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Parity-violation signatures in the large-scale distribution of galaxies

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Abstract

Last year, two separate experimental groups [1, 2] reported claims of parity-violating signatures in the connected four-point correlation function of the matter over-density field $\delta(t, \mathbf{x})$, which is often referred to as the galaxy-four point correlation function. The matter over-density field parametrizes the excess of matter-energy density with respect to the homogeneous background and it's defined as $\delta(t, \mathbf{x}) \equiv \frac{\delta\rho(t,\mathbf{x})-\bar{\rho}(t)}{\bar{\rho}(t)}$, where $\rho(t, \mathbf{x})$ is the energy density field and $\bar{\rho}(t)$ is the average density field. The four-point correlation function is the most simple statistic capable of detecting parity violation in the case of a scalar field. The galaxy four-point correlation function basically quantifies the excess of quartets of galaxies compared to a random distribution. If these measurements are confirmed, they would have two fundamental implications. First of all, the detection of a connected correlator beyond the two-point statistic serves as possible evidence of primordial non-Gaussianity. Moreover, since we know that weak forces play no role in the formation of large-scale structures it would be evidence of new physics.

The goal of this work is to investigate the possibility to formulate an inflationary model capable of leaving parity-violating imprints on late-times observables. Consequently, we examine the inflationary phase within the framework of parity-violating theories of modified gravity. We need to modify the standard scenario of single-field slow-roll Inflation with standard gravitational interaction since it is parity-conserving. We examine the dynamical Chern-Simons (dCS) theory of modified gravity, which extends the standard inflationary model with the lowest-order parity-violating coupling between the inflaton and the graviton. In this context, we demonstrate that dCS leaves a distinctive parity-violating signature in the primordial trispectrum of scalar perturbations. However, the signal is too weak to account for the observed parity violation. Thus, we explore alternative theories where, a priori, an enhancement is feasible due to gravitational waves' birefringence, i.e. the two chiral polarizations, left and right, of the graviton propagate at different speeds. We present an original analysis of the graviton-mediated trispectrum in the case of the chiral-scalar tensor theories of gravity. The graviton-mediated trispectrum is the leading parity-violating contribution to the Fourier transform of the four-point correlation function. The theories we analyze extend Chern-Simons gravity by including parity-violating operators containing first and second derivatives of the non-minimally coupled inflaton field. We manage to generate a parity-violating signal, but we don't observe any significant enhancement compared to the previous scenario.

Contents

Introduction

The exploration of symmetries holds a crucial place in comprehending the fundamental laws of the universe. Within the Standard Model, accounting for the observed P and CP violation in the weak sector is essential to building a consistent particle physics model. In Cosmology, there is ongoing research to investigate parity-violating signatures, both theoretically and experimentally.

From an experimental point of view, there are evidence in the data of parity-violating signatures that suggest that something different from the Lambda cold dark matter (ΛCDM) model could be at work. Evidence of parity violation in the Cosmic Microwave Background (CMB) emerges at a significance level of 3.6σ in the observed pattern of linear polarization among cosmic microwave background photons [3]. This phenomenon is referred to as "cosmic birefringence". From a theoretical point of view, this effect can be explained by means of a modification to electromagnetism through a coupling [4]

$$\mathcal{L}_{int} \propto g(\chi) F_{\mu\nu} \tilde{F}^{\mu\nu}, \tag{0.1}$$

where $g(\chi)$ is a dimensionless function of a scalar field and $F_{\mu\nu}$ is the electromagnetic field-strength tensor while $\tilde{F}_{\mu\nu}$ its dual version. If χ is constant in space and time, then the term the Lagrangian written above can be written as a total derivative. However, if χ depends on space-time, the plane of linear polarization of photons rotates.

Moreover, in Large Scale Structure (LSS) claims of parity violation in the galaxy four-point correlation function of the matter over density field have been recently (2022) reported by [1, 2]. In the experimental analysis, the approach proposed in [5] was employed to explore parity-violation signatures in the $3D$ large-scale structure. This method is based on the concept that the lowest-order shape, of which the parity-transformed version cannot be rotated back into its original form, is a tetrahedron. Using the final release of $BOSS$ galaxy survey [6], the two experimental groups found evidence of parity violation in the large-scale distribution of galaxies respectively at 7.1σ and 2.9σ in the $CMASS$ sample of the survey. Moreover, [2] found parity-violating signatures in the $LOWZ$ sample at 3.1σ. In the papers, multiple tests have been conducted to determine whether the signal is genuine or not. These measurements are the starting point of this master's thesis. We present in chapter 3 the main ideas used in the analysis performed in [1] and [2].

The goal of this work is to reproduce these observations as a relic signature of parity violation that took place in the Early Universe within the inflationary paradigm. Such parity violations can arise as signatures of new physics in the gravitational sector, e.g. the Chern-Simons theory of gravity [7] or the chiral scalar-tensor theories of gravity [8]. For example in the Chern-Simons modification of gravity [7, 9], a new parity-breaking coupling term between the inflaton and the metric is introduced

$$\mathcal{L}_{int} = f(\phi) \epsilon^{\alpha\beta\rho\sigma} R^{\mu\nu}{}_{\alpha\beta} R_{\mu\nu\rho\sigma}, \tag{0.2}$$

where ϕ is the inflaton and $R_{\mu\nu\alpha\beta}$ is the Reimann tensor.

As we have said we work in the so-called Inflationary scenario, which is a period of accelerated expansion before the standard radiation-dominated epoch. Inflation can be achieved by employing a scalar field, denoted as $\phi(t, \mathbf{x})$, which can be separated into two components: the background and the quantum fluctuations. The behavior of the background, $\phi(\tau)$, follows that of a single degree of freedom subjected to a potential, $V(\phi)$. If the potential has a sufficiently flat region, the scalar field behaves like a cosmological-constant like component causing the universe to pass through an accelerated expansion phase, during which the scale factor is stretched by more than sixty e-folds. As a first approximation, this mechanism makes today's universe very homogeneous, isotropic, and flat. However, the inflaton has quantum fluctuations, which got stretched and imprinted at super-horizon scales. Later, during radiation, matter, or cosmological constant domination, they reenter the horizon and provide the seeds for CMB anisotropies and LSS formation.

In the thesis, we review the shortcomings of the Hot Big Bang (HBB) model and their inflationary solution. Then, we discuss the inflationary background dynamics in the case of single-field slow-roll model of inflation [10], and we introduce the three fundamental formalisms we use throughout this work. The first one is perturbation theory in general relativity [11, 12] and, in this context, we focus, particularly, on the role played by the curvature perturbation ζ in Cosmology. Then, we discuss the so-called *Arnowitt-Deser-Misner formalism* (ADM) [13], which is a Hamiltonian formulation of GR in which we can isolate the "true" dynamical variables out of the ten characterizing the metric tensor. Finally, we present the Schwinger-Keldysh diagrammatics rules for primordial perturbations [14], which enable us to compute primordial correlation function in the context of the *In-In formalism* in the same way as $S-$matrix elements are evaluated using Feynmann diagrams in particle physics and in flat space-times.

We compute the primordial power spectra for the curvature perturbation ζ and gravitation waves (GWs) in the framework of single-field slow-roll model of inflation [15, 16] by means of perturbation theory and ADM formalism. Then, using also the $In - In$ formalism, we proceed to the central calculation of this thesis work: the *inflaton graviton-mediated trispectrum* in theories of modified gravity, which is the Fourier transform of the parity-violating contribution to the four-point correlation function.

We start this analysis with the Chern-Simons theory of gravity, which is constructed as an effective field theory (EFT) for gravitation. Basically, we modify the standard inflationary scenario by adding a parity-breaking term, $f(\phi)W\tilde{W}$, which couples a generic function of the Inflaton field with the contraction between the Weyl tensor and its dual. Thus, we present a detailed analysis of the computation of the graviton-mediated trispectrum in this model following the one presented in [9]. The main feature of this computation is that parity violation occurs in the gravitational sector throughout the graviton exchange between two pairs of scalars. The trispectrum violates parity since the equation of motions (EoM) for the two chiral polarizations, left and right, of the gravitational waves (GWs) are different. This results in a different propagator for left and right gravitons. More specifically the EoM depends on a chirality parameter $\mu = \frac{H}{M_{CS}} << 1$, where H is the Hubble constant and M_{CS} is the scale mass of the theory. As we'll see, the signal generated in this particular model is too weak due to chirality suppression. The chirality parameter is taken much lower than one in order to avoid ghost modes in the theory [9].

Therefore, we explore alternative theories where, a priori, an enhancement is feasible due to GWs birefringence. Thus, we present an original computation of the graviton-mediated trispectrum in the so-called chiral scalar-tensor theories of gravity [8] which are theories that extend Chern-Simons gravity by including parity-violating operators containing first and second derivatives of the non-minimally coupled inflaton field. For example, we consider terms such as

$$\epsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}R_{\mu\lambda}{}^{\rho}{}_{\sigma}\nabla^{\sigma}\phi\nabla^{\lambda}\phi, \quad \epsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}R^{\mu\lambda\rho\sigma}\nabla^{\rho}\phi\phi_{\mu}\nabla^{\sigma}\nabla_{\lambda}\phi, \tag{0.3}$$

where ∇_{α} is the covariant derivative operator. Unfortunately, we found that no particular enhancement is produced in this case with respect to the Chern-Simons case.

The work is organized in the following way.

Chapter one: In section 1.1 we review the most important results and concepts of the Friedmann-Lemaître-Roberston-Walker Universe. In Section 1.2, we delve into the topic of perturbation theory within the context of General Relativity (GR), with a detailed examination of the matter of gauge transformations. In section 1.3 we apply what we have done in the previous section in the case of a spatially flat $FLRW$ metric and in 1.4 we introduce the main "character" of the thesis, i.e. the *gauge-invariant curvature perturbation on uniform energy-density hypersurfaces.* The two last sections, 1.5 and 1.6, are devoted to a generic introduction to random fields in Cosmology.

Chapter two: In section 2.1 we discuss and solve throughout the Inflationary mechanism the shortcomings of the HBB model: the horizon problem in section 2.1.1, the flatness problem in section 2.1.2 and, the monopole problem in section 2.1.3. In Section 2.2, we explore the underlying dynamics of Inflation and the concept of slow-roll approximations. In section 2.2.3, we mention something about different models of Inflation and the cosmic no-hair principle. Then, we switch to the description of the quantum fluctuations of the Inflaton field. We work in the so-called Arnowitt-Deser-Misner (ADM) which we fully discuss in appendix C.3. In section 2.3 we introduce the necessary tools to fully derive the scalar and tensor power spectrum of primordial perturbation respectively in section 2.4 and 2.5. Then, we present a very general description, in section 2.6, of the mechanism that allows us to relate the primordial power spectra of scalar and tensor perturbations with the ones that we can observe today. Finally, in section 2.7 we discuss the fundamental theoretical and experimental role of the stochastic background of GWs predicted by the Inflationary model.

Chapter three: In this chapter, we present the main ideas that are used in the analysis of the galaxy four-point correlation function. We present a general discussion on the need to adopt the formalism introduced in [5]. Then in section 3.1, using techniques of quantum mechanics, we discuss how to decompose a generic isotropic function of N vector variables in a convenient orthonormal basis, which is the one used in data analysis. In section 3.2 we show why, considering a scalar field, we need the four-point correlation function to search for parity violation signature. In section 3.3 we briefly introduce the algorithm used in the analysis and in the final section 3.3, we discuss the claims of measurements of parity violation signature found in the galaxy four-point correlation function by [1] and [2].

Chapter four: In this chapter, we summarize the standard way to deal with the $In - In$ formalism in section 4.1 while in section 4.2 we explain the procedure for conducting computations of primordial correlation functions throughout Schwinger-Keldysh Diagrammatics for Primordial Perturbations [14, 17], which are the cosmological analogous to Feynman rules in quantum field theories.

Chapter five: In section 5.1 we introduce, following [18], the EFT approach to modify the inflationary Lagrangian. In section 5.2 we discuss the parity-breaking Chern-Simons term, $f(\phi)W\tilde{W}$, which we are going to consider throughout the rest of the chapter. We also show that this term doesn't modify the background dynamics. Then, in section 5.3 we present an extensive discussion on the spatially flat gauge at second order. Moreover, we demonstrate that up to the first order we do not have to modify the solution for the lapse and shift function with respect to the standard GR case even in the presence of the parity-breaking Chern-Simons term. Then, in section 5.4 we present a very detailed derivation of the equation of motion for the Inflaton and tensor perturbation in Fourier space. We also discuss how to solve these equations in order to find the primordial power spectra in section 5.5. Finally, in section 5.6 we perform the computation of the graviton-mediated trispectrum.

Chapter six: First of all, we discuss the main features of the chiral-scalar tensor theories of gravity introduced in [19]. We observe that also in that case we do not have modification of the background dynamics. Subsequently, we move forward to calculate the primordial power spectra for tensor and scalar perturbations in section 6.2. Notably, similar to the Chern-Simons case, we detect modifications in both the overall tensor power spectrum and the spectral index when compared to the standard single-field slow-roll model of Inflation. Then, in section 6.3 and 6.4 we make an original computation regarding the so-called graviton-mediated trispectrum. Unfortunately, we do not find any enhancement factor with respect to the Chern-Simons case and the signal we are able to generate is too weak to explain the claims made by [1, 2]. In the last section, 6.5, we make some comments on possible ideas for future works.

1 The standard cosmological model

1.1 The Friedmann-Lemaître-Roberston-Walker Universe

Modern cosmology aims to understand the history of our Universe from the beginning until today. The publication of Einstein's general relativity in 1915 [20, 21, 22, 23] enabled us to come up with a testable theory of the universe. Combining notions of fundamental physics we end up considering the so-called Hot Big Bang model which is based on the pioneering works of Friedmann [24], Lemaître[25], Roberston [26] and Walker[27], which offers a comprehensive explanation for a broad range of observed phenomena: the expansion of the universe discovered by the Hubble in 1929 [28], the abundance of light elements, which goes under the name of Big Bang nucleosynthesis (BBN) made by Alpher, Bethe, and Gamow in 1948 [29], and the cosmic microwave background (CMB) radiation discovered by Penzias and Wilson in 1965 [30].

In making physical predictions about observables, as in all other fields of physics, we must rely on simplifying assumptions derived from observations and intuition. First of all, the cosmic microwave background displays a remarkable degree of *isotropy*, except for tiny temperature fluctuations of approximately $\frac{\Delta T}{T} \sim 10^{-5}$. When formulating the assumption of isotropy, one should stress that the universe appears the same in all directions to a family of "privileged" observers: those at rest with respect to the cosmic fluid. Furthermore, adopting the *Copernican principle* is justifiable, which posits that Earth is not a privileged observer in the Universe. These two assumptions together lead to the cornerstone of modern cosmology, the *Cosmological principle*:

Every *comoving* observer observes the Universe around him at a fixed time as *homogeneous* and *isotropic* on sufficiently large scales.

In order to clarify this statement we have to make a few comments on the model used in Cosmology. First of all, we assume that it's possible to describe the dynamics of the universe throughout the metric tensor of a four-dimensional Lorentzian manifold [13] with signature (1,3). Furthermore, even if the local distribution of matter lacks homogeneity and isotropy, we idealize the real universe by homogenizing its matter distribution and redistributing it uniformly to match the observed average density and motion throughout. We recover isotropy and homogeneity on large scale, which are distances bigger than $100Mpc$. Subsequently, we introduce the additional assumption that the motion and geometry of this ideally standardized model universe, influenced by its own gravitational forces, mimic the average motion pattern and geometry observed in the actual universe. Each geometrical point of the manifold is a potential "center" of mass of a cluster of galaxies in the real world, and it's imagined to carry a fundamental observer, the so-called *comoving* observer. These "points" correspond to the kinematic substratum of the model and constitute the *cosmic fluid*. This construction allows us to describe the background dynamics of the Universe. In order to recover the deviation from this background we adopt perturbation theory in General relativity.

While we won't delve into deriving the $FLRW$ metric and its associated geometric framework, we will discuss the fundamental properties that will be employed in this thesis.

As stated in the Cosmological principle, the concept of homogeneity and isotropy remain valid at a fixed time, this implies that the universe exhibits spatial homogeneity and isotropy. For example, translational symmetry over time is absent due to the expansion of the Universe. Thus the manifold can be foliated as $\mathcal{R} \times \Sigma$, where Σ are spatially hypersurfaces that due to the homogeneity and isotropy assumptions are maximally symmetric spaces [31]. $\mathcal{R}$ represents the time direction, and the time variable associated is also called *cosmic time* since every comoving observer measures the same proper time. Roughly speaking spatial homogeneity and isotropy imply that every point in Σ is the equivalent and there are no privileged directions. It's possible to provide a more formal definition of these concepts using differential geometry and group theory as done in [31]. The most general metric tensor satisfying these hypotheses is the $FLRW$ one

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right), \tag{1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of a 2−sphere and we have used polar coordinates (r, θ, ϕ), which are also called comoving coordinates since they specify the so-called comoving distance between points in the manifold as shown in the figure. κ can be normalized to take the following values

$$\kappa = \begin{cases} +1 & \text{spherical} \\ 0 & \text{Euclidean} \\ -1 & \text{hyperspherical} \end{cases} . \tag{1.2}$$

In what follows, according to Planck data [32] we take $\kappa = 0$, thus we assume that the hypersurfaces are spatially flat. In eq.(1.1) $a(t)$ is the Robertson-Walker scale factor which is the dynamical variable in the tensor metric. Friedmann [24] and Lemaître [25] originally discovered the metric tensor of eq.(1.1) but Robertson [26] and [27] showed that this is the most general metric tensor satisfying spatial homogeneity and isotropy.

There are other two important coordinate transformations that are often used in cosmology. The first one is the switching to *confomal* time defined as

$$d\tau = \frac{dt}{a(t)}, \tag{1.3}$$

which allows us to put the metric as

$$ds^2 = a^2(\tau)\left(-d\tau^2 + dx^2 + dy^2 + dz^2\right), \tag{1.4}$$

which is conformal to the Minkowski metric (see [13] for conformal transformation). The other one, which makes sense only if $\kappa \neq 0$, is the one with which the metric becomes

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + S_\kappa^2(\chi)d\Omega^2\right), \tag{1.5}$$

where we have used that

$$d\chi = \frac{dr}{\sqrt{1-\kappa r^2}}, \quad S_\kappa(\chi) = \begin{cases} \sin\chi & \kappa = +1 \\ \chi & \kappa = 0 \\ \sinh\chi & \kappa = -1 \end{cases} . \tag{1.6}$$

In what we have discussed we have adopted natural units $c = \hbar = 1$, and in what follows sometimes we'll also use $M_{pl} = \sqrt{8\pi G}^{-1} = 1$.

1.1.1 The background dynamics

The dynamic of an expanding Universe is described by Einstein's field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.7}$$

where $G_{\mu\nu}$ is the Einstein's tensor, while

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\alpha\mu} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\beta\alpha}, \tag{1.8}$$

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{1.9}$$

are respectively the Ricci tensor and scalar in the convection we adopted[1] defined as functions of the Christoffel symbols

$$\Gamma^\alpha_{\mu\nu} = \frac{g^{\alpha\lambda}}{2}\left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right). \tag{1.10}$$

[1]Different signs' convection are present in the literature.

While in section 1.1 we have discussed the metric tensor which allows us to evaluate the LHS of eq.(1.7), in order to compute the RHS we need to specify the explicit form for the energy-stress tensor. Accordingly, for isotropy and homogeneity, we can't have off-diagonal components and the spatial components must be equal. Thus, the most simple form satisfying these requirements is the energy-stress tensor of a perfect fluid

$$T^{\mu}{}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + p\delta^{\nu}{}_{\mu} = diag(-\rho, p, p, p), \tag{1.11}$$

where ρ is the energy density of the fluid, p is the pressure of the fluid while $u^{\mu} = (1, 0, 0, 0)$ is the four-velocity of the cosmological fluid. Now plug in the metric and energy-stress tensor in eq.(1.7) to obtain the Friedmann equations [33]

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.12}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{1.13}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble constant, which is called "constant" since it doesn't depend on the space variables. Now using the Bianchi identity we can write the continuity equation

$$\nabla^{\alpha} T_{\alpha\beta} = 0, \tag{1.14}$$

which in the case of $FLRW$ cosmology gives

$$\dot{\rho} = -3H(\rho + p). \tag{1.15}$$

This equation doesn't add information since it can be derived from eq.(1.12) and (1.13). Thus, our variables are ρ, p and $a(t)$ and we have two equations. In order to solve the system we specify the equation of state (EoS) for the cosmic fluid and we adopt a barotropic EoS, which is

$$p = w\rho, \quad w \neq w(t, x, y, z). \tag{1.16}$$

In this way, it's possible to completely describe the dynamic of a spatially flat Universe. In fact from eq.(1.15) we get

$$\rho = \rho_0 \left(\frac{a(t)}{a_0}\right)^{-3(1+w)}, \tag{1.17}$$

where the subscript specifies the quantities evaluated at the present time. Plugging this in eq.(1.12) we get

$$a(t) = a_0 \begin{cases} \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} & w \neq -1 \\ e^{H(t-t_0)} & w = -1 \end{cases}. \tag{1.18}$$

Now we have to comment on the possible values that w can assume and their physical interpretations:

- $w = 0$ describes pressureless matter as dark matter and non-relativistic matter. In this case we have

$$\begin{cases} \rho & \propto a^{-3} \\ a & \propto t^{\frac{2}{3}} \\ H & = \frac{2}{3t} \end{cases}. \tag{1.19}$$

  The energy density gets diluted basically because since the universe expands and the volume goes as $V \propto a^{-3}$.

- $w = \frac{1}{3}$ describes radiation and we get

$$\begin{cases} \rho & \propto a^{-4} \\ a & \propto t^{\frac{1}{2}} \\ H & = \frac{1}{2t} \end{cases}, \tag{1.20}$$

  where the energy density gets an extra contribution with respect to the $w = 0$ case since the radiation frequency is redshifted.

- $w = -\frac{1}{3}$ doesn't correspond to any particular state of matter but it's an important point since it's the turning point for the second derivative of $a(t)$, see eq.(1.13). If $w < -\frac{1}{3}$ the universe goes under a phase of accelerated expansion which can not be driven by ordinary matter, i.e. photons or baryonic matter.

- $w = -1$ corresponds to a dark energy contribution which leads to an exponential expansion since

$$\begin{cases} \rho & = const \\ a & \propto e^{Ht} \\ H & = const \end{cases}, \tag{1.21}$$

  which corresponds to the so-called de-Sitter stage. This is as a first approximation of what is used to describe the background dynamics characterizing Inflation.

1.1.2 The Hubble radius and cosmological horizon

In this section, we discuss the issue of cosmological horizons and the *Hubble radius*. The latter is a fundamental quantity which we'll use throughout the rest of this work and is defined as in [34]:

The Hubble radius is the maximum distance over which particles can travel in the course of one expansion time.

Thus, the Hubble radius in natural units is defined as

$$R_c(t) \equiv \frac{c}{H} = \frac{1}{H}, \tag{1.22}$$

which describes the maximum distance since anything can travel faster than light. $\frac{1}{H}$ is the characteristic scale time that describes the evolution of the Universe so we expect light to travel a distance $\frac{1}{H}$ in a characteristic expansion time. Thus, the Hubble radius allows us to understand at a given time in the history of the Universe on what scales information can be exchanged. If the distance between two points is larger than the Hubble radius they cannot exchange information at the particular time we are considering. Otherwise, they are causally disconnected. We can also introduce the *comoving* Hubble radius as

$$r_c(t) = \frac{1}{aH}. \tag{1.23}$$

In the following, we'll frequently use that a given comoving scale λ or the associated wave vector $\mid \mathbf{k} \mid = \frac{2\pi}{\lambda}$ “enters" or “exits" the Horizon. We say that

$$\frac{1}{k} < \frac{1}{aH} \quad \text{the scale is sub-horizon,} \tag{1.24}$$

$$\frac{1}{k} > \frac{1}{aH} \quad \text{the scale is super-horizon,} \tag{1.25}$$

$$\frac{1}{k} = \frac{1}{aH} \quad \text{the scale is at horizon's exit or entrance.} \tag{1.26}$$

In the $FLRW$ Universe the Hubble radius reads

$$R_c(t) = \frac{1}{H} = \begin{cases} \frac{3(1+w)t}{2} & w \neq -1 \\ const & w = -1 \end{cases} \tag{1.27}$$

To conclude this section we discuss the issue of horizons in Cosmology. We introduce the concept of particle's horizon in the $FLRW$ universe with line element as in eq.(1.5)

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k^2(\chi)d\Omega\right]. \tag{1.28}$$

If the model has a beginning time (take $t_{beginning} = 0$ for simplicity) we can introduce the maximum distance over which "information" could have traveled, which is one of light signals. Considering radial geodesic light motion we have that $ds^2 = d\Omega = 0$, thus we get the following differential equation

$$\frac{dt}{a(t)} = \pm d\chi, \tag{1.29}$$

where, the choice of $\pm$ depends on the interval of integration. Specifically, if we integrate from (t_1, r_1) to $(t_2 > t_1, r_2 > r_1)$ or $(t_2 < t_1, r_2 < r_1)$, we use the plus sign; otherwise, we use the minus sign. By integrating from $(t = 0, \chi(r))$ to the origin of the polar coordinate system we are adopting, where, due to homogeneity, we put ourselves as observers $(t, \chi(0))$, we arrive at

$$\int_0^t \frac{d\tilde{t}}{a(\tilde{t})} = \chi(r). \tag{1.30}$$

Thus, using eq.(1.6), we obtain the so-called comoving particle's horizon

$$d_H(t) = S_\kappa\left[\int_0^t \frac{d\tilde{t}}{a(\tilde{t})}\right], \tag{1.31}$$

while its physical version, the particle's horizon, we have

$$D_H(t) = a(t)S_\kappa\left[\int_0^t \frac{d\tilde{t}}{a(\tilde{t})}\right] = a(t)\left[\int_0^t \frac{d\tilde{t}}{a(\tilde{t})}\right], \tag{1.32}$$

where in the last passage we have set $\kappa = 0$. Now in a Friedmann Universe, we have with $w \neq -1$

$$a(t) \propto t^\alpha, \quad \alpha = \frac{2}{3(1+w)}, D_H(t) = \frac{3(1+w)}{(1+3w)}t, \tag{1.33}$$

from which, since $\frac{1}{H} = \frac{2}{3(1+w)t}$, we get

$$D_H(t) \sim R_c(t). \tag{1.34}$$

Nevertheless, these two quantities possess fundamentally distinct conceptual natures. Specifically, the particle's horizon encompasses the entire past history of the Universe, whereas the Hubble radius characterizes the Universe at a specific moment in its evolutionary timeline. Another way to see this is that

$$d_H(t) = \int_0^t \frac{d\tilde{t}}{a(\tilde{t})} = \int_0^{a(t)} \frac{da}{a^2 H} = \int_0^{a(t)} d\ln a[r_c], \tag{1.35}$$

which states that the particle's horizon is the logarithmic integral of the Hubble radius. This is another way to say that the Hubble radius is local (with respect to time) properties while the particle's horizon encompasses all the past history of the Universe. We mention that if the model has an ending time there is another kind of horizon which we are not going to discuss since we do not use it in this work, see [35] for a complete discussion about this point.

1.2 Perturbation theory

As previously discussed in the preceding section, the $FLRW$ model serves as an approximation for the overall evolution of the Universe. However, when aiming to account for deviations from homogeneity and isotropy, we must turn to a perturbation approach. This becomes absolutely essential when

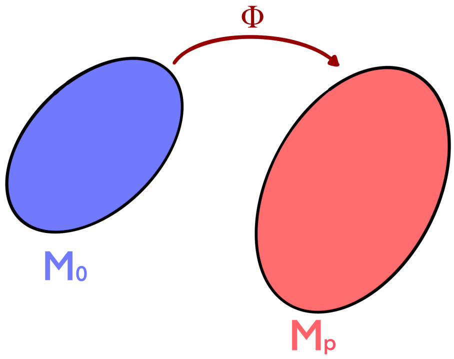


Figure 1: $\phi : M_0 \to M_p$

analyzing the anisotropies in the Cosmic Microwave Background (CMB) and the formation of large-scale structures (LSS).

Within the realm of General Relativity (GR), only a handful of exact solutions to Einstein's field equations exist, and these solutions often prove to be excessively idealized for capturing the complexities of natural phenomena. Consequently, the perturbative approach stands as an essential tool. Thus, we introduce two differential manifolds: the physical space M_p and background one M_0. Roughly speaking the perturbation in some tensor T is defined as the "difference" between the tensor in M_p and the one in M_0. However, within the context of differential geometry, the comparison of tensors is meaningful when they are evaluated at the same point within a given manifold. Thus, we need to introduce a diffeomorphism $\phi : M_0 \to M_p$ which identifies points of the physical space with points of the background as shown in figure 1. The "gauge" issue becomes apparent as there is no reason for favoring one diffeomorphism over another. The choice of the diffeomorphism represents the *gauge problem*.

After an intuitive explanation of why gauge problems arise in the theory of perturbations, we tackle the problem in a formal way as presented in [11, 12]. We introduce a family of space-times model

$$\{\mathcal{M}_\lambda\} \equiv \{(\mathcal{M}, g_\lambda, \tau_\lambda)\}, \tag{1.36}$$

where $\mathcal{M}$ is a four-dimensional Lorentzian manifold[2], g_λ is the tensor metric and τ_λ are generic matter fields, which satisfies EFE, i.e.

$$\varepsilon\{g_\lambda, \tau_\lambda\} = 0, \tag{1.37}$$

where ε represents Einstein's field equations. Then, we assume that $\{\mathcal{M}, g_0, \tau_0\}$ represents the background around which we want to expand and we also take as a hypothesis that g_λ and τ_λ are *smooth* in λ. As we'll see λ is the expansion parameter for the metric and stress-energy tensor and it's the same that labels the family of space-times models we have introduced. Basically, we can think of the leaves $\mathcal{M}_\lambda$ as the space-times where the quantities are of order λ.

The most natural way to study the problem is introducing a five-dimensional manifold,

$$\mathcal{N} = \mathcal{R} \times \mathcal{M} \tag{1.38}$$

which can be foliated by submanifolds diffeomorphic to $\mathcal{M}$, figure 2. We can introduce a chart in which $x^\mu(\mu = 0, 1, 2, 3)$ are coordinates on each $\mathcal{M}_\lambda$, and $x^4 = \lambda$. Now, given a tensor field T_λ defined on each $\mathcal{M}_\lambda$ we can automatically define the tensor field on $\mathcal{N}$ as

$$T(p, \lambda) := T_\lambda(p), \quad p \in \mathcal{M}_\lambda, \tag{1.39}$$

[2]This is not essential but we directly specialize to the case of interest.

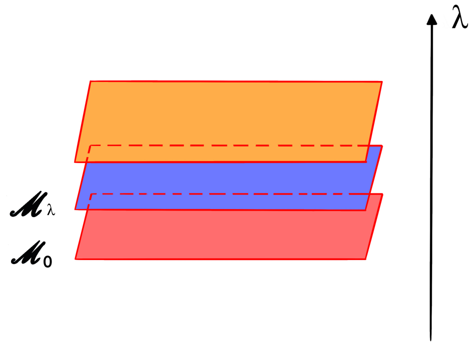


Figure 2: The manifold $\mathcal{N}$.

where P is a point in $\mathcal{M}$. While if we have a tensor defined on $\mathcal{N}$ we can define its restriction on each leaf inverting what we have written above.

Now, in order to define the perturbations in some quantity we need to introduce a way to compare tensors between each leaf and the zero-order version in $\mathcal{M}_0$. Thus we introduce a generic vector field $\mathcal{X}$ which is smooth, i.e.there exist the integral curves, and $\mathcal{X}^4 = 1$. The latter condition assures that $\mathcal{X}$ has always a component perpendicular to each leaf in such a way that the integral curves of $\mathcal{X}$ induce $\forall\lambda$ a diffeomorphism between $\mathcal{M}_0$ and $\mathcal{M}_\lambda$. The numerical value is completely irrelevant for our purpose so we put it to one as done in [11]. So we have introduced a one-parameter group of diffeomorphism which we call

$$\Psi_\lambda : \mathcal{N} \to \mathcal{N}, \tag{1.40}$$

which have $\forall\lambda$ we have an associated diffeomorfism

$$\Psi_\lambda|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{M}_\lambda. \tag{1.41}$$

In this way, we can compare tensor in $\mathcal{M}_\lambda$ and $\mathcal{M}_0$ using the pullback operation. Thus we define the perturbation in some tensor T, defined on $\mathcal{N}$, as

$$\Delta T_\lambda := \Psi^*_\lambda T|_{\mathcal{M}_0} - T_0, \tag{1.42}$$

where T_0 is the tensor evaluated on the background while $\Psi^*_\lambda T|_{\mathcal{M}_0}$ is the pull back of T using Ψ_λ. Since everything is smooth we can write

$$\Delta T_\lambda = \Psi^*_\lambda T|_{\mathcal{M}_0} - T_0 = \sum_{k=1}^{+\infty} \frac{\lambda^k}{k!} \delta^k T, \quad \delta^k T := \frac{d^k}{d\lambda^k} \Psi^*_\lambda T|_{\lambda=0,\mathcal{M}_0}. \tag{1.43}$$

This expression highlights two significant characteristics. The first one underscores that the perturbations remain defined on the background space-time throughout the pullback operation. Furthermore, if we can interpret λ as a perturbative parameter, we gain clarity on the notion of perturbation of the λ^{th} order.

1.2.1 Gauge transformations

At this point, we can delve into the matter of gauge transformations as presented in [11]. As we previously mentioned, there exists no inherent justification for favoring one vector field over another[3].

[3] In literature authors also indicate the vector field itself as the gauge.

This liberty in selecting the diffeomorphism is at the heart of the gauge issue. Our goal is to understand how perturbations behave under a gauge transformation and to determine the authentic physical degrees of freedom that come into play when we perturb the space-time of interest. Thus, let's take two smooth vector fields $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}^4 = \mathcal{Y}^4 = 1$. If we fixed λ the integral curves of $\mathcal{X}$ and $\mathcal{Y}$ define two diffeomorphisms

$$\phi_\lambda|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{M}_\lambda \tag{1.44}$$

$$\psi_\lambda|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{M}_\lambda \tag{1.45}$$

as represented in figure 3. Now given a generic tensor field T defined on $\mathcal{N}$ we can define its pulled-back

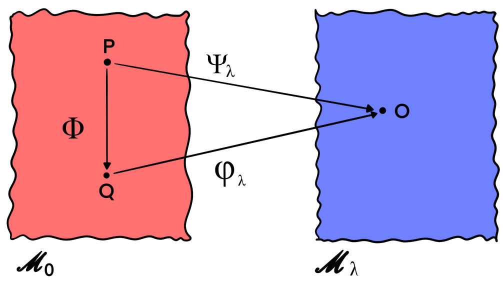


Figure 3: $\phi : M_0 \to M_p$

version evaluated on $\mathcal{M}_0$ with the associated perturbations according to ϕ_λ and ψ_λ (where we omit the subscript because it's irrelevant in what we are going to discuss)

$$T_\lambda^{\mathcal{X}} := \psi_\lambda^* T|_0 = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \pounds_{\mathcal{X}}^k T \Big|_0 = T_0 + \Delta^\psi T_{\mathcal{X}}, \tag{1.46}$$

$$T_\lambda^{\mathcal{Y}} := \phi_\lambda^* T|_0 = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \pounds_{\mathcal{Y}}^k T \Big|_0 = T_0 + \Delta^\phi T_{\mathcal{Y}}, \tag{1.47}$$

where the we have replaced $\mathcal{M}_0$ with 0 for notational clarity, we have used theorem A.1 presented in appendix A.4 and where $\pounds_{\mathcal{X}}^k = \pounds_{\mathcal{X}} \circ .. \circ \pounds_{\mathcal{X}}$. Now we want to understand how the different definitions of the perturbations of the tensor T are related. As we have previously said the perturbations we have written, eq.(1.46) and (1.47), correspond to the perturbations in two different gauges since they are obtained using the two diffeomorphisms of eq.(1.44) and (1.45). As shown in figure 3 we have a natural way of describing how the perturbations are related by introducing

$$\Phi_\lambda = \phi_\lambda^{-1} \circ \psi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0, \tag{1.48}$$

which is a one-parameter family of diffeomorphism on $\mathcal{M}_0$. Thus, we can write

$$T_\lambda^{\mathcal{Y}} = \psi_\lambda^* T|_{\mathcal{M}_0} = \left(\psi_\lambda^* \phi_{-\lambda}^* \phi_\lambda^*\right) T\big|_{\mathcal{M}_0} = \left(\phi_{-\lambda} \circ \psi_\lambda\right)^* \phi_\lambda^* T|_{\mathcal{M}_0} = \Phi_\lambda^* T_\lambda^{\mathcal{X}}. \tag{1.49}$$

Now, according to theorem A.2 we can decompose Φ_λ using a family of *knight diffeomorphism* with generator $\xi_{(1)}, .., \xi_{(k)}, ...$ These smooth vector fields have an associated flow, $\chi^{(1)}, .., \chi^{(k)}, ..$, which are simply the one-parameter group associated with their integral curves. Thus, we can write

$$\Phi_\lambda = .. \circ \chi^{(k)}_{\frac{\lambda^k}{k!}} \circ .. \circ \phi^{(2)}_{\frac{\lambda^2}{2}} \circ \phi^{(1)}_\lambda, \tag{1.50}$$

where the subscripts specify the parameter associated with the relative one-parameter group of diffeomorphism. Now, using theorem A.3 we get

$$T_\lambda^{\mathcal{Y}} = \Phi_\lambda^* T_\lambda^{\mathcal{X}} = \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} .. \sum_{l_k=0}^{+\infty} .. \frac{\lambda^{l_1+2l_2+..+kl_k+..}}{2^{l_2}..(k!)^{l_k}..l_1!l_2!..l_k!..} \pounds_{\zeta_{(1)}^{l_1}} .. \pounds_{\zeta_{(k)}^{l_k}} .. T_\lambda^{\mathcal{X}}. \tag{1.51}$$

Now we can get order by order the transformation rules. For example up to second order we get

$$T_0 + \lambda \pounds_{(\mathcal{Y})} T\big|_0 + \frac{\lambda^2}{2}\pounds^2_{\mathcal{Y}}\Big|_0 = \left[T_0 \lambda\, \pounds_{(\mathcal{X})} T\big|_0 + \frac{\lambda^2}{2}\pounds^2_{\mathcal{Y}}\Big|_0\right] + \left[\lambda \pounds_{\xi_{(1)}} + \frac{\lambda^2}{2}\left(\pounds^2_{\xi_{(1)}} + \pounds_{\xi_{(2)}}\right)\right]\left[T_0 \lambda\, \pounds_{(\mathcal{X})} T\big|_0 + \frac{\lambda^2}{2}\pounds^2_{\mathcal{Y}}\Big|_0\right], \tag{1.52}$$

where the subscript 0 is an abbreviation for $\mathcal{M}_0$. Now, using what we have introduced in eq.(1.43) we can write

$$\delta T^{(1)}_{\mathcal{Y}} = \delta T^{(1)}_{\mathcal{X}} + \pounds_{\xi_{(1)}} T_0, \tag{1.53}$$

$$\delta T^{(2)}_{\mathcal{Y}} = \delta T^{(2)}_{\mathcal{X}} + \left[\pounds^2_{\xi_{(1)}} + \pounds_{\xi_{(2)}}\right] T_0 + 2\pounds_{\xi_{(1)}} \delta T^{(1)}_{\mathcal{X}}. \tag{1.54}$$

In general, λ is set to 1 and we interpret the component of $\xi_{(k)}$ as of the k^{th} order.

The situation resembles a scenario in which we have a contravariant tensor field C defined on a general manifold $\mathcal{K}$, along with a diffeomorphism $\zeta : \mathcal{K} \to \mathcal{K}$. We are faced with the inquiry of how the tensor changes under this diffeomorphism. To define the "new" tensor, we can either pull back the original tensor using the diffeomorphism or push forward it via the inverse map. In order to draw a comparison between these two tensor fields, let's consider a point P and its corresponding image $Q \equiv \zeta(P)$. At point P, we have the initial tensor $C(P)$, and the pullback of the original tensor in Q evaluated in P, denoted as $\zeta^* C|_P$. In this way, we can compare the two tensor fields at the same point.

Analogously, in the context of gauge transformations, we find ourselves in a similar scenario. We can conceptualize the two gauges as the original tensor and its pullback counterpart in the aforementioned example. In the pursuit of gauge transformation, we are in search of the connection between these two gauges. This linkage is established by the diffeomorphism of eq.(1.48).

This approach is called *active* since we have a map that "moves" points on the background as shown in figure 3. However, it's possible to work using coordinates in the so-called *passive* approach [11] but we do not touch this point since what is presented is fully consistent and coordinate-independent.

1.3 Cosmological perturbation

In cosmology, as mentioned in section 1.1 we can work with a spatially flat Robertson-Walker background. Adopting conformal time we now can perturb the background as

$$g_{00} = -a^2(\tau)\left(1 + 2\sum_{r=1}^{+\infty}\frac{\psi^{(r)}}{r!}\right), \tag{1.55}$$

$$g_{0i} = a^2(\tau)\sum_{r=1}^{+\infty}\frac{w_i^{(r)}}{r!}, \tag{1.56}$$

$$g_{ij} = a^2(\tau)\left[\left(1 - 2\sum_{r=1}^{+\infty}\frac{\phi^{(r)}}{r!}\right)\delta_{ij} + \sum_{r=1}^{+\infty}\frac{\chi_{ij}^{(r)}}{r!}\right], \tag{1.57}$$

where we have used Latin indices to label spatial components and where $\chi^i{}_i = 0$. It's standard to split vectors and traceless tensors using a generalization of the Helmholtz theorem [36]. In our case each order of the shift function g_{0i} can be written as

$$w_i^{(r)} = \partial_i w^{(r)||} + w_i^{(r)\perp}, \quad \partial^i w_i^{(r)\perp} = 0. \tag{1.58}$$

While the traceless part of the spatial metric can be written as

$$\chi_{ij}^{(r)} = D_{ij}\chi^{(r)||} + \partial_i \chi_j^{(r)\perp} + \partial_j \chi_i^{(r)\perp} + \chi_{ij}^{(r)T}, \tag{1.59}$$

where we have

$$D_{ij} = \partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2, \quad \partial^i \chi_i^{(r)\perp} = 0, \quad \partial^i \chi_{ij}^{(r)T} = 0, \quad \chi^{(r)T i}{}_i = 0. \tag{1.60}$$

To consistently introduce perturbations to Einstein's field equations, it is imperative to also address the perturbation of the stress-energy tensor. In the context of a fluid, this can be expressed in its most general form [37] as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} + \Pi_{\mu\nu}, \tag{1.61}$$

where ρ and p are the energy density and the pressure, u_μ is the four-velocity while $\Pi_{\mu\nu}$ is the anisotropic stress tensor which is subjected to

$$\Pi^\mu{}_\mu = 0, \quad u^\mu \Pi_{\mu\nu} = 0. \tag{1.62}$$

In the case of a perfect fluid or a scalar field $\Pi_{\mu\nu} = 0$. Now in order to perturb it we have to perturb each quantity in eq.(1.61). Now, regarding perturbations of the stress-energy tensor we can summarize the results as

- The **energy density** is a scalar and we can perturb it as

$$\rho = \rho_0 + \sum_{r=1}^{+\infty} \frac{\delta^r \rho}{r!}, \tag{1.63}$$

  where r specifies the perturbation order. In Cosmology often the so-called *overdensity* matter field is used and it's defined as

$$\delta \equiv \frac{\rho(\tau, \mathbf{x}) - \rho_0(\tau)}{\rho_0(\tau)}. \tag{1.64}$$

- Concerning the **pressure**, which, when it comes to rotations, is also scalar, we can write

$$\tag{1.65}$$

$$p = p_0 + \sum_{r=1}^{+\infty} \frac{\delta^r p}{r!}. \tag{1.66}$$

  The pressure is assumed to be linked to the energy density through the EoS,

$$p = p(\rho, S), \tag{1.67}$$

  where S is the entropy. Thus a perturbation in the pressure can always be written as

$$\delta p = \left.\frac{\partial p}{\partial \rho}\right|_S \delta\rho + \left.\frac{\partial p}{\partial S}\right|_\rho \delta S = c_s^2 \delta\rho + \left.\frac{\partial p}{\partial S}\right|_\rho \delta S, \tag{1.68}$$

  where c_s is the sound speed and where the second term is always called a *non-adiabatic* perturbation.

- The **four-velocity** it's generally perturbed as

$$u^\mu = \frac{1}{a}\left(\delta^\mu{}_0 + \sum_{r=1}^{+\infty} \frac{v^\mu_{(r)}}{r!}\right), \tag{1.69}$$

  where we have to remember that $u^\mu u_\mu = -1$; thus at any order the velocity perturbation it's linked to the lapse perturbation, $\psi_{(r)}$. Moreover, the usual decomposition in scalar and vector is adopted for the spatial component of the four-velocity

$$v^i_{(r)} = \partial^i v^{||}_{(r)} + v^i_{(r)\perp}. \tag{1.70}$$

- The **tensor of anisotropic stress** solely possesses spatial components, with $\Pi_{0\mu} = 0$ [37]. Thus, we can write as

$$\Pi_{ij} = \sum_{r=0}^{+\infty} \Pi_{ij}^{(r)}, \tag{1.71}$$

and we can use the same decomposition adopted for the spatial metric

$$\Pi_{ij}^{(r)} = D_{ij}\chi^{(r)||} + \partial_i \chi_j^{(r)\perp} + \partial_j \chi_i^{(r)\perp} + \chi_{ij}^{(r)T}. \tag{1.72}$$

We will abstain from deriving the perturbed form of Einstein's equation, as their use will be confined to section 1.4. The results in the case of minimally coupled scalar field are reported in [10]. In the rest of the thesis, we'll work directly at the Lagrangian level and we'll derive the *EoM* using a variational principle (we will also adopt the notation introduced in [13]).

1.3.1 Gauge transformations and cosmological gauges at first order

Within this section, we explore several aspects of cosmological gauges and discuss the procedure of gauge-fixing at the first order, offering insights into the methodology. Subsequently, in section 5.3, we will extend our analysis to the second order for the two gauges employed in calculating the scalar trispectrum graviton-mediated within the context of theories of modified gravity.

At first order, the metric tensor has the following components

$$g_{00} = -a^2(\tau)\left(1 + 2\psi^{(1)}\right), \tag{1.73}$$

$$g_{0i} = a^2(\tau) w_i^{(1)}, \tag{1.74}$$

$$g_{ij} = a^2(\tau)\left[\left(1 - 2\phi^{(1)}\right)\delta_{ij} + \chi_{ij}^{(1)}\right], \tag{1.75}$$

while regarding the stress-energy tensor at first order we get [37]

$$T^0{}_0 = -\left(\rho_{(0)} + \delta^{(1)}\rho\right), \tag{1.76}$$

$$T^i{}_0 = -\left(\rho_{(0)} + p_{(0)}\right) v^i_{(1)}, \tag{1.77}$$

$$T^0{}_i = \left(w_i^{(1)} + v_i^{(1)}\right)\left(\rho_{(0)} + p_{(0)}\right), \tag{1.78}$$

$$T^i{}_j = \left(p_{(0)} + \delta^{(1)}p\right) + p_0 \Pi^i{}_j{}^{(1)}. \tag{1.79}$$

$$\tag{1.80}$$

Now, given the perturbation in two gauges, $\left\{\tilde{g}^{(1)}_{\mu\nu}, \delta\tilde{T}^\mu{}_\nu\right\}$ and $\left\{g^{(1)}_{\mu\nu}, \delta T^\mu{}_\nu\right\}$, we have that they are related according to eq.(1.53)

$$\tilde{g}^{(1)}_{\mu\nu} = g^{(1)}_{\mu\nu} + \pounds_{\zeta_{(1)}} g^{(0)}_{\mu\nu}, \tag{1.81}$$

$$\delta\tilde{T}^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu} + \pounds_{\zeta_{(1)}} T^{(0)}_{\mu\nu}, \tag{1.82}$$

where $\zeta^\mu_{(1)}$ denotes a smooth vector field characterized by components of first-order and where we have defined $\delta T^\mu{}_\nu \equiv T^\mu{}_\nu \equiv -T^\mu_{(0)}{}_\nu$. It's customary to decompose $\zeta_{(1)}$ in two scalars and one divergence-free vector as

$$\zeta^0_{(1)} = \alpha_{(1)} \tag{1.83}$$

$$\zeta^i_{(1)} = \partial^i \beta_{(1)} + d^{i\perp}_{(1)}, \quad \partial_i d^{i\perp}_{(1)} = 0. \tag{1.84}$$

At each order, this vector decomposition linked to the gauge transformation is commonly utilized [11, 12]. Now, in order to establish the gauge transformation rules for the quantities we have defined in the metric, we have to compute the Lie derivative of the background metric [38]

$$\pounds g^{(0)}_{\mu\nu} = \nabla_\mu \zeta^{(1)}_\nu + \nabla_\nu \zeta^{(1)}_\mu, \tag{1.85}$$

which in components reads

$$\pounds g^{(0)}_{00} = -2a(\tau)a^{'}(\tau)\alpha_{(1)} - 2a^2(\tau)\alpha^{'}(\tau), \tag{1.86}$$
$$\pounds g^{(0)}_{0i} = a^2(\tau)\left(\partial\beta^{'}_{(1)} + d^{(1)\perp\prime}_i\right) - a^2(\tau)\partial_i\alpha_{(1)}, \tag{1.87}$$
$$\pounds g^{(0)}_{ij} = a^2(\tau)\left[\partial_i\left(\partial_j\beta_{(1)} + d^{(1)\perp}_j\right) + \partial_j\left(\partial_i\beta_{(1)} + d^{(1)\perp}_i\right)\right] + 2a(\tau)a^{'}(\tau)\alpha\delta_{ij}, \tag{1.88}$$

where $'$ is the derivative with respect to conformal time, i.e. $\frac{d}{d\tau}$. Thus, we get the linear transformation rules at first order [11]

$$\tilde{\psi}_{(1)} = \psi_{(1)} + \alpha^{'}_{(1)} + \frac{a^{'}}{a}\alpha_{(1)} \tag{1.89}$$
$$\tilde{w}^{(1)||} = w^{(1)||} + \beta^{'}_{(1)} - \alpha_{(1)}, \quad \tilde{w}^{(1)\perp}_i = w^{(1)\perp}_i + d^{(1)\perp\prime}_i, \tag{1.90}$$
$$\tilde{\phi}^{(1)} = \phi^{(1)} - \frac{1}{3}\nabla^2\beta^{(1)} - \frac{a^{'}}{a}\alpha^{(1)}, \tag{1.91}$$
$$\tilde{\chi}^{(1)||} = \chi^{(1)||} + 2\beta_{(1)}, \quad \tilde{\chi}^{(1)\perp}_i = \chi^{(1)\perp}_i + d^{(1)\perp}_i, \quad \tilde{\chi}^T_{ij} = \chi^T_{ij}, \tag{1.92}$$

from which we immediately realize that tensors are gauge invariant at first order.

For the energy-momentum tensor, we have to compute the Lie derivative of its background version, which in components reads

$$\pounds_{\zeta_{(1)}} T^0_{(0)0} = -\alpha\rho^{'}, \tag{1.93}$$
$$\pounds_{\zeta_{(1)}} T^0_{(0)0} = \left(\partial^i\beta^{(1)} + d^{i\perp}_{(1)}\right)^{'}\left(\rho_{(0)} + p_{(0)}\right), \tag{1.94}$$
$$\pounds_{\zeta_{(1)}} T^0_{(0)i} = -\left(\rho_{(0)} + p_{(0)}\right)\partial_i\alpha_{(1)}, \tag{1.95}$$
$$\pounds_{\zeta_{(1)}} T^i_{(0)j} = \alpha p^{'}_0 \delta^i{}_j. \tag{1.96}$$

Thus, we can easily deduce the gauge transformation rules for the quantities we have introduced in perturbing the stress-energy tensor

$$\delta^{(1)}\tilde{\rho} = \delta^{(1)}\rho + \alpha_{(1)}, \quad \tilde{v}^i_{(1)} = v^i_{(1)} - \left(\partial^i\beta_{(1)} + d^{i\perp}_{(1)}\right)^{'} \tag{1.97}$$
$$\delta\tilde{p}_{(1)} = \delta p_{(1)} + \alpha p^{'}_0, \quad \tilde{\Pi}^i_{(1)j} = \Pi^i_{(1)j}. \tag{1.98}$$

Presently, when considering gauge fixing, the question arises whether it is possible to pass from a completely arbitrary gauge to the desired one. We will illustrate this process for a specific gauge, while for the other gauges, we will solely present the outcome. Basically, since we have two scalars and one vector in $\zeta_{(1)}$ we can set to zero two scalars and one vector in metric perturbations. We provide a brief overview of the most frequently employed gauge choices in Cosmology without a deep analysis of their specific physical properties.

- **Poisson gauge**
  It's defined by the following conditions

$$w^{(1)||} = 0, \quad , \chi^{(1)||} = 0, \quad \chi^{(1)||}_i = 0. \tag{1.99}$$

Now, in order to verify that we can impose this gauge we have to start from an arbitrary gauge, i.e. $\delta g_{\mu\nu}$, and verify if we can achieve what we have written. Thus, in the “new" gauge, $\tilde{g}_{\mu\nu}$, we wonder if we can impose

$$\begin{cases} \tilde{w}^{(1)||} &= 0 \\ \tilde{\chi}^{(1)||} &= 0 \; . \\ \tilde{\chi}_i^{(1)\perp} &= 0 \end{cases} \tag{1.100}$$

Using the gauge transformation rules we arrive at the following system of equations

$$0 = w^{(1)||} + \beta_{(1)}^{'} - \alpha_{(1)}, \tag{1.101}$$

$$0 = \chi^{(1)||} + 2\beta_{(1)}, \tag{1.102}$$

$$0 = \chi_i^{(1)\perp} + d_i^{(1)\perp}, \tag{1.103}$$

which it's algebraic and can be solved in order to find $\alpha_{(1)}, \beta_{(1)}, d_i^{(1)\perp}$. Thus, starting from a very general gauge we can always arrive at the Poisson gauge setting particular values for the components of $\zeta_{(1)}$.

- **Synchronous and time-orthogonal gauge**
  It's defined by the following conditions

$$\tilde{\psi}_{(1)} = 0, \quad \tilde{w}^{(1)||} = 0, \quad \tilde{w}_i^{(1)||} = 0. \tag{1.104}$$

  This gauge is plagued by residual gauge freedoms and is referred to as the synchronous gauge, as the proper time measured by an observer at rest relative to the spatial coordinates it's equal and corresponds to the unperturbed proper time of a Robertson-Walker universe.

- **Comoving gauge**
  The conditions we impose are

$$v^{(1)||} = 0, \quad v_i^{(1)\perp} = 0, \quad w^{(1)||} = 0. \tag{1.105}$$

  This gauge is called comoving since we do not have any spatial components of the perturbed four-velocity.

- **Spatially-flat gauge**
  In this case, we set to zero the scalars and vectors of the perturbed spatial part of the metric, i.e.

$$\phi_{(1)} = 0, \quad \chi^{(1)||} = 0, \quad \chi_i^{(1)\perp} = 0. \tag{1.106}$$

  This is called spatially flat gauge since the Ricci scalar on the spatial hypersurface is zero. It's possible to show that [37]

$$^{(3)}R = \frac{6\kappa}{a^2} + \frac{12\kappa}{a^2}\hat{\phi}_{(1)} + \frac{4}{a^2}\nabla^2\hat{\phi}_{(1)}, \tag{1.107}$$

  where κ is the curvature constant and $\hat{\phi}_{(1)} = \phi_{(1)} + \frac{\nabla^2\chi^{(1)||}}{6}$. In the case of a spatially flat universe and, from $\hat{\phi}_{(1)} = 0$ we immediately get a vanishing Ricci scalar on the spatial hypersurfaces.

- **Uniform energy density gauge**
  This gauge is defined simply by setting

$$\delta^{(1)}\rho = 0, \tag{1.108}$$

  without specifying any other conditions.

1.4 The gauge invariant curvature perturbation

In this section, we present the gauge-invariant curvature perturbation known as ζ. Firstly, we emphasize that a gauge-invariant quantity remains unchanged under a gauge transformation. Various examples of this concept exist, including the *Bardeen gauge-invariant potentials* [39]. As previously mentioned, tensors also maintain gauge invariance at the first order. ζ is defined as

$$\zeta \equiv -\hat{\phi}_{(1)} - \frac{a'}{a}\frac{\delta^{(1)}\rho}{\rho_0'}, \tag{1.109}$$

which is gauge invariant since

$$\tilde{\hat{\phi}}_{(1)} = \hat{\phi}_{(1)} - \frac{a'}{a}\alpha_{(1)}, \tag{1.110}$$

$$\delta^{(1)}\tilde{\rho} = \delta^{(1)}\rho + \alpha_{(1)}\rho_{(0)}'. \tag{1.111}$$

It is alternatively referred to as the *gauge-invariant curvature perturbation on the uniform energy-density hypersurfaces*, as it takes on a resemblance to the Ricci scalar on spatial hypersurfaces when considering the uniform density gauge ($\delta^{(1)}\rho = 0$). In fact, the Ricci scalar on three-dimensional hypersurfaces can be computed [37, 10] as

$${}^{(3)}R = \frac{4}{a^2}\nabla^2\hat{\phi}_{(1)}. \tag{1.112}$$

This quantity is crucial in inflationary computations since it's conserved on *super-horizon scales.* In fact, the energy conservation equation at first order for scalar quantities reads [10]

$$\delta^{(1)}\rho' + 3\frac{a'}{a}\left(\delta^{(1)}\rho + \delta^{(1)}p\right) - 3\left(\rho_0 + p_0\right)\hat{\phi}_{(1)}' + \left(\rho_0 + p_0\right)\nabla^2(V^{(1)} + \sigma^{(1)}) = 0, \tag{1.113}$$

where $V^{(1)} \equiv v^{(1)||} + w^{(1)||}$ and $\sigma \equiv \frac{\chi^{||'}}{2} - w^{||}$. The last term on super-horizon scales can be neglected and, adopting the uniform density gauge, we get

$$\frac{a'}{a}\delta^{(1)}p - \left(\rho_0 + p_0\right)\hat{\phi}_{(1)}' = 0. \tag{1.114}$$

Now, in this gauge, we have $\hat{\phi}_{(1)} = -\zeta$ and we also can split the pressure perturbation in an adiabatic contribution and in a non-adiabatic part, $\delta^{(1)}p = c_s^2\delta^{(1)}\rho + \delta^{(1)}p_{n-a}$. Finally, we can write

$$\zeta' = -\frac{a'}{a}\frac{\delta^{(1)}p_{n-a}}{\rho_0 + p_0}. \tag{1.115}$$

Now, it's possible to show that, in the case of single-field of slow-roll inflation $\delta^{(1)}p \propto \frac{k^2}{a^2}\Phi_H$, which goes to zero on super-horizon scales and where $2\Phi_H \equiv -2\phi_{(1)} - \frac{1}{3}\nabla^2\chi^{||} + 2\frac{a'}{a}w^{||} - \frac{a'}{a}\chi^{||'}$ is one of the Bardeen gravitational potential [39]. Thus, we have that on super-horizon scales the gauge invariant curvature perturbation on uniform energy-density hypersurfaces it's conserved.

1.5 Random fields and correlation functions

We conclude this chapter with a brief introduction to the concept of random fields and correlation functions and their application in Cosmology. First of all, we start by introducing some general concepts in a $D-$dimensional Euclidean manifold, and then in section 1.6 we generalize them to the spatially-flat $FLRW$ case.

Random fields are ubiquitous in physics and the most powerful tool for their analysis is the set of N-point correlation functions (hereafter $NPCF$s). Analyses with $NPCF$s are widely used in different fields [40]: molecular physics [41], materials science [42], field theory [43], diffusive systems [44], statistical mechanics [45] and Cosmology [5].

Now, building upon the works of [40] and [33], we introduce the formal concept of correlation functions in $\mathbb{R}^D$.

Let $\phi(x) : \mathbb{R}^D \longrightarrow \mathbb{R}$ denote a random real scalar field dependent on D-dimensional Euclidean coordinates x. Here, x represents a D-dimensional vector that signifies the absolute position of a specific point in R^D. Formally the N-point correlation functions are defined as

$$\begin{cases} \zeta : (\mathbb{R}^D)^{\otimes N} \longmapsto \mathbb{R}, \\ \zeta(r_1, .., r_{N-1}, s) = \langle \phi(s), \phi(s + r_1, .., \phi(r_{N-1} + s) \rangle, \end{cases} \tag{1.116}$$

where $s, r_1, .., r_{N-1}$ indicates absolute and relative positions, $\otimes$ stands for tensor product and $\langle .. \rangle$ is the ensemble average, i.e. the statistical average over realizations of ϕ. Ideally, we should have $K \longmapsto \infty$ of realizations of the same process and the ensemble average consists of a statistical average over the K copies. For a random field, we can think of a realization as the determination of the scalar field's value at every point within R^D. In general, we don't have access to $K \longmapsto \infty$ copies of the process we are considering. For example in cosmology, the universe is assumed to be an isotropic and homogeneous random process [46] (see section 1.6); i.e. fields, such as the overdensity matter field δ, are stochastic, homogeneous and isotropic fields. The problem is that we don't have other copies of the universe over which we can average. However, under suitable hypotheses, we can apply the Ergodic theorem, eq.(1.122), to evaluate the $NPCF$s using spatial averages.

We conclude this section by introducing some basic concepts about random fields and proving the Ergodic theorem, which will be useful in constructing the galaxy four-point correlation function estimator (section 3.3). We note that the definition of correlation functions has the potential for generalization in both Euclidean and Riemannian manifolds. However, since we adopt $\mathbb{R}^3$ as a simplified representation of the low-redshift universe in section 3 and a flat Friedmann-Lemaître-Robertson-Walker ($FLRW$) universe in section 1.6, our focus lies elsewhere and we do not delve into this generalization.

Presently, we formally introduce the notions of a Gaussian field, as well as a homogeneous and isotropic field. Consider a random field $\phi(x) : \mathbb{R}^D \longmapsto \mathbb{R}$ depending on a D-dimensional Euclidean coordinate x and let's take $\langle \phi(x) \rangle = 0$. The distribution function governing $\phi(x)$ is said to be Gaussian if:

$$\langle \phi(x_1), .., \phi(x_N) \rangle = 0 \qquad \text{if } N \text{ is odd,} \tag{1.117}$$

$$\langle \phi(x_1), .., \phi(x_N) \rangle = \sum_{\text{pairings}} \prod_{pairs} \langle \phi\phi \rangle \qquad \text{if } N \text{ is even,} \tag{1.118}$$

with the sum over pairings not distinguishing those that interchange coordinates in a pair, or which merely interchange pairs. For example, the four-point correlation function becomes

$$\begin{aligned} \langle \phi(x_1), \phi(x_2), \phi(x_3), \phi(x_4) \rangle = \langle \phi(x_1), \phi(x_2) \rangle \langle \phi(x_3), \phi(x_4) \rangle + \\ \langle \phi(x_1), \phi(x_3) \rangle \langle \phi(x_2), \phi(x_4) \rangle + \langle \phi(x_1), \phi(x_4) \rangle \langle \phi(x_2), \phi(x_3) \rangle. \end{aligned} \tag{1.119}$$

In a path integral formulation of correlation functions for random fields, we mention that the probability density functional (PDF) for a Gaussian random field is a Gaussian.

The random field we have introduced is said to be statistically homogeneous if

$$\forall N \in \mathbb{N}, \forall z \in \mathbb{R}^D \quad \zeta(r_1 + z, .., r_N + z) = \zeta(r_1, .., r_N), \tag{1.120}$$

while is statistically isotropic if

$$\forall \hat{R} \in SO(D) \quad \zeta(\hat{R}r_1, .., \hat{R}r_N) = \zeta(r_1, .., r_N), \tag{1.121}$$

where $SO(D) \equiv \{O \in mat_{D\times D}(\mathbb{R}) \mid O^T O = OO^T = \mathbb{I}\}$.

We conclude this section by proving the Ergodic theorem in the case of a homogeneous field as stated in Weinberg's book ([33]).

Theorem 1.1 (Ergodic Theorem). *Consider a homogeneous random field $\phi(x) : \mathbb{R}^D \longmapsto \mathbb{R}$ depending on a D-dimensional Euclidean coordinate x. Let's take that ϕs at distant arguments are uncorrelated, i.e.*

$$u \in \mathbb{R}, \langle \phi(x_1+u)\phi(x_2+u)..\phi(y_1-u)\phi(y_2-u)..\rangle \xrightarrow{|u| \longmapsto \infty} \langle \phi(x_1+u)\phi(x_2+u)..\rangle\langle \phi(y_1-u)\phi(y_2-u)..\rangle. \tag{1.122}$$

If the limit in eq.(1.122) *is approached sufficiently rapidly, then the root mean square difference between any product $\phi(x_1+z)\phi(x_2+z)...$, averaged over a range R of z values around an arbitrary point z_0, and the ensemble average of the same product vanishes as $R^{-\frac{D}{2}}$ for large R. That is if we define*

$$\Delta_R^2(x_1, x_2, ..) = \langle \left[\left(\int d^D z N_R(z)\phi(x_1+z)\phi(x_2+z).. \right) - \langle \phi(x_1)\phi(x_2)..\rangle \right]^2 \rangle, \tag{1.123}$$

where

$$N_R(z) \equiv (\sqrt{\pi}R)^{-D} exp(-\frac{|z-z_0|^2}{R^2}), \tag{1.124}$$

then $\Delta_R \xrightarrow{|u| \longmapsto \infty} O(R^{-\frac{D}{2}})$.

Proof. First of all, notice that the specific form of the function of eq.(1.124) it's not important but we require that

- $\int d^D z N_R(z) = 1$,
- $\begin{cases} N_R(z) \approx const & \text{if } |z-z_0|^2 \ll R \\ N_R(z) \to 0 & \text{if } |z-z_0|^2 \gg R \end{cases}$.

In the analysis of the galaxy $NPCF$ a window function can be adopted

$$\begin{cases} N_R(z) = \frac{1}{R^D} = \frac{1}{V_D} & \text{if } |z-z_0|^2 \leq R \\ N_R(z) = 0 & \text{if } |z-z_0|^2 \geq R \end{cases}. \tag{1.125}$$

Eq.(1.123) can be rewritten using the normalization condition of $N_R(z)$

$$\Delta_R^2(x_1, x_2, ..) = \langle \left[\int d^D z N_R(z) \left[(\phi(x_1+z)\phi(x_2+z)..) - \langle \phi(x_1)\phi(x_2)..\rangle \right] \right]^2 \rangle, \tag{1.126}$$

$$\tag{1.127}$$

which expanding the square becomes

$$\begin{aligned} \Delta_R^2(x_1, ..) &= \langle \left[\int d^D z N_R(z) \int d^D w N_R(w) [(\phi(x_1+z)..\phi(x_1+w)..) + \langle \phi(x_1)\phi(x_2)..\rangle^2] \right] \rangle \\ &- 2\langle \left[\int d^D z N_R(z) \int d^D w N_R(w) [\phi(x_1+z)..] \langle \phi(x_1)\phi(x_2)..\rangle \right] \rangle \qquad (1.128) \\ &= \langle \left[\int d^D z N_R(z) \int d^D w N_R(w) \left[(\phi(x_1+z)..\phi(x_1+w)..) + \langle \phi(x_1)\phi(x_2)..\rangle^2 \right] \right] + \\ &- 2 \int d^D z N_R(z) \int d^D w N_R(w) [\langle (\phi(x_1+z)..)\rangle \langle \phi(x_1)\phi(x_2)..\rangle] \qquad (1.129) \\ &= \int d^D z N_R(z) \int d^D w N_R(w) [\langle \phi(x_1+z)..\phi(x_1+w)..\rangle - \langle \phi(x_1)\phi(x_2)..\rangle^2], \qquad (1.130) \end{aligned}$$

where in the third equality we have used that

$$\begin{aligned} &\int d^D z N_R(z) \int d^D w N_R(w) [\langle (\phi(x_1+z)..)\rangle \langle \phi(x_1)\phi(x_2)..\rangle] = \\ &\int d^D z N_R(z) \int d^D w N_R(w) [\langle (\phi(x_1)..)\rangle \langle \phi(x_1)\phi(x_2)..\rangle] = \langle (\phi(x_1)..)\rangle^2, \qquad (1.131) \end{aligned}$$

which is true only if the field is homogeneous. Now we introduce new integration variables $u \equiv \frac{z-w}{2}$ and $v \equiv \frac{z+w}{2}$ and we get

$$\Delta_R^2(x_1,..) = \int d^D u d^D v |J| (\sqrt{\pi}R)^{-2D} e^{(-\frac{|u+v-z_0|^2}{R^2} - \frac{|u-v-z_0|^2}{R^2})} [\langle\phi(x_1+u+v)..\phi(x_1+v-u))..\rangle - \langle\phi(x_1)\phi(x_2)..\rangle^2] \tag{1.132}$$

$$= (\frac{2}{\pi R^2})^D \int d^D u d^D v e^{(-\frac{2|u-z_0|^2}{R^2} - \frac{2|v|^2}{R^2} -)} [\langle\phi(x_1+u+v)..\phi(x_1+v-u))..\rangle - \langle\phi(x_1)\phi(x_2)..\rangle^2] \tag{1.133}$$

$$= (\frac{2}{\pi R^2})^D \int d^D u d^D v e^{(-\frac{2|u-z_0|^2}{R^2} - \frac{2|v|^2}{R^2} -)} [\langle\phi(x_1+u)..\phi(x_1-u))..\rangle - \langle\phi(x_1)\phi(x_2)..\rangle^2], \tag{1.134}$$

where in the second equality we use the Jacobian for the transformation, $|J| = 2^D$, and in the last one we exploit homogeneity. Now we can integrate on the v variable since the integrand doesn't depend on v

$$\Delta_R^2(x_1,x_2,..) = (\frac{2}{\pi R^2})^{\frac{D}{2}} \int d^D u e^{-\frac{2|u-u_0|^2}{R^2}} [\langle\phi(x_1+u)..\phi(x_1-u)..\rangle - \langle\phi(x_1)\phi(x_2)..\rangle^2]). \tag{1.135}$$

If the limit of eq.(1.122) is sufficiently rapid the integral would converge even without the exponential factor. Since

$$\int d^D u e^{-\frac{2|u-u_0|^2}{R^2}} [\langle\phi(x_1+u)..\phi(x_1-u)..\rangle - \langle\phi(x_1)\phi(x_2)..\rangle^2]), \tag{1.136}$$

is finite, for $R \longmapsto \infty$ we can take the integral of order unity and this proves the statement of the Ergodic theorem, i.e.

$$\Delta_R^2(x_1,x_2,..) \xrightarrow{R\longmapsto\infty} O(R^{-\frac{D}{2}}). \tag{1.137}$$

□

Now if we imagine having K copies of the stochastic process we are considering we can write

$$\Delta_R^2(x_1,x_2,..) = \langle \left[\left(\int d^D z N_R(z)\phi(x_1+z)\phi(x_2+z).. \right) - \langle\phi(x_1)\phi(x_2)..\rangle \right]^2 \rangle \tag{1.138}$$

$$= \sum_K \left[\left(\int d^D z N_R(z)\phi_K(x_1+z)\phi_K(x_2+z).. \right) - \langle\phi(x_1)\phi(x_2)..\rangle \right]^2, \tag{1.139}$$

where the subscript K specifies the field in one of the K realization. Since eq.(1.139) is the sum of positive terms, in order to get 0 when $R \longmapsto \infty$ we require each term to be 0. This means that if we have at our disposal only one copy of the stochastic process[4], provided that the volume of integration is sufficiently large, we can calculate the $NPCF$s as

$$\langle\phi(x_1)\phi(x_2)..\phi(x_N)\rangle = \frac{1}{V_D} \int d^D z \phi(x_1+z)\phi(x_2+z)..\phi(x_N), \tag{1.140}$$

where we have used a window function for $N_R(z)$.

1.6 Cosmological correlation functions

In the preceding section, we introduced random fields on $\mathbb{R}^D$. However, when describing the universe (in its 0th order approximation), we utilize a four-dimensional homogeneous and isotropic spacetime denoted as $(\mathcal{M}, g)$. This spacetime is characterized by a Lorentzian manifold M endowed with a

[4] This is the case in cosmology.

metric g [13], as elaborated upon in section 1.1. As we discussed earlier, homogeneous and isotropic spacetimes can be sliced into homogeneous and isotropic maximally symmetric 3-spaces [13], leading to the metric tensor g taking the form described by equation (1.1). However, in the thesis, we work by setting $\kappa = 0$, i.e. the spatially flat case.

Therefore, given a random real field scalar field $\phi(t, \mathbf{x}) : \mathcal{M} \longmapsto \mathbb{R}$ we define the $NPCF$s as

$$\begin{cases} \zeta : (\mathcal{M})^{\otimes N} \longmapsto \mathbb{R}, \\ \zeta(t, \mathbf{r}_1, .., \mathbf{r}_{N-1}, \mathbf{s}) = \langle \phi(t, \mathbf{s}), \phi(t, \mathbf{s} + \mathbf{r}_1), .., \phi(t, \mathbf{r}_{N-1} + \mathbf{s}) \rangle, \end{cases} \tag{1.141}$$

where we stress that the N fields are evaluated at the same cosmic time. We mention that we need to modify the definition of statistically isotropy and homogeneity with respect to the one introduced in section 1.5; basically we have to restrict these concepts to the spatial hypersurfaces. We say that a field $\phi(t, \mathbf{x}) : \mathcal{M} \longmapsto \mathbb{R}$ is statistically isotropic if

$$\forall \hat{R} \in SO(3) \ \zeta(t, \hat{R}\mathbf{r}_1, .., \hat{R}\mathbf{r}_N) = \zeta(t, \mathbf{r}_1, .., \mathbf{r}_N). \tag{1.142}$$

We state that a field $\phi(t, \mathbf{x}) : \mathcal{M} \longmapsto \mathbb{R}$ is statistically homogeneous if

$$\forall \mathbf{r} \ \zeta(t, \mathbf{r}_1, .., \mathbf{r}_N) = \zeta(t, \mathbf{r}_1 + \mathbf{r}, .., \mathbf{r}_N + \mathbf{r}). \tag{1.143}$$

Now, we are prepared to explore the fundamental cosmological assumptions that underlie the treatment of random fields:

- the universe is *statistically homogeneous*,
- the universe is *statistically isotropic*,
- well-separated patches of the universe are uncorrelated.

We stress that homogeneity and isotropy in this contest are to be interpreted as "spatial" isotropy and homogeneity, as defined in eq.(1.142) and eq.(1.143). As discussed in [46] this hypothesis, the *fair simple hyphotesis*, has successfully met the available tests. Surely matter in the universe is not distributed in a homogeneous and isotropic way but in complex structures such as stars, galaxies, and clusters of galaxies. So homogeneity and isotropy can be applied in a statistical sense, i.e. in the spatial average over large enough regions. One can imagine that the matter distribution in each place is determined by a long sequence of position-dependent physical events which can't be influenced by very distant points. Since well-separated patches of the universe are uncorrelated one can imagine constructing an ensemble by splitting the universe into such patches. We recall that the hypothesis the universe is homogeneous and isotropic with a metric of the form of eq.(1.1) and the fair simple hypothesis are related to the Cosmological principle. We don't discuss the relationship between the two concepts but we present them as different hypothesis.

The fair simple hypothesis is crucial since allows us to conclude that: given any random cosmological field

$$\phi(t, \mathbf{x}) : \mathcal{M} \longmapsto R \tag{1.144}$$

with $\langle \phi(t, \mathbf{x}) \rangle = 0$, the $NPCF$s are isotropic functions of $N - 1$ vector variables. Indeed, homogeneity allows us to write

$$\zeta(t, \mathbf{r}_1, .., \mathbf{r}_N) = \zeta(t, \mathbf{s}, \mathbf{s} + \mathbf{r}_1, .., \mathbf{s} + \mathbf{r}_{N-1}) = \zeta(t, \mathbf{r}_1, .., \mathbf{r}_{N-1}), \tag{1.145}$$

which, using isotropy, we know it must be invariant under simultaneous rotations of $\mathbf{r}_1, .., \mathbf{r}_{N-1}$. This is the starting point for the analysis of parity violation in the galaxy four-point correlation function, presented in section 3 .

