

UNIVERSITÀ DEGLI STUDI DI PADOVA

DIPARTIMENTO DI MATEMATICA
“TULLIO LEVI-CIVITA”

CORSO DI LAUREA MAGISTRALE IN MATEMATICA

**ON THE EMBEDDINGS OF
QUASI-CATEGORIES INTO
PREDERIVATORS**

RELATORI
Prof. Tim Van der Linden
Prof.ssa Silvana Bazzoni

LAUREANDO
Nicola Di Vittorio
1162936

5 LUGLIO 2019

ANNO ACCADEMICO 2018/2019

Contents

Introduction	i
1 Overview of quasi-categories	1
1.1 Simplicial sets	1
1.2 Quasi-categories	13
1.3 Model structure for quasi-categories	16
2 Prederivators	22
2.1 Preliminary results	22
2.2 The 2-category \mathbf{PDer}	24
2.3 The simplicially enriched category \mathbf{PDer}_\bullet	27
3 Embedding theorems	29
3.1 The simplicial embedding	29
3.2 The 2-categorical embedding	32
3.3 A new model for $(\infty, 1)$ -categories	34
Appendix A Model categories	39
A.1 Generalities	39
A.2 Bousfield localization	41
Appendix B 2-categories	44
B.1 Setting the stage	44
B.2 Basics on 2-categories	47

Introduction

Higher categories arise in a lot of different situations, from stable homotopy theory to (derived) algebraic geometry and mathematical physics. Many authors have tried to tame technical difficulties involved in the definition and application of higher categories through the introduction of different models of these structures. This process has, however, tended to lead to a proliferation of complexity rather than progress towards simplicity. A major step forward towards a more inclusive theory of higher categories was made in [Toë05], which contains a very important result that can be rephrased as:

Theorem 1. All models of $(\infty, 1)$ -categories define fibrant objects of Quillen equivalent model categories.

This implies, in particular, that all of their homotopy categories are equivalent. In this spirit, we focus on two of those models. The first is given by quasi-categories, which are presumably the most famous model of $(\infty, 1)$ -categories. The second is represented by prederivators and is not actually a true model, at least not in the classical literature. However, as we will see at the end of this dissertation, prederivators can be endowed with a suitable model structure with respect to which they are really a model of $(\infty, 1)$ -categories.

The first chapter of this thesis is devoted to the basic definitions and properties regarding simplicial sets and quasi-categories, which are particular simplicial sets. At the end of the first chapter we introduce a model structure in which the fibrant objects are exactly the quasi-categories. Then, in the second chapter we present the basics on prederivators. The third and last chapter is dedicated to the interactions between these two structures. First, following [Car16], we explain how to construct two kinds of embeddings – a simplicial and a 2-categorical one – of the theory of quasi-categories into that of prederivators. As a consequence, following [FKKR18], we describe the prederivators that emerge as images of quasi-categories and we put a model structure on prederivators which is equivalent to the model structure on quasi-categories presented in the first chapter. At the end of the dissertation there are two appendices, which are intended to give an overview of

the notions from model category theory and 2-category theory needed in the text.

Notations and conventions

Categories are assumed to be small throughout this text, unless otherwise specified. We denote with \mathbf{Cat} the (2-)category of small categories and with \mathbf{CAT} the (2-)“category” of categories¹. Functor categories are usually denoted with $\mathbf{D}^{\mathbf{C}}$, while the class (or the set) of morphisms between two objects $C, C' \in \mathbf{C}$ is denoted with $\mathbf{C}(C, C')$. We write $\widehat{\mathbf{C}}$ for the category of presheaves over the category \mathbf{C} , and we use the classical notation $h_C := \mathrm{Hom}(-, C)$ (resp. $h^C := \mathrm{Hom}(C, -)$) for the contravariant (resp. covariant) hom-functor. In order to avoid any confusion, following [Car16], we use \mathbf{C} to denote a category, \mathbf{C}_\bullet a simplicially enriched category and $\underline{\mathbf{C}}$ a 2-category. However, we drop this notation when the context makes clear the nature of the category we are dealing with (for instance, in the appendices).

¹There are some set-theoretical technicalities here. Nevertheless, we use Grothendieck universes to avoid the issues involved in this definition.

Chapter 1

Overview of quasi-categories

In this chapter we discuss briefly the theory of simplicial sets, from which we will find quasi-categories as a special case. Quasi-categories first appeared in literature with the name *weak Kan complexes*, in an article written by Boardman and Vogt. As we said in the Introduction, one should think of quasi-categories as a *model* of $(\infty, 1)$ -categories, namely ∞ -categories in which every k -morphism is invertible for $k > 1$. In fact, an actual ∞ -category should have all sorts of higher morphisms, with unitality and associativity holding only up to higher coherences. These coherences give rise to a possibly infinite number of diagrams, making the theory very difficult.

1.1 Simplicial sets

Let Δ be the *simplex category*, whose objects are finite, non-empty, totally ordered sets

$$[n] = (0 < \dots < n), \quad n \geq 0$$

and morphisms are order preserving functions $[m] \rightarrow [n]$, i.e. maps of underlying sets $f: \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, n\}$, such that, for $0 \leq i \leq j \leq m$, we have $f(i) \leq f(j)$. Note that we may interpret $[n]$ as a category and a morphism $[m] \rightarrow [n]$ as a functor. This defines a functor $i: \Delta \rightarrow \mathbf{Cat}$, which embeds the simplex category as a full subcategory of \mathbf{Cat} .

Definition 1.1.1. A *simplicial set* is a functor $\Delta^{\text{op}} \rightarrow \mathbf{Set}$. In general, if \mathbf{C} is a category then a *simplicial object* is just a functor $\Delta^{\text{op}} \rightarrow \mathbf{C}$.

The functor category with simplicial sets as objects and natural transformations as morphisms (usually called *simplicial morphisms*) is denoted by \mathbf{sSet} . The category Δ is endowed with two special sets of morphisms:

$$\begin{array}{ll} d^k: [n-1] \rightarrow [n] & s^k: [n+1] \rightarrow [n] \\ j \mapsto \begin{cases} j, & j < k \\ j+1, & j \geq k \end{cases} & j \mapsto \begin{cases} j, & j \leq k \\ j-1, & j > k \end{cases} \end{array}$$

respectively the unique injective map which does not have k in its image (*coface map*) and the unique surjective map which hits k twice (*codegeneracy map*), for every $0 \leq k \leq n$.

Remark 1.1.2. These maps satisfy the following relations (*cosimplicial identities*):

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, \quad i < j \\ s^j s^i &= s^i s^{j+1}, \quad i \leq j \\ s^j d^i &= \begin{cases} \text{id}, & i = j, j + 1 \\ d^i s^{j-1}, & i < j \\ d^{i-1} s^j, & i > j + 1 \end{cases} \end{aligned}$$

These identities may be verified directly. For example, the first one says that $d^j d^i = d^i d^{j-1}: [n-1] \rightarrow [n+1]$, so one checks that each side of this equation is a monotone injection, and that both sides have the same image. Besides, these sets of morphisms generate *every* morphism of $\mathbf{\Delta}$, in the sense of the following lemma.

Lemma 1.1.3. In $\mathbf{\Delta}$, any arrow $f: [m] \rightarrow [n]$ can be uniquely written as

$$f = d^{i_1} \circ \dots \circ d^{i_k} \circ s^{j_1} \circ \dots \circ s^{j_h}$$

where h, k satisfy $m - h + k = n$, while the strings of subscripts i and j satisfy

$$n > i_1 > \dots > i_k \geq 0, \quad 0 \leq j_1 < \dots < j_h < m - 1$$

Proof. By induction on $i \in [m]$, any monotone f is determined by its image, a subset of $[n]$, and by the set of those $j \in [m]$ at which it does not increase, i.e. $f(j) = f(j+1)$. Putting i_1, \dots, i_k , in reverse order, for those elements of $[n]$ not in the image and j_1, \dots, j_h , in order, for the elements j of $[m]$ where f does not increase, it follows that the functions on both sides of the lemma are equal. In particular, the composite of any two d^k or s^k may be put into the canonical form. \square

We write $X_n = X([n])$ for the set of n -simplices of the simplicial set X and $d_k = X(d^k)$, $s_k = X(s^k)$ for the *face* and *degeneracy* maps, which satisfy the following relations (called *simplicial identities*) by functoriality.

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, \quad i < j \\ s_i s_j &= s_{j+1} s_i, \quad i \leq j \\ d_i s_j &= \begin{cases} \text{id}, & i = j, j + 1 \\ s_{j-1} d_i, & i < j \\ s_j d_{i-1}, & i > j + 1 \end{cases} \end{aligned}$$

From their definition and the lemma above, it follows that simplicial morphisms are exactly the ones which commute with face and degeneracy maps.

Definition 1.1.4. A simplex $x \in X_n$ is called *degenerate* if there exists $i \in [n]$ and $y \in X_{n-1}$ s.t. $x = s_i y$, otherwise it is called *non-degenerate*.

This definition is important because it allows us to work with only few simplices, the non-degenerate ones, and then reconstruct the information on the others by means of degeneracy maps. Because of this, in the following we will always work with non-degenerate simplices, unless otherwise specified.

Definition 1.1.5. Let \mathbf{C} be a category. The *nerve* of \mathbf{C} is the simplicial set $N\mathbf{C}$ whose n -simplices are

$$(N\mathbf{C})_n = \mathbf{Cat}(i[n], \mathbf{C})$$

for every $n \geq 0$.

Namely, 0-simplices of the nerve are the objects of the category, 1-simplices are morphisms, 2-simplices are pairs of composable morphisms and, in general, elements of $(N\mathbf{C})_n$ are given by strings of composable morphisms

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \rightarrow \dots \xrightarrow{f_n} C_n$$

that we can identify with n -tuples (f_1, \dots, f_n) of morphisms of \mathbf{C} such that, for every $0 < i < n$, $\text{dom} f_{i+1} = \text{cod} f_i$. Face maps d_i compose morphisms at the i th object (or remove the first/last morphism if $i = 0, n$) and degeneracy maps s_i insert an identity morphism at the i th object. If $f: X \rightarrow Y$ is a morphism in \mathbf{C} , regarded as an edge of its nerve, then the faces of f are given by the codomain $d_0 f = Y$ and the domain $d_1 f = X$, respectively. If $X \in \mathbf{C}$, regarded as a vertex of its nerve, then $s_0(X) = \text{id}_X: X \rightarrow X$. Finally, given a diagram $C_0 \xrightarrow{\phi} C_1 \xrightarrow{\psi} C_2$, the edge of $N\mathbf{C}$ corresponding to $\psi \circ \phi$ may be uniquely characterized by the fact that there exists a 2-simplex $\sigma \in (N\mathbf{C})_2$ with $d_2(\sigma) = \phi$, $d_0(\sigma) = \psi$, and $d_1(\sigma) = \psi \circ \phi$. Note that the assignment $\mathbf{C} \mapsto N\mathbf{C} = \mathbf{Cat}(i-, \mathbf{C})$ is obviously functorial.

Lemma 1.1.6. The nerve functor $N: \mathbf{Cat} \rightarrow \mathbf{sSet}$ is fully faithful.

Proof. We claim that $\mathbf{Cat}(\mathbf{C}, \mathbf{D}) \xrightarrow{\psi} \mathbf{sSet}(N\mathbf{C}, N\mathbf{D})$ is bijective for all \mathbf{C} and \mathbf{D} , where we set $\psi := N_{\mathbf{C}, \mathbf{D}}$ (the function on hom-sets induced by the nerve functor).

This map is clearly injective because a functor is determined by its behavior on objects and morphisms, which is precisely the behavior of the induced simplicial morphism on 0- and 1-simplices of the nerve.

Let us prove now that ψ is surjective. This means that for every simplicial morphism $f: N\mathbf{C} \rightarrow N\mathbf{D}$ we have to find a functor $F: \mathbf{C} \rightarrow \mathbf{D}$ such that $f = \psi(F)$. For each $n \geq 0$, f determines a map of sets $(N\mathbf{C})_n \rightarrow (N\mathbf{D})_n$, also denoted by f . When $n = 0$, this map sends each object $C \in \mathbf{C}$ to an object of \mathbf{D} , which we will denote by $F(C)$. For every pair of objects

$C, C' \in \mathbf{C}$, the map f carries each morphism $u: C \rightarrow C'$ to a morphism $f(u)$ in the category \mathbf{D} . Since f commutes with face maps, the morphism $f(u)$ has domain $F(C)$ and codomain $F(C')$, and can therefore be regarded as an element of $\mathbf{D}(F(C), F(C'))$; we denote this element by $F(u)$. It remains to show that F is indeed a functor and that $f = \psi(F)$. In order to prove the first claim, we note that the compatibility of f with degeneracy maps implies that we have $F(\text{id}_C) = \text{id}_{F(C)}$ for each $C \in \mathbf{C}$. It will therefore suffice to show that for every pair of composable morphisms $u: C \rightarrow C'$ and $v: C' \rightarrow C''$ in the category \mathbf{C} , we have $F(v) \circ F(u) = F(v \circ u)$ as elements of the set $\mathbf{D}(F(C), F(C''))$. For this, we observe that the diagram $C \xrightarrow{u} C' \xrightarrow{v} C''$ can be identified with a 2-simplex σ of $N\mathbf{C}$. Using the equality $d_i(f(\sigma)) = f(d_i(\sigma))$ for $i = 0, 2$, we see that $f(\sigma)$ corresponds to the diagram $F(C) \xrightarrow{F(u)} F(C') \xrightarrow{F(v)} F(C'')$ in \mathbf{D} . We now compute

$$F(v) \circ F(u) = d_1(f(\sigma)) = f(d_1(\sigma)) = F(v \circ u).$$

This completes the proof of the first statement. To prove the second one, we must show that $f(\tau) = \psi(F)(\tau)$ for each n -simplex τ of $N\mathbf{C}$. This follows by construction in the case $n \leq 1$, and follows in general since an n -simplex of $N\mathbf{D}$ is determined by its 1-dimensional faces. \square

Hence we can regard the category of small categories as a full subcategory of the category of simplicial sets. Actually, it is possible to go backwards and define a functor from simplicial sets to small categories, so that this functor and N form an adjoint pair.

Definition 1.1.7. Let \mathbf{C} be a category and $P: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf over \mathbf{C} . The *category of elements* of P , denoted by $\text{Elts}(P)$ is the category defined as follows:

- objects are pairs (C, s) , where $C \in \mathbf{C}$ and $s \in P(C)$,
- morphisms $(C, s) \rightarrow (C', t)$ are arrows $f: C \rightarrow C'$ such that $Pf(t) = s$.

Theorem 1.1.8. Every presheaf is a colimit of representable presheaves, indexed on its category of elements.

Proof. See [Mac71, Theorem III.7.1] for a reference on the covariant case. The same proof holds for presheaves, by duality. \square

Proposition 1.1.9. Let $F: \mathbf{C} \rightarrow \mathbf{D}$ be a functor from a small category to a locally small category which has small colimits. Then the functor

$$\begin{aligned} \mathbf{D} &\rightarrow \widehat{\mathbf{C}} \\ D &\mapsto \mathbf{D}(F(-), D) \end{aligned}$$

has a left adjoint $G: \widehat{\mathbf{C}} \rightarrow \mathbf{D}$.

Proof. Let $\text{Elts}(X)$ denote the category of elements of a presheaf $X \in \widehat{\mathbf{C}}$. Consider the functor

$$\begin{aligned} \text{Elts}(X) &\rightarrow \mathbf{D} \\ (C, s) &\mapsto FC \end{aligned}$$

and define $G(X) = \text{colim}_{(C,s) \in \text{Elts}(X)} FC$. So we have the following chain of natural isomorphisms

$$\begin{aligned} \mathbf{D}(G(X), D) &= \mathbf{D} \left(\text{colim}_{(C,s) \in \text{Elts}(X)} FC, D \right) \\ &\cong \lim_{(C,s) \in \text{Elts}(X)} \mathbf{D}(FC, D) \\ (Y) &\cong \lim_{(C,s) \in \text{Elts}(X)} \widehat{\mathbf{C}}(h_C, \mathbf{D}(F(-), D)) \\ &\cong \widehat{\mathbf{C}} \left(\text{colim}_{(C,s) \in \text{Elts}(X)} h_C, \mathbf{D}(F(-), D) \right) \\ &\cong \widehat{\mathbf{C}}(X, \mathbf{D}(F(-), D)) \end{aligned}$$

and the last isomorphism holds thanks to Theorem 1.1.8. This proves the claim. \square

Corollary 1.1.10. The nerve functor has a left adjoint τ_1 , called the *fundamental category functor*.

