Università degli studi di Padova

Dipartimento di Fisica e Astronomia

Corso di Laurea Magistrale in

Fisica

Tesi di Laurea Magistrale

The AdS/CFT correspondence for higher spin theories

Relatore:
Prof. Dmitri Sorokin
Laureando:
Francesco Azzurli

Anno Accademico 2014/2015

Contents

Introduction

The aim of this thesis is the study of the main features and peculiarities of the theory of higher spin fields in a 4-dimensional Anti de Sitter space and its relation via holography with a conformal theory of scalar fields in a 3-dimensional Minkowski space-time. The holographic principle in the theory of fundamental interactions originates from the idea that 3-dimensional physics could be described by a theory in 2 spatial dimensions, which emerged after the computation of the black-hole Bekenstein-Hawking entropy [1]. It was proven that such entropy is the highest attainable for a given region of space containing a certain mass, according to General Relativity and Quantum Mechanics [2], and depends only on the surface of the region and not on its volume. Since entropy is an extensive quantity, it has been suggested by Susskind in [3] that physics in a volume of space, that we call bulk, could be described by a dual theory that lived only on the boundary surface, thus explaining the entropy dependence on its extension. This is analogous to what happens with holograms: an object appears to have 3 dimensions but all its information is actually encoded in the surface of the screen that projects it. This idea has been therefore called holographic principle ever since.

The AdS/CFT correspondence is a class of conjectured dualities that realize this principle. The bulk is a curved space in $(d+1)$ dimensions which is asymptotically the Anti de Sitter (AdS_{d+1}) space, namely the vacuum solution of Einstein equations when the cosmological constant is negative. This space is a vacuum of a specific gravitational theory set in the bulk. Its boundary, located at infinity, is the d dimensional flat space, where a Conformal Field Theory (CFT), dual to the bulk theory, is set. The correspondence identifies the asymptotic values of the fields in the bulk near the boundary as sources for fields in the CFT and its partition function with the one of the theory in the bulk. Thus the physical content of the two theories is the same and one can be used to prove statements about the other. In particular, correlation functions of the CFT can be found by using the partition function of the bulk. This amounts to the computation of the so-called Witten diagrams, which depict the (regularized) calculation of a correlator of the bulk theory when the base points are moved to the boundary. Witten diagrams can thus be seen as Feynman diagrams in such limit.

The first example of this correspondence has been the Maldacena conjecture [41], which was motivated by string theory arguments as a duality between open and closed strings and between two interpretations of branes, respectively as Dirichlet branes where open strings are attached and black-hole-like solutions of the gravitation theory provided by the closed strings. Under certain limits, this duality involves a weakly coupled supergravity theory in $\mathrm{AdS}_5 \times S^5$ and a certain supersymmetric quantum gauge field theory in the boundary that is conformal and strongly coupled. Moreover, its gauge group is $SU(N)$ with N large. This has important phenomenological implications: gauge theories with a high coupling constant are difficult to treat, since perturbation theory does not work and lattice methods have some drawbacks. On the other hand the duality allows to perform computations on the bulk with a weakly coupled classical theory, that is easier to handle. Many interesting results have been achieved this way (see for example [4] for a comprehensive review).

The Maldacena duality could help to shed some light on how to quantize gravity and on string theory. However, no proof has yet been given due to lack of a non-perturbative formulation of string theory in AdS. In order to better understand the AdS/CFT framework it could be useful to analyze it in simpler cases. Since the paper by Maldacena, several other dualities have been proposed.

In [43], Klebanov and Polyakov formulated an especially simple version of the correspondence. In the 4-dimensional bulk we have a theory of interacting massless fields, one for each even spin, whose classic limit is called minimal Vasiliev type A model. This theory has been found after a long sequence of attempts at formulating a consistent theory of interacting gauge fields with spin greater than two that could extend General Relativity and is thought to be linked to a tensionless limit of a string theory. As for now, even though its equation of motion are known, no satisfactory and complete action has been constructed. The dual CFT is the so-called $O(N)$ model, first introduced in [5] to describe magnets, whose content is a real massless scalar field in 3 dimensions with a global $O(N)$ symmetry. Depending on certain conditions on the bulk fields, the duality is realized with the free theory or the critical point of the interacting theory with a quartic potential. It is the fact that the dual theory can be free and thus extremely simple, that constitutes the principal attraction of this particular version of the correspondence.

A particular goal of this thesis is to analyze the Klebanov-Polyakov duality with the so-called ambient formalism, that consists in embedding the $(d+1)$-dimensional AdS space AdS_{d+1} and its boundary as submanifolds in a $(d+2)$-dimensional flat ambient space $\mathcal{A}_{d+2}$. The two spaces are then realized as a pseudo-hyperboloid $\mathscr{S}_{d+1}$ and a section of the light-cone $\mathscr{L}_{d+1}$, respectively. Consequently, one extends the tensor fields defined in these submanifolds to the ambient space[1]. This allows to use the ambient tensor fields in place of the original ones, which significantly simplifies computations. In particular, by these means we will set up a framework to test the correspondence for what concerns the computation of the 3-point functions.

On the CFT side, the ambient formalism implements conformal transformations as rotations of the vectors in $\mathcal{A}_{d+2}$ that represent the points of the boundary. In this way, imposing conformal invariance on the relevant physical quantities amounts to requiring invariance with respect to the orthogonal symmetry group of $\mathcal{A}_{d+2}$. We will employ this symmetry to characterize almost completely the 3-point correlators without making explicit computations. This will allow us to treat both the critical and the free $O(N)$ vector model at the same time.

On the bulk side, we will circumvent the difficulty of not having at our disposal an action for Vasiliev's theory by constructing its cubic vertices, which are determined by the higher spin gauge symmetry up to some coupling constants that are not constrained at that order. In this task the ambient formalism is determinant to avoid the complications derived from the curvature of the AdS space, since the ambient fields are defined in a flat space. All the ingredients necessary to compute Witten diagrams will be found in such way.

The work is structured as follows. In the first chapter, we follow the historical development of the theory of higher spin fields and show how the full non-linear interacting theory is constructed. Then, in the second chapter we present the general structure of the AdS/CFT correspondence by starting with the first known and best understood example, namely the Maldacena conjecture. We then describe the Klebanov-Polyakov proposal. The third chapter is devoted to the ambient formalism and its applications to both the boundary and the bulk theories. In the final chapter we show how to use the framework developed so far to compute the correlator between two scalars and one higher spin field, which will confirm the correspondence in that particular example.

[1] Actually this is done in some portions of $\mathcal{A}_{d+2}$ that contain the submanifolds representing AdS_{d+1} and its boundary.

Chapter 1

Higher spins theories

The first time in which particles of arbitrarily high spin appeared in scientific literature was in a paper by Majorana of 1932 (see [6, 7, 8]). The purpose of that work was to find a wave equation that possessed only positive energy solutions, in order to solve the dilemma posed by the Dirac equation about the physical meaning of its negative energy solutions[1]. He discovered one of the unitary (and thus infinite dimensional) representations of the Lorentz group and formulated an equation of motion for fields taking values in that representation. Its solutions were found to describe particles with an arbitrary spin that was related to the mass, a feature that reminds what happens in string theory, as we will explain in subsection 2.1.3.

Majorana's results have been ignored at the time and the investigation on theories of higher spin fields has been resumed by Dirac, who, after the discovery of the equation for a relativistic particle of spin $\frac{1}{2}$, wrote a seminal paper [9] in which he faced the problem of finding the most general form of the wave equation of a relativistic particle of arbitrary spin in view of possible discoveries of such particles or composite systems that could be approximately treated in that way. His results, though, were not compatible with the interaction with an external electromagnetic field, as noticed in a work by Fierz and Pauli [10], in that the minimal prescription of replacing the usual space-time derivative with a covariant one led to a contradiction, as we will explain in the next subsection.

The two authors then proposed to implement gauge invariance through an action principle, that would guarantee its compatibility with the equation of motion. The goal of finding a proper Lagrangian has been pursued in [11, 10] and successfully completed in full generality by Singh and Hagen in [12], by means of the introduction of a certain number of auxiliary fields that are found to vanish on shell.

Unfortunately, even if the program started by Fierz and Pauli had been thus completed, their original aim of consistently adding an electromagnetic interaction to higher spin fields was not achieved. Indeed, even if the equations of motion were compatible with an electromagnetic field, their solutions were still unphysical because they described particles that could move faster than light, as pointed out by Velo and Zwanziger in [13].

Later, a massless limit of the Singh-Hagen Lagrangian has been investigated by Fronsdal and Fang in [14, 15]. The result was that free massless higher integer spin fields are described by completely symmetric tensors whose double trace vanishes and that combine all the Fierz Pauli auxiliary fields. These so-called Fronsdal fields are subjected to a gauge symmetry.

In the paper about integer spins [15], Fronsdal proposed a Gupta program for higher spin fields, whose aim was to find a theory that describes their interaction with other fields. Such theory was expected to be non-linear, since in the spin 2 case it should coincide with General Relativity. Similarly, the original linear gauge symmetry should be deformed in a non linear way. However, some results obtained by S-matrix techniques showed that, under certain general physical hypotheses, no such theory existed at the quantum level. We will review them in subsection 1.2.1. Investigations [27, 28] at the classical level also showed how

[1] This problem had been solved the year before with the introduction of the positron, unbeknownst to Majorana

the required deformation of the gauge symmetry could not be consistent at the next to linear orders. In this context, several proposals on how to circumvent the assumptions of these No-Go theorems have been made, as the presence of infinite higher spin particles, unusual higher derivative couplings and a non flat background geometry. We review them in the subsection 1.2.2.

In [34] Vasiliev proposed a different formulation of the Fronsdal equations based on higher spin connection 1-forms, called frame-like description because it generalizes the frame-like language proposed in [33] by Cartan for General Relativity. We will present such formalism in subsection 1.3.1 and its extension to higher spin fields in 1.4.3.

In the Anti de Sitter space, the frame-like formulation led to the discovery in [31] by Fradkin and Vasiliev of cubic vertices of interactions that were compatible with gauge symmetry and featured higher derivative couplings. These were allowed because of the presence of the dimensional cosmological constant Λ that makes it possible for the vertices to have the right dimensions. Moreover, the flat space limit $\Lambda \to 0$, was found to be singular because of the negative power of Λ, thus explaining the difficulties in the Minkowski space and the necessity of a curved background.

At the same time, the formalism introduced by Vasiliev unveiled in [35] an infinite dimensional gauge symmetry algebra behind the equations of motion of higher spin fields. This higher spin algebra will be reviewed in subsection 1.4.4 and has eventually led to the completion of the Fronsdal program, at least for what concerns massless fields, with the Vasiliev non-linear equations of motion of the interacting theory in 4 dimensions [37] and later in any dimension [38].

In this thesis we will examine briefly only the higher spin equations in AdS_4 in subsection 1.5 and will present a conjectured duality of this higher spin gravity with a 3-dimensional Quantum Field Theory in chapter 2.

1.1 The original problem

In general it is assumed that the laws of physics are covariant under the Poincaré group $ISO(1,3)$, which extends the Lorentz symmetry with the translational one. The corresponding algebra $\mathfrak{iso}(1,3)$ is given by the generators of translations P^μ and of Lorentz transformations $M^{\mu\nu} = -M^{\nu\mu}$. They obey the following commutation rules:

$$\begin{gathered} \left[P^\mu, P^\nu\right] = 0, \quad \left[P^\mu, M^{\alpha\beta}\right] = i\left(P^\alpha \eta^{\mu\beta} - P^\beta \eta^{\mu\alpha}\right) \\ \left[M^{\mu\nu}, M^{\alpha\beta}\right] = i\left(M^{\alpha\nu}\eta^{\mu\beta} - M^{\beta\nu}\eta^{\mu\alpha} - M^{\alpha\mu}\eta^{\nu\beta} + M^{\beta\mu}\eta^{\nu\alpha}\right). \end{gathered} \tag{1.1.1}$$

Thus all particles are represented by fields which sit in irreducible representations of $ISO(1,3)$. These are labeled by the eigenvalues of the two Casimir operators of $\mathfrak{iso}(1,3)$

$$C_1 = P^2, \quad C_2 = W^2,$$

where the Pauli-Lubanski operator has been defined as

$$W^\mu = \frac{1}{2}\mathcal{E}^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}.$$

It can be shown that this operator generates the transformations which belong to the stability groups of $ISO(1,3)$. For a particle of mass m and spin s,

$$C_1 = m^2, \quad C_2 = m^2 s\,(s+1)\,.$$

To generate an irreducible integer higher spin finite-dimensional representation one can start from tensorial products of the vector representation of the Lorentz subgroup, i.e. by considering fields with many indices $\varphi_{\mu_1 \ldots \mu_s}(x)$. A necessary condition for irreducibility is that the φ are traceless, since their traceful component is proportional to the metric, which does not change under Lorentz transformations. The spin of $\varphi_{\mu_1\ldots\mu_s}(x)$ is not yet well-defined, because it includes subspaces characterized by all the spins from 0 to s.

In order to fix the spin, let us consider first a massive particle. There always exists a frame in which it is motionless, i.e. $p^\mu = (m, 0, 0, 0)$. The stability group of this frame is given by three-dimensional rotations, that is elements of $SO(3) < SO(1,3)$, for which C_2 is the associated Casimir. For this reason we decompose the representation of a four-vector in irreducible representations of $SO(3)$. Under such perspective the 0-component of a four-vector is a scalar, while the others form a three-vector. So, in order to obtain the highest spin possible in our construction, we must have that the tensor product of representations of four-vectors involves only the three-vectors, or, in other words, that all the 0 components of φ are zero in the considered frame. A covariant way to express this condition is the following

$$p^{\mu_i}\varphi_{\mu_1\ldots\mu_i\ldots\mu_s}(p) = 0 \implies \partial^{\mu_i}\varphi_{\mu_1\ldots\mu_i\ldots\mu_s}(x),$$

where a Fourier transform has been performed in the first relation. We say that φ is transverse. Finally, tensor products of three-vectors have spins that range from 0 to s according to the symmetry of the permutations of the indices. The highest spin is always associated with the totally symmetric tensor, since it has the greatest number of independent components[2].

All these conditions lead to the following definition of a massive higher spin field:

$$\varphi_{\mu_1\ldots\mu_s}(x) \equiv \varphi_{(\mu_1\ldots\mu_s)}(x) \tag{1.1.2}$$
$$\eta^{\mu_1\mu_2}\varphi_{\mu_1\ldots\mu_s}(x) = 0 \tag{1.1.3}$$
$$\partial^{\mu_1}\varphi_{\mu_1\ldots\mu_s}(x) = 0. \tag{1.1.4}$$

If we considered higher dimensions, the stability group for the rest-frame would have been some $SO(n)$ with $n > 3$. This group has more than one Casimir operator and we would have had also tensor with a mixed symmetry for their indices.

The equations of motion for this field can be then easily derived by a straightforward generalization of the Proca equation

$$\left(\Box + m^2\right) V^\mu - \partial^\mu \left(\partial \cdot V\right) = 0 \tag{1.1.5}$$

by considering that its first member must have the same symmetries of the indices of φ as pointed out, for example, in [12]:

$$\left(\Box + m^2\right)\varphi_{\mu_1\ldots\mu_s} - \partial_{(\mu_1}\left(\partial\cdot\varphi\right)_{\mu_2\ldots\mu_s)} + \frac{2}{D+2s-4}\eta_{(\mu_1\mu_2}\left(\partial\cdot\left(\partial\cdot\varphi\right)\right)_{\mu_3\ldots\mu_s)} = 0, \tag{1.1.6}$$

where D is the dimension of space-time, 4 in our case. Contrary to what happens for (1.1.5), by taking the four-divergence of (1.1.6), it is not possible to derive the constraint of transversality (1.1.4). For this reason it has to be imposed by hand and it reduces (1.1.6) to the expected Klein-Gordon equation.

Suppose now that we want to describe a charged higher spin particle. The minimal prescription suggests to replace every space-time derivative with a covariant one. If we do this with (1.1.6), we get

$$\left(D^\alpha D_\alpha + m^2\right)\varphi_{\mu_1\ldots\mu_s} = 0, \quad D^{\mu_1}\varphi_{\mu_1\ldots\mu_s} = 0$$

but then

$$\begin{aligned} 0 &= \left[\left(D^\alpha D_\alpha + m^2\right), D^{\mu_1}\right]\varphi_{\mu_1\ldots\mu_s} \\ &= 2\left[D^\alpha, D^{\mu_1}\right] D_\alpha\varphi_{\mu_1\ldots\mu_s} \\ &= 2ieF^{\alpha\mu_1}D_\alpha\varphi_{\mu_1\ldots\mu_s}. \end{aligned}$$

[2]Note that any couple of anti-symmetric spatial indices can always be converted to one vector index by means of the totally antisymmetric Levi-Civita tensor ε^{ijk}.

Now, this represents an additional constraint on φ, so that its degrees of freedom reduce and this is not acceptable. This kind of obstructions to the implementation of interactions for higher spins has been first pointed out by Fierz and Pauli in [10].

A way to avoid this difficulty would be to start from a Lagrangian description of the theory, that would immediately translate the $U(1)$ symmetry of the action to the Euler-Lagrange equations. As (1.1.5), also (1.1.6) is associated to an action, which does not give rise to the transversality constraint (1.1.4), though. In order to complete this program, it has been suggested in [10] and [11] to add to the Lagrangian some auxiliary fields whose equations of motion would reduce to (1.1.4). These fields, moreover, are found to be zero on-shell. Let us consider a simple example of this technique: the equations for a massive spin 2 particle. We start from

$$\left(\Box+m^2\right)\varphi_{\mu\nu}-\partial_\mu\left(\partial\cdot\varphi\right)_\nu-\partial_\nu\left(\partial\cdot\varphi\right)_\mu+\frac{1}{2}\eta_{\mu\nu}\left(\partial\cdot\left(\partial\cdot\varphi\right)\right)=0 \tag{1.1.7}$$

whose Lagrangian would be the generalization of the Proca one:

$$\mathcal{L}_2(x)=\frac{1}{2}\partial_\mu\varphi_{\alpha\beta}\left(\partial^\mu\varphi^{\alpha\beta}-\partial^\alpha\varphi^{\mu\beta}-\partial^\beta\varphi^{\alpha\mu}\right)-\frac{m^2}{2}\varphi_{\alpha\beta}\varphi^{\alpha\beta}. \tag{1.1.8}$$

Now, we take the four divergence of (1.1.7) and get

$$m^2\left(\partial\cdot\varphi\right)_\nu+\frac{1}{2}\partial_\nu\left(\partial\cdot\left(\partial\cdot\varphi\right)\right)=0.$$

As anticipated, contrarily to what happens for spin 1, a term with three derivatives appears. In order to eliminate it, we could introduce a scalar field $a(x)$ whose equations of motion eliminate such term. The most generic action for this field is

$$\mathcal{L}_{2aux}(x)=-\frac{1}{2}\partial_\mu a\partial^\mu a-\frac{\alpha}{2}a^2+\beta a\left(\partial\cdot\left(\partial\cdot\varphi\right)\right) \tag{1.1.9}$$

so that, when $a(x)\equiv 0$ as it must eventually turn out, its Euler-Lagrange equations will still eliminate the unwanted contribution by $\partial\cdot(\partial\cdot\varphi)$. The new equations of motion are then

$$\begin{cases}\left(\Box+m^2\right)\varphi_{\mu\nu}-\partial_\mu\left(\partial\cdot\varphi\right)_\nu-\partial_\nu\left(\partial\cdot\varphi\right)_\mu+\frac{1}{2}\eta_{\mu\nu}\left(\partial\cdot\left(\partial\cdot\varphi\right)\right)-\beta\left(\partial_\mu\partial_\nu a-\frac{1}{4}\eta_{\mu\nu}\Box a\right)=0\\ \left(-\Box+\alpha\right)a-\beta\left(\partial\cdot\left(\partial\cdot\varphi\right)\right)=0\end{cases} \tag{1.1.10}$$

Now, we take the four-divergence of the first equation, substitute the $\Box a$ term by using the second:

$$\begin{aligned}2m^2\left(\partial\cdot\varphi\right)_\nu-\partial_\nu\left(\partial\cdot\left(\partial\cdot\varphi\right)\right)-\frac{3}{2}\beta\partial_\nu\Box a &= 0\\ m^2\left(\partial\cdot\varphi\right)_\nu-\partial_\nu\left(\partial\cdot\left(\partial\cdot\varphi\right)\right)-\frac{3}{2}\alpha\beta\partial_\nu a+\frac{3}{2}\beta^2\partial_\nu\left(\partial\cdot\left(\partial\cdot\varphi\right)\right) &= 0.\end{aligned}$$

If we put $\beta=\sqrt{\frac{2}{3}}$, and apply ∂^ν to both members we come to

$$\Box a=\sqrt{\frac{2}{3}}\frac{m^2}{\alpha}\left(\partial\cdot\left(\partial\cdot\varphi\right)\right) \tag{1.1.11}$$

Now, we insert (1.1.11) into (1.1.10) and we arrive at

$$\alpha a=\sqrt{\frac{2}{3}}\left(+\frac{m^2}{\alpha}+1\right)\left(\partial\cdot\left(\partial\cdot\varphi\right)\right)$$

so that, for $\alpha=-m^2$, the a field is set to zero, while (1.1.11) implies that $\partial\cdot(\partial\cdot\varphi)$ vanishes and the transversality constraint follows from (1.1.9). The ugly square root $\sqrt{\frac{2}{3}}$ appearing throughout the equations and the Lagrangian can be removed by rescaling the auxiliary field a into $\sqrt{\frac{3}{2}}a$.

This procedure has been generalized to all spins by Singh and Hagen in [12]. Their final result is that for a particle with spin s it is necessary to introduce $s-1$ auxiliary fields, one for each spin strictly lesser than $s-1$. They share with φ the same properties (1.1.3), (1.1.4) and (1.1.2) and on-shell they vanish, so that φ simply obeys the Klein-Gordon equation:

$$\left(\Box + m^2\right)\varphi_{\mu_1\ldots\mu_s} = 0, \quad (\partial\cdot\varphi)_{\mu_2\ldots\mu_s} = 0. \tag{1.1.12}$$

We are ready to examine the case of massless particles, the ones we will be interested in throughout this thesis. Following the spin 2 example, we set $m=0$. After the suggested rescaling, (1.1.10) becomes

$$\begin{cases}\Box\varphi_{\mu\nu} - \partial_\mu(\partial\cdot\varphi)_\nu - \partial_\nu(\partial\cdot\varphi)_\mu + \frac{1}{2}\eta_{\mu\nu}(\partial\cdot(\partial\cdot\varphi)) - \partial_\mu\partial_\nu a + \frac{1}{4}\eta_{\mu\nu}\Box a = 0 \\ -\frac{3}{2}\Box a - (\partial\cdot(\partial\cdot\varphi)) = 0\end{cases} \tag{1.1.13}$$

Substituting in (1.1.13) the second equation into the first, after some reordering, we get

$$\begin{aligned}\Box\varphi_{\mu\nu} - \partial_\mu(\partial\cdot\varphi)_\nu - \partial_\nu(\partial\cdot\varphi)_\mu - \partial_\mu\partial_\nu a - \frac{1}{2}\eta_{\mu\nu}\Box a &= 0\\ \Box\left(\varphi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}a\right) - \partial_{(\mu}\partial^\alpha\left(\varphi_{\alpha\nu)} - \frac{1}{2}\eta_{\alpha\nu)}a\right) - 2\partial_\mu\partial_\nu a &= 0\\ \Box\left(\varphi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}a\right) - \partial_{(\mu}\partial^\alpha\left(\varphi_{\alpha\nu)} - \frac{1}{2}\eta_{\alpha\nu)}a\right) + \partial_\mu\partial_\nu\eta^{\alpha\beta}\left(\varphi_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}a\right) &= 0\end{aligned} \tag{1.1.14}$$

We see that φ and a can be combined into a new field

$$\Phi_{\mu_1\mu_2}(x) = \varphi_{\mu_1\mu_2}(x) - \frac{1}{2}\eta_{\mu_1\mu_2}a(x) \tag{1.1.15}$$

whose trace is non-vanishing and proportional to a. It obeys the following equation of motion:

$$\Box\Phi_{\mu_1\mu_2} - \partial_{(\mu_1}\partial^\alpha\Phi_{\alpha\mu_2} + \partial_{\mu_1}\partial_{\mu_2}\Phi^\alpha{}_\alpha = 0 \tag{1.1.16}$$

Equation (1.1.16) is the so-called Fronsdal equation for spin 2 because it is a particular case of what happens for every other higher spin, as showed in [14] by Fronsdal. Indeed, he took the limit for $m\to 0$ of the Singh-Hagen Lagrangian and found that all the auxiliary fields decouple, except the one with the highest spin: $a_{\mu_1\ldots\mu_{s-2}}(x)$. It can be combined together with φ with the obvious generalization of (1.1.15)

$$\Phi_{\mu_1\ldots\mu_s}(x) = \varphi_{\mu_1\ldots\mu_s} - \frac{1}{2}\eta_{(\mu_1\mu_2}a_{\mu_3\ldots\mu_s)}.$$

Therefore, $\Phi_{\mu_1\ldots\mu_s}(x)$ describes in full generality a massless field of spin s. Again, a represents the trace of Φ, but, being itself traceless, leads us to the only constraint on Φ

$$\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\Phi_{\mu_1\ldots\mu_s}(x) = 0. \tag{1.1.17}$$

The Fronsdal action for the field $\Phi_{\mu_1\ldots\mu_s}(x)$ has the following form

$$\begin{aligned}S_{Fronsdal} = \int\Bigg(&\frac{1}{2}\partial_\alpha\Phi_{\mu_1\ldots\mu_s}\partial^\alpha\Phi^{\mu_1\ldots\mu_s} - \frac{s}{2}\partial_\alpha\Phi^\alpha_{\mu_2\ldots\mu_s}\partial^\beta\Phi^{\mu_2\ldots\mu_s}_\beta - \frac{s(s-1)}{2}\Phi^\alpha{}_{\alpha\mu_3\ldots\mu_s}\partial_\beta\partial_\gamma\Phi^{\beta\gamma\mu_3\ldots\mu_s}\\ &-\frac{s(s-1)}{4}\partial_\beta\Phi^\alpha{}_{\alpha\mu_3\ldots\mu_s}\partial^\beta\Phi_\gamma{}^{\gamma\mu_3\ldots\mu_s} - \frac{s(s-1)(s-2)}{8}\partial^\beta\Phi^\alpha{}_{\alpha\beta\mu_4\ldots\mu_s}\partial_\delta\Phi_\gamma{}^{\gamma\delta\mu_4\ldots\mu_s}\Bigg)d^4x\end{aligned} \tag{1.1.18}$$

It gives rise to the Fronsdal equation

$$\mathcal{F}_{\mu_1\ldots\mu_s} \equiv \Box\Phi_{\mu_1\ldots\mu_s} - \partial_{(\mu_1}(\partial\cdot\Phi)_{\mu_2\ldots\mu_s)} + \partial_{(\mu_1}\partial_{\mu_2}\Phi^\nu{}_{\nu\mu_3\ldots\mu_s)} = 0, \tag{1.1.19}$$

where $\mathcal{F}_{\mu_1\ldots\mu_s}(x)$ is the so-called Fronsdal tensor. As it is expected for a massless field, (1.1.19) exhibits a gauge invariance under the transformation

$$\delta\Phi_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\Lambda_{\mu_2\ldots\mu_s)}, \tag{1.1.20}$$

where Λ must be traceless. Indeed, under (1.1.20)

$$\begin{aligned}\delta\mathcal{F}_{\mu_1\ldots\mu_s} &= \Box\delta\Phi_{\mu_1\ldots\mu_s} - \partial_{(\mu_1}\left(\partial\cdot\delta\Phi\right)_{\mu_2\ldots\mu_s)} + \partial_{(\mu_1}\partial_{\mu_2}\delta\Phi^{\nu}{}_{\nu\mu_3\ldots\mu_s)}\\ &= \Box\partial_{(\mu_1}\Lambda_{\mu_2\ldots\mu_s)} - \partial_{(\mu_1}\Box\Lambda_{\mu_2\ldots\mu_s)} - 2\partial_{(\mu_1}\partial_{\mu_2}\left(\partial\cdot\Lambda\right)_{\mu_3\ldots\mu_s)}\\ &\quad +2\partial_{(\mu_1}\partial_{\mu_2}\left(\partial\cdot\Lambda\right)_{\mu_3\ldots\mu_s)} + \partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\Lambda^{\nu}{}_{\nu\mu_3\ldots\mu_s)}\\ &= 0\end{aligned} \tag{1.1.21}$$

only when Λ is traceless.

For the free spin 2, we recognize in

$$\delta\Phi_{\mu\nu} = \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu$$

the linearized variation of a metric Φ under infinitesimal diffeomorphisms of the type

$$\delta x^\mu = \Lambda^\mu(x).$$

We expect then that (1.1.16) is the linearized Einstein equation for a quasi-flat space, which describes gravitational waves or, from a quantum perspective, the graviton. This amounts to saying that (1.1.16) should be equivalent to the wave equation. We can show this in general for every spin. Indeed gauge invariance allows us to eliminate some terms in (1.1.20) by the following gauge-fixing.

First of all, we take a gauge parameter $\Lambda^{(1)}$ which obeys

$$\partial\cdot\Lambda^{(1)}_{\mu_2\ldots\mu_s} = -\frac{1}{2}\Phi^{\nu}{}_{\nu\mu_3\ldots\mu_s}.$$

so that we can put $\Phi^{\nu}{}_{\nu\mu_3\ldots\mu_s} = 0$. Now we can perform a second gauge transformation with $\Lambda^{(2)}$ satisfying the equations

$$\begin{cases}\partial\cdot\Lambda^{(2)}_{\mu_2\ldots\mu_s} = 0\\ \Box\Lambda^{(2)}_{\mu_2\ldots\mu_s} = -\left(\partial\cdot\Phi\right)_{\mu_2\ldots\mu_s}\end{cases}$$

that does not spoil the previous result and that gives the following gauge fixing

$$\left(\partial\cdot\Phi\right)_{\mu_2\ldots\mu_s} = 0, \quad \Phi^{\nu}{}_{\nu\mu_3\ldots\mu_s} = 0 \tag{1.1.22}$$

which is referred to as transverse traceless (TT) gauge. Thus, (1.1.19) becomes simply

$$\Box\Phi_{\mu_1\ldots\mu_s} = 0. \tag{1.1.23}$$

Now that we have a free theory capable of describing every massless particle of arbitrary spin, we would like to turn on interactions between them and other particles.

1.2 Constraints on the theory

The program outlined above clashes almost immediately with a series of theorems that show how an interacting theory would be inconsistent in many realistic scenarios. In the first part of this subsection we will review the most important and general ones. After that we will list some ways to bypass each of them by altering slightly their hypotheses. We will choose one, namely the employment of a curved space-time with a non-zero cosmological constant, even though some of the ideas for the Minkowski space will come back when the Ambient Space formalism will be treated.

1.2.1 No-go theorems

1.2.1.1 The Weinberg argument

Consider in full generality a process involving n particles of momenta p_i^μ and spins s_i, which causes the emission of a soft massless particle of momentum $q^\mu \to 0$ and helicity s as shown in figure (1.2.1).

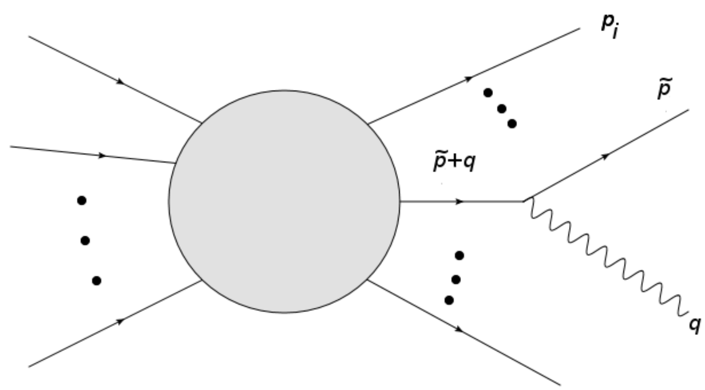


Figure 1.2.1: A generic scattering with an emission of a massless particle with momentum q.

Lorentz invariance then imposes on the S-matrix associated to the process the following form as shown in [19]:

$$S(q, s, p_i^\mu, s_i) = \mathcal{E}^{\mu_1 \ldots \mu_s}(q) M_{\mu_1 \ldots \mu_s}(q, p_i) \tag{1.2.1}$$

where $\mathcal{E}$ is the polarization tensor of the massless particle and, as the field it represents, it is traceless, totally symmetric and transverse:

$$q^\mu \mathcal{E}_{\mu\mu_2 \ldots \mu_s}(q) = 0.$$

Due to gauge fixing performed on $\mathcal{E}$, it should not behave as a proper tensor, in that under Lorentz transformations the gauge fixing that determines $\mathcal{E}$ breaks and so longitudinal components appear. Therefore one should add additional (longitudinal) terms to the usual transformation law. They must have the form of

$$\delta\mathcal{E}^{\mu_1 \ldots \mu_s}(q) = q^{(\mu_1}\mathcal{D}^{\mu_2 \ldots \mu_s)}(q), \quad q^\mu \mathcal{D}_{\mu_2 \ldots \mu_s}(q) = 0, \quad \mathcal{D}^\mu{}_{\mu\mu_4 \ldots \mu_s}(q) = 0 \tag{1.2.2}$$

to fulfill all the algebraic requirements of $\mathcal{E}$. Since S should be a scalar, the spurious components (1.2.2) should not contribute to its value, and this is achieved by requiring that:

$$q^\mu M_{\mu\mu_2 \ldots \mu_s}(p_i) = 0. \tag{1.2.3}$$

Moreover, due to the symmetry of $\mathcal{E}$, we can take M to be totally symmetric too.

Let us now give a closer look at M. Consider first the case in which the emission starts from the one of the external outgoing legs of the diagram. We denote its four-momentum temporarily with $\tilde{p}$ to distinguish it from the set of the momenta p_i. A propagator with momentum $p = \tilde{p} + q$ is then involved. Since every free field, whatever spin it possesses, must ultimately obey a Klein-Gordon equation as we have shown in the previous section (see (1.1.12) and (1.1.23)), this propagator is proportional to the usual double pole for $p^0 = \pm\sqrt{|\vec{p}|^2 + m^2}$ which, for $q \to 0$ becomes (remember that $q^2 = 0$)

$$\frac{1}{(\tilde{p}+q)^2 - m^2 + i\epsilon} \xrightarrow{q \to 0} \frac{1}{2\tilde{p}\cdot q + i\epsilon}.$$

This kind of term dominates over every other, including the emission from internal lines, where p is off-shell and m^2 is not canceled. Notice that if the external leg is incoming, conservation of the four-momentum gives $p = \tilde{p} - q$ and so we get a minus sign in front of $2\tilde{p} \cdot q$. We denote this sign with a factor $\sigma_i = \pm 1$.

We still have to determine which terms contribute to the index structure of M. The tracelessness of $\mathcal{E}$ makes $\eta^{\mu\nu}$ useless as building block. The same happens for q^μ, this time because of the transversality of $\mathcal{E}$. Moreover, since the emission can only involve one of the n particles at a time, we may assume that M is the sum of the contributions of the type $M^i_{\mu_1 \dots \mu_s}(\tilde{p}, q) \equiv M^i_{(\mu_1 \dots \mu_s)}(\tilde{p}, q)$ from each kind of $s_i - s_i - s$ vertex. Then the only way in which we may build a symmetric tensor is by means of products of $\tilde{p}^\mu$. Finally, if we call g_i the coupling constant for the $s_i - s_i - s$ vertex, we also get that $M^i \propto g_i$. (We include in g_i the sign due to the positive or negative charge of the ith particle.) We arrive eventually at an expression of the following form

$$M_{\mu_1 \dots \mu_s}(q, p_i) \propto \sum_i \sigma_i g_i \frac{p_i^{\mu_1} \cdots p_i^{\mu_s}}{p_i \cdot q}. \tag{1.2.4}$$

where any other scalars involved in each M^i can be incorporated in g_i or vanish if $q \to 0$. Let us impose on (1.2.4) the transversality condition (1.2.3):

$$\sum_i \sigma_i g_i p_i^{\mu_2} \cdots p_i^{\mu_s} = 0. \tag{1.2.5}$$

We see that when the emitted particle is a photon, condition (1.2.5) just implies the conservation of the charge

$$\sum_{i \in outgoing} g_i - \sum_{i \in incoming} g_i = 0 \tag{1.2.6}$$

in the process. For spin 2, instead, we find

$$\sum_i \sigma_i g_i p_i^\mu = 0$$

i.e. the principle of equivalence that states that every particle interacts with the graviton with the same strength,

$$g_i \equiv g$$

and the conservation of the four-momentum

$$\sum_{i \in outgoing} p_i^\mu - \sum_{i \in incoming} p_i^\mu = 0.$$

For higher spins there is no such a general solution for every n, p_i and g_i except $g_i \equiv 0$ and we conclude that it is not possible for them to take part in a process that implies their emission to infinity.

There are several ways to circumvent this argument. First we observe that it might be possible that higher spin interactions are at short range and so they cannot appear in the asymptotic states in which the S-matrix projects. Moreover we have considered only vertices of the kind $s - s_i - s_i$, namely the minimal coupling to massless gauge fields: we do not know what could happen in more general cases.

1.2.1.2 The Weinberg-Witten theorem

This no-go example improves the previous one in that it does not rely on the fact that higher spins have long range interactions. Indeed, even if we do not know which kind of interactions our particles may experience, (1.2.6) shows that every particle should interact with gravity in the same way as the others. This exposes our theory to the consequences of the Weinberg-Witten theorem, derived in [21]. Its content (at least the part we will be interested in) is the following:

A theory with a Poincaré-covariant gauge-invariant conserved energy momentum tensor $T^{\mu\nu}$ can not allow particles with spin greater than 1 whose four-momentum is $p^\mu = \int T^{0\mu} d^3x$.

We will give just a sketch of the proof, without delving too much in unneeded technicalities. First, consider a massless particle with spin s that scatters off a graviton, so that its momentum changes from p^μ to $p^\mu + q^\mu$. Then

$$\begin{aligned} \langle p+q\,|P^\mu|\,p\rangle &= p^\mu \langle p+q|p\rangle \\ &= (2\pi)^3 \left(p^0\right) p^\mu \delta^3(\vec{q}) \end{aligned} \tag{1.2.7}$$

where we used a Lorentz-invariant normalization for the single particle states.

Here we assume that we are dealing with physical states whose momentum is never truly determined with infinite precision. For this reason, the reader should consider all the Dirac delta functions as heavily peaked functions with finite width a that, in the distributional limit for $a \to 0$, become true deltas. These functions have basically the same algebraic properties of Dirac delta functions, so that we can forget about the difference. Analogously, every integration is performed on a finite volume, rather than on the whole space. (see [22] for a discussion on these matters) This assumption allows us to make expressions as (1.2.7) non-zero for a finite (insofar small) range of momenta q and therefore meaningful in our setup. We will therefore be forced to implicitly take the limit for $q \to 0$ in every expression throughout our derivation.

Another way to express equality (1.2.7) is

$$\begin{aligned} \langle p+q\,|P^\mu|\,p\rangle &= \left\langle p+q \left| \int T^{0\mu}(t,\vec{x}) d^3x \right| p \right\rangle \\ &= \int \left\langle p+q \left| e^{-i\vec{P}\cdot\vec{x}} T^{0\mu}(t,\vec{0}) e^{i\vec{P}\cdot\vec{x}} \right| p \right\rangle d^3x \\ &= \int e^{-i\vec{q}\cdot\vec{x}} d^3x \left\langle p+q \left| T^{0\mu}(t,\vec{0}) \right| p \right\rangle \\ &= (2\pi)^3 \delta^3(\vec{q}) \left\langle p+q \left| T^{0\mu}(t,\vec{0}) \right| p \right\rangle \end{aligned}$$

and thus

$$\left\langle p+q \left| T^{0\mu}(t,\vec{0}) \right| p \right\rangle = p^0 p^\mu. \tag{1.2.8}$$

Lorentz covariance then imposes

$$\mathcal{T}^{\mu\nu} \equiv \left\langle p+q \left| T^{\mu\nu}(t,\vec{0}) \right| p \right\rangle \propto p^\mu p^\nu. \tag{1.2.9}$$

We notice that for $q^\mu \to 0$ the matrix element (1.2.9) does not vanish.

Now, let $p'^\mu = p^\mu + q^\mu$ and θ be the angle between $\vec{p}'$ and $\vec{p}$. Then

$$\begin{aligned} \left(p'+p\right)^2 &= p'^2 + 2p'\cdot p + p^2 \\ &= 2p'^0 p^0 \left(1-\cos\theta\right) \geq 0 \end{aligned}$$

so that $p'+p$ is time-like. Then, there exists a frame in which $\vec{p}' + \vec{p} = 0$, the so-called brick-wall frame, where

$$p' = (E, 0, 0, E)\,, \quad p = (E, 0, 0, -E)$$

for some $E > 0$. Apply now a rotation of angle α around the x^3 axis in this frame. Since the momenta of the two particles are opposite, the rotation is clockwise for one and anticlockwise for the other.

We may choose to apply this unitary transformation to the states, and then we get

$$|p\rangle \to e^{i\alpha s}\,|p\rangle\,, \quad |p'\rangle \to e^{-i\alpha s}\,|p'\rangle$$

because p and p' are invariant under such rotation (that belongs to the stability group for both) and therefore $|p\rangle$ and $|p'\rangle$ are eigenvectors. (1.2.9) becomes thus

$$\mathcal{T}'^{\mu\nu} = e^{2i\alpha s} \left\langle p' \left| T^{\mu\nu}(t,\vec{0}) \right| p \right\rangle \tag{1.2.10}$$

If, on the contrary, we apply the rotation to the operators, it behaves as a Lorentz transformation and so

$$\mathcal{T}'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} \left\langle p' \left| T^{\alpha\beta}(t,\vec{0}) \right| p \right\rangle. \tag{1.2.11}$$

Equating (1.2.10) we get

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} \left\langle p' \left| T^{\alpha\beta}(t,\vec{0}) \right| p \right\rangle = e^{2i\alpha s} \left\langle p' \left| T^{\mu\nu}(t,\vec{0}) \right| p \right\rangle$$

Now, since the Λ are rotation matrices, their eigenvalues can be only $e^{\pm i\alpha}$ and 1. We deduce that, for $s > 1$, (1.2.11) holds only if $\mathcal{T}^{\mu\nu} = 0$ and in particular in the limit of vanishing q. But this is not possible, as shown before. Therefore we have proved the thesis.

The first thing that stands out about this theorem is that not even gravity appears to be allowed! This, though, is not in contradiction with physics, because one of the hypotheses is not satisfied in such case: gravity, being sourced by the energy-momentum tensor and therefore also by itself, does not possess a conserved stress-energy tensor which is also invariant under diffeomorphisms, i.e. gauge invariant. Thus, under Lorentz transformations $T^{\mu\nu}$ does not behave as a tensor. (The reason is the same that led us to say that the polarization is not a proper tensor in the the previous subsection.) This argument, though, relies on this precise property to work.

One may then wonder whether this happens also in the higher spin case and if it is legitimate to apply this theorem to prove its inconsistency. Porrati found how to avoid this empasse in [23], where instead of the matrix element (1.2.9), the following has been employed:

$$\tilde{\mathcal{T}}^{\mu\nu} = \left\langle p', \lambda' \left| T^{\mu\nu}(t,\vec{0}) \right| p, \lambda \right\rangle. \tag{1.2.12}$$

Here, λ and λ' label spurious polarizations of the higher spin, so that (1.2.12) behaves as a true Lorentz tensor. Then an argument that shows how $\tilde{\mathcal{T}}^{\mu\nu}$ should vanish for $s > 2$ is produced, in contrast with (1.2.9), so that one can complete the proof just the way we have done here.

1.2.1.3 An Aragone-Deser-like argument

The analysis we carried on for the equations of motion and the action for massless particles with spin $s > 1$ can be performed in a similar way for fermionic fields with semi-integer spin $s > \frac{3}{2}$. The result is that they are described by a totally symmetric spin-tensor $\Psi_{\mu_1 \ldots \mu_{s-\frac{1}{2}}}$ such that

$$\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\Psi_{\mu_1 \ldots \mu_{s-\frac{1}{2}}} = 0$$

subject to a gauge invariance under the following transformation

$$\delta\Psi_{\mu_1 \ldots \mu_{s-\frac{1}{2}}}(x) = \partial_{(\mu_1}\Lambda_{\mu_2 \ldots \mu_{s-\frac{1}{2}})}(x) \tag{1.2.13}$$

with

$$\gamma^{\mu_2}\Lambda_{\mu_2 \ldots \mu_{s-\frac{1}{2}}} = 0.$$

In their paper [24], Aragone and Deser considered the action for the spin $\frac{5}{2}$ massless particle and tried to add the interaction with an external gravitational field by minimal coupling. They showed how the minimal prescription, that is replacing each derivative ∂ with a covariant one D, rendered such action non-gauge invariant, because of the non-commutative nature of the covariant derivatives arising from (1.2.13).