Before discussing the measurements of the galaxy four-point correlation function, we want to introduce some fundamental concepts about the two-point correlation function in a spatially-flat $FLRW$

universe. Given a statistically homogeneous and isotropic field $\phi(\mathbf{x})$ which has zero mean, $\langle\phi(\mathbf{x})\rangle = 0$, the two-point correlation function $\zeta(\mathbf{x},\mathbf{y})$ is defined as

$$\zeta(\mathbf{x},\mathbf{y}) \equiv \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle, \tag{1.146}$$

which can be rewritten as

$$\zeta(\mathbf{x},\mathbf{y}) = \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle = \langle\phi(\mathbf{x})\phi(\mathbf{x}+\mathbf{r})\rangle = \langle\phi(\mathbf{0})\phi(\mathbf{r})\rangle = \zeta(\mathbf{r}) = \zeta(r), \tag{1.147}$$

where we have written $\mathbf{y} = \mathbf{x} + \mathbf{r}$, used homegeneity and isotropy and defined $r = |\mathbf{r}|$. So we can say that the two-point correlation function is an isotropic function of one vector variable. It's convenient to write $\phi(\mathbf{x})$[5] in Fourier space as

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{+i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{k}), \tag{1.148}$$

where $\mathbf{k}$ is a comoving wave vector related to the physical one $\mathbf{q}$ throughout $\mathbf{q} = a\mathbf{k}$. We should indicate the field in Fourier space with a tilde, $\tilde{\phi}(\mathbf{k})$, but we omit it since the space of definition of the field is always clear from the contest. If we now consider the ensemble average in Fourier space we get

$$\langle\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\rangle = \langle\int d^3x \int d^3y \phi(\mathbf{x})\phi(\mathbf{y}) e^{-i\mathbf{k}_1\cdot\mathbf{x}} e^{-i\mathbf{k}_2\cdot\mathbf{y}}\rangle = \int d^3x \int d^3r e^{-i\mathbf{k}_1\cdot\mathbf{x}} e^{-i\mathbf{k}_2\cdot(\mathbf{x}+\mathbf{r})}\langle\phi(\mathbf{x})\phi(\mathbf{x}+\mathbf{r})\rangle \tag{1.149}$$

$$= \int d^3x \int d^3r e^{-i\mathbf{k}_1\cdot\mathbf{x}} e^{-i\mathbf{k}_2\cdot(\mathbf{x}+\mathbf{r})}\zeta(r) = (2\pi)^2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\int d^3r e^{-i\mathbf{k}_2\cdot\mathbf{r}}\zeta(r) \tag{1.150}$$

$$= (2\pi)^2\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)P(|\mathbf{k}_2|), \tag{1.151}$$

which is called the Wiener-Kintchine theorem. In the last line, we have introduced the *power spectrum* $P(|\mathbf{k}|)$ which is the anti-Fourier transform of the two-point correlation function,i.e.

$$P(|\mathbf{k}|) = \int d^3r e^{-i\mathbf{k}\cdot\mathbf{x}}\zeta(r). \tag{1.152}$$

The $\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)$ in eq.(1.151) arise from the assumption of homogeneity while the dependence on the modulus of $|\mathbf{k}_2|$ stems from isotropy. In cosmology, it's mostly used the *adimensional power spectrum*, $\Delta(|\mathbf{k}|) = \Delta(k)$, which is the contribution to the variance per logarithmic integral. The variance can be written as

$$\langle\phi^2(\mathbf{x},t)\rangle = \zeta(0) = \int \frac{d^3k}{(2\pi)^3} e^{+i\mathbf{k}\cdot\mathbf{r}}P(k)\bigg|_{\mathbf{r}=\mathbf{0}} = \int_0^\infty \frac{dk 4\pi k^2}{(2\pi)^3}P(k) = \int_0^\infty \frac{dk}{k}\frac{k^3}{2\pi^2}P(k) = \int_0^\infty \frac{dk}{k}\Delta(k), \tag{1.153}$$

where we have introduced the *adimensional power spectrum*

$$\Delta(k) \equiv \frac{k^3}{2\pi^2}P(k). \tag{1.154}$$

Despite $P(k)$ and $\Delta(k)$ are different quantities, they are often referred to as the power spectrum. We can also define another useful quantity, the *spectral index*, defined as

$$n(k) \equiv \frac{d[\ln\Delta(k)]}{d\ln(k)}, \tag{1.155}$$

which describes the shape of the power spectrum.

- If $n = 1$ we get the Harrison-Zeldovich spectrum which means cosmological scale invariance.
- If $n = const$, $\Delta(k)$ is usually written with respect to a pivot scale k_0, $\Delta(k) = \Delta(k_0)(\frac{k}{k_0})^{n-1}$. If $n > 1$ we speak of a blue-tilted spectrum while if $n < 1$ of a red-tilted one.

Note that this kind of reasoning can also be applied to metric perturbations and stress-energy tensor perturbations since these perturbations live on a background space-time which is spatially isotropic and homogeneous [33].

[5] We assume fall-off conditions at infinity for all cosmological quantities which assure that the functions are square-integrable, so the Fourier transform (FT) of ϕ exists.

2 The Inflationary paradigm

Cosmologists strongly believed that before the radiation-dominated epoch, during which the Robertson-Walker scale factor $a(t)$ grows as $\sqrt{t}$, a period of accelerated expansion took place, i.e. *Inflation*. This idea was introduced by Alan Guth in 1981 to solve the flatness (section 2.1.2), the horizon (section 2.1.1) and the monopole (setion 2.1.3) problems of the standard BigBang cosmology. Inflation can be achieved if the energy density of the universe is dominated by the vacuum energy density of a scalar field ϕ, the inflaton. In this way, the universe is dominated by dark energy like component and passes through an accelerated expansion phase, during which the scale factor is stretched by more than 60 efolds, $a_E = e^{60} a_B$ where a_E and a_B are the scale factor respectively at the end and the beginning of Inflation. Inhomogeneities and anisotropies are washed away, making today's universe very homogeneous, isotropic and flat. Moreover, the fields, which drive Inflation, have quantum fluctuations. These fluctuations also got stretched and imprinted at super-horizon scales. Later, during radiation, matter or cosmological constant domination, they reenter the horizon and provide the seeds for cosmic microwave background (CMB) anisotropies and the large-scale structures today (LSS). Inflation predicts some general features on the properties of the density perturbations:

- They are primordial, i.e. they are quantum fluctuations of the fields driving Inflation. They go on super-horizon scales during Inflation and re-enter the horizon during the Big Bang cosmology.
- They are approximately scale-invariant, as confirmed by the Planck's measurement [32] of the spectral index, $n_s = 0.9649 \pm 0.0042$, of the Power Spectrum of the primordial matter density field. Basically, during the accelerated expansion, each k-mode experiences a similar expansion.
- They are approximately Gaussian.

The fluctuations are nearly Gaussian but the detection of connected N-point correlation functions with $N \geq 3$ could unveil some features of the primordial universe, so in recent years these measurements have become crucial in cosmology. Theoretically, even if Inflation remains a paradigm, it's commonly accepted in the standard cosmological model.

This chapter is constructed in the following way. In section 2.1 we discuss and solve throughout the Inflationary mechanism the shortcomings of the HBB model. We present an accurate description of the horizon problem in section 2.1.1 and the flatness problem in section 2.1.2. While regarding the monopole problem we provide a qualitative explanation in section 2.1.3. Then, we focus on the description of the dynamics of Inflation in the case of single-field slow-roll models. The scalar field driving Inflation, the Inflaton $\phi(\tau, \mathbf{x})$, can be split into two parts a homogeneous and isotropic background, $\phi(\tau)$, and the quantum fluctuations, $\delta\phi(\tau, \mathbf{x})$, of the Inflaton itself. Hence, in Section 2.2, we explore the underlying dynamics of the inflaton and the concept of slow-roll approximations. These approximations enable the potential to possess a flat region, where the behavior of $\phi(\tau)$ closely resembles that of an effective cosmological constant, thereby providing accelerated expansion. In section 2.2.3, we mention something about different models of Inflation and the cosmic no-hair principle. Then, we switch to the description of the quantum fluctuations of the Inflaton field. We work in the so-called Arnowitt-Deser-Misner (ADM) which we fully discuss in appendix C.3. In section 2.3 we introduce the necessary tools to fully derive the scalar and tensor power spectrum of primordial perturbation respectively in section 2.4 and 2.5. Then, we present a very general description, in section 2.6, of the mechanism that allows us to relate the primordial power spectra of scalar and tensor perturbations with the ones that we can observe today. Finally, in section 2.7 we discuss the fundamental theoretical and experimental role of the stochastic background of GWs predicted by the Inflationary model.

2.1 The shortcomings of the Hot Big Bang model

In this section, we outline the so-called shortcomings of the Big Bang model, the horizon 2.1.1, the flatness 2.1.2 and the monopole 2.1.3 problems, and we work out how Inflation can solve each of them. For the sake of simplicity, we make the following assumptions for describing the evolution of the universe in the HBB model:

- the HBB model is correct up to the Planck scale ($t \sim t_{pl}$),
- if $t \in [t_{pl}, t_{eq}]$, the universe is radiation dominated, i.e.

$$\begin{cases} a(t) & = a_{eq} \left[\frac{t}{t_{eq}}\right]^{\frac{1}{2}} \\ H(t) & = \frac{1}{2t} \\ \rho(t) & = \rho_{eq} \left[\frac{a_{eq}}{a(t)}\right]^{3} \end{cases}, \tag{2.1}$$

- if $t \in [t_{eq}, t_0]$, the universe is matter dominated (we disregard the recent period of dark energy domination), i.e.

$$\begin{cases} a(t) & = a_{eq} \left[\frac{t}{t_{eq}}\right]^{\frac{2}{3}} = a_0 \left[\frac{t}{t_0}\right]^{\frac{2}{3}} \\ H(t) & = \frac{2}{3t} \\ \rho(t) & = \rho_0 \left[\frac{a_0}{a(t)}\right]^{4} = \rho_{eq} \left[\frac{a_{eq}}{a(t)}\right]^{4} \end{cases}, \tag{2.2}$$

- we do not take smooth transition between different epochs.

Regarding the evolution in the Inflationary case, we assume that the universe went through an early period of exponential expansion which we take, for simplicity, a De-sitter expansion phase:

- if $t \in [t_B, t_E]$, the universe goes through an acceletated expansion

$$\begin{cases} a(t) & = a_B \exp(H(t - t_B)) = a_E \exp(H(t - t_E)) \\ H(t) & = const \\ \rho(t) & = const \end{cases}, \tag{2.3}$$

- the reheating is instantaneous at t_E,
- starting from t_E, we apply the same assumption made in the previous case, the Hot Big Bang (HBB).

2.1.1 The Horizon problem

In simple terms, we can summarize the problem as follows: within the HBB model, at the last scattering surface (when protons and electrons merged into neutral hydrogen with the emission of the CMB), we observe regions that are not causally connected but exhibit similar physical properties, such as the temperature T. The horizon problem emerges when calculating $d_H(t)$ at the last scattering, resulting in a finite value that is smaller than the physical distance between regions exhibiting very similar properties, like temperature. This implies that regions of the universe that were not causally connected at the time of last scattering share remarkably similar characteristics. These similarities can be achieved either by imposing specific initial conditions (though this isn't a physical resolution) or through mechanisms like inflation, which enable the observable universe today to occupy a tiny region of the space where information exchange could have occurred.

This can be quantified in the following way. First, we need to calculate the physical horizon distance at the last scattering. This can be accomplished by employing the formula for computing the proper distance as presented in eq.(A.16), while adjusting the interval of integration accordingly:

$$d_H(t_{ls}) \equiv a(t_{ls}) \int_0^{t_{ls}} \frac{dt}{a(t)} = a(t_{ls}) S_k \left[\int_{z_{ls}}^{\infty} \frac{dz}{a_0 H(z)}\right] \tag{2.4}$$

$$= a(t_{ls}) S_k \left[\int_{z_{ls}}^{\infty} \frac{1}{a_0 H_0} \int_0^z \frac{dz}{\sqrt{\left[\Omega_{0M}\left(1+z\right)^3 + \Omega_{0R}\left(1+z\right)^4 + \Omega_{0\Lambda} + \Omega_{0\kappa}\left(1+z\right)^2\right]}}\right], \tag{2.5}$$

where $t = 0$ corresponds to the Big Bang. Then, we can compute the angular diameter distance at last scattering as

$$d_A(z_{ls}) = a(t_{ls})r(t_{ls}) = a(t_{ls})S\left[\int_0^{z_{ls}} \frac{dz}{a_0 H(z)}\right] \tag{2.6}$$

$$= a(t_{ls})S\left[\int_0^{z_{ls}} \frac{1}{a_0 H_0}\int_0^{z} \frac{dz}{\sqrt{\left[\Omega_{0M}(1+z)^3 + \Omega_{0R}(1+z)^4 + \Omega_{0\Lambda} + \Omega_{0\kappa}(1+z)^2\right]}}\right]. \tag{2.7}$$

Now, we can determine the maximum angle at the surface of last scattering that separates points which have been in causal contact:

$$\theta_{max} = \frac{d_H(t_{ls})}{d_A(t_{ls})} = \frac{a(t_{ls})S_k\left[\int_{z_{ls}}^{\infty} \frac{dz}{a_0 H(z)}\right]}{a(t_{ls})S_k\left[\int_0^{z_{ls}} \frac{dz}{a_0 H(z)}\right]} = \frac{S_k\left[\int_{z_{ls}}^{\infty} \frac{dz}{a_0 H(z)}\right]}{S_k\left[\int_0^{z_{ls}} \frac{dz}{a_0 H(z)}\right]}, \tag{2.8}$$

which for a spatially flat universe becomes

$$\theta_{max} = \frac{\int_{z_{ls}}^{\infty} \frac{dz}{H(z)}}{\int_0^{z_{ls}} \frac{dz}{H(z)}} \approx 0.02 \approx 1.2^\circ, \tag{2.9}$$

where the result can be obtained by numerical integration using the results of the Planck mission [32]. Eq.(2.9) tells us that in principle regions which have angular separation larger than 1.2° have never been in causal contact before. However, this is in contradiction with the nearly perfect isotropy of CMB at large angular scales which was observed ever since CMB's discovery.

Inflation can provide a solution to this problem. If we compute the physical particle's horizon at last scattering we get

$$d_H(t_{ls}) \equiv a(t_{ls})\int_{t_B}^{t_{ls}} \frac{dt}{a(t)} = a(t_{ls})\int_{t_B}^{t_E} \frac{dt}{a(t)} = \frac{a(t_{ls})}{H_E a_E}\exp(H(t_E - t))|_{t_B}^{t_E} \approx \frac{a(t_{ls})}{H_E a_E}e^N, \tag{2.10}$$

where we have neglected the contribution from the HBB phase, we have taken $e^N >> 1$ and we have used the approximation of spatial flatness. In order to solve the horizon problem we have to impose

$$d_H(t_{ls}) > \theta_{max} d_A(t_{ls}), \tag{2.11}$$

where θ_{max} is the maximum angle over which we observe isotropy. We can take $\theta_{max} \approx O(1)$ since its precise value has negligible impact on the minimum number of e-folds N. So we can write

$$\frac{a(t_{ls})}{H_E a_E}e^N > d_A(t_{ls}) \approx \frac{a(t_{ls})}{a_0 H_0}, \tag{2.12}$$

where we have used numerical integration of eq.(2.7) for a spatially flat universe. Thus we get a lower bound on the minimum number of e-folds required to solve the horizon problem:

$$N > \ln\left[\frac{a_E H_E}{a_0 H_0}\right]. \tag{2.13}$$

Now using Tolman's law, $Ta = const$, in the HBB model phase we can write

$$\ln\left[\frac{a_E H_E}{a_0 H_0}\right] = \ln\left[\frac{H_E T_0}{H_0 T_E}\right] = \ln\left[\frac{H_E}{T_E}\right] + \ln\left[\frac{T_0}{H_0}\right]. \tag{2.14}$$

From Planck's data we know that $T_0 \approx 2.728K$ and $H_0 \approx 0.67019 \times \frac{100km}{sMpc}$, so we can estimate $\ln\left[\frac{T_0}{H_0}\right]$ coming back to SI units

$$\ln\left[\frac{T_0}{H_0}\right] = \ln\left[\frac{k_B T_0}{\hbar H_0}\right] = \ln\left[\frac{1.38 \times 10^{-23}\frac{J}{K}2.7K}{1.0545 \times 10^{-34}Js0.67019 \times \frac{100km}{3.085\times 10^{19}km}s^{-1}}\right] \tag{2.15}$$

$$= \ln\left[16.26 \times 10^{28}\right] \approx 28\ln 10 + ln16.26 \approx 68, \tag{2.16}$$

where we have used that

$$\begin{cases} \text{Boltzmann's constant} & k_B = 1.38 \times 10^{-23} \frac{J}{K} \\ \text{Planck's constant} & h = 6.626 \times 10^{-34} Js \\ \text{Mega parsec conversion in km} & 1Mpc = 3.085 \times 10^{19} km \end{cases} . \tag{2.17}$$

To estimate the first term on RHS of eq.(2.14) we have to recall that we have taken an istantaneous reheating, thus we can write write

$$H_E^2 \approx \frac{8\pi G}{3} \left[\frac{\pi^2}{30} g_* T_E^4 \right] \approx \frac{T_E^4}{M_{pl}^2}. \tag{2.18}$$

Using this result we get

$$\ln \left[\frac{H_E}{T_E} \right] \approx \ln \left[\frac{T_E^2}{M_{pl} T_E} \right] \approx \ln \left[\frac{T_E}{M_{pl}} \right], \tag{2.19}$$

Different Inflationary models that have been proposed, make predictions within the range of $10^{-5} \leq \frac{T_E}{M_{pl}} \leq 1$. Therefore the only condition on the Inflationary period we can derive from the solution of the horizon problem is a lower bound on the number of e-folds N,

$$N > 68 + [-5 \ln 10, 0] \approx 66 + [-11, 0]. \tag{2.20}$$

There is another interesting way to obtain the bound of eq.(2.13). Recalling the definition of the comoving Hubble radius eq.(1.23)

$$r_H(t) = \frac{1}{a(t)H(t)}, \tag{2.21}$$

which basically individuates the maximum comoving distances that at a particular time can exchange information. If two points are located beyond the Hubble radius, they are unable to exchange information with each other. We know that in the HBB model it's always increasing as explained in section 1.1.2 and graphically presented in figure 4. The sketch shows that in the Hot Big Bang (HBB)

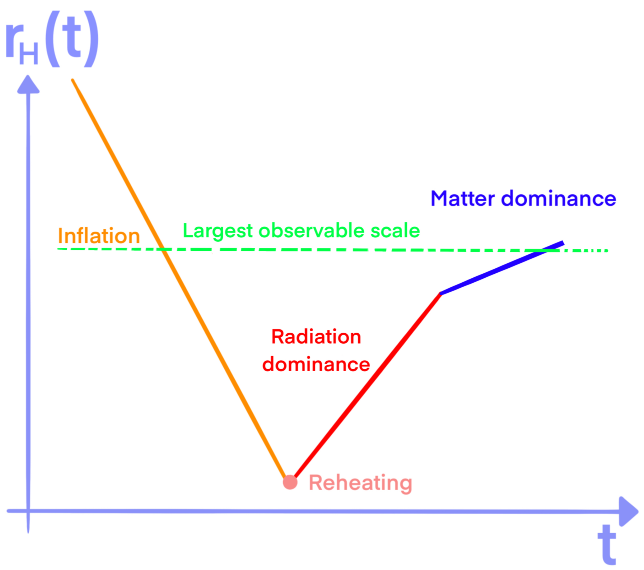


Figure 4: The comoving Hubble radius.

model, regions causally connected with an observer increase over time. Initially, this might not be problematic, as it only implies that a specific length scale λ enters the horizon at a particular time[6]. The issue arises when we observe causal connections between regions that are separated by length scales well beyond the particle horizon predicted by the HBB model. The idea to solve this issue revolves around an early period during which the comoving Hubble radius decreases in a manner that ensures all the scales we observe today were well within the horizon. However, if we want

$$\dot{r}_H(t) = -\frac{\ddot{a}(t)}{\dot{a}(t)^2} < 0, \tag{2.22}$$

we must require $\ddot{a} > 0 \iff w < -\frac{1}{3}$. Thus, we need a period of accelerated expansion, i.e. Inflation, as shown in figure 4. In order to solve the horizon problem we must require that the large scale we can probe today was under the horizon during Inflation

$$r_H(t_B) \geq r_H, \leftrightarrow \exp N > \frac{a_E H_E}{a_0 H_0}, \tag{2.23}$$

which is exactly the same condition we have previously imposed.

2.1.2 The Flatness problem

The *flatness* problem is a fine-tuning problem of the HBB model which can be discussed by using the first Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \tag{2.24}$$

which can be written in the form of eq.(A.6)

$$1 = \Omega(t) - \Omega_k, \tag{2.25}$$

where $\Omega(t) = \Omega_m(t) + \Omega_\gamma(t) + \Omega_\Lambda(t)$. A priori κ could take any value according to some sort of probability density function; so the probability to get a spatially flat Universe has zero measure. The problem is that

$$\mid \Omega_k(t_0) \mid = \mid \Omega(t_0) - 1 \mid = 4 \times 10^{-3} \quad (95\%\text{CL}), \tag{2.26}$$

which consists of saying that the universe is compatible with being spatially flat today. This presents a fine-tuning problem, as assuming the correctness of the HBB model up to the Planck scale ($t_{pl} \approx 10^{-43}$) leads to

$$\mid \Omega(t_{pl}) - 1 \mid \approx \mid \Omega(t_{pl}) - 1 \mid 10^{-60} < 10^{-62}, \tag{2.27}$$

which constraints an adimensional quantity to get a very “fine-tuned" value, even though we lack any physical arguments a priori to support such a statement. In physics, this is referred to as a fine-tuning problem, as we would expect this quantity, i.e. $\mid \Omega(t) - 1 \mid$ at initial time, to be of $O(1)$.

Now we show how to get the bound of eq.(2.27) within the context of the Big Bang model, using that

$$\left[1 - \Omega^{-1}(t)\right] \rho a^2 = const. \tag{2.28}$$

Considering $t \in [t_p, t_{eq}]$ we can write

$$\left[1 - \Omega^{-1}(t)\right] \rho(t) a^2(t) = \left[1 - \Omega^{-1}(t)\right] \rho_{eq} \left[\frac{a_{eq}}{a(t)}\right]^4 a^2(t) = \left[1 - \Omega^{-1}(t)\right] \rho_0 \left[\frac{a_0}{a_{eq}}\right]^3 \left[\frac{a_{eq}^4}{a^2(t)}\right], \tag{2.29}$$

[6]When a length scale λ enters the horizon, it signifies that we can exchange information with points up to a distance of λ from our location.

which, using eq.(2.28) evaluated today, allows us to get the following

$$\left[1-\Omega^{-1}(t)\right]\rho_0\left[\frac{a_0}{a_{eq}}\right]^3\left[\frac{a_{eq}^4}{a^2(t)}\right]=\left[1-\Omega_0^{-1}\right]\rho_0 a_0^2. \tag{2.30}$$

Finally, we can get the upper bound of eq.(2.27)

$$\left[1-\Omega^{-1}(t)\right]=\left[1-\Omega_0^{-1}\right]\frac{a^2(t)}{a_0^2}\frac{a_0}{a_{eq}}=\left[1-\Omega_0^{-1}\right]\frac{a^2(t)}{a_0^2}\left[1+z_{eq}\right]=\left[1-\Omega_0^{-1}\right]\frac{T_0^2}{T^2(t)}\left[1+z_{eq}\right], \tag{2.31}$$

where we have used the definition of the redshift and Tolman's law in the context of the HBB model, i.e. $T\ \alpha\ a^{-1}$. If we take $t=t_{pl}$ in eq.(2.31) we obtain

$$\left[1-\Omega^{-1}(t_p)\right]=\left[1-\Omega_0^{-1}\right]\frac{T_0^2}{T_{t_{pl}}^2}\left[1+z_{eq}\right]\approx\left[1-\Omega_0^{-1}\right]\times 10^{-64}\times 10^4\approx\left[1-\Omega_0^{-1}\right]\times 10^{-60}, \tag{2.32}$$

where we have used that $T_{pl}\equiv T_{t_{pl}}\approx 10^{32}K$, $T_0\approx 2.7K$ and $(1+z_{eq})\approx 10^4$. Plugging the bound of eq.(2.26) in eq.(2.32) we get the desired upper bound for $\left[1-\Omega_{pl}^{-1}\right]$,

$$\mid \Omega_{pl}^{-1}-1\mid=\mid\frac{\Omega_{pl}-1}{\Omega_{pl}}\approx\mid\Omega_0^{-1}-1\mid 10^{-60}=\mid\frac{\Omega_0-1}{\Omega_0}\mid 10^{-60}<10^{-62}, \tag{2.33}$$

which using that $\Omega_0,\Omega_{pl}\sim O(1)$ can be put in the desired expression of eq.(2.27).

A phase of accelerated expansion provides a solution to the flatness problem, i.e. *Inflation.* As mentioned earlier, in order to address the fine-tuning problem, we make the assumption that Ω_k during the onset of Inflation is of order unity, i.e. $\Omega_k(t_B)=\frac{|\kappa|}{a_B^2H_B^2}\sim O(1)$. Under these hypotheses we get

$$\mid\Omega_k(t_0)\mid=\frac{\mid\kappa\mid}{a_0^2H_0^2}=\frac{\mid\kappa\mid}{a_0^2H_0^2}\frac{a_E^2H_E}{a_B^2H_B}\frac{a_B^2H_B}{a_E^2H_E}=\frac{\mid\kappa\mid}{a_B^2H_B^2}\left(\frac{a_B}{a_E}\right)^2\left(\frac{a_EH_E}{a_0H_0}\right)^2\sim\exp(-2N)\left(\frac{a_EH_E}{a_0H_0}\right)^2. \tag{2.34}$$

In order to explain today's parameter density for the spatial curvature parameter, eq.(2.27) tells us that $\Omega_k<1$, so that we have

$$\exp(-2N)\left(\frac{a_EH_E}{a_0H_0}\right)^2<1, \tag{2.35}$$

which translates into the following requirement for the number of e-folds

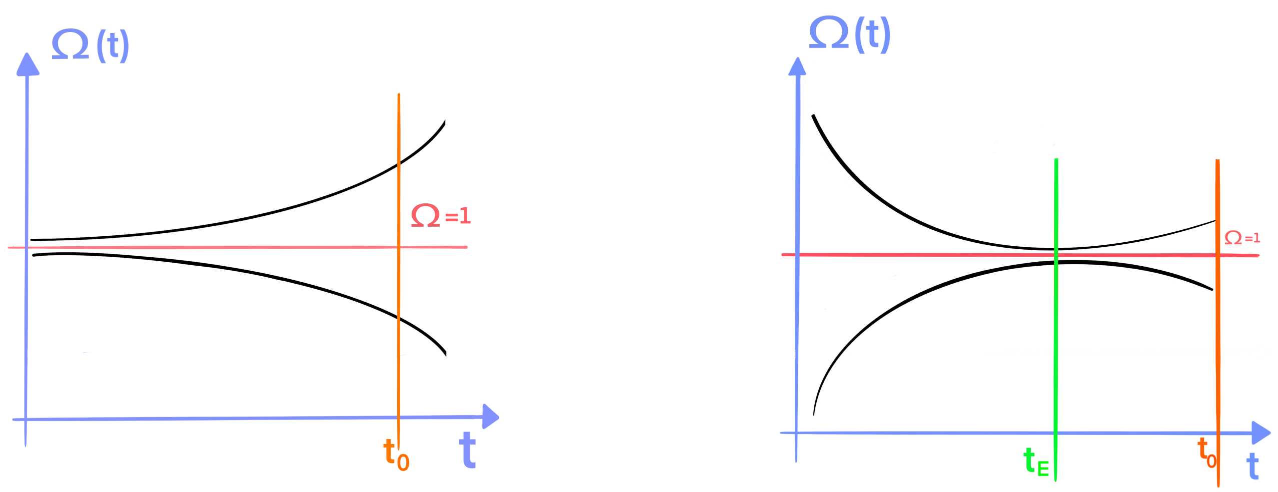
$$N>\ln\left(\frac{a_EH_E}{a_0H_0}\right). \tag{2.36}$$

Note that we have used $\Omega_k<1$ and not the real upper bound $\Omega_k<4\times10^{-3}$ since it has not a relevant impact on the number of e-folds.

There is a nice graphical interpretation of the flatness problem presented in figure 5a and 5b. In the HBB, figure 5a, we have that

$$\Omega_k=\frac{\kappa}{H^2a^2}=\kappa r_H^2(t), \tag{2.37}$$

which it's always increasing since $\dot{r}_H>0$. This implies, as we have seen, the need of very fine-tuned initial conditions in order to reproduce the data. However, if a period of accelerated expansion occurs, during this phase, the value of Ω decreases. If this accelerated expansion lasts for a sufficient duration, it becomes possible to push Ω closer to 1 with the desired accuracy, figure 5b.



(a) The Ω parameter in the *HBB* model.

(b) The Ω parameter in the Inflationary scenario.

Figure 5

2.1.3 The Monopole problem

Historically, this was the first shortcoming of the *HBB* model and was one factor leading to interest in Inflationary models. Basically, the issue involves the generation of massive particles during the Universe's early stages, resulting from the spontaneous symmetry breaking (*SSB*) of some gauge symmetry in "beyond" the standard model (*BSM*) theories. We generally refer to massive relics produced in the early universe after the spontaneous symmetry breaking of some gauge symmetries, for example, we can get [47]:

- *magnetic monopole* (0-dimension) arise from the breaking of the GUT at $T_{GUT} \sim 10^{14} \div 10^{16} GeV$ into a lower gauge symmetry including $U(1)$ gauge group,
- *cosmic strings* (1-dimensions) from the SSB of $U(1)$ gauge symmetry,
- *domain walls* (2-dimension) from the SSB of a discrete symmetry such as the shift symmetry,
- *textures* (3-dimension) from the SSB of an $SU(2)$ gauge symmetry.

In this section, we provide a qualitative example in order to briefly illustrate how this mechanism works. We discuss the formation of *domain walls* which arise from the *SBB* symmetry breaking of a shift symmetry, i.e. $\psi \longmapsto -\psi$, in a model described by the following Lagrangian

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \frac{\lambda}{4}(\psi^2 - \sigma^2)^2, \tag{2.38}$$

whose potential is sketched in blue in figure 6 with λ and σ constants. The main feature of this potential is that the choice of the vacuum, i.e. $\psi = \pm\sigma$, breaks the shift symmetry which is said to be spontaneously broken. When the scalar field is interacting with a thermal bath the potential receives temperature correction of the form [47]

$$V(\psi, T) = \frac{\lambda}{4}(\psi^2 - \sigma^2)^2 + \frac{1}{2}\alpha T^2 \psi^2, \tag{2.39}$$

where α is constant. At the critical temperature $T = T_c \equiv \frac{\sigma}{\sqrt{\alpha}}$, the second derivative of $V(\psi, T)$ at $\psi = 0$ undergoes a change in sign, causing the point to become a local maximum. As shown in the figure, we can see that if $T >> T_c$ the shift symmetry is restored conversely when $T < T_c$ the field rolls down the potential toward one of the two minima, thus it dynamically breaks the symmetry. Upon the occurrence of the Spontaneous Symmetry Breaking (SSB), the fields opt for $+\sigma$ in certain regions and $-\sigma$ in others. Since the field must vary smoothly we have regions in which the field is

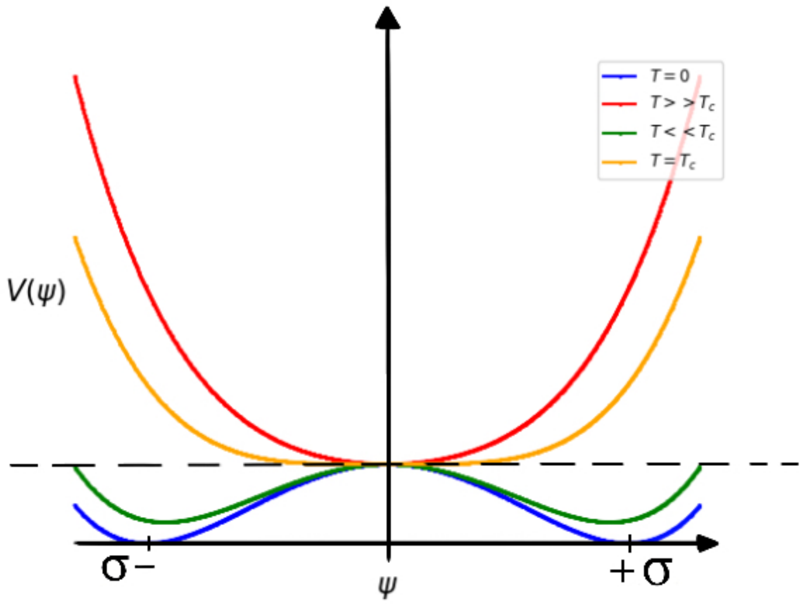


Figure 6: $V(\psi)$.

trapped in the "false vacuum" at $\psi = 0$. Thus in this region the phase transition doesn't take place and we have topological imperfections in the field configuration called *domain walls*.

The basic idea is to introduce ζ which is the typical length over which the scalar field has the same value, as shown in figure 7. If these objects are inside the horizon when produced we have that

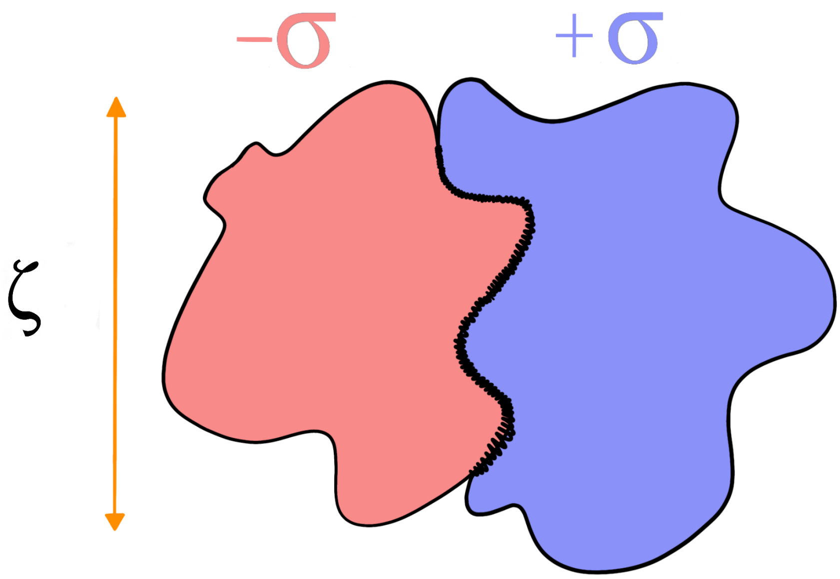


Figure 7: Correlation length ζ.

$\zeta < d_H(t)$. Since we are working in the HBB model at high temperatures the universe is radiation-dominated when the scalar field undergoes the SSB. Thus, using the particle number density we have

$$n_X \propto \zeta^{-3} < \left(\frac{1}{2t}\right)^3 \simeq H^3, \tag{2.40}$$

where Xs are the massive particle produced via the mechanism we are discussing. In radiation dominance, $\rho = \rho_\gamma$, we know [47] that

$$H \approx 1.66 g_*^{\frac{1}{2}} \frac{T^2}{M_{pl}}, \tag{2.41}$$

where g_* is the total effective degrees of freedom of particles coupled to the thermal plasma. Plugging eq.(2.41) in eq.(2.40) we get

$$n_X \leq \left(\frac{g_*^{\frac{1}{2}}}{0.6}\right)^3 \frac{T^6}{M_{pl}^3} \simeq \left(\frac{g_*^{\frac{1}{2}}}{0.6}\right)^3 \left(\frac{T}{M_{pl}}\right)^3 n_\gamma, \tag{2.42}$$

where we have used that $n_\gamma \simeq T^3$ [47]. When we evaluated this at the temperature of the phase transition, in this case, we can take T_{GUT} since this is the typical scale we have in mind we get

$$\left.\frac{n_X}{n_\gamma}\right|_{T=T_{GUT}} \leq \left(\frac{g_*^{\frac{1}{2}} T_{GUT}}{M_{pl}}\right)^3 \approx 10^{-10} \div 10^{-9} \simeq \eta = \frac{n_b}{n_\gamma}, \tag{2.43}$$

where we have take as order of magnitude $g_* \approx O(100)$ and we have introduced η, the baryon asymmetry [47]. If neither interaction nor decay occurs, the number density of domain walls scales as $a^{-3}(t)$ because of the expansion of the universe. Since the photon density scales as $a^{-3}(t)$ as well, the above result of eq.(2.43) it still holds today and in terms of energy budget it reads

$$\Omega_{X0} = \frac{\rho_{0x}}{\rho_{0c}} = \frac{m_X n_{X0}}{\rho_{0c}} = \frac{m_X n_{b0}}{\rho_{0c}} \frac{m_b}{m_b} = \Omega_{0b} \frac{m_X}{m_b}, \tag{2.44}$$

where we have used that $n_{X0} = n_{b0}$. Typically, $m_X \sim 10^{14} \div 10^{15} GeV$ [47] (the GUT scale) we immediately realize that $\Omega_{0x} >> 1$ which would overclose the universe. Without delving into specific details, it is evident that an accelerated expansion lasting more than 60 e-folds would wash away any potential contribution arising from this mechanism.

2.2 The dynamics of Inflation and the slow-roll approximation

Several authors [48] had previously suggested the possibility of an era of exponential expansion preceding the well-known radiation-dominated period. However, it was Alan Guth [49] who first emphasized the fundamental implications this epoch could entail. Guth was working at the time (early 80s) on grand unified theories (GUT) and he soon realized that in these kinds of models, scalar fields could be trapped in a local minimum of their potential. In case one of these fields dominates the energy density of the Universe, it can lead to a phase of accelerated exponential expansion, driven by a slowly varying vacuum energy density. This phase would eventually end when the scalar field starts rolling down its potential toward the true minimum. Guth's realization is that this type of mechanism simultaneously resolves the horizon, flatness, and monopole problems and it also provides the seeds, sourced by the quantum fluctuations of the scalar field itself, for the subsequent evolution of the universe.

Soon Alan Guth realized that its version of Inflation, what it's called "Old Inflation", failed to explain the formation of today's Universe. Therefore, a novel concept was put forth by Linde[50], Albrecht, and Steinhardt[51]. This idea, known as "new inflation," is based on the idea that one or more scalar fields drive an accelerated expansion phase through their gradual rolling down a potential as presented in fig.8. There are various alternatives to the standard scenario, i.e. the single-field slow-roll

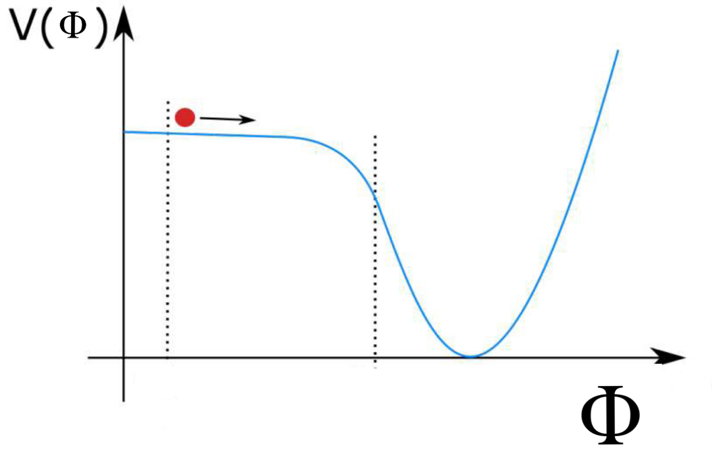


Figure 8: The scalar field ϕ rolling down its potential $V(\phi)$.

Inflation model. However, in this context, we will only focus on describing this particular possibility quantitatively. The main idea is that we need a region in which the potential is quite large but sufficiently flat as shown in fig.8. In this way, we can think of the scalar field as a single degree of freedom that slowly rolls down its "classical" potential. Quantitatively, we work with the following action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\nu\phi\partial_\mu\phi - V(\phi)\right], \tag{2.45}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, and where the first term is the standard Hilbert-Einstein action, the second term is the kinetic term for the inflaton ϕ and the last one is its potential. The main idea is that since, in today's universe, the deviation from perfect homogeneity and isotropy are small, $\frac{\Delta T}{T} \sim 10^{-5}$ in the CMB, we can split the field as

$$\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x}), \tag{2.46}$$

where $\langle\delta\phi^2(t, \mathbf{x})\rangle << \phi_0^2(t)$[7]. Thus, $\phi_0(t)$ is the homogeneous and isotropic background which we can interpret as the vacuum expectation value of the field ϕ since we take $\langle\delta\phi(t, \mathbf{x})\rangle = 0$. While $\delta\phi(t, \mathbf{x})$ are

[7]When the fluctuations are quantum fluctuations $\langle..\rangle$ has to be interpreted as correlation functions in quantum field theory (QFT) while when we observe the effects of the observables universe they are statistical averages as introduced in section 1.5.

its quantum fluctuations which provide the seed for the subsequent evolution of the Universe and it's why we expect that the splitting of eq.(2.46) to hold. We now direct our attention to the background dynamics, leaving the discussion of the quantum nature of the fluctuations for the remainder of the thesis. We stress that since we take the scalar field homogeneous and isotropic, the background dynamics is taken to be the $FLRW$. This might appear non-sense since what we want to show is that independently of the initial conditions if inflation starts we end up with today's observable universe. We postpone this discussion to section 2.2.3.

2.2.1 Background dynamics

To characterize the dynamics of a scalar field in an expanding universe, we evaluate the ϕ's stress-energy momentum tensor $T^{\mu\nu}$, which, in GR, can be evaluated by varying the action with respect to $g_{\mu\nu}$,

$$\delta \mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{1}{\sqrt{-g}}\delta\frac{\left(\frac{M_{pl}^2}{2}\sqrt{-g}R\right)}{\delta g_{\mu\nu}} + \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_\phi\right)}{\delta g_{\mu\nu}}\right]\delta g_{\mu\nu}, \tag{2.47}$$

from which we recover the Einstein field equation in the form

$$G_{\mu\nu} = \frac{2}{M_{pl}^2}\frac{\delta\left(\sqrt{-g}\mathcal{L}_\phi\right)}{\delta g_{\mu\nu}}. \tag{2.48}$$

Thus we recognize the stress-energy momentum tensor of the scalar field as

$$T^{\mu\nu} = -2\sqrt{-g}\frac{\delta S}{\delta g_{\mu\nu}} = -2\frac{\partial \mathcal{L}_\phi}{\partial g_{\mu\nu}} + g^{\mu\nu}\mathcal{L}_\phi = \partial^\mu\phi\partial^\nu\phi + g^{\mu\nu}\left(-\frac{1}{2}g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - V(\phi)\right), \tag{2.49}$$

where we have used $\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}$. Extracting the background value of eq.(2.49) we immediatly get

$$(T_0)^0{}_0 = -\bar{\rho} = -\dot{\phi}_0^2 + \left(\frac{\dot{\phi}_0^2}{2} - V(\phi)\right) = -\left(\frac{1}{2}\dot{\phi}_0^2 + V(\phi)\right), \tag{2.50}$$

$$(T_0)^i{}_j = \bar{p}\delta^i{}_j = \delta^i{}_j\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right), \tag{2.51}$$

from which we get the w as

$$w = \frac{\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)}{\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)}. \tag{2.52}$$

Thus we can write the Friedmann equation in the form

$$H^2 = \frac{1}{3M_{pl}^2}\left[\frac{1}{2}\dot{\phi}_0^2 + V(\phi)\right], \tag{2.53}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{pl}^2}(1+3w)\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right). \tag{2.54}$$

Please take note that in the action given by equation (2.45), we have “two" degrees of freedom: $\phi(t,\mathbf{x})$ and $g_{\mu\nu}(t,\mathbf{x})$. As a result, we can obtain another equation for the background by varying the action with respect to $\phi_0(t)$. Thus, we obtain

$$\delta \mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - \frac{\partial V}{\partial\phi}\delta\phi\right), \tag{2.55}$$

from which we can obtain the Klein-Gordon equation of motion (EoM)

$$\Box\phi - \frac{\partial V(\phi)}{\partial\phi} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - \frac{\partial V(\phi)}{\partial\phi} = 0. \tag{2.56}$$

The background EoM reads

$$\Box\phi_0 \left.\frac{\partial V(\phi)}{\partial\phi}\right|_{\phi=\phi_0} = -\frac{1}{a^3}\frac{\partial}{\partial_t}\left(a^3\dot{\phi}\right) - \left.\frac{\partial V(\phi)}{\partial\phi}\right|_{\phi=\phi_0} = -\ddot{\phi}_0 - 3H\dot{\phi}_0 - \left.\frac{\partial V(\phi)}{\partial\phi}\right|_{\phi=\phi_0} = 0. \tag{2.57}$$

2.2.2 The slow roll conditions

The so-called *slow-roll* conditions are the one that allows the potential to be sufficiently flat in order for Inflation to take place. The **first** slow-roll parameter condition is

$$\epsilon = -\frac{\dot{H}}{H^2} << 1, \tag{2.58}$$

which basically states that the variation of the Hubble parameter, $\dot{H}$, in a Hubble time, $\frac{1}{H}$, is negligible with respect to H. This means that the Hubble constant is almost constant during Inflation. This condition can be expressed in a more illuminating way using that

$$2H\dot{H} = \frac{1}{3M_{pl}^2}\left[\ddot{\phi}_0\dot{\phi}_0 + \left.\frac{\partial V(\phi)}{\partial\phi}\right|_{\phi=\phi_0}\dot{\phi}_0\right] = -\frac{H}{M_{pl}^2}\dot{\phi_0}^2, \tag{2.59}$$

from which we understand the $\dot{H} < 0$ and this explain the minus sign in eq.(2.58). Finally, we can write

$$\epsilon = \frac{1}{2M_{pl}^2}\dot{\phi_0}^2 3M_{pl}^2\frac{1}{\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)} = \frac{\dot{\phi_0}^2 3}{2\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)} << 1, \tag{2.60}$$

which implies $\dot{\phi_0}^2 << V(\phi)$. This allows us to get

$$\epsilon \approx \frac{3}{2}\frac{\dot{\phi}_0^2}{V(\phi)}. \tag{2.61}$$

The **second** slow roll condition corresponds to the requirement

$$\eta = -\frac{\ddot{\phi}_0}{H\dot{\phi}} << 1, \tag{2.62}$$

which is equivalent to saying that the fractional change of $\dot{\phi}_0$ is negligible in an expansion time, i.e. the potential is sufficiently flat. Plugging this condition in the equation of motion we get

$$\dot{\phi}_0 \approx -\frac{V^{'}(\phi_0)}{3H}, \text{ with } V^{'}(\phi_0) = \left.\frac{\partial V(\phi)}{\partial\phi}\right|_{\phi=\phi_0}. \tag{2.63}$$

Now let's see how this condition is related to the potential by taking

$$\ddot{\phi}_0 = -\frac{V^{''}(\phi_0)\dot{\phi}_0}{3H} + \frac{V^{'}(\phi_0)}{3}\frac{\dot{H}}{H^2} = -\frac{V^{''}(\phi_0)\dot{\phi}_0}{3H} + \frac{V^{'}(\phi_0)}{3}\frac{\dot{H}}{H^2}, \tag{2.64}$$

which we can plug in eq.(2.62) to get

$$\eta = \frac{V^{''}(\phi_0)}{3H^2} + \frac{V^{'}(\phi_0)}{3H\dot{\phi}_0}\epsilon \approx \eta_V - \epsilon, \tag{2.65}$$

where we have used that $\frac{V^{'}(\phi_0)}{3H\dot{\phi}_0} \approx -1$ and where we have defined

$$\eta_V \equiv \frac{V^{''}(\phi_0)}{3H^2}, \tag{2.66}$$

which clearly satisfies $\eta_V << 1$. We conclude this section by mentioning a series of important results which we are not going to discuss:

- The condition $\eta << 1$ is imposed [33] to ensure that inflation lasts for a significant number of e-folds in order to solve the shortcomings of the HBB model.
- Inflation ends when $\epsilon, \eta, \eta_V \sim O(1)$; the last condition is equivalent to have $V'' \sim H^2$. From this moment, the **reheating phase** starts and the scalar field goes to the true minimum [10]. V'' dominates over H^2, the scalar field behaves like non-relativistic ordinary matter and it decays into radiation, with a decay rate Γ_ϕ. Then, we recover the standard radiation-dominated epoch of the HBB model.
- In principle one can construct an infinite tower of slow-roll parameters, which ϵ and η are the first ones, using successive derivatives of the potential. For example, we can define $\xi^2 = \left(\frac{1}{4\pi G}\right)^2 \left(\frac{V'V'''}{V^2}\right)^2$, which is second order in the slow-roll parameters. We do not discuss these parameters since we'll work in this thesis at the lowest order in slow-roll parameters.
- It can be demonstrated that the temporal derivatives of ϵ, η, and η_V are of second order in slow-roll. As a result, we can treat them as constants at first order.
- In the next section we'll use that $H << M_{pl}$, which can be derived from the fact that, in order to avoid quantum gravity, we impose $V << M_{pl}^4$.

2.2.3 Final remarks

The single-field slow-roll models are divided into different classes based on the excursion of the scalar field during Inflation:

- *Large-field models* characterized by $\Delta\phi > M_{pl}$,
- *Small-field models* characterized by $\Delta\phi < M_{pl}$,
- *Hybrid models* which share some features of the two previous classes.

While we won't delve into an extensive discussion on this topic, it's worth mentioning that there are observables associated with the scalar field's excursion during inflation. This approach can efficiently provide insights into the shape of the potential involved.

Lastly, we mention the concept known as the *cosmic no-hair principle*, which essentially states that regardless of the initial conditions (which could be anisotropic and inhomogeneous), the universe will still undergo inflation and end up in a Robertson-Walker (RW) scenario. A priori it may not be apparent that inflation can occur when starting with a non-FLRW metric and it's not obvious that the anisotropies and inhomogeneities are washed away by this mechanism. Nonetheless, in a wide range of diverse scenarios, it is possible to demonstrate that everything works as expected. An illustrative example regarding Bianchi models can be found in [47]. However, due to the absence of a completely general proof, this is regarded as a principle.

2.3 Adopted formalism to the computation of primordial power spectra

Now, the goal of the remaining part of the chapter is the computation of the primordial power spectra of the gauge invariant curvature perturbation on uniform energy density hypersurfaces ζ and the gravitational waves (GWs) in a fully consistent manner, considering both inflaton and metric perturbations. For the scalar perturbations, one could choose not to consider metric perturbations and instead work in a De-Sitter background, calculating the primordial power spectrum of the inflaton perturbation. Then, using $\delta\phi = -\frac{\dot{\phi}}{H}\zeta$ [15], one can switch to the curvature perturbation and get its power spectrum on super-horizon scales, which is what we are interested in. In this scenario, the field can be canonically quantized in a De-Sitter background, following the explanation provided in [52]. Although the method of working solely with inflaton perturbations in a De-Sitter background is insightful and commonly the first approach used for explaining the guiding principles of these analyses, our preference is to directly address the completely consistent case, wherein we also incorporate metric perturbations.

In this context, we work in the ADM formalism, introduced in appendix C.3, and we write the metric tensor in the form of eq.(C.38)

$$g_{\alpha\beta} = \begin{pmatrix} -(N^2 - N_i N^i) & N_i \\ N_i & h_{ij} \end{pmatrix}, \quad g^{\alpha\beta} = \begin{pmatrix} -N^{-2} & \frac{N^i}{N^2} \\ \frac{N^i}{N^2} & h^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}, \tag{2.67}$$

where N and N^i are respectively the lapse and shift functions introduce in section C.3. In the **unitary gauge**, we have

$$\delta\phi(t, \mathbf{x}) = 0, \quad h_{ij} = a^2(t) e^{2\zeta(t,\mathbf{x})} \left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma^k{}_j \right), \quad \partial_i \gamma^i{}_j = 0, \quad \gamma^i{}_i = 0, \tag{2.68}$$

where ζ and γ are first order quantities and are the "true" degrees of freedom we are dealing with. I want to stress here that in this gauge ζ coincides with the gauge invariant curvature perturbation. The notation can clearly lead to a misunderstanding since we are calling with the same name two different quantities. If we recall the definition of the gauge invariant curvature perturbation at linear order we have

$$\zeta = -\phi - \frac{\nabla^2 \chi^{||}}{6} - \frac{\dot{a}}{a}\frac{\delta\rho}{\rho_0}. \tag{2.69}$$

In the unitary gauge, we have

$$\phi = -\zeta, \quad \chi^{||} = 0, \quad \delta\rho = \dot{\phi}_0 \delta\dot{\phi} + V' \delta\phi = 0, \tag{2.70}$$

where the expression of the matter density perturbation has been derived from the perturbed version of eq.(2.49). Thus it's clear that the ζ we have introduced in the metric is the gauge invariant curvature perturbation on uniform energy density hypersurfaces. The other gauge we are going to adopt in the master's thesis is the spatially flat gauge[8],

$$\delta\phi(t, \mathbf{x}), \quad h_{ij} = a^2(t) \left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma^k{}_j \right), \quad \partial_i \gamma^i{}_j = 0, \quad \gamma^i{}_i = 0. \tag{2.71}$$

In this particular gauge, it is possible to demonstrate that the gauge-invariant curvature perturbation is $\delta\phi = -\frac{\dot{\phi}}{H}\zeta$ [15]. It is important to emphasize that this ζ corresponds to the gauge-invariant curvature perturbation, which coincidentally, in the spatially flat gauge, equals the scalar field presented in the metric perturbation. We make this clarification because, without it, there might be a misconception that the field ζ in the spatially flat gauge is the gauge invariant curvature perturbation. It's the gauge invariant curvature perturbation which in that specific gauge is equal to the scalar field ζ introduced in the metric perturbation. We stress that if ζ in eq.(2.68) is the "true" gauge invariant curvature perturbation, we can set that ζ to zero, since it's gauge-invariant and in the spatially flat gauge can be set to zero; thus it would always be zero in any other gauge. Even if this can lead to confusion we are going to call ζ in (2.68) as the gauge-invariant curvature perturbation having mind what we have discussed here. In what follows we are going to derive the powers spectrum of the curvature perturbation in the unitary gauge but in section 5.5 we present the same computation in the spatial flat gauge and we'll show that we obtain the same results. This is not particularly insightful but it's useful to verify that by adopting two different gauges we obtain the same result.

Before going to the actual computation we would like to comment on the procedure used to fix the gauge up to the n^{th} order. Eq.(2.68) is the first example of a second-order gauge, and now we proceed to make certain observations pertaining to gauge fixing up to the second order. First of all, the Inflaton perturbation it's a scalar and can be decomposed as

$$\delta\phi(t, \mathbf{x}) = \delta^{(1)}\phi(t, \mathbf{x}) + \delta^{(2)}\phi(t, \mathbf{x}) + ... \tag{2.72}$$

When we establish a gauge up to the n^{th} order, we can conceive of the ability to nullify, at any order up to n^{th}, two scalars and one vector. This is feasible due to the introduction of a new vector ζ related

[8] We use this gauge in chapter 5

to gauge transformations at each order, as elaborated in section 1.2. Please note that this ζ is the vector associated with the gauge transformation. Therefore, in the unitary gauge scenario, we have nullified all the $\delta^{(r)}\phi(t,\mathbf{x})$, $\chi^{(r)\|}$, and $\chi_i^{(r)\perp}$ components, following the notation introduced in equation (1.57). Furthermore, the metric perturbations involving the trace of the spatial metric, i.e. $e^{2\zeta(t,\mathbf{x})}$, have not been expanded. This choice is driven by the convenience of expanding these perturbations once we have completed the computations, as it leads to simplifications in the intermediate steps. This expansion of the metric is more convenient as we'll see when we have to quantize the system. Regarding tensor perturbations, we have written the transverse and traceless part of the metric as $\exp(\gamma)_{ij}$, which up to second order gives what we have written in eq.(2.68). This parametrization of the metric tensor both for scalar and tensor [15] is convenient when we have to quantize the system.

The action of the problem is one of the single-field slow-roll model which can be written in the unitary gauge within the ADM formalism as

$$S = \frac{1}{2}\int d^4x\sqrt{h}N\left[{}^{(3)}R + 2X - 2V\right] + \frac{1}{2}\int d^4x\sqrt{h}N^{-1}\left[E^{ij}E_{ij} - E^2\right], \tag{2.73}$$

where $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the kinetic term of the inflaton field and where we set $M_{pl}^{-2} = 8\pi G = 1$. In the unitary gauge, we have

$$X = \frac{1}{2}N^{-2}\dot{\phi}^2, \tag{2.74}$$

where $\dot{\phi}$ represents the time derivative of the inflaton field with respect to cosmic time. For simplicity, we will omit the subscript 0 going forward. It is worth noting that the inflaton background value is simply a temporal function that appears in the metric as $a(t)$. We recall that

$$E_{ij} = NK_{ij} = \frac{1}{2}\left[\dot{h}_{ij} - D_iN_j - D_jN_i\right], \tag{2.75}$$

where D_i is the covariant derivative operator on the spatial hypersurfaces (see appendix C.3).

2.3.1 Constraint equations

In order to find the action for dynamical degrees of freedom, ζ and γ, it's necessary to solve for N and N^i, which are constraints as explained in section C.3, and plug the results back into the action. We start by computing the constraint equation for the lapse function. Firstly, it is important to observe that there are no terms in the Lagrangian that depend on derivatives of the lapse function. Consequently, our focus will be solely on computing the derivative of the Lagrangian with respect to N, as it is the only thing we need for obtaining the constraint equations

$$\frac{\partial\mathcal{L}}{\partial N} = 0. \tag{2.76}$$

The three-tensor E_{ij}, its trace, the three-dimensional Ricci scalar, the inflaton potential, and the determinant of the three metric do not vary with respect to N. Therefore, the only non-straightforward derivative we need to compute is the one corresponding to the kinetic term, which is given by:

$$\frac{\partial X}{\partial N} = \frac{\partial}{\partial N}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] = \frac{\partial}{\partial N}\left[-\frac{1}{2}g^{00}\partial_0\phi\partial_0\phi\right] = \frac{\partial}{\partial N}\left[\frac{\dot{\phi}^2}{2N^2}\right] = -\frac{\dot{\phi}^2}{N^3}. \tag{2.77}$$

Thus we can obtain the constraint equation as

$$0 = \frac{\partial\mathcal{L}}{\partial N} = \frac{\partial}{\partial N}\left[\frac{1}{2}\sqrt{h}N\left[{}^{(3)}R + 2X - 2V\right] + \frac{1}{2}\sqrt{h}N^{-1}\left[E^{ij}E_{ij} - E^2\right]\right] \tag{2.78}$$

$$= \frac{1}{2}\sqrt{h}\left[{}^{(3)}R + 2X - 2V\right] + \frac{1}{2}\sqrt{h}N\left[-2\frac{\dot{\phi}^2}{N^3}\right] - \frac{1}{2N^2}\sqrt{h}\left[E^{ij}E_{ij} - E^2\right]\Big], \tag{2.79}$$

which, using that

$$2X + 2N\frac{\partial X}{\partial N} = \frac{\dot{\phi}^2}{2N^2} - 2N\frac{\dot{\phi}^2}{N^3} = 0, \tag{2.80}$$

can be simplified as

$$^{(3)}R - 2V - \frac{\dot{\phi}^2}{N^2} - \frac{1}{N^2}\left[E^{ij}E_{ij} - E^2\right] = 0. \tag{2.81}$$

Now, our attention turns to the computation of the constraint equation for the shift function N^i, taking into account that $^{(3)}R$, V, N, and $\dot{\phi}$ are all independent of the shift function. Thus, applying the variational principle we get

$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = \frac{1}{2}\int d^3xdt\sqrt{h}N^{-1}\left\{E_{ij}E_{kl}h^{ik}h^j - h^{ij}E_{ij}\right\}\Big|_{\lambda=0} = \int d^3xdt\sqrt{h}N^{-1}(E^{ij} - Eh^{ij})\left.\frac{dE_{ij}}{d\lambda}\right|_{\lambda=0}, \tag{2.82}$$

where we have used the notation for the variational principle described in [13] and where $\left.\frac{dE_{ij}}{d\lambda}\right|_{\lambda=0} = \frac{1}{2}\left[-D_i\left.\frac{dN_j}{d\lambda}\right|_{\lambda=0} - D_j\left.\frac{dN_i}{d\lambda}\right|_{\lambda=0}\right]$. Finally, using that both E^{ij} and h^{ij} are symmetric, we can write up to a surface term

$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = -2\int d^3xdt\left.\frac{dN_i}{d\lambda}\right|_{\lambda=0} D_j\left\{\frac{\sqrt{h}}{N}(E^{ij} - Eh^{ij})\right\}. \tag{2.83}$$

Thus, we can write the two constraints equations as [15]

$$\begin{cases} 0 & = {}^{(3)}R - 2V - \frac{\dot{\phi}^2}{N^2} - \frac{1}{N^2}\left[E^{ij}E_{ij} - E^2\right] \\ 0 & = D_j\left\{\frac{\sqrt{h}}{N}(E^{ij} - Eh^{ij})\right\} \end{cases}. \tag{2.84}$$

Now, the task at hand is to solve these equations. However, an analytical solution is not readily available, necessitating a step-by-step approach to solve them order by order. As we proceed, it's essential to note that we are interested in expanding actions up to the third order, thus as noted in [15] and demonstrated in [53] we only need first-order solutions to the constraint equations. This is applicable to the single-field slow-roll model, Chern-Simons, and scalar chiral tensor theories of gravity.

At this stage, we need to compute all the terms present in the equations. It is crucial to emphasize that, at the first order, any non-zero scalar or vector arising from the tensor part cannot form, as the tensor indices must be contracted either with a spatial derivative or the Kronecker delta. Thus we can disregard the tensor in what follows. Since we are interested in the solution for the lapse and the shift function at first order we introduce their expansions

$$N = 1 + 2\phi, \quad N^i = \partial^i B + \tilde{N}^i \quad \text{with } \partial_j\tilde{N}^j = 0, \tag{2.85}$$

where we have decomposed the shift function in a scalar and vector part which we can be always done as shown in [36].

With all the necessary components in place, we can now proceed to solve for the desired values. To begin, we compute the three-dimensional Christoffel symbols as

$$^{(3)}\Gamma^k_{ij} = \frac{1}{2}h^{km}(\partial_j h_{mi} + \partial_i h_{mj} - \partial_m h_{ij}) = \partial_j\zeta\delta^k{}_i + \partial_i\zeta\delta^k{}_j - \partial^k\zeta\delta_{ij}, \tag{2.86}$$

where we have retained only first order terms in tensor perturbation while we do not expand the scalar part for simplicity. Please note that the symbols are of first order. Now we can evaluate the Ricci tensor as

$$R_{ij} = \partial_k{}^{(3)}\Gamma^k_{ij} - \partial_j{}^{(3)}\Gamma^k_{ik} + {}^{(3)}\Gamma^k_{ij}{}^{(3)}\Gamma^l_{kl} - {}^{(3)}\Gamma^l_{ik}{}^{(3)}\Gamma^k_{jl}. \tag{2.87}$$

Evaluating each term in eq.(2.87) we get

$$\partial_k {}^{(3)}\Gamma^k_{ij} = 2\partial_i\partial_j\zeta - \delta_{ij}\nabla^2\zeta, \tag{2.88}$$

$$\partial_j {}^{(3)}\Gamma^k_{ik} = 3\partial_i\partial_j\zeta, \tag{2.89}$$

$${}^{(3)}\Gamma^k_{ij}{}^{(3)}\Gamma^l_{kl} = 6\partial_i\zeta\partial_j\zeta - \delta_{ij}3\partial^k\partial_k\zeta, \tag{2.90}$$

$${}^{(3)}\Gamma^l_{ik}{}^{(3)}\Gamma^k_{jl} = 5\partial_i\zeta\partial_j\zeta - 2\delta_{ij}\partial^k\partial_j\zeta, \tag{2.91}$$

and summing everything together we obtain

$${}^{(3)}R_{ij} = -\partial_i\partial_j\zeta + \partial_i\zeta\partial_j\zeta - \delta_{ij}\nabla^2\zeta - \delta_{ij}\partial^k\zeta\partial_k\zeta. \tag{2.92}$$

Now we can easily evaluate the Ricci scalar up to first order as

$${}^{(3)}R = h^{ij\,(3)}R_{ij} = -\frac{4}{a^2}\nabla^2\zeta, \tag{2.93}$$

from which it becomes evident that ζ in this gauge invariant curvature perturbation introduced in section 1.4. Next, we focus on computing the E_{ij} tensor and we get

$${}^{(1)}E_{ij} = \frac{1}{2}\left[\dot{h}_{ij} - D_i\tilde{N}_j - D_j\tilde{N}_i\right] = \frac{1}{2}\left[\dot{h}_{ij} - \partial_i\tilde{N}_j - \partial_j\tilde{N}_i\right] = \frac{1}{2}\left[(2a\dot{a}e^{2\zeta} + 2\dot{\zeta}a^2e^{2\zeta})\delta_{ij} - \partial_i\tilde{N}_j - \partial_j\tilde{N}_i\right] \tag{2.94}$$

$$= a^2\left[\left(\frac{\dot{a}}{a}(1+2\zeta) + \dot{\zeta}\right)\delta_{ij} - \partial_i\partial_j B - \frac{1}{2}\left(\partial_j\tilde{N}_i + \partial_i\tilde{N}_j\right)\right], \tag{2.95}$$

where in the first passage we have considered that the Christoffel symbols are first order. Regarding the trace of this tensor, we immediately get

$$E = e^{-2\zeta}\delta^{ij}\left[(\frac{\dot{a}}{a} + \dot{\zeta})e^{2\zeta}\delta_{ij} - \partial_i\partial_j B - \frac{1}{2}\left(\partial_j\tilde{N}_i + \partial_i\tilde{N}_j\right)\right] = e^{-2\zeta}\left[3(\frac{\dot{a}}{a} + \dot{\zeta})e^{2\zeta} - \nabla^2 B\right] \approx \left[3(\frac{\dot{a}}{a} + \dot{\zeta}) - \nabla^2 B\right], \tag{2.96}$$

where in the last passage only first order terms have been considered. Since in eq.(2.84) we need E^2 we need

$$E^2 = 9\left(\frac{\dot{a}}{a}\right)^2 + 18\frac{\dot{a}}{a}\dot{\zeta} - 6\frac{\dot{a}}{a}\nabla^2 B. \tag{2.97}$$

Now we can evaluate the E_{ij} tensor with one index covariant and the other one contravariant

$$E^i{}_j = a^{-2}e^{-2\zeta}\delta^{im}a^2\left[\left(\frac{\dot{a}}{a}(1+2\zeta) + \dot{\zeta}\right)\delta_{mj} - \partial_m\partial_j B - \frac{1}{2}\left(\partial_j\tilde{N}_m + \partial_m\tilde{N}_j\right)\right] \tag{2.98}$$

$$= \left[\left(\frac{\dot{a}}{a} + \dot{\zeta}\right)\delta^i{}_j - \partial^i\partial_j B - \frac{1}{2}\left(\partial_j\tilde{N}^i + \partial^i\tilde{N}_j\right)\right], \tag{2.99}$$

while the completely covariant version reads

$$E^{ij} = a^{-4}e^{-4\zeta}\delta^{il}\delta^{jm}a^2\left[\left(\frac{\dot{a}}{a}(1+2\zeta) + \dot{\zeta}\right)\delta_{lm} - \partial_l\partial_m B - \frac{1}{2}\left(\partial_l\tilde{N}_m + \partial_m\tilde{N}_l\right)\right] \tag{2.100}$$

$$= a^{-2}\left[\left((\frac{\dot{a}}{a}(1-2\zeta) + \dot{\zeta})\right)\delta^{ij} - \partial^i\partial^j B - \frac{1}{2}\left(\partial^i\tilde{N}^j + \partial^j\tilde{N}^i\right)\right]. \tag{2.101}$$

Now, we can evaluate $E^{ij}E_{ij}$ while considering that $(\partial_i\tilde{N}_j + \partial_j\tilde{N}_i)$ part cannot contribute at first order since they must be contracted with a Kronecker delta

$$E^{ij}E_{ij} = \left[\left(\frac{\dot{a}}{a}(1-2\zeta) + \dot{\zeta}\right)\delta^{ij} - \partial^i\partial^j B\right]\left[\left(\frac{\dot{a}}{a}(1+2\zeta) + \dot{\zeta}\right)\delta_{ij} - \partial_{ij}B\right] \tag{2.102}$$

$$= 3\left(\frac{\dot{a}}{a}(1-2\zeta) + \dot{\zeta}\right)\left(\frac{\dot{a}}{a}(1+2\zeta) + \dot{\zeta}\right) - 2\left(\frac{\dot{a}}{a} + \dot{\zeta}\right)\nabla^2 B \tag{2.103}$$

$$\approx 3\left(\frac{\dot{a}}{a}\right)^2 + 6\dot{\zeta}\frac{\dot{a}}{a} - 2\frac{\dot{a}}{a}\nabla^2 B. \tag{2.104}$$

Before proceeding in solving eqs.(2.84), we evaluate two fundamental contributions which we need to rewrite the constraint equations. The first one is

$$E^{ij}E_{ij} - E^2 = 3\left(\frac{\dot{a}}{a}\right)^2 + 6\dot{\zeta}\frac{\dot{a}}{a} - 2\frac{\dot{a}}{a}\nabla^2 B - \left[9\left(\frac{\dot{a}}{a}\right)^2 + 18\frac{\dot{a}}{a}\dot{\zeta} - 6\frac{\dot{a}}{a}\nabla^2 B\right] \tag{2.105}$$

$$= -6\left(\frac{\dot{a}}{a}\right)^2 - 12\frac{\dot{a}}{a}\dot{\zeta} + 4\frac{\dot{a}}{a}\nabla^2 B, \tag{2.106}$$

while the second is

$$E^i{}_j - E\delta^i{}_j = \left[\left(\frac{\dot{a}}{a} + \dot{\zeta}\right)\delta^i{}_j - \partial^i\partial_j B - \frac{1}{2}\left(\partial_j \tilde{N}^i + \partial^i \tilde{N}_j\right)\right] - 3\left[\frac{\dot{a}}{a} + \dot{\zeta}\right]\delta^i{}_j + \nabla^2 B\delta^i{}_j \tag{2.107}$$

$$= -2\left[\left(\frac{\dot{a}}{a} + \dot{\zeta}\right)\delta^i{}_j\right] - \frac{1}{2}\left(\partial_j \tilde{N}^i + \partial^i \tilde{N}_j\right) - \partial^i\partial_j B + \nabla^2 B\delta^i{}_j. \tag{2.108}$$

Now we are ready to rewrite the second eq of eqs.(2.84) considering that the metric and its determinant are covarianlty constant

$$0 = D_i\left[\frac{1}{\tilde{N}}(E^i{}_j - E\delta^i{}_j)\right] \tag{2.109}$$

$$= \partial_i\left[(1-2\phi)\left\{-2\left[\left(\frac{\dot{a}}{a} + \dot{\zeta}\right)\delta^i{}_j\right] - \frac{1}{2}\left(\partial_j \tilde{N}^i + \partial^i \tilde{N}_j\right) - \partial^i\partial_j B + \nabla^2 B\delta^i{}_j\right\}\right] \tag{2.110}$$

$$= \partial_i\left\{-2\left[\left(\frac{\dot{a}}{a}(1-2\phi) + \dot{\zeta}\right)\delta^i{}_j\right] - \frac{1}{2}\nabla^2 \tilde{N}_j\right\} \tag{2.111}$$

$$= \partial_i\left\{-2\left[\left(-2\frac{\dot{a}}{a}\phi + \dot{\zeta}\right)\delta^i{}_j\right] - \frac{1}{2}\nabla^2 \tilde{N}_j\right\}. \tag{2.112}$$

While the first constraint equation, using that $\frac{\dot{a}}{a} = H$, becomes

$$0 = -\frac{4}{a^2}\nabla^2\zeta - 2(3H^2 - \frac{1}{2}\dot{\phi_0}^2) - (1-4\phi)\left[-6H^2 - 12H\dot{\zeta} + 4H\nabla^2 B\right] - (1-4\phi)\dot{\phi}^2 \tag{2.113}$$

$$= -\frac{4}{a^2}\nabla^2\zeta - \left[-12H\dot{\zeta} + 4H\nabla^2 B\right] - 24\phi H^2 + 4\phi\dot{\phi_0}^2, \tag{2.114}$$

which dividing by four becomes

$$0 = -\nabla^2\left[a^{-2}\zeta + HB\right] + 3H\dot{\zeta} - 6\phi H^2 + \phi\dot{\phi_0}^2. \tag{2.115}$$

Thus, we have to solve the following system of equations

$$\begin{cases} 0 & = -\nabla^2\left[a^{-2}\zeta + HB\right] + 3H\dot{\zeta} - 6\phi H^2 + \phi\dot{\phi_0}^2 \\ 0 & = \partial_i\left\{-2\left[\left(-2\frac{\dot{a}}{a}\phi + \dot{\zeta}\right)\delta^i{}_j\right] - \frac{1}{2}\nabla^2\tilde{N}_j\right\} \end{cases}. \tag{2.116}$$

The solution of this system is

$$\begin{cases} \phi & = \frac{1}{2}\frac{\dot{\zeta}}{H} \\ \tilde{N}^i & = 0 \\ B & = -a^{-2}H^{-1}\zeta + \chi \\ \nabla^2\chi & = \frac{\dot{\phi_0}^2\zeta}{2H^2} \end{cases}, \tag{2.117}$$

which is identical to the one presented in [15].

2.4 The scalar power spectrum

The goal of this section is to evaluate the primordial scalar power spectrum of the curvature perturbation ζ in a fully consistent manner. To start, we must assess the equations of motion (EoM), which requires computing the action at the second order in ζ. The constraints can be substituted with their first-order versions, as explained in the previous section.

2.4.1 The second order action

We recall that the action is

$$S = \frac{1}{2}\int d^4x\sqrt{h}\left[N\left({}^{(3)}R + 2X - 2V\right) + N^{-1}\left(E^{ij}E_{ij} - E^2\right)\right]. \tag{2.118}$$

To begin, we compute the three-dimensional Ricci tensor with an analogous computation tot the one presented in the previous section

$${}^{(3)}R_{ij} = -\partial_i\partial_j\zeta + \partial_i\zeta\partial_j\zeta - \delta_{ij}\nabla^2\zeta - \delta_{ij}\partial^k\zeta\partial_k\zeta. \tag{2.119}$$

The Ricci scalar becomes

$${}^{(3)}R = a^{-2}e^{-2\zeta}\left[-\nabla^2\zeta + \partial_i\zeta\partial^i\zeta - 3\nabla^2\zeta - 3\partial^k\zeta\partial_k\zeta\right] = a^{-2}e^{-2\zeta}\left[-4\nabla^2\zeta - 2\partial^k\zeta\partial_k\zeta\right]. \tag{2.120}$$

Regarding the extrinsic curvature-related tensor we have

$$E_{ij} = \frac{1}{2}\left[(2\dot{\zeta}a^2 + 2\dot{a}a)e^{2\zeta}\delta_{ij} - 2a^2\partial_i\partial_j B + 2a^2\partial_k B^{(3)}\Gamma^k_{ij}\right] \tag{2.121}$$

$$= a^2\left[(\dot{\zeta} + \frac{\dot{a}}{a})e^{2\zeta}\delta_{ij} - \partial_i\partial_j B + \partial_k B(\partial_j\zeta\delta^k{}_i + \partial_i\zeta\delta^k{}_j - \partial^k\zeta\delta_{ij})\right] \tag{2.122}$$

$$= a^2\left[(\dot{\zeta} + \frac{\dot{a}}{a})e^{2\zeta}\delta_{ij} - \partial_i\partial_j B + (\partial_i B\partial_j\zeta + \partial_j B\partial_i\zeta - \partial_k B\partial^k\zeta\delta_{ij})\right], \tag{2.123}$$

where we have used that $N_i = {}^{(3)}g_{ij}\partial^j B = a^2\partial_i B$ at first order. The fully contravariant version of the extrinsic curvature tensor reads

$$E^{ij} = a^{-2}\left[(\dot{\zeta} + \frac{\dot{a}}{a})\delta^{ij}e^{-2\zeta} - \partial^i\partial^j Be^{-4\zeta} + e^{-4\zeta}(\partial^i B\partial^j\zeta + \partial^j B\partial^i\zeta - \partial_k B\partial^k\zeta\delta^{ij})\right]. \tag{2.124}$$

Thus we can compute at second-order

$$E^{ij}E_{ij} \approx 3(\dot{\zeta} + \frac{\dot{a}}{a})^2 + \partial_i\partial_j B\partial^i\partial^j B - 2e^{-2\zeta}(\dot{\zeta} + \frac{\dot{a}}{a})\nabla^2 B - \frac{\dot{a}}{a}2\partial_k B\partial^k\zeta, \tag{2.125}$$

and, we can evaluate

$$E^2 = \left\{e^{-2\zeta}\left[(\dot{\zeta} + \frac{\dot{a}}{a})e^{2\zeta}3 - \nabla^2 B - (2\partial^k B\partial_k\zeta - 3\partial_k B\partial^k\zeta)\right]\right\}^2 \tag{2.126}$$

$$\approx +9(\dot{\zeta} + \frac{\dot{a}}{a})^2 + \left(\nabla^2 B\right)^2 - 6(\dot{\zeta} + \frac{\dot{a}}{a})\frac{\nabla^2 B}{e^{2\zeta}} - 6\partial^k B\partial_k\zeta H. \tag{2.127}$$

Finally, we can compute

$$E^{ij}E_{ij} - E^2 \approx -6(\dot{\zeta} + \frac{\dot{a}}{a})^2 + \partial_i\partial_j B\partial^i\partial^j B - \frac{2\nabla^2 B}{e^{2\zeta}}(\dot{\zeta} + \frac{\dot{a}}{a}) - \left(\nabla^2 B\right)^2 + 6(\dot{\zeta} + \frac{\dot{a}}{a})\frac{\nabla^2 B}{e^{2\zeta}} + 4\partial^k B\partial_k\zeta H \tag{2.128}$$

$$\approx -6(\dot{\zeta} + H)^2 + \partial_i\partial_j B\partial^i\partial^j B + 4e^{-2\zeta}\left[\dot{\zeta} + \frac{\dot{a}}{a}\right]\nabla^2 B - \left(\nabla^2 B\right)^2 + 4\partial^k B\partial_k\zeta H. \tag{2.129}$$

Now, using that $\sqrt{h} = a^3e^{3\zeta}$ we can rewrite the action as

$$S^{\zeta\zeta} = \frac{1}{2}\int d^4x a^3e^{3\zeta}\Big[(1 - \frac{\dot{\zeta}}{H})(-a^{-2}e^{-2\zeta}(4\nabla^2\zeta + 2\partial_k\zeta\partial^k\zeta) + N^{-2}\dot{\phi}_0 - 2V)$$
$$+ N^{-1}\left(-6(\dot{\zeta} + H)^2 + \partial_i\partial_j B\partial^i\partial^j B + 4\left[\dot{\zeta} + \frac{\dot{a}}{a}(1 - 2\zeta)\right]\nabla^2 B - \left(\nabla^2 B\right)^2 + 4\partial^k B\partial_k\zeta H\right)\Big], \tag{2.130}$$

which can be manipulated to be presented in a more illuminating form.