Proof. It suffices to apply Proposition 1.1.9 to the inclusion functor $i: \mathbf{\Delta} \rightarrow \mathbf{Cat}$, noticing that $\widehat{\mathbf{\Delta}} = \mathbf{sSet}$ and $\mathbf{Cat}(i(-), \mathbf{C}) = N\mathbf{C}$. \square

Remark 1.1.11. The category of elements of a simplicial set X can be also found in literature under the name *category of simplices* ($\mathbf{\Delta} \downarrow X$), since we can think of it as the category with objects the simplicial morphisms $\sigma: \Delta^n \rightarrow X$ and with morphisms the commutative triangles

$$\begin{array}{ccc} \Delta^n & & \\ \downarrow \theta & \searrow \sigma & \\ & & X \\ & \nearrow \tau & \\ \Delta^m & & \end{array}$$

Remark 1.1.12. More explicitly, given $X \in \mathbf{sSet}$ we can describe $\tau_1(X)$ as the category whose objects are the vertices of X (i.e. $\text{Ob}(\tau_1(X)) = X_0$) and such that the morphisms are freely generated by X_1 , modulo the relations given by X_2 . Basically, to construct $\tau_1(X)$ we take the free graph on X_0 generated by the arrows in X_1 and define the composition in the following way. Given $f, g, h \in X_1$ we put $h = g \circ f$ if there exists $\sigma \in X_2$ such that $d_2(\sigma) = f$, $d_0(\sigma) = g$, and $d_1(\sigma) = h$.

$$\begin{array}{ccc}
 & y & \\
 f \nearrow & & \searrow g \\
 x & \xrightarrow{h} & z
 \end{array}$$

It is straightforward to check that the composition defined above is associative since it already is at the level of the free graph and that identity arrows behave like identities with respect to this composition (just take the degenerate 2-simplices in the image of the degeneracy maps). Thus $\tau_1(X)$ is actually a category. It is not difficult to verify that this construction gives a left adjoint to the nerve functor, hence it agrees with Corollary 1.1.10.

Definition 1.1.13. The standard n -simplex $\Delta^n := \mathbf{\Delta}(-, [n])$ is the image of $[n]$ under the Yoneda embedding $y : \mathbf{\Delta} \hookrightarrow \mathbf{Set}^{\Delta^{op}} = \mathbf{sSet}$.

Δ^n contains as subcomplexes the boundary $\partial\Delta^n$, namely the simplicial subset generated by non-degenerate simplices in degree less than n and the k^{th} horn Λ_k^n , obtained from the boundary by removing the k -th face, for all $0 \leq k \leq n$. We say that a horn Λ_k^n is *inner* if $0 < k < n$, otherwise we call it *outer horn*.

Proposition 1.1.14. We can express boundary and horns as suitable colimits. In particular it holds that

$$\partial\Delta^n \cong \text{coeq} \left(\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{i=0}^n \Delta^{n-1} \right)$$

and

$$\Lambda_k^n \cong \text{coeq} \left(\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{i \neq k} \Delta^{n-1} \right)$$

Proof. By the first cosimplicial identity, the diagram

$$\begin{array}{ccc}
 [n-2] & \xrightarrow{d^{j-1}} & [n-1] \\
 d^i \downarrow & \lrcorner & \downarrow d^i \\
 [n-1] & \xrightarrow{d^j} & [n]
 \end{array}$$

commutes for each $i < j$. Moreover it is a pullback in $\mathbf{\Delta}$, since the totally ordered set $\{0, \dots, \hat{i}, \dots, \hat{j}, \dots, n\}$ is the intersection of the subsets $\{0, \dots, \hat{i}, \dots, n\}$ and $\{0, \dots, \hat{j}, \dots, n\}$ of $\{0, \dots, n\}$, and this poset is isomorphic to $[n-2]$. Since the bifunctor $\mathbf{\Delta}(-, -)$ is continuous in the covariant component, the previous pullback induces another pullback diagram

$$\begin{array}{ccc}
 \Delta^{n-2} & \longrightarrow & \Delta^{n-1} \\
 \downarrow & \lrcorner & \downarrow \\
 \Delta^{n-1} & \longrightarrow & \Delta^n
 \end{array}$$

The claim is proved once we consider maps induced on the coproducts. An analogous proof holds for the horns. \square

Remark 1.1.15. The category of simplicial sets is a presheaf category, and so in particular a Grothendieck topos. In particular, it is cartesian closed. Indeed, given two simplicial sets X and Y we can construct their product $X \times Y$ componentwise. The exponential (or internal hom) is defined thanks to the Yoneda Lemma and the product-hom adjunction. Its n -simplices are

$$Y_n^X \cong \mathbf{sSet}(\Delta^n, Y^X) \cong \mathbf{sSet}(\Delta^n \times X, Y),$$

and the action on morphisms is the obvious one. This is known as the *simplicial mapping space* (for details see [GJ99, §1.5]).

Remark 1.1.16. \mathbf{Cat} is cartesian closed, with the product given by the usual product of categories and the exponentials being functor categories. The adjunction between products and exponentials in \mathbf{Cat} is the classic tensor-hom adjunction adjusted for small categories.

Proposition 1.1.17. The functor $\tau_1: \mathbf{sSet} \rightarrow \mathbf{Cat}$ preserves finite products.

Proof. Because τ_1 is a left adjoint and \mathbf{sSet} and \mathbf{Cat} are cartesian closed, the bifunctors $(\tau_1 -) \times (\tau_1 -)$ and $\tau_1(- \times -)$ preserve colimits in both variables. Since every presheaf is colimit of representables, it suffices to prove the claim for standard n -simplexes. We know that $\Delta^n = \mathbf{\Delta}(-, [n]) \cong \mathbf{Cat}(i-, i[n]) = N(i[n])$, since i is fully faithful. Then we have

$$\begin{aligned} (\tau_1 \Delta^n) \times (\tau_1 \Delta^m) &\cong (\tau_1 N(i[n])) \times (\tau_1 N(i[m])) \\ &\cong i[n] \times i[m] \\ &\cong \tau_1 N(i[n] \times i[m]) \\ &\cong \tau_1(N(i[n]) \times N(i[m])) \\ &\cong \tau_1(\Delta^n \times \Delta^m). \end{aligned}$$

where the second isomorphism holds because N is fully faithful (hence the counit is a natural isomorphism) and the fourth holds because N commutes with products since it is a right adjoint. \square

For any pair of simplicial sets (X, Y) , we put $\tau_1(X, Y) := \tau_1(Y^X)$. Thus, by Proposition 1.1.17, if we apply the functor τ_1 to the composition map $Z^Y \times Y^X \rightarrow Z^X$ we obtain a composition law $\tau_1(Y, Z) \times \tau_1(X, Y) \rightarrow \tau_1(X, Z)$ for a 2-category $\underline{\mathbf{sSet}}_{\tau_1}$, where we put $\underline{\mathbf{sSet}}_{\tau_1}(X, Y) := \tau_1(X, Y)$. This leads to the following definition.

Definition 1.1.18. We define $\underline{\mathbf{sSet}}_{\tau_1}$ to be the 2-category with

- 0-cells: simplicial sets X, Y, \dots

- 1-cells: simplicial morphisms $f: X \rightarrow Y$
- 2-cells: $f \Rightarrow g: X \rightarrow Y$ morphisms in the category $\tau_1(X, Y)$

Definition 1.1.19. A simplicial morphism $f: X \rightarrow Y$ is a *categorical equivalence* if it is an equivalence in the 2-category \mathbf{sSet}_{τ_1} .

The definition above is the one given by Joyal in [Joy08a]. Lurie (in his book [Lur17]) calls categorical equivalences what we will call *weak categorical equivalences*, following Joyal's notes.

Definition 1.1.20. A simplicial set X is a *Kan complex* if it satisfies the *Kan condition*: every horn $\Lambda_k^n \rightarrow X$, for $0 \leq k \leq n$, has a filler $\Delta^n \rightarrow X$, i.e.

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\forall} & X \\ \downarrow & \nearrow \exists & \\ \Delta^n & & \end{array}$$

commutes.

Example 1.1.21. Given a topological space Y , we define the following functor

$$\begin{aligned} \mathbf{Top} &\rightarrow \mathbf{sSet} \\ Y &\mapsto \mathbf{Sing}(Y) \end{aligned}$$

where

$$\begin{aligned} \mathbf{Sing}(Y): \mathbf{\Delta}^{\text{op}} &\rightarrow \mathbf{Set} \\ [n] &\mapsto \mathbf{Top}(|\Delta^n|, Y) \end{aligned}$$

and

$$|\Delta^n| = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : 0 \leq x_i \leq 1, \sum_i x_i = 1 \right\}$$

is the *geometric n -simplex*. For each morphism $f: [m] \rightarrow [n]$, we define a continuous map $f_*: |\Delta^m| \rightarrow |\Delta^n|$, $(t_0, t_1, \dots, t_m) \mapsto (s_0, s_1, \dots, s_n)$ with

$$s_i = \begin{cases} 0, & \text{if } f^{-1}(i) = \emptyset \\ \sum_{j \in f^{-1}(i)} t_j, & \text{else.} \end{cases}$$

$\mathbf{Sing}(Y)$ is called *the singular set* of Y .

Remark 1.1.22. Note that we can define a functor $\mathbf{\Delta} \rightarrow \mathbf{Top}$, $[n] \mapsto |\Delta^n|$, on which we can apply Proposition 1.1.9, getting a left adjoint to the singular set. This is called *the geometric realization* and can be expressed as the colimit

$$|X| \cong \left| \operatorname{colim}_{\Delta^n \rightarrow X} \Delta^n \right| \cong \operatorname{colim}_{\Delta^n \rightarrow X} |\Delta^n|$$

using that left adjoints commute with colimits.

Proposition 1.1.23. The singular set of a topological space X is a Kan complex.

Proof. For every $0 \leq k \leq n$ the extension problem

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{f} & \text{Sing}(X) \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

is equivalent to the adjoint extension problem

$$\begin{array}{ccc} |\Lambda_k^n| & \xrightarrow{\tilde{f}} & X \\ \begin{array}{c} \uparrow r \\ \downarrow i \end{array} & \nearrow \text{dotted} & \\ |\Delta^n| & & \end{array}$$

which can be solved because horns are retracts of the geometric n -simplex. This means that we get a continuous map $r: |\Delta^n| \rightarrow |\Lambda_k^n|$ with $r \circ i = \text{id}$, hence we can take $\tilde{f} \circ r$ to obtain an extension. \square

Proposition 1.1.24. A simplicial set X is isomorphic to the nerve of a category if and only if every *inner* horn has a *unique* filler.

Proof. First of all, let us take $X \cong N\mathbf{C}$, $\mathbf{C} \in \mathbf{Cat}$, and prove that every inner horn has a unique filler. Let $f_0: \Lambda_k^n \rightarrow X$ be a simplicial morphism, where $0 < k < n$. We wish to show that f_0 can be extended uniquely to a map $f: \Delta^n \rightarrow X$, that is (by Yoneda) a n -simplex of $N\mathbf{C}$. For $0 \leq i \leq n$, let $C_i \in \mathbf{C}$ denote the image under f_0 of the i^{th} vertex of Λ_k^n . We first consider the case where $n \geq 3$. In this case, Λ_k^n contains every edge of Δ^n . For $0 \leq i \leq j \leq n$, let $f_{i,j}: C_i \rightarrow C_j$ denote the 1-simplex of $N\mathbf{C}$ obtained by evaluating f_0 on the edge of Δ^n corresponding to the pair (i, j) . We claim that the construction

$$\begin{aligned} [n] &\rightarrow \mathbf{C} \\ i &\mapsto C_i \\ (i \leq j) &\mapsto (C_i \xrightarrow{f_{i,j}} C_j) \end{aligned}$$

determines a functor $[n] \rightarrow \mathbf{C}$, which we can then identify with an n -simplex of $N\mathbf{C}$ having the desired properties. It is easy to see that $f_{i,i} = \text{id}_{C_i}$ for each $0 \leq i \leq n$, so it will suffice to show that $f_{j,l} \circ f_{i,j} = f_{i,l}$ for every triple $0 \leq i \leq j \leq l \leq n$. The triple (i, j, l) determines a 2-simplex τ of Δ^n . If τ is contained in Λ_k^n , then $\tau' = f_0(\tau)$ is a 2-simplex of $N\mathbf{C}$ satisfying $d_0(\tau') = f_{j,l}$, $d_1(\tau') = f_{i,l}$ and $d_2(\tau') = f_{i,j}$ so that

$$f_{i,l} = d_1(\tau') = d_0(\tau') \circ d_2(\tau') = f_{j,l} \circ f_{i,j}$$

It will therefore suffice to treat the case where the simplex τ does not belong to Λ_k^n . In this case, our assumption that $n \geq 3$ guarantees that we must have $\{i, j, l\} = [n] \setminus \{k\}$. It follows that $n = 3$, so that either $k = 1$ or $k = 2$. We will treat the case $k = 1$ (the case $k = 2$ follows by a similar argument). Note that Λ_1^3 contains all of the non-degenerate 2-simplices of Δ^3 other than τ ; applying the map f_0 , we obtain 2-simplices of $N\mathbf{C}$ which witness the identities

$$f_{0,3} = f_{1,3} \circ f_{0,1}, \quad f_{1,3} = f_{2,3} \circ f_{1,2}, \quad f_{0,2} = f_{1,2} \circ f_{0,1}$$

We now compute

$$f_{0,3} = f_{1,3} \circ f_{0,1} = (f_{2,3} \circ f_{1,2}) \circ f_{0,1} = f_{2,3} \circ (f_{1,2} \circ f_{0,1}) = f_{2,3} \circ f_{0,2}$$

so that $f_{j,l} \circ f_{i,j} = f_{i,l}$, as desired. It remains to treat the case $n = 2$, so that we must also have $k = 1$. In this situation, the map $f_0: \Lambda_k^n \rightarrow N\mathbf{C}$ determines a pair of composable morphisms $f_{0,1}: C_0 \rightarrow C_1$ and $f_{1,2}: C_1 \rightarrow C_2$. This data extends uniquely to a 2-simplex σ of $N\mathbf{C}$ satisfying $d_1(\sigma) = f_{1,2} \circ f_{0,1}$.

For the converse implication, suppose that the simplicial set X satisfies the condition on fillers. We will find a small category \mathbf{C} and an isomorphism of simplicial sets $\phi: X \rightarrow N\mathbf{C}$. Since the nerve functor is fully faithful then the category \mathbf{C} is uniquely determined (up to isomorphism). We construct the category \mathbf{C} , as usual, in the following way

- (i) $\text{Ob}(\mathbf{C}) = X_0$,
- (ii) $\mathbf{C}(C, C') = \{e \in X_1 \mid d_0(e) = C' \text{ and } d_1(e) = C\}$ for every $C, C' \in \mathbf{C}$,
- (iii) $\text{id}_C = s_0(C)$ is the identity on the object C ,
- (iv) if f and g are a pair of composable morphisms in \mathbf{C} , then f and g together determine a map $\Lambda_1^2 \rightarrow X$. By the hypothesis, this map can be extended uniquely to a 2-simplex $\sigma: \Delta^2 \rightarrow X$, satisfying $d_2(\sigma) = f$ and $d_0(\sigma) = g$. We define the composition $g \circ f$ to be the edge $d_1(\sigma)$.

We claim that \mathbf{C} is a category, so let us verify the axioms:

- (a) For every $C \in \mathbf{C}$, the identity id_C is a unit with respect to composition. In fact for every morphism $f: C \rightarrow C'$, we construct two degenerate 2-simplices $\sigma = s_1(f)$ and $\tau = s_0(f)$ s.t. $d_0(\sigma) = d_0 s_1(f) = s_0 d_0(f) = s_0(C') = \text{id}_{C'}$, $d_1(\sigma) = d_1 s_1(f) = f = d_2 s_1(f) = d_2(\sigma)$ and (with analogous calculations based on simplicial identities) $d_0(\tau) = d_1(\tau) = f$ and $d_2(\tau) = \text{id}_C$. Hence

$$\text{id}_{C'} \circ f = d_0(\sigma) \circ d_2(\sigma) = d_1(\sigma) = f = d_1(\tau) = d_0(\tau) \circ d_2(\tau) = f \circ \text{id}_C$$

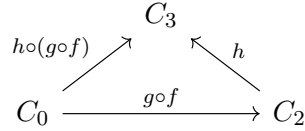
(b) Composition is associative. That is, for every sequence of composable morphisms

$$C_0 \xrightarrow{f} C_1 \xrightarrow{g} C_2 \xrightarrow{h} C_3,$$

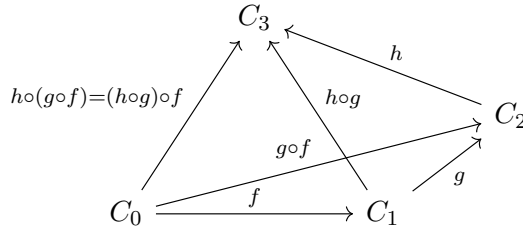
we have $h \circ (g \circ f) = (h \circ g) \circ f$. To prove this, one has to apply repeatedly the hypothesis on existence and unicity of fillers (for inner horns). First of all, let us choose 2-simplices σ_{012} and σ_{123} as below:



Now choose a 2-simplex σ_{023} corresponding to a diagram



These three 2-simplices together define a map $\Lambda_2^3 \rightarrow X$, that we can extend to a 3-simplex $\Delta^3 \rightarrow X$, as in the diagram below



with the 2-simplex σ_{013} that witnesses the associativity axiom $h \circ (g \circ f) = (h \circ g) \circ f$.

