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Master Program in Mathematics

Intrinsic FEM for Vector Laplacian equations

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21 April 2022 - Accademic year 2021/2022

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Introduction

Surface partial differential equation are becoming increasingly important because of their ability to model different phenomena such as earth process [13, 18, 20, 19, 17, 8], biological processes and material science [28, 26, 32] and image processing [12, 35]. PDEs typically describe balance laws of scalar, vector and tensor quantities living on the surface. The detailed mathematical understanding of these PDEs is still limited, and applications are tackled mainly by numerical techniques. One of these is the surface finite element method(SFEM), which dates back to the first study carried out by [15] on the Laplace-Beltrami operator with surface finite elements. Most of this research is, however, focused on *scalar*-valued problems, see [16], and in this case the coupling between the geometry of the surface and the PDE is weak and thus allows to solve these problems with relatively small modifications of established numerical approaches in flat space. But many physical problem and models, like flow on surfaces as well as membranes and shells, see [23, 24], or surface Navier-Stokes equation, see [33, 5], involve *vector unknowns*. For surface vector and tensor PDEs the situation becomes more complicated, because from the modeling and numerical discretization points of view of a much stronger coupling between the solution of the PDE and the geometrical characteristics of the surface. In particular, the extension of the Laplace-Beltrami operator, defined as *div grad*, in well known equation such as Poisson, Heat transfer and the Wave equations, for the vector fields on surfaces can be defined several natural Laplacians. For example, the *rough* and the *Bochner Laplacians*, differing by a minus sign, see [22]; the *Hodge Laplacian* defined through the use of external calculus, see [25], which differs from previous ones by a zero-order term depending only on the curvature of the surface. So it is immediate and natural to ask what is the appropriate definition of vector Laplacian and whether one definition is more appropriate than other for the definition of PDEs arising for example from basic physical principle such as energy minimization. The answer to these questions is not so obvious both in the context of differential geometry and the analysis and development of mathematical and numerical models.

The numerical methods developed and seen so fa for such problems are not many and are often restricted to very special surface cases, such as spherical surfaces [11, 10, 21] or physical problem, see [27, 29, 31]. Although some numerical approaches exist, most of these methods avoid the use of charts and atlases and mainly exploit the possibility of extending functions in the tubular neighborhood of the surface, i.e., exploiting the properties of the embedding space.

These methods based on implicit representation of the surfaces for example via level-set functions have seen significant interest and development. By extending the problem to one higher spatial dimension, these methods do not require explicit discretization of the surface, see [6] as an example of application to the scalar case. On the other hand, methods based on SFEM which are based on projection of the surface quantities from the embedding space, are difficult to extend to the vector/tensor case as additional unknowns and equations must be added to constrain vectors to live in the tangent space, see [30]. On the other hand, the recently developed intrinsic FEM (ISFEM), see [8, 9, 7] ,for scalar parabolic equations and vector hyperbolic systems of balance laws seems to be naturally suited to treat vector quantities. In fact, within this intrinsic approach vector quantities are intrinsically and naturally defined on the tangent planes with the difficulty the the PDEs and their variational formulations must be defined in a covariant or contravariant form to incorporate the geometric information of the domain.

The *purpose* of this thesis is twofold: **i)** to study the nature of the different vector Laplacian forms by showing a case of its derivation from an energy associated to a nematic liquid crystal model. In particular we will study how the Hodge or the Bochner Laplacians develop and are related through Weitzenböck identity; **ii)** to study the discretization of the latter by means of the ISFEM (intrinsic finite element method), by extending the latter to the vector case and using the building blocks developed for the scalar case in [3, 9]. Finally, a few examples on simple surfaces will be numerically solved to test the accuracy and efficiency of the proposed extension of scalar ISFEM to the vector Laplacian. The thesis is structured as follows:

- Chapter 1: we recall some preliminary mathematical notions and definition concerning differential geometry. In particular, we will place great emphasis on the section regarding surfaces and their properties.
- Chapter 2: we give the Hodge and Bochner Laplacian definitions and the Weitzenböck identity, then we show the derivation of the PDE from the minimization of the energy associated to the physical model of the paper [29].
- Chapter 3: we describe the ISFEM method of the paper [6] and then we extend it to the vector case. Finally we give some example of the convergence of the scheme of three different surfaces.

1

Geometrical preliminaries

In this chapter we recall some important notions of differential geometry that will be necessary to define our geometrical setting for PDEs on surfaces. Following [1], we start by the more general definitions of manifolds, functions defined over them, and differentiability. Then we define tangent and cotangent spaces, and extend to the notion of tensors and differential forms. We arrive then to set up relations between this objects, i.e. derivation rules, from the definition of connection to exterior derivatives, Hodge operator and the co-differential, which will come in hand later in the thesis work. In section 1.2, we concentrate on analogous definitions and results on surfaces following [2]. Finally, in section 1.3, we introduce the concepts of local coordinate systems and differential operators on surfaces, and the relation with the classical Cartesian ones, in the setting that will be used later in the main part of the work.

1.1 Differential Geometry

Definition 1.1.1 (Chart)**.** *Let X be a topological space, $U \subset X$ and $V \subset \mathbb{R}^n$ open sets and $\varphi : U \longrightarrow \varphi(U) = V$ homeomorphism, i.e., φ is continuous, bijective with a continuous inverse φ^{-1}. Then, the couple (U, φ) is called chart.*

Given a point $P \in X$ and a chart of X, we call *local coordinates* the image o P through the chart $\varphi(P)$. The inverse of the chart is called *local parametrization.* In general, it is possible to have multiple definitions of this kind of maps. When a subset $V \subset \mathbb{R}^n$ is defined through multiple charts, feels natural to ask for some *compatibility*, i.e., the same structure needs to be defined at the intersection between the charts. For this reason, we give the following definition:

Definition 1.1.2 (Compatibility of charts)**.** *Let (U_i, φ_i), (U_j, φ_j) be two charts of X. The charts are compatible if*

- $U_i \cap U_j = \emptyset$ *or*
- 1. $U_i \cap U_j \neq \emptyset$
  2. $\varphi_i(U_i \cap U_j)$ *and* $\varphi_j(U_i \cap U_j)$ *are open sets of* $\mathbb{R}^n$
  3. $\eta_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$ *is a $\mathcal{C}^\infty$-diffeomorphism ($\mathcal{C}^\infty$ invertible maps with its inverse that is also $\mathcal{C}^\infty$).*

The diffeomorphism η_{ij} is called *coordinates changing/charts changing* or *transition function.*

Definition 1.1.3 (Atlas). *A family collection of compatible charts $\mathcal{A} = \{(U_i, \varphi_i)\}$ such that $X = \cup_{i \in I} U_i$ is called an Atlas.*

With this definitions, it is now possible to define a first type of manifold.

Definition 1.1.4 (Topological manifold). *A topological manifold is a couple $(X, \mathcal{A})$, where X is a topological space and $\mathcal{A}$ is an atlas defined on X.*

In the case of a chart with image contained in $\mathbb{R}^n$, the integer n is called the *dimension of the locals chart.* If the manifold is connected then all the charts must have the same dimension, thus the value of n defines the *manifold dimension.*

To define the class of regularity of a manifold, we first recall that a function f is of class $\mathcal{C}^s$ if it is continuous and its derivatives $f^{(1)}, \dots, f^{(s)}$ are continuous. In particular, we define $\mathcal{C}^s_P$ as the set of functions that are defined in a neighborhood of P and are of class $\mathcal{C}^s$.

Definition 1.1.5 (Differentiable manifold). *A topological manifold is $\mathcal{C}^s$ if all the transition functions η_{ij} are $\mathcal{C}^s$. A manifold is said to be differentiable if η_{ij} are $\mathcal{C}^\infty$.*

From the definition of chart it is also possible to define and study functions over the manifolds. Let X be a manifold, $\mathcal{A}$ be an atlas of X and (U, φ) a chart, the function $f : X \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^s$ if and only if the function

$$\hat{f} = f|_U \circ \varphi^{-1}$$

is of class $\mathcal{C}^s$ as a function in $\mathbb{R}^n$. Thus, it is possible to study a function f on manifolds through the study of the corresponding $\hat{f}$ function defined over an open set of $\mathbb{R}^n$.

The next step concerns the regularity of functions that operate between different manifolds. The idea is to use the previous definition of regularity of a function defined over a manifold and extend it to the case of functions between manifolds. Given two different manifolds X, Y and a function $F : X \longrightarrow Y$, such that for a point $P \in X$ there is $Q = F(P) \in Y$, we consider two charts of X and Y respectively and the composition of those maps with F:

- (U, φ) local chart of X with $P \in U$,
- (V, ψ) local chart of Y with $Q \in V$,
- $\hat{F} = \psi \circ F \circ \varphi^{-1}$ local representation of F.

Then, we say that F is $\mathcal{C}^s$ in a neighborhood of $P \in U$ if $\hat{F}$ is $\mathcal{C}^s$ in a neighborhood of $P \in \varphi(P)$.

Definition 1.1.6 (Differentiable function). *We say that F is differentiable if it is $\mathcal{C}^\infty$ in every point of X.*

Now, given a manifold X, we define the space of centered differentiation Der_P as the space of all operators satisfying the following properties in a point $P \in X$

- $D_P(f + g) = D_P(f) + D_P(g)$;
- $D_P(f) = 0$ if f is a constant function;

- $D_P(f \cdot g) = D_P(f) \cdot g(P) + f(P) \cdot D_P(g)$.

for every function $f, g \in C^\infty_P$. Then we are ready to define the tangent space of a manifold.

Definition 1.1.7 (Tangent space)**.** *The tangent space of a manifold X in a point $P \in X$ is defined as $T_P X = Der_P$.*

With these notions we can define the *differential*, necessary to define a surface. Recalling that given a function between manifolds $F : X \longrightarrow Y$ and scalar function $f : Y \longrightarrow \mathbb{R}$, the *pull-back* is the map

$$\begin{aligned} F^* : \mathcal{C}^\infty_Y &\longrightarrow \mathcal{C}^\infty_X \\ f &\longmapsto f \circ F \end{aligned}$$

with $\mathcal{C}^\infty_Y$ the space of the functions $f : Y \longrightarrow \mathbb{R}$ which are $\mathcal{C}^\infty$.

Definition 1.1.8 (Differential)**.** *The differential of a function $F : X \longrightarrow Y$ in a point $P \in X$ is the map*

$$\begin{aligned} dF_P : T_P X &\longrightarrow T_{F(P)} Y \\ D &\longmapsto dF_P(D) = D \circ F^*_P \end{aligned}$$

Definition 1.1.9 (Tangent bundle)**.** *Let X be a smooth manifold. The tangent bundle of X is the couple (TX, π), with*

$$TX = \bigsqcup_{P \in X} T_P X = \{(P, v) | P \in X, v \in T_P X\},$$

where $T_P X$ is the tangent space of X at the point P, and the projection

$$\begin{aligned} \pi : TX &\to X \\ (P, v) &\longmapsto P. \end{aligned}$$

In analogous way, we can define the *cotangent bundle*:

Definition 1.1.10 (Cotangent bundle)**.** *The cotangent bundle T^*X of X is the dual of the tangent bundle TX,*

$$T^*X = \bigsqcup_{P \in X} T^*_P X = \{(P, \alpha) | P \in X, \alpha \in T^*_P X\},$$

where $T^*_P X = Hom(T_P X, \mathbb{R})$ is the space of the homeomorphism from $T_P X$ to $\mathbb{R}$.

Definition 1.1.11 (Vector bundle)**.** *Let X be a smooth manifold. A vector bundle of rank r on X, is a smooth manifold E with a surjective differentiable function $\pi : E \to X$ such that:*

- *$\forall\, P \in X$ the fiber $E_P = \pi^{-1}(P)$ is a vectorial space of dimension r,*
- *$\forall\, P \in X$ exists an open neighborhood $U \subset X$ of P and a diffeomorphism $\chi : E|_U = \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that $\pi = p_1 \circ \chi$, with $p_1 : U \times \mathbb{R}^r \to U$ the projection along U.*

Definition 1.1.12 (Section)**.** *A section of a vector bundle E on a open set $U \subset X$ is a differentiable function $\sigma : U \to E$ such that $\pi \circ \sigma = id_U$.*

This means that $\sigma(P) = (P, v)$ with $\omega \in E$ and $\forall\, P \in U$. IN particular this definition will be necessary later to define the differential forms.

From the product between tangent and cotangent spaces, we can extend to the definition of tensors.

Definition 1.1.13 (Tensor). *A tensor* $t \in T^p_q(V) = \underbrace{V \otimes \cdots \otimes V}_{p} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{q}$ *is a multilinear function*

$$t : \underbrace{V^* \times \cdots \times V^*}_{p} \times \underbrace{V \times \cdots V \times V}_{q} \xrightarrow{\sim} \mathbb{R}$$

Observation 1. *If* $\{v_1, \ldots, v_n\}$ *is a base of* V *and* $\{v^1, \ldots, v^n\}$ *is the dual base of* V^**, then the tensors*

$$v_{i_1} \otimes \cdots \otimes v_{i_p} \otimes v^{j_1} \otimes \cdots \otimes v^{j_q}, \qquad (i_1, \ldots, i_p = 1, \ldots, n \quad \textit{and} \quad j_1, \ldots, j_q = 1, \ldots, n)$$

form a base of $T^p_q(V)$*, which has a dimension of* n^{p+q}*. Thus, a generic tensor can be express as*

$$t = a^{i_1,\ldots,i_p}_{j_1,\ldots,j_q}\, v_{i_1} \otimes \cdots \otimes v_{i_p} \otimes v^{j_1} \otimes \cdots \otimes v^{j_q},$$

where the $a^{i_1,\ldots,i_p}_{j_1,\ldots,j_q}$ *are* $\mathcal{C}^\infty$ *functions.*

Among the tensors $t \in T^p(V)$, i.e., among the multilinear application $t : \underbrace{V^* \times \cdots \times V^*}_{p} \rightarrow \mathbb{R}$, we consider only the alternating ones, so such that the following relation holds:

$$t(\alpha^{\sigma(1)}, \ldots, \alpha^{\sigma(p)}) = sgn(\sigma) \cdot t(\alpha^1, \ldots, \alpha^p)\,,$$

where $\alpha^j \in V^* = \mathrm{Hom}(V, \mathbb{R})$, and $\sigma(j)$ a permutation on the indices j of the elements in V^*. We indicate with $\Lambda^p(V) \subset T^p(V)$ the subspace of alternating tensors. If $\dim(V) = n < +\infty$, it follows that

$$\dim \Lambda^p(V) = \begin{cases} \binom{n}{q} & \text{if} \quad 0 \leq p \leq n \\ 0 & \text{if} \quad p > n \end{cases}.$$

Given then definition of a tensor, now the state the following operation between tensors:

Definition 1.1.14 (External product). *Let* V *be a vector space, and* $t_1 \in \Lambda^p(V)$ *and* $t_2 \in \Lambda^q(V)$ *alternating tensors, we define the external product* $\wedge$ *as*

$$\begin{aligned} \Lambda^p(V) \times \Lambda^q(V) &\longrightarrow \Lambda^{p+q}(V) \\ (t_1 \quad , \quad t_2) &\longmapsto t_1 \wedge t_2 \end{aligned}$$

such that

$$t_1 \wedge t_2 = \frac{(p+q)!}{p!q!} A(t_1 \otimes t_2)$$

Definition 1.1.15 (Volume form). *Let* M *be a* n*-dimensional manifold. A volume form is a non-vanishing* n*-dimensional form* $\nu \in \Lambda^n(M)$.

This means that $\forall P \in M$ and $\forall \{v_1, \ldots v_n\}$ basis of $T_P M$, $\nu_P(v_1 \ldots v_n) \neq 0$.

Definition 1.1.16 (k-differential form). *Let* M *be a smooth manifold. A k-differential form is a section of* $\Lambda^k(T^*_P M)$*. The set of all k-forms is indicated with* $\Lambda^k(M) = \bigsqcup_{P \in M} \Lambda^k(T^*_P M)$.

Definition 1.1.17 (Exterior derivative). *Let M be a smooth manifold. The exterior derivative is an operator that sends a k-differential form $w \in \Lambda^k(M)$ into a (k+1)-differential form:*

$$d : \Lambda^k(M) \longrightarrow \Lambda^{(k+1)}(M)$$
$$w \longmapsto dw$$

In local coordinates, i.e. using the local we have that

$$w \in \Lambda^k(M) \Longrightarrow w = \sum_{i_1<\cdots<i_r} f_{i_1\ldots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r} \,,$$

where the $f_{i_1\ldots i_r}$ are $\mathcal{C}^\infty$ functions. Then the exterior derivative is defined by:

$$dw = \sum_{i_1<\cdots<i_r} \left(\sum_{j=1}^{n} \frac{\partial f_{i_1\ldots i_r}}{\partial x^j} dx^j \right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \,.$$

Definition 1.1.18 (Connection). *Let M be a smooth manifold. A connection on a vector bundle E is a function:*

$$\nabla : \mathcal{T}(M) \times E(M) \longrightarrow E(M)$$
$$(X, s) \longmapsto \nabla_X s$$

such that the following holds:

1. $\forall\, X_1, X_2 \in \mathcal{T}(M)$, $\forall f_1, f_2 \in \mathcal{C}^\infty(M)$, $\forall \sigma \in E(\sigma)$

$$\nabla_{f_1 X_1 + f_2 X_2} = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$$

2. $\forall\, X \in \mathcal{T}(M)$, $\forall s_1, s_2 \in E(M)$, $\forall a_1, a_2 \in \mathbb{R}$

$$\nabla_X (a_1 s_1 + a_2 s_2) = a_1 \nabla_X s_1 + a_2 \nabla_X s_2$$

3. $\forall\, X \in \mathcal{T}(M)$, $\forall s \in E(M)$, $\forall f \in \mathcal{C}^\infty$

$$\nabla_X (fs) = X(f)s + f \nabla_X s$$

In particular, the section $\nabla_X s \in E(M)$ is called *covariant derivative* of s along the vector field X. If $E = TM$, the connection on E is called *linear connection* on M.

Now we recall some definitions regarding the theory of the Riemannian manifold that will be necessary later to define some important operators.

Definition 1.1.19 (Riemannian metric). *A Riemannian metric on a manifold M is a 2-covariant tensor $g \in \mathcal{T}_2(M)$ symmetric and positive definite, i.e.:*

- $g_P(w, v) = g_P(v, w) \qquad \forall P \in M, \forall v, w \in T_P M$
- $g_P(v, v) > 0 \qquad \forall v \neq 0, v \in T_P M$

In other words, a Riemannian metric associates with each point $P \in M$ a scalar product (positive definite) $g_P : T_P M \times T_P M \longrightarrow \mathbb{R}$, which smoothly depends on the point P.

Definition 1.1.20 (Riemannian manifold)**.** *We call the couple (M, g) Riemannian manifold.*

Theorem 1.1.21. *Every differentiable manifold M admits a Riemann metric.*

In local charts (U, φ), we can express the symmetric tensor g as:

$$g = g_{ij} dx^i \otimes dx^j, \qquad i, j = 1, 2$$

where $g_{ij} \in \mathcal{C}^\infty(U)$ are the coefficients of the metric. Since g is symmetric and positive definite, we can also define its inverse as the *2-contravariant tensor* g^{-1} such that

$$\delta_i^k = g_{ij} g^{jk} .$$

It is now possible to define an explicit connection between the tangent and cotangent spaces of a Riemann manifold.

Definition 1.1.22 ($\flat$ and $\sharp$)**.** *Let (M, g) be a Riemann manifold. For each $P \in M$ there exists an isomorphism:*

$$\begin{aligned} \flat_P : T_P M &\longrightarrow T_P^* M = hom(T_P M, \mathbb{R}) \\ v &\longmapsto \flat(v) \end{aligned}$$

where

$$\begin{aligned} \flat_P(v) : T_P M &\longrightarrow \mathbb{R} \\ w &\longmapsto (\flat(v))(w) = g_P(v, w) \end{aligned}$$

We denote by the symbol $\sharp$ the inverse of $\flat$:

$$\begin{aligned} (\flat_P)^{-1} = \sharp_P : T_P^* M &\longrightarrow T_P M \\ \alpha &\longmapsto \sharp(\alpha) \end{aligned}$$

We want to find an expression for $\flat$ and $\sharp$ in local coordinates. Let $g = g_{ij} dx^i \otimes dx^j$ be the Riemann metric and $X = X^h \partial_h$ a local section of TM. Since $\flat$ is the map from the tangent space to the cotangent space, $\flat(X) = \alpha_j dx^j$. From the definition of $\flat$ we get that:

$$\begin{aligned} (\flat(X))(\partial_k) = g(X, \partial_k) &= \left(g_{ij} dx^i \otimes dx^j\right) \left(X^h \partial_h, \partial_k\right) \\ &= g_{ij} X^h dx^i(\partial_h) dx^j(\partial_k) = g_{ij} X^h \delta_h^i \delta_k^j = g_{hk} X^k . \end{aligned}$$

Moreover, from the definition of $\flat$ we have that:

$$(\flat(X))(\partial_k) = \alpha_j dx^j(\partial_k) = \alpha_j \delta_k^j = \alpha_k .$$

Comparing these two expressions we get:

$$\alpha_k = g_{hk} X^h \qquad \text{and} \flat\left(X^h \partial_h\right) = g_{hk} X^h dx^k$$

So, we identify the isomorphism $\flat$ as the isomorphism that lowers the indices using the Riemann metric. Analogously, it can be proved that the inverse isomorphism $\sharp$ uppers the indices, namely:

$$X^h = g^{kj} g_{hk} X^h = g^{kj} \alpha_k .$$

For example, if we consider a function $f \in \mathcal{C}^\infty(M)$, the differential df is a section of T^*M, thus a 1-differential form. If we apply the isomorphism $\sharp$ to df we get a vector field over M:

$$\nabla f := \sharp(df) \in TM \tag{1.1}$$

which is called *gradient of f*. In local coordinates, we have $df = \frac{\partial f}{\partial x^j} dx^j$, and we get

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i} \, ,$$

since $\sharp : \frac{\partial f}{\partial x^j} \longmapsto g^{ij} \frac{\partial f}{\partial x^j}$.

We define now an operator between differential forms that will be necessary for the calculus on the following chapter.

Definition 1.1.23 (Hodge operator)**.** *Let M be a n-Riemannian manifold with a volume form ν_g. The Hodge operator $\star$ is the unique linear map, for $0 \leq k \leq n$, defined as*

$$\star : \Lambda^k(M) \to \Lambda^{n-k}(M)$$
$$\omega \longmapsto \star\omega$$

and such that

$$\omega \wedge (\star\eta) = \frac{1}{k!} \langle \omega, \eta \rangle \nu_g, \qquad \forall \omega, \eta \in \Lambda^k(M),$$

where $\langle \cdot, \cdot \rangle$ is the metric defined along the fibers of $\Lambda^k(M)$ as $\langle \omega, \eta \rangle = \langle \omega_1 \wedge \cdots \wedge \omega_k, \eta_1 \wedge \cdots \wedge \eta_k \rangle = det(\langle \omega_i, \eta_j \rangle_g)$.

