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Sistemi dinamici in Cosmologia: l’accoppiamento del fattore di scala con i campi cosmologici

Cosmology and dynamical systems: the relationship between the scale factor and cosmological fields

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"Se i corpi luminosi sono carichi di incertezza, non resta che affidarsi al buio."

I.Calvino

Summary

In order to put a spotlight on the fundamental fields that shape the cosmic dynamics, this thesis discusses in an introductive manner the connection between the realms of dynamical systems theory and Cosmology. The primary objective is to discuss how mathematical models can be employed to comprehend the complex dynamics governing the observable universe, by yielding a nonlinear coupling between the scale factor, on one hand, and a possible cosmological field, permeating the space at cosmological scale, on the other.

This thesis begins introducting the Lagrangian formalism applicable to cosmological fields, processing towards an exploration of the Einstein Field Equations and Friedmann solutions, and finally, delving into the dynamics of scalar-tensor theories and how they affect the scale factor's evolution. A fundamental reference for this exploration, especially in the final chapter, is the ELSEVIER-published article titled 'Dynamical Systems Applied to Cosmology: dark energy and modified gravity' (2018). This article serves as a keystone, guiding the main considerations on cosmological models influenced by dark energy.

Sommario

Al fine di mettere in evidenza i campi fondamentali che plasmano la dinamica cosmica, questa tesi discute in modo introduttivo la connessione tra i domini della teoria dei sistemi dinamici e della cosmologia. L'obiettivo principale è esaminare come modelli matematici possano essere impiegati per comprendere le dinamiche complesse che governano l'universo osservabile, generando un'accoppiata non lineare tra il fattore di scala, da un lato, e un possibile campo cosmologico che permea lo spazio a scala cosmologica, dall'altro.

Il percorso di questa tesi inizia con un'introduzione al formalismo lagrangiano applicabile ai campi cosmologici,procedendo poi con l'esplorazione delle equazioni di campo di Einstein e delle soluzioni di Friedmann, e infine approfondendo la dinamica delle teorie scalari-tensoriali e il modo in cui influenzano l'evoluzione del fattore di scala. Un riferimento fondamentale per questa indagine, specialmente nel capitolo finale, è l'articolo pubblicato da ELSEVIER intitolato "Dynamical Systems Applied to Cosmology: dark energy and modified gravity" (2018). Questo articolo fornisce il fondamento principale guidando le principali considerazioni sui modelli cosmologici influenzati dall'energia oscura.

Contents

Introduction

The vast array of galaxies, stars, and cosmic structures in the universe offers a captivating tale of an intricate dynamical system. At the intersection of theoretical physics and cosmology, this thesis focuses on using dynamical systems theory to understand the dynamical evolution of the cosmos. Dark energy dynamics, and complex interactions between cosmic elements (radiation, baryonic matter, dark matter, and dark energy) are the main cosmological study topics examined in this dissertation.

In the first chapter, we delve into the specifics of the Lagrangian formalism for classical fields. Adopting the Lagrangian formalism in Field Theory offers several advantages and intuitions that make it a powerful and wisely used framework, mostly because the Lagrangian approach provides a unified and elegant description of a wide range of physical systems: from scalar fields to vector fields and beyond. This uniformity simplifies the theoretical structure, providing a systematic way to obtain the equations of motions for fields. The *Euler-Lagrange equations* are at first constructed analytically. Afterwards, we use the Principle of Least Action to derive the equations more simply and directly. According to this principle, the configuration or evolution pursued by a dynamical system is the one for which the action is minimized. We explain how a system is defined in both cases by constructing a Lagrangian density and by deriving the Euler-Lagrange equations. Using the Electromagnetic Field and the Schrödinger Field as examples, we utilize this formalism to derive, by the Euler-Lagrange equations, the Maxwell equations in the first case, and the Schrödinger equation in the second. The Quantum Electrodynamics Lagrangian (QED) is presented in the final section. It characterizes a system in which the electromagnetic field affects the wave function, it's derived by expressing the current density and charge density in terms of the wave function.

In the second chapter, we explore the application of the Lagrangian formalism to cosmological field models. We provide a list of the primary fields in our universe according to the Standard Model of particle physics. We begin with the fundamental particles, which are represented by

fermionic fields, and their interactions defined via bosonic field. In sec.2.1 we determine the potential role of gravitation in this collection of fields. The *scale factor* becomes significant in the context of the Cosmological Field because it dictates how the cosmological model, and the related universe, evolve over time. General Relativity establishes a close relationship between the cosmic scale factor and gravity through the Einstein Equations, which translate into the Euler-Lagrange equations in the Lagrangian formalism. This relationship is represented by the *stress-energy tensor*, which is an energy density of matter and radiation. The features of this tensor are analyzed and connected to the Einstein Field Equations in sections 2.2 and 2.3, where we derive the Friedmann solutions of the Einstein equations for a homogeneous and isotropic cosmological model (all derivations where verified using the symbolic manipulator *Wolfram Mathematica*). The Riemann and Ricci tensor are computed whose mathematical formulation has been provided by Weinberg [SWe72] and Hartle [BHa03].
The last chapter investigates how the choice of different cosmological models influences the evolution of the scale factor, establishing its dependence over time in distinct energy domination epochs. Specifically, within the dark energy domination era we observe three primary modelizations of its origin.

Chapter 1

The Lagrangian Formulation for Continuous Systems

In this chapter we delve into the framework of the *Lagrangian formulation* of Field Theory, the starting point for the discussion over cosmological fields. At the heart of this formalism lies the principle of least action, which provides a unified and intuitive perspective on the behaviour of physics fields. We explore how the *Lagrangian density*, a function of field values and their derivatives, captures the dynamics of particles and fields, allowing us to derive the fundamental equations of motion of the physical field in question.

First we will clarify the physical meaning and the correct terminology to understand what is meant by the terms *"field"* and *"density"*, key words in the Lagrangian formulation.

Given a mechanical system of N material points with masses m_i, $i = 1, ..., N$ and coordinates r_i subject to m constraints, let $q_1, ...q_n$ with $n = 3N - m$ be the free coordinates that define the parameterization of the system's coordinates on the constrained manifold. According to the Lagrangian formalism, the mechanical system energy is represented by the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where $\mathbf{q}$ is the generalized coordinate $\mathbf{q} = (q_1, q_2, ..., q_n)$ and $\dot{\mathbf{q}}$ is a shortcut for a time derivative. The Lagrangian is universally defined as $L = K(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}, t)$, with $K(\mathbf{q}, \dot{\mathbf{q}}, t)$ the kinetic energy and $U(\mathbf{q}, t)$ the potential energy. From L can be derived the *Equations of Euler-Lagrange* [C E22]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{1.1}$$

The equations of motion derived using the Euler-Lagrange method turn out to be second-order differential equations. This property of the Lagrangian formalism is due to the fact that, in the Lagrangian, the kinetic energy term is generally a scalar quantity that depends on the square of the velocity components (for scalar coordinates i.e. $K = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$.

To better understand, now, the passing to a continuous system, we consider a system of particles undergoing oscillations, where each particle is characterized by its position $x^{(j)}$, expressed as a sum of its fixed equilibrium position x_j and a displacement $\phi_j(t)$.

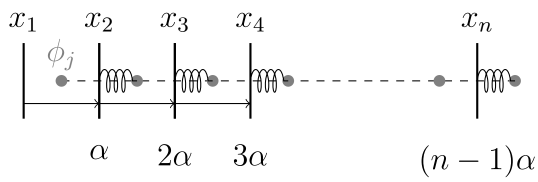

Figure 1.1: System of j particles subject to a small perturbation ϕ_j

The kinetic energy of the system is given by:

$$K = \frac{1}{2} m \sum_j \dot{\phi}_j^2$$

while

$$V = \sum_{j=2}^{n} V^I \left(\phi_j - \phi_{j-1}\right)$$

is the potential energy. From these formulas we can write

$$L = \frac{1}{2} m \sum_{j=1}^{n} \dot{\phi}_j^2 - \frac{1}{2} \sum_{j=2}^{n} V^I \left(\phi_j - \phi_{j-1}\right)$$

With the addition of an external potential the Lagrangian takes the form:

$$L = \sum_{j=1}^{n} \frac{1}{2} m \dot{\phi}_j^2 - \sum_{j=2}^{n} V^I(\phi_j - \phi_{j-1}) - \sum_{j=1}^{n} V^E(\phi_j) \tag{1.2}$$

When the distance between the particles tends to zero, the system becomes describable in a continuous manner, and $\phi(x_j, t)$ transforms into $\phi(x, t)$, becoming a function of both time and space: a *field*. We are formalising the transition form a classical system with a finite number of degrees of freedom $q_j(t)$ to a classical field theory in terms of a scalar field $\phi(\mathbf{r}, t)$.