First of all, we have

$$\frac{1}{2}\int d^4xa^3e^{3\zeta}N^{-1}\Big[\Big(\partial_i\partial_jB\partial^i\partial^jB-\left(\nabla^2B\right)^2\Big)\Big]\approx\frac{1}{2}\int d^4xa^3\Big[\Big(+\nabla^2B\nabla^2B-\left(\nabla^2B\right)^2\Big)\Big]=0, \quad (2.131)$$

where we have disregarded total derivatives. Then, there is another term that can be rewritten up a total derivative

$$\int dtd^3x\frac{2Ha^3}{N}\left[(1+\frac{\dot{\zeta}}{H})\nabla^2Be^{\zeta}+\partial^kB\partial_k\zeta\right]\approx\int dtd^3x2Ha^3\left[\nabla^2Be^{\zeta}+\partial^kB\partial_k\zeta\right] \quad (2.132)$$

$$\approx\int dtd^3x2Ha^3\left[\nabla^2B\zeta+\partial^kB\partial_k\zeta\right] \quad (2.133)$$

$$=\int dtd^3x2Ha^3\left[-\partial^gB\partial_g\zeta+\partial^kB\partial_k\zeta\right]=0. \quad (2.134)$$

where note that at up to second this term is identical to the sum of the third and fifth one in the second line of eq.(2.130). Thus, we get the following action

$$S=\frac{1}{2}\int d^4xa^3e^{3\zeta}\Big[N\left(a^{-2}e^{-2\zeta}\left[-4\nabla^2\zeta-2\partial^k\zeta\partial^k\zeta\right]+N^{-2}\dot{\phi}_0-2V\right)-6N^{-1}\left(\dot{\zeta}+H\right)^2\Big] \quad (2.135)$$

$$=S_1+S_2, \quad (2.136)$$

where for convenience we have introduced

$$S_1=\frac{1}{2}\int d^4xa^3e^{3\zeta}\Big[N\left(a^{-2}e^{-2\zeta}\left[-4\nabla^2\zeta-2\partial^k\zeta\partial^k\zeta\right]-2V\right)\Big], \quad (2.137)$$

$$S_2=\frac{1}{2}\int d^4xa^3e^{3\zeta}N^{-1}\Big[-6\left(\dot{\zeta}+H\right)^2+\dot{\phi}^2\Big]. \quad (2.138)$$

Now we solve the first action

$$S_1=\frac{1}{2}\int d^4xae^{\zeta}\Big[\left(1+\frac{\dot{\zeta}}{H}\right)\left(\left[-4\nabla^2\zeta-2\partial^k\zeta\partial^k\zeta\right]-2\left(3H^2-\frac{\dot{\phi_0}^2}{2}\right)a^2e^{2\zeta}\right)\Big] \quad (2.139)$$

$$\approx\frac{1}{2}\int d^4xa\Big[-4\nabla^2\zeta-\frac{\dot{\zeta}}{H}4\nabla^2\zeta-4\zeta\nabla^2\zeta-2\partial^k\zeta\partial^k\zeta+a^2\left(1+3\zeta+\frac{9}{2}\zeta^2+\frac{\dot{\zeta}}{H}+3\zeta\frac{\dot{\zeta}}{H}\right)\left(-6H^2+\dot{\phi_0}^2\right)\Big] \quad (2.140)$$

$$=\frac{1}{2}\int d^4xa\Big[-\frac{\dot{\zeta}}{H}4\nabla^2\zeta-4\zeta\nabla^2\zeta-2\partial^k\zeta\partial^k\zeta+a^2\left(1+3\zeta+\frac{9}{2}\zeta^2+\frac{\dot{\zeta}}{H}+3\zeta\frac{\dot{\zeta}}{H}\right)\left(-6H^2+\dot{\phi_0}^2\right)\Big], \quad (2.141)$$

where in the first line we have used the first Friedmann equation, i.e. eq.(2.54) and in the last passage we have eliminated a total derivative. Switching to the second action we get

$$S_2=\frac{1}{2}\int d^4xa^3\left(1+3\zeta+\frac{9}{2}\zeta^2-\frac{\dot{\zeta}}{H}-3\frac{\dot{\zeta}\zeta}{H}+\frac{\dot{\zeta}^2}{H^2}\right)\left[-6\left(\dot{\zeta}^2+H^2+2\dot{\zeta}H\right)+\dot{\phi}^2\right] \quad (2.142)$$

$$=\frac{1}{2}\int d^4xa^3\left(1+3\zeta+\frac{9}{2}\zeta^2-\frac{\dot{\zeta}}{H}-3\frac{\dot{\zeta}\zeta}{H}+\frac{\dot{\zeta}^2}{H^2}\right)\left[-\left(6H^2-\dot{\phi}^2\right)-\left(6\dot{\zeta}^2+12\dot{\zeta}H\right)\right] \quad (2.143)$$

$$=\frac{1}{2}\int dtd^3xa^3\left\{-(3H^2-\frac{\dot{\phi_0}^2}{2})9\zeta^2-3(6H+\frac{\dot{\phi_0}^2}{H})\zeta\dot{\zeta}+\frac{\dot{\phi_0}^2\dot{\zeta}^2}{H^2}-12H\dot{\zeta}-2\left(3H^2-\frac{\dot{\phi_0}^2}{2}\right)\left(1-\frac{\dot{\zeta}}{H}+3\zeta\right)\right\}. \quad (2.144)$$

Now we can extract the $0th$ order lagrangian and we get

$$S^{(0)} = \frac{1}{2}\int dtd^3xa^3\left[-2V - 6H^2 - \dot{\phi_0}^2\right], \tag{2.145}$$

where the superscript indicates the order of the action. This is just a test to verify that what we have done is correct. Now, we proceed to validate the vanishing of the first order action

$$\begin{aligned}
S^{(1)} &= \frac{1}{2}\int dtd^3xa^3\left[\left(-6H^2+\dot{\phi_0}^2\right)\left(3\zeta+\frac{\dot\zeta}{H}\right) - 12H\dot\zeta + \left(-6H^2+\dot{\phi_0}^2\right)\left(-\frac{\dot\zeta}{H}+3\zeta\right)\right] && (2.146)\\
&= \frac{1}{2}\int dtd^3xa^3\left[\left(-6H^2+\dot{\phi_0}^2\right)6\zeta - 12H\dot\zeta\right] && (2.147)\\
&= \frac{1}{2}\int dtd^3x\left[a^3\left(-6H^2+\dot{\phi_0}^2\right)6\zeta + \frac{d}{dt}\left(12Ha^3\right)\zeta\right] && (2.148)\\
&= \frac{1}{2}\int dtd^3x\left[a^3\left(-6H^2+\dot{\phi_0}^2\right)6\zeta + 12\left(\dot Ha^3+3H^2a^3\right)\zeta\right] && (2.149)\\
&= \frac{1}{2}\int dtd^3x\left[a^3\left(-6H^2+\dot{\phi_0}^2\right)6\zeta + 12\left(-\frac{\dot{\phi_0}^2}{2}a^3+3H^2a^3\right)\zeta\right] && (2.150)\\
&= 0, && (2.151)
\end{aligned}$$

where we have used that $\dot H = -\frac{\dot{\phi_0}^2}{2}$. Now we are ready to evaluate the second-order action

$$S^{(2)} = \frac{1}{2}\int d^4xa\left\{-\frac{\dot\zeta}{H}4\nabla^2\zeta - 4\zeta\nabla^2\zeta - 2\partial^k\zeta\partial^k\zeta + a^2\left[-(3H^2-\frac{\dot{\phi_0}^2}{2})18\zeta^2 + \frac{\dot{\phi_0}^2\dot\zeta^2}{H^2}\right] - 36H\zeta\dot\zeta a^3\right\}, \tag{2.152}$$

which, using the following integration by parts

$$\text{first} = \frac{1}{2}\int dtd^3xa\left(-4\zeta\nabla^2\zeta\right) = \frac{1}{2}\int dtd^3xa\left(4\partial^i\zeta\partial_i\zeta\right), \tag{2.153}$$

$$\text{blue} = \frac{1}{2}\int dtd^3xa\left(-\frac{4}{H}\dot\zeta\nabla^2\zeta\right) = \frac{1}{2}\int dtd^3xa\left[\frac{2}{H}\frac{d}{dt}\left(\partial^i\zeta\partial_i\zeta\right)\right] = -\frac{1}{2}\int dtd^3x2a(1+\epsilon)(\partial_i\zeta)^2, \tag{2.154}$$

$$\text{third} = \frac{1}{2}\int dtd^3xa^3\left(-36H\zeta\dot\zeta\right) = \frac{1}{2}\int dtd^3xa^3\left(-18H\frac{d}{dt}\zeta^2\right) = \frac{1}{2}\int dtd^3x18a^3\left(3H^2-\frac{\dot{\phi_0}^2}{2}\right)\zeta^2, \tag{2.155}$$

becomes

$$S^{(2)} = \int dtd^3x\epsilon\left\{a^3\dot\zeta^2 - a\partial^k\zeta\partial_k\zeta\right\}, \tag{2.156}$$

where we are working in cosmic time and where in the previous computations we have used that

$$M_{pl} = 1, \quad \epsilon = \frac{\dot{\phi_0}^2}{2H^2}, \quad \dot H = -\frac{\dot{\phi_0}^2}{2}. \tag{2.157}$$

Now using conformal time and reintroducing the Planck mass we can rewrite the action as

$$S^{(2)} = \int d\tau d^3xa^2(\tau)M_{pl}^2\epsilon\left\{\zeta'^2 - \partial_k\zeta\partial^k\zeta\right\}. \tag{2.158}$$

2.4.2 The *EoM* and the power spectrum

In order to get the *EoM* we reintroduce the M_{pl} and we make a field redefinition by defining the so-called Mukhanov-Sasaki variable

$$v \equiv z\zeta, \quad z = \sqrt{2\epsilon} a M_{pl}. \tag{2.159}$$

Thus, the action eq.(2.158) becomes

$$S^{(2)} = \frac{1}{2}\int d^4x z^2 \left\{ \left(\frac{v^{'}}{z} - \frac{z^{'}}{z^2} v \right)^2 - \partial^k\zeta\partial^k\zeta \right\} = \frac{1}{2}\int d^4x z^2 \left\{ \frac{v^{'2}}{z^2} + \left(\frac{z^{'}}{z^2} v\right)^2 - 2\frac{v^{'}}{z}\frac{z^{'}}{z^2}v - \partial^k \frac{v}{z}\partial^k\frac{v}{z} \right\} \tag{2.160}$$

$$= \frac{1}{2}\int d^4x \left\{ v^{'2} + \left(\frac{z^{'}}{z}v\right)^2 - 2v^{'}v\frac{z^{'}}{z} - \partial^k v\partial_k v \right\} \tag{2.161}$$

$$= \frac{1}{2}\int d^4x \left\{ v^{'2} - \partial^k v \partial_k v + \frac{z^{''}}{z}v^2, \right\} \tag{2.162}$$

where we have integrated by parts in the last step. Now going to Fourier space

$$v(\tau, \mathbf{x}) = \int \frac{d^3k e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^2} v(\tau, \mathbf{k}), \tag{2.163}$$

and, varying the action we get the following equation of motion for the mode functions $v_k(\tau) \equiv v(\tau, \mathbf{k})$

$$v_k^{''} + \left(k^2 - \frac{z^{''}}{z}\right) v_k = 0. \tag{2.164}$$

Before solving this equation, we must obtain an explicit expression for $\frac{z''}{z}$, and we can achieve this by using the definition of z in the following way

$$\frac{z''}{z} = \frac{d}{d\tau}\left(\frac{z^{'}}{z}\right) + \left(\frac{z^{'}}{z}\right)^2. \tag{2.165}$$

Thus, we can compute

$$\frac{z^{'}}{z} = a\left(\frac{d}{dt}\frac{\sqrt{2\epsilon}a(t)M_{pl}}{\sqrt{2\epsilon}a(t)M_{pl}}\right) = \frac{\dot{\epsilon}}{2\epsilon}a + aH = aH(1-\eta_V+2\epsilon) \approx -\frac{1}{\tau}(1-\eta_V+3\epsilon), \tag{2.166}$$

where we exploit the result of the appendix on quasi De-Sitter expansion (appendix C.1) and, we use that

$$\frac{\dot{\epsilon}}{\epsilon} = \frac{-\frac{\ddot{H}}{H^2} + 2\frac{\dot{H}^2}{H^3}}{-\frac{\dot{H}}{H^2}} = +\frac{\ddot{H}}{\dot{H}} + 2H\epsilon = \frac{2\dot{\phi}_0\ddot{\phi}_0}{\dot{\phi_0}^2} + 2H\epsilon = 2(-\eta_V + 2\epsilon)H. \tag{2.167}$$

Now eq.(2.165) reads

$$\frac{z''}{z} \approx \frac{1}{\tau^2}(1-\eta_V+3\epsilon) + \frac{1}{\tau^2}(1-2\eta_V+6\epsilon) = \frac{1}{\tau^2}(1-3\eta_V+9\epsilon), \tag{2.168}$$

which allows us to rewrite the equation of motion as

$$v_k^{''} + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right) v_k = 0, \quad \nu^2 \equiv \frac{9}{4} + 9\epsilon - 3\eta_V, \tag{2.169}$$

which can be solved as done in appendix C.2 as

$$v_k(\tau) = \frac{\sqrt{\pi}}{2}\sqrt{-\tau}H^{(1)}_{\nu}\exp i\left(\frac{\nu\pi}{2}+\frac{\pi}{4}\right). \tag{2.170}$$

We recall that in order to obtain the solution we have assumed Bunch-Davies' initial condition [54]

$$v_k(\tau) \xrightarrow{-k\tau>>1} \frac{1}{\sqrt{2k}}e^{-ik\tau}, \tag{2.171}$$

and we required the following

$$W\{v_k, v_k^*\} = v_k v_k^{*'} - v_k^* v_k' = -i, \tag{2.172}$$

which as we'll see allows us to recover the standard commutation relation for the creation and annihilation operators. Since the *EoM* is of second order we can always impose two conditions on the solutions.

Next, we are ready to move forward with the canonical quantization of the system. Our first step is to determine the conjugate momentum to the variable v, denoted as $\pi_v \equiv \frac{\partial\mathcal{L}}{\partial v'} = v'$. In this context, the subscript indicates that π_v is the conjugate momentum with respect to the variable v. Thus, we can impose the canonical commutation relation [43, 52, 17]

$$[v(\tau,\mathbf{x}_1), v'(\tau,\mathbf{x}_2)] = \delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2), \tag{2.173}$$
$$[v(\tau,\mathbf{x}_1), v(\tau,\mathbf{x}_2)] = [v'(\tau,\mathbf{x}_1), v'(\tau,\mathbf{x}_2)] = 0, \tag{2.174}$$

and write the fields as a combination of creation and annihilation operators

$$v(\tau,\mathbf{x}_1) = \int\frac{d^3k}{(2\pi)^3}\left[v_k(\tau,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}(k) + v_k^*(\tau,\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^\dagger(k)\right], \tag{2.175}$$
$$v'(\tau,\mathbf{x}_1) = \int\frac{d^3k}{(2\pi)^3}\left[v_k'(\tau,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}(k) + v_k'^*(\tau,\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^\dagger(k)\right]. \tag{2.176}$$

Thus, inverting the relation we get

$$\hat{a}(q) = i\int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}}\left[v_q'^*(\tau,q)\hat{v}(\tau,\mathbf{x}) - v_q^*(\tau,q)v'(\tau,\mathbf{x})\right], \tag{2.177}$$
$$\hat{a}^\dagger(q) = -i\int d^3x e^{+i\mathbf{q}\cdot\mathbf{x}}\left[v_q'(\tau,q)\hat{v}^\dagger(\tau,\mathbf{x}) - v_q(\tau,q)\hat{v}'^\dagger(\tau,\mathbf{x})\right], \tag{2.178}$$

from which we can obtain the commutation relation for the creation and annihilation operators

$$[\hat{a}(k), \hat{a}^\dagger(q)] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{q}), \tag{2.179}$$
$$[\hat{a}(k), \hat{a}(q)] = [\hat{a}^\dagger(k), \hat{a}^\dagger(q)] = 0. \tag{2.180}$$

In what we have done is crucial that the condition on the Wronskian 2.172 holds in order to recover eq.(2.179) and (2.180).

Finally, we are ready to compute the power spectrum which is simply related to the two-point correlation function in Fourier space

$$\langle 0|\hat{v}_k(\tau,k)\hat{v}_q(\tau,q)|0\rangle = \langle 0|\left(v_k(\tau,k)\hat{a}(k) + v_{-k}^*(\tau,-k)\hat{a}^\dagger(-k)\right)\left(v_q(\tau,q)\hat{a}(q) + v_{-q}^*(\tau,-q)\hat{a}^\dagger(-q)\right)|0\rangle \tag{2.181}$$
$$= \langle 0|v_k(\tau,k)\hat{a}(k)v_{-q}^*(\tau,-q)\hat{a}^\dagger(-q)|0\rangle \tag{2.182}$$
$$= \mid v_k(\tau,k)\mid^2 \delta^{(3)}(\mathbf{k}+\mathbf{q})(2\pi)^3, \tag{2.183}$$

from which we can extract the power spectrum and the adimensional power spectrum (see section 1.6)

$$P_\zeta = \frac{\mid v_k(\tau,k)\mid^2}{2\epsilon a^2 M_{pl}^2}, \quad \Delta_\zeta = \frac{k^3}{2\pi^2}\frac{\mid v_k(\tau,k)\mid^2}{2\epsilon a^2 M_{pl}^2}, \tag{2.184}$$

where the subscript recalls that we are computing the gauge invariant curvature perturbation power spectrum. Using the asymptotic expansion for the Hankel functions [55]

$$v_k(\tau) \xrightarrow{-k\tau<<1} e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}}\frac{(-k\tau)^{-\nu+\frac{1}{2}}}{\sqrt{2k}}\frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)}2^{\nu-\frac{3}{2}}, \tag{2.185}$$

we immediately get the two power spectra on super-horizon scales

$$P_\zeta \approx \frac{1}{4\epsilon k^3}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}, \quad \Delta_\zeta \approx \frac{1}{2\epsilon}\left(\frac{H}{2\pi M_{pl}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}. \tag{2.186}$$

Now, we can immediately evaluate the spectral index (see 1.6) of the scalar perturbation and we get

$$n_s - 1 = \frac{d\ln\Delta_\zeta(k)}{d\ln k} = 2\eta_V - 6\epsilon, \tag{2.187}$$

which corresponds to an almost scale-invariant power spectrum. Since, from section 1.4 we know that the gauge invariant curvature perturbation is conserved on superhorizon scales roughly we can say that on super-horizon scales the power spectrum retains the value it has at horizon crossing, i.e. $k \sim aH$,

$$P_\zeta \approx \frac{1}{4k^3}\left(\frac{H}{\epsilon M_{pl}}\right)^2\Bigg|_{k=aH}, \quad \Delta_\zeta \approx \left(\frac{H}{2\sqrt{2}\epsilon\pi M_{pl}}\right)^2\Bigg|_{k=aH}. \tag{2.188}$$

2.5 The tensor power spectrum

This section aims to evaluate the tensor power spectrum within the *ADM* formalism in a fully consistent way. The calculation presented here may not reveal any particularly enlightening insights, but it serves as a warm-up exercise for what we need to do in the next section. Additionally, this computation enables us to obtain the correct normalization factor for the power spectrum. Independently on the gauge we adopt, since we are interested in tensor perturbations we can set to zero all the scalars in the metric and write

$$g_{\mu\nu} = a^2\begin{pmatrix}1 & 0\\ 0 & (\delta_{ij}+\gamma_{ij}+\frac{1}{2}\gamma_{il}\gamma^l{}_j)\end{pmatrix}, \quad g^{\mu\nu} = a^{-2}\begin{pmatrix}1 & 0\\ 0 & (\delta^{ij}-\gamma^{ij}+\frac{1}{2}\gamma^{il}\gamma^j{}_l)\end{pmatrix}. \tag{2.189}$$

Note that we are interested only in deriving the *EoM* to compute the Power spectrum so we consider terms in the Lagrangian up to second order in h. We recall that the Lagrangian we are interested in is

$$\mathcal{L}_{HE} = \frac{M_{pl}^2}{2}\sqrt{h}\left[{}^{(3)}R + K_{\alpha\beta}K^{\alpha\beta} - K^2\right], \tag{2.190}$$

where we have set $N=1$ consistently to what we have said. Therefore, we can start by evaluating the extrinsic curvature up to second order

$$K_{\alpha\beta} = \frac{1}{2N}\left[\dot{h}_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu\right] = \frac{1}{2}\dot{h}_{\mu\nu}. \tag{2.191}$$

All the computations that we do not repeat here are performed in detail in appendix C.4. Since as shown in section C.3.4 we only need to evaluate the spatial components of the extrinsic curvature we can use Latin indices and obtain

$$K_{ij} = \frac{1}{2}\dot{g}_{ij} = \dot{a}a\delta_{ij} + \dot{a}a\gamma_{ij} + \frac{1}{2}a^2\dot{\gamma}_{ij} + \frac{1}{2}\dot{a}a\gamma_{ik}\gamma^k_j + \frac{1}{4}a^2\dot{\gamma}_{ik}\gamma^k_j + \frac{1}{4}a^2\gamma_{ik}\dot{\gamma}^k_j. \tag{2.192}$$

The fully contravariant extrinsic curvature reads

$$K^{ij} = \frac{\dot{a}}{a^3}\delta^{ij} - \frac{\dot{a}}{a^3}\gamma^{ij} + \frac{1}{2}a^{-2}\dot{\gamma}^{ij} + \frac{1}{2}\frac{\dot{a}}{a^3}\gamma^{ik}\gamma_k^j - \frac{1}{4}a^{-2}\dot{\gamma}^{ik}\gamma_k^j - \frac{1}{4}a^{-2}\gamma^{ik}\dot{\gamma}_k^j. \tag{2.193}$$

Thus we are ready to compute the necessary ingredients to rewrite the second and third terms in the action

$$K^{ij}K_{ij} = 3\frac{\dot{a}^2}{a^2} + \frac{1}{4}\dot{\gamma}^{ij}\dot{\gamma}_{ij}, \tag{2.194}$$

$$g^{ij}K_{ij} = 3\frac{\dot{a}}{a}. \tag{2.195}$$

Now we have to evaluate the three-dimensional Ricci scalar which can be evaluated as we have done in appendix C.4

$${}^{(3)}R = \frac{1}{a^2}\left[\partial_i D^i + \frac{1}{2}\gamma^{li}\gamma_{il}{}^{,k}{}_{,k} + \frac{1}{4}\gamma^{li}{}_{,k}\gamma_{il}{}^{,k}\right], \tag{2.196}$$

where, D_i is a second-order function in tensor perturbation, and its specific expression is not significant. What matters is that it can be represented as a spatial total derivative. As we observe in appendix C.4 $\sqrt{h} \approx a^3(1 + O(\gamma^3))$. Now, inserting everything in eq.(2.190), eliminating total derivatives and switching to conformal time we find the following action

$$\mathcal{S}_T^{(2)} = \int d\tau d^3x \frac{M_{pl}^2}{8}a^2\left\{\gamma'^{ij}\gamma'_{ij} - \gamma^{li}{}_{,k}\gamma_{il}{}^{,k}\right\}. \tag{2.197}$$

Using the variational principle we immediately get the *EoM* for the tensors perturbation

$$\gamma''_{ij} + 2\frac{a'}{a}\gamma'_{ij} - \nabla^2\gamma_{ij} = 0. \tag{2.198}$$

Now, going into Fourier space,

$$\gamma_{ij}(\tau, \mathbf{x}) = \int \frac{d^3k e^{i\mathbf{x}\cdot\mathbf{k}}}{(2\pi)^3}\sum_s \left[\epsilon_{ij}^s(\hat{\mathbf{k}})u_s(\tau, \mathbf{k})b_s(\mathbf{k}) + \left(\epsilon_{ij}^s\right)^*(\hat{\mathbf{k}})u_s^*(\tau, -\mathbf{k})b_s^*(-\mathbf{k})\right], \tag{2.199}$$

where $\epsilon_{ij}^s(\hat{\mathbf{k}})$ are the polarization tensors defined in section D.1, $u_s(\tau, \mathbf{k})$ are the mode functions while $b_s(\mathbf{k})$ are functions of $\mathbf{k}$. The basis we adopt for the polarization is completely equivalent. The equation in Fourier space for the mode functions are

$$u_s''(\tau, \mathbf{k}) + 2\frac{a'}{a}u_s'(\tau, \mathbf{k}) + k^2u_s(\tau, \mathbf{k}) = 0, \tag{2.200}$$

which, using this field redifinition $u(\tau, \mathbf{k}) = \frac{v(\tau,\mathbf{k})}{a(\tau)}$,

$$0 = \left(\frac{v''(\tau, \mathbf{k})}{a} - 2\frac{v'(\tau, \mathbf{k})}{a}\frac{a'}{a} - v(\tau, \mathbf{k})\left(\frac{a''}{a^2} - 2\frac{a'^2}{a^3}\right)\right) + 2\frac{a'}{a}\left(\frac{v'(\tau, \mathbf{k})}{a} - \frac{v(\tau, \mathbf{k})}{a}\frac{a'}{a}\right) + k^2\frac{v(\tau, \mathbf{k})}{a} \tag{2.201}$$

$$= v''(\tau, \mathbf{k}) + \left(k^2 - \frac{a''}{a^2}\right)v(\tau, \mathbf{k}) = \frac{v''(\tau, \mathbf{k})}{a} + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right)v(\tau, \mathbf{k}), \tag{2.202}$$

where we have exploited the results of appendix C.1 and where $\nu^2 = \frac{9}{4} + 3\epsilon$. Thus, we see that equations for the mode functions are identical to the one of the scalar field and we do not repeat the calculations but we report the results

$$P_T^s(k) = \frac{16\pi}{k^3}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon}, \quad \Delta_T^s(k) = \frac{8}{\pi}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon}, \tag{2.203}$$

where the subscript T stands for tensor, the subscript s refers to the polarization. The tensor spectral index is

$$n_T = -2\epsilon. \tag{2.204}$$

At horizon crossing we have

$$P_T^s(k) = \frac{16\pi}{k^3}\left(\frac{H}{M_{pl}}\right)^2, \quad \Delta_T^s(k) = \frac{8}{\pi}\left(\frac{H}{M_{pl}}\right)^2. \tag{2.205}$$

The total tensor power spectrum, which is the sum of the two polarization states contribution, is

$$P_T^s(k) = \frac{32\pi}{k^3}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon}, \quad \Delta_T^s(k) = \frac{16}{\pi}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon}. \tag{2.206}$$

2.6 From quantum fluctuations to initial conditions

Now, we have to understand how from primordial quantum fluctuations we can set the initial condition for the subsequent evolution of the fluctuations. This can be done using that the gauge invariant curvature perturbation on uniform energy density hypersurfaces,

$$\zeta = -\hat{\phi} - \frac{H\delta^{(1)}\rho}{\dot{\rho}_0}, \tag{2.207}$$

is constant on super-horizon scales. Hence, the concept is that the fluctuations generated during inflation go on super-horizon scales. Subsequently, they become frozen and remain so until re-entering within the horizon during the epochs of radiation or matter dominance. As an illustration, let's take two scales, denoted as λ and $\lambda^{'}$, as depicted in the figure 9. These scales are associated with wave vectors $\mathbf{k}$ and $\mathbf{k}^{'}$, respectively. Throughout the inflationary phase, λ and $\lambda^{'}$ go on super-horizon scales at times $t_1(\mathbf{k})$ and $t_1(\mathbf{k}^{'})$, respectively. Subsequently, during the epochs of radiation and matter dominance, these scales re-enter under the horizon at times $t_2(\mathbf{k})$ and $t_2(\mathbf{k}^{'})$, respectively. Thus, if

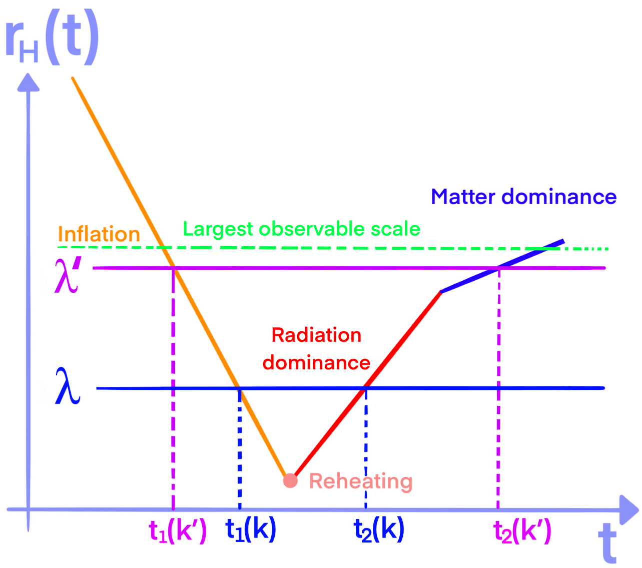


Figure 9: The scales λ and $\lambda^{'}$.

we want to evaluate the power spectrum of matter or radiation perturbation we need to evaluate the relation between these perturbations and ζ. It's possible to show that

$$\zeta_{rad} = \frac{\delta\rho_\gamma}{4\rho_\gamma} = \frac{\Delta T}{T}, \quad \zeta_{mat} = \frac{\delta\rho_{mat}}{3\rho_{mat}}, \tag{2.208}$$

where we have used the standard relation between radiation energy density and temperature, $\rho_\gamma \propto T^4$. It's worth noting that we have computed these quantities within the spatially flat gauge, where $\hat{\phi} = 0$. This choice of gauge is permissible because ζ remains gauge invariant, allowing us to choose any gauge we desire to evaluate its expression. Now, if we want to evaluate the power spectra for temperature and matter perturbations we simply have to use that ζ is conserved on super-horizon scales

$$\zeta(t_1, \mathbf{k}) = \zeta(t_2, \mathbf{k}) = \frac{\delta\rho_{mat}}{3\rho_{mat}}, \tag{2.209}$$

$$\zeta(t_1, \mathbf{k}^{'}) = \zeta(t_2, \mathbf{k}^{'}) = \frac{\Delta T}{T}, \tag{2.210}$$

from which we can write

$$\Delta\zeta|_{t_1(\mathbf{k})} = \Delta\zeta|_{t_2(\mathbf{k})}. \tag{2.211}$$

2.7 The Importance of the stochastic background of GWs

In this section, we mention something about the importance of detection of GWs in constraining Inflationary models. First of all, we have to introduce the so-called *scalar to tensor ratio* r, which is defined as

$$r \equiv= \left.\frac{\Delta_T}{\Delta_\zeta}\right|_{k=aH} = \frac{16}{\pi}\left.\left(\frac{\frac{H^2}{M_{pl}^2}}{\frac{H^2}{M_{pl}^2}}\right)\right|_{k=aH} \pi\epsilon = 16\epsilon = -8n_T. \tag{2.212}$$

This relation is often called *consistency relation.* The actual bound on the scalar-tensor ratio is [32]

$$r < 0.036\ (95\%C.L.). \tag{2.213}$$

The scalar to tensor ratio would provide useful insights into the dynamics of Inflation since it allows us to evaluate:

- **The energy scale of Inflation**
  We say the energy scale at which Inflation took place basically depends on the slow-roll potential $E_{inf} \simeq V^{\frac{1}{4}}$. Now, we can write

$$E_{inf} \simeq V^{\frac{1}{4}} \simeq [HM_{pl}]^{\frac{1}{2}} \simeq \left[\Delta_T^{\frac{1}{2}} M_{pl}\right]^{\frac{1}{2}} \simeq \left[\frac{r}{10^{-2}}\right]^{\frac{1}{4}} 10^{16} GeV, \tag{2.214}$$

  where it's clearly an estimate which allows us to individuate the order of magnitude of the quantity of interest. From this relation, it's evident that a detection of the background of stochastic GWs would give us insights into the energy scale of Inflation.

- **The distinction between large and small field models**
  A measurement of r would provide useful insights also on the kind of model we have to consider. In fact, we can write the excursion of the scalar field during Inflation as

$$\Delta_\phi = \int_{\phi_i}^{\phi_f} \dot{\phi} dt \frac{H}{H} \simeq \frac{\dot{\phi}}{H}\int_{t_1}^{t_2} H dt = \epsilon^{\frac{1}{2}} M_{pl} N_{CMB} = \frac{r^{\frac{1}{2}}}{4} M_{pl} N_{CMB}. \tag{2.215}$$

3 Parity violation in the galaxy four-point correlation function

In this chapter we discuss the issue of the measurement of parity violation in the galaxy four-point correlation function as presented in [2] and [1]. This is the starting point of the work since the goal of the thesis is to try to reproduce these kinds of signals as relic signatures of parity violation in the Early Universe. We are going to do this in section 5 and in section 6 in which we study models of modified gravity which can imprint parity-violating signatures in the trispectrum of the curvature perturbation ζ.

Before entering theoretical computations regarding the four-point correlation function or its analogous in Fourier space, the trispectrum, we would like in this chapter to illustrate the basic ideas underlying the measurement of parity violation in the galaxy four-point correlation function. So for the sake of simplicity, space curvature and expansion are ignored. This allows us to work in three-dimensional euclidean space, $\mathbb{R}^3$. We disregard that galaxies are observed along the back-light cone, not at a fixed instant of cosmic time t. This is a good approximation [46] because most of the available data samples only a small fraction of the Hubble distance. A more detailed and complete analysis can be carried out taking into account additional effects, such as spatial curvature [40]. So we can say that position-dependent fields are real random fields that are statistically homogeneous and isotropic in the sense specified in section 1.5. Moreover, they satisfy the hypothesis of the ergodic theorem eq.(1.122).

In particular, the field we analyze is the fractional matter density field defined on $\mathbb{R}^3$ as

$$\delta(t,\mathbf{r}) \equiv \frac{\delta\rho(t,\mathbf{r})}{\rho_0(t)} \tag{3.1}$$

where $\rho_0(t)$ is the spatial mean of $\rho(t,\mathbf{x}) = \rho_0(t) + \delta\rho(t,\mathbf{x})$. For the sake of simplicity, in the following, we omit the time dependence of the density fields. The galaxy four-point correlation function is defined as

$$\zeta(\mathbf{s},\mathbf{r}_1,\mathbf{r_2},\mathbf{r}_3) \equiv \langle\delta(\mathbf{s})\delta(\mathbf{s}+\mathbf{r}_1)\delta(\mathbf{s}+\mathbf{r_2})\delta(\mathbf{s}+\mathbf{r}_3)\rangle, \tag{3.2}$$

where $\langle..\rangle$ denotes the ensemble average and $\mathbf{s},\mathbf{r}_1,\mathbf{r_2},\mathbf{r}_3$ indicate absolute and relative positions on $\mathbb{R}^3$. Invoking the ergodic theorem, eq.(1.122), the galaxy $4PCF$ estimator [1] becomes

$$\begin{aligned}\hat{\zeta}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) &\equiv \langle\delta(\mathbf{s})\delta(\mathbf{s}+\mathbf{r}_1)\delta(\mathbf{s}+\mathbf{r_2})\delta(\mathbf{s}+\mathbf{r}_3)\rangle &(3.3)\\ &= \frac{1}{V}\int d\mathbf{s}\delta(\mathbf{s})\delta(\mathbf{s}+\mathbf{r}_1)\delta(\mathbf{s}+\mathbf{r_2})\delta(\mathbf{s}+\mathbf{r}_3), &(3.4)\end{aligned}$$

provided that V, the volume over which we measure the $4PCF$, is sufficiently large. In eq.(3.4) we denote with a hat the quantity measured from data and we opt for this notation throughout the entire chapter. We stress that in data analysis we have to take into account the range of validity of eq.(1.140)[9] and we simply assume that the volume is sufficiently large. In eq.(3.4) we have introduced the $4PCF$ estimator but, in practice, we cannot use it and we group the data in bins of finite size, $B(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$, for example, angular and radial bins. For example, given a sample of N_g galaxies at positions

$$\mathbf{x}_{i=1,..,N_g}, \text{ such that } \forall i \in [1,N_g], \quad r_i \in [20,120]h^{-1}Mpc, \tag{3.5}$$

we can divide this interval in 10 identical parts with $\Delta r = 10h^{-1}Mpc$. We label each bin or interval with an index $b = 1,..,b_{max=10}$ and we identify $\Delta_b = [10(b+1), 10(b+2)]h^{-1}Mpc$. In this way, we can define $\theta^b(r)$

$$\theta^b(r) : \mathbb{R} \to \mathbb{R} \tag{3.6}$$

$$\begin{cases}\theta^b(r) = 1 & \text{if } r \in \Delta_b,\\ \theta^b(r) = 0 & \text{otherwise}\end{cases}. \tag{3.7}$$

[9]If the volume is too small we are not computing the $NPCF$

In the same fashion we can introduce the angular binning. Therefore the problem reduces in estimating the coefficients of such decomposition

$$\hat{\zeta}_B = \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) B(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \tag{3.8}$$

where the $\hat{\zeta}_B$ is a $\mathbb{R}$ number and the subscript B refers to one of the bins we are using. We stress that unavoidably we have to average over the space otherwise we are not measuring the $4PCF$ as explained in sec.1.5. In general we have access to a discrete field of N_g particles at positions $\mathbf{x}_{i=1,..,N_g}$ with weights w_i, which is defined as

$$\delta(\mathbf{s}) \equiv \sum_{i=1}^{N_g} w_i \delta^{(3)}(\mathbf{s} - \mathbf{x}_i). \tag{3.9}$$

Using eq.(3.9), the estimator becomes

$$\begin{aligned}
\hat{\zeta}_B &= \frac{1}{V} \int d\mathbf{s} d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \delta(\mathbf{s}) \delta(\mathbf{s} + \mathbf{r}_1) \delta(\mathbf{s} + \mathbf{r_2}) \delta(\mathbf{s} + \mathbf{r}_3) B(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) && (3.10)\\
&= \frac{1}{V} \int d\mathbf{s} d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 B(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \left[\sum_{J_0=1}^{N_g} w_{J_0} \delta^{(3)}(\mathbf{s} - \mathbf{x}_{j_0}) \right] \left[\sum_{J_1=1}^{N_g} w_{J_1} \delta^{(3)}(\mathbf{s} + \mathbf{r}_1 - \mathbf{x}_{j_1}) \right] \times && (3.11)\\
&\times \left[\sum_{J_2=1}^{N_g} w_{J_2} \delta^{(3)}(\mathbf{s} + \mathbf{r}_2 - \mathbf{x}_{j_2}) \right] \left[\sum_{j_3=1}^{N_g} w_{j_3} \delta^{(3)}(\mathbf{s} + \mathbf{r}_3 - \mathbf{x}_{j_3}) \right] && (3.12)\\
&= \sum_{j_0, j_1, j_2, j_3=1}^{N_g} w_{j_0} w_{j_1} w_{j_2} w_{j_3} B(\mathbf{x}_{j_1} - \mathbf{x}_{j_0}, \mathbf{x}_{j_2} - \mathbf{x}_{j_0}, \mathbf{x}_{j_3} - \mathbf{x}_{j_0}), && (3.13)
\end{aligned}$$

which is a sum over a triplet of particles, and it has a complexity $O(N_g^4)$. If we consider the generic $NPCF$, the estimator has complexity $O(N_g^N)$ which, unless N is small ($N = 2$) it's useless due to computationally costs [40]. To solve the problem, instead of projecting on angular bins, we use a basis for isotropic functions of $N-1$ vector variables[10], introduced in section 3.1, whose elements are products of $N-1$ spherical harmonics:

$$\int d\mathbf{s} \delta(\mathbf{s}) \int \left[\prod_{i=1}^{3} d\mathbf{r}_i \theta^{b_i}(r_i) \delta(\mathbf{s} + \mathbf{r}_i) Y_{\Lambda_i m_i}(\hat{\mathbf{i}}_i) \right], \tag{3.14}$$

where $Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i) = Y_{\Lambda_i m_i}(\theta_1, \phi_1)$ is the spherical harmonics with indices Λ_i and m_i (appendix B.1). We are not going to use eq.(3.14) but a sum of terms like this weighted by Wigner symbols, as explained in section 3.3. This trick allows us to obtain an algorithm that has complexity $O(N_g^2)$ for every N, as explained in section 3.3.

The remainder of this chapter is structured as follows. In section 3.1, using techniques of quantum mechanics, we discuss how to decompose a generic isotropic function of N vector variables in a convenient orthonormal basis, which is the one used in data analysis. In section 3.2 we show why, considering a scalar field, we need the four-point correlation function to search for parity violation signature. We give both an intuitive geometrical argument and a more formal one, based on the properties of the basis' elements introduced in section 3.2. In section 3.3 we briefly introduce the algorithm used in the analysis and in the final section 3.3, we discuss the claims of measurements of parity violation signature found in the galaxy $4PCF$ by [2] and [1].

3.1 Isotropic N-point basis functions and their properties

As discussed in section 1.6 we know that the $NPCF$s of a real scalar field are isotropic functions of $N-1$ vector variables. In this section, following [5], we present how to provide an orthonormal basis

[10] As explained in section 1.6 the $NPCF$ are isotropic functions of $N-1$ vector variables.

for square-integrable isotropic functions of positions $\mathbf{R} \equiv [\mathbf{r_1}, .., \mathbf{r_N}]$, where $\mathbf{R}$ stand for the collection of $\mathbf{r_1}, .., \mathbf{r_N}$. To do that in a manageable way we use the quantum mechanics (hereafter QM) formalism for the addition of angular momentum. In appendix B.2 we summarize the most important features of the procedure used for the addition of two angular momenta following [56]. In QM we have that the orbital angular momentum is defined as $\hat{\mathbf{L}} = (\hat{L}_z, \hat{L}_y, \hat{L}_z) = \mathbf{r} \times \hat{\mathbf{p}}$ and it's possible to diagonalize simultaneously $\hat{L}^2$ and $\hat{L}_z$ since $[\hat{\mathbf{L}}^2, \hat{L}_z] = 0$[11]. This procedure can be found in any QM textbook and we have that the spherical harmonics $Y_{lm}(\theta, \phi)$ of order l, m[12] are common eigenfunctions of L^2 and L_z with eigenvalues $l(l+1)$ and m respectively

$$
\begin{aligned}
&\hat{L}^2 Y_{lm}(\theta, \phi) = l(l+1) Y_{lm}(\theta, \phi), \\
&\hat{L}_z Y_{lm}(\theta, \phi) = m Y_{lm}(\theta, \phi),
\end{aligned} \tag{3.15}
$$

where $l \geq 0, l \in \mathbb{N}$ and $m \in [-l, l]$. As stated in [57] we know that

$$
L^2(\mathbb{R}^3) = L^2([0, \infty), r^2 dr) \otimes L^2(S^2) \tag{3.16}
$$

and that the spherical harmonics are an orthonormal basis of $L^2(S^2)$. In eq.(3.16) we have introduced

$$
L^2([0, \infty), r^2 dr) \equiv \{f(r) : \mathbb{R} \to \mathbb{R} \mid \int_0^\infty dr r^2 |f(r)|^2, < \infty\}. \tag{3.17}
$$

Thus it's always possible to decompose any $f(\mathbf{r}) \in L^2(\mathbb{R}^3)$ in the following way

$$
f(\mathbf{r}) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} f_{lm}(r) Y_{lm}(\theta, \phi), \tag{3.18}
$$

where we $r = |\mathbf{r}|$.

We introduce the method starting from the case $N = 2$, i.e. the decomposition of a square-integrable isotropic function $f(\mathbf{r}_1, \mathbf{r}_2) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ of two vector variables, $\mathbf{r}_1$ and $\mathbf{r}_2$. We seek an orthonormal basis for isotropic functions of $\mathbf{r}_1$ and $\mathbf{r}_2$. We can achieve this by using some basic facts of QM. The basic idea is to think of $L^2(\mathbb{R}^3)$ as the state space of a free particle with spin 0 with an associated angular momentum operator $\mathbf{L}$. So the functions we want to decompose belong to the state space of a composite system of two subsystems of particles with 0 spin with space states ε_1 and ε_2 and angular momentum operator $\mathbf{L}_1$ and $\mathbf{L}_2$. We indicate the composite space state as

$$
\varepsilon \equiv \varepsilon_1 \otimes \varepsilon_2, \tag{3.19}
$$

and its relative angular momentum operator with $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$.

Now we want to make contact with the notation introduced in appendix B.2. Since the angular momentum operator acts only on the angular part of the function[13], the one which can be decomposed on S^2, we write

$$
\varepsilon_1 = \varepsilon_2 = L^2(\mathbb{R}^3) = L^2([0, \infty), r^2 dr) \otimes L^2(S^2) = L^2([0, \infty), r^2 dr) \otimes \chi_{1\backslash 2}, \tag{3.20}
$$

where $\chi_{1\backslash 2}$ is the space where the angular momentum operator has no trivial dependence, i.e. $L^2(S^2)$. According to eq.(B.15) we can write

$$
\begin{aligned}
\chi &= (L^2([0, \infty), r^2 dr) \otimes L^2(S^2)) \otimes (L^2([0, \infty), r^2 dr) \otimes L^2(S^2)) && (3.21) \\
&= (L^2([0, \infty), r^2 dr))^{\otimes 2} \otimes (L^2(S^2))^{\otimes 2} && (3.22) \\
&= (L^2([0, \infty), r^2 dr))^{\otimes 2} \otimes (\sum_{\oplus} \chi(L, m)), && (3.23)
\end{aligned}
$$

[11] In what follows, we omit the operator symbol, i.e. the hat, for notational simplicity.

[12] See appendix for definition and useful properties.

[13] We can formally write $\mathbf{L} = \mathbb{I}_r \otimes \mathbf{L}$ in order to emphasize that it doesn't depend on r.

where $\chi(L,m)$ are subspaces with definite total angular momentum. We stress that $\sum_{\oplus}\chi(L,m)$ is a convenient way to express $L^2(S^2)\otimes L^2(S^2)$. We recall that if we choose the spherical harmonics as basis for $L^2(S^2)$ according to eq.(3.22), we can always decompose a generic square-integrable function of two vector variables as

$$f(\mathbf{r}_1,\mathbf{r}_2)=\sum_{l_1,l_2}\sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2} f_{l_1m_1l_2m_2}(r_1,r_2)Y_{l_1m_1}(\theta_1,\phi_1)Y_{l_2m_2}(\theta_2,\phi_2). \tag{3.24}$$

We are not interested in generic functions of $L^2(\mathbb{R}^3\otimes\mathbb{R}^3)$ but in the isotropic ones. Now the isotropic functions are eigenvectors of $\mathbf{L}$ with eigenvalue $L=0$. So the relative state space is

$$(L^2([0,\infty),r^2dr))^{\otimes 2}\otimes\chi(L=0,m=0). \tag{3.25}$$

Now we can express the states of $\chi(L=0,m=0)$ either in the basis in which the states are common eigenfunctions of $L_1^2,(L_1)_z,L_2^2,(L_2)_z$ or L_1^2,L_2^2,L^2,L_z. Our goal is to use products of spherical harmonics, which are common eigenfunctions of $L_1^2,(L_1)_z,L_2^2,(L_2)_z$ and in order to do that we'll use the Clebsch-Gordan coefficient. To do that, we have to sum $\mathbf{L}_1$ and $\mathbf{L}_2$, into $\mathbf{L}=\mathbf{L}_1+\mathbf{L}_2=\mathbf{0}$. An element with $L=0$ can be obtain only if $L_1=L_2=\Lambda$ since $|L_1-L_2|\leq L\leq L_1+L_2$. We denote, according to the notation of appendix B.2[14] this state as $P_{\Lambda,\Lambda}\equiv|\Lambda,\Lambda,0,0\rangle$, which is expressed in the basis of common eigenfunctions of L_1^2,L_2^2,L^2,L_z. The Clebsch-Gordan coefficients (see appendix B.2) allow us to get an analytic expression for $P_{\Lambda,\Lambda}$ as a product of spherical harmonics

$$P_{\Lambda,\Lambda}=\sum_{m_1,m_2}\langle\Lambda m_1,\Lambda m_2|00\rangle|\Lambda m_1\rangle|\Lambda m_2\rangle=\sum_{m=-\Lambda}^{\Lambda}\frac{(-1)^{\Lambda-m}}{\sqrt{2\Lambda+1}}Y_{\Lambda m}(\hat{\mathbf{r}}_1)Y_{\Lambda -m}(\hat{\mathbf{r}}_2)=\sum_{m=-\Lambda}^{\Lambda}\frac{(-1)^{\Lambda}}{\sqrt{2\Lambda+1}}Y_{\Lambda m}(\hat{\mathbf{r}}_1)Y^*_{\Lambda m}(\hat{\mathbf{r}}_2), \tag{3.26}$$

where $|\Lambda,m\rangle$ denotes a single particle state with quantum numbers $\Lambda(\Lambda+1)$ and m respectively for L^2 and $\mathbf{L}_z$ and $\hat{\mathbf{r}}_{1\backslash 2}\equiv\frac{\mathbf{r}_{1\backslash 2}}{r_{1\backslash 2}}$ is used to specify the angular dependence. In the first step of eq.(3.26) we use the definition of the Clebsch-Gordan coefficients, in the second one we use their explicit expression while in the last one, we use the expression for the conjugate of $Y_{lm}(\theta,\phi)$ (see appendix B.1). Using eq.(3.25) every isotropic function of two vector variables can be decomposed as follow

$$f(\mathbf{r}_1,\mathbf{r}_2)=\sum_{\Lambda=0}^{\infty}f(r_1,r_2)P_{\Lambda,\Lambda}(\hat{\mathbf{r}}_1,\hat{\mathbf{r}}_2). \tag{3.27}$$

Now we can generalize this decomposition to square-integrable functions of N vector variables, $\mathbf{R}\equiv[\mathbf{r}_1,..,\mathbf{r}_N]$, which we denote as

$$L^2(\mathbb{R}^3)^{\otimes N}\equiv\{f:(\mathbb{R}^3)^{\otimes N}\longrightarrow\mathbb{R}\mid\int_{(\mathbb{R}^3)^{\otimes N}}\prod_{i=1}^{N}d^3r_i|f(\mathbf{R})|^2<\infty\}, \tag{3.28}$$

which can be written as

$$L^2(\mathbb{R}^3)^{\otimes N}=(L^2([0,\infty),r^2dr)\otimes L^2(S^2))^{\otimes N}=(L^2([0,\infty),r^2dr))^{\otimes N}\otimes(L^2(S^2))^{\otimes N}. \tag{3.29}$$

Now we seek an orthonormal basis of isotropic functions, i.e. functions that are invariant under simultaneous rotations of the position's vectors,

$$\forall\hat{R}\in SO(3),\ f(\hat{R}\mathbf{R})\equiv f(\hat{R}\mathbf{r_1},..,\hat{R}\mathbf{r_N})=f(\mathbf{r_1},..,\mathbf{r_N})=f(\mathbf{R}). \tag{3.30}$$

We proceed in a similar way to what was done in the case $N=2$. We identify

$$\forall i=1,..,N\ \varepsilon_i=L^2(\mathbb{R}^3)=(L^2([0,\infty),r^2dr))_i\otimes\chi_i, \tag{3.31}$$

[14] We adopt this notation throughout the rest of the chapter.

as the state space of a free particle of spin 0 and relative angular momentum operator $\mathbf{L}_i$. In a similar way to eq.(3.20) we have renamed $L^2(S^2)$ with χ in order to make contact with the notation of the appendix B.2. We also have introduced a subscript $i = 1, .., N$ in eq.(3.31) to identify the N vector spaces we have to consider. Therefore we can interpret

$$\varepsilon \equiv (L^2(\mathbb{R}^3))^{\otimes N}, \tag{3.32}$$

as the resulting vector space for a system of N particles with spin 0 and total angular momentum operator

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{L}_i. \tag{3.33}$$

In similar way to eq.(3.23), we get

$$\varepsilon = (L^2([0,\infty), r^2 dr)_1) \otimes \chi_1 \otimes .. \otimes (L^2([0,\infty), r^2 dr)_N) \otimes \chi_N = (L^2([0,\infty), r^2 dr)^{\otimes N} \otimes \sum_{\oplus} \chi(L, m), \tag{3.34}$$

where in the second passage we have dropped the subscript i for simplicity and where $\chi(L, m)$ are subspaces with definite *total* angular momentum. Since the spherical harmonics are basis of χ_i, according to eq.(3.34), we can decompose any function $f(\mathbf{R}) \in L^2(\mathbb{R}^3)^{\otimes N}$ as

$$f(\mathbf{R}) = \sum_{l_1,..l_N} \sum_{m_1,..,m_N} f_{l_1,..,l_N,m_1,..,m_N}(r_1, ., r_N) \prod_{i=1}^{N} Y_{l_i m_i}(\theta_i, \phi_i). \tag{3.35}$$

We are not interested in projecting onto a basis the radial part of the functions but in working only with the angular part. What we mean is that we are not interested in writing $f_{l_1,..,l_N,m_1,..,m_N}(r_1, .., r_N)$ of eq.(3.35) as

$$f_{l_1,..,l_N,m_1,..,m_N}(r_1, .., r_N) = \sum_{k_1} .. \sum_{k_N} a_{k_1,..,k_N,l_1,..,l_N,m_1,..,m_N} \prod_{i=1}^{N} \tilde{f}_{k_i}(r_i), \tag{3.36}$$

where $\{\tilde{f}_1, .., \tilde{f}_j..\}$ is a basis for $L^2([0,\infty), r^2 dr)$[15] and where $a_{k_1,..,k_N,l_1,..,l_N,m_1,..,m_N}$'s are numerical coefficients. Our goal is to decompose the isotropic functions using products of spherical harmonics but if we want to reconstruct the full function we always have to remember that the coefficients multiplying the product of spherical harmonics won't be real numbers but functions of the moduli of the radii. Now we are interested in isotropic functions, which are eigenfunctions of $\mathbf{L}$ with eigenvalue $L = 0$ since we know that rotation invariant states are the ones with 0 angular momentum. A basis of the space can be obtained by taking the tensor product of the single particle's basis, i.e. a basis for $L^2(\mathbb{R}^3)$. But we are interested in obtaining a basis consisting of spherical harmonics products. So, as in the case $N = 2$, we want to express our basis elements as linear combinations of common eigenfunctions $L_1^2, (L_z)_1, .., L_N^2, (L_z)_N$. The states in $\chi(L, m)$ can be written as common eigenfunctions of the operators $L_1^2, .., L_N^2, L^2, L_z$ or common eigenfunctions of $L_1^2, (L_1)_z, ..., L_N^2, (L_N)_z$ and we can link them using Clebsch-Gordan coefficients (see appendix B.2).

Now advancing to $N \geq 3$ vectors we need to specify a scheme for the addition of angular momenta. If $N = 3$, the sum of the the angular momenta associated with the first two directions, Λ_1 and Λ_2, give rise to Λ_{12}, which, due to the triangular rule for addition of angular momenta must satisfy

$$|\Lambda_1 - \Lambda_2| \leq \Lambda_{12} \leq \Lambda_1 + \Lambda_2. \tag{3.37}$$

Now to have a state with total angular momentum we need $\Lambda_3 = \Lambda_{12}$. Therefore in the case $N = 3$, we don't need to specify any intermediate state. However, once we go beyond three vectors, there are

[15] The index k_j can be continuous but for the sake of simplicity we use a discrete one.

choices to be made in the sense that we can have different intermediate combinations which lead to a state with 0 angular momentum. So we need to specify a scheme with which we sum the various angular momenta. We adopt a convention where we decide to sum the momenta as cumulants, i.e. summing Λ_1 with Λ_2 we get Λ_{12} which combined with Λ_3 give rise to Λ_{123} and so on. For the sake of simplicity, we indicate the full angular momenta with $\Lambda \equiv (\Lambda_1, \Lambda_2, \Lambda_{12}, ..)$ and with $m \equiv (m_1, m_2, m_3, ..)$. We do not need to specify m_{12} since $m_{12} = m_1 + m_2$. It's important to specify the intermediate momenta because we get different elements of the basis. Let's suppose that

$$\Lambda_1 = 1,\ \Lambda_2 = 2,\ \Lambda_3 = 3,\ \Lambda_4 = 1, \tag{3.38}$$

we have different possibilities to get status with 0 angular momentum. For example

$$\Lambda_{12} = 2 \longrightarrow \Lambda_{123} = 1, \quad \Lambda_{12} \qquad\qquad = 1 \longrightarrow \Lambda_{123} = 1, \tag{3.39}$$

give rise to two different elements of the basis which are respectively P_Λ and $P_{\Lambda'}$ where $\Lambda = (1, 2, 2, 3, 1, 1)$ and $\Lambda' = (1, 2, 1, 3, 1, 1)$. According to what we have done in the case $N = 2$ we have introduced the elements of the basis as P_Λ. So any isotropic function can be decomposed as

$$f(\mathbf{R}) = \sum_{\Lambda} f(r_1, .., r_N) P_\Lambda(\hat{\mathbf{r}}_1, .., \hat{\mathbf{r}}_N), \tag{3.40}$$

where the sum is over all the possible combinations of N angular momenta that give a final state of 0 angular momentum. The basis' elements can be written as

$$P_\Lambda(\hat{\mathbf{R}}) = \sum_{m_1, m_2..} C_m^\Lambda \prod_{i=1}^{N} Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i), \tag{3.41}$$

where the weights are

$$C_m^\Lambda \equiv C_{m_1..m_N}^{\Lambda_1..\Lambda_N} = \sum_{m_{12}, m_{123}..} \langle \Lambda_1, m_1, \Lambda_2, m_2 | \Lambda_{12}, m_{12} \rangle \times .. \times$$
$$\langle \Lambda_{1..N-1}, m_{1..N-2}, \Lambda_{N-1}, m_{N-1} | \Lambda_{1..N-1}, m_{1..N-1} \rangle \langle \Lambda_{1..N-1}, m_{1..N-1}, \Lambda_N, m_N | \Lambda, 0, 0 \rangle, \tag{3.42}$$

where $|\Lambda, 0, 0\rangle$ is the state in the basis of common eigenfunctions of $L_1^2, L_2^2, ., , L_N^2, L^2, L_z$. Following the same reasoning of eq.(3.26) we can demonstrate the eq.(3.42) using iteratively the addition formula for angular momentum. Indeed we can start decomposing P_Λ as if it was the state resulting from the linear combination of $|\Lambda_{1..N-1}, m_{1..N-1}\rangle$ and $|\Lambda_N, m_N\rangle$:

$$|\Lambda_{1..N-1}, m_{1..N-1}, \Lambda_N, m_N\rangle = \sum_{m_{1..N-1}, m_N} \langle \Lambda_{1..N-1}, m_{1..n-1}, \Lambda_N, m_N || \Lambda, 0, 0 \rangle |\Lambda_{1..N-1}, m_{1..N-1}\rangle Y_{\Lambda_N m_N}, \tag{3.43}$$

where $|\Lambda_{1..N-1}, m_{1..N-1}\rangle$ is the state resulting from the addition of $N - 1$ angular momenta. Now using the addition formula for

$$\mathbf{\Lambda_{1..N-2}} + \mathbf{\Lambda_{N-1}} = \mathbf{\Lambda_{1..N-1}}, \tag{3.44}$$

we can proceed in an iterative way decomposing $|\Lambda_{1..N-1}, m_{1..N-1}\rangle$ as

$$|\Lambda_{1..N-1}, m_{1..N-1}\rangle = \sum_{m_{1..N-2}, m_{N-1}} \langle \Lambda_{1..N-2}, m_{1..n-2}, \Lambda_{N-1}, m_{N-1} | \Lambda_{1..N-1}, m_{1..N-1} \rangle \times$$
$$\times |\Lambda_{1..N-2}, m_{1..N-2}\rangle Y_{\Lambda_{N-1} m_{N-1}}(\hat{\mathbf{r}}_{N-1}). \tag{3.45}$$

So we get

$$P_\Lambda = |\Lambda_1, \Lambda_2, \Lambda_{12}, ..\Lambda_N\rangle = \sum_{m_1, .., m_N} \sum_{m_{12}, m_{123}..} \langle \Lambda_1, m_1, \Lambda_2, m_2 | \Lambda_{12}, m_{12} \rangle \times .. \times$$
$$\langle \Lambda_{1..N-1}, m_{1..N-1}, \Lambda_N, m_N, | \Lambda, 0, 0 \rangle \prod_{i=1}^{N} Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i), \tag{3.46}$$

which can be recast in the desired expression of eq.(3.41).

We conclude this section by deriving the explicit expression for eq.(3.42) and by analyzing some useful properties of the basis elements. Now we focus on deriving the explicit expression for eq.(3.42) in the case $N = 3$, which is the one corresponding to the four-point correlation functions. Then we'll generalize to the case $N > 3$. So we have

$$
\begin{aligned}
C_m^\Lambda &= \sum_{m_{12}} \langle \Lambda_1, m_1, \Lambda_2, m_2 | \Lambda_{12}, m_{12} \rangle \langle \Lambda_{12}, m_{12}, \Lambda_3, m_3 | \Lambda, 0, 0 \rangle && (3.47) \\
&= \sqrt{2\Lambda_{12}+1} \sum_{m_{12}} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} \begin{pmatrix} \Lambda_{12} & \Lambda_3 & 0 \\ m_{12} & m_3 & 0 \end{pmatrix} (-1)^{-\Lambda_{12}+\Lambda_3-m_{12}+\Lambda_2-\Lambda_1} \\
&= \sqrt{2\Lambda_3+1} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \begin{pmatrix} \Lambda_3 & \Lambda_3 & 0 \\ m_3 & -m_3 & 0 \end{pmatrix} (-1)^{-\Lambda_3+\Lambda_3-m_{12}+\Lambda_2-\Lambda_1} \\
&= \sqrt{2\Lambda_3+1} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \frac{(-1)^{\Lambda_3-m_3}}{\sqrt{2\Lambda_3+1}} (-1)^{-\Lambda_3+\Lambda_3-m_3+\Lambda_2-\Lambda_1} && (3.48) \\
&= \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix} (-1)^{\Lambda_3+\Lambda_2+\Lambda_1}, && (3.49)
\end{aligned}
$$

where in the first step we have used the definition of the $3-j$ symbols (appendix B.3), in the second we have exploited that $\begin{pmatrix} \Lambda_{12} & \Lambda_3 & 0 \\ m_{12} & m_3 & 0 \end{pmatrix}$ is different from 0 only if $m_{12} = -m_3$[16] and that $\Lambda_{12} = \Lambda_3$. In the third passage, we have employed eq.(B.24) while in the last one, we have changed the signs of the integers[17]. When $N \geq 4$ we have

$$
\begin{aligned}
C_m^\Lambda = &\sum_{m_{12}, m_{123}, ..} \langle \Lambda_1, m_1, \Lambda_2, m_2 | \Lambda_{12}, m_{12} \rangle \times .. \\
&\times \langle \Lambda_{1..N-1}, m_{1..N-2}, \Lambda_{N-1}, m_{N-1} | \Lambda_{1..N-1}, m_{1..N-1} \rangle \langle \Lambda_{1..N-1}, m_{1..N-1} \Lambda_N, m_N | \Lambda, 0, 0 \rangle && (3.50) \\
&= \prod_{j=2}^{N-1} \sqrt{2\Lambda_{1..j}+1} \sum_{m_{12}, m_{123}, ..} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} .. \begin{pmatrix} \Lambda_{1..N-2} & \Lambda_{N-1} & \Lambda_{1..N-1} \\ m_{1..N-1} & m_{N-1} & -m_{1..N-1} \end{pmatrix} \times \\
&\times \begin{pmatrix} \Lambda_{1..N-1} & \Lambda_N & 0 \\ m_{1..N-1} & m_N & 0 \end{pmatrix} (-1)^{-\Lambda_{12}-..-\Lambda_{1..N-1}-m_{12}-..-m_{1..N-1}+\Lambda_2+..+\Lambda_N-\Lambda_1} && (3.51) \\
&= \prod_{j=2}^{N-1} \sqrt{2\Lambda_{1..j}+1} \sum_{m_{12}, m_{123}, ..} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} .. \begin{pmatrix} \Lambda_{1..N-2} & \Lambda_{N-1} & \Lambda_N \\ m_{1..N-2} & m_{N-1} & m_N \end{pmatrix} \times \\
&\times \frac{(-1)^{\Lambda_N-m_N}}{\sqrt{2\Lambda_{1..N-1}+1}} (-1)^{-\Lambda_{12}-..-\Lambda_{1..N-1}-m_{12}-..-m_{1..N-1}+\Lambda_2+..+\Lambda_N-\Lambda_1} && (3.52) \\
&= \prod_{j=1}^{N-2} \sqrt{2\Lambda_{1..j}+1} \sum_{m_{12}, m_{123}, ..} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} .. \begin{pmatrix} \Lambda_{1..N-2} & \Lambda_{N-1} & \Lambda_{1..N-1} \\ m_{1..N-2} & m_{N-1} & -m_{1..N-1} \end{pmatrix} \times \\
&\times (-1)^{+\Lambda_{12}+..+\Lambda_{1..N-2}-m_{12}-..-m_{1..N-2}} (-1)^{\Lambda_1+\Lambda_2+..+\Lambda_N}, && (3.53)
\end{aligned}
$$

where in the second step we have used the definition of the $3-j$ eq.(B.22), in the third we have used the property eq.(B.24) and in the last step we have used that

$$
\begin{aligned}
&(-1)^{\Lambda_N-m_N}(-1)^{-\Lambda_{12}-..-\Lambda_{1..N-1}-m_{12}-..-m_{1..N-1}+\Lambda_2+..+\Lambda_N-\Lambda_1} = \\
&(-1)^{\Lambda_1+\Lambda_2+..+\Lambda_N}(-1)^{+\Lambda_{12}+..+\Lambda_{1..N-1}-m_{12}-..-m_{1..N-1}}(-1)^{\Lambda_{1..N-1}-\Lambda_N}(-1)^{-m_{1..N-1}-m_N} = \\
&(-1)^{\Lambda_1+\Lambda_2+..+\Lambda_N}(-1)^{+\Lambda_{12}+..+\Lambda_{1..N-1}-m_{12}-..-m_{1..N-1}}, && (3.54)
\end{aligned}
$$

where we have used that $\Lambda_{1..N-1} = \Lambda_N$, $m_{1..N-1} = -m_N$ and the fact that since all the angular momenta are integers their sign in the exponential of (-1) doesn't matter. Note that in an analogous

[16]This can be seen from the definition of the Clebsh-Gordan coefficients.
[17]If $a, b \in \mathbb{Z}$ $(-1)^{a+b} = (-1)^{a-b} = (-1)^{-a+b} = (-1)^{-a-b}$.

way to the case $N=3$, the sum in eq.(3.51) goes from m_{12} to $m_{1..N-1}$ while in eq.(3.52) from m_{12} to $m_{1..N-2}$.

We conclude this section by demonstrating some useful properties of the basis we have introduced.

We start analyzing *parity.* Since the parity of the spherical harmonics $Y_{lm}(\mathbf{r})$ is $(-1)^l$ (appendix B.1) we get

$$\mathcal{P}[P_\Lambda(\hat{\mathbf{R}})] = \mathcal{P}[\sum_{m_1,m_2,..} C^\Lambda_m \prod_{i=1}^N Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i)] = \sum_{m_1,m_2,..} C^\Lambda_m \prod_{i=1}^N \mathcal{P}[Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i)] = \sum_{m_1,m_2,..} C^\Lambda_m \prod_{i=1}^N Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i)(-1)^{\Lambda_i} \tag{3.55}$$

$$= \varepsilon(\Lambda) \sum_{m_1,m_2,..} C^\Lambda_m \prod_{i=1}^N Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i) = \varepsilon(\Lambda) P^\Lambda_m(\hat{\mathbf{R}}), \tag{3.56}$$

where $\varepsilon(\Lambda) = (-1)^{\sum_{i=1}^N \Lambda_i}$ and where we have used that C^Λ_m are numbers which do not transform under parity.

Now we analyze the behavior under complex conjugation

$$\begin{cases} P^\Lambda_m = [P^\Lambda_m]^* & \text{if } \sum_{i=1}^N \Lambda_i \text{ is even} \\ P^\Lambda_m = -[P^\Lambda_m]^* & \text{if } \sum_{i=1}^N \Lambda_i \text{ is odd} \end{cases}. \tag{3.57}$$

We now prove this last claim. First, the spherical harmonics (appendix B.1) satisfy

$$Y^*_{l,m}(\hat{\mathbf{r}}) = (-1)^m Y^*_{l,-m}(\hat{\mathbf{r}}) \tag{3.58}$$

and we can also write that

$$\langle L_1, -m_1, L_2, -m_2 | L, -m\rangle = (-1)^{L_1+L_2-L} \langle L_1, m_1, L_2, m_2 | L, m\rangle, \tag{3.59}$$

since

$$\langle L_1, -m_1, L_2, -m_2 | L, -m\rangle = \begin{pmatrix} L_1 & L_2 & L \\ -m_1 & -m_2 & m \end{pmatrix} (-1)^{-L_1+L_2+m}\sqrt{2L+1} \tag{3.60}$$

$$= \begin{pmatrix} L_1 & L_2 & L \\ m_1 & m_2 & -m \end{pmatrix} (-1)^{-L_1+L_2+m}\sqrt{2L+1}(-1)^{L_1+L_2+L} \tag{3.61}$$

$$= \langle L_1, m_1, L_2, m_2 | L, m\rangle (-1)^{L_1+L_2+L} \tag{3.62}$$

$$= \langle L_1, m_1, L_2, m_2 | L, m\rangle (-1)^{L_1+L_2-L}, \tag{3.63}$$

where we have used eq.(B.22), eq.(B.28) and we have flipped the sign of L in the last equality. Then we have $[C^\Lambda_m]^* = C^\Lambda_m$ since the Clebsch-Gordan coefficients can be chosen to be real ([56]). Using eq.(3.63) we have

$$C^\Lambda_{-m} = \varepsilon(\Lambda) C^\Lambda_m \tag{3.64}$$

since

$$\begin{aligned}
C^\Lambda_{-m} &= \sum_{m_{12},..} \langle \Lambda_1, -m_1, \Lambda_2, -m_2 | \Lambda_{12}, -m_{12}\rangle \times .. \times \langle \Lambda_{1..N-1}, -m_{1..N-1} \Lambda_N, -m_N | \Lambda, 0, 0\rangle \\
&= \sum_{m_{12},..} \langle \Lambda_1, m_1, \Lambda_2, m_2 | \Lambda_{12}, m_{12}\rangle \times .. \times \\
&\langle \Lambda_{1..N-1}, m_{1..N-2}, \Lambda_{N-1}, m_{N-1} | \Lambda_{1..N-1}, m_{1..N-1}\rangle \langle \Lambda_{1..N-1}, m_{1..N-1}, \Lambda_N, m_N | \Lambda, 0, 0\rangle \times \\
&\times (-1)^{\Lambda_1+\Lambda_2-\Lambda_{12}+\Lambda_{12}+\Lambda_3-\Lambda_{123}+..+\Lambda_{1..N-2}+\Lambda_{N-1}-\Lambda_{1..N-1}+\Lambda_{1..N-1}+\Lambda_N} \\
&= \sum_{m_{12},..} \langle \Lambda_1, m_1, \Lambda_2, m_2 | \Lambda_{12}, m_{12}\rangle \times .. \times \\
&\langle \Lambda_{1..N-1}, m_{1..N-2}, \Lambda_{N-1}, m_{N-1} | \Lambda_{1..N-1}, m_{1..N-1}\rangle \langle \Lambda_{1..N-1}, m_{1..N-1}, \Lambda_N, m_N | \Lambda, 0, 0\rangle \times \\
&\times (-1)^{\Lambda_1+\Lambda_2+..+\Lambda_N} \\
&= \varepsilon(\Lambda) C^\Lambda_m,
\end{aligned}$$

where in the first step since $m_{12},.. \in [-\Lambda_{12}, \Lambda_{12}],..$ we directly flip the sign of $m_{12},...$ Now we can prove the claim of eq.(3.57) in the following way

$$[P_m^\Lambda]^* = \sum_{m_1,m_2,..} C_m^\Lambda \prod_{i=1}^N Y^*_{\Lambda_i m_i}(\hat{\mathbf{r}}_i) = \sum_{m_1,m_2,..} C_m^\Lambda \prod_{i=1}^N Y_{\Lambda_i,-m_i}(\hat{\mathbf{r}}_i)(-1)^{m_i} \tag{3.65}$$

$$= \varepsilon(\Lambda) \sum_{m_1,m_2,..} C_{-m}^\Lambda (-1)^{\sum_{i=1}^N m_i} \prod_{i=1}^N Y_{\Lambda_i,-m_i}(\hat{\mathbf{r}}_i) = \varepsilon(\Lambda) \sum_{m_1,m_2,..} C_m^\Lambda \prod_{i=1}^N Y_{\Lambda_i,m_i}(\hat{\mathbf{r}}_i) = \varepsilon(\Lambda) P_m^\Lambda, \tag{3.66}$$

where we have used that $\sum_i m_i = 0$ and since $m_i \in [-\Lambda_i, \Lambda_i]$, in the fourth equality we have flipped the sign of $m = (m_1, .., m_N)$.

From the definition, it's clear that P_Λs are a complete basis of $\chi(0,0)$. Now we discuss the orthonormality condition of P_Λ, which simply follows from the orthonormality of the spherical harmonics eq.(B.3) and of Clebsch-Gordan coefficients, appendix B.2,

$$\int d\hat{\mathbf{R}} P_\Lambda(\hat{\mathbf{R}}) P^*_{\Lambda'}(\hat{\mathbf{R}}) = \delta_{\Lambda_1,\Lambda_1'} \delta_{\Lambda_2,\Lambda_2'} \delta_{\Lambda_{12},\Lambda_{12}'} .., \tag{3.67}$$

where we recall that $\hat{\mathbf{R}} = [\hat{\mathbf{r}}_1, .., \hat{\mathbf{r}}_N]$.

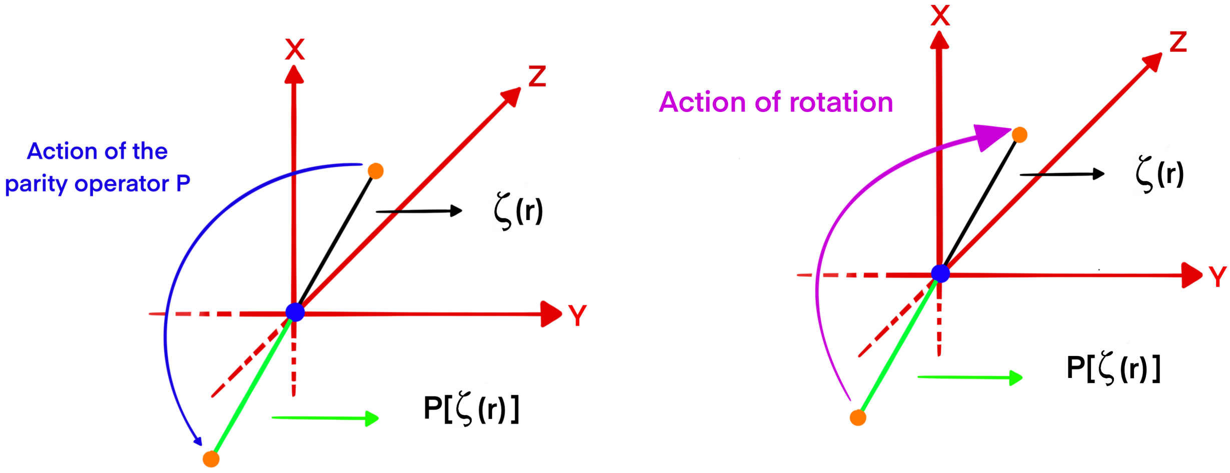
3.2 Parity violation with the four-point scalar correlation function

In this section, we explore the necessity of using $NPCF$s (N-Point Correlation Functions) with $N \geq 4$ to search for parity violation signatures. We consider a random real scalar field $\phi(\mathbf{x})$ defined on $\mathbb{R}^3$, and denote all the correlation functions with the Greek letter ζ, specifying their dependence to clarify the context.