Since here we are dealing with integer spins, we will present in detail an analogous argument that shows this inconsistency for a spin 3 particle. First of all, from (1.1.18), we get the action

$$\begin{aligned} S_3 &= \int \left(\frac{1}{2}\partial_\alpha \Phi_{\mu_1\mu_2\mu_3}\partial^\alpha \Phi^{\mu_1\mu_2\mu_3} - \frac{3}{2}\partial_\alpha \Phi^\alpha_{\mu_2\mu_3}\partial^\beta \Phi^{\mu_2\mu_3}_\beta - 3\Phi^\alpha{}_{\alpha\mu_3}\partial_\beta\partial_\gamma \Phi^{\beta\gamma\mu_3} \right. \\ & \left. -\frac{3}{2}\partial_\beta \Phi^\alpha{}_{\alpha\mu_3}\partial^\beta \Phi_\gamma{}^{\gamma\mu_3} - \frac{3}{4}\partial^\beta \Phi^\alpha{}_{\alpha\beta}\partial_\delta \Phi_\gamma{}^{\gamma\delta} \right) d^4x. \end{aligned}$$

Under the minimal prescription, it becomes

$$\begin{aligned} \mathcal{S}_3 &= \int \left(\frac{1}{2}D_\alpha \Phi_{\mu_1\mu_2\mu_3}D^\alpha \Phi^{\mu_1\mu_2\mu_3} - \frac{3}{2}D_\alpha \Phi^\alpha_{\mu_2\mu_3}D^\beta \Phi^{\mu_2\mu_3}_\beta - 3\Phi^\alpha{}_{\alpha\mu_3}D_\beta D_\gamma \Phi^{\beta\gamma\mu_3} \right. \\ & \left. -\frac{3}{2}D_\beta \Phi^\alpha{}_{\alpha\mu_3}D^\beta \Phi_\gamma{}^{\gamma\mu_3} - \frac{3}{4}D^\beta \Phi^\alpha{}_{\alpha\beta}D_\delta \Phi_\gamma{}^{\gamma\delta} \right) d^4x \end{aligned} \tag{1.2.14}$$

while the gauge transformation for Φ is adapted in a similar way:

$$\delta\Phi_{\mu_1\mu_2\mu_3} = D_{(\mu_1}\Lambda_{\mu_2\mu_3)}. \tag{1.2.15}$$

Let us compute the variation of (1.2.14) under (1.2.15). First of all, we derive a useful identity:

$$\begin{aligned} \left[D^\nu, D^\alpha D_\alpha\right] V^{\mu_1\cdots\mu_n} &= D^\alpha \left[D^\nu, D_\alpha\right] V^{\mu_1\cdots\mu_n} + \left[D^\nu, D^\alpha\right] D_\alpha V^{\mu_1\cdots\mu_n} \\ &= D^\alpha \left(R^\nu{}_\alpha{}^{(\mu_1}{}_\gamma V^{\mu_2\cdots\mu_n)\gamma} \right) + R^{\nu\alpha}{}_\alpha{}^\gamma D_\gamma V^{\mu_1\cdots\mu_n} + R^{\nu\alpha(\mu_1}{}_\gamma D_\alpha V^{\mu_2\cdots\mu_n)\gamma}. \end{aligned}$$

The variation then reads

$$\begin{aligned} \delta\mathcal{L}_3 &= \int (3D_\alpha D_{\mu_1}\Lambda_{\mu_2\mu_3}D^\alpha \Phi^{\mu_1\mu_2\mu_3} \\ & -3D_\alpha D^\alpha \Lambda_{\mu_2\mu_3}D^\beta \Phi^{\mu_2\mu_3}_\beta - 6D_\alpha D_{\mu_2}\Lambda^\alpha_{\mu_3}D^\beta \Phi^{\mu_2\mu_3}_\beta \\ & -6D^\alpha \Lambda_{\alpha\mu_3}D_\beta D_\gamma \Phi^{\beta\gamma\mu_3} - 3\Phi^\alpha{}_{\alpha\mu_3}D_\beta D_\gamma D^\beta \Lambda^{\gamma\mu_3} - 3\Phi^\alpha{}_{\alpha\mu_3}D_\beta D_\gamma D^\gamma \Lambda^{\beta\mu_3} - 3\Phi^\alpha{}_{\alpha\mu_3}D_\beta D_\gamma D^{\mu_3}\Lambda^{\gamma\beta} \\ & -6D_\beta D^\alpha \Lambda_{\alpha\mu_3}D^\beta \Phi_\gamma{}^{\gamma\mu_3} \\ & \left. -3D^\beta D^\alpha \Lambda_{\alpha\beta}D_\delta \Phi_\gamma{}^{\gamma\delta} \right) d^4x \\ &= \int (-3D^\alpha D_\alpha D_{\mu_1}\Lambda_{\mu_2\mu_3}\Phi^{\mu_1\mu_2\mu_3} \\ & +3D_\alpha D^\alpha D^\beta \Lambda_{\mu_2\mu_3}\Phi^{\mu_2\mu_3}_\beta - 3\left[D_\alpha D^\alpha, D^\beta\right] \Lambda_{\mu_2\mu_3}\Phi^{\mu_2\mu_3}_\beta - 6\left[D_\alpha, D_{\mu_2}\right] \Lambda^\alpha_{\mu_3}D^\beta \Phi^{\mu_2\mu_3}_\beta + 6D_\alpha \Lambda^\alpha_{\mu_3}D_{\mu_2}D^\beta \Phi^{\mu_2\mu_3}_\beta \\ & -6D^\alpha \Lambda_{\alpha\mu_3}D_\beta D_\gamma \Phi^{\beta\gamma\mu_3} - 3\Phi^\alpha{}_{\alpha\mu_3}\left[D_\beta, D_\gamma\right] D^\beta \Lambda^{\gamma\mu_3} - 6\Phi^\alpha{}_{\alpha\mu_3}D_\gamma D_\beta D^\beta \Lambda^{\gamma\mu_3} - 3\Phi^\alpha{}_{\alpha\mu_3}D_\beta D_\gamma D^{\mu_3}\Lambda^{\gamma\beta} \\ & +6D^\alpha D^\beta D_\beta \Lambda_{\alpha\mu_3}\Phi_\gamma{}^{\gamma\mu_3} - 6\left[D^\alpha, D^\beta D_\beta\right] \Lambda_{\alpha\mu_3}\Phi_\gamma{}^{\gamma\mu_3} \\ & \left. +3\left[D_\delta, D^\beta D^\alpha\right] \Lambda_{\alpha\beta}\Phi_\gamma{}^{\gamma\delta} + 3D^\beta D^\alpha D_\delta \Lambda_{\alpha\beta}\Phi_\gamma{}^{\gamma\delta} \right) d^4x \\ &= \int \left(6D_\alpha \left(R^{\beta\alpha}{}_{\mu_2}{}^\gamma \Lambda_{\gamma\mu_3} \right) \Phi^{\mu_2\mu_3}_\beta + 6R^{\beta\alpha}{}_{\mu_2}{}^\gamma D_\alpha \Lambda_{\gamma\mu_3}\Phi^{\mu_2\mu_3}_\beta - 3R^{\beta\gamma}D_\gamma \Lambda_{\mu_2\mu_3}\Phi^{\mu_2\mu_3}_\beta \right. \\ & -6R_{\mu_2\gamma}\Lambda^\gamma_{\mu_3}D^\beta \Phi^{\mu_2\mu_3}_\beta - 6R_{\alpha\mu_2\mu_3}{}^\delta \Lambda^\alpha_\delta D^\beta \Phi^{\gamma\mu_3}_\beta \\ & -6D^\beta \left(R^\delta_\beta \Lambda_{\delta\mu_3} \right) \Phi_\gamma{}^{\gamma\mu_3} - 6D^\beta \left(R^\alpha{}_{\beta\mu_3}{}^\delta \Lambda_{\delta\alpha} \right) \Phi_\gamma{}^{\gamma\mu_3} - 6R^\delta_\beta D^\beta \Lambda_{\delta\mu_3}\Phi_\gamma{}^{\gamma\mu_3} - 3R^\alpha{}_{\beta\mu_3}{}^\delta D^\beta \Lambda_{\delta\alpha}\Phi_\gamma{}^{\gamma\mu_3} \\ & +6R^{\alpha\delta}D_\delta \Lambda_{\alpha\mu_3}\Phi_\gamma{}^{\gamma\mu_3} \\ & \left. -3R_{\delta\epsilon}D^\alpha \Lambda^\epsilon_\alpha \Phi_\gamma{}^{\gamma\delta} + 3D^\beta \left(R_{\delta\alpha\beta}{}^\epsilon \Lambda^\alpha_\epsilon - R_{\delta\epsilon}\Lambda^\epsilon_\beta \right) \Phi_\gamma{}^{\gamma\delta} \right) \end{aligned} \tag{1.2.16}$$

As we can see, the variation is now proportional to the Riemann and Ricci tensors and is not zero in a curved space. In particular, assuming a free gravitational field, even though the $R_{\mu\nu}$ vanishes, $R_{\mu\nu\rho\sigma}$ does not, because it is proportional to the Weyl tensor that is zero only in a flat space. Again, we see that, if gravity is coupled minimally to higher spins, the theory is inconsistent.

1.2.1.4 The Coleman-Mandula theorem

As we have seen, massless higher spin fields possess a gauge symmetry. This symmetry is rather different from the one of a Yang-Mills theory because, contrarily to what happens in that case, the gauge parameters are tensors, rather than scalars, and therefore are affected by Lorentz transformations. This means that the associated conserved charge is also a tensor. We are going to find it, but first we compute the traces of the Fronsdal tensor (1.1.19) that we will need later:

$$\begin{aligned}
\mathcal{F}^{\alpha}{}_{\alpha\mu_3\ldots\mu_s} &= \Box\Phi^{\alpha}{}_{\alpha\mu_3\ldots\mu_s} - 2\left(\partial\cdot(\partial\cdot\Phi)\right)_{\mu_3\ldots\mu_s} - \partial_{(\mu_3}\left(\partial\cdot\Phi\right)^{\alpha}{}_{\alpha\mu_4\ldots\mu_s)} + \Box\Phi^{\alpha}{}_{\alpha\mu_3\ldots\mu_s} + 2\partial_{(\mu_3}\left(\partial\cdot\Phi\right)^{\alpha}{}_{\alpha\mu_4\ldots\mu_s)} \\
&= 2\left(\Box\Phi^{\alpha}{}_{\alpha\mu_3\ldots\mu_s} + \frac{1}{2}\partial_{(\mu_3}\left(\partial\cdot\Phi\right)^{\alpha}{}_{\alpha\mu_4\ldots\mu_s)} - \left(\partial\cdot(\partial\cdot\Phi)\right)_{\mu_3\ldots\mu_s}\right)
\end{aligned}$$

$$\mathcal{F}^{\alpha\beta}{}_{\alpha\beta\mu_4\ldots\mu_s} = 0.$$

The Fronsdal action (1.1.18) can be rewritten integrating by parts in this way:

$$\begin{aligned}
S_{Fronsdal} &= \int\Bigg(-\frac{1}{2}\Phi_{\mu_1\ldots\mu_s}\partial_\alpha\partial^\alpha\Phi^{\mu_1\ldots\mu_s} + \frac{s}{2}\Phi_{\mu_1\ldots\mu_s}\eta^{\mu_1\alpha}\partial_\alpha\partial^\beta\Phi^{\mu_2\ldots\mu_s}_{\beta} - \frac{s\,(s-1)}{2}\Phi_{\mu_1\ldots\mu_s}\eta^{\mu_1\mu_2}\partial_\beta\partial_\gamma\Phi^{\beta\gamma\mu_3\ldots\mu_s} \\
&\quad + \frac{s\,(s-1)}{4}\Phi_{\mu_1\ldots\mu_s}\eta^{\mu_1\mu_2}\partial_\beta\partial^\beta\Phi_{\gamma}{}^{\gamma\mu_3\ldots\mu_s} + \frac{s\,(s-1)\,(s-2)}{8}\Phi_{\mu_1\ldots\mu_s}\eta^{\mu_1\mu_2}\eta^{\mu_3\beta}\partial_\beta\partial_\delta\Phi_{\gamma}{}^{\gamma\delta\mu_4\ldots\mu_s}\Bigg)\,d^4x \\
&= \int -\frac{1}{2}\Phi_{\mu_1\ldots\mu_s}\Big(\Box\Phi^{\mu_1\ldots\mu_s} - \partial^{(\mu_1}\left(\partial\cdot\Phi\right)^{\mu_2\ldots\mu_s)} + \partial^{(\mu_1}\partial^{\mu_2}\Phi_{\nu}{}^{\nu\mu_3\ldots\mu_s)} - \partial^{(\mu_1}\partial^{\mu_2}\Phi_{\nu}{}^{\nu\mu_3\ldots\mu_s)} \\
&\quad + 2\eta^{(\mu_1\mu_2}\left(\partial\cdot(\partial\cdot\Phi)\right)^{\mu_3\ldots\mu_s)} - \eta^{(\mu_1\mu_2}\Box\Phi_{\gamma}{}^{\gamma\mu_3\ldots\mu_s)} - \frac{1}{2}\eta^{(\mu_1\mu_2}\partial^{\mu_3}\left(\partial\cdot\Phi\right)_{\gamma}{}^{\gamma\mu_4\ldots\mu_s)}\Big)\,d^4x \\
&= \int -\frac{1}{2}\Phi_{\mu_1\ldots\mu_s}\Big(\mathcal{F}^{\mu_1\ldots\mu_s} - \frac{1}{2}\eta^{(\mu_1\mu_2}\mathcal{F}_{\alpha}{}^{\alpha\mu_3\ldots\mu_s)} \qquad (1.2.17) \\
&\quad - \partial^{(\mu_1}\partial^{\mu_2}\Phi_{\nu}{}^{\nu\mu_3\ldots\mu_s)} + \eta^{(\mu_1\mu_2}\left(\partial\cdot(\partial\cdot\Phi)\right)^{\mu_3\ldots\mu_s)}\Big)\,d^4x \\
&= \int -\frac{1}{2}\Phi_{\mu_1\ldots\mu_s}\left(\mathcal{F}^{\mu_1\ldots\mu_s} - \frac{1}{2}\eta^{(\mu_1\mu_2}\mathcal{F}_{\alpha}{}^{\alpha\mu_3\ldots\mu_s)}\right)d^4x
\end{aligned}$$

where in step (1.2.17) the last two terms disappear after integrating by parts one of them. Now, consider the variation of this expression under a gauge transformation (1.1.20), which leaves the Fronsdal tensor (1.1.19) unchanged as showed in (1.1.21):

$$\begin{aligned}
\delta S_{Fronsdal} &= \int -\frac{s}{2}\partial_{\mu_1}\Lambda_{\mu_2\ldots\mu_s}\left(\mathcal{F}^{\mu_1\ldots\mu_s} - \frac{1}{2}\eta^{(\mu_1\mu_2}\mathcal{F}_{\alpha}{}^{\alpha\mu_3\ldots\mu_s)}\right)d^4x \\
&= \int \frac{s}{2}\Lambda_{\mu_2\ldots\mu_s}\partial_{\mu_1}\left(\mathcal{F}^{\mu_1\ldots\mu_s} - \frac{1}{2}\eta^{(\mu_1\mu_2}\mathcal{F}_{\alpha}{}^{\alpha\mu_3\ldots\mu_s)}\right)d^4x \\
&\equiv \int \frac{s}{2}\Lambda_{\mu_2\ldots\mu_s}\partial_{\mu_1}\mathcal{G}^{\mu_1\ldots\mu_s}d^4x \qquad (1.2.18)
\end{aligned}$$

where

$$\mathcal{G}^{\mu_1\ldots\mu_s} = \mathcal{F}^{\mu_1\ldots\mu_s} - \frac{1}{2}\eta^{(\mu_1\mu_2}\mathcal{F}_{\alpha}{}^{\alpha\mu_3\ldots\mu_s)}$$

and

$$\mathcal{G}_{\alpha}{}^{\alpha\mu_3\ldots\mu_s} = (s-1)\,\mathcal{F}_{\alpha}{}^{\alpha\mu_3\ldots\mu_s}, \qquad \mathcal{G}_{\alpha\beta}{}^{\alpha\beta\mu_4\ldots\mu_s} = 0.$$

Gauge invariance of the action then implies that (1.2.18) vanishes and that the current[3]

$$J^{\mu_1\ldots\mu_s}(x) = \mathcal{G}^{\mu_1\ldots\mu_s} - \frac{1}{2s}\eta^{(\mu_1\mu_2}\mathcal{G}_{\alpha}{}^{\alpha\mu_3\ldots\mu_s)}$$

is conserved. The associated charge is

$$Q^{\mu_2\ldots\mu_s} = \int J^{0\mu_2\ldots\mu_s}(x)d^4x.$$

If interactions with other particles are turned on, gauge invariance must still be preserved and therefore there will be a conserved charge $Q^{\mu_2\cdots\mu_s}$. This, however, is prohibited by the Coleman-Mandula theorem [25] which states[4] that if G is the connected group of symmetries of the S-matrix of a theory such that

1. $ISO(1,3) < G$, i.e. Poincaré invariance is a symmetry of the theory

2. All particles have a positive-defined energy and for each $M > 0$ there is a finite number of particle of mass $m < M$

3. Scattering amplitudes are analytic in the Mandelstam variables s and t

then G is locally isomorphic to the direct product of $ISO(1,3)$ and an internal symmetry group whose generators are scalars. Conserved charges such as $Q^{\mu_2\cdots\mu_s}$ are thus ruled out. Even if a generalization of Lie algebras is employed, namely graded Lie algebras, a similar conclusion can be drawn. This has been done in [26], where it has been shown that the only allowed generators that are not scalars, are spinors and give rise to supersymmetry.

1.2.2 Loop-holes and yes-go examples

1.2.2.1 An infinite number of particles

When an interaction term for a higher spin particle is added to its free Lagrangian, the overall gauge invariance under (1.1.20) usually breaks and it is necessary to add a corresponding transformation to the field the higher spin interacts with. If such a field carries the higher spin itself, i.e. when self-interactions are investigated, it is necessary to deform the gauge parameter algebra.

This can be done order by order in a coupling constant g, enumerating all the possible self-interactive terms, the respective gauge transformations and imposing at the end that the Lie product closes in the so extended algebra. This procedure is described, for example, in [27], where it is shown that the result for a spin 1 particle is a Yang-Mills theory, whereas for spin 2 Einstein gravity is ultimately recovered. In these two examples, it is sufficient to stop at the first perturbative order in g to reach a closure of the gauge algebra. On the contrary, starting from $s = 3$, this can not happen anymore, and the commutator between two deformed gauge transformations can never be put in the form of a new gauge transformation at the second order in g.

A possible solution suggested in that paper (but also in the seminal work [14]) is to consider that self-interaction for a higher spin may be consistent only in presence of other higher spins capable of compensating the unwanted terms in gauge transformations. Those fields should interact with each other, or with themselves. A whole infinite tower of higher spin particles may be required in order to complete this program, which suggests the presence of a higher spin symmetry algebra that acts on this set of fields.

[3]Here this current is not given by just $\mathcal{G}^{\mu_1\cdots\mu_s}$, because the tracelessness of Λ removes its traceful part.

[4]Here we are not giving a rigorous statement, nor we enlist the technical hypotheses with no particular relation to the main topic

Even though it has been shown in [28] that actually this proposal does not solve the problem for spin 3 particles, the presence (and necessity) of infinitely many spins is found a particular feature of higher spin interactive theory that is known as for now. However, such topic will be addressed in the next subsection. In this subsection we limit ourselves to notice that if a consistent higher spin theory actually requires an infinite set of massless fields, the Coleman-Mandula no-go theorem (et similia), do not apply, because they require a finite number of particles under a finite mass-shell.

1.2.2.2 From minimal coupling to general couplings

The gravitational interaction has been our main investigative tool in the first three examples. Indeed, by the Weinberg theorem, we deduced that higher-spins should behave as all the other particles when dealing with gravity. To do so, though, we employed a very specific interaction vertex, namely the one between the graviton (spin 2) and two higher spins (of spin s) of the same type. In general, for a Lagrangian polynomial in the fields, this vertex is proportional to

$$\frac{\delta \mathcal{L}_{matter}}{\delta g_{\mu\nu}} g_{\mu\nu} = T^{\mu\nu} g_{\mu\nu}$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the matter and is covariantly conserved as a consequence of the diffeomorphism invariance of the theory: indeed, the variation of S_{matter} under a diffeomorphism parametrized by $\xi^\mu(x)$ is ($\delta g_{\mu\nu} = D_{(\mu}\xi_{\nu)}$)

$$\begin{aligned} \delta S_{matter} &= \frac{1}{2}\int \sqrt{g}\frac{\delta \mathcal{L}_{matter}}{\delta g_{\mu\nu}} D_\mu \xi_\nu d^4x \\ &= -\frac{1}{2}\int \sqrt{g} D_\mu T^{\mu\nu}\xi_\nu d^4x \end{aligned}$$

and thus

$$D_\mu T^{\mu\nu} = 0. \tag{1.2.19}$$

In Weinberg's theorem we have therefore assumed in general that $T^{\mu\nu}$ for a higher spin φ is quadratic in φ and contains two derivatives, that appear in the Feynman rule as momenta of the external leg of φ. This led us to derive the following relation for M^i (see (1.2.1) and (1.2.4))

$$M^i_{\mu\nu} \propto p_\mu p_\nu.$$

We now recognize (1.2.3) as the energy-momentum tensor conservation (1.2.19) in disguise (notice that at spatial infinity D is replaced by ∂ and, in Fourier transform, by q).

This assumption is actually equivalent to the minimal prescription for the gravitational coupling. Only in that case the energy-momentum tensor $\theta^{\mu\nu}$ that one finds for the free theory by the Nöther theorem is indeed (equivalent by the Belinfante-Rosenfeld construction to) $T^{\mu\nu}$. The form of $\theta^{\mu\nu}$ is constrained by the kinetic term of the free Lagrangian, that is quadratic in φ and contains two derivatives, namely what we assumed for $T^{\mu\nu}$.

In the Weinberg-Witten theorem, on the contrary, we made no assumptions on the kind of coupling with gravitons, but we showed anyway that $\langle p\,|T^{\mu\nu}|\,p\rangle \propto p^\mu p^\nu$ (see (1.2.9)) by considering the limit of a scattering of a soft off-shell graviton on the higher spin particle (a particular case of the one examined in the proof of Weinberg's theorem). This result, equivalent to minimal coupling, needed only the very general hypothesis of Lorentz covariance of $T^{\mu\nu}$ to infer from (1.2.8) equation (1.2.9).

However, if interacting higher spins do not possess a covariant energy-momentum tensor (and this is very likely the case, as already is shown by gravity for $s = 2$), such argument fails and again the equivalence principle has to be postulated in order to complete the proof as done by Porrati in [23]. This principle plays a fundamental role in the argument by Aragone and Deser.

We now see that if we choose to abandon the minimal coupling in favor of more general vertices, including non-minimal ones and others between three particles with different spin, the first three no-go arguments cease to apply.

The research on cubic vertices for higher spins has been performed following mainly two paths. In both a perturbative term is introduced in the form of an interaction with a coupling constant g. The analysis is then carried at the first order in this parameter.

The first approach is the non-covariant one, where only the physical degrees of freedom of the fields are taken in consideration. This amounts to fixing some convenient gauge, for example the light-cone gauge. Then, among all the possible deformations of the free Lagrangian, only those that lead to consistent field-theoretic generators of the Poincaré algebra are selected. This has ultimately allowed to find a complete classification of the interactions among any three arbitrary spins that correctly reduces to the known cases for the smallest spins. For bosonic massless fields, the ones we are interested in, the list of cubic vertices can be found in [29]. In chapter (3) we will explain how to list all the possible vertices in a curved background by employing the flat space ones for fields in a different gauge: the so-called Transverse Traceless gauge, i.e. the one imposed in (1.1.22).

A covariant approach is also possible, by the BRST method (see [30]). As already explained, it consists in finding all the possible deformations of the gauge algebra that allow to include interaction terms while maintaining the whole action gauge-invariant.

These results show a remarkable feature: the number of derivatives n needed to build each vertex between particles of spins s_1, s_2 and s_3 is bounded by the relation

$$\left(\sum_i s_i\right) - 2\min\left(s_1, s_2, s_3\right) \leq n \leq \left(\sum_i s_i\right). \tag{1.2.20}$$

In particular when one of the spins is 2 and all the others are greater than 2, we find that

$$n \geq 2$$

and the equivalence holds only if $s_1 = s_2 = s_3 = 2$. This simply tells us that the graviton ceases to couple minimally to particles starting from spin 3, as we anticipated. If instead the lowest spin is 1 and the others are 2 or greater, $n > 2$: we find the equivalent limit for minimal couplings of electromagnetism. Indeed, electromagnetism minimal prescription follows the same rules as the gravitational version and one could prove, along the very same lines of the Aragone-Deser argument, that gauge invariance breaks for electrically charged higher spins due to the non commutativity of covariant derivatives.

1.2.2.3 A curved space-time with a non-zero cosmological constant

Until now, we have assumed that the space where higher spins propagate was flat. Even when turning on an interaction with gravity, we considered solutions with an asymptotically flat metric, so that we could deal with free gravitons at the infinity and therefore use the S-matrix formalism. This is fairly reasonable, since the universe that we observe is approximately flat. Beyond its naturalness, this hypothesis is crucial to make all the no-go theorems we have discussed work, except (apparently) the Aragone-Deser argument.

The most direct conclusion that we drew from the latter was that there can be no minimal coupling with gravity (or electromagnetism, as noted before). Actually there is a remedy to this troublesome fact and it involves the either Anti de Sitter (in the following AdS) or de Sitter (dS) spaces, the maximally symmetric solution for a gravitation theory with a non-vanishing cosmological constant $\Lambda \lesseqgtr 0$ respectively and therefore negative (positive) curvature. In such settings, none of the no-gos is of any use. From now on we will consider only the AdS case.

In their paper [31], Fradkin and Vasiliev showed that it is possible to construct an action in AdS that describes particles for every spin, which reduces to the Fronsdal action reformulated for a curved background in the quadratic approximation and that, at the cubic order, contains higher derivative couplings and the minimal one. In the limit for a vanishing cosmological constant, the free theory becomes the one described by the flat Fronsdal action that we wrote in (1.1.18). On the contrary, when interactions are taken into account, Λ plays a fundamental role. Its physical dimension is the inverse of a square length and therefore it appears in every vertex with the role of compensating the unwanted extra length dimensions introduced by the derivatives. For this reason flat space limit $\Lambda \to 0$ is singular for some of the vertices, where the cosmological constant appears in negative powers. These are the higher derivative ones and are proportional to the (linearized) Riemann tensor, so that they can compensate the unwanted terms that we found in our Aragone-Deser-like argument. This explains why gauge-invariance is broken in the flat limit.

As shown for spin 2 by Zinoviev [32], a similar mechanism makes it possible to couple minimally electromagnetism with higher spins only in AdS.

1.3 Gravitation and gauge theories

The yes-go examples provided so far suggest that an non-linear interacting theory for higher spin fields should exist in the Anti de Sitter space (see subsection 1.2.2). The remainder of this chapter is devoted to the construction of such theory, starting from the description of its linearization in sections 1.3 and 1.4 and then of the complete theory in 1.5.

As we saw in section 1.1, massless higher spin fields are gauge fields. However their gauge transformation (1.1.20) looks rather different from the one present in Yang-Mills theories that describe the other fundamental interactions, in that the gauge parameters are not scalars but tensors. The goal of this section is then to develop a unified formalism for these two examples of gauge theory in order to later implement interactions that are compatible with this symemtry. In particular we will focus on the spin 2 case, namely the gravitational field, leaving the general higher spin case to the next section. In the first subsection we present the frame-like formulation of General Relativity, that we will later generalize to higher spin fields in subsection 1.4.3. Then, in 1.3.2, we will employ vector bundles to treat general gauge theories in a similar way. In the last subsection 1.3.3 we come back to gravity and consider the special case of the AdS space.

1.3.1 Cartan formulation of the General Relativity

In this section we review the so-called Cartan formulation of Einstein gravity, that makes contact with other gauge theories more explicit and will be later generalized to describe a higher spin gravity theory in AdS_4.

Let us start from the principle of equivalence, which says that for each point x_0 in a curved space-time $\mathcal{M}$ of dimension d there always exists a system of coordinates $\bar{x}$ in which physics is described at $\bar{x}_0$ as in flat space. In particular this applies to the metric[5], so that

$$g_{\mu\nu}(x_0) = \frac{\partial \bar{x}^a}{\partial x^\mu}\frac{\partial \bar{x}^b}{\partial x^\nu}\eta_{ab}. \tag{1.3.1}$$

Here we use, as in the rest of this subsection the following notation: Greek indices, called world indices, refer to tensorial objects living in a general and possibly curved manifold, while Latin ones refer to its flat tangent space and are called fiber indices. From (1.3.1) it is clear that $\bar{x}(x)$ is far from being unique, since any

$$\bar{x}'^a = \Lambda^a{}_b \bar{x}^b \tag{1.3.2}$$

leaves (1.3.1) invariant if Λ is a Lorentz transformation, defined by the property

$$\Lambda^a{}_m \eta^{mn} \Lambda^b{}_n = \eta^{ab}. \tag{1.3.3}$$

[5] In this system of coordinates, the first derivatives of the metric vanish and therefore (see (1.3.26)) also the Christoffel symbol $\Gamma^\rho_{\mu\nu}$ is zero at x.

Let us call vielbein

$$e_\mu^a(x) \equiv \frac{\partial \bar{x}^a}{\partial x^\mu}, \qquad \bar{x} \text{ defined for each } x \text{ by (1.3.1)}$$

one of the coordinate changes that transforms the Minkowski metric into the curved one at each point x. One can regard $e_\mu^a(x)$ as a local change of basis for the tangent space at x $T_x(\mathcal{M}) \simeq \mathbb{R}^{1,d-1}$ that converts the local basis ∂_μ into a standard one that one may choose for $\mathbb{R}^{1,d-1}$, which is called fiber or tangent space. With this picture in mind it becomes obvious that

$$\det(e) \neq 0 \; \forall x \tag{1.3.4}$$

and therefore there exists

$$\left(e^{-1}\right) \equiv e_a^\mu = \frac{\partial x^\mu}{\partial \bar{x}^a}, \quad e_a^\mu e_\mu^b = \delta_b^a, \quad e_a^\mu e_\nu^a = \delta_\nu^\mu \tag{1.3.5}$$

the inverse of the vielbein.

e encodes all the information contained in the usual metric g and shares the same degrees of freedom. Indeed e is a $d \times d$ matrix subjected to an invariance (1.3.3) with $\frac{d(d-1)}{2}$ degrees of freedom (from (1.3.3)) and therefore has

$$d^2 - \frac{1}{2}d^2 + \frac{1}{2}d = \frac{d(d+1)}{2} \tag{1.3.6}$$

independent degrees of freedom, the same as the metric g.

Given some tensor field $T_{\mu_1 \ldots \mu_k}(x)$ with rank k, we define its fiber version as

$$T_{a_1 \ldots a_k}(x) = e_{a_1}^{\mu_1} \cdots e_{a_k}^{\mu_k} T_{\mu_1 \ldots \mu_k}(x),$$

i.e. the tensor field expressed in the fiber basis given by e. The fiber indices are lowered and lifted by means of the standard Minkowski metric as follows from (1.3.1): for example,

$$\begin{aligned} V_a &= e_a^\mu \left(g_{\mu\nu} V^\nu\right) \\ &= e_a^\mu g_{\mu\nu} e_b^\nu V^b \\ &= \eta_{ab} V^b. \end{aligned}$$

We would like now to define a covariant derivative acting on fiber tensors. Let us do this for a contravariant vector, the generalization to general tensors being straight-forward. First of all we have

$$\begin{aligned} D_\mu V^\nu &= \partial_\mu V^\nu + \Gamma_{\mu\alpha}^\nu V^\alpha \\ &= \partial_\mu \left(e_a^\nu V^a\right) + \Gamma_{\mu\rho}^\nu e_a^\rho V^a \\ &= e_a^\nu \partial_\mu V^a + \left(\partial_\mu e_a^\nu + \Gamma_{\mu\rho}^\nu e_a^\rho\right) V^a, \end{aligned} \tag{1.3.7}$$

where Γ is the Christoffel symbol associated to g. Converting the ν index to a fiber one (n), relation (1.3.7) becomes

$$\begin{aligned} D_\mu V^n &= e_\nu^n e_a^\nu \partial_\mu V^a + e_\nu^n \left(\partial_\mu e_a^\nu + \Gamma_{\mu\alpha}^\nu e_a^\alpha\right) V^a \\ &= \partial_\mu V^n + \omega_\mu{}^n{}_a V^a. \end{aligned} \tag{1.3.8}$$

where we used (1.3.5) and defined

$$\omega_\mu{}^a{}_b(x) = e_\nu^a(x) \left(\partial_\mu e_b^\nu(x) + \Gamma_{\mu\rho}^\nu(x) e_b^\rho(x)\right), \tag{1.3.9}$$

the so called spin connection. For tensors with rank greater than one, the same rules of the usual covariant differentiation of contravariant tensors are followed.

In the language of p-forms, we can define a differential operator D acting on (tensor-valued) 0-forms, i.e. objects without world indices, so that (1.3.8) can be expressed as

$$DV^a = dV^a + \omega^a{}_b V^b, \tag{1.3.10}$$

and we treat $\omega^{ab}(x)$ as a matrix of 1-forms or, more precisely, as member of the space $T^*_x(\mathcal{M})\otimes\text{Skew}\left(\mathbb{R}^{1,d-1}\right)$, where Skew $\left(\mathbb{R}^{1,\mathrm{d}-1}\right)$ is the vector spaces of antisymmetric matrices mapping $\mathbb{R}^{1,d-1}$ onto itself. Indeed, from

$$D_\mu g^{\alpha\beta} = D_\mu\left(e^\alpha_a \eta^{ab} e^\beta_b\right) = 0 \tag{1.3.11}$$

it follows that

$$D\eta^{ab} = 0 \iff \omega^a{}_c \eta^{cb} + \omega^b{}_c \eta^{ac} = 0 \iff \omega^{ab} = -\omega^{ba} \tag{1.3.12}$$

if $De^a = 0$, but this is implied by equation (1.3.9), which can be then written using (1.3.5) as

$$D_\mu e^a_\nu \equiv \partial_\mu e^a_\nu + \omega^a_{\mu b} e^b_\nu - \Gamma^\rho_{\mu\nu} e^b_\rho = 0 \tag{1.3.13}$$

where the spin connection ω acts on fiber indices, while Γ acts on the world ones. Property (1.3.12) is an important because

$$\text{Skew}\left(\mathbb{R}^{1,d-1}\right) \simeq \mathfrak{so}(1,d-1),$$

i.e. the Lie algebra of the symmetry group of the fiber tensors, so that (1.3.8) looks like the Yang-Mills covariant derivative for the Lorentz group $SO(1,d-1)$. We will indeed delve on this similarity in the next subsection to make contact between the theory of General Relativity and gauge theories.

The vanishing of $D\eta^{ab}$ allows us to find easily the rule for covariant tensors:

$$DT_{a_1\dots a_k} = dT_{a_1\dots a_k} + \sum_{i=1}^k \omega_{a_i}{}^b T_{a_1..b\dots a_n},$$

namely it is sufficient to lower and raise indices by means of η.

As for now, we defined D as a differential operator acting on indices of 0-forms, the fiber tensors, that do not carry any world index. We can generalize this definition to arbitrary p-forms guided by the resemblance to the standard differential d: if $F^{a_1\dots a_k}_{\boldsymbol{p}}$ is some tensor-valued p-form belonging to $\Omega^p(\mathcal{M})\otimes\mathscr{X}^k\left(\mathbb{R}^{1,d-1}\right)$, then

$$DF^{a_1\dots a_k}_{\boldsymbol{p}} = dF^{a_1\dots a_k}_{\boldsymbol{p}} + \sum_{i=1}^k \omega^{a_i}{}_b \wedge F^{a_1\dots b\dots a_k}_{\boldsymbol{p}}, \tag{1.3.14}$$

where $\wedge$ is the usual antisymmetric wedge product for p-forms[6]. Contrarily to d, D^2 is not zero and is given by the following expression (again, we treat vectors for simplicity):

$$\begin{aligned} D^2V^a &= D\left(dV^a + \omega^a{}_b V^b\right) \\ &= d\omega^a{}_b V^b + \omega^a{}_c \wedge \omega^c{}_b V^b \end{aligned} \tag{1.3.15}$$

We define then the curvature 2-form

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{1.3.16}$$

which is the Riemann tensor with two indices converted into fiber ones. Indeed, the fiber version of the known identity

$$\left[D_\mu, D_\nu\right] V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma$$

[6]Notice that in this definition all the world indices are antisymmetrized and therefore no Christoffel symbols appear, because their lower indices are symmetric (see (1.3.17)).

is

$$dx^\mu \wedge dx^\nu \left[D_\mu, D_\nu\right] e^\rho_r V^r = 2e^\rho_r D^2 V^r = dx^\mu \wedge dx^\nu R_{\mu\nu}{}^\rho{}_\sigma e^\sigma_s V^s$$

or

$$R_{rs} = \frac{1}{2} R_{\mu\nu\rho\sigma} e^\rho_r e^\sigma_s dx^\mu \wedge dx^\nu.$$

Here we see that the symmetry properties of the indices of the Riemann tensor arise naturally from the ones of the connection and of 2-forms.

Christoffel symbols are not unconstrained. They must respect the following relation:

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} = 0. \tag{1.3.17}$$

$T^\rho_{\mu\nu}$ is called torsion. Let us see how (1.3.17) reflects on connections. From (1.3.9) we get

$$\Gamma^\rho_{\mu\nu} = e^\rho_a \omega_\mu{}^a{}_b e^b_\nu - \partial_\mu e^\rho_b e^b_\nu \tag{1.3.18}$$

that plugged into (1.3.17) gives (we use (1.3.5) to write $\partial_\mu e^\rho_b e^b_\nu = -e^\rho_b \partial_\mu e^b_\nu$)

$$\begin{aligned} T^\rho_{\mu\nu} &= e^\rho_a \omega_\mu{}^a{}_b e^b_\nu - \partial_\mu e^\rho_b e^b_\nu - e^\rho_a \omega_\nu{}^a{}_b e^b_\mu + \partial_\nu e^\rho_b e^b_\mu \\ &= e^\rho_b \partial_\mu e^b_\nu - e^\rho_b \partial_\nu e^b_\mu + e^\rho_b \omega_\mu{}^b{}_a e^a_\nu - e^\rho_b \omega_\nu{}^b{}_a e^a_\mu \end{aligned}$$

or, by (1.3.14),

$$e^\rho_r T^r_{\mu\nu} dx^\mu \wedge dx^\nu = e^\rho_r D e^r.$$

We may then define the torsion fiber tensor as a 2-form that vanishes

$$T^a \equiv De^a = 0, \tag{1.3.19}$$

as follows from (1.3.13).

The formalism we have developed is especially useful to deal with spinors in General Relativity. Indeed, there is no intuitive way to use the Christoffel symbol to build a covariant derivative suitable for them, since $GL(d)$ does not have a spinorial representation. In our formalism, though, the connection is just an element of $\mathfrak{so}(1, d-1)$ and we may write it in the representation we prefer and in particular the spinorial one. Let us take for example a Dirac spinor $\psi^{\mathtt{a}}(x)$ (letters in roman font as $\mathtt{a}, \mathtt{b}, \mathtt{c}$ are used for spinor indices here). Its covariant derivative will be then

$$D_\mu \psi_{\mathtt{a}} = \partial_\mu \psi_{\mathtt{a}} + \omega^{ab}_\mu \left(\sigma_{ab}\right)_{\mathtt{a}}{}^{\mathtt{b}} \psi_{\mathtt{b}}$$

where σ_{ab} is a representation of the M_{ab} generator of the Lorentz group (see (1.1.1))

$$\sigma_{ab} = \frac{1}{4} \left[\gamma_a, \gamma_b\right]$$

and γ_a are the Dirac matrices that obey to

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}.$$

Let us now see how the vielbein changes under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \epsilon^\mu(x)$. As it is well known,

$$\delta g_{\mu\nu} = D_\mu \epsilon_\nu + D_\nu \epsilon_\mu$$

while from (1.3.1) we get

$$\delta g_{\mu\nu} = \delta e^a_\mu \eta_{ab} e^b_\nu + e^a_\mu \eta_{ab} \delta e^b_\nu$$

so that $D_\mu \epsilon_\nu = \delta e^a_\mu \eta_{ab} e^b_\nu$ and thus

$$\delta e^a = D\epsilon^a.$$

However, we saw in (1.3.2) that the vielbein is defined up to a Lorentz transformation, that can be in principle defined point by point, realizing thus an additional local symmetry for the e field. Then the full infinitesimal gauge transformation of e reads

$$\delta e^a = D\epsilon^a(x) + \epsilon^{ab}(x)e_b(x), \tag{1.3.20}$$

where $\epsilon^{ab}(x) = -\epsilon^{ba}(x) \in \mathfrak{so}(1, d-1)$ represents the infinitesimal version of (1.3.2). Equation (1.3.19) requires that also ω^{ab} changes under local Lorentz transformations: if we call D' the corresponding transformation of D,

$$0 \stackrel{!}{=} D'e'^a = D'(e^a + \epsilon^{ab}e_b)$$

is satisfied at the first order if

$$D'e^a = De^a - D\epsilon^{ab}e_b$$

equivalent to

$$\delta\omega^{ab} = -D\epsilon^{ab}. \tag{1.3.21}$$

More in general, any fiber tensor may be transformed according to a local Lorentz transformation, but the covariant derivative D is invariant under such transformations if and only if the connection transforms as (1.3.21).

1.3.2 Gauge symmetries and vector bundles

Diffeomorphisms in General Relativity are local symmetries and indeed we showed in the previous subsection some resemblance between the associated covariant derivative and the Yang-Mills one. Equation (1.3.8) suggested that the theory of gravitation can be interpreted as a gauge theory based on a local Lorentz invariance. In this subsection we generalize the formalism developed in the preceding one to describe a general gauge theory. The result will be that General Relativity shares many features with gauge theories, but not all. We will also employ these result to find a suitable way to describe higher spin fields and their equations of motion, from which a higher spin symmetry algebra will arise.

Let $\mathcal{G}$ be a Lie group with Lie algebra $\mathfrak{g}$. If we choose a vector space F such that $\mathcal{G}$ acts on F as a group of matrices, we may represent the matter fields associated to a gauge invariance under $\mathcal{G}$ as vectors in F defined at each point x of $\mathcal{M}$. For example, in QCD, $\mathcal{G}$ is $SU(3)$ and the quarks are represented by a vector in $F = \mathbb{C}^3$ on which $SU(3)$ acts in the fundamental representation. If we choose some other representation, we would employ some other F.

Let us thus define a new manifold with (real) dimension

$$\dim(E) = \dim(\mathcal{M}) + \dim(F)$$

called vector bundle E on $\mathcal{M}$ associated to F such that there exists a surjective differentiable projection function $\pi : E \to \mathcal{M}$ that obeys two requirements:

1. Fiber isomorphism: for each $x \in \mathcal{M}$, $\pi^{-1}(x)$ is a vector space isomorphic to F, or simply

$$\pi^{-1}(x) = F.$$

2. Local triviality: $\mathcal{M}$ can be covered by open sets u_i and

$$\pi^{-1}(u_i) \cong u_i \times \mathbb{R}^{\dim(F)},$$

where $\cong$ means diffeomorphic.

These properties tell us that locally E is just a collection of the points of $\mathcal{M}$ to each of which a copy of F is attached. From a physical point of view, E represents all the possible configurations of the matter field that lives in F. It would be a mistake, though, to deduce that E is simply $\mathcal{M} \times F$. In such a case E is said trivial. A familiar vector bundle is the tangent bundle $T\mathcal{M}$, the collection of all the tangents spaces $T_x\mathcal{M}$. Another vector bundle we have encountered is the one of p-forms $\Omega^p(\mathcal{M})$, for which the fiber is the vector space of p-forms $\Lambda^p(T_x^*\mathcal{M})$.

Since we now have a setting for our gauge theory, we would like to specify a particular configuration of our fields. We call a smooth section of E a differentiable function, $\phi: \mathcal{M} \to E$, that is one of the inverses of π, namely

$$\pi(\phi(x)) = x.$$

The set of all the possible sections on E is $\Gamma(E)$. From a physical point of view, $\phi(x)$ would be the matter field.

Our goal now is to define a covariant derivative on E. This is usually done in Physics by demanding that the result behaves as a (contravariant) vector. In our case we prefer to employ the language of p-forms, i.e. we require that D acts on sections of E, seen as 0-forms with values in F, to give 1-forms also with values in F.

The covariant derivative is specified by the so-called connection ∇ on E, defined as

$$\nabla: \Gamma(E) \to \Omega^1(E),$$

where we used the notation

$$\Gamma(\Omega^1(\mathcal{M}) \otimes E) \equiv \Omega^p(E).$$

Moreover, ∇ must obey the Leibniz rule for any differentiable function $f: \mathcal{M} \to \mathbb{R}$:

$$\nabla(f\phi) = \phi df + f\nabla\phi.$$

It is clear that $\nabla = d$, the so-called trivial connection, satisfies these requirements and if ∇ is a connection, so is $\nabla + \omega$, for $\omega \in \Omega^1(\mathrm{End}(E))$, a 1-form with values in the space of endomorphisms on F. It can be shown that actually every connection can be obtained from a known one by summing elements from $\Omega^1(\mathrm{End}(E))$. For this reason these connections are called affine, namely, from a given base point in their space, one can reach any other by summing with an element from a proper vector space, here represented by $\mathrm{End}(E)$.

We may take the trivial connection as a base point and in components we can thus write

$$\nabla\phi^a = d\phi^a + \omega^a{}_b\phi^b$$

so that if we choose a particular ω (also called connection with abuse of language), we get the covariant derivative associated to ω:

$$D\phi^a = d\phi^a + \omega^a{}_b\phi^b.$$

This result can be intuitively understood in this formalism in the following way. Suppose that we know the value of ϕ at x. If we move a little away from x we expect that ϕ changes as consequence of the different point in which we compute it, but we also have to take into account how the fibers F are attached one to the other to build E. Indeed we expect that for non-trivial bundles, moving from one fiber to the other implies a local change of basis and therefore the need of a compensating endomorphism to compare ϕ in the two locations.

One can extend the map ∇ to forms with higher rank: $\nabla: \Omega^i(E) \to \Omega^{i+1}(E)$ and this extension is unique if we require that it behaves following the Leibniz rule. It reads

$$D\phi^a_{\boldsymbol{p}} = d\phi^a_{\boldsymbol{p}} + \omega^a{}_b \wedge \phi^b_{\boldsymbol{p}}.$$

In particular, it is possible to prove that

$$D^2\phi^a = R^a{}_b\phi^b, \quad R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

where $R^a{}_b \in \Omega^2(\mathrm{End}(E))$ is a 2-form with values in the space of endomorphisms on F. It is called curvature associated to the connection.

Let us consider for example electromagnetism. In this case, $\mathcal{G} = U(1)$ and, taking as a matter field some complex scalar field, $F = \mathbb{C}$. Then $\omega = ieA$ is proportional to the potential 1-form A. As for the curvature 2-form, it is simply given by (the proof consists in a computation similar to the one made in (1.3.15))

$$(D\omega)\,\phi$$

but

$$D\omega = d\omega + \omega \wedge \omega = iedA = ieF_{\mu\nu}dx^\mu \wedge dx^\nu$$

with the usual definition of the fields strength.