Formally, the transition to the continuum occurs as follows: we first rewrite the Lagrangian as

$$L = \alpha \left[\sum_{j=1}^{n} \frac{1}{2} \frac{m}{\alpha} \dot{\phi}_j^2 - \sum_{j=2}^{n} \frac{1}{\alpha} V^I(\phi_j - \phi_{j-1}) - \sum_{j=1}^{n} \frac{1}{\alpha} V^E(\phi_j) \right]$$

where α is the distance between particles. Then, adopting the following rule for the passing to the continuous case

$$\frac{\partial V^I}{\alpha \partial(\xi_{j+1} - \xi_j)} \longrightarrow \left. \frac{\partial V^I}{\partial r} \right|_{r=\phi_{j+1}-\phi_j}$$

where $r = \alpha(\xi_{j+1} - \xi_j)$ has been rescaled with $\xi(x,t) = \frac{\phi}{\alpha}$, as

$$\lim_{\alpha \to 0} \frac{\phi_{j+1} - \phi_j}{\alpha} = \left. \frac{\partial \phi}{\partial x} \right|_{x=x_j}$$

we can determine $V^I(\phi_{j+1} - \phi_j) \to V^I(\frac{\partial \phi}{\partial x})$. The mass function becomes $\rho_j = \frac{m_j}{\alpha}$ which represents the local density at the nominal point x_j, and this transforms into $\rho(x)$ in the continuous limit. Thus, in the transition to the continuum, it follows that:

$$K = \sum_{j=1}^{n} \alpha \rho_j \dot{\phi}_j^2 \to \int dx \rho(x) \dot{\phi}_j^2$$

with $\alpha = \Delta x$ infinitesimal, leading to

$$U = \sum_{j=2}^{n} \alpha \frac{1}{\alpha} V^I(\phi_j - \phi_{j-1}) \to \int dx \widetilde{V^I} \left(\frac{\partial \phi}{\partial x} \right)$$

Finally the external potential is scaled as:

$$\alpha \sum_{j=1}^{n} \frac{1}{\alpha} V^E(\phi_j) = \sum_{j=1}^{n} \Delta x \widetilde{V^E}(\phi_j) \to \int dx \widetilde{V^E}(\phi_j)$$

In the continuum, we then define the Lagrangian as :

$$L = \int_{space} dx \left[\frac{1}{2} \rho(x) \left(\frac{\partial \phi}{\partial t} \right)^2 - \widetilde{V^I} \left(\frac{\partial \phi}{\partial x} \right) - \widetilde{V^E}(\phi) \right] = \int_{space} dx \mathcal{L}$$

where $\mathcal{L}\left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \phi, x, t\right)$ is referred to as the *Lagrangian density*, a function that depends on the field variable ϕ, its spatial derivatives and time. While the Lagrangian (L) carries units of energy, the Lagrangian density ($\mathcal{L}$) has the units of energy per unit volume. From the continuum we have that:

$$L = \sum_{j=1}^{N} \alpha \frac{1}{2} \frac{m_j}{\alpha} \dot{\phi}_j^2 - \sum_{j=2}^{N} \alpha \frac{1}{\alpha} V^I(\phi_j - \phi_{j-1}) - \sum_{j=1}^{N} \alpha \frac{1}{\alpha} V^E(\phi_j)$$

$$\begin{aligned} \Longrightarrow L = \alpha \frac{m_1}{2\alpha} \dot{\phi}_1^2 - \alpha \frac{1}{\alpha} V^E(\phi_1) + \alpha \frac{m_N}{\alpha} \dot{\phi}_N^2 - \alpha \frac{1}{\alpha} V^I(\phi_N - \phi_{N-1}) - \alpha \frac{1}{\alpha} V^E(\phi_N) + \\ + \sum_{j=2}^{N-1} \alpha \left(\frac{1}{2} \frac{m_j}{\alpha} \dot{\phi}_j^2 - \frac{1}{\alpha} V^I(\phi_j - \phi_{j-1}) - \frac{1}{\alpha} V^E(\phi_j) \right) \end{aligned}$$

The first terms represent the boundaries of the problem, the outcomes of which are readily determined by the boundary conditions, and in the continuous case, they vanish. Therefore, for the purpose of this discussion, we will limit our study to the Euler-Lagrange equations for the values of $j = 2, ..., n-1$; where the Lagrangian density (last term of the equation above) is:

$$\mathcal{L} = \sum_{j=2}^{N-1} \alpha \left(\frac{1}{2} \frac{m_j}{\alpha} \dot{\phi}_j^2 - \frac{1}{\alpha} V^I(\phi_j - \phi_{j-1}) - \frac{1}{\alpha} V^E(\phi_j) \right)$$

The Euler-Lagrange equations, for the j-th body (from eq. 1.4):

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}_j} - \frac{\partial L}{\partial \phi_j} = 0 \Longrightarrow \frac{d}{dt} \left(\alpha \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j} \right) = \frac{\partial L}{\partial \phi_j}$$

where:

$$\begin{aligned} \frac{\partial L}{\partial \phi_j} &= -\alpha \frac{1}{\alpha} \frac{\partial}{\partial \phi_j} \left[V^I(\phi_j - \phi_{j-1}) + V^I(\phi_{j+1} - \phi_j) \right] = \\ &= -\alpha \frac{\partial}{\partial \phi_j} \left[\frac{1}{\alpha} V^I(\alpha \phi'_j) + \frac{1}{\alpha} V^I(\alpha \phi'_{j+1}) \right] \end{aligned}$$

with $\frac{1}{\alpha} V^I = \widetilde{V^I}$, and

$$\frac{\partial \phi}{\partial x} = \lim_{\alpha \to 0} \frac{\phi_j - \phi_{j-1}}{\alpha} = \phi'_j$$

$$\frac{\partial}{\partial \phi_j} \widetilde{V^I}(\alpha \phi'_j) = \frac{\partial \widetilde{V^I}(\alpha \phi'_j)}{\partial (\alpha \phi'_j)} \frac{\partial (\alpha \phi'_j)}{\partial \phi_j}$$

where the second term $\frac{\partial (\alpha \phi'_j)}{\partial \phi_j} = 1$. Working similarly for the term in $j+1$, we obtain:

$$\Longrightarrow \frac{\partial L}{\partial \phi_j} = -\alpha \left[\frac{\partial \widetilde{V^I}}{\partial (\alpha \phi'_j)} - \frac{\partial \widetilde{V^I}}{\partial (\alpha \phi'_{j+1})} \right] - \alpha \frac{\partial \widetilde{V^E}}{\partial \phi_j}$$

The Euler-Lagrange equations thus take the form:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_j} - \left(\frac{\partial \widetilde{V^I}}{\partial \phi'_{j+1}} - \frac{\partial \widetilde{V^I}}{\partial \phi'_j}\right)\frac{1}{\alpha} + \frac{\partial \widetilde{V^E}}{\partial \phi_j} = 0$$

For a continuous system [VBa04]:

$$\begin{aligned} &\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_j} \longrightarrow \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial t}\right)}\right) \\ &\frac{1}{\alpha}\left(\frac{\partial \tilde{V}^I}{\partial \phi'_{j+1}} - \frac{\partial \tilde{V}^I}{\partial \phi'_j}\right) \longrightarrow \frac{\partial}{\partial x}\frac{\partial \tilde{V}^I}{\partial \phi'} \\ &\frac{\partial \tilde{V}^E}{\partial \phi_j} \longrightarrow -\frac{\partial \mathcal{L}}{\partial \phi_j} \end{aligned} \tag{1.3}$$

Therefore,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial t}\right)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)}\right) - \frac{\mathcal{L}}{\partial \phi} = 0 \tag{1.4}$$

Thit is the general equation of motion for natural fields.

The Lagrangian density serves as the starting point for deriving the equations of motion for physical fields, whether in classical field theory, electromagnetism, fluid dynamics, or quantum field theory. It is essentially a generalized Lagrangian, appropriate for continuous systems and fields, and it encapsulates the dynamics of the field. By specifying the Lagrangian density $\mathcal{L}$ appropriately for a particular field, we obtain the specific equations of motion that describe the behavior of that field in the given physical context.

1.1 Least Action Principle

In the section above, we addressed the classification of physical systems into discrete material points (particles), characterized by dynamic variables represented through coordinates $\mathbf{r}$, and continuous fields (waves), which constitute dynamic systems described by one or more continuous functions of coordinates and time $\phi = \phi(\mathbf{r}, t)$. Furthermore, we examined the transition in formalism from the treatment of discrete particles to the broader field-based approach. We can derive the same outcome in a more straightforward and direct way, as elaborated in the forthcoming section.

In classical mechanics, we formalize the transition from a classical system with a finite number of degrees of freedom $q_i(t)$ to a classical field theory in terms of a scalar field $\phi(\mathbf{r}, t)$ through the Principle of Least Action. This principle states that the actual path or configuration followed by a physical system between two points is the one for which the action S becomes stationary. We define the action S as the integral over time of the Lagrangian, between $t_1 < t_2$:

$$S = \int_{t_1}^{t_2} \mathbf{L} dt \tag{1.5}$$

In the context of a system of particles, the Lagrangian is expressed as a sum over the various degrees of freedom. However, when dealing with fields, these degrees of freedom are distributed at each point in space. Therefore, as described previously ([LMa22]):

$$\mathbf{L} = \int d^3x \mathcal{L}(\phi', \phi, x) \tag{1.6}$$

where with ϕ' we mean the derivatives of the fields with respect to the coordinates:

$$\phi'(x) = \frac{\partial \phi}{\partial x^\mu}$$

To deduce the differential equations that determine the evolution of the field, consider a functional defined as

$$I = \int_{\alpha}^{\beta} \mathcal{L}(f', f, x) dx \tag{1.7}$$

and let's discuss the values of f so that I becomes stationary. Let's examine an arbitrary perturbation of the paths linking the two points in question: $f_\epsilon(x) = f(x) + \epsilon\eta(x)$ where $\eta(\alpha) = \eta(\beta) = 0$. To determine the location where the action is stationary, we examine the partial derivative of the action with respect to ϵ at $\epsilon = 0$. This quantifies how the action's variation behaves as the ϵ-parametrized path converges to the original path. If the limit equals zero, it implies that the action remains relatively unchanged for minor ϵ variations. On the other hand, a non-zero limit indicates that introducing a perturbation leads to a substantial alteration in the action.

$$\lim_{\epsilon \to 0} \frac{I(\epsilon) - I(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{I[f(x) + \epsilon\eta(x)] - I[(f(x)]}{\epsilon} = 0$$

We can write

$$\mathcal{L}(f'_\epsilon, f_\epsilon, x) = \mathcal{L}(f' + \epsilon\eta', f + \epsilon\eta, x) = \mathcal{L}(f'(x), f(x), x) + \epsilon \left(\frac{\partial \mathcal{L}}{\partial f'}\eta' + \frac{\partial \mathcal{L}}{\partial f}\eta \right) + \sigma(\epsilon^2)$$

using Taylor's serie. Replacing the expression found in eq.1.7:

$$\delta I = \int_{\alpha}^{\beta} \left[\frac{\partial \mathcal{L}}{\partial f'} \eta' + \frac{\partial \mathcal{L}}{\partial f} \eta \right] dx$$

solving the first limit by parts:

$$\int \frac{\partial \mathcal{L}}{\partial f'} \eta' dx = \frac{\partial \mathcal{L}}{\partial f'} \eta - \int \eta \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial f} dx$$

$$\Rightarrow \delta I = \int_{\alpha}^{\beta} \left(\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial f'} \right) \eta dx + \frac{\partial \mathcal{L}}{\partial f'} \eta \Big|_{\alpha}^{\beta}$$

where the last fraction is the contribution from the interval limits and therefore equal to zero. Hence, it is observed that by imposing the derivative of the action to be zero, and thus applying the principle of least action, one obtains the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial f'} = 0 \tag{1.8}$$

Graphically, in the configuration space, the actual path represents a curve. The stationarity condition asserts that this actual path is an extremum. For every physical system, $\mathcal{L}(\dot{\mathbf{q}}, \mathbf{q}, t)$ with $\mathbf{q} = (q_1, ...q_n)$, there will be an optimization problem solved by the Euler-Lagrange equation 1.4, and this problem is formulated based on the action eq.1.5.