It follows that \mathbf{C} is a well-defined category. Note that every n -simplex $\sigma: \Delta^n \rightarrow X$ determines a functor $[n] \rightarrow \mathbf{C}$, given on objects by the values of σ on the vertices of Δ^n and on morphisms by the values of σ on the edges of Δ^n . This construction determines a map of simplicial sets $\phi: X \rightarrow N\mathbf{C}$, which is clearly bijective on simplices of dimension ≤ 1 . To complete the proof, we will show by induction on $n \geq 0$, that ϕ induces a bijection $\mathbf{sSet}(\Delta^n, X) \rightarrow \mathbf{sSet}(\Delta^n, N\mathbf{C})$. For $n = 0$ and $n = 1$, as we said, this is obvious from the construction. Assume therefore that $n \geq 2$ and choose an integer k such that $0 < k < n$. We have a commutative diagram

$$\begin{array}{ccc} \mathbf{sSet}(\Delta^n, X) & \longrightarrow & \mathbf{sSet}(\Delta^n, N\mathbf{C}) \\ \downarrow & & \downarrow \\ \mathbf{sSet}(\Lambda_k^n, X) & \longrightarrow & \mathbf{sSet}(\Lambda_k^n, N\mathbf{C}) \end{array}$$

Since X and $N\mathbf{C}$ both satisfy the extension condition, the vertical maps are bijective and the lower horizontal map is bijective by virtue of our inductive hypothesis, knowing that inner horns are particular colimits of lower dimensional standard simplices and having in mind that contravariant hom functors send colimits in limits. It follows that the upper horizontal map is also bijective, as desired. \square

Proposition 1.1.25. A simplicial set X is isomorphic to the nerve of a groupoid if and only if every horn has a *unique* filler.

Proof. Let us prove the “if” first. By the previous proposition, if every horn has a *unique* filler then, in particular this holds true for inner ones, so that $X \cong N\mathbf{C}$, for some $\mathbf{C} \in \mathbf{Cat}$. It remains to show that \mathbf{C} is indeed a groupoid. Let $f: C \rightarrow C'$ be a 1-simplex in $N\mathbf{C}$. Using the surjectivity of the map $\mathbf{sSet}(\Delta^2, N\mathbf{C}) \rightarrow \mathbf{sSet}(\Lambda_2^2, N\mathbf{C})$, we see that there exists a 2-simplex σ of $N\mathbf{C}$ satisfying $d_0(\sigma) = f$ and $d_1(\sigma) = \text{id}_{C'}$. Setting $g = d_2(\sigma)$, we conclude that $f \circ g = \text{id}_{C'}$: that is, g is a left inverse to f . Similarly, the surjectivity of the map $\mathbf{sSet}(\Delta^2, N\mathbf{C}) \rightarrow \mathbf{sSet}(\Lambda_0^2, N\mathbf{C})$ allows us to construct a map $h: C' \rightarrow C$ satisfying $h \circ f = \text{id}_C$. The calculation

$$g = \text{id}_C \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ \text{id}_{C'} = h$$

shows that $g = h$ is an inverse of f , so that f is invertible as desired.

Now suppose that \mathbf{C} is a groupoid. We wish to show that, for $0 \leq k \leq n$, every map $\sigma_0: \Lambda_k^n \rightarrow N\mathbf{C}$ can be extended to an n -simplex $\sigma: \Delta^n \rightarrow N\mathbf{C}$. For inner horns, this follows from Proposition 1.1.24. We will treat the case where $k = 0$; the case $k = n$ follows by similar reasoning. We consider several cases:

- In the case $n = 0$, we have $\Lambda_0^0 = \Delta^0$, so we can take $\sigma = \sigma_0$.
- In the case $n = 1$, the map $\sigma_0: \Lambda_0^1 \rightarrow N\mathbf{C}$ can be identified with an object $C \in \mathbf{C}$. In this case, we can take σ to be an edge of $N\mathbf{C}$ corresponding to any morphism with codomain C (e.g. id_C).
- In the case $n = 2$, we can identify σ_0 with a pair of morphisms in \mathbf{C} having the same domain, which we can depict as a diagram

$$\begin{array}{ccc} & D & \\ f \nearrow & & \dashrightarrow \\ C & \xrightarrow{g} & E \end{array}$$

Our assumption that \mathbf{C} is a groupoid guarantees that we can extend this diagram to a 2-simplex of \mathbf{C} , by means of the morphism $g \circ f^{-1}: D \rightarrow E$.

- In the case $n \geq 3$, the map σ_0 determines a collection of objects $\{C_i\}_{0 \leq i \leq n}$ and morphisms $f_{i,j} : C_i \rightarrow C_j$ for $i \leq j$ (as in the proof of Proposition 1.1.24). We wish to show that these morphisms determine a functor $[n] \rightarrow \mathbf{C}$ (which we can then identify with an n -simplex σ of NC satisfying $\sigma|_{\Lambda_0^n} = \sigma_0$). For this, we must verify the identity $f_{j,l} \circ f_{i,j} = f_{i,l}$ for $0 \leq i \leq j \leq l \leq n$. Note that this identity is satisfied whenever the triple $(i \leq j \leq l)$ determines a 2-simplex of Δ^n belonging to the horn Λ_0^n . This is automatic unless $n = 3$ and $(i, j, l) = (1, 2, 3)$. To handle this exceptional case, we compute

$$\begin{aligned} (f_{2,3} \circ f_{1,2}) \circ f_{0,1} &= f_{2,3} \circ (f_{1,2} \circ f_{0,1}) \\ &= f_{2,3} \circ f_{0,2} \\ &= f_{0,3} \\ &= f_{1,3} \circ f_{0,1}. \end{aligned}$$

Since \mathbf{C} is a groupoid, composing with $f_{0,1}^{-1}$ on the right yields the desired identity $f_{2,3} \circ f_{1,2} = f_{1,3}$. \square

Remark 1.1.26. In particular, a category is a groupoid if and only if its nerve is a Kan complex.

Remark 1.1.27. Uniqueness of the filler for nerves of categories or groupoids reflects the definition of a category, in which the composition of two morphisms is *uniquely* defined.

1.2 Quasi-categories

We would like to have a notion generalizing both Kan complexes and (nerve of) categories. In order to do that we have to drop the unicity of the filler and the extension condition for outer horns.

Definition 1.2.1. A simplicial set \mathcal{C} is a *quasi-category* or, with a little abuse of notation, an ∞ -*category* if every inner horn has a filler.

Definition 1.2.2. Let \mathcal{C}, \mathcal{D} be two quasi-categories. A *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ is just a map of simplicial sets.

Definition 1.2.3. If \mathcal{C} and \mathcal{D} are two quasi-categories and $F, G: \mathcal{C} \rightarrow \mathcal{D}$ are two functors, a *natural transformation* from F to G is a morphism $H: \mathcal{C} \times \Delta^1 \rightarrow \mathcal{D}$ such that $H|_{\mathcal{C} \times \{0\}} = F$ and $H|_{\mathcal{C} \times \{1\}} = G$.

Mimicking what we have done for nerves of small categories, if we have a quasi-category \mathcal{C} we say that an element of \mathcal{C}_0 is an *object* of \mathcal{C} and that an element of \mathcal{C}_1 is a *morphism*. Face maps $s = d_1, t = d_0 : \mathcal{C}_1 \rightrightarrows \mathcal{C}_0$ are the source and target map. In analogy with ordinary category theory we write $f: x \rightarrow y$ if $s(f) = x$ and $t(f) = y$. Hence we can define the set of morphisms from x to y as the pullback

$$\begin{array}{ccc}
 \mathrm{Hom}_{\mathcal{C}}(x, y) & \longrightarrow & \mathcal{C}_1 \\
 \downarrow & \lrcorner & \downarrow (s, t) \\
 * & \xrightarrow{(x, y)} & \mathcal{C}_0 \times \mathcal{C}_0
 \end{array}$$

The identity map is $\mathrm{id} = s_0: \mathcal{C}_0 \rightarrow \mathcal{C}_1$, and in fact from the simplicial identities $d_0 s_0 = d_1 s_0 = \mathrm{id}_{\mathcal{C}_0}$ it follows that $\mathrm{id}_x = s_0 x: x \rightarrow x$. Compositions of morphisms are defined thanks to the Kan condition for inner horns, but the choice of a composition is not unique in general. This issue is fixed by asking that the space of all such choices (which is a Kan complex thanks to Proposition 1.2.2.3 of [Lur17]) has to be contractible, namely homotopic (in the simplicial sense of Definition 1.2.9) to the point Δ^0 .

As one would expect, quasi-categories are a meaningful tool for encoding higher dimensional informations, e.g. we can define homotopies between morphisms.

Definition 1.2.4. Two edges $f, g: x \rightarrow y$ in a quasi-category \mathcal{C} are said to be *homotopic* ($f \sim g$) if there exists a 2-simplex $\sigma: \Delta^2 \rightarrow \mathcal{C}$ such that $d_2(\sigma) = \mathrm{id}_x, d_0(\sigma) = g$ and $d_1(\sigma) = f$.

$$\begin{array}{ccc}
 & x & \\
 \mathrm{id}_x \nearrow & & \searrow g \\
 x & \xrightarrow{f} & y
 \end{array}$$

There is an analogous definition of homotopy with the identity on the right side.

Proposition 1.2.5. Suppose that there exist 2-simplices as in the following pictures.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & x & \\
 \mathrm{id}_x \nearrow & & \searrow f \\
 x & \xrightarrow{g} & y
 \end{array} & & \begin{array}{ccc}
 & y & \\
 f \nearrow & & \searrow \mathrm{id}_y \\
 x & \xrightarrow{g} & y
 \end{array} \\
 \\
 \begin{array}{ccc}
 & y & \\
 g \nearrow & & \searrow \mathrm{id}_y \\
 x & \xrightarrow{f} & y
 \end{array}
 \end{array}$$

Each of these 2-simplices defines a relation on the edges appearing in the boundary, but all these relations coincide with the homotopy relation.

Proof. See [Cis18, Lemma 1.6.4]. □

Proposition 1.2.6. The homotopy relation is an equivalence relation on hom-sets.

Proof. Let $\phi: \Delta^1 \rightarrow \mathcal{C}$ be an edge. Then $s_1(\phi)$ is a homotopy from ϕ to itself. Thus homotopy is a reflexive relation. Suppose next that $\phi, \phi', \phi'': \mathcal{C} \rightarrow \mathcal{C}'$ are edges with the same source and target. Let σ be a homotopy from ϕ to ϕ' , and σ' a homotopy from ϕ to ϕ'' . Let $\sigma'': \Delta^2 \rightarrow \mathcal{C}$ denote the constant map at the vertex \mathcal{C}' . These three maps determine a simplicial morphism $\Lambda_1^3 \rightarrow \mathcal{C}$. Since \mathcal{C} is a quasi-category, there exists a 3-simplex $\tau: \Delta^3 \rightarrow \mathcal{C}$ extending the map defined before. It is easy to see that $d_1(\tau)$ is a homotopy from ϕ' to ϕ'' . If we take $\phi = \phi''$ we recover the symmetry of the homotopy relation. By a similar argument it follows that this relation is also transitive. Then it is an equivalence relation. \square

The homotopy class of a morphism $f: x \rightarrow y$ will be denoted $[f]$. Hence we can define the homotopy category $\text{Ho}(\mathcal{C})$ of a quasi-category \mathcal{C} by passing to homotopy classes of morphisms.

Proposition 1.2.7. Let \mathcal{C} be a quasi-category. If we define

$$[g] \circ [f] := [g \circ f] \quad \text{and} \quad \text{id}_x := [\text{id}_x] = [s_0x]$$

for a couple of morphisms f, g and an object x in \mathcal{C} , then $\text{Ho}(\mathcal{C})$ is a category. Moreover, there exists an isomorphism of categories $\tau_1(\mathcal{C}) \cong \text{Ho}(\mathcal{C})$.

Proof. The proof is a straightforward verification (see for example [Cis18, Theorem 1.6.6]). \square

Definition 1.2.8. A morphism $f: x \rightarrow y$ in a quasi-category \mathcal{C} is an *equivalence* if $[f]: x \rightarrow y$ is an isomorphism in $\text{Ho}(\mathcal{C})$.

We can also define homotopy between maps of simplicial sets.

Definition 1.2.9. Let $f, g: X \rightarrow Y$ be simplicial maps. A simplicial homotopy $H: f \simeq g$ is a map making the following diagram

$$\begin{array}{ccc} X \times \Delta^0 \cong X & & \\ 1 \times d^1 \downarrow & \searrow f & \\ X \times \Delta^1 & \xrightarrow{H} & Y \\ 1 \times d^0 \uparrow & \nearrow g & \\ X \times \Delta^0 \cong X & & \end{array}$$

commute.

In particular we are interested in relative homotopy, a fundamental requirement for the definition of simplicial homotopy groups.

Definition 1.2.10. Let $i: L \subset X$ be an inclusion such that the restrictions of f and g to L coincide. We say that there is a simplicial homotopy from f to g relative to L (notation: $f \xrightarrow{\simeq} g \text{ (rel } L)$) if the diagram above exists and the following

$$\begin{array}{ccc} L \times \Delta^1 & \xrightarrow{pr_L} & L \\ i \times 1 \downarrow & & \downarrow f|_L = g|_L \\ X \times \Delta^1 & \xrightarrow{H} & Y \end{array}$$

is commutative as well.

Homotopy relation between simplicial maps is not an equivalence relation in general, but it is in a lot of interesting cases. Suppose we have chosen a model structure on simplicial sets (see next section for details).

Proposition 1.2.11. If Y is fibrant, then homotopy of maps $X \rightarrow Y$ and $X \rightarrow Y \text{ (rel } L)$ are equivalence relations.

Proof. See [GJ99, Corollary 6.2]. □

Definition 1.2.12. Let X be a fibrant simplicial set and $v \in X_0$ a vertex of X . We define $\pi_n(X, v)$, $n \geq 0$, as the set of homotopy classes of maps $\alpha: \Delta^n \rightarrow X \text{ (rel } \partial\Delta^n)$ for α fitting into the diagram

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow v \\ \Delta^n & \xrightarrow{\alpha} & X \end{array}$$

In particular, $\pi_0(X)$ is the set of homotopy classes of vertices of X , that is the set of path components of X .

Proposition 1.2.13. With these definitions, $\pi_n(X, v)$ is a group for $n \geq 1$, which is abelian if $n \geq 2$.

Proof. See [GJ99, Theorem 7.2]. □

1.3 Model structure for quasi-categories

Definition 1.3.1. A map of simplicial sets $f: X \rightarrow Y$ is a *Kan fibration* if it has the right lifting property (from now on RLP) with respect to the inclusion $\Lambda_k^n \hookrightarrow \Delta^n$, for every $n > 0$ and $k \in [n]$.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

A map of simplicial sets is called *inner Kan fibration* if it has the RLP with respect to the inclusion of inner horns.

Definition 1.3.2. A map of simplicial sets is a *trivial Kan fibration* if it has the RLP with respect to the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for every $n \geq 0$.

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

In the following, we introduce an important class of morphisms, very useful to determine if a simplicial morphism is a fibration.

Definition 1.3.3. A morphism $i: A \rightarrow B$ which has the left lifting property (LLP) with respect to every Kan fibration is said to be *anodyne* (or even anodyne extension). It is called *inner anodyne* if it has LLP with respect to inner ones.

There exists another equivalent manner to define anodyne morphisms, but first we need the following definition.

Definition 1.3.4. A class of morphisms in a cocomplete category is called *saturated* if it closed under pushouts, retracts (in the arrow category) and transfinite compositions.

It is trivial to show that the class M_p of all morphisms which have the LLP with respect to a fixed simplicial map $p: X \rightarrow Y$ is saturated. If we have a class of monomorphisms B , we can define its *saturation* M_B or *the saturated class generated by B* as the intersection of all saturated classes M containing B . This leads to an alternative definition of anodyne extensions.

Definition 1.3.5. The class of anodyne extensions is the saturated class generated by the set of all inclusions $\Lambda_k^n \subset \Delta^n$, $0 \leq k \leq n$.

The equivalence between the two definitions is straightforward. The importance of anodyne extensions lies on the following proposition.

Proposition 1.3.6. A map of simplicial sets is a fibration if it has the right lifting property with respect to all anodyne extensions.

Proof. See [GJ99, Corollary 4.3] □

Remark 1.3.7. This specializes to the case of inner Kan fibrations. In this case one proves that a simplicial set X is a quasi-category if and only if the projection $X \rightarrow \Delta^0$ has the RLP with respect to every inner anodyne map of simplicial sets.

Proposition 1.3.8. A simplicial set X is a quasi-category if and only if the inclusion of simplicial sets $\Lambda_1^2 \hookrightarrow \Delta^2$ induces a trivial Kan fibration

$$p: X^{\Delta^2} \rightarrow X^{\Lambda_1^2}$$

Proof. We know that p is a trivial Kan fibration if and only if every lifting problem

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & X^{\Delta^2} \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ \Delta^m & \longrightarrow & X^{\Lambda_1^2} \end{array}$$

admits a solution. Using the fact that \mathbf{sSet} is cartesian closed it is easy to show that the previous lifting problem is equivalent to

$$\begin{array}{ccc} (\Delta^m \times \Lambda_1^2) \amalg_{\partial\Delta^m \times \Lambda_1^2} (\partial\Delta^m \times \Delta^2) & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^m \times \Delta^2 & \longrightarrow & \Delta^0 \end{array}$$

which is obtained by the universal property of the pushout. Let T be the collection of all morphisms of simplicial sets which have the left lifting property with respect to the projection $X \rightarrow \Delta^0$. Then p is a trivial Kan fibration if and only if T contains each of the inclusion maps

$$(\Delta^m \times \Lambda_1^2) \amalg_{\partial\Delta^m \times \Lambda_1^2} (\partial\Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$$

Since T is saturated by a previous observation, this is equivalent to the requirement that T contains all inner anodyne morphisms (see [Lur17, Proposition 2.3.2.1]), which is in turn equivalent to the requirement that X is a quasi-category (by the Remark 1.3.7). \square

Classically one gives a model structure on the category of small categories in a way such that weak equivalences are categorical equivalences. This allows us to recover information also on equivalence of simplicial sets, using a suitable Quillen equivalence between the two model structures. On simplicial sets there are two “natural” model structures, one having Kan complexes as fibrant objects and the other whose fibrant objects are quasi-categories.