Proposition 1.1.24. *Let M be a n-dimensional Riemannian manifold with the volume form ν_g. The Hodge star operator satisfies the following properties for $\alpha, \beta \in \Lambda^k(M)$:*

- $\alpha \wedge \star\beta = \beta \ \wedge \star\alpha = \langle \alpha, \beta \rangle \nu_g$
- $\star 1 = \nu_g; \quad \star\nu_g = 1$
- $\star \star \alpha = (-1)^{k(n-k)} \alpha$
- $\langle \alpha, \beta \rangle = \langle \star\alpha, \star\beta \rangle$
- Given a vector field F and the associated 1-form ω_F: $\mathrm{curl}(F) = \star \mathbf{d} \omega_F$ and $\mathrm{div}(F) = \star \mathbf{d} \star \omega_F$

Definition 1.1.25 (External codifferential)**.** *Let M be a n-Riemannian manifold, the external codifferential $\delta : \Lambda^k(M) \to \Lambda^{k-1}(M)$ is defined for every $k = 1, \dots, n$ by*

$$\delta = (-1)^{n(k+1)+1} \star \boldsymbol{d} \star,$$

with $\boldsymbol{d}$ the external differential.

We define $(\cdot, \cdot)$ to be the positive definite inner product over $\Lambda^k(M)$:

$$(\omega, \eta) = \frac{1}{k!} \int_M \langle \omega, \eta \rangle \nu_g, \qquad \forall \omega, \eta \in \Lambda^k(M),$$

then the following proposition holds:

Proposition 1.1.26. *Let M be a n-dimensional Riemannian manifold. Then $\boldsymbol{d}$ and δ are adjoints*

$$(\boldsymbol{d}\eta, \omega) = \int_M \boldsymbol{d}\eta \wedge \star\omega = \int_M \eta \wedge \star\delta\omega = (\eta, \delta\omega)$$

$\forall \omega \in \Lambda^k(M)$ and $\eta \in \Lambda^{k-1}(M)$.

Finally, we recall the definition of connection and covariant derivative, deriving the so called *Christoffel symbols* and their expressions on a Riemann manifold. Let (U, φ) be a local chart of a smooth manifold M. Let E be a vector bundle of rank r, with the diffeomorphism $\chi : E_U = \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^r$. The canonical basis of $\mathbb{R}^r$ defines the associated local basis of E_U, i.e. the sections $e_1, \cdots, e_r \in E(U)$ such that, for all $P \in U$, the set $\{e_1(P), \cdots e_r(P)\}$ is a local basis for the fiber E_P. The local chart determines the local basis $\{\partial_1 = \frac{\partial}{\partial_, x^1}, \cdots, \partial_n = \frac{\partial}{\partial x^n}\}$ of TM. So, from definition 1.1.18, we get that

$$\nabla_{\partial_j} e_h = \Gamma^k_{jh} e_k \qquad j = 1, \cdots, n \quad h, k = 1, \cdots, r, \tag{1.2}$$

where $\Gamma^k_{jh} \in \mathcal{C}^\infty(U)$.

We call these function *connection coefficients*. In the case of a linear connection, $E = TM$, the functions Γ^k_{jh} are called *Christoffel symbols*. In particular, these coefficients determine completely the connection, allowing us to express the connection of a general section $s = s^k e_k$ as the following:

$$\nabla_X(s^k) = X(s^k) + \Gamma^k_{jh} X^j s^h, \qquad k = 1, \cdots, r.$$

where $X = X^j \partial_j \in \mathcal{T}(M)$ and $\Gamma^k_{jh}, s^k, X^j \in \mathcal{C}^\infty(U)$.

We are now ready to recall one of the most important results of Riemannian manifolds, the *Levi-Civita* theorem. Thanks to this theorem, an explicit expression for the Christoffel symbols is available.

Theorem 1.1.27. *On every Riemannian manifolds (M, g), there exists a unique symmetric connection ∇ compatible with the metric. This connection satisfies the following:*

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle)$$

for all $X, Y, Z \in \mathcal{T}(M)$. The Christoffel symbols of ∇ have the following expression:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \tag{1.3}$$

1.2 Surfaces

In this section, we will focus on the theory concerning surfaces, following [2]. We begin with the definition of an immersed, or parametrized, surface.

Definition 1.2.1 (Immersed(or parametrized) surface)**.** *An immersed (or parametrized) surface in $\mathbb{R}^3$ is a map $\varphi : U \longrightarrow \mathbb{R}^3$ of class $\mathcal{C}^\infty$, where $U \subseteq \mathbb{R}^2$ is an open set, such that the differential $d\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is injective (i.e., of rank 2) in every point $x \in U$. The image $\varphi(U)$ of φ is the support of the immersed surface.*

For example, given the map $\varphi : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, the differential $\mathrm{d}\varphi_x$ of $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ in $\hat{x} \in U$ is represented by the Jacobian matrix:

$$\mathrm{Jac}\ \varphi(\hat{x}) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial x^1}(\hat{x}) & \frac{\partial \varphi_1}{\partial x^2}(\hat{x}) \\ \frac{\partial \varphi_2}{\partial x^1}(\hat{x}) & \frac{\partial \varphi_2}{\partial x^2}(\hat{x}) \\ \frac{\partial \varphi_3}{\partial x^1}(\hat{x}) & \frac{\partial \varphi_3}{\partial x^2}(\hat{x}) \end{bmatrix} \in M_{3,2}(\mathbb{R})$$

and it is of rank 2 in $\hat{x}$ if there exist a 2×2 minor with non zero determinant.

In this definition the emphasis is on the map rather than on its image. Note that, we are not asking for the immersed surfaces to be a homeomorphism with their images or to be injective. For example, if $U = (-1, +\infty) \times \mathbb{R}$, and $\varphi : U \longrightarrow \mathbb{R}^3$ is given by

$$\varphi(x^1, x^2) = \left(\frac{3x^1}{1 + (x^1)^3}, \frac{3{x^1}^2}{1 + (x^1)^3}, x^2 \right),$$

we have that φ is an injective immersed surface, but is not a homeomorphism with its image, as $\varphi\left((-1,1) \times (-1,1)\right)$ is not open in $\varphi(U)$. However, both these properties are locally true. To prove this fact, we state the following corollary:

Corollary 1.2.2. *Let $\varphi : U \longrightarrow \mathbb{R}^3$ be an immersed surface. Then every $\hat{x} \in U$ has a neighborhood $U_1 \subseteq U$ such that $\varphi_{|U_1} : U_1 \longrightarrow \mathbb{R}^3$ is a homeomorphism with its image.*

Now we are ready to state the definition of surface:

Definition 1.2.3 (Surface)**.** *A connected subset $\mathcal{S} \subset \mathbb{R}^3$ is a (regular) surface if $\forall P \in \mathcal{S}$ there exists a $\mathcal{C}^\infty$-map $\varphi : U \longrightarrow \mathbb{R}^3$, where $U \subset \mathbb{R}^2$, such that:*

- *$\varphi(U) \subset \mathcal{S}$ is an open neighborhood of P;*
- *φ is an homeomorphism with its image;*
- *$\forall\, Q \in U$, the differential $\boldsymbol{d}\varphi_Q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is injective.*

An easy example of surface is given by the plane through a point $s_0 \in \mathbb{R}^3$ and parallel to the linearly independent vectors $v_1, v_2 \in \mathbb{R}^3$. It has only a single local parametrization $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $\varphi(x^1, x^2) = s_0 + x^1 v_1 + x^2 v_2$.

Definition 1.2.4 (Critical point, critical value, level set)**.** *Let $V \subset \mathbb{R}^3$ an open set and $F : V \longrightarrow \mathbb{R}$ a differentiable function. We say that:*

- *$P \in V$ is a regular point of F if dF_P is surjective.*
- *$P \in V$ is a critical point of F if dF_P is not surjective.*
- *A critical value is the image $F(P)$ of a critical point $P \in V$.*

In this setting we get that, for a point $P \in V$, $dF_P : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is not surjective, that means that P is critical *if and only if* dF_P is everywhere zero. Note that, this happens when the gradient ∇f is zero.

As a consequence, the following proposition gives us another way to define a regular surface:

Proposition 1.2.5. *Let $U \subset \mathbb{R}^3$ be an open set and $f \in C^\infty(U)$. If $a \in \mathbb{R}$ is a regular value of f, then every connected component of the level set $f^{-1}(a) = \{P \in U | f(P) = a\}$ is a regular surface.*

Then, we have the following definition:

Definition 1.2.6 (Level set surface)**.** *Let $U \subset \mathbb{R}^3$ be an open set, and $f \in C^\infty(U)$. Every connected component of the level set $f^{-1}(a)$, where a is a regular value for f, is a regular surface.*

For example, the elliptic paraboloid $x^3 - (x^1)^2 - (x^2)^2 = a$, for $a \in \mathbb{R}$, is a regular surface. In fact, its expression is of the form $f^{-1}(a)$, where $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the the function $f(x^1, x^2, x^3) = x^3 - (x^1)^2 - (x^2)^2 \in \mathcal{C}^\infty$, and the gradient $\nabla f = (-2x^1, -2x^2, 1)$ is never vanishing. This means that $f^{-1}(a)$ is a level set surface $\forall a \in \mathbb{R}$, and so it is a regular surface.

The last characterization of a surface that we recall here is given by the following:

Proposition 1.2.7. *Every regular surface is locally a graph. In particular, if $\mathcal{S} \subset \mathbb{R}^3$ is a regular surface and $P \in \mathcal{S}$, then there exists a local parametrization $\varphi : U \longrightarrow \mathcal{S}$ in P which takes one of the following forms:*

$$\varphi(x^1, x^2) = \begin{cases} (x^1, x^2, f(x^1, x^2)), & \textit{or} \\ (x^1, f(x^1, x^2), x^2), & \textit{or} \\ (f(x^1, x^2), x^1, x^2), & \end{cases}$$

for a given function $f \in \mathcal{C}^\infty(U)$.

Let $U \subseteq \mathbb{R}^2$ be an open set and $f \in \mathcal{C}^\infty(U)$ an arbitrary function, then the parametrization:

$$\varphi(x^1, x^2) = (x^1, x^2, f(x^1, x^2)), \tag{1.4}$$

is called *Monge parametrization*. This is the only parametrization for the so called *graph* $\Gamma_F = \{(x^1, x^2, f(x^1, x^2)) \in \mathbb{R}^3 | (x^1, x^2) \in U\}$ of f, which is a regular surface.

Another important result regarding maps over a regular surface is the following.

Proposition 1.2.8. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a regular surface, $U \subseteq \mathbb{R}^2$ an open subset, and $\varphi : U \longrightarrow \mathbb{R}^3$ an immersed surface with support contained in $\mathcal{S}$. Then:*

- *$\varphi(U)$ is open in $\mathcal{S}$;*
- *if φ is injective then for all $P \in \varphi(U)$ there exist a neighborhood $W \subset \mathbb{R}^3$ of $P \in \mathbb{R}^3$ with $W \cap S \subseteq \varphi(U)$, and a map $\phi : W \longrightarrow \mathbb{R}^2$ of class $\mathcal{C}^\infty$ such that $\phi(W) \subseteq U$ and $\phi_{|W \cap S} \equiv \varphi^{-1}_{|W \cap S}$. In particular, $\varphi^{-1} : \varphi(U) \longrightarrow U$ is continuous, so φ is a local parametrization of $\mathcal{S}$.*

In other words, if we already know that $\mathcal{S}$ is a surface, to verify whether a map $\varphi : U \longrightarrow \mathbb{R}^3$ from an open subset U of $\mathbb{R}^2$ to $\mathcal{S}$ is a local parametrization. it is sufficient to check that φ is injective and that $\mathrm{d}\varphi_x$ has rank 2 for all $x \in U$. Summarizing, we may deduce the continuity of the inverse of a globally injective immersed surface φ only if we already know that the image of φ lies within a regular surface.

The following theorem will be useful later in discussing the orthogonalization of tangent vectors to the surface S and the associated parametrization:

Theorem 1.2.9. *Let $\mathcal{S}$ be a surface, and let $\varphi : U \longrightarrow \mathcal{S}$, $\psi : V \longrightarrow \mathcal{S}$ be two local parametrizations with $\Omega = \varphi(U) \cap \psi(V) \neq \emptyset$. Then the map $h = \varphi^{-1} \circ \psi_{|\psi^{-1}(\Omega)} : \psi^{-1}(\Omega) \longrightarrow \varphi^{-1}(\Omega)$ is a diffeomorphism.*

Given the characterization of a surface, we now focus on its tangent vectors and their variation on the surface, arriving to the definition of first and second fundamental form.

Definition 1.2.10 (Tangent vector). *Let $S \subset \mathbb{R}^3$ be a regular surface and $P \in \mathcal{S}$. A tangent vector to S at P is a vector of the form $\gamma'(0)$, where $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^3$ is a curve of class $\mathcal{C}^\infty$, whose support lies in S and such that $\gamma(0) = P$. The set of all possible tangent vectors to S at P is the tangent cone $T_P S$ to S at P.*

In this definition we remark that the cone C is a subset of a vector space V such that $av \in C$, $\forall a \in \mathbb{R}$ and $\forall v \in C$.

Proposition 1.2.11. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface, $P \in \mathcal{S}$ and $\varphi : U \subset \mathbb{R}^2 \longrightarrow \mathcal{S}$ a local parametrization at P with $\varphi(\hat{x}) = P$. Then $d\varphi_{\hat{x}}$ is an isomorphism between $\mathbb{R}^2$ and $T_P\mathcal{S}$. In particular, $T_P\mathcal{S} = d\varphi_{\hat{x}}(\mathbb{R}^2)$ is always a vector space of dimension 2, and $d\varphi_{\hat{x}}(\mathbb{R}^2)$ does not depend on φ but only on $\mathcal{S}$ and P.*

With this result we can define

Definition 1.2.12 (Tangent plane). *Let $S \subset \mathbb{R}^3$ be a surface and $P \in S$. The vector space $T_P S \subset \mathbb{R}^3$ is the tangent plane to S at P.*

We remark that with this definition the tangent plane is a vector subspace of $\mathbb{R}^3$, so it contains always the origin no matter where the point $P \in S$. When we want to draw the tangent plane on a point P on a surface, we actually are considering the *affine tangent plane* given by $P + T_P S$. Also, it can be shown that the affine tangent plane is the best plane approximating the surface at the point P. The previous proposition provides us a direct connection between $\mathbb{R}^2$ and $T_P S$, in sense that, since they are isomorphic, the canonical basis $\{e_1, e_2\}$ is mapped to $\{v_1, v_2\}$, which spans $T_P S$. In particular, we give the definition of tangent vector in the case of the surfaces:

Definition 1.2.13 (Tangent vectors). *Let $S \subset \mathbb{R}^3$ be a set and $P \in S$. If $\varphi : U \longrightarrow S$ is a local parametrization centered at P, and $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$, then the tangent vectors $\left.\frac{\partial}{\partial x^1}\right|_P, \left.\frac{\partial}{\partial x^2}\right|_P \in T_P S$ are defined by:*

$$\left.\frac{\partial}{\partial x^j}\right|_P = d\varphi_O(e_j) = \frac{\partial \varphi}{\partial x^j}(O) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial x^j}(O) \\ \frac{\partial \varphi_2}{\partial x^j}(O) \\ \frac{\partial \varphi_3}{\partial x^j}(O) \end{bmatrix}.$$

Proposition 1.2.14. *Let $U \subset \mathbb{R}^3$ an open set, and $a \in \mathbb{R}$ a regular value of a function $F \in \mathcal{C}^\infty(U)$. If S is a connected component of $F^{-1}(a)$ and $P \in S$, the tangent plane $T_P S$ is the subspace of $\mathbb{R}^3$ orthogonal to $\nabla F(P)$.*

Recalling the example (1.4), we obtain that the two tangent vectors:

$$\left.\frac{\partial}{\partial x^1}\right|_P = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x^1}(x^1,x^2) \end{bmatrix}, \qquad \left.\frac{\partial}{\partial x^2}\right|_P = \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial x^2}(x^1,x^2) \end{bmatrix} \tag{1.5}$$

span the tangent plane at the point P of the regular surface given by the Monge parametrization.

Definition 1.2.15 (Differential on surface)**.** *If $F : S \longrightarrow \mathbb{R}^n$ is a $\mathcal{C}^\infty$ map, and $P \in S$, the differential $dF_P : T_PS \longrightarrow \mathbb{R}^n$ of F at P is defined by setting $dF_P(v) = (F \circ \sigma)(0)$ for all $v \in T_PS$, where $\sigma : (-\varepsilon, \varepsilon)$ is an arbitrary curve in S with $\sigma(0) = P$ and $\sigma(0) = v$.*

1.2.1 Properties of Surfaces

After giving the generalities of what a surface is, we now explore its properties, in particular about measuring lengths, areas and curvatures. We begin this section with the *first fundamental form*. The euclidean $\mathbb{R}^3$ space is defined with the usual scalar product. If we consider a regular surface $\mathcal{S} \in \mathbb{R}^3$ and a point $P \in \mathcal{S}$, the tangent plane $T_P\mathcal{S}$ can be considered as a subspace of $\mathbb{R}^3$ and for this reason we can consider the inner product between two vector in the point P in this space as the scalar product induced by $\mathbb{R}^3$.

Definition 1.2.16 (First fundamental form)**.** *Let $\mathcal{S}$ be a regular surface. For all $P \in \mathcal{S}$ we denote by $\langle\cdot,\cdot\rangle_P$ the positive definite scalar product on $T_P\mathcal{S}$ induced by the canonical product of $\mathbb{R}^3$. The first fundamental form $I_P : T_P\mathcal{S} \longrightarrow \mathbb{R}$ is the positive definite quadratic form associated with this scalar product:*

$$\forall v \in T_P\mathcal{S} \qquad I_P(v) = \langle v, v\rangle_P \geq 0.$$

We know that, if a parametrization is given, it is possible to obtain the basis of the tangent plane at a point P by deriving the parametrization. The following definition is then direct:

Definition 1.2.17 (Metric coefficients)**.** *Let $\mathcal{S}$ be a regular surface and $\varphi : U \in \mathbb{R}^2 \longrightarrow \mathcal{S}$ a local parametrization of $\mathcal{S}$. Then, the metric coefficients of $\mathcal{S}$ with respect to φ are the functions $E, F, G : U \longrightarrow \mathbb{R}$ given by*

$$E(x) = \langle\partial_1, \partial_1\rangle_{\varphi(x)}, \quad F(x) = \langle\partial_1, \partial_2\rangle_{\varphi(x)} \quad \textit{and} \quad G(x) = \langle\partial_2, \partial_2\rangle_{\varphi(x)} \qquad \forall x \in U\,.$$

Since, given a parametrization φ of a surface $\mathcal{S}$, $\langle\partial_1, \partial_2\rangle_{\varphi(x)} = T_{\varphi(x)}\mathcal{S}$, the first fundamental form is completely determined by the metric coefficients, which are $\mathcal{C}^\infty$ function because they come from the differentiations of the parametrization of S. For this reason, it follows that the metric coefficients and so the first fundamental form depend strongly on the local parametrization. For this reason I_P it is an intrinsic object of the surface $\mathcal{S}$. Recalling the example (1.4) and its tangent vectors, we can calculate the metric coefficients

$$E(x) = 1 + \left(\frac{\partial f(x)}{\partial x^1}\right)^2, \quad f(x) = \frac{\partial f(x)}{\partial x^1}\frac{\partial f(x)}{\partial x^2}, \quad G(x) = 1 + \left(\frac{\partial f(x)}{\partial x^2}\right)^2,$$

and so the first fundamental form

$$I_P = \begin{bmatrix} 1 + \left(\frac{\partial f(x)}{\partial x^1}\right)^2 & \frac{\partial f(x)}{\partial x^1}\frac{\partial f(x)}{\partial x^2} \\ \frac{\partial f(x)}{\partial x^1}\frac{\partial f(x)}{\partial x^2} & 1 + \left(\frac{\partial f(x)}{\partial x^2}\right)^2 \end{bmatrix}.$$

Definition 1.2.18 (Angle). *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface, and $P \in \mathcal{S}$. A determination of the angle between two tangent vectors $v_1, v_2 \in T_P\mathcal{S}$ is a $\theta \in \mathbb{R}$ such that*

$$\cos\theta = \frac{\langle v_1, v_2 \rangle}{\sqrt{I_P(v_1) I_P(v_2)}} = \frac{F}{\sqrt{EG}}.$$

Moreover, if $\sigma_1, \sigma_2 : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{S}$ are curves with $\sigma_1(0) = \sigma_2(0) = P$, we shall call the angle between σ_1 and σ_2 at P the angle between $\sigma_1'(0)$ and $\sigma_2'(0)$.

With the concept of angle, we can give here important definitions regarding *orthogonality*. In the canonical Euclidean system the axes are orthogonal. We can extend this property to the basis of tangent planes on a surface.