Suppose now that we want to change locally the basis in the fiber and we do that by acting on F with an element $g \in \mathcal{G}$ defined point by point:

$$\phi'^a(x) = g^a{}_b(x)\phi^b(x).$$

Then

$$\begin{aligned} D'\phi'^a &= d\phi'^a + \omega'^a{}_b\phi'^b \\ &= dg^a{}_b\phi^b + g^a{}_b d\phi^b + \omega'^a{}_b g^b{}_c\phi^c \end{aligned}$$

but $D\phi$ should transform just as ϕ, namely

$$D'\phi'^a = g^a{}_b D\phi^b$$

and therefore we get

$$\omega'^a{}_b = g^a{}_c\omega^c{}_d\left(g^{-1}\right)^d{}_b - \left(dg^a{}_c\right)\left(g^{-1}\right)^c{}_b\,. \tag{1.3.22}$$

This is not the linear transformation we would expect for a matrix, because of the last term. This tells us that ω does not behave as a tensor under local transformations belonging to $\mathcal{G}$. This is caused by the presence inside D of the spurious term d that is not a local endomorphism of F. This problem is not present for the curvature tensor. In order to perform the computation we shorten the notation by considering all the quantities as matrices with the usual (wedge) product rows by columns, therefore dropping all the indices. We get

$$\begin{aligned} R' &= d\omega' + \omega'\omega' \\ &= d\left(g\omega g^{-1} - dgg^{-1}\right) + \left(g\omega g^{-1} - dgg^{-1}\right)\left(g\omega g^{-1} - dgg^{-1}\right) \\ &= dg\omega g^{-1} + gd\omega g^{-1} - g\omega d\left(g^{-1}\right) + dgd\left(g^{-1}\right) \\ &\quad + g\omega\omega g^{-1} - g\omega g^{-1}dgg^{-1} - dg\omega g^{-1} + dgg^{-1}dgg^{-1} \\ &= gRg^{-1} + dg\omega g^{-1} - g\omega d\left(g^{-1}\right) + dgd\left(g^{-1}\right) \\ &\quad + g\omega d\left(g^{-1}\right) - dg\omega g^{-1} - dgd\left(g^{-1}\right) \\ &= gRg^{-1} \end{aligned} \tag{1.3.23}$$

where we used the fact that

$$gg^{-1} = 1 \implies dgg^{-1} + gd\left(g^{-1}\right) = 0 \implies d\left(g^{-1}\right) = -g^{-1}dgg^{-1}.$$

In the electromagnetic case (1.3.22) expresses just the appearance of the derivative of the gauge parameter under gauge transformations of A, which is not present in the transformation law of $F_{\mu\nu}$.

Having at our disposal a covariant derivative, we may define a notion of parallel transport. Given any curve $\gamma(s):\ [0,1]\to\mathcal{M}$, and the value of ϕ at $\gamma(0)$, we want to determine what is the value of ϕ at each point of the curve in order for its covariant derivative along the tangent of the curve $\gamma'(s)$ to vanish. This is expressed as

$$\gamma'^{\mu}(s)D_\mu\phi^a(\gamma(s))=0,$$

or, in an integral form,

$$\phi^a(\gamma(s))=\phi^a(\gamma(0))-\int_0^s\gamma'^{\mu}\omega_\mu{}^a{}_b\phi^b(\gamma(t))dt.$$

A generic solution to this equation is given by

$$\phi^a(\gamma(s))=\mathcal{P}\left[e^{-\int_0^s\gamma'^{\mu}(t)\omega_\mu(\gamma(t))dt}\right]^a{}_b\phi^b(\gamma(0)), \tag{1.3.24}$$

where $\mathcal{P}$ stands for path ordering of the matrices in the series expansion of the exponential, i.e.

$$\mathcal{P}\left[\omega(\gamma(s))\omega(\gamma(t))\right]=\begin{cases}\omega(\gamma(s))\omega(\gamma(t)) & s<t\\ \omega(\gamma(t))\omega(\gamma(s)) & s\geq t\end{cases}.$$

Now, we may introduce an additional requirement on our connection. We want to admit only fields that under parallel transport transform according to an element of $\mathcal{G}$. Therefore it is clear from (1.3.24) that $\omega_\mu\in\mathfrak{g}$. In light of this result we compute the infinitesimal version of the gauge transformations of the connection (1.3.22). Let $g^a{}_b(x)=\delta^a{}_b+\epsilon^a{}_b(x)$ where $\epsilon^a{}_b(x)\in\mathfrak{g}$ is an infinitesimal element of the Lie algebra in the same representation as ω and is a 0-form. Then (1.3.22) can be written at the first order in the matricial notation as

$$\delta\omega=([\epsilon,\omega]-d\epsilon)=-D\epsilon \tag{1.3.25}$$

where ω acts on $\epsilon\in\mathfrak{g}$ in the adjoint representation, i.e. via the Lie product. The differential in (1.3.25) does not appear in the infinitesimal transformation of the curvature:

$$\delta R=[\epsilon,R]\,.$$

Returning to our example for a $U(1)$ gauge symmetry,

$$\phi(\gamma(s))=e^{-ie\int_{\gamma(0)}^{\gamma(s)}A_\mu dx^\mu}\phi(\gamma(0))$$

is simply the expression of the phase that a field acquires along a curve and the exponential is the so called Wilson line. Here path ordering is not necessary, being A_μ a real number, as expected, since $\mathfrak{u}(1)=\mathbb{R}$.

1.3.3 Gravity as a gauge theory in AdS

As we saw in (1.3.1), General Relativity seems to be based on the gauging of the Lorentz group, which acts on the vector bundle $T\mathcal{M}$, i.e. the tangent bundle and, more generally, on $\mathscr{X}^{k,l}(\mathcal{M})$, the set of all the smooth tensor fields with k contravariant and l covariant indices, and $\Psi(\mathcal{M})$, the set of all possible smooth spinorial fields on $\mathcal{M}$. However, the torsion constraint (1.3.19) puts some restriction on the connection that is not present in gauge theories such as electromagnetism. This relation is just the fiber form of the known relation between Christoffel symbols and the metric. Indeed by substituting $g_{\mu\nu}=e_\mu^m e_\nu^n\eta_{mn}$ into

$$\Gamma^\rho_{\mu\nu}=\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu}\right) \tag{1.3.26}$$

we can write Γ as a function solely of the vielbein and therefore we can do the same for ω by using (1.3.9).

So, even though e appears to be a matter field, it is actually the fundamental field of the theory. It is this feature that makes General Relativity different from a conventional gauge theory. In this subsection, our aim is to restore at least partially the similarity we hinted before. To do that, we will have to abandon some of the characteristics of General Relativity.

First of all, we notice that the vielbein is a 1-form that carries fiber indices and has its own gauge variation (1.3.20). We are thus led to identify it with some connection that in turn should belong to some Lie algebra. Its vectorial nature makes it natural to associate e^a with the translations generator P^a of the Poincaré group.

Consider now a generalization of the Poincaré algebra[7] (1.1.1) in which we replace the commutator of the momenta with

$$\left[P^a, P^b\right] = -\Lambda M^{ab}, \tag{1.3.27}$$

where $\Lambda \in \mathbb{R}$ is some parameter and the right hand side is determined by the symmetry of the indices. The Poincaré algebra is restored when $\Lambda = 0$. When $\Lambda \neq 0$, (1.3.27) gives the $SO(1,d)$ or $SO(2,d-1)$, the symmetry group of a space-time with one more dimension $0'$ (either space-like or time-like). The signature of the metric of this space depends on the sign of Λ. Using proper indices $\mathrm{A}, \mathrm{B}, ... = 0', 0, 1, ..., d-1$, we can decompose M^{AB}, the generators of the extended space symmetry group, into

$$M^{ab}, \quad M^{a0'} \equiv \frac{1}{\sqrt{|\Lambda|}} P^a,$$

so that, by using the $\mathfrak{so}$ commutation rules, we come to

$$\left[M^{a0'}, M^{b0'}\right] = -M^{b0'}\eta^{a0'} + M^{0'0'}\eta^{ab} + M^{ba}\eta^{0'0'} - M^{0'a}\eta^{0'b} = -M^{ab}\eta^{0'0'}$$

and therefore $\eta^{0'0'} = \mathrm{sign}(\Lambda)$.

If we define a connection with values in this algebra

$$\Omega = \frac{1}{2}\omega^{ab} M_{ab} + e^a P_a \tag{1.3.28}$$

the associated covariant derivative is $D_\Omega = d + \Omega$ and the corresponding curvature reads

$$\begin{aligned}
R_\Omega &\equiv d\Omega + \Omega \wedge \Omega \\
&= \frac{1}{2} d\omega^{ab} M_{ab} + de^a P_a + \left(\frac{1}{2}\omega^{ab} M_{ab} + e^i P_i\right) \wedge \left(\frac{1}{2}\omega^{cd} M_{cd} + e^j P_j\right) \\
&= \frac{1}{2} d\omega^{ab} M_{ab} + de^a P_a + \frac{1}{8}\omega^{ab} \wedge \omega^{cd} \left[M_{ab}, M_{cd}\right] + \frac{1}{2} e^i \wedge e^j \left[P_i, P_j\right] \\
&\quad \frac{1}{4} e^i \wedge \omega^{ab} \left[P_i, M_{ab}\right] + \frac{1}{4}\omega^{ab} \wedge e^i \left[M_{ab}, P_i\right] \\
&= \frac{1}{2}\left(d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} - \Lambda e^a \wedge e^b\right) M_{ab} + \left(de^a + \omega^a{}_b \wedge e^b\right) P_a \\
&= \frac{1}{2}\left(R^{ab} - \Lambda e^a \wedge e^b\right) M_{ab} + T^a P_a
\end{aligned} \tag{1.3.29}$$

where $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ is the curvature in the usual sense of General Relativity, i.e. when only the Lorentz spin-connection is used. Notice, though, that for $\Lambda \neq 0$ there is an additional term proportional to Λ that can be understood once we rewrite it with world indices:

$$\Lambda g_{\mu[\nu} g_{\rho]\sigma} = R^\Lambda_{\mu\nu\rho\sigma},$$

[7] Here we prefer the real version of the algebra given in (1.1.1) that can be obtained by the substitutions $P^a \to -iP^a$ and $M^{ab} \to -iM^{ab}$, that causes all the commutators to change sign.

the Riemann tensor of the (A)dS space, namely the maximally symmetric solution of Einstein equations in presence of a cosmological constant Λ. This is in accordance with the symmetry group associated to Λ.

We see that if we enforce

$$R_\Omega = 0$$

we get simultaneously

$$R^{ab} = R^{ab}_\Lambda, \quad T^a = 0, \tag{1.3.30}$$

that is, the equations that describe a maximally symmetric space and the torsion constraint. If instead we impose only the latter, we will have Einstein gravity around the AdS background.

Gauge symmetry is realized at the infinitesimal level (see (1.3.25)) by

$$\delta\Omega = D_\Omega \epsilon, \quad \epsilon \equiv \frac{1}{2}\epsilon^{ab} M_{ab} + \epsilon^a P_a, \quad \epsilon^{ab} = -\epsilon^{ba}$$

or, in components,

$$\begin{aligned}
\delta\Omega &= d\epsilon + [\Omega, \epsilon] \\
&= \frac{1}{2} d\epsilon^{ab} M_{ab} + d\epsilon^a P_a + \left[\frac{1}{2}\omega^{ab} M_{ab} + e^i P_i, \frac{1}{2}\epsilon^{cd} M_{cd} + \epsilon^j P_j\right] \\
&= \frac{1}{2} d\epsilon^{ab} M_{ab} + d\epsilon^a P_a + \frac{1}{4}\omega^{ab}\epsilon^{cd}\left[M_{ab}, M_{cd}\right] + e^i \epsilon^j \left[P_i, P_j\right] \\
& \quad \frac{1}{2} e^i \epsilon^{ab}\left[P_i, M_{ab}\right] + \frac{1}{2}\omega^{ab}\epsilon^i \left[M_{ab}, P_i\right] \\
&= \frac{1}{2}\left(d\epsilon^{ab} + \omega^a{}_c \epsilon^{cb} + \omega^b{}_c \epsilon^{ac} - \Lambda e^a \epsilon^b\right) M_{ab} + \left(d\epsilon^a + \omega^a{}_b \epsilon^b - e^b \epsilon_b{}^a\right) P_a. \\
&= \frac{1}{2}\left(D\epsilon^{ab} - 2\Lambda e^a \epsilon^b\right) M_{ab} + \left(D\epsilon^a - e^b \epsilon_b{}^a\right) P_a.
\end{aligned} \tag{1.3.31}$$

or, in components

From now on we will set our theory in the AdS space in 4 dimensions and employ both the connections $\mathcal{D}$ and D. However, since we would like to restore a dynamical gravity, at least at the perturbative level, we change a bit our notation. We call

$$h^a_\mu(x), \quad \varpi^{ab}_\mu(x)$$

the vielbein and the spin-connection for AdS respectively, while we reserve the old symbols for their dynamical perturbation:

$$e^a_\mu(x), \quad \omega^{ab}_\mu(x).$$

Our definition of the covariant derivatives becomes thus

$$D = d + \left(\varpi^{ab} + \omega^{ab}\right) L_{ab}, \quad D_\Omega = d + \left(\varpi^{ab} + \omega^{ab}\right) L_{ab} + \left(h^a + e^a\right) P_a, \tag{1.3.32}$$

and we denote with D_0 the background derivative

$$D_0 = d + \varpi^{ab} L_{ab}.$$

The two relations in (1.3.30) then can be written as

$$D_0 h^a = 0, \quad D_0 \varpi^{ab} = R^{ab}_\Lambda = \Lambda h^a \wedge h^b. \tag{1.3.33}$$

More in general, we assume that every higher spin field is a small fluctuation around the gravitational background of order $O(h)$.

1.4 Unfolding and frame-like formalism of higher spin fields

In this section we reformulate the Fronsdal equations in the frame-like formalism, namely by using 1-forms and 0-forms analogous to those employed for the gravitation equations linearized around AdS presented in the previous section. To do so, we will make also use of the so-called unfolding procedure. It consists in introducing infinitely many auxiliary fields that parametrize the derivatives of the fundamental ones, so that the equations of motion are always of first order. This will render manifest the gauge algebra of higher spin fields, which will be infinite dimensional and require the presence of all integer spins to be closed, as hinted by one of the yes-go examples.

In the first subsection we introduce unfolding by applying it to the gravitational field. Then, with the aid of the spinor formalism to describe tensors with complicated symmetry properties, we employ it to higher spins in subsection (1.4.3). In the last subsection we will show that the resulting equations are just the linearized version of a zero curvature condition of the higher spin connection.

1.4.1 Unfolding gravity

In this subsection we will analyze the spin 2 field $\Phi_{\mu\nu}$ and its equations of motion employing the techniques that we will use to describe higher spin fields in the AdS_4 space. In our setting, where the torsion constraint is an equation of motion, this field is encoded by the perturbation of the vielbein, $e^a_\mu(x)$, and the connection, ω^{ab}_μ, to which we associated two gauge transformations under (infinitesimal) diffeomorphisms

$$\delta e^a_\mu = D_\mu \epsilon^a - h^b_\mu {\epsilon_b}^a \tag{1.4.1}$$

$$\delta \omega^{ab}_\mu = D_\mu \epsilon^{ab} - \Lambda h^{[a}_\mu \epsilon^{b]} \tag{1.4.2}$$

that come from (1.3.31).

It is interesting to compare these results with the ones that we obtained in subsection (1.3.1). There we saw that if we wanted to represent the field $g_{\mu\nu}$ with h^a_μ we would have had the problem that the second is in principle just a 4×4 matrix, while the first has a symmetry property that reduces its independent components (see (1.3.6)). Local Lorentz invariance, though, reduced the count of the degrees of freedom. Here this role for $\Phi_{\mu\nu}$ is played by the second term in (1.3.20) $h^b_\mu {\epsilon_b}^a$. Indeed ϵ^{ab} is antisymmetric in a and b, so that, thanks to the vielbein that converts fiber indices into world ones, $h^b_\mu {\epsilon_b}^a$ is antisymmetric in a and μ. This allows to gauge away any unwanted antisymmetric contributions to e^a_μ.

Now let us write down the equations of motion associated to e^a, namely the vanishing of the torsion. From (1.3.33) we get[8]

$$0 = D\left(e^a + h^a\right) = D_0 e^a + {\omega^a}_b \wedge h^b. \tag{1.4.3}$$

We may now perform a consistency check on (1.4.3): applying D_0 do both the members we get

$$\begin{aligned} 0 &= D_0^2 e^a + D_0 {\omega^a}_b \wedge h^b \\ &= -\Lambda h^a \wedge e_b \wedge h^b + D_0 {\omega^a}_b \wedge h^b. \end{aligned} \tag{1.4.4}$$

In order to interpret this equation, let us write down the equation of motion for ω^{ab}:

$$\begin{aligned} R^{ab} &= d\left(\omega^{ab} + \varpi^{ab}\right) + \left({\omega^a}_c + {\varpi^a}_c\right) \wedge \left(\omega^{cb} + \varpi^{cb}\right) - \Lambda h^a \wedge e^b - \Lambda e^a \wedge h^b - \Lambda h^a \wedge h^b \\ &= d\omega^{ab} + d\varpi^{ab} + \omega^{ac} \wedge {\varpi_c}^b + \varpi^{ac} \wedge {\varpi_c}^b + \varpi^{ac} \wedge {\omega_c}^b \\ &\quad -\Lambda h^a \wedge e^b - \Lambda e^a \wedge h^b - \Lambda h^a \wedge h^b \\ &= D_0 \omega^{ab} - \Lambda h^{[a} \wedge e^{b]}. \end{aligned} \tag{1.4.5}$$

[8]Here and in the rest of the section, we will work at the first order in perturbations

We recognize then (1.4.4) as the first Bianchi identity,

$$0 = R^{ab} \wedge h_b = dx^\mu \wedge dx^\nu \wedge dx^\rho R_{\mu\nu\rho}{}^\sigma h^a_\sigma, \tag{1.4.6}$$

which simply fixes the symmetry properties of the four world indices of the Riemann tensor. We could have come to the same conclusion directly from (1.4.4). First of all, we need to write down the curvature form appearing in (1.4.4) as a rank 4 fiber tensor, by means of the vielbein, doing in a certain sense the opposite of what we did in (1.4.6). For this purpose we define

$$R^{a,b|c,d} h_c \wedge h_d \equiv D_0 \omega^{a,b} - \Lambda h^{[a} \wedge e^{b]}. \tag{1.4.7}$$

Let us explain the notation used here. Every irreducible representation of $SO(1,3)$ is related to a Young tableau that describes the symmetry properties of the indices of tensors sitting there. We group these indices by separating them with commas, so that each group corresponds to a row of the related Young diagram. If, instead, we deal with tensor products, we juxtapose the additional indices by separating them with a vertical bar. For instance

$$T^{a,b|c} = T^{a,b} \otimes T'^{c}.$$

Using (1.4.7), (1.4.4) reads

$$R^{a,b|c,d} h_c \wedge h_d \wedge h_b = 0. \tag{1.4.8}$$

Equation (1.4.8) gives some constraints on the symmetry properties of $R^{a,b|c|d}$. We can always write

$$R^{a,b|c|d} \equiv R^{a,b,c,d} + R^{ac,b,d} + R^{ac,bd}$$

and it is clear that in general only the last term is such that (1.4.8) is always satisfied. In our case, the energy-momentum tensor of the higher spin fields and the gravitational perturbation e itself, being (more than) quadratic in the fields, is negligible at first order and therefore the Einstein equations imply that both the Ricci tensor and scalar vanish, so that the Riemann tensor coincides with the Weyl tensor, i.e. its traceless part. We call it $C^{ab,cd} \equiv R^{ab,cd}$.

In the following we will employ the so-called unfolding procedure to solve the equations of motion of the Weyl tensor. It consists in the definition of infinite auxiliary fields that represent the derivatives of $C^{ab,cd}$. These auxiliary fields obey equations that are derived by the one involving $C^{ab,cd}$ by differentiating it with D_0 a proper number of times. The resulting infinite system of equations, when solved, determines all the derivatives of $C^{ab,cd}$ and hence the Weyl tensor itself. Even though this seems to complicate the description of the theory, it will unveil in the general case a gauge symmetry involving all the spins together.

Let us start from (1.4.5). Applying D_0 to both members we find

$$\begin{aligned} D_0 R^{a,b} - \Lambda h^a \wedge h_c \wedge \omega^{c,b} - \Lambda h^b \wedge h_c \wedge \omega^{a,c} + \Lambda h^a \wedge \omega^b{}_c \wedge h^c - \Lambda h^b \wedge \omega^a{}_c \wedge h^c &= 0 \\ D_0 R^{a,b} &= 0, \end{aligned}$$

which we recognize as the (linearized) second Bianchi identity for the Riemann tensor

$$D^0_{[\mu} R_{\alpha\beta]}{}^{\gamma\delta} h^a_\gamma \wedge h^b_\delta = 0.$$

We can solve in general this equation by defining an auxiliary field that represents $D_0 R^{a,b}$. Let us rewrite this equation by using $C^{ab,cd}$:

$$D_0 C^{a_1 a_2, b_1 b_2} \wedge h_{a_1} \wedge h_{b_1} = 0. \tag{1.4.9}$$

Now, let us call

$$C^{a_1 a_2, b_1 b_2|c} \equiv h^{c\mu} D^0_\mu C^{a_1 a_2, b_1 b_2}$$

so that (1.4.9) reads

$$C^{a_1 a_2, b_1 b_2|c} h_c \wedge h_{a_1} \wedge h_{b_1} = 0. \tag{1.4.10}$$

Again, since

$$\yng(2,2) \otimes \yng(1) = \yng(2,2,1) \oplus \yng(2,1) \oplus \yng(3,2)$$

we can rewrite $C^{a_1a_2,b_1b_2|c}$ as

$$C^{a_1a_2,b_1b_2|c} \equiv C^{a_1a_2,b_1b_2,c} + traces + C^{a_1a_2c,b_1b_2} \tag{1.4.11}$$

where $traces$ denotes the traceful terms (i.e. proportional to η^{ac} or η^{bc}), while $C^{a_1a_2,b_1b_2,c}$ and $C^{a_1a_2c,b_1b_2}$ are not. Only the last term in (1.4.11) has the right symmetry properties that make (1.4.10) vanish identically. We can thus rewrite (1.4.9) as

$$D_0C^{a_1a_2,b_1b_2} = \mathcal{P}_{\yng(2,2)}\left[h_c C^{a_1a_2c,b_1b_2}\right], \tag{1.4.12}$$

where $\mathcal{P}_{\yng(2,2)}$ is some projector that imposes on the right hand side of (1.4.12) the symmetry properties of the Weyl tensor. Here we avoid to derive precise expressions for such projections, because they are quite complicated if we use the 4-dimensional notation. Since, on the contrary, with the formalism that we develop in the next subsection these results are written in a much simpler form, we prefer to focus on the general structure of the equations of motion for the C tensors.

Following our unfolding algorithm, we solve (1.4.12) by applying to both its sides D_0 to get

$$\Lambda\mathcal{P}_{\yng(2,2)}\left[\left(h^{a_1}\wedge h_d\right)C^{a_2d,b_1b_2} + \left(h^{b_1}\wedge h_d\right)C^{a_1a_2,b_2d}\right] = \mathcal{P}_{\yng(2,2)}\left[h_c D_0 C^{a_1a_2c,b_1b_2}\right] \tag{1.4.13}$$

If we write

$$C^{a_1a_2a_3,b_1b_2|d} \quad \equiv \quad h^{\mu d}D^0_\mu C^{a_1a_2a_3,b_1b_2}, \tag{1.4.14}$$

equation (1.4.13) reads

$$\mathcal{P}_{\yng(2,2)}\left[h_c\wedge h_d C^{a_1a_2c,b_1b_2|d}\right] = \mathcal{P}_{\yng(2,2)}\left[\left(\Lambda h^{a_1}\wedge h_d\right)C^{a_2d,b_1b_2} + \left(\Lambda h^{b_1}\wedge h_d\right)C^{a_1a_2,b_2d}\right]. \tag{1.4.15}$$

By a reasoning analogous to the previous one with Young diagrams, (1.4.15) implies

$$D_0C^{a_1a_2a_3,b_1b_2} = \mathcal{P}_{\yng(3,2)}\left[h_d C^{a_1a_2a_3d,b_1b_2} + \Lambda h_d\eta^{a_1a_2}C^{a_3d,b_1b_2}\right]. \tag{1.4.16}$$

where both the C tensors appearing in the r.h.s are traceless, i.e. reside in an irreducible representation of the Lorentz group. The first term in (1.4.16) makes the left member of (1.4.15) vanish, while the other one gives the contributions proportional to the Weyl tensor found in the right hand side of (1.4.15)[9].

If we now unfold equation (1.4.16) we get

$$\begin{aligned}
D_0^2C^{a_1a_2a_3,b_1b_2} \quad &= \quad \mathcal{P}_{\yng(3,2)}\left[h_dD_0C^{a_1a_2a_3d,b_1b_2} + \Lambda h_d\eta^{a_1a_2}D_0C^{a_3d,b_1b_2}\right] \\
&= \quad \mathcal{P}_{\yng(3,2)}\left[h_dD_0C^{a_1a_2a_3d,b_1b_2} + \Lambda\eta^{a_1a_2}h_d\wedge h_cD_0C^{a_3dc,b_1b_2}\right] \\
&= \quad \mathcal{P}_{\yng(3,2)}\left[h_dD_0C^{a_1a_2a_3d,b_1b_2}\right],
\end{aligned}$$

which looks like (1.4.13), but this time involves a tensor $C^{a_1a_2a_3a_4,b_1b_2}$ with one more index of the a type. It is now clear that by induction one finds that all the unfolded equations associated to the Weyl tensor are constraints on traceless fields of type

$$C^{a_1\dots a_k,b_1b_2} \in \begin{array}{|c|c|c|c|}\hline & & k-3 & \\ \hline & \\ \cline{1-2}\end{array}$$

They read

$$\begin{cases} D_0\omega^{a_1,b_1} = h_{a_2}\wedge h_{b_2}C^{a_1a_2,b_1b_2} \\ D_0C^{a_1\dots a_k,b_1b_2} = \mathcal{P}_{\yng(5,2)}\left[h_{a_{k+1}}C^{a_1\dots a_ka_{k+1},b_1b_2} + \Lambda\eta^{a_1a_2}h_dC^{a_3\dots a_kd,b_1b_2}\right]\end{cases} \tag{1.4.17}$$

[9]It is the projector that makes it possible for both of them to appear.

1.4.2 The spinorial notation

Unfolding the linearized Einstein equations required the definition of an infinite number of tensorial 0-forms with arbitrary rank. These tensors should sit in irreducible representations of the Lorentz group, i.e. they must be traceless and with a definite symmetry of the indices. In order to put them in such form, we had to employ some projectors in equations (1.4.17). This task can be simplified if we work with another representation of the Lorentz group: the spinorial one.

As it is well known, $SL(2,\mathbb{C})$ is a double covering of $SO^+_\uparrow(1,3)$, the subgroup of $SO(1,3)$ in which lie the orthocronous proper Lorentz transformations. The homomorphism between the two is realized in the following way. A vector $v^\mu \in \mathbb{R}^{1,3}$ is represented by the 2×2 Hermitean matrix

$$V = v^\mu \sigma_\mu = \begin{pmatrix} v_0 + v_3 & v_1 - iv_2 \\ v_0 + iv_2 & v_0 - v_3 \end{pmatrix} = V^\dagger \tag{1.4.18}$$

where the σ_μ are the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We see from (1.4.18) that

$$\det(V) = v^2,$$

so that Lorentz transformations are given by left and right multiplication[10] with 2×2 matrices that preserve the determinants, i.e. $SL(2,\mathbb{C})$:

$$V \to V' = SVS^\dagger, \quad S \in SL(2,\mathbb{C}). \tag{1.4.19}$$

It is clear that this representation is 2 to 1, for S and $-S$ give the same Lorentz transformation. The spinorial representation of the Lorentz group is associated with the fundamental representation of $SL(2,\mathbb{C})$ acting on vectors in $\mathbb{C}^2$ that we call Weyl spinors. From now on, we use the first Greek letters $\alpha, \beta, ...$ to denote spinor indices that can be equal only to 1 or 2, while dotted indices $\dot\alpha, \dot\beta, ... = 1, 2$ are used for the the conjugated spinors, i.e. the elements of the dual of $\mathbb{C}^2$. Then we have that ψ_α transforms according to the fundamental representation (denoted by 2)

$$\psi'_\alpha = S_\alpha{}^\beta \psi_\beta$$

while for $\psi_{\dot\alpha}$ we need to use the conjugate of S (in the so-called anti-fundamental representation, denoted by $\bar 2$)

$$\psi'_{\dot\alpha} = (S^*)_{\dot\alpha}{}^{\dot\beta}\,\psi_{\dot\beta}.$$

The two representations are inequivalent and, by tensorial products, produce every irreducible representation of the Lorentz group. For instance, we see from (1.4.19) that vectors sit in $2 \otimes \bar 2$.

It is also possible to introduce a "metric", namely a way to contract spinorial indices in order to get a scalar product of the vectors written in the spinorial form (1.4.18). Indeed,

$$\begin{aligned} \det(v_{\alpha\dot\alpha}) &= v_{11}v_{22} - v_{12}v_{21} \\ &= \frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} v_{\alpha\dot\alpha} v_{\beta\dot\beta}, \end{aligned} \tag{1.4.20}$$

and we see that

$$\varepsilon^{\alpha\beta} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \varepsilon^{\dot\alpha\dot\beta}, \quad \varepsilon^{\alpha\beta}\varepsilon_{\gamma\beta} = \delta^\alpha_\gamma \tag{1.4.21}$$

[10] To see that it is necessary to take two (and not one) $SL(2,\mathbb{C})$ matrices and that they must be multiplied with V in the order given by (1.4.19), it suffices to impose $V'^\dagger = V'$

is the sought "metric", which, contrarily to the usual one, is anti-symmetric. This implies that when we lower or raise indices by means of ε in the following way (see 1.4.21)

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta = -\psi_\beta\varepsilon^{\beta\alpha}, \quad \psi_\alpha = \varepsilon_{\gamma\alpha}\varepsilon^{\gamma\beta}\psi_\beta = \psi^\beta\varepsilon_{\beta\alpha} = -\varepsilon_{\alpha\beta}\psi^\beta$$

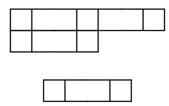
we need to use the convention that indices are raised by left multiplication and lowered by right multiplication with ε, which is also coherent with

$$\varepsilon_{\alpha\beta} = \varepsilon^{\gamma\delta}\varepsilon_{\gamma\alpha}\varepsilon_{\delta\beta}.$$

From (1.4.20) we also recover the spinorial version of η, the four-dimensional metric:

$$\eta^{\alpha\dot{\alpha},\beta\dot{\beta}} = \frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}. \tag{1.4.22}$$

In the following we will need only a particular class of irreducible representations of the Lorentz group, namely those represented by the following Young tableaux:



Let us see what they look like in the spinor notation. First, we derive a useful identity. Any spin-tensor of rank 2 can be written as

$$T_{\alpha\beta} = \frac{1}{2}T_{(\alpha\beta)} + \frac{1}{2}T_{[\alpha\beta]}$$

and

$$\varepsilon^{\alpha\beta}T_{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta}T_{[\alpha\beta]} = \frac{1}{2}T^\gamma{}_\gamma,$$

so that

$$T_{\alpha\beta} = \frac{1}{2}T_{(\alpha\beta)} + \frac{1}{4}\varepsilon_{\alpha\beta}T^\gamma{}_\gamma. \tag{1.4.23}$$

Given a tensor $T^{a,b}$ which is antisymmetric in two indices, applying repeatedly (1.4.23), its spinorial version reads[11]

$$\begin{aligned} T^{a,b}\sigma_a^{\alpha\dot{\alpha}}\sigma_b^{\beta\dot{\beta}} &= T^{a,b}\left(\frac{1}{2}\sigma_a^{(\alpha\dot{\alpha}}\sigma_b^{\beta)\dot{\beta}} + \frac{1}{4}\varepsilon^{\alpha\beta}\sigma_a^{\alpha\dot{\gamma}}\sigma^{\beta}_{b\dot{\gamma}}\right) \\ &= T^{a,b}\left(\frac{1}{4}\sigma_a^{(\alpha(\dot{\alpha}}\sigma_b^{\beta)\dot{\beta})} + \frac{1}{8}\varepsilon^{\alpha\beta}\sigma_a^{(\alpha\dot{\gamma}}\sigma^{\beta)}_{b\dot{\gamma}} + \frac{1}{8}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma_a^{(\alpha\dot{\gamma}}\sigma^{\beta)}_{b\dot{\gamma}} + \frac{1}{16}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma_a^{\delta\dot{\gamma}}\sigma_{b\delta\dot{\gamma}}\right) \\ &= T^{a,b}\left(\varepsilon^{\alpha\beta}T^{\dot{\alpha}\dot{\beta}} + \varepsilon^{\dot{\alpha}\dot{\beta}}T^{\alpha\beta}\right) \end{aligned}$$

where we used the antisymmetry of $T^{a,b}$ to rule out the first and the last term and defined the tensors

$$T^{\alpha\beta} = \frac{T^{a,b}}{8}\sigma_a^{(\alpha\dot{\gamma}}\sigma^{\beta)}_{b\dot{\gamma}}, \quad T^{\dot{\alpha}\dot{\beta}} = \frac{T^{a,b}}{8}\sigma_a^{\gamma(\dot{\alpha}}\sigma^{\dot{\beta})}_{b\gamma}.$$

Since $T^{a,b}$ is real, $T^{\alpha\beta} = \left(T^{\dot{\alpha}\dot{\beta}}\right)^*$. This procedure can be applied similarly to the case of a traceless tensor $T^{a_1\dots a_{k+l},b_1\dots b_k}$, which is represented by

$$T^{\alpha_1\dots\alpha_{2k+l}\dot{\alpha}_{2k+1}\dots\dot{\alpha}_{2k+l}}\varepsilon^{\dot{\alpha}_1\dot{\alpha}_{k+1}}\dots\varepsilon^{\dot{\alpha}_k\dot{\alpha}_{2k}} + T^{\dot{\alpha}_1\dots\dot{\alpha}_{2k+l}\alpha_{2k+1}\dots\alpha_{2k+l}}\varepsilon^{\alpha_1\alpha_{k+1}}\dots\varepsilon^{\alpha_k\alpha_{2k}}, \tag{1.4.24}$$

where both dotted and undotted indices of the T spin-tensors are symmetric, so that and the first is the complex conjugated of the second. The reason why no mixed terms with both $\varepsilon^{\alpha\alpha}$ and $\varepsilon^{\dot{\alpha}\dot{\alpha}}$ appear is that they do not vanish when contracted with $\eta_{\alpha\dot{\alpha},\alpha\dot{\alpha}}$, while $T^{a_1\dots a_{k+l},b_1\dots b_k}$ should be traceless.

[11]Dotted and undotted indices are symmetrized separately.

Completely symmetric traceless tensors $T^{a_1\dots a_k}$ have the following spinorial form:

$$T^{\alpha_1\dots\alpha_k\dot\alpha_1\dots\dot\alpha_k} = T^{a_1\dots a_k}\sigma_{a_1}^{\alpha_1\dot\alpha_1}\cdots\sigma_{a_k}^{\alpha_k\dot\alpha_k}, \tag{1.4.25}$$

that is symmetric in both the dotted and undotted indices and therefore traceless under contractions with $\eta_{\alpha\dot\alpha,\alpha\dot\alpha}$.

We conclude by applying these results to some of the tensors encountered in the previous subsections. The vielbein and the spin connection in the spinorial formalism read, using (1.4.23),

$$h^{\alpha\dot\alpha}, \quad \varpi^{\alpha\dot\alpha,\beta\dot\beta} \equiv \varpi^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} + \varpi^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}$$

and therefore D_0 acts on spin tensors as if there were two separate spin connections $\varpi^{\alpha\beta}$ and $\varpi^{\dot\alpha\dot\beta}$ for each type of index. For example, if we consider a vector $V^{\alpha\dot\alpha}$,

$$\begin{aligned} D_0V^{\alpha\dot\alpha} &= dV^{\alpha\dot\alpha} + \varpi^{\alpha\dot\alpha,\beta\dot\beta}V_{\beta\dot\beta} \\ &= dV^{\alpha\dot\alpha} + \varpi^{\alpha\beta}V_\beta^{\dot\alpha} + \varpi^{\dot\alpha\dot\beta}V_{\dot\beta}^{\alpha}. \end{aligned} \tag{1.4.26}$$

The same happens for the curvature form, that acts on undotted and dotted indices respectively with

$$R^{\alpha\beta} = h^{\alpha\dot\gamma}\wedge h^{\beta}_{\dot\gamma}, \quad \text{and} \quad R^{\dot\alpha\dot\beta} = h^{\gamma\dot\alpha}\wedge h^{\dot\beta}_{\gamma}. \tag{1.4.27}$$

Analogously, the symmetry generators of AdS$_4$ M^{ab} and P^a are substituted by

$$P^{\alpha\dot\alpha}, \quad M^{\alpha\dot\alpha,\beta\dot\beta} \equiv M^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} + M^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}.$$

In the following we will also need two of their commutators:

$$\left[P^{\alpha\dot\alpha}, P^{\beta\dot\beta}\right] = -\Lambda\left(M^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} + M^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\right) \tag{1.4.28}$$
$$\left[P^{\alpha\dot\alpha}, M^{\beta\gamma}\varepsilon^{\dot\beta\dot\gamma} + M^{\dot\beta\dot\gamma}\varepsilon^{\beta\gamma}\right] = -\left(P^{\beta\dot\beta}\varepsilon^{\alpha\gamma}\varepsilon^{\dot\alpha\dot\gamma} - P^{\gamma\dot\gamma}\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\right),$$

which implies

$$\left[P^{\alpha\dot\alpha}, M^{\beta\gamma}\right] = -P^{\beta\dot\alpha}\varepsilon^{\alpha\gamma} - P^{\gamma\dot\alpha}\varepsilon^{\alpha\beta}, \quad \left[P^{\alpha\dot\alpha}, M^{\dot\beta\dot\gamma}\right] = -P^{\alpha\dot\beta}\varepsilon^{\dot\alpha\dot\gamma} - P^{\alpha\dot\gamma}\varepsilon^{\dot\alpha\dot\beta}. \tag{1.4.29}$$

1.4.3 Unfolding of an arbitrary spin field

Let us consider now a generic spin s field. Our goal is to reproduce for this case the analysis carried out in section 1.4.1. Analogy suggests to promote $\Phi^{\mu_1\dots\mu_s}$ to a vielbein-like field $e_\mu^{a_1\dots a_{s-1}}$, that is completely symmetric in the fiber indices, can be interpreted as a tensor-valued 1-form and is related to Φ by

$$h_{a_1(\mu_1}\cdots h_{a_{s-1}\mu_{s-1}}e_{\mu_s)}^{a_1\dots a_{s-1}} \equiv \Phi_{\mu_1\dots\mu_s}. \tag{1.4.30}$$

The double tracelessness constraint (1.1.17) is replaced by

$$e^{a_1\dots a_{s-1}}\eta_{a_1a_2} = 0, \tag{1.4.31}$$

which simply states that e is traceless, as well as symmetric, and therefore sits in an irreducible representation of $SO(1,3)$, a more natural condition with respect to double tracelessness, which is in turn implied by (1.4.31).

In a curved background, Φ undergoes the gauge transformation

$$\delta\Phi^{\mu_1\dots\mu_s} = D_0^{(\mu_1}\epsilon^{\mu_2\dots\mu_s)}, \tag{1.4.32}$$

which (1.4.30) suggests that $e^{a_1 \dots a_{s-1}}$ transforms in the following way:

$$\delta e^{a_1 \dots a_{s-1}} = D_0 \epsilon^{a_1 \dots a_{s-1}}, \tag{1.4.33}$$

where $\epsilon^{a_1 \dots a_{s-1}}$ is a completely symmetric 0-form. Equation (1.4.32) is the immediate generalization of (1.3.20).

Now, let us compare the degrees of freedom of Φ and e. There is no way to express the symmetry between the world index μ_s and the fiber ones, so that e comes with some additional non-physical degrees of freedom with respect to the Fronsdal field. This is depicted by Young tableaux as follows

$$e = \ydiagram{3} \otimes \ydiagram{1} = \ydiagram{3,1} \oplus \ydiagram{4} \tag{1.4.34}$$

We already encountered a similar situation for the gravity vielbein in section 1.4.1. There the issue was solved by the gauge transformation (1.4.1). Let us adopt that solution also in our case and generalize (1.4.33) to

$$\delta e_\mu^{a_1 \dots a_{s-1}} = D^0_\mu \epsilon^{a_1 \dots a_{s-1}} + \rho h_{\mu b} \epsilon^{a_1 \dots a_{s-1}, b}, \tag{1.4.35}$$

where we introduced a dimensional quantity

$$\rho \equiv \sqrt{-\Lambda}$$

and a new gauge parameter $\epsilon^{a_1 \dots a_{s-1}, b}$ that has the symmetry of the first term in (1.4.34) in the b and a indices and, therefore, thanks to the background vielbein h, in the μ and a indices. This new gauge invariance allows us to subtract the unwanted degrees of freedom. From an analogy with gravity, we expect that the new gauge parameter is associated to a 1-form field

$$\omega^{a_1 \dots a_{s-1}, b} \tag{1.4.36}$$

that plays the role of the spin-connection whose gauge transformation is a generalization of (1.4.2) and reads

$$\delta \omega^{a_1 \dots a_{s-1}, b} = D_0 \epsilon^{a_1 \dots a_{s-1}, b} - \rho \left(h^b \epsilon^{a_1 \dots a_{s-1}} - h^{(a_1} \epsilon^{a_2 \dots a_{s-1})b} \right) \tag{1.4.37}$$

Before proceeding further, let us convert all the quantities into spin-tensors:

$$\begin{gathered} e^{a_1 \dots a_{s-1}} \to e^{\alpha_1 \dots \alpha_{s-1} \dot\alpha_1 \dots \dot\alpha_{s-1}}, \\ \epsilon^{a_1 \dots a_{s-1}} \to \epsilon^{\alpha_1 \dots \alpha_{s-1} \dot\alpha_1 \dots \dot\alpha_{s-1}}, \;\; \epsilon^{a_1 \dots a_{s-1}, b} \to \varepsilon^{\dot\alpha_{s-1} \dot\beta} \epsilon^{\alpha_1 \dots \alpha_{s-1} \beta \dot\alpha_1 \dots \dot\alpha_{s-1}} + \varepsilon^{\alpha_{s-1} \beta} \epsilon^{\alpha_1 \dots \alpha_{s-1} \dot\alpha_1 \dots \dot\alpha_{s-1} \dot\beta} \\ \omega^{a_1 \dots a_{s-1}, b} \to \varepsilon^{\dot\alpha_{s-1} \dot\beta} \omega^{\alpha_1 \dots \alpha_{s-1} \beta \dot\alpha_1 \dots \dot\alpha_{s-1}} + \varepsilon^{\alpha_{s-1} \beta} \omega^{\alpha_1 \dots \alpha_{s-1} \dot\alpha_1 \dots \dot\alpha_{s-1} \dot\beta} \end{gathered}$$

so that (1.4.35) and (1.4.37) become respectively

$$\delta e^{\alpha_1 \dots \alpha_{s-1} \dot\alpha_1 \dots \dot\alpha_{s-1}} = D_0 \epsilon^{\alpha_1 \dots \alpha_{s-1} \dot\alpha_1 \dots \dot\alpha_{s-1}} - \rho h_\beta^{\dot\alpha_{s-1}} \epsilon^{\alpha_1 \dots \alpha_{s-1} \beta \dot\alpha_1 \dots \dot\alpha_{s-2}} - \rho h^{\alpha_{s-1}}_{\dot\beta} \epsilon^{\alpha_1 \dots \alpha_{s-2} \dot\alpha_1 \dots \dot\alpha_{s-1} \dot\beta}. \tag{1.4.38}$$

and

$$\delta \omega^{\alpha_1 \dots \alpha_s \dot\alpha_1 \dots \dot\alpha_{s-2}} = D_0 \epsilon^{\alpha_1 \dots \alpha_s \dot\alpha_1 \dots \dot\alpha_{s-2}} - \rho h^{\alpha_s}_{\dot\beta} \epsilon^{\alpha_1 \dots \alpha_{s-1} \dot\alpha_1 \dots \dot\alpha_{s-2} \dot\beta} \tag{1.4.39}$$

Here we introduced a convention that we will employ throughout this subsection: spinor indices denoted by the same Greek letter and of the same (un)dotted kind are implicitly symmetrized, with the rules explained in the "Notation" appendix. Moreover, when we use the shorthand notation $T^{\alpha(k)\dot\alpha(l)}$ to mean $T^{\alpha_1 \dots \alpha_k \dot\alpha_1 \dots \dot\alpha_l}$ in the text.