Definition 1.3.9. Let \mathbf{C} and \mathbf{D} be small categories and $F: \mathbf{C} \rightarrow \mathbf{D}$ a functor between them. We call F an *isofibration* if every isomorphism $u: FC \rightarrow D$ in \mathbf{D} can be lifted to an isomorphism $v: C \rightarrow C'$ in \mathbf{C} with $F(v) = u$.

With this definition in mind we can construct the following model structure, which is actually the unique model structure on \mathbf{Cat} such that the weak equivalences are the categorical equivalences (see [SP12]).

Proposition 1.3.10. The category \mathbf{Cat} of small categories and functors between them admits a model structure $(\mathcal{W}, \text{Cof}, \text{Fib})$, where:

- \mathcal{W} are equivalences of categories,
- Cof are functors injective on the sets of objects,
- Fib are isofibrations.

This is called the *folk* (or *canonical*) *model structure*.

Proof. See [Bal12, Proposition 5.4] □

Proposition 1.3.11. \mathbf{sSet} admits a model structure $(\mathcal{W}, \text{Cof}, \text{Fib})$ in which

- \mathcal{W} are weak homotopy equivalences, i.e. morphisms whose geometric realization is a weak homotopy equivalence of topological spaces¹,
- Cof are monomorphisms, i.e. morphisms of simplicial sets $f: X \rightarrow Y$ such that $f_n: X_n \rightarrow Y_n$ is an injection of sets for all $n \in \mathbb{N}$,
- Fib are Kan fibrations.

This is called the *Quillen model structure*.

Proof. See [GJ99, Theorem 11.3] □

In this model structure fibrant objects are Kan complexes. The following result explains the reason why we called “trivial Kan fibrations” exactly like that.

Proposition 1.3.12. Trivial Kan fibrations are trivial fibrations in the Quillen model structure.

Proof. See [GJ99, Theorem 11.2]. □

In Remark A.2.10 we say that \mathbf{sSet} admits left Bousfield localization. Notably, we will see that Quillen model structure is the localization of another model structure in which fibrant objects are quasi-categories, known as the *Joyal model structure*. Let us first define $\tau_0: \mathbf{sSet} \rightarrow \mathbf{Set}$ to be the functor that sends a simplicial set X to the set of isomorphism classes of

¹Namely, a morphism that induces isomorphisms between the homotopy groups of the two topological spaces.

objects in the category $\tau_1(X)$. We know that τ_1 preserves finite products (Proposition 1.1.17), and moreover τ_0 is easily shown to be the composite

$$\mathbf{sSet} \xrightarrow{\tau_1} \mathbf{Cat} \xrightarrow{J} \mathbf{Gpd} \hookrightarrow \mathbf{Cat} \xrightarrow{N} \mathbf{sSet} \xrightarrow{\pi_0} \mathbf{Set}$$

where π_0 takes a simplicial set X to the set of path components of vertices and J sends a category to its groupoid of isomorphisms. Since π_0 clearly respects products and all the other functors are right adjoints, we deduce that τ_0 preserves products too. Then, with a construction similar to the one presented in Definition 1.1.18, we can define the category \mathbf{sSet}_{τ_0} whose objects are simplicial sets and such that, for any two objects A, B there is an hom-set $\tau_0(A, B) := \tau_0(B^A)$.

Definition 1.3.13. A *weak categorical equivalence* is a morphism of simplicial sets $u: A \rightarrow B$ such that the map

$$\tau_0(u, X): \tau_0(B, X) \rightarrow \tau_0(A, X)$$

is bijective for every quasi-category X .

Proposition 1.3.14. There is a model structure on \mathbf{sSet} for which

- \mathcal{W} are weak categorical equivalences,
- \mathcal{Cof} are monomorphisms,
- fibrant objects are quasi-categories.

This is called the *Joyal model structure*.

Proof. See [Ste18, Theorem 6.6]. □

Remark 1.3.15. Both in the Quillen and in the Joyal model structure every simplicial set is cofibrant.

What follows is a particular case of the so-called *Leibniz construction* (see [Rie14, Construction 11.1.7]).

Definition 1.3.16. Let \mathbf{A} be a category enriched over a monoidal category \mathcal{V} . Assume that \mathcal{V} has pullbacks and consider two morphisms $j: A \rightarrow A'$ and $f: B \rightarrow B'$ of \mathbf{A} . We define the *pullback-hom* $\mathrm{Hom}(j, f)$ to be the morphism obtained by the UP of the following pullback in \mathcal{V}

$$\begin{array}{ccc}
 & & \xrightarrow{j^*} \\
 \mathrm{Hom}(A', B) & \xrightarrow{\mathrm{Hom}(j, f)} & P \longrightarrow \mathrm{Hom}(A, B) \\
 \downarrow f_* & & \downarrow f_* \\
 & \mathrm{Hom}(A', B') & \xrightarrow{j^*} \mathrm{Hom}(A, B')
 \end{array}$$

In this context \mathbf{A} is just \mathbf{sSet} enriched over itself, since it is monoidal with respect to the product of simplicial sets (it is cartesian closed).

Proposition 1.3.17. The pullback-hom of a cofibration with an inner fibration is an inner fibration.

Proof. See [Rie14, Remark 15.2.2] □

As a corollary of this technical result, we are able to prove that quasi-categories form a cartesian closed category.

Corollary 1.3.18. \mathbf{qCat} , the full subcategory of \mathbf{sSet} spanned by quasi-categories, is cartesian closed.

Proof. We have to prove that if A is a simplicial set and X is a quasi-category, then X^A is a quasi-category. This is true because the pullback-hom of $\emptyset \rightarrow A$ and $X \rightarrow \Delta^0 \cong *$ is $X^A \rightarrow *$. □

Since every Kan complex is a quasi-category, it follows that Joyal model structure has more fibrant objects. Furthermore both model structures have the same cofibrations. It follows that Joyal model structure has a smaller class of weak equivalences. This means that the Quillen model structure is a left Bousfield localization of the Joyal model structure. In particular, a weak categorical equivalence is necessarily a weak homotopy equivalence.

Lemma 1.3.19. The nerve and its left adjoint define a Quillen adjunction $\tau_1: \mathbf{sSet} \rightleftarrows \mathbf{Cat}: N$ between the Joyal model structure on \mathbf{sSet} and the folk model structure on \mathbf{Cat} .

Proof. See [Rie14, Lemma 15.3.8] □

Corollary 1.3.20. If $f: X \rightarrow Y$ is a categorical equivalence, then the induced functor $\tau_1(f): \tau_1(X) \rightarrow \tau_1(Y)$ is an equivalence of categories. If $F: \mathbf{C} \rightarrow \mathbf{D}$ is an equivalence of categories, then $NF: N\mathbf{C} \rightarrow N\mathbf{D}$ is a categorical equivalence.

Proof. This follows from Lemma 1.3.19 and Ken Brown's lemma (Proposition A.1.6). □

Chapter 2

Prederivators

The first appearance of the notion of derivator (i.e., roughly speaking, “bicomplete prederivator”) dates back to the well-known 1983 manuscript *Pursuing Stacks* by Alexander Grothendieck. Derivator theory was later developed by him in the 1991 manuscript *Les Dérivateurs* and by several other people since then. In the following we won’t need to work with derivators, but instead it will suffice to use prederivators that satisfy only part of the axioms defining derivators (see [Gro12, §1]).

2.1 Preliminary results

In the following, for the sake of simplicity, we will confuse $[n]$ with its image through the inclusion functor $\Delta \xrightarrow{i} \mathbf{Cat}$. From the context will be clear if we consider $[n]$ as a set or as a category.

Definition 2.1.1. A *prederivator* is a strict 2-functor $\mathbb{D} : \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{CAT}$.

For each functor $J \xrightarrow{u} K$ between two small categories we use u^* to denote its image through \mathbb{D} and we call $\mathbb{D}([0])$ *the underlying category* of \mathbb{D} . The very first example of prederivator comes from ordinary category theory.

Example 2.1.2. Let \mathbf{C} be a category, then we can consider the following prederivator

$$\begin{aligned} \mathbf{y}_{\mathbf{C}} : \mathbf{Cat}^{\text{op}} &\rightarrow \mathbf{CAT} \\ J &\mapsto \mathbf{C}^J \end{aligned}$$

called the *prederivator represented by \mathbf{C}* .

Example 2.1.3. In Corollary 1.3.18 we showed that the simplicial mapping space $\mathcal{C}_{\bullet}^{\mathcal{K}} := \mathbf{sSet}(\Delta^{\bullet} \times K, \mathcal{C})$ is a quasi-category whenever \mathcal{C} is so. Hence we can define the *prederivator $\mathbf{Ho}_{\mathcal{C}}$ associated to the quasi-category \mathcal{C}* as follows

$$\begin{aligned} \mathbf{Ho}_{\mathcal{C}} : \mathbf{Cat}^{\text{op}} &\rightarrow \mathbf{CAT} \\ J &\mapsto \mathbf{Ho}(\mathcal{C}^{N(J)}) \end{aligned}$$

where $N(J)$ is the nerve of J and the functoriality of this construction is ensured by [Joy08b, Theorem 5.14].

Example 2.1.4. Let \mathbb{D} be a prederivator and let M be a fixed category. Then

$$\begin{aligned} \mathbb{D}^M : \mathbf{Cat}^{\text{op}} &\rightarrow \mathbf{CAT} \\ J &\mapsto \mathbb{D}^M(J) = \mathbb{D}(M \times J) \end{aligned}$$

is again a prederivator, called the *shifted prederivator*.

Remark 2.1.5. Let $K \in \mathbf{Cat}$, $k \in K$ and consider the (unique) functor $k: [0] \rightarrow K$ picking k . After we apply the prederivator \mathbb{D} to it we obtain the evaluation functor

$$\begin{aligned} k^* : \mathbb{D}(K) &\rightarrow \mathbb{D}([0]) \\ X &\mapsto X_k \\ X \xrightarrow{f} Y &\mapsto X_k \xrightarrow{f_k} Y_k. \end{aligned}$$

Furthermore, a natural transformation of the form

$$\begin{array}{ccc} & j & \\ & \curvearrowright & \\ [0] & \alpha \Downarrow & K \\ & \curvearrowleft & \\ & k & \end{array}$$

is just a map $\alpha: j \rightarrow k$ (where j and k are the objects of K chosen by the morphisms with their same name), which in turn determines a morphism $X_\alpha := \alpha_X^*: X_i \rightarrow X_j$ in the underlying category of \mathbb{D} .

Definition 2.1.6. We define the *underlying diagram functor* of \mathbb{D} at K as the following functor

$$\begin{aligned} \text{dia}_K : \mathbb{D}(K) &\rightarrow \mathbb{D}([0]^K) \\ X &\mapsto \text{dia}_K X \end{aligned}$$

where

$$\begin{aligned} \text{dia}_K X : K &\rightarrow \mathbb{D}([0]) \\ j &\mapsto X_j \\ j \xrightarrow{\alpha} k &\mapsto X_j \xrightarrow{X_\alpha} X_k \end{aligned}$$

Remark 2.1.7. Given another category J and $j \in J$, we can consider the corresponding functor

$$j \times \text{id}_K : K \cong [0] \times K \longrightarrow J \times K$$

giving a *partial underlying diagram functor*

$$\text{dia}_{J,K} : \mathbb{D}(J \times K) \longrightarrow \mathbb{D}(K)^J.$$

This definition is basically the same as the previous one, with the only difference being that this time we use the shifted prederivator. We require that prederivators satisfy the following axioms, which are the first two axioms of derivators and a strictification of the axiom defining strong derivators (see [Gro12]).

(Der 1) for every set of small categories $\{J_i\}_{i \in I}$ the canonical map

$$\mathbb{D} \left(\coprod_{i \in I} J_i \right) \rightarrow \prod_{i \in I} \mathbb{D}(J_i)$$

is an equivalence of categories,

(Der 2) $f: X \rightarrow Y$ is an isomorphism in $\mathbb{D}(J)$ if and only if $f_j: X_j \rightarrow Y_j$ is an isomorphism in $\mathbb{D}([0]) \forall j \in J$, that is

$$\text{dia}_J: \mathbb{D}(J) \rightarrow \mathbb{D}([0])^J$$

is conservative,

(Der 5') the partial underlying diagram functor

$$\text{dia}_{[1],J}: \mathbb{D}([1] \times J) \longrightarrow \mathbb{D}(J)^{[1]}$$

is full and surjective on objects for each category J .

Proposition 2.1.8. $\mathbf{Ho}_{\mathcal{C}}$ satisfies the list of axioms introduced in the Remark 2.1.7.

Proof. The first axiom is satisfied since $\mathbf{sSet}(N(-), \mathcal{C})$ sends coproducts in products, which are preserved by \mathbf{Ho} . The surjectivity of $\text{dia}_{[1],J}$ follows from the definition of the homotopy category and the other axioms are proven in Lemma 1.3 and Lemma 1.6 of [Car16]. \square

2.2 The 2-category \mathbf{PDer}

Definition 2.2.1. Let \mathbb{D} and \mathbb{D}' be prederivators. A *morphism of prederivators* $F: \mathbb{D} \rightarrow \mathbb{D}'$ is a pseudonatural transformation between them.

Remark 2.2.2. Unraveling the definition, this morphism is given by a collection of functors

$$F_J: \mathbb{D}(J) \rightarrow \mathbb{D}'(J), \quad J \in \mathbf{Cat}$$

and a family of natural isomorphisms

$$\gamma_u: u^* \circ F_K \xrightarrow{\cong} F_J \circ u^*, \quad u: J \rightarrow K.$$

This means that the following square commutes up to (natural) isomorphism.

$$\begin{array}{ccc}
J & \mathbb{D}(K) & \xrightarrow{F_K} \mathbb{D}'(K) \\
u \downarrow & u^* \downarrow & \swarrow \gamma_u \quad \downarrow u^* \\
K & \mathbb{D}(J) & \xrightarrow{F_J} \mathbb{D}'(J)
\end{array}$$

These morphisms have to satisfy the following conditions:

- (i) $\gamma_{\text{id}_J} = \text{id}_{F_J}$,
- (ii) for every $w: K \rightarrow L$ we have $\gamma_{wu} = \gamma_u \circ \gamma_w$,
- (iii) for every $u, v: J \rightarrow K$ and $\alpha: u \Rightarrow v$ we have

$$\begin{array}{ccc}
u^* \circ F_K & \xrightarrow{\gamma_u} & F_J \circ u^* \\
\alpha^* \Downarrow & & \Downarrow \alpha^* \\
v^* \circ F_K & \xrightarrow{\gamma_v} & F_J \circ v^*
\end{array}$$

Definition 2.2.3. A morphism of prederivators is called *strict* if $\gamma_u = \text{id}$ for every u .

Definition 2.2.4. Let $F, G: \mathbb{D} \rightarrow \mathbb{D}'$ be morphisms of prederivators. A *natural transformation* $\tau: F \Rightarrow G$ is a modification of the pseudonatural transformations F and G .

This means we have a family $\{\tau_I: F_I \rightarrow G_I\}_{I \in \mathbf{Cat}}$ of natural transformations such that, for every functor $u: J \rightarrow K$, the following diagram

$$\begin{array}{ccc}
u^* \circ F_K & \xrightarrow{\gamma_u} & F_J \circ u^* \\
\tau_k \Downarrow & & \Downarrow \tau_J \\
u^* \circ G_K & \xrightarrow{\gamma_u} & G_J \circ u^*
\end{array}$$

commutes. At this point, we can define the 2-category **PDer**.

Definition 2.2.5. **PDer** is the 2-category with

- 0-cells: prederivators $\mathbb{D}, \mathbb{D}', \dots$
- 1-cells: morphisms of prederivators $F: \mathbb{D} \rightarrow \mathbb{D}'$
- 2-cells: natural transformations $\tau: F \Rightarrow G$ between 1-cells.

The sub-2-category of prederivators, strict morphisms and natural transformations between them is usually denoted by **PDerst**. As an example, we can see that there is a 2-categorical embedding of **Cat** into it.

Remark 2.2.6. Every functor $F: \mathbf{C} \rightarrow \mathbf{D}$ determines a strict morphism of prederivators $\mathbf{y}_{\mathbf{C}} \rightarrow \mathbf{y}_{\mathbf{D}}$ by precomposition. The morphisms in the image are 2-natural transformations. Thus, as a special case of the 2-categorical Yoneda lemma (see Theorem B.2.12), we get a fully faithful 2-functor

$$\begin{aligned} y: \underline{\mathbf{Cat}} &\rightarrow \underline{\mathbf{PDer}}^{\text{st}} \\ \mathbf{C} &\mapsto \mathbf{y}_{\mathbf{C}} \\ \mathbf{C} \xrightarrow{F} \mathbf{D} &\mapsto F \circ - \\ F \xrightarrow{\alpha} G &\mapsto \alpha \circ - \end{aligned}$$

Remark 2.2.7. Every functor between two categories induces a morphism of the prederivators defined in Example 2.1.4. In fact, given a prederivative \mathbb{D} and a functor $v: L \rightarrow M$, we consider the functor

$$v \times \text{id}_K: L \times K \rightarrow M \times K,$$

for K a generic category. Then we apply \mathbb{D} to this functor and we obtain the morphism

$$\mathbb{D}(v \times \text{id}_K): \mathbb{D}(M \times K) = \mathbb{D}^M(K) \rightarrow \mathbb{D}^L(K) = \mathbb{D}(L \times K).$$

These data assemble into a morphism of prederivators $\mathbb{D}^M \rightarrow \mathbb{D}^L$. A similar argument holds for natural transformations, hence every prederivative \mathbb{D} induces a 2-functor $\mathbb{D}^{(-)}: \underline{\mathbf{Cat}}^{\text{op}} \rightarrow \underline{\mathbf{PDer}}$.