In summary, the Lagrangian density, along with the principle of least action and the Euler-Lagrange equation, provides a powerful and general framework for describing the dynamics of physical fields in physics. This formalism has been successfully applied to understand a wide range of phenomena, from classical electromagnetic waves to quantum particles in the Standard Model of particle physics.

1.2 The Lagrangian Formulation in Field Theory

In this section, we will further discuss how to apply the Lagrangian formalism using the example of two known fields: the electromagnetic field and a material field.

1.2.1 Electromagnetic Field formulation

The most significant example of a field in classical physics is the electromagnetic field, described by two vectors at each point, corresponding to the values of the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$. The purpose of the following section is to deduce the Lagrangian[1] within the context of an electromagnetic field. We begin the discussion considering the Maxwell equations in terms of $(\mathbf{E}, \mathbf{B})$. Given a charge distribution with current density $\mathbf{j}(\mathbf{r}, t)$ and charge density $\rho(\mathbf{r}, t)$:

$$\begin{aligned}
&\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 4\pi\rho(\mathbf{r}, t) \qquad \text{Gauss' Law for the electric field} \\
&\nabla \wedge \mathbf{E}(\mathbf{r}, t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \qquad \text{Faraday's Law} \\
&\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \qquad \text{Gauss' Law for magnetism} \\
&\nabla \wedge \mathbf{B}(\mathbf{r}, t) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r}, t) + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{Ampère-Maxwell Law}
\end{aligned} \tag{1.9}$$

And the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{1.10}$$

that derives from the first and the last of eq.1.9 and expresses the conservation of charge. To make these equations more tractable, it's possible to reformulate them in terms of electromagnetic potentials. Due to the magnetic field having zero divergence, there exists a function $\mathbf{A}(\mathbf{r}, t)$ referred to as the *vector potential* (or *magnetic potential*) [Jan20], such that

$$\mathbf{B} = \nabla \wedge \mathbf{A} \tag{1.11}$$

replacing eq.1.11 into Faraday's Law:

$$\nabla \wedge \mathbf{E} = -\frac{1}{c}\frac{\partial(\nabla \wedge \mathbf{A})}{\partial t} \iff \nabla \wedge \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{1.12}$$

Since the equation represents the curl of an irrotational quantity, there exists a function $\Phi(\mathbf{r}, t)$ known as the *scalar potential* (or *electric potential*) such that:

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{1.13}$$

Eq.1.13 and eq.1.11 verify the equations of Faraday's Law and Gauss' Law for magnetism, as $\nabla \wedge (-\nabla\Phi) = 0$ and $\nabla \cdot (\nabla \wedge \mathbf{A}) = 0$.

Consequently, we derive a pair of generalized Maxwell's equations formulated in the context of

[1]From now on, whenever we refer to the *Lagrangian* of the system we will actually mean the Lagrangian density $\mathcal{L}$, since the Lagrangian does not play a pivotal role in the equations of motion.

the vector potential $\mathbf{A}$, and the scalar potential Φ:

$$\begin{aligned} &\nabla^2\Phi + \frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = -4\pi\rho \\ &\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{j} \end{aligned} \tag{1.14}$$

To ascertain the second equation, the following relation has been utilized:

$$\nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \boldsymbol{\nabla^2}\mathbf{A}$$

Φ and $\mathbf{A}$ make up four functions in total (one for the scalar function, one for each spatial component of $\mathbf{A}$), which can be solved and their solutions can be used to find $\mathbf{E}$ and $\mathbf{B}$, this is a simplification over the original equations that make up six functions.
We can achieve additional simplification by applying the *Lorenz gauge*, which represents a partial gauge-fixing procedure. A general gauge transformation determines two new fields that have the same physical significance as the initial ones:

$$\begin{aligned} &\mathbf{A}' = \mathbf{A} + \nabla f(\mathbf{r}, t) \\ &\Phi' = \Phi - \frac{\partial f(\mathbf{r}, t)}{\partial t} \end{aligned} \tag{1.15}$$

The *Lorenz condition* imposes constraints on the choice of the function f, such that the scalar and vector potential are connected by:

$$\nabla \cdot \mathbf{A}' + \frac{1}{c}\frac{\partial \Phi'}{\partial t} = 0 \tag{1.16}$$

hence, for a general gauge transformation[2]:

$$\nabla \cdot \mathbf{A} + \nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \Leftrightarrow -\nabla^2 f + \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$

[2]Given

$$-\nabla^2 f + \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = g(\mathbf{r}, t)$$

the general solutions are the so-called *green functions*:

$$f(\mathbf{r}, t) = \int_{space} dr' \frac{g(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} + f_0(\mathbf{r}, t)$$

where $t_r = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$.

As a result of a Lorenz gauge transformation, we consequently derive:

$$\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{j} \end{aligned} \tag{1.17}$$

given $\mathbf{j} = \rho\mathbf{v}$.

Considering the four-potential $A^\mu = \left(\frac{\Phi}{c}, \mathbf{A}\right)$, the four-current $J^\mu = (\rho c, \mathbf{j})$ the electromagnetic Maxwell tensor[3] $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$:

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i \qquad F^{ij} = \partial^i A^j - \partial^j A^i = \epsilon^{ijk} B^k$$

The Maxwell equations in terms of the field tensor are:

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= -\frac{4\pi}{c} J^\mu \\ \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} &= 0 \end{aligned} \tag{1.18}$$

where ϵ^{ijk} and $\epsilon^{\mu\nu\rho\sigma}$ are the Levi-Civita symbols.
We can establish the structure of the Lagrangian for the electromagnetic field by making use of our prior knowledge of the equations of motion derived from it: the Maxwell equations above. Given that these equations are first-order partial differential equations concerning the field tensor $F^{\mu\nu}$, the Lagrangian $\mathcal{L}_{em}$ must necessarily take a quadratic form in $F^{\mu\nu}$ [VBa04].

$$\mathcal{L}_{em} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \tag{1.19}$$

When the electromagnetic field is coupled with a current J^μ, the Lagrangian characterizing the system introduces an interaction term, alongside the inherent kinetic contribution:

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \tag{1.20}$$

where the first term is the kinetic term and $J_\mu A^\mu$ is the interaction term (in vacuum is zero). As [VBa04]

$$\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(B^2 - \frac{E^2}{c^2}\right)$$

[3]The Maxwell equations become $\Box A^\mu = \mu_0 J^\mu$, with the D'Alambert operator $\Box = \sum_a \partial^a \partial_a = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

the Lagrangian density can be written also in terms of the fields:

$$\mathcal{L} = \frac{\epsilon_0}{2}\left|\mathbf{E}^2\right| - \frac{1}{2\mu_0}\left|\mathbf{B}^2\right| - \rho\Phi + \mathbf{j}\cdot\mathbf{A} \tag{1.21}$$

1.2.2 Schrödinger Field formulation

The identical procedure can be replicated in the scenario of a scalar field, in particular for the Schrödinger Field. In Classical Mechanics, a particle's dynamics are governed by the time evolution of its associated *wave-function* $\psi(\mathbf{r}, t)$, according to the non-relativistic Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = H\psi(\mathbf{r}, t) \tag{1.22}$$

where the Hamiltonian is given by $H = \frac{\hat{\mathbf{p}^2}}{2m} + V(\hat{x}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t)$. Field Theory addresses the challenges posed by multi-particle frameworks through a radical change of perspective[4]. The concept of this theory describes a particle as an oscillation of an abstract field [TWe]. The application of the Lagrangian formulation introduces us to the concept of fields as the fundamental entities, with particles being construed as localized excitations within these fields.

In the context of Field Theory, the wave function is treated as a field, just like the electric field, but it's a field of probability amplitudes. Therefore, ψ is considered a field because it assigns a complex number (amplitude) to every point in space, which determines the likelihood of finding a particle there. We will now focus on formulating the Lagrangian that characterizes a Schrödinger Field.

This Lagrangian contains terms that govern how the field evolves in space-time and how it interacts with other fields. In the freely evolving case, the Lagrangian takes the form:

$$\mathcal{L}_{Schr} = \frac{i}{2}\hbar\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi \tag{1.23}$$

The field's evolution and particles interactions are determined by the Euler-Lagrange equations. These equations give back the Schrödinger equation. In ψ the Euler-Lagrange equations are:

$$\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\frac{\partial\psi}{\partial x}} + \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\frac{\partial\psi}{\partial y}} + \frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{\frac{\partial\psi}{\partial z}} + \frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{\frac{\partial\psi}{\partial z}} + \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\frac{\partial\psi}{\partial t}} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \tag{1.24}$$

[4]i.e. the fundamental entities are fields and not particles; particles are regarded as excitations of these fields

yielding

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* = i\hbar\frac{\partial\psi^*}{\partial t} \tag{1.25}$$

which is the Schrödinger equation.

