}$; the space transverse to this direction will be $\mathbb{R}^4 \times T^4$ with coordinates $(x_1, \ldots, x_4)$ in $\mathbb{R}^4$, while T^4 is a four torus we will mostly ignore.

4.2.1 The Supergravity side

Consider N_5 NS5 branes placed in a symmetric arrangement on circle of radius a in the (x^1, x^2) plane at positions $\boldsymbol{x}_m$ with $m = 1, \dots, N_5$. The metric for this configuration is given by [24]

$$ds^2 = -dt^2 + d\tilde{y}^2 + H_5(\boldsymbol{x})(dx_1^2 + \cdots + dx_4^2) + ds^2_{T^4} \tag{4.11}$$

$$H_5(\boldsymbol{x}) = 1 + \sum_{m=1}^{N_5} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_m|^2}. \tag{4.12}$$

Introducing adapted coordinates $(\rho, \theta, \phi, \psi)$ with $\theta \in \left[0, \frac{\pi}{2}\right]$ and $\rho \geq 0$ such that[1]

$$x_1 + ix_2 = a \cosh\rho \sin\theta e^{i\phi} \quad \text{and} \quad x_3 + ix_4 = a \sinh\rho \cos\theta e^{i\psi}, \tag{4.13}$$

the positions of the NS5 branes are given by $(x_1)_m + i(x_2)_m = ae^{i\phi_m}$ with $\phi_m = \frac{2\pi m}{N_5}$ and both $(x_3)_m$ and $(x_4)_m$ equal to zero. Inserting this in (4.12), we get

$$\begin{aligned} H_5(\boldsymbol{x}) =& 1 + \sum_{m=1}^{N_5} \frac{1}{|x_1 + ix_2 - ae^{i\phi_m}|^2 + |x_3 + ix_4|^2} \\ =& 1 + \frac{N_5}{a^2(\cosh^2\rho - \sin^2\theta)} \left(\frac{\sinh(N_5\chi)}{\cosh(N_5\chi) - \cos(N_5\chi)} \right) = 1 + \frac{N_5}{a^2(\cosh^2\rho - \sin^2\theta)} \Lambda_{N_5} \end{aligned} \tag{4.14}$$

where $e^\chi = \frac{\cosh\rho}{\sin\theta}$; for the details on the intermediate steps see Appendix A.2.

The factor Λ_{N_5} contains the information about the precise location of the NS5 branes; so, to simplify the metric, we will smear them along the circle in the ϕ angle. This amounts to computing an integral on the angle ϕ rather than a sum on the location of the branes to compute H_5 (see (A.22)). Actually, the same result you get from the integral can be simply obtained by sending $\Lambda_{N_5} \to 1$ or, equivalently, by taking the large N_5 limit as, indeed, in this limit we have many sources that get progressively closer to one another as more and more are added. In the limit $N_5 \to \infty$ we effectively have a continuous distribution of NS5 branes. So, in the $N_5 \gg 1$ limit, H_5 is given by

$$H_5 = 1 + \frac{N_5}{a^2(\cosh^2\rho - \sin^2\theta)} \tag{4.15}$$

The last simplification we are going to take is the aforementioned *decoupling limit*; in this limit we neglect the 1 in the expression of H_5 and this coincides to taking the near-horizon limit for the metric. In this limit the NS5 metric reads

$$\begin{aligned} ds^2 &= -dt^2 + d\tilde{y}^2 + N_5(d\rho^2 + d\theta^2) + a^2 H_5(\sin^2\theta \cosh^2\rho \, d\phi^2 + \cos^2\theta \sinh^2\rho^2 \, d\psi^2) \\ &= -dt^2 + d\tilde{y}^2 + ds^2_\perp \end{aligned} \tag{4.16}$$

$$H_5 = \frac{N_5}{a^2(\cosh^2\rho - \sin^2\theta)}. \tag{4.17}$$

4.2.2 Gauged WZW model

To reproduce the transverse part of the NS5 metric (namely, looking at (4.16), the $ds^2_\perp$ piece) we start from a gauged WZW model with group structure

$$\frac{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)}{\mathrm{U}(1)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{R}}, \tag{4.18}$$

where the U(1)s will be the gauge groups.

We parametrize the two groups elements $g \equiv (g_\mathrm{sl}, g_\mathrm{su}) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ as

$$g_\mathrm{sl} = e^{\frac{i}{2}(\tau+\sigma)\sigma_3} e^{\rho\sigma_1} e^{\frac{i}{2}(\tau-\sigma)\sigma_3}, \qquad g_\mathrm{su} = e^{\frac{i}{2}(\psi+\phi)\sigma_3} e^{i\theta\sigma_1} e^{\frac{i}{2}(\psi-\phi)\sigma_3}, \tag{4.19}$$

[1] Basically, we think of $\mathbb{R}^4$ as the product of the two $\mathbb{R}^2$ factors that can then each be seen as a $\mathbb{C}$ factor.

where σ_i are Pauli's matrices. In line with what has been said in Section 4.1, the action for the WZW model with target space $G \equiv \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ from where we start from is

$$S_{\text{wzw}} = \frac{k}{4\pi}\int_{S^2} d^2x \ \mathrm{Tr}\left[g^{-1}\partial_\mu g g^{-1}\partial^\mu g\right] + \frac{ik}{6\pi}\int_B d^3y \, \epsilon_{\alpha\beta\gamma}\,\mathrm{Tr}\left[g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\right] \tag{4.20}$$

where there is a hidden minus sign in front of the trace on the SU(2) component compared to the trace on SL(2) to ensure that the correct signature in G (i.e. one timelike and five spacelike dimensions) is achieved. Substituting the explicit form of the two group elements from (4.19), the final action turns out to be (see Appendix A.3 for the intermediate steps)

$$\begin{aligned} S_{\text{wzw}} =& \frac{k}{\pi}\int_{S^2} d^2z\left(-\cosh^2\rho\,\partial\tau\bar\partial\tau + \partial\rho\bar\partial\rho + \sinh^2\rho\,\partial\sigma\bar\partial\sigma + \cos^2\theta\,\partial\psi\bar\partial\psi + \partial\theta\bar\partial\theta + \sin^2\theta\,\partial\phi\bar\partial\phi\right) \\ &+ \frac{k}{\pi}\int_{S^2} d^2z\left(-\cosh^2\rho\,(\partial\tau\bar\partial\sigma - \partial\sigma\bar\partial\tau) - \cos^2\theta\,(\partial\phi\bar\partial\psi - \partial\psi\bar\partial\phi).\right) \end{aligned} \tag{4.21}$$

The one above is the action for a string theory in a non-flat background with non-zero B field, see (2.68) plus (2.69). In this case the background metric is the one of G; to get the background we wish, meaning to get the $ds^2_\perp$ part of the metric from (4.16), we must quotient G by two U(1) gauge transformations, so that all the points along the two gauged direction will be physically equivalent. As done in [25], we gauge the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ subgroup such that

$$(g_{\text{sl}}, g_{\text{su}}) \mapsto \left(e^{i\alpha\sigma_3}g_{\text{sl}}e^{i\beta\sigma_3}, e^{-i\alpha\sigma_3}g_{\text{su}}e^{i\beta\sigma_3}\right) \tag{4.22}$$

namely, the transformation acting on the parameters from (4.19) as follows

$$\begin{aligned} &\tau \mapsto \tau + \alpha + \beta && \phi \mapsto \phi - \alpha - \beta \\ &\sigma \mapsto \sigma + \alpha - \beta && \psi \mapsto \psi - \alpha + \beta \\ &\rho \mapsto \rho && \theta \mapsto \theta \end{aligned} \tag{4.23}$$

The expected number of gauge fields for two U(1) groups in a two-dimensional space (the Riemann sphere) would be four: a two-component vector for each U(1); however, in this case, since the currents being gauged are null, it turns out that just two gauge fields survive and are thus necessary. Indicating the two gauge fields as $\mathcal{A}$ and $\bar{\mathcal{A}}$, the two couple to a combination of the currents ($T_3 = \frac{1}{2}\sigma_3$)

$$\begin{aligned} J_3^{\text{sl}} =& k\,\mathrm{Tr}\left[(-iT_3)\partial g_{\text{sl}}g_{\text{sl}}^{-1}\right] = k\left(\cosh^2\rho\,\partial\tau - \sinh^2\rho\,\partial\sigma\right) \\ \bar J_3^{\text{sl}} =& k\,\mathrm{Tr}\left[(-iT_3)g_{\text{sl}}^{-1}\bar\partial g_{\text{sl}}\right] = k\left(\cosh^2\rho\,\bar\partial\tau + \sinh^2\rho\,\bar\partial\sigma\right) \\ J_3^{\text{su}} =& k\,\mathrm{Tr}\left[(-iT_3)\partial g_{\text{su}}g_{\text{su}}^{-1}\right] = k\left(\cos^2\theta\,\partial\psi + \sin^2\theta\,\partial\phi\right) \\ \bar J_3^{\text{su}} =& k\,\mathrm{Tr}\left[(-iT_3)g_{\text{su}}^{-1}\bar\partial g_{\text{su}}\right] = k\left(\cos^2\theta\,\bar\partial\psi - \sin^2\theta\,\bar\partial\phi\right) \end{aligned} \tag{4.24}$$

given by $J_3^{\text{sl}} + J_3^{\text{su}}$ for $\bar{\mathcal{A}}$, and $\bar J_3^{\text{sl}} - \bar J_3^{\text{su}}$ for $\mathcal{A}$ as these are null combinations with respect to the Killing metric. Putting all of this together, the action for the *gauged* WZW model is given by

$$\begin{aligned} S_{\text{tot}} =& S_{\text{wzw}} + S_{\text{gauge}} \\ =& S_{\text{wzw}} + \frac{1}{\pi}\int_{S^2} d^2z\left(2\bar{\mathcal{A}}(J_3^{\text{sl}} + J_3^{\text{su}}) + 2\mathcal{A}(\bar J_3^{\text{sl}} - \bar J_3^{\text{su}}) - 4k(\sinh^2\rho + \cos^2\theta)\mathcal{A}\bar{\mathcal{A}}\right), \end{aligned} \tag{4.25}$$

with the gauge fields transforming under a gauge transformation as

$$\mathcal{A} \mapsto \mathcal{A} + \partial\beta; \qquad \bar{\mathcal{A}} \mapsto \bar{\mathcal{A}} + \bar\partial\alpha. \tag{4.26}$$

The action S_{tot} with gauge transformations from (4.22) and (4.26) is gauge invariant.

To get the $ds^2_\perp$ metric, we need to integrate out the gauge fields with the help of their equations of motions which have solutions

$$\begin{aligned} \mathcal{A} =& \frac{\cosh^2\rho\,\partial\tau + \cos^2\theta\,\partial\phi + \sin^2\theta\,\partial\phi - \sinh^2\rho\,\partial\sigma}{\cosh(2\rho) + \cos(2\theta)} \\ \bar{\mathcal{A}} =& \frac{\cosh^2\rho\,\bar\partial\tau - \cos^2\theta\,\bar\partial\phi + \sin^2\theta\,\bar\partial\phi + \sinh^2\rho\,\bar\partial\sigma}{\cosh(2\rho) + \cos(2\theta)}. \end{aligned} \tag{4.27}$$

Inserting these in (4.25), performing all computations and renaming $k \to N_5$ yields exactly what we are after

$$
\begin{aligned}
S_{\text{tot}} = \frac{N_5}{\pi} \int d^2z \Bigg(\partial\rho\bar{\partial}\rho + \partial\theta\bar{\partial}\theta + & \frac{N_5}{\Sigma}\left(\sin^2\theta\, \cosh^2\rho\, \partial\phi\bar{\partial}\phi + \cos^2\theta\, \sinh^2\rho\, \partial\psi\bar{\partial}\psi\right) \\
& + \frac{N_5\cos^2\theta\cosh^2\rho}{\Sigma}\left(\partial\psi\bar{\partial}\phi - \partial\phi\bar{\partial}\psi\right)\Bigg)
\end{aligned}
\tag{4.28}
$$

with

$$
\Sigma = N_5(\cosh^2\rho - \sin^2\theta). \tag{4.29}
$$

The one in (4.28), together with (4.29), is the correct action of string theory with background given by the $ds^2_\perp$ piece from (4.16).

4.3 Supertubes

A more general solution than the one above is the supertube solution. In this case the NS5 branes spiral around the $\tilde{y}$ direction into a helix, giving rise to a momentum and angular momentum charge, as we will see below. The amount of the spiraling is controlled by a new parameter $k \in \mathbb{Z}$ (this is a different k with respect to the one from Section 4.1).

4.3.1 Supergravity limit

The supergravity metric is retrieved by slightly modifying the positions of the NS5 branes in the previous solution by introducing a new term in the definition of the ϕ_m angle, namely the angle determining the placement of the branes as $(x_1)_m + i(x_2)_m = ae^{i\phi_m}$; the new term gets us from $(m = 1, \dots, N_5)$

$$
\phi_m = \frac{2\pi m}{N_5} \qquad \text{to} \qquad \phi_m = \frac{k}{N_5}\frac{t+\tilde{y}}{R_{\tilde{y}}} + \frac{2\pi m}{N_5}. \tag{4.30}
$$

As can be clearly seen, the NS5 solution is simply given by setting $k = 0$.

The metric when $k \neq 0$, i.e. the metric for the NS5-P system, is [23] ($v = t - \tilde{y}$ and $u = t + \tilde{y}$)

$$
\begin{aligned}
ds^2 = &- dudv + N_5\left[d\rho^2 + d\theta^2 + \frac{1}{\Sigma_0}\left(\sin^2\theta\cosh^2\rho\, d\phi^2 + \cos^2\theta\sinh^2\rho\, d\psi^2\right)\right] \\
&+ \frac{1}{\Sigma_0}\left[\frac{2k}{R_{\tilde{y}}}\sin^2\theta\, dvd\phi + \frac{k^2}{N_5 R_{\tilde{y}}^2}dv^2\right] + ds^2_{T^4}
\end{aligned}
\tag{4.31}
$$

$$
e^{-2\Phi} = \frac{N_{\text{p}}\Sigma_0}{N_5 k^2 V_4}, \qquad B = \frac{N_5^2\cos^2\theta\cosh^2\rho}{\Sigma_0}d\phi\wedge d\psi + \frac{k\cos^2\theta}{R_{\tilde{y}}\Sigma_0}dv\wedge d\psi \tag{4.32}
$$

where $\Sigma_0 = \sinh^2\rho + \cos^2\theta$ and V_4 is the volume of the four torus.

Following the method from [26], we can compute the linear and angular momentum associated to this solution; let's start with the linear momentum. Following the general recipe from the above reference, if we wish to compute the momentum along a translation-invariant direction $\tilde{y}$ chosen between the set X of such directions, we must compute the asymptotic value of the metric component $g_{t\tilde{y}}$. If this goes as

$$
g_{t\tilde{y}} \sim \frac{q}{r^{D-3}}, \tag{4.33}
$$

where q is some constant, D is the number of spacetime dimensions without the ones in X and r is the radius in these D dimensions, then the modulus of the momentum along the direction $\tilde{y}$ per unit of volume in the space spanned by the X directions, is given by

$$
\left|\mathcal{P}_{\tilde{y}}\right| = \frac{(D-3)\Omega_{D-2}q}{16\pi G}, \tag{4.34}
$$

where Ω_{D-2} is the surface of the unit radius S^{D-2}, and $G = 8\pi^6 g_s^2$ is Newton's constant[2]. Using the metric from (4.31), we find

$$g_{t\tilde{y}} \sim \frac{k^2 a^2}{N_5 R_{\tilde{y}}^2 r^2} \quad \Rightarrow \quad q = \frac{k^2 a^2}{N_5 R_{\tilde{y}}^2} \tag{4.35}$$

where $r = a \sinh \rho$, and in this case $D = 5$. Hence, we find

$$|\mathcal{P}_{\tilde{y}}| = \frac{\Omega_3}{8\pi G} \frac{k^2 a^2}{N_5 R_{\tilde{y}}^2} = \frac{2\pi^2}{64\pi^7 g_s^2} \frac{k^2 a^2}{N_5 R_{\tilde{y}}^2} = \frac{k^2 a^2}{32\pi^5 N_5 g_s^2 R_{\tilde{y}}^2}; \tag{4.36}$$

and, multiplying by the volume of the four torus and the $\tilde{y}$-coordinate's one, we find

$$|P_{\tilde{y}}| = |\mathcal{P}_{\tilde{y}}| (2\pi R_{\tilde{y}} V_4) = \frac{k^2 a^2 V_4}{16\pi^4 N_5 g_s^2 R_{\tilde{y}}}. \tag{4.37}$$

A similar reasoning leads us to the computation of the angular momentum of the metric. This time, we look at the component $g_{t\phi}$ which should behave at infinity as

$$g_{t\phi} \sim \frac{\tilde{q}}{r^{D-3}} \quad \text{for (4.31)} \quad g_{t\phi} \sim \frac{k a^2}{R_{\tilde{y}} r^2} \quad \Rightarrow \quad \tilde{q} = \frac{k a^2}{R_{\tilde{y}}} \tag{4.38}$$

from where we find, together with the help of the definition of the angular momentum density $\mathcal{J}_\phi = \frac{\Omega_{D-2}\tilde{q}}{8\pi G}$,

$$\mathcal{J}_\phi = \frac{\Omega_3 a^2 k}{8\pi G R_{\tilde{y}}} = \frac{\pi a^2 k}{4 G R_{\tilde{y}}} \quad \Rightarrow \quad J_\phi = \mathcal{J}_\phi (2\pi R_{\tilde{y}} V_4) = \frac{\pi^2 a^2 k V_4}{2G} = \frac{a^2 k V_4}{16\pi^4 g_s^2}. \tag{4.39}$$

It is also possible to relate the angular momentum J_ϕ to N_5, N_p and k using the fact that $|P_{\tilde{y}}| = \frac{N_\mathrm{p}}{R_{\tilde{y}}}$ together with (4.37) that yields

$$N_\mathrm{p} = \frac{k^2 a^2 V_4}{16\pi^4 N_5 g_s^2} \tag{4.40}$$

which, once inserted in (4.39), gives

$$J_\phi = \frac{N_5 N_\mathrm{p}}{k} \tag{4.41}$$

Since it was introduced at the beginning of this section, little has been said about the parameter k; we now rectify this issue. The parameter k has a simple meaning: say you start circling around the $\tilde{y}$ axis from the mth brane; then, once the circle around the $\tilde{y}$ axis is completed, you will find yourself sitting at the $(m+k)$th brane's position. Hence, k gives the number of branes' positions that are jumped as a circle in $\tilde{y}$ is completed. If k and N_5 admit a common divisor different from 1, i.e. if $\gcd(k, N_5) = d \neq 1$, than there will be d strands of NS5 branes winding around the $\tilde{y}$ circle; on the other hand, if $\gcd(k, N_5) = 1$, then there will be a single brane winding N_5 times around the $\tilde{y}$ circle. See Figure 4.1.