Definition 2.2.8. Given a prederivative \mathbb{D} , we define the functor

$$\begin{aligned} \text{Ob}(\mathbb{D}): \underline{\mathbf{Cat}}^{\text{op}} &\rightarrow \underline{\mathbf{Set}} \\ J &\mapsto \text{Ob}(\mathbb{D}(J)) \\ I \xrightarrow{u} J &\mapsto \text{Ob}(\mathbb{D}(J)) \xrightarrow{\mathbb{D}(u)} \text{Ob}(\mathbb{D}(I)) \end{aligned}$$

where by $\mathbb{D}(u)$ we mean the associated function on objects.

Remark 2.2.9. As a matter of fact, the target of the previous functor is not a small category, since the image of a small category with respect to the action of a prederivative is no longer a small category. However, we follow the general practice to avoid size issues invoking a bigger Grothendieck universe with regard to which it is actually small.

As we said in Remark 2.1.7, we will assume implicitly that prederivators satisfy axioms (Der 1-2) and (Der 5'). Hence, with a little abuse of notation, we will also write \mathbf{PDer} to denote the category having them as objects. We can define a functor which acts by restricting a prederivative (satisfying the aforementioned axioms) to its action on the objects of the categories in its image as follows.

Definition 2.2.10. The *restriction functor* is defined as

$$\begin{aligned} \text{Ob: } \mathbf{PDer}^{\text{st}} &\rightarrow \mathbf{Set}^{\mathbf{Cat}^{\text{op}}} \\ \mathbb{D}_1 &\mapsto \text{Ob}(\mathbb{D}_1) \\ \mathbb{D}_1 \xrightarrow{F} \mathbb{D}_2 &\mapsto \text{Ob}(\mathbb{D}_1) \xrightarrow{\text{Ob}(F)} \text{Ob}(\mathbb{D}_2) \end{aligned}$$

where $\text{Ob}(F)$ is given componentwise by the function

$$\text{Ob}(\mathbb{D}_1(J)) \rightarrow \text{Ob}(\mathbb{D}_2(J))$$

induced on objects from the functors $F_J: \mathbb{D}_1(J) \rightarrow \mathbb{D}_2(J)$ for every $J \in \mathbf{Cat}$.

Proposition 2.2.11. The restriction functor $\text{Ob: } \mathbf{PDer}^{\text{st}} \rightarrow \mathbf{Set}^{\mathbf{Cat}^{\text{op}}}$ is faithful.

Proof. A strict morphism $F: \mathbb{D} \rightarrow \mathbb{D}'$ is determined by its value on $J \in \mathbf{Cat}$, since coherence conditions are trivially fulfilled. To show that the functor is faithful, let us take a family of functions $r_J: \text{Ob}(\mathbb{D}(J)) \rightarrow \text{Ob}(\mathbb{D}'(J))$ and prove that there is *at most* one strict morphism of prederivators with components $F_J: \mathbb{D}(J) \rightarrow \mathbb{D}'(J)$ such that $\text{Ob}(F_J) = r_J$. Indeed if we take F as above, and we consider a morphism $f: X \rightarrow Y$ in $\mathbb{D}(J)$, by (Der 5') we can find $\hat{f} \in \mathbb{D}([1] \times J)$ such that $f = \text{dia}_{[1],J}(\hat{f})$. Since $\text{dia}_{[1],J}$ is defined as the action of a prederivator on the unique natural transformation between the functors $[0] \rightrightarrows [1]$, each of which chooses one of the two objects of $[1]$, we get a commutative square

$$\begin{array}{ccc} \mathbb{D}([1] \times J) & \xrightarrow{F_{[1] \times J}} & \mathbb{D}'([1] \times J) \\ \text{dia}_{[1],J} \downarrow & & \downarrow \text{dia}_{[1],J} \\ \mathbb{D}(J)^{[1]} & \xrightarrow{F_J} & \mathbb{D}'(J)^{[1]} \end{array}$$

corresponding componentwise to the coherence conditions for 2-cells of Remark 2.2.2, (iii). It follows that $F_J(f) = F_J(\text{dia}_{[1],J}(\hat{f})) = \text{dia}_{[1],J}(r_{[1] \times J}(\hat{f}))$. Therefore two strict morphisms of derivators with the same restrictions must coincide. \square

2.3 The simplicially enriched category \mathbf{PDer}_\bullet

We can define a simplicially enriched category having prederivators as objects in the following way.

Definition 2.3.1. We define \mathbf{PDer}_\bullet as the simplicially enriched category such that, for every pair of prederivators $\mathbb{D}_1, \mathbb{D}_2$, the simplicial set of mor-

phisms is

$$\begin{aligned} \mathbf{PDer}(\mathbb{D}_1, \mathbb{D}_2): \mathbf{\Delta}^{\text{op}} &\rightarrow \mathbf{Set} \\ [n] &\mapsto \mathbf{PDer}^{\text{st}}(\mathbb{D}_1, \mathbb{D}_2^{[n]}) \\ ([m] \rightarrow [n]) &\mapsto \mathbf{PDer}^{\text{st}}(\mathbb{D}_1, \mathbb{D}_2^{[n]}) \rightarrow \mathbf{PDer}^{\text{st}}(\mathbb{D}_1, \mathbb{D}_2^{[m]}) \end{aligned}$$

Notice that we are using the result found in Example 2.2.7 and then taking the morphism on hom-sets induced by the hom-functor $\mathbf{PDer}(\mathbb{D}_1, -)$. We ask ourselves what the composition of two morphisms should be. Thus we want to construct a map

$$\begin{aligned} \mathbf{PDer}_n(\mathbb{D}_1, \mathbb{D}_2) \times \mathbf{PDer}_n(\mathbb{D}_2, \mathbb{D}_3) &\rightarrow \mathbf{PDer}_n(\mathbb{D}_1, \mathbb{D}_3) = \mathbf{PDer}^{\text{st}}(\mathbb{D}_1, \mathbb{D}_3^{[n]}) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

This can be built from f and g by means of the composition

$$\mathbb{D}_1 \xrightarrow{f} \mathbb{D}_2^{[n]} \xrightarrow{g^{[n]}} (\mathbb{D}_3^{[n]})^{[n]} \cong \mathbb{D}_3^{[n] \times [n]} \xrightarrow{\text{diag}^{[n]}} \mathbb{D}_3^{[n]}$$

where $\text{diag}_I: I \times I \rightarrow I$, $I \in \mathbf{Cat}$ is the diagonal functor.

Chapter 3

Embedding theorems

In this chapter we discuss the results proved in [Car16] and [FKKR18].

3.1 The simplicial embedding

Definition 3.1.1. Let us define the functor

$$\begin{aligned} \mathbf{Ho}: \mathbf{qCat} &\rightarrow \mathbf{PDer}^{\text{st}} \\ \mathcal{C} &\mapsto \mathbf{Ho}_{\mathcal{C}} \\ (\mathcal{C} \rightarrow \mathcal{C}') &\mapsto \mathbf{Ho}_{\mathcal{C}} \rightarrow \mathbf{Ho}_{\mathcal{C}'} \end{aligned}$$

where the action on morphisms is obtained componentwise by applying the functor $(-)^{N(J)}$ and then the functor \mathbf{Ho} , for every $J \in \mathbf{Cat}$.

Definition 3.1.2. We define \mathbf{qCat}_{\bullet} as the simplicially enriched category such that, for every pair of quasi-categories $\mathcal{C}, \mathcal{C}'$, the simplicial set of morphisms is

$$\begin{aligned} \mathbf{qCat}(\mathcal{C}, \mathcal{C}') &: \Delta^{\text{op}} \rightarrow \mathbf{Set} \\ [n] &\mapsto \mathbf{qCat}(\mathcal{C}, \mathcal{C}'^{\Delta^n}) \\ ([m] \rightarrow [n]) &\mapsto \mathbf{qCat}(\mathcal{C}, \mathcal{C}'^{\Delta^n}) \rightarrow \mathbf{qCat}(\mathcal{C}, \mathcal{C}'^{\Delta^m}) \end{aligned}$$

Lemma 3.1.3. \mathbf{Ho} extends to a simplicial functor $\mathbf{Ho}_{\bullet}: \mathbf{qCat}_{\bullet} \rightarrow \mathbf{PDer}_{\bullet}$.

Proof. The proof is straightforward. The action on objects is given by that of \mathbf{Ho} , and to define the action on the simplicial sets of morphisms we just consider the action of the functor \mathbf{Ho} between the set of morphisms of the categories \mathbf{qCat} and \mathbf{PDer} as follows:

$$\mathbf{qCat}(\mathcal{C}, \mathcal{C}'^{\Delta^n}) \xrightarrow{\mathbf{Ho}} \mathbf{PDer}(\mathbf{Ho}_{\mathcal{C}}, \mathbf{Ho}_{\mathcal{C}'^{\Delta^n}}).$$

Then we notice that $\mathbf{qCat}(\mathcal{C}, \mathcal{C}'^{\Delta^n}) = \mathbf{qCat}_n(\mathcal{C}, \mathcal{C}')$ by definition and moreover there is a chain of isomorphisms

$$\begin{aligned} \mathbf{Ho}_{\mathcal{C}'^{\Delta^n}} &= \mathbf{Ho}((\mathcal{C}'^{\Delta^n})^{N(-)}) \\ &\cong \mathbf{Ho}(\mathcal{C}'^{N(-) \times \Delta^n}) \\ &\cong \mathbf{Ho}(\mathcal{C}'^{N(-) \times N([n])}) \\ &\cong \mathbf{Ho}(\mathcal{C}'^{N(- \times [n])}) \\ &= \mathbf{Ho}_{\mathcal{C}'}(- \times [n]) \\ &= \mathbf{Ho}_{\mathcal{C}'}^{[n]} \end{aligned}$$

which gives an isomorphism $\mathbf{PDer}(\mathbf{Ho}_{\mathcal{C}}, \mathbf{Ho}_{\mathcal{C}'^{\Delta^n}}) \cong \mathbf{PDer}(\mathbf{Ho}_{\mathcal{C}}, \mathbf{Ho}_{\mathcal{C}'}^{[n]}) = \mathbf{PDer}_n(\mathbf{Ho}_{\mathcal{C}}, \mathbf{Ho}_{\mathcal{C}'})$. Hence it suffices to define the action of the simplicial functor \mathbf{Ho}_\bullet in such a way that at the level of the sets of n -simplices it agrees with \mathbf{Ho} . \square

Recall the following two important results (see [Bor94a, §3.7] or [Rie14, §1.3]) in the theory of Kan extensions.

Proposition 3.1.4. Let $F: \mathbf{C} \rightarrow \mathbf{E}$ and $G: \mathbf{C} \rightarrow \mathbf{D}$ be two functors. If \mathbf{C} is small and \mathbf{E} is complete, then the right Kan extension of F along G exists and can be computed using the following formula

$$\mathbf{Ran}_G F(d) \cong \lim((d \downarrow G) \xrightarrow{\pi_d} \mathbf{C} \xrightarrow{F} \mathbf{E})$$

where $\pi_d: (d \downarrow G) \rightarrow \mathbf{C}$, $(c, f: d \rightarrow Gc) \mapsto c$ is the projection functor.

Corollary 3.1.5. When G is fully faithful, the counit $\mathbf{Ran}_G F \circ G \Rightarrow F$ is an isomorphism.

These considerations are especially helpful in proving the next facts.

Lemma 3.1.6. Let \mathcal{C} be a quasi-category, and $j: \mathbf{\Delta}^{\text{op}} \rightarrow \mathbf{Cat}^{\text{op}}$ be the opposite of the inclusion functor. Then $\mathbf{Ob}(\mathbf{Ho}_{\mathcal{C}})$ is the right Kan extension of \mathcal{C} along j .

Proof. We know that $\mathbf{\Delta}^{\text{op}}$ is small and \mathbf{Set} is complete, hence it suffices to show that

$$\mathbf{Ob}(\mathbf{Ho}_{\mathcal{C}})(J) \cong \lim((J \downarrow j) \xrightarrow{\pi_J} \mathbf{\Delta}^{\text{op}} \xrightarrow{\mathcal{C}} \mathbf{Set}).$$

As an immediate consequence of Definition 2.2.8,

$$\begin{aligned} \mathbf{Ob}(\mathbf{Ho}_{\mathcal{C}})(J) &= \mathbf{Ob}(\mathbf{Ho}_{\mathcal{C}}(J)) \\ &= \mathbf{Ob}(\mathbf{Ho}(\mathcal{C}^{N(J)})) \\ &= \mathbf{sSet}(N(J), \mathcal{C}) \end{aligned}$$

since the objects of the homotopy category $\mathbf{Ho}(\mathcal{C}^{N(J)})$ are the simplicial morphisms between $N(J)$ and \mathcal{C} . The result follows easily once we express $N(J)$ as the colimit of its simplices, i.e. the colimit indexed over its category of elements (see [Car16, Lemma 2.8]). \square

Proposition 3.1.7. $\mathbf{Ho}: \mathbf{qCat} \rightarrow \mathbf{PDer}^{\text{st}}$ is fully faithful.

Proof. The claim is equivalent to $\mathbf{qCat}(\mathcal{C}, \mathcal{C}') \cong \mathbf{PDer}^{\text{st}}(\mathbf{Ho}(\mathcal{C}), \mathbf{Ho}(\mathcal{C}'))$. We show that \mathbf{Ho} induces a bijection on hom-sets by finding an inverse. First notice that

$$\text{Ob}(\mathbf{Ho}_{\mathcal{C}}) \circ j \cong \text{Ran}_j \mathcal{C} \circ j \cong \mathcal{C}: \mathbf{\Delta}^{\text{op}} \rightarrow \mathbf{Set}$$

since j is fully faithful (Corollary 3.1.5). Therefore, we define a functor Res which sends a prederivator of the form $\mathbf{Ho}(\mathcal{C})$ to its restriction to \mathcal{C} (using Ob and j as above) and a map of prederivators to a map of quasi-categories in a similar manner. For every $f: \mathcal{C} \rightarrow \mathcal{C}'$ in \mathbf{qCat} , we know that $\mathbf{Ho}(f) = \text{Ho} \circ (f^{N(-)})$. Hence $\text{Res}(\mathbf{Ho}(f)) = \text{Ob}(\mathbf{Ho}(f)) \circ j \in \mathbf{sSet}$ and its n -simplices are

$$\begin{aligned} \text{Res}(\mathbf{Ho}(f))[n] &= \text{Ob}(\mathbf{Ho}(f)) \circ j[n] \\ &= \text{Ob}(\text{Ho}(f^{N[n]})) \\ &= \text{Ob}(\text{Ho}(f^{\Delta^n})) \end{aligned}$$

with $\text{Ob}(\text{Ho}(f^{\Delta^n})): \text{Ob}(\text{Ho}(\mathcal{C}^{\Delta^n})) \rightarrow \text{Ob}(\text{Ho}(\mathcal{C}'^{\Delta^n}))$. We can rewrite the former as $\text{Ob}(\text{Ho}(\mathcal{C}^{\Delta^n})) = \mathbf{sSet}(\Delta^n, \mathcal{C})$, while the latter is equal to $\mathbf{sSet}(\Delta^n, \mathcal{C}')$. The Yoneda lemma implies that the morphism between those two are in bijection with the simplicial maps $\mathcal{C} \rightarrow \mathcal{C}'$, so that we find $\text{Res}(\mathbf{Ho}(f)) = f$. On the other hand, we have to prove that $\mathbf{Ho}(\text{Res}(F)) = F$ for every morphism $F: \mathbf{Ho}_{\mathcal{C}} \rightarrow \mathbf{Ho}_{\mathcal{C}'}$. By Proposition 2.2.11, it suffices to show that the restrictions of $\mathbf{Ho}(\text{Res}(F))$ and F to functors in $\mathbf{Set}^{\mathbf{Cat}^{\text{op}}}$ do coincide. This follows from Lemma 3.1.6 and the adjunction $j^* \dashv \text{Ran}_j$, since

$$\begin{aligned} \mathbf{Set}^{\mathbf{Cat}^{\text{op}}}(\text{Ob}(\mathbf{Ho}_{\mathcal{C}}), \text{Ob}(\mathbf{Ho}_{\mathcal{C}'})) &\cong \mathbf{Set}^{\mathbf{Cat}^{\text{op}}}(\text{Ran}_j \mathcal{C}, \text{Ran}_j \mathcal{C}') \\ &\cong \mathbf{sSet}(j^* \text{Ran}_j \mathcal{C}, \mathcal{C}') \\ &\cong \mathbf{sSet}(\mathcal{C}, \mathcal{C}') \end{aligned}$$

Hence a morphism between $\text{Ob}(\mathbf{Ho}_{\mathcal{C}})$ and $\text{Ob}(\mathbf{Ho}_{\mathcal{C}'})$ is completely determined by its restriction to a morphism $\mathcal{C} \rightarrow \mathcal{C}'$. It remains to check that $\text{Res}(\mathbf{Ho}(\text{Res}(F))) = \text{Res}(F)$, but this is true since $\text{Res} \circ \mathbf{Ho}$ is the identity on $\mathbf{sSet}(\mathcal{C}, \mathcal{C}')$. \square

Theorem 3.1.8. $\mathbf{Ho}_\bullet: \mathbf{qCat}_\bullet \rightarrow \mathbf{PDer}_\bullet$ is simplicially fully faithful.