Both the charge density of matter in space, denoted as $\rho(\mathbf{r}, t)$, and the current density $\mathbf{j}(\mathbf{r}, t)$ can be described in terms of the wave function [Shu82].

$$\begin{aligned}\rho(\mathbf{r}, t) &= \psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t)\\ \mathbf{j}(\mathbf{r}, t) &= \frac{\hbar}{2mi}\left(\psi^*(\mathbf{r}, t)\nabla\psi(\mathbf{r}, t) - \psi(\mathbf{r}, t)\nabla\psi^*(\mathbf{r}, t)\right)\end{aligned} \tag{1.26}$$

definitions that satisfy the equation of continuity given by Eq.1.10. These new definitions (Eq.1.26) can be substituted into the Maxwell's equations Eq.1.9.

The resulting Lagrangian (in a static universe) is a function of the two wave functions (ψ,ψ^*) and their spatial-temporal derivatives, as well as the electromagnetic fields ($\mathbf{A}$ and Φ) and their spatial-temporal derivatives, describing a system in which the wave function is influenced by the electromagnetic field. To confirm this we demonstrate that the Euler-Lagrange equations resulting from this Lagrangian align with the Maxwell and Schrödinger equations. Given Eq.1.20 for the electromagnetic field and Eq.1.23 above:

$$\mathcal{L}_{QED} = \frac{i}{2}\hbar\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - \psi\psi^*\Phi + \left(\frac{\hbar^2}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)\right)\cdot\mathbf{A} - \frac{1}{4\mu_0}F^{\mu\nu}_{\mu\nu} \tag{1.27}$$

This defines the Lagrangian for Quantum Electrodynamics (QED), a specialized branch of electrodynamics that explores the interactions between charged particles, it is a comprehensive Quantum Field Theory that elucidates electromagnetic interactions among charged particles. QED provides a detailed, quantum-mechanical description of how charged particles interact through the electromagnetic field. The detailed process of quantization of the fields themselves, involves replacing classical fields with field operators. This is outside the scope of this thesis and will be left to more detailed texts on the subject.

Chapter 2

The Lagrangian Formulation in Cosmological Fields

2.1 Fields in our Universe: from the Standard Model to the Gravitational Field

In the opening chapter, we examined the application to field theory, showing two examples of fields described by a Lagrangian density.

The present chapter starts with a concise introduction to the diverse fields observed in our universe. Following this, in the final segment of the initial section, we delve into the role of gravity and the crucial concept of the *scale factor*. Transitioning to the second section, we introduce the fundamental notion of the *stress-energy tensor*, providing essential insights into the mass-energy content of a system. The concluding section of this chapter describes the tensor and space-time curvature within a model of a perfect fluid in a homogeneous and isotropic cosmological setting. Here, we establish the connection between the two through the Einstein Field Equations, ultimately deriving the temporal evolution of the scale factor.

The Standard Model of particle physics is a remarkably successful theoretical framework that classifies the fundamental particles and their interactions, developed in the mid-20th century, it provides a comprehensive understanding of the microscopic world. At its core, the Standard Model views particles not as small spheres but as excitations of fields that permeate all of space-time: they are little vibrations in their particular field. The interaction between particle is then regarded as a coupled vibration between the corresponding fields [PSu19]. Each fundamental particle (such as quarks, leptons and bosons) corresponds to a unique field.

The Standard Model categorizes elementary particles into two main groups: fermions and bosons. Fermions, which include quarks and leptons, constitute the building blocks of matter. Quarks combine to form protons and neutrons, while leptons like electrons and neutrinos play essential roles in various processes. Bosons, on the other hand, are force carriers that mediate interactions between particles. Photons γ, for instance, mediate electromagnetism, while $W^{\pm}$ and Z bosons are responsible for the weak nuclear force.

In the upcoming discussion, we will only examine *free* fields with no sources or interactions.

2.1.1 Fermionic Field

Fermionic fields describe the behaviour of quarks and leptons: elementary particles of spin $\frac{1}{2}$, each with their own antiparticle. Quarks (up-down, top-bottom, charm-strange) combine to form the more complex hadrons, that, with leptons (electron-neutrino, muon-muon neutrino, tau-tau neutrino), depic the atom with the "Thomson model" of an electron orbiting around a positive nucleus. For this reason, fermions (or fermionics field) are the elementary constituents of what we call matter.

The dynamics of fermions are governed by the Dirac equation, which is determined by the Dirac Lagrangian for a Spinor Field. This relativistic wave equation accounts for the behavior of fermions in the presence of electromagnetic fields. Consider a spinor field (Spin-$\frac{1}{2}$) ψ, the Lagrangian [DGr08]:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi \tag{2.1}$$

where ψ and the ajoint spinor $\bar{\psi}$ are indipendent field variables, γ^{μ} are the Dirac gamma matrices and ∂_{μ} is the partial derivative with respect to spacetime coordinates. Note that the Lagrangian for a particular system is not unique: one can simply multiply $\mathcal{L}$ by a constant, or add a constant, and see that such terms cancel out when it comes to the application of the Euler-Lagrange equations, so they do not affect the field equations.

Applying the Euler-Lagrange equation to $\bar{\psi}$, we find:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

and

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = i\gamma^\mu \partial_\mu \psi - m\psi$$

so that

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \tag{2.2}$$

this is the Dirac equation describing a particle of spin $\frac{1}{2}$ and mass m.

$$i\gamma^\mu \partial_\mu \bar{\psi} - m\bar{\psi} = 0 \tag{2.3}$$

is the adjoint of the Dirac equation.

2.1.2 Scalar-boson Field

Scalar particles are those without intrinsic angular momentum, their spin-0 nature is described by scalar fields. The Lagrangian density for a free scalar field ϕ is given by the *Klein-Gordon Lagrangian* [DGr08]:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \tag{2.4}$$

in this case

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \phi = \partial^\mu \phi \tag{2.5}$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \tag{2.6}$$

hence the Euler-Lagrangian formula from Eq.1.4 gives back an equation known as *the Klein-Gordon equation*:

$$\partial_\mu \partial^\mu \phi + m^2\phi = 0 \tag{2.7}$$

which describes a particle of spin 0 and mass m, hence the dynamic governing a scalar field. The solutions to the Klein-Gordon equation describe the evolution of the scalar field over space-time.

The only known example of an existing scalar particle in the Standard Model is the Higgs boson, which is a fundamental particle discovered at CERN in 2012. The associated Higgs Field is the reason why particles acquire mass in the Standard Model of particle physics ([RMa10]).

2.1.3 Gauge-boson Fields

Gauge bosons are particles of integer spin, they are the elementary particles that govern what we describe as a force in the everyday world [RMa10]. They are the ones that determine the interactions, from what we know interactions are governed by some combination of the four basic forces: gravity, electromagnetism, nuclear (the strong force) and radioactive (the weak force). The Standard Model successfully unifies three of this four fundamental forces: gravity, described by general relativity, is not integrated into this "Theory of Almost Everything".

Electromagnetism Electromagnetism is one of the fundamental forces of nature and is mediated between particles by the exchange of photons. The behave of the electromagnetic field is encapsulated in the Lagrangian derived in the last chapter Eq.1.20. Its Lagrangian is a particular case of the Proca Lagrangian for a vector (spin-1). [RMa10]

The interaction of charged particles with the electromagnetic field is embodied in the Dirac equation and expressed through the Lagrangian QED derived in the first chapter Eq.1.27.

Weak interaction force The weak force is mediated by three massive bosons: W^+, W^- and Z. These exchange particles are responsible for transmitting the weak force between particles that interact through it. The Lagrangian term that represents the weak force involves interactions between these fields and fermions under the model of the electroweak theory. It provides a unified framework for describing both the electromagnetic force and the weak force within a single theoretical structure. In the electroweak theory, the Lagrangian incorporates the kinetic terms and interaction terms for both the electromagnetic and weak fields.

Strong interaction force The strong interaction force, also known as the strong nuclear force is responsible for binding quarks together to form protons, neutrons, and other hadrons, as well as for holding these hadrons together within atomic nuclei. The force carrier particles associated with the strong force are called gluons. The Yang-Mills Lagrangian is the one that describes the strong interaction force, though for the purpose of this dissertation, my analysis will not concentrate on this topic. Their interactions are described by a Lagrangian that includes terms representing each of the fundamental forces.

To sum up, in the Standard Model of particle physics we have a roster of fundamental particles and fields that govern their interactions. Breaking down the smallest components allowed by relativity and gauge invariance [RMa10], we find: the Lepton Fields including the electron, muon, and tau, each paired with its associated neutrino field; the Quark Fields, of six types that are confined within hadrons, the Higgs Scalar Field, with his unique role to provide mass to other particles through the Higgs mechanism[1]. Weak Boson Fields, that include the W^+, W^- and Z bosons, responsible for weak interactions; the Electromagnetic Field, described by the photon field and Gluon Field, responsible for the strong force, which binds quarks inside protons and neutrons.

While quarks and electrons (of three types) interact electromagnetically through the photon field, all particles, including neutrinos (of three types), engage via the weak force mediated by the $W^\pm$ and Z fields [DGr08].

It's worth noting that, beyond these known fields, there might be additional fields and particles awaiting discovery. While all particles are assumed to experience gravitational interactions, our understanding of gravity is not yet incorporated into the Standard Model due to the absence of a complete "quantum theory of gravity". The Standard Model does not include a description of gravity and falls short in accounting for dark matter and dark energy (more details will be given in Chapter 3), which together constitute the majority of the universe's mass and energy. The quest for a more comprehensive theory, such as theories of supersymmetry or grand unification, continues to drive modern research in particle physics.