Definition 1.2.19 (Orthogonal parametrization). *We say that a local parametrization φ of a surface $\mathcal{S}$ is orthogonal if its coordinate curves meet at a right angle, that is, if $\partial_1|_P$ and $\partial_2|_P$ are orthogonal for each P in the image of φ.*

The parametrization of example (1.4) is not orthogonal, in fact in general the metric coefficient $F \neq 0$. An important definition that follows from the concept of orthogonality is the normal vector or in general normal vector field:

Definition 1.2.20 (Normal vector field). *A normal vector field on a surface $\mathcal{S} \subset \mathbb{R}^3$ is a map $N : \mathcal{S} \longrightarrow \mathbb{R}^3$ of class $\mathcal{C}^\infty$ such that $N(P)$ is orthogonal to $T_P S \quad \forall P \in \mathcal{S}$; If, moreover, $||N|| = 1$ we shall say that N is normal versor field to $\mathcal{S}$.*

In particular, we note that since we are in $\mathbb{R}^3$, the knowledge of one between the tangent plane or the normal vector field, completely defines the other one through the condition of orthogonality. For example, recalling example (1.4), from the tangent vectors we calculate the normal vector field in the following way:

$$N \circ \varphi = \frac{\partial_1 \times \partial_2}{||\partial_1 \times \partial_2||} = \frac{1}{\sqrt{1 + \left(\dfrac{\partial f}{\partial x^1}\right)^2 + \left(\dfrac{\partial f}{\partial x^2}\right)^2}} \begin{bmatrix} -\dfrac{\partial f}{\partial x^1} \\ -\dfrac{\partial f}{\partial x^2} \\ 1 \end{bmatrix} \tag{1.6}$$

In addition, with the definition of normal vector field we can determine an external and an internal face of the surface, by considering the sign of the normal vector, thus creating an oriented object whenever we can identify the normal vector field. Since N depends on the parametrization of the surface, if another chart has a different N, the orientability of the surface is not guaranteed anymore. In fact:

Proposition 1.2.21. *A surface $\mathcal{S} \subset \mathbb{R}^3$ is orientable if and only if there exists a normal versor field on $\mathcal{S}$.*

Moreover, it is possible to define the orientation by:

Definition 1.2.22 (Gauss map). *Let $\mathcal{S} \subset \mathbb{R}^3$ be an oriented surface. The Gauss map of $\mathcal{S}$ is the normal versor field $N : \mathcal{S} \longrightarrow \mathcal{S}^2$ that identifies the given orientation.*

When such a normal vector field exists, it can give a measure the curvature of the surface. In fact, the curvature is an indicator of how much the tangent plane varies between close points on the surface, and this variation can also be determined by studying the variation of the normal versor. Then, directly from the Gauss map, we have:

Definition 1.2.23 (Shape Operator/Weingarten map)**.** *Let $\mathcal{S} \subset \mathbb{R}^3$ be an oriented surface and $P \in \mathcal{S}$. Then for each tangent vector $v \in T_P\mathcal{S}$, we define the Shape operator/Weingarten map $\mathcal{B}_P : T_P\mathcal{S} \longrightarrow T_P\mathcal{S}$:*

$$\mathcal{B}_P(v) = -dN_P(v). \tag{1.7}$$

As said before, the Gauss map determines uniquely the tangent planes to the surface, in the same way, the shape operator tells us how the normal to the tangent plane, thus the tangent plane itself, varies in every direction. So, it tells us the way S is curving in all directions at P. From example in eq. (1.4), we calculate the differential of the Gauss map by computing:

$$\begin{aligned} dN_P(\partial_j) &= \frac{\partial(N\circ\varphi)}{\partial x^j}(x) \\ &= \frac{1}{(1+\|\nabla f\|^2)^{3/2}} \begin{bmatrix} \frac{\partial f}{\partial x^1}\frac{\partial f}{\partial x^2}\frac{\partial^2 f}{\partial x^j \partial x^2} - \left(1+\left(\frac{\partial f}{\partial x^2}\right)^2\right)\frac{\partial^2 f}{\partial x^j\partial x^1} \\ \frac{\partial f}{\partial x^1}\frac{\partial f}{\partial x^2}\frac{\partial^2 f}{\partial x^j \partial x^1} - \left(1+\left(\frac{\partial f}{\partial x^1}\right)^2\right)\frac{\partial^2 f}{\partial x^j\partial x^2} \\ 0 \end{bmatrix} \end{aligned}$$

Definition 1.2.24 (Second fundamental form)**.** *Let $\mathcal{S} \subset \mathbb{R}^3$ be an oriented surface, with Gauss map $N : \mathcal{S} \longrightarrow \mathcal{S}^2$. The second fundamental form of $\mathcal{S}$ is the quadratic form $II_P : T_P\mathcal{S} \longrightarrow \mathbb{R}$ given by*

$$II_P(v) = -\langle dN_P(v), v\rangle_P, \qquad \forall v \in T_P\mathcal{S}.$$

In particular, given a local parametrization $\varphi : U \subset \mathbb{R}^2 \longrightarrow \mathcal{S}$ of the surface $\mathcal{S}$ and a point $\varphi(x^1, x^2) = P \in \mathcal{S}$, it is known that every vector of the tangent space can be written as a linear combination of the two tangent vectors obtained from the parametrization: $v = v^{(1)}\partial_1 + v^{(2)}\partial_2, \quad \forall v \in T_P\mathcal{S}$. So, we can define the *form coefficient* of the second fundamental form by using the basis vectors in the following way:

$$L(x) = -\langle dN_{\varphi(x)}(\partial_1), \partial_1\rangle_{\varphi(x)}, \quad M(x) = -\langle dN_{\varphi(x)}(\partial_1), \partial_2\rangle_{\varphi(x)}, \quad N(x) = -\langle dN_{\varphi(x)}(\partial_2), \partial_2\rangle_{\varphi(x)}$$

with $L, M, N : U \longrightarrow \mathbb{R}$. Thus, the second fundamental form is completely determined. In fact, $\forall x \in U, v \in T_P\mathcal{S}$ we get:

$$II_{\varphi(x)}(v) = L(x)\left(v^{(1)}\right)^2 + 2M(x)v^{(1)}v^{(2)} + N(x)\left(v^{(2)}\right)^2.$$

For example, considering again the Monge parametrization from eq. (1.4), the coefficients of the second fundamental form are:

$$L(x) = \frac{1}{\sqrt{1+||\nabla f||^2}}\frac{\partial^2 f}{\partial(x^1)^2}, \quad M(x) = \frac{1}{\sqrt{1+||\nabla f||^2}}\frac{\partial^2 f}{\partial x^1 \partial x^2}, \quad N(x) = \frac{1}{\sqrt{1+||\nabla f||^2}}\frac{\partial^2 f}{\partial(x^2)^2},$$

so, for a generic point $\varphi(x) = P \in T_P\mathcal{S}$ and a generic vector $v \in T_P\mathcal{S}$, the second fundamental form is

$$\begin{aligned} II_P(v) &= -\langle dN_P(v), v\rangle_P \\ &= -\langle dN_P(\partial_1), \partial_1\rangle_P \left(v^{(1)}\right)^2 - 2\langle dN_P(\partial_1), \partial_2\rangle_P v^{(1)} v^{(2)} - \langle dN_P(\partial_2), \partial_2\rangle_P \left(v^{(2)}\right)^2 \\ &= -\langle dN_P(\partial_1), \partial_1\rangle_P \left(v^{(1)}\right)^2 - 2\langle dN_P(\partial_1), \partial_2\rangle_P v^{(1)} v^{(2)} - \langle dN_P(\partial_2), \partial_2\rangle_P v \\ &= L(x) \left(v^{(1)}\right)^2 + 2M(x) v^{(1)} v^{(2)} + N(x) \left(v^{(2)}\right)^2 \\ &= \frac{1}{\sqrt{1 + ||\nabla f||^2}} \left[\frac{\partial^2 f}{(\partial x^1)^2} \left(v^{(1)}\right)^2 + 2\frac{\partial^2 f}{\partial x^1 \partial x^2} v^{(1)} v^{(2)} + \frac{\partial^2 f}{(\partial x^2)^2} \left(v^{(2)}\right)^2\right] \end{aligned}$$

Finally, we define here how to compute the area of bounded regions of the surface, and the extension to the definition of integrals over regions of the surface. We begin with some definitions about the regions over the surface which will allow us to define the integrals.

Definition 1.2.25 (Regular region). *A regular region $R \subset \mathcal{S}$ is a connected compact subset of $\mathcal{S}$ obtained as the closure of its interior $\mathring{R}$ and whose boundary is parametrized by finitely many curvilinear polygons with disjoint supports. If $\mathcal{S}$ is compact, then $R = \mathcal{S}$ is a regular region without boundary.*

Definition 1.2.26 (Partition of regular region). *Let $R \subseteq \mathcal{S}$ be a regular region of a surface $\mathcal{S}$. A partition of R is a finite family $\mathcal{R} = \{R_1, \dots R_n\}$ of regular regions contained in R, such that $R = \cup_{i=1} R_i$ and $R_i \cap R_j \subseteq \partial R_i \cap \partial R_j$, for $i, j = 1, \dots, n$ and $i \neq j$. The diameter diam R of a partition is the maximum of the diameters of the elements of $\mathcal{R}$. A pointed partition of R is a pair $(\mathcal{R}, P)$ given by a partition $\mathcal{R}$ of R and a n-tuple $P = \{p_1, \dots, p_n\}$ of points of R such that $p_i \in R_i$, $i = 1, \dots, n$.*

Definition 1.2.27 (Orthogonal projection). *Let $R \subseteq \mathcal{S}$ be a regular region of a regular surface $\mathcal{S}$ and $(\mathcal{R}, P)$ a pointed partition of R. For all $R_i \in \mathcal{R}$, denote by $\pi_i(R_i)$ the orthogonal projection of R_i on the affine tangent plane $p_i + T_{p_i}\mathcal{S}$. The area of the pointed partition is defined as:*

$$\text{Area}(\mathcal{R}, P) = \sum_i \text{Area}(\pi_i(R_i)).$$

The region R is rectifiable if the limit

$$\mathcal{A}_R = \lim_{diam\ R \to 0} \text{Area}(\mathcal{R}, P)$$

exists and is finite. If it is the case, the limit is the area of R.

Thanks to these definitions, we can now state some results regarding the integration of a function over a surface.

Theorem 1.2.28. *Let $R \subseteq \mathcal{S}$ be a regular region contained in the image of a local parametrization $\varphi : U \longrightarrow \mathcal{S}$. Then, R is rectifiable and its area is*

$$\mathcal{A}_R = \int_{\varphi^{-1}(R)} \sqrt{EG - F^2} dx.$$

Moreover we have that:

Lemma 1.2.29. *Given $\varphi : U \longrightarrow \mathcal{S}$ a local parametrization of a surface $\mathcal{S}$, then:*

$$\|\partial_1 \times \partial_2\| = \sqrt{EG - F^2}.$$

Furthermore, if $\psi : V \longrightarrow \mathcal{S}$ is another local parametrization with $W = \psi(V) \cap \varphi(U) \neq \emptyset$, and $f = \psi^{-1} \circ \varphi|_{\varphi(U)^{-1}}$, then

$$(\partial_1 \times \partial_2)|_{\varphi(x)} = \det(Jf)(x)(\tilde{\partial}_1 \times \tilde{\partial}_2)|_{\psi \circ f(x)},$$

for all $x \in \varphi^{-1}(W)$, where $\{\tilde{\partial}_1, \tilde{\partial}_2\}$ is the basis induced by ψ.

Now it is possible to define the integral of a function defined over a regular region of a surface:

Definition 1.2.30 (Integral on a surface)**.** *Let $R \subseteq \mathcal{S}$ be a regular region contained in the image of a local parametrization $\varphi : U \longrightarrow \mathcal{S}$ of a regular surface $\mathcal{S}$, and $f : R \longrightarrow \mathbb{R}$ a continuous function. The integral of f on R is given by:*

$$\int_R f = \int_{\varphi^{-1}(R)} (f \circ \varphi)\sqrt{EG - F^2} dx.$$

From this the famous *Stokes theorem* follows. In fact, recalling definition 1.1.16 of a differential form, we get that:

Theorem 1.2.31. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface with smooth boundary $\partial\mathcal{S}$ and ω a 1-differential form with compact support on $\mathcal{S}$. Then:*

$$\int_{\partial\mathcal{S}} \omega = \int_{\mathcal{S}} d\omega.$$

1.3 Coordinate Systems

Before proceeding, we make a parenthesis regarding the system frames that can be used to express quantities on the surface, as they will come in handy later. In particular, we need to explore the difference between the knowledge of physical, covariant and contravariant vectors and tensors, so that we can move from one to the other exploiting and ensuring the invariance of the coordinates.

First, we recall that given a n-dimensional vector space V and a covariant basis of V $\{v_1, v_2, \ldots v_n\}$, every vector $v \in V$ can be expressed in a unique way as a linear combination of the elements of the basis, i.e., $v = \sum_{i=1}^n \alpha^i v_i$. We call the coefficients α^i the *contravariant components* of the vector v, and we denote them with the superscript. Whereas, if we consider a contravariant vector v^*, the dual of v, its components are called *covariant components* and we denote them with the subscript: $v^* = \sum_{i=1}^n \beta_i (v^*)^i$. So, with this identifications, given a regular surface $\mathcal{S}$ and a point $P \in \mathcal{S}$, we will write vectors of the tangent space with coefficient with superscripts. Instead, vectors of the cotangent space will have the coefficients with the subscripts:

$$\begin{aligned} v &= v^1 t_1 + v^2 t_2, \qquad \forall v \in T_P\mathcal{S}, \\ \omega &= \omega_1 \alpha^1 + \omega_2 \alpha^2, \qquad \forall \alpha \in T_P^*\mathcal{S}. \end{aligned}$$

with $\{t_1, t_2\}$ and $\{\alpha^1, \alpha^2\}$ basis of $T_P\mathcal{S}$ and $T_P^*\mathcal{S}$ respectively.

At this point, we explore the connection between the canonical reference frame and the local one. Every point of $\mathbb{R}^3$ can be written through the canonical basis $\{e_1, e_2, e_3\}$, in the usual *Global Cartesian coordinate System* (GCS) in the coordinates (x^1, x^2, x^3). The same can be said for vectors: $\forall \mathbf{v} \in \mathbb{R}^3$, $\mathbf{v} = \sum_{i=1}^{3} v_{(i)} e_i$, and we call $v_{(i)}$ the *physical components* of $\mathbf{v}$ relative to the canonical basis, e.g. these components would be the one that a physical instrument would detect. If we consider a regular surface $\mathcal{S}$ embedded in $\mathbb{R}^3$, each point $P \in \mathcal{S}$ can be described through the the parametrization of the surface $\varphi : U \subset \mathbb{R}^2 \longrightarrow \mathcal{S}$, with respect to local coordinates (s_1, s_2). As said before, we can compute the reference basis vector for the tangent space to the surface at the point P as $\left\{\frac{\partial \varphi}{\partial s^1}, \frac{\partial \varphi}{\partial s^2}\right\}$, and we can complete this set of vectors to a basis of $\mathbb{R}^3$ with $\frac{\partial_1 \times \partial_2}{\|\partial_1 \times \partial_2\|}$. We call this reference as *Local Curvilinear coordinate System* (LCS) and we will associate it with the local coordinates (s_p^1, s_p^2, s_p^3), as we can think of them as three vector "attached" to a any point P on the surface which depend on the local parametrization. Now, to derive the connection between these two systems, we follow the work of Dziuk and Elliott[16] to arrive at the so called *Fermi Coordinates*, which represents the global coordinates in a neighborhood of the surface. First, we define a distance function and a lemma for which it will be direct the result. Assuming that there exists $G \subset \mathbb{R}^3$ bounded and open with exterior normal ν, such that $\mathcal{S} = \partial G$, we define the *Oriented distance* for $\mathcal{S}$ as

$$d(x) = \begin{cases} \inf_{y \in \mathcal{S}} |x - y| & x \in \mathbb{R}^3 \setminus \bar{G}, \\ -\inf_{y \in \mathcal{S}} |x - y| & x \in G. \end{cases}$$

We remark that d is globally Lipschitz-continuous with Lipschitz constant 1.

Lemma 1.3.1. *We define $W_\varepsilon = \{x \in \mathbb{R}^3 | \, |d(x)| < \varepsilon\}$. Then $d \in \mathcal{C}^k(W_\varepsilon)$, and for every point $x \in W_\varepsilon$ there exists a unique point $a(x) \in \mathcal{S}$ such that*

$$x = a(x) + d(x)\nu(a(x)).$$

Moreover, we have that

$$\nabla d(x) = \nu(a(x)), \qquad |\nabla d(x)| = 1, \qquad \text{for } x \in W_\varepsilon.$$

With this lemma we get a connection between points in the so call *tubular* neighborhood W_ε of $\mathcal{S}$ and points on the surface. In particular, for every point in W_ε there exists only one correspondent point in the surface. This is possible as long as the normal lines to the surface do not have any intersection that belongs to the tubular neighborhood. The following proposition from [14] ensure this fact:

Proposition 1.3.2. *Let $\mathcal{S}$ be a regular surface and $\varphi : U \subset \mathbb{R}^2 \longrightarrow \mathcal{S}$ a local parametrization centered at $P \in \mathcal{S}$. Then, there exists a neighborhood $W_\varepsilon \subset \varphi(U)$ of P in $\mathcal{S}$ and positive constant $\varepsilon > 0$ such that the segments of the normal lines passing through points $Q \in W$, centered at Q and with length 2ε, are disjoint.*

Therefore, we have obtained a connection between the GCS and the LCS which result to be a diffeomorphism thank to the previous two lemmas:

$$\begin{aligned} \text{GCS} &\longrightarrow \text{LCS} \\ \varphi_P : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x_P &\longmapsto s_P \end{aligned} \qquad\qquad \begin{aligned} \text{LCS} &\longrightarrow \text{GCS} \\ \psi_P := \varphi_P^{-1} : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3, \\ s_P &\longmapsto x_P \end{aligned}$$

In addition, to get an explicit connection between these two systems that will be useful for the intrinsic definitions, we introduce the notation of the *scale factor*. Given a $\mathbb{R}^3$ reference systems different from the usual Cartesian one, for example the previous LCS, for each coordinate is associated a *scale factor* h_i which gives a measure of how a change in coordinate changes the position of a point. More specifically, given a set of orthogonal curvilinear coordinates, for which each direction of the coordinates is orthogonal to the others:

$$\begin{cases} u = u(x^1, x^2, x^3) \\ v = v(x^1, x^2, x^3) \\ w = w(x^1, x^2, x^3) \end{cases}$$

the scale factor is calculated as

$$\begin{aligned} h_u &= \sqrt{\left(\frac{\partial x^1}{\partial u}\right)^2 + \left(\frac{\partial x^2}{\partial u}\right)^2 + \left(\frac{\partial x^3}{\partial u}\right)^2} \\ h_v &= \sqrt{\left(\frac{\partial x^1}{\partial v}\right)^2 + \left(\frac{\partial x^2}{\partial v}\right)^2 + \left(\frac{\partial x^3}{\partial v}\right)^2} \\ h_w &= \sqrt{\left(\frac{\partial x^1}{\partial w}\right)^2 + \left(\frac{\partial x^2}{\partial w}\right)^2 + \left(\frac{\partial x^3}{\partial w}\right)^2} \end{aligned}$$

For example, given the *spherical polar coordinates*

$$\begin{cases} x^1 = \rho \sin\varphi \cos\theta \\ x^2 = \rho \sin\varphi \sin\theta \\ x^3 = \rho \cos\varphi \end{cases}$$

the scale factors are:

$$\begin{aligned} h_\rho &= \sqrt{(\sin\varphi\cos\theta)^2 + (\sin\varphi\sin\theta)^2 + (\cos\varphi)^2} = 1 \\ h_\varphi &= \sqrt{(\rho\cos\varphi\cos\theta)^2 + (\rho\cos\varphi\sin\theta)^2 + (-\rho\sin\varphi)^2} = \rho \\ h_\theta &= \sqrt{(-\rho\sin\varphi\sin\theta)^2 + (\rho\sin\varphi\cos\theta)^2 + (0)^2} = \rho\sin\varphi \end{aligned}$$

Therefore, any vector in $\mathbb{R}^3$ can be read through the Cartesian system, $e_{(i)}$, with the physical components or any other different reference system, $\hat{e}_{(i)}$, for which the components change according with the scale factors:

$$v = v_{(i)} e_{(i)} = h_{(i)} v^i \hat{e}_{(i)}.$$

So we get the relation between the physical and contravariant components:

$$v_{(i)} = h_{(i)} v^i. \tag{1.8}$$

Using (1.1.22), it follows in a direct way the relationship between the physical and covariant coordinates:

$$v_{(i)} = h_{(i)} g^{ij} v_j. \tag{1.9}$$

In the setting of a regular surface $\mathcal{S} \subset \mathbb{R}^3$, we want to write the differential operators not with the usual canonical system, GCS, but with the LCS given by the parametrization of the surface. So, we have that:

Proposition 1.3.3. *Let* (s^1, s^2) *be the curvilinear coordinates on* $\mathcal{S}$ *and* $\mathcal{G}_{\mathcal{S}}$ *the associated metric tensor. Let* $f : \mathcal{S} \longrightarrow \mathbb{R}$ *a scalar differentiable function on* $\mathcal{S}$*,* $\mathbf{u} : \mathcal{S} \longrightarrow \mathbb{R}^2$ *a contravariant (tangent) vector field on* $\mathcal{S}$*, given by* $\mathbf{u} = u^1\mathbf{t}_1 + u^2\mathbf{t}_2$ *and* $\mathbb{T} : \Omega \longrightarrow \mathbb{R}^{2\times2}$ *a rank-2 contravariant (tangent) tensor given by,* $\mathbb{T} = \{\tau^{ij}\}$*. Then, the* ***intrinsic differential operators*** *on* $\mathcal{S}$ *expressed in the local curvilinear coordinate system are given by the following expressions:*

- *The* ***intrinsic gradient*** *of* f *is:*

$$\nabla_{\mathcal{G}_S} f = \mathcal{G}_S^{-1} \nabla f = g^{ij} \frac{\partial f}{\partial s^i}. \tag{1.10}$$

- *The* ***intrinsic divergence*** *of* f *is:*

$$\nabla_{\mathcal{G}_S} \cdot f = \frac{1}{\sqrt{\det \mathcal{G}_S}} \nabla \cdot \left(\sqrt{\det \mathcal{G}_S} f \right). \tag{1.11}$$

- *The k-th component of the* ***intrinsic divergence*** *of* $\mathbb{T}$ *is:*

$$(\nabla_{\mathcal{G}_S} \mathbb{T})^j = \left(\nabla_{\mathcal{G}_S} \right)_i \tau^{ik} = \frac{1}{\sqrt{\det \mathcal{G}_S}} \frac{\partial}{\partial s^i} \left(\det \mathcal{G}_S \tau^{ik} \right) + \Gamma^k_{ij} \tau^{ij} \tag{1.12}$$

- *The k-th component of the* ***intrinsic curl*** *of* $\mathbf{u}$ *is:*

$$\begin{aligned} (\nabla \times_{\mathcal{G}_S} \mathbf{v})^k &= \frac{1}{\sqrt{\det \mathcal{G}_S}} \sum_{ij} \varepsilon^{ijk} \partial_i \left(h_{(j)} u_{(j)} \right) \\ &= \frac{1}{\sqrt{\det \mathcal{G}_S}} \sum_{ij} \varepsilon^{ijk} \partial_i \left(h^2_{(j)} u^j \right). \end{aligned} \tag{1.13}$$

- *The* ***intrinsic Laplace-Beltrami operator*** *of* f *is:*

$$\Delta_{\mathcal{G}_S} f = \nabla_{\mathcal{G}_S} \cdot \nabla_{\mathcal{G}_S} f = \frac{1}{\sqrt{\det \mathcal{G}_S}} \frac{1}{\partial s^i} \left(\sqrt{\det \mathcal{G}_S}\, g^{ij} \frac{\partial f}{\partial s^j} \right) \tag{1.14}$$

We recall that the $h_{(i)}$ are the elements of the first fundamental form, which depends only on the parameters of the parametrization. Therefore, the $h_{(i)}$ are function of s_1 and s_2.

We state two important results concerning intrinsic divergence of theorem 1.2.31 and the intrinsic version of the Green's lemma, that will be useful later when solving PDEs on surfaces.