Proof. This is indeed a consequence of the previous Proposition, once we extend \mathbf{Ho} to a simplicially enriched functor, following the construction presented in the proof of Lemma 3.1.3. In fact, we have isomorphisms of simplicial sets which specialise as follows at the level of n -simplices:

$$\begin{aligned} \mathbf{PDer}_n(\mathbf{Ho}_\mathcal{C}, \mathbf{Ho}_{\mathcal{C}'}) &= \mathbf{PDer}^{\text{st}}(\mathbf{Ho}_\mathcal{C}, \mathbf{Ho}_{\mathcal{C}'}^{[n]}) \\ &\cong \mathbf{qCat}(\mathcal{C}, \mathcal{C}'^{\Delta^n}) \\ &= \mathbf{qCat}_n(\mathcal{C}, \mathcal{C}') \end{aligned}$$

In particular, this construction respects simplicial composition laws of \mathbf{qCat}_\bullet and \mathbf{PDer}_\bullet . \square

3.2 The 2-categorical embedding

Lemma 3.2.1. \mathbf{Ho} extends to a 2-functor $\underline{\mathbf{Ho}}: \underline{\mathbf{qCat}} \rightarrow \underline{\mathbf{PDer}}$.

Proof. The idea is to prove that the assignment $\mathcal{C} \mapsto \mathbf{Ho}(\mathcal{C}^{N(-)})$ is a 2-functor looking at the image, or in other words to give an explicit and componentwise construction of the 2-functors $\mathbf{Ho}(\mathcal{C}^{N(J)})$, $J \in \mathbf{Cat}$, assuming that at the level of 0-cells $\underline{\mathbf{Ho}}$ is defined just like \mathbf{Ho} . Therefore, the next step is to define a functor

$$\underline{\mathbf{qCat}}(\mathcal{C}, \mathcal{C}') \rightarrow \underline{\mathbf{PDer}}(\mathbf{Ho}(\mathcal{C}), \mathbf{Ho}(\mathcal{C}'))$$

To this end, we first construct a 2-functor $\underline{\mathbf{Ho}}: \underline{\mathbf{qCat}} \rightarrow \underline{\mathbf{Cat}}$ that sends a quasi-category to its homotopy category. The action on morphism categories is given by the product-hom adjunction. In particular, we consider the counit

$$ev: (-)^{\mathcal{C}} \times \mathcal{C} = (- \times \mathcal{C}) \circ (-)^{\mathcal{C}} \Rightarrow \text{id}_{\underline{\mathbf{qCat}}}$$

of the adjunction $- \times \mathcal{C} \dashv (-)^{\mathcal{C}}$, then we take the component at \mathcal{C}' obtaining

$$ev_{\mathcal{C}'}: \mathcal{C}'^{\mathcal{C}} \times \mathcal{C} \rightarrow \mathcal{C}'.$$

Now we apply the usual functor \mathbf{Ho} , which commutes with finite products,

$$\mathbf{Ho}(ev_{\mathcal{C}'}): \mathbf{Ho}(\mathcal{C}'^{\mathcal{C}} \times \mathcal{C}) \cong \mathbf{Ho}(\mathcal{C}'^{\mathcal{C}}) \times \mathbf{Ho}(\mathcal{C}) \rightarrow \mathbf{Ho}(\mathcal{C}').$$

Then

$$\underline{\mathbf{qCat}}(\mathcal{C}, \mathcal{C}') = \mathbf{Ho}(\mathcal{C}'^{\mathcal{C}}) \rightarrow \mathbf{Ho}(\mathcal{C}')^{\mathbf{Ho}(\mathcal{C})} = \underline{\mathbf{Cat}}(\underline{\mathbf{Ho}}(\mathcal{C}), \underline{\mathbf{Ho}}(\mathcal{C}'))$$

is the image of $\mathbf{Ho}(ev_{\mathcal{C}'})$ through the natural isomorphism defining the product-hom adjunction in \mathbf{Cat} . It remains to show that the nerve functor

can be enhanced to a 2-functor $\underline{N}: \mathbf{Cat} \rightarrow \mathbf{qCat}$. We define it on morphisms through the following chain of natural isomorphisms

$$\begin{aligned} \underline{\mathbf{Cat}}(K, J) &= J^K \\ &\cong \mathbf{Ho}(N(J^K)) \\ &\cong \mathbf{Ho}(N(J)^{N(K)}) \end{aligned}$$

in which the first isomorphism is the inverse of the counit of the adjunction $\mathbf{Ho} \dashv N$ (invertible since N is fully faithful) and the second one holds since for any two categories \mathbf{A} and \mathbf{B} the canonical map $N(\mathbf{B}^{\mathbf{A}}) \rightarrow N(\mathbf{B})^{N(\mathbf{A})}$ is an isomorphism (see [Joy08b, Proposition B.0.16]). \square

Definition 3.2.2. We say that a simplicial set is *small* if it is isomorphic to a small colimit of standard n -simplices. A quasi-category is called small if it is a small simplicial set.

Remark 3.2.3. For example, this is the case of the quasi-categories arising as the nerve of small categories.

The following result, whose proof can be found in [Car16], is important since it extends the embedding of quasi-categories into prederivators to the world of 2-categories.

Theorem 3.2.4. The restriction of the 2-functor \mathbf{Ho} to the 2-category of small quasi-categories is bicategorically fully faithful.

Definition 3.2.5. A category is called *homotopy finite* if its nerve has finitely many nondegenerate simplices.

Remark 3.2.6. The condition of being homotopy finite is stronger than the one of being finite. For instance, we can consider a finite group G seen as a groupoid with only one object. It is a finite category since it has one object and a finite number of morphisms, namely the group of endomorphisms of its only object. However, its nerve has infinitely many nondegenerate simplices. In fact, the nondegenerate simplices of this simplicial set are strings of non-identity arrows corresponding to the non-identity elements of the group. Hence n -simplices coincide with the elements of size n of the free group on G . Face maps act as a composition according to the group law of G , and degeneracy maps add copies of the identity element of G . Fortunately, a more workable characterization of homotopy finite categories is available. Indeed, a category is homotopy finite if and only if it is finite, skeletal and admits no nontrivial endomorphisms (see [Car16, Definition 0.3]).

Remark 3.2.7. Homotopy finite categories assemble into a 2-category which we will denote by \mathbf{HFin} .

Example 3.2.8. Any ordinal category $[n]$ is homotopy finite.

Proposition 3.2.9. \mathbf{HFin} is a small 2-category.

Proof. Notice that for small 2-category we mean that n -cells assemble into a set, for $n = 0, 1, 2$. First of all, we notice that the class of objects (in the sense of 0-cells) is countable and hence a set. In fact, one can write

$$\mathrm{Ob}(\mathbf{HFin}) = \coprod_{m,n} C_{m,n}$$

where $C_{m,n}$ is the set of homotopy finite categories with exactly m objects and n morphisms. Given that this is a coproduct of finite sets indexed by a countable set, it has to be countable. In a similar way we see that the class of functors between two homotopy finite categories J and K and strict natural transformations between functors F and G are actually sets, being subsets of, respectively, $\mathbf{Set}(\mathrm{Mor}(J), \mathrm{Mor}(K))$ and $\mathbf{Set}(\mathrm{Ob}(J), \mathrm{Mor}(K))$. \square

In the following, we replace the target 2-category \mathbf{PDer} with the 2-category of prederivators defined on homotopy finite categories. Another deep insight about the behaviour of the functor realizing the 2-categorical embedding is witnessed by the following theorem (which is Theorem 4.1 of [Car16]).

Theorem 3.2.10. The restriction of \mathbf{Ho} to the 2-category of Kan complexes reflects equivalences.

3.3 A new model for $(\infty, 1)$ -categories

In this section we present the results achieved by the authors in the paper [FKKR18]. Throughout this section we consider $\mathbf{PDer}_{\mathrm{hfin}}^{\mathrm{st}}$ as the ordinary category of 2-functors $\mathbf{HFin}^{\mathrm{op}} \rightarrow \mathbf{Cat}$ and strict natural transformations.

Proposition 3.3.1. $\mathbf{PDer}_{\mathrm{hfin}}^{\mathrm{st}}$ is a locally small category.

Proof. It is well known that \mathbf{Cat} is locally small and, moreover, \mathbf{HFin} is a small 2-category by Proposition 3.2.9. A standard result in enriched category theory ensures that the functor category $\mathbf{PDer}_{\mathrm{hfin}}^{\mathrm{st}}$ is locally small. \square

Remark 3.3.2. From its very definition it follows that $\mathbf{PDer}_{\mathrm{hfin}}^{\mathrm{st}}$ is complete and cocomplete since this is the case for \mathbf{Cat} and limits and colimits are computed pointwise in \mathbf{Cat} .

From now on \mathbf{PDer} will be used to denote prederivators defined on homotopy finite categories instead of $\mathbf{PDer}_{\mathrm{hfin}}$, hence in particular we will use the notation $\mathbf{PDer}^{\mathrm{st}}$ instead of $\mathbf{PDer}_{\mathrm{hfin}}^{\mathrm{st}}$.

Definition 3.3.3. We define the underlying simplicial set functor

$$\begin{aligned} R: \mathbf{PDer}^{\mathrm{st}} &\rightarrow \mathbf{sSet} \\ \mathbb{D} &\mapsto \mathrm{Ob}(\mathbb{D}) \end{aligned}$$

sending a prederivator \mathbb{D} to the composite $\text{Ob} \circ \mathbb{D} \circ j: \mathbf{\Delta}^{\text{op}} \rightarrow \mathbf{Set}$, where j is the opposite of the functor which includes the simplex category into the category of homotopy finite categories and Ob sends¹ a small category to its set of objects².

Proposition 3.3.4. R has both a left and a right adjoint.

Proof. First of all, notice that R can be written as the composition

$$\mathbf{PDer}^{\text{st}} = \text{Hom}(\mathbf{HFin}^{\text{op}}, \mathbf{Cat}) \xrightarrow{j^*} \text{Hom}(\mathbf{\Delta}^{\text{op}}, \mathbf{Cat}) \xrightarrow{\text{Ob}_*} \text{Hom}(\mathbf{\Delta}^{\text{op}}, \mathbf{Set}) = \mathbf{sSet}$$

in which $\mathbf{\Delta}^{\text{op}}$ and \mathbf{Set} are regarded as discrete 2-categories (i.e. if the only 2-cells are identities). Formal arguments based on the completeness and cocompleteness of \mathbf{Cat} and \mathbf{Set} show that each functor has both a left and a right adjoint, given by suitable Kan extensions. These compose into a right and a left adjoint for R . \square

Remark 3.3.5. Using the formula for calculate left Kan extensions, we can explicitly define the left adjoint $L: \mathbf{sSet} \rightarrow \mathbf{PDer}^{\text{st}}$ of R as

$$LX = \text{colim}_{\mathbf{\Delta}^n \rightarrow X} \mathbf{Cat}(-, [n]), \quad X \in \mathbf{sSet}$$

Lemma 3.3.6. The unit of the adjunction $L \dashv R$ is an isomorphism.

Proof. We have to prove that $\eta_X: X \cong RL(X)$ for every $X \in \mathbf{sSet}$. An easy calculation (see [FKKR18, Proposition 1.18]) shows that the isomorphism holds for representable presheaves. Hence, since both L and R are left adjoints and then preserve colimits we have

$$\begin{aligned} RL(X) &\cong RL(\text{colim } \mathbf{\Delta}^n) \\ &\cong \text{colim}(RL\mathbf{\Delta}^n) \\ &\cong \text{colim } \mathbf{\Delta}^n \\ &\cong X. \end{aligned} \quad \square$$

Remark 3.3.7. There is an isomorphism of prederivators $y_J \cong \mathbf{Ho}(NJ)$, which is a consequence of the isomorphism $\text{Ho}(NJ) \cong J$ in \mathbf{Cat} . Indeed, as mentioned in Lemma 3.2.1, $N(J^K) \cong (NJ)^{NK}$ holds, hence

$$y_J(K) = J^K \cong \text{Ho}(N(J^K)) \cong \text{Ho}((NJ)^{NK}) = \mathbf{Ho}(NJ)(K),$$

naturally in K .

Lemma 3.3.8. R is a left inverse for \mathbf{Ho} , when we restrict the image of R to quasi-categories.

¹This is not to be confused with the functor introduced in Definition 2.2.10.

²Notice that we are treating \mathbb{D} as an ordinary functor by ignoring its action on 2-cells. Otherwise the composition $\text{Ob} \circ \mathbb{D} \circ j$ would not make sense.

Proof. This is a simple computation. In fact, at the level of n -simplices we have

$$\begin{aligned}
 (R\mathbf{Ho}(X))_n &= \text{Ob}(\mathbf{Ho}(X)[n]) \\
 &= \text{Ob}(\text{Ho}(X^{N[n]})) \\
 &= \text{Ob}(\text{Ho}(X^{\Delta^n})) \\
 &= (X^{\Delta^n})_0 \\
 &= X_n
 \end{aligned}$$

in which the last step is the Yoneda lemma. From this one gets an isomorphism $R\mathbf{Ho}(X) \cong X$. \square

In summary, there exists a diagram (which is not commutative) of categories and functors as follows.

$$\begin{array}{ccc}
 & \xrightarrow{L} & \\
 \mathbf{sSet} & \perp & \mathbf{PDer}^{\text{st}} \\
 & \xleftarrow{R} & \\
 \uparrow & & \uparrow \\
 \mathbf{qCat} & & \mathbf{Ho}
 \end{array}$$

We would like to transfer the Joyal model structure on simplicial sets to prederivators, via the functor R . That is to say, we define a morphism $F: \mathbb{D} \rightarrow \mathbb{D}'$ in $\mathbf{PDer}^{\text{st}}$ to be a weak equivalence (or a fibration) if and only if $RF: R\mathbb{D} \rightarrow R\mathbb{D}'$ is a weak equivalence (or a fibration) in the Joyal model structure and we show that $L \dashv R$ is a Quillen equivalence. In this case one says that the model structure obtained using R is *transferred* from the previous one, and that R *creates* weak equivalences and fibrations.

Theorem 3.3.9. $\mathbf{PDer}^{\text{st}}$ with the transferred model structure is Quillen equivalent to \mathbf{sSet} with the Joyal model structure.

Proof. First we have to prove that $L \dashv R$ is a Quillen adjunction. This can be done using a classic result due to Kan, which can be found e.g. in [Hir09, Theorem 11.3.2], that allows us to transfer a model structure when a set of conditions are met. In order to use Kan's theorem it is enough to show that

- the Joyal model structure on simplicial sets is *cofibrantly generated* (see [Hov07, §2] for the definition and [Ste18, Theorem 5.12] for a proof of this fact),
- $\mathbf{PDer}^{\text{st}}$ is complete and cocomplete, which is the case since it is a presheaf category,

- R preserves filtered colimits, since it preserves all colimits by Proposition 3.3.4,
- the unit of the adjunction is an isomorphism, by Proposition 3.3.6.

Therefore this adjunction is actually a Quillen one. It remains to prove that it is also a Quillen equivalence, hence a morphism $f: LX \rightarrow Y$ is a weak equivalence if and only if the corresponding morphism through the adjunction $f^\sharp: X \rightarrow RY$ is a weak equivalence. The double implication holds because, using the definition of right adjoints via universal arrows, we see that $f^\sharp = R(f) \circ \eta_X$. But since η_X is an isomorphism, f^\sharp is a w.e. iff $R(f)$ is a w.e. iff f is a w.e. (since R creates w.e.). \square

Although this result is in itself meaningful, it would be nice to know which prederivators are images of quasi-categories via the functor \mathbf{Ho} , at least up to isomorphism. In the attempt to study the essential image of the above-mentioned functor, the authors of [FKKR18] introduce the following class of prederivators.

Definition 3.3.10. A prederivator is *quasi-representable* if it satisfies the following conditions.

1. Given an homotopy finite category J , the counit of the adjunction $L \dashv R$ evaluated at $\mathbf{Ho}(NJ)$,

$$LNJ \cong LR \mathbf{Ho}(NJ) \xrightarrow{\epsilon_{\mathbf{Ho}(NJ)}} \mathbf{Ho}(NJ) \cong y_J$$

induces a bijection

$$\mathbf{PDer}^{\text{st}}(y_J, \mathbb{D}) \xrightarrow{\epsilon_{\mathbf{Ho}(NJ)}^*} \mathbf{PDer}^{\text{st}}(LNJ, \mathbb{D})$$

2. For any homotopy finite category J , there is a coequalizer

$$\text{Ob}(\mathbb{D}(J \times [2]) \times_{\mathbb{D}(J \times [1])} \mathbb{D}(J)) \rightrightarrows \text{Ob}(\mathbb{D}(J \times [1])) \rightarrow \text{Ob}(\mathbb{D}(J)^{[1]})$$

3. $R\mathbb{D}$ is a quasi-category.

Remark 3.3.11. This definition is quite technical, but roughly speaking we are asking that these prederivators act on objects in a way that commutes with certain colimits. On morphisms they are determined by the value that they assume on objects in a suitable sense (i.e. in a way that is compatible with homotopies). Furthermore, we ask that their underlying simplicial sets are quasi-categories.