Following the analogy with gravity, we need to eliminate the unphysical degrees of freedom of the spin connection (1.4.36), which is a purely auxiliary field. To do so, we impose a torsion-like constraint:

$$D_0 e^{\alpha_1 \dots \alpha_{s-1} \dot\alpha_1 \dots \dot\alpha_{s-1}} + \rho \omega^{\alpha_1 \dots \alpha_{s-1} \beta \dot\alpha_1 \dots \dot\alpha_{s-2}} \wedge h_\beta^{\dot\alpha_{s-1}} + \rho \omega^{\alpha_1 \dots \alpha_{s-2} \dot\alpha_1 \dots \dot\alpha_{s-1} \dot\beta} \wedge h^{\alpha_{s-1}}_{\dot\beta} = 0. \tag{1.4.40}$$

where the appearance of $h^{\alpha\dot{\alpha}}$ instead of $e^{\alpha\dot{\alpha}}$ is due to the fact that we are making a first order approximation, where terms like $e^{\alpha\dot{\alpha}}\omega^{\alpha(s)\dot{\alpha}(s-2)}$ are of second order. Relation (1.4.40) is meant to allow us to determine $\omega^{\alpha(s)\dot{\alpha}(s-2)}$ and $\omega^{\alpha(s-2)\dot{\alpha}(s)}$ as functions of $e^{\alpha(s-1)\dot{\alpha}(s-1)}$, just like happens for spin 2. We need to check, though, if (1.4.40) contains enough independent components to constrain completely $\omega^{\alpha(s)\dot{\alpha}(s-2)}$ and $\omega^{\alpha(s-2)\dot{\alpha}(s)}$. By using the spinorial version of (1.3.5)

$$h^{\alpha\dot{\alpha}}_{\mu}h_{\nu\alpha\dot{\alpha}} = 2g_{\mu\nu},$$

we can decompose $\omega^{\alpha(s)\dot{\alpha}(s-2)}$ in the following way

$$\begin{aligned}
\omega_{\mu}^{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-2}} &= \frac{1}{2}\omega_{\nu}^{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-2}}h^{\nu\gamma\dot{\gamma}}h_{\mu\gamma\dot{\gamma}} \\
&= \frac{1}{4}\omega_{\nu}^{(\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-2}}h^{\nu\gamma)\dot{\gamma}}h_{\mu\gamma\dot{\gamma}} + \frac{1}{8}\omega_{\nu}^{\alpha_1\dots\alpha_{s-1}\beta\dot{\alpha}_1\dots\dot{\alpha}_{s-2}}h^{\nu\dot{\gamma}}_{\beta}\varepsilon^{\gamma\alpha_s}h_{\mu\gamma\dot{\gamma}} \\
&= \frac{1}{8}\omega_{\nu}^{(\alpha_1\dots\alpha_s(\dot{\alpha}_1\dots\dot{\alpha}_{s-2}}h^{\nu\gamma)\dot{\gamma})}h_{\mu\gamma\dot{\gamma}} + \frac{1}{16}\omega_{\nu}^{(\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-3}\dot{\beta}}h^{\nu\gamma)}_{\dot{\beta}}\varepsilon^{\dot{\alpha}_{s-2}\dot{\gamma}}h_{\mu\gamma\dot{\gamma}} \\
&\quad + \frac{1}{16}\omega_{\nu}^{\alpha_1\dots\alpha_{s-1}\beta(\dot{\alpha}_1\dots\dot{\alpha}_{s-2}}h^{\nu\dot{\gamma})}_{\beta}\varepsilon^{\gamma\alpha_s}h_{\mu\gamma\dot{\gamma}} + \frac{1}{32}\omega_{\nu}^{\alpha_1\dots\alpha_{s-1}\beta\dot{\alpha}_1\dots\dot{\alpha}_{s-3}\dot{\beta}}h^{\nu}_{\beta\dot{\beta}}\varepsilon^{\dot{\gamma}\dot{\alpha}_{s-3}}\varepsilon^{\gamma\alpha_s}h_{\mu\gamma\dot{\gamma}} \\
&\equiv \tilde{\omega}^{\alpha_1\dots\alpha_{s+1}\dot{\alpha}_1\dots\dot{\alpha}_{s-1}}h_{\mu\alpha_{s+1}\dot{\alpha}_{s-1}} + \tilde{\omega}^{\alpha_1\dots\alpha_{s-1}\dot{\alpha}_1\dots\dot{\alpha}_{s-1}}h^{\alpha_s}_{\mu\dot{\alpha}_{s-1}} \\
&\quad + \tilde{\omega}^{\alpha_1\dots\alpha_{s+1}\dot{\alpha}_1\dots\dot{\alpha}_{s-3}}h^{\dot{\alpha}_{s-2}}_{\mu\alpha_{s+1}} + \tilde{\omega}^{\alpha_1\dots\alpha_{s-1}\dot{\alpha}_1\dots\dot{\alpha}_{s-3}}h^{\alpha_s\dot{\alpha}_{s-2}}_{\mu}
\end{aligned}$$

and we see that the third term can not be determined by (1.4.40), since

$$\tilde{\omega}^{\alpha_1\dots\alpha_{s-1}\beta\gamma\dot{\alpha}_1\dots\dot{\alpha}_{s-3}}h^{\dot{\alpha}_{s-2}}_{\gamma}\wedge h^{\dot{\alpha}_{s-1}}_{\beta} \equiv 0.$$

An analogous issue involves $\omega^{\alpha(s-2)\dot{\alpha}(s)}$. We may solve this problem by introducing a generalization of the gauge transformation (1.4.39):

$$\delta\omega^{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-2}} = D_0\epsilon^{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-2}} - \rho h^{\dot{\alpha}_{s-2}}_{\beta}\epsilon^{\alpha_1\dots\alpha_s\beta\dot{\alpha}_1\dots\dot{\alpha}_{s-3}} - \rho h^{\alpha_s}_{\dot{\beta}}\epsilon^{\alpha_1\dots\alpha_{s-1}\dot{\alpha}_1\dots\dot{\alpha}_{s-2}\dot{\beta}}. \tag{1.4.41}$$

so that the new gauge parameter $\epsilon^{\alpha(s+1)\dot{\alpha}(s-3)}$ is able to remove the unwanted fictitious degrees of freedom encoded in $\tilde{\omega}^{\alpha(s+1)\dot{\alpha}(s-3)}$. Moreover, we expect that this gauge transformation is associated to a new spin connection-like field $\omega^{\alpha(s+1)\dot{\alpha}(s-3)}$ that descends from $\omega^{\alpha(s)\dot{\alpha}(s-2)}$ and transforms according to

$$\delta\omega^{\alpha_1\dots\alpha_{s+1}\dot{\alpha}_1\dots\dot{\alpha}_{s-3}} = D_0\epsilon^{\alpha_1\dots\alpha_{s+1}\dot{\alpha}_1\dots\dot{\alpha}_{s-3}} - \rho h^{\alpha_{s+1}}_{\dot{\gamma}}\epsilon^{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-1}\dot{\gamma}}. \tag{1.4.42}$$

$\omega^{\alpha(s+1)\dot{\alpha}(s-3)}$, being auxiliary, should be expressed as a function of $e^{\alpha(s-1)\dot{\alpha}(s-1)}$, $\omega^{\alpha(s)\dot{\alpha}(s-2)}$ and $\omega^{\alpha(s-2)\dot{\alpha}(s)}$. The equation that specifies this relation, should also be gauge invariant under (1.4.42), (1.4.41) and (1.4.38). It is then clear that it should have a form that generalizes (1.4.40):

$$D_0\omega^{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-2}} + \rho\omega^{\alpha_1\dots\alpha_s\beta\dot{\alpha}_1\dots\dot{\alpha}_{s-3}}\wedge h^{\dot{\alpha}_{s-2}}_{\beta} + \rho e^{\alpha_1\dots\alpha_{s-1}\dot{\alpha}_1\dots\dot{\alpha}_{s-2}\dot{\beta}}\wedge h^{\alpha_s}_{\dot{\beta}} = 0, \tag{1.4.43}$$

where the fields involved have been identified according to the indices they carry. Again, just like (1.4.40), (1.4.43) is invariant also under the following transformation

$$\delta'\omega^{\alpha_1\dots\alpha_{s+1}\dot{\alpha}_1\dots\dot{\alpha}_{s-3}} = -\rho h^{\dot{\alpha}_{s-3}}_{\gamma}\epsilon^{\alpha_1\dots\alpha_{s+1}\gamma\dot{\alpha}_1\dots\dot{\alpha}_{s-4}},$$

which signals the fact that (1.4.43) is not able to constrain all the components of $\omega^{\alpha(s+1)\dot{\alpha}(s-3)}$. Those that are unaffected by (1.4.43) can be anyway gauged away if we deform (1.4.42) into

$$\delta\omega^{\alpha_1\dots\alpha_{s+1}\dot{\alpha}_1\dots\dot{\alpha}_{s-3}} = D_0\epsilon^{\alpha_1\dots\alpha_{s+1}\dot{\alpha}_1\dots\dot{\alpha}_{s-3}} - \rho h^{\alpha_{s+1}}_{\dot{\gamma}}\epsilon^{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-1}\dot{\gamma}} - \rho h^{\dot{\alpha}_{s-3}}_{\gamma}\epsilon^{\alpha_1\dots\alpha_{s+1}\gamma\dot{\alpha}_1\dots\dot{\alpha}_{s-4}}.$$

We can repeat this argument inductively and define step by step a $\omega^{\alpha(k+1)\dot\alpha(l-1)}$ descending from $\omega^{\alpha(k)\dot\alpha(l)}$. The same can be done with $\omega^{\alpha(s-2)\dot\alpha(s)}$, and leads to the introduction of $\omega^{\alpha(k-1)\dot\alpha(l+1)}$ as the gauge field associated to the gauge parameter $\epsilon^{\alpha(k-1)\dot\alpha(l+1)}$. The spin s field is then represented by the following 1-forms

$$\omega^{\alpha(k)\dot\alpha(l)}, \quad k+l=2s-2, \quad k,l\geq 0,$$

where we used $\omega^{\alpha(s-1)\dot\alpha(s-1)}$ in place of $e^{\alpha(s-1)\dot\alpha(s-1)}$ to simplify our notation, as we will often do in the following. The Young tableau[12] associated to the four dimensional form of $\omega^{\alpha(k)\dot\alpha(l)}$ is

$$\begin{array}{l} \square\square \;.\;.\;.\; k+l \;.\;.\;.\;.\;.\; \square\square \\ \square\square \;.\;.\;.\; k-l \;.\;.\; \square\square \end{array} \tag{1.4.44}$$

The equations for the connections $\omega^{\alpha(k)\dot\alpha(l)}$ that we have thus derived is

$$D_0\omega^{\alpha_1\ldots\alpha_{s-1+k}\dot\alpha_1..\dot\alpha_{s-1-k}} = \rho h_\beta^{\dot\alpha_{s-1-k}}\wedge\omega^{\alpha_1\ldots\alpha_{s-1+k}\beta\dot\alpha_1\ldots\dot\alpha_{s-2-k}} + \rho h_{\dot\beta}^{\alpha_{s-1-k}}\wedge\omega^{\alpha_1\ldots\alpha_{s-2+k}\dot\alpha_1\ldots\dot\alpha_{s-1-k}\dot\beta} \tag{1.4.45}$$

and the related gauge transformations that leave them invariant are

$$\begin{aligned}\delta\omega^{\alpha_1\ldots\alpha_{s-1+k}\dot\alpha_1..\dot\alpha_{s-1-k}} &= D_0\epsilon^{\alpha_1\ldots\alpha_{s-1+k}\dot\alpha_1\ldots\dot\alpha_{s-1-k}} \\ &\quad -\rho h_\beta^{\dot\alpha_{s-1-k}}\epsilon^{\alpha_1\ldots\alpha_{s-1+k}\beta\dot\alpha_1\ldots\dot\alpha_{s-2+k}} - \rho h_{\dot\beta}^{\alpha_{s-1}}\epsilon^{\alpha_1\ldots\alpha_{s-2+k}\dot\alpha_1\ldots\dot\alpha_{s-1+k}\dot\beta},\end{aligned} \tag{1.4.46}$$

where $1-s<k<s-1$ is some integer. It is clear that we could not write (1.4.45) for $\omega^{\alpha(2s-2)}$ or $\omega^{\dot\alpha(2s-2)}$, since they can not have contributions from both the terms proportional to ρ. These two 1-forms represent a tensor with the following symmetry

$$\begin{array}{l} \square\square \;.\;.\;.\; s-1 \;.\;.\;.\; \square\square \\ \square\square \;.\;.\;.\; s-1 \;.\;.\;.\; \square\square \end{array} \tag{1.4.47}$$

so that the 2-form

$$D_0\omega^{\alpha_1\ldots\alpha_{2s-2}} + \rho\omega^{\alpha_1\ldots\alpha_{2s-3}\dot\beta}\wedge h_{\dot\beta}^{\alpha_{2s-2}}$$

and its conjugate, once one converts world indices to fiber ones, represent a tensor with a Young diagram of the kind

$$\begin{array}{l} \square\square \;.\;.\;.\; s \;.\;.\;.\; \square\square \\ \square\square \;.\;.\;.\; s \;.\;.\;.\; \square\square \end{array} \tag{1.4.48}$$

that we interpret as the higher spin curvature, since it reduces to the Weyl tensor in the gravitational case. We may therefore write for $\omega^{\alpha(2s-2)}$

$$D_0\omega^{\alpha_1\ldots\alpha_{2s-2}} + \rho\omega^{\alpha_1\ldots\alpha_{2s-3}\dot\beta}\wedge h_{\dot\beta}^{\alpha_{2s-2}} = h^{\dot\gamma}_{\alpha_{2s}}\wedge h_{\dot\gamma\alpha_{2s-1}}C^{\alpha_1\ldots\alpha_{2s}}. \tag{1.4.49}$$

This equation corresponds to (1.4.4) for spin 2. We will solve it as in section 1.4.1, by the unfolding algorithm. Applying D_0 to (1.4.49), by using (1.4.27) and (1.4.45) we find

$$0 = h^{\dot\gamma}_{\alpha_{2s}}\wedge h_{\dot\gamma\alpha_{2s-1}}\wedge D_0C^{\alpha_1\ldots\alpha_{2s}}. \tag{1.4.50}$$

Equation (1.4.50) is solved[13] by

$$D_0C^{\alpha_1\ldots\alpha_{2s}} = -\rho h_{\alpha_{2s+1}\dot\alpha_1}C^{\alpha_1\ldots\alpha_{2s+1}\dot\alpha_1}. \tag{1.4.51}$$

[12] In this section all the Young tableaux are referred to the vector-tensor version of the fields.

[13] In order to see this, it is useful to go back to the vector-tensor formalism and write (1.4.50) as

$$h_{a_s}\wedge h_{b_s}\wedge D_0C^{a_1\ldots a_s,b_1\ldots b_s},$$

that is solved by

$$-\rho h_{a_{s+1}}\wedge C^{a_1\ldots a_{s+1},b_1\ldots b_s} = D_0C^{a_1\ldots a_s,b_1\ldots b_s}.$$

This equation is invariant under the higher spin gauge transformations.

Let us now apply D_0 to (1.4.51):

$$\rho h^{\alpha_{2s}\dot{\gamma}} \wedge h_{\beta\dot{\gamma}} C^{\alpha_1\ldots\alpha_{2s-1}\beta} = h_{\alpha_{2s+1}\dot{\alpha}_1} \wedge D_0 C^{\alpha_1\ldots\alpha_{2s+1}\dot{\alpha}_1}. \tag{1.4.52}$$

Equation (1.4.52) is solved by

$$D_0 C^{\alpha_1\ldots\alpha_{2s+1}\dot{\alpha}_1} = -\rho h_{\alpha_{2s+2}\dot{\alpha}_2} C^{\alpha_1\ldots\alpha_{2s+2}\dot{\alpha}_1\dot{\alpha}_2} - \rho h^{\alpha_{2s}\dot{\alpha}_1} C^{\alpha_1\ldots\alpha_{2s-1}\alpha_{2s+1}}. \tag{1.4.53}$$

If we apply again D_0 to (1.4.53), we get

$$\begin{gathered}\rho h^{\alpha_{2s+1}\dot{\gamma}} \wedge h_{\beta\dot{\gamma}} C^{\alpha_1\ldots\alpha_{2s}\beta\dot{\alpha}_1} + \rho h^{\gamma\dot{\alpha}_1} \wedge h_{\gamma\dot{\beta}} C^{\alpha_1\ldots\alpha_{2s+1}\dot{\beta}} = \\ h_{\alpha_{2s+2}\dot{\alpha}_2} \wedge D_0 C^{\alpha_1\ldots\alpha_{2s+2}\dot{\alpha}_1\dot{\alpha}_2} + \rho h^{\alpha_{2s+1}\dot{\alpha}_1} \wedge h_{\beta\dot{\alpha}_1} C^{\alpha_1\ldots\alpha_{2s-1}\alpha_{2s}\beta\dot{\alpha}_1},\end{gathered}$$

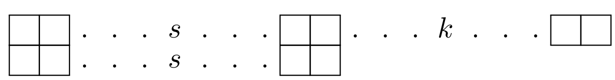
and thus

$$\rho h^{\gamma\dot{\alpha}_1} \wedge h_{\gamma\dot{\beta}} C^{\alpha_1\ldots\alpha_{2s+1}\dot{\beta}} = h_{\alpha_{2s+2}\dot{\alpha}_2} \wedge D_0 C^{\alpha_1\ldots\alpha_{2s+2}\dot{\alpha}_1\dot{\alpha}_2},$$

i.e. an equation of the same form as (1.4.52). We see then that the unfolding procedure goes on for infinitely many steps, in each of which a tensor with one more couple of dotted and undotted indices appears:

$$D_0 C^{\alpha_1\ldots\alpha_{2s+k}\dot{\alpha}_1\ldots\dot{\alpha}_k} = -\rho h_{\alpha_{2s+k+1}\dot{\alpha}_{k+1}} C^{\alpha_1\ldots\alpha_{2s+k+1}\dot{\alpha}_1\ldots\dot{\alpha}_{k+1}} - \rho h^{\alpha_{2s+k}\dot{\alpha}_k} C^{\alpha_1\ldots\alpha_{2s+k-1}\dot{\alpha}_1\ldots\dot{\alpha}_{k-1}}. \tag{1.4.54}$$

These correspond to the descendants of the Weyl tensor encountered in section 1.4.1 and, by (1.4.24) are represented by a Young tableau of the following kind:



(1.4.55)

Our final result is then that to a Fronsdal field Φ with spin s we can associate a vielbein-like field e with $s-1$ dotted and undotted indices. e has some additional fictitious degrees of freedom, that can be eliminated by some gauge transformations that involve e and spin connection-like 1-forms $\omega^{\alpha(k)\dot{\alpha}(l)}$ such that

$$k + l = 2(s-1). \tag{1.4.56}$$

These obey equations that allow to determine systematically every auxiliary field by torsion-like constraints and eventually to define a gauge-invariant Weyl-like tensor $C^{\alpha(2s)}$ whose derivatives are parametrized by the unfolding procedure as a series of tensors $C^{\alpha(k)\dot{\alpha}(l)}$ with

$$|k - l| = 2s. \tag{1.4.57}$$

The Fronsdal equations in AdS can be retrieved by solving the first two equations for the connections, namely (1.4.40) and (1.4.43), to get a differential equation of the second order, which, in the four dimensional notation, reads

$$\Box\Phi^{\mu_1\ldots\mu_s} - m^2\Phi^{\mu_1\ldots\mu_s} - D_0^{(\mu_1} D_{0\nu}\Phi^{\mu_2\ldots\mu_s)\nu} + \frac{1}{2} D_0^{(\mu_1} D_0^{\mu_2}\Phi_\nu{}^{\nu\mu_3\ldots\mu_s)} + 2\Lambda g^{(\mu_1\mu_2}\Phi_\nu{}^{\nu\mu_3\ldots\mu_s)} = 0. \tag{1.4.58}$$

$$m^2 = -\Lambda\left((s-2)(d+s-3)-2\right) \tag{1.4.59}$$

Equation (1.4.58) is not simply obtained by replacing all the derivatives in (1.1.19) with the covariant ones. Indeed, the non-commutativity of the latter implies that one should add a factor $\frac{1}{2}$ to the fourth term in (1.4.58) to get the same amount of terms after symmetrization. Moreover, gauge invariance under (1.4.32) requires a mass-like term m^2 given by (1.4.59) that cancels all the unwanted contributions from commutations of covariant derivatives. This does not imply that the fields are massless. Indeed, since in the AdS space translations are not commutative anymore by (1.3.27), they cannot define a Casimir by $P^\mu P_\mu = m^2$, so that mass is not connected with the irreducible representation of the symmetry group in which the fields we are considering sit. The value of m^2 is fixed by the gauge and space-time symmetries satisfied by the field equation.

The formulation of the Fronsdal theory we described here and in general of higher spin fields is called frame-like, because it is a generalization of the Cartan description of gravitation by local inertial frames e^a_μ. The formalism involving $\Phi_{\mu_1\ldots\mu_s}$, which generalizes the metric field, is called metric-like formulation.

1.4.4 A hidden symmetry

Our analysis of a spin s field produces two sets of equations involving 0- and 1-forms, that take values in the space of completely symmetric spin-tensors. Each set is characterized by equations that share the same form, even though applied to tensors with different ranks. Moreover, there is no particular difference between those involving two different spins. This hints that it is possible to pack them all into two equations involving respectively the ω and C fields for all spins.

Let us thus define

$$\omega(x,y,\bar{y}) = \sum_{\substack{k=0\\l=0}}^{+\infty} \frac{1}{k!l!}\omega^{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}(x)y_{\alpha_1}\cdots y_{\alpha_k}\bar{y}_{\dot{\alpha}_1}\cdots\bar{y}_{\dot{\alpha}_l} \tag{1.4.60}$$

$$C(x,y,\bar{y}) = \sum_{\substack{k=0\\l=0}}^{+\infty} \frac{1}{k!l!}C^{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}(x)y_{\alpha_1}\cdots y_{\alpha_k}\bar{y}_{\dot{\alpha}_1}\cdots\bar{y}_{\dot{\alpha}_l} \tag{1.4.61}$$

$$\epsilon(x,y,\bar{y}) = \sum_{\substack{k=0\\l=0}}^{+\infty} \frac{1}{k!l!}\epsilon^{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}(x)y_{\alpha_1}\cdots y_{\alpha_k}\bar{y}_{\dot{\alpha}_1}\cdots\bar{y}_{\dot{\alpha}_l}, \tag{1.4.62}$$

where y_α and $\bar{y}_{\dot{\alpha}}$ are two auxiliary spinors that have the purpose of incorporating every tensor we encountered into a scalar function of y and $\bar{y}$[14]. It is obvious that these functions are $\mathcal{C}^\infty$ in y and $\bar{y}$ and form a vector space, that we call $\mathscr{F}(y,\bar{y})$. We see then that if one considers all spins together, all the fields that describe them belong to the same vector space $\mathscr{F}(y,\bar{y})$, that may thus be assumed as the fiber space of the higher spin theory. On the other hand, if $f\in\mathscr{F}(y,\bar{y})$, it can be written in its Taylor expansion around $y=\bar{y}=0$ as

$$f(y,\bar{y}) = \sum_{\substack{k=0\\l=0}}^{+\infty} f^{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}y_{\alpha_1}\cdots y_{\alpha_k}\bar{y}_{\dot{\alpha}_1}\cdots\bar{y}_{\dot{\alpha}_l}$$

for some suitable tensor coefficients $f^{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}$. They are symmetric in every (un)dotted index, since, if it were not so, by means of (1.4.23) we could write for example

$$f^{\alpha_1\alpha_2}y_{\alpha_1}y_{\alpha_2} = \left(\frac{1}{2}\varepsilon^{\alpha_1\alpha_2}f^\gamma{}_\gamma + \frac{1}{2}f^{(\alpha_1\alpha_2)}\right)y_{\alpha_1}y_{\alpha_2} = \frac{1}{2}f^{(\alpha_1\alpha_2)}y_{\alpha_1}y_{\alpha_2},$$

[14]These functions must be even under $y,\bar{y}\to -y,-\bar{y}$, namely

$$f(y,\bar{y}) = f(-y,-\bar{y}), \tag{1.4.63}$$

because of (1.4.56) and (1.4.57) and thus belong to a subspace of $\mathscr{F}(y,\bar{y})$. However, for simplicity, we will leave this implicit in the following. It is interesting to notice that this restriction comes from the fact that the vielbeins e^k_l are built from tensors Φ by (1.4.30). If they came from by spin-tensors, namely if we were considering also higher spin fermions, we would have had also odd functions. The resulting theory would not be much different from the one we are going to develop.

and only the symmetric part survives. These tensors span thus a basis for $\mathscr{F}(y,\bar{y})$ and in the four-dimensional formalism, have the symmetry properties of one- and two-row tableaux, as shown by (1.4.25) and (1.4.24).

In this setting[15] equations (1.4.45), (1.4.49) can be reformulated in the following way

$$\mathscr{D}\omega \equiv \mathcal{D}_0\omega - h^{\dot\alpha}_{\beta}\wedge \bar{y}_{\dot\alpha}\frac{\partial}{\partial y_\beta}\omega - h^{\alpha}_{\dot\beta}\wedge y_\alpha \frac{\partial}{\partial \bar{y}_{\dot\beta}}\omega = h^{\dot\gamma}_{\alpha_1}\wedge h_{\dot\gamma\alpha_2}\frac{\partial^2 C(x,y,0)}{\partial y_{\alpha_1}\partial y_{\alpha_2}} + h^{\gamma}_{\dot\alpha_1}\wedge h_{\dot\alpha_2\gamma}\frac{\partial^2 C(x,0,\bar{y})}{\partial \bar{y}_{\dot\alpha_1}\partial \bar{y}_{\dot\alpha_2}}, \quad (1.4.64)$$

where

$$\mathcal{D}_0 = d - \varpi^{\alpha}_{\beta} y_\alpha \frac{\partial}{\partial y_\beta} - \varpi^{\dot\alpha}_{\dot\beta}\bar{y}_{\dot\alpha}\frac{\partial}{\partial \bar{y}_{\dot\beta}}$$

acts as the covariant derivative and $\mathscr{D}$ is a differential operator acting on p-forms that are functions of x, y, and $\bar{y}$. Indeed, (1.4.64) can be expanded in a power series of y and $\bar{y}$ and imposes on each tensorial coefficient the proper equation from (1.4.45).

Let us clarify this point by an example. Consider the contribution from $\frac{1}{2}\omega^{\alpha_1\alpha_2}y_{\alpha_1}y_{\alpha_2}$. The covariant derivative acts directly on the yy term of $\omega(x,y,\bar{y})$

$$\mathcal{D}_0\left(\frac{1}{2}\omega^{\alpha_1\alpha_2}y_{\alpha_1}y_{\alpha_2}\right) = \frac{1}{2}\left(d\omega^{\alpha_1\alpha_2} + \varpi^{\alpha_1\beta}\omega^{\alpha_2}_{\beta} + \varpi^{\alpha_2\beta}\omega^{\alpha_1}_{\beta}\right)y_{\alpha_1}y_{\alpha_2},$$

while derivatives $\frac{\partial}{\partial y_\alpha}$ and multiplications by y select terms with respectively one more and and one less undotted index, so that, in order to get tensors with exactly two undotted indices, we have to refer to the term proportional to $y\bar{y}$:

$$h^{\alpha}_{\dot\beta}\wedge y_\alpha \frac{\partial}{\partial \bar{y}_{\dot\beta}}\left(\omega^{\alpha_1\dot\alpha_2}y_{\alpha_1}\bar{y}_{\dot\alpha_2}\right) = \frac{1}{2}\left(h^{(\alpha_1}_{\dot\beta}\wedge\omega^{\alpha_2)\dot\beta}\right)y_{\alpha_1}y_{\alpha_2}.$$

Finally, $C(x,y,0)$ selects the C tensors with only undotted indices and therefore we recognize (1.4.40) for spin 2 particles in the component of (1.4.64) proportional to $\frac{1}{2}y_{\alpha_1}y_{\alpha_2}$.

Similarly, the gauge transformations (1.4.46) can be encoded into

$$\delta\omega = \mathscr{D}\epsilon = \mathcal{D}_0\epsilon - h^{\dot\alpha}_{\beta}\wedge\bar{y}_{\dot\alpha}\frac{\partial}{\partial y_\beta}\epsilon - h^{\alpha}_{\dot\beta}\wedge y_\alpha\frac{\partial}{\partial\bar{y}_{\dot\beta}}\epsilon \quad (1.4.65)$$

Since $\omega(x,y,\bar{y})$ is associated to a gauge invariance under a parameter that lives in $\mathscr{F}(y,\bar{y})$ too, it should be related to a connection taking values there, so that (1.4.65) plays the role of (1.3.25). We may then see $\omega(x,y,\bar{y})$ as a perturbation of the background AdS connection Ω and write the complete connection $W(x,y,\bar{y})\in\mathscr{F}(y,\bar{y})$ as

$$W(x,y,\bar{y}) \equiv \Omega(x,y,\bar{y}) + \omega(x,y,\bar{y}), \quad (1.4.66)$$

where the first term is the zero order contribution to W (1.3.28) and ω is of first order. We recognize then $\mathscr{D}$ as the zero order action on $\mathscr{F}(y,\bar{y})$ of $D_W = d + W \approx D_\Omega$, the covariant derivative associated to W, and (1.4.60) as the linearization of

$$R_W = dW + W\wedge W = d\Omega + \Omega\wedge\Omega + d\omega + [\Omega,\omega] + O(\omega^2) = d\omega + [\Omega,\omega] + O(\omega^2) \quad (1.4.67)$$

where we used the fact that $R_\Omega = 0$. R_W is then the curvature form, represented by the r.h.s of (1.4.64), where it is thus natural for the Weyl tensor to appear. Moreover, the terms proportional to the background vielbein in (1.4.64) are those that come from the action of the $h^\mu P_\mu$ part of Ω on ω.

Our aim, now, is to find a suitable product that renders $\mathscr{F}(y,\bar{y})$ a Lie algebra, that we call higher spin algebra $\mathfrak{hs}(1,3)$. This algebra must contain as a subalgebra $\mathfrak{so}(2,3)$, defined by M^{ab}, P^a and the commutator relations (1.1.1) and (1.3.27).

[15]Here we put $\rho = 1$, for simplicity of notation.

First of all we need to define a product $\star$ on this space that is not commutative. Given $f(y,\bar{y}), g(y,\bar{y}) \in \mathfrak{hs}(1,3)$, the right choice is

$$f \star g = f \exp\left(-\frac{\overleftarrow{\partial}}{\partial y_\alpha}\varepsilon_{\alpha\beta}\frac{\overrightarrow{\partial}}{\partial y_\beta}\right)\exp\left(\frac{\overleftarrow{\partial}}{\partial \bar{y}_{\dot\alpha}}\varepsilon_{\dot\alpha\dot\beta}\frac{\overrightarrow{\partial}}{\partial \bar{y}_{\dot\beta}}\right) g, \tag{1.4.68}$$

where the arrows indicate on which factor the derivative acts. (1.4.68) defines the Lie bracket on $\mathfrak{hs}(1,3)$ as

$$[f,g]_\star = f\star g - g\star f. \tag{1.4.69}$$

It is clear that this definition transforms elements of $\mathscr{F}(y,\bar{y})$ in other members of that space. (1.4.69) is bilinear, as derivatives and multiplication in (1.4.68) are, while the alternating property follows from the fact that $[\cdot,\cdot]_\star$ is built as a commutator. This also implies the Jacobi identity, because it can be shown that $\star$ is associative. $\mathfrak{hs}(1,3)$ is therefore a Lie algebra and, contrarily to the ones usually used in Physics, is infinite dimensional.

We now show that this definition reproduces the known commutation relations for the $\mathfrak{so}(2,3)$ sub-algebra. If we use the following representation for the $SO(2,3)$ generators[16]

$$M_{\alpha\beta} = \frac{1}{2}y_\alpha y_\beta, \quad M_{\dot\alpha\dot\beta} = -\frac{1}{2}\bar{y}_{\dot\alpha}\bar{y}_{\dot\beta}, \quad P_{\alpha\dot\beta} = \frac{1}{2}y_\alpha\bar{y}_{\dot\beta}$$

$[P_{\alpha\dot\alpha}, P_{\beta\dot\beta}]_\star$ reads

$$\begin{aligned}\frac{1}{4}y_\alpha\bar{y}_{\dot\alpha}\star y_\beta\bar{y}_{\dot\beta} - \frac{1}{4}y_\beta\bar{y}_{\dot\beta}\star y_\alpha\bar{y}_{\dot\alpha} &= \frac{1}{4}y_\alpha\bar{y}_{\dot\alpha}y_\beta\bar{y}_{\dot\beta} - \frac{1}{4}y_\beta\bar{y}_{\dot\beta}y_\alpha\bar{y}_{\dot\alpha} + \frac{1}{4}\varepsilon_{\dot\alpha\dot\beta}y_\alpha y_\beta - \frac{1}{4}\varepsilon_{\alpha\beta}\bar{y}_{\dot\alpha}\bar{y}_{\dot\beta}\\ &\quad -\frac{1}{4}\varepsilon_{\dot\beta\dot\alpha}y_\beta y_\alpha - \frac{1}{4}\varepsilon_{\beta\alpha}\bar{y}_{\dot\beta}\bar{y}_{\dot\alpha} - \frac{1}{4}\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} + \frac{1}{4}\varepsilon_{\dot\beta\dot\alpha}\varepsilon_{\beta\alpha}\\ &= \varepsilon_{\alpha\beta}M_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}M_{\alpha\beta}\end{aligned}$$

while

$$\begin{aligned}[P_{\alpha\dot\alpha}, M_{\dot\beta\dot\gamma}]_\star &= -\frac{1}{4}y_\alpha\bar{y}_{\dot\alpha}\star\bar{y}_{\dot\beta}\bar{y}_{\dot\gamma} + \frac{1}{4}\bar{y}_{\dot\beta}\bar{y}_{\dot\gamma}\star y_\alpha\bar{y}_{\dot\alpha}\\ &= -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}y_\alpha\bar{y}_{\dot\gamma} - \frac{1}{2}\varepsilon_{\dot\alpha\dot\gamma}y_\alpha\bar{y}_{\dot\beta}\\ &= -\varepsilon_{\dot\alpha\dot\beta}P_{\alpha\dot\gamma} - \varepsilon_{\dot\alpha\dot\gamma}P_{\alpha\dot\beta}\end{aligned}$$

and similarly

$$[P_{\alpha\dot\alpha}, M_{\beta\gamma}]_\star = -\varepsilon_{\alpha\beta}P_{\gamma\dot\alpha} - \varepsilon_{\alpha\gamma}P_{\beta\dot\alpha}.$$

They correspond to (1.4.28) and (1.4.29) and therefore, also the $[M,M]$ commutators are correctly reproduced thanks to the Jacobi identity.

In order to use Ω as a connection we need the following bracket:

$$[P_{\alpha\dot\alpha}, f(y,\bar{y})]_\star = -\bar{y}_{\dot\alpha}\varepsilon_{\alpha\beta}\frac{\partial f}{\partial y_\beta} + y_\alpha\varepsilon_{\dot\alpha\dot\beta}\frac{\partial f}{\partial\bar{y}_{\dot\beta}}.$$

We then see that we can rewrite $\mathscr{D}$ as[17]

$$\mathscr{D}f = df + \varpi^{\alpha\beta}\left[M_{\alpha\beta}, f\right]_\star + \varpi^{\dot\alpha\dot\beta}\left[M_{\dot\alpha\dot\beta}, f\right]_\star + h^{\alpha\dot\alpha}\left[P_{\alpha\dot\alpha}, f\right]_\star = df + [\Omega, f]_\star\,,$$

[16]Their form is determined by the indices they carry.

[17]It is important to remember that for general p-forms, the wedge products give additional signs to the star products and this can transform some commutators in anti-commutators. For example, $\varpi^{\alpha\beta}\left[M_{\alpha\beta},\omega\right]_\star = \{\varpi,\omega\}_\star$.

i.e. the first order approximation of the covariant derivative associated to $\mathfrak{hs}(1,3)$ acting on p-forms with values in $\mathfrak{hs}(1,3)$.

Let us now write the equations associated to C (1.4.54) in the new formalism:

$$\tilde{\mathscr{D}}C = \mathcal{D}_0 C + h_{\alpha\dot\alpha}\frac{\partial^2 C}{\partial y_\alpha \partial \bar{y}_{\dot\alpha}} + h^{\dot\alpha\alpha}\bar{y}_{\dot\alpha}y_\alpha C = 0. \tag{1.4.70}$$

We see that this time we had to define a different operator on $\mathfrak{hs}(1,3)$ $\tilde{\mathscr{D}}$, that acts differently in the part involving $P_{\alpha\dot\alpha}$. However, if we consider the anti-commutator between P and some $f \in \mathfrak{hs}(1,3)$,

$$\{P_{\alpha\dot\alpha}, f\}_\star = y_\alpha \bar{y}_{\dot\alpha} f - \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\frac{\partial^2 f}{\partial \bar{y}_{\dot\beta}\partial y_\beta},$$

we find that

$$\tilde{\mathscr{D}}f = df + \varpi^{\alpha\beta}\left[M_{\alpha\beta}, f\right]_\star + \varpi^{\dot\alpha\dot\beta}\left[M_{\dot\alpha\dot\beta}, f\right]_\star + \rho h^{\alpha\dot\alpha}\left\{P_{\alpha\dot\alpha}, f\right\}_\star . \tag{1.4.71}$$

The representation in which Ω acts in (1.4.71) is called twisted adjoint representation and can be realized as

$$\Omega \star f - f \star \pi(\Omega),$$

where $\pi : \mathfrak{hs}(1,3) \to \mathfrak{hs}(1,3)$ is an automorphism of the higher spin algebra defined by

$$\pi(f(y,\bar{y})) = f(-y,\bar{y}). \tag{1.4.72}$$

We can thus interpret (1.4.70) as the Bianchi identity of the curvature C.

In light of these results we understand now why it is necessary to have infinite spins in the theory, as hinted by one of the yes-go examples in subsection 1.2.2: $\mathfrak{h}(1,3)$ does not have any finite subalgebra besides $\mathfrak{so}(2,3)$.

1.5 Turning on interactions: the Vasiliev equations

As for now, we treated all higher spin fields as perturbations around an AdS background, so that the resulting equations of motion are linear. Here we present their extension to a full non-linear theory, which was found by Vasiliev in [37] for the four dimensional case and later extended to any dimension in [38].

In general, the full non-linear equations of motion should be of the same form of (1.4.67) and (1.4.70):

$$dW + W * W = R_W \tag{1.5.1}$$
$$dR_W + W * R_W - R_W * \pi(W) = 0, \tag{1.5.2}$$

where (1.5.1) encodes the higher spin curvature 2-form associated with the connection 1-form[18]. An integrability check, namely the verification that (1.5.1) is consistent with $d^2 \equiv 0$ gives

$$dW * W - W * dW = dR_W.$$

Inserting (1.5.1) in this expression, one gets, remembering that the representation of R_W is the twisted adjoint one,

$$dR_W = R_W * \pi(W) - W * W * W - W * R_W + W * W * W,$$

i.e. (1.5.2), the Bianchi identity for the curvature.

[18] Here the $*$ product incorporates also the $\wedge$ product.

The curvature was implemented in the previous subsection as a 0-form C that was promoted to a 2-forms by the two vielbeins contracted with y derivatives in (1.4.64). This complicates the treatment in the interacting case, because h is part of W.

A solution to this problem is to provide our space with additional spinorial coordinates z^α and $\bar{z}^{\dot{\alpha}}$, to which the 1-forms $dz^\alpha, d\bar{z}^{\dot{\alpha}}$ are associated. These coordinates, along with y and $\bar{y}$, are called twistors. We can use them to produce 2-forms

$$dz^\alpha \wedge dz_\alpha, \quad d\bar{z}^{\dot{\alpha}} \wedge d\bar{z}_{\dot{\alpha}}. \tag{1.5.3}$$

with which we can construct the curvature without the necessity of derivatives in y, since the 2-forms (1.5.3) carry no indices. We then define

$$R_W = R + \bar{R} \equiv B(x,y,z) dz^\alpha \wedge dz_\alpha + B(x,y,z) d\bar{z}^{\dot{\alpha}} \wedge d\bar{z}_{\dot{\alpha}} \tag{1.5.4}$$

by a 0-form $B(x,y,z)$ on the extended space. Also the connection has to be coherently replaced with a new one that depends also on $z, \bar{z}$:

$$W(x,y) \to \mathcal{W}(x,y,z) \equiv \mathcal{W}^\mu_x dx_\mu + \mathcal{A}^\alpha dz_\alpha + \bar{\mathcal{A}}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}}. \tag{1.5.5}$$

The differential d must similarly be extended to $d \equiv d_x + d_z + d_{\bar{z}}$, where we denote the one acting on x as d_x and those on the coordinates z and $\bar{z}$ as $d_z, d_{\bar{z}}$. Finally, we define

$$\pi(f(y,\bar{y},z,\bar{z})) \equiv f(-y,\bar{y},-z,\bar{z}), \quad \bar{\pi}(f(y,\bar{y},z,\bar{z})) \equiv f(y,-\bar{y},z,-\bar{z}), \tag{1.5.6}$$

the generalization of the twisted automorphism (1.4.72) and its complex conjugate.

In order for (1.5.1) and (1.5.2) to make sense, it is necessary to give a definition for the Weyl product that involves also z and $\bar{z}$ and that we denote by $\star$. It turns out that $z, \bar{z}$ should be dual to $y, \bar{y}$, namely

$$[z,y]_\star = [\bar{z},y]_\star = 0, \quad \left[z^\alpha, \bar{z}^\beta\right]_\star = 0, \quad \left[z^\alpha, z^\beta\right]_\star = 2\varepsilon^{\alpha\beta}, \quad \left[\bar{z}^{\dot{\alpha}}, \bar{z}^{\dot{\beta}}\right]_\star = -2\varepsilon^{\dot{\alpha}\dot{\beta}}$$

whereas (1.4.68) gives

$$\left[y^\alpha, \bar{y}^\beta\right] = 0, \quad \left[y^\alpha, y^\beta\right] = -2\varepsilon^{\alpha\beta}, \quad \left[\bar{y}^{\dot{\alpha}}, \bar{y}^{\dot{\beta}}\right] = 2\varepsilon^{\dot{\alpha}\dot{\beta}}.$$

The correct extension of (1.4.68) is thus

$$f \star g = f \exp\left(-\left(\frac{\overleftarrow{\partial}}{\partial y_\alpha} + \frac{\overleftarrow{\partial}}{\partial z_\alpha} \right) \varepsilon_{\alpha\beta} \left(\frac{\overrightarrow{\partial}}{\partial y_\beta} - \frac{\overrightarrow{\partial}}{\partial z_\beta} \right) + \left(\frac{\overleftarrow{\partial}}{\partial \bar{y}_{\dot{\alpha}}} + \frac{\overleftarrow{\partial}}{\partial \bar{z}_{\dot{\alpha}}} \right) \varepsilon_{\dot{\alpha}\dot{\beta}} \left(\frac{\overrightarrow{\partial}}{\partial \bar{y}_{\dot{\beta}}} - \frac{\overrightarrow{\partial}}{\partial \bar{z}_{\dot{\beta}}} \right) \right) g, \tag{1.5.7}$$

where $f \equiv f(y,\bar{y},z,\bar{z})$ and $g \equiv g(y,\bar{y},z,\bar{z})$

By (1.5.7) we can realize the automorphisms $\pi, \bar{\pi}$ in (1.5.6) as[19]

$$\begin{gathered} \pi(f) = \kappa \star f \star \kappa, \quad \bar{\pi}(f) = \bar{\kappa} \star f \star \bar{\kappa} \\ \kappa \star \kappa = 1 = \bar{\kappa} \star \bar{\kappa} \\ \kappa \equiv e^{y^\alpha z_\alpha}, \quad \bar{\kappa} \equiv e^{\bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}}. \end{gathered} \tag{1.5.8}$$

[19]In order to prove these identities it is better to employ an alternative definition of $\star$, equivalent to (1.5.7):

$$f \star g = \int d^2u d^2\bar{u} d^2v d^2\bar{v} f(y+u, \bar{y}+\bar{u}, z+u, \bar{z}+\bar{u}) g(y+v, \bar{y}+\bar{v}, z-v, \bar{z}-\bar{v}) e^{u^\alpha v_\alpha - \bar{u}^{\dot{\alpha}} \bar{v}_{\dot{\alpha}}},$$

where integration is performed along $(1+i)\mathbb{R}$ for u, v and $(1-i)\mathbb{R}$ for $\bar{u}, \bar{v}$ for convergence reasons.

Then the proper equations for the interacting theory read

$$\begin{aligned} d\mathcal{W} + \mathcal{W} \star \mathcal{W} &= R + \bar{R} \\ dR + \mathcal{W} \star R - R \star \pi(\mathcal{W}) &= 0 \\ d\bar{R} + \mathcal{W} \star \bar{R} - \bar{R} \star \bar{\pi}(\mathcal{W}) &= 0 \end{aligned} \tag{1.5.9}$$

or, by defining $\mathcal{R} \equiv R \star \kappa, \bar{\mathcal{R}} \equiv \bar{R} \star \bar{\kappa}$, $\star$-multiplying the last two equations in (1.5.9) by κ[20] and using (1.5.8)

$$\begin{aligned} d\mathcal{W} + \mathcal{W} \star \mathcal{W} &= \mathcal{R} \star \kappa + \bar{\mathcal{R}} \star \bar{\kappa} \\ d\mathcal{R} + [\mathcal{W}, \mathcal{R}]_\star &= 0 \\ d\bar{\mathcal{R}} + \left[\mathcal{W}, \bar{\mathcal{R}}\right]_\star &= 0 \end{aligned} \tag{1.5.10}$$

The connection $\mathcal{W}$ can be gauge transformed by

$$\delta\mathcal{W} = d\epsilon + [\mathcal{W}, \epsilon]_\star, \quad \epsilon \equiv \epsilon(x, y, \bar{y}, z, \bar{z})$$

and the related transformation of B reads

$$\delta B = B \star \pi(\epsilon) - \epsilon \star B.$$

Equations (1.5.10) acquire a simpler form under the redefinition

$$\mathcal{A}^\alpha = \frac{1}{2}\left(S^\alpha + z^\alpha\right), \quad \bar{\mathcal{A}}^{\dot{\alpha}} = \frac{1}{2}\left(\bar{S}^{\dot{\alpha}} + \bar{z}^{\dot{\alpha}}\right),$$

that allows us to eliminate the derivatives on $z, \bar{z}$ coming from the differentials by

$$[f, z^\alpha]_\star = -2\partial_z^\alpha f$$

and its complex conjugate. The result of this operation is

$$\begin{aligned} &d_x\mathcal{W}_x + \mathcal{W}_x \star \mathcal{W}_x = 0 && d_x B + \mathcal{W}_x \star B - B \star \pi(\mathcal{W}_x) = 0 \\ &d_x S^\alpha + [\mathcal{W}_x, S^\alpha]_\star = 0 && \left[S^\alpha, S^\beta\right] = 4\varepsilon^{\alpha\beta}\left(1 + B \star \kappa\right) \\ &[S^\alpha, B \star \bar{\kappa}] = 0 \\ &\left[S^\alpha, \bar{S}^{\dot{\alpha}}\right] = 0 \end{aligned} \tag{1.5.11}$$

and their complex conjugates.