2.1.4 The role of gravity

Gravity plays a fundamental role in the fabric of the universe, shaping the structure and behavior of celestial bodies and governing the dynamics of the cosmos [SMC04] [CWM73].

Gravity, a fundamental force shaping the cosmos, was first conceptualized by Sir Isaac Newton in the form of the law of universal gravitation. According to Newton, every particle of matter attracts every other particle with a force proportional to their masses and inversely proportional to the square of the distance between them. Albert Einstein later redefined gravity through General Relativity, depicting it not merely as a force but as a curvature in space-time caused by mass and energy \cite{hartle}. Massive objects, like planets and stars, create ripples in the fabric of space-time, causing other objects to follow the curves created by this curvature.

This force of gravity intricately governs celestial dynamics. It governs the orbits of planets around stars, moons around planets, and the motion of stars within galaxies. On cosmic scales,

[1] The Higgs boson is the particle associated with the excitation of this field.

gravity influences the distribution of galaxies and the formation of celestial bodies, building the cosmic web that defines the large-scale structure of the cosmos.

The profound connection between gravity and the Cosmological Field[2], a hypothetical field which regulates the rate and exphansion of the Universe, is exemplified by the gravitational interplay between matter that tends to slow down the universe's expansion, influencing the dynamics of the field. The impact of gravity extends further, influencing phenomena such as early universe density fluctuations, gravitational lensing, and the propagation of gravitational waves.

In the context of the Cosmological Field the *scale factor* is introduced, a quantitative measure of how the universe (particularly one of its models) changes as time progresses, capturing the expansion or contraction of cosmic structures. Studying the behavior of the scale factor is essential for understanding the large-scale evolution and fate of the universe in cosmological models. The evolution of the scale factor is governed by the *Friedmann equations*, which describe the dynamics of the universe in the framework of general relativity and will be deeply analyzed in the last section. The Friedmann equations relate the expansion rate of the universe to its energy content, including matter, radiation, and dark energy. Gravity enters in these equations represented by the energy density of matter and radiation.

In the next sections we will dig into the principal components of the Cosmological Field, giving a detailed overview of their physical meanings and implications.

2.2 Stress-Energy Tensor

In the framework of General Relativity, the concept of mass and energy is unified into the energy-momentum four-vector p^μ. To illustrate, consider a three-dimensional surface denoted as S, oriented perpendicular to the x direction. If we aim to examine the z component of momentum, we can assess the density flux of p^z specifically in the x direction, denoted as T^{zx}. The matrix T, referred to as the *stress-energy tensor*, is a key element in relativity and plays a central role.

The stress-energy tensor encodes the flux densities of both momentum and energy. This tensor becomes particularly significant in the realm of General Relativity, where it acts as the source

[2]In a general term *Cosmological Field* refers to any field associated with the large-scale structure and evolution of the universe.

of gravitational fields.

Given a Lagrangian density $\mathcal{L}$, that we recall is a function of a set of fields ϕ_ρ and their derivatives but explicitly not of any of the space-time coordinates, we can construct the energy-stress tensor by looking at the total derivative of $\mathcal{L}$ with respect to x^μ [HGoon], one of the generalized coordinates of the system:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial \phi_\rho}\phi_{\rho,\mu} + \frac{\partial \mathcal{L}}{\partial \phi_{\rho,\nu}}\phi_{\rho,\mu\nu} + \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{2.8}$$

where we use the following notation:

$$\phi_{\rho,\nu} = \frac{d\phi_\rho}{dx^\nu}$$

The equation of motion is:

$$\frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\rho,\nu}}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\rho} = 0 \tag{2.9}$$

Substituing the term $\frac{\partial \mathcal{L}}{\partial \phi_\rho}\phi_{\rho,\mu}$ into the equation 2.8, we obtain:

$$\frac{d\mathcal{L}}{dx^\mu} = \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\rho,\nu}}\right)\phi_{\rho,\mu} + \frac{\partial \mathcal{L}}{\partial \phi_{\rho,\nu}}\frac{d\phi_{\rho,\nu}}{dx^\nu} + \frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\rho,\nu}}\phi_{\rho,\mu}\right) + \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{2.10}$$

this last can be written

$$\frac{d}{dx^\nu}\left(\frac{\partial \mathcal{L}}{\partial \eta_{\rho,\nu}}\phi_{\rho,\mu} - \mathcal{L}\delta_{\mu\nu}\right) = -\frac{\partial \mathcal{L}}{\partial x^\mu}$$

In the case where $\mathcal{L}$ represents a free field, i.e., no interaction between the field and point particles moving in space and time through the field, this last equation takes on the form of a set of divergence conditions [HGoon],

$$T^\nu_{\mu,\nu} = 0 \tag{2.11}$$

On a quantity with the form of a 4-tensor[3] if the second rank:

$$T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial \phi_{\rho,\nu}}\phi_{\rho,\mu} - \mathcal{L}\delta^\nu_\mu \tag{2.13}$$

The formalization of the equations that will be followed ensures the preservation of general

[3] In Relativity (both general and special) all the equations are expressed in their tensorial form. In general a tensor is a quantity that transform as follows, under a change in coordinates from x^i to x^j:

$$A'^{kl\ldots}_{pq\ldots} = \frac{\partial x'^k}{\partial x^m}\frac{\partial x'^l}{\partial x^n}\ldots\frac{\partial x^r}{\partial x'^p}\frac{\partial x^s}{\partial x'^q}\ldots A^{mn\ldots}_{rs\ldots} \tag{2.12}$$

[AFr16]

covariance[4].

Even in General Relativity there are laws of conservation of mass, energy, and momentum, expressed in general terms in covariant form[5] from Eq.2.11: the stress-energy tensor allows us to express the conservation of energy-momentum tensor.

Physical meaning of T^{ν}_{μ}

The time-time component of the tensor is associated with energy density, while the spatial components T^{ik} represent flux of the i-th component of linear momentum across the x^k surface and behave like vectors in ordinary space. They are associated with pressures and momentum flux. Within the framework of General Relativity (GR), this tensor characterizes the flow and distribution of energy and momentum across space-time and serves as the fundamental source of the gravitational field, determining the overall curvature of space-time [CWM73]. Understanding the energy-momentum tensor is crucial for getting an insight of the geometry of space-time in the presence of matter and energy. In more details:

- T^0_0 represents energy density
- T^0_i signifies the momentum density
- T^i_0 indicates the rate of energy transfer
- T^j_i reflects the momentum flux

The stress-energy tensor is symmetrical, i.e., $T_{\mu\nu} = T_{\nu\mu}$ [SMC04].

Note that the divergenceless property of the tensor (Eq.2.11) when expressed in terms of spatial components, can be related to the equations of continuity.

The form of the energy-stress tensor for a perfect fluid provides a simplified yet effective representation for understanding the essential characteristics of a dynamic system, making it a valuable tool for modeling in physics and astrophysics.

Since for a scalar field the Lagrangian does not depend on time or any other coordinate:

$$\frac{\partial \mathcal{L}}{\partial x^{\nu}} = 0 \tag{2.14}$$

[4] the equations must remain valid regardless of the reference system (inertial or non-inertial) with respect to which various quantities are defined.

[5] The covariant derivative with respect to the coordinate j is, for example, in the case of a two-index tensor: $A_{ik,j} = \frac{\partial A_{ik}}{\partial x^j} - \Gamma^l_{ij} A_{lk} - \Gamma^l_{kj} A_{il}$

we have a specific form for the equations of continuity $\partial_\mu j^\mu = 0$ and for the energy-momentum conservation requirements (Eq.2.11). Hence, for a perfect fluid:

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p g_{\mu\nu} \tag{2.15}$$

p and ρ are the pressure and energy density of matter and u_i is the four-velocity of the fluid:

$$u_i = g_{ik}\frac{dx^k}{ds}$$

with x^k being the trajectory followed by the object in space-time, and g_{ik} the metric.

The stress tensor $T_{\mu\nu}$ for a fluid is diagonal, in an inertial frame where it's at rest, a perfect fluid is characterized[6] by [BHa03]:

$$T_{\mu\nu} = \mathrm{diag}(\rho, -p, -p, -p) \tag{2.16}$$

In the context of fluid dynamics, the stress-energy tensor embodies both the fluid's intrinsic motion and its response to external forces. While for a gravitational field, it describes the mass-energy content of the cosmic fluid: the rest mass, total momentum, energy, and pressure [AFr16]. Einstein as a consequence constructed a relationship between the metric and matter by equating $T_{\mu\nu}$ to the tensor obtained from the metric $g_{\mu\nu}$, which contains only first and second derivatives of itself. This will be detailed in the following section. In essence, the stress-energy tensor is an instrument that describes the shape and the matter dynamics within Einstein's Field Equations, defining how space-time deforms and curves in response to the distribution of matter and energy.

2.3 Einstein Field Equations

The comprehensive connection between the gravitational field sources, namely the matter and energy distribution in the Universe, and the metric characterizing the space-time geometry is encapsulated by the field equations of General Relativity [AFr16]. According to GR, a free particle traverses along a geodesic influenced by the matter distribution. In essence, the metric itself, denoted as $g_{\mu\nu}$, is determined by the presence of matter. This fundamental principle

[6] Pressure and density are related by an equation of state: for example the gas of galaxies can be modeled by $p = 0$, the cosmic microwave background radiation by $p = \rho/3$ [BHa03]

forms the foundation for the derivation of the field equations, representing the desired link that Einstein aimed to establish. To delve into the Einstein Field Equations, tensors play a central role as essential components for articulating and describing these equations.