Firstly, we approach this issue from a geometric perspective. We can envision the $NPCF$s as solid figures: the $2PCF$ corresponds to a segment (see Fig. 10a), the $3PCF$ to a triangle (see Fig. 11a), the $4PCF$ to a tetrahedron (see Fig. 12a), and so on. Geometrically, a parity transformation is a reflection through the origin (as depicted in Fig. 10a, 11a, and 12a). $NPCF$s violate parity if the reflected solid figure differs from the original one, i.e., they cannot be rigidly rotated to overlap. In a 3D space, parity transformation can also be interpreted as a reflection about a plane (such as the $y = 0$ plane depicted in Fig. 10a, 11a, and 12a), followed by a 180° rotation about the vector perpendicular to that plane, $\hat{\mathbf{y}} = (0, 1, 0)$ in our case. Since the $NPCF$s are isotropic, only the mirroring effect is significant.

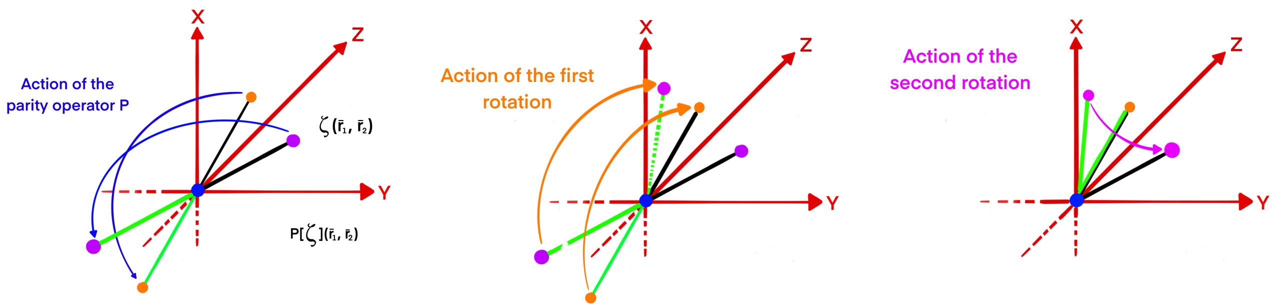
If $N = 2$, there always exists a rotation, represented in Fig. 10b, that can map the transformed segment back to the original one. In the same way the parity-transformed $3PCF$ (fig.11a) can be transformed into the original one through the two rotations sketched in fig.11b and 11c. While in the case $N = 4$ we can rotate as in fig.12b but as we can see in fig.12c the two tetrahedrons are different. Choosing the blue dot as our primary vertex, the $4PCF$ is defined by the remaining three vertices, the orange ($\mathbf{r}_1$), red ($\mathbf{r}_2$) and violet ($\mathbf{r}_3$). For a given tetrahedron, $\zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, we can order the argument from the smallest to the larger one, i.e. $r_1 \leq r_2 \leq r_3$. When viewing the tetrahedron from the blue dot, i.e. the primary one, looking down along each vector r_i, the direction in which one reads going from smallest to largest side defines a handedness, clockwise (right) or counterclockwise (left). In this fashion, the tetrahedron of fig.12a is right-handed while the parity transformed is left-handed (see fig.12c). So regarding the $4PCF$ parity converts the clockwise tetrahedrons to the counterclockwise ones. This kind of argument can't be applied in the case $N = 2\backslash 3$, i.e. we aren't able to define a handedness that is equivalent to saying that the $2PCF$ and $3PCF$ are parity symmetric. The same line of reasoning can be applied in the cases in which $N\rangle 4$.

We can demonstrate why parity violation could occur only with $N \geq 4$. The $NPCF$ are isotropic functions of $N-1$ vector variables. Therefore they must be function of scalar quantities only. In the case $N = 2$ we have only one available vector, $\mathbf{r}$. The only scalar we can create is r. While if $N = 3$ we



(a) $\zeta(r)$ and $\mathcal{P}\zeta(r)$. (b) The rotation.

Figure 10: $2PCF$



(a) $\zeta(\mathbf{r}_1, \mathbf{r}_2)$ and $\mathcal{P}\zeta(\mathbf{r}_1, \mathbf{r}_2)$. (b) The first rotation. (c) The second rotation.

Figure 11: $3PCF$

have r_1, r_2 and $\mathbf{r}_1 \cdot \mathbf{r}_2$. So we can't violate parity since $|\mathcal{P}[\mathbf{r}_1]| = |\mathbf{r}_1|$ and $P[\mathbf{r}_1 \cdot \mathbf{r}_2] = (-\mathbf{r}_1) \cdot (-\mathbf{r}_2) = \mathbf{r}_1 \cdot \mathbf{r}_2$. If we have three vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ we can form scalar quantities from an odd number of vectors, such as $\mathbf{r}_1 \times \mathbf{r}_2 \cdot \mathbf{r}_3$, which are not parity invariant.

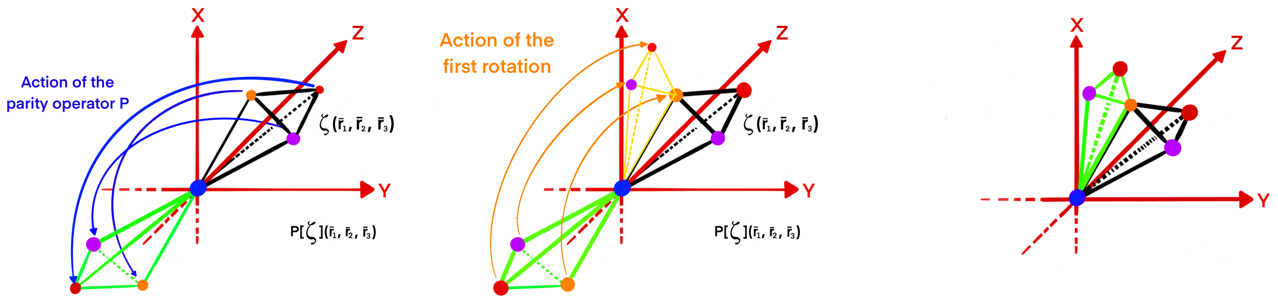
We can also tackle the problem using the transformation law of P_Λ introduced in section 3.1, i.e. eq.(3.56). The case $N = 2$ is trivial since we have not angular dependence. While in general, eq.(3.56) tells us that

$$\mathcal{P}[P_\Lambda] = \varepsilon(\Lambda) P_\Lambda \quad \text{with } \varepsilon(\Lambda) = (-1)^{\sum_{i=1}^{N} \Lambda_i}. \tag{3.68}$$

If $N = 3$ we have to sum two angular momenta, $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, to obtain 0. To get 0, inevitably, we need $\Lambda_1 = \Lambda_2$, which implies

$$\varepsilon(\Lambda) = (-1)^{\Lambda_1 + \Lambda_2} = (-1)^{2\Lambda_1} = 1 \; \forall \Lambda_1 \in \mathcal{N}. \tag{3.69}$$

Since every element of the basis is parity even, we have that any isotropic function of two variables is



(a) $\zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ and $\mathcal{P}\zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$.

(b) The first rotation.

(c) The parity even and parity odd tetrahedons.

Figure 12: $4PCF$

parity even. In the case $N \geq 4$ we have to sum $N-1$ angular momenta to obtain 0,

$$\sum_{i=1}^{N-1} \mathbf{\Lambda_i} = \mathbf{0}. \tag{3.70}$$

For example if $N = 4$ we can obtain a P_Λ with $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_{12}, \Lambda_3) = (1, 2, 2, 2)$ which has $\varepsilon(\Lambda) = (-1)^{1+2+2} = -1$. So if the isotropic function of N vector variables gets non 0 contribution from the elements of the basis with $\varepsilon(\Lambda) = -1$ the $NPCF$ violates parity.

3.3 The galaxy four-point correlation function estimator, data analysis and results

In this section, we discuss the estimator of the galaxy $4PCF$ (eq.(3.84)) projected on the basis of section 3.1, we outline the algorithm used in data analysis (eq.(3.97), eq.(3.98) and eq.(3.99) and lastly we report the claim of parity violation of [2] and [1], which use the final galaxy catalog of Baryon Oscillation Spectroscopy Survey ($BOSS$), from the twelfth data release ($DR12$) [6] of the Sloan Digital Survey-III ($SDSS-III$).

The galaxy $4PCF$ estimator has been introduced in eq.(3.2) as

$$\zeta(\mathbf{s}, \mathbf{r}_1, \mathbf{r_2}, \mathbf{r}_3) \equiv \langle \delta(\mathbf{s})\delta(\mathbf{s} + \mathbf{r}_1)\delta(\mathbf{s} + \mathbf{r_2})\delta(\mathbf{s} + \mathbf{r}_3)\rangle. \tag{3.71}$$

From the discussion of section 1.6 we learn that the galaxy $4PCF$ is homogeneous and isotropic, so it can be decomposed into the basis introduced in section 3.1:

$$\zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \sum_{\Lambda=(\Lambda_1,\Lambda_2,\Lambda_3)} \zeta_\Lambda(r_1, r_2, r_3) P_\Lambda(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3), \tag{3.72}$$

where $\Lambda \equiv (\Lambda_1, \Lambda_2, \Lambda_3)$; in the case $N = 3$ we don't need to specify any intermediate state of angular momenta since Λ_{12} must be equal to Λ_3. $\zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, using eq.(3.49), becomes

$$\zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \sum_\Lambda \zeta_\Lambda(r_1, r_2, r_3) \sum_{m_1,m_2,m_3} C_m^\Lambda Y_{\Lambda_1 m_1}(\hat{\mathbf{r}}_1) Y_{\Lambda_2 m_2}(\hat{\mathbf{r}}_2) Y_{\Lambda_3 m_3}(\hat{\mathbf{r}}_3) \tag{3.73}$$

$$= \sum_\Lambda \zeta_\Lambda(r_1, r_2, r_3)\varepsilon(\Lambda) \sum_{m_1,m_2,m_3} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix} Y_{\Lambda_1 m_1}(\hat{\mathbf{r}}_1) Y_{\Lambda_2 m_2}(\hat{\mathbf{r}}_2) Y_{\Lambda_3 m_3}(\hat{\mathbf{r}}_3). \tag{3.74}$$

The coefficients (hereafter denoted "multiplets") $\zeta_\Lambda(r_1, r_2, r_3)$ can be obtained through the orthonormality relation eq.(3.67):

$$\zeta_\Lambda(r_1, r_2, r_3) = \int d\hat{\mathbf{r}}_1 d\hat{\mathbf{r}}_2 d\hat{\mathbf{r}}_3 \zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) P^*_\Lambda(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \tag{3.75}$$

$$= (-1)^{\Lambda_1+\Lambda_2+\Lambda_3} \int d\hat{\mathbf{r}}_1 d\hat{\mathbf{r}}_2 d\hat{\mathbf{r}}_3 \zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) P_\Lambda(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \tag{3.76}$$

$$= \sum_{m_1,m_2,m_3} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \int d\hat{\mathbf{r}}_1 d\hat{\mathbf{r}}_2 d\hat{\mathbf{r}}_3 \zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) Y_{\Lambda_1 m_1}(\hat{\mathbf{r}}_1) Y_{\Lambda_2 m_2}(\hat{\mathbf{r}}_2) Y_{\Lambda_3 m_3}(\hat{\mathbf{r}}_3), \tag{3.77}$$

where in the second equality we have used eq.(3.57), $P^*_\Lambda = \varepsilon(\Lambda) P_\Lambda$, while in the last equality the explicit expression for P_Λ which is eq.(3.41) in the case $N = 3$. Now we recall that from eq.(3.56) and (3.57) we have

$$\mathcal{P}[P_{(\Lambda_1,\Lambda_2,\Lambda_3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)] = \varepsilon(\Lambda) P_{(\Lambda_1,\Lambda_2,\Lambda_3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \tag{3.78}$$

$$P^*_{(\Lambda_1,\Lambda_2,\Lambda_3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \varepsilon(\Lambda) P_{(\Lambda_1,\Lambda_2,\Lambda_3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \tag{3.79}$$

which allows us to find a natural split of $\zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ into parity-even and parity-odd parts:

$$\zeta_+(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \sum_{\Lambda_1+\Lambda_2+\Lambda_3=\text{even}} \zeta_\Lambda(r_1, r_2, r_3) P_\Lambda(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3), \tag{3.80}$$

$$\zeta_-(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \sum_{\Lambda_1+\Lambda_2+\Lambda_3=\text{odd}} \zeta_\Lambda(r_1, r_2, r_3) P_\Lambda(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3), \tag{3.81}$$

which satisfy

$$\mathcal{P}[\zeta_\pm((\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3))] = \pm[\zeta_\pm((\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3))], \tag{3.82}$$

$$\zeta^*_\pm((\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)) = \pm[\zeta_\pm((\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3))]. \tag{3.83}$$

We stress that since P_Λs are orthonormal and the ones with $\Lambda_1 + \Lambda_2 + \Lambda_3$ odd are immaginary, to obtain a real $4PCF$ $\zeta_{\Lambda_1+\Lambda_2+\Lambda_3=odd}(r_1, r_2, r_3)$ must be purely immaginary.

Now we briefly explain the algorithm used in data analysis. In eq.(3.4) we have introduced the galaxy $4PCF$ estimator

$$\hat{\zeta}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{V} \int d\mathbf{s} \delta(\mathbf{s}) \delta(\mathbf{s} + \mathbf{r}_1) \delta(\mathbf{s} + \mathbf{r}_2) \delta(\mathbf{s} + \mathbf{r}_3), \tag{3.84}$$

where we recall that we denote with a hat the quantity measured from data. If we now insert eq.(3.84) into eq.(3.77) we obtain an estimator for the basis coefficients of the galaxy $4PCF$

$$\hat{\zeta}_\Lambda(r_1, r_2, r_3) = \frac{1}{V} \int d\mathbf{s} \delta(\mathbf{s}) \int \left[\sum_{m_1,m_2,m_3} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left(\prod_{i=1}^{3} d\hat{\mathbf{r}}_i \delta(\mathbf{s} + \mathbf{r}_i) Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i) \right) \right] \tag{3.85}$$

$$= \sum_{m_1,m_2,m_3} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \frac{1}{V} \int d\mathbf{s} \delta(\mathbf{s}) a_{\Lambda_1,m_1}(\mathbf{s}, r_1) a_{\Lambda_2,m_2}(\mathbf{s}, r_2) a_{\Lambda_3,m_3}(\mathbf{s}, r_3), \tag{3.86}$$

where we define the harmonics coefficients as

$$a_{\Lambda_i,m_i}(\mathbf{s}, r_i) \equiv \int d\hat{\mathbf{r}}_i \delta(\mathbf{s} + \mathbf{r}_i) Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i). \tag{3.87}$$

Then we bin the $4PCF$ in radii, the appeal of this decomposition is clear since eq.(3.85) is exactly separable in $\hat{\mathbf{r}}_i$. Similarly to what we have done in eq.(3.7), we introduce a binning function $\theta^b(r)$

which is unity if r is in radial bin b and zero else. The radially-averaged $4PCF$ becomes

$$\hat{\zeta}^B_\Lambda = \frac{1}{\nu_B}\int\left[\prod_{i=1}^{3} r_i^2 dr_i \theta^{b_i}(r_i)\right]\hat{\zeta}_\Lambda(r_1, r_2, r_3) \tag{3.88}$$

$$= \frac{1}{\nu_B}\frac{1}{V}\int d\mathbf{s}\delta(\mathbf{s})\int\left[\sum_{m_1,m_2,m_3}\begin{pmatrix}\Lambda_1 & \Lambda_2 & \Lambda_3\\ m_1 & m_2 & m_3\end{pmatrix}\left(\prod_{i=1}^{3} d\mathbf{r}_i\theta^{b_i}(r_i)Y_{\Lambda_i m_i}(\hat{\mathbf{r}}_i)\delta(\mathbf{s}+\mathbf{r}_i)\right)\right] \tag{3.89}$$

$$= \sum_{m_1,m_2,m_3}\begin{pmatrix}\Lambda_1 & \Lambda_2 & \Lambda_3\\ m_1 & m_2 & m_3\end{pmatrix}\frac{1}{V}\int d\mathbf{s}\delta(\mathbf{s})a^{b_1}_{\Lambda_1,m_1}(\mathbf{s})a^{b_2}_{\Lambda_2,m_2}(\mathbf{s})a^{b_3}_{\Lambda_3,m_3}(\mathbf{s}), \tag{3.90}$$

where we have introduced the radially averaged harmonic coefficients

$$a^{b_i}_{\Lambda_i,m_i}(\mathbf{s}) \equiv \frac{1}{\nu_{b_i}}\int d\mathbf{r}_i\theta^{b_i}(r_i)Y_{\Lambda_i m_1}(\hat{\mathbf{r}}_i)\delta(\mathbf{s}+\mathbf{r}_i), \tag{3.91}$$

and the bin volume

$$\nu_B = \prod_{i=1}^{3}\nu_{b_i} = \left[\prod_{i=1}^{3} r_i^2 dr_i\theta^{b_i}(r_i)\right]. \tag{3.92}$$

In eq.(3.90) we have introduced a bin index $B = [b_1, b_2, b_3]$ where b_i specify in which bin we are analyzing the variable r_i. In the example of eq.(3.7), $b_i = 1, .., 10$. So in that case we have 10^3 possibilities for the bin index; $B = [1,1,1], [1,1,2]..$ etc. We underline that $\hat{\zeta}^B_\Lambda$ doesn't depend on (r_1, r_2, r_3), it's the coefficient of $\hat{\zeta}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ projected in the basis' element denoted by Λ for the angular variables and B for the radial one.

As mentioned in section 3 in general we have access to a discrete field of N_g particles at positions $\mathbf{x}_{i=1,..,N_g}$ with weights w_i

$$\delta(\mathbf{s}) = \sum_{i=1}^{N_g}\left[w_i\delta^{(3)}(\mathbf{s}-\mathbf{x}_i)\right], \tag{3.93}$$

where $\delta^{(3)}(..)$ is the three dimensional Dirac delta function. Using eq.(3.93) we can replace the integral of eq.(3.90) with a sum over the N_g galaxies:

$$\hat{\zeta}^B_\Lambda = \sum_{i=1}^{N_g} w_i \sum_{m_1,m_2,m_3}\begin{pmatrix}\Lambda_1 & \Lambda_2 & \Lambda_3\\ m_1 & m_2 & m_3\end{pmatrix}\frac{1}{V}a^{b_1}_{\Lambda_1,m_1}(\mathbf{x}_i)a^{b_2}_{\Lambda_2,m_2}(\mathbf{x}_i)a^{b_3}_{\Lambda_3,m_3}(\mathbf{x}_i). \tag{3.94}$$

Strictly, this decomposition is correct only if the bin indices b_1, b_2 and b_3 are not coincident (due to shot noise effects, [58]). Therefore, in data analysis, we guarantee this by enforcing $b_1\langle b_2\langle b_3$.

Now in practice, we assume a fixed maximum multipole Λ_{max} and a number of bins N_b. We take Λ_{max} relatively low, which gives an angular resolution of $\theta_{min} \simeq \frac{2\pi}{\Lambda_{max}}$ for the internal angles of the $4PCF$ tetrahedron [1]. This approximation ensures that only a finite number of $a^{b_i}_{\Lambda_i m_i}(\mathbf{x}_i)$ coefficients (asymptotically [40], $N_b \times \Lambda^3_{max}$) need to be estimated at each position $\mathbf{x}_i$. If we want to reconstruct the $4PCF$, $\zeta(\mathbf{s}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ from the set of measured basis coefficients, this truncation would lead to an approximation error. We can avoid this issue by projecting the theory model in the same way. In data analysis we are not interested in all the possible Λ with $\Lambda_i \leq \Lambda_{max}$ but only in the ones that can violate parity, i.e. $\Lambda_1 + \Lambda_2 + \Lambda_3 = odd$.

The full procedure for estimating the $4PCF$ from a discrete set of N_g particles, eq.(3.93), is thus:

1. For a given primary particle $i = 1, .., N_g$, compute $a^{b_i}_{\Lambda_i m_i}(\mathbf{x}_i)$ as a weighted sum of spherical harmonics:

$$a^{b_i}_{lm}(\mathbf{x}_i) = \frac{1}{\nu_{b_i}} \int d\mathbf{r}\theta^{b_i}(r)\delta(\mathbf{r}+\mathbf{x}_i)Y_{\Lambda_i m_i}(\hat{\mathbf{r}}) \tag{3.95}$$

$$= \frac{1}{\nu_{b_i}} \sum_{j=1}^{N_g} \left[w_j \int d\mathbf{r}\delta^{(3)}(\mathbf{r}+\mathbf{x}_i-\mathbf{x}_j)Y_{\Lambda_i m_i}(\hat{\mathbf{r}})\theta^{b_i}(r) \right] \tag{3.96}$$

$$= \frac{1}{\nu_{b_i}} \sum_{j=1}^{N_g} \left[w_j Y_{\Lambda_i m_i}(\widehat{\mathbf{x}_j-\mathbf{x}_i})\theta^{b_i}(|\mathbf{x}_j-\mathbf{x}_i|) \right] . \tag{3.97}$$

2. For each Λ multiplet and bin index B, sum over m_1, m_2, m_3 the product of $a^{b_1}_{\Lambda_1,m_1}(\mathbf{x}_i)a^{b_2}_{\Lambda_2,m_2}(\mathbf{x}_i)a^{b_3}_{\Lambda_3,m_3}(\mathbf{x}_i)$ weighted by $\varepsilon(\Lambda)C^{\Lambda}_m$ to obtain

$$\sum_{m_1,m_2,m_3} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \frac{1}{V} a^{b_1}_{\Lambda_1,m_1}(\mathbf{x}_i)a^{b_2}_{\Lambda_2,m_2}(\mathbf{x}_i)a^{b_3}_{\Lambda_3,m_3}(\mathbf{x}_i). \tag{3.98}$$

3. Repeat for each primary particle and sum to get

$$\hat{\zeta}^{B}_{\Lambda} = \sum_{i=1}^{N_g} w_i \sum_{m_1,m_2,m_3} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \frac{1}{V} a^{b_1}_{\Lambda_1,m_1}(\mathbf{x}_i)a^{b_2}_{\Lambda_2,m_2}(\mathbf{x}_i)a^{b_3}_{\Lambda_3,m_3}(\mathbf{x}_i). \tag{3.99}$$

We note that the $4PCF$ contains also a *disconnected* piece sourced by two copies of the $2PCF$. Whilst this can be subtracted at the estimator level directly [59], it does not contribute to parity-odd multiplets since the $2PCF$ can't violate parity as explained in sec.3.2. As anticipated in section 3 the algorithm we implement has a complexity $O(N_g^2)$. Each coefficient $a^{b_i}_{\Lambda_i m_i}$ of eq.(3.97) involves a sum over N_g and the algorithm require to evaluate $a^{b_i}_{\Lambda_i m_i}$ at the location of each of the N_g particles, thus we get complexity order $O(N_g^2)$. Note that eq.(3.98) can be at most of order $O((1+2\Lambda_{max})^3)$ while eq.(3.99) is of $O(N_g)$, therefore their contribution can be completely disregarded in evaluating the efficiency of the method. We mention that in the analysis we do not have access to the overdensity matter field δ directly, and we must work instead with a set of data and random particles. This allows an estimation of the full $4PCF$ via a generalization [60] which we are not going to discuss.

We conclude this section by reporting the results of the analysis of [2] and [1], which use the final galaxy catalog of the Baryon Oscillation Spectroscopic Survey ($BOSS$), from the twelfth data release ($DR12$) [6] of the Sloan Digital Sky Survey-III ($SDSS-III$). The survey contains two samples, $CMASS$ and $LOWZ$. The former sample which contains 587 071 (216 041) galaxies in the Northern (Southern) galactic cap, across a redshift range $z \in [0.43, 0.7]$, while the latter contains 280 067 Luminous Red Galaxies ($LRGs$), across a redshift range $z \in [0.2, 0.4]$. Basically, the analysis used the method, we have discussed in this section, restricted to the parity odd multiplets, i.e. $\Lambda_1 + \Lambda_2 + \Lambda_3 = odd$. [2] find in $LOWZ$ 3.1σ evidence for a non-zero parity-odd $4PCF$, and in $CMASS$ a parity-odd $4PCF$ at 7.1σ. [1] find a a detection probability of 99.6% (2.9σ) using the $CMASS$ sample. This provides significant evidence for parity violation signature, either from cosmological sources or systematics. The authors perform various systematic tests which do not reveal any observational artifacts. However, at the moment we cannot exclude this possibility.

4 The diagrammatic rules for the $In - In$ formalism

In the previous sections, we have seen how to deal with the computation of the power spectrum for the single-field slow-roll model of Inflation. The procedure we have described it's standard and can be applied to every model of Inflation. In order to compute the power spectrum for tensor and scalar perturbation one has to derive the equation of motions, solve them with Bunch-Davies initial conditions, impose the canonical commutation relations, and perform the computation of the two-point correlation function in Fourier space. The power spectrum is the only correlator we need to consider if the field is *Gaussian* (see section 1.6). However, a detection of a non-zero three-point correlation function would rule out automatically the Gaussianity of the field under consideration. This line of reasoning clearly applies to all the other statistics. So, Non-Gaussianity constitutes a key observable in Cosmology since it allows us to discriminate among competing scenarios for the generation of cosmological perturbations [10]. Furthermore, concerning scalar fields and the quest for parity violation, it becomes essential to examine at least the four-point correlation function, which is the goal of this master's thesis.

Consequently, it becomes imperative to develop methodologies enabling the efficient computation of higher-order correlation functions in the early universe. According to the Inflationary paradigm, quantum fluctuations of the Inflaton field can be transferred to matter and radiation as explained in section 2.6.

In this section, we delve into the computations of correlation functions beyond the two-point statistics of quantum fluctuation during Inflation. These are "quantum averages", not averages over a classical ensemble of a stochastic field (see section 1.5). We are interested in reaching the latter situation since every cosmological field, the over-density matter field δ or the CMB temperature field T, from which we build observables today is assumed to be a stochastic classical field described by the set of $N-$point correlation functions. The field configuration in the early universe must be locked into one of an ensemble of classical configurations with ensemble averages given by the quantum correlators computed in the early Universe. It's suggested that it happens on superhorizon scales [61]. Once the fields under consideration are classical we can apply the Ergodic theorem, eq.(1.122) and interpret these ensemble averages as spatial averages.

The formalism we are going to introduce in section 4.1 is the $In - In$ formalism [17], which is very similar to what is done in quantum field theory (QFT) in the computation of $S-$matrix elements. Our goal here is not to provide a complete derivation of this formalism, but rather to explain the procedure for conducting such computations throughout Schwinger-Keldysh Diagrammatics for Primordial Perturbations [14], which are the cosmological analogous to Feynman rules in QFT.

4.1 The $In - In$ formalism

In this section, we want to sketch the derivation of the "Master formula" of the $In - In$ formalism for the computations of cosmological correlation function during Inflation following the treatment made in the appendix of [17].

4.1.1 The Hamiltonian time dependence

Therefore, consider a general Hamiltonian system involving canonical variables $\phi_a(t, \mathbf{x})$ and their corresponding conjugate momenta $\pi_a(t, \mathbf{x})$, where the index a serves to identify the fields and their spin components. The system satisfies the canonical commutation relations

$$[\phi_a(t, \mathbf{x}), \pi_b(y, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi_a(t, \mathbf{x}), \phi_b(t, \mathbf{y})] = [\pi_a(t, \mathbf{x}), \pi_b(t, \mathbf{y})] = 0, \tag{4.1}$$

and the equations of motion

$$\dot{\phi}_a(t, \mathbf{x}) = i\left[H[\phi(t), \pi(t)], \phi_a(t, \mathbf{x})\right], \quad \dot{\pi}_a(t, \mathbf{x}) = i\left[H[\phi(t), \pi(t)], \pi_a(t, \mathbf{x})\right], \tag{4.2}$$

where H is the Hamiltonian which is a functional of the fields and the functional dependence does not depend on the time t considered.

Keeping in mind the explanation of the Inflationary dynamics and the distinction between classical and quantum behavior of the inflaton as outlined in section 2.2.1, we assume the existence of a time-varying solution denoted by complex numbers $\bar{\phi}_a(\mathbf{x},t)$ and $\bar{\pi}_a(\mathbf{x},t)$ that satisfies the classical equations of motion. During Inflation, this corresponds to the Robertson-Walker dynamic determined by the homogeneous part of the scalar field. The background equation of motion reads

$$\dot{\bar{\phi}}_a(\mathbf{x},t) = \frac{\delta H[\bar{\phi}(t),\bar{\pi}(t)]}{\delta\bar{\pi}_a(\mathbf{x},t)} \ , \quad \dot{\bar{\pi}}_a(\mathbf{x},t) = -\frac{\delta H(\bar{\phi}(t),\bar{\pi}(t)]}{\delta\bar{\phi}_a(\mathbf{x},t)} \ . \tag{4.3}$$

The next step is to expand the Hamiltonian around this background in the same way we have done in section 2 and 2.4

$$\phi_a(t,\mathbf{x}) = \bar{\phi}_a(t,\mathbf{x}) + \delta\phi_a(t,\mathbf{x}) \ , \quad \pi_a(t,\mathbf{x}) = \bar{\pi}_a(t,\mathbf{x}) + \delta\pi_a(t,\mathbf{x}) \ . \tag{4.4}$$

Now, we can plug eq.(4.4) into eq.(4.1) to obtain the commutation relations for the perturbations

$$\Big[\delta\phi_a(t,\mathbf{x}),\delta\pi_b(t,\mathbf{y})\Big] = i\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}) \ , \quad \Big[\delta\phi_a(t,\mathbf{x}),\delta\phi_b(t,\mathbf{x})\Big] = \Big[\delta\pi_a(t,\mathbf{x}),\delta\pi_b(t,\mathbf{x})\Big] = 0 \ , \tag{4.5}$$

where we have used that the background solutions are numbers that commute with everything. The next goal is finding evolution equations for the perturbations using eq.(4.2). Thus, we need to expand the Hamiltonian in powers of the perturbations $\delta\phi_a(t,\mathbf{x})$ and $\delta\pi_a(t,\mathbf{x})$:

$$H[\phi(t),\pi(t)] = \quad H[\bar{\phi}(t),\bar{\pi}(t)] + \sum_a \frac{\delta H[\bar{\phi}(t),\bar{\pi}(t)]}{\delta\bar{\phi}_a(\mathbf{x},t)}\delta\phi_a(\mathbf{x},t] + \sum_a \frac{\delta H[\bar{\phi}(t),\bar{\pi}(t)]}{\partial\bar{\pi}_a(\mathbf{x},t)}\delta\pi_a(\mathbf{x},t) + \tilde{H}[\delta\phi(t),\delta\pi(t);t] \ , \tag{4.6}$$

where $\tilde{H}[\delta\phi(t),\delta\pi(t);t]$ is the sum of all terms in the Hamiltonian beyond first order in the perturbations. The Hamiltonian's zero-order term, being a function of time and possessing commutative properties with all other elements, can be ignored in all the following discussions. At first order, we observe that

$$i\left[\sum_b \int d^3y\, \frac{\delta H[\bar{\phi}(t),\bar{\pi}(t)]}{\delta\bar{\phi}_b(\mathbf{y},t)}\delta\phi_b(\mathbf{y},t) + \sum_b \int d^3y\, \frac{\delta H[\bar{\phi}(t),\bar{\pi}(t)]}{\delta\bar{\pi}_b(\mathbf{y},t)}\delta\pi_b(\mathbf{y},t),\delta\phi_a(\mathbf{x},t)\right] = \dot{\bar{\phi}}_a(\mathbf{x},t), \tag{4.7}$$

$$i\left[\sum_b \int d^3y\, \frac{\delta H[\bar{\phi}(t),\bar{\pi}(t)]}{\delta\bar{\phi}_b(\mathbf{y},t)}\delta\phi_b(\mathbf{y},t) + \sum_b \int d^3y\, \frac{\delta H[\bar{\phi}(t),\bar{\pi}(t)]}{\delta\bar{\pi}_b(\mathbf{y},t)}\delta\pi_b(\mathbf{y},t),\delta\pi_a(\mathbf{x},t)\right] = \dot{\bar{\pi}}_a(\mathbf{x},t) \ . \tag{4.8}$$

Thus, we can rewrite the evolution equations

$$\dot{\bar{\phi}}_a(\mathbf{x},t) + \delta\dot{\phi}_a(\mathbf{x},t) = i\Big[H[\phi(t),\pi(t)],\delta\phi_a(\mathbf{x},t)\Big] \ , \qquad \dot{\bar{\pi}}_a(\mathbf{x},t) + \delta\dot{\pi}_a(\mathbf{x},t) = i\Big[H[\phi(t),\pi(t)],\delta\pi_a(\mathbf{x},t)\Big] \ , \tag{4.9}$$

using the observations we have made as

$$\delta\dot{\phi}_a(\mathbf{x},t) = i\Big[\tilde{H}[\phi(t),\pi(t);t],\delta\phi_a(\mathbf{x},t)\Big] \ , \qquad \delta\dot{\pi}_a(\mathbf{x},t) = i\Big[\tilde{H}[\phi(t),\pi(t);t],\delta\pi_a(\mathbf{x},t)\Big] \ . \tag{4.10}$$

However H generates the time-dependence of $\phi_a(t,\mathbf{x})$ and $\pi_a(t,\mathbf{x})$, it is $\tilde{H}$ that determines the time dependence of the perturbations. Thus, in order to find the evolution equations for the perturbations one has to expand the original Hamiltonian H in powers of fluctuations disregarding the zeroth and first order in these fluctuations. This construction gives $\tilde{H}$ an explicit dependence on time.

4.1.2 The master's formula

In this section, we derive the master formula of the $In-In$ formalism throughout an operator formalism as presented in [17]. We are interested in the computation of the expectation value of an operator, which we generically call Q, which is a product of $\delta\phi_a$ and $\delta\pi_a$ evaluated at the same time t but different spatial position in general. These fields are, for example, Inflaton fluctuations, scalars, and tensor fluctuations of the metric. As we'll see we can compute correlation functions in the Heisenberg picture as

$$\langle Q(t)\rangle = \langle\Omega|Q(t)|\Omega\rangle, \tag{4.11}$$

where t is the end of inflation since we are interested in computing correlation function when Inflation ends. $|\Omega\rangle$ is the vacuum state in the far past which we indicate as t_0, which can be taken to be $-\infty$. This is the reason why the formalism is referred to as *In-In*. In this formalism, both vacuum states are considered in the distant past, while in quantum field theory, the states under consideration are the initial state $|IN\rangle$ in the remote past and the final state $|OUT\rangle$ in the distant future.

Now, since $Q(t)$ it's a generic product of fields we need to understand how this field evolves in time. Using eq.(4.10), we can express the fluctuations at the time t in relation to those of the same operators at an initial time t_0 by means of a unitary transformation

$$\delta\phi_a(t) = U^{-1}(t,t_0)\delta\phi_a(t_0)\,U(t,t_0)\ , \qquad \delta\pi_a(t) = U^{-1}(t,t_0)\delta\pi_a(t_0)\,U(t,t_0)\ , \tag{4.12}$$

where $U(t,t_0)$ is the evolution operator defined throughout the differential equation

$$\frac{d}{dt}U(t,t_0) = -i\,\tilde{H}[\delta\phi(t_0),\delta\pi(t_0);t]\,U(t,t_0), \quad U(t_0,t_0)=1\ . \tag{4.13}$$

It's possible to show that what we have written is correct but we postpone the demonstration since it's equal to the one we present to demonstrate that eq.(4.17) is solution of the differential equation (4.15). To evaluate $U(t,t_0)$, we decompose $\tilde{H}$ into a quadratic part H_0, which refers to the kinematic part of $\tilde{H}$, and an interaction term H_I

$$\tilde{H}[\delta\phi(t),\delta\pi(t);t] = H_0[\delta\phi(t),\delta\pi(t);t] + H_I[\delta\phi(t),\delta\pi(t);t]\ . \tag{4.14}$$

Our goal is to find a similar decomposition as the one made in QFT to evaluate the Dyson evolution operator [43]. To this end, we introduce an analogous of an "interaction picture" in which the time dependence of the fluctuation operators $\delta\phi_a^I(t)$ and $\delta\pi_a^I(t)$ is governed by the quadratic part of the Hamiltonian

$$\delta\dot{\phi}_a^I(t) = i\Big[H_0[\delta\phi^I(t),\delta\pi^I(t);t],\delta\phi_a^I(t)\Big]\ , \qquad \delta\dot{\pi}_a^I(t) = i\Big[H_0[\delta\phi^I(t),\delta\pi^I(t);t],\delta\pi_a^I(t)\Big]\ , \tag{4.15}$$

and the initial conditions

$$\delta\phi_a^I(t_0) = \delta\phi_a(t_0)\ , \qquad \delta\pi_a^I(t_0) = \delta\pi_a(t_0)\ . \tag{4.16}$$

The solution of eq.(4.15) can be written as

$$\delta\phi_a^I(t) = U_0^{-1}(t,t_0)\delta\phi_a(t_0)U_0(t,t_0)\ , \qquad \delta\pi_a^I(t) = U_0^{-1}(t,t_0)\delta\pi_a(t_0)U_0(t,t_0)\ , \tag{4.17}$$

with U_0 defined by the following differential equation

$$\frac{d}{dt}U_0(t,t_0) = -i\,H_0[\delta\phi(t_0),\delta\pi(t_0);t]\,U_0(t,t_0) \tag{4.18}$$

with initial condition

$$U_0(t_0,t_0) = 1\ . \tag{4.19}$$

It's easy to show that $U_0(t,t_0)$ is unitary. In fact, we have

$$0 = \frac{d}{dt}\mathbb{1} = \frac{d}{dt}\left(U_0(t,t_0)U_0^{-1}(t_0,t_0)\right) = U_0(t,t_0)\frac{d}{dt}U_0^{-1}(t_0,t_0) - i\,H_0[\delta\phi(t_0),\delta\pi(t_0);t]\,U_0(t,t_0)U_0^{-1}(t_0,t_0), \tag{4.20}$$

from which we can obtain

$$\frac{d}{dt}U_0^{-1}(t,t_0) = +i\, U_0^{-1}(t,t_0) H_0[\delta\phi(t_0), \delta\pi(t_0); t]\,. \tag{4.21}$$

which is identical to the hermitian conjugate of eq.(4.18). Notice that the fields in Hamiltonian $H_0[\delta\phi(t_0), \delta\pi(t_0); t]$ are evaluated at the time t_0, thus the time dependence is encoded in the explicit time dependence $[..; t]$. Now, we show that we can write the solution as we have stated. In fact, let's define the fields as

$$\delta\phi^I(t) = U_0^{-1}(t,t_0)\delta\phi^I(t_0)U_0(t,t_0), \quad \delta\pi^I(t) = U_0^{-1}(t,t_0)\delta\pi^I(t_0)U_0(t,t_0), \tag{4.22}$$

where we omit the subscript a for simplicity. If we now take the time derivative of the $\delta\phi^I$ (the reasoning is identical for the conjugate momenta)

$$\begin{aligned}
\delta\dot{\phi}^I(t) &= \left(\frac{d}{dt}U_0^{-1}(t,t_0)\right)\delta\phi^I(t_0)U_0(t,t_0) + U_0^{-1}(t,t_0)\delta\phi^I(t_0)\left(\frac{d}{dt}U_0(t,t_0)\right) && (4.23)\\
&= \left(iU_0^{-1}(t,t_0)H_0\left[\delta\phi(t_0), \delta\pi(t_0)\right]\right)\delta\phi^I(t_0)U_0(t,t_0) + U_0^{-1}(t,t_0)\delta\phi^I(t_0)\left(-iH_0\left[\delta\phi(t_0), \delta\pi(t_0)\right]U_0(t,t_0)\right) && (4.24)\\
&= +iU_0^{-1}(t,t_0)\left[H_0\left[\delta\phi(t_0), \delta\pi(t_0)\right]\delta\phi^I(t_0) - \delta\phi^I(t_0)H_0\left[\delta\phi(t_0), \delta\pi(t_0)\right]\right]U_0(t,t_0) && (4.25)\\
&= +iU_0^{-1}(t,t_0)\left[H_0\left(\delta^I\phi(t_0), \delta\pi^I(t_0), t\right), \delta\phi^I(t_0)\right]U_0(t,t_0) && (4.26)\\
&= +i\left[H_0\left(\delta^I\phi(t), \delta\pi^I(t), t\right), \delta\phi^I(t)\right], && (4.27)
\end{aligned}$$

where in the last step we have used the fact that the Hamiltonian is a polynomial in $\delta\phi^I$ and $\delta\pi^I$ and the definition of the interaction picture fields.

Now, we can proceed in deriving the form of the unitary operator $U(t,t_0)$. Now, from eq.(4.12) we can write

$$\begin{aligned}
\delta\phi_a(t) &= U^{-1}(t,t_0)\delta\phi_a U(t,t_0) = U^{-1}(t,t_0)U_0(t,t_0)U_0^{-1}(t,t_0)\delta\phi_a U_0(t,t_0)U_0^{-1}(t,t_0)U(t,t_0) && (4.28)\\
&= F^{-1}(t,t_0)\delta\phi_a^I(t)F(t,t_0), && (4.29)
\end{aligned}$$

where

$$F(t,t_0) = U_0^{-1}(t,t_0)U(t,t_0), \tag{4.30}$$

is unitary since the product of unitary operators. Now, we seek a differential equation for $F(t,t_0)$

$$\begin{aligned}
\frac{d}{dt}F(t,t_0) &= \frac{d}{dt}U_0^{-1}(t,t_0)U(t,t_0) + U_0^{-1}(t,t_0)\frac{d}{dt}U(t,t_0) && (4.31)\\
&= +i\, U_0^{-1}(t,t_0)H_0[\delta\phi(t_0), \delta\pi(t_0); t]U(t,t_0) - U_0^{-1}(t,t_0)i\,\tilde{H}[\delta\phi(t_0), \delta\pi(t_0); t]U(t,t_0) && (4.32)\\
&= +i\, H_0[\delta\phi(t), \delta\pi(t); t]F(t,t_0) - U_0^{-1}(t,t_0)i\,\tilde{H}[\delta\phi(t_0), \delta\pi(t_0); t]U_0(t,t_0)F(t,t_0) && (4.33)\\
&= +i\, H_0[\delta\phi(t), \delta\pi(t); t]U_0^{-1}(t,t_0)U(t,t_0) - i\,\tilde{H}[\delta\phi(t), \delta\pi(t); t] && (4.34)\\
&= -iH_I(t)F(t,t_0) && (4.35)
\end{aligned}$$

where, we have used that all the Hamiltonians employed are polynomials in the perturbations and where we define $H_I(t)$ as the interaction Hamiltonian in the interaction picture

$$H_I(t) \equiv U_0(t,t_0)H_I[\delta\phi(t_0), \delta\pi(t_0); t]U_0^{-1}(t,t_0) = H_I[\delta\phi^I(t), \delta\pi^I(t); t]. \tag{4.36}$$

The solution of equations like eq.(4.36) is well known [43]

$$F(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t)\,dt\right), \tag{4.37}$$

where T is the time-ordering operator [43]. Thus, a generic operator $Q(t)$ product of $\delta\phi$s and $\delta\pi$s, can be written in terms of the free fields of the interaction picture as

$$\begin{aligned} Q(t) &= F^{-1}(t,t_0)\, Q^I(t) F(t,t_0) \\ &= \left[\bar{T}\exp\left(i\int_{t_0}^{t} H_I(t)\,dt\right)\right]\, Q^I(t)\, \left[T\exp\left(-i\int_{t_0}^{t} H_I(t)\,dt\right)\right] \,, \end{aligned} \tag{4.38}$$

where $\bar{T}$ denotes anti-time-ordering. Now, we are almost done but we would like to substitute the vacuum of the interacting theory with the one of the free theory in the correlation functions eq.(4.11). We can do this because of the identity $F^{-1}F = \mathbb{1}$ [17, 62]. Thus, we finally arrive at the master formula of the $In - In$ formalism

$$\langle Q(t)\rangle = \langle 0|\left[\bar{T}\exp\left(i\int_{t_0}^{t} H_I(t)\,dt\right)\right]\, Q^I(t)\, \left[T\exp\left(-i\int_{t_0}^{t} H_I(t)\,dt\right)\right]|0\rangle. \tag{4.39}$$

4.2 The diagrammatic formalism

Now, throughout the rest of the thesis, we do not work with the master formula of the $In - In$ formalism, eq(4.39), but rather with a set of diagrammatic rules which are the cosmological equivalent of Feynman rules in QFT. This set of rules is the so-called *Schwinger-Keldysh Diagrammatics for Primordial Perturbations* [14] which can be derived in a path integral formulation of the problem. While we abstain from presenting the derivation of these rules, we offer a concise overview of the essential steps required for evaluating correlation functions based on the given Lagrangian of the system.

The problem we are confronted with, the computation of correlation function during Inflation, involves just three degrees of freedom. One of these is represented by a scalar field, which can either be the ζ field or the Inflaton, contingent on the gauge chosen. The remaining two degrees of freedom pertain to the tensor perturbation. The dynamic is governed by a Lagrangian density $\mathcal{L}$, wherein, due to our perturbative scheme, exclusively "polynomials" of the fields are incorporated, and the couplings are allowed to possess a time dependency, given the time-varying nature of the background.

Even if the basic ideas are very similar to the Feynman formalism, there are a series of properties that are different with respect to the QFT case. First of all, there exist two kinds of vertices which we call plus + or minus -, which originate from the time-ordered product and the anti-time-ordered product, correspondingly. Consequently, when evaluating a diagram containing N vertices, it becomes necessary to sum all the possible configurations (a total of $2N$ options) involving the assignment of each vertex as either a plus vertex or a minus vertex. Each plus or minus vertex contributes a factor of i or $+i$ along with any factors from the vertex itself.

Moreover, we encounter four distinct types of propagators, depending on the four possible vertices combinations at disposal

$$++,\quad +-,\quad -+,\quad --\,. \tag{4.40}$$

To compute the propagators at tree level, we require the mode functions that fulfill both the Bunch-Davies initial condition and the canonical commutation relation. Refer to section 2.4 for an illustrative instance in the context of the single-field slow-roll model of Inflation. However, if we aim to accurately retrieve the complete outcome, these factors should be incorporated. Then, the tree-level propagators in the 3−momentum space are [14]

$$G_{++}(k;\tau_1,\tau_2) = G_{>}(k;\tau_1,\tau_2)\theta(\tau_1-\tau_2) + G_{<}(k;\tau_1,\tau_2)\theta(\tau_2-\tau_1), \tag{4.41}$$

$$G_{+-}(k;\tau_1,\tau_2) = G_{<}(k;\tau_1,\tau_2), \tag{4.42}$$

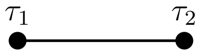
$$G_{-+}(k;\tau_1,\tau_2) = G_{>}(k;\tau_1,\tau_2), \tag{4.43}$$

$$G_{--}(k;\tau_1,\tau_2) = G_{<}(k;\tau_1,\tau_2)\theta(\tau_1-\tau_2) + G_{>}(k;\tau_1,\tau_2)\theta(\tau_2-\tau_1), \tag{4.44}$$

where

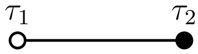
$$G_>(k;\tau_1,\tau_2) \equiv u(\tau_1,k)u^*(\tau_2,k), \tag{4.45}$$
$$G_<(k;\tau_1,\tau_2) \equiv u^*(\tau_1,k)u(\tau_2,k). \tag{4.46}$$

In diagrammatic representation, a black dot signifies a positive vertex (+), while a white dot represents a negative vertex (−). Therefore, we have



$$= \quad G_{++}(k;\tau_1,\tau_2), \tag{4.47}$$

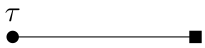
$$= \quad G_{+-}(k;\tau_1,\tau_2), \tag{4.48}$$



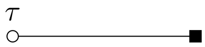
$$= \quad G_{-+}(k;\tau_1,\tau_2), \tag{4.49}$$

$$= \quad G_{--}(k;\tau_1,\tau_2). \tag{4.50}$$

Within a diagram, there are internal legs that link points within the diagram, and external legs that connect an internal point with an external one. The external points are where we position the fields for which we intend to compute correlation functions. The propagators we have written can be used both to evaluate *internal* legs (*bulk propagators*) and *external* legs (*bulk-to-boundary* propagators). The external legs terminated at the final conformal time $\tau = \tau_0$, which is the time at which we are interested in computing the correlation functions. We'll set $\tau = 0$, in the next chapter. It's important to emphasize that a boundary point does not differentiate between the positive (+) and negative (−) designations, resulting in just two categories of bulk-to-boundary propagators.



$$= G_+(k;\tau) \equiv G_{++}(k;\tau,\tau_f),$$



$$= G_-(k;\tau) \equiv G_{-+}(k;\tau,\tau_f).$$

Notice that we have indicated the external points with a black square. Here we have discussed the propagator for a scalar field. When we want to deal with tensor fields we also have to take into account the polarization portion of the propagators. We are not going to discuss this point but the polarization portion is identical to all four propagators and takes the form

$$\sum_h \epsilon^h_{ij}(\mathbf{k}) \left[\epsilon^h\right]^*_{ab}(\mathbf{k}), \tag{4.51}$$

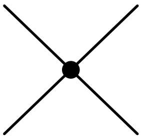
if the propagator connects the vertex ij with the vertex ab. We can think of these two vertices as the τ_1 and τ_2 in eq.(4.50). The fact that the polarization doesn't change depends explicitly on how we compute the propagators; for more details see [14].

The final aspect to address pertains to deriving the vertex factors. The process bears resemblance to the methodology employed in QFT. Nevertheless, in this context, we do not make the Fourier transform with respect to conformal time. Consequently, time integrals will manifest in the eventual results, and they often add complexity to the computations involved. As previously mentioned, for every individual interaction vertex present in the original Lagrangian, we must formulate two distinct vertices, representing the + and − types, respectively. We provide a series of examples in order to clarify the procedure. We represent scalars using a solid line and tensors using a wavy line. We don't differentiate between internal and external lines, as this becomes evident based on the vertices that the line connects.

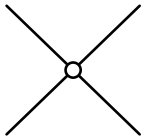
Non-derivative couplings. Non-derivative couplings are easy to derive and one just proceeds in a completely analogous way to the QFT case with the differences mentioned above. Let's consider an interaction term of the form

$$\mathcal{L}_{int} = -\frac{\lambda}{24} a^4(\tau)\varphi^4(t, \mathbf{x}). \tag{4.52}$$

Then in three-momentum space, we get the following rules



$$= -i\lambda \int_{\tau_0}^{\tau_f} d\tau\, a^4(\tau) \cdots ,$$



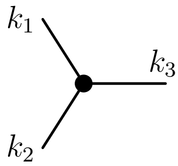
$$= +i\lambda \int_{\tau_0}^{\tau_f} d\tau\, a^4(\tau) \cdots , \tag{4.53}$$

where $\cdots$ means that we have this time integral acting on every part of the correlation function which is time dependent.

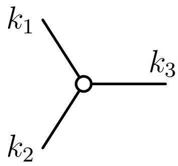
Derivative couplings. Derivative coupling has to be treated in a different way. First of all, we have to distinguish spatial and temporal derivatives since we do not Fourier transform with respect to the time variable. Regarding spatial derivatives, no additional explanations are required. For instance, let's examine the diagrammatic rule for the subsequent interaction:

$$\mathcal{L}_{int} \propto -\frac{\lambda}{6} a^2(\tau)\varphi(\partial^i\varphi)(\partial_i\varphi), \tag{4.54}$$

which is given by,



$$\propto +\frac{i\lambda}{3}(\mathbf{k}_1\cdot\mathbf{k}_2 + \mathbf{k}_2\cdot\mathbf{k}_3 + \mathbf{k}_3\cdot\mathbf{k}_1)\int_{\tau_0}^{\tau_f} d\tau\, a^2(\tau)\cdots , \tag{4.55}$$

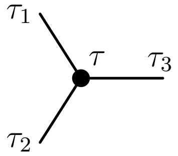


$$\propto -\frac{i\lambda}{3}(\mathbf{k}_1\cdot\mathbf{k}_2 + \mathbf{k}_2\cdot\mathbf{k}_3 + \mathbf{k}_3\cdot\mathbf{k}_1)\int_{\tau_0}^{\tau_f} d\tau\, a^2(\tau)\cdots . \tag{4.56}$$

Now we can analyze the time-derivative couplings in the Lagrangian. The time derivatives within a vertex should be directly applied to the propagators connected to it. Let's take the following interaction,

$$\mathcal{L}_{int} \propto -\frac{\lambda}{6} a^2(\tau)\varphi\varphi'^2. \tag{4.57}$$

The diagrammatic rule is given by,



$$\propto -\frac{i\lambda}{3}\int_{\tau_0}^{\tau_f} d\tau\, a^2(\tau)\left[\partial_\tau G_{+a_1}(k_1;\tau,\tau_1)\right]\left[\partial_\tau G_{+a_2}(k_2;\tau,\tau_2)\right] G_{+a_3}(k_3;\tau,\tau_3) \tag{4.58}$$

$$+ 2 \text{ permutations}, \tag{4.59}$$

where permutations refer to various combinations in which the two temporal derivatives can be arranged and, where $a_1, a_2, a_3 = \pm$. The corresponding rule for the minus type vertex is derived from the plus type by introducing an additional negative sign.

Now, let's provide an overview of the steps required to calculate a general correlation function:

- Initially, the action needs to be expanded up to the second order concerning both tensor and scalar perturbations. This expansion is essential for determining the EoM. Typically, this process is carried out separately. While evaluating, for example, the scalar component, any other factor apart from the scalar function under consideration is set to zero (refer to section 2.4). If we are considering only traceless and transverse tensor and scalar the EoM are decoupled since we are not able to form a non-zero bilinear involving the scalar, the tensor, derivatives, and the Kronecker delta and the Levi-Civita tensor[18] (see section 5.3 for a complete discussion about this point).
- Subsequently, after establishing the correlation function being computed, the next step involves identifying the specific type of diagrams that could contribute to it. Once we have established the "kind" of coupling we need, for example, a tensor-scalar-scalar non-derivative coupling, we have to expand the action and find every term that contributes to it. Then, we have to recover the diagrammatic rules for this interaction.
- Then, we have to sketch all the possible diagrams of interest labeling each vertex with a black dot (plus-type vertex) or a white dot (minus-type vertex), in all possible ways. Therefore, we have 2^N different diagrams if we have N vertices. Then, we label external points as square dot.
- Then, using propagators and rules vertices we have to evaluate each diagram and sum everything together. Integrate each vertex over time from the initial time which we'll set to be $\tau = -\infty$ to a final time, which is set to be zero $\tau_f = 0$. Please note that each vertex has an associated different time variable.
- Finally, we obtain the correlation function in Fourier space by multiplying the result by $(2\pi)^3\delta^{(3)}\left(\sum_i \mathbf{k}_i\right)$, where $\mathbf{k}_i$ are the three-momenta involved in the diagram.

[18] We do not possess any other tools for index contraction.

5 Chern-Simons theory of modified gravity

5.1 Modifying Einstein's GR

The actual Inflationary paradigm, standard slow-roll models of Inflation, provides a satisfactory explanation for various experimental observations, including those related to the CMB and LSS. These include the homogeneity, isotropy, and flatness of today's Universe and the origin of the initial density perturbations. Einstein's General Relativity (GR) is the paradigm used for the description of gravitational interactions. However, since Inflation could take place at very high energies, it's possible and interesting to consider models which go beyond standard gravity. For example, the first model of Inflation [63] was based on R^2-higher order gravitational terms.

Besides these considerations, there are other reasons, not strictly related to Inflation, that suggest a modification of General Relativity could take place at high energies. In particle physics, we lack a quantum description of gravitational interactions, i.e. "quantum gravity", which can't be achieved by quantizing GR with standard techniques. Moreover in cosmology, explaining the late-time cosmic acceleration of the universe with GR is full of difficulties. In the $FLRW$ model, which is based on Einstein's gravity, it's assumed that this phase of accelerated expansion is driven by a cosmological constant Λ which is interpreted as vacuum energy. This assumption comes with two challenging issues: the inability to account for the gravitational properties of the vacuum energy and the coincidence problem. The former, basically, consists of the incredibly small and highly fine-tuned value of the vacuum energy [64]. The latter corresponds to the fact the densities of dark energy and dark matter are measured to be of the same order of magnitude, i.e $\frac{\rho_{\Lambda 0}}{\rho_{Dm0}} \sim O(1)$. This implies that we are living in a very special period of cosmic history, which is the result of specific initial conditions in the Early Universe. The so-called "coincidence problem" is trying to find an answer to "why now" this specific period occurs. Due to these reasons, numerous alternatives to standard gravity have been explored in recent years.

5.2 Effective field theory approach

In order to modify gravity we follow the argument presented in [18] in order to illustrate the basic ideas of this kind of approach without adventuring into extensive computations. First of all, we focus only on the case of a single-field slow-roll Inflation model, i.e.

$$\mathcal{L} = \sqrt{-g}\left[\left(\frac{M_{pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)\right], \tag{5.1}$$

where we recall that $g = det(g_{\mu\nu})$ and $M_{pl}^2 = \frac{1}{8\pi G}$. The first term corresponds to the Hilbert-Einstein action which is the unique, up to a cosmological constant contribution, diffeomorphism invariant action for a four-dimensional metric, whose EoM are at most "second order" [19][19]. Hence, the expression of eq.(5.1) represents the most general Lagrangian density for gravity and a scalar field, whose EoM involve a maximum of two space-time derivatives. The basic idea of this approach is considering the action of eq.(5.1) as the first term of the expansion of the "true" Lagrangian in effective field theory approach (EFT). The terms with higher derivatives are suppressed by some undefined large mass parameter M, which characterizes the true field theory which produces the EFT Lagrangian. Here we assume that all the coupling constants in the higher-derivatives terms are powers of M multiplied with coefficients roughly of order unity, where the number of powers required is fixed by dimensional analysis. Since we can't produce any term with three-derivative[20], the first correction can be written

[19] We have up to second derivatives; the Einstein's field equation contains up to second derivatives.

[20] We have to construct scalar quantities using the Riemann tensor, the Ricci tensor, the Ricci scalar, the metric tensor, derivatives of the scalar field, and generic scalar function of the scalar field. Since the curvature tensors contain two "powers" of derivatives and have an even number of indices we can't write a scalar term with an odd number of derivatives.

[7] as

$$\delta\mathcal{L} = L = \sqrt{g}\Bigg[\frac{1}{2}M_P^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$
$$+ f_3(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + f_4(\phi)g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\Box\phi + f_5(\phi)(\Box\phi)^2 + f_7(\phi)R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + f_8(\phi)Rg^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$+ f_9(\phi)R\Box\phi + f_{10}(\phi)R^2 + f_{11}(\phi)R^{\mu\nu}R_{\mu\nu} + f_{12}(\phi)C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + f_{13}(\phi)\tilde{\varepsilon}^{\mu\nu\rho\sigma}C^{\kappa\lambda}_{\mu\nu}C_{\rho\sigma\kappa\lambda}\Bigg], \tag{5.2}$$

where $\tilde{\varepsilon}^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor density defined as $\tilde{\varepsilon}^{\mu\nu\rho\sigma} \equiv \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}$ with $\varepsilon^{\mu\nu\rho\sigma}$ corresponding to the Levi-Civita symbol [38]. Moreover, the various f_n coefficients are dimensionless, i.e. they contain the necessary numbers of power of the scale mass M, and they can depend on a generic function of the Inflation field. In eq.(5.2) we also have introduced the Weyl tensor which is defined as [13]

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}) + \frac{R}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}), \tag{5.3}$$

which is the traceless part of the Riemann curvature tensor. The last term of eq.(5.2)

$$f_{13}(\phi)\tilde{\varepsilon}^{\mu\nu\rho\sigma}C^{\kappa\lambda}{}_{\mu\nu}C_{\rho\sigma\kappa\lambda}, \tag{5.4}$$

is the so-called Chern-Simons term, which can also be written as replacing the Weyl tensor with the Riemann tensor (see appendix D.2) as

$$f_{13}(\phi)\tilde{\varepsilon}^{\mu\nu\rho\sigma}R^{\kappa\lambda}{}_{\mu\nu}R_{\rho\sigma\kappa\lambda}. \tag{5.5}$$

The Lagrangian described in equation (5.2) consists of various terms, but only the Chern-Simons term can exhibit parity-violating signatures. Assuming that this is the "right" approach to Inflation, only data that can tell us which coefficients f_n have to be kept in order to reproduce the dynamics. Since at the moment, with the current data, we are not able to do this we can study the effects of the Chern-Simons and set to zero all the other terms. So we are left with the following action

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \frac{\phi}{4f}{}^*RR\right], \tag{5.6}$$

where we have introduced the Pontryagin density defined as

$${}^*RR = {}^*R^{\sigma\rho\mu\nu}R_{\sigma\rho\mu\nu}, \tag{5.7}$$

where

$${}^*R^{\sigma\rho\mu\nu} = \frac{1}{2}\tilde{\epsilon}^{\mu\nu\alpha\beta}R^{\sigma\rho}{}_{\alpha\beta} = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}R^{\sigma\rho}{}_{\alpha\beta}. \tag{5.8}$$

Note that in eq.(5.6) we have set $f_{13}(\phi) = \frac{\phi}{4f}$ with f a dimensional constant, it contains one power of M.

We conclude this paragraph by mentioning two important properties of the Chern-Simons action. Firstly, we note that the Pontryagin is zero when evaluated on the background since the Weyl tensor is conformally invariant [38]. It is worth recalling that a conformal transformation maps

$$g_{\mu\nu} \longrightarrow \tilde{g}\mu\nu = \Omega^2(x)g\mu\nu, \tag{5.9}$$

where $g_{\mu\nu}$ is a generic metric and, where $\Omega(x)$ is an arbitrary non-vanishing function of spacetime. Therefore, the spatially flat $FLRW$ metric is conformally equivalent to the Minkowski metric, in which all curvature tensors vanish. Consequently, there are no modifications to the background equation of motion for the Inflaton, i.e. we obtain the same results we have shown in section 2.2.1.

Furthermore, the Pontryagin density can be written as total derivative [65] in the following way

$$\nabla_\alpha \tilde{K}^\alpha = \frac{1}{2} {}^*RR, \tag{5.10}$$

where

$$\tilde{K}^\alpha := \tilde{\epsilon}^{\alpha\beta\delta\gamma} \left(\Gamma^\chi_{\beta\sigma} \partial_\delta \Gamma^\sigma_{\gamma\chi} + \frac{2}{3} \Gamma^\chi_{\beta\sigma} \Gamma^\sigma_{\delta\epsilon} \Gamma^\epsilon_{\gamma\chi} \right), \tag{5.11}$$

is the CS topological current. In what follows we often use CS to indicate quantities related to the Chern-Simons theory of modified gravity. Notice that the $\epsilon^{\alpha\beta\delta\gamma}$ is the Levi-Civita "symbol" which is related to the Levi-Civita "tensor" via $\tilde{\epsilon}^{\alpha\beta\delta\gamma} = \frac{\tilde{\epsilon}^{\alpha\beta\delta\gamma}}{\sqrt{-g}}$ (see [38] to a complete explanation on the convection). For the sake of convenience in what follows, we will also use the symbol K^α for the topological CS current, which corresponds to $\tilde{K}$ with the replacement of the Levi-Civita tensor with its symbol equivalent.

5.3 Metric perturbations and gauge fixing

We work in the so-called Arnowitt-Deser-Misner (ADM) formalism of the metric, section C.3, in the generalization of the spatially flat gauge [15]. The reasoning that allows us to impose this gauge is analogous to the one we have done in section 2.3. In Cartesian coordinates, the metric tensor and its inverse become

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(N^2 - N_i N^i) & N_i \\ N_i & g_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = a^2 \begin{pmatrix} -\frac{1}{N^2} & \frac{N_i}{N^2} \\ \frac{N_i}{N^2} & (g^{ij} - \frac{N^i N^j}{N^2}) \end{pmatrix}, \tag{5.12}$$

where $a^2(\tau)$ is the scale factor, N is the lapse function, and, $N_i = \partial_i \psi + E_i$ is the shift function which is decomposed in a scalar part ψ and in a vector part, the divergence-free three vector E_i, i.e. $\partial_i E^i = 0$. In this section, we omit the use of the superscript (3) for spatial tensors because it becomes evident from the context whether a tensor is associated with the spatial hypersurface. The spatial metric receives a contribution from the background and from transverse and traceless symmetric tensor and can be written as [15]

$$g_{ij} = a^2 \exp h_{ij} = a^2 \left(\delta_{ij} + h_{ij} + \frac{1}{2} h_{ik} h^k{}_j + .. \right), \quad h^i{}_i = 0, \quad \partial_i h^i{}_j = 0. \tag{5.13}$$

In the generalization of the spatially flat gauge, we don't have considered perturbation of first, $h^{(1)}_{ij}$, second, $h^{(2)}_{ij}$, order and so on as we have explained in section 1.3. Following [15] we have written the metric in the following way

$$g_{ij} = (\exp h)_{ij} \approx \delta_{ij} + h_{ij} + \frac{1}{2} h_{ik} h^k{}_j + ..+, \tag{5.14}$$

which, since g_{ij} is symmetric is well defined. We recall that the lapse and shift functions are auxiliary fields in standard gravity, section C.3.2, which can be removed by solving the EoM as we have done in section 2.3.1. However, considering that we have modified the gravity sector of our theory through the Chern-Simons term, it is necessary to determine whether these fields become dynamical or not. Now we show that they remain as constraints and that the CS term doesn't modify the first-order solutions of their Euler-Lagrnage equations which are what we need since we are interested in expanding the action up to third order. We are going to compute the graviton-mediated trispectrum in section 5.6 which is computed using two vertices of the kind scalar-scalar-tensor, constituting a third-order term. Thus, as explained in [8, 7, 15], we need the solution of the constraints equation at first order. Now since the CS term can be written as

$$S^{(2)}_{CS} = \int d^4x \sqrt{-g} \frac{\phi}{4f} {}^*RR = \int d^4x \sqrt{-g} \frac{\phi}{2f} \nabla_\alpha \tilde{K}^\alpha = \int d^4x \frac{\phi}{2f} \nabla_\alpha K^\alpha \tag{5.15}$$

$$= \int d^4x \left[-\frac{\phi'}{2f} K^0 - \partial_i \left(\frac{\phi}{2f} \right) K^i \right] = \int d^4x \left[-\frac{\phi'}{2f} K^0 \right] = \int d^4x \left[-\frac{\phi'}{2f} \epsilon^{0ijk} \left(\Gamma^\alpha_{i\sigma} \partial_j \Gamma^\sigma_{k\alpha} + \frac{2}{3} \Gamma^\alpha_{i\sigma} \Gamma^\sigma_{j\chi} \Gamma^\chi_{\sigma k} \right) \right], \tag{5.16}$$

we observe that we have a Levi-Civita symbol multiplying everything. So a possible contribution to the Lagrangian is a scalar quantity constructed contracting the Levi-Civita symbol with a product constructed using the two scalars, N and ψ, the solenoidal vector, E_i, a traceless and transverse tensor, h_{ij}, and their derivatives. The maximum number of spatial derivatives that can appear is six since we have the multiplication of three Christoffel symbols which can contain one "power" of spatial derivatives and because the quantity D appears already in the form $\partial_i D$ in the metric tensor eq.(5.12). Now since we are interested in first-order constraint equations for the lapse and shift function, we only need to discuss all the bilinear terms we can construct; we are considering the action up to second order since higher order terms produce non-linear term in Euler-Lagrange equations.

We directly discuss the problem in Fourier space since it's easier. Since we have only two fields in the bilinear we can express everything as a function of only one vector variable $\mathbf{k}$ because of the three-dimensional Dirac delta arising from the $\mathbf{x}$ integral. For example in the case of a product of two generic scalar fields A and B we get

$$\int d\tau d^3x \frac{d^3k e^{i\mathbf{x}\cdot\mathbf{k}}}{(2\pi)^3}\frac{d^3q e^{i\mathbf{x}\cdot\mathbf{q}}}{(2\pi)^3} A(\tau,\mathbf{k})B(\tau,\mathbf{q}) = \int d\tau \frac{d^3k}{(2\pi)^3} A(\tau,\mathbf{k})B(\tau,-\mathbf{k}). \tag{5.17}$$

If we have derivatives in Fourier space we'll have a correspondent k_is, but we can have only one k_i contracted with the Levi-Civita symbol otherwise we get zero because of symmetry. Thus if we have more than one derivative we must perform contractions in order to have only one ks contracted with the symbol.

Starting from the combination of two scalar fields, $S_1 = N/\psi$ and $S_2 = N/\psi$, and considering three and five derivatives [21] we only can get

$$S_1 S_2 k_i k_j k_l \epsilon^{ijl},\ S_1 S_2 k_i k_j k_l k^2 \epsilon^{ijl}, \tag{5.18}$$

which clearly are all zeros for what we have anticipated before. In what follows we completely disregard temporal derivatives since the results are unchanged. Regarding the scalar-tensor bilinear, without considering the ones with more than one "free" k appears, there are only the following possibilities

$$S h_{jl} k_i \epsilon^{ijl}, \tag{5.19}$$

which is zero and, where $S = \psi, N$. In principle, we have to consider three or five derivatives but in these cases, we have two ks contracted with the Levi-Civita symbol. Now going to the vector-scalar combinations we need at least two free derivatives to get non-trivial contractions but these terms are zero, as explained above (we can't contract the ks with the vector because of transversality). For example, we could write

$$S N_l k_i k_l \epsilon^{ijl}, \tag{5.20}$$

which clearly is zero. Considering tensor-vector contribution we have to contract one of the two indices of the metric with a derivative index or with the index of the vector

$$N_l k_i h^l{}_j k_f \epsilon^{ijf},\ N_f k_i h^l{}_j k_l \epsilon^{ijf},\ N_f k^f k_i h^l{}_j k_l \epsilon^{ijf}, \tag{5.21}$$

which are zeros because of symmetries or transversality of the vector and tensor. Regarding tensor-tensor and vector-vector bilinears, we can create non-vanishing terms which, in real space, can be for example

$$h_{li}{}^{,m} h^l{}_{k,mj} \epsilon^{ijk},\ N_{i,j} N_k \epsilon^{ijk}. \tag{5.22}$$

This discussion teaches us that the equation for the lapse function is unchanged and thus it remains a non-dynamical d.o.f. However since terms as the ones in eq.(5.22), exist we have to show that the field E_i does not become dynamical.

[21] It's pointless to consider one derivative since this means that two indices of the Levi-Civita symbol must be contracted to form a scalar quantity.

This can be proved by observing that the only Christoffel symbol which receives a non-zero contribution from $\dot{E}_i$ is Γ^i_{00}. Therefore, given the form of eq.(5.16), we conclude that the shift function remains a non-dynamical d.o.f. For completeness, we demonstrate the statement we have done regarding the Christoffel symbols reporting only the contributions which contain temporal derivatives of the shift function

$$\Gamma^0_{00} = \frac{1}{2}\left[g^{00}\left(g_{00,0}\right) + g^{0i}\left(2g_{0i,0} - g_{00,i}\right)\right] = \frac{1}{2}\left[N^{-2}\left(2N\dot{N} - 2N_i\dot{N}^{,i}\right) + \frac{N^i}{N^2}\left(2\dot{N}_{,i} - 2NN_i + 2N_{j,i}N^i\right)\right] \to 0, \tag{5.23}$$

$$\Gamma^0_{0i} = \frac{1}{2}g^{00}g_{00,i} + g^{0j}(g_{0j,i} + g_{ij,0} - g_{0i,j}) \to 0, \tag{5.24}$$

$$\Gamma^i_{00} = \frac{1}{2}\left[g^{i0}g_{00,0} + g^{ij}(2g_{0j,0} - g_{00,j})\right] \to \left[-\frac{N^i}{N^2}\left(2N\dot{N} - 2N_i\dot{N}^{,i}\right) + (h^{ij} - \frac{N^iN^j}{N^2})\dot{N}_{,j}\right] \to h^{ij}\dot{N}_{,j}, \tag{5.25}$$

$$\Gamma^i_{0j} = \frac{1}{2}\left[g^{i0}(g_{j0,0} + g_{00,j} - g_{j0,0}) + g^{ik}(g_{jk,0} + g_{k0,j} - g_{j0,k})\right] \to 0, \tag{5.26}$$

$$\Gamma^i_{jk} = \frac{1}{2}\left[g^{i0}(g_{j0,k} + g_{k0,j} - g_{jk,0} + g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l})\right] \to 0. \tag{5.27}$$

Now from the previous results, it's clear that the constraint equation for N is the same as in standard gravity while the one for the E_i receives additional contributions. However, these contributions are linear terms in the transverse vector field E^i (they can contain spatial derivatives clearly) so, as in the case of standard gravity, we can find a solution by setting E_i to zero, as we have done in section 2.3.1. Thus we can compute the constraints in a similar way to what we have done in section 2.3.1 and write the solution for the constraints up to first-order [15] as

$$N = 1 + \frac{\dot{\phi}\delta\phi}{2HM_{pl}^2}, \quad N_i = \partial_i\psi + E_i, \quad E_i = 0 \quad \psi = \chi, \quad \partial^2\chi = \frac{\dot{\phi}^2}{2H^2M_{pl}^2}\frac{d}{dt}\left(-\frac{H}{\dot{\phi}}\delta\phi\right), \tag{5.28}$$

where we have reintroduce the Planck mass to recover the correct prefactor with respect to the result presented in [15].

5.4 The equation of motion for the Inflaton and tensor perturbations

In order to obtain the EoM we need to expand the action of eq.(5.6) up to second order considering the constraints we have previously solved for the scalar sector. We work in conformal time. Regarding the Inflaton, at second order, the action doesn't receive any contribution from the Hilbert-Einstein action since

$$K_{ij} = -\partial_{ij}\psi, \quad K^{ij} = -a^{-4}\partial^{ij}\psi, \quad K^i{}_j = -a^{-2}\partial^i\partial_j\psi, \tag{5.29}$$

from which we understand that both K^2 and $K^{ij}K_{ij}$ can be written as total derivatives. Notice that in evaluating eq.(5.29) we have considered that the three-dimensional Christoffel symbols do not depend on any scalars. The three-dimensional Ricci tensor clearly can't contribute since we do not have scalars in the three-dimensional metric.

Thus we obtain, using the constraints. the scalar Lagrangian up to second order

$$S^{(2)} = \int d^4x \left[\frac{a^2}{2}\left(\delta\phi'^2 - \delta\phi_{,i}\delta\phi^{,i}\right) + \frac{a^2(t)}{2\left(1+\frac{\dot\phi}{2HM_{pl}^2}\delta\phi\right)^2}\delta\dot\phi\dot\phi - a\partial^i\chi\partial_i\delta\phi\phi' - \frac{a^4}{2}\frac{\partial^2 V(\phi)}{\partial\phi^2}\bigg|_{\phi(\tau,\mathbf{x})=\phi(\tau)}\delta\phi^2\right] \tag{5.30}$$

$$\approx \int d^4x \left[\frac{a^2}{2}\left(\delta\phi'^2 - \delta\phi_{,i}\delta\phi^{,i}\right) - \frac{1}{a(\tau)}\frac{\phi'^2}{2HM_{pl}^2}\delta\phi\delta\phi' + a\partial^2\chi\delta\phi\phi' - \frac{a^4}{2}\frac{\partial^2 V(\phi)}{\partial\phi^2}\bigg|_{\phi(\tau,\mathbf{x})=\phi(\tau)}\delta\phi^2\right] \tag{5.31}$$

$$\approx \int d^4x \left[\frac{a^2}{2}\left(\delta\phi'^2 - \delta\phi_{,i}\delta\phi^{,i}\right) - \frac{1}{a(\tau)}\frac{\phi'^2}{2HM_{pl}^2}\delta\phi\delta\phi' - a(\tau)H\epsilon\delta\phi'\delta\phi - \frac{a^4}{2}\frac{\partial^2 V(\phi)}{\partial\phi^2}\bigg|_{\phi(\tau,\mathbf{x})=\phi(\tau)}\delta\phi^2\right] \tag{5.32}$$

$$\approx \int d^4x \left[\frac{a^2}{2}\left(\delta\phi'^2 - \delta\phi_{,i}\delta\phi^{,i}\right) - 2a^2H\epsilon\delta\phi\delta\phi' - \frac{a^4}{2}\frac{\partial^2 V(\phi)}{\partial\phi^2}\bigg|_{\phi(\tau,\mathbf{x})=\phi(\tau)}\delta\phi^2\right], \tag{5.33}$$

where we have disregarded a first-order piece and, at lowest order in the slow parameter we have used that

$$\partial^2\chi = \frac{\dot\phi}{2H^2M_{pl}^2}\frac{d}{dt}\left(-\frac{H}{\dot\phi}\delta\phi\right) = \epsilon\frac{d}{dt}\left(-\frac{H}{\dot\phi}\delta\phi\right) = -\frac{H}{\dot\phi}\epsilon\delta\dot\phi - \epsilon\frac{d}{dt}\left(\frac{\sqrt{\epsilon}}{\sqrt{2}M_{pl}}\right) \approx -\frac{H}{\dot\phi}\epsilon\delta\dot\phi. \tag{5.34}$$

We need the equation of motion for the scalar in order to write down the scalar propagators and to compute the scalar power spectrum. This computation yields identical outcomes to those derived in chapter 2.4. However, it's valuable to verify that when employing two distinct gauges, we arrive at the same outcome.