Lemma 1.3.4. *Let* $\mathcal{S} \subset \mathbb{R}^3$ *be a surface with smooth boundary* $\partial\mathcal{S}$ *and* X *be a continuous differentiable vector field. Then:*

$$\int_{\mathcal{S}} \nabla_{\mathcal{G}} \cdot X \, ds = \int_{\partial\mathcal{S}} \langle X, \mu \rangle_{\mathcal{G}} \, d\sigma$$

where $\mu : \mathcal{S} \longrightarrow \mathbb{R}^2$ *denotes the vector tangent to* $\mathcal{S}$ *and normal to* $\partial\mathcal{S}$ *with components written with respect to the local reference frame (i.e.* $\mu = \mu^1\partial_1 + \mu^2\partial_2$*), and* ds *and* $d\sigma$ *are the surface area measure and the curve length measure, respectively.*

Lemma 1.3.5. *Let* $\mathcal{S} \subset \mathbb{R}^3$ *be a surface with smooth boundary* $\partial\mathcal{S}$ *and* $f, g \in \mathcal{C}^2(\overline{\mathcal{S}})$ *be continuously differentiable functions over* $\overline{\mathcal{S}}$*. Then:*

$$\int_{\mathcal{S}} \langle \nabla_{\mathcal{G}} f, \nabla_{\mathcal{G}} g \rangle_{\mathcal{G}} \, ds = -\int_{\mathcal{S}} \Delta_{\mathcal{G}} f \, g \, ds + \int_{\partial\mathcal{S}} \langle \nabla_{\mathcal{G}} f, \mu \rangle_{\mathcal{G}} \, g \, d\sigma$$

where $\mu : \mathcal{S} \longrightarrow \mathbb{R}^2$ *denotes the vector tangent to* $\mathcal{S}$ *and normal to* $\partial\mathcal{S}$ *with components written with respect to the local reference frame, and* ds *and* $d\sigma$ *are the surface area measure and the curve length measure, respectively.*

1.3.1 Orthogonal reference system

In this last section, we will set-up the geometrical framework used in the following chapters. We make use of a global Monge parametrization to obtain a set of tangent vectors that we orthogonalize, in order to obtain an orthogonal local reference system. As a consequence, the resulting metric is diagonal, yielding to simplified expressions for the intrinsic differential operators.

Recalling the *Monge parametrization* given in eq. (1.4), in the local coordinates $(x^1, x^2) \subset \mathbb{R}^2$, we first compute the tangent vectors $\{\hat{t_1}, \hat{t_2}\}$ by eq. (1.5), and then we orthogonalize them by fixing $\hat{t_1}$ and applying Gram-Schmidt to $\hat{t_2}$. The two orthogonal tangent vectors $\{t_1, t_2\}$ form a basis of the tangent plane at the point $P \in \mathcal{S}$. We can extend this basis to be reference frame in $\mathbb{R}^3$, our LCS, by extending the basis with the normal vector ν to the surface. The vector ν is calculated in eq. (1.6). Explicitly we have:

$$t_1(P) = \hat{t_1} = \left[1; 0; \partial_{x^1} f\right] \tag{1.15}$$

$$t_2(P) = \hat{t_2} - \frac{\langle t_1, \hat{t_2}\rangle}{\langle t_1, t_1\rangle} t_1 = \left[-\frac{\partial_{x^1} f\, \partial_{x^2} f}{1 + (\partial_{x^1} f)^2}; 1; \frac{\partial_{x^2} f}{1 + (\partial_{x^1} f)^2}\right] \tag{1.16}$$

$$\nu(P) = \left[\frac{-\partial_{x^1} f}{\|t_1 \times t_2\|}; \frac{-\partial_{x^2} f}{\|t_1 \times t_2\|}; \frac{1}{\|t_1 \times t_2\|}\right] \tag{1.17}$$

with $\partial_{x^1} f = \dfrac{\partial f}{\partial x^1}$, $\partial_{x^2} f = \dfrac{\partial f}{\partial x^2}$ and $\|t_1 \times t_2\| = \sqrt{1 + (\partial_{x^1} f)^2 + (\partial_{x^2} f)^2}$. We will denote the $\mathbb{R}^3$ local coordinates (s^1, s^2, s^3).

The associated metric tensor is the diagonal matrix:

$$\mathcal{G}_3 = \begin{bmatrix} \|t_1(P)\|^2 & 0 & 0 \\ 0 & \|t_2(P)\|^2 & 0 \\ 0 & 0 & \|\nu(P)\|^2 \end{bmatrix} = \begin{bmatrix} h_{(1)}^2 & 0 & 0 \\ 0 & h_{(2)}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which can be simplified to the first 2×2 block

$$\mathcal{G}_2 = \begin{bmatrix} \|t_1(P)\|^2 & 0 \\ 0 & \|t_2(P)\|^2 \end{bmatrix} = \begin{bmatrix} h_{(1)}^2 & 0 \\ 0 & h_{(2)}^2 \end{bmatrix}$$

if only tangential quantities are considered.

The intrinsic differential operators of proposition 1.3.3 can be simplified when written with respect to the previous metric tensors:

- The *k-th* component of the *gradient* of a scalar function f is:

$$(\nabla_{\mathcal{G}} f)^k = \frac{1}{h_{(k)}^2} \frac{\partial f}{\partial s^k}. \tag{1.18}$$

- The *divergence* of a vector $\mathbf{u}$ is:

$$\nabla_{\mathcal{G}} \cdot \mathbf{u} = \frac{1}{h_{(1)} h_{(2)}} \nabla \cdot \left(h_{(1)} h_{(2)} \mathbf{u}\right). \tag{1.19}$$

- The *k-th* component of the *divergence* of the *2-tensor* $\mathbb{T}$ is:

$$(\nabla_{\mathcal{G}} \mathbb{T})^k = \nabla_{\mathcal{G}} \cdot \tau^{(\cdot k)} + \frac{1}{h_{(k)}} \sum_i \left((\tau^{ik} + \tau^{ki}) \frac{\partial h_{(k)}}{\partial s^i} - \tau^{ii} \frac{h_{(i)}}{h_{(k)}} \frac{\partial h_{(i)}}{\partial s^k} \right). \tag{1.20}$$

- The *k-th* component of the intrinsic curl of a vector **u** is:

$$(\nabla \times_{\mathcal{G}} \mathbf{v})^k = \frac{h_{(k)}}{h_{(1)}h_{(2)}} \sum_{ij} \varepsilon^{ijk} \partial_i \left(h_{(j)}^2 u^j \right). \tag{1.21}$$

- The *Laplace-Beltrami* operator of a scalar function f is:

$$\Delta_{\mathcal{G}} f = \frac{1}{h_{(1)}h_{(2)}} \left[\sum_i \frac{1}{\partial s^i} \left(\frac{h_{(1)}h_{(2)}}{h_{(i)}^2} \frac{\partial f}{\partial s^i} \right) \right]. \tag{1.22}$$

In addition we get a simplified formula for the gradient of a vector field:

$$(\nabla_{\mathcal{G}} \mathbf{u})_{(ij)} = h_{(i)} h_{(j)} (\nabla_{\mathcal{G}} \mathbf{u})^{ij} = \begin{cases} \dfrac{\partial u^i}{\partial s^i} + \dfrac{1}{h_{(i)}} \langle \mathbf{u}, \nabla_{\mathcal{G}} h_{(i)} \rangle_{\mathcal{G}} & \text{if } i = j \\ \dfrac{1}{h_{(i)}} \left(\dfrac{\partial (h_{(j)} u^j)}{\partial s^i} - \dfrac{u^i h_{(i)}}{h_{(j)}} \dfrac{\partial h_{(i)}}{\partial s^j} \right) & \text{if } i \neq j \end{cases} \tag{1.23}$$

Remark 1. *In our work we will use theMonge parametrization* $\phi : (x^1, x^2) \in U \subseteq \mathbb{R}^2 \longrightarrow \phi(U) \subset \mathcal{S} \subset \mathbb{R}^3$*, but as said before, we will consider the reference system obtained by orthogonalizing the tangent vectors. This implies that we will obtain derivations associated with the tangent vectors that are different from the initial ones, which are identified by the canonical derivatives* $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$*, recalling* (1.2.11)*. These new differentiations are associated to a different local chart, so to a different local parametrization. Then, recalling* (1.2.9)*, we get that the map between these two local parametrization is a diffeomorphism, which is given by the restriction of the two orthogonal tangent vector to their first 2 coordinates, obtaining a matrix that represents the Jacobian of the changes through the orthogonalization of the two coordinate axis* x^1, x^2 *to the local coordinates* s^1, s^2*:*

$$W := \begin{bmatrix} 1 & -\dfrac{\partial_{x^1} f \, \partial_{x^2} f}{1 + (\partial_{x^1} f)^2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x^1}{\partial s^1} & \dfrac{\partial x^1}{\partial s^2} \\ \dfrac{\partial x^2}{\partial s^1} & \dfrac{\partial x^2}{\partial s^2} \end{bmatrix}.$$

So, W acts as the Jacobian of the diffeomorfism between these two local maps, that we call

$$\begin{aligned} \bar{\psi} : V \subset \mathbb{R}^2 &\longrightarrow U \subset \mathbb{R}^2 \\ (s^1, s^2) &\longmapsto \bar{\psi}(s^1, s^2) = (x^1, x^2). \end{aligned}$$

This will be essential when we will need to calculate the partial differentiation of a function depending on the local coordinates s^1, s^2*, for example the metric, and we will use this change within the chain-rule:*

$$\frac{\partial h_1(s^1, s^2)}{\partial s^1} = \frac{\partial \tilde{h}_1(\bar{\psi}(s^1, s^2))}{\partial x^1} \frac{\partial x^1}{\partial s^1} + \frac{\partial \tilde{h}_1(\bar{\psi}(s^1, s^2))}{\partial x^2} \frac{\partial x^2}{\partial s^1}$$

where $\tilde{h}_i$*,* $i = 1, 2$*, is the metric defined through the initial Monge parametrization.*

2

Derivation of the Vector Laplacian

The purpose of this section is to show the derivation of the vector Laplacian that generalizes the scalar Laplacian. Most of the results are, however, focused on problems with scalar functions, as for example Dziuk work [16], which simplifies the differential calculus since the covariant derivative of a vector field, or more generally a tensor field, is not needed. In this case the coupling between the geometry of the surface and the PDE is weak and thus allows to solve these problems with small modifications of established numerical approaches in flat space. But there are also many physical models on surfaces that involve vector functions and in these cases, the relation between the surface and and the vector or tensor-valued surface PDEs is much stronger and much more difficult to study.

So, following the paper [29], we show an example of derivation of a vector Laplacian PDE, but unlike the paper we will use an intrinsic approach, starting from the physical model of the Nematic Liquid Crystal, whose model and energy can be found in [34]. In particular, this derivation turns out to be equivalent to the one carried out in the paper [29], in which however is used an embedded approach that makes use of the projection on the surface to define all the necessary operators.

We start considering the Frank-Oseen energy associated to the system in a domain of $\Omega \subset \mathbb{R}^3$, and after applying initial simplifications to the formula, we will proceed by writing all the operators contained in the energy in terms of the local coordinates of the LCS, and then obtain the formulation on the surface by sending the third coordinate of the LCS to zero, i.e. applying the limit for the normal direction to the surface tending to zero.

Then we minimize the energy on the surface, through the method of the L^2 gradient flow, obtaining the equilibrium equation, which will be a PDE containing the vector Laplacian. Finally, while there are several natural Laplacians acting on vector fields on surfaces, we consider the *deRham Laplacian*, which will be derived from the energy analysis, and through the Weitzenböck identity we will switch to the *Bochner Laplacian*, as it is better known and easier to deal with, and then we will study the obtained PDE through the ISFEM method in the next chapter.

Before we begin, we give below the definitions of *deRham*, *Bochner Laplacian* and the relationship between them, *Weitzenböck identity*, which will be used at the end of the derivation of the equation on the surface.

Vector Laplacians and their relationship

Following [4], we give the definition of *Bochner Laplacian*:

Definition 2.0.1 (Bochner Laplacian)**.** *Let $\mathcal{S}$ be a compact, oriented manifold equipped with a metric. Let E be a vector bundle over $\mathcal{S}$ equipped with a fiber metric and a compatible connection ∇. This connection gives rise to a differential operator*

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*\mathcal{S} \otimes E)$$

where $\Gamma(E)$ denotes the smooth sections of E, and $T^*\mathcal{S}$ is the cotangent bundle of $\mathcal{S}$. Taking the L^2-adjoint of ∇, we define the diffential operator

$$\nabla : \Gamma(T^*\mathcal{S} \otimes E) \longrightarrow \Gamma(E).$$

Then, the Bochner Laplacian is given by

$$\Delta^B = \nabla^*\nabla$$

Definition 2.0.2 (DeRham Laplacian)**.** *Let $(\mathcal{S}, g)$ be a Riemannian manifold. Then, given the external differential $\boldsymbol{d}$ and the external codifferential δ defined over $\mathcal{S}$, we define the DeRham Laplacian as*

$$\Delta^{dR} = \boldsymbol{d}\delta - \delta\boldsymbol{d}$$

Then, we can state the *Weitzenböck identity* that connects the previous Laplacians defined over a surface:

$$\Delta^B \mathbf{v} = \Delta^{dR}\mathbf{v} + \mathcal{K}\mathbf{v} \qquad \forall\, \mathbf{v} \in T\mathcal{S}, \tag{2.1}$$

where $\mathcal{K}$ is the curvature of $\mathcal{S}$. In particular, we notice that the two Laplacians differs only by a term that depends only on the curvature of the surface.

Now we introduce formulation of the model.

2.1 Formulation of the surface model

Among the liquid crystal models, i.e. physical systems consisting of elongated rod-like molecules with a preferred local average direction, we consider the model of Nematic liquid crystals. In this case, the long axes of the constituent molecules tend to be parallel to each other choosing some common preferred direction, usually called *anisotropic axis*. We start by introducing a unit vector $\mathbf{u}$, called *director*, to describe the local direction of average molecular alignment in liquid crystals.

We consider a surface bound system of densely packed rod-like particles that tend to align tangentially to the director, and following Frank-Oseen theory, we take in consideration a free energy that depends on the director $\mathbf{u}$ as follows:

$$F_{\mathrm{F}}[\mathbf{u}, \Omega] = \frac{1}{2}\int_\Omega K_1(\nabla \cdot \mathbf{u})^2 + K_2(\mathbf{u} \cdot (\nabla \times \mathbf{u}))^2 + K_3||\mathbf{u} \times (\nabla \times \mathbf{u})||^2 \,\mathrm{dV}. \tag{2.2}$$

This energy describes the spatial variation on a domain $\Omega \subset \mathbb{R}^3$ of the director field and represents the energy costs due to spatial distortions and deformations, such as splay, twist and bend, modeled through the three different terms with K_1, K_2 and K_3 coefficients. We approximate these constants with a same factor $K = K_1 = K_2 = K_3$ to get the following:

$$\begin{aligned} F_{OC}[\mathbf{u}, \Omega] &= \frac{K}{2} \int_\Omega (\nabla \cdot \mathbf{u})^2 + (\mathbf{u} \cdot (\nabla \times \mathbf{u}))^2 + ||\mathbf{u} \times (\nabla \times \mathbf{u})||^2 \mathrm{dV} \\ &= \frac{K}{2} \int_\Omega (\nabla \cdot \mathbf{u})^2 + (\mathbf{u} \cdot (\nabla \times \mathbf{u}))^2 + ||\mathbf{u}||^2 \cdot ||(\nabla \times \mathbf{u})||^2 - (\mathbf{u} \cdot (\nabla \times \mathbf{u}))^2 \ \mathrm{dV} \\ &\overset{||\mathbf{u}||=1}{=} \frac{K}{2} \int_\Omega (\nabla \cdot \mathbf{u})^2 + ||\nabla \times \mathbf{u}||^2 \ \mathrm{dV}. \end{aligned}$$

Since we want to formulate this energy on a Riemannian surface $\mathcal{S}$, we consider a tubular neighborhood $\mathcal{S} \subset \Omega_\delta$, with $\mathbf{u}$ tangent to the surface, so $\mathbf{u} \in T\mathcal{S}$, and $\delta > 0$ small enough to ensure that every point in this domain, can be projected along the normal direction of the surface $\mathcal{S}$ in a unique way. The idea to restrict this energy to be a surface energy is to proceed by performing a thin limit approximation, by considering $\delta \searrow 0$. Due to the fact that a smooth vector field with **unit norm** exists on a surface S if and only if $\chi(S) = 0$, we start by relaxing the constraint $||\mathbf{u}|| = 1$. We translate this properties by enforcing the condition adding a quartic state potential to the free energy, with a penalty term $\omega_\mathrm{n} \gg K$:

$$F_{w_\mathrm{n}}[\mathbf{u}, \Omega_\delta] = \frac{K}{2} \int_{\Omega_\delta} (\nabla \cdot \mathbf{u})^2 + ||\nabla \times \mathbf{u}||^2 \ \mathrm{dV} + \frac{\omega_\mathrm{n}}{4} \int_{\Omega_\delta} (||\mathbf{u}||^2 - 1)^2 \ \mathrm{dV}. \tag{2.3}$$

We now analyze the previous energy using the intrinsic setting, i.e. using the LCS basis in the space $\mathbb{R}^3$, with which we can write the intrinsic operators (1.3.3) to derive the equation containing the vector Laplacian. From now on we will consider a simplified version of energy (2.3), in which the last penalty term will not be present. In this way, we will deal with a more general equation, obtaining a method that can be applied in contexts where the vector Laplacian is present. The simplified energy we will consider in the intrinsic setting is given by:

$$F[\mathbf{u}, \Omega_\delta] = \frac{K}{2} \int_{\Omega_\delta} (\nabla \cdot_{\mathcal{G}_3} \mathbf{u})^2 + ||\nabla \times_{\mathcal{G}_3} \mathbf{u}||^2 \ \mathrm{dV}. \tag{2.4}$$

Next, we analyze this formula in the space Ω_δ with the aim of arriving at the corresponding energy on the surface $\mathcal{S}$ via the limit for δ tending to zero.

2.1.1 Intrinsic Approach

As said before, we apply our intrinsic framework to eq. (2.4). In particular, we will refer to operators defined on the surface $\mathcal{S}$ through the tensor $\mathcal{G}_2$ and to quantities in the subspace $\Omega_\delta \subset \mathbb{R}^3$ with the tensor $\mathcal{G}_3$ associated to the LCS. In particular we will refer to quantities on the tubular neighborhood Ω_{delta} with the variable $\tilde{\mathbf{u}}$ and to quantities on the surface $\mathcal{S}$ with the variable $\mathbf{u}$.

We now begin to write all quantities explicitly, starting with the gradient of the vector $\tilde{\mathbf{u}} \in T_P\Omega_\delta$, for which we use the expression (1.18):

$$\begin{aligned}
(\nabla_{\mathcal{G}_3} \cdot \tilde{\mathbf{u}})^2 &= \frac{1}{h_{(1)}^2 h_{(2)}^2} \left[\sum_{i=1}^{3} \frac{\partial (h_{(1)} h_{(2)} u^i)}{\partial s^i} \right]^2 = \frac{1}{h_{(1)}^2 h_{(2)}^2} \left[\sum_{i=1}^{2} \frac{\partial (h_{(1)} h_{(2)} u^i)}{\partial s^i} + h_{(1)} h_{(2)} \frac{\partial u^3}{\partial s^3} \right]^2 \\
&= \frac{1}{h_{(1)}^2 h_{(2)}^2} \left[\sum_{i=1}^{2} \frac{\partial (h_{(1)} h_{(2)} u^i)}{\partial s^i} \right]^2 + \frac{2}{h_{(1)} h_{(2)}} \sum_{i=1}^{2} \frac{\partial (h_{(1)} h_{(2)} u^i)}{\partial s^i} \frac{\partial u^3}{\partial s^3} + \left(\frac{\partial u^3}{\partial s^3} \right)^2 \\
&= (\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \frac{2}{h_{(1)} h_{(2)}} \left(\sum_{i=1}^{2} \frac{\partial (h_{(1)} h_{(2)} u^i)}{\partial s^i} \right) \frac{\partial u^3}{\partial s^3} + \left(\frac{\partial u^3}{\partial s^3} \right)^2 \\
&= (\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \frac{2}{h_{(1)} h_{(2)}} \left(\sum_{i=1}^{2} \frac{\partial (h_{(1)} h_{(2)} u^i)}{\partial s^i} \right) \mathcal{O}(\xi) + \mathcal{O}(\xi^2) \\
&= (\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \mathcal{O}(\xi)
\end{aligned}$$

where we consider the fact that $\frac{\partial u^3}{\partial s^3} = \mathcal{O}(\xi)$. This is because our purpose is to take the limit for $\delta \to 0$ to get the formulation on the surface $\mathcal{S}$, and on S the quantities are defined in terms of tangential variables while there is no variation along the normal direction ν of the LCS. It means that the value ξ goes to zero when δ goes to zero. Moreover, since we assumed that every $\tilde{\mathbf{u}} \in T\Omega_\delta$ is parallel to the surface and is obtained through a parallel transport of $\mathbf{u}$ along the normal direction to the surface ν, it means that $\tilde{\mathbf{u}}$ is an extension of $\mathbf{u}$, i.e. $\tilde{\mathbf{u}}|_{\mathcal{S}} = \mathbf{u} \in T\mathcal{S}$. This has the direct consequence that the covariant derivative of the third component of every vector $\tilde{\mathbf{u}} \in T\Omega_\delta$, in the direction of ξ, is zero. Hence, recalling the definition of connection in (1.1.18):

$$0 = \nabla_\xi \, \tilde{u}^I = \partial_\xi \, \tilde{u}^i + \widetilde{\Gamma}^i_{\xi k} \, \tilde{u}^k \,,$$

where we use the lower-case indices when we indicate quantities on the surface and upper-case indices when when we consider quantities on Ω_δ.