Riemann curvature tensor. The initial tensor introduced in this context is the Riemann curvature tensor, also referred to as the Riemann-Christoffel tensor. This tensor serves to characterize the intrinsic curvature of space and plays a crucial role in discerning whether the space is flat or exhibits curvature [AFr16].The tensor is defined in terms of the Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$, that are related to the metric itself [ENe14]:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left[\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right] \tag{2.17}$$

and

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} \tag{2.18}$$

Ricci tensor. From the Riemann curvature tensor, the Ricci tensor can be derived by a contraction of the Riemann tensor: summing over one index of the Riemann tensor. An important property of the Ricci tensor is that it's symmetric, meaning $R_{\mu\nu} = R_{\nu\mu}$ [SMC04].

$$R_{ik} = R^l_{ilk} \tag{2.19}$$

Physically, it represents the local gravitational effects caused by the distribution of matter and energy in space-time. Ricci scalar derives from the tensor [CWM73]:

$$R = g^{ik} R_{ik} \tag{2.20}$$

Einstein tensor. The combination of the Ricci tensor and scalar denote the Einstein tensor as follows [CWM73][SMC04]:

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R \tag{2.21}$$

From the equations above we can see that the Einstein tensor G_{ik} has the first and second derivatives of the metric g_{ik}.

Field Equations. The complete set of Einstein's field equations involves the Ricci tensor, the scalar curvature (or equivalently the Einstein tensor), and the cosmological constant, offering a mathematical framework to clarify how matter and energy impact the curvature of space-time. This equations relate curvature to the stress-energy tensor: a measure of matter energy density [BHa03],discussed in the previous sec. 2.2.

The *Einstein Field Equations* for the cosmological field are defined as follows[7] [SWe72][BHa03]:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.22}$$

$G = 6.67\,10^{-11}$ $\mathrm{Nm}^2/\mathrm{kg}^2$ is Newton's constant [BNT16],

The symbol Λ represents the cosmological constant, a constant parameter that will be explored in the subsequent chapter. For the present discussion, it is assumed to be a constant unaffected by any of the four space-time coordinates, particularly cosmic time.
Initially, Einstein formulated his equations without including the term of the cosmological constant, but this extension of the field equations maintains the generally covariant nature of the equation [AFr16]. This is contingent on Λ being a universal constant, as we just mentioned. The most stringent requirement for the value of this constant is to be sufficiently small not to modify, for instance, the laws of planetary motion. In the following section 2.3.1, we will illustrate an example of applying the Einstein Field Equation to derive the Friedmann dynamic equation. As a result of what has just been stated, we can omit the cosmological constant in this calculation; we will instead focus on its treatment in the next chapter.

The components of the Einstein equation consist of ten partial differential equations for the metric coefficient $g_{\mu\nu}$, given the matter sources $T_{\mu\nu}$, of which only six are independent.[8]. They are analogous to Maxwell's equations, except for their nonlinearity [SMC04]. The Einstein Field Equations represent the Euler-Lagrange of the Cosmological Field.

2.3.1 Friedmann solutions for a Homogeneous, Isotropic Cosmological Model

The Robertson-Walker, or more entirely the Friedmann-Lemaître-Robertson-Walker (FLRW), metric (Eq.2.24) is formulated to accommodate various scenarios of the scale factor $a(t)$. The

[7]The velocity of light, that should contribute to the equation as a $\frac{1}{c^4}$ on the second term, is assumed to be $c = 1$ throughout this thesis

[8]due to Bianchi identities whose topic will be left out for this thesis work

next step is to apply this metric to Einstein's equations in order to get the Friedmann equations, which establish the relationship between the universe's energy-momentum content and the scale factor. Although isotropy and homogeneity are essential to the FLRW metric, they don't provide insights on the scale factor's dynamic behavior. The metric is entered into the Einstein Field Equations (EFE) in order to get this information. Two coupled equations that are referred to as the Friedmann-Lemaître equations (FLE) or just Friedmann equations are the result. Together, these equations describe how the scale factor evolves all over time. Both equations are related to one other, with the equation of state establishing a link between pressure and density within the cosmic medium[9] [SWe08].

In GR, space-time is generally curved due to the presence of gravitating matter and energy. Based on the metric tensor, the generalized element of space-time distance can be expressed [AFr16].

$$ds^2 = g_{ik} dx^i dx^j \tag{2.23}$$

with $i, j = 0,1,2,3$.

The space-time geometry of a homogeneous and isotropic cosmological model is accurately described by the Robertson-Walker metric, aligning with these symmetries [SWe08]:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2) \right] \tag{2.24}$$

where we used the notation of the spherical coordinates r, θ and ϕ as comoving coordinates, t is the proper time and $a(t)$ is the *cosmic scale factor* that describes the expansion or contraction of the universe; $k = \pm 1, 0$ is a constant prescribing the spatial curvature.

The derivation of the Robertson-Walker metric can be approached through two distinct methods. One of them employs the tensor formalism within the framework of General Relativity, utilizing Christoffel symbols and other tensor-related concepts [SWe72]. Alternatively, the FLRW can be derived by generalizing the metric for an isotropic 3D space (see [AFr16]). In both cases, we ascertain that this metric originates directly from the symmetrical properties of the Cosmological Principle[10] and remains independent of the General Relativity theory articulated by the Field Equations.

[9]The equation of state in the context of cosmology is often related to the properties of the cosmic fluid, where the pressure p and density ρ are connected. The nature of this connection depends on the components of the cosmic fluid, such as matter, radiation, or dark energy. Discussions over this topic are not part of this work but can be found in [SWe08] for instance.

[10]The universe, as seen by fundamental observers, appears homogeneous and isotropic.

For the purpose of this calculation, we will consider the curvature parameter $k = 1$.

The FLRW metric is diagonal, i.e. the mixed space-time terms are equal to zero:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t)\frac{1}{1-r^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{pmatrix} \tag{2.25}$$

Setting $\dot{a} = \frac{da}{dt}$, using the Christoffel symbols given by Eq.2.17 and knowing that

$$g^{im}g_{mk} = \delta^i_k \tag{2.26}$$

where δ^i_k is the Kronecker delta, equals to 1 for $i = k$ and zero for $i \neq k$, we find that:

$$\begin{aligned}
&\Gamma^0_{11} = \frac{a\dot{a}}{1-r^2} && \Gamma^2_{12} = \frac{1}{r} = \Gamma^3_{13} \\
&\Gamma^0_{22} = a\dot{a}r^2 && \Gamma^2_{33} = -\sin\theta\cos\theta \\
&\Gamma^0_{33} = a\dot{a}r^2\sin^2\theta && \\
&\Gamma^1_{01} = \frac{\dot{a}}{a} = \Gamma^2_{02} = \Gamma^3_{03} && \\
&\Gamma^1_{11} = \frac{r}{1-r^2} && \Gamma^3_{23} = \cot\theta \\
&\Gamma^1_{22} = -r(1-r^2) && \\
&\Gamma^1_{33} = -r(1-r^2)\sin^2\theta &&
\end{aligned} \tag{2.27}$$

We determine first the components of the Riemann tensor R^i_{klm} given by the Eq.2.18 , from which we can contract the upper index and find Ricci tensor as in Eq.2.19:

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2}{1-r^2} \\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2) \\
R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2)\sin^2\theta
\end{aligned} \tag{2.28}$$

The result that $R_{12} = R_{13} = R_{02} = R_{03} = R_{23} = 0$ arises from the rotational symmetry of the metric. Meanwhile, $R_{10} = 0$ is due to our choice of setting the clocks such that the metric remains invariant under time reversal transformations, as deeply discussed in [SWe72]. We recall the relationship between the Einstein tensor and the curvature Eq.2.21, and we find that all the components of the Einstein tensor vanish except for:

$$\begin{aligned} G_{00} &= \frac{3}{a^2}\left(1+\dot{a}^2\right) \\ G_{ii} &= -\left[2\frac{\ddot{a}}{a}+\frac{1}{a^2}(1+\dot{a}^2)\right] \end{aligned} \tag{2.29}$$

where $i = 1,2,3$ are the spatial coordinates.

Matter inside a homogeneous and isotropic universe can be described, at large scales and with high precision, as a perfect fluid [NTa18]. In a cosmological setting, where the geometry is modeled using the Robertson-Walker metric, the conservation equation of the stress-energy tensor takes a specific form. In this case, it translates into an equation of continuity that relates the rate of change of energy density and pressure to the expansion of the universe [SWe72]. For a perfect fluid, the energy-momentum tensor is written as eq.2.15. We deduce that gravity is influenced not only by the rest mass of the gravitating object but also by all terms of an energetic nature, including pressure.

As we discussed in sec.2.2, for a fluid at rest in comoving coordinates, the four-velocity is[11]:

$$u^\mu = (1,0,0,0) \tag{2.30}$$

and the energy-momentum tensor, recalling Eq.2.15 becomes, in its matricial form:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \tag{2.31}$$

Two set of equations have therefore been derived: one for the time component and one for the space ones, corresponding to the first and second dynamical equations for a homogeneous and isotropic cosmological model [AFr16][BHa03]:

$$\begin{aligned} &\frac{3}{a^2}\left(1+\dot{a}^2\right) = 8\pi\rho \\ &2\frac{\ddot{a}}{a}+\frac{1}{a^2}(1+\dot{a}^2) = -8\pi p \end{aligned} \tag{2.32}$$

[11] As already stated, $c = 1$ through all the work of this thesis

So:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{1}{a^2} \tag{2.33}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.34}$$

The first equation, widely recognized as *the Friedmann equation*, is the one that allows one to express the time derivative $\dot{a}(t)$ as the scale factor and calculate how it evolves.

The last equation yields a crucial condition: a criterion that distinguishes an accelerating universe characterized by an increasing expansion rate ($\ddot{a} > 0$ and so $\rho + 3p < 0$) from a decelerating one with a decreasing expansion rate ($\ddot{a} < 0$ and hence $\rho + 3p > 0$) [NTa18].

Chapter 3

Dynamical systems and Cosmology

3.1 The Evolution of the Scale Factor

The present chapter focuses on the different evolutions of the scale factor $a(t)$, influenced by the choice of different cosmological models.