Regarding tensor perturbations, the Hilbert-Einstein term becomes as in section 2.5

$$S_{HE}^{(2)} = \int d^4x \frac{M_{pl}^2}{8}\left[(h^i{}_j)'(h^j{}_i)' - (\partial_k h^i{}_j)(\partial^k h^j{}_i)\right], \tag{5.35}$$

while we need to expand the Pontryagin density and we achieve this using a smart trick. Since the Weyl tensor is conformally invariant we have that also the Pontryagin density is conformally invariant, thus we can perform the expansion using the Minkowski metric as the background. Furthermore, we know from eq.(5.11) that we can write

$$S_{CS}^{(2)} = \int d^4x\sqrt{-g}\frac{\phi}{4f}{}^*RR = \int d^4x\sqrt{-g}\frac{\phi}{2f}\nabla_\alpha\tilde K^\alpha = \int d^4x\frac{\phi}{f}\nabla_\alpha K^\alpha \tag{5.36}$$

$$= \int d^4x\left[-\frac{\phi'}{2f}K^0 - \partial_i\left(\frac{\phi}{2f}\right)K^i\right] = \int d^4x\left[-\frac{\phi'}{2f}K^0\right], \tag{5.37}$$

where we have used the Leibniz rule for covariant derivatives [13] and we have eliminated surface contributions. Thus we are left to expand K^0 in a Minkowski background, where the Christoffel symbols and all the curvature tensors are zero. Thus the relative perturbed quantities are at least of first order. Thus, as explained in the appendix D.2 the action we get

$$S_{CS}^{(2)} = \int d^4x\left[-\frac{\phi'}{4f}K^0\right] = \int d^4x\epsilon^{ijl}\frac{\phi'}{4f}\left[-h'^l{}_i\partial_j h'_{kl} + h_{li}{}^{,m}h^l{}_{k,mj}\right]. \tag{5.38}$$

Collecting everything together we get

$$S^{(2)} = \int d^4x\bigg\{\frac{a^2}{2}\left[(\delta\phi')^2 - \delta\phi_{,i}\delta\phi^{,i}\right] - 2a^2H\epsilon\delta\phi\delta\phi' - \frac{a^4}{2}\frac{\partial^2V(\phi)}{\partial\phi^2}\bigg|_{\phi(\tau,\mathbf{x}}\delta\phi^2 +$$

$$+\frac{M_{pl}^2}{8}a^2\left[(h^i{}_j)'(h^j{}_i)' - (\partial_k h^i{}_j)(\partial^k h^j{}_i)\right] + \epsilon^{ijl}\frac{\phi'}{4f}\left[-(h^l{}_i)'\partial_j(h_{kl})' + h_{li}{}^{,m}h^l{}_{k,mj}\right]\bigg\}. \tag{5.39}$$

Our action coincides with the one presented in [9, 66, 67]. Since the main reference of the computation of the trispectrum in section 5.6 is [9] we briefly verify that the action reported in that paper coincides with eq.(5.39). We report this computation since it's not completely trivial. The only difference presented is in the expansion of the action is in the CS term which they write as

$$\int d^4x \frac{a^4\phi}{4f}\delta^2({}^*RR) = \int d^4x \frac{\phi}{2f}\epsilon^{ijk}\left[h''_{il}(\partial_j h^l{}_k)' + (\partial_m h'_{li})(\partial_j\partial^l h^m{}_k + \partial_k\partial^m h^l{}_j)\right]. \tag{5.40}$$

The first term in eq.(5.40) can be reformulated such that only two (instead of the original three) "powers" of metric temporal derivatives appear

$$\begin{aligned}
\text{first term} &= \int d^4x \frac{\phi\epsilon^{ijk}}{2f}\left[h''_{il}(\partial_j h^l{}_k)'\right] &(5.41)\\
&= \int d^4x \frac{\phi\epsilon^{ijk}}{2f}\frac{1}{2}\left[h''_{il}(\partial_j h^l{}_k)' - h''_{kl}(\partial_j h^l{}_i)'\right] &(5.42)\\
&= \int d^4x \frac{1}{2}\frac{\epsilon^{ijk}}{2f}\left\{-h'_{il}\left[\phi(\partial_j h^l{}_k)'\right]' - \phi h''_{kl}(\partial_j h^l{}_i)'\right\} &(5.43)\\
&= \int d^4x \frac{\epsilon^{ijk}}{2f}\left\{-h'_{il}\phi(\partial_j h^l{}_k)'' - h'_{il}\phi'(\partial_j h^l{}_k)' - \phi h''_{kl}(\partial_j h^l{}_i)'\right\} &(5.44)\\
&= \int d^4x \frac{\epsilon^{ijk}}{4f}\left\{\partial_j h'_{il}\phi(h^l{}_k)'' - h'_{il}\phi'(\partial_j h^l{}_k)' - \phi h''_{kl}(\partial_j h^l{}_i)'\right\} &(5.45)\\
&= \int d^4x \frac{\epsilon^{ijk}}{4f}\left\{-h'_{il}\phi'(\partial_j h^l{}_k)'\right\}, &(5.46)
\end{aligned}$$

where we have eliminated total derivatives due to boundary terms and in the second equality we have symmetrized the integrand in i, k. The second term in eq.(5.39) can be rewritten as

$$\begin{aligned}
\text{second term} &= \int d^4x \frac{\phi\epsilon^{ijk}}{2f}\left[(\partial_m h'_{li})(\partial_j\partial^l h^m{}_k + \partial_k\partial^m h^l{}_j)\right] &(5.47)\\
&= \int d^4x \frac{\phi\epsilon^{ijk}}{2f}\left[(\partial_m h'_{li})(\partial_j\partial^l h^m{}_k + \partial_k\partial^m h^l{}_j)\right] &(5.48)\\
&= \int d^4x \frac{\phi\epsilon^{ijk}}{2f}\frac{1}{2}\left[-(h'_{li})(\partial_m\partial_j\partial^l h^m{}_k) + (\partial_m h'_{li})(\partial_k\partial^m h^l{}_j) - (\partial_m h'_{lj})(\partial_k\partial^m h^l{}_i)\right] &(5.49)\\
&= \int d^4x \frac{\epsilon^{ijk}}{2f}\frac{1}{2}\left[+\phi(\partial_m h'_{li})(\partial_k\partial^m h^l{}_j) - (\partial_k\partial_m h_{lj})(\partial^m h^l{}_i\phi)'\right] &(5.50)\\
&= \int d^4x \frac{\epsilon^{ijk}}{2f}\frac{1}{2}\left\{-\phi(\partial_m h'_{li})(\partial_k\partial^m h^l{}_j) + (\partial_k\partial_m h_{lj})\left[\partial^m h^l{}_i\phi' + (\partial^m h^l{}_i)'\phi\right]\right\} &(5.51)\\
&= \int d^4x \frac{\epsilon^{ijk}}{4f}\left\{(\partial_k\partial_m h_{lj})\partial^m h^l{}_i\phi'\right\}, &(5.52)
\end{aligned}$$

where we have disregarded a total derivative and we have used that h_{ij} is transverse, i.e. eq.(D.12).

Now we compute the *EoM* for scalar and tensor in Fourier space. Thus we expand the scalar and tensor perturbation as

$$\delta\phi(\tau,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[u(\tau,\mathbf{k})b_0(\mathbf{k}) + u^*(\tau,-\mathbf{k})b_0^*(-\mathbf{k})\right], \tag{5.53}$$

$$h_{ij}(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^3}\sum_s\left[\epsilon_{ij}^{(s)}(\hat{\mathbf{k}})u_s(\mathbf{k},\tau)b_s(\mathbf{k}) + \left(\epsilon_{ij}^{(s)}\right)^*(\hat{\mathbf{k}})u_s^*(-\mathbf{k},\tau)b_s^*(-\mathbf{k})\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.54}$$

where $\epsilon_{ij}^{(s)}$ is an arbitrary polarization basis introduced in appendix D.1. In the following discussion, we adopt the *chiral* polarization basis, i.e. the one we indicate as $\{R, L\}$ in appendix D.1. For the scalar part, varying the action[22]

$$\delta S_{scalar} = \int d^4x \left\{ a^2 \left[\delta\phi' \frac{d(\delta\phi')}{d\lambda} - \delta\phi_{,i}\delta\frac{d\phi^{,i}}{d\lambda} \right] - 2a(\tau)H(t)\epsilon \frac{d}{d\lambda}\left(\delta\phi'\delta\phi\right) - a^4\frac{\partial^2 V}{\partial\phi^2}\delta\phi\delta\frac{d\phi}{d\lambda} \right\} \quad (5.55)$$

$$= \int d^4x \frac{d(\delta\phi)}{d\lambda}\left[-\left(a^2\delta\phi'\right)' + a^2\nabla^2(\delta\phi) + 2\left(a(\tau)H(t)\epsilon\right)'\delta\phi - a^4\frac{\partial^2 V}{\partial\phi^2}\delta\phi \right], \quad (5.56)$$

and going in Fourier space, we get the *EoM* for the mode functions, $u(\tau, \mathbf{k})$,

$$u''(\tau,\mathbf{k}) + 2\frac{a'(\tau)}{a(\tau)}u'(\tau,\mathbf{k}) + \left(k^2 + a^2\frac{\partial^2 V}{\partial\phi^2}\right)u(\tau,\mathbf{k}) - 2\left(a(\tau)H(t)\epsilon\right)'u(\tau,\mathbf{k}) = 0, \quad (5.57)$$

which introducing the new variable $\chi_u(\tau,\mathbf{k}) = a(\tau)u(\tau,\mathbf{k})$ becomes

$$0 = \left(\frac{\chi_u''}{a} - \frac{\chi_u' a'}{a^2} - \frac{a'\chi_u'}{a^2} - \left(\frac{a''}{a^2} - 2\frac{a'^2}{a^3}\right)\chi_u\right) - 2\frac{a'}{a}\left(\frac{\chi_u'}{a} - \frac{a'\chi_u}{a^2}\right) + \left(k^2 + a^2\frac{\partial^2 V}{\partial\phi^2}\right)\frac{1}{a}\chi_u + \left(\frac{a''}{a} - 2\frac{a'^2}{a^2}\right)\epsilon\frac{\chi_u}{a} \quad (5.58)$$

$$= \left(\frac{\chi_u''}{a} - \frac{a''}{a^2}\chi_u\right) + \left(k^2 + a^2\frac{\partial^2 V}{\partial\phi^2}\right)\frac{1}{a}\chi_u - 2\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)\epsilon\frac{\chi_u}{a}, \quad (5.59)$$

which using that $\frac{a''}{a} = \frac{2}{\tau^2}(1 + \frac{3}{2}\epsilon)$, appendix C.1, and

$$a^2\frac{\partial^2 V}{\partial\phi^2} \approx \frac{1}{H^2\tau^2}\frac{\partial^2 V}{\partial\phi^2} = \frac{3\eta_V}{\tau^2}, \qquad \left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)\epsilon \approx \left[\frac{2}{\tau^2} + \frac{1}{\tau^2}\right]\epsilon = \frac{3\epsilon}{\tau^2} \quad (5.60)$$

where we have used eq.(2.66), can be put in the following form

$$\chi_u'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2}\right)\chi_u = 0, \quad (5.61)$$

where $\nu_T^2 = \frac{9}{4} + 9\epsilon - 3\eta_V$, and which as shown in appendix C.2 can be put in the form of a Bessel equation. A solution satisfying the canonical commutation relation[23] and initial Bunch-Davies vacuum [54] can be found as, appendix C.2,

$$\chi_u(\tau,\mathbf{k}) = \frac{\sqrt{\pi}}{2}\exp i\left(\frac{\nu_T}{2}\pi + \frac{\pi}{4}\right)\sqrt{-\tau}H^{(1)}_{\nu_T}(-k\tau), \quad (5.62)$$

where $H^{(1)}_{\nu_T}(x)$ is the Hankel function of the first kind of index ν_T [55].

Regarding tensors, we use the variational principle directly on the Lagrangian expressed in Fourier space. In order to proceed in a systematic way we split the action into three pieces:

$$S_1 = \int d^4x \frac{M_{pl}^2}{8}a^2\left[(h^i{}_j)'(h^j{}_i)' - (\partial_k h^i{}_j)(\partial^k h^j{}_i)\right], \quad (5.63)$$

$$S_2 = -\int d^4x \frac{\epsilon^{ijk}}{4f}\left\{h_{il}'\phi'(\partial_j h^l{}_k)'\right\}, \quad (5.64)$$

$$S_3 = \int d^4x \frac{\epsilon^{ijk}}{4f}\left\{(\partial_k\partial_m h_{lj})\partial^m h^l{}_i\phi'\right\}. \quad (5.65)$$

[22]Here we adopt the same notation used in [13].

[23]Please notice that with the change of variable used, we have put the kinetic term in the canonical form [43]

As shown in appendix D.1, going to Fourier space we get

$$S_1 = \int \frac{d\tau M_{pl}^2 a^2}{4} \frac{d^3k}{(2\pi)^3} \left\{ u_R^{'}(\mathbf{k}) u_R^{'}(-\mathbf{k}) + u_L^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) - k^2 \left[u_R(\mathbf{k}) u_R(-\mathbf{k}) + u_L(\mathbf{k}) u_L(-\mathbf{k}) \right] \right\}, \tag{5.66}$$

$$S_2 = -\int d\tau \frac{k\phi^{'}}{2f} \int \frac{d^3k}{(2\pi)^3} \left\{ u_R^{'}(\mathbf{k}) u_R^{'}(-\mathbf{k}) - u_L^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) \right\}, \tag{5.67}$$

$$S_3 = +\int d\tau \frac{\phi^{'}}{2f} \frac{d^3k}{(2\pi)^3} k^3 \left\{ u_R(\mathbf{k}) u_R(-\mathbf{k}) - u_L(\mathbf{k}) u_L(-\mathbf{k}) \right\}, \tag{5.68}$$

where for the sake of simplicity we eliminated the time dependence of the mode function. Now if we perform a variation over $u_R(\mathbf{k})$, which is one of the two independent degrees of freedom, of the three actions we get

$$\delta S_1 = -\int d\tau \frac{M_{pl}^2 a^2}{2} \frac{d^3k}{(2\pi)^3} \left\{ \left[\frac{1}{a^2} \left(a^2 u_R^{'}(-\mathbf{k}) \right)^2 + k^2 u_R(-\mathbf{k}) \right] \delta u_R(\mathbf{k}) \right\}, \tag{5.69}$$

$$\delta S_2 = +\int d\tau \frac{k}{f} \int \frac{d^3k}{(2\pi)^3} \left\{ \phi^{'} u_R^{'}(-\mathbf{k}) \right\}^{'} \delta u_R(\mathbf{k}), \tag{5.70}$$

$$\delta S_3 = +\int d\tau \frac{\phi^{'}}{f} \frac{d^3k}{(2\pi)^3} k^3 u_R(-\mathbf{k}) \delta u_R(\mathbf{k}). \tag{5.71}$$

So putting everything together we get the *EoM*

$$0 = -\frac{M_{pl}^2 a^2}{2} \left[\frac{1}{a^2} \left(a^2 u_R^{'}(-\mathbf{k}) \right)^2 + k^2 u_R(-\mathbf{k}) \right] + \frac{k}{f} \left\{ \phi^{'} u_R^{'}(-\mathbf{k}) \right\}^{'} + \frac{\phi^{'}}{f} k^3 u_R(-\mathbf{k}), \tag{5.72}$$

which sending $\mathbf{k} \longmapsto -\mathbf{k}$ becomes

$$0 = u_R^{''}(\mathbf{k}) + 2\frac{a^{'}}{a} u_R(\mathbf{k})^{'} + k^2 u_R(\mathbf{k}) - \frac{2k}{M_{pl}^2 f a^2} \left\{ \phi^{''} u_R^{'}(\mathbf{k}) + \phi^{'} u_R^{''}(\mathbf{k}) + k^2 \left\{ u_R(\mathbf{k}) \right\} \phi^{'} \right\} \tag{5.73}$$

$$= u_R^{''}(\mathbf{k}) \left(1 - \frac{2k}{M_{pl}^2 f a^2} \phi^{'} \right) + u_R(\mathbf{k})^{'} \left(2\frac{a^{'}}{a} - \frac{2k}{M_{pl}^2 f a^2} \phi^{''} \right) + k^2 u_R(\mathbf{k}) \left(1 - \frac{2k}{M_{pl}^2 f a^2} \right), \tag{5.74}$$

where we recall that $k =\mid \mathbf{k} \mid$. Now if we apply the variational principle with respect the other degree of freedom, i.e. $u_L(\mathbf{k})$, we get

$$\delta S_1 = -\int d\tau \frac{M_{pl}^2 a^2}{2} \frac{d^3k}{(2\pi)^3} \left\{ \left[\frac{1}{a^2} \left(a^2 u_L^{'}(-\mathbf{k}) \right)^2 + k^2 u_L(-\mathbf{k}) \right] \delta u_L(\mathbf{k}) \right\}, \tag{5.75}$$

$$\delta S_2 = \int d\tau \frac{\mathbf{k}}{f} \int \frac{d^3k}{(2\pi)^3} \left\{ \phi^{'} u_L^{'}(-\mathbf{k}) \right\}^{'} \delta u_L(\mathbf{k}), \tag{5.76}$$

$$\delta S_3 = \int d\tau \frac{\phi^{'}}{f} \frac{d^3k}{(2\pi)^3} k^3 u_L(-\mathbf{k}) \delta u_L(\mathbf{k}), \tag{5.77}$$

When we combine the three contributions, we obtain

$$0 = -\frac{M_{pl}^2 a^2}{2} \left\{ \left[\frac{1}{a^2} \left(a^2 u_L^{'}(-\mathbf{k}) \right)^2 + k^2 u_L(-\mathbf{k}) \right] \right\} + \frac{k}{f} \left\{ \phi^{'} u_L^{'}(-\mathbf{k}) \right\}^{'} + \frac{\phi^{'}}{f} k^3 u_L(-\mathbf{k}), \tag{5.78}$$

which sending $\mathbf{k} \longmapsto -\mathbf{k}$ becomes

$$0 = u_L^{''}(\mathbf{k}) + 2\frac{a^{'}}{a} u_L(\mathbf{k})^{'} + k^2 u_L(\mathbf{k}) + \frac{2k}{M_{pl}^2 f a^2} \left\{ \phi^{''} u_L^{'}(\mathbf{k}) + \phi^{'} u_L^{''}(\mathbf{k}) + k^2 \left\{ u_L(\mathbf{k}) \right\} \phi^{'} \right\} \tag{5.79}$$

$$= u_L^{''}(\mathbf{k}) \left(1 + \frac{2k}{M_{pl}^2 f a^2} \phi^{'} \right) + u_L(\mathbf{k})^{'} \left(2\frac{a^{'}}{a} + \frac{2k}{M_{pl}^2 f a^2} \phi^{''} \right) + k^2 u_L(\mathbf{k}) \left(1 + \frac{2k\phi^{'}}{M_{pl}^2 f a^2} \right). \tag{5.80}$$

It's possible to write eq.(5.74) and (5.80) in a more compact way as

$$0 = u_s^{''}(\mathbf{k}) \left(1 - \lambda_s \frac{2k}{M_{pl}^2 f a^2}\phi^{'}\right) + u_s(\mathbf{k})^{'} \left(2\frac{a^{'}}{a} - \lambda_s \frac{2k}{M_{pl}^2 f a^2}\phi^{''}\right) + k^2 u_s(\mathbf{k}) \left(1 - \lambda_s \frac{2k\phi^{'}}{M_{pl}^2 f a^2}\right), \quad (5.81)$$

which is identical to the one obtained in [66]. In eq.(5.81) $\lambda_s = \pm 1$ respectively for R and L polarization (see appendix D.1). Now we introduce the following quantity

$$z_s(\mathbf{k}, \tau) \equiv a(\tau)\sqrt{1 - \lambda_s \frac{2k}{M_{pl}^2 f a^2}\phi'}, \quad (5.82)$$

and a related new variable $\chi_s(\tau, \mathbf{k})$ which is defined as

$$\chi_s(\mathbf{k}, \tau) \equiv z_s(\mathbf{k}, \tau) u_s(\mathbf{k}, \tau). \quad (5.83)$$

Before proceeding in manipulating the *EoM* we show how to write the function multiplying $u^{'}(\tau, \mathbf{k})$ in a smarter way. If we take the time derivative of $z_s(\tau, \mathbf{k})$

$$z_s^{'}(\mathbf{k}, \tau) = \frac{1}{2z_s}\left(2aa^{'} - \lambda_s \frac{2k}{M_{pl}^2 f}\phi^{''}\right) = \frac{a^2}{2z_s}\left(2\frac{a^{'}}{a} - \lambda_s \frac{2k}{M_{pl}^2 a^2 f}\phi^{''}\right), \quad (5.84)$$

which becomes

$$z_s^{'}(\mathbf{k}, \tau)\frac{2z_s}{a^2} = \left(2\frac{a^{'}}{a} - \lambda_s \frac{2k}{M_{pl}^2 a^2 f}\phi^{''}\right). \quad (5.85)$$

In order to get the *EoM* in the new variable we need to compute the first and second derivatives of $u(\tau, \mathbf{k})$ in order to substitute their values in eq.(5.81)

$$u_s(\mathbf{k}, \tau)^{'} = \frac{\chi_s^{'}(\mathbf{k}, \tau)}{z_s(\mathbf{k}, \tau)} - \frac{z_s^{'}(\mathbf{k}, \tau)}{z_s^2(\mathbf{k}, \tau)}\chi_s(\mathbf{k}, \tau), \quad (5.86)$$

$$u_s(\mathbf{k}, \tau)^{''} = \frac{\chi_s^{''}(\mathbf{k}, \tau)}{z_s(\mathbf{k}, \tau)} - 2\chi_s^{'}(\mathbf{k}, \tau)\frac{z_s^{'}(\mathbf{k}, \tau)}{z_s^2(\mathbf{k}, \tau)} + \chi_s(\mathbf{k}, \tau)\left[2\frac{\left(z_s^{'}(\mathbf{k}, \tau)\right)^2}{z_s^3(\mathbf{k}, \tau)} - \frac{z_s^{''}(\mathbf{k}, \tau)}{z_s^2(\mathbf{k}, \tau)}\right]. \quad (5.87)$$

The equation of motion in the chiral basis, suppressing the time and $\mathbf{k}$ dependence and using eq.(5.85), (5.86) and (5.87), becomes

$$0 = \left[\frac{\chi_s^{''}}{z_s} - 2\chi_s^{'}\frac{z_s^{'}}{z_s^2} + \chi_s\left(2\frac{\left(z_s^{'}\right)^2}{z_s^3} - \frac{z_s^{''}}{z_s^2}\right)\right]\frac{z_s^2}{a^2} + \left[\frac{\chi_s^{'}}{z_s} - \frac{z_s^{'}}{z_s^2}\chi_s\right]z_s^{'}(\mathbf{k}, \tau)\frac{2z_s}{a^2} + k^2\frac{\chi_s}{z_s}\frac{z_s^2}{a^2} \quad (5.88)$$

$$= \left[\frac{\chi_s^{''}}{z_s} - \chi_s\frac{z_s^{''}}{z_s^2}\right]\frac{z_s^2}{a^2} + k^2\frac{\chi_s}{z_s}\frac{z_s^2}{a^2}, \quad (5.89)$$

which we can write in a very simple form as

$$\chi_s^{''} + \left[k^2 - \frac{z_s^{''}}{z_s}\right]\chi_s = 0. \quad (5.90)$$

Finally, if we expand keeping the lowest order in the slow-roll parameters and we introduce the "chemical potential" μ as

$$\mu \equiv \frac{\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H}{M_{pl}}\right)^2 = \frac{\dot{\phi}H}{fM_{pl}^2}, \quad (5.91)$$

we arrive at the desired expression [7, 68]

$$\chi_s^{''} + \left[k^2 - \left(\frac{2+3\epsilon}{\tau^2} + \lambda_s \frac{2k\mu}{\tau}\right)\right] \chi_s = 0. \tag{5.92}$$

Note that our result differs from the one presented in [9] since we have changed the sign of the Pontryagin density in the action; this sign can be reabsorbed in the coupling and we recover the expression presented in the paper. Before proceeding we discuss a technical point. The coefficient λ_s is equal to ± 1 respectively for R and L modes. This implies that there exist some values of the physical wave number $k_{phys} = \frac{k}{a}$ for which the factor $z_R^2(\tau, \mathbf{k})$ becomes negative. This happens when

$$\frac{2k}{M_{pl}^2 f a^2} \phi^{'} = \frac{2k}{M_{pl}^2 f a} \dot{\phi} > 1, \tag{5.93}$$

which, introducing the Chern-Simons mass,

$$M_{CS} = \frac{M_{pl}^2 f}{2\dot{\phi}} = \frac{H}{2\mu}, \tag{5.94}$$

can be written as

$$\frac{k_{phys}}{M_{CS}} \dot{\phi} > 1. \tag{5.95}$$

These modes acquire a negative kinetic becoming ghost fields [7, 9, 68]. Since this kind of behavior can be problematic [69] we assume that only gravitons with $k_{phys} < M_{CS}$ are at work at the beginning of Inflation. This condition holds also during the rest of the Inflationary period since the universe expands k_{phys} decreases. Moreover, at the beginning of Inflation, we need also gravitons with $k_{phys} \gg H$, which implies that the mode function is deep inside the horizon, i.e.

$$\lambda_{phys} << \frac{1}{H}. \tag{5.96}$$

This condition is crucial to have quantum tensor perturbations produced from a Bunch-Davies vacuum state [7]. This condition is preserved if $M_{CS} >> H$, which, in the language of the chemical potential reads

$$\frac{H}{M_{CS}} = \mu << 1. \tag{5.97}$$

Now, the effective potential $\pm \frac{2k\mu}{\tau}$ approaches 0 in the asymptotic past, i.e. $\tau \to -\infty$, we can use the Bunch-Davies vacuum state as the initial condition for (5.92):

$$\chi_s(\tau, \mathbf{k}) = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{5.98}$$

Thus, introducing $\tilde{\nu}_T = \frac{3}{2} + \epsilon$, the solution of the EoM for the mode functions is [7, 68]

$$\chi_s(\tau, \mathbf{k}) = (-2k\tau)^{\tilde{\nu}_T} \sqrt{-\tau} e^{-ik\tau} e^{-i(\frac{\pi}{4} + \pi\frac{\tilde{\nu}_T}{2})} U\left(\frac{1}{2} + \tilde{\nu}_T - \lambda_s i\mu, 1 + 2\tilde{\nu}_T, 2ik\tau\right) \exp\left(\lambda_s \frac{\pi}{2}\mu\right), \tag{5.99}$$

where U is the Tricomi's confluent hypergeometric function [55]. The solution for the tensor mode function due to canonical normalization becomes

$$u_s(\tau, \mathbf{k}) = \frac{\sqrt{2}}{M_{pl}} \frac{1}{z_s(\tau, \mathbf{k})} (-2k\tau)^{\tilde{\nu}_T} \sqrt{-\tau} e^{-ik\tau} e^{-i(\frac{\pi}{4} + \pi\frac{\tilde{\nu}_T}{2})} U\left(\frac{1}{2} + \tilde{\nu}_T - \lambda_s i\mu, 1 + 2\tilde{\nu}_T, 2ik\tau\right) \exp\left(\lambda_s \frac{\pi}{2}\mu\right), \tag{5.100}$$

where the correct normalization is recovered from the action presented in [7]. We observe that $\frac{\sqrt{2}}{M_{pl}}$ can be incorporated a priori into the definition of $z_s(\tau, \mathbf{k})$. However, since it is not required for the solution of the equations of motion (EoM), we have set it to one in the previous computation.

5.4.1 The scalar and graviton propagators.

In this section, we introduced the different propagators we have to employ in the following sections in order to apply the Schwinger-Keldysh diagrammatic rules, section 4.2.

As explained in section 4.2 we have four different kinds of propagators both for the scalar and tensor. Starting with the scalar sector, the tree-level propagators in momentum space are

$$G_{++}(k,\tau_1,\tau_2) = u(k,\tau_1)u^*(k,\tau_2)\theta(\tau_1-\tau_2) + u^*(k,\tau_1)u(k,\tau_2)\theta(\tau_2-\tau_1), \tag{5.101}$$
$$G_{+-}(k,\tau_1,\tau_2) = u^*(k,\tau_1)u(k,\tau_2), \tag{5.102}$$
$$G_{-+}(k,\tau_1,\tau_2) = u(k,\tau_1)u^*(k,\tau_2), \tag{5.103}$$
$$G_{--}(k,\tau_1,\tau_2) = u^*(k,\tau_1)u(k,\tau_2)\theta(\tau_1-\tau_2) + u(k,\tau_1)u^*(k,\tau_2)\theta(\tau_2-\tau_1) \tag{5.104}$$

where $u(k,\tau_1) = a(\tau)\chi_u(k,\tau)$ is the scalar mode function. The bulk propagators are

$$G_{+b}(k,\tau,0) \equiv \mathcal{G}_+(k,\tau) = u^*(k,\tau)u(k,0), \approx \frac{H^2}{2k^3}(1-ik\tau)e^{+ik\tau}, \tag{5.105}$$
$$G_{-b}(k,\tau,0) \equiv \mathcal{G}_-(k,\tau) = u(k,\tau)u^*(k,0) \approx \frac{H^2}{2k^3}(1+ik\tau)e^{-ik\tau}, \tag{5.106}$$

where the index b stands for bulk, and where we have taken the second conformal time to be zero since we are interested in computing the correlation function outside the horizon. In the last step, we have taken the lowest order in slow-roll parameters since in section 5.6 we work within this hypothesis.

Regarding tensors we have

$$G_{++,s}(k,\tau_1,\tau_2) = u_s(k,\tau_1)u_s^*(k,\tau_2)\theta(\tau_1-\tau_2) + u_s^*(k,\tau_1)u_s(k,\tau_2)\theta(\tau_2-\tau_1), \tag{5.107}$$
$$G_{+-,s}(k,\tau_1,\tau_2) = u_s^*(k,\tau_1)u_s(k,\tau_2), \tag{5.108}$$
$$G_{-+,s}(k,\tau_1,\tau_2) = u_s(k,\tau_1)u_s^*(k,\tau_2), \tag{5.109}$$
$$G_{--,s}(k,\tau_1,\tau_2) = u_s^*(k,\tau_1)u_s(k,\tau_2)\theta(\tau_1-\tau_2) + u_s(k,\tau_1)u_s^*(k,\tau_2)\theta(\tau_2-\tau_1), \tag{5.110}$$

where the index s represents the polarization states, and where $u_s(\tau,k)$ is what we have computed in eq.(5.100). We do not report the tensor bulk propagators since we are not going to use them.

5.5 Scalar and tensor Power Spectrum

In order to proceed to the computation of the primordial power spectra, we must canonically quantize the scalar and tensor fields. In order to recover the canonical commutation relation we have to quantize the rescaled fields both for scalar, $\chi_u(\tau,\mathbf{k}) = a(\tau)u(\tau,\mathbf{k})$, and tensor, $\chi_s(\tau,\mathbf{k}) = \frac{M_{pl}z_s(\tau,\mathbf{k})}{\sqrt{2}}u_s(\tau,\mathbf{k})$, which are

$$\hat{\chi}_s(\tau,\mathbf{k}) = \chi_s(\tau,\mathbf{k})\hat{b}_s(\mathbf{k}) + \chi_s^*(\tau,\mathbf{k})\hat{b}_s^\dagger(-\mathbf{k}), \tag{5.111}$$
$$\hat{\chi}_u(\tau,\mathbf{k}) = \chi_u(\tau,\mathbf{k})\hat{b}_0(\mathbf{k}) + \chi_u^*(\tau,\mathbf{k})\hat{b}_0^\dagger(-\mathbf{k}). \tag{5.112}$$

where $\hat{b}_0(\mathbf{k})$, $\hat{b}_R(\mathbf{k})$ and $\hat{b}_L(\mathbf{k})$ are the respectively creation and annihilation operators which obey the usual relations

$$\langle 0|\hat{b}_i^\dagger = 0, \quad \hat{b}_i|0\rangle = 0, \quad [\hat{b}_i(\mathbf{k}),\hat{b}_j^\dagger(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{ij}, \quad [\hat{b}_i(\mathbf{k}),\hat{b}_j(\mathbf{k}')] = [\hat{b}_i^\dagger(\mathbf{k}),\hat{b}_j^\dagger(\mathbf{k}')] = 0, \tag{5.113}$$

where $i = 0, R, L$. Now, in order to derive the Power spectrum we must evaluate

$$\langle 0|\hat{u}(\tau,\mathbf{k})\hat{u}(\tau,-\mathbf{k}')|0\rangle = a^{-2}(\tau)\langle 0|\left(\chi_u(\mathbf{k})\hat{b}_0(\mathbf{k}) + \chi_u^*(\mathbf{k})\hat{b}_0^\dagger(-\mathbf{k})\right)\left(\chi_u(-\mathbf{k}')\hat{b}_0(-\mathbf{k}') + \chi_u(-\mathbf{k}')^*\hat{b}_0^\dagger(-\mathbf{k}')\right)|0\rangle \tag{5.114}$$
$$= a^{-2}(\tau)\chi_u(\tau,\mathbf{k})\chi_u^*(\tau,\mathbf{k}')\langle 0|\hat{b}_0(-\mathbf{k}')\hat{b}_0^\dagger(-\mathbf{k}')|0\rangle \tag{5.115}$$
$$= a^{-2}(\tau)\chi_u(\tau,\mathbf{k})\chi_u^*(\tau,\mathbf{k})(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}'), \tag{5.116}$$

where in the first step we have suppressed the temporal dependence for clarity. Now, from this, we can derive the inflaton adimensional power spectrum, introduced in eq.(1.154)

$$\Delta_{\delta\phi}(k) = \frac{k^3}{2\pi^2}P(k) = \frac{k^3}{2\pi^2}a^{-2}(\tau)\chi_u(\tau,\mathbf{k})\chi_u^*(\tau,\mathbf{k}), \tag{5.117}$$

where S stands for scalar. Since we are interested in super-horizon scales, we can use the asymptotic expansion for the Hankel function for $-k\tau << 1$

$$H_\nu^{(1)}(x) \approx \left[-\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu i\right], \tag{5.118}$$

to approximate the scalar mode function as

$$\chi_\mu(\tau,\mathbf{k}) \approx \frac{\sqrt{\pi}}{2}\exp\left(i\frac{\nu_T}{2}\pi + i\frac{\pi}{4}\right)\sqrt{-\tau}\left[-\frac{\Gamma(\nu_T)}{\pi}\left(\frac{2}{-k\tau}\right)^{\nu_T} i\right] = \exp\left(i\frac{\pi}{2}\left(\nu_T - \frac{1}{2}\right)\frac{(-k\tau)^{-\nu_T+\frac{1}{2}}}{\sqrt{2k}}\frac{\Gamma(\nu_T)}{\Gamma(\frac{3}{2})}2^{\nu_T-\frac{3}{2}}\right. \tag{5.119}$$

Thus, the adimensional power spectrum becomes

$$\Delta_{\delta\phi}(k) \approx \frac{k^3}{2\pi^2}\left(H\tau(1-\epsilon)\right)^2\frac{(-k\tau)^{-2\nu_T+1}}{2k}\left[\frac{\Gamma(\nu_T)}{\Gamma(\frac{3}{2})}\right]^2 2^{2\nu_T-3} \tag{5.120}$$

$$= \frac{k^3}{2\pi^2}\left(H\tau(1-\epsilon)\right)^2\frac{(-k\tau)^{-2-2(3\epsilon-\eta_V)}}{k}\left[\frac{\Gamma(\frac{3}{2}+\epsilon-\eta_V)}{\Gamma(\frac{3}{2})}\right]^2 2^{2(3\epsilon-\eta_V)}, \tag{5.121}$$

where we have used that $\nu_T \approx \frac{3}{2} + 9\epsilon - \eta_V$. At first order in slow-roll parameters the adimensional power spectrum $\Delta_{\delta\phi}(k)$, i.e. Eq.(5.121), and the power spectrum $P_{\delta\phi}(k)$ become

$$\Delta_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{-2(3\epsilon-\eta_V)}, \tag{5.122}$$

$$P_{\delta\phi}(k) = \frac{1}{k^3}\frac{H^2}{2}\left(\frac{k}{aH}\right)^{-2(3\epsilon-\eta_V)}, \tag{5.123}$$

which are identical to the ones we have derived in the standard scenario. Now passing to the curvature perturbation with [15]

$$\zeta(\mathbf{k}) = -\frac{\delta\phi(\mathbf{k})}{\sqrt{2\epsilon}M_{pl}}, \tag{5.124}$$

we can write the adimensional power spectrum and power spectrum as

$$\Delta_\zeta(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{-2(3\epsilon-\eta_V)}\frac{1}{2\epsilon M_{pl}^2} = \left(\frac{H}{2\pi M_{pl}}\right)^2\left(\frac{k}{aH}\right)^{-2(3\epsilon-\eta_V)}\frac{1}{2\epsilon}, \tag{5.125}$$

$$P_\zeta(k) = \frac{1}{k^3}\frac{H^2}{2}\left(\frac{k}{aH}\right)^{-2(3\epsilon-\eta_V)}\frac{1}{2\epsilon M_{pl}^2} = \left(\frac{H}{M_{pl}}\right)^2\left(\frac{k}{aH}\right)^{-2(3\epsilon-\eta_V)}\frac{1}{4\epsilon k^3}. \tag{5.126}$$

Concerning tensors, we can follow the same procedure with a slight modification to account for the polarization tensor. Therefore we compute

$$\langle 0|\hat{h}_{ij}^s(\mathbf{k})\hat{h}_s^{ij*}(-\mathbf{k}')|0\rangle = \epsilon_{ij}^s(\mathbf{k})\epsilon_s^{ij*}(\mathbf{k})\langle 0|\hat{u}_s(\mathbf{k})\hat{u}_s^*(-\mathbf{k}')|0\rangle = 2u_s(\mathbf{k})u_s^*(\mathbf{k})(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}'), \tag{5.127}$$

where we have used that $\epsilon_{ij}^s(\mathbf{k})\epsilon_s^{ij*}(\mathbf{k}) = 2$ for both $s = L/R$, see appendix D.1. In order to proceed we need a super-horizon expression for the solutions of the EoM. Using the asymptotic form of the hypergeometric confluent functions we can write [68]

$$u_s(\tau,\mathbf{k}) \approx \frac{\sqrt{2}}{M_{pl}}\frac{1}{z_s(\tau,\mathbf{k})}\sqrt{\frac{-\tau}{2(-k\tau)^3}}e^{i(-\frac{\pi}{4}+\frac{\pi}{2}\tilde{\nu}_T)}\frac{\Gamma(\nu_T)}{\Gamma(\frac{3}{2})}\left(\frac{-k\tau}{2}\right)^{\frac{3}{2}-\tilde{\nu}_T}e^{\lambda_s\frac{\pi}{2}\mu}. \tag{5.128}$$

Therefore the adimensional power spectrum reads

$$\Delta_T^s = \frac{k^3}{2\pi^2}\frac{2}{M_{pl}^2}\frac{1}{z_s^2(\tau,\mathbf{k})}\frac{\tau}{2(k\tau)^3}\left[\frac{\Gamma(\nu_T)}{\Gamma(\frac{3}{2})}\right]^2\left(\frac{-k\tau}{2}\right)^{3-2\tilde{\nu}_T} e^{\lambda_s\pi\mu} \tag{5.129}$$

$$= \frac{1+\lambda_s\pi\mu}{\pi^2 M_{pl}^2 z_s^2(\tau,\mathbf{k})\tau^2}\left(\frac{-k\tau}{2}\right)^{-2\epsilon} \tag{5.130}$$

$$\approx \frac{H^2}{\pi^2 M_{pl}^2}e^{\lambda_s\pi\mu}\left(\frac{-k\tau}{2}\right)^{-2\epsilon} \tag{5.131}$$

$$= \frac{\Delta_T}{2}e^{\lambda_s\pi\mu}, \tag{5.132}$$

where Δ_T is the adimensional tensor power spectrum we have computed in section 2.5, where we have taken the lowest order in the slow roll parameters and in μ and, in the third step we have considered

$$z_s^2(\tau,\mathbf{k}) = a(\tau)\sqrt{1-\lambda_s 2\frac{k_{phys}}{M_{CS}}} \approx a(\tau). \tag{5.133}$$

We can approximate in this way because we initially consider a scenario where $k_{phys} < M_{CS}$ at the beginning of Inflation, and additionally, in eq.(5.132), we are considering the value of z at the end of Inflation. Thus

$$\left.\frac{k}{a}\right|_E = \left.\frac{k}{a}\right|_B \frac{a_B}{a_E} \approx \left.\frac{k}{a}\right|_B e^{-N}, \tag{5.134}$$

where the subscripts stand for the beginning and the end of Inflation while N is the minimum number of e-folds required to solve the shortcomings of the HBB model. Thus, it's clear that any correction to $a(\tau)$ in z_s is completely washed away by the accelerated expansion.

The disparity between the L and R power spectra at the linear level offers a potential observable, which is often called the chirality parameter

$$\Theta = \frac{\Delta_T^R - \Delta_T^L}{\Delta_T^R + \Delta_T^L} \approx \frac{(1+\pi\mu)-(1-\pi\mu)}{(1+\pi\mu)+(1-\pi\mu)} = \pi\mu, \tag{5.135}$$

which we expect to be small and as discussed in [7], only weak constraints could be put with future CMB experiments. This quantity is the degree of gravitational circular polarization introduced in [9] which is

$$\Pi_{circ} = \pi\mu = \pi\frac{\sqrt{2\epsilon M_{pl}}}{f}\left(\frac{H}{M_{pl}^2}\right)^2, \tag{5.136}$$

which quantitatively takes values according to [9]

$$\Pi_{circ} \approx 0.9\left(\frac{\epsilon}{10^{-2}}\right)^{\frac{1}{2}}\left(\frac{H}{10^{14}GeV}\right)^2\left(\frac{10^9 GeV}{f}\right) + O(\mu^2). \tag{5.137}$$

Another important feature is the total tensor ratio is modified in the Chern-Simons theory of modified gravity even if the effect is unobservably small

$$\Delta_T^{CS} = \Delta_T^R + \Delta_T^L \approx \Delta_T\left(1+\Theta^2)\right). \tag{5.138}$$

Therefore, at linear order in μ, we do not observe any modifications in the total power spectrum or in the relationship between scalar and tensor quantities, $r = 16\epsilon$. Going to higher orders in μ, it's possible to modify the total power spectrum with respect to the standard single-field slow-roll model [7]. Exploring the modification of the spectral index is intriguing, but we refrain from conducting the calculation here since a similar computation is carried out in the subsequent chapter (the result is identical once we exchange the two chirality parameters).

5.6 The scalar trispectrum graviton mediated

The goal of this section is the calculation of the scalar trispectrum, building upon the approach introduced in [9], and incorporating an additional observation that was not discussed in the paper, section 5.6.3. We restrict ourselves only to the lowest contribution in slow-roll parameters that can present parity-violating signatures, which is the zeroth order.

Before proceeding with the calculation, it is essential to elucidate why an imaginary part of the trispectrum leads to parity violation, as emphasized in [9], whereas the real part cannot produce such an effect. In fact, we know that the four-point correlation function, i.e.

$$\zeta(\tau, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \equiv \langle \delta\phi(\tau, \mathbf{x}_1)\delta\phi(\tau, \mathbf{x}_2)\delta\phi(\tau, \mathbf{x}_3)\delta\phi(\tau, \mathbf{x}_4)\rangle, \tag{5.139}$$

is real and in Fourier space can be written as

$$\zeta(\tau, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \int \frac{d^3k_1 e^{i\mathbf{k}_1\cdot\mathbf{x}_1}}{(2\pi)^3}\frac{d^3k_2 e^{i\mathbf{k}_2\cdot\mathbf{x}_2}}{(2\pi)^3}\frac{d^3k_3 e^{i\mathbf{k}_3\cdot\mathbf{x}_3}}{(2\pi)^3}\frac{d^3k_4 e^{i\mathbf{k}_4\cdot\mathbf{x}_4}}{(2\pi)^3}\zeta(\tau, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4). \tag{5.140}$$

Since $\zeta^*(\tau, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \zeta(\tau, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ we have $\zeta^*(\tau, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \zeta(\tau, -\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3, -\mathbf{k}_4)$. Hence, if the trispectrum is real, it cannot violate parity

$$\Re\left[\zeta(\tau, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4\right] = \zeta^*(\tau, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \zeta(\tau, -\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3, -\mathbf{k}_4), \tag{5.141}$$

while if it has an imaginary component

$$\Im\left[\zeta(\tau, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4))\right]^* = \Im\left[\zeta(\tau, -\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3, -\mathbf{k}_4))\right] = -\Im\left[\zeta(\tau, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)\right], \tag{5.142}$$

it clearly presents parity-violating signatures.

Now we delve into the actual calculation of the trispectrum, which is performed using the Schwinger-Keldysh diagrammatic rules [14] for the In-In formalism's master formula [17] (see section 4.2). The initial step involves identifying the types of diagrams that contribute at the lowest order in slow-roll to the scalar trispectrum. First of all, we notice that the scalar contribution to the trispectrum coming from all the possible diagrammatic possibilities of terms coming from the expansion of the potential can't violate parity. For example the following interaction vertex

$$\int d^4x \frac{1}{4!}\,\frac{\partial^4 V}{\partial\phi^4}\delta\phi^4(\tau, \mathbf{x})\bigg|_{\phi(\tau,\mathbf{x})=\phi(\tau)}, \tag{5.143}$$

which diagrammatically in Fourier space has two forms[24]

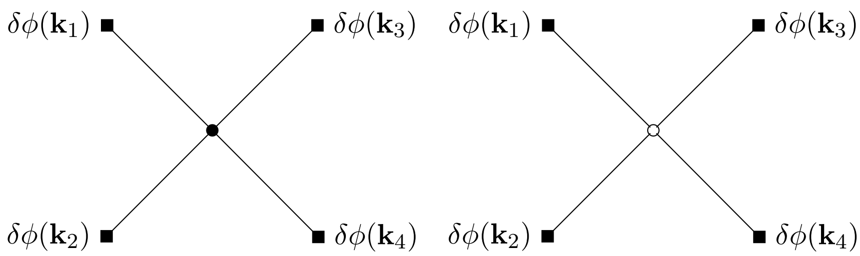


Figure 13: Quartic interaction from the potential expansion.

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can't produce any contribution to the parity-violating part of the trispectrum[25]. We notice that the possible contributions to the parity-violating trispectrum must be the ones, shown in fig.14, which are mediated by gravitons since the parity violation arises from the modification of standard *GR*. Clearly, there are contributions coming from much more complicated diagrams involving loops but here we consider only the dominant terms which are the tree-level contributions. Thus at lowest order, we are seeking the following diagram shape:

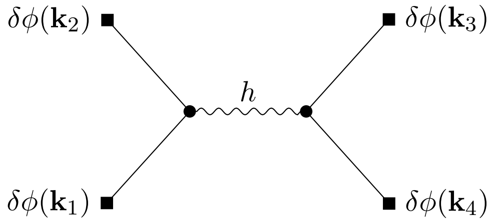


Figure 14: The diagram for the graviton-mediated trispectrum.

Based on the preceding discussion, it becomes evident that we require interaction vertices of the type depicted in fig. 15. These vertices involve two scalars and one tensor.

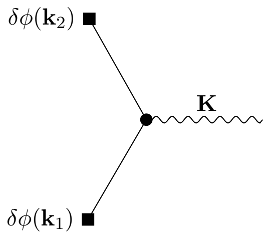


Figure 15: scalar-scalar-tensor vertex

This kind of contribution can come up only from the kinetic term and the CS one. Expanding the kinetic term we obtain the following contribution to the action in position space

$$S_{int}^{(1)} = -\int d^4x \frac{1}{2} a^2(\tau) h^{ij} \partial_i \delta\phi(\tau, \mathbf{x}) \partial_j \delta\phi(\mathbf{x}, \tau). \tag{5.144}$$

which in Fourier space becomes

$$S_{int}^{(1)} = -\frac{1}{2} \int \frac{d^4x}{(H\tau)^2} \frac{d^3k e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^3} \frac{d^3q e^{i\mathbf{q}\cdot\mathbf{x}}}{(2\pi)^3} \frac{d^3K e^{i\mathbf{K}\cdot\mathbf{x}}}{(2\pi)^3} ik_f iq_l u(k,\tau) u(q,\tau) \sum_h \epsilon^{lf}(\mathbf{K}) u_h(K,\tau) \tag{5.145}$$

$$= \frac{1}{2}(2\pi)^3 \int \frac{d\tau}{(H\tau)^2} \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3K}{(2\pi)^3} k_f q_l u(k,\tau) u(q,\tau) \sum_h \epsilon^{lf}(\mathbf{K}) u_h(K,\tau) \delta^3(\mathbf{k}+\mathbf{q}+\mathbf{K}), \tag{5.146}$$

from which, according to [14] we can write the relative vertex diagrammatic rule in Fourier space as

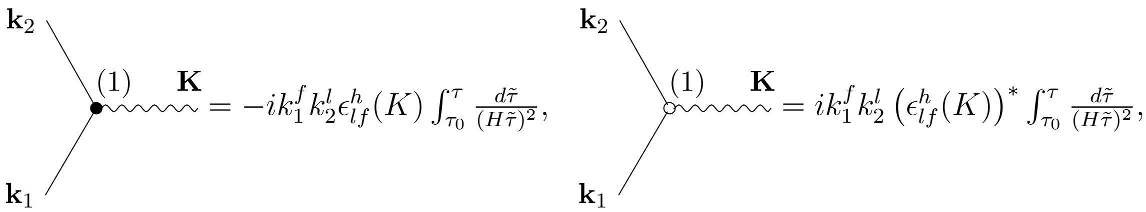


Figure 16: Diagrammatic rule for the interaction vertex

[24] each vertex can be of the + or − type which is respectively represented with a dot and an empty dot, see section 4.2.

[25] See section 4.2 for the diagrammatic formalism.

where (1) on the vertex is used to refer to the interaction of eq.(5.144), $\epsilon^h_{ij}(\mathbf{K})$ are the polarization tensors introduced in appendix D.1 with $h = L/R$, and $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$.

Now we need to expand the CS term in order to obtain the scalar-scalar-tensor vertex. Using the conformal invariance of the Weyl tensor [13] and retaining the leading order in slow roll it's possible to get [7]

$$S^{(2)}_{int} = -\int d^4x \frac{2}{fM_{pl}} \sqrt{\epsilon} \left(\partial^l \delta\phi\right) \epsilon^{ijk} \left[(\partial_k \delta\phi)\, \partial_i h'_{lj}\right], \tag{5.147}$$

which up to a surface term can be written in Fourier space as

$$S^{(2)}_{int} = \frac{2}{fM_{pl}} \sqrt{\epsilon}\epsilon^{ijk} \int d^4x \frac{d^3k e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^3} \frac{d^3q e^{i\mathbf{q}\cdot\mathbf{x}}}{(2\pi)^3} \frac{d^3K e^{i\mathbf{K}\cdot\mathbf{x}}}{(2\pi)^3} ik_f iq_l \left(u(k,\tau)u(q,\tau)\right)' \sum_h \epsilon^{lf}(K) u_h(K,\tau) \tag{5.148}$$

$$= \frac{2}{fM_{pl}} \sqrt{\epsilon}\epsilon^{ijk} (2\pi)^3 \int d\tau \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3K}{(2\pi)^3} k_f q_l \left(u(k,\tau)u(q,\tau)\right)' \sum_h \epsilon^{lf}(K) u_h(K,\tau) \delta^3(\mathbf{k}+\mathbf{q}+\mathbf{K}), \tag{5.149}$$

where we have reintroduced the Planck mass by dimensional analysis. Now, since temporal derivatives appear [14], we directly write down the expression of the vertex in which the external lines are contracted with scalars. The associated diagrammatic rule becomes

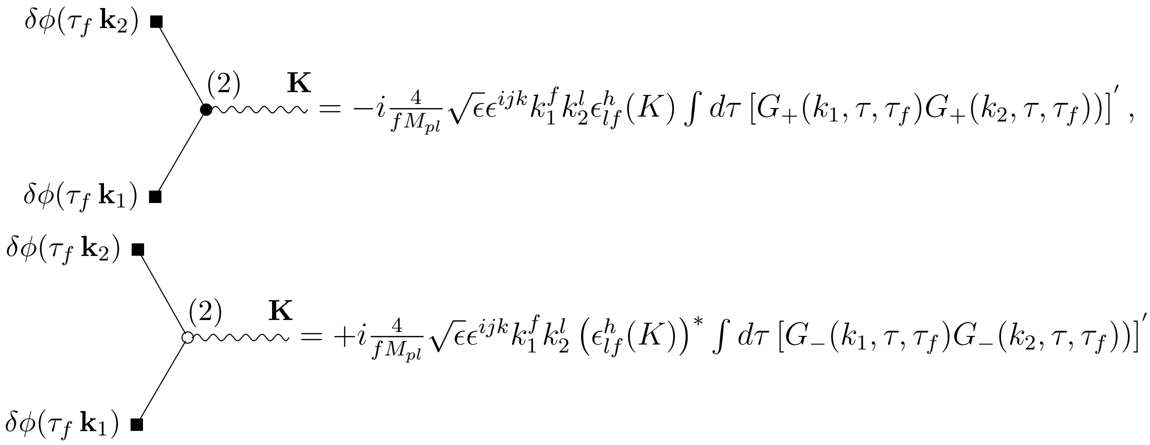


Figure 17: Diagrammatic rule for the CS interaction vertex

Thus, using the two interaction verteces of eq.(5.147) and eq.(5.149) we can get four diagram combinations:

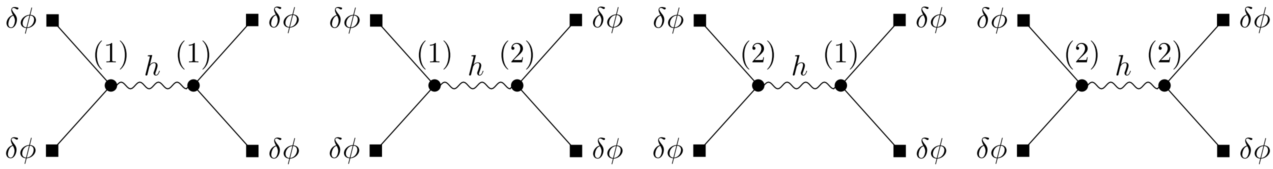


Figure 18: The possible diagrammatic combination using the two vertices.

where we haven't specified the momenta of the scalar field and the kind of vertices (see section 4.2) for simplicity but in the calculations all the possible permutations must be considered in order to get the correct result. Notice that the second and the third diagrams can be obtained one from the other by exchanging the momenta.

Thus we start the computation of the first possibility in which the two vertices came from the kinetic term. Now we have to compute twelve diagrams since we have three channels

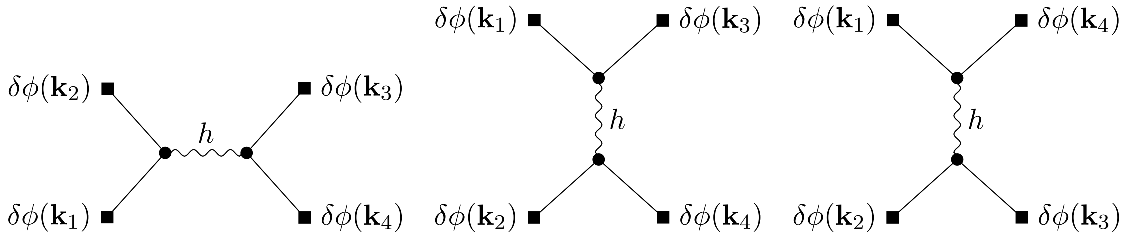


Figure 19: s,t and, u channel

and each channel comes in four combinations since we have two vertices (each vertex can be of the $+$ or $-$ type which are respectively represented with a dot and an empty dot)

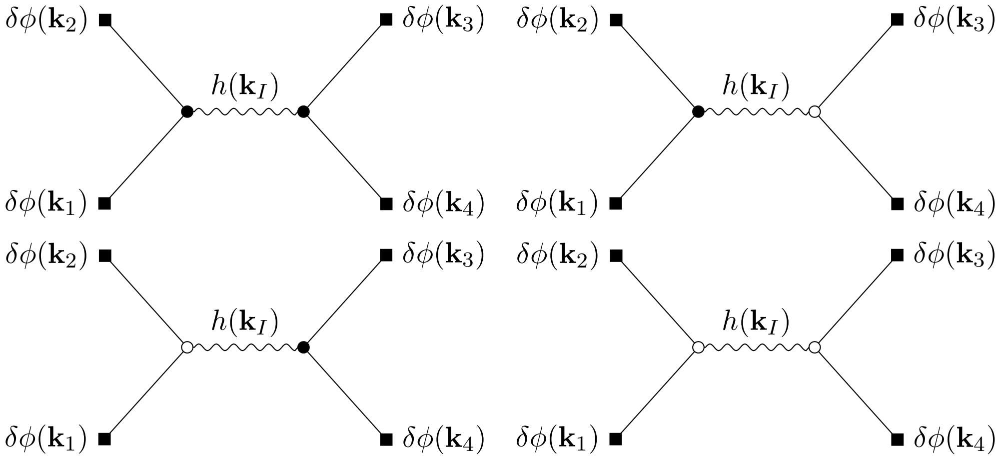


Figure 20: The four possibilities for the s channel

Since the computation for each channel is identical we introduce $\mathbf{K}_I = (\mathbf{k}_I^1, \mathbf{k}_I^2, \mathbf{k}_I^3, \mathbf{k}_I^4)$ with $I = s, t, u$

$$\mathbf{K}_s = (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4), \qquad \mathbf{K}_t = (\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_4), \qquad \mathbf{K}_u = (\mathbf{k}_1, \mathbf{k}_4, \mathbf{k}_3, \mathbf{k}_2), \tag{5.150}$$

and $\mathbf{k}_I = \mathbf{k}_I^1 + \mathbf{k}_I^2 = \mathbf{k}_I^3 + \mathbf{k}_I^4$. Therefore, we can determine the contribution from the four diagrams for the generic channel I using the rules we have determined above.

$$\text{first} = -\sum_h \int_{-\infty}^0 \frac{d\tau_1 d\tau_2}{(H^2\tau_1\tau^2)^2}\epsilon^h_{ij}(k_I)(k_I^1)^i(k_I^2)^j \mathcal{G}_+(k_I^1,\tau_1)\mathcal{G}_+(k_I^2,\tau_1)$$
$$\times\,[\epsilon^h]^*_{ab}(k_I)(k_I^3)^a(k_I^4)^b\mathcal{G}_+(k_I^3,\tau_2)\mathcal{G}_+(k_I^4,\tau_2)u_h(k_I,max[\tau_1,\tau_2])u_h^*(k_I,min[\tau_1,\tau_2])] \tag{5.151}$$
$$= -\sum_h P_h(\mathbf{K}_I)\mathcal{J}_h^{(2)}(\mathbf{k}), \tag{5.152}$$

$$\text{second} = \sum_h P_h(\mathbf{K}_I)\int_{-\infty}^0 \frac{d\tau_1}{(H\tau_1)^2}\frac{d\tau_2}{(H\tau_2)^2}[\mathcal{G}_-(k_I^1,\tau_1)\mathcal{G}_-(k_I^2,\tau_1)[\mathcal{G}_+(k_I^3,\tau_2)\mathcal{G}_+(k_I^4,\tau_2)u_h^*(k_I,\tau_1)u_h(k_I,\tau_2)] \tag{5.153}$$
$$= \sum_h P_h(\mathbf{K}_I)\bar{\mathcal{J}}_h^{(1)}(\mathbf{k}_1,\mathbf{k}_2)J_h^{(1)}(\mathbf{k}_3,\mathbf{k}_4), \tag{5.154}$$

$$\text{third} = \sum_h P_h(\mathbf{K}_I)\int_{-\infty}^0 \frac{d\tau_1}{(H\tau_1)^2}\frac{d\tau_2}{(H\tau_2)^2}[u_{+b}(k_I^1,\tau_1)u_{+b}(k_I^2,\tau_1)[u_{-b}(k_I^3,\tau_2)u_{-b}(k_I^4,\tau_2)u_h(k_I,\tau_1)u_h^*(k_I,\tau_2)] \tag{5.155}$$
$$= \sum_h P_h(\mathbf{K}_I)\int_{-\infty}^0 \frac{d\tau_1}{(H\tau_1)^2}[u_{+b}(k_I^1,\tau_1)u_{+b}(k_I^2,\tau_1)u_h(k_I,\tau_1)\int \frac{d\tau_2}{(H\tau_2)^2}[u_{-b}(k_I^3,\tau_2)u_{-b}(k_I^4,\tau_2)u_h^*(k_I,\tau_2)] \tag{5.156}$$
$$= \sum_h P_h(\mathbf{K}_I)\mathcal{J}_h^{(1)}(\mathbf{k}_1,\mathbf{k}_2)\bar{\mathcal{J}}_h^{(1)}(\mathbf{k}_3,\mathbf{k}_4), \tag{5.157}$$
$$\text{fourth} = -\sum_h P_h(\mathbf{K}_I)\bar{\mathcal{J}}_h^{(2)}(\mathbf{k}), \tag{5.158}$$

where we have introduced

$$P_h(\mathbf{K}_I) = \epsilon^h_{ij}(k_I)(k_I^1)^i(k_I^2)^j[\epsilon^h]^*_{ab}(k_I)(k_I^3)^a(k_I^4)^b, \tag{5.159}$$

and the following notation for the time integrals

$$\mathcal{J}_h^{(2)}(\mathbf{K}_I) = \int \frac{d\tau_1 d\tau_2}{(H^2\tau_1\tau_2)^2}\mathcal{G}_+(\mathbf{k}_I^1,\tau_1)\mathcal{G}_+(\mathbf{k}_I^2,\tau_1)\mathcal{G}_+(\mathbf{k}_I^3,\tau_2)\mathcal{G}_+(\mathbf{k}_I^4,\tau_4)u_h[\mathbf{k}_I,\max(\tau_1,\tau_2)]u_h^*[\mathbf{k}_I,\min(\tau_1,\tau_2)], \tag{5.160}$$
$$\mathcal{J}_h^{(1)}(\mathbf{k}_I^i,\mathbf{k}_I^j) = \int \frac{d\tau}{(H\tau)^2}\mathcal{G}_+(\mathbf{k}_I^i,\tau)\mathcal{G}_+(\mathbf{k}_I^j,\tau)u_h^*(\mathbf{k}_I^i+\mathbf{k}_I^j,\tau), \tag{5.161}$$
$$T_{var_h}(\mathbf{K}_I) = \Re\left(\mathcal{J}_h^{(2)}(\mathbf{K}_I) - \mathcal{J}_h^{(1)}(\mathbf{k}_I^1,\mathbf{k}_I^2)\mathcal{J}_h^{(1)*}(\mathbf{k}_I^3,\mathbf{k}_I^4)\right). \tag{5.162}$$

Summing everything together we get

$$\langle\delta\phi(\mathbf{k}_1)\delta\phi(\mathbf{k}_2)\delta\phi(\mathbf{k}_3)\delta\phi(\mathbf{k}_4)\rangle = -\sum_{I,s} 2P_h(\mathbf{K}_I)T_{var_h}(\mathbf{K}_I), \tag{5.163}$$

Now using eq.(D.12) that

$$P_R(\mathbf{K}_I) = \epsilon^R_{ij}(\mathbf{K})(k_I^1)^i(k_I^2)^j[\epsilon^R]^*_{ab}(K_I)(k_I^3)(k_I^4)^b = [\epsilon^L_{ij}]^*(\mathbf{K})(k_I^1)^i(k_I^2)^j\epsilon^L_{ab}(K_I)(k_I^3)^a(k_I^4)^b = P_L^*(\mathbf{K}_I). \tag{5.164}$$

we can write, from eq.(5.163), that

$$\Re\langle\delta\phi(\mathbf{k}_1)\delta\phi(\mathbf{k}_2)\delta\phi(\mathbf{k}_3)\delta\phi(\mathbf{k}_4)\rangle = -2\sum_I \Re\left(P_R(\mathbf{K}_I)\right)\left[T_{var_R}(\mathbf{K}_I) + T_{var_L}(\mathbf{K}_I)\right], \tag{5.165}$$
$$\Im\langle\delta\phi(\mathbf{k}_1)\delta\phi(\mathbf{k}_2)\delta\phi(\mathbf{k}_3)\delta\phi(\mathbf{k}_4)\rangle = -2\sum_I \Im\left(P_R(\mathbf{K}_I)\right)\left[T_{var_R}(\mathbf{K}_I) - T_{var_L}(\mathbf{K}_I)\right]. \tag{5.166}$$

Thus the computation of the graviton-mediated trispectrum proceeds in two steps: the computation of the polarization sums and the time integral.

5.6.1 Polarization sum

First of all, we introduce a Cartesian system in Fourier space such that the $z-$axis is parallel to $\mathbf{k}_I$ as shown in figure 21 (for clarity we haven't reported all the vectors). In this particular reference frame,

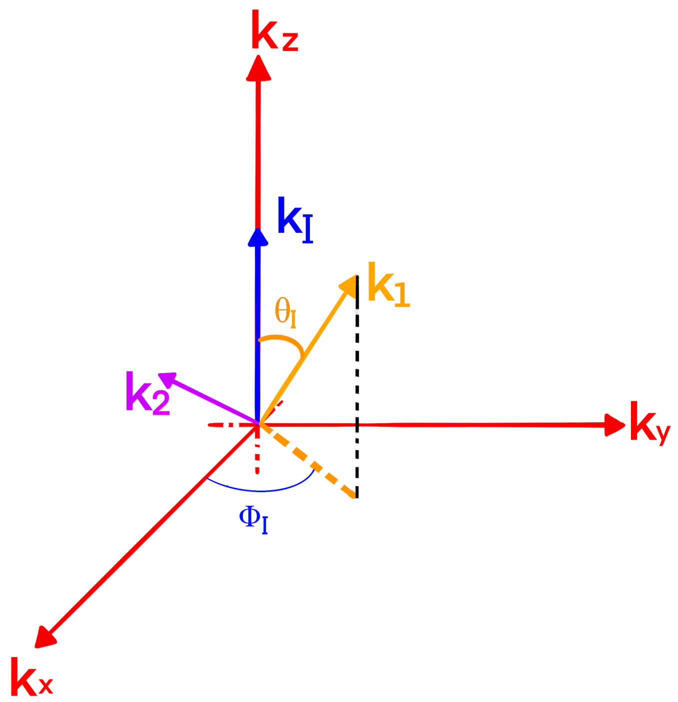


Figure 21: The basis.

the two polarizations read

$$\epsilon^R_{ij}(\mathbf{k}_I) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i & 0\\ i & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \epsilon^L_{ij}(\mathbf{k}_I) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i & 0\\ -i & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{5.167}$$

and we can write in polar coordinates the $\mathbf{K}^i_I$s as

$$\mathbf{K}^i_I = K^i_I\left[\cos\left(\phi^i_I\right)\sin\left(\theta^i_I\right), \sin\left(\theta^i_I\right)\sin\left(\phi^i_I\right), \cos\left(\theta^i_I\right)\right]. \tag{5.168}$$

Now we want to compute the polarization factor

$$\mathcal{P}_h(\mathcal{K}_I) = \left[\epsilon^h_{ab}(\mathbf{k}_I)\left(K^1_I\right)^a\left(K^2_I\right)^b\right]\left[\epsilon^{h\,*}_{cd}(\mathbf{k}_I)\left(K^3_I\right)^c\left(K^4_I\right)^d\right], \tag{5.169}$$

which is the product of two similar expressions; we can obtain the second from the first by exchanging momenta and taking the complex conjugate. Using that $\mathbf{K}^2_I = \mathbf{k}_I - \mathbf{K}^2_I$ and that $\epsilon^{L/R}_{ij}(k_I)k^i_I = 0$, the first term for the right polarization can be written as

$$\left[\epsilon^R_{ab}(\mathbf{k}_I)\left(K^1_I\right)^a\left(K^2_I\right)^b\right] = -K^1_I\begin{pmatrix}\cos\left(\phi^1_I\right)\sin\left(\theta^1_I\right)\\ \sin\left(\theta^1_I\right)\sin\left(\phi^1_I\right)\\ \cos\left(\theta^1_I\right)\end{pmatrix}^T\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}K^1_I\begin{pmatrix}\cos\left(\phi^i_I\right)\sin\left(\theta^1_I\right)\\ \sin\left(\theta^1_I\right)\sin\left(\phi^1_I\right)\\ \cos\left(\theta^1_I\right)\end{pmatrix} \tag{5.170}$$

$$= -K^1_I K^1_I\begin{pmatrix}\cos\left(\phi^1_I\right)\sin\left(\theta^1_I\right)\\ \sin\left(\theta^1_I\right)\sin\left(\phi^1_I\right)\\ \cos\left(\theta^1_I\right)\end{pmatrix}^T\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\left(\phi^i_I\right)\sin\left(\theta^1_I\right) + i\sin\left(\theta^1_I\right)\sin\left(\phi^1_I\right)\\ +i\cos\left(\phi^i_I\right)\sin\left(\theta^1_I\right) - \sin\left(\theta^1_I\right)\sin\left(\phi^1_I\right)\\ 0\end{pmatrix} \tag{5.171}$$

$$= -\frac{K^1_I K^1_I}{\sqrt{2}}\sin^2\left(\theta^1_I\right)e^{+i(2\phi^1_I)}, \tag{5.172}$$

where $K_I^1 \equiv\mid \mathbf{K}_I^1 \mid$. Now sending $\mathbf{K}_I^{1/2} \longmapsto \mathbf{K}_I^{3/4}$ and taking the complex conjugate we get for the second factor in eq.(5.169)

$$\left[\epsilon_{ab}^{R\,*}(\mathbf{k}_I)\left(K_I^4\right)^a\left(K_I^2\right)^b\right]=\left[\epsilon_{ab}^{L}(\mathbf{k}_I)\left(K_I^4\right)^a\left(K_I^2\right)^b\right]==-\frac{K_I^3K_I^3}{\sqrt{2}}\sin^2\left(\theta_I^3\right)e^{-i2\phi_I^3}. \tag{5.173}$$

Taking the product of eq.(5.172) and eq.(5.173) we obtain

$$\mathcal{P}_R(\mathcal{K}_I)=\frac{1}{2}\left[K_I^1K_I^3\sin\left(\theta_I^1\right)\sin\left(\theta_I^3\right)e^{+i\phi_I^1}e^{-i\phi_I^3}\right]^2, \tag{5.174}$$

$$\mathcal{P}_L(\mathcal{K}_I)=\frac{1}{2}\left[K_I^1K_I^3\sin\left(\theta_I^1\right)\sin\left(\theta_I^3\right)e^{-i\phi_I^1}e^{+i\phi_I^3}\right]^2. \tag{5.175}$$

5.6.2 Time integral

We now compute the time integral following the procedure adopted in [9]. What we want to compute is

$$T_{var_h}(\mathbf{K}_I)=\Re\left(\mathcal{J}_h^{(2)}(\mathbf{K}_I)-\mathcal{J}_h^{(1)}(\mathbf{k}_I^1,\mathbf{k}_I^2)\mathcal{J}_h^{(1)*}(\mathbf{k}_I^3,\mathbf{k}_I^4)\right), \tag{5.176}$$

where we recall that from the computation of section 5.4

$$u(\tau,\mathbf{k})=\frac{-i}{a(\tau)}\frac{\sqrt{\pi}}{2}\exp\left(i\frac{\nu_T}{2}\pi+i\frac{\pi}{4}\right)\sqrt{-\tau}H_{\nu_T}^{(1)}(-k\tau), \tag{5.177}$$

$$u_s(\tau,\mathbf{k})=\frac{1}{z_s(\tau,\mathbf{k})}2\sqrt{\frac{(-k\tau)^3}{k}}e^{-ik\tau}e^{-i(\frac{\pi}{4}+\pi\frac{\nu_T}{2})}U(\frac{1}{2}+\nu_T-\lambda_si\mu,1+2\nu_T,2ik\tau)e^{\lambda_s\frac{\pi}{2}\mu}. \tag{5.178}$$

First of all, we are interested in keeping only the dominant term both for the real part and the imaginary part of the scalar trispectrum. We have to retain μ since the parity-violating part of the trispectrum, i.e. the imaginary piece, has to be proportional to μ. Regarding the real part, this is different from zero and contains terms of zeroth order in the slow-roll parameters because the solution of the *EoM* contains zeroth order terms in the slow-roll parameters. Keeping this assumption in mind we have

$$a(\tau)=-\frac{1}{H\tau},\quad \nu_T=\frac{3}{2},\quad H_{\frac{3}{2}}^{(1)}(-k\tau)=-\frac{\sqrt{\frac{2}{\pi}}i(1+ik\tau)e^{-ik\tau}}{(-k\tau)^{\frac{3}{2}}},\quad z_s(\tau,\mathbf{k})\approx a(\tau), \tag{5.179}$$

where in expanding z we have also considered that being an effective field theory we have $k_{phys} < M_{CS}$. Thus for the scalar we obtain

$$u(\tau,\mathbf{k})=(+iH\tau)\frac{\sqrt{\pi}}{2}(-1)\sqrt{-\tau}\left(-\frac{\sqrt{\frac{2}{\pi}}i(1+ik\tau)e^{-ik\tau}}{(-k\tau)^{\frac{3}{2}}}\right)=\frac{H}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau}, \tag{5.180}$$

while for the tensor we get

$$u_s(\tau,\mathbf{k})=\frac{\sqrt{2}}{M_{pl}}(-H\tau)(-2k\tau)^{\frac{3}{2}}\sqrt{-\tau}e^{-ik\tau}(-1)U(2-\lambda_si\mu,4,2ik\tau)e^{\lambda_s\frac{\pi}{2}\mu}\left(i\frac{\frac{1+ik\tau}{4k^3\tau^3}}{U(2,4,2ik\tau)}\right) \tag{5.181}$$

$$=\left(\frac{H}{M_{pl}\sqrt{k^3}}\right)e^{-ik\tau}\left(\frac{U(2-\lambda_si\mu,4,2ik\tau)}{U(2,4,2ik\tau)}\right)i\left(1+ik\tau\right)e^{\lambda_s\frac{\pi}{2}\mu}. \tag{5.182}$$

Before continuing, we introduce an additional approximation to enable us to analytically evaluate the time integral. We assume that

$$U(2-i\lambda_s\mu,4,2ik\tau)\approx C_sU(2,4,2ik\tau), \tag{5.183}$$

where we can find the amplitude at first order in μ by imposing the equality in $\tau = 0$

$$C_R \approx 1 - i(-1+\gamma)\mu, \quad C_L \approx 1 + i(-1+\gamma)\mu, \tag{5.184}$$

where $\gamma = 0.577216$ is the Euler constant. Using the HypergeometricU[...] function in Mathematica, we can derive this result. To assess the accuracy of our approximation, it is crucial to emphasize that the primary contribution to the graviton mode function arises near horizon crossing, where $|\tau k_i| \approx 1$. Furthermore, if $-k\tau < -1$ the integrand is highly oscillatory and we expect a subdominant contribution [17]. Moreover, we can plot the

$$\left| \frac{C_s U(2-\lambda_s\mu, 4, 2iz)}{U(2,4,2iz)} \right|^2, \tag{5.185}$$

as show in figure 22. In the first subplot, we display the outcome for the right polarization, whereas, in the second subplot, we show the result for the left polarization. From the plot we can appreciate that if $\mu << 1$ and $z = k\tau < 1$ it's reasonable that the approximation works.