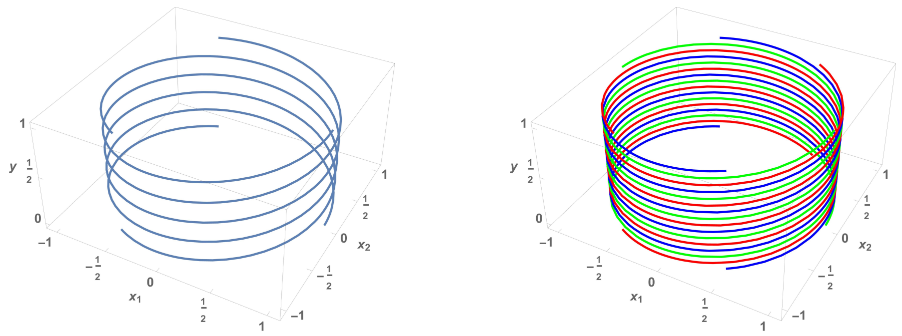
4.3.2 Supertubes from WZW model

To reproduce the metric (4.31) using a similar technique to the one used in Section 4.2.2, we need to enlarge the target space to accommodate for the branes' tilting. The target space is then

$$\mathrm{G} = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SU}(2) \times \mathbb{R}_t \times \mathbb{S}_{\tilde{y}} \times T^4. \tag{4.42}$$

G has signature equal to $(10, 2)$, while we would need a signature equal to $(9, 1)$ to correctly reproduce the NS5-P supertube. To get the correct signature and dimensionality, we again take the coset G/H, with H being the gauge group gauging two null isometries of G. Introducing the two gauge fields, and integrating them out using their equations of motions, we effectively reach the desired supergravity

[2]Recall that we set $\alpha' = 1$; if it was not, we would have $G = 8\pi^6 g_s^2 \alpha'^4$.



(a) $N_5 = 3$, $k = 5$. *One* NS5 *brane wounding three times around* $\tilde{y}$.

(b) $N_5 = 6$, $k = 15$. *Three* NS5 *strands wounding around* $\tilde{y}$.

Figure 4.1: Two examples of supertubes with different windings and strands numbers.

background: the NS5-P supertube in the decoupling limit. The most general null embeddings of $H = \mathrm{U}(1) \times \mathrm{U}(1)$ will be considered first.

The group element g will be now given by (the ϕ and σ used here are opposite to the ones of Section 4.2.2)

$$g = (g_{\rm sl}, g_{\rm su}, e^{it}, e^{i\tilde{y}}) = \big(e^{\frac{i}{2}(\tau-\sigma)\sigma_3} e^{\rho\sigma_1} e^{\frac{i}{2}(\tau+\sigma)\sigma_3}, e^{\frac{i}{2}(\psi-\phi)\sigma_3} e^{i\theta\sigma_1} e^{\frac{i}{2}(\psi+\phi)\sigma_3}, e^{it}, e^{i\tilde{y}}\big). \tag{4.43}$$

The action of the most general null embedding of $\mathrm{U}(1) \times \mathrm{U}(1)$ on g is

$$g \mapsto (e^{il_1\alpha} g_{\rm sl} e^{ir_1\beta}, e^{-il_2\alpha} g_{\rm su} e^{-ir_2\beta}, e^{il_3\alpha} e^{it} e^{ir_3\beta}, e^{-il_4\alpha} e^{i\tilde{y}} e^{-ir_4\beta}), \tag{4.44}$$

and this implies the following transformations rules for the parameters from (4.43)

$$\begin{aligned} &\tau \mapsto \tau + l_1\alpha + r_1\beta \qquad && \psi \mapsto \psi - l_2\alpha - r_2\beta \qquad && t \mapsto t + l_3\alpha + r_3\beta \\ &\sigma \mapsto \sigma - l_1\alpha + r_1\beta && \phi \mapsto \phi + l_2\alpha - r_2\beta && \tilde{y} \mapsto \tilde{y} - l_4\alpha - r_4\beta \end{aligned} \tag{4.45}$$

and the general conserved currents

$$\begin{aligned} \mathrm{U}(1)_{\rm L}: &\quad J = l_1 J_3^{\rm sl} + l_2 J_3^{\rm su} + l_3 \hat{P}_{t,\rm L} + l_4 \hat{P}_{\tilde{y},\rm L} \\ \mathrm{U}(1)_{\rm R}: &\quad \bar{J} = r_1 \bar{J}_3^{\rm sl} + r_2 \bar{J}_3^{\rm su} + r_3 \hat{P}_{t,\rm R} + r_4 \hat{P}_{\tilde{y},\rm R} \end{aligned} \tag{4.46}$$

where $J_3^{\rm sl,su}$ and $\bar{J}_3^{\rm sl,su}$ are the ones from (4.24) and

$$\hat{P}_{t,\rm L} \equiv \partial t, \quad \hat{P}_{t,\rm R} \equiv \bar{\partial} t, \quad \hat{P}_{\tilde{y},\rm L} \equiv \partial \tilde{y}, \quad \hat{P}_{\tilde{y},\rm R} \equiv \bar{\partial} \tilde{y}. \tag{4.47}$$

For the two isometries that are gauged to be null isometries, the norms of the above currents J and $\bar{J}$ must be zero with respect to the Killing metric; this amounts to require that

$$\langle J, J \rangle = N_5(-l_1^2 + l_2^2) - l_3^2 + l_4^2 = 0 \quad \text{and} \quad \langle \bar{J}, \bar{J} \rangle = N_5(-r_1^2 + r_2^2) - r_3^2 + r_4^2 = 0. \tag{4.48}$$

The action for such a general null gauging is equal to

$$S_{\rm tot} = S_{\rm wzw} + \frac{1}{2\pi} \int_{S^2} d^2z \, (-\partial u \bar{\partial} v - \partial v \bar{\partial} u) + \frac{1}{\pi} \int_{S^2} d^2z \, (2\mathcal{A}\bar{J} + 2\bar{\mathcal{A}}J - 4N_5 \Sigma \mathcal{A}\bar{\mathcal{A}}) \tag{4.49}$$

where

$$N_5\Sigma = \frac{1}{2}\big(N_5(l_1 r_1 \cosh 2\rho - l_2 r_2 \cos 2\theta) + l_3 r_3 - l_4 r_4\big). \tag{4.50}$$

Choosing the values such that $|l_1| = |l_2|$ and $|l_3| = |l_4|$ (and similarly for the right component of the current) tilts the null currents J and $\bar{J}$ in the $\mathbb{R}_t \times \mathbb{S}_{\tilde{y}}$ direction. The action $S_{\rm tot}$ from (4.49), after integrating out the gauge field, takes the form of the action of a string theory with background metric and B field given by the supergravity supertubes solutions. Specifically, selecting the values

$$l_1 = l_2 = 1, \quad l_3 = -l_4 = -\frac{k}{R_{\tilde{y}}}, \quad r_1 = -r_2 = 1, \quad r_3 = -r_4 = -\frac{k}{R_{\tilde{y}}}, \tag{4.51}$$

leads us to the NS5-P supertube of the previous section as, indeed, integrating out the gauge fields via their equations of motion and selecting the gauge $\tau = \sigma = 0$, S_{tot} becomes

$$\begin{aligned} S_{\text{tot}} =& -\frac{1}{2\pi}\int d^2z\,(\partial u\bar{\partial}v + \partial v\bar{\partial}u) + \frac{N_5}{\pi}\int d^2z\,\Big[\partial\rho\bar{\partial}\rho + \partial\theta\bar{\partial}\theta \\ & \frac{1}{\Sigma_0}(\sin^2\theta\cosh^2\rho\,\partial\phi\bar{\partial}\phi + \cos^2\theta\sinh^2\rho\,\partial\psi\bar{\partial}\psi) + \frac{\cos^2\theta\cosh^2\rho}{\Sigma_0}(\partial\phi\bar{\partial}\psi - \partial\psi\bar{\partial}\phi) \\ & + \frac{k}{N_5\Sigma_0 R_{\tilde{y}}}\big[\sin^2\theta(\partial\phi\bar{\partial}v + \partial v\bar{\partial}\phi) + \cos^2\theta(\partial\psi\bar{\partial}v - \partial v\bar{\partial}\psi)\big] + \frac{k^2}{R_{\tilde{y}}^2 N_5^2\Sigma_0}\partial v\bar{\partial}v\Big]. \end{aligned} \tag{4.52}$$

To understand the role played by the parameter k in this picture it is useful to focus on the gauge action for $\rho = 0$ and $\theta = \frac{\pi}{2}$, i.e. to focus the gauge action while being at the branes' locus. In this case the gauge action becomes ($\tau = 0$)

$$S_{\text{gauge}} = \frac{1}{\pi}\int d^2z\Big[\mathcal{A}\Big(-\frac{k}{R_{\tilde{y}}}dv - N_5 d\phi\Big) + \bar{\mathcal{A}}\Big(-\frac{k}{R_{\tilde{y}}}d\bar{v} - N_5 d\bar{\phi}\Big)\Big] \tag{4.53}$$

i.e., the quadratic term in $\mathcal{A}$ and $\bar{\mathcal{A}}$ vanishes, so the two gauge fields act as Lagrange multipliers imposing the constraint

$$N_5 d\phi + \frac{k}{R_{\tilde{y}}}dv = 0. \tag{4.54}$$

This is nothing but the condition that the branes wind in helices around the $\tilde{y}$-ϕ torus as can be checked by taking $d\phi = -\frac{2\pi k}{N_5}$, which amounts to translate by k's branes position along the ϕ circle, namely the one on the (x_1, x_2) plane along which the branes are positioned. By definition of the parameter k, this should mean that a whole lap of the $\tilde{y}$ circle has been completed; and indeed, substituting $d\phi = -\frac{2\pi k}{N_5}$ in (4.54) yields (take $t = 0$)

$$d\tilde{y} = 2\pi R_{\tilde{y}}. \tag{4.55}$$

Taking a T-duality of the NS5-P metric (4.31) along the $\tilde{y}$ direction, leads to the NS5-F1 supertube with metric given by (we refer to the T-dual of $\tilde{y}$ as y and, consequently, to the y-coordinate-circle's radius as $R_y = \frac{1}{R_{\tilde{y}}}$)

$$\begin{aligned} ds^2 =& -dudv + N_5\Big[d\rho^2 + d\theta^2 + \frac{1}{\Sigma}\big(\cosh^2\rho\sin^2\theta\,d\phi^2 + \sinh^2\rho\cos^2\theta\,d\psi^2\big)\Big] \\ & + \frac{2\nu}{\Sigma}\big(\sin^2\theta\,dtd\phi + \cos^2\theta\,dyd\psi\big) + \frac{\nu^2}{N_5\Sigma}\big(N_5\sin^2\theta\,d\phi^2 + N_5\cos^2\theta\,d\psi^2 + dudv\big) + ds^2_{T^4} \end{aligned} \tag{4.56}$$

where $\nu = kR_y$ and the B field and dilaton are given by

$$\begin{aligned} B =& \frac{\cos^2\theta\big(\nu^2 + \cosh^2\rho\big)}{\Sigma}d\phi\wedge d\psi - \frac{\nu^2}{N_5\Sigma}dt\wedge dy \\ & + \frac{\nu\cos^2\theta}{\Sigma}dt\wedge d\psi + \frac{\nu\sin^2\theta}{\Sigma}dy\wedge d\phi \end{aligned} \qquad e^{-2\Phi} = \frac{N_1\Sigma}{k^2R_y^2V_4}, \qquad \Sigma = \frac{\nu^2}{N_5} + \Sigma_0 \tag{4.57}$$

with $u = t + y$ and $v = t - y$. The metric for this configuration of charges is related to the parameters choice

$$l_1 = l_2 = 1, \quad l_3 = l_4 = -kR_y, \quad r_1 = -r_2 = 1, \quad r_3 = -r_4 = -kR_y, \tag{4.58}$$

We highlight that the NS5-F1 supertube is the same system we already studied in Section 3.4.1, namely the D1-D5 fuzzball, just in a different duality frame.

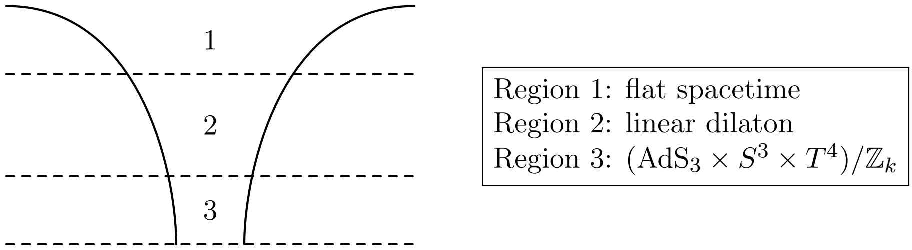


Figure 4.2: Structure of spacetime for the supergravity system of NS5-F1 branes from the end of Section 4.3.2. Basically, the decoupling limit, which amounts to neglecting the 1 in the H_5 harmonic function, decouples the asymptotic flat spacetime that we would have at $\rho \to \infty$ if the 1 was in place, leaving behind an asymptotic linear dilaton spacetime.

Before moving on with the next chapter, we take the chance to spend a few words on the structure of spacetime resulting from taking the decoupling limit; a cartoon representing the situation for the NS5-F1 system can be seen in Figure 4.2. Essentially, the decoupling limit affects the asymptotic spacetime that we get by taking the $\rho \to \infty$ limit: with the 1 present in the H_5 harmonic function we would get an asymptotically flat spacetime, while in the decoupling limit we get an asymptotic *linear dilaton* spacetime

$$ds^2 \sim -dt^2 + dy^2 + N_5(d\rho^2 + d\Omega_3^2) + ds^2_{T^4} \quad \text{with} \quad \Phi \sim -\rho \tag{4.59}$$

where Φ is the dilaton. This is also the asymptotic limit for a stack of NS5 branes in the decoupling limit, as from far away distinguishing between the branes placed on a circle and the branes placed on top of each other is difficult. Finally, the structure indicated by $(\mathrm{AdS}_3 \times S^3 \times T^4)/\mathbb{Z}_k$ in Figure 4.2 is known as *orbifold*; if we set $k = 1$ the orbifold reduces to the simpler $\mathrm{AdS}_3 \times S^3 \times T^4$ throat. The emerging of the orbifold can be naively seen by taking the $R_y \to \infty$ limit in (4.56) to look at the throat; in this limit, the metric becomes

$$\begin{aligned} ds^2 =& N_5\left[-\frac{\cosh^2\rho}{k^2R_y^2}dt^2 + \frac{\sinh^2\rho}{k^2R_y^2}dy^2 + d\rho^2\right] \\ &+ N_5\left[\left(\frac{dt}{kR_y} + d\phi\right)^2 \sin^2\theta + \left(\frac{dy}{kR_y} + d\psi\right)^2 \cos^2\theta + d\theta^2\right] + ds^2_{T^4}. \end{aligned} \tag{4.60}$$

As we can see, to put the metric in a true $\mathrm{AdS}_3 \times S^3 \times T^4$ form, we would need to implement a coordinate change such as

$$d\phi \to d\phi' = \frac{dt}{kR_y} + d\phi \quad \text{and} \quad d\psi \to d\psi' = \frac{dy}{kR_y} + d\psi \tag{4.61}$$

However, ϕ' and ψ' are only angles if they satisfy periodic boundary conditions, and while there are no problems for ϕ' to satisfy this, ψ' does pose some problems since both y and ϕ are periodic coordinates. In the case of $k = 1$ this problem is promptly solved because the periodicity in ϕ and in y match each other, but for $k \neq 1$ the orbifold structure emerges. We will not go in more detail about this because it is beyond the scope of this thesis.