A cosmological model is a theoretical framework that describes the structure, evolution, and dynamics of the universe as a whole. It serves as a mathematical and conceptual representation, defining a dynamical system influenced by all the energy components (matter, radiation and dark energy), as will be seen further on. Every cosmological model respects the Cosmological Principle (homogeneity[1] and isotropy[2]), they all utilise the FLRW metric (Eq.2.24) to describe the spatial geometry. Finally a cosmological model derives its own equations of motion by adjusting the Friedmann equations. The cosmological models we will present share identical initial conditions, aligning with the initial singularity commonly referred to as the *Big Bang*.

By first determining the energy densities' dependency on $a(t)$, we may obtain the scale factor's immediately apparent evolutions. From the conservation of the energy-momentum tensor: multiplying Eq.2.33 by a^2, differentiating and using Eq.2.34; or similarly applying the first law of thermodynamics (as explained in [AFr16]) we derive the energy conservation equation for

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[1] The universe appears the same to all the fundamental observers on large scales without preferred location.

[2] The universe looks the same in all directions when viewed from any fundamental observer.

matter fluid [NTa18]:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{3.1}$$

The mass-energy terms include both the gravitational ρ_m and relativistic ρ_γ particles.
For a dust-like (or gravitational matter dominated) universe, i.e., with $p_m = 0$, we obtain:

$$\rho_m(t) \propto a(t)^{-3} \tag{3.2}$$

while for a radiation dominated universe $p = p_r = \rho_r/3$:

$$\rho_r(t) \propto a^{-4}(t) \tag{3.3}$$

Consequently, by substituting the equations mentioned above in the Friedmann equation, a specific simple set of solutions for $\dot{a}(t)$ can be obtained.

In the last chapter, we explained the derivation of the Friedmann solutions imposing $k = +1$, but the cosmological Friedmann equation can be expressed as a function of the general curvature parameter k and with $8\pi G = 1$:

$$\frac{k}{a^2} + H^2 = \frac{k^2}{3}\rho \tag{3.4}$$

H is Hubble parameter $H = \dot{a}/a$.

Depending on the various interpretations of cosmological models, in particular relaying on the curvature value k, there are different write-ups of the dynamic equations and thus different scale factor trends. Some are illustrated in [AFr16], we will report the essentials.

For the case of a flat universe ($k = 0$) the scale factor evolves as a power-law function of time, in particular:

1. For a gravitational matter dominated universe:

$$a(t) \propto t^{2/3} \tag{3.5}$$

   This is known as the **Einstein-De Sitter model**: for $t \to \infty$ $\dot{a}(t)$ becomes zero, therefore the universe tends to a static model.

2. For a radiation domination:

$$a(t) \propto t^{1/2} \tag{3.6}$$

Another important model is the **Milne Universe**, for whom $k = -1$ and $a(t) = \pm t$: the universe expands and contracts uniformly and homogeneously.

To generalize the form of the evolution of the scale factor, we consider the case in which the Universe is dominated by a single component with a non-vanishing matter contribution and a linear equation of state:

$$p = w\rho \tag{3.7}$$

$\rho > 0$ and w is the equation of state parameter that characterizes the ratio of pressure to energy density. With w constant, we find the evolution of ρ and a, replacing the EoS in Eq.3.1:

$$\rho \propto a^{-3(1+w)} \tag{3.8}$$

then, solving the differential equations for $\dot{a}/a$ [STs10]:

$$a(t) \propto (t - t_0)^{\frac{2}{3(w+1)}} \tag{3.9}$$

where t_0 is a constant. The general evolution described by Eqs.3.7 to 3.9 corresponds to the relations found earlier, with $w = 1/3$ for the radiation-dominated era giving a radiation density as Eq.3.3 and a scale factor evolution in Eq.3.6.

Non-relativistic matter ("dust") corresponds to the case with a negligible pressure relative to its energy density, i.e. $w \simeq 0$, and the evolution during the matter-dominated era given by Eqs.3.2 and3.5.

As a result, we have established the dependence of the scale factor on the two primary components and generators of energy in the universe. However, as discussed in the last chapter, these are not the only contributors: 70% of the energy budget of the universe is composed by *dark energy*. The remaining part is therefore dominated by a gravitating matter component, divided into standard baryonic matter (about 4% [AFr16]) and another invisible entity called *dark matter* ($\sim$ 26%), which is needed to explain discrepancies in the observed rotation curves of galaxies [NTa18]. The relativistic component (radiation) includes the Cosmic Microwave Background, photons and neutrinos and contributes 1% of the total energy budget. The concept of *dark energy* was introduced to account for the observed accelerated expansion. It is defined as a type of matter characterized by negative pressure $p_{darken} < -\rho_{darken}/3$ [NTa18][AFr16].

As of our current knowledge, the following is the most widely accepted interpretation of the evolution of the scale factor. The early universe is characterized by a radiation-dominated era[3] where relativistic particles dominated the energy density and the scale factor evolved as Eq.3.6. As the universe continued to expand and cool, it entered a gravitational-matter dominated era, where the energy density was dominated by matter, including dark matter and baryonic matter. During this era the scale factor continued to increase as Eq.3.5, and the expansion rate gradually slowed. In the more recent cosmic history, the universe has entered a phase where dark energy dominates the energy budget, with an accelerated expansion. This expansion can be modeled by the *cosmological constant* Λ, as detailed in sec.3.2, through what's known as the ΛCDM model, returning a scale factor trend that follows the Eq.3.9. There are otherwise several models that parameterise this acceleration, we will list some of them below. These various interpretations on the generation of the dark energy lead to different dynamical scenarios, determining different outcomes on the evolution of the scale factor in the expansive era dominated by dark energy.

3.2 ΛCDM model

The most straightforward model of dark energy is embodied by the *cosmological constant* Λ, initially introduced by Einstein to formulate a cosmological model that would lead to a static universe. Its physical interpretation is that of a constant repulsive energy term of the vacuum that opposes gravity, filling space homogeneously.

In the ΛCDM model, while dark energy is represented by Λ, dark matter is assumed to be a non-relativistic matter component, specifically cold dark matter (CDM). With the inclusion of the cosmological constant, the Einstein Field Equations take on the following form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{3.10}$$

and the Friedmann equations, with $k = +1$:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{1}{a^2} + \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \end{aligned} \tag{3.11}$$

From these equations we can derive the evolutions of the cosmic scale factor if the cosmological

[3]excluding the rapid expansion that followed the Big Bang known as cosmic inflation that is out of the purpose of this thesis

constant completely dominates the evolution:

$$a(t) \propto e^{\eta t} \quad \text{with} \quad \eta = \sqrt{\frac{\Lambda}{3}} \tag{3.12}$$

Hence, Eq.3.9 presents the expected behavior associated with dominance by matter at early times and dominance by the cosmological constant at late times. In mathematical terms, we have that:

$$a(t) \propto t^{\frac{2}{3(w+1)}} \quad \text{as} \quad t \to 0$$
$$a(t) \propto \exp\left[\frac{2t}{3(w+1)}\right] \quad \text{as} \quad t \to \infty$$

Reading Λ as a vacuum energy and writing $\Lambda/3$ in the Friedmann equations in terms of vacuum density and pressure, i.e., we add a ρ_v and $3p_v = -\rho_v$, the dynamical equation becomes:

$$\ddot{a} = -\frac{4\pi G a(t)}{3}(\rho_m - 2\rho_v) \tag{3.13}$$

where ρ_m and ρ_v are respectively the density of gravitating matter and the density equivalent mass density of vacuum energy, we can deduce:

$$\Lambda = 8\pi G \rho_v \tag{3.14}$$

Einsten's cosmological model is an unstable solution, since any small deviation, even if only local, of the density will cause that portion of the universe to fall into a state of exponential expansion or exponential contraction [AFr16].

We note from the Friedmann solutions (Eq.2.34) that, in order to give rise to the observed acceleration $\ddot{a} > 0$, the cosmological constant has a negative pressure $p_v = -\rho_v/3$, as previously mentioned. Therefore it corresponds with an equation of state parameter $w = -1$, implying that the dark energy behaves like a vacuum energy exerting negative pressure that drives the accelerated expansion of the universe.

The cosmological equations with both matter ($p_m = 0$) and radiation ($p_r = \rho_r/3$) are written as [NTa18]:

$$\begin{aligned} 3H^2 &= k^2\rho_m + k^2\rho_r + \Lambda \\ 2\dot{H} + 3H^2 &= -\frac{k^2}{3}\rho_r + \Lambda \end{aligned} \tag{3.15}$$

By introducing the relative energy densities of matter, radiation and cosmological constant as:

$$\Omega_m = \frac{k^2 \rho_m}{3H^2}, \quad \Omega_r = \frac{k^2 \rho_r}{3H^2} \quad \text{and} \quad \Omega_\Lambda = \frac{k^2 \rho_\Lambda}{3H^2} \tag{3.16}$$

we can express the Friedmann equations in their most simplified form:

$$1 = \Omega_m + \Omega_r + \Omega_\Lambda \tag{3.17}$$

To obtain the dynamical system for this cosmological model one can differentiate $\Omega_m = x$ and $\Omega_r = y$ with respect to a variable η so that $d\eta = Hdt$. This approach facilitates an examination of the associated phase portrait, as elaborated in the Physical Report [NTa18], which is recommended for a more in-depth exploration.

Despite its potential role in explaining cosmic acceleration, the observed value of Λ is significantly smaller than expected from theoretical considerations [Wei00]. There are alternative interpretations of dark energy that, instead of manifesting as a cosmological constant, exhibit dynamics; for the purpose of this dissertation, the analysis will now concentrate on these models.

In the next sections therefore, scalar-tensor theories will be presented. These gravitational theories extend General Relativity by introducing a scalar field in addition to the tensor field that describes the gravitational interaction. In these theories, gravity is mediated not only by the curvature of space-time (Ricci tensor) but also by a scalar field that can vary in space and time.