Finally, thanks to the next theorem, we are able to conclude that these prederivators are precisely those in the essential image of \mathbf{Ho} .

Theorem 3.3.12. A prederivator is quasi-representable if and only if it is isomorphic to one of the form $\mathbf{Ho}(\mathcal{C})$, for a quasi-category \mathcal{C} .

Proof. First of all, let us prove that if a prederivator \mathbb{D} is quasi-representable, then there is an isomorphism of prederivators $\mathbb{D} \cong \mathbf{Ho}(R\mathbb{D})$. From the definition of a quasi-representable prederivator, we know that $R\mathbb{D}$ is a quasi-category. By Lemma 3.3.8 we have an isomorphism of simplicial sets $R\mathbf{Ho}(R\mathbb{D}) \cong R\mathbb{D}$, then we get $\mathbf{Ho}(R\mathbb{D}) \cong \mathbb{D}$ using that R reflects isomorphisms between quasi-representable prederivators (see [FKKR18, Lemma 2.17]). For the other implication, see [FKKR18, Proposition 2.9]. \square

Appendix A

Model categories

Model categories were first introduced by Daniel Quillen in 1967 in his book *Homotopical algebra*. They provide a setting for homotopy theory which generalize many of the classical features of the homotopy theory of (good) topological spaces. In this appendix we recall the very basics of model category theory. More advanced topics, such as the small object argument or cofibrantly generated model categories or even model structures on functor categories (for instance, the Reedy model structure), can be found in e.g. [Dyc19, §2], [Rie19] or in the classics [Hov07] and [Hir09].

A.1 Generalities

First, let us recall the definition of the *arrow category* $\mathbf{C}^{[1]}$ of a category \mathbf{C} . This is the category whose objects are morphisms of \mathbf{C} and whose morphisms are commutative squares.

Definition A.1.1. A *functorial factorization* is an ordered pair (α, β) of functors $\mathbf{C}^{[1]} \rightarrow \mathbf{C}^{[1]}$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in \mathbf{C}^{[1]}$.

Definition A.1.2. A *model structure* $(\mathcal{W}, \text{Cof}, \text{Fib})$ on \mathbf{C} consists of three subcategories of $\mathbf{C}^{[1]}$ called *weak equivalences* (w.e. for short), *cofibrations* and *fibrations*, and two functorial factorizations (α, β) and (γ, δ) satisfying the following properties.

1. (*2-out-of-3*) If the following diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow h & \swarrow f \\ & & Z \end{array}$$

commutes in \mathbf{C} and any two of f, g and h are weak equivalences, then so is the third.

2. (*Retracts*) \mathcal{W} , Cof and Fib are stable under retracts. Namely, if $f: X \rightarrow Y$ is a weak equivalence (resp. cofibration, fibration) and the following diagram

$$\begin{array}{ccccc}
 & \overset{\curvearrowright}{\longrightarrow} & & \overset{\curvearrowright}{\longrightarrow} & \\
 A & \longrightarrow & X & \longrightarrow & A \\
 g \downarrow & & f \downarrow & & \downarrow g \\
 B & \longrightarrow & Y & \longrightarrow & B \\
 & \underset{\curvearrowright}{\longrightarrow} & & \underset{\curvearrowright}{\longrightarrow} &
 \end{array}$$

commutes, then g is a weak equivalence (resp. cofibration, fibration).

3. (*Lifting*) Suppose we have a commutative diagram of solid arrows

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 i \downarrow & \nearrow \text{dotted} & \downarrow p \\
 B & \longrightarrow & Y
 \end{array}$$

where $i \in \text{Cof}$ and $p \in \text{Fib}$. Then the dotted arrow exists and makes the diagram commute if either i or p is also a weak equivalence.

4. (*Factorization*) For every morphism f , we have that $\alpha(f) \in \text{Cof}$, $\beta(f) \in \mathcal{W} \cap \text{Fib}$, $\gamma(f) \in \mathcal{W} \cap \text{Cof}$ and $\delta(f) \in \text{Fib}$.

Definition A.1.3. A *model category* is a complete and cocomplete category endowed with a model structure.

Given a model category, we will say that a morphism is a *trivial fibration* (resp. trivial cofibration) if it is both a weak equivalence and a fibration (resp. cofibration). We will say, moreover, that an object X is *cofibrant* if the unique morphism $\emptyset \rightarrow X$ from the initial object to X is a cofibration and we will call it *fibrant* if the unique morphism $X \rightarrow *$ from X to the terminal object is a fibration.

Definition A.1.4. Let \mathbf{C} and \mathbf{D} be model categories.

1. A functor $F: \mathbf{C} \rightarrow \mathbf{D}$ (resp. $G: \mathbf{D} \rightarrow \mathbf{C}$) is called *left Quillen* (resp. *right Quillen*) if it is a left adjoint and preserves cofibrations and trivial cofibrations (resp. if it is a right adjoint and preserves fibrations and trivial fibrations).
2. An adjunction between two model categories in which the left adjoint is a left Quillen functor is called a *Quillen adjunction*.
3. A Quillen adjunction is a *Quillen equivalence* if a morphism $FX \rightarrow Y$ is a w.e. if and only if the corresponding morphism $X \rightarrow GY$ through the adjunction is a w.e., for all cofibrants $X \in \mathbf{C}$ and fibrants $Y \in \mathbf{D}$.

Given a model category \mathbf{M} , we would like to treat weak equivalences as they were isomorphisms. This task is overcome by “formally inverting” weak equivalences, constructing the homotopy category $\mathrm{Ho}(\mathbf{M})$ which comes with a localization functor $\gamma_{\mathbf{M}}: \mathbf{M} \rightarrow \mathrm{Ho}(\mathbf{M})$ (see [Hov07, §1.2]).

Definition A.1.5. Suppose we have given two model categories \mathbf{M}, \mathbf{N} and a category \mathbf{D} . Let $F: \mathbf{M} \rightarrow \mathbf{D}$ and $G: \mathbf{M} \rightarrow \mathbf{N}$ be functors.

1. The *left derived functor* $\mathbb{L}F: \mathrm{Ho}(\mathbf{M}) \rightarrow \mathbf{D}$ of F (if it exists) is the right Kan extension of F along $\gamma_{\mathbf{M}}$.
2. The *total left derived functor* $\mathbb{L}(G): \mathrm{Ho}(\mathbf{M}) \rightarrow \mathrm{Ho}(\mathbf{N})$ of G is the left derived functor of $\gamma_{\mathbf{N}} \circ G$.

Dually, we can define right derived functors by means of left Kan extensions and so on. It is a classical result in homotopical algebra that left (resp. right) Quillen functors always admit total left (resp. total right) derived functors.

Proposition A.1.6 (Ken Brown’s lemma). Suppose \mathbf{M} is a model category and \mathbf{N} is a category equipped with a subcategory of arrows which satisfies the 2-out-of-3 axiom. We call these arrows weak equivalences, for simplicity.

- i) If $F: \mathbf{M} \rightarrow \mathbf{N}$ is a functor which takes trivial cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences.
- ii) If F takes trivial fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.

Proof. The result is classical. A proof is contained in the first chapter of [Hov07]. \square

A.2 Bousfield localization

In the following we will see how the Bousfield localization can be used to produce new model structures out of given ones. The theory is quite complicated and lies on the so-called Reedy model structure, a well-behaved model structure one can put on certain functor categories (see [Rie14, Chapter 14] for details).

Definition A.2.1. A *fibrant approximation* to an object Y of a model category \mathbf{M} is a couple (\widehat{Y}, j) where \widehat{Y} is a fibrant object and $j: Y \rightarrow \widehat{Y}$ is a weak equivalence. Similarly, a *cofibrant approximation* of $X \in \mathbf{M}$ is the datum of (\widetilde{X}, i) , where \widetilde{X} is cofibrant and $i: \widetilde{X} \rightarrow X$ is a weak equivalence.

Definition A.2.2. Let \mathbf{M} be a model category and $X, Y \in \mathbf{M}$. A *left homotopy function complex* from X to Y is a triple $(\widetilde{X}, \widehat{Y}, \mathbf{M}(\widetilde{X}, \widehat{Y}))$ where:

- (i) \tilde{X} is a cosimplicial resolution¹ of X , namely a cofibrant approximation $\tilde{X} \rightarrow cc_*X$ in the Reedy model structure on \mathbf{M}^Δ .
- (ii) \hat{Y} is a fibrant approximation to Y
- (iii) $\mathbf{M}(\tilde{X}, \hat{Y})$ is the simplicial set defined by $\mathbf{M}(\tilde{X}, \hat{Y})_n = \mathbf{M}(\tilde{X}([n]), \hat{Y})$

One can define also right homotopy function complexes by means of cofibrant approximation of the first object and simplicial resolution of the second one and two-sided homotopy function complexes which are a combination of the two (an explicit description can be found in [Hir09]).

Remark A.2.3. Once we choose a type of homotopy function complex, it is customary to denote $\text{Map}(X, Y) := \mathbf{M}(\tilde{X}, \hat{Y})$. The idea behind this construction is to associate a simplicial set of morphisms to every couple of objects in the model category, just as in ordinary (locally small) categories one has a set of morphisms for each pair of objects. This analogy goes on, in the sense that (see [Hir09, §17.5]).

- 1) we can define *functorial* homotopy function complexes, i.e. functors $\text{Map}(-, -): \mathbf{M}^{\text{op}} \times \mathbf{M} \rightarrow \mathbf{sSet}$ sending a couple of objects to their (left, right or two-sided) homotopy function complex,
- 2) for every morphism $g: X \rightarrow Y$ in \mathbf{M} and object $A \in \mathbf{M}$ there exist, by functoriality, two induced maps $g^*: \text{Map}(Y, A) \rightarrow \text{Map}(X, A)$ and $g_*: \text{Map}(A, X) \rightarrow \text{Map}(A, Y)$.

In the rest of this subsection \mathbf{M} will denote a model category and \mathcal{C} a class of maps in \mathbf{M} .

Definition A.2.4. An object $W \in \mathbf{M}$ is called \mathcal{C} -local if it is fibrant and for every $f: A \rightarrow B$ of \mathcal{C} the induced map $f^*: \text{Map}(B, W) \rightarrow \text{Map}(A, W)$ is a weak equivalence.

Definition A.2.5. A morphism $g: X \rightarrow Y$ of \mathbf{M} is a \mathcal{C} -local equivalence if for every \mathcal{C} -local object W the induced map of homotopy function complexes $g^*: \text{Map}(Y, W) \rightarrow \text{Map}(X, W)$ is a weak equivalence.

As usual one can dualize the previous two definitions. Nevertheless we won't need these dual notions, since we will treat only the left Bousfield localization of a model category. Colocal objects and colocal equivalences lead to the definition of *right* Bousfield localization, for which dual versions of the results given for the left one hold true.

Definition A.2.6. The *left Bousfield localization* of \mathbf{M} with respect to \mathcal{C} (if it exists) is a model category structure $\mathbf{L}_{\mathcal{C}}\mathbf{M}$ on the underlying category of \mathbf{M} such that

¹A cosimplicial object in a category \mathbf{C} is just a functor $\Delta \rightarrow \mathbf{C}$. The constant cosimplicial object at X will be denoted by cc_*X .

- (a) the class of weak equivalences of $L_{\mathcal{C}}\mathbf{M}$ equals the class of \mathcal{C} -local equivalences of \mathbf{M} ,
- (b) the class of cofibrations of $L_{\mathcal{C}}\mathbf{M}$ equals the class of cofibrations of \mathbf{M} and
- (c) the class of fibrations of $L_{\mathcal{C}}\mathbf{M}$ is the class of maps with the right lifting property with respect to those maps that are both cofibrations and \mathcal{C} -local equivalences.

Roughly speaking, the (left) Bousfield localization allows us to “invert” more maps, i.e. construct a model structure with a class of weak equivalences that strictly contains the class of weak equivalences of the original model category, but that has the same cofibrations (the fibrations being determined by RLP with respect to trivial cofibrations), in the sense of the following proposition.

Proposition A.2.7. If $L_{\mathcal{C}}\mathbf{M}$ is the left Bousfield localization of \mathbf{M} with respect to \mathcal{C} , then every weak equivalence of \mathbf{M} is a weak equivalence of $L_{\mathcal{C}}\mathbf{M}$.

Proof. See [Hir09, Proposition 3.3.3.] □

Bousfield localizations are localizations in the sense that they can be characterized with a universal property that is similar to the one satisfied by localizations of categories.

Proposition A.2.8. The left Bousfield localization $L_{\mathcal{C}}\mathbf{M}$ is such that there is a left Quillen functor $\mathbf{M} \rightarrow L_{\mathcal{C}}\mathbf{M}$ whose total left derived functor sends all morphisms in \mathcal{C} to weak equivalences and any left Quillen functor $\mathbf{M} \rightarrow \mathbf{N}$ whose total left derived functor sends \mathcal{C} to weak equivalences factors uniquely through $\mathbf{M} \rightarrow L_{\mathcal{C}}\mathbf{M}$.

Proof. See [Hir09, Proposition 3.3.18.] □

As we said, not every model category admits a Bousfield localization, in general. Anyway, many model categories used in practice can be localized in the sense of Bousfield, as the following theorem shows.

Theorem A.2.9. Let \mathbf{M} be a *left proper cellular model category* (see [Hir09, Definition 13.1.1 and Definition 12.1.1] for details) and let \mathcal{C} be a *set* of maps in \mathbf{M} . Then the left Bousfield localization of \mathbf{M} with respect to \mathcal{C} does exist and the fibrant objects of $L_{\mathcal{C}}\mathbf{M}$ are the \mathcal{C} -local objects of \mathbf{M} .

Proof. See [Hir09, Theorem 4.1.1.] □

Remark A.2.10. \mathbf{sSet} , \mathbf{Top} , \mathbf{sSet}_* and \mathbf{Top}_* are left proper cellular model categories ([Hir09, Proposition 4.1.4.]), hence all these categories admit a (left) Bousfield localization for every set of maps \mathcal{C} .

Appendix B

2-categories

Already at the level of ordinary category theory, there are plenty of examples of categories in which the hom-sets have a richer structure. For instance, they can be abelian groups (as in additive categories), or topological spaces or simplicial sets. Even more, as it happens for the category of small categories, each hom-set can be a category itself (e.g. functor categories). These examples are all instances of the same concept, i.e. they are all *enriched categories*. The last example is particularly significant, since it extends in a suitable sense the notion of a category by adding “morphisms between morphisms”. These will be called *2-categories*.

B.1 Setting the stage

Definition B.1.1. A *monoidal category* \mathcal{V} is the datum of

- 1) a category \mathcal{V}_0 , called *the underlying category* of \mathcal{V} ;
- 2) a bifunctor $\otimes: \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathcal{V}_0$, called *the tensor product* (the image of the pair (X, Y) under the action of \otimes will be denoted by $X \otimes Y$);
- 3) an object $I \in \mathcal{V}_0$, called *the unit*;
- 4) a family of isomorphisms (called *associativities*) α with components

$$\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$$

which are natural in all their arguments and two families of natural isomorphisms l and r with components respectively $l_X: I \otimes X \rightarrow X$ and $r_X: X \otimes I \rightarrow X$ (called *left and right identities*).

These data must satisfy the following *coherence conditions* on associativities and identities.

$$\begin{array}{ccc}
& ((X \otimes Y) \otimes Z) \otimes T & \\
\alpha_{X,Y,Z} \otimes 1 \swarrow & & \searrow \alpha_{X \otimes Y, Z, T} \\
(X \otimes (Y \otimes Z)) \otimes T & & (X \otimes Y) \otimes (Z \otimes T) \\
\alpha_{X,Y \otimes Z, T} \searrow & & \swarrow \alpha_{X,Y,Z \otimes T} \\
X \otimes ((Y \otimes Z) \otimes T) & \xrightarrow{1 \otimes \alpha_{Y,Z,T}} & X \otimes (Y \otimes (Z \otimes T))
\end{array}$$

$$\begin{array}{ccc}
(X \otimes I) \otimes Y & \xrightarrow{\alpha_{A,I,Y}} & X \otimes (I \otimes Y) \\
r_X \otimes 1 \searrow & & \swarrow 1 \otimes l_Y \\
& X \otimes Y &
\end{array}$$

We refer to [Bor94b, §6.1] for further details on monoidal categories as well as for the definition of symmetric monoidal closed category. Crucial examples of symmetric monoidal closed categories are the cartesian closed categories (e.g. **Set** or any elementary topos), with the cartesian product as a tensor product.

Definition B.1.2. Let \mathcal{V} be a monoidal category. A *category enriched over \mathcal{V}* or *\mathcal{V} -category* \mathbf{C} consists of:

1. a class $\text{Ob}(\mathbf{C})$ of objects;
2. for every pair of objects $X, Y \in \text{Ob}(\mathbf{C})$, an object $\mathbf{C}(X, Y) \in \mathcal{V}$;
3. for every triple $X, Y, Z \in \text{Ob}(\mathbf{C})$ of objects, a *composition* morphism

$$c_{XYZ}: \mathbf{C}(X, Y) \otimes \mathbf{C}(Y, Z) \rightarrow \mathbf{C}(X, Z)$$

in \mathcal{V} ;

4. for every object $X \in \text{Ob}(\mathbf{C})$, a *unit* morphism $u_X: I \rightarrow \mathbf{C}(X, X)$ in \mathcal{V} .