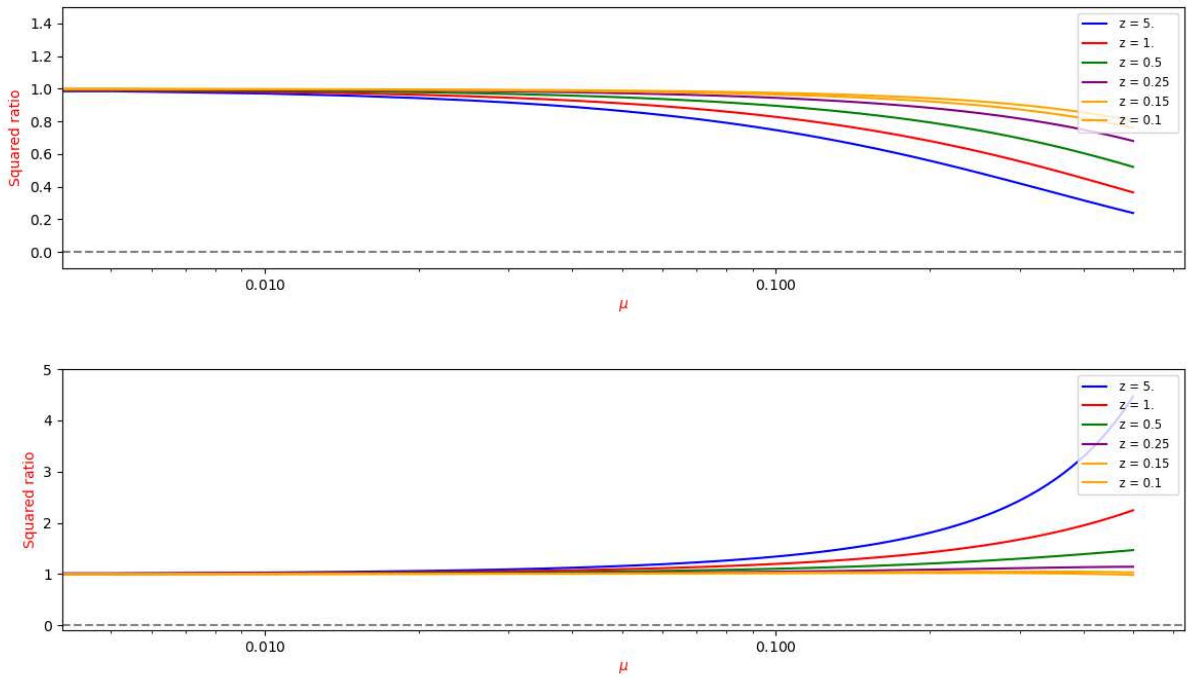


Figure 22: $\left| \frac{C_s U(2-\lambda_s\mu,4,2iz)}{U(2,4,2iz)} \right|^2$ vs μ for different value of $z \equiv k\mu$.

This assumption allows us to evaluate the time integral taking the $0th$ order in slow-roll and μ solutions of the EoM,

$$u(\tau, \mathbf{k}) = \frac{iH}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau}, \tag{5.186}$$

$$u_s(\tau, \mathbf{k}) = \frac{iH}{M_{pl}\sqrt{k^3}}(1+ik\tau)e^{-ik\tau}. \tag{5.187}$$

since μ has the only effect to modify the amplitude of the mode function. Thus we can simplify the scalar propagator as

$$\mathcal{G}_\pm(k,\tau) = \frac{H^2}{2k^3}(1-\pm k\tau)e^{\pm ik\tau}. \tag{5.188}$$

We do not report the explicit expression for the modification of the tensor propagators since in the time integral we report the explicit expression for this propagator as product of mode functions. Since in the time integral, i.e. eq.(5.176), the graviton mode function appears always multiplied by its complex conjugate we have that

$$T_{var_s}(\mathbf{K}_I) = \Re\left(C_s C_s^* e^{\lambda_s \mu\pi}\left[\mathcal{J}_h^{(2)}(\mathbf{K}_I) - \mathcal{J}_h^{(1)}(\mathbf{k}_I^1,\mathbf{k}_I^2)\mathcal{J}_h^{(1)*}(\mathbf{k}_I^3,\mathbf{k}_I^4)\right]_{\mu=0}\right), \tag{5.189}$$

which using that up to first order $C_s C_s^* \approx 1$ we get

$$T_{var_s}(\mathbf{K}_I) = e^{\lambda_s \pi\mu}\, T_{var_s}(\mathbf{K}_I)|_{\mu=0}\,. \tag{5.190}$$

Therefore using Mathematica we finally obtain using the $i\epsilon$-prescription in the asymptotic past [15]

$$\begin{aligned}T_{var_h}(\mathbf{K}_I)|_{\mu=0} = -\frac{\bar{A}_t}{2k_I^3}\frac{H^4}{\prod_{i=1}^4 2k_i^3}\Bigg\{&\frac{K_I^1+K_I^2}{[a_{34}^I]^2}\left[\frac{1}{2}(a_{34}^I+k_I)([a_{34}^I]^2-2b_{34}^I)+k_I^2(K_I^3+K_I^4)\right]+(1,2)\leftrightarrow(3,4)\\ &+\frac{K_I^1K_I^2}{k_t}\left[\frac{b_{34}^I}{a_{34}^I}-k_I+\frac{k_I}{a_{12}^I}\left(K_I^3K_I^4-k_i\frac{b_{34}^I}{a_{34}^I}\right)\left(\frac{1}{k_t+\frac{1}{a_{12}^I}}\right)\right]+(1,2)\leftrightarrow(3,4)\\ &-\frac{k_I}{a_{12}^I a_{34}^I k_t}\left[b_{12}^I b_{34}^I+2k_I^2\left(\prod_{i=1}^4 k_i\right)\left(\frac{1}{k_t^2}+\frac{1}{a_{12}^I a_{34}^I}+\frac{k_I}{k_t a_{12}^I a_{34}^I}\right)\right]\Bigg\},\end{aligned} \tag{5.191}$$

with

$$k_t = \sum_{i=1}^4 |\mathbf{K}_I^i|,\ \ a_{ij}^I = \Big[|\mathbf{K}_I^i| + |\mathbf{K}_I^j| + k_I\Big],\ \ b_{ij}^I = \Big[(|\mathbf{K}_I^i| + |\mathbf{K}_I^j)k_I + |\mathbf{K}_I^i||\mathbf{K}_I^j|\Big],\ \ \bar{A}_t = 4\left(\frac{H}{M_{pl}}\right)^2. \tag{5.192}$$

It is worth noting that the outcome of eq. (5.191) remains independent of the polarization, as evident from our choice to set μ equal to zero. Consequently, we omit the chiral index when discussing the time integral evaluated at $\mu = 0$. The program used to compute the integral is available on this link. Now we can finally compute the graviton-mediated trispectrum as

$$\Re\langle\delta\phi(\mathbf{k}_1)\delta\phi(\mathbf{k}_2)\delta\phi(\mathbf{k}_3)\delta\phi(\mathbf{k}_4)\rangle = -4\sum_I \Re\left(P_R(\mathbf{K}_I)\right) T_{var_R}(\mathbf{K}_I)|_{\mu=0}\,, \tag{5.193}$$

$$\Im\langle\delta\phi(\mathbf{k}_1)\delta\phi(\mathbf{k}_2)\delta\phi(\mathbf{k}_3)\delta\phi(\mathbf{k}_4)\rangle = -4\pi\mu\sum_I \Im\left(P_R(\mathbf{K}_I)\right) T_{var_R}(\mathbf{K}_I)|_{\mu=0}\,. \tag{5.194}$$

We now focus on one of the collapsed limits of the trispectrum, i.e. the one in which $|\mathbf{K}_I^i| >> |\mathbf{k}_I| \approx 0$, with $i \in 1,2,3,4$. Operating with this approximation is much easier, and our goal is to demonstrate that the trispectrum's amplitude in the collapsed limit is too small for detection. In this way, we only want to highlight the difficulty of detecting the trispectrum. Certainly, a more comprehensive examination becomes necessary in alternative configurations, as we encounter numerous simplifications in the collapsed limit. The collapsed limit also implies that $\mathbf{K}_I^1 \approx -\mathbf{K}_I^2$ and $\mathbf{K}_I^3 \approx -\mathbf{K}_I^4$, and the fact that $\mathbf{k}_I \approx 0$ makes our previous assumption for the tensor mode function exact. Unless the conditions $\mathbf{K}_I^1 \approx -\mathbf{K}_I^2 \approx -\mathbf{K}_I^3 \approx \mathbf{K}_I^4$ are met, we can select a specific channel, and the influence of the other two channels becomes negligible, due to the factor $\frac{1}{k_I^3}$. By excluding this possibility and narrowing our focus to one channel, the resulting outcome is

$$T_{var_\pm}(\mathbf{K}_I)\Big|_{\mu=0}^{Collapsed} = -\frac{\bar{A}_t}{2k_I^3}\left(\frac{H^4}{\prod_{i=1}^4 2K_I^i}\right)\frac{9K_I^1K_I^3}{4}, \tag{5.195}$$

and the corresponding trispectrum reads

$$\Re\langle\delta\phi(\mathbf{k}_1)\delta\phi(\mathbf{k}_2)\delta\phi(\mathbf{k}_3)\delta\phi(\mathbf{k}_4)\rangle = \left[K_I^1K_I^3\sin\left(\Theta_I^1\right)\sin\left(\Theta_I^3\right)\right]^2\cos\left(+2i\left(\phi_I^1-\phi_I^3\right)\right)\frac{\bar{A}_t}{2k_I^3}\left(\frac{H^4}{\prod_{i=1}^4 2K_I^i}\right)\frac{9K_I^1K_I^3}{4}, \tag{5.196}$$

$$\Im\langle\delta\phi(\mathbf{k}_1)\delta\phi(\mathbf{k}_2)\delta\phi(\mathbf{k}_3)\delta\phi(\mathbf{k}_4)\rangle = \pi\mu\left[K_I^1K_I^3\sin\left(\Theta_I^1\right)\sin\left(\Theta_I^3\right)\right]^2\sin\left(+2i\left(\phi_I^1-\phi_I^3\right)\right)\frac{\bar{A}_t}{2k_I^3}\left(\frac{H^4}{\prod_{i=1}^4 2K_I^i}\right)\frac{9K_I^1K_I^3}{4}. \tag{5.197}$$

Now recalling that $\delta\phi = -\frac{\dot{\phi}}{H}\zeta = -\sqrt{2\epsilon}M_{pl}\zeta$ we can switch to the trispectrum of the curvature perturbation and we get

$$\Re\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle = \frac{1}{\left(2\epsilon M_{pl}^2\right)^2}\left[K_I^1 K_I^3\right]^2\left[\sin\left(\Theta_I^1\right)\sin\left(\Theta_I^3\right)\right]^2\cos\left(+2i\left(\phi_I^1-\phi_I^3\right)\right)\frac{9\bar{A}_t}{2k_I^3}\left(\frac{H^4}{\prod_{i=1}^4 2K_I^i}\right), \tag{5.198}$$

$$\Im\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle = \frac{\pi\mu}{\left(2\epsilon M_{pl}^2\right)^2}\left[K_I^1 K_I^3\right]^3\left[\sin\left(\Theta_I^1\right)\sin\left(\Theta_I^3\right)\right]^2\sin\left(+2i\left(\phi_I^1-\phi_I^3\right)\right)\frac{9\bar{A}_t}{2k_I^3}\left(\frac{H^4}{\prod_{i=1}^4 2K_I^i}\right). \tag{5.199}$$

Now we can use the expression of the power spectrum of the curvature perturbation at 0^{th} order in slow-roll

$$P_\zeta(k) = \frac{1}{4\epsilon k^3}\left(\frac{H}{M_{pl}}\right)^2, \tag{5.200}$$

we can rewrite the real and complex part of the scalar trispectrum as

$$\Re\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle = \frac{9}{16}r\left[\sin\left(\Theta_I^1\right)\sin\left(\Theta_I^3\right)\right]^2\cos\left(+2i\left(\phi_I^1-\phi_I^3\right)\right)P_\zeta(k_I)P_\zeta(k_I^1)P_\zeta(k_I^3), \tag{5.201}$$

$$\Im\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle = \frac{9}{16}\Pi_{circ}r\left[\sin\left(\Theta_I^1\right)\sin\left(\Theta_I^3\right)\right]^2\sin\left(+2i\left(\phi_I^1-\phi_I^3\right)\right)P_\zeta(k_I)P_\zeta(k_I^1)P_\zeta(k_I^3), \tag{5.202}$$

where we have used that

$$\bar{A}_T = 4\left(\frac{H}{M_{pl}}\right)^2, \quad r = 16\epsilon. \tag{5.203}$$

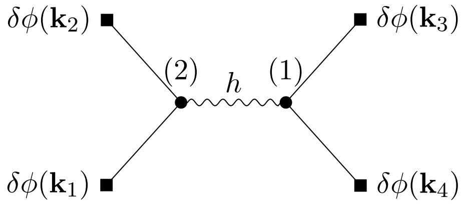
We conclude this section with some final remarks about the graviton-mediated trispectrum. The natural smallness of the trispectrum is due to the r suppression factor and it makes it unobservable given the current limit on the sensitivity [9]. The factor r originates from a consistency relation within bispectra [9], as it can be demonstrated that the trispectrum emerges as a product of two distinct bispectra in the collapsed limit. It is possible to incorporate enhancement factors through adjustments to the model, such as in quasi-single field models of inflation [9] but we are not going to discuss this point. Nevertheless, caution must be exercised to avoid compromising the bispectrum measurement, given the interconnections between the trispectrum and the bispectrum in the collapsed limit.

5.6.3 The scalar trispectrum with the Chern-Simons interaction vertex

In this section, we provide a brief explanation as to why the interference diagrams (refer to fig. 18) and the one involving two vertices from the CS term do not contribute to the scalar trispectrum. We can discuss both terms in the same way by repeating all the steps we have done previously:

- write down the four amplitudes coming from the vertices combinations,
- evaluate the polarization portion,
- use the simplifying assumption to treat the hypergeometric confluent functions,
- derive the time integral.

The two contributions are 0 since the time integral is 0. In the case of the interference term, which diagrammatically is



the time integral becomes

$$T_{var_h}(\mathbf{K}_I) = \Re\left(\mathcal{J}^{(2)}_h(\mathbf{K}_I) - \tilde{\mathcal{J}}^{(1)}_h(\mathbf{k}^1_I,\mathbf{k}^2_I)\mathcal{J}^{(1)*}_h(\mathbf{k}^3_I,\mathbf{k}^4_I)\right), \tag{5.204}$$

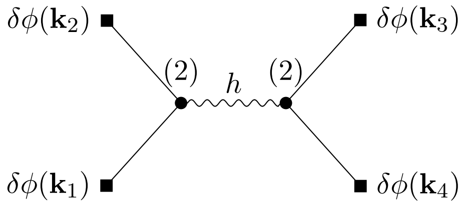
where

$$\mathcal{J}^{(2)}_h(\mathbf{K}_I) = \int \frac{d\tau_1 d\tau_2}{(H^2\tau_1\tau_2)^2}\left[u_+(\mathbf{k}^1_I,\tau_1)u_+(\mathbf{k}^2_I,\tau_1)\right]' u_+(\mathbf{k}^3_I,\tau_2)u_+(\mathbf{k}^4_I,\tau_4)u_h[\mathbf{k}_I,\max(\tau_1,\tau_2)]u^*_h[\mathbf{k}_I,\min(\tau_1,\tau_2)], \tag{5.205}$$

$$\mathcal{J}^{(1)}_h(\mathbf{k}^i_I,\mathbf{k}^j_I) = \int \frac{d\tau}{(H\tau)^2}u_+(\mathbf{k}^i_I,\tau)u_+(\mathbf{k}^j_I,\tau)u^*_h(\mathbf{k}^i_I+\mathbf{k}^j_I,\tau), \tag{5.206}$$

$$\tilde{\mathcal{J}}^{(1)}_h(\mathbf{k}^i_I,\mathbf{k}^j_I) = \int \frac{d\tau}{(H\tau)^2}\left[u_+(\mathbf{k}^i_I,\tau)u_+(\mathbf{k}^j_I,\tau)\right]' u^*_h(\mathbf{k}^i_I+\mathbf{k}^j_I,\tau). \tag{5.207}$$

With an implementation of the previous code, we can show that this integral is exactly 0. While regarding the diagram with two CS interaction vertices, which is



we can write the time integral as

$$T_{var_h}(\mathbf{K}_I) = \Re\left(\mathcal{J}^{(2)}_h(\mathbf{K}_I) - \tilde{\mathcal{J}}^{(1)}_h(\mathbf{k}^1_I,\mathbf{k}^2_I)\tilde{\mathcal{J}}^{(1)*}_h(\mathbf{k}^3_I,\mathbf{k}^4_I)\right), \tag{5.208}$$

which can be evaluated using this code and the result of the trispectrum and it approximately scales, as order of magnitude, as

$$\langle\delta\phi(\mathbf{k_1}\delta\phi(\mathbf{k_2}\delta\phi(\mathbf{k_3}\delta\phi(\mathbf{k_4}\rangle \approx \frac{\epsilon}{f^2M^2_{pl}}\left(\frac{H}{M_{pl}}\right)^2\frac{H^8}{k^9}, \tag{5.209}$$

where with k we are simply counting the power of ks which enter in the expression. Eq.(5.209) is largely suppressed with respect to the standard case which scales as

$$\langle\delta\phi(\mathbf{k_1}\delta\phi(\mathbf{k_2}\delta\phi(\mathbf{k_3}\delta\phi(\mathbf{k_4}\rangle \approx \left(\frac{H}{M_{pl}}\right)^2\frac{H^4}{k^9}. \tag{5.210}$$

The suppression factor is

$$\frac{\epsilon H^4}{f^2M^2_{pl}} = \frac{H^4M^2_{pl}}{M^2_{CS}8\epsilon H^2M^2_{pl}} \approx \frac{\epsilon H^2}{M^2_{CS}} << 1, \tag{5.211}$$

where we have use that $f = \frac{M_{CS}2\dot{\phi}}{M^2_{pl}} = \frac{M_{CS}2\sqrt{2\epsilon}H}{M_{pl}}$.

6 Chiral scalar-tensor theories of gravity

In this section, following the idea of an effective field theory approach to modify the standard Inflationary scenario, section 5.2, we introduce the so-called chiral scalar-tensor theories of gravity proposed in [19]. As the Chern-Simons theory of gravity falls short in explaining the signal detected in *LSS*, the idea being explored is whether these types of theories can amplify the signal through the birefringence effect they induce on left and right-handed gravitons. As we'll see this is not the case. Within this theory, we introduce covariant *parity-breaking* terms which have more derivatives both of the metric and of the Inflaton field, which, in this context, is not minimally coupled to gravity. However, these theories lead to equations of motion with higher-order derivatives and, Ostrogradsky shows that this may make the theory pathological [70] with the appearance of ghost's modes, for example. However, under suitable conditions, it's possible to show that this occurrence is not verified [19]. We are not going to discuss this point in depth but simply we list which hypotheses are needed.

The actions of the theories we are considering have the following form

$$S_{PV_1} = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R + \mathcal{L}_{PV_1} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{6.1}$$

$$S_{PV_2} = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R + \mathcal{L}_{PV_2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{6.2}$$

where, $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$ represent Lagrangians that incorporate parity-violating operators involving up to only first and second derivatives of the inflaton field, ϕ, respectively. The explicit expression of the $\mathcal{L}_{PV_1}$ is [19]

$$\mathcal{L}_{\rm PV1} = \sum_{A=1}^{4} a_A L_A\,, \tag{6.3}$$

where, with $M_{pl}=1$, we have

$$L_1 = \tilde{\epsilon}^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}R_{\mu\nu\;\;\lambda}^{\;\;\;\rho}\phi^\sigma\phi^\lambda\,, \qquad L_3 = \tilde{\epsilon}^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}R^{\sigma}{}_{\nu}\phi^\rho\phi_\mu\,,$$
$$L_2 = \tilde{\epsilon}^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}R_{\mu\lambda}^{\;\;\;\rho\sigma}\phi_\nu\phi^\lambda\,, \qquad L_4 = \tilde{\epsilon}^{\mu\nu\rho\sigma}R_{\rho\sigma\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}\phi^\lambda\phi_\lambda\,, \tag{6.4}$$

where $\tilde{\epsilon}_{\rho\sigma\alpha\beta}$ is the Levi-Civita tensor (see section 5), and $\phi^\mu = \nabla^\mu\phi$. The couplings a_A in (6.3) are adimensional functions of the scalar field and its kinetic term, i.e. $a_A = a_A(\phi,\phi^\mu\phi_\mu)$. In order to avoid the Ostrogradsky modes, we have to work in the unitary gauge 2.3 and we have to set

$$4a_1 + 2a_2 + a_3 + 8a_4 = 0\,, \tag{6.5}$$

from which we understand that we have only three independent coefficients.

Now, switching to the second Lagrangian $\mathcal{L}_{\rm PV2}$ we have [19]

$$\mathcal{L}_{\rm PV2} = \sum_{A=1}^{7} b_A M_A\,, \tag{6.6}$$

where

$$M_1 = \varepsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}\phi^\rho\phi_\mu\phi^\sigma_\nu\,, \qquad M_4 = \varepsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}\phi_\nu\phi^\rho{}_\mu\phi^\sigma{}_\lambda\phi^\lambda,$$
$$M_2 = \varepsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}\phi^\rho{}_\mu\phi^\sigma{}_\nu\,, \qquad M_5 = \varepsilon^{\mu\nu\alpha\beta}R_{\alpha\rho\sigma\lambda}\phi^\rho\phi_\beta\phi^\sigma{}_\mu\phi^\lambda{}_\nu,$$
$$M_3 = \varepsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}\phi^\sigma\phi^\rho{}_\mu\phi^\lambda{}_\nu\phi_\lambda\,, \qquad M_6 = \varepsilon^{\mu\nu\alpha\beta}R_{\beta\gamma}\phi_\alpha\phi^\gamma{}_\mu\phi^\lambda{}_\nu\phi^\lambda,$$
$$M_7 = (\Box\phi)M_1\,, \tag{6.7}$$

where $\phi^\sigma{}_\nu = \nabla^\sigma\nabla_\nu\phi$ and $b_A = b_A(\phi,\phi^\mu\phi_\mu)$. Also in this case, to avoid the Ostrogradsky pathological modes, we have to work in the unitary gauge with the following four conditions [19]

$$b_7 = 0\,, \qquad b_6 = 2(b_4+b_5)\,, \qquad b_2 = -A_*^2(b_3-b_4)/2\,, \tag{6.8}$$

where $A_* = \dot{\phi}(t)/N$ and N is the lapse function of the spacetime.

6.1 The actions in the ADM framework

We work in the ADM formalism of the metric, section C.3, and in the unitary gauge, section 2.3, since we have to avoid pathological modes. In Cartesian coordinates and conformal time, we write

$$g_{\mu\nu} = a^2(\tau)\begin{pmatrix} -(N^2 - N_i N^i) & N_i \\ N_i & g_{ij} \end{pmatrix}, \tag{6.9}$$

where $a^2(\tau)$ is the scale factor, N is the lapse function, and, $N_i = \partial_i \psi + E_i$ the shift function and, the spatial metric is

$$g_{ij} = a^2\left(e^{2\zeta}\delta_{ij} + \exp h_{ij}\right) = a^2\left(e^{2\zeta}\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h^k{}_j + ..\right), \quad h^i{}_i = 0, \quad \partial_i h^i{}_j = 0. \tag{6.10}$$

According to the Hamiltonian analysis performed in [19] we can write the parity-violating actions in the following way

$$\begin{aligned}\sqrt{-g}\,\mathcal{L}_{PV_1} = {} & \frac{4\epsilon}{N}\left(\frac{H}{M_{pl}}\right)^2 \epsilon^{ijl}\Big[2(2a_1 + a_2 + 4a_4)\left(KK_{mi}D_lK_j^m + {}^{(3)}R_{mi}D_lK_j^m - K_{mi}K^{mn}D_lK_{jn}\right) \\ & - (a_2 + 4a_4)\left(2K_{mi}K_j^n D_n K_l^m + {}^{(3)}R_{jlm}{}^n D_n K_i^m\right)\Big], \end{aligned} \tag{6.11}$$

$$\begin{aligned}\sqrt{-g}\mathcal{L}_{PV_2} = {} & \frac{\dot\phi^3}{N^4}\epsilon^{ijl}\Big[2N\bigg(\frac{b_1}{M_{pl}^5}NK_{mi}D_lK^m{}_j + \frac{(b_4 + b_5 - b_3)}{M_{pl}^6}K_{mi}K^n{}_jD_nK^m{}_l\bigg) \\ & + \dot\phi\bigg(\frac{b_3}{M_{pl}^8}{}^{(3)}R_{jlm}{}^nK^m{}_iD_nN - \frac{2(b_4 + b_5)}{M_{pl}^8}{}^{(3)}R_{ml}K^m{}_jD_iN\bigg)\Big], \end{aligned} \tag{6.12}$$

where we have reintroduced the Planck mass.

We conclude this section by discussing equations of motion of the lapse and shift functions. We can apply exactly the same discussion of the Chern-Simons case, section 5.3. Thus, since we do not need to expand the action to fourth order in scalar and tensor perturbations, we only need first-order solutions for the constraint equations. We can set the solution of the constraint equations of motion as in standard gravity

$$N = 1 + \frac{\dot\zeta}{H}, \quad N^{i\perp} = 0, \quad \psi = -a^{-2}\frac{\zeta}{H} + \chi, \quad \nabla^2\chi = \frac{\dot\phi^2}{2H^2M_{pl}^2}\dot\zeta = \epsilon\dot\zeta \tag{6.13}$$

6.2 The EoM and the Power spectrum

In this section, we investigate the chirality introduced by these theories in the primordial power spectrum of tensor modes. Here, "chirality" refers to the distinction between left-handed and right-handed gravitons. We won't provide a detailed derivation of the results but will provide brief insights into the key aspects. The primary references for this subsection are [8] and [71].

Before going into the details of the tensor computations I want to stress that the *purely scalar* part does not receive any contribution at quadratic order for both Lagrangians. Thus, since we are working in the unitary gauge we can copy what we have obtained in section 2.4. Thus, we have at quadratic order the following Lagrangian

$$S_{(2)}^{\zeta\zeta} = \int d\tau d^3x a^2 \epsilon M_{pl}^2 h_{ij}\Big\{\zeta'^2 - \partial^i\zeta\partial^j\zeta\Big\}, \tag{6.14}$$

from which we can derive the solution to EoM in Fourier space with Bunch-Davies initial condition and at lowest order in slow-roll parameter as

$$u_\zeta(\tau, \mathbf{x}) = -\frac{1}{\sqrt{2\epsilon k}aM_{pl}}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau} = \frac{iH}{\sqrt{2\epsilon k^3}M_{pl}}\left(ik\tau + 1\right)e^{-ik\tau}, \tag{6.15}$$

where we have used $a(\tau) \approx -\frac{1}{H\tau}$ and we also have to recall that in order to recover the canonical commutation relation we have also to impose the condition on the Wronskian, eq.(2.172), of the normalized field, which in this case is $aM_{pl}\sqrt{2\epsilon}\zeta$. We rewrite the mode functions in the form of eq.(6.15) since we are going to compute the graviton-mediated trispectrum in section 6.4 at lowest order in slow-roll parameters[26]. The full solution is the one presented in 2.4 from which we recover the power spectrum for scalar perturbations.

The initial step for calculating the power spectrum is to derive the equations of motion, which we accomplish in Fourier space. Therefore, we express the tensor modes in Fourier space as follows:

$$h_{ij}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=L,R} h_s(\mathbf{k}, t)\epsilon_{ij}^{(s)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{6.16}$$

where $\epsilon_{ij}^{(s)}(\mathbf{k})$ are the polarization tensor in the $\{R, L\}$ basis, appendix D.1, $h_s(\mathbf{k}, t)$ are the so-called mode function of tensors. The action derived by Lagrangian $\mathcal{L}_{PV_1}$ (6.11) at leading order in slow-roll and at quadratic order in h_{ij} is given by [8]

$$S_{hh}^{PV_1} = \sum_{s=L,R} \int d\tau \int \frac{d^3k}{(2\pi)^3} \left[A_{T,s}^2 |h_s'(\mathbf{k}, \tau)|^2 - B_{T,s}^2 k^2 |h_s(\mathbf{k}, \tau)|^2\right]\,, \tag{6.17}$$

where $h_s{}' = \frac{d}{d\tau}h_s$, and we have introduced

$$A_{T,s}^2 \equiv \frac{M_{Pl}^2}{2}a^2\left(1 - \lambda_s\frac{k_{phys}}{M_{\mathrm{PV1}}}\right), \quad B_{T,s}^2 \equiv \frac{M_{Pl}^2}{2}a^2\left[1 - \frac{4}{M_{Pl}^6}\frac{\dot{\phi}^2}{a}(\dot{f} + \dot{g})\lambda_s k\right]\,, \tag{6.18}$$

with $\lambda_s = \pm 1$ for right and left-handed gravitons respectively, and with

$$M_{PV_1} \equiv \frac{M_{Pl}^6}{8}\frac{1}{\dot{\phi}^2}\frac{1}{(f + g)H}\,, \tag{6.19}$$

and

$$f \equiv a_1 + \frac{a_2}{2} + 2a_4\,, \qquad g \equiv \frac{a_2}{2} + 2a_4\,. \tag{6.20}$$

Now, everything works in a very similar way to what we have done in the Chern-Simons of modified gravity. First of all, we notice that the right-handed graviton modes, i.e. $\lambda_R = +1$, can acquire a negative kinetic term if $k_{phys} > M_{PV_1}$, where $k_{phys} \equiv \frac{k}{a}$ with k co-moving wave number. In such cases, these modes become unstable, potentially leading to critical issues within the theory, such as a breach of unitarity or the propagation of negative energy modes forward in time[8]. To avoid such circumstances, we introduce a cutoff scale denoted as $\Lambda \leq M_{PV_1}$. At the beginning of Inflation, we restrict our consideration to gravitons for which $k_{phys} < \Lambda$. Moreover, at the beginning of inflation, we need also gravitons with $k_{phys} >> H$. This follows from the fact that we impose Bunch-Davies initial conditions. Thus, by putting everything together we have that

$$\chi_1 \equiv \frac{H}{M_{PV_1}} << 1, \tag{6.21}$$

where we have introduced the *chirality parameter* χ_1 in analogy to what we have done in section 5.4. Now, we observe that this condition is verified throughout the inflationary epoch since the scale factor exponentially increases. For example at the end of inflation, we have

$$\frac{k_{phys}}{a_E} = \frac{k_{phys}}{a_B}\frac{a_B}{a_E} = \frac{k_{phys}}{a_B}e^{-60}, \tag{6.22}$$

where a_E and a_B are respectively the scale factor at the end and beginning of inflation.

[26]We seek the lowest order in slow-roll parameters since we expect that the parity-violating trispectrum to be proportional to the chirality parameters which are much lower that one.

If we now introduce a new quantity, the graviton speed, as

$$c_{T,s}^2 \equiv \frac{B_{T,s}^2}{A_{T,s}^2}\,, \tag{6.23}$$

the action (6.17) becomes

$$S_{\gamma\gamma}^{\rm PV1} = \sum_{s=L,R} \int d\tau \int \frac{d^3k}{(2\pi)^3} A_{T,s}^2 \left[|\gamma_s'(\mathbf{k},\tau)|^2 - c_{T,s}^2 k^2 |\gamma_s(\mathbf{k},\tau)|^2\right]. \tag{6.24}$$

In order to canonically quantize the field we have to write the action in the canonical form. Thus, we make the following field redefinition

$$\mu_s \equiv A_{T,s} h_s\,. \tag{6.25}$$

This allows us to rewrite the action as

$$S_{\gamma\gamma}^{\rm PV1} = \sum_{s=L,R} \int d\tau \int \frac{d^3k}{(2\pi)^3} \left[|\mu_s'(\mathbf{k},\tau)|^2 - c_{T,s}^2 k^2 |\mu_s(\mathbf{k},\tau)|^2 + \frac{A_{T,s}''}{A_{T,s}} |\mu_s(\mathbf{k},\tau)|^2\right]. \tag{6.26}$$

Now, applying the variational principle in Fourier space we get the following equations of motion for μ_s

$$\mu_s'' + \left(c_{T,s}^2 k^2 - \frac{A_{T,s}''}{A_{T,s}}\right)\mu_s = 0\,, \tag{6.27}$$

where the we can rewrite the last term using

$$\frac{A_{T,s}''}{A_{T,s}} = \frac{d}{d\tau}\left(\frac{A_{T,s}'}{A_{T,s}}\right) + \left(\frac{A_{T,s}'}{A_{T,s}}\right)^2 = \frac{2+3\epsilon}{\tau^2} - \frac{\lambda_s k}{\tau}\chi_1 + \mathcal{O}\left(\epsilon^2, \chi_1^2, \epsilon\chi_1\right)\,, \tag{6.28}$$

where ϵ is the first slow-roll parameter, section 2.2. Thus, the *EoM* at lowest order in χ_1 and slow roll parameters are [7]

$$\mu_s'' + \left(c_{T,s}^2 k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau}\chi_1\right)\mu_s = 0\,, \quad \nu_T \simeq \frac{3}{2} + \epsilon\,, \quad c_{T,s}^2 \simeq 1 - \lambda_s k\tau\chi_1. \tag{6.29}$$

We promptly observe that in this theory, a birefringence effect occurs, implying that the two chiral polarizations propagate at distinct velocities that deviate from the speed of light. This marks the primary difference between chiral-scalar tensor theories of gravity and the Chern-Simons one. As we'll see this particular feature does not influence the calculation of the graviton-mediated trispectrum. No enhancement is observed compared to the Chern-Simons theory.

It's important to highlight that when setting $f + g = 0$, we fall back to the standard scenario. Now we can rewrite the *EoM* for the mode function as

$$\mu_s'' + \left[k^2\left(1 - \lambda_s k \frac{H}{M_{PV_1}}\tau\right) - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s \frac{k}{\tau}\frac{H}{M_{PV_1}}\right]\mu_s = 0\,. \tag{6.30}$$

It's possible to find an approximate analytical solution [8, 71] in terms of Airy functions [55]

$$\mu_s(y) = \alpha\left(\frac{\xi(y)}{g(y)}\right)^{1/4} \mathrm{Ai}(\xi) + \beta\left(\frac{\xi(y)}{g(y)}\right)^{1/4} \mathrm{Bi}(\xi)\,, \tag{6.31}$$

where $y = -k\tau$, α and β are constants to be determined, and

$$g(y) = \frac{\nu_T^2}{y^2} - 1 - \lambda_s y \frac{H}{M_{PV_1}} + \lambda_s \frac{H}{M_{PV_1}}\frac{1}{y}\,, \tag{6.32}$$

$$\xi(y) = \begin{cases} \left(-\frac{3}{2}\int_{y_0^s}^{y} \sqrt{g(y')}\,dy'\right)^{2/3} & y \le y_0^s\,, \\ -\left(\frac{3}{2}\int_{y_0^s}^{y} \sqrt{g(y')}\,dy'\right)^{2/3} & y \ge y_0^s\,, \end{cases} \tag{6.33}$$

with

$$y_0^s = -\frac{1 - 2^{1/3}\left[1 + 3\left(\frac{H}{M_{PV_1}}\right)^2\right]/Y - 2^{-1/3}Y}{3\lambda_s \frac{H}{M_{PV_1}}}\,, \tag{6.34}$$

where

$$Y = \left(Y_1 + \sqrt{-4\left[1 + 3\left(\frac{H}{M_{PV_1}}\right)^2\right]^3 + Y_1^2}\right)^{1/3}\,, \quad Y_1 = -2 + 27\nu_T^2\left(\frac{H}{M_{PV_1}}\right)^2 - 9\left(\frac{H}{M_{\rm PV1}}\right)^2\,. \tag{6.35}$$

By assuming that the Universe was initially in an adiabatic vacuum state we can set the value of the integration constants [71]

$$\alpha = \sqrt{\frac{\pi}{2k}}e^{i\pi/4}\,, \qquad \beta = i\sqrt{\frac{\pi}{2k}}e^{i\pi/4}\,. \tag{6.36}$$

Before passing to the computation of the power spectrum we stress that the solution is a sort of “expansion" in slow-roll parameters and χ_1 of the de-Sitter mode functions [8].

Now, we can compute the super-horizon power spectra for left and right gravitons in the same way we have done previously so we do not repeat the computation. Finally, we get [71]

$$P_T^L = 2\frac{|u_L(y)_{y\ll 1}|^2}{A_{T,L}^2}\,, \qquad P_T^R = 2\frac{|u_R(y)_{y\ll 1}|^2}{A_{T,R}^2}\,, \tag{6.37}$$

where an expansion in $y = 0$ of the solution to the *EoM* is employed [71]. Retaining only the leading order in slow-roll parameters, we finally obtain

$$P_T^L = \frac{P_T}{2}\exp\left[\frac{\pi}{16}\frac{H}{M_{PV_1}}\right]\,, \qquad P_T^R = \frac{P_T}{2}\exp\left[-\frac{\pi}{16}\frac{H}{M_{PV_1}}\right]\,, \tag{6.38}$$

where here P_T is the *GR*'s power spectrum eq.(2.206). We notice that, in order to find the solution we have used

$$A_{T,s}^{-2} = \frac{2}{M_{pl}^2 a^2}\frac{1}{\left(1 - \lambda_s \frac{k_{phys}}{M_{PV_1}}\right)} \approx \frac{2}{M_{pl}^2 a^2}\left(1 + \lambda_s \frac{k_{phys}}{M_{PV_1}}\right) \approx \frac{2}{M_{pl}^2 a^2}, \tag{6.39}$$

where in the last step we have used the idea of eq.(6.22), i.e. we have completely disregarded $\frac{k_{phys}}{M_{PV_1}}$ since at the end of inflation we have

$$\left.\frac{k_{phys}}{M_{PV_1}}\right|_E = \left.\frac{k_{phys}}{M_{PV_1}}\right|_B \frac{a_B}{a_E} = \left.\frac{k_{phys}}{M_{PV_1}}\right|_B e^{-N}, \tag{6.40}$$

where the subscripts B and E stand for the beginning and the end (of the inflationary period) while N is the number of e-folds. Thus, it's clear that this term is completely negligible at the end of Inflation when we compute the power spectrum on super horizon scales.

Now, we can appreciate how the chirality of this kind of theory impacts on the power spectrum. First of all, we have a modification at the linear level of the two tensor power spectra

$$P_T^L \approx \frac{P_T}{2}\exp\left(1 + \frac{\pi}{16}\frac{H}{M_{PV_1}}\right)\,, \qquad P_T^R \approx \frac{P_T}{2}\left(1 - \frac{\pi}{16}\frac{H}{M_{PV_1}}\right) \tag{6.41}$$

This effect is encoded in the so-called chirality parameter χ which is defined as

$$\chi \equiv \frac{P_T^R - P_T^L}{P_T^R + P_T^L} =\approx -\frac{\pi}{16}\frac{H}{M_{PV_1}}\,. \tag{6.42}$$

where in the last step we have stopped at first order. The procedure is exactly identical to what we have done in the case of the Chern-Simons theory. As we have previously done another interesting quantity to compute is r_{PV_1}, i.e. the tensor-to-scalar ratio in the chiral-scalar tensor theories of gravity. The first modification to the total tensor power spectrum is at second order in χ_1

$$P_T^{PV_1} = P_T^R + P_T^L = P_T \left[1 + \frac{\pi^2}{256}\left(\frac{H}{M_{\rm PV1}}\right)^2\right] = P_T\left(1+\chi^2\right) , \tag{6.43}$$

from which we can immediately compute r_{PV_1} since the scalar power spectrum get no contributions from the chiral-scalar tensor theories

$$r_{PV_1} \equiv \frac{\Delta_T^{PV_1}}{\Delta_S} = r\left(1+\chi^2\right) , \tag{6.44}$$

where r is the tensor-to-scalar ratio eq.(2.212), in the standard GR slow-roll single-field model of Inflation. We immediately understand this correction is unobservably small since the chirality χ is $\ll 1$.

The chiral-scalar tensor theories of gravity also modify the spectral index of tensor perturbation. We apply the same line of reasoning presented in [15] and we derive the index at horizon crossing

$$k = a(t_*)H(t_*), \tag{6.45}$$

where k is the comoving wave vector and t_* is horizon crossing time. In what follows we suppress the star for notational convenience. As shown in [15] the tensor modes become constant after horizon crossing and get frozen on superhorizon scales. Thus, we can compute the spectral index in the following way

$$n_T \equiv \frac{d\ln\Delta_T^{PV_1}}{d\ln k} \simeq -2\epsilon + \frac{d\ln\chi^2}{d\ln k} = -2\epsilon + 2\frac{d\ln\left(1+\chi^2\right)}{dt}\frac{dt}{d\ln k} \approx -2\epsilon + 2\chi\dot{\chi}\frac{1+\epsilon}{H} \tag{6.46}$$

$$= -2\epsilon + \frac{\pi^2}{128}\frac{H}{M_{PV_1}}\left[-\epsilon\frac{H^2}{M_{PV_1}} - \frac{H\dot{M}_{PV_1}}{M_{PV_1}^2}\right]\frac{1+\epsilon}{H} \tag{6.47}$$

$$\simeq -2\epsilon + \frac{\pi^2}{128}\left(\frac{H}{M_{PV_1}}\right)\left[-\epsilon\left(\frac{H}{M_{PV_1}}\right) - \frac{\dot{M}_{PV_1}}{M_{PV_1}^2}\right] , \tag{6.48}$$

where we have used that the spectral index of standard GR is -2ϵ, eq.(2.204), we have used a change of variable, we have expanded the logarithm at linear order and we have used that

$$\frac{d\ln k}{dt} = \frac{d}{dt}\ln\left(a(t)H(t)\right) = \frac{1}{aH}\left(\dot{a}H + a\dot{H}\right) = H(1-\epsilon). \tag{6.49}$$

Therefore, even in this scenario, the deviations from the standard spectral index ($n_T = -2$) are typically small, making the detection of primordial gravitational waves ($PGWs$) predicted by this model well beyond the capabilities of gravitational wave interferometers.

Now, we switch to the second Lagrangian, denoted as $\mathcal{L}_{PV_2}$, which, when expanded to second order in tensor perturbations, is described as [8]

$$S_{PV_2}^{\gamma\gamma} = \sum_{s=L,R}\int d\tau \int \frac{d^3k}{(2\pi)^3}\left[\tilde{A}_{T,s}^2|h_s'(\mathbf{k},\tau)|^2 - \frac{M_{Pl}^2}{2}a^2k^2|h_s(\mathbf{k},\tau)|^2\right] , \tag{6.50}$$

where we have introduced

$$\tilde{A}_{T,s}^2 \equiv \frac{M_{Pl}^2}{2}a^2\left(1 - \lambda_s\frac{k_{phys}}{M_{PV_2}}\right) , \quad M_{PV_2} \equiv \frac{M_{Pl}}{2}\left(\tilde{b}_1 - b\frac{H}{M_{Pl}}\right)^{-1} , \tag{6.51}$$

where the two couplings $\tilde{b}_1$ and b are defined as function of the indipedent parameter

$$\tilde{b}_1 \equiv \frac{\dot{\phi}^3}{M_{Pl}^6}b_1 , \quad b \equiv \frac{\dot{\phi}^4}{M_{Pl}^8}(b_4 + b_5 - b_3) . \tag{6.52}$$

Now everything is identical to the previous case once we have introduced a second chirality parameter

$$\chi_2 \equiv \frac{H}{M_{PV_2}} << 1, \tag{6.53}$$

where we have taken $\chi_2 << 1$ in order to avoid ghosts in the model. Then, one can obtain the same results we have derived in the previous case simply by making the following substitution

$$\begin{aligned} M_{PV_1} &\longrightarrow M_{PV_2}, \\ \chi_1 &\longrightarrow \chi_2. \end{aligned} \tag{6.54}$$

6.3 The scalar-scalar-tensor vertices

Now, we turn our attention toward the calculation of the leading contribution to the trispectrum, which can give rise to parity-violating signals. The idea is quite similar to what was described in the Chern-Simons theory, chapter 5. Thus, we are not going to repeat all the conceptual steps. We seek diagrams of the form

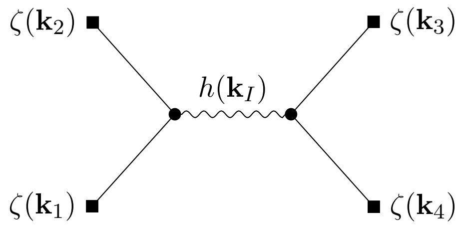


Figure 23: The graviton-mediated trispectrum

Thus, as a first step, we have to write down all the possible vertex of the type scalar-scalar-tensor. The first contribution arises from standard GR and on superhorizon scales it has the form [15]

$$S_{int} = \int d\tau d^3x \epsilon M_{pl}^2 a^2(\tau) h^{ij} \partial_i \zeta \partial_j \zeta, \tag{6.55}$$

where we have switched to conformal time and we have reintroduced the Planck mass with respect to what is presented in [15].

Next, we need to calculate all the potential scalar-scalar-tensor vertices that stem from the parity-violating terms in the Lagrangians $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$. Before going into the actual computations, presented in section 6.3.1 and 6.3.2, we introduce some basic quantities that we are going to use throughout the actual computations.

First of all, we recall that the metric tensor in cartesian coordinates and cosmic time can be written as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^2 e^{2\zeta}(\delta_{ij} + h_{ij}) \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^{-2} e^{-2\zeta}(\delta^{ij} - h^{ij}) \end{pmatrix}, \tag{6.56}$$

where we do not consider second-order tensor perturbations since we are seeking scalar-scalar-tensor vertices. Moreover, we do not expand directly the scalar part since retaining the exponential simplifies the computations.

Then, we can compute the three-dimensional Christoffel symbols up to second-order as

$$\Gamma^i_{jk} = \left[\left(\delta^i{}_j \zeta_{,k} + \delta^i{}_k \zeta_{,j} - \delta_{kj} \zeta^{,i} \right) + \frac{1}{2} \left(h^i{}_{j,k} + h^i{}_{k,j} - h_{jk}{}^{,i} \right) - \left(h^i{}_j \zeta_{,k} + h^i{}_k \zeta_{,j} - h^{il} \delta_{jk} \zeta_{,l} \right) \right]. \tag{6.57}$$

As apparent from eq.(6.11) and eq.(6.12) we also need the expression of the extrinsic curvature tensor, the three-dimensional Reimann tensor, and three-dimensional Ricci tensor Then, we compute the fully covariant expression of the extrinsic curvature

$$K_{ij} = \frac{1}{2}\left[\dot{g}_{ij} - D_i N_j - D_j N_i\right] \tag{6.58}$$

$$= a^2 \left\{ e^{2\zeta} \left[\left(H + \dot{\zeta}\right)(\delta_{ij} + h_{ij}) + \frac{\dot{h}_{ij}}{2} \right] - \psi_{,ij} - \psi_{,[i}\zeta_{,j]} + \delta_{ij}\psi_{,l}\zeta^{,l} - \frac{\psi_{,l}}{2}\left[h^l{}_{[i,j]} + - h_{ij}{}^{,l} \right] \right\}, \tag{6.59}$$

where $\psi_{,[i}\zeta_{,j]} = \psi_{,i}\zeta_{,j} + \psi_{,j}\zeta_{,i}$, which is the symmetrization operation. Moreover, we have used that at first order we have

$$N_i = {}^{(3)}g_{ij}N^j = a^2\delta_{ij}N^j. \tag{6.60}$$

Incorporating the accurate factor is essential from a dimensional perspective. We'll discuss about this point in section 6.3.2.

If we compute the $(1,1)$ version of the extrinsic curvature tensor we immediately get

$$K^m{}_j = \left(H + \dot{\zeta}\right)\delta^m{}_j + \frac{\dot{h}^m{}_j}{2} - e^{-2\zeta}\left(\psi^{,m}{}_{,j} - \psi_{,ij}h^{mi}\right) + \left[-\psi_{,[j}\zeta^{,m]} + \psi_{,l}\zeta^{,l}\delta^m{}_j - \frac{\psi_{,l}}{2}\left(h^{l[m}{}_{,j]} - h^m{}_j{}^{,l} \right) \right]. \tag{6.61}$$

The trace of the extrinsic curvature up to second order is

$$K = 3\left(H + \dot{\zeta}\right) - \left(\psi^{,j}{}_{,j} - \psi_{,fj}h^{fj}\right)e^{-2\zeta} + \psi^{,j}\zeta_{,j}. \tag{6.62}$$

We also need the three-dimensional covariant derivative of eq.(6.61) in evaluating eq.(6.11) and (6.12) but because of symmetries we get zero contribution from terms proportional to the Kronecker delta. Thus, we write the covariant derivative of $k^m{}_i$ putting to zero $\left(H + \dot{\zeta}\right)\delta^m{}_j$ and we obtain

$$D_l K^m{}_j = \frac{\dot{h}^m{}_{j,l}}{2} + \left(2\zeta_{,l}\psi^{,m}{}_{,j} + \psi_{,fj}h^{fm}{}_{,l}\right) + \left[-\psi_{,j}\zeta^{,m}{}_{,l} + \frac{\psi_{,fl}}{2}\left(h^{f[m}{}_{,j]} - h^m{}_j{}^{,f} \right) \right.$$
$$\left. + \frac{\psi_{,f}}{2}\left(h^{f[m}{}_{,j]l} - h^m{}_{j,l}{}^{,f} \right) \right] - \frac{\dot{h}^m{}_l\zeta_{,j}}{2} - \frac{\psi^{,f}{}_{,j}}{2}\left(h^m{}_{[l,f]} - h_{fl}{}^{,m} \right). \tag{6.63}$$

The first order expression for the Riemann tensor and Ricci tensor are

$$R_{ijk}{}^m = \partial_k \left[\delta^m{}_{\{i}\zeta_{,j\}} + \frac{1}{2}h^m{}_{\{i,j\}} \right] - \partial^m \left[\delta_{k\{}\zeta_{,j\}} + \frac{1}{2}h_{k\{i,j\}} \right], \tag{6.64}$$

$$R_{ij} = -\zeta_{,ij} - \delta_{ij}\nabla^2\zeta - \frac{1}{2}\nabla^2 h_{ij}, \tag{6.65}$$

where $\{i,j\}$ is the antisymmetrization of the indices. As we'll see we need the expression of the curvature tensors up to second order but only in scalar quantities. Thus, we obtain

$$R_{jlm}{}^n = \delta^n{}_{\{j}\zeta_{,l\}m} + \zeta_{,m}\delta^n{}_{[l}\zeta_{,j]} + \delta^n{}_l\zeta_{,j}\zeta_{,m} - \delta^n{}_j\zeta_{,m}\zeta_{,l} - \zeta_{,m}\delta^n{}_{[l}\zeta_{,j]}, \tag{6.66}$$

$$R_{mi} = \zeta_{,m}\zeta_{,i} + \delta_{mi}\zeta_{,f}\zeta^{,f}. \tag{6.67}$$

6.3.1 The expansion of $\mathcal{L}_{PV_2}$

We start seeking the scalar-scalar-tensor vertices in the $\mathcal{L}_{PV_2}$ Lagrangian, which we can write as [19]

$$\mathcal{L} = \frac{\dot{\phi}^3}{N^4}\epsilon^{ijl}\left\{ 2N\left[\frac{b_1}{M_{pl}^5}NK_{mi}D_lK^m{}_j + \frac{(b_4 + b_5 - b_3)}{M_{pl}^6}K_{mi}K^n{}_jD_nK^m{}_l \right] \right. \tag{6.68}$$

$$\left. + \dot{\phi}\left[\frac{b_3}{M_{pl}^8}{}^{(3)}R_{jlm}{}^nK^m{}_iD_nN - \frac{2(b_4 + b_5)}{M_{pl}^8}{}^{(3)}R_{ml}K^m{}_jD_iN \right] \right\} \tag{6.69}$$

We start by expanding the third and fourth terms in the Lagrangian

$$\text{third} = \frac{\dot{\phi}^4}{M_{pl}^8} b_3 \left[\epsilon^{ijl\,(3)}R_{jlm}{}^{n}K^{m}{}_{i}D_nN\right] = \frac{\dot{\phi}^4 b_3}{HM_{pl}^8}\left[\epsilon^{ijl\,(3)}R_{jlm}{}^{n}K^{m}{}_{i}\partial_n\dot{\zeta}\right], \tag{6.70}$$

$$\text{fourth} = -\frac{\dot{\phi}^4}{M_{pl}^8} 2(b_4+b_5)\left[{}^{(3)}R_{ml}K^{m}{}_{j}D_iN\right] = -\frac{\dot{\phi}^4 2(b_4+b_5)}{HM_{pl}^8}\left[{}^{(3)}R_{ml}K^{m}{}_{j}\partial_i\dot{\zeta}\right], \tag{6.71}$$

where we have used the expression for N, i.e. eq.(6.13). Evidently, in order to perform the computations, we require the first-order expressions of the Riemann tensor eq.(6.64), the Ricci tensor eq.(6.65), and the extrinsic curvature tensor, which reads

$$K^{m}{}_{j} = \frac{\dot{h}^{m}{}_{j}}{2} - \psi^{,m}{}_{,j}, \tag{6.72}$$

where in the extrinsic curvature tensor we have set to zero the Kronecker delta. This necessity arises from the fact that the zeroth-order contributions within the extrinsic curvature are proportional to the identity, and as we have explained above they do not contribute. Thus, the third term reads

$$\text{third} = \frac{\dot{\phi}^4 b_3}{2HM_{pl}^8}\epsilon^{ijl}\left\{\dot{\zeta}_{,j}\zeta_{,lm}\dot{h}^{m}{}_{i} - \zeta_{,jm}\dot{\zeta}_{,l}\dot{h}^{m}{}_{i} + \left[h^{n}{}_{\{l,j\}m} + h_{m\{,l\}}{}^{,n}\right]\left(-\frac{\zeta^{,m}{}_{,i}}{Ha^2} + \epsilon\nabla^{-2}\dot{\zeta}^{,m}{}_{,i}\right)\dot{\zeta}_{,n}\right\}. \tag{6.73}$$

Because the scale factor's powers have an impact on the outcomes, it is worthwhile to invest time to verify that no errors have been made. This can be made by a dimensional analysis. Indeed, we have the flexibility to decide whether the comoving coordinates (x, y, z) possess dimensions or whether we attribute dimensions to the scale factor since we are working with a spatially flat RW metric, i.e.

$$ds^2 = -dt^2 + a^2(t)\left(dx^2+dy^2+dz^2\right) = a^2(\tau)\left(-d\tau^2+dx^2+dy^2+dz^2\right). \tag{6.74}$$

If we work in cosmic time it's clear that t has the dimension of a length while switching to conformal time we have the option to decide whether to assign dimensions to τ or $a(\tau)$. Working with $a(\tau)$ with the dimension of a length we can verify if we have the right number of its powers in eq.(6.73). We want that the action, i.e.

$$S_{PV_2}^{\text{third term}} = 2\epsilon^2\frac{H^3}{M_{pl}^4}\int dtd^3x b_3\epsilon^{ijl}\left\{\dot{\zeta}_{,j}\zeta_{,lm}\dot{h}^{m}{}_{i} - \zeta_{,jm}\dot{\zeta}_{,l}\dot{h}^{m}{}_{i} + \left[h^{n}{}_{\{l,j\}m} + h_{m\{,l\}}{}^{,n}\right]\left(-\frac{\zeta^{,m}{}_{,i}}{Ha^2} + \epsilon\nabla^{-2}\dot{\zeta}^{,m}{}_{,i}\right)\dot{\zeta}_{,n}\right\}. \tag{6.75}$$

posses no dimension. Let's take the first term for example in eq.(6.75) which in conformal time becomes

$$S = 2\epsilon^2\frac{H^3}{M_{pl}^4}\int d\tau d^3x b_3 a^{-1}(\tau)\epsilon^{ijl}\zeta'{}_{,j}\zeta_{,lm}h'^{m}{}_{i}. \tag{6.76}$$

Now, the line element $d\tau d^3x$, ζ, h_{ij}, and their spatial and temporal derivatives have no dimension in these units. Moreover, $H = \frac{\dot{a}}{a}$ clearly has mass dimension one since t has the dimension of a length. Consequently, when considering the integrand in eq.(6.76), which includes $\frac{1}{a(\tau)}$, it becomes apparent that it possesses a mass dimension of one. Consequently, the total action exhibits a dimension of zero. Thus, we have confirmed that the correct number of scale factor powers has been incorporated. We can carry out this type of analysis for each term we are going to write.

Now, we can expand the fourth term as

$$\text{fourth} = -8\epsilon^2(b_4+b_5)\epsilon^{ijl}\frac{H^3}{M_{pl}^4}\left\{\nabla^2 h_{ml}\left(-\frac{\zeta^{,m}{}_{,j}}{Ha^2} + \epsilon\nabla^{-2}\dot{\zeta}^{,m}{}_{,j}\right) - \dot{h}^{m}{}_{j}\zeta_{,ml}\right\}\dot{\zeta}_{,i}. \tag{6.77}$$

Now, we focus on the expansion of the first term, which is the more complicated one

$$\text{first} = \frac{\dot{\phi}^3}{M_{pl}^5 N^2}\epsilon^{ijl} 2b_1 K_{mi} D_l K^m{}_j = \epsilon^{\frac{3}{2}} \frac{1}{N^2}\left(\frac{H^3}{M_{pl}^2}\right)\epsilon^{ijl} 2^{\frac{5}{2}} b_1 K_{mi} D_l K^m{}_j. \tag{6.78}$$

First of all we notice that $N^2 = 1 + 2\frac{\dot{\zeta}}{H} + \left(\frac{\dot{\zeta}}{H}\right)^2$. The second order term in N^2 can't contribute since we would need one of the two extrinsic curvature tensors to be of zeroth order, i.e. proportional to a Kronecker delta. However, we can't disregard a priori the first order contribution. Thus, we divide the computation in two steps:

- the first one in which $N^{-2} = 1$,
- the second one in which we take $N^{-2} \approx -2\frac{\dot{\zeta}}{H}$.

1. Thus, we start from the first computation in which we can write the Lagrangian as

$$\epsilon^{\frac{3}{2}}\left(\frac{H^3}{M_{pl}^2}\right)\epsilon^{ijl} 2^{\frac{5}{2}} b_1 K_{mi} D_l K^m{}_j. \tag{6.79}$$

   In order to perform the expansion we need the expression of the extrinsic curvature, eq.(6.59), and its covariant derivative, eq.(6.63), up to second order by setting to zero all the terms proportional to a Kronecker delta. The calculation is quite lengthy and we report the final result omitting the prefactor

$$\begin{aligned}\epsilon^{ijl} K_{mi} D_l K^m{}_j = \epsilon^{ijl}\Bigg\{ & a^2\left[H h_{mi} + \frac{\dot{h}_{mi}}{2}\right]\left[2\zeta_{,l}\left(-\frac{\zeta^{,m}{}_{,l}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}^{,m}{}_{,l}\right)\right] - \frac{1}{2}\left(-\frac{\zeta_{,im}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,im}\right)\times \\ & \left\{a^2\left[-\left(-\frac{\zeta_{,fl}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,fl}\right)\left(h^f{}_j{}^{,m} - h^{m,f}_{j}\right) - \left(-\frac{\zeta_{f}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{f}\right)\left(h^f{}_j{}^{,m} - h^{m,f}_{j}\right)\right]\right. \\ & - a^2\dot{h}^m{}_l\zeta_{,j} - a^2\left(-a^{-2}\frac{\zeta}{H} + \epsilon\partial^{-2}\dot{\zeta}\right)^{,f}{}_{,j} h_l{}^{\{m}{}_{,f\}}\Bigg\} + a^2\Bigg[\left(-\frac{\zeta_{,i}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,i}\right)\zeta_{,m} \\ & + \left(-\frac{\zeta_{,m}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,m}\right)\zeta_{,i}\Bigg]\frac{\dot{h}^m{}_{j,l}}{2}\Bigg\},\end{aligned} \tag{6.80}$$

   where we have have used eq.(6.13).

2. The second piece, which we can call B, is easy to compute

$$B = -\epsilon^{\frac{3}{2}} 2^{\frac{5}{2}}\left(\frac{H^3}{M_{pl}^2}\right) b_1 \epsilon^{ijl} K_{mi} D_l K^m{}_j 2\frac{\dot{\zeta}}{H} = \epsilon^{\frac{3}{2}} 2^{\frac{5}{2}}\left(\frac{H}{M_{pl}}\right)^2 b_1 \dot{\zeta}\epsilon^{ijl}\dot{h}^m{}_{j,l} a^2\left(-a^{-2}\frac{\zeta_{,mi}}{H} + \epsilon\partial^{-2}\dot{\zeta}_{,mi}\right). \tag{6.81}$$

Thus, collecting the two pieces we immediately obtain the expression for the first piece of the first term of the Lagrangian

$$\begin{aligned}\text{first} = \epsilon^{\frac{3}{2}} 2^{\frac{5}{2}}\left(\frac{H^3}{M_{pl}^2}\right) b_1 \epsilon^{ijl}\Bigg\{ & a^2\left[H h_{mi} + \frac{\dot{h}_{mi}}{2}\right]\left[2\zeta_{,l}\left(-\frac{\zeta^{,m}{}_{,l}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}^{,m}{}_{,l}\right)\right] - \frac{1}{2}\left(-\frac{\zeta_{,im}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,im}\right)\times \\ & \left\{a^2\left[-\left(-\frac{\zeta_{,fl}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,fl}\right)\left(h^f{}_j{}^{,m} - h^{m,f}_{j}\right) - \left(-\frac{\zeta_{f}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{f}\right)\left(h^f{}_j{}^{,m} - h^{m,f}_{j}\right)\right]\right. \\ & - a^2\dot{h}^m{}_l\zeta_{,j} - a^2\left(-a^{2}\frac{\zeta}{H} + \epsilon\partial^{-2}\dot{\zeta}\right)^{,f}{}_{,j} h_l{}^{,\{m}{}_{,f\}}\Bigg\} + a^2\Bigg[\left(-\frac{\zeta_{,i}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,i}\right)\zeta_{,m} \\ & + \left(-\frac{\zeta_{,m}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,m}\right)\zeta_{,i}\Bigg]\frac{\dot{h}^m{}_{j,l}}{2}\Bigg\} + \epsilon^{\frac{3}{2}} 2^{\frac{5}{2}}\left(\frac{H}{M_{pl}}\right)^2 b_1 \dot{\zeta}\epsilon^{ijl}\dot{h}^m{}_{j,l} a^2\left(-a^{-2}\frac{\zeta_{,mi}}{H} + \epsilon\partial^{-2}\dot{\zeta}_{,mi}\right).\end{aligned} \tag{6.82}$$

In the end, we have to evaluate the expansion of the second term, which we can write as

$$\text{second} = \frac{\dot{\phi}^3}{M_{pl}^6}\frac{2}{N^3}\epsilon^{ijl} f_1 K_{mi} K^n{}_j D_n K^m{}_l = \epsilon^{\frac{3}{2}}\frac{H^3}{M_{pl}^3}\frac{2^{\frac{5}{2}}}{N^3}\epsilon^{ijl} f_1 K_{mi} K^n{}_j D_n K^m{}_l, \tag{6.83}$$

where, for the sake of simplicity, we have defined $f_1 \equiv = b_4 + b_5 - b_3$. First of all, we notice that the terms proportional to the Kronecker delta in K_{mi} and $K^m{}_l$ can be set to zero since they produce vanishing contributions. Now, to advance further, we observe that we can take $K^n{}_j$ up to first order. Consequently, we divide our calculations into three distinct parts and we'll ignore the couplings for the sake of simplicity:

1. the initial part A in which $K^n{}_j$ is of zeroth order and we set $N = 1$,
2. the subsequent part B in which we take $K^n{}_j$ of first order and we set $N = 1$,
3. the last term D in which we consider higher orders in N. Since K_{mi} and $D_n K^m{}_l$ are at least of first order we need to consider $K^n{}_j$ of zeroth order.

Notice that the terms proportional to Kronecker's delta contain all the zero-order terms.

1. We already compute the first term since we can write

$$A = \epsilon^{\frac{3}{2}} 2^{\frac{5}{2}}\frac{H^3}{M_{pl}^3}\epsilon^{ijl} H f_1 K_{mi} D_j K^m{}_l \tag{6.84}$$

$$\begin{aligned}
&= \epsilon^{\frac{3}{2}} 2^{\frac{5}{2}}\frac{H^4}{M_{pl}^3} f_1 \epsilon^{ijl}\Big\{ a^2\left[H h_{mi} + \frac{\dot{h}_{mi}}{2}\right]\left[2\zeta_{,l}\left(-\frac{\zeta^{,m}{}_{,l}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}^{,m}{}_{,l}\right)\right] - \frac{1}{2}\left(-\frac{\zeta_{,im}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,im}\right)\times \\
&\Big\{a^2\Big[-\left(-\frac{\zeta_{,fl}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,fl}\right)\left(h^f{}_j{}^{,m} - h^{m,f}_{j}\right) - \left(-\frac{\zeta_f}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_f\right)\left(h^f{}_j{}^{,m} - h^{m,f}_{j}\right)\Big] \\
&- a^2\dot{h}^m{}_l\zeta_{,j} - a^2\left(-a^{-2}\frac{\zeta}{H} + \epsilon\partial^{-2}\dot{\zeta}\right)^{,f}{}_{,j} h_l{}^{,\{m}{}_{,f\}}\Big\} + a^2\Big[\left(-\frac{\zeta_{,i}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,i}\right)\zeta_{,m} \\
&+ \left(-\frac{\zeta_{,m}}{Ha^2} + \epsilon\partial^{-2}\dot{\zeta}_{,m}\right)\zeta_{,i}\Big]\frac{\dot{h}^m{}_{j,l}}{2}\Big\}
\end{aligned} \tag{6.85}$$

2. We are not going to explicitly perform this computation since there are no particular technical parts worth to be discussed

$$B = f_1\epsilon^{\frac{3}{2}}\frac{H^3}{M_{pl}^3}2^{\frac{5}{2}}a^2\left(-a^{-2}\frac{\zeta}{H} + \epsilon\nabla^{-2}\dot{\zeta}\right)_{,im}\left(-\dot{\zeta}\dot{h}^m{}_{l,j} + \left(-a^{-2}\frac{\zeta}{H} + \epsilon\nabla^{-2}\dot{\zeta}\right)^{,n}{}_{,j}\dot{h}^m{}_{j,l}\right). \tag{6.86}$$

3. The result of the last computation reads

$$C = \epsilon^{\frac{3}{2}} 2^{\frac{5}{2}}\frac{H^3}{M_{pl}^3}\epsilon^{ijl} H f_1 K_{mi} D_j K^m{}_l = 3\epsilon^{\frac{3}{2}} 2^{\frac{5}{2}}\frac{H^4}{M_{pl}^3}\epsilon^{ijl} a^2\dot{\zeta}\left(-a^{-2}\frac{\zeta}{H} + \epsilon\nabla^{-2}\dot{\zeta}\right)_{,mi}\frac{\dot{h}^m{}_{j,l}}{2} \tag{6.87}$$

Thus, by collecting everything together we have the scalar-scalar-tensor vertices from the Lagrangian of eq.(6.12)

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}^{\zeta\zeta h}_{PV_2} &= 2\epsilon^2\frac{H^3}{M_{pl}^4}b_3\epsilon^{ijl}\left\{\dot{\zeta}_{,j}\zeta_{,lm}\dot{h}^m{}_i - \zeta_{,jm}\dot{\zeta}_{,l}\dot{h}^m{}_i + \left[h^n{}_{\{l,j\}m} + h_{m\{j,l\}}{}^{,n}\right]\left(-\frac{\zeta}{a^2H}+\epsilon\nabla^2\dot{\zeta}\right)^{,m}{}_{,i}\dot{\zeta}_{,n}\right\}+ \\
&- 8\epsilon^2(b_4+b_5)\epsilon^{ijl}\frac{H^3}{M_{pl}^4}\left\{\nabla^2 h_{ml}\left(-\frac{\zeta^{,m}{}_{,j}}{Ha^2}+\epsilon\nabla^{-2}\dot{\zeta}^{,m}{}_{,j}\right) - \dot{h}^m{}_j\zeta_{,ml}\right\}\dot{\zeta}_{,i}+ \\
&+ \epsilon^{\frac{3}{2}}2^{\frac{5}{2}}\left(\frac{H^3}{M_{pl}^2}\right)b_1\epsilon^{ijl}\left\{a^2\left[Hh_{mi}+\frac{\dot{h}_{mi}}{2}\right]\left[2\zeta_{,l}\left(-\frac{\zeta^{,m}{}_{,l}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}^{,m}{}_{,l}\right)\right] - \frac{1}{2}\left(-\frac{\zeta_{,im}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta_{,im}}\right)\times\right. \\
&\left\{a^2\left[-\left(-\frac{\zeta_{,fl}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}_{,fl}\right)\left(h^f{}_j{}^{,m} - h^{m,f}_j\right) - \left(-\frac{\zeta_f}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}_f\right)\left(h^f{}_j{}^{,m} - h^{m,f}_j\right)\right]+\right. \\
&- a^2\dot{h}^m{}_l\zeta_{,j} - a^{-2}\left(-a^2\frac{\zeta}{H}+\epsilon\partial^{-2}\dot{\zeta}\right)^{,f}{}_{,j}h_l{}^{,\{m}{}_{,f\}}\bigg\} + a^2\left[\left(-\frac{\zeta_{,i}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}_{,i}\right)\zeta_{,m}+\right. \\
&+\left.\left(-\frac{\zeta_{,m}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}_{,m}\right)\zeta_{,i}\right]\frac{\dot{h}^m{}_{j,l}}{2}\bigg\} + \epsilon^{\frac{3}{2}}2^{\frac{5}{2}}\left(\frac{H}{M_{pl}}\right)^2 b_1\dot{\zeta}\epsilon^{ijl}\dot{h}^m{}_{j,l}a^2\left(-a^{-2}\frac{\zeta_{,mi}}{H}+\epsilon\partial^{-2}\dot{\zeta}_{,mi}\right) \\
&+ \epsilon^{\frac{3}{2}}2^{\frac{5}{2}}\frac{H^4}{M_{pl}^3}f_1\epsilon^{ijl}\left\{a^2\left[Hh_{mi}+\frac{\dot{h}_{mi}}{2}\right]\left[2\zeta_{,l}\left(-\frac{\zeta^{,m}{}_{,l}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}^{,m}{}_{,l}\right)\right] - \frac{1}{2}\left(-\frac{\zeta_{,im}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta_{,im}}\right)\times\right. \\
&\left\{a^2\left[-\left(-\frac{\zeta_{,fl}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}_{,fl}\right)\left(h^f{}_j{}^{,m} - h^{m,f}_j\right) - \left(-\frac{\zeta_f}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}_f\right)\left(h^f{}_j{}^{,m} - h^{m,f}_j\right)\right]+\right. \\
&- a^2\dot{h}^m{}_l\zeta_{,j} - a^{2}\left(-a^{-2}\frac{\zeta}{H}+\epsilon\partial^{-2}\dot{\zeta}\right)^{,f}{}_{,j}h_l{}^{,\{m}{}_{,f\}}\bigg\} + a^2\left[\left(-\frac{\zeta_{,i}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}_{,i}\right)\zeta_{,m}+\right. \\
&+\left.\left(-\frac{\zeta_{,m}}{Ha^2}+\epsilon\partial^{-2}\dot{\zeta}_{,m}\right)\zeta_{,i}\right]\frac{\dot{h}^m{}_{j,l}}{2}\bigg\}+ \\
&+ f_1\epsilon^{\frac{3}{2}}\frac{H^3}{M_{pl}^3}2^{\frac{5}{2}}a^2\left(-a^{-2}\frac{\zeta}{H}+\epsilon\nabla^{-2}\dot{\zeta}\right)_{,im}\left(-a^2\dot{\zeta}\dot{h}^m{}_{l,j} + \left(-a^{-2}\frac{\zeta}{H}+\epsilon\nabla^{-2}\dot{\zeta}\right)^{,n}{}_{,j}\dot{h}^m{}_{l,n}\right) \\
&+ 3f_1\epsilon^{\frac{3}{2}}2^{\frac{5}{2}}\frac{H^4}{M_{pl}^3}a^2\epsilon^{ijl}\dot{\zeta}\left(-a^{-2}\frac{\zeta}{H}+\epsilon\nabla^{-2}\dot{\zeta}\right)_{,mi}\frac{\dot{h}^m{}_{l,n}}{2},
\end{aligned}
\tag{6.88}
$$

which is a quite long Lagrangian but as we'll see it has no impact on the leading order of the graviton-mediated trispectrum.

6.3.2 The expansion of $\mathcal{L}_{PV_1}$

Let's now focus towards $\mathcal{L}_{PV_1}$ as represented by equation (6.11). Since many of the terms overlap with those already computed in the preceding section, we won't repeat those calculations. Instead, we'll present the computations different from our previous analysis. Furthermore, we are not providing a complete representation of all vertices stemming from the Lagrangian because, as we will demonstrate in section 6.4, these vertices are subject to suppression relative to the coupling we derive from the Hilbert-Einstein term. This can be understood also by dimensional analysis however for completness we report this computation.