Chapter 5

Geodesics

In this chapter we are going to analyze how the geometry of some string theory solutions presented in the previous chapters influence the motion of massless probes. The analysis will be carried out for different supergravity backgrounds, starting from the D1-D5 circular fuzzball, and moving to both the elliptical and circular NS5 branes arrangement, to finally end with the elliptical and circular NS5-F1 fuzzball. The main goal is looking for closed trajectories orbiting around the said solutions, indicating the possibility for the above branes configurations to trap light similarly to more ordinary, GR black holes. In this context, the main results presented in this chapter suggest that as the microstates parameters are changed, the absorption properties of some fuzzball configuration change as well. Indeed, it seems that the fuzzballs under study only absorb light with a specific impact parameter, while deviating all other light rays with different impact parameters. However, since the impact parameter giving absorption depends on some microstate-characterizing parameter, one can think that considering the full ensemble will generate absorption for all impact parameters below a certain critical one.

5.1 Geodesics in D1-D5 fuzzballs

We already know the metric for the D1-D5 fuzzball with circular profile from Section 3.4.1, equation (3.46); we rewrite it here (in a more compact form and with some minor changes of notation: $h_{\text{Sec. 3.4.1}} \equiv H_{\text{here}}$, $Q'_{1,5} \equiv Q_{1,5}$, $r \equiv \rho$ plus neglecting the $ds^2_{T^4}$ component of the metric) for completeness

$$\begin{aligned} ds^2 =& H^{-1}\left[-(dt+\omega_\phi d\phi)^2 + (dy+\omega_\psi d\psi)^2\right] \\ &+ H\left[f\left(\frac{d\rho^2}{\rho^2+a^2}+d\theta^2\right)+\rho^2\cos^2\theta d\psi^2+(\rho^2+a^2)\sin^2\theta d\phi^2\right], \end{aligned} \tag{5.1}$$

where

$$H = \sqrt{\left(1+\frac{Q_1}{\rho^2+a^2\cos^2\theta}\right)\left(1+\frac{Q_5}{\rho^2+a^2\cos^2\theta}\right)}, \qquad f=\rho^2+a^2\cos^2\theta, \qquad a=\frac{\sqrt{Q_1Q_5}}{R} \tag{5.2}$$

$$\omega_\phi = \frac{a\sqrt{Q_1Q_5}\sin^2\theta}{\rho^2+a^2\cos^2\theta}, \qquad \omega_\psi = -\frac{a\sqrt{Q_1Q_5}\cos^2\theta}{\rho^2+a^2\cos^2\theta},$$

Please notice that, compared to [27], along with other variable renaming, we have $(\omega_\psi)_{\text{here}} = -(\omega_\psi)_{\text{from [27]}}$.

To simplify the study that follows, we will restrict to the $\theta = 0$ surface, i.e. to the plane orthogonal to the branes in the $\boldsymbol{x}$ space; the metric (5.1) becomes

$$ds^2 = H^{-1}\left[-dt^2+(dy+\omega_\psi d\psi)^2\right] + H\left[(\rho^2+a^2)\frac{d\rho^2}{\rho^2+a^2}+\rho^2 d\psi^2\right]. \tag{5.3}$$

Wishing to study the null paths in the D1-D5 fuzzball, we introduce the Lagrangian

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{5.4}$$

where $\dot{x}^\mu$ represents the derivative with respect to an affine parameter λ. From it, taking into consideration that $\mathcal{L}$ does not depend on t, y, ϕ and ψ, we can compute the corresponding conserved quantities[1]

$$
\begin{aligned}
E &= -\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{\dot{t}}{H} = \frac{(a^2+\rho^2)\dot{t}}{\sqrt{(a^2+Q_1+\rho^2)(a^2+Q_5+\rho^2)}},\\
P &= \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\dot{y}+\omega_\psi\dot{\psi}}{H} = \frac{(a^2+\rho^2)\dot{z} - a\sqrt{Q_1Q_5}\dot{\psi}}{\sqrt{(a^2+Q_1+\rho^2)(a^2+Q_5+\rho^2)}},\\
J_\psi &= \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{\omega_\psi\left(\dot{y}+\omega_\psi\dot{\psi}\right)}{H} + H\rho^2\dot{\psi} = \frac{\dot{\psi}(a^2\rho^2 + (Q_1+\rho^2)(Q_5+\rho^2) - a\sqrt{Q_1Q_5}\dot{z}}{\sqrt{(a^2+Q_1+\rho^2)(a^2+Q_5+\rho^2)}}
\end{aligned}
\tag{5.5}
$$

and the velocities related to them

$$
\begin{aligned}
\dot{t} &= EH = \frac{E\sqrt{(a^2+Q_1+\rho^2)(a^2+Q_5+\rho^2)}}{a^2+\rho^2},\\
\dot{y} &= \frac{PH^2\rho^2 - J_\psi\omega_\psi + P\omega_\psi^2}{H\rho^2} = \frac{aJ_\psi\sqrt{Q_1Q_5} + a^2P\rho^2 + P(Q_1+\rho^2)(Q_5+\rho^2)}{\rho^2\sqrt{(a^2+Q_1+\rho^2)(a^2+Q_5+\rho^2)}},\\
\dot{\psi} &= \frac{J_\psi - P\omega_\psi}{H\rho^2} = \frac{a^2J_\psi + a\sqrt{Q_1Q_5}P + \rho^2J_\psi}{\rho^2\sqrt{(a^2+Q_1+\rho^2)(a^2+Q_5+\rho^2)}}.
\end{aligned}
\tag{5.6}
$$

With the help of these, starting from the null condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$ with $g_{\mu\nu}$ from (5.3), we can obtain the following equation for $\dot{\rho}$

$$
\dot{\rho}^2 = E^2 - P^2 - \frac{(J_\psi - \omega_\psi P)^2}{H^2\rho^2}
\tag{5.7}
$$

that we later recast, with the help of the chain rule $\dot{\rho} = \frac{d\rho}{d\psi}\dot{\psi}$, into ($x \equiv \rho^2$)

$$
\left(\frac{d\rho}{d\psi}\right)^2 = \frac{x\left(Ax^3+Bx^2+Cx+D\right)}{\left(a^2J_\psi + a\sqrt{Q_1Q_5}P + J_\psi x\right)^2} \equiv \frac{x\mathcal{P}(x)}{\left(a^2J_\psi + a\sqrt{Q_1Q_5}P + J_\psi x\right)^2};
\tag{5.8}
$$

the parameters $A, \dots, D$'s explicit definitions in terms of the metric's charges are

$$
\begin{aligned}
A &= E^2 - P^2\\
B &= -J_\psi^2 + \left(2a^2+Q_1+Q_5\right)\left(E^2-P^2\right)\\
C &= -2aJ_\psi\sqrt{Q_1Q_5}P + Q_1Q_5\left(E^2-P^2\right) + a^2\left(-2J_\psi^2 + \left(Q_1+Q_5+a^2\right)\left(E^2-P^2\right)\right)\\
D &= -a^2\left(aJ_\psi + \sqrt{Q_1Q_5}P\right)^2.
\end{aligned}
\tag{5.9}
$$

In the remainder of this section we are interested in both the values x_i such that $\mathcal{P}(x_\mathrm{i}) = 0$, and radii x_* such that $\mathcal{P}(x_*) = \mathcal{P}'(x_*) = 0$, i.e. zeros of $\mathcal{P}(x)$ as well as of its first derivative. The x_i represent *inversion radii*, meaning radii at which the particle inverts its motion, switching from approaching the fuzzball to moving away from it. The x_*, on the other hand, represent radii at which the massless particle starts to "circle around" the fuzzball, moving along a close trajectory. These second kind of trajectories are the most interesting among the two because they signal the possibility for the specific fuzzball we are considering to capture massless particles, meaning that it would possess the property after which black holes get their name.

We will save the more interesting solutions for later; now we focus on the study of inversion radii. To do this we rewrite (5.8) in a more explicit way as

$$
\left(\frac{d\rho}{d\psi}\right)^2 = \frac{(\rho^2+a^2+Q_1)(\rho^2+a^2+Q_5)\rho^4}{b^2\left(\rho^2+a^2+\frac{av\sqrt{Q_1Q_5}}{b_\psi}\right)} - \rho^2
\tag{5.10}
$$

[1]In our notation, the conserved quantity with respect to a variable x such that $\frac{\partial \mathcal{L}}{\partial x} = 0$ is $C = \frac{\partial \mathcal{L}}{\partial \dot{x}}$.

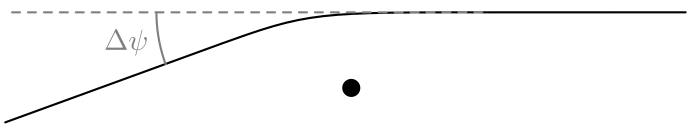


Figure 5.1: Deflection angle for a massless particle approaching the D1-D5 fuzzball in the $\theta = 0$ slice of the metric. The black dot represents the fuzzball location.

where

$$v = \frac{P}{E}, \qquad b_\psi = \frac{J_\psi}{E}, \qquad b = \frac{b_\psi}{\sqrt{1-v^2}} \tag{5.11}$$

with b_ψ being the impact parameter of the particle in the $\theta = 0$ section. For large ρ, it is possible to expand the RHS of (5.10) to find that the inversion radius ρ_i is given by

$$\rho_\mathrm{i} = b\left(1 - \frac{f_2}{2b^2}\right) \quad \text{with} \quad f_2 = Q_1 + Q_5 - \frac{2av\sqrt{Q_1 Q_5}}{b_\psi}. \tag{5.12}$$

So, we learn that for large impact parameters the massless particle will not be captured by the fuzzball, but its trajectory will be perturbed nonetheless; actually, as done in [27], it is possible to compute the deflection angle due to the fuzzball. In our case, it turns out to be (see the reference for more details on how to compute it and Figure 5.1)

$$\Delta\psi = \frac{\pi(1-v^2)}{2b_\psi^2}\left(Q_1 + Q_5 - \frac{2av\sqrt{Q_1Q_5}}{b_\psi}\right) + \dots \tag{5.13}$$

We now turn our attention to the circling paths; to do this we are now going to study the polynomial $\mathcal{P}(x)$ from (5.8) as done in [28]. The sign of A and D is fixed: A can only be positive, since $E^2 > P^2$; while D is bound to be negative. As a consequence, $\mathcal{P}(x \gg 1) > 0$ and $\mathcal{P}(0) < 0$, implying that at least one of $\mathcal{P}(x)$'s three zeroes is positive. Of the three, we should concentrate on the largest one, since this will be the first-one a particle meets as it approaches the fuzzball. Requiring the largest zero to be positive and double, together with the conditions $A > 0$ and $D < 0$, we find that[2] $C > 0$ and $B < 0$.

The largest solution to the equation $\mathcal{P}'(x) = 0$, meaning the large solution to the equation

$$3Ax^2 + 2Bx + C = 0 \tag{5.14}$$

is

$$x_* = \frac{-B + \sqrt{B^2 - 3AC}}{3A}, \tag{5.15}$$

and it exists iff $B^2 \geq 3AC$. Plugging it inside the equation $\mathcal{P}(x) = 0$, D can be seen to be equal to (look at (B.3))

$$D = \frac{2\big(B^2 - 3AC\big)^{\frac{3}{2}} - B\big(2B^2 - 9AC\big)}{27A^2}. \tag{5.16}$$

Imposing the requirement $D \leq 0$, we find that $B \leq 4AC$ and this, in conjunction with the former bound on B, combines into the bound $4AC \geq B^2 \geq 3AC$. These bounds are quite tricky to be solved in the most general case; however, choosing $B = -2\sqrt{AC}$, which is equivalent to imposing $D = 0$, we find

$$x_* = \frac{C}{A} = \sqrt{\frac{-2aJ_\psi\sqrt{Q_1Q_5}P + Q_1Q_5\big(E^2 - P^2\big) + a^2\big(-2J_\psi^2 + \big(Q_1 + Q_5 + a^2\big)\big(E^2 - P^2\big)\big)}{E^2 - P^2}}. \tag{5.17}$$

[2]To see this, it is sufficient to write $\mathcal{P}(x) = (x - z_*)^2(x - z_1)$, where z_* is the largest, double, positive zero, and expand the expression to find $\mathcal{P}(x) = x^3 + (-z_1 - 2z_*)x^2 + 2z_*z_1x - z_1z_*^2$. Using the constraint $D > 0$, we deduce $z_1 > 0$; now all other conclusions follow immediately.

Again, this condition, with the many free parameters it has, is rather unintelligible so, to get a comprehensible result, it is easier to fix some parameters. For instance, specializing to the case $Q_1 = Q_5 = Q$ and $P = -\frac{aJ_\psi}{\sqrt{Q_1 Q_5}}$, the one above simply reduces to (see (B.5))

$$x_* = Q + a^2. \tag{5.18}$$

At this point it is interesting to look at (5.10) where the above values for the charges and momentum have been selected, the upshot is

$$\begin{aligned}\left(\frac{d\rho}{d\psi}\right)^2 =& \frac{(\rho^2 + a^2 + Q)^2}{b^2} - \rho^2 \\ =& \frac{(\rho^2 + a^2 + Q - b\rho)(\rho^2 + a^2 + Q + b\rho)}{b^2}.\end{aligned} \tag{5.19}$$

The turning-point equation is now easy to solve, and returns that there exist turning points if $b > b_{\text{crit}} = 2\sqrt{a^2 + Q}$; on the other hand, if $b = b_{\text{crit}}$, the null path ends up in a closed trajectory with radius $\rho_*^2 = x_*$ around the fuzzball, hence being trapped forever. Finally, for $b < b_{\text{crit}}$ the particle reaches $\rho = 0$ and after possibly looping around few times, escapes back to infinity. Summing up, for the above configurations of charges and momentum, we find that the massless probe will be sent back to infinity by the fuzzball if $b > b_{\text{crit}}$ and if $b < b_{\text{crit}}$; in the first case it stops short of $\rho = 0$, whereas in the second case it does arrive at $\rho = 0$ to then continue its journey towards infinity. Conversely, for $b = b_{\text{crit}}$ the particle gets trapped in a closed trajectory around the fuzzball.

If we wanted to compare this to the behavior of a massless probe in the Schwarzschild geometry, keeping in mind that the fuzzball under consideration is a two-charge solution in five spacetime dimensions, so not really a physical solution, we would immediately notice a striking difference: in the case of the fuzzball the particle ends up being trapped, and hence never coming back from its trip, just for a specific impact parameter (i.e., b_{crit}), while in the Schwarzschild black hole all particles hitting the event horizon are doomed to never emerge again from the hole. A plausible idea to cure this issue, is to recall that up to now we have been analyzing *one specific* fuzzball configuration, whereas a true black hole would be composed by a statistical ensemble of such configurations. If then every configuration had a slightly different b_{crit}, once the average over the ensemble is carried out, we would find a picture agreeing with the absorbing behavior of a typical black hole. One way in which we could explore some different fuzzball configurations is by the introduction of the parameter k from Section 4.3.1. In the case of the NS5-P supertube we have seen that k indicates the number of branes positions shifted as a turn of the y circle is completed; performing an S duality from the NS5-F1 metric it is possible to implement the dependence on k in the D1-D5 metric as well. Doing the S duality, it is possible to see that k will appear wherever there is a y-coordinate's radius R; given this fact, and looking at the definition of the parameter a in (5.2), we have that $a_{k\neq 1} = \frac{a_{k=1}}{k}$. With this substitution inserted in (5.18) and in $b_{\text{crit}} = 2\sqrt{a^2 + Q}$, we immediately see that changing k, which again corresponds to considering different circular supertubes, the critical radius, as well as the value for b_{crit}, change, giving us hope that the discussion given above on the ensemble average is at least plausible.

5.2 Geodesics in NS5-F1 supertubes

5.2.1 Circular supertube

Moving to the S-dual system to the just considered, which we already met in Section 4.3.2, the metric is given by

$$\begin{aligned}ds^2 =& -\left(1 - \frac{\nu^2}{N_5\Sigma}\right)(dt^2 - dy^2) + N_5\left[d\rho^2 + d\theta^2 + \frac{1}{\Sigma}(\cosh^2\rho\sin^2\theta d\phi^2 + \sinh^2\rho\cos^2\theta d\psi^2)\right] + \\ &+ \frac{2\nu}{\Sigma}(\sin^2\theta dtd\phi + \cos^2\theta dyd\psi) + \frac{\nu^2}{N_5\Sigma}(N_5\sin^2\theta d\phi^2 + N_5\cos^2\theta d\psi^2)\end{aligned} \tag{5.20}$$

with $\nu = kR$, $\Sigma = \frac{\nu^2}{N_5} + \Sigma_0 = \frac{\nu^2}{N_5} + \sinh^2\rho + \cos^2\theta$. Actually, this is not the S-dual to the D1-D5 fuzzball because the metric above is the metric in the decoupling limit for the NS5 branes, and we have already seen at the end of Section 4.3.2 how this influences the asymptotic limit.