These data must satisfy the following axioms:

- the composition is associative, i.e. the following diagram

$$\begin{array}{ccc}
& (\mathbf{C}(X, Y) \otimes \mathbf{C}(Y, Z)) \otimes \mathbf{C}(Z, T) & \\
\alpha_{\mathbf{C}(X, Y), \mathbf{C}(Y, Z), \mathbf{C}(Z, T)} \swarrow & & \searrow c_{XYZ \otimes 1} \\
\mathbf{C}(X, Y) \otimes (\mathbf{C}(Y, Z) \otimes \mathbf{C}(Z, T)) & & \mathbf{C}(X, Z) \otimes \mathbf{C}(Z, T) \\
1 \otimes c_{YZT} \searrow & & \swarrow c_{XZT} \\
\mathbf{C}(X, Y) \otimes \mathbf{C}(Y, T) & \xrightarrow{c_{XYT}} & \mathbf{C}(X, T)
\end{array}$$

commutes for every object X, Y, Z, T in \mathbf{C} .

- u_X behaves like a unit for the composition, i.e. the following diagram

$$\begin{array}{ccccc}
I \otimes \mathbf{C}(X, Y) & \xrightarrow{l_{\mathbf{C}(X, Y)}} & \mathbf{C}(X, Y) & \xleftarrow{r_{\mathbf{C}(X, Y)}} & \mathbf{C}(X, Y) \otimes I \\
u_X \otimes 1 \downarrow & & \parallel & & \downarrow 1 \otimes u_Y \\
\mathbf{C}(X, X) \otimes \mathbf{C}(X, Y) & \xrightarrow{c_{XXY}} & \mathbf{C}(X, Y) & \xleftarrow{c_{XYX}} & \mathbf{C}(X, Y) \otimes \mathbf{C}(Y, Y)
\end{array}$$

commutes for every object X, Y in \mathbf{C} .

Example B.1.3. Enriched categories are found in nature with a lot of different names. For instance, a category enriched over \mathbf{Top} (where \mathbf{Top} is a nice category of topological spaces, e.g. the category of CW complexes or the category of compactly generated and weakly Hausdorff spaces) is called a *topological category*.

One kind of enriched category, used throughout the text, is the following.

Definition B.1.4. A category enriched over \mathbf{sSet} is called a *simplicially enriched category* or also *simplicial category*¹.

Definition B.1.5. A \mathcal{V} -functor $F: \mathbf{A} \rightarrow \mathbf{B}$ (also called *enriched functor*) between two \mathcal{V} -categories \mathbf{A} and \mathbf{B} consist of

- a map which assigns to every object $X \in \mathbf{A}$ an object $FX \in \mathbf{B}$,
- for every pair of objects $X, Y \in \mathbf{A}$, a morphism

$$F_{XY}: \mathbf{A}(X, Y) \rightarrow \mathbf{B}(FX, FY)$$

which makes following diagrams

$$\begin{array}{ccc}
\mathbf{A}(X, Y) \otimes \mathbf{A}(Y, Z) & \xrightarrow{c_{XYZ}} & \mathbf{A}(X, Z) \\
F_{XY} \otimes F_{YZ} \downarrow & & \downarrow F_{XZ} \\
\mathbf{B}(FX, FY) \otimes \mathbf{B}(FY, FZ) & \xrightarrow{c_{FXFYFZ}} & \mathbf{B}(FX, FZ)
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{u_X} & \mathbf{A}(X, X) \\
u_{FX} \searrow & & \downarrow F_{XX} \\
& & \mathbf{B}(FX, FX)
\end{array}$$

¹The latter name can be confusing, since simplicially enriched categories and simplicial objects of \mathbf{Cat} are different in general. Hence we prefer to stick with the former name.

commute.

Definition B.1.6. A *simplicial functor* is precisely defined to be a \mathbf{sSet} -functor.

Definition B.1.7. We say that a \mathcal{V} -functor is *fully faithful* when F_{XY} is an isomorphism in \mathcal{V} for every X and Y .

We can also define \mathcal{V} -natural transformations in a similar fashion. They are subject to an hexagonal coherence diagram and they can be composed in an “horizontal” and in a “vertical” way, just like ordinary natural transformations (see [Bor94b, §6.2]).

B.2 Basics on 2-categories

With the previous section in mind, it is really easy to say what a 2-category should be.

Definition B.2.1. A 2-category is a \mathbf{Cat} -enriched category, with the tensor product given by the product of categories and the terminal category as unit.

Remark B.2.2. Spelling out the definition above, a 2-category \mathbf{C} consist of

- a class of objects $\text{Ob}(\mathbf{C})$,
- for any pair $X, Y \in C$ a small category $\mathbf{C}(X, Y)$,
- for any triple $X, Y, Z \in C$ a functor $\mu: \mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) \rightarrow \mathbf{C}(X, Z)$ called composition law,
- a unit functor $[0] \rightarrow \mathbf{C}(X, X)$, that is to say an object $\text{id}_X \in \mathbf{C}(X, X)$ for every object $X \in \mathbf{C}$.

Furthermore, these data are subject to the following conditions

1. for every $X \in \mathbf{C}$, the functors

$$\mu(-, \text{id}_Y): \mathbf{C}(X, Y) \rightarrow \mathbf{C}(X, Y)$$

and

$$\mu(\text{id}_X, -): \mathbf{C}(X, Y) \rightarrow \mathbf{C}(X, Y)$$

are the identity functors,

2. for every $X, Y, Z, W \in \mathbf{C}$, the diagram

$$\begin{array}{ccc} \mathbf{C}(X, Y) \times \mathbf{C}(Y, Z) \times \mathbf{C}(Z, W) & \xrightarrow{\mu \times \text{id}} & \mathbf{C}(X, Z) \times \mathbf{C}(Z, W) \\ \text{id} \times \mu \downarrow & & \downarrow \mu \\ \mathbf{C}(X, Y) \times \mathbf{C}(Y, W) & \xrightarrow{\mu} & \mathbf{C}(X, W) \end{array}$$

commutes.

The elements of $\text{Ob}(\mathbf{C})$ are called 0-cells, the objects of $\mathbf{C}(X, Y)$ are called 1-cells and the morphisms of $\mathbf{C}(X, Y)$ are called 2-cells. The notations are the same as for categories, functors and natural transformations in \mathbf{Cat} , which is the prototypical example of 2-category.

Remark B.2.3. The vertical composition of 2-cells is defined by means of the composition law of the hom-categories, while the functor μ recovers the composition of 1-cells and the horizontal composition of 2-cells. Moreover, the functoriality of μ can be used to prove the interchange law for 2-cells (see [VP19, §3] for further details).

Remark B.2.4. Since in 2-categories there are two kinds of arrows, namely 1-cells and 2-cells, we have three different ways to construct a dual category of a given 2-category \mathbf{C} . We can indeed reverse only 1-cells, only 2-cells or both, obtaining respectively the 2-categories \mathbf{C}^{op} , \mathbf{C}^{co} and \mathbf{C}^{coop} .

Definition B.2.5. A 1-cell $f: X \rightarrow Y$ inside a 2-category \mathbf{C} is said to be an *equivalence* if there exists another 1-cell $g: Y \rightarrow X$ together with two invertible 2-cells $1_X \Rightarrow gf$ and $fg \Rightarrow 1_Y$.

As we have seen, a 2-category is a particular enriched category, when $\mathcal{V} = \mathbf{Cat}$. Naturally, the next step is to describe how the notion of enriched functor specializes to this case.

Definition B.2.6. Let \mathbf{C}, \mathbf{D} be two 2-categories. A 2-functor $F: \mathbf{C} \rightarrow \mathbf{D}$ is the datum of

- (1) an assignment $X \mapsto FX$ for every $X \in \text{Ob}(\mathbf{C})$,
- (2) a functor $F_{XY}: \mathbf{C}(X, Y) \rightarrow \mathbf{D}(FX, FY)$ subject to the coherence conditions of Definition B.1.5

Remark B.2.7. The previous is also known as *strict 2-functor*, to distinguish it from other weak versions of 2-functors between 2-categories. Indeed, the second point of Definition B.2.6 tells us that a 2-functor sends identities to identities and respect compositions, just as an ordinary functor does, with an extra action on 2-cells. Nevertheless, it makes sense to ask for a weak version of the coherences, whose diagrams commute only up to a 2-cell. If these 2-cells are invertible we get a *pseudofunctor*, otherwise we have a *lax* or *colax functor* (depending on the direction of the 2-cell). Further details can be found in [Bor94a, §7.5]

Example B.2.8. The very first example of 2-functors are the 2-dimensional analogues of the hom-functors. For instance, let us take the covariant case $\mathbf{C}(X, -): \mathbf{C} \rightarrow \mathbf{Cat}$, with $X \in \text{Ob}(\mathbf{C})$. The action of this functor is really simple:

- (i) it sends every 0-cell Y to the small category $\mathbf{C}(X, Y)$,
- (ii) it maps every 1-cell $f: Y \rightarrow Z$ to the functor

$$\begin{aligned} f \circ -: \mathbf{C}(X, Y) &\rightarrow \mathbf{C}(X, Z) \\ g &\mapsto f \circ g \\ (\gamma: g \Rightarrow g') &\mapsto 1_f * \gamma \end{aligned}$$

where $*$ is the horizontal composition of 2-cells.

- (iii) and finally it sends every 2-cell $\alpha: f \Rightarrow g$ to the horizontal post-composition $\alpha * -$.

The contravariant case $\mathbf{C}(-, Y): \mathbf{C}^{\text{op}} \rightarrow \mathbf{Cat}$ is completely analogous.

Definition B.2.9. A 2-functor is called *fully faithful* if it induces isomorphisms of hom-categories. We call a 2-functor *bicategorically fully faithful* if it induces *equivalences* of hom-categories.

Definition B.2.10. Let $F, G: \mathbf{C} \rightarrow \mathbf{D}$ be 2-functors between 2-categories. A 2-natural transformation $\alpha: F \Rightarrow G$ is the datum of a 1-cell

$$\alpha_C: FC \rightarrow GC$$

for every $C \in \mathbf{C}$, in such a way that the following diagram

$$\begin{array}{ccc} \mathbf{C}(C, C') & \xrightarrow{F_{CC'}} & \mathbf{D}(FC, FC') \\ G_{CC'} \downarrow & & \downarrow \alpha_{C'} \circ - \\ \mathbf{D}(GC, GC') & \xrightarrow[-\circ \alpha_C]{} & \mathbf{D}(FC, GC') \end{array}$$

commutes.

In Remark B.2.7 we said that 2-functors have weaker counterparts, namely pseudofunctors and lax functors. So it happens for 2-natural transformations, which in turn can be weakened into *pseudonatural transformations* and *lax natural transformations*. For the sake of simplicity, we prefer to give the definitions in the strict case. An explicit definition can be found, again, in [Bor94a, §7.5]. As a particular case of a pseudonatural transformation, for instance, one recovers the notion of morphism of prederivators, whose definition can be found in details in Remark 2.2.2. Finally, we have a notion of “morphism between 2-natural transformations”.

Definition B.2.11. Let $F, G: \mathbf{C} \rightarrow \mathbf{D}$ be 2-functors and $\alpha, \beta: F \Rightarrow G$ 2-natural transformations. A *modification* $\Xi: \alpha \Rrightarrow \beta$ is a family of 2-cells

$$\Xi_C: \alpha_C \Rightarrow \beta_C$$

such that for any two 1-cells $f, g: C \rightrightarrows C'$ and any 2-cell $\gamma: f \Rightarrow g$, we have that

$$\Xi_{C'} * F\gamma = G\gamma * \Xi_C$$

The definition above applies, basically unchanged, also to pseudonatural and lax natural transformations. By its very definition, a modification is a kind of “morphism of order three”. Hence, it is natural to “go higher” and introduce the notion of 3-category, in which morphisms are now adding up to a 2-category (see [Bor94a, §7.3]). In this setting it holds a more general version of Yoneda lemma (see [Hed16, §6.9]).

Theorem B.2.12. Let \mathbf{C} be a 2-category, $F: \mathbf{C} \rightarrow \mathbf{Cat}$ a 2-functor and C a 0-cell in \mathbf{C} , then there exist an isomorphism of categories

$$\mathrm{Hom}(\mathbf{C}(C, -), F) \cong FC$$

where $\mathrm{Hom}(\mathbf{C}(C, -), F)$ is the category whose objects are the 2-natural transformations $\mathbf{C}(C, -) \Rightarrow F$ and with the modifications between those 2-natural transformations as morphisms.

From here we get a 2-categorical Yoneda embedding $y: \mathbf{C} \rightarrow \mathrm{Hom}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})$, sending an object of the 2-category \mathbf{C} into the representable contravariant 2-functor and acting in the natural way at the level of 1-cells and 2-cells. When $\mathbf{C} = \mathbf{Cat}$, this embedding reduces to the embedding of the 2-category of small categories into the 2-categories of prederivators and strict morphisms (see Remark 2.2.6).

Bibliography

- [Bal12] E. Balzin. Lectures on homotopical and model categories. *Lecture notes available at <https://www.mccme.ru/iun/postscript/s12/balzin-MC-Lectures.pdf>*, 2012.
- [Bor94a] F. Borceux. *Handbook of Categorical Algebra: volume 1, Basic category theory*, volume 1. Cambridge University Press, 1994.
- [Bor94b] F. Borceux. *Handbook of Categorical Algebra: Volume 2, Categories and Structures*, volume 2. Cambridge University Press, 1994.
- [BSP11] C. Barwick and C. Schommer-Pries. On the unicity of the homotopy theory of higher categories. *arXiv preprint [arXiv:1112.0040](https://arxiv.org/abs/1112.0040)*, 2011.
- [Car16] K. Carlson. On the ∞ -categorical Whitehead theorem and the embedding of quasicategories in prederivators. *arXiv preprint [arXiv:1612.06980](https://arxiv.org/abs/1612.06980)*, 2016.
- [Cis18] D.-C. Cisinski. Higher categories and homotopical algebra. *Preprint available at www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf*, 2018.
- [Dyc19] T. Dyckerhoff. Introduction to Higher Category Theory. *Lecture notes available at <https://www.math.uni-hamburg.de/home/dyckerhoff/higher/notes.pdf>*, 2019.
- [FKKR18] D. Fuentes-Keuthan, M. Kedziorek, and M. Rovelli. A model structure on prederivators for $(\infty, 1)$ -categories. *arXiv preprint [arXiv:1810.06496](https://arxiv.org/abs/1810.06496)*, 2018.
- [GJ99] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*. Progress in Mathematics. Birkhäuser Basel, 1 edition, 1999.
- [Gro12] M. Groth. Derivators, pointed derivators, and stable derivators. *arXiv preprint [arXiv:1112.3840v2](https://arxiv.org/abs/1112.3840v2)*, 2012.

- [Gro15] M. Groth. A short course on ∞ -categories. *arXiv preprint arXiv:1007.2925*, 2015.
- [Hed16] J. Hedman. 2-categories and Yoneda lemma, 2016.
- [Hir09] P. S. Hirschhorn. *Model categories and their localizations*. Number 99 in Mathematical Surveys and Monographs. American Mathematical Soc., 2009.
- [Hov07] M. Hovey. *Model categories*. Number 63 in Mathematical Surveys and Monographs. American Mathematical Soc., 2007.
- [Jar15] J. F. Jardine. *Local Homotopy Theory*. Springer, 2015.
- [Joy08a] A. Joyal. Notes on quasi-categories. *Preprint*, 2008.
- [Joy08b] A. Joyal. The theory of quasi-categories and its applications, 2008.
- [Lur17] J. Lurie. Higher Topos Theory, online version April, 2017.
- [Lur18] J. Lurie. Kerodon. <https://kerodon.net>, 2018.
- [Mac71] S. MacLane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.
- [Rez17] C. Rezk. Stuff about quasicategories. *Unpublished notes available online at <https://faculty.math.illinois.edu/~rezk/595-fal16/quasicats.pdf>*, 2017.
- [Rie08] E. Riehl. A model structure for quasi-categories. *Preprint available at <http://www.math.jhu.edu/~eriehl/topic.pdf>*, 2008.
- [Rie11] E. Riehl. A leisurely introduction to simplicial sets. *Unpublished expository article available online at <http://www.math.jhu.edu/~eriehl/ssets.pdf>*, 2011.
- [Rie14] E. Riehl. *Categorical Homotopy Theory*. Number 24 in New Mathematical Monographs. Cambridge University Press, 2014.
- [Rie19] E. Riehl. Homotopical categories: from model categories to $(\infty, 1)$ -categories. *arXiv preprint arXiv:1904.00886*, 2019.
- [RV18] E. Riehl and D. Verity. Elements of ∞ -category theory. *Preprint available at www.math.jhu.edu/~eriehl/elements.pdf*, 2018.

- [SP12] C. Schommer-Pries. The canonical model structure on **Cat**. *Blog post available at <https://sbseminar.wordpress.com/2012/11/16/the-canonical-model-structure-on-cat/>*, 2012.
- [Ste18] D. Stevenson. Notes on the Joyal model structure. *arXiv preprint arXiv:1810.05233*, 2018.
- [Toë05] B. Toën. Vers une axiomatisation de la théorie des catégories supérieures. *K-theory*, 34(3):233–263, 2005.
- [VP19] M. Volpe and P. Polesello. The Fundamental Costack. *Thesis available at <http://tesi.cab.unipd.it/62105/>*, 2019.
- [VPE15] M. Vergura, M. Prasma, and B. Edixhoven. A Giraud-type theorem for model topoi, 2015.