Until now we did not impose conditions (1.4.56) and (1.4.57) that insured the fact that $\mathcal{W}$ and B describe bosons, namely

$$\mathcal{W}(-y, -\bar{y}, -z, -\bar{z}) = \mathcal{W}(y, \bar{y}, z, \bar{z}), \quad B(-y, -\bar{y}, -z, -\bar{z}) = B(y, \bar{y}, z, \bar{z}), \tag{1.5.12}$$

(see (1.4.63) in a note of the previous subsection). By (1.5.8) we can write (1.4.63) in a simple way:

$$\kappa \star \bar{\kappa} \star \mathcal{W} \star \bar{\kappa} \star \kappa = \mathcal{W}, \quad \kappa \star \bar{\kappa} \star B \star \bar{\kappa} \star \kappa = B. \tag{1.5.13}$$

[20]Notice that from

$$dz^\alpha \wedge dz^\beta \wedge dz^\gamma \equiv 0, \quad d_x\kappa = 0$$

follows

$$dR \star \kappa = d(R \star \kappa)$$

and the analogous statement for the conjugated quantities.

Equation (1.5.13) for S^α, which is present in (1.5.5) multiplied to dz^α, is equivalent to the following condition:

$$\kappa \star \bar{\kappa} \star S^\alpha \star \bar{\kappa} \star \kappa = -S^\alpha. \tag{1.5.14}$$

We then get from the third of (1.5.11)

$$\begin{aligned} 0 &= S^\alpha \star B \star \bar{\kappa} \star \kappa - B \star \bar{\kappa} \star S^\alpha \star \kappa \\ 0 &= S^\alpha \star B \star \bar{\kappa} \star \kappa + B \star \kappa \star S^\alpha \star \kappa \star \bar{\kappa} \star \kappa \\ 0 &= \{S^\alpha, B \star \kappa\}_\star , \end{aligned} \tag{1.5.15}$$

where we used (1.5.14) in the second step and $\star$-multiplied both sides for $\kappa \star \bar{\kappa} \star \kappa$ in the third one. System (1.5.11) supplemented by constraint (1.5.15) constitute the Vasiliev equations, that describe non-linearly interacting higher spin fields.

It is possible to generalize these equations, for example by substituting B in the first of (1.5.9) with $f(B)$ for some analytic odd function f that can be expanded in a $\star$-product series. It can be shown that the only theories that are parity-invariant are, up to redefinitions, those with $f \equiv 1, i$. They are called respectively A and B type. However, in the following we will be interested in a model that is simpler than the one presented so far, namely the restriction to even spins. The latter is achieved by defining the inversions $\iota_\pm$

$$\iota_\pm(f(y,\bar{y},z,\bar{z})) \equiv f(iy, \pm i\bar{y}, -iz, \mp i\bar{z}), \quad \iota(f \star g) = \iota(g) \star \iota(f)$$

and imposing

$$\iota_+(\mathcal{W}) = -\mathcal{W}, \quad \iota_-(B) = B. \tag{1.5.16}$$

We can understand why it is so by examining the linearized case, where ω, the first order contribution to $\mathcal{W}$, was restricted to contain in its expansion only terms of the kind

$$(y)^k (\bar{y})^l , \quad k + l = 2s - 2$$

for a spin s by (1.4.56). Under ι_+ these terms transform as

$$(y)^k (\bar{y})^l \to \begin{cases} (y)^k (\bar{y})^l & k+l = 4j \\ -(y)^k (\bar{y})^l & k+l = 4j-2 \end{cases}, \quad j \in \mathbb{N}$$

and we see that we fall in the first case for odd spins $s = 2j + 1$ and in the second for $s = 2j$, for which then $\omega \to -\omega$. Similarly, C, the linearized version of B, was allowed by (1.4.57) to contain only terms with $|k - l| = 2s$ and, under ι_- they transform as

$$(y)^k (\bar{y})^l \to \begin{cases} (y)^k (\bar{y})^l & k-l = 4j \\ -(y)^k (\bar{y})^l & k-l = 4j-2 \end{cases}, \quad j \in \mathbb{N}$$

and we get $C \to C$ for even spins.

Let us conclude with a comment on the twistorial extension of the space. Notice that the first equation of (1.5.11) is a zero curvature condition

$$d_x \mathcal{W}_x + \mathcal{W}_x \star \mathcal{W}_x = 0 \tag{1.5.17}$$

and thus its solution can be put in the following form:

$$g^{-1}(x, y, \bar{y}) \star d_x g(x, y, \bar{y}),$$

for some arbitrary function g. Indeed, from

$$d_x \left(g^{-1} \star g\right) = 0$$

we get

$$dg^{-1} = -g^{-1} \star d_x g \star g^{-1}$$

and then (1.5.17) follows directly. These solutions are called pure gauge, because there exists a gauge transformation that makes them vanish[21], namely the one with parameter g: using (1.3.22) we get indeed

$$\begin{aligned} \mathcal{W}'_x &= g \star \mathcal{W}_x \star g^{-1} - dg \star g^{-1} \\ &= g \star g^{-1} \star dg \star g^{-1} - dg \star g^{-1} \\ &= 0. \end{aligned}$$

These transformations do not eliminate the dynamical degrees of freedom, that are just shifted in the twistorial space. Its function is then to encode part of the physics in order to render the Vasiliev equations as simple as possible.

[21]This is a general fact: no curvature means that the space is $\mathcal{W}_x$-flat and therefore describable by a vanishing connection.

Chapter 2

AdS/CFT for higher spin theories

This chapter is dedicated to the Klebanov-Polyakov conjecture that relates the Vasiliev higher spin field theory in AdS_4 with a conformal quantum field theory of scalars in a flat 3-dimensional space. In particular, correlation functions of the latter can be found by computations that involve the fields of the former. This correspondence between two field theories, one classical living in an AdS_{d+1} space and the other quantized, set in a d-dimensional flat space, is just an example of a more general class of conjectures that have been verified in numerous examples but not yet proven. The aim of this thesis is to present a test of the Klebanov-Polyakov conjecture.

In the first section, we give a short review of the first and best known of the so-called AdS/CFT conjectures, the Maldacena correspondence. We will focus on the most salient features and we will specify the formalism that realizes the duality in the second section. Then, we will present in 2.2 a field theory that shows some remarkable properties that will lead us to the formulation of the Klebanov-Polyakov conjecture. We dedicate the last part of the chapter to some checks of this duality.

2.1 The original Maldacena conjecture

The Maldacena conjecture is a duality between a classical supergravity (or, more generally, string) theory set in $\mathrm{AdS}_5 \times S^5$ space and a supersymmetric quantum theory of gauge fields living in 4 dimensions and possessing conformal symmetry. For this reason, in the next two sections we will treat some basic aspects of conformal theories, that will be later investigated in more detail in chapter 3, and the geometric properties of the AdS space.

Since the duality stems from string theory, in subsection 2.1.3 we give a very brief introduction to strings that will be focused on the aspects that brought Maldacena to formulate his conjecture, which will be finally stated in the last subsection.

2.1.1 Conformal symmetry

In the following, we will consider theories in flat space that share an enhanced symmetry: conformal invariance. Conformal Field Theories are generally labeled CFT_d, where d stands for the dimensions of the space in which they are set.

Let us start with the simplest example of a CFT: a free theory of massless scalars in d dimensions:

$$S_{\varphi free} = \int \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi d^dx. \tag{2.1.1}$$

The field φ has the mass dimension $\Delta = \frac{d-2}{2}$. Now, if we rescale all lengths according to

$$x \to \lambda x, \tag{2.1.2}$$

we expect that

$$\varphi(x) \to \varphi'(x') = \lambda^{-\Delta}\varphi(x) \tag{2.1.3}$$

and therefore

$$\begin{aligned} S'_{\varphi free} &= \int \frac{1}{2}\partial'_\mu\varphi'(x')\partial'^\mu\varphi'(x')d^dx' \\ &= \int \frac{1}{2}\lambda^{2\frac{2-d}{2}-2}\partial_\mu\varphi(x)\partial^\mu\varphi(x)\lambda^d d^dx \\ &= S_{\varphi free}, \end{aligned}$$

namely the free theory is invariant under rescalings. More in general, if $\varphi(x)$ is a generic field sitting in some arbitrary irreducible representation of the Poincaré group, the generic action that describes it is

$$S[\varphi] = \int \mathcal{L}(\varphi(x), \partial\varphi(x))d^dx. \tag{2.1.4}$$

After a rescaling (2.1.2), (2.1.4) takes the form

$$S'[\varphi'] = \int \mathcal{L}(\varphi'(\lambda x), \lambda^{-1}\partial\varphi'(\lambda x))\lambda^d d^dx.$$

Since we consider only local actions, namely those that are polynomial in the field and its derivatives, it is clear that in order for (2.1.4) to be invariant under rescalings, the λ factors arising from derivatives and the measure of integration must be compensated by the transformation law of φ, which has to rescale according to equation (2.1.3). In this context, we call conformal dimension (or weight) of φ the parameter Δ appearing in (2.1.3). For example, in an interaction term like

$$\int d^dx g\varphi^p(x),$$

the power p is fixed by conformal invariance and the weight Δ as

$$p = \frac{d}{\Delta}. \tag{2.1.5}$$

Scaling invariance is often associated to a broader symmetry, that involves transformations whose result is to rescale the metric of a space by a factor that may depend on the point.

$$g_{\mu\nu}(x) \to c(x)g_{\mu\nu}(x), \iff \frac{\partial x^\mu}{\partial x'^\alpha}g_{\mu\nu}(x)\frac{\partial x^\nu}{\partial x'^\beta} = c(x)g_{\alpha\beta}(x) \tag{2.1.6}$$

Transformations of this kind include rescalings $x^\mu \to \lambda x^\mu$, for which $c(x) = \lambda^2$ and Lorentz transformations, for which $c(x) = 1$. A less trivial example is the inversion $x^\mu \to \frac{x^\mu}{x^2}$ that acts on $g_{\mu\nu}$ as

$$\begin{aligned} g'_{\alpha\beta}(x') &= \frac{\partial\left(\frac{x'^\mu}{x'^2}\right)}{\partial x'^\alpha}g_{\mu\nu}(x)\frac{\partial\left(\frac{x'^\nu}{x'^2}\right)}{\partial x'^\beta} \\ &= \left(\frac{\delta^\mu_\alpha}{x^2} - 2x^\mu x_\alpha\right)g_{\mu\nu}(x)\left(\frac{\delta^\nu_\beta}{x^2} - 2x^\nu x_\beta\right) \\ &= \frac{1}{(x^2)^2}g_{\alpha\beta}(x). \end{aligned} \tag{2.1.7}$$

These symmetries form the Conformal group $Conf(1, d-1)$, the in-depth analysis of which we postpone to section 3.1. Here we just anticipate that

$$Conf(1, d-1) \simeq SO(2, d) \tag{2.1.8}$$

(see subsection 3.1.2).

Fields are called primary if under the action $x \to x'(x)$ of the conformal group they transform as follows:

$$\varphi(x) \to \left|\det \frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \varphi(x). \tag{2.1.9}$$

Equation (2.1.9) generalizes (2.1.3). The free scalar theory presented at the beginning is a typical example in which scale invariance is extended to a full conformal invariance. Taking the determinant of (2.1.6) we see that

$$c(x) = \left|\det \frac{\partial x}{\partial x'}\right|^{\frac{2}{d}}$$

and therefore the transformation of (2.1.1) under (2.1.9) can be written as

$$\begin{aligned} S'_{\varphi free} &= \int \frac{1}{2}\partial'_\mu\varphi'(x')\partial'^\mu\varphi'(x')d^dx' \\ &= \int \frac{1}{2}c^{-1+\frac{d}{2}}\eta^{\mu\nu}\frac{\partial x^\alpha}{\partial x'^\mu}\partial_\alpha\varphi\frac{\partial x^\beta}{\partial x'^\nu}\partial_\beta\varphi c^{-\frac{d}{2}}d^dx. \qquad (2.1.10) \\ & \int \frac{1}{2}\left(2c^{\frac{d-2}{4}}\varphi\partial'_\mu c^{\frac{d-2}{4}}\partial'^\mu\varphi + \varphi^2\partial'_\mu c^{\frac{d-2}{4}}\partial'^\mu c^{\frac{d-2}{4}}\right)d^dx' \qquad (2.1.11) \end{aligned}$$

If we now consider the matrix

$$\Lambda^\alpha{}_\mu \equiv \frac{1}{\left|\det \frac{\partial x}{\partial x'}\right|^{\frac{1}{d}}}\frac{\partial x^\alpha}{\partial x'^\mu},$$

we see from (2.1.6), using $g \equiv \eta$, that Λ is a Lorentz transformation by definition

$$\Lambda^\alpha{}_\mu\eta_{\alpha\beta}\Lambda^\beta{}_\nu = \eta_{\mu\nu}$$

and thus line (2.1.10) coincides with $S_{\varphi free}$ in (2.1.1). For $S'_{\varphi free}$ to be conformally invariant, then, line (2.1.11) should vanish. Let us perform an integration by parts of the first term to get

$$(2.1.11) = \int \frac{1}{2}\left(-c^{\frac{d-2}{4}}\varphi^2\partial'^\mu\partial'_\mu c^{\frac{d-2}{4}}\right)d^dx'$$

and we see that (2.1.11) is zero if

$$\partial'^\mu\partial'_\mu c^{\frac{\Delta}{2}} = 0. \tag{2.1.12}$$

We see that the so-called Weyl factor c is constrained by (2.1.12). We will find a condition equivalent to (2.1.12) when we will analyze in full generality the conformal group[1].

These considerations allow us also to better understand (2.1.9) in the case of tensors:

$$\begin{aligned} \varphi_{\mu_1..\mu_k}(x) &\to \left|\det \frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}}\cdots\frac{\partial x^{\nu_k}}{\partial x'^{\mu_k}}\varphi_{\mu_1..\mu_k}(x) \\ &= \left|\det \frac{\partial x'}{\partial x}\right|^{-\frac{\Delta+k}{d}} \Lambda^{\nu_1}{}_{\mu_1}\cdots\Lambda^{\nu_1}{}_{\mu_1}\varphi_{\mu_1..\mu_k}(x) \qquad (2.1.13) \end{aligned}$$

and we see that conformal transformations act as rescaling plus a Lorentz transformation due to the tensorial nature of the object.

[1] Indeed equation (3.1.9) is just (2.1.12) when one considers infinitesimal transformations $x'^\mu = x^\mu + \varepsilon^\mu$ and $c(x) = 1 - \frac{2}{d}\partial\cdot\varepsilon$, namely

$$\frac{\Delta}{d}\partial^\mu\partial_\mu\partial\cdot\varepsilon = 0.$$

2.1.2 The Anti-de Sitter space

In presence of a negative cosmological constant Λ, a maximally symmetric solution of the Einstein equations in $d+1$ dimensions[2] in vacuum

$$R^{MN} - \frac{1}{2} g^{MN} R = \Lambda g^{MN}$$

can be found by embedding the $d+1$ dimensional space-time into a $d+2$ flat space as a (pseudo)hyperboloid that obeys

$$\left(X^0\right)^2 + \left(X^{d+1}\right)^2 - \sum_{i=1}^{d} \left(X^i\right)^2 = L^2 \tag{2.1.14}$$

where X^{d+1} is an additional coordinate and L is the curvature radius related to Λ by

$$\Lambda = -\frac{(d-1)\,d}{2L^2}.$$

This solution[3] is called Anti de Sitter space AdS$_{d+1}$. As we have shown in section 1.3.3 the isometry group of AdS$_{d+1}$ is $SO(2,d)$. Note that the antipodal map, that acts by sending X^M, X^{d+1} into $-X^M, -X^{d+1}$, belongs to this group.

Our goal is to find a suitable set of coordinates to describe the AdS space that makes the metric form as simple as possible. In general the AdS metric is given by the pull-back of the flat $(d+2)$-dimensional metric

$$ds^2 = \left(dX^0\right)^2 + \left(dX^{d+1}\right)^2 - \sum_{i=1}^{d} \left(dX^i\right)^2, \tag{2.1.15}$$

i.e. by expressing one of the coordinates as a function of the other by means of (2.1.14) and substituting this expression into (2.1.15). However all the coordinates in (2.1.14) appear quadratically, so that each of them is related to the others by a square root, which complicates all computations. Due to the time-like nature of the new coordinate X^{d+1}, it proves useful to employ "light-cone" coordinates

$$u \equiv X^d + X^{d+1} \quad v \equiv -X^d + X^{d+1}. \tag{2.1.16}$$

Indeed, (2.1.14) becomes

$$uv + \left(X^0\right)^2 - \sum_{i=1}^{d-1} \left(X^i\right)^2 = L^2$$

and solving for v now involves no square root

$$v = \frac{L^2 - \left(X^0\right)^2 + \sum_{i=1}^{d-1} \left(X^i\right)^2}{u}. \tag{2.1.17}$$

[2] From now on we will employ capital letters like M, N to denote indices of tensors in AdS$_{d+1}$ and Greek letters as μ, ν, ρ for indices that can go only from 0 to $d-1$. Analogously, the $(d+1)$-dimensional coordinates are denoted by X^M, while the d-dimensional ones are x^μ. These conventions are listed in the "Notation" appendix.

[3] Actually (2.1.14) features time-like closed world-lines, which would allow to a particle to travel backwards in time. This is not physically acceptable, because it spoils causality. The existence of these curves can be seen by considering a curve $\gamma(\tau)$ in the $(d+2)$-dimensional space which obeys $\left(\gamma^0\right)^2 + \left(\gamma^{d+1}\right)^2 = L^2$ and $\gamma^M \equiv 0$ for $M \neq 0$, namely travels on a circle. The tangent vectors of γ are time-like and, from (2.1.14) it follows that $\gamma(\tau)$ is in AdS$_{d+1}$. In general (2.1.14) tells us that topologically AdS$_{d+1}$ is $S^1 \times \mathbb{R}^d$ and therefore has a non-trivial first group of homotopy, which corresponds to closed time-like loops. A solution to this issue is to take the universal cover of this space, that is then considered to be the true physical solution to (2.1.14), since locally it is indistinguishable from the pseudohyperboloid.

Equation (2.1.17) is not well defined for $u = 0$ and therefore we have to assume either $u > 0$ or $u < 0$. Since AdS is naturally divided in two patches related by the antipodal map, we would like to identify them according to the sign of u. This can be done by performing a rescaling on X^μ

$$X^\mu \equiv \frac{u}{L} x^\mu \quad 0 \leq \mu \leq d-1,$$

so that the antipodal map is realized by $u \to -u$. From now on we restrict ourselves to $u > 0$ and use the standard flat metric $\eta_{\mu\nu}$ to contract x^μ. Using

$$dv = \left(-\frac{L^2}{u^2} - \frac{x^\mu x_\mu}{L^2} \right) du - \frac{2u}{L^2} x^\mu dx_\mu$$

the pull-back of (2.1.15) is then

$$\begin{aligned} ds^2 &= dudv + \left(\frac{u}{L} dx^\mu + \frac{x^\mu}{L} du \right) \left(\frac{u}{L} dx_\mu + \frac{x_\mu}{L} du \right) \\ &= \left(-\frac{L^2}{u^2} - \frac{x^\mu x_\mu}{L^2} \right) du^2 - \frac{2u}{L^2} x^\mu dx_\mu du + \frac{u^2}{L^2} dx^\mu dx_\mu + 2\frac{x^\mu}{L^2} u dx_\mu du + \frac{x^\mu x_\mu}{L^2} du^2 \\ &= -\frac{L^2}{u^2} du^2 + \frac{u^2}{L^2} dx^\mu dx_\mu \end{aligned} \tag{2.1.18}$$

It is also useful to define an alternative form of (2.1.18) by the change of variable

$$z \equiv \frac{L^2}{u}, \quad z > 0,$$

The resulting metric is

$$ds^2 = \frac{L^2}{z^2}(-dz^2 + dx^\mu dx_\mu) \tag{2.1.19}$$

We will refer to (z, x^μ) as Poincaré patch coordinates.

Metric (2.1.19) is said to be conformally flat because it can be turned into a flat metric by a local rescaling and therefore by a conformal transformation. The spatial infinity, namely $X^i \to \infty$ $(i = 1, ..., d)$ or $X^{d+1} \to \infty$ by (2.1.14), is situated at $u \to \pm\infty$ by (2.1.16) or $z \to 0$. In this limiting point, (2.1.19) assumes the form of the usual d-dimensional flat metric and therefore the boundary at infinity of the AdS space is given by the d-dimensional flat space. This is the geometrical ground for the AdS$_{d+1}/CFT_d$ correspondence.

2.1.3 Some basic facts about superstrings and branes

The Maldacena conjecture has its roots in Superstring theory. Strings are a generalization of point particles, in that they are 1-dimensional physical objects that one can picture as short deformable lines. At fixed time, they are described in some space-time with dimension D $\mathcal{M}^D$ by the coordinates X^m of each of their points, identified by some parameter $\sigma \in [0, 1]$. During their motion, they span a 2-dimensional manifold in space, called world-sheet, that generalizes the world line. Therefore we need an additional parameter that plays the role of proper time for the string: τ. The world-sheet is thus determined by a function $X :\to [0, 1] \times \mathbb{R} \to \mathcal{M}^D$:

$$X^m(\sigma, \tau). \tag{2.1.20}$$

There are two types of strings, the closed ones, for which

$$X^m(0, \tau) = X^m(1, \tau)$$

and the open ones, for which the endpoints do not coincide. One may then impose two kinds of boundary conditions on each endpoint coordinate $X^m(0, \tau), X^m(1, \tau)$ independently: Neumann, for which they are free to move but $\partial_\sigma X^m = 0$, and Dirichlet, namely $\partial_\tau X^m = 0$. In this latter case, the ending points are constrained to belong to some $(p+1)$-dimensional hyperplanes of $\mathcal{M}^D$, the so-called Dp-branes (the D stands for Dirichlet, here).

The world-sheet can be also seen as a 2-dimensional space-time, where σ is the spatial coordinate and τ the temporal one, and the X^m coordinates as d scalar fields. By this identification, a suitable action for strings can be written with a dimensional parameter α' is thus introduced, a squared length, the so-called the Regge slope. Upon quantization of this bosonic world-sheet field theory, it can be shown that the excited states of the X^m fields represent in the physical space-time particles with masses proportional to α'^{-1} growing with the spin s and that belong to a discrete infinite set, the Regge trajectory. Closed strings give rise to particles represented by tensor fields with even rank. Rank 2 tensors are the only massless ones[4]: one is antisymmetric and is called Kalb-Ramond field, the other is associated with the graviton. Similarly, open strings represent every integer spin and $s = 1$ is the only massless mode.

Strings can interact with each other and the coupling constant is universal and denoted by g_s. It is then possible to analyze the theory perturbatively in powers of g_s and derive transition amplitudes in a way that is analogous to the computation of amplitudes in Quantum Field Theory by Feynman diagrams.

As for now we have considered only the bosonic string, that gives rise only to bosons. Fermions can be introduced too, by adding to the world-sheet D fermionic fields Ψ^m_α $(\alpha = 1, 2)$. Then the world-sheet action is supersymmetric and so is the resulting space-time theory, called Superstring Theory. It is possible to show that the world-sheet quantum theory is anomaly-free and hence consistent only if $D = 10$.

The Maldacena conjecture consists in a duality between two specific open and closed string theories. Let us analyze them separately.

2.1.3.1 Closed strings and black 3-branes

Closed strings generate spin 2 massless modes that are therefore associated with a gravitational interaction propagating in the whole $\mathcal{M}^D$. In the supersymmetric case, there are several consistent nonequivalent closed string theories. Here we consider just the IIB type. In the limit $\alpha' \to 0$, this theory is equivalent to the type IIB supergravity theory, a classical theory that is one of the supersymmetric extensions of General Relativity. Its field content consists in a graviton, its superpartner gravitino with spin $\frac{3}{2}$, the Kalb-Ramond field mentioned previously, the Ramond-Ramond fields with even rank, namely gauge fields that are p-forms with p even, a scalar dilaton and its superpartner dilatino. In this setting, a possible solution of the equations of motion of the metric is the black p-brane, namely a higher dimensional analog of a black hole. Let us consider a stack of N coincident 3-branes. The resulting metric reads

$$ds^2 = \frac{1}{\sqrt{1+\left(\frac{L}{y}\right)^4}} dx^\mu dx_\mu - \sqrt{1+\left(\frac{L}{y}\right)^4}\, d\vec{y}\cdot d\vec{y}, \tag{2.1.21}$$

where x^μ are the 4-dimensional coordinates of the brane contracted with the usual metric $\eta_{\mu\nu}$, $\vec{y}$ are the 6 directions perpendicular to the brane contracted with the euclidean metric, $y \equiv \sqrt{\vec{y}\cdot\vec{y}}$ and L is a length given by

$$L \equiv \left(4\pi N g_s \alpha'^2\right)^{\frac{1}{4}}. \tag{2.1.22}$$

The isometries of (2.1.21) are the rotations in the subspaces that are parallel or orthogonal to the brane, namely elements of $SO(1, 3)$ and $SO(6)$ respectively. Far away from the black 3-brane, at $y \to +\infty$ the metric (2.1.21) becomes constant, so that in such region the space is asymptotically flat.

Near the horizon, namely where $y \to 0$, we get

$$ds^2 \xrightarrow{y\to 0} \frac{y^2}{L^2} dx^\mu dx_\mu - \frac{L^2}{y^2} dy^2 - L^2 d\Omega_5^2, \tag{2.1.23}$$

[4] There is also a massless scalar, the dilaton, that we will not be concerned with.

where we used the 6-dimensional spherical coordinates and wrote $d\Omega_5^2$ to denote the line element of the 5-sphere S^5. We recognize in the first two terms of (2.1.23) the AdS metric (2.1.18) and therefore we see that near the horizon of a $D3$-brane the space looks like $\mathrm{AdS}_5 \times S^5$. Particles situated there are trapped in the gravitational well, so that they decouple from those propagating at $y \to \infty$.

2.1.3.2 Open strings on $D3$-branes

Due to the fact that they can generate massless particles with spin 1, open strings are naturally associated to gauge fields. For instance, for the bosonic string, the ending points attached to N coinciding Dp-branes behave as $U(N)$ gauge fields living in a $(p+1)$-dimensional flat space-time, i.e. the subspace of $\mathcal{M}^D$ given by the brane. Actually, it is possible to write an action for the fields in the brane, the Dirac-Born-Infeld (DBI) action, that in the limit of $\alpha' \to 0$ reduces to Yang-Mills one.

In the supersymmetric case, on $D3$-branes and for a small α', one can obtain the super Yang-Mills $\mathcal{N}=4$ $SU(N)$ theory, that is a conformal gauge theory invariant also under supersymmetric transformations with 4 fermionic supersymmetry generators. This implies an additional $SU(4)$ global invariance, called R symmetry. Since $SU(4) \simeq SO(6)$, this corresponds to the fact that rotations in the 6-dimensional sub-space orthogonal to the 4-dimensional brane leave the latter invariant ($SO(6)$ is also the isometry group of S^5).

The spin 1 fields generated by open strings living on the brane, interact with closed strings that propagate in the whole $\mathcal{M}^D$. In the limit $\alpha', g_s \to 0$, the DBI action prescribes a coupling with the (super)gravitational background given by

$$G_{10} \propto g_s^2 \alpha'^4,$$

the 10-dimensional Newton constant. This small coupling implies that, away from the branes, supergravity behaves as a free theory, decoupled from the particles on the branes.

Let us now consider a further contact point between strings and quantum field theories. 't Hooft showed in [40] that the Feynman diagrams of a gauge theory with symmetry group $SU(N)$ and coupling constant g admit a limit for $N \to \infty$ if the so-called 't Hooft coupling

$$\lambda \equiv g^2 N \tag{2.1.24}$$

is kept fixed. In this limit, they can be put in correspondence with those representing scattering amplitudes in a certain string theory with

$$g_s = g^2. \tag{2.1.25}$$

2.1.4 The Maldacena argument

We have presented two theories, each involving N 3-branes. In the limit for $\alpha' \to 0$ they share two symmetries:

1. A $SO(2,4)$ invariance, realized as the conformal group (see (2.1.8)) for $\mathcal{N}=4$ $SU(N)$ Super Yang-Mills living on the 3-branes and as the isometry group of AdS_5 near the 3-branes.

2. A $SO(6)$ global symmetry, realized as the R symmetry and as the isometry group of the S^5 coordinates of the space near the branes.

Moreover, in both cases we saw that the region far from the branes is governed by free supergravity around a flat space, decoupled from the particles that propagate near the branes. To reach this conclusion for open strings, we assumed that also

$$g_s \to 0. \tag{2.1.26}$$

On the other hand, if we keep fixed L, $\alpha' \to 0$ imposes

$$g_s N \gg 1, \tag{2.1.27}$$

by (2.1.22). Limits (2.1.26) and (2.1.27) together imply $N \to \infty$ and therefore, in the open string side we are considering the 't Hooft limit of the gauge theory in presence of a strong 't Hooft coupling $\lambda \equiv g_s N$ (2.1.27) (see (2.1.24) and (2.1.25)).

The fact that in both settings away from the branes physics is the same, while near them it is governed by two theories with the same symmetries, led Maldacena in [41] to conjecture that those two theories should be dual as prescribed by the famous AdS_5/CFT_4 correspondence:

> A super Yang-Mills $\mathcal{N} = 4$ $SU(N)$ theory in 4 dimensions with a strong coupling λ in the large N limit is dual to type IIB supergravity in a $\mathrm{AdS}_5 \times S^5$ background.

In particular this suggests that the 't Hooft limit relates this gauge theory with type IIB superstrings. Moreover, one can think that this CFT lives at the boundary of $\mathrm{AdS}_5 \times S^5$, where supergravity is set. We will refer to this space as bulk in the following.

One of the reasons for which this duality is so important is that it relates a perturbative and classical theory in the gravitational side, with a strongly coupled quantum field theory that cannot be treated perturbatively. This allows one to study properties of the field theory that would be otherwise much more difficult to find.

2.1.5 Holography basics and Witten diagrams

As for now, we just noticed the similarities between the theory at the boundary and the one living in the bulk and stated that they should be dual. In this section we specify further this point and develop a formalism that allows us to translate some statements of one side of the correspondence into the formalism of the other side. We will do this by considering a more general case of a correspondence between a classical theory that lives in a space that is asymptotically AdS_{d+1} and a conformal quantum field theory in d dimensions, without any further specification. Our results are a realization of the holographic principle of which AdS_5/CFT_4 is an example. We will later apply them to a bulk theory of interacting higher spin fields to formulate the Klebanov-Polyakov conjecture.

To simplify our computations, we will use a Wick rotation on the metric of the boundary theory. This implies that $\eta_{\mu\nu} = -\delta_{\mu\nu}$ and that the signature of the AdS metric varies accordingly: $\mathrm{sign}(g_{MN}) = \mathrm{diag}(-..., -)$.

2.1.6 Bulk-to-Boundary propagators

The simplest theory that we can have in the bulk is the one that contains only some free scalar field $\phi(X)$ with mass m. As for now, we do not try to specify what is its dual on the boundary. ϕ obeys the Klein-Gordon equation

$$\frac{1}{\sqrt{g}}\partial_M \sqrt{g} g^{MN} \partial_N \phi + m^2 \phi = 0. \tag{2.1.28}$$

where g_{MN} is the AdS_{d+1} metric, while g is its determinant. If we employ the coordinates (z, x) and insert the relative metric (2.1.19) into (2.1.28) we obtain

$$\left(\frac{z}{L}\right)^2 \partial^\mu \partial_\mu \phi - \left(\frac{z}{L}\right)^{d+1} \partial_z \left(\left(\frac{L}{z}\right)^{d-1} \partial_z \phi\right) + m^2 \phi = 0. \tag{2.1.29}$$

The solutions of this equation are determined by the boundary conditions. As we showed in subsection 2.1.2, the boundary of AdS_{d+1} is $\mathbb{R}^d$ at $z = 0$. This space can be compactified into S^d by adding a point at infinity that we denote by ∞. We identify such point in the AdS coordinates as $z = +\infty$. In this way, we compactified AdS_{d+1} too.

Let us analyze the behavior of ϕ near the boundary. By Fourier transforming the x variables, we set $\phi(z, x) = \tilde{\phi}(z) e^{ip^\mu x_\mu}$ in (2.1.29). The first term in (2.1.29) is proportional to $\frac{z^2 p^2}{L^2}$ and is negligible at $z \sim 0$, so that we are left to solve

$$-\left(\frac{z}{L}\right)^{d+1}\partial_z\left(\left(\frac{L}{z}\right)^{d-1}\partial_z\tilde{\phi}\right)+m^2\tilde{\phi}=0. \tag{2.1.30}$$

The presence of powers of z in (2.1.30) suggests a solution of the kind $\tilde{\phi}=Cz^{\Delta}$ for some $C,\Delta\in\mathbb{R}$ given by (2.1.30):

$$-\frac{\Delta\left(\Delta-d\right)}{L^2}Cz^{\Delta}+m^2Cz^{\Delta}=0.$$

This equation implies

$$\Delta\left(\Delta-d\right)=m^2L^2 \tag{2.1.31}$$

solved by

$$\Delta_{\pm}=\frac{d\pm\sqrt{d^2+4m^2L^2}}{2}\gtrless 0. \tag{2.1.32}$$

We see therefore that there are two possible asymptotic behaviors for ϕ: one that diverges at $z=0$ as z^{Δ_-} and the other that is regular and asymptotic to z^{Δ_+}.

If we want to express the values of ϕ in the interior of AdS by means of the boundary conditions, it is clear that we need to consider only solutions that approach Cz^{Δ_-} as $z\to 0$, so that their value at $z\sim 0$ is not vanishing. However, we still have the problem that at $z=0$ every such solution diverges. We can anyway circumvent this difficulty by considering the behavior at, say, $z=\epsilon$ for some arbitrarily small ϵ, namely by requiring that

$$\phi(z,x)\overset{z\sim 0}{\sim}c(z,x)^{\Delta_-}\bar{\phi}(x), \tag{2.1.33}$$

where $\bar{\phi}(x)$ is a function defined on the boundary and $c(z,x)$ is some function with the only requirement that $c(z,x)\sim z$ at $z\sim 0$.

This function is related to the conformal symmetry of the boundary. Indeed, suppose we wanted to extend the AdS metric to the region with $z=0$. It is clear that we cannot do this by using g_{MN} from (2.1.19), which is singular at the boundary. However

$$ds^2=\frac{c(z,x)^2}{z^2}\left(-dz^2+dx^{\mu}dx_{\mu}\right) \tag{2.1.34}$$

is not and, by restricting it to the x coordinates, can be our sought extension $g_{\mu\nu}(x)$. The simplest possible choice for $c(x,z)=z$, for example, gives $g_{\mu\nu}=\eta_{\mu\nu}$, as follows from (2.1.19). The arbitrariness in choosing $c(z,x)$ just reflects the conformal symmetry of the boundary.

We would then like to express the values of ϕ in every point of AdS$_{d+1}$ in terms of $\bar{\phi}$ by means of (2.1.29). Suppose indeed that we can define a "Green function" $K_{\Delta_-}(z,x)$ such that it solves (2.1.29)

$$\left(\frac{z}{L}\right)^2\partial^{\mu}\partial_{\mu}K_{\Delta_-}-\left(\frac{z}{L}\right)^{d+1}\partial_z\left(\left(\frac{L}{z}\right)^{d-1}\partial_zK_{\Delta_-}\right)+m^2K_{\Delta_-}=0 \tag{2.1.35}$$

and that

$$\lim_{z\to 0}z^{-\Delta_-}K_{\Delta_-}(z,x)=\delta^d(x), \tag{2.1.36}$$

then

$$\phi(z,x)=\int K_{\Delta_-}(z,x-y)\bar{\phi}(y)d^dy \tag{2.1.37}$$

solves (2.1.29) and obeys (2.1.33). We call K_{Δ_-} bulk-to-boundary propagator.

The simplest way to find K_{Δ_-} is by exploiting the compactification explained above as done by Witten in [42]. The idea is to find K_{Δ_-} at some particular point P of the boundary. Then, we can perform on K_{Δ_-} a transformation that brings P to some other Q also in the boundary. If this transformation is a symmetry of the boundary, we get another valid form of the propagator. We take P to be the point at infinity ∞. Since in this case K_{Δ_-} should not depend on x, (2.1.35) becomes equivalent to (2.1.30) and we find again the solutions $z^{\Delta_\pm}$. As we show later, to get the correct behavior at $z = 0$ of K_{Δ_-} (2.1.36), we have to assume that the propagator at infinity is

$$K_{\Delta_-}(z, \infty) = C_{\Delta_-} z^{\Delta_+}. \tag{2.1.38}$$

Now, consider the transformation

$$z, x^\mu \to \frac{z}{z^2 - x^2}, \frac{x^\mu}{z^2 - x^2} \tag{2.1.39}$$

that brings ∞ to the point on the boundary $z = 0, x^\mu = 0$. At $z = 0$, (2.1.39) acts as $x^\mu \to \frac{x^\mu}{x^2}$ and is a conformal transformation as we showed in (2.1.7) and therefore a symmetry of the boundary. It transforms (2.1.38) into

$$K_{\Delta_-}(z, x) = C_{\Delta_-} \left(\frac{z}{z^2 - x^2} \right)^{\Delta_+} \tag{2.1.40}$$

where C_{Δ_-} is yet to be determined.

Let us verify the fundamental property (2.1.36). It is clear that (2.1.40) for $z = 0$ is zero everywhere except where $x = 0$, since we are using a metric with a definite sign $-\delta_{\mu\nu}$. There, it diverges, as a Dirac delta function. To prove this formally, we need to show that

$$\mathcal{S}' - \lim_{z \to 0} z^{-\Delta_-} K_{\Delta_-}(z, x) = \mathcal{S}' - \lim_{z \to 0} z^{-\Delta_-} C_{\Delta_-} \left(\frac{z}{z^2 - x^2} \right)^{\Delta_+} = \delta^d(x),$$

where $\mathcal{S}' - \lim$ denotes a limit in the space of distributions $\mathcal{S}'$. Performing a limit in $\mathcal{S}'$ on the distribution K_{Δ_-} means computing an ordinary limit of its application to a test function ψ:

$$\begin{aligned}
\lim_{z \to 0} \int K_0(z, x)\psi(x) d^d x &= \lim_{z \to 0} \int z^{-\Delta_-} C_{\Delta_-} \left(\frac{z}{z^2 + r^2} \right)^{\Delta_+} \psi(x) r^{d-1} dr d\Omega_d \\
&= \lim_{z \to 0} \int C_{\Delta_-} \left(\frac{1}{1 + r^2} \right)^{\Delta_+} \psi(zx) r^{d-1} dr d\Omega_d \qquad (2.1.41) \\
&= I_{\Delta_-} \psi(0) \qquad (2.1.42) \\
&= I_{\Delta_-} \int \delta^d(x) \psi(x) d^d x
\end{aligned}$$

where in the first step we used spherical coordinates and the solid angle measure Ω_d; in (2.1.41) we performed a rescaling $r \to rz$ to make z appear only inside ψ. This happens only if one chooses Δ_+ in (2.1.38). Finally in (2.1.42) we defined the always-converging integral

$$I_{\Delta_-} = C_{\Delta_-} \int \left(\frac{1}{1 + r^2} \right)^{\Delta_+} r^{d-1} dr d\Omega_d \tag{2.1.43}$$

and we used the dominated convergence theorem and the fact that ψ is bounded to perform the limit inside the integral.

Equation (2.1.43) can be solved and allows to determine $C_{\Delta_-} = I_{\Delta_-}^{-1}$, which results in

$$C_{\Delta_-} = \frac{\Gamma(\Delta_+)}{\pi^{d/2} \Gamma\left(\Delta_+ - \frac{d}{2}\right)}. \tag{2.1.44}$$

In the following we will write just K_Δ, C_Δ and Δ in place of $K_{\Delta_-}, C_{\Delta_-}$ and Δ_+, if it is not ambiguous.

As for now, we analyzed the simplest possible field: the scalar one. What happens with a generic tensorial field $\phi_{M_1 \dots M_s}$? Like before, we consider only free fields and interactions perturbations. In general these fields will solve a Klein-Gordon equation similar to (2.1.29)

$$\left(D_N D^N + m^2\right) \phi_{M_1 \dots M_s} = 0 \tag{2.1.45}$$

either after a gauge fixing for massless fields[5], or after removing all the auxiliary fields coming from the Singh-Hagen Lagrangian in AdS (see subsection 1.1).

As before, we would like to analyze the behavior of this field at $z \sim 0$, where we set $\phi_{zM_2 \dots M_s} \to 0$. Using (1.3.26) one finds that the only non-vanishing components of the Christoffel symbol are

$$\Gamma^z_{\mu\nu} = \frac{1}{z}\delta_{\mu\nu}, \quad \Gamma^\nu_{\mu z} = -\frac{1}{z}\delta^\nu_\mu, \quad \Gamma^z_{zz} = \frac{1}{z}$$

and therefore the expansion of (2.1.45)

$$\begin{aligned}
\left(D_N D^N + m^2\right) \phi_{M_1 \dots M_s} &= D_N \left(-\frac{z^2}{L^2} \delta^{PN} \left(\partial_P \phi_{M_1 \dots M_s} - \Gamma^Q_{P(M_1} \phi_{QM_2 \dots M_s)} \right) \right) + m^2 \phi_{M_1 \dots M_s} \\
&= -\left(\frac{z}{L}\right)^{d+1} \partial_N \left(\left(\frac{L}{z}\right)^{d-1} \left(\partial^N \phi_{M_1 \dots M_s} - \delta^{PN} \Gamma^Q_{P(M_1} \phi_{QM_2 \dots M_s)} \right) \right) + m^2 \phi_{M_1 \dots M_s} \\
&\quad + \frac{z^2}{L^2} \Gamma^R_{N(M_1} \delta^{PN} \left(\partial_P \phi_{RM_2 \dots M_s)} - \Gamma^Q_{PR} \phi_{QM_2 \dots M_s)} - \Gamma^Q_{P(M_2} \phi_{QRM_3 \dots M_s)} \right)
\end{aligned}$$

becomes for the components of $\phi_{M_1 \dots M_s}$ tangential to the boundary

$$0 = -\left(\frac{z}{L}\right)^{d+1} \partial_N \left(\left(\frac{L}{z}\right)^{d+1} \partial^N \phi_{\mu_1 \dots \mu_s} \right) + \frac{sd}{L^2} \phi_{\mu_1 \dots \mu_s} - \frac{2sz}{L^2} \partial_z \phi_{\mu_1 \dots \mu_s} + \left(m^2 - \frac{s(s-1)}{L^2} \right) \phi_{\mu_1 \dots \mu_s} \tag{2.1.46}$$

We can then find the behavior at $z \sim 0$ of the solutions of (2.1.46) as we did for (2.1.29), namely by a Fourier transform and the ansatz $\phi_{\mu_1 \dots \mu_s}(z, x) \sim z^\delta$. We find the relation

$$(\delta + s - d)(\delta + s) - s = m^2 L^2 \tag{2.1.47}$$

solved by

$$\delta_\pm = \frac{d - 2s \pm \sqrt{d^2 + 4s + 4m^2 L^2}}{2}$$

If we replace m^2 with its value for Fronsdal fields (1.4.59), we get

$$\delta_\pm = \frac{d - 2s \pm 2\left(\frac{d}{2} + s - 2\right)}{2}. \tag{2.1.48}$$

We choose the asymptotic solution with δ_- and thus set the boundary conditions

$$\bar{\phi}_{\mu_1 \dots \mu_s}(x) = \lim_{z \to 0} z^{-\delta_-} \phi_{\mu_1 \dots \mu_s}(z, x), \tag{2.1.49}$$

The bulk-to-boundary propagator for higher spin fields will then be

$$K^{M_1 \dots M_s \mu_1 \dots \mu_s}_{\delta_-}(z, x), \quad \phi^{M_1 \dots M_s}(z, x) = \int K^{M_1 \dots M_s \mu_1 \dots \mu_s}_{\delta_-}(z, x - y) \bar{\phi}_{\mu_1 \dots \mu_s}(y) d^d y. \tag{2.1.50}$$

Equations (2.1.49) and (2.1.50) impose that

$$\lim_{z \to 0} z^{-\delta_-} K^{M_1 \dots M_s \mu_1 \dots \mu_s}_{\delta_-}(z, x) = \delta^d(x) \delta^{M_1(\mu_1} \cdots \delta^{M_s \mu_s)}.$$

In section 3.6.2 we will derive the precise form of these propagators by means of the ambient formalism, that simplifies greatly the computations.

[5] The transverse traceless gauge that we found for the flat space case (1.1.22) can be imposed also on Fronsdal fields in AdS obeying (1.4.58), which contains also a mass-like term (1.4.59) due to the non-vanishing curvature.

2.1.7 Witten diagrams

The content of a generic quantum field theory involving the fields φ_i that sit in some irreducible representation of the Poincaré group is completely specified by its classical action

$$S[\phi_i] = \int \mathcal{L}(\varphi_i, \partial\varphi_i) d^d x.$$

that determines all the correlators by

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \frac{\int \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S[\varphi_i]} \mathcal{D}\varphi_i}{\int e^{-S[\varphi_i]} \mathcal{D}\varphi_i}, \tag{2.1.51}$$

where $\mathcal{O}_k$ are some observables that are functions of φ_i. Equation (2.1.51) can also be rewritten as

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \left. \frac{\delta^n W[\bar{\phi}_k]}{\delta\bar{\phi}_1(x_1) \cdots \delta\bar{\phi}_n(x_n)} \right|_{j_k=0},$$

where

$$W[\bar{\phi}_j] = \log \int e^{-S[\varphi_i] + \sum_k \int \bar{\phi}_k(x) \mathcal{O}_k(x) d^d x} \mathcal{D}\varphi_i \tag{2.1.52}$$

is the generating functional of connected diagrams and we defined some suitable auxiliary sources $\bar{\phi}_k$ that are coupled to the various observables $\mathcal{O}_k$. Since we are dealing with a conformal theory, also $\mathcal{O}_k$ and $\bar{\phi}_k$ should be primary fields. We label their weights with Δ_k and $d - \Delta_k$ respectively, so that (2.1.52) is conformally invariant. In the classical limit,

$$W[\bar{\phi}_j] \xrightarrow{\hbar \to 0} -S[\varphi_i] + \sum_k \int \bar{\phi}_k(x) \mathcal{O}_k(x) d^d x$$

and therefore we recognize W as the quantum version of the classical functional S.