Once again, we will take $\theta = 0$, which gives us the metric

$$ds^2 = -\left(1 - \frac{\nu^2}{n_5\Sigma}\right)(dt^2 - dy^2) + n_5 d\rho^2 + \frac{2\nu}{\Sigma} dy d\psi + \frac{\nu^2 + n_5\sinh^2\rho}{\Sigma} d\psi^2, \tag{5.21}$$

and repeat similar steps to the ones performed in the previous section, i.e. introduce the Lagrangian $\mathcal{L}$ for the metric (5.20), compute the conserved quantities

$$\begin{aligned}
E &= -\frac{\partial\mathcal{L}}{\partial\dot{t}} = \dot{t}\left(1 - \frac{\nu^2}{n_5\Sigma}\right) = \frac{\dot{t} n_5\cosh^2\rho}{n_5 + k^2R^2 + n_5\sinh^2\rho} \\
P &= \frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{\nu(\dot{\psi}n_5 - \dot{y}\nu)}{n_5\Sigma} + \dot{y} = \frac{n_5[\dot{y} + 2\dot{\psi}kR + \dot{y}\cosh(2\rho)]}{n_5 + 2k^2R^2 + n_5\cosh(2\rho)} \\
J_\psi &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \frac{\dot{\psi}n_5\sinh^2\rho + \nu(\dot{y} + \dot{\psi}\nu)}{\Sigma} = n_5\left(\dot{\psi} + \frac{kR\dot{y} - n_5\dot{\psi}}{n_5 + k^2R^2 + n_5\sinh^2\rho}\right)
\end{aligned} \tag{5.22}$$

and solve them to find the velocities

$$\begin{aligned}
\dot{t} &= \frac{n_5 E\Sigma}{n_5\Sigma - \nu^2} = E + \frac{k^2ER^2}{n_5\cosh^2\rho} \\
\dot{y} &= \frac{n_5\Sigma\left[\nu(-J_\psi + P\nu) + n_5P\sinh^2\rho\right]}{n_5(-\nu^2 + n_5\Sigma)\sinh^2\rho - \nu^2(n_5 + \nu^2 - n_5\Sigma)} = P + \frac{kR(-J_\psi + kPR)}{n_5\sinh^2\rho} \\
\dot{\psi} &= \frac{\Sigma[-\nu(n_5P + J_\psi\nu) + n_5J_\psi\Sigma]}{n_5(-\nu^2 + n_5\Sigma)\sinh^2\rho - \nu^2(n_5 + \nu^2 - n_5\Sigma)} = \frac{J_\psi\cosh^2\rho - kPR}{n_5\sinh^2\rho}.
\end{aligned} \tag{5.23}$$

Substituting these inside the condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, it is possible to find

$$\begin{aligned}
\dot{\rho}^2 = \frac{\left(-J_\psi^2 + n_5\left(E^2 - P^2\right)\right)\sinh^4\rho}{n_5^2\sinh^2\rho\cosh^2\rho} + \\
+ \frac{\left(-J_\psi^2 + n_5\left(E^2 - P^2\right) - (J_\psi - kPR)^2 + E^2k^2R^2\right)\sinh^2\rho - (J_\psi - kPR)^2}{n_5^2\sinh^2\rho\cosh^2\rho}
\end{aligned} \tag{5.24}$$

that can later be recast into ($x \equiv \sinh^2\rho$)

$$\left(\frac{d\rho}{d\psi}\right)^2 = \frac{x\left(Ax^2 + (A - B + C)x - B\right)}{(J_\psi(1 + x) - kPR)(1 + x)} = \frac{x\mathcal{P}(x)}{(J_\psi(1 + x) - kPR)(1 + x)}, \tag{5.25}$$

with

$$A = -J_\psi^2 + N_5\left(E^2 - P^2\right) \qquad B = (J_\psi - kPR)^2 \qquad C = E^2k^2R^2. \tag{5.26}$$

Now, the way to proceed to look for closed null paths is the same as before: find x_* such that $\mathcal{P}(x_*) = \mathcal{P}'(x_*)$; solving the equation $\mathcal{P}(x) = 0$ we find the discriminant $\Delta = A^2 + 2A(B+C) + (B-C)^2$ whose sign is

$$\Delta \geq 0 \quad \text{for} \quad A \in \left(-\infty, -(\sqrt{B} + \sqrt{C})^2\right] \cup \left[-(\sqrt{B} - \sqrt{C})^2, +\infty\right). \tag{5.27}$$

For sake of brevity, we rename $A_1 \equiv -(\sqrt{B} + \sqrt{C})^2$ and $A_2 \equiv -(\sqrt{B} - \sqrt{C})^2$. For A such that $\Delta \geq 0$, the solutions to $\mathcal{P}(x) = 0$ are

$$x_1 = \frac{B - A - C + \sqrt{\Delta}}{2A} \quad \text{and} \quad x_2 = \frac{B - A - C - \sqrt{\Delta}}{2A} \tag{5.28}$$

Keeping in mind that $\mathcal{P}(x)$, being proportional to $\left(\frac{d\rho}{d\psi}\right)^2$, must be positive, we divide the analysis in three cases and restrict to $\sqrt{C} > \sqrt{B}$ (so that $x_* > 0$, see below).

- Case 1: $A > 0$. In this case $x_1 > 0$ and $x_2 < 0$; $\mathcal{P}(x)$ is non-negative iff $x \geq x_1$, given that $x > 0$ by definition. The massless particle can arrive from infinity at $x = x_1$ where it inverts its motion.
- Case 2: $A \in [A_2, 0)$. In this case $x_2 > x_1 > 0$ and $\mathcal{P}(x) \geq 0$ for $x \in [x_1, x_2]$; so these values are isolated from infinity. As A approaches $-\big(\sqrt{B}-\sqrt{C}\big)^2$ the two solutions converge to $x_* = \frac{\sqrt{B}}{\sqrt{C}-\sqrt{B}}$.
- Case 3: $A \in (-\infty, A_1]$. In this case x_1 and x_2 are both negative so no inversion point exist.

Since they are the most interesting cases to us, let us spend a few more words on the x_*s coming from $A = -\big(\sqrt{B}-\sqrt{C}\big)^2$ and $A = \big(\sqrt{B}+\sqrt{C}\big)^2$ starting from the first one. For $A = -\big(\sqrt{B}-\sqrt{C}\big)^2$ we have

$$\mathcal{P}(x) = -\Big(\big(\sqrt{B}-\sqrt{C}\big)x + \sqrt{B}\Big)^2 \quad \Rightarrow \quad x_* = \frac{\sqrt{B}}{\sqrt{C}-\sqrt{B}} = \frac{|J_\psi - kPR|}{EkR - |J_\psi - kPR|},$$

and, as hinted above, this is acceptable (i.e. positive) only if $\sqrt{C} > \sqrt{B}$. It cannot be reached from infinity since, with A being negative, we would have $\big(\frac{d\rho}{d\psi}\big)^2 < 0$ at $\rho \to \infty$, as we can see from (5.25). On the other hand, the x_* coming from $A = \big(\sqrt{B}+\sqrt{C}\big)^2$ has to be discarded because it is negative. Similarly to the case of the D1-D5 fuzzball, the condition on A for the existence of a circular null path imposes a condition on the impact parameter with which the particle approaches the hole. This is seen most easily for $P = 0$, when the condition $A = -\big(\sqrt{B}-\sqrt{C}\big)^2$ is easily solvable in terms of J_ψ/E. Most of the similarities with the D1-D5 fuzzball seem to end here though, as this solution cannot be reached from infinity, meaning that a massless probe would only be able to travel along the trajectory if it spawned on it. This is related to the fact that, while in D1-D5 fuzzball the critical solution arises from trajectories that arrive from infinity and are deviated back to infinity, in the case of the NS5-F1 supertube the critical solution emerges from a trajectory that is already limited between two finite radii, as we said in “Case 2” above.

AdS limit

Taking the $R \to \infty$ limit, the metric (5.20) transforms into

$$ds^2 = N_5\Big[-\frac{\cosh^2\rho}{k^2R^2}dt^2 + \frac{\sinh^2\rho}{k^2R^2}dy^2 + \Big(\frac{dt}{kR} + d\phi\Big)^2\sin^2\theta + \Big(\frac{dy}{kR} + d\psi\Big)^2\cos^2\theta + d\theta^2 + d\rho^2\Big], \tag{5.29}$$

and taking the $\theta = 0$ section returns

$$ds^2 = n_5\Big[-\frac{\cosh^2\rho}{k^2R^2}dt^2 + \frac{\sinh^2\rho}{k^2R^2}dy^2 + \Big(\frac{dy}{kR} + d\psi\Big)^2 + d\rho^2\Big]. \tag{5.30}$$

To this, we can associate the conserved charges

$$\begin{aligned} E &= \dot{t}\frac{n_5\Sigma_0}{k^2R^2} = \dot{t}\frac{n_5\cosh^2\rho}{k^2R^2} \\ P &= \frac{n_5(\dot{\psi}kR + \dot{y}\Sigma_0)}{k^2R^2} = \frac{n_5(\dot{y} + \dot{\psi}kR + \dot{y}\sinh^2\rho)}{k^2R^2} \\ J_\psi &= n_5\Big(\dot{\psi} + \frac{\dot{y}}{kR},\Big) \end{aligned} \tag{5.31}$$

and velocities

$$\begin{aligned} \dot{t} &= \frac{Ek^2R^2}{n_5\Sigma_0} = \frac{Ek^2R^2}{n_5\cosh^2\rho} \\ \dot{y} &= \frac{kR(kPR - J_\psi)}{n_5(\Sigma_0 - 1)} = \frac{kR(kPR - J_\psi)}{n_5\sinh^2\rho} \\ \dot{\psi} &= \frac{kPR - J_\psi\Sigma_0}{n_5(1 - \Sigma_0)} = \frac{J_\psi\cosh^2\rho - kPR}{n_5\sinh^2\rho}. \end{aligned} \tag{5.32}$$

The equation for $\dot\rho^2$ determined from the null condition as already done for the previous systems is

$$\dot\rho^2 = -\frac{(J_\psi - kPR)^2\cosh^2\rho + J_\psi\cosh^2\rho\sinh^2\rho - E^2k^2R^2\sinh^2\rho}{n_5^2\sinh^2\rho\cosh^2\rho}, \tag{5.33}$$

and can be turned into

$$\left(\frac{d\rho}{d\psi}\right)^2 = \frac{x\left[-Ax^2 + (-B - A + C)x - B\right]}{(1+x)\left(J_\psi x + J_\psi - kPR\right)^2} = \frac{x\mathcal{P}(x)}{(1+x)(J_\psi x + J_\psi - kPR)^2}, \tag{5.34}$$

where $x \equiv \sinh^2\rho$, and we introduced

$$A = J_\psi^2, \quad B = \left(J_\psi - kPR\right)^2, \quad C = E^2k^2R^2. \tag{5.35}$$

Studying $\mathcal{P}(x) = 0$, we find that the requirement for the discriminant to be non-negative, once combined with the constraint $A = J_\psi^2 > 0$, translates into $A \in \left[0, \left(\sqrt{B} - \sqrt{C}\right)^2\right] \cup \left[\left(\sqrt{B} + \sqrt{C}\right)^2, +\infty\right)$. Further requiring that the two solution given by

$$x_1 = \frac{C - A - B - \sqrt{\Delta}}{2A} \quad \text{and} \quad x_2 = \frac{C - A - B + \sqrt{\Delta}}{2A} \tag{5.36}$$

are both non-negative, together with $\sqrt{C} > \sqrt{B}$, imply that $A \in \left[0, \left(\sqrt{B} - \sqrt{C}\right)^2\right]$. Finally, x_* is again computed by selecting A so that $\mathcal{P}(x)$ is a perfect square; this occurs for $A = \left(\sqrt{B} - \sqrt{C}\right)^2$, and x_* is given by (it is positive iff $\sqrt{C} > \sqrt{B}$)

$$x_* = \frac{\sqrt{B}}{\sqrt{C} - \sqrt{B}} = \frac{|J_\psi - kPR|}{EkR - |J_\psi - kPR|} \tag{5.37}$$

Once again, x_*, as well as x_1 and x_2, cannot be reached from infinity.

Considering now the $k = 1$ case, and defining $d\phi' = \frac{dt}{R} + d\phi$ and $d\psi' = \frac{dy}{R} + d\psi$, the metric (5.29) turns into ($\phi' \equiv \phi$, $\psi' \equiv \psi$)

$$ds^2 = N_5\left(-\frac{\cosh^2\rho}{R^2}dt^2 + \frac{\sinh^2\rho}{R^2}dy^2 + \sin^2\theta d\phi + \cos^2\theta d\psi^2 + d\theta^2 + d\rho^2\right). \tag{5.38}$$

Starting from this, and following identical steps as before, we arrive at

$$\left(\frac{d\rho}{d\psi}\right)^2 = \frac{-Ax^2 + (-A - B + C)x - B}{x(1+x)J_\psi^2} = \frac{\mathcal{P}(x)}{x(1+x)J_\psi^2}. \tag{5.39}$$

The study of $\mathcal{P}(x)$ is now the equal to the one performed for $k \neq 1$ with A, B and C from (5.35) replaced by

$$A = J_\psi^2, \quad B = R^2P^2, \quad C = R^2E^2. \tag{5.40}$$

In particular, x_* is now given by

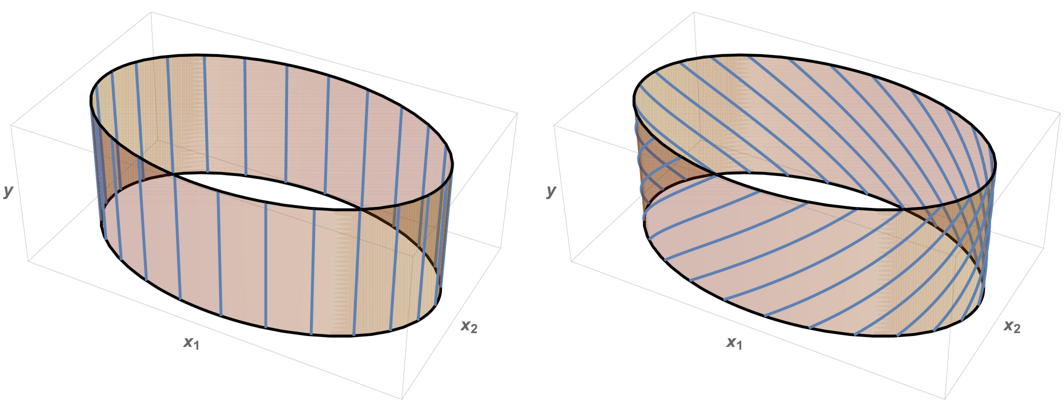
$$x_* = \frac{\sqrt{B}}{\sqrt{C} - \sqrt{B}} = \frac{P}{E - P} \tag{5.41}$$

for $A = \left(\sqrt{B} - \sqrt{C}\right)^2$.

5.2.2 Elliptical supertube

Up to this point, the NS5 branes have always been positioned on a circle in the (x^1, x^2)-plane; we now consider the situation where the branes sit on an ellipse in the same plane, with a_1 being the semi-major axis and a_2 the semi-minor one. The coordinates used to parametrize this solution in the $\mathbb{R}^4$ orthogonal to the y-direction are

$$x_1 = \sqrt{r^2 + a_1^2}\sin\theta\,\cos\phi, \quad x_2 = \sqrt{r^2 + a_2^2}\sin\theta\,\sin\phi, \quad x_3 = r\cos\theta\,\cos\psi, \quad x_4 = r\cos\theta\,\sin\psi, \tag{5.42}$$



(a) *Elliptical* NS5 *branes* (b) *Elliptical supertube*

Figure 5.2: Representations for the branes positioned along an ellipse. On the left-hand-side we have the configuration with $k = 0$ and $N_5 = 25$, while on the right-hand-side we have $k = 4$ and $N_5 = 25$.

and, while we are at it, we also define the two quantities

$$a^2 = \frac{1}{2}(a_1^2 + a_2^2), \qquad \epsilon = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}, \tag{5.43}$$

along with

$$\Delta_i = r^2 + a_i^2, \qquad \Delta = \Delta_1 \Delta_2 \cos^2\theta + r^2 \sin^2\theta \,(\Delta_1 \sin^2\phi + \Delta_2 \cos^2\phi) \tag{5.44}$$

where $i = 1, \ldots, 4$ and $a_3 = a_4 = 0$.