Now we can express the Christoffel symbols in terms of *shape operator* and terms depending on the normal direction. We recall that every point in the tubular neighborhood $\tilde{x} \in \Omega_\delta$ can be written in the LCS by 1.3.1:

$$\tilde{x}(s^1, s^2, \xi) = x(s^1, s^2) + \xi \nu(s^1, s^2).$$

Here, using the definition of shape operator (1.7), we get that the coefficients of the metric can be written as follow:

$$\begin{aligned}
\tilde{g}_{ij} &= (\partial_i x \cdot \partial_j x) = \partial_i x \cdot \partial_j x + \xi \, \partial_i x \cdot \partial_j \nu + \xi \, \partial_j x \cdot \partial_i \nu + \xi^2 \partial_i \nu \cdot \partial_j \nu; \\
&= g_{ij} + 2\xi \, \partial_i x \cdot \partial_j \nu + O(\xi^2) = g_{ij} - 2\xi \, \mathcal{B}_{ij} + \mathcal{O}(\xi^2) \\
\tilde{g}_{\xi\xi} &= \nu \cdot \nu = 1; \\
\tilde{g}_{i\xi} &= \tilde{g}_{\xi i} = 0.
\end{aligned}$$

And then, we use the definition of Christoffel symbols (1.3) to get:

$$\widetilde{\Gamma}^i_{\xi k} = \frac{1}{2} \tilde{g}^{il} \left(\frac{\partial \tilde{g}_{lk}}{\partial \xi} + \frac{\partial \tilde{g}_{\xi l}}{\partial s^k} - \frac{\partial \tilde{g}_{\xi k}}{\partial s^l} \right) = \frac{1}{2} \tilde{g}^{il} \left(-2\mathcal{B}_{kl} + \mathcal{O}(\xi) \right) = \mathcal{B}^k_i + \mathcal{O}(\xi)^k_i.$$

Finally, we have the following result:

$$\partial_\xi \, \tilde{u}^i = -\widetilde{\Gamma}^i_{\xi k} \, \tilde{u}^k = \mathcal{B}^k_i \tilde{u}^k + \mathcal{O}(\xi)^k_i. \tag{2.5}$$

The next step is to compute the curl of $\tilde{\mathbf{u}} \in T\Omega_\delta$ recalling the formula in eq. (1.21):

$$\nabla \times_{\mathcal{G}_3} \tilde{\mathbf{u}} = \frac{1}{\sqrt{\det(\mathcal{G}_3)}} \varepsilon^{ijk} \partial_i \left(h^2_{(j)} v^j \right) = \frac{1}{h_{(1)} h_{(2)}} \begin{bmatrix} \partial_2 \left(h^2_{(3)} u^3 \right) - \partial_3 \left(h^2_{(2)} u^2 \right) \\ \partial_3 \left(h^2_{(1)} u^1 \right) - \partial_1 \left(h^2_{(3)} u^3 \right) \\ \partial_1 \left(h^2_{(2)} u^2 \right) - \partial_2 \left(h^2_{(1)} u^1 \right) \end{bmatrix},$$

and taking the squared norm of this term, we obtain:

$$\begin{aligned} \|\nabla \times_{\mathcal{G}_3} \tilde{\mathbf{u}}\|^2 &= (\nabla \times_{\mathcal{G}_3} \tilde{\mathbf{u}})^T \mathcal{G}_3 (\nabla \times_{\mathcal{G}_3} \tilde{\mathbf{u}}) \\ &= \frac{1}{h^2_{(1)} h^2_{(2)}} \Big\{ h^2_{(1)} \left[\partial_2 \left(h^2_{(3)} u^3 \right) - \partial_3 \left(h^2_{(2)} u^2 \right) \right]^2 + h^2_{(2)} \left[\partial_3 \left(h^2_{(1)} u^1 \right) - \partial_1 \left(h^2_{(3)} u^3 \right) \right]^2 + \\ &\qquad + h^2_{(3)} \left[\partial_1 \left(h^2_{(2)} u^2 \right) - \partial_2 \left(h^2_{(1)} u^1 \right) \right]^2 \Big\} \\ &= \frac{1}{h^2_{(1)} h^2_{(2)}} \Big\{ h^2_{(1)} \left[\partial_2 (u^3) - \partial_3 \left(h^2_{(2)} u^2 \right) \right]^2 + h^2_{(2)} \left[\partial_3 \left(h^2_{(1)} u^1 \right) - \partial_1 \left(u^3 \right) \right]^2 \Big\} + \\ &\qquad + \frac{1}{h^2_{(1)} h^2_{(2)}} \left[\partial_1 \left(h^2_{(2)} u^2 \right) - \partial_2 \left(h^2_{(1)} u^1 \right) \right]^2 \\ &= \frac{1}{h^2_{(2)}} \left[\left(\partial_3 \left(h^2_{(2)} u^2 \right) \right)^2 - 2 \partial_2 (u^3) \partial_3 \left(h^2_{(2)} u^2 \right) + \left(\partial_2 (u^3) \right)^2 \right] + \\ &\quad + \frac{1}{h^2_{(1)}} \left[\left(\partial_3 \left(h^2_{(1)} u^1 \right) \right)^2 - 2 \partial_1 (u^3) \partial_3 \left(h^2_{(1)} u^1 \right) + \left(\partial_1 (u^3) \right)^2 \right] + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}_{|\mathcal{S}}\|^2_{\mathcal{S}} \\ &= \frac{1}{h^2_{(2)}} \left[\left(\partial_3 \left(h^2_{(2)} u^2 \right) \right)^2 - 2 \mathcal{O}(\xi) \partial_3 \left(h^2_{(2)} u^2 \right) + \mathcal{O}(\xi^2) \right] + \\ &\quad + \frac{1}{h^2_{(1)}} \left[\left(\partial_3 \left(h^2_{(1)} u^1 \right) \right)^2 - 2 \mathcal{O}(\xi) \partial_3 \left(h^2_{(1)} u^1 \right) + \mathcal{O}(\xi^2) \right] + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}_{|\mathcal{S}}\|^2_{\mathcal{S}} \\ &= \frac{1}{h^2_{(2)}} \left[\partial_3 \left(h^2_{(2)} u^2 \right) \right]^2 + \frac{1}{h^2_{(1)}} \left[\partial_3 \left(h^2_{(1)} u^1 \right) \right]^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}_{|\mathcal{S}}\|^2_{\mathcal{S}} + \mathcal{O}(\xi). \end{aligned}$$

By analizing the first two terms we get:

$$\begin{aligned} & \frac{1}{h^2_{(2)}} \left[\partial_3 \left(h^2_{(2)} u^2 \right) \right]^2 + \frac{1}{h^2_{(1)}} \left[\partial_3 \left(h^2_{(1)} u^1 \right) \right]^2 \\ =& \frac{1}{h^2_{(2)}} \left[\partial_3 \left(h^2_{(2)} \right) u^2 + h^2_{(2)} \partial_3 u^2 \right]^2 + \frac{1}{h^2_{(1)}} \left[\partial_3 \left(h^2_{(1)} \right) u^1 + h^2_{(1)} \partial_3 u^1 \right]^2 \\ =& \frac{1}{h^2_{(2)}} \left[h^2_{(2)} \partial_3 u^2 \right]^2 + \frac{1}{h^2_{(1)}} \left[h^2_{(1)} \partial_3 u^1 \right]^2 = h^2_{(2)} \left[\partial_3 u^2 \right]^2 + h^2_{(1)} \left[\partial_3 u^1 \right]^2 \\ \overset{(2.5)}{=}& h^2_{(2)} \left[\mathcal{B}^1_k u^k \right]^2 + h^2_{(1)} \left[\mathcal{B}^2_k u^k \right]^2 + \mathcal{O}(\xi) = (\mathcal{B}\mathbf{u})^T \mathcal{G}_2 \mathcal{B}\mathbf{u} + \mathcal{O}(\xi) \\ =& \|\mathcal{B}\mathbf{u}\|^2_{\mathcal{S}} + \mathcal{O}(\xi) \end{aligned}$$

So, putting everything together, we obtain that (2.4) can be re-written as:

$$
\begin{aligned}
F[\tilde{\mathbf{u}}, \Omega_\delta] &= \int_{\mathcal{S}} \int_{-\delta/2}^{\delta/2} \left[\frac{K}{2} \left((\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}\|_{\mathcal{S}}^2 + \|\mathcal{B}\mathbf{u}\|_{\mathcal{S}}^2 \right) + \mathcal{O}(\xi) \right] d\xi d\mathcal{S} \\
&= \int_{\mathcal{S}} \left[\frac{K}{2} \left((\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}\|_{\mathcal{S}}^2 + \|\mathcal{B}\mathbf{u}\|_{\mathcal{S}}^2 \right) \left(\int_{-\delta/2}^{\delta/2} d\xi \right) d\mathcal{S} \right] + \int_{\mathcal{S}} \int_{-\delta/2}^{\delta/2} \mathcal{O}(\xi) d\xi d\mathcal{S} \\
&= \delta \int_{\mathcal{S}} \left[\frac{K}{2} \left((\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}\|_{\mathcal{S}}^2 + \|\mathcal{B}\mathbf{u}\|_{\mathcal{S}}^2 \right) \right] d\mathcal{S} + \delta \mathcal{O}(\delta) \\
&= \delta \left(\int_{\mathcal{S}} \left[\frac{K}{2} \left((\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}\|_{\mathcal{S}}^2 + \|\mathcal{B}\mathbf{u}\|_{\mathcal{S}}^2 \right) \right] d\mathcal{S} + \mathcal{O}(\delta) \right).
\end{aligned}
$$

Finally, dividing by δ and taking the limit $\delta \to 0$, we have that:

$$
\begin{aligned}
\lim_{\delta \to 0} \frac{F[\tilde{\mathbf{u}}, \Omega_\delta]}{\delta} &= \lim_{\delta \to 0} \left(\int_{\mathcal{S}} \left[\frac{K}{2} \left((\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}\|_{\mathcal{S}}^2 + \|\mathcal{B}\mathbf{u}\|_{\mathcal{S}}^2 \right) \right] d\mathcal{S} + \mathcal{O}(\delta) \right) \\
&= \frac{K}{2} \int_{\mathcal{S}} \left((\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}\|_{\mathcal{S}}^2 + \|\mathcal{B}\mathbf{u}\|_{\mathcal{S}}^2 \right) d\mathcal{S}
\end{aligned}
$$

and we define the *weak surface Frank-Oseen energy* as

$$
F^{\mathcal{S}}[\mathbf{u}] = \frac{K}{2} \int_{\mathcal{S}} \left((\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}\|_{\mathcal{S}}^2 + \|\mathcal{B}\mathbf{u}\|_{\mathcal{S}}^2 \right) d\mathcal{S}. \tag{2.6}
$$

We can observe that, this energy is the sum of an intrinsic and an extrinsic contributions:

$$
F^{\mathcal{S}}[\mathbf{u}] = F_I^{\mathcal{S}}[\mathbf{u}] + F_E^{\mathcal{S}}[\mathbf{u}]\,,
$$

which are respectively:

$$
F_I^{\mathcal{S}}[\mathbf{u}] = \frac{K}{2} \int_{\mathcal{S}} \left((\nabla_{\mathcal{G}_2} \cdot \mathbf{u})^2 + \|\nabla \times_{\mathcal{G}_2} \mathbf{u}\|_{\mathcal{S}}^2 \right) \, \mathrm{d}\mathcal{S}, \tag{2.7}
$$

and

$$
F_E^{\mathcal{S}}[\mathbf{u}] = \frac{K}{2} \int_{\mathcal{S}} ||\mathcal{B}\mathbf{u}||_{\mathcal{S}}^2 \, \mathrm{d}\mathcal{S}, \tag{2.8}
$$

where we recall that the shape operator is defined by $\mathcal{B} = -\mathrm{grad}\ \nu$, with ν the outer normal to the surface $\mathcal{S}$.

2.2 Energy Minimization

Having obtained the energy on the surface, we now try to minimize this functional $F^{\mathcal{S}}[\mathbf{u}]$. Lets start by the definition of the functional spaces:

$$
\begin{aligned}
H(\mathrm{div}, \mathcal{S}, \mathrm{T}\mathcal{S}) &:= \left\{ \mathbf{u} \in L^2(\mathcal{S}; \mathrm{T}\mathcal{S}) : \mathrm{div}\, \mathbf{u} \in L^2(\mathcal{S}) \right\} \\
H(\mathrm{rot}, \mathcal{S}, \mathrm{T}\mathcal{S}) &:= \left\{ \mathbf{u} \in L^2(\mathcal{S}; \mathrm{T}\mathcal{S}) : \mathrm{rot}\, \mathbf{u} \in L^2(\mathcal{S}) \right\} \\
H^{\mathrm{DR}}(\mathrm{div}, \mathcal{S}, \mathrm{T}\mathcal{S}) &:= H(\mathrm{div}, \mathcal{S}, \mathrm{T}\mathcal{S}) \cap H(\mathrm{rot}, \mathcal{S}, \mathrm{T}\mathcal{S}).
\end{aligned}
$$

We need now to define our operators in a proper way on the surface not only with the local coordinates, but also with the exterior calculus that will be necessary to derive the final expression of the Laplacian. We collect the most important definitions in the following table, which derive from the proposition of the intrinsic operators (1.3.3) and from the theory of the Hodge operator 1.1.23 and (1.1.24):

Symbolic	Local coordinates	Exterior Calculus
$\langle \mathbf{u}, \mathbf{v} \rangle$	$u^i g_{ij} v^i$	$\star(\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta})$
$\text{grad}\, f$	$g^{ij}\partial_j f \partial_i \varphi$	$\mathbf{d}f$
$\text{rot}\, f$	$\frac{1}{\sqrt{\lvert\mathcal{G}\rvert}}\left(\partial_{s^1} f \partial_{s^2}\varphi - \partial_{s^2} f \partial_{s^1}\varphi\right)$	$\star\mathbf{d}f$
$\text{div}\, \mathbf{v}$	$\partial_i v^i + \frac{1}{\sqrt{\lvert\mathcal{G}\rvert}} v^i \partial_i \sqrt{\lvert\mathcal{G}\rvert}$	$\star\mathbf{d}\star\boldsymbol{\alpha}$
$\text{rot}\, \mathbf{v}$	$\frac{1}{\sqrt{\lvert\mathcal{G}\rvert}}\left(\partial_{s^1} v_{s^2} - \partial_{s^2} v_{s^1}\right)$	$\star\mathbf{d}\boldsymbol{\alpha}$

In particular, we recall that φ is the parametrization of the surface through the local coordinates (s^1, s^2), and we observe that in our context $f : \mathcal{S} \longrightarrow \mathbb{R}$ is a scalar function, $\mathbf{u}, \mathbf{v} : \mathcal{S} \longrightarrow T\mathcal{S}$ are vector field and $\boldsymbol{\alpha}, \boldsymbol{\beta} : \mathcal{S} \longrightarrow T^*\mathcal{S}$ are the 1-forms associated to the vector fields $\mathbf{u}, \mathbf{v}$ through the $\sharp$-$\flat$ isomorphism.

We want to find

$$\mathbf{u}^* = \text{argmin}\left\{F^{\mathcal{S}}[\mathbf{u}] : \mathbf{u} \in H^{\text{DR}}(\mathcal{S}; \text{T}\mathcal{S})\right\}$$

through an L^2-gradient flow approach, in which the gradient of $F^{\mathcal{S}}$ has to be interpreted with respect to the $L^2(\mathcal{S}; T\mathcal{S})$ inner product. So, we impose that the variation in time of the variable $\mathbf{u}$ is equivalent to the variational derivative of the functional of the energy (2.6):

$$-\partial_t \mathbf{u} = \frac{\delta F^{\mathcal{S}}}{\delta \mathbf{u}}[\mathbf{u}].$$

Multiplying each member by a test function $\mathbf{v} \in H^{\text{DR}}(\mathcal{S}; T\mathcal{S})$ and integrating over $\mathcal{S}$ we get:

$$\int_{\mathcal{S}} \langle -\partial_t \mathbf{u}, \mathbf{v}\rangle \, \text{d}\mathcal{S} = \int_{\mathcal{S}} \left\langle \frac{\delta F^{\mathcal{S}}}{\delta \mathbf{u}}[\mathbf{u}], \mathbf{v} \right\rangle \text{d}\mathcal{S} \;=\; \lim_{\varepsilon \to 0} \frac{F^{\mathcal{S}}[\mathbf{u} + \varepsilon \mathbf{v}] - F^{\mathcal{S}}[\mathbf{u}]}{\varepsilon} \;=\; \left[\frac{d}{d\varepsilon} F^{\mathcal{S}}[\mathbf{u} + \varepsilon \mathbf{v}]\right]\Bigg|_{\varepsilon=0}$$

for which, substituting $\mathbf{u} + \varepsilon \mathbf{v}$ in $F^{\mathcal{S}}$ we obtain:

$$\begin{aligned}
F^{\mathcal{S}}[\mathbf{u} + \varepsilon \mathbf{v}] &= \frac{K}{2} \int_{\mathcal{S}} \Big(\text{div}(\mathbf{u} + \varepsilon \mathbf{v})\Big)^2 + \Big(\text{rot}(\mathbf{u} + \varepsilon \mathbf{v})\Big)^2 + ||\mathcal{B} \cdot (\mathbf{u} + \varepsilon \mathbf{v})||^2 \, \text{d}\mathcal{S} \\
&= \frac{K}{2} \int_{\mathcal{S}} \Big(\text{div}(\mathbf{u}) + \varepsilon \text{div}(\mathbf{v})\Big)^2 + \Big(\text{rot}(\mathbf{u}) + \varepsilon \text{rot}(\mathbf{v})\Big)^2 + \Big((\mathbf{u} + \varepsilon \mathbf{v})^T \cdot \mathcal{B}^T\Big)\Big(\mathcal{B} \cdot (\mathbf{u} + \varepsilon \mathbf{v})\Big) \, \text{d}\mathcal{S} \\
&= \frac{K}{2} \int_{\mathcal{S}} \Big(\text{div}(\mathbf{u})^2 + 2\varepsilon \, \text{div}(\mathbf{u})\text{div}(\mathbf{v}) + \varepsilon^2 \text{div}(\mathbf{v})^2 + \text{rot}(\mathbf{u})^2 + 2\varepsilon \, \text{rot}(\mathbf{u})\text{rot}(\mathbf{v}) + \varepsilon^2 \text{rot}(\mathbf{v})^2 + \\
&\qquad + \mathbf{u}^T \mathcal{B}^T \mathcal{B} \mathbf{u} + 2\varepsilon \, \mathbf{u}^T \mathcal{B}^T \mathcal{B} \mathbf{v} + \varepsilon^2 \mathbf{v}^T \mathcal{B}^T \mathcal{B} \mathbf{v}\Big) \; \text{d}\mathcal{S}
\end{aligned}$$

and since the shape operator is symmetric, i.e. $\mathcal{B} = \mathcal{B}^T$, we have that $\mathcal{B}^T \mathcal{B} = \mathcal{B}^2$. Now we can proceed differentiating and then evaluating in $\varepsilon = 0$:

$$\left[\frac{d}{d\varepsilon}F^{\mathcal{S}}[\mathbf{u}+\varepsilon\mathbf{v}]\right]\Bigg|_{\varepsilon=0} = \left[\frac{d}{d\varepsilon}\left(\frac{K}{2}\int_{\mathcal{S}}\Big(\mathrm{div}(\mathbf{u})^2 + 2\varepsilon\,\mathrm{div}(\mathbf{u})\mathrm{div}(\mathbf{v}) + \varepsilon^2\mathrm{div}(\mathbf{v})^2 + \mathrm{rot}(\mathbf{u})^2 + 2\varepsilon\,\mathrm{rot}(\mathbf{u})\mathrm{rot}(\mathbf{v}) + \right.\right.$$

$$\left.\left. + \varepsilon^2\mathrm{rot}(\mathbf{v})^2 + \mathbf{u}^T\mathcal{B}^2\mathbf{u} + 2\varepsilon\,\mathbf{u}^T\mathcal{B}^2\mathbf{v} + \varepsilon^2\mathbf{v}^T\mathcal{B}^2\mathbf{v}\Big)\ \mathrm{d}\mathcal{S}\right)\right]\Bigg|_{\varepsilon=0}$$

$$= \left[\frac{K}{2}\int_{\mathcal{S}}\Big(2\mathrm{div}(\mathbf{u})\mathrm{div}(\mathbf{v}) + 2\varepsilon\,\mathrm{div}(\mathbf{v})^2 + 2\mathrm{rot}(\mathbf{u})\mathrm{rot}(\mathbf{v}) + 2\varepsilon\,\mathrm{rot}(\mathbf{v})^2 + 2\mathbf{u}^T\mathcal{B}^2\mathbf{v} + \right.$$

$$\left. + 2\varepsilon\,\mathbf{v}^T\mathcal{B}^2\mathbf{v}\Big)\ \mathrm{d}\mathcal{S}\right]\Bigg|_{\varepsilon=0}$$

$$= K\int_{\mathcal{S}}\Big(\mathrm{div}(\mathbf{u})\mathrm{div}(\mathbf{v}) + \mathrm{rot}(\mathbf{u})\mathrm{rot}(\mathbf{v}) + \Big\langle\mathcal{B}^2\mathbf{u},\mathbf{v}\Big\rangle\Big)\mathrm{d}\mathcal{S}\,. \tag{2.9}$$

Now, we want to analyze the surface expression for the terms $\mathrm{div}(\mathbf{u})\mathrm{div}(\mathbf{v})$ and $\mathrm{rot}(\mathbf{u})\mathrm{rot}(\mathbf{v})$. We begin with the first, by recalling that:

- from the definition of the Riemannian metric we have that $g_{ij}g^{jk} = \delta_i^k$;
- for every vector field $\mathbf{b}$ there exists an associated 1-form α through the isomorphism $\sharp$-$\flat$ (see eq. (1.1)), and in particular for every scalar function f we have the association between $\mathrm{grad}f$-$\mathbf{d}f$. Explicitly:

$$(\mathrm{grad}f)^i = \mathbf{d}f_j\, g^{ij} \tag{2.10}$$

$$(b)^i = \alpha_j\, g^{ij} \tag{2.11}$$

For all scalar function f and all vector field $\mathbf{v}$ we have that:

$$-\int_{\mathcal{S}}\langle\,\mathrm{grad}\,f,\mathbf{b}\rangle\ \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}}(\mathrm{grad}\,f)^i g_{ij}v^j\ \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}}(\mathbf{d}f)_l\, g^{il}\, g_{ij}\,\alpha_m g^{jm}\ \mathrm{d}\mathcal{S}$$
$$= -\int_{\mathcal{S}}(\mathbf{d}f)_l\,\delta_j^l\,\alpha_m g^{jm}\ \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}}(\mathbf{d}f)_l\,\alpha_m g^{lm}\ \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}}\langle\mathbf{d}f,\boldsymbol{\alpha}\rangle\ \mathrm{d}\mathcal{S}. \tag{2.12}$$

Then, we recall some definitions and properties regarding *k-differential forms* , *Hodge operator*, *codifferential* over the surface $\mathcal{S}$ (i.e., a manifold with dimension 2):

- from definition 1.1.23 of the Hodge Operator, we have that $\star : \Lambda^k(\mathcal{S}) \longrightarrow \Lambda^{2-k}(\mathcal{S})$;
- from definition 1.1.25 of the codifferential, we have that $\delta : \Lambda^k(\mathcal{S}) \longrightarrow \Lambda^{k-1}(\mathcal{S})$ and in particular
$$\delta = (-1)^{n(k+1)+1} \star \mathbf{d}\star = (-1)^{2(k+1)+1} \star \mathbf{d}\star = - \star \mathbf{d}\star,$$
since $2(k+1)+1$ will always be odd for all $k = 0, 1, 2$;
- from proposition 1.1.26, we have that the codifferential and the Hodge operator are adjoint:
$$\langle\mathbf{d}\eta,\omega\rangle = \langle\eta,\delta\omega\rangle$$
for all $\eta \in \Lambda(\mathcal{S})^{k-1}$ and $\omega \in \Lambda(\mathcal{S})^k$;