The AdS/CFT correspondence then states that the W functional of the theory at the boundary should be equal to S_{AdS} of the theory in the bulk. However, we need to associate the sources $\bar{\phi}_k(x)$ with fields in the gravitational side of the correspondence. Let us first assume that $\mathcal{O}_k$ and $\bar{\phi}_k$ are scalars. Since our aim is to express S_{AdS} as a functional of fields that live on the boundary, it is natural to associate to $\bar{\phi}_k(x)$ the source for a scalar field $\phi_k(z, x)$ in the bulk through (2.1.37). However, to make this identification, we need to verify that $\bar{\phi}_k(x)$ behaves as a primary field with weight $d - \Delta_k$. Indeed, from (2.1.33) we saw that $\bar{\phi}_k(x)$ is defined up to a factor $c(z, x)^{\Delta_{k-}}$. Rescaling the function $c(z, x)$ by λ rescales the boundary metric (2.1.34) with a factor λ^2 and we can thus interpret it as conformal rescaling. At the same time $\bar{\phi}_k(x)$ rescales too by a coefficient equal to $\lambda^{-\Delta_{k-}}$, so that it behaves as a primary field with dimension $\Delta_{k-} = d - \Delta_k$. Therefore the weight of $\mathcal{O}_k$ is $\Delta_k = d - \Delta_{k-} = \Delta_{k+}$.

If we consider tensorial fields, then both $\bar{\phi}_k^{\mu_1 \ldots \mu_s}$ and $\mathcal{O}_k^{\mu_1 \ldots \mu_s}$ will be tensors of the same rank and symmetry type and we can still write (2.1.52) by assuming that $\bar{\phi}_k^{\mu_1 \ldots \mu_s}$ and $\mathcal{O}_k^{\mu_1 \ldots \mu_s}$ have been implicitly completely contracted one with each other. Then we can identify $\bar{\phi}_k^{\mu_1 \ldots \mu_s}(x)$ with the boundary source of a tensor field $\phi^{M_1 \ldots M_s}$ by (2.1.50). Notice that if $\phi^{M_1 \ldots M_s}$ is a gauge field in the bulk, then the coupling between $\bar{\phi}_k^{\mu_1 \ldots \mu_s}$ and $\mathcal{O}_k^{\mu_1 \ldots \mu_s}$ should be invariant under the transformations induced by the bulk gauge symmetry. Its variation would be, for instance in the case of a completely symmetric Fronsdal field,

$$\begin{aligned} \delta \int \bar{\phi}^{\mu_1 \ldots \mu_s} \mathcal{O}_{\mu_1 \ldots \mu_s}(x) d^d x &= \int \partial^{\mu_s} \bar{\Lambda}^{\mu_1 \ldots \mu_{s-1}} \mathcal{O}_{\mu_1 \ldots \mu_s}(x) d^d x \\ &= -\int \bar{\Lambda}^{\mu_1 \ldots \mu_{s-1}} \partial^{\mu_s} \mathcal{O}_{\mu_1 \ldots \mu_s}(x) d^d x \end{aligned} \tag{2.1.53}$$

and therefore $\mathcal{O}$ should be a conserved current on the boundary, namely

$$\partial^{\mu_s}\mathcal{O}_{\mu_1\ldots\mu_s} = 0.$$

To find the weight of $\mathcal{O}_{\mu_1\ldots\mu_s}$, consider as before a rescaling of the $c(z,x)$ function, corresponding to a scale transformation on the boundary CFT. In order for it to leave invariant the source term in the action, the fields should transform as follows

$$\int \lambda^{-\delta_-}\bar{\phi}^{\mu_1\ldots\mu_s}\lambda^{-d+\delta_-}\mathcal{O}_{\mu_1\ldots\mu_s}(x)\lambda^d d^d x.$$

Using (2.1.13) and remembering that contractions of tensors are invariant under Lorentz transformations, we find

$$\Delta + s = d - \delta_-,$$

which, for higher spins obeying (2.1.48), implies

$$\Delta = \delta_+ - s = d + s - 2. \tag{2.1.54}$$

With these identifications, the AdS/CFT correspondence is realized by conjecturing that

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \left.\frac{\delta^n S_{grav}\left[\bar{\phi}_k\right]}{\delta\bar{\phi}_1(x_1)\cdots\delta\bar{\phi}_n(x_n)}\right|_{\bar{\phi}_k=0}$$

Let us make an example. Consider a theory in the bulk with the simplest action of a free scalar:

$$S_{grav}[\phi] = \int \frac{1}{2}\partial_M\phi(x)g^{MN}(x)\partial_N\phi(x)\sqrt{g}d^d x dz. \tag{2.1.55}$$

We now rewrite (2.1.55) as a function of $\bar{\phi}(x)$ integrating by parts and using the equations of motion (2.1.29)

$$\begin{aligned}
S_{grav}[\bar{\phi}] &= \int \frac{1}{2}\partial_M\left(\phi g^{MN}\partial_N\phi\sqrt{g} - m^2\phi^2\right)d^d x dz - \int\frac{1}{2}\phi\partial^M\left(\partial_M\phi\sqrt{g}\right)d^d x dz && (2.1.56)\\
&= C_\Delta \int \lim_{z\to 0}\left[\phi(z,x)\partial_z\left(\int \frac{z^{\Delta_+}\bar{\phi}(y_2)d^d y_2}{\left(z^2-(x-y)^2\right)^{\Delta_+}}\right)\frac{L^{d-1}}{z^{d-1}}\right]d^d x\\
&= C_\Delta L^{d-1}\int \lim_{z\to 0}\left[z^{\Delta_-}\bar{\phi}(x)\frac{\Delta_+ z^{\Delta_+-1}\bar{\phi}(y)}{\left(z^2-(x-y)^2\right)^{\Delta_+}}\frac{1}{z^{d-1}}\right]d^d x d^d y\\
&= \Delta C_\Delta L^{d-1}\int \frac{\bar{\phi}(x)\bar{\phi}(y)}{|x-y|^{2\Delta}}d^d x d^d y && (2.1.57)
\end{aligned}$$

and therefore

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{\delta^2 S_{grav}[\bar{\phi}]}{\delta\bar{\phi}(x)\delta\bar{\phi}(y)} = \frac{2\Delta C_\Delta L^{d-1}}{|x-y|^{2\Delta}} \tag{2.1.58}$$

As we will see in (3.5.5) this is the form of a correlator between two primary fields with the same weight Δ. We can rewrite (2.1.58) as

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \lim_{z\to 0}\Delta L^{d-1}K_\Delta(z,x-y), \tag{2.1.59}$$

namely the 2 point Green function at x, y of the field $\mathcal{O}$ is given by the bulk-to-boundary propagator of its dual that connects the two points on the boundary x and y.

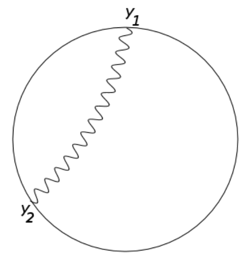


Figure 2.1.1: A Witten diagram connecting two boundary points where $\mathcal{O}$ is inserted.

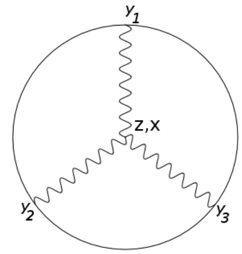


Figure 2.1.2: A Witten diagram for a 3 point correlator of the operator $\mathcal{O}$ inserted in the boundary points y_1, y_2 and y_3. The vertex is at the bulk point (z, x) that is integrated over the whole AdS space.

Let us then consider a perturbation to the bulk theory and supplement our original action with a cubic vertex given by

$$S_3 = \int \frac{\lambda}{3!}\phi^3(z,x)\sqrt{g}d^dxdz. \tag{2.1.60}$$

Using (2.1.40) we can rewrite S_3 as

$$S_3 = \frac{\lambda}{3!}\int \bar{\phi}(y_1)\bar{\phi}(y_2)\bar{\phi}(y_3)K_\Delta(z,x-y_1)K_\Delta(z,x-y_2)K_\Delta(z,x-y_3)\sqrt{g}d^dy_1d^dy_2d^dy_3d^dxdz.$$

We can now compute the three point correlator

$$\begin{aligned}\langle\mathcal{O}(y_1)\mathcal{O}(y_2)\mathcal{O}(y_3)\rangle &= \frac{\delta^2 S_3[\bar{\phi}]}{\delta\bar{\phi}(y_1)\delta\bar{\phi}(y_2)\delta\bar{\phi}(y_3)} \\ &= \lambda\int K_\Delta(z,x-y_1)K_\Delta(z,x-y_2)K_\Delta(z,x-y_3)\sqrt{g}d^dxdz \end{aligned}\tag{2.1.61}$$

As we see it amounts to an integral over the AdS space of three propagators from the integration variable and the three external points $y_{1,2,3}$. This happens for every vertex of the bulk theory.

These examples can be generalized to a standard procedure to compute boundary correlators by means of bulk calculations. It can be depicted by Feyman-like diagrams, called Witten diagrams and introduced in [42]. The rules to compute them are the following:

1. The boundary is represented by a circle that encloses the bulk.

2. The points $x_1, ..., x_n$ at the boundary, where operators $\mathcal{O}_k$ with weight Δ_k are inserted, are attached to the outer circle and connected to inner points by bulk-to-boundary propagators derived from proper fields ϕ_k that at $z \sim 0$ are asymptotic to $z^{d-\Delta_k}$.

3. According to the vertices in the bulk action, these propagators are attached to common points (z, x) in the bulk.

4. The sought correlator is given by the integration over all the bulk interaction points.

For instance the computation performed in (2.1.61) is represented graphically by figure 2.1.2, while the one of the 2-point function (2.1.59) is represented by (2.1.1).

These are the correlators which involve only $\mathcal{O}$ that we can compute if we consider only the classical limit of the theory in the bulk defined by the action (2.1.60). If we go back and focus on the original Maldacena conjecture, however, we must remember that it does not state simply a duality between the Super Yang-Mills $\mathcal{N}=4$ $SU(N)$ gauge theory and type IIB supergravity, but between two string theories that reduce to them in certain limits. These limits, in the gravitational side, allowed us to consider quantum corrections as lower order contributions to the action. If we want to make computations at the next-to-leading order, we have to include such corrections. Witten diagrams will then also employ the so-called bulk-to-bulk propagators, namely the Green functions of the bulk theory that one uses to compute Feynman diagrams in the bulk. Their usage is the same as bulk-to-boundary propagators with the only difference that they are attached to two points in the bulk. Since we will not be dealing with them in this thesis, we will not delve in further details.

2.1.8 Expectation values of the dual operators

The computation of the 2 point correlator that we just performed allows us to give an interesting interpretation to the class of solutions for the scalar field that we ignored, namely those that at $z\sim 0$ scale as z^{Δ_+}. Let the complete solution of the equations of motion (2.1.29) be

$$\phi(z,x) \overset{z\sim 0}{\sim} \bar{\phi}(x)z^{\Delta_-} + E(x)z^{\Delta_+}. \tag{2.1.62}$$

Then, the free action is in general divergent, because of the pole given by $\bar{\phi}(x)z^{\Delta_-}$ at $z=0$. However, we can define

$$\phi(z,x) = z^{\Delta_-}\chi(z,x),$$

so that χ is regular on the boundary. Action (2.1.55) can be rewritten as

$$\begin{aligned} S_{grav}[\chi] &= -\int \frac{1}{2}\delta^{MN}\partial_M\left(z^{\Delta_-}\chi\right)\partial_N\left(z^{\Delta_-}\chi\right)\frac{L^{d-1}}{z^{d-1}}d^dxdz \\ &= L^{d-1}\int\frac{1}{2}\left(-\Delta_-^2 z^{-2}\chi^2 - \Delta_- z^{-1}2\chi\partial_z\chi - \left(\partial_z\chi\right)^2 + \partial_\mu\chi\partial^\mu\chi\right) z^{2\Delta_- - d+1}d^dxdz. \end{aligned}$$

To regolarize this action, we can subtract the first two terms and assume

$$\Delta_- > \frac{d-2}{2}, \tag{2.1.63}$$

so that we are left with the finite action

$$\tilde{S}_{grav}[\chi] = L^{d-1}\int\frac{1}{2}\left(-\left(\partial_z\chi\right)^2 + \partial_\mu\chi\partial^\mu\chi\right)z^{2\Delta_- - d+1}d^dxdz.$$

Now we can extract by an integration by parts a boundary term, analogous to (2.1.56):

$$\begin{aligned} \tilde{S}_{grav}[\chi] &= \lim_{z\to 0}\frac{z^{2\Delta_- - d+1}L^{d-1}}{2}\int \chi\partial_z\chi d^dx \\ &= \lim_{z\to 0}\frac{z^{2\Delta_- - d+1}L^{d-1}}{2}\int\left(\bar{\phi}(x) + E(x)z^{\Delta_+ - \Delta_-}\right)\partial_z\left(\bar{\phi}(x) + E(x)z^{\Delta_+ - \Delta_-}\right)d^dx. \end{aligned} \tag{2.1.64}$$

If we assume

$$\frac{d-1}{2} < \Delta_- < \frac{d}{2} \tag{2.1.65}$$

the only term that survives the limit in (2.1.64) is

$$\tilde{S}_{grav} = \frac{\Delta_+ - \Delta_-}{2} L^{d-1} \int \bar{\phi}(x) E(x) d^d x \tag{2.1.66}$$

and we can extract $E(x)$ by a functional derivative:

$$E(x) = \frac{2L^{1-d}}{\Delta_+ - \Delta_-} \frac{\delta \tilde{S}_{grav}}{\delta \bar{\phi}(x)} = \langle \mathcal{O}(x) \rangle \,.$$

We can therefore interpret $E(x)$ as the expected value of the dual CFT operator. Another possible way to obtain $E(x)$ from (2.1.66) is to assume

$$\bar{\phi}(x) \equiv \delta^d(x).$$

This equation tells us that an operator insertion on the boundary theory at x is equivalent to a point-like source of the field ϕ located there. This relation holds also for a more general action: the $E(x)$ field that solves the equations of motion in presence of sources located at $x_1, ..., x_n$ is

$$E(x) \propto \langle \mathcal{O}(x) \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle \,. \tag{2.1.67}$$

Moreover the limitation (2.1.65) can be removed by suitable further regularizations of the action.

2.2 Higher spin AdS/CFT

After having developed a formalism of a generic AdS/CFT duality, we will now consider an example of AdS/CFT holography which involves a higher spin generalization of gravity, the Vasiliev theory. We start by describing a simple CFT that we will relate to a minimal model of the higher spin field theory in the second part of this section.

2.2.1 The $O(N)$ model

Let us consider a theory in 3 dimensions with N real scalar fields $\varphi^a(x)$ with a global $O(N)$ symmetry. The action for this field reads

$$S\left[\varphi^a\right] = \int \partial_\mu \varphi^a(x) \partial^\mu \varphi^a(x) d^3 x \tag{2.2.1}$$

and the relative equation of motion is

$$\Box \varphi^a(x) = 0. \tag{2.2.2}$$

This theory is conformal, as we saw in (2.1.1)[6] if the conformal dimension of the scalar is $\Delta = \frac{1}{2}$.

We can use equation (2.2.2) to construct an infinite number of conserved currents that are $O(N)$ singlets, the simplest one being[7]

$$\mathcal{J}^{\mu_1 \mu_2} \equiv \varphi^a \partial^{\mu_1} \partial^{\mu_2} \varphi^a - \partial^{\mu_1} \varphi^a \partial^{\mu_2} \varphi^a.$$

[6]The fact that this field is an $O(N)$ vector does not influence the proof.

[7]There are only currents with even spin. Indeed, suppose, for example that we wanted to construct the spin 1 one. It should read

$$\mathcal{J}^\mu \equiv \varphi^a \partial^\mu \varphi^a, \tag{2.2.3}$$

but

$$\partial_\mu \mathcal{J}^\mu = \partial_\mu \varphi^a \partial^\mu \varphi^a$$

that cannot be set to zero by any compensating terms, since (2.2.3) is already the most general form for $\mathcal{J}^\mu$. A similar problem arises when considering general odd spin currents $\mathcal{J}^{\mu_1 \cdots \mu_{2k+1}}$ defined by a relation similar to (2.2.5), in which the coefficient c_k of the term

$$c_k \partial^{\mu_1} \cdots \partial^{\mu_k} \varphi \partial^{\mu_{k+1}} \cdots \partial^{\mu_{2k+1}} \varphi$$

has to be set to zero because

$$c_k \partial_\mu \partial^{\mu_1} \cdots \partial^{\mu_k} \varphi \partial^{\mu_{k+1}} \cdots \partial^{\mu_{2k}} \partial^\mu \varphi$$

can not be compensated by any of the other terms in expansion (2.2.5) of $\mathcal{J}^{\mu_1 \cdots \mu_{2k+1}}$, but, as we will see later, by (2.2.8) this sets all the coefficients c_j to 0.

In general, we can construct an infinite number of symmetric conserved tensors $\mathcal{J}^{\mu_1\cdots\mu_{2k}}$

$$\partial_{\mu_1}\mathcal{J}^{\mu_1\cdots\mu_{2k}} = 0 \tag{2.2.4}$$

with even rank $s = 2k$ ($k \in \mathbb{N}_0$) whose most general form is

$$\mathcal{J}^{\mu_1\cdots\mu_{2k}} \equiv \frac{1}{2}\sum_{j=0}^{2k} c_j \left(\partial^{(\mu_1}\cdots\partial^{\mu_j}\varphi^a\right)\left(\partial^{\mu_{j+1}}\cdots\partial^{\mu_{2k})}\varphi^a\right) \tag{2.2.5}$$

for some c_i. If we contract $\mathcal{J}$ with an auxiliary vector u^μ, we get

$$\begin{aligned}\mathcal{J}_s(x,u) &\equiv u_{\mu_1}\cdots u_{\mu_{2k}}\mathcal{O}^{\mu_1\cdots\mu_{2k}}(x)\\ &= \sum_{j=0}^{k-1}\binom{2k}{j}c_i\,(u\cdot\partial)^j\,\varphi^a\,(u\cdot\partial)^{2k-j}\,\varphi^a + \frac{1}{2}\binom{2k}{k}c_k\,(u\cdot\partial)^k\,\varphi^a\,(u\cdot\partial)^k\,\varphi^a\end{aligned} \tag{2.2.6}$$

and the conservation condition implies

$$\partial_\mu\frac{\partial\mathcal{J}_s(x,u)}{\partial u^\mu} = 0. \tag{2.2.7}$$

Using (2.2.6) and (2.2.2) we can expand the left hand side of (2.2.7) as

$$\begin{aligned}\partial_\mu\frac{\partial\mathcal{J}_s(x,u)}{\partial u_\mu} &= \sum_{j=1}^{k-1}\binom{2k}{j}c_j j\,(u\cdot\partial)^{j-1}\,\partial^\mu\varphi^a\,(u\cdot\partial)^{2k-j}\,\partial_\mu\varphi^a\\ &\quad+\sum_{j=0}^{k-1}\binom{2k}{j}c_j\,(2k-j)\,(u\cdot\partial)^j\,\partial_\mu\varphi^a\,(u\cdot\partial)^{2k-j-1}\,\partial^\mu\varphi^a\\ &\quad+\binom{2k}{k}c_k s\,(u\cdot\partial)^{k-1}\,\partial^\mu\varphi^a\,(u\cdot\partial)^k\,\partial_\mu\varphi^a\\ &= \sum_{j'=0}^{k-1}\binom{2k}{j'+1}c_{j'+1}\left(j'+1\right)(u\cdot\partial)^{j'}\,\partial^\mu\varphi^a\,(u\cdot\partial)^{2k-j'-1}\,\partial_\mu\varphi^a\\ &\quad+\sum_{j=0}^{k-1}\binom{2k}{j}c_j\,(2k-j)\,(u\cdot\partial)^j\,\partial_\mu\varphi^a\,(u\cdot\partial)^{2k-j-1}\,\partial^\mu\varphi^a\end{aligned}$$

where in the second step we replaced j with $j' \equiv j-1$ in the first term and combined it with the third term in the first step. (2.2.7) then imposes

$$c_{j+1} = -\frac{\binom{s}{j}(s-j)}{\binom{s}{j+1}(j+1)}c_j = -c_j, \tag{2.2.8}$$

i.e. all the coefficients c_j are determined up to a normalization c_0 by

$$c_j = (-1)^j\,c_0.$$

Since φ is a primary field with dimension $\Delta = \frac{1}{2}$, as we will show in subsection (2.1.58), in virtue of (3.5.5) and the $O(N)$ global symmetry, the correlator of φ^i has the following form

$$\left\langle\varphi^i(x)\varphi^j(y)\right\rangle = \frac{\delta^{ij}}{|x-y|}$$

where we chose a suitable normalization for φ. Therefore

$$\begin{aligned}\langle \mathcal{J}(x)\mathcal{J}(y)\rangle &= c_0^2 \left\langle \varphi^i(x)\varphi^j(y)\varphi_i(x)\varphi_j(y)\right\rangle \\ &= c_0 \frac{\delta^{ij}}{|x-y|}\frac{\delta_{ij}}{|x-y|} \\ &= \frac{Nc_0^2}{(x-y)^2}\end{aligned}$$

A similar dependence on N holds for the other currents. To make their 2 point function independent of N we then choose

$$c_0 = \frac{1}{\sqrt{N}}.$$

Due to their definition (2.2.5), the currents in $d = 3$ naturally have conformal dimension

$$\Delta_s = (d + s - 2) = s + 1 \tag{2.2.9}$$

where s is the number of derivatives and therefore the rank of $\mathcal{J}^{\mu_1 \cdots \mu_s}$.

2.2.2 The Klebanov-Polyakov conjecture

In the previous chapter, in section 1.5 we saw that it is possible to construct in AdS$_4$ a complete interacting theory of massless particles with every integer spin. They are gauge bosons in a representation of a higher spin symmetry algebra. One may wonder whether this system has a holographic dual on the AdS boundary and what is the dual theory.

First of all, as we saw in section 2.1.7, we expect that every higher spin field $\Phi_{M_1 \dots M_s}$[8] that is non-vanishing on the boundary is a source for some observable in the CFT. Moreover, since $\Phi_{M_1 \dots M_s}(z, x)$ is a gauge field, it should be coupled to a completely symmetric conserved current $\mathcal{J}_{\mu_1 \dots \mu_s}(x)$. So it is natural to identify these currents with those of the $O(N)$ model (2.2.5). This is also hinted by the fact that the conformal dimensions of the dual of $\Phi_{M_1 \dots M_s}$ and of $\mathcal{J}_{\mu_1 \dots \mu_s}$ (2.1.54) and (2.2.9) are the same.

However in the progress of identification of fields in the two sides we immediately encounter two issues. First of all, the $O(N)$ model has currents with only even spin, while in general Vasiliev theory involves also the odd ones. This is solved by considering the minimal model with only even spins as explained in section 1.5. Here we consider the A type theory, see section 1.5. The second difficulty is more subtle and is related to the scalar field. In subsection 2.1.6 we saw that the dual of $\phi(z, x)$ should have dimension

$$\Delta_+ > \frac{d}{2} = \frac{3}{2}$$

by (2.1.32), but in the $O(N)$ theory the scalar current is $\varphi^a\varphi^a(x)$, whose weight is 1. Therefore our identification between boundary and bulk fields can not work, even though only for spin 0.

We can now proceed in two different ways, as suggested by Klebanov and Polyakov in the paper where they introduced the higher spin/$O(N)$ model duality [43]. First of all notice that for the bulk scalar field

$$\Delta_- = 1, \quad \Delta_+ = 2.$$

These relations suggest to change the identification rules (2.1.36) and (2.1.37) and consider the solution that is asymptotic to z^{Δ_+} instead of z^{Δ_-} and define

$$\bar{\phi}(x) \equiv \lim_{z\to 0} z^{-\Delta_+}\phi(z, x)$$

[8]Here the metric-like form of Fronsdal fields is more convenient.

so that the dual operator has dimension $\Delta_- \equiv \Delta = 1$ and the correct bulk-to-boundary propagator is

$$\tilde{K}_\Delta(z,x) \equiv C_{\Delta_+}\left(\frac{z}{z^2+x^2}\right)^{\Delta_-}.$$

This restores the duality rules. Indeed, it has been shown in [46] that, if[9]

$$\Delta_\pm > \frac{d-2}{2}$$

both choices of the conformal weight for the dual operator $\Delta = \Delta_\pm$ lead to a consistent AdS/CFT duality between two related theories. As a consequence of our identification of $\bar{\phi}$ and E in (2.1.62) as the source and the expectation value of the dual operator, we see that choosing $\Delta = \Delta_-$ amounts to exchanging the roles of $\bar{\phi}$ and E. Since $\mathcal{J}$ and $\bar{\phi}$ can be seen as conjugated variables, we can expect that the CFT partition function of the theory associated to Δ_- is the Legendre transform of the usual one where the scalar operator has dimension Δ_+. Indeed it has been shown in [46] that this transformation leads to the correct bulk-to-boundary propagator K_{Δ_-}.

If to z^{Δ_+} we associated the free $O(N)$ model, we should find by choosing z^{Δ_-} another CFT with a scalar operator with weight 2. Consider then the following deformation of our original action

$$\delta S\left[\varphi^i\right] = \int \frac{\lambda}{2N}\left(\varphi^i\varphi^i\right)^2 d^3x, \tag{2.2.10}$$

namely an interaction vertex with a 't Hooft-like coupling $\frac{\lambda}{2N}$. The resulting action $S_{int}\left[\varphi^i\right]$ is not conformally invariant from the classical point of view, since the dimension of φ is $\frac{1}{2}$ and therefore (2.2.10) is not even scale invariant (see (2.1.5)). However quantization can alter the symmetry properties of a theory. In this case the renormalization group flow brings the interacting theory to an IR critical point where it is conformal. Currents $\mathcal{J}^{\mu_1\cdots\mu_s}$ are not conserved exactly, but only in the large N limit. Similarly, the (anomalous) dimension of $\mathcal{J}$ is $2 + O\left(\frac{1}{N}\right)$, coinciding thus with Δ_+ only for $N \to +\infty$. The weights of every other spin is the same as in the free theory up to terms of order $\frac{1}{N}$. These results are more general, as it has been proven in [44], where the addition of a multitrace deformation to the original action on the boundary can lead an operator to change its dimension from Δ_- to Δ_+ , by a renormalization group flow.

We may then summarize the so-called Klebanov-Polyakov conjecture in the following way

> The A-type minimal Vasiliev theory in AdS_4 is dual to a Conformal Field Theories of scalars with global $O(N)$ symmetry through the identification of the Fronsdal fields in the bulk with sources of conserved currents on the boundary. If one takes as dimension of the dual scalar operator Δ_+, the theory at the boundary is a free CFT, while if Δ_- is taken, it is the critical point of the theory at large N with a $\left(\varphi^i\varphi^i\right)^2$ interaction.

This conjecture can be further generalized. Type B minimal Vasiliev theory has been conjectured by Sezgin and Sundell to be dual to an $O(N)$ model where the real scalar is replaced by a real fermion field. When considering the Δ_+ dimension, the boundary theory is the free one, while in the other case it is the critical point of the one with an interaction of the kind $\left(\bar{\psi}^i\psi^i\right)^2$. Non-minimal theories appear to be linked with a CFT endowed with a global $U(N)$ symmetry that allows currents of all spins. For example, the spin 1 current is

$$\mathcal{J}^\mu = \varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*.$$

[9] This limitation comes from the fact that it is possible to regularize the action in such a way that it is finite for both the possible asymptotic behaviors of ϕ, see (2.1.63) in the previous subsection for the free case.

2.3 Tests of the KP conjecture

The AdS/CFT correspondence is an equivalence that involves the actions of the theories on both sides. For this reason, the way in which we formulated the interacting theory of higher spin field in subsection 1.5 is not very suitable to test the Klebanov-Polyakov conjecture, in that it is based solely on the equations of motion. Indeed the most direct test would be to compute correlators by Feynman diagrams on the boundary and by Witten diagrams in the bulk and verify that they coincide. However, without a higher spin action this is not possible. A possible solution to this issue would be to construct a part of the action by carrying out a perturbative analysis of the Vasiliev equations and derive the interaction vertices order by order. This would allow to compute Witten diagrams in the way described in subsection 2.1.7. We will pursue this approach in chapter 4.

In their paper [48], Giombi and Yin proposed an alternative approach, based on the equations of motion and the results of subsection 2.1.8. Namely, the idea is to compute the correlator[10]

$$\left\langle \mathcal{J}^{(s_1)}(x_1)\mathcal{J}^{(s_2)}(x_2)\mathcal{J}_2^{(s_3)}(x_3)\right\rangle \tag{2.3.1}$$

by solving the equations of motion in presence of point-like sources located at x_2 and x_3 and computing the limit value $E(x_1)$, which represents by (2.1.67) the sought correlator (2.3.1). More precisely

$$E(x_1) = \mathcal{C}_{s_1} a_{s_2} a_{s_3} \left\langle \mathcal{J}^{(s_1)}(x_1)\mathcal{J}^{(s_2)}(x_2)\mathcal{J}_2^{(s_3)}(x_3)\right\rangle .$$

where $\mathcal{C}_s, a_{s_1}, a_{s_2}, a_{s_3}$ are some unknown normalization factors. Comparing the different results coming from $E(x_2)$ and $E(x_3)$ and from $\left\langle \mathcal{J}^{(s_j)}(x)\mathcal{J}^{(s_j)}(y)\right\rangle$ one can determine also the unknown normalizations up to a common factor, which is interpreted as the coupling constant associated to that particular cubic vertex. In this way a verification of the conjecture for 3-point correlators has been carried out for the case $s_3 = 0$ and $s_2 = s_3 = 0$.

As we saw in subsection 1.5, the dynamics of interacting higher spin fields can be reduced to the twistorial space parametrized by $y, \bar{y}, z,$ and $\bar{z}$, since the space-time connection is flat and can be gauged away. Naturally, it is always in principle possible to do the opposite, namely to use the equations of motion to reformulate the theory in AdS and eliminate the dependence on the twistors. Through the unfolding formalism, Vasiliev noticed in [49] that this means that a certain system can have the same description in twistorial space but different realizations in space-time. In particular, one can link theories living in different dimensions, thus giving rise to a holographic duality. In [49] it has been shown that in 3 dimensions conserved currents are described by unfolded equations that are equivalent to the Vasiliev ones in 4 dimensions.

The goal of the thesis is to present the 3-point correlator test with a different formalism, that employs the standard way to compute Witten diagrams by differentiating an action. To do so, we will need to find the cubic interaction vertices between three higher spins in a particular gauge. To do so, we will introduce in the next chapter the so-called ambient space, that will also allow us to find the functional form of correlators in the CFT without computing them by Feynman diagrams and thus to make a comparison in a simpler way.

[10] Here we use a shorthand notation to denote currents with spin s_1, s_2 and s_3.

Chapter 3

Ambient space formalism

As we have seen in chapter 2, the symmetries of the theories in the bulk and the boundary are one of the fundamental reasons to conjecture the AdS_4/CFT_3 duality. We would like to employ them to simplify our computations, but we encounter some difficulties in doing so. In the bulk, the $SO(2,3)$ symmetry is realized by isometries of the AdS_4 background space, which is a curved manifold. Therefore, we have to employ covariant derivatives, which are non-commutative and hence difficult to handle. On the other hand, $SO(2,3)$ appears in the theory at the boundary as the conformal group (see 3.1.2), that acts non-linearly on the coordinates.

These two problems have actually a common solution: the so-called ambient space formalism. In this chapter we will explain how to describe both theories in a greatly simplified way by embedding them in an extended space-time that will be flat, and thus will allow us to get rid of covariant derivatives and employ the usual ones, and whose symmetry group is $SO(2,3)$ acting as the usual Lorentz group.

3.1 The conformal group

3.1.1 Definition of the group

A conformal transformation between two open sets $U, V \subseteq \mathbb{R}^n$ is a continuous and differentiable map $\mathcal{C}(x) : U \to V$ that preserves the angles between the tangent vectors of any two curves in their intersection. The angle between two vectors $v(x)$ and $w(x)$ is computed from their scalar products in the following way (we omit the arguments of v and w not to clutter the notation and we denote the standard metric as $\eta_{ij} \equiv \delta_{ij}$)

$$\cos(\theta) = \frac{v^i w^j \eta_{ij}}{\sqrt{(v^i v^j \eta_{ij})\,(w^i w^j \eta_{ij})}} \tag{3.1.1}$$

Therefore, imposing invariance of $\cos(\theta)$ under the transformation $\mathcal{C}(x)$, by (3.1.1) we find

$$\frac{v^i w^j \eta_{ij}}{\sqrt{(v^i v^j \eta_{ij})\,(w^i w^j \eta_{ij})}} = \frac{v^a w^b \frac{\partial \mathcal{C}^i}{\partial x^a}\frac{\partial \mathcal{C}^j}{\partial x^b}\eta_{ij}}{\sqrt{\left(v^a v^b \frac{\partial \mathcal{C}^i}{\partial x^a}\frac{\partial \mathcal{C}^j}{\partial x^b}\eta_{ij}\right)\left(w^a w^b \frac{\partial \mathcal{C}^i}{\partial x^a}\frac{\partial \mathcal{C}^j}{\partial x^b}\eta_{ij}\right)}}. \tag{3.1.2}$$

Now, if we take $v \perp w$, (3.1.2) becomes simply

$$v^a w^b \frac{\partial \mathcal{C}^i}{\partial x^a}\frac{\partial \mathcal{C}^j}{\partial x^b}\eta_{ij} = 0 \tag{3.1.3}$$

and therefore, since (3.1.3) holds for every v and w (perpendicular to each other),

$$\frac{\partial \mathcal{C}^i}{\partial x^a}\frac{\partial \mathcal{C}^j}{\partial x^b}\eta_{ij} = f(x)\eta_{ab} \tag{3.1.4}$$

for some $f(x)$, the so-called conformal factor, that in general may be different from point to point. This expression resembles the defining relation of a rotation. Indeed, it would be such if $f \equiv 1$. Here, though, vectors appear not only to be rotated, but also rescaled, so that we deduce that a conformal transformation is equivalent to a local rescaling and rotation of the metric.

In a more general setting, a conformal transformation $\mathcal{C}$ is a diffeomorphism between two differentiable manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ endowed with two metrics (or pseudo-metrics) $g_1(x)$ and $g_2(x)$ such that

$$f(x) g_1(x) = g_2^*(x) \tag{3.1.5}$$

where $\cdot^*$ denotes the pull-back associated to $\mathcal{C}$. Conformal transformations between a manifold $\mathcal{M}$ and itself are then a sub-group $Conf(\mathcal{M})$ of the diffeomorphisms that have the following properties:

- The identical map belongs trivially to $Conf(\mathcal{M})$ with $f \equiv 1$
- Each conformal transformation has an inverse for which the conformal factor is $1/f$
- The composition of $\mathcal{C}_1, \mathcal{C}_2 \in Conf(\mathcal{M})$ with conformal factors f_1 and f_2 is a new conformal transformation with $f = f_1 \cdot f_2$

$Conf(\mathcal{M})$ is a Lie group.

Our aim is now to classify all possible conformal transformations associated with the Minkowski d-dimensional space-time and its usual pseudo-metric $\eta_{\mu\nu}$. Since we are interested in the ones connected with the identity, we can start from their infinitesimal version $\mathcal{C}^\mu(x) = x^\mu + \varepsilon^\mu(x)$ $(f(x) = 1 + c(x))$. Equation (3.1.4) becomes then

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = c\eta_{\mu\nu}. \tag{3.1.6}$$

Now, we can eliminate c by taking the trace of (3.1.6) to obtain

$$c = \frac{2}{d} \partial \cdot \varepsilon \tag{3.1.7}$$

and inserting this relation back into (3.1.6):

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d} (\partial \cdot \varepsilon) \eta_{\mu\nu}. \tag{3.1.8}$$

Take twice the divergence of (3.1.8):

$$\Box \varepsilon_\nu = -\frac{2}{d} \partial_\nu (\partial \cdot \varepsilon) \tag{3.1.9}$$

$$\left(1 - \frac{2}{d}\right) \Box (\partial \cdot \varepsilon) = 0. \tag{3.1.10}$$

Notice that (3.1.10) is non-trivial only for $d \neq 2$. We will assume $d > 2$ for the rest of the discussion. From (3.1.10) we can deduce that c is a function at most quadratic in x and thus we can parametrize it as

$$\varepsilon^\mu(x) = a^\mu + q^{\mu\nu} x_\nu + r^{\mu\nu\rho} x_\nu x_\rho.$$

where $b^{\mu\nu\rho} = b^{\mu\rho\nu}$. Then, from (3.1.8), choosing $r = 0$, we get

$$q^{(\mu\nu)} = \frac{1}{2} q^\alpha{}_\alpha \eta^{\mu\nu}$$

that is, we can express q as a sum of its symmetric and antisymmetric parts

$$q^{\mu\nu} = \Delta \eta^{\mu\nu} + \omega^{\mu\nu}, \quad \omega^{\mu\nu} = -\omega^{\nu\mu}.$$

Applying a derivative on (3.1.8) and permuting indices we find that $r_{\mu\nu\sigma}$ is expressed in terms of its trace $r^{\alpha}{}_{\alpha\rho}$, namely:

$$\begin{array}{rll}
\partial_\rho\partial_\mu\varepsilon_\nu + \partial_\rho\partial_\nu\varepsilon_\mu = & \frac{2}{d}\eta_{\mu\nu}\partial_\rho\left(\partial\cdot\varepsilon\right) & |+ \\
\partial_\mu\partial_\nu\varepsilon_\rho + \partial_\mu\partial_\rho\varepsilon_\nu = & \frac{2}{d}\eta_{\nu\rho}\partial_\mu\left(\partial\cdot\varepsilon\right) & |- \\
\partial_\nu\partial_\rho\varepsilon_\mu + \partial_\nu\partial_\mu\varepsilon_\rho = & \frac{2}{d}\eta_{\rho\mu}\partial_\nu\left(\partial\cdot\varepsilon\right) & |+ \\
 & & \Downarrow \\
2\partial_\nu\partial_\rho\varepsilon_\mu = & \frac{2}{d}\left(\eta_{\mu\nu}\eta_{\alpha\rho} - \eta_{\nu\rho}\eta_{\alpha\mu} + \eta_{\rho\mu}\eta_{\alpha\nu}\right)\partial^\alpha\left(\partial\cdot\varepsilon\right) &
\end{array}$$

or

$$r_{\mu\nu\rho} = \frac{1}{d}\left(\eta_{\mu\nu}r^{\alpha}{}_{\alpha\rho} - \eta_{\nu\rho}r^{\alpha}{}_{\alpha\mu} + \eta_{\rho\mu}r^{\alpha}{}_{\alpha\nu}\right)$$

so that, introducing the vector $b_\mu = r^{\alpha}{}_{\alpha\mu}$, we finally get the most general form of a conformal transformation:

$$\begin{aligned}
\varepsilon^\mu(x) &= a^\mu + \Delta x^\mu + \omega^{\mu\nu}x_\nu + \frac{1}{d}\left(x^\mu r^{\alpha}{}_{\alpha\rho}x^\rho - x^2 r_{\alpha}{}^{\alpha\mu} + x^\mu r^{\alpha}{}_{\alpha\nu}x^\nu\right) \\
&= a^\mu + \Delta x^\mu + \omega^{\mu\nu}x_\nu - x^2 b^\mu + 2\left(x\cdot b\right)x^\mu.
\end{aligned}$$

We see that c comprises

- Translations with parameter a^μ ($x^\mu \to x^\mu + a^\mu$) generated by $P^\mu = -i\partial^\mu$.
- Lorentz transformation with parameter $\omega^{\mu\nu}$ ($x^\mu \to \omega^{\mu}{}_{\nu}x^\nu$) generated by $M_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)$.
- Dilations with parameter Δ ($x^\mu \to \lambda x^\mu$), generated by $D = -ix^\alpha\partial_\alpha$.
- Special Conformal Transformations with parameter b^μ generated by $K_\mu = -i\left(2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu\right)$.

The finite version of the latter transformation is

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2\left(bx\right) + b^2x^2} = \frac{x^\mu + b^\mu x^2}{\left(x^\nu + b^\nu x^2\right)^2}x^2 \tag{3.1.11}$$

that can be obtained as an inversion (i.e. $x^\mu \to \frac{x^\mu}{x^2}$), followed by a translation with parameter b^μ, followed by another inversion:

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + b^\mu = \frac{x^\mu + b^\mu x^2}{x^2} \to \frac{x^\mu + b^\mu x^2}{\left(x^\mu + b^\mu x^2\right)^2}x^2.$$

We would like now to determine the Lie algebra of $Conf\left(\mathbb{R}^{1,d-1}\right)$. It is obvious that $SO(1,d-1) < Conf(\mathbb{R}^{1,d-1})$. Therefore, in addition to the known commutators of $\mathfrak{so}(1,d-1)$ presented in (1.1.1), we have

$$[D, M_{\alpha\beta}] = 0$$

$$[K_\alpha, M_{\beta\gamma}] = i\eta_{\alpha\gamma}K_\beta - i\eta_{\alpha\beta}K_\gamma,$$

determined by the Lorentz algebra of a scalar D and a vector K_α and

$$[D, P_\alpha] = \left(\partial_\alpha\left(x^\mu\partial_\mu\right) - x^\mu\partial_\mu\partial_\alpha\right) = \partial_\alpha = iP_\alpha$$

$$\begin{aligned}
[P_\alpha, K_\beta] &= -\partial_\alpha\left(2x_\beta x^\gamma\partial_\gamma - x^\gamma x_\gamma\partial_\beta\right) + \left(2x_\beta x^\gamma\partial_\gamma - x^\gamma x_\gamma\partial_\beta\right)\partial_\alpha \\
&= -2\eta_{\alpha\beta}x^\gamma\partial_\gamma - 2x_\beta\partial_\alpha + 2x_\alpha\partial_\beta \\
&= -2i\eta_{\alpha\beta}D - 2iM_{\alpha\beta}
\end{aligned} \tag{3.1.12}$$

$$
\begin{aligned}
[D, K_\alpha] &= -x^\mu \partial_\mu \left(2x_\alpha x^\gamma \partial_\gamma - x^\gamma x_\gamma \partial_\alpha\right) + \left(2x_\alpha x^\gamma \partial_\gamma - x^\gamma x_\gamma \partial_\alpha\right) x^\mu \partial_\mu \\
&= -2x_\alpha x^\gamma \partial_\gamma - 2x_\alpha x^\gamma \partial_\gamma + 2x^\gamma x_\gamma \partial_\alpha + 2x_\alpha x^\mu \partial_\mu - x^\gamma x_\gamma \partial_\alpha \\
&= +x^2 \partial_\alpha - 2x_\alpha x^\gamma \partial_\gamma \\
&= iK_\alpha
\end{aligned}
$$

$$
\begin{aligned}
[K_\alpha, K_\beta] = &- \left(2x_\alpha x^\gamma \partial_\gamma - x^\gamma x_\gamma \partial_\alpha\right)\left(2x_\beta x^\delta \partial_\delta - x^\delta x_\delta \partial_\beta\right) + \left(2x_\beta x^\delta \partial_\delta - x^\delta x_\delta \partial_\beta\right)\left(2x_\alpha x^\gamma \partial_\gamma - x^\gamma x_\gamma \partial_\alpha\right) \\
= &-8x_\alpha x_\beta x^\delta \partial_\delta + 2x_\alpha x^2 \partial_\beta + 2x^2 \eta_{\alpha\beta} x^\delta \partial_\delta + 2x^2 x_\beta \partial_\alpha + 2x^2 x_\alpha \partial_\beta \\
&+ 8x_\alpha x_\beta x^\delta \partial_\delta - 2x_\beta x^2 \partial_\alpha - 2x^2 \eta_{\alpha\beta} x^\delta \partial_\delta - 2x^2 x_\alpha \partial_\beta - 2x^2 x_\beta \partial_\alpha \\
= &\ 0.
\end{aligned}
$$

3.1.2 Embedding the conformal group in the ambient space

The representation of the conformal group that led us to its definition is not linear. Therefore, we would like to find a space where $Conf\left(\mathbb{R}^{1,d-1}\right)$ acts linearly, namely an extension of the Minkowski space with extra coordinates. We call this manifold $\mathcal{A}_{d+2}$, the ambient space, for which $\mathbb{R}^{1,d-1}$ is just a sub-manifold. The simplest extension possible is a vector space that contains $\mathbb{R}^{1,d-1}$ as a sub-space. We can therefore endow $\mathcal{A}_{d+2}$ with a diagonal metric η_{mn} $(\mathrm{m}, \mathrm{n} = 0, 1, ..., N)$ and parametrize it with N coordinates $\boldsymbol{X}^{\mathrm{m}}$ so that η_{mn} reduces to $\eta_{\mu\nu}$ in the Minkowski sub-space. To keep things as simple as possible, we may simply set $\boldsymbol{X}^\mu = x^\mu$ and add a number of coordinates starting from $\boldsymbol{X}_d$.

$Conf\left(\mathbb{R}^{1,d-1}\right)$ is defined by the property of acting on the $\mathbb{R}^{1,d-1}$ metric just as a rescaling, that is not always constant, but depends on the coordinates when the transformation considered is non-linear. On the other hand, we want to realize every element of the conformal group linearly on $\mathcal{A}_{d+2}$. Thus conformal transformations may cause the ambient metric to rescale only by a constant coefficient. From

$$
\boldsymbol{X}^{\mathrm{m}} \boldsymbol{X}_{\mathrm{m}} = x^2 + \sum_{\tilde{\mathrm{m}} > d-1} \left(\boldsymbol{X}^{\tilde{\mathrm{m}}}\right)^2 \eta_{\tilde{\mathrm{m}}\tilde{\mathrm{m}}}
$$

we see that this implies that also x^2 acquires just a constant coefficient, that is in contrast to what we expect to happen. If, on the other hand, we applied a local rescaling to $\boldsymbol{X}$ after the linear transformation, we would solve our problem. This rescaling can be justified only if we say that a point x of the physical space is represented by a class of vectors $\boldsymbol{X}$ that differ one from each other only by a scale factor that may depend on x. In this class obviously a vector such that $\boldsymbol{X}^\mu = x^\mu$ has to be present.