The metric generated by this branes configuration is equal to [29]

$$ds^2 = \mathsf{K}^{-1}\big[-(dt + \mathsf{A})^2 + (dy + \mathsf{B})^2\big] + \mathsf{H}\, d\boldsymbol{x} \cdot d\boldsymbol{x} + ds^2_{T^4}, \tag{5.45}$$

where

$$\begin{aligned}
\mathsf{H} &= \frac{N_5 \sqrt{\Delta_1 \Delta_2}}{\Delta} \\
\mathsf{K} &= 1 + \frac{k^2 \mathsf{H}}{N_5^2}\left[a^2 + \frac{1}{2}\big[(\Delta_1 + \Delta_2)\cos^2\theta + (2r^2 - (\Delta_1 - \Delta_2)\cos 2\phi)\sin^2\theta\big] - N_5 \mathsf{H}^{-1}\right] \\
\mathsf{A} &= -\frac{a_1 a_2 k \sin\theta}{2\Delta}\Big[\big((\Delta_1 + \Delta_2) - (\Delta_1 - \Delta_2)\cos 2\phi\big)\sin\phi \, d\phi - (\Delta_1 - \Delta_2)\cos\theta \, \sin 2\phi \, d\theta\Big] \\
\mathsf{B} &= a_1 a_2 k \frac{\sqrt{\Delta_1 \Delta_2}}{\Delta}\cos^2\theta \, d\psi
\end{aligned} \tag{5.46}$$

with k being the same parameter we already encountered in Section 4.3.1, and we are still in the decoupling limit for the NS5 branes, which means linear dilaton asymptotics.

Actually, the metric for the elliptical NS5-F1 supertube in the decoupling limit can still be obtained by a similar procedure to the one used in Chapter 4, namely the gauging of a WZW model, however, there are some differences with respect to said procedure. In particular, in this case, the target space G of which we then form the coset via the two (still null) gauge transformations, has not a precise group structure, but rather is a *Lunin-Mathur* geometry, and in particular is the Lunin-Mathur geometry for the AdS decoupling limit of the elliptical NS5-F1 supertube, meaning the metric of the form

$$ds^2_{\mathrm{LM}} = -\mathsf{K}'^{-1}(d\hat{v} + \beta)(d\hat{u} + \omega) + \mathsf{H}\, d\boldsymbol{x} \cdot d\boldsymbol{x} + ds^2_{\mathcal{M}}, \tag{5.47}$$

where $\mathcal{M}$ will be T^4 in our case, $\mathsf{K}' = \mathsf{K} - 1$ and $\hat{v} = \tau + \sigma$, while $\hat{u} = \tau - \sigma$. To this, the $\mathbb{R} \times S^1$ piece $-dt^2 + dy^2$ is then added, and then two null isometries are gauged away; we will not perform these steps here; see [29] for that. See Figure 5.2b for a representation of this brane arrangement.

We now wish to perform the same analysis done in the previous section, hence we restrict to the $\theta = 0$ section, compute the conserved quantities equal to

$$E = \frac{\dot{t}}{\mathsf{K}}, \quad P = \frac{\Delta \dot{y} + a_1 a_2 k \sqrt{\Delta_1 \Delta_2}\dot{\psi}}{\mathsf{K}\Delta}, \quad J_\psi = \frac{a_1 a_2 k \Delta \sqrt{\Delta_1 \Delta_2}\dot{y} + \big(\mathsf{H}\mathsf{K} r^2 \Delta^2 + a_1^2 a_2^2 k^2 \Delta_1 \Delta_2\big)\dot{\psi}}{\Delta^2 \mathsf{K}} \tag{5.48}$$

from which we extract the velocities

$$\dot{t} = E\mathsf{K}, \quad \dot{y} = \mathsf{K}P + \frac{a_1a_2k(a_1a_2kP\Delta_1\Delta_2 - J_\psi\Delta\sqrt{\Delta_1\Delta_2})}{\mathsf{H}r^2\Delta}, \quad \dot{\psi} = \frac{J_\psi\Delta - a_1a_2kP\sqrt{\Delta_1\Delta_2}}{\mathsf{H}r^2\Delta}. \tag{5.49}$$

Substituting the velocities from (5.49) in the condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$ we extract an equation for $\dot{r}$ equal to

$$\begin{aligned}-\big(N_5^2r^2\big)\dot{r}^2 =& -2a_1a_2J_\psi P\sqrt{(a_1^2+r^2)(a_2^2+r^2)} + a_1^2\big[a_2^2\big(J_\psi^2 - k^2\big(E^2-P^2\big)\big)r^2\big]\\ &+ r^2\Big[a_2^2\big(J_\psi^2 - k^2\big(E^2-P^2\big)\big) + J_\psi^2r^2 - \big(E^2-P^2\big)N_5\sqrt{(a_1^2+r^2)(a_2^2+r^2)}\Big]\\ &+ r^2\big(E^2-P^2\big)k^2\Big(\sqrt{(a_1^2+r^2)(a_2^2+r^2)} - r^2\Big).\end{aligned} \tag{5.50}$$

The square roots in this equation render it difficult to study it, so, to simplify the treatment, we will make use of the ϵ parameter from (5.43) and redefine $a_2 = a_1\sqrt{(1-\epsilon)/(1+\epsilon)}$; we can then expand up to linear order in ϵ equation (5.50) to get

$$\dot{r}^2 = \frac{A(1+\epsilon)r^4 + a_1^2(A-B+C)r^2 + a_1^4B(\epsilon-1)}{r^2N_5^2} = \frac{\mathcal{P}}{r^2N_5} \tag{5.51}$$

where A, B and C are[3]

$$A = -J_\psi^2 + N_5\big(E^2-P^2\big), \quad B = (J_\psi - kP)^2, \quad C = E^2k^2. \tag{5.52}$$

The analysis now goes through in the same way as the one performed for the NS5-F1 circular supertube, meaning that, renaming $x = r^2$, we compute the discriminant of $\mathcal{P}(x)$, $\Delta_\mathcal{P}$, to be

$$\Delta_\mathcal{P} = a_1^4\big(A^2 + (B-C)^2 + 2A(B+C)\big) + O(\epsilon^2). \tag{5.53}$$

Imposing it to be zero in order to look for radii solving both $\mathcal{P}(x) = 0$ and $\mathcal{P}'(x) = 0$, gives back the two solutions for A

$$A = (\sqrt{B}+\sqrt{C})^2 \qquad \text{and} \qquad A = -(\sqrt{B}-\sqrt{C})^2; \tag{5.54}$$

for the first value above, the critical radius is negative, hence we discard this solution, but, for the second one we actually find

$$\mathcal{P}(x) = -\Big(\big(\sqrt{B}-\sqrt{C}\big)\sqrt{1+\epsilon}\,x + a_1^2\sqrt{B}\sqrt{1-\epsilon}\Big)^2 \tag{5.55}$$

and the upshot is that the critical radius r_* is such that

$$r_*^2 = \frac{a_1^2\sqrt{B}}{\sqrt{C}-\sqrt{B}}\sqrt{\frac{1-\epsilon}{1+\epsilon}}. \tag{5.56}$$

From this, we notice that once we modify the shape of the supertube (from a circular section to an elliptical one) the critical radius changes. Furthermore, as the ellipse is changed by acting on ϵ, again the critical radius gets modified. This holds true for the impact parameter as well since the condition from which it is possible to obtain the impact parameter, namely the $\Delta_\mathcal{P} = 0$ condition, now contains terms of order ϵ^2. Now, the typical branes arrangement in the ensemble will differ strongly from a circular supertube or even an elliptical one, so it would be interesting to study how the critical radius of more typical microstates is affected by the structure of the given microstate. Nevertheless, the results obtained thus far indeed give us hope that the same behavior will manifest in the case of typical microstates-determining parameters.

[3] These are quite similar to the ones in (5.26) except some missing factors of R, the y-coordinate radius. This is because in this system the coordinates are rescaled to be dimensionless i.e. $y_{\text{ellipse}} = y_{\text{circle}}/R$.

5.3 Geodesics for NS5 branes solutions

5.3.1 Elliptical array

If we set $k = 0$ in the metric (5.45) we get a system of NS5 branes placed, rather than along a circle, like we saw in Section 4.2.1, along an ellipse with metric given by

$$\begin{aligned} ds^2 =& - dt^2 + dy^2 + N_5\left(\frac{dr^2}{\sqrt{\Delta_1\Delta_2}} + \frac{\sqrt{\Delta_1\Delta_2}\sin^2\theta\,(\Delta_1\sin^2\phi + \Delta_2\cos^2\phi)}{\Delta}\,d\phi^2\right) \\ &+ N_5\sqrt{\Delta_1\Delta_2}\left(\frac{r^2\cos^2\theta}{\Delta}\,d\psi^2 + \frac{r^2\sin^2\theta + \cos^2\theta\,(\Delta_1\cos^2\phi + \Delta_2\sin^2\phi)}{\Delta}\,d\theta^2\right) \\ &+ 2N_5\sqrt{\Delta_1\Delta_2}\,\frac{(\Delta_2 - \Delta_1)\cos\theta\,\cos\phi\,\sin\theta\,\sin\phi}{\Delta}\,d\theta\,d\phi. \end{aligned} \tag{5.57}$$

In this case we get (again expanding up to linear order in ϵ and having selected $\theta = 0$)

$$\left(\frac{dr}{d\psi}\right)^2 = \frac{r^2\Big[\big(-J_\psi^2 + N_5\big(E^2 - P^2\big)\big)r^4 + a_1^2 r^2\big(-2J_\psi^2 + N_5\big(E^2 - P^2\big)(1+\epsilon)\big) - a_1^4 J_\psi^2\Big]}{J_\psi^2(a_1^2 + r^2)^2}. \tag{5.58}$$

Addressing the terms inside the square bracket as $\mathcal{P}$, we will again look for critical radii r_* by looking at the discriminant of $\mathcal{P}$; before doing that, allow us to rewrite $\mathcal{P}$ as ($x \equiv r^2$)

$$\mathcal{P} = Ar^4 + (A(1+\epsilon) + B(\epsilon - 1))a_1^2 r^2 - a_1^4 B \tag{5.59}$$

with

$$A = -J_\psi^2 + N_5\big(E^2 - P^2\big), \quad \text{and} \quad B = J_\psi^2. \tag{5.60}$$

The discriminant is then equal to

$$\Delta_{\mathcal{P}} = a_1^4(A+B)(A + B + 2\epsilon(A - B)) + O(\epsilon^2), \tag{5.61}$$

and requiring it to be null, together with the help of the coefficients from (5.60), we find the condition (we omit the $O(\epsilon^2)$ for brevity)

$$a_1^4 N_5\big(E^2 - P^2\big)\Big(N_5\big(E^2 - P^2\big) + 2\epsilon\big[-2J_\psi^2 + N_5\big(E^2 - P^2\big)\big]\Big) = 0 \tag{5.62}$$

from where we can obtain the solution

$$J_\psi^2 = \frac{N_5\big(E^2 - P^2\big)(1 + 2\epsilon)}{4\epsilon}. \tag{5.63}$$

Inserting this in (5.59) and solving for x_* we get

$$-\frac{N_5\big(E^2 - P^2\big)(1 - 2\epsilon)}{4\epsilon}x^2 + \left(N_5\big(E^2 - P^2\big)\epsilon - \frac{N_5\big(E^2 - P^2\big)}{2\epsilon}\right)a_1^2 x - a_1^4\frac{N_5\big(E^2 - P^2\big)(1 + 2\epsilon)}{4\epsilon} \tag{5.64}$$

and this is equal to

$$-\left(\sqrt{\frac{N_5\big(E^2 - P^2\big)(1 - 2\epsilon)}{4\epsilon}}x + a_1^2\sqrt{\frac{N_5\big(E^2 - P^2\big)(1 + 2\epsilon)}{4\epsilon}}\right)^2. \tag{5.65}$$

From this we deduce that we should have

$$r_*^2 = -a_1^2\sqrt{\frac{1 + 2\epsilon}{1 - 2\epsilon}} \tag{5.66}$$

which is negative and so has to be discarded. So, in the end, no critical solution is available in the case of the elliptical array of NS5 branes.

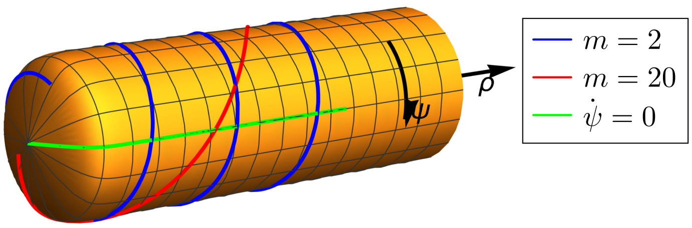


Figure 5.3: Cigar with three examples of incoming geodesics. As the dimensionless parameter m gets larger, the geodesic gets nearer to the $\rho = 0$ point in the cigar, see Figure 5.4 for a better representation of this. The green line is just a geodesic for which $\dot{\psi} = 0$; notice that it can reach the point $\rho = 0$.

5.3.2 Circular array

We now return to the simpler metric of the NS5 branes with the hope of understanding from where in G do the null paths in the coset G/H descend from. To simplify things even more, we will focus on the $\theta = 0$ section; the metric in this section can either come from the coset used in Section 4.2.2 where we set $\theta = 0$, or, alternatively, from the $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$ coset where we gauge away the τ direction; we will follow this last approach. The metric for the NS5 branes is the same from Section 4.2.1, i.e.

$$ds^2 = -dt^2 + dy^2 + ds^2_{T^4} + N_5\left[d\rho^2 + d\theta^2 + \frac{1}{\Sigma_0}\left(\cosh^2\rho\sin^2\theta d\phi^2 + \sinh^2\rho\cos^2\theta d\psi^2\right)\right] \tag{5.67}$$

with $\Sigma_0 = \sinh^2\rho + \cos^2\theta$ as for the NS5-F1 solution. We now focus again on the $\theta = 0$ slice.

Cigar metric

The metric for the $\theta = 0$ slice, known as *cigar metric* because of its shape in the (ρ, ψ) variables, see Figure 5.3, is equal to (we neglect the $ds^2_{T^4}$)

$$ds^2 = -dt^2 + dy^2 + N_5 d\rho^2 + N_5\tanh^2\rho d\psi^2. \tag{5.68}$$

The conserved charges and velocities related to this metric are given in Appendix B.2. Employing them we find (following the now usual steps)

$$\left(\frac{d\rho}{d\psi}\right)^2 = \frac{\tanh^2\rho\left[\sinh^2\rho\left(N_5\left(E^2 - P^2\right) - J_\psi^2\right) - J_\psi^2\right]}{J_\psi^2\cosh^2\rho}. \tag{5.69}$$

For this metric there exist no closed null path: the massless probe reaches radius ρ_* such that

$$\sinh^2\rho_* = \frac{J_\psi^2}{N_5\left(E^2 - P^2\right) - J_\psi^2} \tag{5.70}$$

and then inverts its motion. Moreover, given that if we want ρ_* to be positive we must require $J_\psi^2 < N_5\left(E^2 - P^2\right)$, the point of inversion can be reached from infinity since

$$\lim_{\rho\to\infty}\left(\frac{d\rho}{d\psi}\right)^2 = \frac{N_5\left[\left(E^2 - P^2\right) - J_\psi^2\right]}{J_\psi^2}. \tag{5.71}$$

Setting $P = 0$ and $N_5 E^2 = m J_\psi^2$, with m some number that must be greater than 1, given the condition for the positivity of ρ_*, (5.69) transforms into

$$\left(\frac{d\rho}{d\psi}\right)^2 = \frac{\tanh^2\rho\left[(m-1)\sinh^2\rho - 1\right]}{\cosh^2\rho} \quad\to\quad \frac{d\rho}{d\psi} = \pm\sqrt{\frac{\tanh^2\rho\left[(m-1)\sinh^2\rho - 1\right]}{\cosh^2\rho}}. \tag{5.72}$$

Note that with $P = 0$ and the introduction of m, the turning point ρ_* is given by the equation

$$\sinh^2\rho_* = \frac{1}{m-1}, \tag{5.73}$$

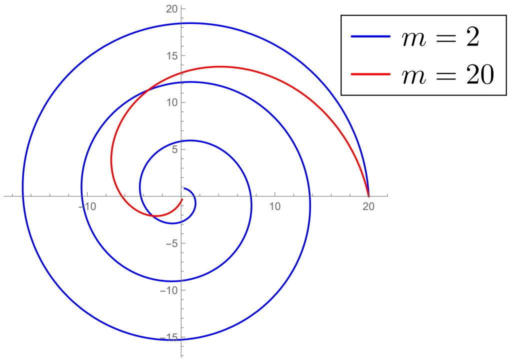


Figure 5.4: Two-dimensional representation of the incoming geodesics with $m = 2$ and $m = 20$. From this plot we clearly see that the red line gets much closer to the point $\rho = 0$ than the blue line.

hence the bigger m is the closer the geodesic gets to $\rho = 0$.

The equation on the right side of the arrow in (5.72) is still too complicated to be solved analytically; however, a numerical solution can still be found. Specifically, selecting the minus sign, which amounts to focus on the incoming geodesic, two solutions, respectively for $m = 2$ and $m = 20$, are presented in Figure 5.3 and Figure 5.4.