We write down once again the expression of the $\mathcal{L}_{PV_1}$ Lagrangian in the ADM formalism and within the unitary gauge as [19]

$$
\begin{aligned}
\sqrt{-g}\,\mathcal{L}_{PV_1} = \frac{4\epsilon}{N}\left(\frac{H}{M_{pl}}\right)^2\epsilon^{ijl}&\left[2(2a_1+a_2+4a_4)\left(KK_{mi}D_lK^m_j + {}^{(3)}R_{mi}D_lK^m_j - K_{mi}K^{mn}D_lK_{jn}\right)\right. \\
&\left.-(a_2+4a_4)\left(2K_{mi}K^n_jD_nK^m_l + {}^{(3)}R_{jlm}{}^nD_nK^m_i\right)\right].
\end{aligned}
\tag{6.89}
$$

Now, we can start by expanding the first term in the Lagrangian in order to find the scalar-scalar-tensor vertex

$$
\text{first} = \frac{4\epsilon}{N}\left(\frac{H}{M_{Pl}}\right)^2\epsilon^{ijl}2(2a_1+a_2+4a_4)KK_{mi}D_lK^m{}_j, \tag{6.90}
$$

where ϵ is the first slow-roll parameter, eq.(2.58). First of all, we notice that K_{mi} and $K^m{}_j$ can't contain terms proportional to a Kronecker delta because of symmetry the result would be zero. Thus, we immediately understand that we need the expression of K up to the first order, which is

$$K = 3\left(H + \dot{\zeta}\right) - \nabla^2\psi. \tag{6.91}$$

Now we can split the expansion into three parts:

- The first one, A, in which $\frac{1}{N} = 1$ and $K = 3\dot{\zeta} - a^{-2}\nabla^2\psi$. Thus we immediately get

$$\begin{aligned} A &= 4\epsilon\left(\frac{H}{M_{pl}}\right)^2 \epsilon^{ijl} 2(2a_1 + a_2 + 4a_4)\left(3\dot{\zeta} - \nabla^2\psi\right) K_{mi} D_l K^m{}_j & (6.92)\\ &= -4\epsilon\left(\frac{H}{M_{pl}}\right)^2 \epsilon^{ijl} 2(2a_1 + a_2 + 4a_4)\left(3\dot{\zeta} - \nabla^2\psi\right)\epsilon^{ijl} a^2 \psi_{,mi} \frac{\dot{h}^m{}_{j,l}}{2} & (6.93)\\ &= -4\epsilon\left(\frac{H}{M_{pl}}\right)^2 2(2a_1 + a_2 + 4a_4)\left[3\dot{\zeta} + \frac{\nabla^2\zeta}{Ha^2} - \epsilon\dot{\zeta}\right]\epsilon^{ijl} a^2 \left(-a^{-2}\frac{\zeta_{,mi}}{H} + \epsilon\partial^{-2}\dot{\zeta}_{,mi}\right)\frac{\dot{h}^m{}_{j,l}}{2}. & (6.94) \end{aligned}$$

- The second one, B, in which we consider the first order expansion of $\frac{1}{N} \approx -\frac{\dot{\zeta}}{H}$ and $K = 3H$. We get

$$\begin{aligned} B &= -\frac{\dot{\zeta}}{H} 4\epsilon\left(\frac{H}{M_{pl}}\right)^2 \epsilon^{ijl} 2(2a_1 + a_2 + 4a_4) 3H K_{mi} D_l K^m{}_j & (6.95)\\ &= -\dot{\zeta} a^2 \epsilon\left(\frac{H}{M_{pl}}\right)^2 24(2a_1 + a_2 + 4a_4)\epsilon^{ijl}\left(-a^{-2}\frac{\zeta_{,mi}}{H} + \epsilon\partial^{-2}\dot{\zeta}_{,mi}\right)\frac{\dot{h}^m{}_{j,l}}{2} & (6.96) \end{aligned}$$

- The third piece, C, in which we set $N = 1$ and $K = 3H$. Thus we have

$$C = 4\epsilon\left(\frac{H}{M_{pl}}\right)^2 2(2a_1 + a_2 + 4a_4) 3H\epsilon^{ijl} K_{mi} D_l K^m{}_j \tag{6.97}$$

$$\tag{6.98}$$

where we can use eq.(6.80) to obtain the final result. We do not collect everything together since we'll make a general discussion

Now, we can focus on the expansion of the second term of eq.(6.11), which is

$$\text{second} = \frac{8\epsilon}{N}\left(\frac{H}{M_{pl}}\right)^2 (2a_1 + a_2 + 4a_4)\,\epsilon^{ijl\,(3)}R_{mi} D_l K^m{}_j, \tag{6.99}$$

where the $^{(3)}$ indicates that the quantity we are considering belongs to the spatial section. For simplicity, we'll omit it. First of all, we notice that $D_l K^m{}_j$ can't contain terms proportional to a delta since the result would be zero because of symmetries. Moreover, the only first-order term it contains is proportional to h. Thus we need the expression of the Ricci tensor up to the second order in scalar perturbations, which is given by eq.(6.67). We do not need to consider tensor perturbations in this expansion. The final result is the sum of three pieces:

- In the first piece, we set $N = 1$, we take R_{mi} of second order in scalar perturbations and we fix $D_l K^m{}_j = \frac{\dot{h}^m{}_{j,l}}{2}$. The result is

$$A = 8\epsilon\left(\frac{H}{M_{pl}}\right)^2 (2a_1 + a_2 + 4a_4)\,^{(3)}\epsilon^{ijl}\zeta_{,m}\zeta_{,i}\frac{\dot{h}^m{}_{j,l}}{2}. \tag{6.100}$$

- In the second piece, A, we take the Ricci tensor at first order, the covariant derivative of the extrinsic curvature up to second order and we set $N=1$. The final result is

$$
\begin{aligned}
B = {} & 8\epsilon\left(\frac{H}{M_{pl}}\right)^2(2a_1+a_2+4a_4)\,\epsilon^{ijl}\Bigg\{ \\
& -\zeta_{,mi}\left[\left(-\frac{\zeta}{a^2H}+\epsilon\partial^{-2}\dot{\zeta}\right)_{,fl}\left(h^{fm}{}_{,j}+h^{f}{}_{j}{}^{,m}-h^{m}{}_{j}{}^{,f}\right)+\left(-\frac{\zeta}{a^2H}+\epsilon\partial^{-2}\dot{\zeta}\right)_{,f}\left(h^{f}{}_{j}{}^{,m}{}_{,l}-h^{m}{}_{j}{}^{,f}{}_{,l}\right)\right] \\
& +\zeta_{,mi}\frac{\dot{h}^{m}{}_{l}\zeta_{,j}}{2}-\zeta_{,mi}\Bigg[-\left(-\frac{\zeta}{a^2H}+\epsilon\partial^{-2}\dot{\zeta}\right)^{,f}{}_{,j}\left(h^{m}{}_{l,f}+h^{m}{}_{f,l}-h_{fl}{}^{,m}\right) \\
& +\left(-\frac{\zeta}{a^2H}+\epsilon\partial^{-2}\dot{\zeta}\right)^{,m}{}_{,f}\left(h^{f}{}_{l,j}+h^{f}{}_{j,l}\right)\Bigg] \\
& -\frac{1}{2}\nabla^2 h_{mi}\left(2\zeta_{,l}\left(-\frac{\zeta}{a^2H}+\epsilon\partial^{-2}\dot{\zeta}\right)^{,m}{}_{,j}-\left(-\frac{\zeta}{a^2H}+\epsilon\partial^{-2}\dot{\zeta}\right)_{,j}\zeta^{,m}{}_{,l}\right)\Bigg\}.
\end{aligned}
\tag{6.101}
$$

- The last piece, C, we consider N of first order, R_{mi} of first order in scalar perturbations, and we take $D_l K^m{}_j = \frac{\dot{h}^m{}_{j,l}}{2}$. Higher order terms in N produce quartic vertex interaction. Thus, we have

$$
C = 8\epsilon\left(\frac{H}{M_{pl}}\right)^2(2a_1+a_2+4a_4)\,\frac{\dot{\zeta}}{H}\epsilon^{ijl}\zeta_{,im}\frac{\dot{h}^m{}_{j,l}}{2}. \tag{6.102}
$$

Now we turn our attention to the third piece of the Lagrangian

$$
\text{third} = \frac{4\epsilon}{N}\left(\frac{H}{M_{pl}}\right)^2\epsilon^{ijl}2(2a_1+a_2+4a_4)\left(-K_{mi}K^m{}_n D_l K^n{}_j\right) \tag{6.103}
$$

We can split the computation into three parts

- To start, we can set N equal to one, and then we can examine the two terms that emerge when we consider the zeroth-order values for $K_{mi}=a^2H\delta_{mi}$ and $K^m{}_n = H\delta^m{}_n$, respectively. We can't have both of them of zeroth order at the same time because we would get zero because of symmetries. Since we fall back to something proportional to $K_{in}D_l K^n{}_j$ we do not repeat the actual computation (see eq.(6.80)).
- Then, we have to consider all three extrinsic curvature tensors at first order and we get

$$
-2\epsilon\left(\frac{H}{M_{pl}}\right)^2\epsilon^{ijl}2(2a_1+a_2+4a_4)a^2\left[-\dot{\zeta}\psi_{,in}\dot{h}^n{}_{j,l}+\psi_{,im}\psi^{,m}{}_{,n}\dot{h}^n{}_{j,l}\right]. \tag{6.104}
$$

- Finally we have to consider the possibility of expanding N. We can't expand N up to the second order since $D_l K^n{}_j$ is of the first order. So we would have to take the other two extrinsic curvatures of zeroth order and we get zero. The final result is

$$
2\epsilon\left(\frac{H}{M_{pl}}\right)^2\epsilon^{ijl}2(2a_1+a_2+4a_4)a^2\dot{\zeta}\psi_{,in}\dot{h}^n{}_{j,l}. \tag{6.105}
$$

Then, we can turn our attention to the fourth term in the Lagrangian

$$
\text{fourth} = \frac{4\epsilon}{N}\left(\frac{H}{M_{pl}}\right)^2\epsilon^{ijl}\left[-(a_2+4a_4)2K_{mi}K_j^n D_n K_l^m\right]. \tag{6.106}
$$

However, we do not perform this computation since it is identical to what we have already computed in expanding the second term in the Lagrangian $\mathcal{L}_{PV_2}$.

Now, we can expand the fifth term in the Lagrangian

$$\text{fifth} = -4\frac{\epsilon}{N}\left(\frac{H}{M_{pl}}\right)^2 (a_2+4a_4)\, {}^{(3)}R_{jlm}{}^{n} D_n K^{m}{}_{i}. \tag{6.107}$$

The method to perform this computation is identical to the expansion of the second term of the Lagrangian $\mathcal{L}_{PV_1}$. Thus, we report the result

$$\begin{aligned}
\text{fifth} = &-4\epsilon\left(\frac{H}{M_{pl}}\right)^2 (a_2+4a_4)\frac{\epsilon^{ijl}}{2}\left[\dot{h}^{m}{}_{i,j}\zeta_{,ml} - \dot{h}^{m}{}_{i,l}\zeta_{,jm} + \dot{h}^{m}{}_{i,l}\zeta_{,j}\zeta_{,m} - \dot{h}^{m}{}_{i,j}\zeta_{,l}\zeta_{,m}\right] \\
&- 4\epsilon\left(\frac{H}{M_{pl}}\right)^2 (a_2+4a_4)\,\epsilon^{ijl}\Bigg\{\zeta_{,lm}\Bigg[\psi_{fi}h^{fm}{}_{,j} + \frac{\psi_{,fj}}{2}\left(h^{fm}{}_{,i} + h^{f}{}_{i}{}^{,m} - h^{m}{}_{i}{}^{,f}\right) \\
&+ \frac{\psi_{,f}}{2}\left(h^{f}{}_{i,j}{}^{,m} - h^{m}{}_{i}{}^{,f}{}_{,j}\right) - \frac{\dot{h}^{m}{}_{j}}{2}\zeta_{,i} - \frac{\psi^{,f}{}_{,i}}{2}\left(h^{m}{}_{j,f} + h^{m}{}_{f,j} - h_{fj}{}^{,m}\right)\Bigg] - (j\leftrightarrow l)\Bigg\} \\
&- 2\epsilon\left(\frac{H}{M_{pl}}\right)^2 (a_2+4a_4)\,\epsilon^{ijl}\left\{h^{n}{}_{\{j,l\}m} + h_{m\{l}{}^{,n}{}_{,\}j}\right\}\left[2\zeta_{,n}\psi^{,m}{}_{,i} - \psi_{,i}\zeta^{,m}{}_{,n}\right] \\
&+ 4\epsilon\left(\frac{H}{M_{pl}}\right)^2 (a_2+4a_4)\,\epsilon^{ijl}\zeta^{,n}{}_{,l}\Bigg\{\psi_{,fi}h^{f}{}_{j,n} - \frac{\dot{h}_{jn}}{2}\zeta_{,i} + \psi_{,fn}\left[h^{f}{}_{[i,j]} - h_{ij}{}^{,f}\right] \\
&- \psi^{,f}{}_{,i}\left(h_{j[n,f]} - h_{fn,j}\right) + \frac{\psi_{,f}}{2}h_{i}^{\{f}{}_{,i\}n} - (j\leftrightarrow l)\Bigg\} \\
&+ 2\epsilon\left(\frac{H}{M_{pl}}\right)^2 (a_2+4a_4)\,\epsilon^{ijl}\dot{h}^{m}{}_{i,\{j}\zeta_{,l\}m},
\end{aligned} \tag{6.108}$$

where we do not have substituted the actual value of ψ for simplicity.

6.4 The graviton-mediated trispectrum

In this section, we want to illustrate the main ideas revolving around the computation of the graviton-mediated trispectrum. In this section, we'll show that we do not obtain an enhancement factor with respect to what we have obtained in Chern-Simons even if here we have a birefringence effect. The reason is that the correction to the velocity of the chiral polarization of the gravitational waves,

$$C_{T,s}^2 \approx 1 - \lambda_s k\tau\chi_{1/2} \approx 1 + \lambda_s\frac{k}{aH}\chi_{1/2}, \tag{6.109}$$

scales with the chirality parameter. It is evident that as $-k\tau$ approaches infinity, the sound speed can grow significantly. However, when we examine the behavior of the trispectrum for $-k\tau << -1$, we encounter a highly oscillatory integrand during the computation of the trispectrum, leading to significant cancellations [17].

In order to proceed, we are going to use the same scheme we have applied in the Chern-Simons case. First of all, we want to show that any contributions arising from the vertices we have computed in section 6.3.1 and 6.3.2 can be disregarded with respect to the case in which we consider two standard GR vertices in trispectrum. The contributions under discussion refer to the interference components involving a standard GR vertex and a vertex derived from the Lagrangians (6.11) and (6.12), as well as diagrams featuring two vertices from these two Lagrangians. First of all, given the explicit expression of the Lagrangian $\mathcal{L}_{PV_2}$, the contributions stemming from that Lagrangian are slow-roll suppressed if we take $\epsilon \simeq 10^{-2}$ [27]. This is different for $\mathcal{L}_{PV_1}$ since it has the same slow-roll prefactor with respect to the standard GR vertex. Then, they have a suppression factor given by the ratio

$$\left(\frac{H}{M_{pl}}\right)^n, \quad \text{with } n \in N, n \geq 2. \tag{6.110}$$

[27] There are models of inflation for which $\epsilon \sim O(1)$.

From Planck data [72], we know that at the horizon exit of a given physical wavenumber $\frac{k}{a}$, the Hubble parameter, which is equal to $\frac{k}{a}$, has the value [18]

$$H = \frac{k}{a} = \sqrt{\epsilon}2 \times 10^{14}\text{Gev}, \tag{6.111}$$

where $\epsilon = -\frac{\dot{H}}{H^2}$ is evaluated at Horizon exit for any given wave numer k. So it's clear that if we take $\epsilon \approx O(1)$ we have that

$$\left(\frac{H}{M_{pl}}\right) \approx 10^{-4}, \tag{6.112}$$

which results in suppression factors of at least 10^{-8} with respect to the standard GR vertex. In this discussion, we are considering the coupling parameters which are functions of the scalar field (see eq.(6.11) and eq.(6.12)) as order one quantities [18] according to the idea that we are building an EFT of Inflation (see section 5.2). We want also to discuss deeply about this point. Since the theory is built as an EFT of Inflation the vertices stemming from the Lagrangians, $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$, contain more derivative with respect to the standard GR vertex term. So their contribution is suppressed by powers of $\frac{k}{\Lambda}$ [28], where Λ is a cut-off scale. Thus, because of everything we have said we can conclude that we can safely disregard all the contributions we have computed.

Up to this point, we have provided a preliminary dimensional analysis argument. However, our next step is to validate this analysis and confirm that there is no enhancement factor involved when we examine the scalar-scalar-tensor vertex stemming from the Lagrangians $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$. The fact that there are no enhancement factors due to the sound speed is clear from the explicit expression of the vertices presented in section 6.3.1 and 6.3.2. Furthermore, one should consider whether there exist enhancement factors of the form $\frac{1}{k}$ that lead to enhancements as k approaches zero. We ask this question since we have to employ the solution for shift function

$$N^i = \partial^i \psi, \quad \psi = -a^{-2}\frac{\zeta}{H} + \epsilon \nabla^{-2}\dot{\zeta}, \tag{6.113}$$

in which appears the operator ∇^{-2}. In Fourier space this results in a factor of $\frac{1}{k^2}$ where k is the comoving wavevector associated to the field ζ in ψ. Thus a priori one can expect an enhancement factor of the form $\frac{1}{k}$ since we have a temporal derivative acting on ζ. The general idea of this kind of analysis is that a spatial or a temporal derivative is equivalent to k factor in Fourier space. However, as one can verify in all the vertices written in section 6.3.1 and 6.3.2 the scalar field ψ appears at least with one spatial derivative. So, we do not produce any enhancement factor.

Now we want to provide an explicit example in order to clarify why we can disregard all the vertices stemming from $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$. This is not a very rigorous proof but it's reasonable that everything works in this way. Let's consider the two interaction vertices

$$S^{(1)}_{int} = \int d\tau d^3x \epsilon M^2_{pl} a^2(\tau) h^{ij}\partial_i\zeta\partial_j\zeta, \quad S^{(2)}_{int} = -2^{\frac{5}{2}}\epsilon^{\frac{3}{2}}\left(\frac{H^4}{M^3_{pl}}\right)\int dt d^3x f_1 \epsilon^{ijl} H h_{mi} 2\zeta_{,l}\frac{\zeta^{,m}{}_{,l}}{H}, \tag{6.114}$$

where the second vertex is a contribution extracted from eq.(6.88). In order to compute the trispectrum within the $In-In$ formalism we have to switch to conformal time and we get

$$S^{(2)}_{int} = -2^{\frac{7}{2}}\epsilon^{\frac{3}{2}}\left(\frac{H^4}{M^3_{pl}}\right)\int d\tau d^3x a(\tau) f_1 \epsilon^{ijl} h_{mi}\zeta_{,l}\zeta^{,m}{}_{,l}. \tag{6.115}$$

Now, the idea is to make a comparison of the two vertices at the horizon crossing. We recall that given a comoving wavector k the definition of horizon crossing is

$$k = aH. \tag{6.116}$$

[28] The idea is the same presented in [9].

First of all, we make this comparison at horizon crossing since the most important contribution to the time integral part of trispectrum arises at horizon crossing [17]. Certainly, each momentum entering in the trispectrum crosses the horizon at distinct intervals, but for simplicity, we approximate them to be within a similar magnitude, k. Moreover we disregard any directional contribution form the polarization portion. Thus, if we make a comparison of the two vertices at the horizon crossing we get

$$|\frac{-a(\tau)2^{\frac{5}{2}}\epsilon^{\frac{3}{2}}\left(\frac{H^4}{M_{pl}^3}\right)f_1\epsilon^{ijl}h_{mi}2\zeta_{,l}\zeta^{,m}{}_{,l}}{\epsilon M_{pl}^2a^2(\tau)h^{ij}\partial_i\zeta\partial_j\zeta}|\approx\epsilon^{\frac{1}{2}}\left(\frac{H^4}{M_{pl}^3}\right)f_1\frac{H^3a^4}{M_{pl}^2a^4H^2}\approx f_1\epsilon^{\frac{1}{2}}\left(\frac{H}{M_{pl}}\right)^5, \tag{6.117}$$

from which we understand the large suppression we get. In this computation, we also used that f_1 is roughly of order one. However, it will be interested to understand to compute the trispectrum with an explicit form of the coupling parameters. We are not going to discuss this issue but we'll make some comments in section 6.5.

Now, we focus on the computation of the graviton-mediated trispectrum with two standard GR vertices. We are going to use the same approximation we employed in the Chern-Simons case; we approximate the tensor mode functions with the one in de-Sitter and we match the approximate solutions with the real ones in $\tau=0$. Actually, we do not need to match the approximate solution with the real one. In fact, since the graviton mode functions appear in the propagator we need only to match the expression of the square modulus of the mode function in $\tau=0$ with the real value. But two times the modulus of the mode function in $\tau=0$ is the Power spectrum of tensor perturbation thus we can use the result reported in [71]. Moreover, we notice that everything works in exactly the same way as what we have done in section 5.6 to the computation of the Chern-Simons graviton mediated trispectrum. The polarization portion of the diagram is identical since the coupling between the two ζs to the graviton in the chiral scalar-tensor theories of gravity is identical to the inflaton-inflaton-coupling in the Chern-Simons case. Moreover, the time integral is the same since the time-dependent part of the mode function is identical in eq.(6.15) and eq.(5.180). The difference between this case and the Chern-Simons case arises from the normalization factor in the amplitudes of the mode functions, denoted as $u_\zeta(\tau,\mathbf{k})$ and $u_{\delta\phi}(\tau,\mathbf{k})$, as well as the prefactor in the vertices. However, it's easy to show that once every factor is taken into account the result is identical to the one we have computed in Chern-Simons once we have switched to the new chirality parameter

$$\Re\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle=-\frac{4}{\left(2\epsilon M_{pl}^2\right)^2}\sum_I\Re\left(P_R(\mathbf{K}_I)\right)T_{var_R}(\mathbf{K}_I)|_{\chi_i=0}, \tag{6.118}$$

$$\Im\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle=-\frac{\pi\chi_i}{\left(2\epsilon M_{pl}^2\right)^2}\sum_I\Im\left(P_R(\mathbf{K}_I)\right)T_{var_R}(\mathbf{K}_I)|_{\chi_i=0}, \tag{6.119}$$

where χ_i with $i=1,2$ is the chirality parameter introduced before and, where,

$$\mathcal{P}_R(\mathcal{K}_I)=\frac{1}{2}\left[K_I^1K_I^3\sin\left(\theta_I^1\right)\sin\left(\theta_I^3\right)e^{+i\phi_I^1}e^{-i\phi_I^3}\right]^2, \tag{6.120}$$

$$\mathcal{P}_L(\mathcal{K}_I)=\frac{1}{2}\left[K_I^1K_I^3\sin\left(\theta_I^1\right)\sin\left(\theta_I^3\right)e^{-i\phi_I^1}e^{+i\phi_I^3}\right]^2. \tag{6.121}$$

Moreover, as we have computed in chapter 5 we have

$$\begin{aligned}T_{var_h}(\mathbf{K}_I)|_{\chi_i=0}=-\frac{\bar{A}_t}{2k_I^3}\frac{H^4}{\prod_{i=1}^4 2k_i^3}\Bigg\{&\frac{K_I^1+K_I^2}{[a_{34}^I]^2}\left[\frac{1}{2}(a_{34}^I+k_I)([a_{34}^I]^2-2b_{34}^I)+k_I^2(K_I^3+K_I^4)\right]+(1,2)\leftrightarrow(3,4)\\&+\frac{K_I^1K_I^2}{k_t}\left[\frac{b_{34}^I}{a_{34}^I}-k_I+\frac{k_I}{a_{12}^I}\left(K_I^3K_I^4-k_i\frac{b_{34}^I}{a_{34}^I}\right)\left(\frac{1}{k_t+\frac{1}{a_{12}^I}}\right)\right]+(1,2)\leftrightarrow(3,4)\\&-\frac{k_I}{a_{12}^Ia_{34}^Ik_t}\left[b_{12}^Ib_{34}^I+2k_I^2\left(\prod_{i=1}^4k_i\right)\left(\frac{1}{k_t^2}+\frac{1}{a_{12}^Ia_{34}^I}+\frac{k_I}{k_ta_{12}^Ia_{34}^I}\right)\right]\Bigg\},\end{aligned} \tag{6.122}$$

with

$$k_t = \sum_{i=1}^{4} \mid \mathbf{K}_I^i \mid, \;\; a_{ij}^I \;\; = \Big[\mid \mathbf{K}_I^i \mid + \mid \mathbf{K}_I^j \mid + k_I\Big], \;\; b_{ij}^I = \Big[(\mid \mathbf{K}_I^i \mid + \mid \mathbf{K}_I^j)k_I + \mid \mathbf{K}_I^i \mid\mid \mathbf{K}_I^j \mid\Big], \;\; \bar{A}_t \;\; = 4\left(\frac{H}{M_{pl}}\right)^2. \tag{6.123}$$

All the conventions we are using here are identical to the ones we have adopted in chapter 5.

As stressed many times in chapter 5 it's reasonable that the assumption we made works since the real mode functions are a “sort" of expansion in the slow-roll and chirality parameters around the de-Sitter mode functions. However, it's important to note that providing a numerical verification of this assumption in this particular case is significantly more intricate, and as a result, we have chosen to omit it. Moreover, we stress once again that this assumption is exact in the case of the collapsed trispectrum when $\mathbf{K}_I \approx 0$.

6.5 The graviton-mediated trispectrum with de-Sitter mode functions

The next step one could be interested in is trying to understand if it's possible to generate a parity-violating signal using the vertices we have written in section 6.3.1 and 6.3.2 without the suppression introduced by the chirality parameters. We won't delve deeply into this aspect, but we aim to provide some comments that could prove valuable for a future analysis. The key point we aim to demonstrate is that, regardless of the specific form of the coupling function in the vertices, it is impossible to violate parity in a trispectrum involving two vertices originating from $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$ and the de-Sitter mode functions. Thus, we must consider a trispectrum in which we consider a standard GR vertex and one of the vertices we have obtained in the expansion of the Lagrangians. We are going to show this with a specific example.

Now, in order to perform the computation we need to write down the diagrammatic rules for the $In - In$ formalism. Thus, we need the vertex contribution for the two terms we are considering. Starting form the the vertex arising from the standard GR we have position space

$$S_{int}^{(1)} = \int d\tau d^3x \epsilon M_{pl}^2 a^2(\tau) h^{ij} \partial_i\zeta\partial_j\zeta, \tag{6.124}$$

which, writing the scalar and tensor fields in Fourier space, becomes

$$S_{int}^{(1)} = M_{pl}^2 \int \frac{d\tau d^3x}{(H\tau)^2} \frac{d^3k e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^3} \frac{d^3q e^{i\mathbf{q}\cdot\mathbf{x}}}{(2\pi)^3} \frac{d^3K e^{i\mathbf{K}\cdot\mathbf{x}}}{(2\pi)^3} ik_f iq_l u_\zeta(k,\tau) u_\zeta(q,\tau) \sum_h \epsilon^{lf}(\mathbf{K}) u_h(K,\tau) \tag{6.125}$$

$$= -(2\pi)^3 M_{pl}^2 \int \frac{d\tau}{(H\tau)^2} \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3K}{(2\pi)^3} k_f q_l u_\zeta(k,\tau) u_\zeta(q,\tau) \sum_h \epsilon^{lf}(\mathbf{K}) u_h(K,\tau) \delta^3(\mathbf{k}+\mathbf{q}+\mathbf{K}). \tag{6.126}$$

According to [14] we can write the relative vertex diagrammatic rule in Fourier space as

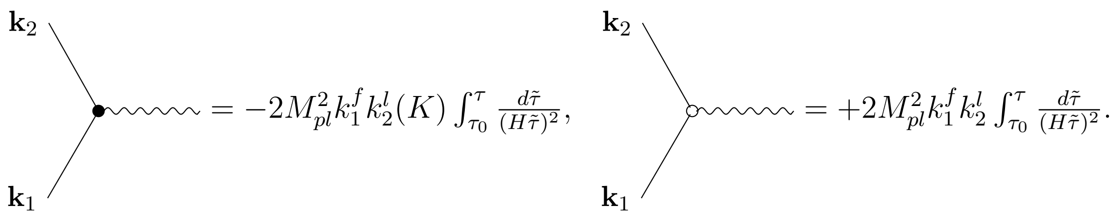


Figure 24: Diagrammatic rule for the interaction vertex

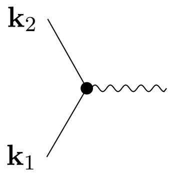
Now, we need the interaction rules for the other vertex

$$S_{int}^{(2)} = -\frac{2\epsilon^2}{M_{pl}}\left(\frac{H^3}{M_{pl}^4}\right)\int d\tau d^3x b_3 \epsilon^{ijl}\dot{\zeta}_{,j}\dot{\zeta}_{,lm}h^m{}_i \tag{6.127}$$

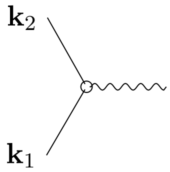
which is the first contribution we have written in eq.(6.88). This is not exactly the first contribution but is one of the terms we get once we integrate by parts to remove the temporal derivative from the tensor perturbations. Writing the scalar and tensor fields in Fourier space, the vertex becomes

$$\begin{aligned} S_{int}^{(2)} = &+(2\pi)^3\, i\frac{2\epsilon^2}{M_{pl}}\left(\frac{H}{M_{pl}}\right)^3\int d\tau d^3x H\tau b_3 \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{d^3K}{(2\pi)^3}k_j q_l q_m u'_\zeta(k,\tau)u'_\zeta(q,\tau)\times \\ &\times\sum_h [\epsilon_h]^m{}_i(\mathbf{K})u'_h(K,\tau)\delta^3(\mathbf{k}+\mathbf{q}+\mathbf{K}), \end{aligned} \tag{6.128}$$

The relative vertex diagrammatic rule in Fourier space can be written as



$$= +i\frac{4\epsilon^2}{M_{pl}}\left(\frac{H}{M_{pl}}\right)^3\epsilon^{ijl}k_j q_l q_m\int d\tau b_3 H\tau u'_\zeta(k,\tau)u'_\zeta(q,\tau),$$



$$= -i\frac{4\epsilon^2}{M_{pl}}\left(\frac{H}{M_{pl}}\right)^3\epsilon^{ijl}k_j q_l q_m\int d\tau b_3 H\tau u'_\zeta(k,\tau)u'_\zeta(q,\tau),$$

Figure 25: Diagrammatic rule for the interaction vertex

where we have also a factor of two due to symmetry factor. The key distinction compared to the Chern-Simons case is the inclusion of an i factor in one of the two vertices. Consequently, in the total polarization factor, we introduce an i factor that is absent when the sum of the momenta in the two vertices is even. It's important to have an overall factor of i in the polarization portion not have a single factor of i in the vertex itself. If we use two vertices of the Lagrangian of $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$ we always have a polarization factor that contains an even number of momenta and we do not get any overall i contribution. This is crucial when we write down the explicit expression of the real and imaginary part of the trispectrum.

Now we proceed in the computation of the trispectrum following what we have done in chapter 5, we know that each channel comes in four combinations since we have two vertices (each vertex can be of the + or − type which are respectively represented with a dot and an empty dot)

Hence, we can calculate the contribution arising from the four diagrams for the general channel I

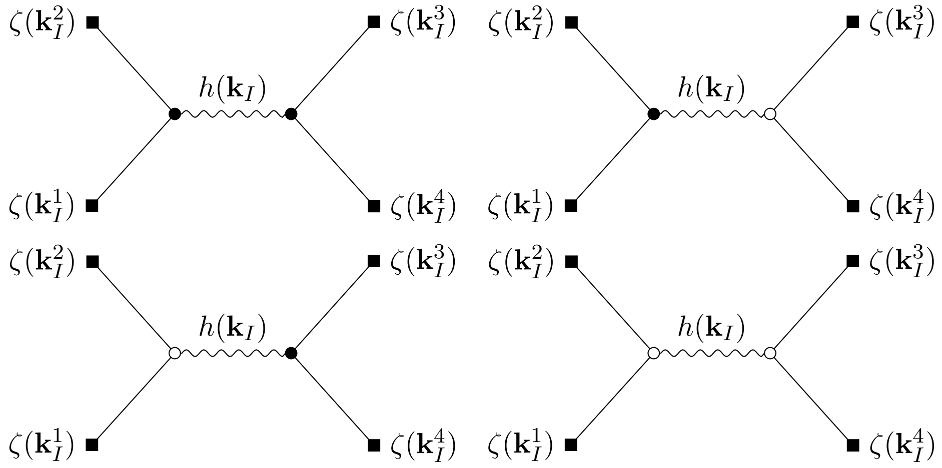


Figure 26: The four possibilities for the s channel

utilizing the established rules:

$$\text{first} = -i8\epsilon^2 M_{pl}\left(\frac{H}{M_{pl}}\right)^3 \sum_h P_h(\mathbf{K}_I)\mathcal{J}_h^{(2)}(\mathbf{k}), \tag{6.129}$$

$$\text{second} = i8\epsilon^2 M_{pl}\left(\frac{H}{M_{pl}}\right)^3 \sum_h P_h(\mathbf{K}_I)\bar{\mathcal{J}}_h^{(1)}(\mathbf{K}_I), \tag{6.130}$$

$$\text{third} = i8\epsilon^2 M_{pl}\left(\frac{H}{M_{pl}}\right)^3 \sum_h P_h(\mathbf{K}_I)\mathcal{J}_h^{(1)}(\mathbf{K}_I), \tag{6.131}$$

$$\text{fourth} = -i8\epsilon^2 M_{pl}\left(\frac{H}{M_{pl}}\right)^3 \sum_h P_h(\mathbf{K}_I)\bar{\mathcal{J}}_h^{(2)}(\mathbf{k}), \tag{6.132}$$

where, the bar is the complex conjugate operation, and where, we have introduced

$$P_h(\mathbf{K}_I) = {\epsilon^h}_{ij}(\mathbf{K}_I)\left(K_I^1\right)^i \left(K_I^2\right)^j \epsilon^{abc}\left[{\epsilon^h}_{af}(\mathbf{K}_I)\right]^* \left(K_I^3\right)^b \left(K_I^4\right)^c \left(K_I^4\right)^f . \tag{6.133}$$

For the time integrals the following notation has been adopted

$$\mathcal{J}_h^{(2)}(\mathbf{K}_I) = \int \frac{d\tau_1 d\tau_2 b_3(\tau_2)\tau_2}{H\tau_1^2}\mathcal{G}_+(\mathbf{k}_I^1,\tau_1)\mathcal{G}_+(\mathbf{k}_I^2,\tau_1)\mathcal{G}'_+(\mathbf{k}_I^3,\tau_2)\mathcal{G}'_+(\mathbf{k}_I^4,\tau_4)u_h[\mathbf{k}_I,\max(\tau_1,\tau_2)]u_h^*[\mathbf{k}_I,\min(\tau_1,\tau_2)], \tag{6.134}$$

$$\mathcal{J}_h^{(1)}(\mathbf{K}_I^1) = \int \frac{d\tau_1 d\tau_2 \tau_2}{H\tau_1^2}\mathcal{G}_+(\mathbf{k}_I^1,\tau_1)\mathcal{G}_+(\mathbf{k}_I^2,\tau_1)u_h^*(\mathbf{k}_I,\tau_1)\mathcal{G}'_+(\mathbf{k}_I^3,\tau_2)\mathcal{G}'_+(\mathbf{k}_I^4,\tau_2)u_h^*(\mathbf{k}_I,\tau_2), \tag{6.135}$$

$$C_{var_h}(\mathbf{K}_I) = \Re\left(\mathcal{J}_h^{(2)}(\mathbf{K}_I) - \mathcal{J}_h^{(1)}(\mathbf{K}_I)\right), \tag{6.136}$$

where the scalar propagator, $\mathcal{G}_+$, and the tensor mode function,u_h, can be computed using the result of section 6.2. Summing everything together we get

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle = -i8\epsilon^2 M_{pl}\left(\frac{H}{M_{pl}}\right)^3 \sum_{I,h} P_h(\mathbf{K}_I)C_{var_h}(\mathbf{K}_I), \tag{6.137}$$

Now, we can put the real part and imaginary part of the trispectrum as in eq.(5.165) and eq.(5.166) but with the crucial difference that we have a i prefactor

$$\Re\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle = 8\epsilon^2 M_{pl}\left(\frac{H}{M_{pl}}\right)^3 \sum_I \Im\left(P_R(\mathbf{K}_I)\right)\left[C_{var_R}(\mathbf{K}_I) - C_{var_L}(\mathbf{K}_I)\right], \tag{6.138}$$

$$\Im\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle = -8\epsilon^2 M_{pl}\left(\frac{H}{M_{pl}}\right)^3 \sum_I \Re\left(P_R(\mathbf{K}_I)\right)\left[C_{var_R}(\mathbf{K}_I) + C_{var_L}(\mathbf{K}_I)\right]. \tag{6.139}$$

Now, it's clear that in order to compute the leading order contribution we can use the de-Sitter mode function; the real one are expansion in slow-roll and chirality parameters around this solution. This is the most important difference with respect to the standard GR vertices. It's possible to compute the integral by taking b_3 constant and we'll also consider it of order unity since we are dealing with an effective field theory and the result is exactly zero for this vertex. It's possible to use other vertices to obtain a non-zero contribution. In this case, we are not going to produce a parity-violation but the idea is the same also for the other vertices.

What we intend to convey is that when we examine the trispectrum with two vertices from the Lagrangians $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$, we do not introduce an additional i contribution in the polarization factor, and both the real and imaginary parts would take a similar form to the one we previously computed in the Chern-Simons scenario

$$\Re\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle \propto \sum_I \Im\left(P_R(\mathbf{K}_I)\right)\left[C_{var_R}(\mathbf{K}_I) + C_{var_L}(\mathbf{K}_I)\right], \tag{6.140}$$

$$\Im\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle \propto \sum_I \Re\left(P_R(\mathbf{K}_I)\right)\left[C_{var_R}(\mathbf{K}_I) - C_{var_L}(\mathbf{K}_I)\right], \tag{6.141}$$

where the explicit form of the polarization factors and the time integrals don't matter. In fact, if the propagators for the chiral polarization are identical we have that

$$C_{var_R}(\mathbf{K}_I) = C_{var_L}(\mathbf{K}_I), \tag{6.142}$$

which implies

$$\Im\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\zeta(\mathbf{k}_4)\rangle = 0. \tag{6.143}$$

Finally, it would be interesting to compute the graviton-mediated trispectrum using an explicit form for the coupling functions as the one proposed in [8] in order to understand if it's possible to produce an enhancement in the signal.

Conclusions

The starting point of this work is the detection of a connected four-point correlation function of the matter over-density field that exhibits parity violation [1, 2]. We would like to stress one more time that if these measurements are confirmed and proven to be primordial in origin, they will have two significant implications. Firstly, discovering a connected correlator beyond the two-point statistic provides evidence for primordial non-Gaussiany. Furthermore, considering that weak forces are not involved in the formation of large-scale structures, such evidence would indicate the presence of new physics.

Given our current absence of primordial tensor mode detections, the scalar sector remains the sole domain in which we can investigate parity violation. Concerning scalars, as we have seen, the first correlator capable of exhibiting parity violation is the four-point correlation function. Moreover, using the approach introduced in [5], we have built the estimator for the galaxy four-point correlation function in chapter 3. This estimator enables us to significantly decrease the computational expense, reducing it from $O(N_g^4)$ to $O(N_g^2)$, where N_g represents the number of galaxies in the survey. Consequently, it empowers us to measure and compute this correlation function, which would otherwise be challenging due to computational costs.

Moreover, I'd like to emphasize that the signal may be spurious, and there is no a priori reason to assume that this signal originates from early universe physics. It could have its roots in astrophysical sources or late-time cosmological effects. However, the huge amount of data we expect from future experiments such as $DESI$, $EUCLID$, and $RUBIN$, galaxy surveys, would provide a natural environment to study scalar correlation functions and in which to hunt for parity-violating signatures. These upcoming surveys will provide larger samples and different systematics concerning the $BOSS$ catalog. This will enable us to gain a better understanding of the true origin of the signal. If the evidence is of cosmological origin, it would likely be detectable in $DESI$. Discovering evidence in $DESI$ would serve as a compelling indicator that either the signal is authentic, or it is caused by some yet-undiscovered parity-breaking systematic shared between BOSS and $DESI$. However, considering the technical differences in the instrumentation used in the two surveys, the latter possibility also appears unlikely.

I also would like to stress that this effect of parity violation in the CMB is not observed [73]. This could imply two possibilities: either the signal identified in the BOSS galaxy catalog is spurious, or parity violation occurs at significantly different scales. We observe parity violation in the formation of large-scale structures, but we do not detect parity violation on the scales of the cosmic microwave background. The second scenario would be very interesting and challenging from a theoretical point of view.

Then, we try to provide an Inflationary model that can leave a parity-violating signature in the scalar trispectrum of the curvature perturbation. Thus, following an introduction to cosmological fundamentals, including $FLRW$ dynamics, perturbation theory, correlation functions, and the gauge-invariant curvature perturbation on a uniform energy density hypersurface denoted as ζ in chapter 1, we redirect our attention to the inflationary phase. Our primary objective is to explain the observed signal as a remnant signal of parity violation occurring in the early universe. Firstly we review the shortcomings of the HBB model and their inflationary solution. Then, we present the single-field slow-roll model of inflation under the slow-roll hypothesis [10]. The inflationary background dynamics mimic the behavior of a Cosmological constant component giving rise to an accelerated expansion period. Then, we present a detailed and self-consistent analysis within the ADM formalism, the Hamiltonian formulation of GR, of the tensor and ζ power spectra on super-horizon scales. We also stress the importance of a detection of GWs background since it wuold be crucial insights into the dynamics of Inflation.

Prior to delving into the computation of the parity-violating contribution within the framework of modified theories of gravity, we introduce the Schwinger-Keldysh diagrammatic rules for primordial perturbations [14, 17]. These rules enable the computation of primordial correlation functions using the $In - In$ formalism, similar to how $S-$matrix elements are evaluated using Feynman diagrams in particle physics and flat spacetimes.

Then, we start the analysis of modified gravity models in order to see if it's possible to produce a parity-violating signature in today's observables. First of all, we consider the Chern-Simons theory of gravity, which is constructed as an effective field theory (EFT) for gravitation. The interaction Lagrangian we add to the standard Inflationary one is of the form

$$\mathcal{L}_{int} = \sqrt{-g}\frac{\phi}{4f} * RR = \frac{\phi}{8f}\epsilon^{\alpha\beta\sigma\rho}R^{\mu\nu}{}_{\alpha\beta}R_{\mu\nu\rho\sigma}, \tag{6.144}$$

where f is a dimensional constant. We provide a comprehensive and thorough derivation of the equations of motion and power spectra for both scalar and tensor modes within the spatially flat gauge. The main feature of this kind of model is that it introduces a modification of the equation of motion for the left and right polarizations for gravitons. This modification depends on the so-called chirality parameter

$$\mu = \frac{H}{M_{CS}} << 1, \tag{6.145}$$

and in the limit in which μ goes to zero, we recover the standard single-field model of slow-roll Inflation. This modification of the EoM results in different propagators for left and right gravitons. This can source parity violation in the trispectrum, which is the Fourier transform of the four-point correlation function. Furthermore, the chirality parameter introduces variations in the primordial power spectra for tensor modes, resulting in modifications to the scalar-to-tensor ratio, the spectral index, and the overall power spectrum for tensor modes. Nevertheless, as we have observed, these effects remain unobservable due to their suppression by the smallness of the chirality parameter.

Then, we present a detailed analysis of the computation of the graviton-mediated trispectrum in this model working within the ADM formalism in the spatially-flat gauge. The two vertices that contribute to the scalar trispectrum are

$$S^{(1)}_{int} = -\int d\tau d^3x \frac{1}{2}a^2(\tau)h^{ij}\partial_i\delta\phi(\tau,\mathbf{x})\partial_j\delta\phi(\mathbf{x},\tau), \quad S^{(2)}_{int} = -\int d\tau d^3x \frac{2}{fM_{pl}}\sqrt{\epsilon}\left(\partial^l\delta\phi\right)\epsilon^{ijk}\left[\left(\partial_k\delta\phi\right)\partial_i h^{'}_{lj}\right]. \tag{6.146}$$

We have shown that the interference term and the purely Chern-Simons contribution, $S^{(2)}_{int}$, are respectively zero and suppressed by $\frac{\epsilon H^2}{M^2_{CS}}$ with respect to the purely kinetic contribution which we have computed using the approximation made in [9] for the mode functions. Basically, we approximate the Chern-Simons mode functions with the de-Sitter ones multiplied by a constant which is different for left and right gravitons. This constant is obtained by matching these approximate solutions with the original ones in $-k\tau = 0$. This, as we have verified, it's a reasonable assumption. Thus, the main contribution to the trispectrum arises when considering two vertices of the type $S^{(1)}_{int}$. The main feature in this case is that the trispectrum violates parity since the propagators for left and right gravitons are different. The signal we get in this particular model is too weak due to chirality suppression. As discussed in [9], in models that go beyond the standard inflationary scenario with dynamical Chern-Simons term, such as those involving multiple scalar fields or superluminal scalar sound speed, it is possible to encounter a substantial enhancement factor $F \geq 10^6$ for the trispectrum. In the paper, the author does not provide a comprehensive discussion on this topic. Therefore, it would be worthwhile to conduct a thorough investigation into such models to comprehend how they can generate such a significant enhancement, making the signal detectable.

Hence, we investigate alternative theories in which, in principle, there is a possibility of enhancement owing to gravitational waves (GWs) birefringence. These theories extend Chern-Simons gravity by incorporating parity-violating operators that involve the first and second derivatives of the non-minimally coupled inflaton field such as

$$f(\phi)\epsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}R_{\mu\lambda}{}^{\rho}{}_{\sigma}\nabla^{\sigma}\phi\nabla^{\lambda}\phi, \quad g(\phi)\epsilon^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}R^{\mu\lambda\rho\sigma}\nabla^{\rho}\phi\phi_{\mu}\nabla^{\sigma}\nabla_{\lambda}\phi. \tag{6.147}$$

To begin, we calculate the modifications made to the power spectra of tensors, and we discover a situation that mirrors the one discussed in the context of the Chern-Simons theory of gravity. We observe changes in the tensor power spectra, yet these modifications are unobservable. Then, we provide an original calculation of the graviton-mediated trispectrum within the framework of chiral scalar-tensor theories of gravity [8, 19]. The procedure for the scalar trispectrum is identical to the one we have presented for the Chern-Simons theory. All the couplings introduced by these new theories are largely suppressed with respect to the standard gravity term, which is analogous to the kinetic term in the Chern-Simons theory. Given that the suppression of the trispectrum results from the smallness of the chirality parameter and the fact that the theories are constructed as effective field theories, it would be intriguing to further explore the scenario where this chirality parameter goes to zero. For example in the case of $\mathcal{L}_{PV_1}$ it's sufficient to send $f+g=0$ in order to have $M_{PV_1} \longmapsto +\infty$, (see [8] for details). In this case, we can use de-Sitter mode functions in order to assess whether enhancement factors become possible. In this scenario, we have demonstrated the necessity of considering an interference term between the standard GR term and a term originating from the expansion of the Lagrangians $\mathcal{L}_{PV_1}$ and $\mathcal{L}_{PV_2}$. Otherwise, we get a zero parity-violating part of the trispectrum. Another intriguing aspect of this scenario involves attempting to assign specific functional forms to the coupling functions appearing in the vertices, such as f and g in equation (6.147), and trying to understand if it's possible to get an enhancement in this case.

Currently, aside from the modification of the Chern-Simons theory outlined in [9], there are no other instances in the literature of inflationary models that can produce a signal capable of explaining the observations reported in [1, 2]. Nonetheless, as outlined in [74], it has been shown that it's impossible to generate detectable signals for a series of inflationary models.

A Appendix A

A.1 Friedmann equation

Now we derive a useful form for the first Friedmann equation (1.12) with matter, radiation, and cosmological constant components

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} = \frac{8\pi G}{3}\left[\rho_M + \rho_R + \rho_\Lambda\right] - \frac{\kappa}{a^2}. \tag{A.1}$$

This equation will be useful in the derivation of the proper distance and the angular diameter distance in section A.2 and A.3. From the continuity equation (1.15) we know that

$$\rho_M = \rho_{0M}\left(\frac{a_0}{a}\right)^3 (w=0), \quad \rho_R = \rho_{0R}\left(\frac{a_0}{a}\right)^4 (w=\frac{1}{3}), \quad \rho_\Lambda = \rho_{\Lambda 0} = const(w=-1), \tag{A.2}$$

thus we can write

$$H^2 = \frac{8\pi G}{3}\left[\rho_{0M}\left(\frac{a_0}{a}\right)^3 + \rho_{0R}\left(\frac{a_0}{a}\right)^4 + \rho_{0\Lambda} - \frac{3\kappa}{8\pi G}a^{-2}\right] \tag{A.3}$$

$$= \frac{H_0^2}{H_0^2}\frac{8\pi G}{3}\left[\rho_{0M}\left(\frac{a_0}{a}\right)^3 + \rho_{0R}\left(\frac{a_0}{a}\right)^4 + \rho_{0\Lambda} - \frac{3\kappa}{8\pi G}a^{-2}\right] \tag{A.4}$$

$$= H_0^2\left[\Omega_{0M}\left(\frac{a_0}{a}\right)^3 + \Omega_{0R}\left(\frac{a_0}{a}\right)^4 + \Omega_{0\Lambda} + \Omega_{0\kappa}\left(\frac{a_0}{a}\right)^2\right] \tag{A.5}$$

$$= H_0^2\left[\Omega_{0M}(1+z)^3 + \Omega_{0R}(1+z)^4 + \Omega_{0\Lambda} + \Omega_{0\kappa}(1+z)^2\right], \tag{A.6}$$

where we have used that $\frac{a_0}{a} = (1+z)$ with z redshift [33]. Moreover, we have introduced the energy parameter for each component today which is defined as

$$\Omega_{0i} = \frac{\rho_{0i}}{\rho_{0c}}, \tag{A.7}$$

where, the index i represents matter, radiation, or cosmological constant, while ρ_c denotes the present-day critical density. This critical density signifies the density that the universe would possess if it were flat

$$\rho_{0c} = \frac{3H_0^2}{8\pi G}. \tag{A.8}$$

A.2 The proper distance

To begin with, let's consider a light signal traveling from a luminous source to our location, which we take to be at the origin of the polar coordinates system. Due to isotropy the signal travels along radial null geodesic with $ds^2 = d\Omega^2 = 0$, so that we can derive the following differential equation

$$\frac{dt}{a(t)} = \pm\frac{dr}{\sqrt{1-\kappa r^2}}, \tag{A.9}$$

where the choice of $\pm$ depends on the interval of integration. If we integrate from (t_1, r_1) to $(t_2 > t_1, r_2 > r_1)$ or $(t_2 < t_1, r_2 < r_1)$, we use the plus sign; otherwise, we use the minus sign. Thus integrating eq.(A.9) we are able to relate the radial comoving coordinate with the emission time of a light signal

$$\int_{t_0}^{t}\frac{d\tilde{t}}{a(\tilde{t})} = -\int_0^r \frac{d\tilde{r}}{\sqrt{1-\kappa\tilde{r}^2}}, \tag{A.10}$$

which using the following change of coordinates

$$\begin{cases} r = \sinh\chi & \text{if } k=-1 \\ r = \chi & \text{if } k=0 \\ r = \sin\chi & \text{if } k=+1 \end{cases}, \tag{A.11}$$

becomes

$$\int_{t_0}^{t} \frac{d\tilde{t}}{a(\tilde{t})} = -\int_{\chi(0)}^{\chi(r)} d\chi = -\chi(r). \tag{A.12}$$

Now recalling the definition of the redshift, $(1+z) \equiv \frac{a_0}{a(t)}$ we get that

$$\frac{d}{dt}(1+z) = \frac{d}{dt}(\frac{a_0}{a}) = -\frac{a_0 \dot{a}}{a^2}, \tag{A.13}$$

which we can recast in the following way

$$\frac{dz}{H(z)a_0} = -\frac{dt}{a(t)}. \tag{A.14}$$

If we want to compute the radial distance of a source observed with redshift z, eq.(A.12) we can write that

$$r(z) = S\left[\int_{t(z)}^{t_0} -\frac{dt}{a(t)}\right] = S\left[\int_{z}^{0} \frac{dz}{H(z)a_0}\right] \tag{A.15}$$

$$= S\left[\frac{1}{a_0 H_0}\int_0^z \frac{dz}{\sqrt{\left[\Omega_{0M}(1+z)^3 + \Omega_{0R}(1+z)^4 + \Omega_{0\Lambda} + \Omega_{0\kappa}(1+z)^2\right]}}\right], \tag{A.16}$$

where we have used the explicit expression for $H(z)$, i.e. eq.(A.6). Eq.(A.16) can be expressed in a more convenient way considering that $\Omega_\kappa = -\frac{\kappa}{H_0^2 a_0^2}$,

$$a_0 r(z) = \frac{1}{H_0 \Omega_\kappa^{\frac{1}{2}}} \sinh\left\{\Omega_\kappa^{\frac{1}{2}} \int_0^z \frac{dz}{\sqrt{\left[\Omega_{0M}(1+z)^3 + \Omega_{0R}(1+z)^4 + \Omega_{0\Lambda} + \Omega_{0\kappa}(1+z)^2\right]}}\right\}, \tag{A.17}$$

where in the case $\kappa = 0$ the limit $\Omega_\kappa \to 0$ is used.

A.3 The angular-diameter distance

In this section, we introduce another kind of distance, *angular diameter distance*, d_A, which allows us to compare angular sizes with physical dimensions. Let's consider two points A and B at spatial coordinates

$$\begin{cases} (r_A, \theta_A, \phi_A) = (r, \theta_A, 0) \\ (r_B, \theta_B, \phi_B) = (r, \theta_B, 0) \end{cases}, \tag{A.18}$$

which both emit lights at time t_1 and they are observed at present time t_0. They subtend, in the sky, an angle with modulus

$$\mid \theta \mid \equiv \mid \theta_A - \theta_B \mid \tag{A.19}$$

and they are separated by a proper distance given by

$$s = a(t_1) r_1 \theta. \tag{A.20}$$

We define the angular diameter distance so that

$$d_A = \frac{s}{\theta} = a(t_1) r_1. \tag{A.21}$$

This definition makes sense only if the two points are at the same radial coordinate, for example in the case of CMB emission we know that all the points lie on a surface. In general, this definition is used in the small angle approximation to compute the physical extension of a source in the sky.

A.4 Perturbation theory

In this section, we present the theorems' statements, omitting the proofs, which we utilize in deriving the gauge transformation of perturbations.

Theorem A.1. *Let ξ be a vector field on a differential manifold $\mathcal{M}$, generating a flow $\phi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, where $\phi(0,p) = p$, $\forall p \in \mathcal{M}$. $\forall\ \lambda \in \mathbb{R}$ and $\forall p \in \mathcal{M}$, we write $\phi_\lambda(p) := \phi(\lambda, p)$. Let T be a tensor field on $\mathcal{M}$. The map ϕ^*_λ defines a new field $\phi^*_\lambda T$ on $\mathcal{M}$, the pull-back of T , which is thus a function of λ. The field $\phi^*_\lambda T$ admits the following expansion around $\lambda = 0$:*

$$\phi^*_\lambda T = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \left.\frac{d^k}{d\lambda^k}\right|_{\lambda=0} \pounds^k_\xi T. \tag{A.22}$$

Theorem A.2. *Let $\mathcal{M}$ a differential manifold and $\Psi : \mathcal{R} \times \mathcal{M} \longmapsto \mathcal{M}$ a one-parameter family of diffeomorphisms. Then $\exists \phi^{(1)}, .., \phi^{(k)}, ..$ one parameter groups of diffeomorphisms of $\mathcal{M}$ with associated vector fields $\zeta_{(1)}, .., \zeta_{(k)}, ..$ such that*

$$\Psi_\lambda = .. \circ \phi^{(k)}_{\frac{\lambda^k}{k!}} \circ .. \circ \phi^{(2)}_{\frac{\lambda^2}{2}} \circ \phi^{(1)}_\lambda. \tag{A.23}$$

According to what is presented in [11] we call this kind of decomposition a knight diffeomorphism. We stress that Ψ_λ is not a one-parameter group of diffeomorphisms but a one-parameter family of diffeomorphisms. Now we are ready to state the fundamental theorem used in obtaining the gauge transformation rules of section 1.2.

Theorem A.3. *Let $\mathcal{M}$ and T a tensor field defined on it. The pullback of $\Psi^*_\lambda T$ by a one-parameter family of knight diffeomorphisms Ψ with generators $\zeta_{(1)}, .., \zeta_{(k)}, ..$ can be expanded around $\lambda = 0$ as follows*

$$\Psi^*_\lambda T = \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} .. \sum_{l_k=0}^{+\infty} .. \frac{\lambda^{l_1+2l_2+..+kl_k+..}}{2^{l_2}..(k!)^{l_k}..l_1!l_2!..l_k!..} \pounds_{\zeta^{l_1}_{(1)}} .. \pounds_{\zeta^{l_k}_{(k)}} ..T. \tag{A.24}$$

B Appendix B

B.1 Spherical Harmonics

We introduce the spherical harmonics in the Condon-Shortley convection as

$$Y_{lm}(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(cos\theta)e^{im\phi}, \tag{B.1}$$

where $l \in \mathbf{N}$, $m \in Z$ $m \in [-l, l]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ and

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l, \tag{B.2}$$

with $x \in [-1, 1]$. In the form of eq.(B.1) the spherical harmonics are an orthonormal basis of the vector space $L^2(S^2)$, where S^2 is the unit sphere. The orthonormality condition reads

$$\int d\Omega Y_{lm}(\theta,\phi)Y^*_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{B.3}$$

where we have defined the scalar product as

$$\forall f, g \in L^2(S^2) \quad (f(\theta,\phi), g(\theta,\phi)) = \int d\Omega f(\theta,\phi)g^*(\theta,\phi). \tag{B.4}$$

While the fact that they are a basis implies that $\forall f \in L^2(S^2)$ we can write

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi), \tag{B.5}$$

where

$$a_{lm} = \int d\Omega Y^*_{lm}(\theta,\phi)f(\theta,\phi). \tag{B.6}$$

Now we analyze the behavior of spherical harmonics under the parity operator, $\mathbb{P}$, which acts on vectors as $\forall \mathbf{x} \in \mathbb{R}^3$ $\mathbb{P}[\mathbf{x}] = -\mathbf{x}$, which in spherical coordinates (fig.27) reads

$$\mathbb{P}[\mathbf{x}](r,\theta,\phi) = \mathbf{x}(r, \pi-\theta, \phi+\pi). \tag{B.7}$$

So under parity, we get

$$\mathbb{P}[Y_{lm}] = \mathbb{P}[NP_l^m(cos\theta)e^{im\phi}] = NP_l^m(-cos\theta)e^{im(\phi+\pi)} = (-1)^m(-1)^{m+l}Y_{ml}(\theta,\phi) = (-1)^l Y_{ml}(\theta,\phi), \tag{B.8}$$

where we have defined $N = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}$ and we have used

$$P^l_m(-x) = (-1)^l P^l_m(x), \tag{B.9}$$

which can be deduced from

$$P_l^m(z=-x) = \frac{1}{2^l l!}(1-z^2)^{\frac{m}{2}}\frac{d^{l+m}}{dz^{l+m}}(z^2-1)^l = \frac{1}{2^l l!}(1-x^2)^{\frac{m}{2}}(\frac{dx}{dz})^{l+m}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l = (-1)^{l+m}P_l^m(x). \tag{B.10}$$

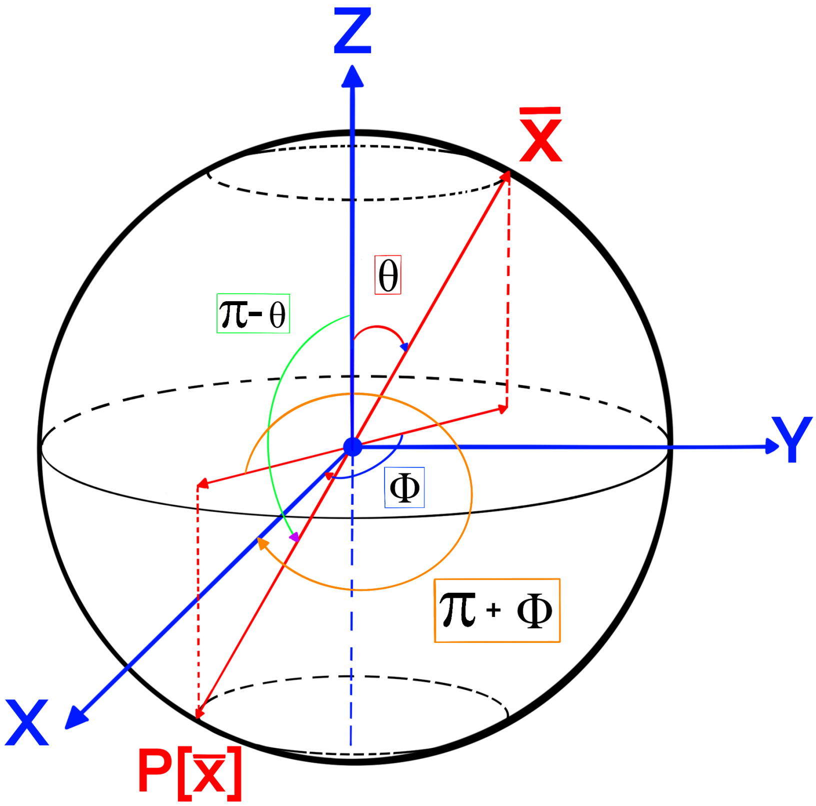


Figure 27: Transformation of the vector $\mathbf{x}$ under parity, $\mathbb{P}$.

B.2 Addition of angular momenta and Clebsh-Gordan coefficients

Consider a system, whose state space is χ, and an angular momentum operator $\mathbf{J}$ relative to this system. Since $[J^2, J_z] = 0$, it's always possible to construct a basis of common eigenvectors of these two operators:

$$J^2|j,m\rangle = j(j+1)|j,m\rangle, \\ J_z|j,m\rangle = m|j,m\rangle, \tag{B.11}$$

with $m \in [-j, j]$ and j could be a fractional or integer number. Since it's a basis we know that $\chi = span\{|j,m\rangle\}$. We denote by $\chi(j)$ the vector space spanned by the set of vectors of the standard basis which correspond to fixed values j. There are $2j+1$ of these vectors in this subspace and we know that the entire space can be considered as the direct sum of these subspaces

$$\chi = \sum_{\oplus} \chi(j). \tag{B.12}$$

We recall that J^2 and J_z are block diagonal with respect to this decomposition of χ.

Now consider a system formed by the union of two subsystems, we refer to the subsystem 1\2 respectively with an index 1\2. The state space of the global system is the tensor product of the individual space states

$$\chi = \chi_1 \otimes \chi_2, \tag{B.13}$$

and we know that a basis of the total system is constructed taking the tensor product of the bases chosen in χ_1 and χ_2:

$$|j_1, m_1, j_2, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle. \tag{B.14}$$

Since the decomposition of eq.(B.12) holds for the vector spaces χ_1 and χ_2 we obtain

$$\chi = \sum_{\oplus} \chi_1(j_1) \otimes \sum_{\oplus} \chi_2(j_2) = \sum_{\oplus} \chi_{1,2}(j_1 j_2). \tag{B.15}$$

The space $\chi_{1,2}(j_1 j_2) = \chi_1(j_1) \otimes \chi_2(j_2)$ has dimensions $(2j_1+1)(2j_2+1)$ and it's invariant under the actions of any functions of $\mathbf{L_1} \equiv L_1 \otimes \mathfrak{I}_1$ and $\mathbf{L_2} \equiv \mathfrak{I}_1 \otimes L_2$.

Typically, since $L^2 \equiv (\mathbf{L}_2 + \mathbf{L}_1)^2, L_1^2, L_2^2$ and $L_z \equiv (L_1)_z + (L_2)_z$ commutes, a change of basis is performed such that

$$L^2|L, m, L_1, L_2\rangle = L(L+1)|L, m, L_1, L_2\rangle, \tag{B.16}$$
$$L_z|L, m, L_1, L_2\rangle = m|L, m, L_1, L_2\rangle, \tag{B.17}$$
$$L_1^2|L, m, L_1, L_2\rangle = L_1(L_1+1)|L, m, L_1, L_2\rangle, \tag{B.18}$$
$$L_2^2|L, m, L_1, L_2\rangle = L_2(L_2+1)|L, m, L_1, L_2\rangle, \tag{B.19}$$

where it can be shown that $|L_1 - L_2| \leq L \leq L_1 + L_2$. The Clebsch-Gordan coefficients are the coefficients that allow us to pass from the standard basis, the tensor product of the two bases in χ_1 and χ_2, to $|L, M, L_1, L_2\rangle$. In fact, we can write

$$|L, M, L_1, L_2\rangle = \sum_{m_1=-L_1}^{L_1} \sum_{m_2=-L_2}^{L_2} \langle L_1, m_1, L_2, m_2|L, M, L_1, L_2\rangle |L_1, m_1, L_2, m_2\rangle, \tag{B.20}$$

where we define $\langle L_1, m_1, L_2, m_2|L, M, L_1, L_2\rangle$ as Clebsch-Gordan coefficients. We haven't summed over the indexes j_1 and j_2 since the change of basis can be performed individually inside each of the $\chi_{1,2}(j_1, j_2)$ since eq.(B.15) holds. We recall that the operators L^2, L_1^2, L_2^2 and L_z are block diagonal with respect to the decomposition of eq.(B.15).

To conclude this section we recall some basic properties that can be easily demonstrated ([56]) or imposed

1. $\langle L_1, m_1, L_2, m_2|L, M, L_1, L_2\rangle = \langle L, M, L_1, L_2|L_1, m_1, L_2, m_2\rangle$,
2. $\sum_{L,m} \langle L_1, m_1, L_2, m_2|L, M, L_1, L_2\rangle \langle L, M, L_1, L_2|L_1^{'}, m_1^{'}, L_2^{'}, m_2^{'}\rangle$
$= \delta_{L_1^{'} L_1} \delta_{L_2^{'} L_2} \delta_{m_1^{'} m_1} \delta_{m_2^{'} m_2} \delta(L, L_1, L_2)$, where

$$\begin{cases} \delta(L, L_1, L_2) = 1 & \text{if } |L_1 - L_2| \leq L \leq L_1 + L_2 \\ \delta(L, L_1, L_2) = 0 & \text{otherwise} \end{cases}, \tag{B.21}$$

3. $\sum_{m_1, m_2} \langle L_1, m_1, L_2, m_2|L, M, L_1, L_2\rangle \langle L^{'}, M^{'}, L_1, L_2|L_1, m_1, L_2, m_2\rangle = \delta_{LL'} \delta_{mm'} \delta(L, L_1, L_2)$.

B.3 Wigner or 3-j symbols

The Wigner symbol are defined in terms of the Clebsch-Gordan coefficient via

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1 j_2 m_2 | j_3 m_{-3} \rangle, \tag{B.22}$$

which can be inverted as

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = \sqrt{2j_3+1} (-1)^{j_1 - j_2 - m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{B.23}$$

In the special case in which $j_3 = m_3 = 0$, $j_1 = j_2 = l$ and $m_1 = -m_2 = m$ is possible to derive ([55])

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = \frac{(-1)^{j-m}}{\sqrt{2j+1}}. \tag{B.24}$$

The orthonormality condition reads ([5])

$$\sum_{m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_3 \\ m_4 & m_2 & m_3 \end{pmatrix} = \frac{\delta_{j_1 j_4}\delta_{m_1 m_4}}{2j_1+1}, \tag{B.25}$$

where δ_{ab} is the Kronecker and summing over m_1 we get

$$\sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_3 \\ m_4 & m_2 & m_3 \end{pmatrix} = 1. \tag{B.26}$$

The symbols have the following symmetries

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} \tag{B.27}$$

$$= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{B.28}$$

The symbols can also be taken to be real since the Clebsch-Gordan coefficients can be taken real ([56]).

C Appendix C

C.1 Quasi de Sitter Universe

In a quasi de Sitter stage of the universe, we consider a slowly varying Hubble parameter, which is

$$\epsilon = -\frac{\dot{H}}{H^2} \neq 0. \tag{C.1}$$

It's possible to show that by switching to conformal time τ we can write

$$\tau \approx -\frac{1}{aH(1-\epsilon)} \approx -\frac{1}{aH}(1+\epsilon). \tag{C.2}$$

Now, we derive some useful expressions. We start by computing at first order in slow the first and second derivatives of the expansion parameter $a(\tau)$

$$a^{'}(\tau) = +\frac{1}{\tau^2 H(1-\epsilon)} - \frac{1}{\tau(1-\epsilon)}a(\tau)\epsilon, \tag{C.3}$$

$$a^{''}(\tau) = -\frac{2\left[1+\frac{3}{2}\epsilon\right]}{\tau^3 H(1-\epsilon)}, \tag{C.4}$$

where we have used that

$$\frac{d}{d\tau}\left(\frac{1}{H}\right) = a(t)\frac{d}{dt}\left(\frac{1}{H}\right) = -a(t)\epsilon. \tag{C.5}$$

We conclude this section by computing the ratio between the second derivative of the scale factor and the scale factor itself

$$\frac{a^{''}(\tau)}{a(\tau)} = \frac{2\left[1+\frac{3}{2}\epsilon\right]}{\tau^3 H(1-\epsilon)}H\tau(1-\epsilon) \approx \frac{2}{\tau^2}\left[1+\frac{3}{2}\epsilon\right]. \tag{C.6}$$

C.2 Bessel's equation

The goal of this section is to show that the following equation

$$u_k(\tau)^{''} + \left[k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right]u_k(\tau) = 0, \quad \nu^2 > 0,\ \tau < 0, \tag{C.7}$$

can be solved by means of Bessel's equation [55]. If we introduce a new variable

$$u_k(\tau) = \sqrt{-\tau}z(-k\tau), \tag{C.8}$$

$$u_k^{'}(\tau) = -k\sqrt{-\tau}z^{'}(-k\tau) - \frac{z(-k\tau)}{2\sqrt{-\tau}}, \tag{C.9}$$

$$u_k^{''}(\tau) = k^2\sqrt{-\tau}z^{''}(-k\tau) + k\frac{z^{'}(-k\tau)}{\sqrt{-\tau}} - \frac{z(-k\tau)}{4(-\tau)^{\frac{3}{2}}}, \tag{C.10}$$

where we also have directly computed the first and second derivatives. Plugging in the result in eq.(C.7) we get

$$0 = k^2\sqrt{-\tau}z^{''}(-k\tau) + k\frac{z^{'}(-k\tau)}{\sqrt{-\tau}} - \frac{z(-k\tau)}{4(-\tau)^{\frac{3}{2}}} + \left[k^2 - \frac{\nu^2-\frac{1}{4}}{\tau^2}\right]\sqrt{-\tau}z(-k\tau) \tag{C.11}$$

$$= k^2\sqrt{-\tau}z^{''}(-k\tau) + k\frac{z^{'}(-k\tau)}{\sqrt{-\tau}} + \left[k^2 - \frac{\nu^2}{\tau^2}\right]\sqrt{-\tau}z(-k\tau), \tag{C.12}$$

which, multiplying by $(-\tau)^{\frac{3}{2}}$ reads

$$0 = (-\tau k)^2 z^{''}(-k\tau) - k\tau\frac{z^{'}(-k\tau)}{\sqrt{-\tau}} + [k^2\tau^2 - \nu^2]z(-k\tau) = x^2 z^{''}(x) + xz^{'}(x) + (x^2-\nu^2)z(x), \tag{C.13}$$

where we have defined $x \equiv -k\tau$ in order to make clear that this a Bessel's equation [55].

As this result is utilized multiple times in the thesis, we solve this equation with the condition that in the sub-horizon regime, i.e., when $-k\tau \gg 1$, the following holds

$$u_k(\tau) \xrightarrow{-k\tau >> 1} \frac{1}{\sqrt{2k}} e^{-ik\tau}, \tag{C.14}$$

and also with the condition on the Wronskian which allows us to recover the standard commutation relation once we quantize,

$$u_k u_k^{*'} - u_k^* u_k^{'} = -i. \tag{C.15}$$

Please note that we can always impose two conditions since the differential equation is of second order.

Thus a general solution can be written using the Hankel functions [55] as follows

$$u_k(\tau) = C_1(k) H_\nu^{(1)}(-k\tau) + C_1(k) H_\nu^{(2)}(-k\tau), \tag{C.16}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are respectively the Hankel's function of the first and second kind. Since [55]

$$\begin{cases} H_\alpha^{(1)}(x) & \approx \left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \exp i\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \\ H_\alpha^{(2)}(x) & \approx \left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \exp i\left(-x + \frac{\alpha\pi}{2} + \frac{\pi}{4}\right) \end{cases}, \quad x >> 1, \tag{C.17}$$

we must impose that

$$C_{(2)}(k) = 0, \tag{C.18}$$

$$C_{(1)}(k) = \frac{\sqrt{\pi}}{2} \exp i\left(\frac{\nu\pi}{2} + \frac{\pi}{4}\right). \tag{C.19}$$

Finally, we can write the solution as

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} \sqrt{-\tau} H_\nu^{(1)}(-k\tau) \exp +i\left(\frac{\nu\pi}{2} + \frac{\pi}{4}\right). \tag{C.20}$$

Using the asymptotic expansion for the Hankel functions [55] we immediately get that on super-horizon we have

$$u_k(\tau) \xrightarrow{-k\tau << 1} e^{i(\nu - \frac{1}{2})\frac{\pi}{2}} \frac{(-k\tau)^{-\nu + \frac{1}{2}}}{\sqrt{2k}} \frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)} 2^{\nu - \frac{3}{2}}, \tag{C.21}$$

where $\Gamma(\nu)$ is the Euler function of ν [55]. In order to verify the condition on the Wronskian we need two useful properties [55]

$$H_\nu^{(1)} = H_\nu^{(2)*}, \quad W\left\{H_\nu^{(1)}(z), H_\nu^{(2)}(z)\right\} = -\frac{4i}{\pi z}. \tag{C.22}$$

Thus, we can write

$$W\left\{u_k(-\tau), u_k^*(-\tau)\right\} = \frac{\pi}{4}(-\tau)\frac{4i}{\pi\tau} = -i, \tag{C.23}$$

which proves the statement.

C.3 *ADM*'s formalism

The *ADM* formalism, proposed by Richard Arnowitt, Stanley Deser, and Charles W. Misner in 1959 [75], presents a Hamiltonian treatment of General Relativity (*GR*). While we won't delve into the derivation of the General Relativity Hamiltonian in this context[29], we will focus on the fact that this formalism brings to light a crucial insight. Within the *ADM* framework, it becomes apparent that four out of ten metric degrees of freedom are not dynamical and can be eliminated from the Lagrangian once we have solved their Euler-Lagrange equation. Moreover, this formalism offers valuable insights into both the quantization program of general relativity and the domain of numerical relativity.

C.3.1 Globally hyperbolic spacetime

First of all, we need to introduce the concept of a globally hyperbolic spacetime which is the cornerstone upon which is based the entire formalism. Indeed, the applicability of this formalism is contingent solely upon the condition that the spacetime is globally hyperbolic. It is worth noting that instances exist where these conditions are not met, thereby rendering the utilization of this formalism unfeasible.

In order to do this we have to introduce some notions regarding the causal structure of a manifold. Considering a manifold M, we define a **causal curve** as one that remains time-like or null at all points. Consequently, for a subset W of M, its **causal future**, denoted as $J^+(W)$, consists of points that can be reached by following a causal curve starting from W. Similarly, the **chronological future**, denoted as $I^+(W)$, includes points that can be reached by following a timelike curve starting from W. Furthermore, we assert that the subset W is considered **achronal** if there is no causal curve connecting any two points within it. Then, we define the **future domain of dependence** of W, $D^+(W)$, as the set of points that can be reached from a future-directed causal curve starting from W. In an analogous way, we can define the **past domain of dependence** of W, $D^-(W)$. There is no a priori reason for which a generic point of the manifold must stay in one of the two domains. We can now introduce the concept of **Cauchy surface**, which is defined as an achronal surface Σ for which $D(\Sigma) \equiv D^+(\Sigma) \cup D^-(\Sigma)$ coincides with the entire manifold M. If a spacetime has a Cauchy surface, it's said to be **globally hyperbolic**.

C.3.2 The *ADM*'s decomposition of the metric

In this section, we provide an interpretation of the concept of time flow in *GR*, and we present an operative way to introduce the *ADM* splitting of the metric throughout the introduction of the lapse and shift functions.

Let be $(M, g_{\alpha\beta})$ a globally hyperbolic spacetime with a Cauchy surface Σ. It can be shown [13] that it's possible to foliate it with Cauchy surfaces Σ_t parametrized by a global time function t. The core of the *ADM* formalism is the splitting between the notions of "time" and "space", creating a framework that enables a description of spacetime's temporal evolution. The idea of globally hyperbolic spacetime suggests that the variables evolving with time pertain to those defined on the spatial hypersurface Σ. These variables are the six independent components of the induced three-dimensional spatial metric, which evolves as one moves within the foliation.

Let us now proceed to formalize this conceptual framework. First of all, we proceed to introduce the induced spatial metric. Thus, let n^α be the normal vector[30] to the Σ_t, which are spatial hypersurfaces since they are achronal [13]. Now it's possible to decompose the metric as

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \tag{C.24}$$

where one can show that $h^\alpha{}_\beta$ and $-n^\alpha n_\beta$ are projectors on the tangent vector space [76]. Then, we discuss the concept of "time flow" by defining a smooth vector field t^α,

$$t^\alpha \nabla_\alpha t = 1, \tag{C.25}$$

[29] The derivation of the Hamiltonian can be promptly obtained subsequent to the forthcoming discussion.
[30] It's orthogonal to the hypersurface and $-n^\alpha n_\alpha = 1$.

this condition amounts to say that this vector field always has a normal component to Σ_t. As emphasized in [77], and as becomes evident from its definition, we can select the vector field t^α as we like. Moreover, we can decompose this vector in its “temporal" component also called **lapse function** N, and its “spatial component" also called **shift function** N^α

$$N = -n^\alpha t_\alpha, \tag{C.26}$$

$$N^\alpha = h^\alpha{}_\beta t^\beta. \tag{C.27}$$

By utilizing the integral curves of t^α, it becomes possible to uniquely introduce a family of diffeomorphisms denoted as $\phi_t : \Sigma_0 \to \Sigma_t$, where Σ_0 and Σ_t represent different spacelike hypersurfaces. Hence, by progressing forward in time, we observe the effect of this “moving forward in time", as we begin from Σ_0 and advance by a parameter t to reach Σ_t. This offers an idea of how to proceed in order to find the Hamiltonian formulation of General Relativity (GR). In fact, we can conceive the entire manifold as the “evolution" of Σ_0 through time. This suggests that we must view the Riemannian metric on the three-dimensional spatial manifold as the “true" dynamical variable in GR. Being N and N_α the component of the tangent vector to the ϕ_t, it's clear that they represent the prescription to move forward in time as presented in figure 28. These variables are not dynamic in nature and can be removed from the Lagrangian once we have solved their constraint equations.

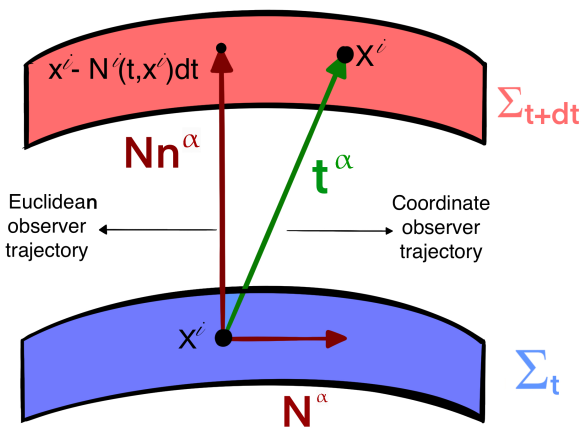


Figure 28: Moving in time with the lapse and shift functions.