3.3 Quintessence models

Quintessence refers to a hypothetical form of dark energy postulated to explain the observed accelerated expansion of the universe. Unlike the cosmological constant in the ΛCDM model, which has a constant energy density, quintessence involves a dynamic, time-varying scalar field with an evolving energy density permeating the cosmos and somehow manifesting itself in dark energy.

In other words, the quintessence field, often represented by a scalar field with a potential energy function, introduces a dynamic component to the dark energy content of the universe. The dynamics of quintessence can lead to variations in the equation of state, influencing the expansion rate of the universe over cosmic time.

As reported in [NTa18], if we consider a scalar field minimally coupled to gravity, the related Lagrangian is:

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \tag{3.18}$$

with $V(\phi)$ being a general self-interaction potential for ϕ. This leads to the gravitational field equations in the form of:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = k^2\left(T_{\mu\nu} + T^{(\phi)}_{\mu\nu}\right) \tag{3.19}$$

where $T^{\phi}_{\mu\nu}$ is the energy-momentum tensor of the scalar field. Considering the FLRW metric Eq.2.24, and a linear equation of state (EoS) $p = w\rho$ assumed for the matter field, the Einstein field equations lead to the following Friedmann equations:

$$\begin{aligned} 3H^2 &= k^2\left(\rho + \frac{1}{2}\dot{\phi}^2 + V\right) \\ 2\dot{H} + 3H^2 &= -k^2\left(w\rho + \frac{1}{2}\dot{\phi}^2 - V\right) \end{aligned} \tag{3.20}$$

The dynamics of quintessence are governed by the Klein-Gordon equation, describing the evolution of the scalar field across space-time:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial\phi} = 0 \tag{3.21}$$

The potential energy plays a crucial role in shaping the behavior of quintessence, allowing for a variety of scenarios: $V(\phi)$ can assume any form, from an exponential potential, for instance $V(\phi) = V_0 e^{-\alpha k\phi}$, to a power-law potential $V(\phi) = M^{\alpha+4}/\phi^\alpha$. Given an explicit form of the potential one can derive the precise dynamical system and the phase portrait of that cosmological model.

Understanding the evolution of quintessence involves investigating the EoS parameter w, which characterizes the ratio of pressure to energy density. Unlike the cosmological constant with a fixed $w = -1$, quintessence allows for a time-varying equation of state. It implies that the dark energy field possesses a dynamic and evolving nature, allowing it to track the dominant energy component in the universe and become the dominant component as the universe expands.

Quintessence fields are in fact characterized by a tracking behavior: as the universe expands, the energy density of various components (matter, radiation, etc.) changes at different rates; the key feature of tracking behavior is that the energy density of the quintessence field adjusts itself to be of the same order of magnitude as the dominant energy component in the universe. It also means that regardless of where the field starts in its potential, it tends to converge to a trajectory that tracks the dominant energy density. In the early stages, when matter or radiation dominates, the quintessence field behaves like them. As the universe expands and the energy density of matter or radiation decreases, the energy density of the slowly rolling quintessence field becomes significant, eventually leading to its dominance [STs10].

In theory, it is possible that dark energy is constituted by more than one scalar field:

$$\mathcal{L}_{\phi_1,\ldots,\phi_N} = -\sum_{i=1}^{N} \partial\phi_i^2 - V(\phi_1, \ldots\phi_N) \tag{3.22}$$

where N represents coupled canonical scalar fields.

3.4 Phantom models

In this section, we consider an alternative model that explores the concept of dark energy from non-canonical scalar fields.

Canonical refers to the standard or usual form of the kinetic term for a scalar field ϕ in the Lagrangian density: typically it's proportional to the square of the first derivative of the field with respect to space-time coordinates. Mathematically, it takes the form $\frac{1}{2}(\partial_\mu\phi)^2$, where ∂_μ as before represents the partial derivative with respect to space-time coordinates. Non-canonical scalar fields involve kinetic terms that deviate from this standard form.

Non-canonical scalar fields introduce new degrees of freedom and dynamics that can lead to unique cosmological consequences. The behavior of these fields can differ significantly from that of canonical scalar fields.

The simplest example of a non-canonical scalar field model differs from the canonical one only for the sign of the kinetic term, this scalar field is known as *Phantom field*:

$$\mathcal{L}_\phi = +\frac{1}{2}\partial\phi^2 - V(\phi) \tag{3.23}$$

The effective equations of state for non-canonical scalar fields can lead to departures from the simple $w = -1$ behavior associated with a cosmological constant, for phantom dark energy $w_{de} < -1$. Recalling Eqs.3.8 and3.4:

$$a(t) \propto (t_0 - t)^{\frac{2}{3(w_{de}+1)}} \tag{3.24}$$

with t_0 set. Knowing that the exponent is negative for $w_{de} < -1$, thus $a(t)$ is indeed expanding as t increases, [NTa18] considers the interesting case where, with a particular set of initial conditions, the scale factor diverges. This leads to a future singularity known as *big rip*, where the universe will expand so fast that everything will be ripped apart.

The cosmological equations arising from this Lagrangian 3.23 are:

$$\begin{aligned} 3H^2 &= k^2\left(\rho - \frac{1}{2}\dot{\phi}^2 + V\right) \\ 2\dot{H} + 3H^2 &= k^2\left(-w\rho + \frac{1}{2}\dot{\phi}^2 + V\right) \end{aligned} \tag{3.25}$$

and the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\partial V}{\partial \phi} = 0 \tag{3.26}$$

As mentioned above, the EoS parameter $w_{de} < -1$ leads, in general, to an accelerated expansion, just like the cosmological constant. However, in the case of a phantom field, this acceleration becomes even more pronounced, leading in extreme cases to the *big rip*.

3.5 Quintom models

Quintom dark energy is a theoretical scenario that combines the features of both quintessence and phantom dark energy models. While quintessence satisfy $w_{de} \geq -1$, and the phantom model $w_{de} < -1$, there's a scenario of dark energy that gives rise to an EoS larger than -1 in the past and less that -1 today, satisfying the current observations. Quintom dark energy can exhibit tracking behavior similar to quintessence, where its energy density tracks that of the dominant component in the universe. As the universe expands and the energy density of other components decreases, quintom dark energy can enter a phantom-like phase with $w < -1$, driving accelerated expansion [Zha05].

The simplest model is represented by the following Lagrangian with two scalar fields, one canonical ϕ and one phantom field σ [NTa18]:

$$\mathcal{L} = -\frac{1}{2}\partial\phi^2 + \frac{1}{2}\partial\sigma^2 - V(\phi, \sigma) \tag{3.27}$$

This quintom scenario can distinguish between interacting and non-interacting potentials for the two scalar fields ϕ and σ, leading to different Friedmann equations and therefore dynamical evolutions. Notably[NTa18]:

1. Non-interacting $V(\phi, \sigma) = V_1(\phi) + V_2(\sigma)$;

2. Interacting potentials $V(\phi, \sigma) = V_0 e^{\lambda_\phi k\phi - \lambda_\sigma k\sigma}$ with $\lambda_\phi = -\frac{1}{kV_1}\frac{\partial V_1}{\partial \phi}$ and $\lambda_\sigma = -\frac{1}{kV_2}\frac{\partial V_2}{\partial \sigma}$

Overall we can conclude that our current understanding of the evolution of the scale factor in the universe, influenced by the presence of dark energy, has been significantly shaped by the framework of the Λ*CDM model*. This model, supported by observational evidence, has provided a robust description of the observed cosmic expansion history. However, despite the success of the ΛCDM model, the exploration of alternative dark energy models, such as *quintessence*, *phantom*, and *quintom* dark energy, provide a new perspective on the evolution of the scale factor and on the fate of the universe beyond the standard framework.

Current and upcoming observational endeavors, including surveys and experiments, are set to deepen our understanding of dark energy. Precise measurements of the Cosmic Microwave Background and Type Ia supernovae, along with an in-depth theoretical investigation of scalar-tensor fields, have the potential to provide crucial insights into the nature of dark energy.

Conclusion

This thesis delves into the intricate dynamics of dynamical systems coupling the evolution of the scale factor with cosmological fields, particularly exploring various models within the cosmological field, by combining principles from theoretical physics and cosmology. Starting with a comprehensive exploration of the Lagrangian formalism, the study extends to the foundations of General Relativity and the description of the Cosmological Field, encompassing its fundamental Einstein Field Equations. The research aims to unravel nuanced relationships between the scale factor and cosmological fields, navigating complexities introduced by scalar-tensor theories such as quintessence and phantom theories.

After a general introduction to the Lagrangian formalism for field theory, we explored the dynamics of cosmological fields, utilizing the Lagrangian formalism to shed light on two well-known examples: the Schrödinger Field and the Electromagnetic Field. We then discussed field theory for the Standard Model of particle physics, providing an overview of the primary fields that define our universe. Moving beyond, we delved into the formulation of gravitational interaction, the only fundamental interaction not covered by the Standard Model. The Cosmological Field was then introduced, elucidating the power of gravity and detailing the elements of Einstein's cosmological field equations. We derived the dynamics equations, specifically the Friedmann equations, which play a fundamental role in describing the evolution of the universe. A detailed analysis was then conducted on the stress-energy tensor, a key component in understanding the role of gravity in these equations.

We then explained how the observed cosmic acceleration can be attributed to dark energy, often modeled as a cosmological constant, and its impact on the scale factor. While the ΛCDM model successfully explains cosmic acceleration, the observed value of the cosmological constant raises questions. As a consequence, the study further delved into alternative models for dark energy, including scalar-tensor theories and quintessence models. Observational tests and future experiments are highlighted as crucial in refining our understanding of dark energy properties and potential deviations from the cosmological constant. The Lagrangian formalism, applied to dynamical systems, emerges as an effective tool for disclosing the interactions that have shaped cosmic evolution.

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