As can be clearly seen from Figure 5.3, once $\dot{\psi}$ is different from zero (or, equivalently, once $J_\psi \neq 0$) the geodesics swirl around the cigar. This is not a surprise given that the cigar metric (5.68) approaches the cylinder one in (ρ, ψ) coordinates for $\rho \gg 1$ and, indeed, the geodesics on a cylinder are helices. As the particle gets closer to its inversion point near the tip of the cigar, where the metric deviates from a cylinder, the solution is not anymore a perfect helix as it can be checked by expanding the square root in (5.72) for $\rho \sim \rho_*$.

Concerning the solution for $\dot{\psi} = 0$, the geodesic equation for ρ is simply given by $\ddot{\rho} = 0$, see (B.11); hence why in this case we obtain a straight line.

Conclusions

In this thesis we have reviewed some classical black hole solutions in GR, starting from the Schwarzschild solution and ending with the Kerr metric, without forgetting the Reissner-Nordström metric. We presented the basic concepts of black hole mechanics: Hawking temperature, Bekenstein-Hawking entropy and the four laws of black holes mechanics, and then finally reviewed the argument for the information paradox.

Successively, we presented some fundamental concepts from String Theory, introducing and quantizing, through lightcone quantization, Bosonic String Theory. We discussed CFTs, defining what a conformal transformation is, and later focussing on the particular case of two-dimensional CFTs, since the worldsheet is a two-dimensional manifold. We have reviewed how the action for a String Theory in a non-flat metric is given, and showed how other background fields outside the metric, such as the B field and the dilaton, can be added to the String Theory action. Finally, we ended the second chapter with the introduction of the (massless) spectrum of Type IIB and IIA string theories and with a section summarizing T and S dualities.

The thesis later dealt with the presentation of some known examples of black hole metrics produced by different arrangements of D and M branes, starting off with a four-charge black hole and then transitioning to a three-charge black hole. In this latter case, the Bekenstein-Hawking entropy has been computed, and later compared with the entropy coming from a microscopic counting of the degrees of freedom in String Theory, showing that the two results agree. Finally, we reviewed the fuzzball proposal, meaning a proposal suggesting that the microstates of a black hole might be structures extending up to the horizon. In this regard, we concentrated on two-charge fuzzballs, first presenting the general idea in the F1-P picture, and later computing the metric in the case of the circular D1-D5 fuzzball.

In its opening section, Chapter 4 has then been devoted to the presentation of WZW models, while following sections focussed on showing how the supergravity metric in the decoupling limit for some fuzzball solutions can be obtained starting from a WZW model with target space G, and realizing the coset G/H by means of two $\mathrm{U}(1)$ gauge transformations. We initially showed how this method produces the desired result in the simpler case of a circular array of NS5 branes, and then performed a similar analysis in case of the NS5-P supertube, starting from the general null-gauge transformation, later specialized to the case under study. We also quoted that the same result holds for a NS5-F1 supertube, reached by a T duality of the NS5-P solution.

In the last chapter we studied the motion of massless particles in some background metrics previously presented throughout the thesis, such as the D1-D5 fuzzball from Chapter 3, the circular NS5-F1 supertube and the circular array of NS5 branes, both from Chapter 4, along with the motion of massless particles on some new background metrics, such as the elliptical NS5-F1 supertube and the elliptical NS5 branes array. The main focus of this study was the presence of null paths orbiting around the various fuzzballs and, if such trajectories existed, how did they depend on the microstates-characterizing parameters present in the given fuzzball solution. Particularly interesting is the case of the elliptical NS5-F1 supertube, as this is an original study that indeed shows how the critical radius changes as we move in (a portion of) the phase space of fuzzballs.

Finally, we took the first steps into the study of how to relate the geodesics in G/H to the ones in G, studying the simple case of (some) null geodesics in the cigar metric for the NS5 solution, and then setting up the study on the ones in AdS_3.

Appendices

Appendix A

Computations for NS5 branes solutions

A.1 WZW model equations of motion

We derive here the equations of motion for the WZW model. Starting with S_0 from (4.1), we find

$$\begin{aligned}\delta S_0 =& \frac{1}{2\lambda^2}\int d^2x\ \mathrm{Tr}\left[(-g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\partial_\mu \delta g)g^{-1}\partial^\mu g\right]\\ =&\frac{1}{2\lambda^2}\int d^2x\ \mathrm{Tr}\left[\partial_\mu(g^{-1}\delta g)g^{-1}\partial^\mu g\right]\\ =&-\frac{1}{2\lambda^2}\int d^2x\ \mathrm{Tr}\left[g^{-1}\delta g\partial_\mu(g^{-1}\partial^\mu g)\right],\end{aligned} \tag{A.1}$$

where, to better see what happened in the second step, it suffices to move the term outside the parenthesis in the very first line, namely $g^{-1}\partial^\mu g$, in front of the first term inside the parenthesis, namely $-g^{-1}\delta g g^{-1}\partial_\mu g$, by means of a cyclic permutation; having done this, it will hopefully be clearer how the second step comes about.

For the WZ term instead we find

$$\begin{aligned}\delta\Gamma =&\frac{i}{4\pi}\int d^3y\,\epsilon_{\alpha\beta\gamma}\ \mathrm{Tr}\left[(-g^{-1}\delta g g^{-1}\partial^\alpha g + g^{-1}\partial^\alpha\delta g)g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\right]\\ =&\frac{i}{4\pi}\int d^3y\,\epsilon_{\alpha\beta\gamma}\ \mathrm{Tr}\left[g^{-1}\partial^\alpha(\delta g g^{-1})\partial^\beta g g^{-1}\partial^\gamma g\right]\\ =&\frac{i}{4\pi}\int d^3y\,\epsilon_{\alpha\beta\gamma}\ \mathrm{Tr}\left[\partial^\alpha(g^{-1}\delta g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g)\right]\\ =&\frac{i}{4\pi}\int d^2x\,\epsilon_{\alpha\beta}\ \mathrm{Tr}\left[g^{-1}\delta g g^{-1}\partial^\alpha g g^{-1}\partial^\beta g\right]\\ =&\frac{-i}{4\pi}\int d^2x\,\epsilon_{\alpha\beta}\ \mathrm{Tr}\left[g^{-1}\delta g\partial^\alpha g^{-1}\partial^\beta g\right] = \frac{-i}{4\pi}\int d^2x\,\epsilon_{\alpha\beta}\ \mathrm{Tr}\left[g^{-1}\delta g\partial^\alpha(g^{-1}\partial^\beta g)\right].\end{aligned} \tag{A.2}$$

In the first step we employed both the trace's and the epsilon-indices' cyclicity. To understand the equivalence between second and third line, it is useful to start from the equivalence

$$\begin{aligned}g^{-1}\partial^\alpha(\delta g g^{-1})\partial^\beta g g^{-1}\partial^\gamma g =&\partial^\alpha(g^{-1}\delta g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g) - \partial^\alpha(g^{-1})\delta g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\\ &- g^{-1}\delta g g^{-1}\partial^\beta g\partial^\alpha(g^{-1})\partial^\gamma g\end{aligned} \tag{A.3}$$

and check that the two last terms cancel each other thanks to ϵ antisymmetry. Finally, in the last step we again used the antisymmetry of the ϵ to collect a derivative.

From the two above-results we deduce that the variation of $S = S_0 + k\Gamma$ is

$$-\frac{1}{2\lambda^2}\partial_\mu(g^{-1}\partial^\mu g) - \frac{ik}{4\pi}\epsilon_{\alpha\beta}\partial^\alpha(g^{-1}\partial^\beta g) \tag{A.4}$$

that, once complex coordinates are employed, transforms into ($g^{z\bar z} = 2\ g^{zz} = g^{\bar z\bar z} = 0, \epsilon_{z\bar z} = \frac{i}{2}$)

$$-\frac{1}{\lambda^2}\left[\partial(g^{-1}\bar\partial g) + \bar\partial(g^{-1}\partial g)\right] + \frac{k}{2\pi}\left[\bar\partial(g^{-1}\partial g) - \partial(g^{-1}\bar\partial g)\right]. \tag{A.5}$$

Rearranging the terms we find

$$\left(-\frac{1}{\lambda^2}-\frac{k}{2\pi}\right)\partial(g^{-1}\bar{\partial}g)+\left(-\frac{1}{\lambda^2}+\frac{k}{2\pi}\right)\bar{\partial}(g^{-1}\partial g), \tag{A.6}$$

from where, imposing the condition that the above variation must be zero, we see that for $\lambda^2=\frac{2\pi}{k}$ the current $\bar{J}=g^{-1}\bar{\partial}g$ is indeed antiholomorphic.

A.2 NS5 branes harmonic function

We start by rewriting the explicit formula for $H_5(\boldsymbol{x})$ given in (4.14); that is, from

$$H_5(\boldsymbol{x})=1+\sum_{m=1}^{N_5}\frac{1}{|x_1+ix_2-ae^{i\phi_m}|^2+|x_3+ix_4|^2}. \tag{A.7}$$

Introducing the coordinates from (4.13) we get (we neglect the 1)

$$\sum_{m=1}^{N_5}\frac{1}{|a\cosh\rho\sin\theta e^{i\phi}-ae^{i\phi_m}|^2+a^2\sinh^2\rho\cos^2\theta}. \tag{A.8}$$

Defining the new variables $\tilde{\phi}\equiv\phi_m-\phi$, $s\equiv\sin\theta$ and $c\equiv\cos\theta$, and opening up the squared-module, results in

$$\frac{1}{a^2}\sum_{m=1}^{N_5}\frac{1}{\cosh^2\rho+s^2-2s\cosh\rho\cos\tilde{\phi}}. \tag{A.9}$$

Collecting the $2s\cosh\rho$ term, and defining χ so that $e^\chi=\frac{\cosh\rho}{s}$, we find

$$\frac{\sinh\chi}{a^2(\cosh^2\rho-s^2)}\sum_{m=1}^{N_5}\frac{1}{\cosh\chi-\cos\tilde{\phi}}, \tag{A.10}$$

where to construct the $\sinh\chi$ we multiplied and divided by $(\cosh^2\rho-s^2)$. Employing the use of the exponential expression of both $\cosh\rho$ and $\cos\tilde{\phi}$, and collecting in the appropriate way the terms, we get

$$\frac{1}{\cosh\chi-\cos\tilde{\phi}}=\frac{2}{e^\chi\left(1-e^{-\chi-i\tilde{\phi}}\right)\left(1-e^{-\chi+i\tilde{\phi}}\right)}=\frac{2}{e^\chi}\sum_{a=0}^{\infty}\left(e^{-\chi-i\tilde{\phi}}\right)^a\sum_{b=0}^{\infty}\left(e^{-\chi+i\tilde{\phi}}\right)^b. \tag{A.11}$$

Plugging this inside (A.10), and renaming $\frac{2\sinh\chi}{e^\chi a^2(\cosh^2\rho-s^2)}\equiv\kappa$ for shortness, we get

$$\kappa\sum_{m=1}^{N_5}\sum_{a,b=0}^{\infty}e^{-(a+b)\chi}e^{-i(a-b)\tilde{\phi}}. \tag{A.12}$$

Rewriting the sums on a and b as a sum on $a=b$, one on $a>b$ and one on $a<b$, we get

$$\kappa\sum_{m=1}^{N_5}\left[\sum_{a=0}^{\infty}e^{-2a\chi}+\left(\sum_{a>b}+\sum_{a<b}\right)\sum_{b=0}^{\infty}e^{-(a+b)\chi-i(a-b)\tilde{\phi}}\right]. \tag{A.13}$$

We collect the two sums on $a<b$ and $a>b$ into a formal sum $\sum_\pm$ where the two signs refer to the signs of the $(a-b)$ term obtaining (we also compute the sums on a and m for the first term inside the squared bracket)

$$\kappa\frac{N_5e^\chi}{2\sinh\chi}+\kappa\sum_{\pm}\sum_{m=1}^{N_5}\sum_{a>b}\sum_{b=0}^{\infty}e^{-(a+b)\chi\pm i(a-b)\tilde{\phi}}. \tag{A.14}$$

We focus for a moment on the sums on a and b to see how we can rewrite them better. Expanding them, we find

$$\sum_{a>b}\sum_{b=0}^{\infty} e^{-(a+b)\chi\pm i(a-b)\tilde{\phi}} = \sum_{a=1}^{\infty} e^{-a\chi\pm ia\tilde{\phi}} + \sum_{a=2}^{\infty} e^{-(a+1)\chi\pm i(a-1)\tilde{\phi}} + \sum_{a=3}^{\infty} e^{-(a+2)\chi\pm i(a-2)\tilde{\phi}} + \dots \tag{A.15}$$

Changing sum variable inside the sums, we obtain

$$\sum_{a>b}\sum_{b=0}^{\infty} e^{-(a+b)\chi\pm i(a-b)\tilde{\phi}} = \sum_{a=1}^{\infty} e^{-a\chi\pm ia\tilde{\phi}} + \sum_{a=1}^{\infty} e^{-(a+2)\chi\pm ia\tilde{\phi}} + \sum_{a=1}^{\infty} e^{-(a+4)\chi\pm ia\tilde{\phi}} + \dots \tag{A.16}$$

Hence, we can now rewrite everything as

$$\kappa\left[\frac{N_5 e^{\chi}}{2\sinh\chi} + \left(\sum_{\pm}\sum_{m=1}^{N_5}\sum_{a=1}^{\infty} e^{-a\chi\pm ia\tilde{\phi}}\right)\sum_{b=0}^{\infty} e^{-2b\chi}\right]. \tag{A.17}$$

Now, substituting $\tilde{\phi}$ by its explicit expression $\tilde{\phi} = \phi_m + \phi = \frac{2\pi m}{N_5} + \phi$, and using the fact that

$$\sum_{m=1}^{N_5} e^{\pm ia\frac{2\pi}{N_5}m} \neq 0 \tag{A.18}$$

only for $a \in N_5 \times \mathbb{N}$, we can rewrite the sum on a from 0 to ∞ as a sum on the multiples of N_5, i.e. as a sum on $a \in \{N_5,\, 2N_5,\, \dots\}$. This yields

$$\kappa\frac{e^{\chi}}{2\sinh\chi}\left[N_5 + \sum_{\pm}\sum_{a=1}^{\infty} e^{-aN_5\pm iaN_5\phi}\sum_{m=1}^{N_5} e^{2\pi aim}\right]. \tag{A.19}$$

Performing the three sums, we get

$$\kappa\frac{N_5 e^{\chi}}{2\sinh\chi}\left(-1 + \frac{1}{1-e^{-N_5\chi+iN_5\phi}} + \frac{1}{1-e^{-N_5\chi+-iN_5\phi}}\right). \tag{A.20}$$

Summing the terms in the parenthesis and substituting the value of κ, we find the result (4.14) (modulo the 1 we neglected at the beginning)

$$\kappa\frac{N_5 e^{\chi}}{2\sinh\chi}\frac{\sinh(N_5\chi)}{\cosh(N_5\chi)-\cos(N_5\phi)} = \frac{N_5}{a^2(\cosh^2\rho-\sin^2\theta)}\frac{\sinh(N_5\chi)}{\cosh(N_5\chi)-\cos(N_5\phi)} \tag{A.21}$$

If we wanted to smear the NS5 branes along the circle on which they sit, the integral we should compute is

$$\begin{aligned}\frac{N_5}{2\pi a^2}\int_0^{2\pi}\frac{d\phi}{\cosh^2\rho+s^2-2s\cosh\rho\cos\phi} &= \frac{N_5}{4\pi a^2 s\cosh\rho}\int_0^{2\pi}\frac{d\phi}{\cosh\chi-\cos\phi}\\ &= \frac{N_5\beta\sinh\chi}{2\pi a^2(\cosh^2\rho-s^2)}\int_0^{2\pi}\frac{d\phi}{1-\beta\cos\phi};\end{aligned} \tag{A.22}$$

namely the integral of the argument of the sum in (A.9), where $\beta = \cosh^{-1}\chi$ and we multiplied everything by a normalization factor $\frac{N_5}{2\pi}$. This is achieved by using (3.41)

$$\frac{N_5\beta\sinh\chi}{2\pi a^2(\cosh^2\rho-s^2)}\frac{2\pi}{\sqrt{1-\cosh^{-2}\chi}} = \frac{N_5}{a^2(\cosh^2\rho-s^2)} \tag{A.23}$$

A.3 WZW action

Starting from the action for the WZW models given by

$$S = \frac{k}{4\pi}\int_{S^2} d^2x \ \mathrm{Tr}\left[g^{-1}\partial_\mu g g^{-1}\partial^\mu g\right] + \frac{ik}{6\pi}\int_B d^3y\, \epsilon_{\alpha\beta\gamma}\,\mathrm{Tr}\left[g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\right], \tag{A.24}$$

we move to complex coordinates in the first integral ($d^2x = \frac{1}{2}d^2z$, $\partial^z = 2\partial_{\bar{z}}$ and $\partial^{\bar{z}} = 2\partial_z$) which results in (again, we write just the first term)

$$\frac{k}{2\pi}\int d^2z \ \mathrm{Tr}\left[g^{-1}\partial g g^{-1}\bar{\partial} g\right]. \tag{A.25}$$