- we recall that for all vector field $\mathbf{v}$ and its 1-form associated α, we have

$$\operatorname{div} \mathbf{v} = \star \mathbf{d} \star \boldsymbol{\alpha}.$$

Thus, from eq. (2.12), we get that:

$$-\int_{\mathcal{S}} \langle \operatorname{grad} f, \mathbf{v} \rangle \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}} \langle \mathbf{d} f, \boldsymbol{\alpha} \rangle \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}} f \delta \boldsymbol{\alpha} \, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} f \star \mathbf{d} \star \boldsymbol{\alpha} \, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} f \operatorname{div}(\mathbf{v}) \, \mathrm{d}\mathcal{S}\,,$$

and substituting $f = \operatorname{div} \mathbf{u}$, we obtain:

$$-\int_{\mathcal{S}} \langle \operatorname{grad} \operatorname{div}(\mathbf{u}), \mathbf{v} \rangle \, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v}) \, \mathrm{d}\mathcal{S}.$$

Now, for the term $\operatorname{rot}(\mathbf{u})\operatorname{rot}(\mathbf{v})$, we start by recalling that:

- for every scalar function f, we have:

$$\operatorname{rot} f = \star \mathbf{d} f;$$

- from property (1.1.24) we have that: for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda^k(\mathcal{S})$

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \langle \star \boldsymbol{\alpha}, \star \boldsymbol{\beta} \rangle.$$

We can then write:

$$-\int_{\mathcal{S}} \langle \operatorname{rot} f, \mathbf{v} \rangle \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}} \langle \star \mathbf{d} f, \boldsymbol{\alpha} \rangle \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}} \langle \star \star \mathbf{d} f, \star \boldsymbol{\alpha} \rangle \, \mathrm{d}\mathcal{S}$$

and recalling that $\mathbf{d}$ and δ are adjoint and that $\star \star \boldsymbol{\alpha} = -\boldsymbol{\alpha}$, we have:

$$\begin{aligned} -\int_{\mathcal{S}} \langle \star \star \mathbf{d} f, \star \boldsymbol{\alpha} \rangle \, \mathrm{d}\mathcal{S} &= \int_{\mathcal{S}} \langle \mathbf{d} f, \star \boldsymbol{\alpha} \rangle \, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} f \, \delta \star \boldsymbol{\alpha} \, \mathrm{d}\mathcal{S} = -\int_{\mathcal{S}} f \, \star \mathbf{d} \star \star \boldsymbol{\alpha} \, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} f \, \star \mathbf{d} \boldsymbol{\alpha} \, \mathrm{d}\mathcal{S} \\ &= \int_{\mathcal{S}} f \, \operatorname{rot}(\mathbf{v}) \, \mathrm{d}\mathcal{S}\,. \end{aligned}$$

Now, substituting $f = \operatorname{rot}(\mathbf{u})$, we obtain

$$-\int_{\mathcal{S}} \langle \operatorname{rot} \operatorname{rot}(\mathbf{u}), \mathbf{v} \rangle \, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} \operatorname{rot}(\mathbf{u}) \, \operatorname{rot}(\mathbf{v}) \, \mathrm{d}\mathcal{S}\,,$$

and putting all together in (2.9), we have:

$$\begin{aligned} \int_{\mathcal{S}} \langle -\partial_t \mathbf{u}, \mathbf{v} \rangle \, \mathrm{d}\mathcal{S} &= \left[\frac{d}{d\varepsilon} F^{\mathcal{S}}[\mathbf{u} + \varepsilon \mathbf{v}] \right] \Bigg|_{\varepsilon=0} = K \int_{\mathcal{S}} \Big(\langle \operatorname{grad} \operatorname{div}(\mathbf{u}), \mathbf{v} \rangle + \langle \operatorname{rot} \operatorname{rot}(\mathbf{u}), \mathbf{v} \rangle + \Big\langle \mathcal{B}^2 \mathbf{u}, \mathbf{v} \Big\rangle \Big) \mathrm{d}\mathcal{S} \\ &= K \int_{\mathcal{S}} \Big\langle -\operatorname{grad} \operatorname{div}(\mathbf{u}) - \operatorname{rot} \operatorname{rot}(\mathbf{u}) + \mathcal{B}^2 \mathbf{u}, \mathbf{v} \Big\rangle \, \mathrm{d}\mathcal{S} \\ &= \int_{\mathcal{S}} \Big\langle K\Big(-\Delta^{\mathrm{dR}} \mathbf{u} + \mathcal{B}^2 \mathbf{u} \Big), \mathbf{v} \Big\rangle \, \mathrm{d}\mathcal{S}, \end{aligned}$$

where Δ^{dR} is the *De-Rham* Laplacian (see definition 2.0.2) associated to the vector fields through the $\sharp$-$\flat$ isomorphisms. In fact, using the previous definitions for *div* and *rot* in terms on exterior calculus, the association between 1-forms and vector fields and the properties of the Hodge Operator, differential and codifferential, we have:

$$\begin{aligned} -\int_{\mathcal{S}} \Big\langle \operatorname{grad} \operatorname{div}(\mathbf{u}) + \operatorname{rot} \operatorname{rot}(\mathbf{u}), \mathbf{v} \Big\rangle \mathrm{d}\mathcal{S} &= -\int_{\mathcal{S}} \Big\langle \mathbf{d}(\star \mathbf{d} \star (\boldsymbol{\alpha})) + \star \mathbf{d}(\star \mathbf{d}(\boldsymbol{\alpha})),\ \boldsymbol{\beta} \Big\rangle \mathrm{d}\mathcal{S} \\ = -\int_{\mathcal{S}} \Big\langle (\mathbf{d} \star \mathbf{d} \star + \star \mathbf{d} \star \mathbf{d}) \boldsymbol{\alpha},\ \boldsymbol{\beta} \Big\rangle \mathrm{d}\mathcal{S} &= \int_{\mathcal{S}} \Big\langle (\mathbf{d}\delta + \delta \mathbf{d}) \boldsymbol{\alpha},\ \boldsymbol{\beta} \Big\rangle \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} \Big\langle \Delta^{\mathrm{dR}} \boldsymbol{\alpha},\ \boldsymbol{\beta} \Big\rangle \mathrm{d}\mathcal{S} \end{aligned}$$

Therefore, we obtain that $\forall \mathbf{v} \in H^{\mathrm{DR}}(\mathcal{S}; T\mathcal{S})$

$$\int_{\mathcal{S}} \Big\langle -\partial_t \mathbf{u}, \mathbf{v} \Big\rangle \, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}} \Big\langle K\Big(-\Delta^{\mathrm{dR}} \mathbf{u} + \mathcal{B}^2 \mathbf{u} \Big), \mathbf{v} \Big\rangle \, \mathrm{d}\mathcal{S}$$

that leads to

$$\partial_t \mathbf{u} + K\Big(-\Delta^{\mathrm{dR}} \mathbf{u} + \mathcal{B}^2 \mathbf{u} \Big) = 0 \qquad \text{for} \quad \mathbf{u} \in H^{\mathrm{DR}}(\mathcal{S}; T\mathcal{S}) \tag{2.13}$$

which is a PDE that contains a vector Laplacian.

Now, using the following geometrical properties from [30, appendix D]:

$$\begin{aligned} 0 &= \mathcal{B}^2 - \mathcal{H}\mathcal{B} + \mathcal{K}\,\mathbb{P}\,, \\ \mathcal{B} &= \mathcal{B}\,\mathbb{P}, \end{aligned}$$

where $\mathcal{K} = \det\mathcal{B}$ is the curvature of $\mathcal{S}$, $\mathcal{H} = \mathrm{tr}(\mathcal{B})$ is the mean curvature of $\mathcal{S}$, $\mathbb{P} = \mathbb{I} - \nu\nu^T$ the projection operator with ν the normal to $\mathcal{S}$ and $\mathbb{P}\,\mathbf{u} = \mathbf{u}$ since $\mathbf{u} \in T\mathcal{S}$. Then, recalling the definitions of the *deRham Laplacian* 2.0.2, *Bochner Laplacian* 2.0.1 and their relationship in eq. (2.1), eq. (2.13) becomes:

$$\begin{aligned} \partial_t \mathbf{u} + K\Big(-\Delta^{\mathrm{dR}} \mathbf{u} + \mathcal{B}^2 \mathbf{u} \Big) &= \partial_t \mathbf{u} + K\Big(-\Delta^{\mathrm{B}} \mathbf{u} + \mathcal{K}\mathbf{u} + (\mathcal{H}\mathcal{B} - \mathcal{K}\,\mathbb{P})\mathbf{u} \Big) \\ &= \partial_t \mathbf{u} + K\Big(-\Delta^{\mathrm{B}} \mathbf{u} + \mathcal{H}\mathcal{B}\mathbf{u} \Big) = 0 \end{aligned}$$

And finally, we arrive to the PDE that contains the Bochner vector Laplacian over a surface:

$$\partial_t \mathbf{u} + K\Big(-\Delta^{\mathrm{B}} \mathbf{u} + \mathcal{H}\mathcal{B}\mathbf{u} \Big) = 0\,, \tag{2.14}$$

which we will solve numerically in the following chapter. We remark that, starting from the energy of the model, we obtain a PDE that contains the Bochner Laplacian and a reaction term.

3

Intrinsic Surface Finite Element Method

In this chapter we look in more detail at the ISFEM applied to solve a PDE on generic surfaces following the work in [6] for scalar equation and then generalizing the approach to vector equations. This method is different from the one presented in [15], the so-called surface finite element methods (SFEM), which define the differential operators on the surface by using the projection operator along the normal. Moreover, SFEM considers a piece-wise polygonal approximation of the surface and introduce a finite element space defined directly on this triangular surface mesh. Our method follows the previous one, but it differs on rewriting all quantities on the surface by making use of *LCS*, and exploiting all the intrinsic geometric information of the surface where the PDE is defined. In particular, our triangulation will be construct directly on surface, instead of using an approximation, and we will assume that tangent planes are known in exact or approximate form at the nodes of the triangulation. This is possible also thanks to the recent developed capabilities for constructing surface triangulation and tangent plane from point data in computer graphics.

Considering the previously derived PDE in eq. (2.14), in which we take for simplicity $K = 1$, we develop the intrinsic method first in the scalar case, then in the vector case.

3.1 Intrisic Surface FEM in the scalar case

Recalling the intrinsic setting with the Monge parametrization, defined in section 1.3.1, we consider a compact surface $\mathcal{S} \subset \mathbb{R}^3$ and the following equation:

$$\frac{\partial u(x,t)}{\partial t} - \Delta_{\mathcal{G}} u(x,t) + c\, u(x,t) = f(x) \qquad \text{on } \mathcal{S}, \tag{3.1}$$

where

- $u : \mathcal{S} \times I_T \longrightarrow \mathbb{R}$ is a scalar function defined on the surface and depending on time in $I_T = (0, T] \subset \mathbb{R}$;
- f is a force function, and $f \in L^2(\mathcal{S})$ to ensure the well posedness of the equation;
- c is the corresponding coefficient of the reaction term of (2.14), $\mathcal{HB}$, in the scalar case.

All the differential operators are written intrinsically in the LCS coordinate system. Recalling the definition of the *Hilbert space*:

$$H^1(\mathcal{S}) = \Big\{ w : \mathcal{S} \longrightarrow \mathbb{R} \mid w \in L^2(\mathcal{S}), \nabla w \in L^2(\mathcal{S}) \Big\} \tag{3.2}$$

we define the space of the test function as $\mathcal{V}(\mathcal{S}) = \{v \in H^1(\mathcal{S}) \mid \int_\Gamma v = 0\}$. Then, recalling the definition of the intrinsic operators in section 1.3.1 and the intrinsic version of the *Green Lemma* in lemma 1.3.5, we can write the *intrinsic variational formulation* of eq. (3.1) by multiplying by a test function $v \in \mathcal{V}(\mathcal{S})$ and then integrating over the surface $\mathcal{S}$. We obtain the following variational formulation:

Problem 1. *Find* $u \in L^2(I_T; H^1(\mathcal{S}))$ *such that*

$$m(\partial_t u, v) + a(u, v) + b(u, v) = F(u) \qquad \forall v \in \mathcal{V}(\mathcal{S})$$

where the bilinear forms are given by:

$$m(\partial_t u, v) = \int_{\mathcal{S}} \partial_t u \; v \; ds, \qquad a(u, v) = \int_{\mathcal{S}} \langle \nabla_{\mathcal{G}} u, \nabla_{\mathcal{G}} v \rangle_{\mathcal{G}} \; ds$$
$$b(u, v) = c \int_{\mathcal{S}} u \; v \; ds$$

and the right-and-side:

$$F(v) = \int_{\mathcal{S}} f \; v \; ds$$

We are now ready to explore the intrinsic formulation of the *finite element method.* We start by considering a surface triangulation $\mathcal{T}(\mathcal{S})$ of the surface S formed by the union of non-intersecting surface triangles T_i, with vertices on $\mathcal{S}$. The edges connecting the vertices are geodesic curves of minimal length. Therefore, we have that $\mathcal{T}(\mathcal{S}) = \cup_{i=1}^{N_T} T_i$, with N_T the total number of nodes of the triangulation, and every non empty intersection between two curved triangles, $\sigma_{ij} = T_i \cap T_j$, is and internal geodesic edge or vertex.

We can define the space of the basis functions generating the finite-dimensional FEM space, over the curved triangulation of the surface as $\mathcal{V}_h(\mathcal{T}(\mathcal{S}))$, and for this we use the lowest order conforming approach, which gives:

$$\mathcal{V}_h(\mathcal{T}(\mathcal{S})) = \{v \in C^0(\mathcal{T}(\mathcal{S})) \mid v_{|T} \in \mathcal{P}_1(T) \qquad \forall\, T \in \mathcal{T}(\mathcal{S})\}, \tag{3.3}$$

where $\mathcal{P}_1(T)$ is the space of the polynomial of first degree. In particular, $\mathcal{V}_h(\mathcal{T}(\mathcal{S}))$ is spanned by the nodal basis functions $\varphi_1, \cdots, \varphi_{N_T}$ defined by requiring the classical interpolation property:

$$\varphi_j \in \mathcal{V}_h(\mathcal{T}(\mathcal{S})), \qquad \varphi_j(P_i) = \delta_{ij}, \qquad i, j = 1, \cdots, N_T \tag{3.4}$$

where P_i are the vertices of the triangulations. Then, every function $v \in \mathcal{V}_h(\mathcal{T}(\mathcal{S}))$ can be written as

$$v(x) = \sum_{j=1}^{N_T} v_j \; \varphi_j(x) \qquad x \in \mathcal{S}$$

with v_j the nodal coefficients. From the practical point of view, to calculate each basis function we proceed as follows. Given the global coordinates $x(P)$ of $P \in T$, we can define the affine function $\tilde{\varphi}(x) = \tilde{a} + \tilde{b}x^1 + \tilde{c}x^2 + \tilde{d}x^3$ as a function in $\mathbb{R}^3$. Then by composition with the local parametrization we can get the basis function in the local coordinates:

$$\varphi_j^T(s^1, s^2) = \tilde{\varphi}_j^T \circ \phi(s^1, s^2).$$

Moreover, assuming that for each cell T there exist an open set $U \subset \mathbb{R}^2$ such that $T \subset \phi_{m_t}(U)$, for a point $m_T \in T$, we have that

$$\phi_{m_T} = T_{m_T}(\mathcal{S}) + \mathcal{O}(h^2).$$

Exploiting this relation, it is enough to re-write $\tilde{\varphi}_j^T$ in the local coordinates of $T_{m_T}(\mathcal{S})$, thus maintaining the "linear" feature of the basis function.

Observation 2. *We remark that, up to the definition of $\mathcal{V}_h(\mathcal{T}(\mathcal{S}))$, everything has defined without any approximations, since every quantities has been defined on $\mathcal{T}(\mathcal{S})$ whose interior coincides with the surface $\mathcal{S}$. Approximations issues arise when we need to compute quantities, and in our case we find them in appropriate quadrature rules for surface integrals and in the calculation of relevant quantities on the cells. So to keep the exact setting as much as possible, we assume that all relevant geometric information related to the surface are known at the vertices of the triangulation, even approximated but in consistent form.*

Therefore, we can formulate the intrinsic SFEM variational formulation of (3.1) written in the LCS as:

Problem 2. *Find $u_h \in L^2(I_T; \mathcal{V}_h(\mathcal{T}(\mathcal{S})))$ such that*

$$m(\partial_t u_h, v) + a(u_h, v) + b(u_h, v) = F(u_h) \qquad \forall v \in \mathcal{V}_h(\mathcal{T}\mathcal{S})$$

where the bilinear forms are given by:

$$m(\partial_t u_h, v) = \int_{\mathcal{S}} \partial_t u_h \; v \; ds \qquad\qquad a(u_h, v) = \int_{\mathcal{S}} \langle \nabla_{\mathcal{G}} u_h, \nabla_{\mathcal{G}} v \rangle \; ds$$

$$b(u_h, v) = c \int_{\mathcal{S}} u_h \; v \; ds$$

and the right and side form:

$$F(v) = \int_{\mathcal{S}} f \; v \; ds$$

We can now express the numerical solution as linear combination of the basis functions, i.e. $u_h = \sum_{j=1}^{N_T} u_j \varphi_j$. For every $i = 1, \cdots, N_T$ we obtain:

$$\sum_{j=1}^{N_T} \partial_t u_j m(\varphi_j, \varphi_i) + \sum_{j=1}^{N_T} \partial_t u_j (a(\varphi_j, \varphi_i) + b(\varphi_j, \varphi_i)) = F(\varphi_i) \,,$$

We can also re-write the scheme in matrix form:

$$\mathbf{M} \partial_t u + (\mathbf{A} + \mathbf{B}) u = \mathbf{F} \,,$$

where $u = \{u_i\}$ is the vector of the coefficients with respect to the basis functions, and $b = \{F(\varphi_i)\}$ is the right-hand-side vector. The above matrices take the following expressions:

$$\mathbf{M_{ij}} = \int_{\mathcal{S}} \varphi_j \varphi_i \; \mathrm{dx} \qquad\qquad \mathbf{A_{ij}} = \int_{\mathcal{S}} \langle \nabla_{\mathcal{G}} \varphi_j, \nabla_{\mathcal{G}} \varphi_i \rangle \; \mathrm{dx}$$

$$\mathbf{B_{ij}} = c \int_{\mathcal{S}} \varphi_j \; \varphi_i \; \mathrm{dx}$$

For the time discretization, we consider an Implicit Euler scheme. Let $\{t^n\}_{n=0}^N$ be a partition of the time interval I_T and $\Delta t^n = t^{n+1} - t^n$ the n-th time step. For $n = 0, \cdots, N-1$, the scheme reads:

$$\left(\frac{1}{\Delta t^n}\mathbf{M} + \mathbf{A} + \mathbf{B}\right) u^{n+1} = \mathbf{F} + \frac{1}{\Delta t^n}\mathbf{M} u^n \tag{3.5}$$

where the superscripts indicate the time-step evaluation.

To calculate the surface integrals we use the Gaussian quadrature rules, so it is necessary to evaluate the nodal basis functions at quadrature points P_j inside the cell T. For this reason, the knowledge of the set of tangent vectors at the point is needed. Since we are considering low order polynomials basis function, the optimal accuracy is still ensured already by considering first order quadrature rules. We consider here the mid-point rule, so tangent vectors and metric needs to be defined at the mid-point of each cell. If geometric quantities are known only at the vertices of the triangulation, a linear interpolation of these at mid-points is enough to ensure the accuracy of the scheme (see [6]).

3.2 Intrinsic Surface FEM in the vector case

We consider now the vector form of the equation (3.1) on a regular surface $\mathcal{S} \subset \mathbb{R}^3$:

$$\frac{\partial \mathbf{u}(x,t)}{\partial t} - \Delta_{\mathcal{G}} \mathbf{u}(x,t) + \tilde{B}\mathbf{u}(x,t) = \mathbf{f}(x) \qquad \text{on } \mathcal{S}, \tag{3.6}$$

where

- $\mathbf{u} : \mathcal{S} \times I_T \longrightarrow \mathbb{R}^3$ is a tangent vector to the surface $\mathbf{u} \in T\mathcal{S}$, depending on time in $I_T = (0, T] \subset \mathbb{R}$;
- $\mathbf{f} \in L^2(\mathcal{S})$ to ensure the well posedness of the equation;
- $\tilde{B}$ is a 1 contravariant-1 covariant tensor, so in our case it maps vectors to vectors.

This can be directly related to the final equation of the previous chapter, eq. (2.14), by considering $\tilde{B} = \mathcal{HB}$. In this case, to ensure the well posedness of the problem, we add to $\tilde{B}$ the term $c\mathbb{I}$ with $c > 0$ and consider $\mathcal{H} > 0$, in this way we get that $\tilde{B} = \mathcal{HB} + c\mathbb{I}$ is positive semi-definite, since $\mathcal{B}$ is symmetric. With this observation, the bilinear form results to be coercive and the existence of the solution is guaranteed by the Lax-Milgram theorem.