This observation suggests to take as a sub-manifold representing $\mathbb{R}^{1,d-1}$ one that is left invariant under rescalings. The simplest non trivial one is the light-cone $\mathscr{L}_{d+1}$ defined by

$$
\boldsymbol{X}^{\mathrm{m}} \boldsymbol{X}_{\mathrm{m}} = x^2 + \sum_{\tilde{\mathrm{m}} > d-1} \left(\boldsymbol{X}^{\tilde{\mathrm{m}}}\right)^2 \eta_{\tilde{\mathrm{m}}\tilde{\mathrm{m}}} = 0. \tag{3.1.13}
$$

Since our x^μ are unconstrained, x^2 can be both positive and negative and thus we have to add at least 2 coordinates to satisfy (3.1.13), where at least one is time-like and the other is space-like. If we stick to this lower limit and assume $\mathcal{A}_{d+2} = \mathbb{R}^{2,d}$, then we can put (3.1.13) in the useful form

$$
\boldsymbol{X}^{\mathrm{m}} \boldsymbol{X}_{\mathrm{m}} = x^2 + 2\boldsymbol{X}_+ \boldsymbol{X}_-
$$

by switching to the light-cone coordinates

$$
\boldsymbol{X}_+ = \frac{\boldsymbol{X}_d + \boldsymbol{X}_{d+1}}{\sqrt{2}}, \quad \boldsymbol{X}_- = \frac{-\boldsymbol{X}_d + \boldsymbol{X}_{d+1}}{\sqrt{2}}
$$

so that the metric is represented by

$$\eta = \begin{pmatrix} \eta_{\mu\nu} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Equation (3.1.13) is just one constraint: we have to find another one in order to have a manifold with dimension d. Since it has to affect only the two additional coordinates, we may take

$$\boldsymbol{X}_+ = 1 \tag{3.1.14}$$

not to spoil (3.1.13). The sub-manifold described by (3.1.13) and (3.1.14) is

$$\mathscr{L}_{d+1}/\mathbb{R} = \left\{\boldsymbol{X}^{\mathrm{m}} | \boldsymbol{X}^2 = 0, \boldsymbol{X}_+ = 1\right\} \tag{3.1.15}$$

the set of light rays or, with abuse of language, simply the light-cone. Here the Lorentz group $SO(1,d-1)$ is implemented trivially by matrices acting only on the first d coordinates and leaves $\mathscr{L}_{d+1}/\mathbb{R}$ invariant.

As it is known, $SO(1,d-1) < SO(2,d)$, the symmetry group of the ambient space. Let us then see what is the effect of the remaining symmetry of $\mathcal{A}_{d+2}$ on $\mathscr{L}_{d+1}/\mathbb{R}$. After a transformation of $SO(2,d)$, in general (3.1.14) does not hold anymore. Since we are free to take rescalings of the light-cone points, we may perform an additional dilation of factor $\lambda(\boldsymbol{X}) = \boldsymbol{X}_+^{-1}$ to return to $\mathscr{L}_{d+1}/\mathbb{R}$: $\boldsymbol{X}^{\mathrm{m}} \to \boldsymbol{X}'^{\mathrm{m}}$. The combination of these two transformations results in the following change of the metric of $\mathscr{L}_{d+1}/\mathbb{R}$:

$$\begin{aligned} d\boldsymbol{X}'^{\mathrm{m}} d\boldsymbol{X}'_{\mathrm{m}} &= \left(\lambda(\boldsymbol{X}) d\boldsymbol{X}^{\mathrm{m}} + \boldsymbol{X}^{\mathrm{m}} \frac{\partial\lambda(\boldsymbol{X})}{\partial \boldsymbol{X}^{\mathrm{n}}} d\boldsymbol{X}^{\mathrm{n}}\right)^2 \\ &= \lambda^2(\boldsymbol{X}) d\boldsymbol{X}^2 + 2\lambda \boldsymbol{X}^{\mathrm{m}} \frac{\partial\lambda(\boldsymbol{X})}{\partial \boldsymbol{X}^{\mathrm{n}}} d\boldsymbol{X}^{\mathrm{n}} d\boldsymbol{X}_{\mathrm{m}} + \boldsymbol{X}^2 \left(\frac{\partial\lambda(\boldsymbol{X})}{\partial \boldsymbol{X}^{\mathrm{n}}} d\boldsymbol{X}^{\mathrm{n}}\right)^2 \\ &= \lambda^2(\boldsymbol{X}) d\boldsymbol{X}^2 \end{aligned} \tag{3.1.16}$$

where we exploited that $\boldsymbol{X}^2 = 0$ and therefore $\boldsymbol{X}^{\mathrm{m}} d\boldsymbol{X}_{\mathrm{m}} = \frac{1}{2} d\left(\boldsymbol{X}^2\right) = 0$. Equation (3.1.16) is just the defining relation of the conformal group, once we restrict ourselves to the metric of the first d coordinates. We may then deduce that $SO(2,d) \cong Conf\left(\mathbb{R}^{1,d-1}\right)$.

It is thus possible to reconstruct the conformal Lie algebra from $\mathfrak{so}(2,d)$. Its generators form an antisymmetric tensor of rank 2 which respects the following commutation relation

$$\left[M^{\mathrm{mn}}, M^{\mathrm{ab}}\right] = i\left(M^{\mathrm{an}}\eta^{\mathrm{mb}} - M^{\mathrm{bn}}\eta^{\mathrm{ma}} - M^{\mathrm{ma}}\eta^{\mathrm{nb}} + M^{\mathrm{bm}}\eta^{\mathrm{na}}\right). \tag{3.1.17}$$

Let us decompose it into relations involving the components $\mu, +, -$. First, consider $M^{+\mu}$ and $M^{-\mu}$

$$\left[M^{+\nu}, M^{+\beta}\right] = 0, \quad \left[M^{-\nu}, M^{-\beta}\right] = 0 \tag{3.1.18}$$

$$\left[M^{+\nu}, M^{-\beta}\right] = i\left(-M^{\beta\nu}\eta^{+-} - M^{+-}\eta^{\nu\beta}\right). \tag{3.1.19}$$

since $\eta^{\pm\mp} = 1$ and $\eta^{\pm\mu} = 0$. P^μ and K^μ introduced earlier generate a two Abelian sub-algebras of the conformal algebra. Therefore we are led by (3.1.18) to recognize them as

$$M^{+\mu} = \frac{1}{\sqrt{2}} P^\mu, \quad M^{-\mu} = \frac{1}{\sqrt{2}} K^\mu.$$

Here the coefficients are determined in order to get from (3.1.19) the same coefficients as in (3.1.12). We therefore see that

$$D = M^{+-}.$$

The other commutation rules are compatible with (3.1.17).

In general it can be shown that there is an isomorphism between the conformal group of $\mathbb{R}^{m,n}$ and the orthogonal group $SO(m+1,n+1)$. This relation is not valid in 2 dimensions, where the conformal group is much larger and its algebra is infinite dimensional.

Finally we write the most general expression for a point in $\mathscr{L}_{d+1}/\mathbb{R}$ for future use:

$$\boldsymbol{X}^{\mathrm{m}} \in \mathscr{L}_{d+1}/\mathbb{R} \iff \boldsymbol{X}^{\mathrm{m}} = \left(x^\mu, 1, -\frac{1}{2}x^2\right) \tag{3.1.20}$$

3.2 Embedding AdS in the ambient space

As it is well known, the $(d+1)$-dimensional Anti de Sitter space, as every maximally symmetric solution to the Einstein equations, can be described as a (pseudo)hyperboloid embedded into a $(d+2)$-dimensional space that we call $\mathcal{A}_{d+2}$ (since it coincides with the one defined in (3.1.2)) as

$$X^M X^N \eta_{MN} + \left(X^{d+1}\right)^2 = L^2 \tag{3.2.1}$$

where X^M are the usual coordinates in $d+1$ dimensions, while X^{d+1} is an additional time-like one and L is the curvature radius. If we combine these coordinates in the form $\mathcal{X}^{\mathrm{m}} = \left(X^M, X^{d+1}\right)$, where $\mathrm{m} = 0, ..., d+1$, equation (3.2.1) can be rewritten as

$$\mathcal{X}^{\mathrm{m}}\mathcal{X}^{\mathrm{n}}\eta_{\mathrm{mn}} = L^2 \tag{3.2.2}$$

where $\eta_{\mathrm{mn}} = \mathrm{diag}\,(1,-1,...,-1,1)$ and coincides with η_{MN} for the first $d+1$ entries. We call the sub-manifold of $\mathcal{A}_{d+2}$ described by (3.2.2) $\mathscr{S}_{d+1}$. Now the symmetries of AdS, namely translational and Lorentz invariance, are both explicitly unified in a general $SO(2,d)$ symmetry that preserves (3.2.2).

We may employ the light-cone coordinates in place of the last two (we keep using $\mathrm{m} = 0, 1, ..., \mathrm{d}-1, +, -$)

$$\mathcal{X}_+ = \frac{\mathcal{X}_d + \mathcal{X}_{d+1}}{\sqrt{2}}, \quad \mathcal{X}_- = \frac{-\mathcal{X}_d + \mathcal{X}_{d+1}}{\sqrt{2}}.$$

Then any point in $\mathscr{S}_{d+1}$ can be expressed by using the Poincaré coordinates z, x^μ (see (2.1.19)) as

$$\mathcal{X}^{\mathrm{m}} \in \mathscr{S}_{d+1} \iff \mathcal{X}^{\mathrm{m}} = \frac{L}{z}\left(x^\mu, 1, -\frac{1}{2}\left(x^\mu x_\mu - z^2\right)\right). \tag{3.2.3}$$

The boundary of the AdS manifold $\mathscr{S}_{d+1}$ is given by the points for which $X^M \to \infty$, that can be obtained by $z \to 0$ in (3.2.3). In this limit equation (3.2.3) may be written (asymptotically) as

$$\mathcal{X}^{\mathrm{m}} \underset{z\to 0}{\propto} \left(x^\mu, 1, -\frac{1}{2}x^\mu x_\mu\right)$$

that we can recognize as a point of the light-cone $\mathscr{L}_{d+1}$ as described by (3.1.20). The fact that the boundary of AdS_{d+1} is a Minkowski space-time with a dimension d is then expressed in the ambient space by saying that $\mathscr{S}_{d+1}$ is asymptotic to $\mathscr{L}_{d+1}$ at infinity.

3.3 Tensors in the Ambient space

In the previous two sections we showed how AdS_{d+1} and its boundary $\mathbb{R}^{1,d-1}$ can be embedded as sub-manifolds of $\mathcal{A}_{d+2}$. The aim of this section is to extend the formalism we developed for the coordinates of these two spaces to every other physical quantity. Namely, we want to convert each tensor field defined in one of the two sub-manifolds into a tensor field of the embedding $\mathcal{A}_{d+2}$.

Let us generalize our task by considering a submanifold $\mathcal{N}$ of $\mathbb{R}^{2,d} = \mathcal{A}_{d+2}$, whose dimensions is n, embedded in $\mathbb{R}^{2,d}$ by an injective differentiable map $\iota: \mathcal{N} \to \mathbb{R}^{2,d}$. It induces a map, the so called pull-back $\iota^* : \mathscr{X}^k(U) \to \mathscr{X}^k(\mathcal{N})$, between tensor fields of rank k defined on some subset $U \subset \mathbb{R}^{2,d}$ containing $\mathcal{N}$, $\mathscr{X}^k(U)$, and those defined on $\mathcal{N}$, $\mathscr{X}^k(\mathcal{N})$:

$$T_{i_1\dots i_k}(p) \equiv (\iota^*\mathcal{T})_{i_1\dots i_k}(p) = d\iota^{j_1}_{\ i_1}(\iota(p))\cdots d\iota^{j_k}_{\ i_k}(\iota(p))\mathcal{T}_{j_1\dots j_k}(\iota(p)) \tag{3.3.1}$$

where $p \in \mathcal{N}$ and $d\iota(p) : T_p(\mathcal{N}) \to \mathbb{R}^{d+2}$ is the differential map induced by ι between the tangent space of $\mathcal{N}$ in p ($T_p(\mathcal{N})$) and $\mathbb{R}^{2,d}$. This allows to express a tensor field in the embedding manifold as a field in the embedded one.

In our case, however, we want to do the opposite, namely to extend a tensor field defined on $\mathcal{N} = \mathscr{S}_{d+1}$ and $\mathscr{L}_{d+1}/\mathbb{R}$ to $\mathbb{R}^{2,d}$. To do so, we need to characterize a subspace of $\mathscr{X}^k(U)$ for which there can be an isomorphism with $\mathscr{X}^k(\mathcal{N})$. First of all we notice that the tangent space of $\mathbb{R}^{2,d}$, namely a copy of $\mathbb{R}^{2,d}$ itself, can be decomposed as

$$\mathbb{R}^{2,d} = T_p(\mathcal{N}) \oplus T_p^{\perp}(\mathcal{N}),$$

where $T_p^{\perp}(\mathcal{N})$ is the space of the vectors orthogonal to $\mathcal{N}$. Therefore we take the ambient tensors that do not have components in $T_p^{\perp}(\mathcal{N})$, namely

$$\mathcal{T} \in \mathscr{X}^k(U) \text{ such that } \mathcal{T}(p) \in \mathscr{T}_p^k \equiv \bigotimes_{i=1}^{k} T_p(\mathcal{N}). \tag{3.3.2}$$

We call this condition ambient transversality[1].

Their domain of definition U should be such that the value of the ambient tensor $\mathcal{T}$ at every point $q \in U$ is uniquely determined by the value of its counterpart T on $\mathcal{N}$ at some $p \in \mathcal{N}$. In this way ambient tensors do not encode more information than those living in the sub-manifold.

In our case, both $\mathscr{L}_{d+1}/\mathbb{R}$ and $\mathscr{S}_{d+1}$ are defined by a condition of the kind

$$\tilde{\mathtt{X}}^2 = \alpha \tag{3.3.3}$$

for some $\alpha \geq 0$, which makes every vector $\tilde{\mathtt{X}}^{\mathrm{m}}$ belong to $T^{\perp}(\mathcal{N})$. We then choose

$$U_{\mathscr{S}} = \left\{\mathtt{X}|\mathtt{X}^2 > 0\right\}, \quad U_{\mathscr{L}} = \left\{\mathtt{X}|\mathtt{X}^2 = 0\right\}$$

as domains of definition. Since the Us are invariant under rescalings, a natural way to determine the values of tensors in $U_{\mathscr{L}}$ and $U_{\mathscr{S}}$ outside $\mathcal{N}$ is to set

$$\mathcal{T}(\lambda\mathtt{X}) = f(\lambda)\mathcal{T}(\mathtt{X})$$

so that the values on $\mathcal{N}$ determine those all over U. We choose $f(\lambda) = \lambda^{-\Delta}$ for some $\Delta \in \mathbb{R}$, that we call degree:

$$\mathcal{T}(\lambda\mathtt{X}) = \lambda^{-\Delta}\mathcal{T}(\mathtt{X}). \tag{3.3.4}$$

Moreover (3.3.3) implies that the vectors in $T^{\perp}(\mathcal{N})$ are those parallel to $\tilde{\mathtt{X}}^{\mathrm{m}}$ in $U_{\mathscr{L}}$ and $U_{\mathscr{S}}$. To them, in the case of $\mathcal{N} = \mathscr{L}_{d+1}/\mathbb{R}$, we should add the linear combinations with $\delta^{\mathrm{m}+}$, which comes from the condition $\boldsymbol{X}^+ = 1$ in (3.1.15). Therefore condition (3.3.2) can be expressed by saying that $\mathscr{X}^k(\mathcal{N})$ is isomorphic to

$$\mathscr{T}^k\left(U_{\mathscr{S}}\right) = \left\{\mathcal{T} \in \mathscr{X}^k\left(U_{\mathscr{S}}\right)|\mathcal{X}_{\mathrm{m}_i}\mathcal{T}^{\mathrm{m}_1\dots\mathrm{m}_k}(\mathtt{X}) = 0\forall i = 1,\dots,k,\ \mathtt{X} \in U_{\mathscr{S}}\right\} \tag{3.3.5}$$

where

$$\mathcal{X}(\mathtt{X}) \equiv \left(\frac{\mathtt{X}^2}{L^2}\right)^{-\frac{1}{2}}\mathtt{X}$$

[1]Not to be confused with transversality in the sense of (1.1.4)

and

$$\mathscr{T}^k(U_{\mathscr{L}}) = \left\{\mathcal{T} \in \mathscr{X}^k(U_{\mathscr{L}}) \,|\, \mathtt{X}_{\mathrm{m}_i}\mathcal{T}^{\mathrm{m}_1\ldots\mathrm{m}_k}(\mathtt{X}) = 0 = \delta^+_{\mathrm{m}_i}\mathcal{T}^{\mathrm{m}_1\ldots\mathrm{m}_k}(\mathtt{X}) \forall i = 1,...,k,\ \mathtt{X} \in U_{\mathscr{L}}\right\} \tag{3.3.6}$$

for $\mathcal{N} = \mathscr{S}_{d+1}$ and $\mathcal{N} = \mathscr{L}_{d+1}/\mathbb{R}$ respectively[2].

In our calculation, we can realize the spaces (3.3.5) and (3.3.6) in the case of $\mathscr{S}_{d+1}$ by

$$\mathcal{T}^{\mathrm{m}_1\ldots\mathrm{m}_k}(\mathtt{X}) = \left(\frac{\mathtt{X}^2}{L^2}\right)^{-\frac{\Delta}{2}} \frac{\partial \mathcal{X}^{\mathrm{m}_1}}{\partial X^{M_1}} \cdots \frac{\partial \mathcal{X}^{\mathrm{m}_k}}{\partial X^{M_k}} T^{M_1\ldots M_k}(\mathcal{X}(\mathtt{X})) \tag{3.3.7}$$

where one should differentiate (3.2.3) and for $\mathscr{L}_{d+1}/\mathbb{R}$ by

$$\mathcal{T}^{\mathrm{m}_1\ldots\mathrm{m}_k}(\mathtt{X}) = \left(\mathtt{X}^+\right)^{-\Delta} \frac{\partial \boldsymbol{X}^{\mathrm{m}_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \boldsymbol{X}^{\mathrm{m}_k}}{\partial x^{\mu_k}} T^{\mu_1\ldots\mu_k}(\boldsymbol{X}(\mathtt{X})) \tag{3.3.8}$$

where

$$\boldsymbol{X}(\mathtt{X}) \equiv \frac{\mathtt{X}}{\mathtt{X}^+}.$$

Let us check that (3.3.7) is consistent with our requirements. We do this just for (3.3.7) and with vectors, for simplicity, the proof for the general case being the same. First of all, (3.3.1) composed with (3.3.7) should be the identity:

$$T_M = \frac{\partial \mathcal{X}^{\mathrm{m}}}{\partial X^M}\mathcal{T}_{\mathrm{m}} = \frac{\partial \mathcal{X}^{\mathrm{m}}}{\partial X^M}\eta_{\mathrm{mn}}\frac{\partial \mathcal{X}^{\mathrm{n}}}{\partial X^N}T^N = g_{MN}T^N = T_M.$$

Here we used (3.3.1) applied to the metric tensor:

$$\eta_{\alpha\beta} = \frac{\partial \boldsymbol{X}^{\mathrm{m}}}{\partial x^\alpha}\frac{\partial \boldsymbol{X}^{\mathrm{n}}}{\partial x^\beta}\eta_{\mathrm{mn}}, \quad g_{AB} = \frac{\partial \mathcal{X}^{\mathrm{m}}}{\partial X^A}\frac{\partial \mathcal{X}^{\mathrm{n}}}{\partial X^B}\eta_{\mathrm{mn}}. \tag{3.3.9}$$

Transversality, on the other hand, follows from

$$\mathcal{X}_{\mathrm{m}}\mathcal{T}^{\mathrm{m}}(\mathcal{X}) = \mathcal{X}_{\mathrm{m}}\frac{\partial \mathcal{X}^{\mathrm{m}}}{\partial X^M}T^M(X) = \frac{1}{2}\frac{\partial\left(\mathcal{X}_{\mathrm{m}}\mathcal{X}^{\mathrm{m}}\right)}{\partial X^M}T^M(X) = 0.$$

Similar considerations are valid for tensors in $\mathscr{L}_{d+1}/\mathbb{R}$. In this case, though, the transversality constraint must be formulated in a stronger way. Indeed, suppose that it was possible to write an ambient tensor as

$$\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\boldsymbol{X}) = \boldsymbol{X}_{\mathrm{m}_1}\mathcal{S}_{\mathrm{m}_2\ldots\mathrm{m}_k}. \tag{3.3.10}$$

When considering contractions with the first index, (3.3.6) is trivially satisfied in virtue of the light-cone defining relation $\boldsymbol{X}^2 = 0$. Let us inspect the $\mathscr{L}_{d+1}$ tensor associated to (3.3.10):

$$\begin{aligned}
T_{\mu_1\ldots\mu_k}(x) &= \frac{\partial \boldsymbol{X}^{\mathrm{m}_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \boldsymbol{X}^{\mathrm{m}_k}}{\partial x^{\mu_k}}\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\boldsymbol{X}(x)) \\
&= \frac{\partial \boldsymbol{X}^{\mathrm{m}_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \boldsymbol{X}^{\mathrm{m}_k}}{\partial x^{\mu_k}}\boldsymbol{X}_{\mathrm{m}_1}\mathcal{S}_{\mathrm{m}_2\ldots\mathrm{m}_k} \\
&= \frac{\partial\left(\boldsymbol{X}^2\right)}{\partial x^{\mu_1}}S_{\mu_2\ldots\mu_k} \\
&= 0.
\end{aligned} \tag{3.3.11}$$

Therefore ambient transversality for the light-cone acquires the further condition that a tensor must never be of the form (3.3.10). We call this requirement strong ambient transversality. Anyway, most of the time, we will just refer to (3.3.6), which is easier to use.

[2] Actually the first condition in (3.3.6) is not enough, as we explain later.

Notice that also the $\eta_{\alpha\beta}$ and g_{AB} have ambient counterparts, but they do not coincide with η_{mn}, because η_{mn} is not transversal and equations (3.3.9) are not invertible. For example, the ambient version of g_{AB} is

$$\mathcal{P}^{\mathrm{mn}} = \frac{\partial \mathcal{X}^{\mathrm{m}}}{\partial X^A}\frac{\partial \mathcal{X}^{\mathrm{n}}}{\partial X^B} g^{AB} = \eta^{\mathrm{mn}} - \frac{\mathcal{X}^{\mathrm{m}}\mathcal{X}^{\mathrm{n}}}{\mathcal{X}^2}, \tag{3.3.12}$$

that is also a projector onto ambient transversal tensors. Indeed, given any $\mathcal{T}^{\mathrm{m}_1\ldots\mathrm{m}_k} \in \mathscr{X}^k(U_{\mathscr{S}})$

$$\bar{\mathcal{T}}^{\mathrm{m}_1\ldots\mathrm{m}_k} \equiv \mathcal{P}^{\mathrm{m}_1}_{\mathrm{n}_1}\cdots\mathcal{P}^{\mathrm{m}_1}_{\mathrm{n}_1}\mathcal{T}^{\mathrm{n}_1\ldots\mathrm{n}_k}$$

is a transversal tensor, since

$$\mathcal{X}_{\mathrm{m}}\mathcal{P}^{\mathrm{m}}_{\mathrm{n}} = \mathcal{P}^{\mathrm{m}}_{\mathrm{n}}\mathcal{X}^{\mathrm{n}} = 0. \tag{3.3.13}$$

Equation (3.3.9) allows us to formulate a rule to automatically convert any expression in $\mathscr{S}_{d+1}$ and $\mathscr{L}_{d+1}/\mathbb{R}$ into an ambient one. Let us focus on AdS. Given any scalar expression $f(X)$, from (3.3.7) and (3.3.8) we know that its ambient version $\mathcal{F}(\mathcal{X})$ is such that

$$f(X) = \mathcal{F}(\mathcal{X}(X)),$$

namely scalars do not transform (except for their homogeneity property away from $\mathscr{S}_{d+1}$). This should also happen with any scalar expression that is obtained by contractions of AdS tensors. We expect that its ambient version is also such a contraction. Indeed

$$\mathcal{V}^{\mathrm{m}}\mathcal{W}_{\mathrm{m}} = \frac{\partial \mathcal{X}^{\mathrm{m}}}{\partial X^M}\eta_{\mathrm{mn}}\frac{\partial \mathcal{X}^{\mathrm{n}}}{\partial X^N} V^M W^N = V^M W_M \tag{3.3.14}$$

by (3.3.9). So whenever we need to convert an expression from AdS we just need to replace every tensor with its ambient version and perform the same contractions. This rule is also compatible with the metric (3.3.12)

$$\mathcal{V}^{\mathrm{m}}\mathcal{P}_{\mathrm{mn}}\mathcal{W}^{\mathrm{n}} = \mathcal{V}_{\mathrm{n}}\mathcal{W}^{\mathrm{n}} = V_N W^N = V^M g_{MN} W^N, \tag{3.3.15}$$

where we used in the first step the transversality of $\mathcal{V}$ that reduces $\mathcal{P}$ to just η, while in the second one we employed (3.3.14).

3.4 Index-free formalism

In this section, we develop a formalism similar to the one used in subsection 1.4.4 to deal with tensors with an arbitrary rank without cluttering our notation with many indices and complicating our computations.

Let U^{m} be an auxiliary vector that in general depends on the coordinates of AdS X^M or the ones of the Minkowski d-dimensional space x^μ. We choose it so that

$$U^{\mathrm{m}}U_{\mathrm{m}} = 0 \tag{3.4.1}$$

$$U^{\mathrm{m}}\mathcal{X}_{\mathrm{m}} = 0, \quad U^{\mathrm{m}}\boldsymbol{X}_{\mathrm{m}} = 0. \tag{3.4.2}$$

Conditions (3.4.2) admits at least one solution. Indeed, if we parametrize U^{m} as $U^{\mathrm{m}} = (u^\mu, U^+, U^-)$, it follows from (3.2.3) and (3.1.20) that

$$\begin{aligned} U^{\mathrm{m}}\mathcal{X}_{\mathrm{m}} &= \frac{L}{z}\left(x^\mu u_\mu + U^- - \frac{1}{2}U^+\left(x^\mu x_\mu - z^2\right)\right) \\ U^{\mathrm{m}}\boldsymbol{X}_{\mathrm{m}} &= x^\mu u_\mu + U^- - \frac{1}{2}U^+ x^\mu x_\mu. \end{aligned}$$

We obtain (3.4.2) if

$$U^{+}=0, \quad U^{-}=-x^{\mu}u_{\mu} \tag{3.4.3}$$

and compatibility with (3.4.1) gives then

$$2U^{+}U^{-}+u^{\mu}u_{\mu}=0 \implies u^{\mu}u_{\mu}=0.$$

Notice that, since U is transversal, it is the ambient version of some world vector and indeed by (3.1.20) and (3.2.3) we can write

$$U^{\mathrm{m}}=u^{\mu}\frac{\partial \boldsymbol{X}^{\mathrm{m}}}{\partial x^{\mu}}, \quad U^{\mathrm{m}}=u^{M}\frac{\partial \mathcal{X}^{\mathrm{m}}}{\partial X^{M}} \tag{3.4.4}$$

where

$$u^{M}=\frac{z}{L}\left(u^{\mu},0\right), \quad u^{M}u_{M}=0.$$

Then, consider a tensor in ambient space that in principle fulfills only the homogeneity condition (3.3.4). Now, contract it with U to obtain a scalar

$$\mathcal{T}(\mathtt{X},U)=\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\mathtt{X})U^{\mathrm{m}_1}\cdots U^{\mathrm{m}_k}. \tag{3.4.5}$$

$\mathcal{T}(\mathtt{X},U)$ uniquely[3] represents a tensor that is

1. Symmetric, because the symmetry of $U^{\mathrm{m}_1}\cdots U^{\mathrm{m}_k}$ removes any antisymmetric contribution to $\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\mathtt{X})$
2. Traceless by virtue of (3.4.1) that removes any traceful part of $\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\mathtt{X})$
3. Transverse in the ambient sense of (3.3.5) and (3.3.6). Indeed the longitudinal part of $\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}$ for $\mathscr{S}_{d+1}$ is given by (if both previous two constraints hold)

$$\frac{1}{L^2}\mathcal{X}_{(\mathrm{m}_1}\left(\mathcal{X}^{\mathrm{m}}\mathcal{T}_{\mathrm{m}\mathrm{m}_2\ldots\mathrm{m}_k)}\right)+\mathit{traces}$$

where *traces* means terms proportional to η_{mn} that is not necessary to specify further. It is easy to see that when (3.4.1) and (3.4.2) are satisfied, this contribution disappears from (3.4.5). As for tensors from $\mathscr{L}_{d+1}$, by (3.4.4), if we consider for the sake of simplicity $k=1$ (but this holds for any k)

$$\begin{aligned} U^{\mathrm{m}}\mathcal{T}_{\mathrm{m}}(\boldsymbol{X}) &= u^{\mu}\frac{\partial \boldsymbol{X}^{\mathrm{m}}}{\partial x^{\mu}}\mathcal{T}_{\mathrm{m}}(\boldsymbol{X}) \\ &= u^{\mu}T_{\mu}(x) \end{aligned}$$

so that $\mathcal{T}(\boldsymbol{X},U)=T(x,u)$, where $T(x,u)$ is the polynomial associated to a tensor in the Minkowski space when contracted with a vector u that obeys the same zero norm condition (3.4.1) as U. Since $T_{\mu}(x)$ is the world version of $\mathcal{T}_{\mathrm{m}}$, the latter must then obey the transversality constraint (3.3.6). Besides, it is obvious that (3.4.2) rules out also the tensors that do not obey the strong transversality condition (see (3.3.10) and the related discussion)

One may also say that a tensor $\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\mathtt{X})$ is represented by the class of equivalence of polynomials in $\mathtt{X}$ and U that differ from each other by terms proportional to U^2 or $(\mathtt{X}\cdot U)$. In order to extract the tensor from the corresponding polynomial, one could in principle differentiate with respect to U[4]. However ∂_U is not compatible with the constraint (3.4.1), since

$$\frac{\partial}{\partial U^{\mathrm{m}}}U^2=2U^{\mathrm{m}}\neq 0$$

[3]As we said, U is the ambient version of some world vector u and therefore world tensors that can be schematically written as $T=uS$ would be represented by $\mathcal{T}=U\mathcal{S}$ and thus by vanishing polynomials. To prevent this, we assume that u is complex and therefore no world or ambient tensors may be proportional to u or U. For further details about this see section 3.1 of [50].

[4]Derivatives $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial \mathtt{X}}$ here and in the following are meant to act formally only on the explicit dependence on U and $\mathtt{X}$ and therefore, even though U is a function of $\mathtt{X}$, derivatives in U and $\mathtt{X}$ have no effect on $\mathtt{X}$ and U respectively.

even when $U^2 = 0$. For this reason polynomials in the same equivalence class would correspond to different tensors. This problem can be solved by using

$$\mathscr{D}_{\mathrm{m}} \equiv \left(\frac{d}{2} - 1 + U \cdot \frac{\partial}{\partial U}\right) \frac{\partial}{\partial U^{\mathrm{m}}} - \frac{1}{2} U_{\mathrm{m}} \frac{\partial^2}{\partial U^{\mathrm{a}} \partial U_{\mathrm{a}}} \tag{3.4.6}$$

instead of ∂_U. Indeed $\mathscr{D}_{\mathrm{m}} U^2 = 0$.

One can show (see [50]) that

$$\mathcal{T}_{\mathrm{m}_1 \ldots \mathrm{m}_k}(\mathtt{X}) = \frac{1}{k! \left(\frac{d}{2} + k - 1\right)_{\frac{d}{2}}} \mathscr{D}_{\mathrm{m}_1} \cdots \mathscr{D}_{\mathrm{m}_k} \mathcal{T}(\mathtt{X}, U),$$

where we used the Pochhammer symbol defined in the "Notation" appendix. By means of $\mathscr{D}_{\mathrm{m}}$ the usual tensor operations can be performed in the index-free formalism. The scalar product

$$\mathcal{U}_{\mathrm{m}_2 \ldots \mathrm{m}_k} \equiv \mathcal{V}^{\mathrm{m}} \mathcal{T}_{\mathrm{m}\mathrm{m}_2 \ldots \mathrm{m}_k}$$

is represented by

$$\mathcal{U}(\mathtt{X}, U) = \frac{1}{k \left(\frac{d}{2} + k - 1\right)} \mathcal{V}^{\mathrm{m}} \mathscr{D}_{\mathrm{m}} \mathcal{T}(\mathtt{X}, U) \tag{3.4.7}$$

The tensor product is trivially obtained by the usual product between scalars: given $\mathcal{T}_{\mathrm{m}_1 \ldots \mathrm{m}_k}(\mathtt{X})$, $\mathcal{S}_{\mathrm{m}_1 \ldots \mathrm{m}_l}(\mathtt{X})$ and their associated polynomials $\mathcal{T}(\mathtt{X}, U)$ and $\mathcal{S}(\mathtt{X}, U)$,

$$\mathcal{U}_{\mathrm{m}_1 \ldots \mathrm{m}_{k+l}}(\mathtt{X}) = \binom{k+l}{l}^{-1} \mathcal{T}_{(\mathrm{m}_1 \ldots \mathrm{m}_k}(\mathtt{X}) \mathcal{S}_{\mathrm{m}_{k+1} \ldots \mathrm{m}_{k+l})}(\mathtt{X})$$

is represented by

$$\begin{aligned} \mathcal{U}(\mathtt{X}, U) &= \mathcal{T}_{\mathrm{m}_1 \ldots \mathrm{m}_k}(\mathtt{X}) \mathcal{S}_{\mathrm{m}_{k+1} \ldots \mathrm{m}_{k+l}}(\mathtt{X}) U^{\mathrm{m}_1} \cdots U^{\mathrm{m}_k} U^{\mathrm{m}_{k+1}} \cdots U^{\mathrm{m}_{k+l}} \\ &= \mathcal{T}(\mathtt{X}, U) \mathcal{S}(\mathtt{X}, U). \end{aligned} \tag{3.4.8}$$

Using

$$\mathscr{D}^{\mathrm{m}} \mathscr{D}_{\mathrm{m}} = -\frac{1}{2} U \cdot \frac{\partial}{\partial U} \frac{\partial^2}{\partial U^{\mathrm{a}} \partial U_{\mathrm{a}}}$$

we find that the contraction $\mathcal{T}^{\mathrm{n}}{}_{\mathrm{n}\mathrm{m}_3 \ldots \mathrm{m}_k}(\mathtt{X})$ is associated to

$$\begin{aligned} \mathcal{T}'(\mathtt{X}, U) &\equiv \mathcal{T}^{\mathrm{n}}{}_{\mathrm{n}\mathrm{m}_3 \ldots \mathrm{m}_k} U^{\mathrm{m}_3} \cdots U^{\mathrm{m}_k} \\ &= -\frac{1}{2k(k-1)\left(\frac{d}{2} + k - 1\right)\left(\frac{d}{2} + k - 2\right)} U \cdot \frac{\partial}{\partial U} \frac{\partial^2}{\partial U^{\mathrm{a}} \partial U_{\mathrm{a}}} \mathcal{T}(\mathtt{X}, U) \end{aligned} \tag{3.4.9}$$

Ambient transversality 3.3.5 and (3.3.6) can be translated in this formalism by (3.4.7) as

$$\frac{1}{k\left(\frac{d}{2} + k - 1\right)} \mathtt{X}^{\mathrm{m}} \mathscr{D}_{\mathrm{m}} \mathcal{T}(\mathtt{X}, U) = 0,$$

equivalent to

$$\mathtt{X}^{\mathrm{n}} \frac{\partial}{\partial U^{\mathrm{n}}} \mathcal{T}(\mathtt{X}, U) = 0. \tag{3.4.10}$$

3.5 Application to the boundary

By means of the ambient formulation of the theory at the boundary of the AdS space we are able to describe the duals of the higher spins that live in the bulk, i.e. some composite fields that share the same properties of higher spins for what concern their tensorial nature. Our goal, now, is to employ our new language to enforce the conformal symmetry on correlators of these fields, thus constraining their possible form. In our formalism, this will amount to just imposing manifest $SO(2,d)$ invariance, along with the ambient constraints on tensors (3.3.6) and (3.3.4). This kind of task is familiar in theoretical physics and is easily solved. We will find that 2-point functions are completely determined by these requirements, while for 3-point correlators this happens only for $s<1$ and for higher spins there are anyway strong restrictions.

In this section, first we consider only primary scalar fields, defined by (2.1.9), and we employ some standard CFT techniques to derive powerful restrictions on 2-point and 3-point correlators. Then, our aim will be to extend these results to primary tensorial fields using the ambient space formalism that will contrast with the lower dimensional derivation for its simplicity.

3.5.1 2-point functions of scalars

As our first example of a CFT correlator we examine the 2-point function between ϕ_1, ϕ_2 two primary scalar fields with dimensions Δ_1, Δ_2. We require that the correlator

$$f(x,y) = \langle \phi_1(x)\phi_2(y)\rangle$$

is invariant under each class of conformal transformations:

1. Translations: if we perform $x^\mu \to x^\mu - y^\mu$ we get

$$f(x,y) = \langle \phi_1(x-y)\phi_2(0)\rangle$$

so that f depends actually only on $x-y$.

2. Lorentz transformations: f must be a scalar and therefore, since the only scalar that one can build with $(x-y)^\mu$ is its norm, we have that

$$f(x,y) \equiv f\left(|x-y|\right)$$

3. Special conformal transformations. This check is a bit more complicated. First of all we have that in (2.1.9)

$$\left|\det \frac{\partial x'}{\partial x}\right| = \left(\frac{1}{1+2b\cdot x + b^2x^2}\right)^d$$

for (3.1.11) with parameter b. Now, let us employ some short-hand notation:

$$\beta_x \equiv 1 + 2b\cdot x + b^2 x^2 \tag{3.5.1}$$

$$\left(x^\mu + b^\mu x^2\right)^2 = x^2 \beta(x) \tag{3.5.2}$$

To perform the transformation of f we avail ourselves of translational invariance to eliminate y and we find

$$f'(x',0) = \frac{1}{\beta_x^{\Delta_1}}\left\langle \phi_1(x')\phi_2(0)\right\rangle \stackrel{!}{=} f(x,0)$$

but, at the same time, from (3.5.4) and (3.5.2) we get

$$\begin{aligned} f(x',y') &= \frac{C_{12}}{\left|\frac{x^\mu+b^\mu x^2}{\beta_x}\right|^{\Delta_1+\Delta_2}} \\ &= \frac{C_{12}}{\sqrt{x^2\frac{\beta_x}{\beta_x^2}}^{\Delta_1+\Delta_2}} \\ &= \frac{C_{12}\beta_x^{\frac{\Delta_1+\Delta_2}{2}}}{|x|^{\Delta_1+\Delta_2}} \end{aligned} \tag{3.5.3}$$

that produces the last constraint $\frac{\Delta_1+\Delta_2}{2}=\Delta_1$ or

$$\Delta_1=\Delta_2.$$

4. Rescalings: under a rescaling we must get

$$f'(\lambda x,\lambda y)=\lambda^{\Delta_1}\lambda^{\Delta_2}\left\langle\phi_1(\lambda x)\phi_2(\lambda y)\right\rangle \stackrel{!}{=} f(x,y)$$

Since the only length that scales with λ that we have at our disposal is $|x-y|$, it follows that

$$f(x,y)=\frac{C_{12}}{|x-y|^{\Delta_1+\Delta_2}} \tag{3.5.4}$$

where C_{12} is some proportionality constant.

The final form of a correlator of scalar primary fields is then

$$\left\langle\phi_1(x)\phi_2(y)\right\rangle=\frac{C_{12}\delta_{\Delta_1\Delta_2}}{|x-y|^{2\Delta_1}}. \tag{3.5.5}$$

3.5.2 3-point functions of scalars

Let us now analyze the following 3-point correlator

$$f(x,y,z)=\left\langle\phi_1(x)\phi_2(y)\phi_3(z)\right\rangle, \tag{3.5.6}$$

where the fields have conformal dimension of Δ_1,Δ_2 and Δ_3 respectively. Analogously as before, translational invariance implies that

$$f(x,y,z)\equiv f\left((x-y),(y-z),(z-x)\right),$$

and by Lorentz invariance, one can use only scalar products of these quantities, i.e.

$$|x-y|,\quad |y-z|,\quad |z-x| \tag{3.5.7}$$

$$B_y\equiv(x-y)\cdot(y-z),\quad B_z\equiv(y-z)\cdot(z-x),\quad B_x\equiv(x-y)\cdot(z-x). \tag{3.5.8}$$

We now notice that, for example

$$\begin{aligned} B_y+B_z &= (y-z)\cdot((x-y)+(z-x)) \\ &= -|y-z|^2, \end{aligned}$$

so that none of the terms in (3.5.8) is independent of (3.5.7). We can thus write

$$f(x,y,z)=\sum_{a_1,a_2,a_3} C_{123}^{a_1,a_2,a_3}\,|x-y|^{a_1}\,|y-z|^{a_2}\,|z-x|^{a_3} \tag{3.5.9}$$

for some constants $C_{123}^{a_1,a_2,a_3}$. Scale invariance, on the other hand imposes

$$\Delta_1 + \Delta_2 + \Delta_3 = -\left(a_1 + a_2 + a_3\right). \tag{3.5.10}$$

Let us now also require the invariance under special conformal transformations with parameter b^μ, again putting one of the arguments to zero by means of a translation to simplify our computation. Referring to (3.5.1), we get from (3.5.6)

$$f'(x',y',0) = \frac{1}{\beta_x^{\Delta_1}}\frac{1}{\beta_y^{\Delta_2}}\left\langle \phi_1(x')\phi_2(y')\phi_3(0)\right\rangle$$

while from (3.5.9) (see (3.5.2) and the computation in (3.5.3))

$$\begin{aligned}
f(x',y',0) &= \sum_{a_1,a_2,a_3} C_{123}^{a_1,a_2,a_3}\left|\frac{x^\mu + b^\mu x^2}{\beta_x} - \frac{y^\mu + b^\mu y^2}{\beta_y}\right|^{a_1} \frac{|y|^{a_2}}{|\beta_y|^{\frac{a_2}{2}}}\frac{|x|^{a_3}}{|\beta_x|^{\frac{a_3}{2}}}\\
&= \sum_{a_1,a_2,a_3} C_{123}^{a_1,a_2,a_3}\left(\frac{x^2}{\beta_x} - 2\frac{\left(x^\mu + b^\mu x^2\right)\left(y_\mu + b_\mu y^2\right)}{\beta_x\beta_y} + \frac{y^2}{\beta_y}\right)^{\frac{a_1}{2}} \frac{|y|^{a_2}}{|\beta_y|^{\frac{a_2}{2}}}\frac{|x|^{a_3}}{|\beta_x|^{\frac{a_3}{2}}}\\
&= \sum_{a_1,a_2,a_3} C_{123}^{a_1,a_2,a_3}\left(\frac{x^2\beta_y - 2\left(x\cdot y + (bx)\,y^2 + (by)\,x^2 + b^2x^2y^2\right) - y^2\beta_y}{\beta_x\beta_y}\right)^{\frac{a_1}{2}} \frac{|y|^{a_2}}{|\beta_y|^{\frac{a_2}{2}}}\frac{|x|^{a_3}}{|\beta_x|^{\frac{a_3}{2}}}\\
&= \sum_{a_1,a_2,a_3} C_{123}^{a_1,a_2,a_3}\frac{|x-y|^{a_1}}{|\beta_x\beta_y|^{\frac{a_1}{2}}}\frac{|y|^{a_2}}{|\beta_y|^{\frac{a_2}{2}}}\frac{|x|^{a_3}}{|\beta_x|^{\frac{a_3}{2}}}
\end{aligned}$$

from which we get

$$\begin{cases} \frac{a_2}{2} + \frac{a_1}{2} = -\Delta_2 \\ \frac{a_3}{2} + \frac{a_1}{2} = -\Delta_1 \end{cases}$$

that, along with (3.5.10), completely determines a_1, a_2, a_3:

$$a_1 = -\left(\Delta_1 + \Delta_2 - \Delta_3\right), \quad a_2 = -\left(-\Delta_1 + \Delta_2 + \Delta_3\right), \quad a_1 = -\left(\Delta_1 - \Delta_2 + \Delta_3\right).$$

We come finally to the following expression for the conformally invariant 3-point correlator of scalar fields:

$$\left\langle \phi_1(x)\phi_2(y)\phi_3(z)\right\rangle = \frac{C_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}\,|y-z|^{-\Delta_1+\Delta_2+\Delta_3}\,|z-x|^{-\Delta_1+\Delta_2+\Delta_3}}. \tag{3.5.11}$$

3.5.3 2-point functions of higher spins and the 0-0-s correlator

In the ambient approach, imposing conformal invariance amounts to requiring invariance under the $SO(2,d)$ linear transformation of the ambient coordinates. At the same time, though, we must remember that our fields obey the homogeneity condition (3.3.4) and the ambient transversality condition (3.3.6).

Let us delve a bit more on the first. From (3.3.4) it is obvious that there is a connection between homogeneity and the defining property of primary fields (2.1.3). To explore it, we write the effect of a dilatation $x^\mu \to x'^\mu = \lambda x^\mu$ on a point $\boldsymbol{X}(x) \in \mathscr{L}_{d+1}/\mathbb{R}$ by (3.1.20):

$$\boldsymbol{X}^{\mathrm{m}}\left(x^\mu, 1, -\frac{1}{2}x^2\right) \to \left(\lambda x^\mu, 1, -\frac{1}{2}\lambda^2 x^2\right) \sim \left(x^\mu, \frac{1}{\lambda}, -\frac{1}{2}\lambda x^2\right) \equiv \boldsymbol{X}'^{\mathrm{m}}. \tag{3.5.12}$$

Now notice that this is a Lorentz transformation in $\mathcal{A}_{d+2}$

$$\mathtt{X}^{\mathrm{m}} = \left(\mathtt{X}^\mu, \mathtt{X}^+, \mathtt{X}^-\right) \to \mathtt{X}'^{\mathrm{m}} = \left(\mathtt{X}^\mu, \frac{1}{\lambda}\mathtt{X}^+, \lambda\mathtt{X}^-\right) \tag{3.5.13}$$

since it obviously preserves scalar products, where the $+$ component is always multiplied by the $-$ one and λ is canceled by $\frac{1}{\lambda}$. This conclusion is obvious in light of our construction in 3.1.2, where we showed how any element of the conformal group is equivalent to an $SO(2,d)$ transformation followed by a rescaling in the light-cone.