We will focus on the SL(2) component since the steps for the SU(2) one are identical. The first step is to compute ($g \equiv g_{\mathrm{sl}}$ for brevity)

$$\partial g = \partial\begin{pmatrix} e^{i\tau}\cosh\rho & e^{i\sigma}\sinh\rho \\ e^{-i\sigma}\sinh\rho & e^{-i\tau}\cosh\rho\end{pmatrix} = \begin{pmatrix} e^{i\tau}(i\partial\tau\cosh\rho + \partial\rho\sinh\rho) & e^{i\sigma}(\partial\rho\cosh\rho + i\partial\sigma\sinh\rho) \\ e^{-i\sigma}(\partial\rho\cosh\rho - i\partial\sigma\sinh\rho) & e^{-i\tau}(-i\partial\tau\cosh\rho + \partial\rho\sinh\rho)\end{pmatrix} \tag{A.26}$$

and similarly for $\bar{\partial} g$. The next step consists in computing

$$g^{-1}\partial g = \begin{pmatrix} i(\partial\sigma\sinh^2\rho + \partial\tau\cosh^2\rho) & e^{i(\sigma-\tau)}[\partial\rho + i(\partial\sigma+\partial\tau)\sinh\rho\cosh\rho] \\ e^{-i(\sigma-\tau)}[\partial\rho - i(\partial\sigma+\partial\tau)\sinh\rho\cosh\rho] & -i(\partial\sigma\sinh^2\rho + \partial\tau\cosh^2\rho)\end{pmatrix}. \tag{A.27}$$

The last step amounts to computing the full $g^{-1}\partial g g^{-1}\bar{\partial} g$ matrix; since we will later compute the trace, we report here just the two diagonal elements

$$\begin{aligned}(g^{-1}\partial g g^{-1}\bar{\partial} g)_{11} =& i\sinh\rho\cosh\rho\left[\bar{\partial}\rho(\partial\sigma+\partial\tau) - \partial\rho(\bar{\partial}\sigma+\bar{\partial}\tau)\right] \\ &+ \partial\sigma\bar{\partial}\sigma\sinh^2\rho - \partial\tau\bar{\partial}\tau\cosh^2\rho + \partial\rho\bar{\partial}\rho\end{aligned} \tag{A.28}$$

$$\begin{aligned}(g^{-1}\partial g g^{-1}\bar{\partial} g)_{22} =& -i\sinh\rho\cosh\rho\left[\bar{\partial}\rho(\partial\sigma+\partial\tau) - \partial\rho(\bar{\partial}\sigma+\bar{\partial}\tau)\right] \\ &+ \partial\sigma\bar{\partial}\sigma\sinh^2\rho - \partial\tau\bar{\partial}\tau\cosh^2\rho + \partial\rho\bar{\partial}\rho.\end{aligned} \tag{A.29}$$

Hence, we have

$$\mathrm{Tr}\left[g^{-1}\partial g g^{-1}\bar{\partial} g\right] = 2\big(-\partial\tau\bar{\partial}\tau\cosh^2\rho + \partial\rho\bar{\partial}\rho + \partial\sigma\bar{\partial}\sigma\sinh^2\rho\big). \tag{A.30}$$

From this we can also retrieve the result for the g_{su} case by simply substituting $\tau\to\psi$, $\sigma\to\phi$ and $\rho\to i\theta$; operating the substitution gives (the minus in front of the trace is there by definition, so that we get the correct signature)

$$-\,\mathrm{Tr}\left[g_{\mathrm{su}}^{-1}\partial g_{\mathrm{su}} g_{\mathrm{su}}^{-1}\bar{\partial} g_{\mathrm{su}}\right] = 2\big(\partial\psi\bar{\partial}\psi\cos^2\theta + \partial\theta\bar{\partial}\theta + \partial\phi\bar{\partial}\phi\sin^2\theta\big). \tag{A.31}$$

Inserting both of the previous results in (A.25) we find

$$\frac{k}{\pi}\int d^2z\left[-\partial\tau\bar{\partial}\tau\cosh^2\rho + \partial\rho\bar{\partial}\rho + \partial\sigma\bar{\partial}\sigma\sinh^2\rho + \partial\psi\bar{\partial}\psi\cos^2\theta + \partial\theta\bar{\partial}\theta + \partial\phi\bar{\partial}\phi\sin^2\theta\right] \tag{A.32}$$

Turning now our attention to the WZ term, we will here follow a different approach with respect to the one used above; we write

$$\begin{aligned}\partial^\alpha g = \partial^\alpha\big(e^{\frac{i}{2}(\tau+\sigma)\sigma_3}e^{\rho\sigma_1}e^{\frac{i}{2}(\tau-\sigma)\sigma_3}\big) =& \left(\frac{i}{2}(\partial^\alpha\tau + \partial^\alpha\sigma)\right)\sigma_3 g + e^{\frac{i}{2}(\tau+\sigma)\sigma_3}(\partial^\alpha\rho)\sigma_1 e^{\rho\sigma_1}e^{\frac{i}{2}(\tau-\sigma)\sigma_3} \\ &+ g\left(\frac{i}{2}(\partial^\alpha\tau - \partial^\alpha\sigma)\right)\sigma_3\end{aligned} \tag{A.33}$$

and consequently

$$\begin{aligned}g^{-1}\partial^\alpha g =& g^{-1}\left(\frac{i}{2}(\partial^\alpha\tau + \partial^\alpha\sigma)\right)\sigma_3 g + e^{-\frac{i}{2}(\tau-\sigma)\sigma_3}(\partial^\alpha\rho)\sigma_1 e^{\frac{i}{2}(\tau-\sigma)\sigma_3} + \left(\frac{i}{2}(\partial^\alpha\tau - \partial^\alpha\sigma)\right)\sigma_3 \\ =& \frac{i}{2}(\partial^\alpha\tau + \partial^\alpha\sigma)M_1 + \partial^\alpha\rho M_2 + \frac{i}{2}(\partial^\alpha\tau - \partial^\alpha\sigma)M_3\end{aligned} \tag{A.34}$$

where we defined

$$M_1 = g^{-1}\sigma_3 g, \qquad M_2 = e^{-\frac{i}{2}(\tau-\sigma)\sigma_3}\sigma_1 e^{\frac{i}{2}(\tau-\sigma)\sigma_3}, \qquad M_3 = \sigma_3. \tag{A.35}$$

We now need to compute the product between three factors like the one in (A.34); in doing this, we also need to keep in mind that the product of these three factors will be contracted with the Levi-Civita tensor ϵ, so we will neglect terms that are symmetric in the derivatives. In the end, this simply implies that just terms proportional to $M_1M_2M_3$ (and permutations of this) will give a different-from-zero contribution. Proceeding, yields

$$\begin{aligned} g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g = \frac{i^2}{4}\big[&(\partial^\alpha\tau+\partial^\alpha\sigma)\partial^\beta\rho(\partial^\gamma\tau-\partial^\gamma\sigma)M_1M_2M_3 \\ &+(\partial^\alpha\tau+\partial^\alpha\sigma)(\partial^\beta\tau-\partial^\beta\sigma)\partial^\gamma\rho M_1M_3M_2+\partial^\alpha\rho(\partial^\beta\tau+\partial^\beta\sigma)(\partial^\gamma\tau-\partial^\gamma\sigma)M_2M_3M_1 \\ &+\partial^\alpha\rho(\partial^\beta\tau-\partial^\beta\sigma)(\partial^\gamma\tau+\partial^\gamma\sigma)M_2M_3M_1+(\partial^\alpha\tau-\partial^\alpha\sigma)(\partial^\beta\tau+\partial^\beta\sigma)\partial^\gamma\rho M_3M_1M_2 \\ &+(\partial^\alpha\tau-\partial^\alpha\sigma)\partial^\beta\rho(\partial^\gamma\tau+\partial^\gamma\sigma)M_3M_2M_1\big]. \end{aligned} \tag{A.36}$$

Using the fact that we will contract the above result with ϵ, we can collect the terms of the kind $\partial^\alpha\sigma\partial^\gamma\tau-\partial^\alpha\tau\partial^\gamma\sigma$ and eliminate those symmetric in the derivates (like $\partial^\alpha\tau\partial^\gamma\tau$). Doing this, we find that the non-zero contribution is

$$\begin{aligned} g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g = \frac{1}{2}\big[&(\partial^\alpha\tau\partial^\beta\rho\partial^\gamma\sigma)M_1M_2M_3+(\partial^\alpha\tau\partial^\beta\sigma\partial^\gamma\rho)M_1M_3M_2 \\ &+(\partial^\alpha\rho\partial^\beta\tau\partial^\gamma\sigma)M_2M_1M_3+(\partial^\alpha\rho\partial^\beta\sigma\partial^\gamma\tau)M_2M_3M_1+(\partial^\alpha\sigma\partial^\beta\rho\partial^\gamma\tau)M_3M_2M_1 \\ &+(\partial^\alpha\sigma\partial^\gamma\rho\partial^\beta\tau)M_3M_1M_2\big]. \end{aligned} \tag{A.37}$$

Employing the fact that the matrices the products of matrices M_1, M_2 and M_3 are inside a trace, we can collect the terms thanks to the cyclic symmetry of the trace; this yields

$$\begin{aligned} g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g =&\frac{M_1M_2M_3}{2}(\partial^\alpha\tau\partial^\beta\rho\partial^\gamma\sigma+\partial^\alpha\rho\partial^\beta\sigma\partial^\gamma\tau+\partial^\alpha\sigma\partial^\gamma\rho\partial^\beta\tau) \\ &+\frac{M_1M_3M_2}{2}(\partial^\alpha\tau\partial^\beta\sigma\partial^\gamma\rho+\partial^\alpha\rho\partial^\beta\tau\partial^\gamma\sigma+\partial^\alpha\sigma\partial^\beta\rho\partial^\gamma\tau). \end{aligned} \tag{A.38}$$

All the terms inside the first parenthesis are equal to one another up to a cyclic permutation of the indices, and the same goes for the terms inside the second parenthesis. The terms inside the first parenthesis differ by an odd number of permutations of the indices with respect to the ones in the second square parenthesis. Employing these two facts, we can write everything as

$$\frac{3}{2}\partial^\alpha\tau\partial^\beta\rho\partial^\gamma\sigma\big(M_1M_2M_3-M_2M_1M_3\big). \tag{A.39}$$

At this point, using the definitions (A.35), and taking the trace, we find that

$$\begin{aligned} \epsilon_{\alpha\beta\gamma}\,\mathrm{Tr}\left[g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\right] =&12\epsilon_{\alpha\beta\gamma}\partial^\alpha\tau\partial^\beta\rho\partial^\gamma\sigma\cosh\rho\sinh\rho \\ =&-6\epsilon_{\alpha\beta\gamma}\partial^\alpha\tau\partial^\beta\sigma\partial^\gamma\big(\cosh^2\rho\big) \end{aligned} \tag{A.40}$$

where in the third step we noticed that what appears in the second step is just a total derivative (the minus sign emerges from the renaming and exchanging of the indices β and γ).

We can now retrieve the result for the SU(2) component as well by using the same substitutions from before, namely $\tau\to\psi$, $\sigma\to\phi$ and $\rho\to i\theta$. Inserting these in (A.40) we find

$$\begin{aligned} -\epsilon_{\alpha\beta\gamma}\,\mathrm{Tr}\left[g_{\mathrm{su}}^{-1}\partial^\alpha g_{\mathrm{su}}g_{\mathrm{su}}^{-1}\partial^\beta g_{\mathrm{su}}g_{\mathrm{su}}^{-1}\partial^\gamma g_{\mathrm{su}}\right] =&-12\epsilon_{\alpha\beta\gamma}\partial^\alpha\psi\partial^\beta\theta\partial^\gamma\phi\cos\theta\sin\theta \\ =&6\epsilon_{\alpha\beta\gamma}\partial^\alpha\psi\partial^\beta\phi\partial^\gamma\big(\cos^2\theta\big) \end{aligned} \tag{A.41}$$

Inserting both results inside the WZ term from (A.24), using Stokes theorem and changing to complex coordinates ($d^2x=\frac{1}{2}d^2z$, $\epsilon_{z\bar z}=\frac{i}{2}$, $g^{z\bar z}=g^{\bar z z}=2$) results in

$$\begin{aligned} k\Gamma[g] =&\frac{ik}{\pi}\int d^2x\,\epsilon_{\alpha\beta}\big(-\cosh^2\rho\,\partial^\alpha\tau\partial^\beta\sigma+\cos^2\theta\,\partial^\alpha\psi\partial^\beta\phi\big) \\ =&\frac{k}{\pi}\int d^2z\left[-\cosh^2\rho\,(\partial\tau\bar\partial\sigma-\partial\sigma\bar\partial\tau)-\cos^2\theta\,(\partial\phi\bar\partial\psi-\partial\psi\bar\partial\phi)\right]. \end{aligned} \tag{A.42}$$

Appendix B

Geodesics

We give here some additional details and results from Chapter 5.

B.1 D1-D5 fuzzball

We now give some details about the computation of the D coefficient from (5.16). Inserting x_* from (5.15), meaning

$$x_* = \frac{-B + \sqrt{B^2 - 3AC}}{3A}, \tag{B.1}$$

inside $\mathcal{P}(x) = 0$ with $\mathcal{P}$ from (5.8) equal to

$$Ax^3 + Bx^2 + Cx + D, \tag{B.2}$$

gets us ($\Delta \equiv B^2 - 3AC$)

$$\begin{aligned} D &= -Ax_*^3 - Bx_*^2 - Cx_* \\ &= \frac{B^3 - \Delta^{3/2} + 3B\Delta - 3B^2\sqrt{\Delta}}{27A^2} + \frac{-2B^3 + 3ABC + 2B^2\sqrt{\Delta}}{9A^2} + \frac{CB - C\sqrt{\Delta}}{3A} \\ &= \frac{-2B^3 + 9ABC - \Delta^{3/2} + 3B^2\sqrt{\Delta} - 9AC\sqrt{\Delta}}{27A^2} = \frac{2\Delta^{3/2} - B\big(2B^2 - 9AC\big)}{27A^2}. \end{aligned} \tag{B.3}$$

For $x_* = \sqrt{\frac{C}{A}}$, using the explicit expressions of the $A, \ldots, D$ parameters from (5.9), that is, using

$$\begin{aligned} A &= E^2 - P^2 \\ B &= -J_\psi^2 + \big(2a^2 + Q_1 + Q_5\big)\big(E^2 - P^2\big) \\ C &= -2aJ_\psi\sqrt{Q_1Q_5}P + Q_1Q_5\big(E^2 - P^2\big) + a^2\big(-2J_\psi^2 + \big(Q_1 + Q_5 + a^2\big)\big(E^2 - P^2\big)\big) \\ D &= -a^2\big(aJ_\psi + \sqrt{Q_1Q_5}P\big)^2 \end{aligned} \tag{B.4}$$

together with the conditions from Section 5.1 $Q_1 = Q_5 = Q$ and $P = -\frac{aJ_\psi}{Q}$, we have

$$\begin{aligned} x_* &= \sqrt{\frac{Q^2\big[2a^2J_\psi^2 + \big(Q^2E^2 - a^2J_\psi^2\big) - 2a^2J_\psi^2 + \frac{1}{Q^2}\big(2Qa^2 + a^4\big)\big(Q^2E^2 - a^2J_\psi^2\big)\big]}{Q^2E^2 - a^2J_\psi^2}} \\ &= \sqrt{\frac{\big(Q^2E^2 - a^2J_\psi^2\big)\big(Q^2 + 2a^2Q + a^4\big)}{Q^2E^2 - a^2J_\psi^2}} = Q + a^2. \end{aligned} \tag{B.5}$$

B.2 Geodesics in the cigar

The conserved charges associated with the Lagrangian coming from the cigar metric (5.68) are

$$E = \dot{t}, \qquad P = \dot{y}, \qquad J_\psi = n_5 \dot{\psi} \tanh^2 \rho \tag{B.6}$$

with velocities

$$\dot{t} = E, \qquad \dot{y} = P, \qquad \dot{\psi} = \frac{J_\psi}{n_5 \tanh^2 \rho}. \tag{B.7}$$

The equation for $\dot{\rho}$ retrieved from the null condition is

$$\dot{\rho}^2 = -\frac{J_\psi^2 \cosh^2 \rho - n_5 \left(E^2 - P^2\right) \sinh^2 \rho}{n_5^2 \sinh^2 \rho}. \tag{B.8}$$

B.2.1 Geodesic equation

For the simple metric

$$ds^2 = -dt^2 + dy^2 + n_5 d\rho^2 + n_5 \tanh^2 \rho d\psi^2 \tag{B.9}$$

the full geodesic equations can be promptly derived. The non-vanishing Christoffel coefficients are

$$\Gamma^4_{24} = \Gamma^4_{42} = \frac{1 - \tanh^2 \rho}{\tanh \rho}, \qquad \Gamma^2_{44} = \tanh \rho (\tanh^2 \rho - 1), \tag{B.10}$$

which result into the (non-trivial) geodesic equations

$$\begin{cases} \ddot{\rho} + \tanh \rho (\tanh^2 \rho - 1) \ddot{\psi}^2 = 0 \\ \ddot{\psi} + 2\left(\frac{1 - \tanh^2 \rho}{\tanh \rho}\right) \dot{\psi} \dot{\rho} = 0 \end{cases} \tag{B.11}$$

The second equation can be promptly integrated, leading to a relation for $\dot{\psi}$ that is equal (up to constants) to the third relation in (B.7), i.e.

$$\frac{\ddot{\psi}}{\dot{\psi}} = -2 \frac{d}{d\lambda} \log(\tanh \rho) \quad \Rightarrow \quad \dot{\psi}(\lambda) = \frac{\dot{\psi}(\lambda_0) \tanh^2(\rho(\lambda_0))}{\tanh^2(\rho(\lambda))} = \frac{\text{const}}{\tanh^2(\rho(\lambda))}. \tag{B.12}$$

Inserting the last result for $\dot{\psi}$ in the equation for ρ you get the same result you would get if you derived with respect to λ the (B.8); again, up to constants.