We consider the solution $\mathbf{u} = u^1\mathbf{t_1} + u^2\mathbf{t_2}$ written in contravariant coordinates, and we need to extend our functional spaces to a two-dimensional form, due to the fact that the solution depends on the tangent space to the surface, which is a two-dimensional space. The corresponding two-dimensional extension of the Hilbert space (3.2) is then:

$$\vec{H}^1(\mathcal{S}) = \left[H^1(\mathcal{S})\right]^2 = \left\{\mathbf{w} : \mathcal{S} \longrightarrow \mathbb{R}^3 \mid \mathbf{w} \in L^2(\mathcal{S}), \nabla\mathbf{w} \in L^2(\mathcal{S})\right\},$$

in which we consider the following $L^2(\mathcal{S})$ scalar products: for every $\mathbf{v}, \mathbf{w} \in T\mathcal{S}$ and every $\nabla_{\mathcal{G}}\mathbf{v}, \nabla_{\mathcal{G}}\mathbf{w} : T\mathcal{S} \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3$ we define

$$(\mathbf{v}, \mathbf{w})_{L^2(\mathcal{S})} := \int_{\mathcal{S}} \langle \mathbf{v}, \mathbf{w} \rangle_{\mathcal{G}} \, \mathrm{ds},$$

$$(\nabla_{\mathcal{G}}\mathbf{v}, \nabla_{\mathcal{G}}\mathbf{w})_{L^2(\mathcal{S})} := \int_{\mathcal{S}} \nabla_{\mathcal{G}}\mathbf{v} :_{\mathcal{G}} \nabla_{\mathcal{G}}\mathbf{w} \, \mathrm{ds}.$$

The term $\nabla_{\mathcal{G}}\mathbf{v} :_{\mathcal{G}} \nabla_{\mathcal{G}}\mathbf{w}$ is a double-scalar product in the LCS, and is defined as the sum of the component-wise product of the two elements:

$$\nabla_{\mathcal{G}}\mathbf{v} :_{\mathcal{G}} \nabla_{\mathcal{G}}\mathbf{w} = \sum_{i,j}(\nabla_{\mathcal{G}}\mathbf{v})_{(ij)}(\nabla_{\mathcal{G}}\mathbf{w})_{(ij)} = \sum_{i,j}(\nabla\mathbf{v})^{ij}(\nabla\mathbf{w})^{ij}h_{(i)}^2 h_{(j)}^2 . \tag{3.7}$$

We also need to extend the test-space to the vector case:

$$\vec{\mathcal{V}}(\mathcal{S}) = \mathcal{V}(\mathcal{S}) \times \mathcal{V}(\mathcal{S}) = \left\{\mathbf{v} \in \left[H^1(\mathcal{S})\right]^2 \Big| \right\}. \tag{3.8}$$

Then testing eq. (3.6) by a test function $\mathbf{v}$, we obtain the *variational formulation* for the vector case:

Problem 3. *Find* $\mathbf{u} \in L^2(I_T; \vec{H}^1(\mathcal{S}))$ *such that*

$$m(\partial_t\mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{v}) = F(\mathbf{u}) \qquad \forall \mathbf{v} \in \vec{\mathcal{V}}(\mathcal{S}) \tag{3.9}$$

where the bilinear forms are given by:

$$m(\partial_t\mathbf{u}, \mathbf{v}) = \int_{\mathcal{S}} \langle \partial_t\mathbf{u}, \mathbf{v}\rangle_{\mathcal{G}} \; ds \qquad\qquad a(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{S}} \nabla_{\mathcal{G}}\mathbf{u} :_{\mathcal{G}} \nabla_{\mathcal{G}}\mathbf{v} \; ds$$

$$b(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{S}} \langle \tilde{B}\mathbf{u}, \mathbf{v}\rangle_{\mathcal{G}} \; ds$$

and the right-and-side linear form:

$$F(\mathbf{v}) = \int_{\mathcal{S}} \langle \mathbf{f}, \mathbf{v}\rangle_{\mathcal{G}} \; ds$$

.

3.2.1 Fulle-discrete scheme in the vector case

We start by the definition of the space of the basis functions generating the FEM space as the extension of the scalar one, presented in the previous section. By considering a surface triangulation, as before, we can write:

$$\vec{\mathcal{V}}_h(\mathcal{T}(\mathcal{S})) = \mathcal{V}_h(\mathcal{T}(\mathcal{S})) \times \mathcal{V}_h(\mathcal{T}(\mathcal{S}))$$

with $\mathcal{V}_h(\mathcal{T}(\mathcal{S}))$ defined in eq. (3.3). We write our numerical solution as $\mathbf{u}_h = u_h^1\mathbf{t_1} + u_h^2\mathbf{t_2}$, where each contravariant component can be written in terms of the scalar basis functions:

$$\mathbf{u}_h = \sum_{j=1}^{N} u_j^1\varphi_j\mathbf{t_1} + u_j^2\varphi_j\mathbf{t_2}. \tag{3.10}$$

Then, we consider all the expression for the differential operators as described in section 1.3.1. The idea is to expand the computations and collect back terms in a block-structured matrix as:

$$\left[\begin{array}{c|c} \text{comp-1 } \mathbf{u},\ \text{comp-1 } \mathbf{v}(N\times N) & \text{comp-2 } \mathbf{u},\ \text{comp-1 } \mathbf{v}(N\times N) \\ \hline \text{comp-1 } \mathbf{u},\ \text{comp-2 } \mathbf{v}(N\times N) & \text{comp-2 } \mathbf{u},\ \text{comp-2 } \mathbf{v}(N\times N) \end{array}\right] \left[\begin{array}{c} \mathbf{u}^1(N\times 1) \\ \hline \mathbf{u}^2(N\times 1) \end{array}\right] = \left[\begin{array}{c} \mathbf{F}^1(N\times 1) \\ \hline \mathbf{F}^2(N\times 1) \end{array}\right]$$

$\mathbf{u}^1 = \{u_j^1\}, \mathbf{u^2} = \{u_j^2\}$, for $j = 1, \cdots, N$, are the coefficients of the two contravariant components in terms of the scalar basis functions.

We organize our work, by considering each bilinear form separately. We start by considering the weak *Laplacian* term:

$$
\left[\begin{array}{c|c}
\begin{gathered}
\frac{\partial\varphi_\ell}{\partial s^1}\frac{\partial\varphi_m}{\partial s^1}+\frac{\partial\varphi_\ell}{\partial s^1}\varphi_m\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}+\varphi_\ell\frac{\partial\varphi_m}{\partial s^1}\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\\
+\varphi_\ell\varphi_m\left[\left(\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\right)^2+\left(\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\right)^2+\frac{1}{h_{(2)}^2}\left(\frac{\partial h_{(1)}}{\partial s^2}\right)^2\right]\\
+\frac{1}{h_{(2)}^2}\frac{\partial h_{(1)}\varphi_\ell}{\partial s^2}\frac{\partial h_{(1)}\varphi_m}{\partial s^2}
\end{gathered}
&
\begin{gathered}
\varphi_\ell\frac{\partial\varphi_m}{\partial s^1}\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}+\frac{\partial\varphi_\ell}{\partial s^2}\varphi_m\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\\
+\varphi_\ell\varphi_m\left(\frac{1}{h_{(1)}^2}\frac{\partial h_{(1)}}{\partial s^2}\frac{\partial h_{(1)}}{\partial s^1}+\frac{1}{h_{(2)}^2}\frac{\partial h_{(2)}}{\partial s^2}\frac{\partial h_{(2)}}{\partial s^1}\right)\\
-\varphi_\ell\frac{\partial\varphi_m}{\partial s^2}\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}-\frac{\partial\varphi_\ell}{\partial s^1}\varphi_m\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\\
-\varphi_\ell\varphi_m\frac{1}{h_{(1)}h_{(2)}}\left(\frac{\partial h_{(2)}}{\partial s^1}\frac{\partial h_{(1)}}{\partial s^2}+\frac{\partial h_{(2)}}{\partial s^1}\frac{\partial h_{(1)}}{\partial s^2}\right)
\end{gathered}
\\ \hline
\begin{gathered}
\frac{\partial\varphi_\ell}{\partial s^1}\varphi_m\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}+\varphi_\ell\frac{\partial\varphi_m}{\partial s^2}\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\\
+\varphi_\ell\varphi_m\left(\frac{1}{h_{(1)}^2}\frac{\partial h_{(1)}}{\partial s^1}\frac{\partial h_{(1)}}{\partial s^2}+\frac{1}{h_{(2)}^2}\frac{\partial h_{(2)}}{\partial s^1}\frac{\partial h_{(2)}}{\partial s^2}\right)\\
-\frac{\partial\varphi_\ell}{\partial s^2}\varphi_m\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}-\varphi_\ell\frac{\partial\varphi_m}{\partial s^1}\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\\
-\varphi_\ell\varphi_m\frac{1}{h_{(1)}h_{(2)}}\left(\frac{\partial h_{(2)}}{\partial s^1}\frac{\partial h_{(1)}}{\partial s^2}+\frac{\partial h_{(2)}}{\partial s^1}\frac{\partial h_{(1)}}{\partial s^2}\right)
\end{gathered}
&
\begin{gathered}
\frac{\partial\varphi_\ell}{\partial s^2}\frac{\partial\varphi_m}{\partial s^2}+\frac{\partial\varphi_\ell}{\partial s^2}\varphi_m\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^2}+\varphi_\ell\frac{\partial\varphi_m}{\partial s^2}\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^2}\\
+\varphi_\ell\varphi_m\left[\left(\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^2}\right)^2+\left(\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\right)^2+\frac{1}{h_{(1)}^2}\left(\frac{\partial h_{(2)}}{\partial s^1}\right)^2\right]\\
+\frac{1}{h_{(1)}^2}\frac{\partial h_{(2)}\varphi_\ell}{\partial s^1}\frac{\partial h_{(2)}\varphi_m}{\partial s^1}
\end{gathered}
\end{array}\right]
\left[\begin{array}{c} u_1^1\\ \vdots\\ u_N^1\\ \hline u_1^2\\ \vdots\\ u_N^2\end{array}\right]
$$

Then, collecting terms together we get:

$$
\begin{bmatrix}\langle\nabla_{\mathcal{G}}\,\varphi_\ell,\nabla_{\mathcal{G}}\,\varphi_m^1\rangle_{\mathcal{G}} & 0\\ 0 & \langle\nabla_{\mathcal{G}}\,\varphi_\ell,\nabla_{\mathcal{G}}\,\varphi_m^2\rangle_{\mathcal{G}}\end{bmatrix}\begin{bmatrix}h_{(1)}^2 & 0\\ 0 & h_{(2)}^2\end{bmatrix}
$$

$$
+\begin{bmatrix}\varphi_\ell\varphi_m & 0\\ 0 & \varphi_\ell\varphi_m\end{bmatrix}\begin{bmatrix}\left(\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\right)^2+\left(\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\right)^2+\frac{2}{h_{(2)}^2}\left(\frac{\partial h_{(1)}}{\partial s^2}\right)^2 & 0\\ 0 & \left(\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\right)^2+\left(\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\right)^2+\frac{2}{h_{(2)}^2}\left(\frac{\partial h_{(1)}}{\partial s^2}\right)^2\end{bmatrix}
$$

$$
+\begin{bmatrix}0 & \varphi_\ell\varphi_m\\ \varphi_\ell\varphi_m & 0\end{bmatrix}\begin{bmatrix}0 & \frac{1}{h_{(1)}^2}\frac{\partial h_{(1)}}{\partial s^1}\frac{\partial h_{(1)}}{\partial s^2}+\frac{1}{h_{(2)}^2}\frac{\partial h_{(2)}}{\partial s^1}\frac{\partial h_{(2)}}{\partial s^2}-\frac{2}{h_{(1)}h_{(2)}}\frac{\partial h_{(1)}}{\partial s^2}\frac{\partial h_{(2)}}{\partial s^1}\\ \frac{1}{h_{(1)}^2}\frac{\partial h_{(1)}}{\partial s^1}\frac{\partial h_{(1)}}{\partial s^2}+\frac{1}{h_{(2)}^2}\frac{\partial h_{(2)}}{\partial s^1}\frac{\partial h_{(2)}}{\partial s^2}-\frac{2}{h_{(1)}h_{(2)}}\frac{\partial h_{(1)}}{\partial s^2}\frac{\partial h_{(2)}}{\partial s^1} & 0\end{bmatrix}
$$

$$
+\begin{bmatrix}
\left\langle\begin{bmatrix}\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\\ \frac{h_{(1)}}{h_{(2)}^2}\frac{\partial h_{(1)}}{\partial s^2}\end{bmatrix},\nabla_{\mathcal{G}}\,\varphi_\ell\right\rangle_{\mathcal{G}}\varphi_m^1 & \left\langle\begin{bmatrix}-\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\\ \frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\end{bmatrix},\nabla_{\mathcal{G}}\,\varphi_\ell\right\rangle_{\mathcal{G}}\varphi_m^1\\
\left\langle\begin{bmatrix}\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\\ -\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\end{bmatrix},\nabla_{\mathcal{G}}\,\varphi_\ell\right\rangle_{\mathcal{G}}\varphi_m^2 & \left\langle\begin{bmatrix}\frac{h_{(2)}}{h_{(1)}^2}\frac{\partial h_{(2)}}{\partial s^1}\\ \frac{1}{h_{(1)}}\frac{\partial h_{(2)}}{\partial s^2}\end{bmatrix},\nabla_{\mathcal{G}}\,\varphi_\ell\right\rangle_{\mathcal{G}}\varphi_m^2
\end{bmatrix}
$$

$$
+\begin{bmatrix}
\left\langle\begin{bmatrix}\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^1}\\ \frac{h_{(1)}}{h_{(2)}^2}\frac{\partial h_{(1)}}{\partial s^2}\end{bmatrix},\nabla_{\mathcal{G}}\,\varphi_m\right\rangle_{\mathcal{G}}\varphi_\ell & \left\langle\begin{bmatrix}\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\\ -\frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\end{bmatrix},\nabla_{\mathcal{G}}\,\varphi_m\right\rangle_{\mathcal{G}}\varphi_\ell\\
\left\langle\begin{bmatrix}-\frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\\ \frac{1}{h_{(2)}}\frac{\partial h_{(2)}}{\partial s^1}\end{bmatrix},\nabla_{\mathcal{G}}\,\varphi_m\right\rangle_{\mathcal{G}}\varphi_\ell & \left\langle\begin{bmatrix}\frac{h_{(2)}}{h_{(1)}^2}\frac{\partial h_{(1)}}{\partial s^1}\\ \frac{1}{h_{(1)}}\frac{\partial h_{(1)}}{\partial s^2}\end{bmatrix},\nabla_{\mathcal{G}}\,\varphi_m\right\rangle_{\mathcal{G}}\varphi_\ell
\end{bmatrix}
$$

So, for $i, j = 1, 2$ that denotes the indices for the vector-components, $l, m = 1, \cdots, N$ the indices for the basis function decomposition, we can define the following quantities:

$$
\begin{aligned}
\mathbf{A}^{ii}_{lm} &= \int_{\mathcal{S}} h^2_{(i)} \langle \nabla_g \varphi_m, \nabla_g \varphi_l \rangle_g ; \\
\mathbf{M}^{ij}_{lm} &= \int_{\mathcal{S}} \mu_{ij} \varphi_m \varphi_l \qquad \text{with} \\
&\qquad \mu_{11} = \left(\frac{1}{h_{(1)}} \frac{\partial h_{(1)}}{\partial s^1} \right)^2 + \left(\frac{1}{h_{(2)}} \frac{\partial h_{(2)}}{\partial s^1} \right)^2 + \frac{2}{h^2_{(2)}} \left(\frac{\partial h_{(1)}}{\partial s^2} \right)^2 , \\
&\qquad \mu_{22} = \left(\frac{1}{h_{(2)}} \frac{\partial h_{(2)}}{\partial s^2} \right)^2 + \left(\frac{1}{h_{(1)}} \frac{\partial h_{(1)}}{\partial s^2} \right)^2 + \frac{2}{h^2_{(1)}} \left(\frac{\partial h_{(2)}}{\partial s^1} \right)^2 , \\
&\qquad \mu_{12} = \frac{1}{h^2_{(1)}} \frac{\partial h_{(1)}}{\partial s^1} \frac{\partial h_{(1)}}{\partial s^2} + \frac{1}{h^2_{(2)}} \frac{\partial h_{(2)}}{\partial s^1} \frac{\partial h_{(2)}}{\partial s^2} - \frac{2}{h_{(1)} h_{(2)}} \frac{\partial h_{(1)}}{\partial s^2} \frac{\partial h_{(2)}}{\partial s^1} ; \\
\mathbf{B}^{ij}_{lm} &= \int_{\mathcal{S}} \langle \mathbf{w}_{ij}, \nabla_g \varphi_m \rangle_g \varphi_l \qquad \text{with} \\
&\qquad \mathbf{w}_{11} = \begin{bmatrix} \frac{1}{h_{(1)}} \frac{\partial h_{(1)}}{\partial s^1} \\ \frac{h_{(1)}}{h^2_{(2)}} \frac{\partial h_{(1)}}{\partial s^2} \end{bmatrix}, \quad \mathbf{w}_{22} = \begin{bmatrix} \frac{h_{(2)}}{h^2_{(1)}} \frac{\partial h_{(1)}}{\partial s^1} \\ \frac{1}{h_{(1)}} \frac{\partial h_{(1)}}{\partial s^2} \end{bmatrix}, \quad \mathbf{w}_{12} = -\mathbf{w}_{21} = \begin{bmatrix} -\frac{1}{h_{(1)}} \frac{\partial h_{(1)}}{\partial s^2} \\ \frac{1}{h_{(2)}} \frac{\partial h_{(2)}}{\partial s^1} \end{bmatrix}.
\end{aligned}
$$

Finally, the double-scalar product term can be expressed in the following block-matrix formulation:

$$
\begin{bmatrix} \mathbb{A}^{11} & \mathbb{B} \\ \mathbb{B}^T & \mathbb{A}^{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{11} + \mathbf{M}^{11} + \mathbf{B}^{11} + \mathbf{B}^{T,11} & \mathbf{M}^{12} + \mathbf{B}^{12} + \mathbf{B}^{T,12} \\ \mathbf{M}^{21} + \mathbf{B}^{21} + \mathbf{B}^{T,21} & \mathbf{A}^{22} + \mathbf{M}^{22} + \mathbf{B}^{22} + \mathbf{B}^{T,22} \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{bmatrix}.
$$

Remark 2. *We remark that all the $N \times N$-matrices are obtained as in the scalar case, with coefficients that depend on the surface through the values of the metric, i.e. $h_{(1)}, h_{(2)}$ and its derivatives $\frac{\partial h_{(i)}}{\partial s^j}$, for $i, j = 1, 2$. In particular, when the metric has constant coefficients, for example if the surface is **flat**, the system becomes a lot easier. In fact, since the metric becomes the identity, the matrix reduces to a diagonal block system in which the two components are completely decoupled:*

$$
\begin{bmatrix} \int_{\mathcal{S}} \langle \nabla_g \varphi_m, \nabla_g \varphi_l \rangle_g & 0 \\ 0 & \int_{\mathcal{S}} \langle \nabla_g \varphi_m, \nabla_g \varphi_l \rangle_g \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{bmatrix}.
$$

Considering now the *reaction term*, we get that:

$$
\langle \tilde{B} \varphi_\ell, \varphi_m \rangle_g = \left\langle \begin{pmatrix} \tilde{B}^1_1 \varphi^1_\ell + \tilde{B}^1_2 \varphi^2_\ell \\ \tilde{B}^2_1 \varphi^1_\ell + \tilde{B}^2_2 \varphi^2_\ell \end{pmatrix}, \varphi_m \right\rangle_g = h^2_{(1)} (\tilde{B}^1_1 \varphi^1_\ell \varphi^1_m + \tilde{B}^1_2 \varphi^2_\ell \varphi^1_m) + h^2_{(2)} (\tilde{B}^2_1 \varphi^1_\ell \varphi^2_m + \tilde{B}^2_2 \varphi^2_\ell \varphi^2_m) ,
$$

and then, defining

$$
\mathbf{R}^{ij}_{\ell m} = \int_{\mathcal{S}} h^2_{(i)} \tilde{B}^i_j \varphi_\ell \varphi_m ,
$$

we can re-write the reaction block-matrix as:

$$
\begin{bmatrix} \mathbf{R}^{11} & \mathbf{R}^{12} \\ \mathbf{R}^{21} & \mathbf{R}^{22} \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{bmatrix}.
$$

Finally, the mass matrix in the time derivative term, and the right-hand-side vector, are given by:

$$\mathbf{T}^i_{\ell m} = \int_{\mathcal{S}} h^2_{(i)} \varphi_\ell \varphi_m \,, \qquad \text{and} \qquad \mathbf{F}^i_l = \int_{\mathcal{S}} h^2_{(i)} f^i \varphi_l .$$

Going back to eq. (3.9), we can express the final system in the following way:

$$\begin{bmatrix} \mathbf{T}^1 & 0 \\ 0 & \mathbf{T}^2 \end{bmatrix} \begin{bmatrix} \partial_t \mathbf{u}^1 \\ \partial_t \mathbf{u}^2 \end{bmatrix} + \left(\begin{bmatrix} \mathbb{A}^{11} & \mathbb{B} \\ \mathbb{B}^T & \mathbb{A}^{22} \end{bmatrix} + \begin{bmatrix} \mathbf{R}^{11} & \mathbf{R}^{12} \\ \mathbf{R}^{21} & \mathbf{R}^{22} \end{bmatrix} \right) \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{bmatrix},$$

Remark 3. *Due to the nature of the linear system that we get, we expect the final matrix to be symmetric. In the numerical tests, we will show that we actually get a symmetric system.*

Concerning the time discretization, we consider an *Implicit Euler* scheme. Then, for $n = 0, \cdots, N-1$, we can write the *fully-discrete ISFEM scheme* as:

Problem 4. *Find* $\mathbf{u} \in L^2(I_T, \vec{\mathcal{V}}_h(\mathcal{T}(\mathcal{S})))$ *such that:*

$$\left(\frac{1}{\Delta t^n} \begin{bmatrix} \mathbf{T}^1 & 0 \\ 0 & \mathbf{T}^2 \end{bmatrix} + \begin{bmatrix} \mathbb{A}^{11} & \mathbb{B} \\ \mathbb{B}^T & \mathbb{A}^{22} \end{bmatrix} + \begin{bmatrix} \mathbf{R}^{11} & \mathbf{R}^{12} \\ \mathbf{R}^{21} & \mathbf{R}^{22} \end{bmatrix} \right) \begin{bmatrix} \mathbf{u}^1_{n+1} \\ \mathbf{u}^2_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{bmatrix} + \frac{1}{\Delta t^n} \begin{bmatrix} \mathbf{T}^1 & 0 \\ 0 & \mathbf{T}^2 \end{bmatrix} \begin{bmatrix} \mathbf{u}^1_{h,n} \\ \mathbf{u}^2_{h,n} \end{bmatrix} \tag{3.11}$$

with $\{t^n\}_{n=0}^N$ *be a partition of* I_T, $\Delta t^n = t^{n+1} - t^n$ *the n-th time step.*

Then, to solve the system, we proceed as in the scalar case (3.5). So, we compute the integrals in space applying the surface mid-point rule, which is a consistent rule with the same accuracy of the linear SFEM.

3.3 Numerical Results

In this section, we verify experimentally the applicability of the previously derived scheme to solve eq. (2.14) from chapter 3, in numerical cases with and without time dependence. For simplicity, we take $\tilde{B} = \mathbb{I}$ to be the reaction coefficient.