Now, we need to derive the explicit form of the metric in ADM's formalism. This can be done as presented in [77] introducing a set of coordinates on the manifold. First of all, we notice that the hypersurfaces are the level sets of the time function t

$$\Sigma_t := \{y^\alpha : t(y^\alpha) = const\}, \tag{C.28}$$

where y^α are generic coordinates on M. Thus, introducing a new coordinate system (t, x^i) in such a way that the coordinates x^a on the hypersurfaces $\Sigma_{t=0}$ are transported along the time flow, ϕ_t, introduced previously. It's clear that the tangent vector field to these curves,

$$y^\alpha_{\bar{x}^a}(t) = y^\alpha(t, \bar{x}^i), \tag{C.29}$$

where $\bar{x}^a$ it's a generic point on $\Sigma_{t=0}$ which is fixed for each curve, corresponds to the vector field

$$t^\alpha = Nn^\alpha + N^\alpha. \tag{C.30}$$

The time flow can be viewed in this sense by following the curve we have introduced. In the language of coordinates, the tangent vector reads

$$(\partial_t)^\alpha \equiv \left(\frac{\partial y^\alpha}{\partial t}\right)_x = t^\alpha. \tag{C.31}$$

Now, the *lapse function* measures the rate of proper time measured by a *Eulerian observer* (fig.28) which is an observer at rest in the spacetime. An observer at rest in the space-time can move along a trajectory whose tangent vector is normal to the hypersurfaces. In fact, as shown in figure 28, the lapse of proper time between Σ_t and Σ_{t+dt}, for the Eulerian observer is

$$d\tau = N(t, x^i)dt. \tag{C.32}$$

While the *shift vector* $N^i(t, x^i)$ represents a “spatial" function that quantifies the displacement between *Eulerian* and *coordinate* observers following a time-lapse dt from Σ_t and Σ_{t+dt}, as shown in figure 28. The coordinate observer is the one that follows the time flow introduced by the vector field t^α. Thus, moving normally (see the Eulerian observer trajectory in figure 28) from Σ_t to Σ_{t+dt} the coordinates of the point on the upper leaf, x^i_{upper}, are related to the coordinate of the starting point on Σ_t, x^i_{low}, in this way

$$x^i_{low} = x^i, \quad x^i_{upper} = x^i - N^i(t, x, y, z)dt, \tag{C.33}$$

as shown in figure 28. Please notice that at the infinitesimal level, it's indifferent to take N^i on Σ_t or Σ_{t+dt}; [78] takes the vector defined on Σ_t for example.

The construction we have done is useful for understanding how to write the four-dimensional metric tensor using the three-dimensional one, the lapse, and shift functions [78]. In order to do this we want to compute the displacement of two infinitesimally adjacent points $y^\mu(t) = (t, x^i)$ and $y^\mu + dy^\mu = (t + dt, x^i + dx^i)$, which respectively belong to Σ_t and Σ_{t+dt}. What we have to do is compute the spacetime distance of the two points. If we want to compute distances in one of the two leaves we are considering, i.e. Σ_t and Σ_{t+dt}, we use respectively the spatial metric ${}^{(3)}g_{ij}(t, x, y, z)$ and ${}^{(3)}g_{ij}(t + dt, x, y, z)$. If we want to move between the two leaves we can use the prescription provided by the lapse and shift functions. Thus, we have everything to understand how to calculate distances between the two points. We can use the generalization of the Pythagorean theorem in GR. The time separation between the two leaves is

$$N(t, x, y, z)dt, \tag{C.34}$$

while the “spatial" separation on the upper leaves between the two points reads

$$(x^i + dx^i) - (x^i_{upper}) = \left[x^i + dx^i\right] - \left[x^i - N^i(t, x, y, z)dt\right] = dx^i + N^i(t, x, y, z)dt. \tag{C.35}$$

Thus using the general relativistic version of the Pythagorean theorem [78] we get

$$ds^2 = -N^2dt^2 + {}^{(3)}g_{ij}(t + dt, x, y, z)\left(dx^i + N^i(t, x, y, z)\right)\left(dx^j + N^j(t, x, y, z)\right) \tag{C.36}$$

$$\approx -\left(N^2 - N_iN^i\right)dt^2 + {}^{(3)}g_{ij}(t, x, y, z)dx^idx^j + 2N_j(t, x, y, z)dx^j, \tag{C.37}$$

where we have disregarded quantities that are infinitesimal of higher orders and where we have defined $N_i = {}^{(3)}g_{ij}N^j$ and $N_iN^i = {}^{(3)}g_{ij}N^jN^i$. Since what we have computed is invariant; the path we've taken does not impact the outcome. Thus, we can read the components of the metric tensor and its inverse as

$$g_{\mu\nu} = \begin{pmatrix} -\left(N^2 - N_iN^i\right) & N_j \\ N_i & {}^{(3)}g_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & \left({}^{(3)}g^{ij} - \frac{N^iN^j}{N^2}\right) \end{pmatrix}, \tag{C.38}$$

where ${}^{(3)}g^{ik(3)}g_{kj} = \delta^i{}_j$ and where the inverse metric is obtained by imposing $g^{\mu\nu}g_{\nu\alpha} = \delta^\mu{}_\alpha$ [78]. The explicit components of the vectors N^α and n^α and the projection tensor $h_{\mu\nu}$ read

$$n^\mu = \frac{1}{N}\left(1, -N^i\right), \quad n_\mu = N(1,0,0,0), \tag{C.39}$$

$$N^\mu = \left(0, N^i\right), \quad N_\mu = h_{\mu\nu}N^\nu, \tag{C.40}$$

$$h_{\mu\nu} = \begin{pmatrix} N_l N^l & N_i \\ N_j & {}^{(3)}g_{ij} \end{pmatrix}, \quad h^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & {}^{(3)}g_{ij} \end{pmatrix}. \tag{C.41}$$

We conclude this section by making remarks about the projection tensor and by introducing the covariant derivative on spatial hypersurfaces. Now, given a “spatial" vector v^α, i.e. $n_\alpha v^\alpha = 0$, if we want to raise and lower indices we can use $h_{\mu\nu}$ or $h^{\mu\nu}$

$$v_\alpha = h_{\alpha\beta}v^\beta = (g_{\alpha\beta} + n_\alpha n_\beta)v^\beta = g_{\alpha\beta}v^\beta, \tag{C.42}$$

and this reasoning can be generalized to each kind of tensor. Moreover, it’s possible to show that

$$D_\alpha = h_\alpha{}^\beta \nabla_\beta, \tag{C.43}$$

it’s the only covariant derivative operator defined on Σ [13] (see the definition of covariant derivative operator). We extensively use this definition in the next section while trying to express four-dimensional quantities in terms of three-dimensional ones.

C.3.3 The extrinsic curvature

Now, before proceeding with the derivation of the ADM’s action, we need a final ingredient, the extrinsic curvature, which is strictly linked to the concept of “temporal" derivative of the spatial metric. As we’ll see it contains a temporal derivative of the spatial metric and so it represents the conjugate momentum with respect to the spatial metric ${}^{(3)}g_{ij}$ which is the canonical variable in this formalism. Thus, the fundamental ingredients we have are the lapse and shift functions which merely prescribe how to move forward in time, the canonical variable ${}^{(3)}g_{ij}$ and its conjugate momentum Π_{ij} which is strictly linked to the definition of the extrinsic curvature tensor.

Whilst all of the information about the intrinsic curvature of a manifold is contained in the single component of the Ricci scalar, the extrinsic curvature describes the way something is embedded in a higher dimensional space[38]. There is a nice example presented in [38] regarding a cylinder, which is $R \times S^1$. It’s always possible to choose coordinates in such a way that the cylinder is flat; We are also free to consider an alternative metric wherein the cylinder isn’t flat. However, the key point we want to underscore is that it’s possible to render the cylinder flat within a certain metric. Thus, it has zero intrinsic curvature but it does exhibit non-zero extrinsic curvature since the cylinder is embedded in higher-dimension space. Extrinsic curvature refers to how the surface curves in dimensions beyond its own, while measurements of lengths and areas within a curved surface only reveal the intrinsic curvature, excluding the extrinsic curvature.

Now, we can turn our attention to the definition of the curvature tensor [13] which is

$$K_{\alpha\beta} = h_\alpha{}^\gamma \nabla_\gamma \zeta_\beta, \tag{C.44}$$

where ζ_β is the unit tangent vector to a geodesic congruence which is orthogonal to the specific hypersurface we are considering, Ω. Since the congruence is hypersurface orthogonal we have that the vorticity tensor is zero, thus the extrinsic curvature is symmetric [13, 76]. Nonetheless, it’s feasible to expand this definition by differentiating it with respect to the normal vector field to Ω. It’s conceivable to conceptualize extrinsic curvature without necessitating the introduction of any geodesic congruence. This is due to the fact that when provided with the normal vector to a spatial hypersurface, it’s always possible to identify a set of geodesics that, on the hypersurface Σ, possess the normal vector as their normalized tangent vector. Thus, we can write

$$K_{\alpha\beta} = h_\alpha{}^\gamma \nabla_\gamma n_\beta = \frac{1}{2}\pounds_n h_{\alpha\beta}, \tag{C.45}$$

where the last equality can be shown in this way. First of all, since the extrinsic curvature is symmetric we have

$$\nabla_\alpha n_\beta = \nabla_\beta n_\alpha. \tag{C.46}$$

Now, using this equation and the normalization condition we get

$$\begin{cases} n_\alpha \nabla_\beta n^\alpha = n^\alpha \nabla_\beta n_\alpha & = 0 \\ n_\alpha \nabla^\beta n^\alpha = n^\alpha \nabla^\beta n_\alpha & = 0 \\ n^\alpha \nabla_\alpha n_\beta = n^\alpha \nabla_\beta n_\alpha & = n_\alpha \nabla_\beta n^\alpha = 0 \end{cases}, \tag{C.47}$$

Now, we can verify what we are interested in by direct computation of the Lie derivative as

$$\pounds_n h_{\alpha\beta} = \left(g_{\alpha\beta} + n_\alpha n_\beta\right)_{;\gamma} n^\gamma + \left(g_{\alpha\gamma} + n_\alpha n_\gamma\right) n^\gamma{}_{;\beta} + \left(g_{\gamma\beta} + n_\gamma n_\beta\right) n^\gamma{}_{;\alpha} \tag{C.48}$$

$$= n_\beta n^\gamma \nabla_\gamma n_\alpha + n_\alpha n^\gamma \nabla_\gamma n_\beta + \nabla_\alpha n_\beta + \nabla_\beta n_\alpha + n_\gamma n_\beta \nabla_\alpha n^\gamma + n_\alpha n_\gamma \nabla_\beta n^\gamma \tag{C.49}$$

$$= 2\nabla_\alpha n_\beta + n_\beta n^\gamma \nabla_\gamma n_\alpha = 2 h_\alpha{}^\gamma \nabla_\gamma n_\beta, \tag{C.50}$$

where ; stands for four-dimansional covariant derivative and where we have used eq.(C.47).

Now, we derive an explicit useful expression for the extrinsic curvature starting from the definition and using the definition of the Lie derivative [38]

$$K_{\alpha\beta} = \frac{1}{2}\pounds_n h_{\alpha\beta} = \frac{1}{2}\left[n^\gamma \nabla_\gamma h_{\alpha\beta} + h_{\alpha\gamma}\nabla_\beta n^\gamma + h_{\beta\gamma}\nabla_\alpha n^\gamma\right] \tag{C.51}$$

$$= \frac{1}{2} h_\alpha{}^\gamma h_\beta{}^\delta \left[n^\mu \nabla_\mu h_{\gamma\delta} + h_{\gamma\nu}\nabla_\delta n^\nu + h_{\delta\mu}\nabla_\gamma n^\mu\right], \tag{C.52}$$

where the last passage can be proven by direct evaluation and using the definition of the projection tensor $h_{\alpha\beta}$ and the properties of eq.(C.47). Now, by multiplying and dividing by the lapse function eq.(C.52) and by using that

$$h_{\gamma\mu} N \nabla_\nu n^\mu = h_{\gamma\nu} \nabla_\nu N n^\mu, \tag{C.53}$$

we arrive at

$$K_{\alpha\beta} = \frac{1}{2N} h_\alpha{}^\gamma h_\beta{}^\delta \left[N n^\mu \nabla_\mu h_{\gamma\delta} + h_{\gamma\nu}\nabla_\delta N n^\nu + h_{\delta\mu}\nabla_\gamma N n^\mu\right] \tag{C.54}$$

$$= \frac{1}{2N} h_\alpha{}^\gamma h_\beta{}^\delta \left[\left(t^\mu - N^\mu\right)\nabla_\mu h_{\gamma\delta} + h_{\gamma\nu}\nabla_\delta\left(t^\nu - N^\nu\right) + h_{\delta\mu}\nabla_\gamma\left(t^\mu - N^\mu\right)\right] \tag{C.55}$$

$$= \frac{1}{2N} h_\alpha{}^\gamma h_\beta{}^\delta \left[\pounds_t h_{\gamma\delta} - \pounds_N h_{\gamma\delta}\right], \tag{C.56}$$

where we have used that $n^\alpha = \frac{1}{N}\left(t^\alpha - N^\alpha\right)$. Now, we need to directly evaluate the second Lie derivative

$$h_\alpha{}^\gamma h_\beta{}^\delta \pounds_N h_{\gamma\delta} = h_\alpha{}^\gamma h_\beta{}^\delta \left[N^\gamma \nabla_\gamma h_{\alpha\beta} + h_{\alpha\gamma}\nabla_\beta N^\gamma + h_{\beta\gamma}\nabla_\alpha N^\gamma\right] \tag{C.57}$$

$$= h_\alpha{}^\gamma h_\beta{}^\delta N^\mu \nabla_\mu h_{\gamma\delta} + h^\mu{}_\alpha h^\nu{}_\beta \nabla_\nu N_\mu + h_\alpha{}^\gamma h_{\beta\delta}\nabla_\gamma\left(h^{\delta\nu} N_\nu\right) \tag{C.58}$$

$$= D_\alpha N_\beta + D_\beta N_\alpha + h_\alpha{}^\gamma h_\beta{}^\delta N^\mu \nabla_\mu h_{\gamma\delta} + h_{\alpha\mu} h_\beta{}^\nu N_\chi \nabla_\nu h^{\chi\nu} + h_\alpha{}^\mu h_{\beta\nu} N_\chi \nabla_\mu h^{\nu\chi} \tag{C.59}$$

$$= D_\alpha N_\beta + D_\beta N_\alpha, \tag{C.60}$$

where in the last passage we have used that

$$h_\alpha{}^\gamma h_\beta{}^\delta N^\mu \nabla_\mu h_{\gamma\delta} = h_\alpha{}^\gamma h_\beta{}^\delta N^\mu \left(n_\gamma \nabla_\mu n_\delta + \nabla_\mu n_\gamma n_\delta\right) = 0, \tag{C.61}$$

$$h_{\alpha\mu} h_\beta{}^\nu N_\chi \nabla_\nu h^{\chi\nu} = h_{\alpha\mu} h_\beta{}^\nu N_\chi \left(\nabla_\nu n^\chi n^\nu + n^\chi \nabla_\nu n^\nu\right) = 0, \tag{C.62}$$

$$h_\alpha{}^\mu h_{\beta\nu} N_\chi \nabla_\mu h^{\nu\chi} = h_\alpha{}^\mu h_{\beta\nu} N_\chi \left(\nabla_\mu \nabla_\mu n^\nu n^\chi + n^\nu \nabla_\mu n^\chi\right) = 0. \tag{C.63}$$

Now, we are ready to express the extrinsic curvature in its final form

$$K_{\alpha\beta} = \frac{1}{2N}\left[h_\alpha{}^\gamma h_\beta{}^\delta \pounds_t h_{\gamma\delta} - D_\alpha N_\beta - D_\beta N_\alpha\right] = \frac{1}{2N}\left[\dot{h}_{\alpha\beta} - D_\alpha N_\beta - D_\beta N_\alpha\right], \tag{C.64}$$

where in the last step we have used $\dot{h}_{\alpha\beta} = \frac{\partial h_{\alpha\beta}}{\partial t}$ [13]. This property can be immediately understood once we recall the coordinates we are adopting; see also the discussion relative to the Lie derivative presented in [38].

C.3.4 The ADM's action

Now, we start by deriving the expression for the ADM's action. First of all, we stress that the philosophy is to relate four-dimensional quantities to three-dimensional ones. For the sake of simplicity, we explicitly indicate when a quantity pertains to three dimensions. As we'll show in a moment we have that

$$G_{\alpha\beta}n^{\alpha}n^{\beta} = R_{\alpha\beta}n^{\alpha}n^{\beta} - \frac{R}{2}g_{\alpha\beta}n^{\alpha}n^{\beta} = R_{\alpha\beta}n^{\alpha}n^{\beta} + \frac{R}{2} = \frac{1}{2}\left\{{}^{(3)}R + K^2 - K^{\alpha\beta}K_{\alpha\beta}\right\}, \tag{C.65}$$

$$R_{\alpha\beta}n^{\alpha}n^{\beta} = K^2 - K^{\alpha\beta}K_{\alpha\beta}, \tag{C.66}$$

where $G_{\mu\nu}$ is the Einstein tensor [13], ${}^{(3)}R$ is the Ricci scalar of the spatial hypersurfaces, and

$$K^2 = \left(K^{\alpha}{}_{\alpha}\right)^2. \tag{C.67}$$

Once we have established these relations we can write the Hilbert-Einstein action as [13]

$$\mathcal{L}_{HE} = \sqrt{-g}R = N\sqrt{h}R = N\sqrt{h}2\left(G_{\alpha\beta}n^{\alpha}n^{\beta} - R_{\alpha\beta}n^{\alpha}n^{\beta}\right) = N\sqrt{h}\left[{}^{(3)}R + K^{\alpha\beta}K_{\alpha\beta} - K^2\right], \tag{C.68}$$

where we do not show that $\sqrt{-g} = N\sqrt{h}$ [13], where $\sqrt{-g}$ and $\sqrt{h}$ are the determinant of the four and three metric. The crucial observation we make is that there are no temporal derivatives of the lapse function $N(t,x,y,z)$ and the shift function $N^i(t,x,y,z)$. Consequently, these components account for four degrees of freedom that act as constraints in the same fashion the temporal component of the vector potential A^{μ} is a constraint in the electromagnetic Lagrangian. We will proceed to solve the relative equations of motions and subsequently plug the solutions into the action. This is the most significant outcome from the perspective of our objective within the framework of the ADM's action.

We now move forward to demonstrate the previously mentioned identities eq.(C.65) and (C.66). To accomplish this, we start with the expression for the three-dimensional Riemann tensor. Thus, we consider a one-dimensional form defined on Σ and how the Riemann tensor acts on it

$${}^{(3)}R_{\alpha\beta\gamma}{}^{\delta}w_{\delta} = \left(D_{\alpha}D_{\beta} - D_{\beta}D_{\alpha}\right)w_{\gamma}. \tag{C.69}$$

First of all, we evaluate the first term and then we obtain the second by symmetry

$$D_{\alpha}D_{\beta}w_{\gamma} = h_{\beta}{}^{\chi}h_{\gamma}{}^{\delta}h_{\alpha}{}^{\mu}\nabla_{\mu}\left(h_{\chi}{}^{\nu}h_{\delta}{}^{\phi}\nabla_{\nu}w_{\phi}\right) = h_{\beta}{}^{\chi}h_{\gamma}{}^{\delta}h_{\alpha}{}^{\mu}\nabla_{\mu}\nabla_{\chi}w_{\delta} + h_{\gamma}{}^{\delta}K_{\alpha\beta}n^{\zeta}\nabla_{\zeta}w_{\delta} - h_{\beta}{}^{\delta}w_{\zeta}K^{\zeta}{}_{\delta}K_{\gamma\alpha}, \tag{C.70}$$

where, in order to get the final result, one has to use

$$\nabla_{\alpha}h^{\beta}{}_{\gamma} = \nabla_{\alpha}\left(n^{\beta}n_{\gamma}\right), \quad h_{\beta}{}^{\gamma}n_{\gamma} = 0, \quad h^{\alpha}{}_{\beta}h^{\beta}{}_{\gamma} = h^{\alpha}{}_{\gamma}, \quad n_{\gamma}w^{\gamma} = 0. \tag{C.71}$$

Now, using eq.(C.70) and its symmetrized version we get

$${}^{(3)}R_{\alpha\beta\gamma}{}^{\delta} = h_{\alpha}{}^{\mu}h_{\beta}{}^{\chi}h_{\gamma}{}^{\delta}h^{\delta}{}_{\nu}R_{\mu\chi\gamma}{}^{\nu} + K_{\gamma\beta}K^{\delta}{}_{\alpha} - K_{\gamma\alpha}K^{\delta}{}_{\beta}, \tag{C.72}$$

where no specific remarks need to be made regarding this calculation. Now, we list a series of properties that we need to perform the actual calculation.

$$R_{\alpha\beta\gamma\delta}g^{\alpha\gamma}g^{\beta\delta} = R_{\alpha\gamma}g^{\beta\delta} = R, \tag{C.73}$$

$$R_{\alpha\beta\gamma\delta}g^{\alpha\beta}n^{\beta}n^{\delta} = R_{\beta\alpha\delta\gamma}g^{\alpha\beta}n^{\beta}n^{\delta} = R_{\beta\delta}n^{\beta}n^{\delta}, \tag{C.74}$$

$$R_{\alpha\beta\gamma\delta}n^{\alpha}n^{\gamma}g^{\beta\delta} = R_{\alpha\gamma}n^{\alpha}n^{\gamma}, \tag{C.75}$$

$$R_{\alpha\beta\gamma\delta}n^{\alpha}n^{\beta}n^{\gamma}n^{\delta} = -R_{\alpha\beta\delta\gamma}n^{\alpha}n^{\beta}n^{\gamma}n^{\delta} = 0, \tag{C.76}$$

where R is the Ricci scalar while $R_{\alpha\beta}$ the Ricci tensor. Thus, in order to proceed we have to compute

$$R_{\alpha\beta\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} = R_{\alpha\beta\gamma\delta}\left(g^{\alpha\gamma} + n^{\alpha}n^{\gamma}\right)\left(g^{\beta\delta} + n^{\beta}n^{\delta}\right) = R + 2R_{\alpha\beta}n^{\alpha\beta} = 2G_{\alpha\beta}n^{\alpha}n^{\beta}. \tag{C.77}$$

Now, if we compute

$$ {}^{(3)}R_{\alpha\beta\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} = R_{\alpha\beta\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} + K^{\alpha\beta}K_{\alpha\beta} - K^2, \tag{C.78} $$

where, no observations are to be made for this computation. Thus, we have proved the first relation. Now, we turn to the second one.

$$ R_{\alpha\beta}n^\alpha n^\beta = R_{\alpha\gamma\beta}{}^{\gamma}n^\alpha n^\beta = -n^\alpha g^{\chi\gamma}R_{\alpha\gamma\chi}{}^{\beta}n_\beta = -n^\alpha g^{\chi\gamma}\left[\nabla_\alpha\nabla_\gamma - \nabla_\gamma\nabla_\alpha\right]n_\chi \tag{C.79} $$

$$ = K^2 - K^{\alpha\beta}K_{\alpha\beta}. \tag{C.80} $$

In arriving at this result, we've exploited the symmetries of the Riemann tensor. Notably, the key observation is that the final outcome is attained through the application of Leibniz rules, along with the properties outlined in equation Eq.(C.47), and the symmetry of the extrinsic curvature tensor.

We conclude this section by observing that since $K_{\mu\nu}$ is a spatial tensor we have that $K^{\mu\nu}n_\nu = 0$. This, implies that $K^{\mu 0} = K^{0}{}_{\mu} = 0$. So, in order to compute the ADM's action we do not need to evaluate $K_{0\mu}$

$$ K^{\alpha\beta}K_{\alpha\beta} = K^{ij}K_{ij}, \quad K = K^{\mu}{}_{\mu} = K^{i}{}_{i}. \tag{C.81} $$

This essentially indicates that we can work as K_{ij} were a tensor defined on a Riemannian manifold and raise its indices using ${}^{(3)}g_{ij}$. This final step is allowed by the explicit expression of $h^{\mu\nu}$, for which we are aware that $h^{\mu 0} = 0$ for all μ. This observation will be useful when expanding the action of the chiral-scalar tensor theories of gravity.

C.4 The tensor power spectrum in single-field slow-roll models.

As mentioned in section 2.5, we have the option to simplify the equations by setting $N = 1$, $N^i = 0$, and all the scalars in the metric to zero, resulting in the following expressions:

$$ g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & a^2(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma^{l}{}_{j}) \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & a^{-2}(\delta^{ij} - \gamma^{ij} + \frac{1}{2}\gamma^{il}\gamma^{j}{}_{l}) \end{pmatrix}. \tag{C.82} $$

Starting from the extrinsic curvature tensor we simply get

$$ K_{ij} = \frac{1}{2}\dot{g}_{ij} = \dot{a}a\delta_{ij} + \dot{a}a\gamma_{ij} + \frac{1}{2}a^2\dot{\gamma}_{ij} + \frac{1}{2}\dot{a}a\gamma_{ik}\gamma^k_j + \frac{1}{4}a^2\dot{\gamma}_{ik}\gamma^k_j + \frac{1}{4}a^2\gamma_{ik}\dot{\gamma}^k_j. \tag{C.83} $$

from which we can obtain the fully contravariant version as follows

$$ K^{ij} = g^{il}g^{jf}K_{lf} \tag{C.84} $$

$$ = a^{-4}(\delta^{il} - \gamma^{il} + \frac{1}{2}\gamma^{ik}\gamma^{l}{}_{k})(\delta^{jf} - \gamma^{jf} + \frac{1}{2}\gamma^{jk}\gamma^f_k)\Big(\dot{a}a\delta_{lf} + \dot{a}a\gamma_{lf} + \frac{1}{2}a^2\dot{\gamma}_{lf} + \frac{1}{2}\dot{a}a\gamma_{lk}\gamma^{k}{}_{f} + \frac{1}{4}a^2\dot{\gamma}_{lk}\gamma^{k}{}_{f} + \frac{1}{4}a^2\gamma_{lk}\dot{\gamma}^{k}{}_{f}\Big) \tag{C.85} $$

$$ = a^{-4}\left(\dot{a}a\delta^{ij} + \dot{a}a\gamma^{ij} + \frac{1}{2}a^2\dot{\gamma}^{ij} + \frac{1}{2}\dot{a}a\gamma^i_k\gamma^{kj} + \frac{1}{4}a^2\dot{\gamma}^i_k\gamma^{kj} + \frac{1}{4}a^2\gamma^i_k\dot{\gamma}^{kj}\right) + a^{-4}\left(-\delta^{il}\gamma^{jf} - \gamma^{il}\delta^{jf}\right)\left(\dot{a}a\delta_{lf} + \dot{a}a\gamma_{lf} + \frac{1}{2}a^2\dot{\gamma}_{lf}\right) + \frac{\dot{a}}{a^3}\left[\gamma^i_f\gamma^{jf} + \frac{1}{2}\delta^{il}\gamma^{jk}\gamma_{lk} + \frac{1}{2}\gamma^{ik}\gamma^j_k\right] \tag{C.86} $$

$$ = \frac{\dot{a}}{a^3}\delta^{ij} - \frac{\dot{a}}{a^3}\gamma^{ij} + \frac{1}{2}a^{-2}\dot{\gamma}^{ij} + \frac{1}{2}\frac{\dot{a}}{a^3}\gamma^{ik}\gamma^j_k - \frac{1}{4}a^{-2}\dot{\gamma}^{ik}\gamma^j_k + \frac{1}{4}a^{-2}\gamma^{ik}\dot{\gamma}^j_k \tag{C.87} $$

$$ = a^{-4}\Big\{\dot{a}a\delta^{ij} + \dot{a}a\gamma^{ij} + \frac{1}{2}a^2\dot{\gamma}^{ij} + \frac{1}{2}\dot{a}a\gamma^i_k\gamma^{kj} + \frac{1}{4}a^2\dot{\gamma}^i_k\gamma^{kj} + \frac{1}{4}a^2\gamma^i_k\dot{\gamma}^{kj} - 2(\dot{a}a\gamma^{ij} + \gamma^{jf}\dot{a}a\gamma^i_f) - \frac{1}{2}a^2\dot{\gamma}^i_f\gamma^{jf} - \frac{1}{2}a^2\dot{\gamma}^{il}\gamma^j_l + \dot{a}a\left[\gamma^i_f\gamma^{jf} + \frac{1}{2}\delta^{il}\gamma^{jk}\gamma_{lk} + \frac{1}{2}\gamma^{ik}\gamma^j_k\right]\Big\} \tag{C.88} $$

$$ = a^{-4}\left(\dot{a}a\delta^{ij} - \dot{a}a\gamma^{ij} + \frac{1}{2}a^2\dot{\gamma}^{ij} + \frac{1}{2}\dot{a}a\gamma^i_k\gamma^{kj} - \frac{1}{4}a^2\dot{\gamma}^i_k\gamma^{kj} - \frac{1}{4}a^2\gamma^i_k\dot{\gamma}^{kj}\right) \tag{C.89} $$

$$ = \frac{\dot{a}}{a^3}\delta^{ij} - \frac{\dot{a}}{a^3}\gamma^{ij} + \frac{1}{2}a^{-2}\dot{\gamma}^{ij} + \frac{1}{2}\frac{\dot{a}}{a^3}\gamma^{ik}\gamma^j_k - \frac{1}{4}a^{-2}\dot{\gamma}^{ik}\gamma^j_k - \frac{1}{4}a^{-2}\gamma^{ik}\dot{\gamma}^j_k. \tag{C.90} $$

Now we can compute what it's needed in order to rewrite the Lagrangian

$$K^{ij}K_{ij} = \left[\frac{\dot a}{a^3}\delta^{ij} - \frac{\dot a}{a^3}\gamma^{ij} + \frac{1}{2}a^{-2}\dot\gamma^{ij} + \frac{1}{2}\frac{\dot a}{a^3}\gamma^{ik}\gamma^j_k - \frac{1}{4}a^{-2}\dot\gamma^{ik}\gamma^j_k - \frac{1}{4}a^{-2}\gamma^{ik}\dot\gamma^j_k\right] \tag{C.91}$$

$$\times\left[\dot a a\delta_{ij} + \dot a a\gamma_{ij} + \frac{1}{2}a^2\dot\gamma_{ij} + \frac{1}{2}\dot a a\gamma_{ik}\gamma^k_j + \frac{1}{4}a^2\dot\gamma_{ik}\gamma^k_j + \frac{1}{4}a^2\gamma_{ik}\dot\gamma^k_j\right] \tag{C.92}$$

$$= \frac{\dot a}{a^3}\left[3\dot a a + \frac{1}{2}\dot a a\gamma_{ik}\gamma^k_i + \frac{1}{2}a^2\dot\gamma_{ik}\gamma^{ki}\right] - \frac{\dot a}{a^3}\gamma^{ij}\left[\dot a a\gamma_{ij} + \frac{1}{2}a^2\dot\gamma_{ij}\right] + \frac{1}{2}a^{-2}\dot\gamma^{ij}\left[\dot a a\gamma_{ij} + \frac{1}{2}a^2\dot\gamma_{ij}\right] + \tag{C.93}$$

$$\left[+\frac{1}{2}\dot a a\gamma_{ik}\gamma^{ki} - \frac{1}{4}a^2\dot\gamma_{ik}\gamma^{ki} - \frac{1}{4}a^2\gamma_{ik}\dot\gamma^{ik}\right]\frac{\dot a}{a^3} \tag{C.94}$$

$$= 3\frac{\dot a^2}{a^2} + \frac{1}{4}\dot\gamma^{ij}\dot\gamma_{ij}. \tag{C.95}$$

Now we can compute the trace of the extrinsic curvature retaining terms up to second order as follows

$$g^{ij}K_{ij} = a^{-2}(\delta^{ij} - \gamma^{ij} + \frac{1}{2}\gamma^{il}\gamma^j_l)\left[\dot a a\delta_{ij} + \dot a a\gamma_{ij} + \frac{1}{2}a^2\dot\gamma_{ij} + \frac{1}{2}\dot a a\gamma_{ik}\gamma^k_j + \frac{1}{4}a^2\dot\gamma_{ik}\gamma^k_j + \frac{1}{4}a^2\gamma_{ik}\dot\gamma^k_j\right] \tag{C.96}$$

$$= \left[3\frac{\dot a}{a} + \frac{1}{2}\frac{\dot a}{a}\gamma_{ik}\gamma^{ki} + \frac{1}{4}\dot\gamma_{ik}\gamma^{ki} + \frac{1}{4}\gamma_{ik}\dot\gamma^{ki}\right] - \gamma^{ij}\left[\frac{\dot a}{a}\delta_{ij} + \frac{\dot a}{a}\gamma_{ij} + \frac{1}{2}\dot\gamma_{ij}\right] + \frac{1}{2}\gamma^{il}\gamma_{li}\frac{\dot a}{a} \tag{C.97}$$

$$= 3\frac{\dot a}{a}. \tag{C.98}$$

Now in order to complete the computation of the $\mathcal{L}_{HE}$ we need the expression of the Ricci scalar. Thus, starting from the Christoffel symbols we have

$${}^{(3)}\Gamma^i_{jk} = \frac{g^{il}}{2}\left[g_{lj,k} + g_{lk,j} - g_{jk,l}\right] \tag{C.99}$$

$$= \frac{1}{2}(\delta^{il} - \gamma^{il} + \frac{1}{2}\gamma^{if}\gamma^l_f)\left[\gamma_{jl,k} + \gamma_{kl,j} - \gamma_{jk,l} + \frac{1}{2}\left((\gamma^f_l\gamma_{fj})_{,k} + (\gamma^f_l\gamma_{fk})_{,j} - (\gamma^f_k\gamma_{fj})_{,l}\right)\right] \tag{C.100}$$

$$= \frac{1}{2}\left(\gamma^i_{j,k} + \gamma^i_{k,j} - \gamma_{jk}{}^{,i} + \frac{1}{2}\left((\gamma^{fi}\gamma_{fj})_{,k} + (\gamma^{fi}\gamma_{fk})_{,j} - (\gamma^f_k\gamma_{fj})^{,i}\right)\right) - \frac{1}{2}\gamma^{il}\left[\gamma_{jl,k} + \gamma_{kl,j} - \gamma_{jk,l}\right]. \tag{C.101}$$

Before we continue, it is worth noting that because the tensor is traceless and transverse, ${}^{(3)}\Gamma^i_{il}$ is of second order in the perturbation. Therefore the Ricci tensor on the three-space reads

$$R_{ij} = \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} - \frac{1}{2}\Gamma^l_{ij}\Gamma^k_{lk} - \Gamma^l_{kj}\Gamma^k_{li} \tag{C.102}$$

$$= \partial_k B^k_{ij} - \partial_j A_i - \frac{1}{2}\gamma_{ij}{}^{,k}{}_{,k} - \frac{1}{4}\left[\gamma^l_{k,j} + \gamma^l_{j,k} - \gamma_{kj}{}^{,l}\right]\left[\gamma^k{}_{l,i} + \gamma^k{}_{i,l} - \gamma_{li}{}^{,k}\right], \tag{C.103}$$

where we have introduced B^f_{ij} and A_f, the explicit forms of which are not crucial for our current discussion. The essential point is that they can both be expressed as spatial divergences. Now since it eq.(C.103) contains both first and second-order terms to get the Ricci scalar is sufficient to contract with the inverse metric expanded up to first order

$$R = a^{-2}\left\{\partial_j D^j + \frac{1}{2}\gamma^{ij}\gamma_{ij}{}^{,k}{}_{,k} - \frac{1}{4}\left[\gamma^l{}_k{}^{,i} + \gamma^{li}{}_{,k} - \gamma^{i}_k{}^{,l}\right]\left[\gamma^k{}_l{}^{,i} + \gamma^k{}_{i,l} - \gamma_{li}{}^{,k}\right]\right\} \tag{C.104}$$

$$= a^{-2}\left\{F^i{}_{,i} + \frac{1}{2}\gamma^{ij}\gamma_{ij}{}^{,k}{}_{,k} - \frac{1}{4}\left[\gamma^l{}_k{}^{,i}\gamma^k{}_l{}^{,i} - \gamma^{li}{}_{,k}\gamma_{li}{}^{,k} - \gamma^i_k{}^{,l}\gamma^k{}_{i,l}\right]\right\} \tag{C.105}$$

$$= \frac{1}{a^2}\left[\partial_i D^i + \frac{1}{2}\gamma^{ij}\gamma_{ij}{}^{,k}{}_{,k} + \frac{1}{4}\gamma^{li}{}_{,k}\gamma_{il}{}^{,k}\right] \tag{C.106}$$

where the explicit form of D_j is not important but it's worth noticing that it's of second order in tensor perturbation. Using this Mathematica code it's possible to verify that

$$\sqrt{h} \approx a^3(1 + O(h^3)). \tag{C.107}$$

Thus we can finally write the action, setting $N = 1$, as follows

$$\mathcal{S}_T^{(2)} = \int \frac{M_{pl}^2}{2} d^4x N\sqrt{h} \left[{}^{(3)}R + K_{ij}K^{ij} - K^2 \right] \tag{C.108}$$

$$= \int \frac{M_{pl}^2}{2} d^4x a^3 \left\{ \frac{1}{a^2} \left[\partial_i D^i + \frac{1}{2}\gamma^{ij}\gamma_{ij}{}^{,k}{}_{,k} + \frac{1}{4}\gamma^{li}{}_{,k}\gamma_{il}{}^{,k} \right] + 3\frac{\dot{a}^2}{a^2} + \frac{1}{4}\dot{\gamma}^{ij}\dot{\gamma}_{ij} - 9\frac{\dot{a}^2}{a^2} \right\}, \tag{C.109}$$

which neglecting the $0th$ order action, integrating by parts, and eliminating total derivatives can be recast as

$$\mathcal{S}_T^{(2)} = \int d^4x \frac{M_{pl}^2}{8} a^3 \left\{ \dot{\gamma}^{ij}\dot{\gamma}_{ij} - \frac{1}{a^2}\gamma^{li}{}_{,k}\gamma_{il}{}^{,k} \right\}. \tag{C.110}$$

The result can be rewritten using conformal time as

$$\mathcal{S}_T^{(2)} = \int d^4x \frac{M_{pl}^2}{8} a \left\{ {\gamma'}^{ij}{\gamma'}_{ij} - \gamma^{li}{}_{,k}\gamma_{il}{}^{,k} \right\}. \tag{C.111}$$

D Appendix D

D.1 Polarization tensors

The focus of this section is to analyze two bases for the polarization tensors along with their respective properties [7]. First of all, we introduce the Fourier transform of the tensor perturbation as

$$h_{ij}(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^3} \sum_s h_s(\mathbf{k},\tau)\epsilon_{ij}^{(s)}(\hat{n}), \tag{D.1}$$

where $\epsilon_{ij}^{(s)}(\hat{n})$ are the two components for the polarization basis and

$$\hat{n} \equiv \frac{\mathbf{k}}{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{D.2}$$

with $k =\mid \mathbf{k} \mid$.

1. The $\{+,\times\}$ basis which is constructed starting from two orthonormal vectors

$$\begin{cases} v &= (\sin\phi, -\cos\phi, 0) \\ w &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \end{cases} \qquad \text{if } \theta \neq \frac{\pi}{2}, \tag{D.3}$$

$$\begin{cases} v &= (1,0,0) \\ w &= (0,1,0) \end{cases} \qquad \text{if } \theta = 0, \tag{D.4}$$

   which are orthogonal to the vector $\hat{n}$. Thus we can introduce the two polarizations tensor as

$$\epsilon_{ij}^{+} = v_i v_j - w_i w_j, \qquad \epsilon_{ij}^{\times} = w_i v_j + v_i w_j, \tag{D.5}$$

   which are real, traceless, and transverse, i.e.

$$\left[\epsilon_{ij}^{s}(\mathbf{k})\right]^* = \left[\epsilon_{ij}^{s}(\mathbf{k})\right], \quad Tr\{\epsilon_{ij}^{s}(\mathbf{k})\} = 0, \quad \epsilon_{ij}^{s}(\mathbf{k})k^i = \epsilon_{ij}^{s}(\mathbf{k})k^j = 0, \quad \text{with } s = +/\times. \tag{D.6}$$

   Sending $\mathbf{k} \longmapsto -\mathbf{k}$ we have that (see fig.27 appendix B.1)

$$v \longmapsto +v, \tag{D.7}$$

$$w \longmapsto -w, \tag{D.8}$$

   which allow us to write

$$\begin{cases} \epsilon_{ij}^{+}(-\mathbf{k}) &= \epsilon_{ij}^{+}(\mathbf{k}) \\ \epsilon_{ij}^{\times}(-\mathbf{k}) &= -\epsilon_{ij}^{\times}(\mathbf{k}) \end{cases}. \tag{D.9}$$

   Moreover, it's easy to show that we have

$$\begin{cases} Tr\{\epsilon_{ij}^{+}(\mathbf{k})\epsilon_{ij}^{+}(\mathbf{k})\} = Tr\{\epsilon_{ij}^{\times}(\mathbf{k})\epsilon_{ij}^{\times}(\mathbf{k})\} &= 2 \\ Tr\{\epsilon_{ij}^{+}(\mathbf{k})\epsilon_{ij}^{\times}(\mathbf{k})\} = Tr\{\epsilon_{ij}^{\times}(\mathbf{k})\epsilon_{ij}^{+}(\mathbf{k})\} &= 0 \end{cases}. \tag{D.10}$$

2. The chiral basis $\{R, L\}$ is defined as

$$\epsilon_{ij}^{R} = \frac{\epsilon_{ij}^{+} + i\epsilon_{ij}^{\times}}{\sqrt{2}}, \qquad \epsilon_{ij}^{L} = \frac{\epsilon_{ij}^{+} - i\epsilon_{ij}^{\times}}{\sqrt{2}}, \tag{D.11}$$

   which are one the complex conjugate of the other, traceless and transverse, i.e.

$$\left[\epsilon_{ij}^{L}(\mathbf{k})\right]^* = \left[\epsilon_{ij}^{R}(\mathbf{k})\right], \quad Tr\{\epsilon_{ij}^{s}(\mathbf{k})\} = 0, \quad \epsilon_{ij}^{s}(\mathbf{k})k^i = \epsilon_{ij}^{s}(\mathbf{k})k^j = 0, \quad \text{with } s = R/L. \tag{D.12}$$

Using eq.(D.9) we can deduce

$$\begin{cases} \epsilon_{ij}^R(-\mathbf{k}) & = \epsilon_{ij}^L(\mathbf{k}) \\ \epsilon_{ij}^L(-\mathbf{k}) & = \epsilon_{ij}^R(\mathbf{k}) \end{cases}. \tag{D.13}$$

Moreover, from eq.(D.10) we can obtain that

$$\begin{cases} Tr\{\epsilon_{ij}^R(\mathbf{k})\epsilon_{ij}^L(\mathbf{k})\} = Tr\{\epsilon_{ij}^L(\mathbf{k})\epsilon_{ij}^R(\mathbf{k})\} & = 2 \\ Tr\{\epsilon_{ij}^R(\mathbf{k})\epsilon_{ij}^R(\mathbf{k})\} = Tr\{\epsilon_{ij}^L(\mathbf{k})\epsilon_{ij}^L(\mathbf{k})\} & = 0 \end{cases}. \tag{D.14}$$

The last property we report is

$$\epsilon^{ijf} k_j \left[\epsilon^s\right]_j^l (\mathbf{k}) = -i\lambda_s k \left[\epsilon^s\right]^{li} (\mathbf{k}), \tag{D.15}$$

where $\lambda_s = +1/-1$ for $s = R/L$ and $k =\mid \mathbf{k} \mid$. In the case $s = R$ we can write the following

$$\epsilon^{ijf} k_j \left[\epsilon^R\right]_j^l (\mathbf{k}) = \epsilon^{ijf} k_j \frac{\left[\epsilon^+\right]_f^l (\mathbf{k}) + i \left[\epsilon^\times\right]_f^l (\mathbf{k})}{\sqrt{2}} \tag{D.16}$$

$$= \epsilon^{ijf} k_j \frac{v_f v^l - w_f w^l + i w_f v^l + i v_f w^l}{\sqrt{2}} \tag{D.17}$$

$$= k \frac{w^i v^l + v^i w^l - i v^i v^l + i w^i w^l}{\sqrt{2}} \tag{D.18}$$

$$= -ik \frac{i w^i v^l + i v^i w^l + v^i v^l - w^i w^l}{\sqrt{2}} \tag{D.19}$$

$$\tag{D.20}$$

$$= -ik\lambda_R \epsilon_R^{li}, \tag{D.21}$$

where we have used that $(\mathbf{a} \times \mathbf{b})^i) = \epsilon^{ijk} a_j b_k$ and that

$$\begin{cases} \mathbf{k} \times \mathbf{v} & = \mathbf{w} \\ \mathbf{k} \times \mathbf{w} & = -\mathbf{v}, \end{cases} \tag{D.22}$$

which can be obtained by the definitions of $\mathbf{v}$ and $\mathbf{w}$. Now taking $\mathbf{k} \longmapsto -\mathbf{k}$ in eq.(D.21) we get

$$\epsilon^{ijf} k_j \left[\epsilon^R\right]_j^l (-\mathbf{k}) = \epsilon^{ijf} k_j \left[\epsilon^L\right]_j^l (\mathbf{k}) = = +ik\lambda_R \epsilon_R^{li}(-\mathbf{k}) = -ik\lambda_L \epsilon_L^{li}(\mathbf{k}). \tag{D.23}$$

This concludes the demonstration of the assertion of eq.(D.15).

D.2 Two equivalent ways to express the Pontryagin density

Now we prove that the Pontryagin density

$$^*RR = {}^*R^{\sigma\rho\mu\nu} R_{\sigma\rho\mu\nu}, \tag{D.24}$$

can be expressed alternatively as

$$^*RR = {}^*C^{\sigma\rho\mu\nu} C_{\sigma\rho\mu\nu}. \tag{D.25}$$

This follows from the definition of the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}) + \frac{R}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}). \tag{D.26}$$

In fact inserting eq.(D.26) into eq.(D.25) we get

$$^*RR = {}^*C^{\sigma\rho\mu\nu} C_{\rho\sigma\mu\nu} = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma} C^{\kappa\lambda}{}_{\mu\nu} C_{\kappa\lambda\rho\sigma} \tag{D.27}$$

$$= \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}\left[R^{\kappa\lambda}{}_{\mu\nu} - \frac{1}{2}(g^{\kappa}{}_{\mu}R^{\nu}_{\lambda} - g^{\kappa}{}_{\nu}R^{\lambda}{}_{\mu} - g^{\lambda}{}_{\mu}R^{\kappa}{}_{\nu} + g^{\lambda}{}_{\nu}R^{\kappa}{}_{\mu}) + \frac{R}{6}(g^{\kappa}{}_{\mu}g^{\lambda}{}_{\nu} - g^{\lambda}{}_{\mu}g^{\kappa}{}_{\nu})\right]$$

$$\times\left[R_{\kappa\lambda\rho\sigma} - \frac{1}{2}(g_{\rho\kappa}R_{\sigma\lambda} - g_{\rho\lambda}R_{\sigma\kappa} - g_{\sigma\kappa}R_{\rho\lambda} + g_{\sigma\lambda}R_{\rho\kappa}) + \frac{R}{6}(g_{\rho\kappa}g_{\sigma\lambda} - g_{\sigma\kappa}g_{\rho\lambda})\right], \tag{D.28}$$

which can be decomposed into three terms that we evaluate separately. The first one reads as

$$1st = \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}\left[R^{\kappa\lambda}{}_{\mu\nu}R_{\kappa\lambda\rho\sigma} - \frac{1}{2}(R_{\rho}{}^{\lambda}{}_{\mu\nu}R_{\sigma\lambda} - R^{\kappa}{}_{\rho\mu\nu}R_{\sigma\kappa} - R_{\sigma}{}^{\lambda}{}_{\mu\nu}R_{\rho\lambda} + R^{\kappa}{}_{\sigma\mu\nu}R_{\rho\kappa}) + \frac{R}{6}(R_{\rho\sigma\mu\nu} - R_{\sigma\rho\mu\nu})\right] \tag{D.29}$$

$$= \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}\left[R^{\kappa\lambda}{}_{\mu\nu}R_{\kappa\lambda\rho\sigma} - \frac{1}{2}(R_{\rho}{}^{\lambda}{}_{\mu\nu}R_{\sigma\lambda} - R^{\kappa}{}_{\rho\mu\nu}R_{\sigma\kappa} - R_{\sigma}{}^{\lambda}{}_{\mu\nu}R_{\rho\lambda} + R^{\kappa}{}_{\sigma\mu\nu}R_{\rho\kappa}) + \frac{R}{6}(R_{\rho\sigma\mu\nu} - R_{\sigma\rho\mu\nu})\right] \tag{D.30}$$

$$= \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}R^{\kappa\lambda}{}_{\mu\nu}R_{\kappa\lambda\rho\sigma}, \tag{D.31}$$

where we have used that $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 0$ and $\epsilon^{\mu\nu\rho\sigma}R_{\lambda\nu\rho\sigma}R^{\lambda}{}_{\mu} = 0$. The first relation can be demonstrated in this way

$$\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \epsilon^{0ijk}R_{0ijk} + \epsilon^{i0jk}R_{i0jk} + \epsilon^{ij0k}R_{ij0k} + \epsilon^{ijk0}R_{ijk0} = 4\epsilon^{0ijk}R_{0ijk} \tag{D.32}$$

$$= 4\left[\epsilon^{0123}R_{0123} + \epsilon^{0132}R_{0132} + \epsilon^{0213}R_{0213} + \epsilon^{0231}R_{0231} + \epsilon^{0312}R_{0312} + \epsilon^{0321}R_{0321}\right] \tag{D.33}$$

$$= 4\left[\epsilon^{0123}(R_{0123} + R_{0231} + R_{0312}) + \epsilon^{0132}(R_{0132} + R_{0213} + R_{0321})\right] = 0, \tag{D.34}$$

where we have used that

$$\begin{cases} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma} \\ R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho} \\ R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu} \end{cases}, \qquad \text{and} \qquad R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0. \tag{D.35}$$

In an analogous way we get

$$\epsilon^{\mu\nu\rho\sigma}R_{\lambda\nu\rho\sigma}R^{\lambda}{}_{\mu} = \epsilon^{0ijk}R_{\lambda ijk}R^{\lambda}{}_{0} + \epsilon^{i0jk}R_{i\lambda jk}R^{\lambda}{}_{0}R^{\lambda}{}_{0} + \epsilon^{ij0k}R_{ij\lambda k}R^{\lambda}{}_{0} + \epsilon^{ijk0}R_{ijk\lambda}R^{\lambda}{}_{0} = 4\epsilon^{0ijk}R_{\lambda ijk}R^{\lambda}{}_{0} \tag{D.36}$$

$$= 4R^{\lambda}{}_{0}\left[\epsilon^{0123}R_{\lambda123} + \epsilon^{0132}R_{\lambda132} + \epsilon^{0213}R_{\lambda213} + \epsilon^{0231}R_{\lambda231} + \epsilon^{0312}R_{\lambda312} + \epsilon^{0321}R_{\lambda321}\right] \tag{D.37}$$

$$= 4R^{\lambda}{}_{0}\left[\epsilon^{0123}(R_{\lambda123} + R_{\lambda231} + R_{\lambda312}) + \epsilon^{0132}(R_{\lambda132} + R_{\lambda213} + R_{\lambda321})\right] = 0. \tag{D.38}$$

The second piece of eq.(D.28) can be written as

$$
\begin{aligned}
2nd =& -\frac{1}{4\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}(g^{\kappa}{}_{\mu}R^{\lambda}_{\nu}-g^{\kappa}_{\nu}R^{\lambda}_{\mu}-g^{\lambda}_{\mu}R^{\kappa}_{\nu}+g^{\lambda}_{\nu}R^{\kappa}_{\mu})\Big[R_{\kappa\lambda\rho\sigma}-\frac{1}{2}(g_{\rho\kappa}R_{\sigma\lambda}-g_{\rho\lambda}R_{\sigma\kappa}-g_{\sigma\kappa}R_{\rho\lambda}+g_{\sigma\lambda}R_{\rho\kappa}) \\
&+\frac{R}{6}(g_{\rho\kappa}g_{\sigma\lambda}-g_{\sigma\kappa}g_{\rho\lambda})\Big] \qquad \text{(D.39)} \\
=& -\frac{\epsilon^{\mu\nu\rho\sigma}}{4\sqrt{-g}}R^{\lambda}_{\nu}\left[R_{\mu\lambda\rho\sigma}-\frac{1}{2}(g_{\rho\mu}R_{\sigma\lambda}-g_{\rho\lambda}R_{\sigma\mu}-g_{\sigma\mu}R_{\rho\lambda}+g_{\sigma\lambda}R_{\rho\mu})+\frac{R}{6}(g_{\rho\mu}g_{\sigma\lambda}-g_{\sigma\mu}g_{\rho\lambda})-(\mu\leftrightarrow\nu)\right] \\
&+\frac{\epsilon^{\mu\nu\rho\sigma}}{4\sqrt{-g}}R^{\kappa}_{\nu}\left[R_{\kappa\mu\rho\sigma}-\frac{1}{2}(g_{\rho\kappa}R_{\sigma\mu}-g_{\rho\mu}R_{\sigma\kappa}-g_{\sigma\kappa}R_{\rho\mu}+g_{\sigma\mu}R_{\rho\kappa})+\frac{R}{6}(g_{\rho\kappa}g_{\sigma\mu}-g_{\sigma\kappa}g_{\rho\mu})-(\mu\leftrightarrow\nu)\right] \qquad \text{(D.40)} \\
=& \, 0, \qquad \text{(D.41)}
\end{aligned}
$$

where we have used that $g_{\alpha\beta}$ and $R_{\alpha\beta}$ are symmetric, $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}=0$ and $\epsilon^{\mu\nu\rho\sigma}R_{\lambda\nu\rho\sigma}R^{\lambda}{}_{\mu}=0$. The third one can be put in the following form

$$
\begin{aligned}
3rd =& \frac{R}{6}\epsilon^{\mu\nu\rho\sigma}(g^{\kappa}{}_{\mu}g^{\lambda}{}_{\nu}-g^{\lambda}{}_{\mu}g^{\kappa}{}_{\nu})\left[R_{\rho\sigma\kappa\lambda}-\frac{1}{2}(g_{\rho\kappa}R_{\sigma\lambda}-g_{\rho\lambda}R_{\sigma\kappa}-g_{\sigma\kappa}R_{\rho\lambda}+g_{\sigma\lambda}R_{\rho\kappa})+\frac{R}{6}(g_{\rho\kappa}g_{\sigma\lambda}-g_{\sigma\kappa}g_{\rho\lambda})\right] \qquad \text{(D.42)} \\
=& \frac{R}{6}\epsilon^{\mu\nu\rho\sigma}\left[R_{\rho\sigma\mu\nu}-\frac{1}{2}(g_{\rho\mu}R_{\sigma\nu}-g_{\rho\nu}R_{\sigma\mu}-g_{\sigma\mu}R_{\rho\nu}+g_{\sigma\nu}R_{\rho\mu})+\frac{R}{6}(g_{\rho\mu}g_{\sigma\nu}-g_{\sigma\mu}g_{\rho\nu})\right] \\
&-\frac{R}{6}\epsilon^{\mu\nu\rho\sigma}\left[R_{\rho\sigma\nu\mu}-\frac{1}{2}(g_{\rho\nu}R_{\sigma\mu}-g_{\rho\mu}R_{\sigma\nu}-g_{\sigma\nu}R_{\rho\mu}+g_{\sigma\mu}R_{\rho\nu})+\frac{R}{6}(g_{\rho\nu}g_{\sigma\mu}-g_{\sigma\nu}g_{\rho\mu})\right] \qquad \text{(D.43)} \\
=& \, 0. \qquad \text{(D.44)}
\end{aligned}
$$

Thus we have proved the statement.

D.3 Expansion of the Pontryagin density in tensor perturbation up to second order

As explained in section 5.4 we don't need second order perturbations but only first order one. Thus we can use the following metric tensor

$$
\tilde{g}_{\alpha\beta}=\begin{pmatrix}-1 & 0\\ 0 & (\delta_{ij}+h_{ij})\end{pmatrix}, \qquad \tilde{g}^{\alpha\beta}=\begin{pmatrix}-1 & 0\\ 0 & (\delta^{ij}-h^{ij}).\end{pmatrix} \qquad \text{(D.45)}
$$

Now we can proceed in computing the Christoffel symbols and we get

$$
\begin{aligned}
\Gamma^0_{00} &= \Gamma^0_{0i}=\Gamma^i_{00}=0, && \text{(D.46)}\\
\Gamma^0_{ij} &= \frac{1}{2}g^{00}(-g_{ij,0})=\frac{1}{2}h'_{ij}, && \text{(D.47)}\\
\Gamma^i_{0j} &= \frac{1}{2}g^{ik}(g_{k0,j}+g_{kj,0}-g_{0j,k})=\frac{1}{2}\delta^{ik}h'{}_{kj}=\frac{1}{2}h'^{i}{}_{j}, && \text{(D.48)}\\
\Gamma^i_{jk} &= \frac{1}{2}\delta^{if}(h_{jf,k}+h_{kf,j}-g_{jk,f})=\frac{1}{2}(h^i_{j,k}+h^i_{k,j}-g^{\,,i}_{jk}), && \text{(D.49)}
\end{aligned}
$$

where, in this context, we have represented the time derivative using the conformal derivative, as we began with a metric where we adopted conformal time. Thus we can expand the zero component of the topological current,

$$
K^0:=\epsilon^{0\beta\delta\gamma}\left(\Gamma^{\chi}_{\beta\sigma}\partial_{\delta}\Gamma^{\sigma}_{\gamma\chi}+\frac{2}{3}\Gamma^{\chi}_{\beta\sigma}\Gamma^{\sigma}_{\delta\epsilon}\Gamma^{\epsilon}_{\gamma\chi}\right), \qquad \text{(D.50)}
$$

up to second order, in the following way

$$K^0 = \epsilon^{0bcd}\left(\Gamma^n{}_{bm}\partial_c\Gamma^m{}_{dn} + \frac{2}{3}\Gamma^n{}_{bm}\Gamma^m{}_{cl}\Gamma^l{}_{dn}\right) \tag{D.51}$$

$$= \epsilon^{0bcd}\left(\Gamma^0{}_{b0}\partial_c\Gamma^0{}_{d0} + \Gamma^0{}_{bf}\partial_c\Gamma^f{}_{d0} + \Gamma^f{}_{b0}\partial_c\Gamma^0{}_{df} + \Gamma^f{}_{bl}\partial_c\Gamma^l{}_{df}\right) \tag{D.52}$$

$$= \epsilon^{0bcd}\left(\frac{1}{4}h'_{bf}\partial_c h'^f{}_d + \frac{1}{4}h'^f{}_b\partial_c h'_{df} + \frac{1}{2}(h^f{}_{b,l} + h^f{}_{l,b} - h_{lb}{}^{,f})\frac{1}{2}(h^l{}_{d,fc} + h^l{}_{f,dc} - h_{fd,c}{}^{,l})\right) \tag{D.53}$$

$$= \epsilon^{0bcd}\frac{1}{2}\left(h'^f{}_b\partial_c h'_{df} - h_{lb}{}^{,f}h^l{}_{d,fc}\right), \tag{D.54}$$

where in the second passage we have disregarded third order terms while in the last one, we have eliminated all the contributions that can be integrated away in the action. Thus we get

$$S^{(2)}_{CS} = \int d^4x\left[-\frac{\phi'}{2f}K^0\right] = \int d^4x\frac{\phi'}{4f}\left[-h'^f{}_b\partial_c h'_{df} + h_{lb}{}^{,f}h^l{}_{d,fc}\right]. \tag{D.55}$$

D.4 The actions S_1, S_2, and S_3 in Fourier space

So starting from the kinetic term, i.e. S_1, using this expression for the tensor in Fourier space[31]

$$h_{ij}(\mathbf{x},\tau) = \sum_{s=R/L}\int\frac{d^3k}{(2\pi)^3}u_s(\mathbf{k},\tau)\epsilon^s_{ij}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{D.56}$$

we can write

$$S_1 = \int d^4x\frac{M^2_{pl}}{8}a^2\sum_{a,z=R,L}\int\frac{d^3q d^3k e^{i(\mathbf{q}+\mathbf{k})\cdot\mathbf{x}}}{(2\pi)^6}\left\{u'_a(\mathbf{k})u'_z(\mathbf{q})[\epsilon_a]^i_j(\mathbf{k})[\epsilon_z]^j_i(\mathbf{q}) + k^f q_f u_a(\mathbf{k})u_z(\mathbf{q})[\epsilon_a]^i_j(\mathbf{k})[\epsilon_z]^j_i(\mathbf{q})\right\} \tag{D.57}$$

$$= \int d\tau\frac{M^2_{pl}}{8}a^2\sum_{a,z=R,L}\int\frac{d^3k}{(2\pi)^3}\left\{u'_a(\mathbf{k})u'_z(-\mathbf{k})[\epsilon_a]^i_j(\mathbf{k})[\epsilon_z]^j_i(-\mathbf{k}) - k^2u_a(\mathbf{k})u_z(-\mathbf{k})[\epsilon_a]^i_j(\mathbf{k})[\epsilon_z]^j_i(-\mathbf{k})\right\} \tag{D.58}$$

$$\begin{aligned}
&= \int\frac{d\tau M^2_{pl}a^2}{8}\frac{d^3k}{(2\pi)^3}\Big\{u'_R(\mathbf{k})u'_R(-\mathbf{k})[\epsilon_R]^i_j(\mathbf{k})[\epsilon_R]^j_i(-\mathbf{k}) + u'_R(\mathbf{k})u'_L(-\mathbf{k})[\epsilon_R]^i_j(\mathbf{k})[\epsilon_L]^j_i(-\mathbf{k})\\
&+ u'_L(\mathbf{k})u'_R(-\mathbf{k})[\epsilon_L]^i_j(\mathbf{k})[\epsilon_R]^j_i(\mathbf{k}) + u'_L(\mathbf{k})u'_L(-\mathbf{k})[\epsilon_L]^i_j(\mathbf{k})[\epsilon_L]^j_i(-\mathbf{k}) - k^2[u_R(\mathbf{k})u_R(-\mathbf{k})[\epsilon_R]^i_j(\mathbf{k})[\epsilon_R]^j_i(-\mathbf{k})\\
&+ u_R(\mathbf{k})u_L(-\mathbf{k})[\epsilon_R]^i_j(\mathbf{k})[\epsilon_L]^j_i(-\mathbf{k}) + u_L(\mathbf{k})u_R(-\mathbf{k})[\epsilon_L]^i_j(\mathbf{k})[\epsilon_R]^j_i(-\mathbf{k}) + u_L(\mathbf{k})u_L(-\mathbf{k})[\epsilon_L]^i_j(\mathbf{k})[\epsilon_L]^j_i(-\mathbf{k})]\Big\}\\
&= \int\frac{d\tau M^2_{pl}a^2}{8}\frac{d^3k}{(2\pi)^3}\Big\{u'_R(\mathbf{k})u'_R(-\mathbf{k})[\epsilon_R]^i_j(\mathbf{k})[\epsilon_L]^j_i(\mathbf{k}) + u'_L(\mathbf{k})u'_L(-\mathbf{k})[\epsilon_L]^i_j(\mathbf{k})[\epsilon_R]^j_i(\mathbf{k})\\
&- k^2[u_R(\mathbf{k})u_R(-\mathbf{k})[\epsilon_R]^i_j(\mathbf{k})[\epsilon_L]^j_i(\mathbf{k}) + u_L(\mathbf{k})u_L(-\mathbf{k})[\epsilon_L]^i_j(\mathbf{k})[\epsilon_R]^j_i(\mathbf{k})]\Big\}
\end{aligned} \tag{D.59}$$

$$= \int\frac{d\tau M^2_{pl}a^2}{4}\frac{d^3k}{(2\pi)^3}\left\{u'_R(\mathbf{k})u'_R(-\mathbf{k}) + u'_L(\mathbf{k})u'_L(-\mathbf{k}) - k^2\left[u_R(\mathbf{k})u_R(-\mathbf{k}) + u_L(\mathbf{k})u_L(-\mathbf{k})\right]\right\}, \tag{D.60}$$

[31]Notice that between this mode functions and the one written in section 5.4 there is a multiplicative factor which doesn't modify on the *EoM*.

where in the second equality we have performed the $\mathbf{q}$ integral using the $\delta^3(\mathbf{k}+\mathbf{q})$ arising from the $\mathbf{x}$ integral while in the fourth we have used eq.(D.14). For the action S_2 we obtain

$$S_2 = -i \sum_{z,a=R/L} \int d^4x \frac{\phi^{'}\epsilon^{ijf}}{4f} \int \frac{d^3k e^{+i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^3} \frac{d^3q e^{+i\mathbf{q}\cdot\mathbf{x}}}{(2\pi)^3} \Big[[\epsilon_z]_{li}(\mathbf{k}) u_z^{'}(\mathbf{k}) q_j [\epsilon_a]^l{}_j(\mathbf{q}) u_a^{'}(\mathbf{q}) \Big] \tag{D.61}$$

$$= -i \int d\tau \frac{\phi^{'}\epsilon^{ijf}}{4f} \int \frac{d^3k}{(2\pi)^3} \sum_{z,a=R/L} \Big\{ [\epsilon_z]_{li}(\mathbf{k}) u_z^{'}(\mathbf{k}) k_f [\epsilon_a]^l{}_j(-\mathbf{k}) u_a^{'}(-\mathbf{k}) \Big\} \tag{D.62}$$

$$= -i \int d\tau \frac{\phi^{'}\epsilon^{ijf}k_f}{4f} \int \frac{d^3k}{(2\pi)^3} \Big\{ u_R^{'}(\mathbf{k}) u_R^{'}(-\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(-\mathbf{k})$$
$$+ u_R^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(-\mathbf{k}) + u_L^{'}(\mathbf{k}) u_R^{'}(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(-\mathbf{k}) + u_L^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_L]^l{}_j(-\mathbf{k}) \Big\} \tag{D.63}$$

$$= -i \int d\tau \frac{\phi^{'}\epsilon^{ijf}k_f}{4f} \int \frac{d^3k}{(2\pi)^3} \Big\{ u_R^{'}(\mathbf{k}) u_R^{'}(-\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) [\epsilon_L]^l{}_j(\mathbf{k}) + u_R^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(\mathbf{k})$$
$$+ u_R^{'}(-\mathbf{k}) u_L^{'}(\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_L]^l{}_j(\mathbf{k}) + u_L^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(\mathbf{k}) \Big\} \tag{D.64}$$

$$= -i \int d\tau \frac{\phi^{'}\epsilon^{ijf}k_f}{4f} \int \frac{d^3k}{(2\pi)^3} \Big\{ u_R^{'}(\mathbf{k}) u_R^{'}(-\mathbf{k}) [\epsilon_L]^l{}_j(\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) + u_L^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(\mathbf{k}) \Big\} \tag{D.65}$$

$$= -\int d\tau \frac{\phi^{'}k}{4f} \int \frac{d^3k}{(2\pi)^3} \Big\{ u_R^{'}(\mathbf{k}) u_R^{'}(-\mathbf{k}) [\epsilon_R]^l{}_j(\mathbf{k}) [\epsilon_L]_{lj}(\mathbf{k}) - u_L^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_R]^l{}_i(\mathbf{k}) \Big\} \tag{D.66}$$

$$= -\int d\tau \frac{\phi^{'}k}{2f} \int \frac{d^3k}{(2\pi)^3} \Big\{ u_R^{'}(\mathbf{k}) u_R^{'}(-\mathbf{k}) - u_L^{'}(\mathbf{k}) u_L^{'}(-\mathbf{k}) \Big\} \tag{D.67}$$

where we have used the same techniques adopted in deriving eq.(D.67), we have used eq.(D.13), in the fifth equality we have exploited the property demonstrated in eq.(D.21) and (D.23) of the appendix D.1. Regarding the action S_3 we get

$$S_3 = -\sum_{z,a=R/L} \int d^4x \frac{\phi^{'}\epsilon^{ijf}}{4f} \int \frac{d^3k e^{+i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^3} \frac{d^3q e^{+i\mathbf{q}\cdot\mathbf{x}}}{(2\pi)^3} \Big[+ik_m [\epsilon_z]_{li}(\mathbf{k}) u_z(\mathbf{k}) \Big(q_f q^m [\epsilon_a]^l{}_j(\mathbf{q}) u_a(\mathbf{q}) \Big) \Big] \tag{D.68}$$

$$= -i \int d\tau \frac{\phi\epsilon^{ijf}}{4f} a^4 \int \frac{d^3k}{(2\pi)^3} k^2 \sum_{z=R/L} \Big\{ \Big[[\epsilon_z]_{li}(\mathbf{k}) u_z(\mathbf{k}) k_f [\epsilon_a]^l{}_j(-\mathbf{k}) u_a(-\mathbf{k}) \Big] \Big\} \tag{D.69}$$

$$= -i \int d\tau \frac{\phi^{'}\epsilon^{ijf}}{4f} \frac{d^3k}{(2\pi)^3} k^2 k_f \Big\{ u_R(\mathbf{k}) u_R(-\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(-\mathbf{k}) + u_R(\mathbf{k}) u_L(-\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) [\epsilon_L]^l{}_j(-\mathbf{k})$$
$$+ u_L(\mathbf{k}) u_R(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(-\mathbf{k}) + u_L(\mathbf{k}) u_L(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_L]^l{}_j(-\mathbf{k}) \Big\} \tag{D.70}$$

$$= -i \int d\tau \frac{\phi^{'}\epsilon^{ijf}}{4f} \frac{d^3k}{(2\pi)^3} k^2 k_f \Big\{ u_R(\mathbf{k}) u_R(-\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) [\epsilon_L]^l{}_j(\mathbf{k}) + u_R(\mathbf{k}) u_L(-\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(\mathbf{k})$$
$$+ u_L(\mathbf{k}) u_R(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_L]^l{}_j(\mathbf{k}) + u_L(\mathbf{k}) u_L(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(\mathbf{k}) \Big\} \tag{D.71}$$

$$= -i \int d\tau \frac{\phi^{'}\epsilon^{ijf}}{4f} \frac{d^3k}{(2\pi)^3} k^2 k_f \Big\{ u_R(\mathbf{k}) u_R(-\mathbf{k}) [\epsilon_R]_{li}(\mathbf{k}) [\epsilon_L]^l{}_j(\mathbf{k}) + u_L(\mathbf{k}) u_L(-\mathbf{k}) [\epsilon_L]_{li}(\mathbf{k}) [\epsilon_R]^l{}_j(\mathbf{k}) \Big\} \tag{D.72}$$

$$= +\int d\tau \frac{\phi^{'}}{2f} \frac{d^3k}{(2\pi)^3} k^3 \Big\{ u_R(\mathbf{k}) u_R(-\mathbf{k}) - u_L^{'}(\mathbf{k}) u_L(-\mathbf{k}) \Big\}, \tag{D.73}$$

where we have exploited the same properties used for deriving the expression of S_2.

D.5 Expansion of $\frac{z''(\tau,\mathbf{k})}{z(\tau,\mathbf{k})}$

The goal of this section is to expand at the lowest order in the slow roll parameters $\frac{z''(\tau,\mathbf{k})}{z(\tau,\mathbf{k})}$, where we recall that

$$z_s(\mathbf{k},\tau) \equiv a(\tau)\sqrt{1-\lambda_s \frac{2k}{M_{pl}^2 f a^2}\phi'} = a(\tau)\sqrt{1-\lambda_s \frac{kg}{a^2}}, \tag{D.74}$$

where in order to simplify the computations, we have introduced

$$g = \frac{2}{M_{pl}^2 f}\phi'. \tag{D.75}$$

The time derivative of z reads

$$z_s' = \frac{a'}{a}z_s - \frac{1}{2}\lambda_s k \frac{a^2}{z_s}\left(\frac{g'}{a^2}\right)', \tag{D.76}$$

while we can write its second derivative as

$$z_s'' = \frac{a''}{a}z_s - \frac{a'^2}{a^2}z_s + \frac{a'}{a}z_s' - \frac{1}{2}\lambda_s k\left[\frac{a^2}{z_s}\left(\frac{g'}{a^2}\right)'' + \left(\frac{2aa'}{z_s} - \frac{a^2 z_s'}{z_s^2}\right)\left(\frac{g'}{a^2}\right)'\right]. \tag{D.77}$$

Now we can write the quantity of interest as

$$\frac{z_s''}{z_s} = \frac{a''}{a} - \frac{a'^2}{a^2} + \frac{a'}{a}\left(\frac{a'}{a} - \frac{1}{2}\lambda_s k\frac{a^2}{z_s^2}\left(\frac{g'}{a^2}\right)'\right) - \frac{1}{2}\lambda_s\frac{k}{z_s}\left[\frac{a^2}{z_s}\left(\frac{g'}{a^2}\right)'' + \left(\frac{2aa'}{z_s} - \frac{a^2 z_s'}{z_s^2}\right)\left(\frac{g'}{a^2}\right)'\right] \tag{D.78}$$

$$= \frac{a''}{a} - \frac{\lambda_s k}{2}\frac{\left(\frac{g'}{a^2}\right)''}{1-\lambda_s k\frac{g'}{a^2}} - \frac{3a'a}{2}\frac{\lambda_s k a^2}{z_s^2}\left(\frac{g'}{a^2}\right)' - \frac{1}{2}\lambda_s\frac{k}{z_s}\left\{-\frac{a^2}{z_s^2}\left[\frac{a'}{a}z_s - \frac{\lambda_s k a^2}{2z_s}\left(\frac{g'}{a^2}\right)'\right]\right\}\left(\frac{g'}{a^2}\right)' \tag{D.79}$$

$$= \frac{a''}{a} - \frac{\lambda_s k}{2}\frac{\left(\frac{g'}{a^2}\right)''}{1-\lambda_s k\frac{g'}{a^2}} - \frac{a'}{a}\lambda_s k\frac{\left(\frac{g'}{a^2}\right)'}{1-\lambda_s k\left(\frac{g'}{a^2}\right)'} - \frac{1}{4}\frac{(\lambda_s k)^2}{z_s}\frac{\left[\left(\frac{g'}{a^2}\right)'\right]^2}{\left[1-\lambda_s k\left(\frac{g'}{a^2}\right)'\right]^4}. \tag{D.80}$$

Now we need the explicit expression of $\frac{g'}{a^2}$ and its derivatives as functions of the slow roll parameters (see section 2.2). Thus, considering the lowest order and taking $\dot{\phi} < 0$[32] [68] we retrieve

$$\left(\frac{g'}{a^2}\right) = \frac{2}{M_{pl}^2 f}\left(\frac{\phi'}{a^2}\right) = \frac{2}{M_{pl}^2 f}\left(\frac{\dot{\phi}}{a}\right) = \frac{2}{M_{pl}^2 f}\frac{\dot{\phi}}{a} = -\frac{2}{M_{pl}^2 f}\sqrt{2\epsilon}M_{pl}H_{inf}^2\tau(1+\epsilon) \approx \frac{2\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H}{M_{pl}}\right)^2\tau, \tag{D.81}$$

where we have used that $\dot{\phi} = -\sqrt{2}\epsilon M_{pl}H$ and $a(\tau) = -\frac{1}{H\tau(1-\epsilon)}$ (see appendix C.1). While regarding its first and second derivatives we respectively get

$$\left(\frac{g'}{a^2}\right)' = \frac{2\sqrt{2}M_{pl}}{f}\left\{\sqrt{\epsilon}\left(\frac{H}{M_{pl}}\right)^2 + \left[\sqrt{\epsilon}\left(\frac{H}{M_{pl}}\right)^2\right]'\right\} = \frac{2M_{pl}}{f}\left(\frac{H}{M_{pl}}\right)^2\left[\sqrt{2\epsilon} + O(\epsilon^{\frac{3}{2}})\right] \approx \frac{2\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H}{M_{pl}}\right)^2, \tag{D.82}$$

[32]We can choose the sign of the derivative of the inflaton field as we want since it depends on the shape of the potential.

where we have used the fact that derivatives of the slow-roll parameters and H produce other powers of these parameters and

$$\left(\frac{g^{'}}{a^2}\right)^{''} \approx \left[\frac{2\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H_{inf}}{M_{pl}}\right)^2\right]^{'} = \frac{2\sqrt{2}M_{pl}}{f}\left(\frac{H_{inf}}{M_{pl}}\right)^2 O(\epsilon^{\frac{3}{2}}) \approx 0. \tag{D.83}$$

Inserting eq.(D.81), (D.82) and (D.83), in eq.(D.80), we obtain

$$\frac{z_s^{''}}{z_s} \approx \frac{a^{''}}{a} - \frac{a^{'}}{a}\lambda_s k \frac{\frac{2\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H_{inf}}{M_{pl}}\right)^2}{1-\lambda_s k\frac{2\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H_{inf}}{M_{pl}}\right)^2} \approx \frac{a^{''}}{a} - \frac{a^{'}}{a}\lambda_s k\frac{2\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H_{inf}}{M_{pl}}\right)^2. \tag{D.84}$$

Now using that what it's shown in appendix C.1 we obtain

$$\frac{z_s^{''}}{z_s} \approx \frac{2}{\tau^2}(1+\frac{3}{2}\epsilon) + \frac{1}{\tau}\lambda_s k\frac{2\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H_{inf}}{M_{pl}}\right)^2. \tag{D.85}$$

Finally, if we introduce the “chemical potential" μ as

$$\mu \equiv \frac{\sqrt{2\epsilon}M_{pl}}{f}\left(\frac{H_{inf}}{M_{pl}}\right)^2 \tag{D.86}$$

we arrive at the desire expression

$$\chi_s^{''} + \left[k^2 - \left(\frac{1+3\epsilon}{\tau^2} + \frac{2k\mu}{\tau}\right)\right]\chi_s = 0, \tag{D.87}$$

which coincides with eq.(4.9) of [7] and with eq.(59) of [68].

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