B.3 Geodesics in AdS$_3$

We present here the very first steps we took with the aim of connecting the geodesics from Section 5.3.2 with the ones in the respective "upstairs" space: AdS$_3$. The idea followed is to take a geodesic in the AdS space and smear it along the τ direction, in order to make it τ-independet; at that point gauging τ away will amount to focus on a given $\tau = \text{const}$ slice.

To approach the study of such geodesics, we are going to follow the approach from [30]. Basically, we start from the parametrization of an element of SL$(2, \mathbb{R})$ given by

$$g = e^{iu\sigma_2} e^{\rho\sigma_3} e^{iv\sigma_2} = \begin{pmatrix} \cos\tau \cosh\rho + \cos\phi \sinh\rho & \sin\tau \cosh\rho - \sin\phi \sinh\rho \\ -\sin\tau \cosh\rho - \sin\phi \sinh\rho & \cos\tau \cosh\rho - \cos\phi \sinh\rho \end{pmatrix} \tag{B.13}$$

with $u = \frac{1}{2}(\tau + \phi)$ and $v = \frac{1}{2}(\tau - \phi)$. The group element g will be the one entering the WZW action, to which we would like to add the gauge fields to perform the gauge transformation. We group in the lightcone coordinates $x^\pm = \lambda \pm \sigma$ the two worldsheet coordinates λ and σ; at which point, the equations of motion coming from the WZW action in lightcone coordinates tell us that we can in general decompose g as

$$g = g_+(x^+) g_-(x^-). \tag{B.14}$$

If we now eliminate the dependence from the worldsheet coordinate σ from the group element g, we will be left with a worldsheet that becomes a worldline, and so we will be describing particles rather than strings. Following the reference, we consider a given solution of the form from (B.14) with

$$g_+ = U e^{iv_+(x^+)\sigma_2} \quad \text{and} \quad g_- = e^{iu_-(x^-)\sigma_2} V \tag{B.15}$$

where U and V are two arbitrary constant matrices in $\mathrm{SL}(2,\mathbb{R})$. If we choose $v_+ = \alpha x^+/2$ and $u_- = \alpha x^-/2$, with $\alpha = \pm\sqrt{4h/k}$ where h is some positive constant and k is the level of the WZW model (namely it is the same k that appeared in Section 4.1), while keeping U and V general, we get

$$g_{\mathrm{tl}} = U \begin{pmatrix} \cos(\alpha\lambda) & \sin(\alpha\lambda) \\ -\sin(\alpha\lambda) & \cos(\alpha\lambda) \end{pmatrix} V; \tag{B.16}$$

as we can see, the dependence on σ is gone, meaning we are now describing the motion of particles as λ changes. Actually, any timelike geodesic can be written in such a way. Parametrizing both U and V as done for g in (B.13) with parameters t_u, ϕ_u and ρ_u for U and t_v, ϕ_v and ρ_v for V, we get

$$\begin{aligned}(g_{\mathrm{tl}})_{11} =& \cosh(\rho_u)\cosh(\rho_v)\cos(\alpha\lambda + t_u + t_v) + \cosh(\rho_u)\sinh(\rho_v)\cos(\alpha\lambda + t_u + \phi_v) \\ &+ \sinh(\rho_u)\cosh(\rho_v)\cos(\alpha\lambda + t_v - \phi_u) + \sinh(\rho_u)\sinh(\rho_v)\cos(\alpha\lambda - \phi_u + \phi_v),\end{aligned} \tag{B.17}$$

$$\begin{aligned}(g_{\mathrm{tl}})_{22} =& \cosh(\rho_u)\cosh(\rho_v)\cos(\alpha\lambda + t_u + t_v) - \cosh(\rho_u)\sinh(\rho_v)\cos(\alpha\lambda + t_u + \phi_v) \\ &- \sinh(\rho_u)\cosh(\rho_v)\cos(\alpha\lambda + t_v - \phi_u) + \sinh(\rho_u)\sinh(\rho_v)\cos(\alpha\lambda - \phi_u + \phi_v).\end{aligned} \tag{B.18}$$

For the diagonal elements, and

$$\begin{aligned}(g_{\mathrm{tl}})_{12} =& \cosh(\rho_u)\cosh(\rho_v)\sin(\alpha\lambda + t_u + t_v) - \cosh(\rho_u)\sinh(\rho_v)\sin(\alpha\lambda + t_u + \phi_v) \\ &+ \sinh(\rho_u)\cosh(\rho_v)\sin(\alpha\lambda + t_v - \phi_u) - \sinh(\rho_u)\sinh(\rho_v)\sin(\alpha\lambda - \phi_u + \phi_v),\end{aligned} \tag{B.19}$$

$$\begin{aligned}(g_{\mathrm{tl}})_{21} =& -\cosh(\rho_u)\cosh(\rho_v)\sin(\alpha\lambda + t_u + t_v) - \cosh(\rho_u)\sinh(\rho_v)\sin(\alpha\lambda + t_u + \phi_v) \\ &+ \sinh(\rho_u)\cosh(\rho_v)\sin(\alpha\lambda + t_v - \phi_u) + \sinh(\rho_u)\sinh(\rho_v)\sin(\alpha\lambda - \phi_u + \phi_v),\end{aligned} \tag{B.20}$$

for the off-diagonal ones. Equating (B.16) and (B.13), one finds that

$$\begin{aligned}\sinh^2\rho =& \frac{1}{4}\left(\left[(g_{\mathrm{tl}})_{11} - (g_{\mathrm{tl}})_{22}\right]^2 + \left[(g_{\mathrm{tl}})_{12} + (g_{\mathrm{tl}})_{21}\right]^2\right) = \cosh^2\rho_u \sinh^2\rho_v + \cosh^2\rho_v \sinh^2\rho_u \\ &+ \cos(t_u + t_v + 2\alpha\lambda - \phi_u + \phi_v)\cosh\rho_v \sinh\rho_v \sinh(2\rho_u)\end{aligned} \tag{B.21}$$

$$\begin{aligned}\cosh^2\rho =& \frac{1}{4}\left(\left[(g_{\mathrm{tl}})_{11} + (g_{\mathrm{tl}})_{22}\right]^2 + \left[(g_{\mathrm{tl}})_{12} - (g_{\mathrm{tl}})_{21}\right]^2\right) = \cosh^2\rho_u \cosh^2\rho_v + \sinh^2\rho_v \sinh^2\rho_u \\ &+ \cos(t_u + t_v + 2\alpha\lambda - \phi_u + \phi_v)\cosh\rho_v \sinh\rho_v \sinh(2\rho_u)\end{aligned} \tag{B.22}$$

and that

$$\begin{aligned}\cos\tau =& \frac{(g_{\mathrm{tl}})_{11} + (g_{\mathrm{tl}})_{22}}{2\cosh\rho} \\ =& \frac{\cos(t_u + t_v + \alpha\lambda)\,\cosh\rho_u\,\cosh\rho_v + \cos(\alpha\lambda - \phi_v + \phi_u)\,\sinh\rho_u\,\sinh\rho_v}{\cosh\rho}\end{aligned} \tag{B.23}$$

$$\begin{aligned}\cos\phi =& \frac{(g_{\mathrm{tl}})_{11} - (g_{\mathrm{tl}})_{22}}{2\sinh\rho} \\ =& \frac{\cos(t_v - \phi_u + \alpha\lambda)\,\sinh\rho_u\,\cosh\rho_v + \cos(\alpha\lambda + \phi_v + t_u)\,\cosh\rho_u\,\sinh\rho_v}{\sinh\rho}.\end{aligned} \tag{B.24}$$

We see from the above equations that ρ oscillates between two values set by ρ_u and ρ_v, and that both of these latter parameter must be different from zero if we wish for the ρ coordinate to oscillate in λ. To simplify things, we select the case in which $t_u = t_v = \phi_u = \phi_v = 0$ and fix $\rho_u = 1$ and $\rho_v = 1.3$; in this case, for $\sinh\rho$, $\cos\tau$ and $\cos\phi$ we get the graph portrayed in Figure B.1.

The geodesic with ρ, ϕ and τ given as in the above figure seems to manifest the correct properties to give, once smeared along τ and once this is gauged away, the geodesic circling around the cigar from Figure 5.3; however, exactly connecting the two would require some deeper study.

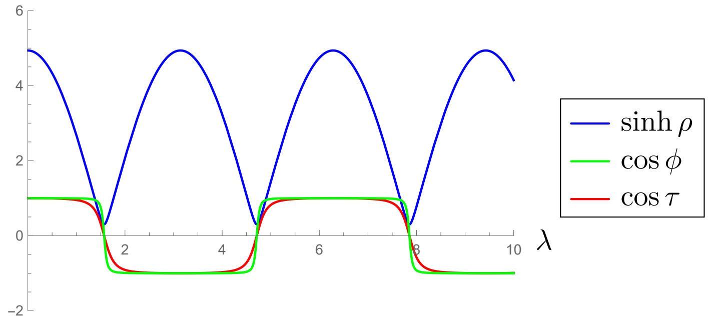


Figure B.1: Timelike geodesic in AdS_3 with $t_u = t_v = \phi_u = \phi_v = 0$, $\rho_u = 1$ and $\rho_v = 1.3$.

Similarly, it would be interesting to take a closer look at the spacelike geodesics, namely the ones represented by the general SL(2) matrix

$$g_{\rm sl} = U \begin{pmatrix} e^{\alpha\lambda} & 0 \\ 0 & e^{-\alpha\lambda} \end{pmatrix} V, \tag{B.25}$$

with elements equal to

$$\begin{aligned}
(g_{\rm sl})_{11} &= \cos(t_u - t_v)\sinh(\alpha\lambda + \rho_u + \rho_v) + \cos(t_u + t_v)\cosh(\alpha\lambda + \rho_u + \rho_v), && \text{(B.26)}\\
(g_{\rm sl})_{12} &= \sin(t_u + t_v)\cosh(\alpha\lambda + \rho_u + \rho_v) - \sin(t_u - t_v)\sinh(\alpha\lambda + \rho_u + \rho_v), && \text{(B.27)}\\
(g_{\rm sl})_{21} &= -\sin(t_u + t_v)\cosh(\alpha\lambda + \rho_u + \rho_v) - \sin(t_u - t_v)\sinh(\alpha\lambda + \rho_u + \rho_v), && \text{(B.28)}\\
(g_{\rm sl})_{22} &= \cos(t_u + t_v)\cosh(\alpha\lambda + \rho_u + \rho_v) - \cos(t_u - t_v)\sinh(\alpha\lambda + \rho_u + \rho_v). && \text{(B.29)}
\end{aligned}$$

Naturally, also in this case, we can derive

$$\begin{aligned}
\cosh^2\rho = \big[&\sinh(\alpha\lambda)[\cosh(\rho_u)\sinh(\rho_v)\cos(t_u - \phi_v) + \sinh(\rho_u)\cosh(\rho_v)\cos(t_v + \phi_u)] \\
&+ \cosh(\alpha\lambda)[\cosh(\rho_u)\cosh(\rho_v)\cos(t_u + t_v) + \sinh(\rho_u)\sinh(\rho_v)\cos(\phi_u - \phi_v)]\big]^2 \\
+\big[&\sinh(\alpha\lambda)[\cosh(\rho_u)\sinh(\rho_v)\sin(t_u - \phi_v) + \sinh(\rho_u)\cosh(\rho_v)\sin(t_v + \phi_u)] \\
&+ \cosh(\alpha\lambda)[\cosh(\rho_u)\cosh(\rho_v)\sin(t_u + t_v) + \sinh(\rho_u)\sinh(\rho_v)\sin(\phi_u - \phi_v)]\big]^2
\end{aligned} \tag{B.30}$$

$$\begin{aligned}
\sinh^2\rho = \big[&\cosh(\alpha\lambda)[\cosh(\rho_u)\sinh(\rho_v)\cos(t_u + \phi_v) + \sinh(\rho_u)\cosh(\rho_v)\cos(t_v - \phi_u)] \\
&+ \sinh(\alpha\lambda)[\cosh(\rho_u)\cosh(\rho_v)\cos(t_u - t_v) + \sinh(\rho_u)\sinh(\rho_v)\cos(\phi_u + \phi_v)]\big]^2 \\
+\big[&\cosh(\alpha\lambda)[\cosh(\rho_u)\sinh(\rho_v)\sin(t_u + \phi_v) - \sinh(\rho_u)\cosh(\rho_v)\sin(t_v - \phi_u)] \\
&+ \sinh(\alpha\lambda)[\cosh(\rho_u)\cosh(\rho_v)\sin(t_u - t_v) + \sinh(\rho_u)\sinh(\rho_v)\sin(\phi_u + \phi_v)]\big]^2,
\end{aligned} \tag{B.31}$$

that can later be used to obtain

$$\begin{aligned}
\cos\tau &= \frac{(g_{\rm sl})_{11} + (g_{\rm sl})_{22}}{2\cosh\rho} \\
&= \frac{\begin{aligned}&\sinh(\alpha\lambda)[\cosh(\rho_u)\sinh(\rho_v)\cos(t_u - \phi_v) + \sinh(\rho_u)\cosh(\rho_v)\cos(t_v + \phi_u)] \\ &+ \cosh(\alpha\lambda)[\cosh(\rho_u)\cosh(\rho_v)\cos(t_u + t_v) + \sinh(\rho_u)\sinh(\rho_v)\cos(\phi_u - \phi_v)]\end{aligned}}{\cosh\rho}
\end{aligned} \tag{B.32}$$

$$\begin{aligned}
\sin\tau &= \frac{(g_{\rm sl})_{12} - (g_{\rm sl})_{21}}{2\cosh\rho} \\
&= \frac{\begin{aligned}&\sinh(\alpha\lambda)[\cosh(\rho_u)\sinh(\rho_v)\sin(t_u - \phi_v) + \sinh(\rho_u)\cosh(\rho_v)\sin(t_v + \phi_u)] \\ &+ \cosh(\alpha\lambda)[\cosh(\rho_u)\cosh(\rho_v)\sin(t_u + t_v) + \sinh(\rho_u)\sinh(\rho_v)\sin(\phi_u - \phi_v)]\end{aligned}}{\cosh\rho}
\end{aligned} \tag{B.33}$$

for the variable τ, while for $\cos\phi$ we get

$$\begin{aligned}
\cos\phi &= \frac{(g_{\rm sl})_{11} - (g_{\rm sl})_{22}}{2\sinh\rho} \\
&= \frac{\begin{aligned}&\cosh(\alpha\lambda)[\cosh(\rho_u)\sinh(\rho_v)\cos(t_u + \phi_v) + \sinh(\rho_u)\cosh(\rho_v)\cos(t_v - \phi_u)] \\ &+ \sinh(\alpha\lambda)[\cosh(\rho_u)\cosh(\rho_v)\cos(t_u - t_v) + \sinh(\rho_u)\sinh(\rho_v)\cos(\phi_u + \phi_v)]\end{aligned}}{\sinh\rho},
\end{aligned} \tag{B.34}$$

and finally $\sin\phi$ is given by

$$\begin{aligned}\sin\phi =&\frac{(g_{\mathrm{sl}})_{12}+(g_{\mathrm{sl}})_{21}}{2\sinh\rho}\\ =&\frac{\begin{array}{c}\cosh(\alpha\lambda)[\sinh(\rho_u)\cosh(\rho_v)\sin(t_v-\phi_u)-\cosh(\rho_u)\sinh(\rho_v)\sin(t_u+\phi_v)]\\ -\sinh(\alpha\lambda)[\cosh(\rho_u)\cosh(\rho_v)\sin(t_u-t_v)+\sinh(\rho_u)\sinh(\rho_v)\sin(\phi_u+\phi_v)]\end{array}}{\sinh\rho}.\end{aligned} \tag{B.35}$$

As can be noticed by comparing the above solutions for $\sinh\rho$, $\cosh\rho$, $\cos\tau$ and $\cos\phi$ with their timelike counterparts, the study of spacelike geodesics promises to be more involved than the one on timelike geodesics.

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