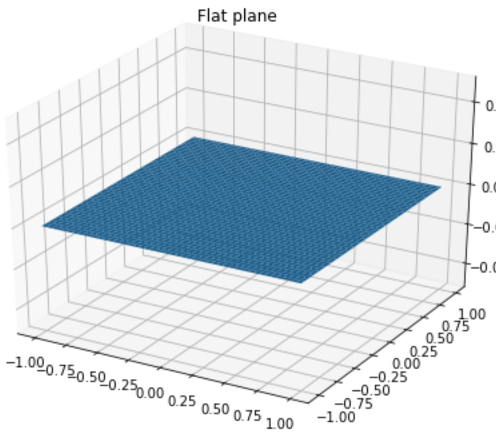
We will start testing the convergence of the scheme on a *stationary case*, i.e. where $\partial_t u = 0$. We observe that in this case, the system (3.11) simplifies to:

$$\left(\begin{bmatrix} \mathbb{A}^{11} & \mathbb{B} \\ \mathbb{B}^T & \mathbb{A}^{22} \end{bmatrix} + \begin{bmatrix} \mathbf{R}^{11} & \mathbf{R}^{12} \\ \mathbf{R}^{21} & \mathbf{R}^{22} \end{bmatrix} \right) \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{bmatrix}. \tag{3.12}$$

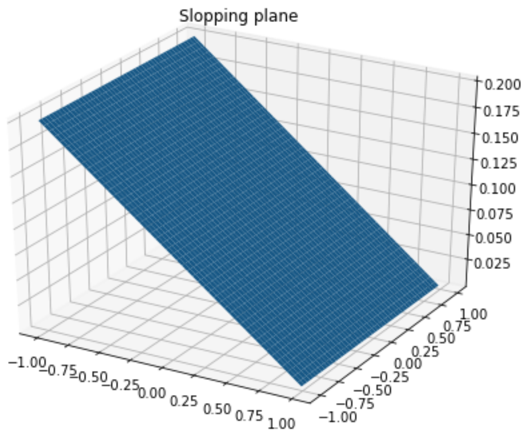
To check the numerical convergence, we consider a *fixed solution* and calculate the corresponding right-hand-side vector from the equation (2.14):

$$\mathbf{f}(x^1, x^2) = -\Delta_{\mathcal{G}} \mathbf{u}_{\text{fix}}(x^1, x^2) + \mathbb{I}\mathbf{u}_{\text{fix}}(x^1, x^2).$$

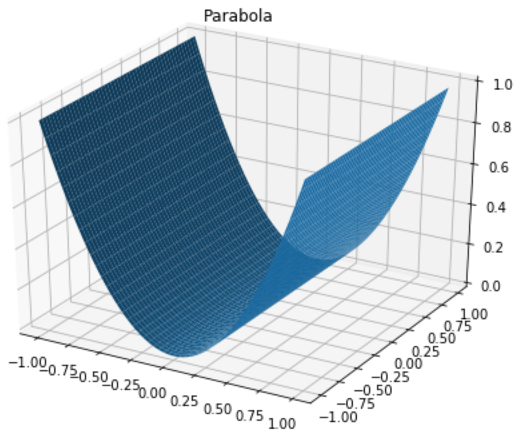
We consider three different domains: a horizontal plane, a sloping plane and a parabola case. Figure 3.1 shows the representation of these surfaces. In particular, we will use the same domain $U = [-1, 1] \subset \mathbb{R}^2$ for the charts needed to define the three different surfaces. The choice follows an increase complexity in terms of the geometric information on the surface. On the horizontal plane, as previously observed in remark 2, the system turns out to be decoupled in terms of the two components. The sloping plane is the first case in which the metric is not the identity



(a)



(b)



(c)

Figure 3.1: Three test surfaces

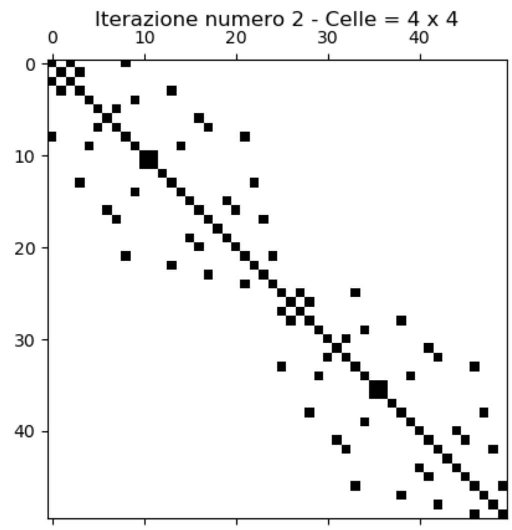


Figure 3.2: Structure of the block matrix of the system for the each case.

matrix, but it is still constant on the whole surface. With a similar argument as in remark 2, we can see that the system simplified to be decoupled in the components. Both these cases do not present the extra matrices $\mathbf{M}^{ij}$ and $\mathbf{B}^{ij}$, because the coefficients depend explicitly on the derivative of the metric tensor. In these cases, we take a non-trivial and non-linear fixed solutions $\mathbf{u}_{fix} = [(x^1)^2, 0]$.

The case of the parabola is the first in which curvatures are considered. The system is no longer a "classical" Laplacian in the two components, but the extra terms are present. The metric is no longer constant and therefore the terms deriving from its derivatives are no longer zero everywhere. In this case the calculation of the right-hand-side from a fixed solution is not trivial anymore, and for this reason, we choose the simple solution $\mathbf{u}_{fix} = [1, 0]$ to be the fixed solution. The right-hand-side would contains all the terms coming from the geometric information of the surface. Note that, the parabola case is still a domain where the the components are decoupled, because the mixed terms in the derivative of the metric, i.e. $\frac{\partial h_i}{\partial s^j} = 0$ with $i \neq j$.

Remark 4. *As first observed in remark 3, we expect the matrix of the system* (4) *to be a block symmetric matrix. To this aim, fig. 3.2 represent the sparsity pattern of the system matrix for the three considered surfaces, on a mesh with* 4×4 *nodes.*

Finally, for *time dependent case*, we simplify the equation (2.14) by assuming $\tilde{B} = 0$. Then:

$$\partial_t \mathbf{u} - \Delta^{\mathrm{B}} \mathbf{u} = 0 \tag{3.13}$$

which is the well-known *Heat Equation.* In this case, there is only the diffusion term and we show the diffusion of the numerical solution. For this purpose, we fix an initial solution and we solve the equation in the time interval $I = [0, 1]$, with a time step of $\Delta t = 0.1$. For simplicity, we consider a parabola case defined on a two-dimensional domain $U = [-4, 4]^2$. We start from an initial solution defined as:

$$\mathbf{u}_{fix} = \begin{cases} 1 & \text{if } \sqrt{(x^1)^2 + (x^2)^2} \leq \varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

x_{nodes}	y_{nodes}	errL2sol	ratio-errL2sol
10	10	1.57869e-02	-
20	20	3.93653e-03	4.0103
40	40	9.8348e-04	4.0026
80	80	2.45829e-04	4.0006
160	160	6.14547e-05	4.0001

Table 3.1: Flat case: L^2 error norms for u and corresponding experimental convergence rates.

x_{nodes}	y_{nodes}	errL2sol	ratio-errL2sol
10	10	1.57897e-02	-
20	20	3.96393e-03	4.0103
40	40	9.90324e-04	4.0026
80	80	2.4754e-04	4.0006
160	160	6.18823e-05	4.0001

Table 3.2: Slope case: L^2 error norms for u and corresponding experimental convergence rates.

with a fixed value of ε. Now we are ready to show the results of the tests.

3.3.1 Flat case

We solve eq. (3.12) on the surface defined by the constant height function $x^3 = f(x^1, x^2) = 0$, in the domain $U = [-1, 1]^2 \subset \mathbb{R}^2$. We consider the smooth solution $\mathbf{u}_{fix} = [(x^1)^2, 0]$ and impose the boundary condition $\mathbf{u} = \mathbf{u}_{fix}$ on $\partial\mathcal{S}$. Figure 3.3 shows the numerical solution in norm, by the color map, and direction by the arrows. We can qualitatively recognize that the behavior is the one of the fixed solution, i.e., the vector field $[1, 0]$ with magnitude $(x^1)^2$. To check numerical convergence we consider a set of meshes and computer the error between the numerical solution and the fixed one. We use a starting mesh with 10 nodes in both x^1 and x^2 directions, and consider 4 level of refinements by doubling the number of nodes on each side. Table 3.1 reports the L^2-norm of the error between the exact and numerical solution on the different refinements of the grid, and the experimental rate of convergence. We can see that we obtain a convergence rate of order 4, as expected.

3.3.2 Slope case

We solveeq. (3.12) in the domain $U = [-1, 1]^2 \subset \mathbb{R}^2$ with the height function $x^3 = f(x^1, x^2) = -0.1 \cdot (x^1 - 1)$. We consider the smooth solution $\mathbf{u}_{fix} = [(x^1)^2, 0]$ and impose the boundary condition $\mathbf{u} = \mathbf{u}_{fix}$ on $\partial\mathcal{S}$. Figure 3.4 shows the numerical solution in norm, by color map, and direction by arrows. We can qualitatively recognize that the behavior is the one of the fixed solution, i.e., the vector field $[1, 0]$ with magnitude $(x^1)^2$. To check the numerical convergence we consider a set of meshes and computer error between the numerical solution and the fixed

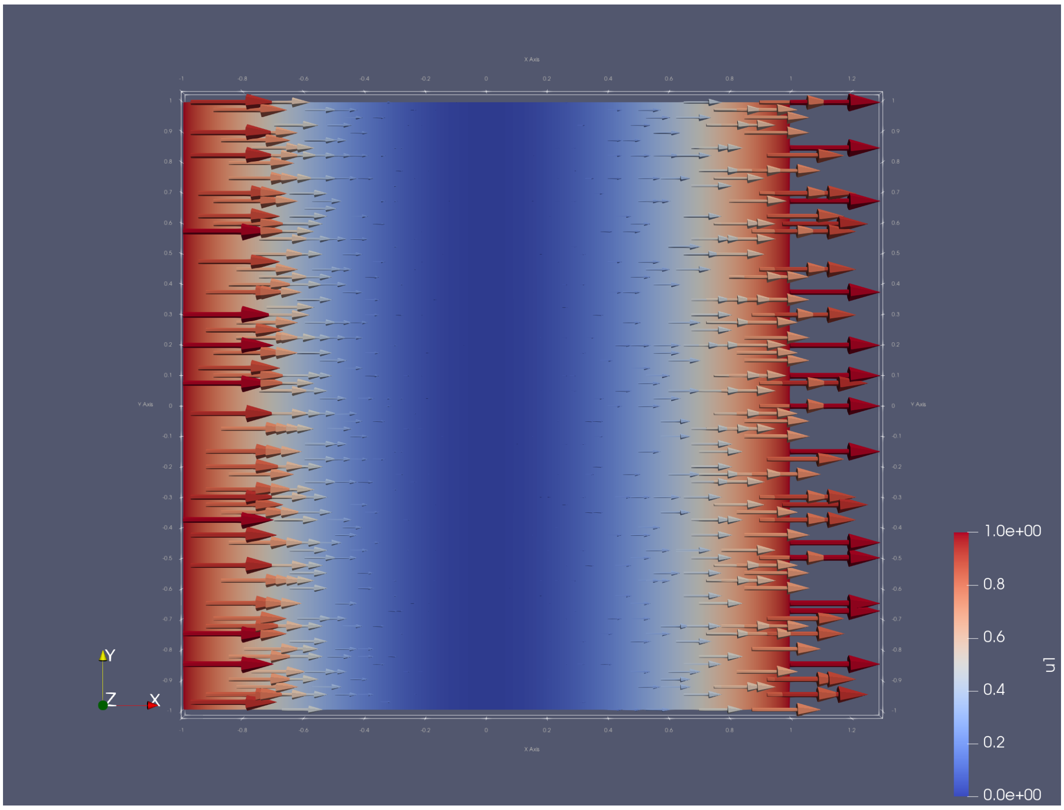


Figure 3.3: Solution on the square domain in the Flat case.

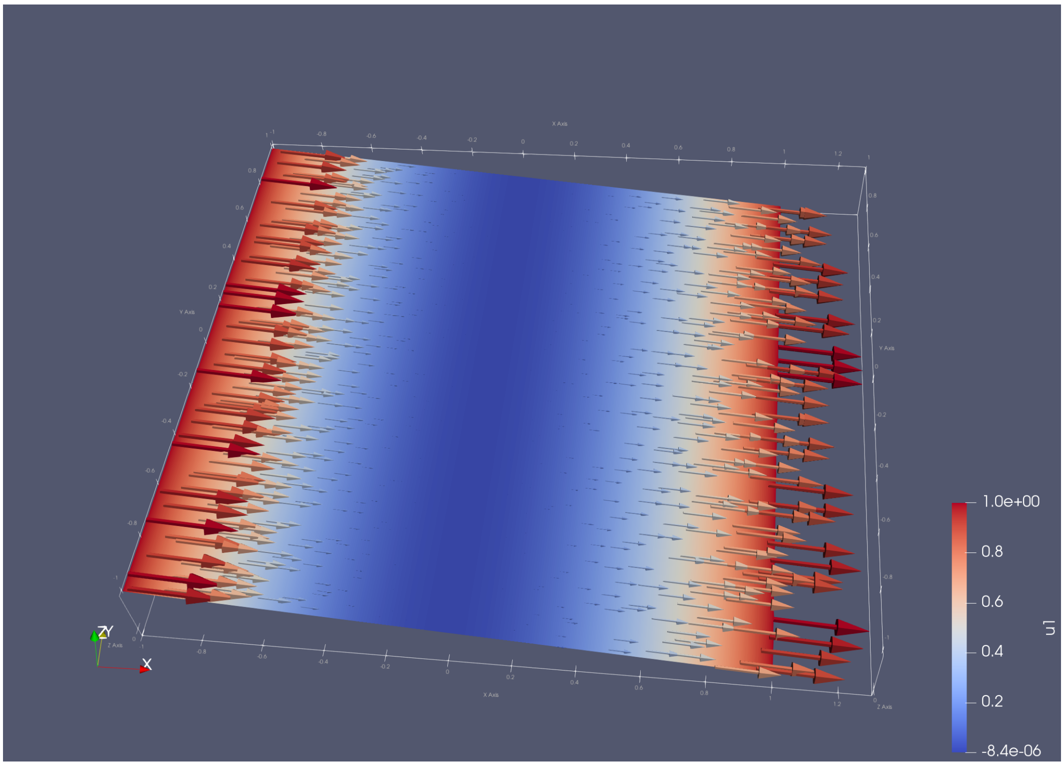


Figure 3.4: Solution on the square domain in the Slope case.

x_{nodes}	y_{nodes}	errL2sol	ratio-errL2sol
10	10	2.22572e-03	-
20	20	5.75757e-04	3.8657
40	40	1.45248-e04	3.9639
80	80	3.63948e-05	3.9908
160	160	9.10391e-06	3.9977

Table 3.3: Parabola case: L^2 error norms for u and corresponding experimental convergence rates.

one. We use a starting mesh with 10 nodes in each x^1 and x^2 directions, and consider 4 level of refinements by doubling the number of nodes on each side. Table 3.2 reports the L^2-norm of the error between the exact solution and the numerical solution on the different refinements of the grid and the experimental rate of convergence. We can see that we obtain a convergence ratio of order 4, as expected.

In particular, we note that the errors for each iteration are very similar to those obtained in the case of the horizontal plane. This is due to the decoupling of the variable in the resolving system. In fact, although the metric is no longer the identity, its contributions are simplified as they are present in both members of the resolving system, obtain an analogous case to the flat one.

3.3.3 Parabola case

We solve eq. (3.12) on the surface defined by the height function $x^3 = f(x^1, x^2) = (x^1)^2$ on the domain $U = [-1, 1]^2 \subset \mathbb{R}^2$. We consider the smooth solution $\mathbf{u}_{fix} = [1, 1]$ and impose the boundary condition $\mathbf{u} = \mathbf{u}_{fix}$ on $\partial\mathcal{S}$. Figure 3.5 shows the numerical solution in norm, by the color map, and direction by the arrows. We can qualitatively recognize that the behavior is the one of the fixed solution, i.e., the vector field $[1, 1]$. We use a starting mesh with 10 nodes in both x^1 and x^2 directions, and consider 4 level of refinements by doubling the number of nodes on each side. Table 3.3 reports the L^2-norm of the error between the exact and numerical solution on the different refinements of the grid, and the experimental rate of convergence. We can see that we obtain a convergence rate of order 4, as expected.

3.3.4 Time dependent problem

We solve (3.13) on the surface defined by the height function $x^3 = f(x^1, x^2) = 0.1 \cdot (x^1)^2$, on the domain $U = [-4, 4]^2 \subset \mathbb{R}^2$. We consider the initial solution as defined in (3.14) with $\varepsilon = 1/50$ and a mesh with 100 nodes for each edge. Figure 3.6 shows the evolution in time of the magnitude of the solution vector $\mathbf{u}$. We decided not to show the solution vector with arrows due to the difficulty in the visualization. Qualitatively the diffusion is as expected: it is faster along the x^2-direction that in the x^1-axis. Due to the surface parametrization, the metric does not depends on x^2, i.e. $h_2 = 1$ and consequently its derivatives are zero. Whereas, in the x^1-direction the metric is not constant, and this affects the diffusion process.

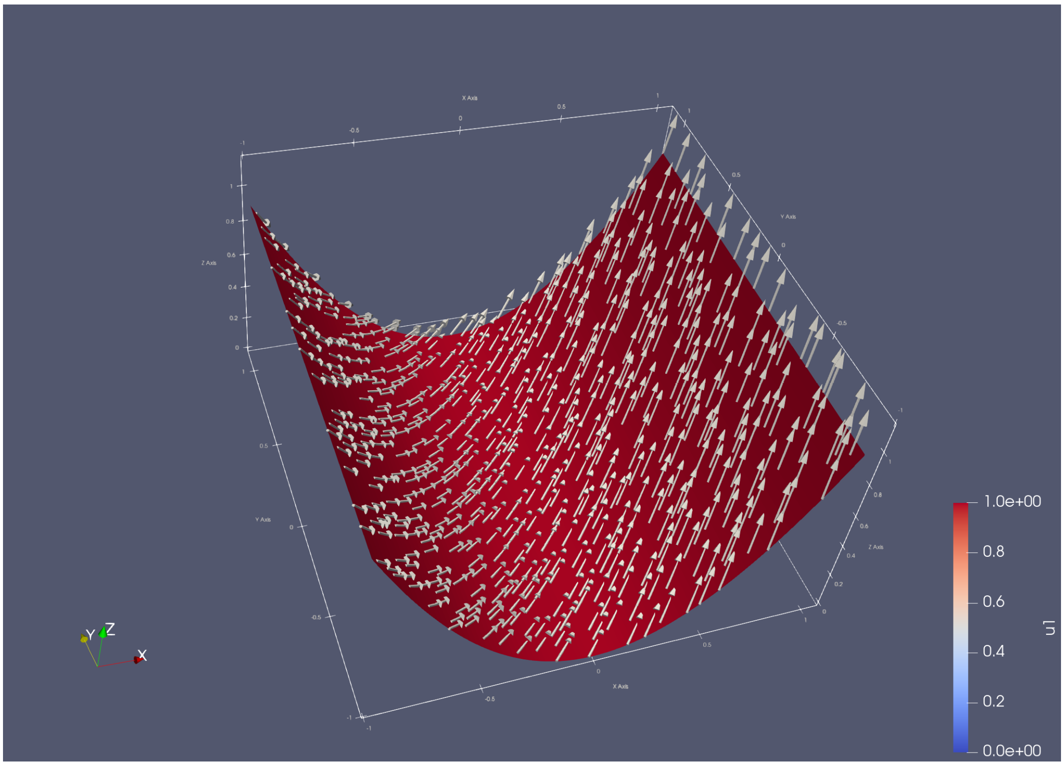


Figure 3.5: Solution on the square domain in the Parabola case. The arrow representing the solution are tangent to the parabola, even though they seem like to "go out" of the surface, due to a visualization problem.

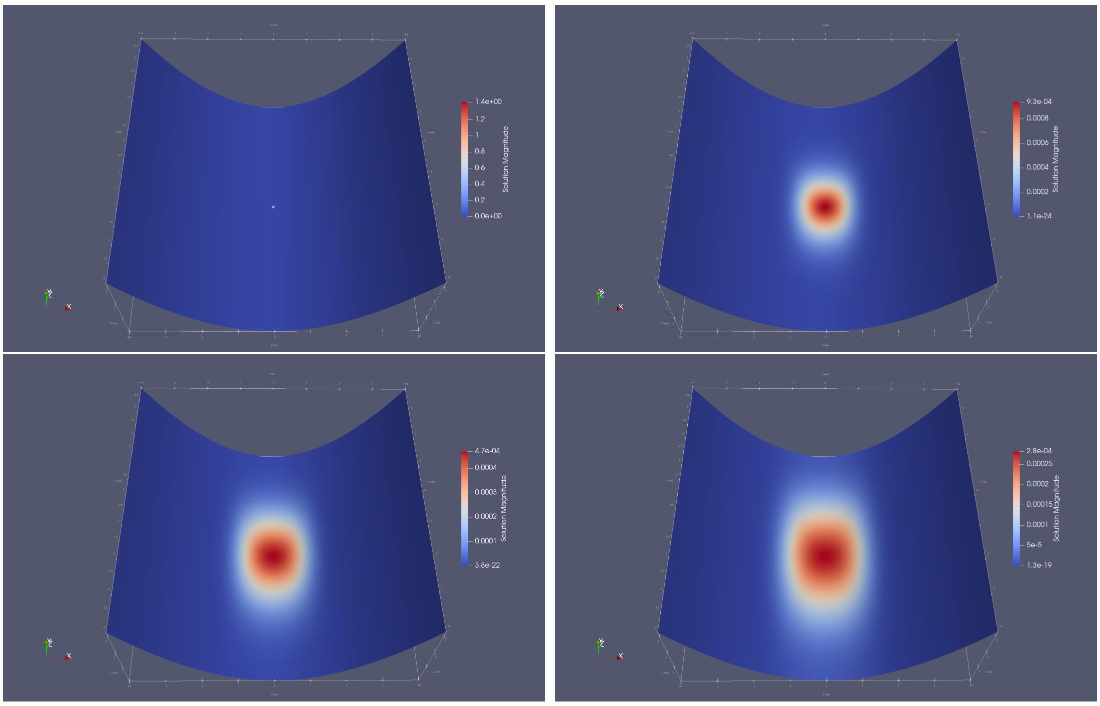


Figure 3.6: Parabola case with time dependence: magnitude of the solution norm of the diffusion of the Heat equation on the parabola surface at the initial time and time step $t = 0.3$, $t = 0.6$ and final time $t = 1$.

3.4 Future work

The next step after what has been shown above will be to test the scheme on more general surfaces, for example by considering the generalization of the parabola into a paraboloid and later on more complicated surfaces. Once the theory and the applicability of the previously defined scheme on general surfaces have been consolidated, the question of whether it is possible to generalize this method not only to cases of vector Laplacians, but also to cases in which tensor Laplacians are taken into account, will directly arise. In such a case, which is certainly really hard, it will also be very interesting to understand the connection between the case of the vector Laplacian and the tensor Laplacian.

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