From (3.5.13) we see that the scale factor does not depend on the coordinates, even though in general it happens. This implies that, from (3.3.1) and (3.5.12),

$$\begin{aligned} T'_{\mu_1\ldots\mu_k}(x') &= \frac{\partial \boldsymbol{X}'^{\mathrm{m}_1}}{\partial x_{\mu_1}}\cdots\frac{\partial \boldsymbol{X}'^{\mathrm{m}_k}}{\partial x_{\mu_k}}\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\lambda\boldsymbol{X}'(x)), \\ T'_{\mu_1\ldots\mu_k}(x') &= \lambda^{-\Delta}\frac{\partial \boldsymbol{X}'^{\mathrm{m}_1}}{\partial x_{\mu_1}}\cdots\frac{\partial \boldsymbol{X}'^{\mathrm{m}_k}}{\partial x_{\mu_k}}\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\boldsymbol{X}'(x)) \end{aligned}$$

i.e. the homogeneity condition on $\mathcal{T}$ with degree Δ coincides with the conformal dimension Δ of T.

A remark is in order here: homogeneity is an algebraic condition on the values of ambient tensors away from the domain of definition of their low dimensional counterparts, not a transformation law, because dilations are not symmetries of $\mathcal{A}_{d+2}$. The fact that the degree agrees with the conformal dimension is just a consistency requirement. Therefore, when we check the homogeneity of correlators, we need to do it *separately* for each coordinate, for we are *not* verifying dilation invariance.

We are now ready to explain how to find the simplest two correlators: the 2-point function of two higher spins and the 3-point function between two scalars and one higher spin.

So, first we compute the correlator between two fields $\Phi_1^{\mathrm{m}_1\ldots\mathrm{m}_{k_1}}(\boldsymbol{X})$ and $\Phi_2^{\mathrm{m}_1\ldots\mathrm{m}_{k_2}}(\boldsymbol{X})$ with degree Δ_1,Δ_2 respectively:

$$\left\langle \Phi_1^{\mathrm{m}_1\ldots\mathrm{m}_{k_1}}(\boldsymbol{X}_1)\Phi_2^{\mathrm{n}_1\ldots\mathrm{n}_{k_2}}(\boldsymbol{X}_2)\right\rangle. \tag{3.5.14}$$

Our first goal is to replicate the index structure of (3.5.14) by tensor multiplication of some fundamental building blocks that are (strongly) ambient transversal with respect to $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ in the m and n indices, respectively.

In order to construct these building blocks, we have only $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and η at our disposal. Then, it is clear that

$$\begin{gathered} \boldsymbol{X}_1^{\mathrm{m}}, \quad \boldsymbol{X}_2^{\mathrm{n}} \\ \mathcal{P}^{\mathrm{m},\mathrm{n}} = \left(\frac{\boldsymbol{X}_1^{\mathrm{n}}\boldsymbol{X}_2^{\mathrm{m}}}{\boldsymbol{X}_1\cdot\boldsymbol{X}_2} - \eta^{\mathrm{mn}}\right) \end{gathered} \tag{3.5.15}$$

are the simplest terms that satisfy the transversality criterion. Actually we are forced to rule out the first two because they violate the strong ambient transversality requirement (see (3.3.10)). Let us show that there are no more general ones.

Proposition 3.5.1. *Every (strongly) ambient transversal tensor that can be built out of $\boldsymbol{X}_1^{\mathrm{m}}$ and $\boldsymbol{X}_2^{\mathrm{n}}$ is a weighted sum of tensor products of $\mathcal{P}^{\mathrm{m},\mathrm{n}}$ defined by* (3.5.15) *and therefore possesses an equal number of* m *and* n *indices.*

Proof. Suppose that it is not so. Then there exists at least a tensor

$$\mathcal{G}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1},\mathrm{n}_1\ldots\mathrm{n}_{k_2}}$$

that obeys our transversality request. Then

$$\boldsymbol{X}_{2\mathrm{m}_1}\boldsymbol{X}_{1\mathrm{n}_1}\mathcal{G}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1},\mathrm{n}_1\ldots\mathrm{n}_{k_2}} = \mathcal{H}^{\mathrm{m}_2\ldots\mathrm{m}_{k_1},\mathrm{n}_2\ldots\mathrm{n}_{k_2}} \tag{3.5.16}$$

for some $\mathcal{H}$ with the same algebraic properties of $\mathcal{G}$. From (3.5.16) it follows that

$$\mathcal{G}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1},\mathrm{n}_1\ldots\mathrm{n}_{k_2}} = B^{\mathrm{m}_1\mathrm{n}_1}\mathcal{H}^{\mathrm{m}_2\ldots\mathrm{m}_{k_1},\mathrm{n}_2\ldots\mathrm{n}_{k_2}} + \mathcal{I}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1},\mathrm{n}_1\ldots\mathrm{n}_{k_2}} \tag{3.5.17}$$

where $\mathcal{I}$ is a tensor that in addition to the algebraic properties of $\mathcal{G}$ obeys

$$\boldsymbol{X}_{2\mathrm{m}_1}\mathcal{I}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1},\mathrm{n}_1\ldots\mathrm{n}_{k_2}} = \boldsymbol{X}_{1\mathrm{m}_1}\mathcal{I}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1},\mathrm{n}_1\ldots\mathrm{n}_{k_2}} = 0, \tag{3.5.18}$$

or the same relation with contractions in the n indices. As we will show later, (3.5.18) is equivalent to

$$\mathcal{I}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1},\mathrm{n}_1\ldots\mathrm{n}_{k_2}} = 0.$$

On the other hand the only rank 2 tensor that one can build out of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ and η that is (strongly) transversal is $\mathcal{P}^{\mathrm{m},\mathrm{n}}$, so that

$$B^{\mathrm{mn}} = -\frac{\mathcal{P}^{\mathrm{m},\mathrm{n}}}{\boldsymbol{X}_1\cdot\boldsymbol{X}_2}$$

and therefore $\mathcal{G}$ is just the product of $\mathcal{H}$ and $\mathcal{P}$. We could repeat the same argument to reduce all possible building blocks to $\mathcal{P}$ or tensors with only m or n indices.

Take then for example $\mathcal{G}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1}}$. It has to be the sum of terms of the schematic form

$$(\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^l\,\eta^{\mathrm{mm}}\cdots\eta^{\mathrm{mm}}\cdot\boldsymbol{X}_2^{\mathrm{m}}\cdots\boldsymbol{X}_2^{\mathrm{m}}$$

for some power $l\in\mathbb{R}$. It is easy to see that if we contract one of these with $\boldsymbol{X}_1^{\mathrm{m}}$, it must never happen with an index coming from a metric, because there can be no compensating term containing $\boldsymbol{X}_1^{\mathrm{m}}$ to cancel it. If no metric appears in $\mathcal{G}$, then it shares the same algebraic properties (3.5.18) of $\mathcal{I}$ (the distinction among m and n is meaningless, since it has to be transverse for both $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.) On the other hand, with only $\boldsymbol{X}_2^{\mathrm{m}}$ at our disposal, it is obvious that we cannot build any transversal tensor with respect to $\boldsymbol{X}_1^{\mathrm{m}}$. Therefore $\mathcal{G}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1}}$ and similarly $\mathcal{G}^{\mathrm{n}_1\ldots\mathrm{n}_{k_2}}$ and $\mathcal{I}^{\mathrm{m}_1\ldots\mathrm{m}_{k_1},\mathrm{n}_1\ldots\mathrm{n}_{k_2}}$ vanish and we conclude our proof. □

We have thus proven that Φ_1 and Φ_2 must have the same rank k. From now on we write

$$\mathcal{S}^{\mathrm{mn}}(\boldsymbol{X}_1,\boldsymbol{X}_2) = \boldsymbol{X}_1^{\mathrm{n}}\boldsymbol{X}_2^{\mathrm{m}} - (\boldsymbol{X}_1\cdot\boldsymbol{X}_2)\,\eta^{\mathrm{mn}}. \tag{3.5.19}$$

and the only tensor that reproduces the algebraic properties of (3.5.14) is the tensor product of k (3.5.15):

$$\frac{\mathcal{S}^{(\mathrm{m}_1|\mathrm{n}_1}\cdots\mathcal{S}^{\mathrm{m}_k)\mathrm{n}_k}}{(\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^k},$$

where the symmetrization acts only on m.

We still have to fix the scaling degree, separately for each tensor[5] as explained before. Under rescalings of factor λ_1 for $\boldsymbol{X}_1$ and λ_2 for $\boldsymbol{X}_2$ (3.5.14) acquires the coefficient

$$\lambda_1^{-\Delta_1}\lambda_2^{-\Delta_2},$$

while (3.5.15) would give none. In general, we need some scalar to compensate this difference in powers of $\lambda_{1,2}$. The only non-vanishing one is $\boldsymbol{X}_1\cdot\boldsymbol{X}_2$ that, however, contributes with $\lambda_1\lambda_2$, i.e with equal powers of λ_1 and λ_2. This implies that

$$\Delta_1 = \Delta_2$$

so that we come to

$$\langle\Phi_1^{\mathrm{m}_1\ldots\mathrm{m}_k}(\boldsymbol{X}_1)\Phi_2^{\mathrm{n}_1\ldots\mathrm{n}_k}(\boldsymbol{X}_2)\rangle = \mathcal{C}_{12}\delta_{\Delta_1\Delta_2}\frac{\mathcal{S}^{(\mathrm{m}_1|\mathrm{n}_1}\cdots\mathcal{S}^{\mathrm{m}_k)\mathrm{n}_k}}{(\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^{\Delta_1+k}}, \tag{3.5.20}$$

[5]Notice that here we are not imposing scaling invariance, that has been already set when we required that our field should be an $SO(2,d)$ tensor. This is an algebraic property that the ambient tensor have to obey and therefore has to be required separately for each of them.

where $\mathcal{C}_{12} \in \mathbb{R}$ is some constant. It is clarifying to convert (3.5.20) into the physical space-time form.

$$\begin{aligned}\langle \phi_1^{\mu_1 \dots \mu_k}(x_1) \phi_2^{\nu_1 \dots \nu_k}(x_2) \rangle &= \mathcal{C}_{12} \delta_{\Delta_1 \Delta_2} \frac{S^{(\mu_1|\nu_1} \dots S^{\mu_k)\nu_k}}{\left|-\frac{1}{2} x_1^\alpha x_{1\alpha} - \frac{1}{2} x_2^\alpha x_{2\alpha} + x_1^\alpha x_{2\alpha}\right|^{\Delta_1 + k}} \\ &= C_{12} \delta_{\Delta_1 \Delta_2} \frac{S^{(\mu_1|\nu_1} \dots S^{\mu_k)\nu_k}}{|x_1 - x_2|^{2\Delta_1 + 2k}},\end{aligned}$$

where we defined $C_{12} \equiv 2^{\Delta_1 + k} \mathcal{C}_{12}$. This form resembles closely (3.5.5) and is clearly its proper tensorial generalization. The ambient approach used here, even though was more straight-forward than the one used in subsection (3.5.1) for imposing the symmetry, required a detailed and lengthy analysis for what concerns the tensorial structure of the correlator.

The index-free formalism simplifies this task. Ambient transversality will be automatic and there will be no distinction between (3.3.6) and its strong form. We show this feature for a slightly more complex task: the derivation of the three point function for two scalars and one higher spin field.

The correlator in question is written in the index-free form as

$$\left\langle \Phi_1(\boldsymbol{X}_1) \Phi_2(\boldsymbol{X}_2) \Phi_3^k(\boldsymbol{X}_3, U_3) \right\rangle \tag{3.5.21}$$

where we wrote U_3 to refer to U computed in relation to x_3^μ (see (3.4.3)), k refers to the rank of Φ_3 and the conformal dimensions of the three fields are Δ_1, Δ_2 and Δ_3. (3.5.21) must be a polynomial in U_3 of degree k.

Analogously as before, we want to build (3.5.21) as a polynomial function of some suitable basic polynomials $\mathcal{B}_i(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, U_3)$. They must obey (3.4.10), i.e.

$$\boldsymbol{X}_3^{\mathrm{m}} \frac{\partial}{\partial U_3^{\mathrm{m}}} \mathcal{B}_i = 0 \tag{3.5.22}$$

and are thus the scalars built from $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3$

$$\boldsymbol{X}_1 \cdot \boldsymbol{X}_2, \quad \boldsymbol{X}_2 \cdot \boldsymbol{X}_3, \quad \boldsymbol{X}_1 \cdot \boldsymbol{X}_3,$$

and the ones in which U_3 is used, $(\boldsymbol{X}_1 \cdot U_3)$ and $(\boldsymbol{X}_2 \cdot U_3)$. Let $f(\alpha, \beta) : \mathbb{R}^2 \to \mathbb{R}$ be any function of these. (3.5.22) requires that

$$\boldsymbol{X}_3^{\mathrm{m}} \frac{\partial}{\partial U_3^{\mathrm{m}}} f(\boldsymbol{X}_1 \cdot U_3, \boldsymbol{X}_2 \cdot U_3) = \boldsymbol{X}_1 \cdot \boldsymbol{X}_3 \frac{\partial f}{\partial \alpha} + \boldsymbol{X}_2 \cdot \boldsymbol{X}_3 \frac{\partial f}{\partial \beta} \stackrel{!}{=} 0$$

or

$$f(\alpha, \beta) \equiv g\left((\boldsymbol{X}_2 \cdot \boldsymbol{X}_3)\, \alpha - (\boldsymbol{X}_1 \cdot \boldsymbol{X}_3)\, \beta\right)$$

for some $g : \mathbb{R} \to \mathbb{R}$, so that we may use as the building block the following expression

$$\mathcal{B}_3 \equiv \left(\frac{\boldsymbol{X}_1}{\boldsymbol{X}_1 \cdot \boldsymbol{X}_3} - \frac{\boldsymbol{X}_2}{\boldsymbol{X}_2 \cdot \boldsymbol{X}_3} \right) \cdot U_3, \tag{3.5.23}$$

that generates any polynomial g. We can then write (3.5.21) as weighted sum of terms of the following form

$$\sum_{e=1}^{k} \binom{k}{e} (\boldsymbol{X}_1 \cdot \boldsymbol{X}_2)^a (\boldsymbol{X}_2 \cdot \boldsymbol{X}_3)^b (\boldsymbol{X}_1 \cdot \boldsymbol{X}_3)^c \left(\frac{\boldsymbol{X}_1 \cdot U_3}{\boldsymbol{X}_1 \cdot \boldsymbol{X}_3} \right)^{k-e} \left(-\frac{\boldsymbol{X}_2 \cdot U_3}{\boldsymbol{X}_2 \cdot \boldsymbol{X}_3} \right)^{e}. \tag{3.5.24}$$

where we have already imposed the degree k with respect to U_3. If we require that (3.5.24) has the correct behavior under rescalings of $\boldsymbol{X}_{1,2,3}$, we come to the following system of equations:

$$\begin{cases} a + c = -\Delta_1 \\ a + b = -\Delta_2 \\ b + c - k = -\Delta_3 \end{cases}$$

from which we find

$$a = -\frac{\Delta_1 + \Delta_2 - \Delta_3 + k}{2}, \quad b = -\frac{-\Delta_1 + \Delta_2 + \Delta_3 - k}{2}, \quad c = -\frac{\Delta_1 - \Delta_2 + \Delta_3 - k}{2}.$$

Inserting these exponents back into (3.5.24) we finally get,

$$\left\langle \Phi_1(\boldsymbol{X}_1)\Phi_2(\boldsymbol{X}_2)\Phi_3^k(\boldsymbol{X}_3, U_3) \right\rangle = \frac{C_{123}\left(\frac{\boldsymbol{X}_1\cdot U_3}{\boldsymbol{X}_1\cdot\boldsymbol{X}_3} - \frac{\boldsymbol{X}_2\cdot U_3}{\boldsymbol{X}_2\cdot\boldsymbol{X}_3}\right)^k}{(\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^{\frac{\Delta_1+\Delta_2-\Delta_3+k}{2}}(\boldsymbol{X}_2\cdot\boldsymbol{X}_3)^{\frac{-\Delta_1+\Delta_2+\Delta_3-k}{2}}(\boldsymbol{X}_1\cdot\boldsymbol{X}_3)^{\frac{\Delta_1-\Delta_2+\Delta_3-k}{2}}}. \tag{3.5.25}$$

for some constant C_{123}.

3.5.4 3-point functions for general higher spins

Here we want to generalize further the result of the previous subsection and consider three higher spin fields of rank k_1, k_2 and k_3 and degree Δ_1, Δ_2 and Δ_3:

$$\left\langle \Phi_1^{k_1}(\boldsymbol{X}_1, U_1)\Phi_2^{k_2}(\boldsymbol{X}_2, U_2)\Phi_3^{k_3}(\boldsymbol{X}_3, U_3) \right\rangle. \tag{3.5.26}$$

Let us list all the basic polynomials. First of all, we use $\boldsymbol{X}_i$ and U_j $(i,j = 1,2,3)$ to find all possible scalars:

$$\boldsymbol{X}_i \cdot \boldsymbol{X}_j, \quad \boldsymbol{X}_i \cdot U_j, \quad U_i \cdot U_j \quad i \neq j. \tag{3.5.27}$$

Now impose (3.4.10)[6]:

$$\boldsymbol{X}_j \cdot \frac{\partial}{\partial U_j} \boldsymbol{X}_i \cdot U_j = \boldsymbol{X}_i \cdot \boldsymbol{X}_j \tag{3.5.28}$$

$$\boldsymbol{X}_j \cdot \frac{\partial}{\partial U_j} U_i \cdot U_j = \boldsymbol{X}_j \cdot U_i, \quad \boldsymbol{X}_i \cdot \frac{\partial}{\partial U_i} U_i \cdot U_j = \boldsymbol{X}_i \cdot U_j. \tag{3.5.29}$$

We see from (3.5.28) and (3.5.29) that only the first scalar in (3.5.27) can be a basic building block. The other two have to be put into a transverse combination suggested by (3.5.28) and (3.5.29),

$$\mathcal{B}_{ij} = (\boldsymbol{X}_i \cdot \boldsymbol{X}_j)(U_i \cdot U_j) - (\boldsymbol{X}_i \cdot U_j)(\boldsymbol{X}_j \cdot U_i), \quad i < j$$

and the generalization of (3.5.23)

$$\mathcal{B}_k = \frac{\boldsymbol{X}_i \cdot U_k}{\boldsymbol{X}_i \cdot \boldsymbol{X}_k} - \frac{\boldsymbol{X}_j \cdot U_k}{\boldsymbol{X}_j \cdot \boldsymbol{X}_k}, \quad i < j, \quad i, j \neq k.$$

The 6 building blocks $\mathcal{B}_{ij}, \mathcal{B}_k$ exhaust all the possible choices for the polynomials containing the U_i. This can be proven in the same way as we proved that $\mathcal{B}_3$ was the unique building block containing U_3 in the previous subsection: one defines a function of 9 arguments[7] $f(\boldsymbol{X}_i \cdot U_j, U_i \cdot U_j)$ and enforces the 3 constraints (3.4.10) to find that actually f must depend on 6 transversal polynomials built with these scalar products. Since we already have found 6 suitable combinations, it is not necessary to repeat that procedure.

Following our scheme, we impose the proper degree in each U_i and $\boldsymbol{X}_i$ to the most general term that can be obtained from our building blocks

$$\mathcal{B}_{12}^{a_1}\mathcal{B}_{13}^{a_2}\mathcal{B}_{23}^{a_3}\mathcal{B}_1^{b_1}\mathcal{B}_2^{b_2}\mathcal{B}_3^{b_3}(\boldsymbol{X}_1 \cdot \boldsymbol{X}_2)^{c_1}(\boldsymbol{X}_2 \cdot \boldsymbol{X}_3)^{c_2}(\boldsymbol{X}_1 \cdot \boldsymbol{X}_3)^{c_3}.$$

[6] Indices i, j... etc. that label different fields are not meant to obey the Einstein convention on summations

[7] Since $\boldsymbol{X}_i \cdot U_i = 0$, there are just $3 \cdot 2 = 6$ non vanishing scalar products between the $\boldsymbol{X}_i$s and the U_js, while $U_i \cdot U_i = 0$ allows only 3 non-zero products of the kind $U_i \cdot U_j$.

Notice that both a_i and b_i must be positive, because the U_i can appear only in positive powers. We come thus to 6 conditions for the 9 unknowns a_i, b_i, c_i:

$$\begin{cases} a_1 + a_2 - b_1 + c_1 + c_3 = -\Delta_1 \\ a_1 + a_3 - b_2 + c_1 + c_2 = -\Delta_2 \\ a_2 + a_3 - b_3 + c_2 + c_3 = -\Delta_3 \\ a_1 + a_2 + b_1 = k_1 \\ a_1 + a_3 + b_2 = k_2 \\ a_2 + a_3 + b_3 = k_3 \\ a_i > 0 \\ b_i > 0 \end{cases} . \tag{3.5.30}$$

We decide to leave unconstrained the b_i and to determine a_i and c_i as their functions. System (3.5.30) then can be put in the following form (we omit the domains for a_i and b_i for brevity)

$$\begin{cases} c_1 + c_3 = -\Delta_1 - k_1 + 2b_1 \\ c_1 + c_2 = -\Delta_2 - k_2 + 2b_2 \\ c_2 + c_3 = -\Delta_3 - k_3 + 2b_3 \\ a_1 + a_2 = k_1 - b_1 \\ a_1 + a_3 = k_2 - b_2 \\ a_2 + a_3 = k_3 - b_3 \end{cases} \tag{3.5.31}$$

and easily solved to get

$$\begin{cases} c_1 = -\frac{\delta_1+\delta_2-\delta_3}{2} + (b_1 + b_2 - b_3) \\ c_2 = -\frac{-\delta_1+\delta_2+\delta_3}{2} + (-b_1 + b_2 + b_3) \\ c_3 = -\frac{\delta_1-\delta_2+\delta_3}{2} + (b_1 - b_2 + b_3) \\ a_1 = \frac{k_1+k_2-k_3}{2} - \frac{b_1+b_2-b_3}{2} \\ a_2 = \frac{k_1-k_2+k_3}{2} - \frac{b_1-b_2+b_3}{2} \\ a_3 = \frac{-k_1+k_2+k_3}{2} - \frac{-b_1+b_2+b_3}{2} \end{cases} ,$$

where we defined

$$\delta_i = \Delta_i + k_1. \tag{3.5.32}$$

We then get the most general form for (3.5.26):

$$\left\langle \Phi_1^{k_1}(\boldsymbol{X}_1, U_1)\Phi_2^{k_2}(\boldsymbol{X}_2, U_2)\Phi_3^{k_3}(\boldsymbol{X}_3, U_3)\right\rangle = \sum_{b_i \in \mathscr{B}} C_{b_1 b_2 b_3} \frac{\mathcal{C}^{b_1 b_2 b_3}}{(\boldsymbol{X}_1 \cdot \boldsymbol{X}_2)^{\frac{\delta_1+\delta_2-\delta_3}{2}} (\boldsymbol{X}_2 \cdot \boldsymbol{X}_3)^{\frac{-\delta_1+\delta_2+\delta_3}{2}} (\boldsymbol{X}_1 \cdot \boldsymbol{X}_3)^{\frac{\delta_1-\delta_2+\delta_3}{2}}} . \tag{3.5.33}$$

Here $C_{b_1 b_2 b_3}$ are some constants and

$$\mathcal{C}^{b_1 b_2 b_3} \equiv \mathcal{B}_{12}^{\frac{k_1+k_2-k_3}{2} - \frac{b_1+b_2-b_3}{2}} \mathcal{B}_{13}^{\frac{k_1-k_2+k_3}{2} - \frac{b_1-b_2+b_3}{2}} \mathcal{B}_{23}^{\frac{k_1-k_2+k_3}{2} - \frac{b_1-b_2+b_3}{2}} \mathcal{D}_1^{b_1} \mathcal{D}_2^{b_2} \mathcal{D}_3^{b_3}$$

is the tensorial part that gives a scale factor of $\lambda_i^{k_i}$ for each $\boldsymbol{X}_i$, in which we used

$$\mathcal{D}_k \equiv \frac{(\boldsymbol{X}_j \cdot \boldsymbol{X}_k)(\boldsymbol{X}_i \cdot U_k) - (\boldsymbol{X}_i \cdot \boldsymbol{X}_k)(\boldsymbol{X}_j \cdot U_k)}{\boldsymbol{X}_i \cdot \boldsymbol{X}_j} \quad i < j, \quad i, j \neq k,$$

while $\mathscr{B}$ is the set where the b_i can vary and is given by the following constraints[8]

$$\begin{cases} \frac{k_1+k_2-k_3}{2} \geq \frac{b_1+b_2-b_3}{2} & (a_1 \geq 0) \\ \frac{k_1-k_2+k_3}{2} \geq \frac{b_1-b_2+b_3}{2} & (a_2 \geq 0) \\ \frac{-k_1+k_2+k_3}{2} \geq \frac{-b_1+b_2+b_3}{2} & (a_3 \geq 0) \\ b_1 \geq 0 \\ b_2 \geq 0 \\ b_3 \geq 0 \end{cases} \implies \begin{cases} 0 \leq b_1 \leq k_1 \\ 0 \leq b_2 \leq k_2 \\ 0 \leq b_3 \leq k_3 \end{cases} . \tag{3.5.34}$$

If we let one of the fields, say the first, to be a scalar, this picture simplifies considerably. Indeed, this assumption is equivalent to imposing

$$a_1 = a_2 = b_1 = k_1 = 0$$

so that (3.5.31) becomes

$$\begin{cases} c_1 + c_3 = -\Delta_1 - k_1 \\ c_1 + c_2 = -\Delta_2 - k_2 + 2b_2 \\ c_2 + c_3 = -\Delta_3 - k_3 + 2b_3 \\ a_3 = k_2 - b_2 \\ a_3 = k_3 - b_3 \end{cases}$$

whose solution is

$$\begin{cases} c_1 = -\frac{\delta_1+\delta_2-\delta_3}{2} + (k_2 - k_3) \\ c_2 = -\frac{-\delta_1+\delta_2+\delta_3}{2} + (k_2 - k_3 + 2b_3) \\ c_3 = -\frac{\delta_1-\delta_2+\delta_3}{2} + (k_3 - k_2) \\ a_3 = k_3 - b_3 \\ b_2 = k_2 - k_3 + b_3 \end{cases} . \tag{3.5.35}$$

The related Green function is then

$$\Big\langle \Phi_1(\boldsymbol{X}_1)\Phi_2^{k_2}(\boldsymbol{X}_2,U_2)\Phi_3^{k_3}(\boldsymbol{X}_3,U_3)\Big\rangle = \sum_{b_3\in\mathscr{B}} C_{b_3} \frac{\mathcal{B}_{23}^{k_3-b_3}\mathcal{D}_2^{k_2-k_3+b_3}\mathcal{D}_3^{b_3}}{(\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^{\frac{\Delta_1+\delta_2-\delta_3}{2}}(\boldsymbol{X}_2\cdot\boldsymbol{X}_3)^{\frac{-\Delta_1+\delta_2+\delta_3}{2}}(\boldsymbol{X}_1\cdot\boldsymbol{X}_3)^{\frac{\Delta_1-\delta_2+\delta_3}{2}}}. \tag{3.5.36}$$

Again the numerator scales as $\lambda_i^{k_i}$. Let us analyze the allowed values for b_3, i.e. the set $\mathscr{B}_3$. From (3.5.35), positivity of a_3, b_2, b_3 provides the following system:

$$\begin{cases} b_3 \leq k_3 \\ b_3 \geq k_3 - k_2 \\ b_3 \geq 0 \end{cases} \implies \max(0, k_3 - k_2) \leq b_3 \leq k_3. \tag{3.5.37}$$

In particular, when the spins k_2 and k_3 are equal, (3.5.36) is completely determined by $b_3 = k_3$.

As we showed in subsection 2.1.7, the fields, whose correlators we will need to compute in the boundary CFT, are conserved currents, that obey (see (2.2.4))

$$\partial_{\mu_1}\phi^{\mu_1\ldots\mu_s} = 0. \tag{3.5.38}$$

[8]Notice that (3.5.33) is redundant in 3 dimensions, because $\boldsymbol{X}_{1,2,3}$ and $U_{1,2,3}$ can not be linearly independent, since the ambient space has only 5 dimensions and therefore the structures $\mathcal{B}_{12}, \mathcal{B}_{13}$ and $\mathcal{B}_{23}$ and $\mathcal{D}_{1,2,3}$ are not independent. Indeed, one can show that

$$(\mathcal{D}_1\mathcal{B}_{23} + \mathcal{D}_2\mathcal{B}_{13} + \mathcal{D}_3\mathcal{B}_{12} - \mathcal{D}_1\mathcal{D}_2\mathcal{D}_3)^2 \propto \mathcal{B}_{23}\mathcal{B}_{13}\mathcal{B}_{12}.$$

This relation can be translated into a constraint for the set $\mathscr{B}$.

Using (3.4.7) this condition is equivalent in Ambient space to

$$\partial \cdot \mathscr{D}\Phi = 0. \tag{3.5.39}$$

Equation 3.5.39 allows to further restrict the set $\mathscr{B}_3$. In the case of the correlator with two scalars and a spin s particle with weight Δ_3, instead, it just imposes

$$\Delta_3 = d + s - 2.$$

We conclude this subsection with a final remark. Both (3.5.36) and (3.5.33) show the same structure at the denominator (and the same can be said for (3.5.25) after some algebraic manipulation) that resembles the one we got for scalars (3.5.11) but with a difference: the replacement of Δ_i with δ_i. This change is fictitious though, because the numerator scales in such a way that it completely cancels this difference.

3.6 Application to the bulk

In section 1.5 we presented the full interacting non-linear theory of higher spins by means of its equations of motion. However, in order to realize the AdS/CFT correspondence described in 2.2.2, namely the Klebanov-Polyakov conjecture, we need to compute Witten diagrams, whose rules are determined by the action of the bulk theory, which is not presently known. Therefore, a possible approach is to consider the perturbation theory of the Vasiliev equations and write a part of its complete action by starting from the free Lagrangian, that gives rise to the standard Fronsdal equations, and adding order by order every vertex of interaction. This can be done by expanding the Vasiliev equations around the AdS background and seeking the proper Lagrangian that produces all the perturbative terms. For the simplest case, though, namely the cubic vertices, imposing invariance under the AdS symmetry group and gauge transformations is sufficient to completely determine the vertices (up to constant factors).

This task is more easily carried out in the ambient space. For this reason, the goal of this section will be to develop the rules to compute Witten diagrams for the Fronsdal fields in this formalism. To do so we will simplify the problem by setting a traceless transverse gauge similar to (1.1.22) in AdS. We start by finding the proper ambient covariant derivative in subsection (3.6.1) and employ it in subsection 3.6.2 to write the general propagators for higher spin fields, solving (2.1.50). After this, we find the most general expression for cubic vertices of higher spins in AdS.

3.6.1 Covariant derivatives in the ambient space

Covariant derivatives assume a specially simple form in the ambient space and this is one of the main advantages of its usage. Indeed they correspond to taking the transverse part of an ambient derivative of the tensor in question:

$$D_B T_{A_1 \ldots A_k} \overset{(3.3.5)}{\longrightarrow} \mathcal{P}^{\mathrm{m}}_{\mathrm{b}} \mathcal{P}^{\mathrm{m}_1}_{\mathrm{a}_1} \cdots \mathcal{P}^{\mathrm{m}_k}_{\mathrm{a}_k} \partial_{\mathrm{m}} \mathcal{T}_{\mathrm{m}_1 \ldots \mathrm{m}_k} \equiv \mathcal{D}_{\mathrm{b}} \mathcal{T}_{\mathrm{a}_1 \ldots \mathrm{a}_k}. \tag{3.6.1}$$

This is actually the first definition of D_M proposed by Levi-Civita in [52] for manifolds embedded in a flat space. In the following we will use the following short-hand notation:

$$\prod_{i=1}^{k} \mathcal{V}^{\mathrm{m}_i} \equiv \mathcal{V}^{\mathrm{m}_1} \cdots \mathcal{V}^{\mathrm{m}_k}$$

for products of an arbitrary number of tensors $\mathcal{V}$.

Let us apply definition (3.6.1) to the computation of the Laplacian operator $g^{MN}D_MD_N$ acting on a symmetric tensor in the ambient formalism, where it becomes $\mathcal{P}^{\rm bc}\mathcal{D}_{\rm b}\mathcal{D}_{\rm c}$ by (3.3.15):

$$\begin{aligned}
\mathcal{P}^{\rm bc}\mathcal{D}_{\rm b}\mathcal{D}_{\rm c}\mathcal{T}_{{\rm a}_1\ldots{\rm a}_k} &= \mathcal{P}^{\rm bc}\mathcal{D}_{\rm b}\left(\mathcal{P}_{\rm c}^{\rm m}\prod_{i=1}^{k}\mathcal{P}_{{\rm a}_i}^{{\rm m}_i}\partial_{\rm m}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k}\right)\\
&= \mathcal{P}^{\rm bc}\mathcal{P}_{\rm b}^{\rm d}\mathcal{P}_{\rm c}^{\rm e}\left(-\frac{1}{L^2}\eta_{\rm de}\mathcal{X}^{\rm m}\prod_{i=1}^{k}\mathcal{P}_{{\rm a}_i}^{{\rm m}_i}\partial_{\rm m}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k}-\sum_{j=1}^{k}\frac{\mathcal{P}_{\rm e}^{\rm m}}{L^2}\eta_{{\rm da}_j}\mathcal{X}^{{\rm m}_j}\prod_{i\neq j}\mathcal{P}_{{\rm a}_i}^{{\rm m}_i}\partial_{\rm m}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k}\right.\\
&\quad\left.+\mathcal{P}_{\rm e}^{\rm m}\prod_{i=1}^{k}\mathcal{P}_{{\rm a}_i}^{{\rm m}_i}\partial_{\rm m}\partial_{\rm d}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k}\right)\\
&= -\frac{d+1}{L^2}\mathcal{X}^{\rm m}\prod_{i=1}^{k}\mathcal{P}_{{\rm a}_i}^{{\rm m}_i}\partial_{\rm m}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k}-\frac{k}{L^2}\mathcal{P}_{{\rm a}_j}^{\rm m}\prod_{i\neq j}\mathcal{P}_{{\rm a}_i}^{{\rm m}_i}\mathcal{X}^{{\rm m}_j}\partial_{\rm m}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k}\\
&\quad+\mathcal{P}^{\rm dm}\prod_{i=1}^{k}\mathcal{P}_{{\rm a}_i}^{{\rm m}_i}\partial_{\rm m}\partial_{\rm d}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k}\\
&= \mathcal{P}\left(\partial_{\rm m}\partial^{\rm m}-\frac{1}{L^2}\left(\mathcal{X}_{\rm m}\partial^{\rm m}\right)\left(\mathcal{X}_{\rm n}\partial^{\rm n}+d\right)+\frac{k}{L^2}\right)\mathcal{T}_{{\rm a}_1\ldots{\rm a}_k} \qquad (3.6.2)
\end{aligned}$$

where we denoted by a generic $\mathcal{P}$ all the transversal projections in the last step. Notice that, due to homogeneity (3.3.4), the operator $\mathcal{X}_{\rm m}\partial^{\rm m}$, which merely counts the powers of $\mathcal{X}$, can be replaced by $-\Delta$.

We can use ambient transversality to simplify considerably expression (3.6.1) for totally symmetric fields. Indeed, consider the following term in the expansion of (3.6.1) that one obtains after substituting $\mathcal{P}$ with the expression given in (3.3.12):

$$\begin{aligned}
\frac{1}{L^2}\mathcal{P}_{\rm b}^{\rm n}\mathcal{X}_{{\rm a}_i}\mathcal{X}^{{\rm m}_i}\partial_{\rm n}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k} &= -\frac{1}{L^2}\mathcal{P}_{\rm b}^{\rm n}\mathcal{X}_{{\rm a}_i}\partial_{\rm n}\mathcal{X}^{{\rm m}_i}\mathcal{T}_{{\rm m}_1\ldots{\rm m}_k}\\
&= -\mathcal{X}_{{\rm a}_i}\frac{1}{L^2}\mathcal{T}_{{\rm m}_1\ldots\hat{\rm m}_i\ldots{\rm m}_k{\rm b}} \qquad (3.6.3)
\end{aligned}$$

where we used the Leibniz rule and ambient transversality. We can use (3.6.3) to prove that in $\prod_{i=1}^{k}\mathcal{P}_{{\rm a}_i}^{{\rm m}_i}$ only the terms with at most one of the $\frac{\mathcal{X}_{{\rm a}_i}\mathcal{X}^{{\rm m}_i}}{L^2}$ factors coming from the projectors survive, the others being ruled out by ambient transversality. On the other hand, using homogeneity (3.3.4),

$$\mathcal{P}_{\rm b}^{\rm n}\partial_{\rm n}\mathcal{T}_{{\rm a}_1\ldots{\rm a}_k}=\partial_{\rm b}\mathcal{T}_{{\rm a}_1\ldots{\rm a}_k}+\frac{\Delta}{L^2}\mathcal{X}_{\rm b}\mathcal{T}_{{\rm a}_1\ldots{\rm a}_k}.$$

Our conclusion is then that we can rewrite (3.6.1) as

$$\mathcal{D}_{\rm b}\mathcal{T}_{{\rm a}_1\ldots{\rm a}_k}=\partial_{\rm b}\mathcal{T}_{{\rm a}_1\ldots{\rm a}_k}+\frac{\Delta}{L^2}\mathcal{X}_{\rm b}\mathcal{T}_{{\rm a}_1\ldots{\rm a}_k}-\frac{1}{L^2}\mathcal{X}_{({\rm a}_1}\mathcal{T}_{{\rm a}_2\ldots{\rm a}_k){\rm b}}. \qquad (3.6.4)$$

The transverse traceless gauge (1.1.22) in the ambient space is expressed as

$$\mathcal{D}_{{\rm m}_1}\Phi^{{\rm m}_1\ldots{\rm m}_s}=0,\quad \Phi^{{\rm m}_1\ldots{\rm m}_s}\eta_{{\rm m}_1{\rm m}_2}=0, \qquad (3.6.5)$$

but, by contracting (3.6.4) with $\mathcal{P}$, transversality can be rewritten as

$$\partial_{{\rm m}_1}\Phi^{{\rm m}_1\ldots{\rm m}_s}=0.$$

In this gauge, the equations of motion for a Fronsdal field are of the Klein Gordon type (2.1.45). If we choose $-\Delta=s-2$ the second and the third term in (3.6.2) are exactly $-\frac{m^2}{L^2}$ with m given by (1.4.59), and therefore in the ambient space the equation of motion is

$$\partial^{\rm n}\partial_{\rm n}\Phi^{{\rm m}_1\ldots{\rm m}_s}=0. \qquad (3.6.6)$$

3.6.2 The propagators

Now that we have at our disposal the expression of the AdS Laplacian $D_M D^M$ in the ambient space (3.6.2) we are able to solve equations (2.1.46) for the higher spin bulk-to-boundary propagators.

3.6.2.1 Scalars

The bulk-to-boundary propagator for scalar fields (2.1.40) from a boundary point x_2 to a bulk point (z_1, x_1)

$$K_\Delta(z_1, x_1, x_2) = C_\Delta \left(\frac{z}{z^2 - (x_1 - x_2)^2} \right)^\Delta \tag{3.6.7}$$

can be rewritten in the ambient formalism as

$$K_\Delta(\mathcal{X}_1, \boldsymbol{X}_2) = C_\Delta L^\Delta \frac{1}{(2\mathcal{X}_1^{\mathrm{m}} \boldsymbol{X}_{2\mathrm{m}})^\Delta} \tag{3.6.8}$$

by using (3.2.3) and (3.1.20) to express (z_1, x_1) and x_2 as $\mathcal{X}_1$ and $\boldsymbol{X}_2$. Notice that the ambient scalar (3.6.8) is homogeneous of degree Δ by (3.3.4). Verifying that (3.6.7) actually obeys the Klein-Gordon equation (2.1.29) is simpler in the ambient formalism[9]. Indeed, by (3.6.2), it amounts to

$$\left(\partial_1^{\mathrm{m}} \partial_{1\mathrm{m}} - \frac{1}{L^2} \left(\mathcal{X}_1^{\mathrm{n}} \partial_{1\mathrm{n}} \right) \left(\mathcal{X}_1^{\mathrm{p}} \partial_{1\mathrm{p}} + d \right) + m^2 \right) K_\Delta(\mathcal{X}_1, \boldsymbol{X}_2) = C_\Delta L^\Delta \frac{-\frac{\Delta^2}{L^2} + \frac{d}{L^2}\Delta + m^2}{(2\mathcal{X}_1^{\mathrm{m}} \boldsymbol{X}_{2\mathrm{m}})^\Delta} = 0 \tag{3.6.9}$$

The numerator of (3.6.9) is zero exactly when $\Delta = \Delta_\pm$ given by (2.1.31). To get the right behavior at the boundary we must take $\Delta \equiv \Delta_+$.

Another form for (3.6.8) that we will need later is obtained by the Schwinger parameter method:

$$K_\Delta(\mathcal{X}_1, \boldsymbol{X}_2) = N_\Delta^0 \int_0^{+\infty} \frac{dt}{t} t^\Delta e^{-2(\mathcal{X}_1^{\mathrm{m}} \boldsymbol{X}_{2\mathrm{m}})t} \tag{3.6.10}$$

where

$$N_\Delta^0 = \frac{C_\Delta L^\Delta}{\Gamma(\Delta)}.$$

3.6.2.2 Vectors

The bulk-to-boundary propagator of a vector field is a function $K_\Delta^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1, \boldsymbol{X}_2)$ that, as every ambient tensor, must be transversal

$$\mathcal{X}_{1\mathrm{m}} K_\Delta^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1, \boldsymbol{X}_2) = 0 = \boldsymbol{X}_{2\mathrm{n}} K_\Delta^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1, \boldsymbol{X}_2) \tag{3.6.11}$$

and obeys the Klein-Gordon-like equation (2.1.45) for vectors

$$\mathcal{P} \left(\partial_1^{\mathrm{m}} \partial_{1\mathrm{m}} - \frac{1}{L^2} \left(\mathcal{X}_1^{\mathrm{n}} \partial_{1\mathrm{n}} \right) \left(\mathcal{X}_1^{\mathrm{p}} \partial_{1\mathrm{p}} + d \right) + m^2 + \frac{1}{L^2} \right) K_\Delta^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1, \boldsymbol{X}_2) = 0 \tag{3.6.12}$$

by (3.6.2).

First of all, due to the similarity between (3.6.12) we decompose $K_\Delta^{\mathrm{m}|\mathrm{n}}$ as

$$K_\Delta^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1, \boldsymbol{X}_2) \equiv S^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1, \boldsymbol{X}_2) K_\Delta. \tag{3.6.13}$$

[9] Here we used ∂_1^{m} for derivatives in $\mathcal{X}_1$

Then we write the most general form for $S^{\mathrm{n}|\mathrm{m}}$

$$S^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1,\boldsymbol{X}_2)\equiv a\eta^{\mathrm{mn}}+b\mathcal{X}_1^{\mathrm{m}}\boldsymbol{X}_2^{\mathrm{n}}+c\mathcal{X}_1^{\mathrm{n}}\boldsymbol{X}_2^{\mathrm{m}}+e\mathcal{X}_1^{\mathrm{m}}\mathcal{X}_1^{\mathrm{n}}+f\boldsymbol{X}_2^{\mathrm{m}}\boldsymbol{X}_2^{\mathrm{n}}$$

where a,b,c,e,f are some undetermined coefficients and we impose (3.6.11) to (3.6.13), finding the conditions

$$\begin{cases} a+(\mathcal{X}_1\cdot\boldsymbol{X}_2)c+\mathcal{X}_1^2e=0\\ L^2b+(\mathcal{X}_1\cdot\boldsymbol{X}_2)f=0\\ (\mathcal{X}_1\cdot\boldsymbol{X}_2)c+a=0\\ (\mathcal{X}_1\cdot\boldsymbol{X}_2)e=0\end{cases}$$

solved by

$$c=-\frac{a}{\mathcal{X}_1\cdot\boldsymbol{X}_2},\quad f=-\frac{\mathcal{X}_1^2}{(\mathcal{X}_1\cdot\boldsymbol{X}_2)}b,\quad e=0.$$

so that, up to $(\mathcal{X}_1\cdot\boldsymbol{X}_2)$ factors that can be absorbed by K_δ, we can write

$$S^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1,\boldsymbol{X}_2)=\alpha\left(\eta^{\mathrm{mn}}-\frac{\mathcal{X}_1^{\mathrm{n}}\boldsymbol{X}_2^{\mathrm{m}}}{\mathcal{X}_1\cdot\boldsymbol{X}_2}\right)+\beta\left(\mathcal{X}_1^{\mathrm{m}}\boldsymbol{X}_2^{\mathrm{n}}-\frac{\mathcal{X}_1^2}{\mathcal{X}_1\cdot\boldsymbol{X}_2}\boldsymbol{X}_2^{\mathrm{m}}\boldsymbol{X}_2^{\mathrm{n}}\right).$$

However, due to strong transversality of the light-cone tensors (see (3.3.10)), we must set $\beta=0$. We get thus

$$S^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1,\boldsymbol{X}_2)\equiv\alpha\left(\eta^{\mathrm{mn}}-\frac{\mathcal{X}_1^{\mathrm{n}}\boldsymbol{X}_2^{\mathrm{m}}}{\mathcal{X}_1\cdot\boldsymbol{X}_2}\right).$$

and we can now impose (3.6.12) to obtain

$$\mathcal{P}\left(\partial_1^{\mathrm{q}}\partial_{1\mathrm{q}}-\frac{1}{L^2}\left(\mathcal{X}_1^{\mathrm{n}}\partial_{1\mathrm{n}}\right)\left(\mathcal{X}_1^{\mathrm{p}}\partial_{1\mathrm{p}}+d\right)+m^2+\frac{1}{L^2}\right)K_\Delta^{\mathrm{n}|\mathrm{m}}\quad=\quad\left(-\frac{1}{L^2}\left(-\Delta\right)\left(-\Delta+d\right)+\frac{1}{L^2}+m^2\right)K_\Delta^{\mathrm{n}|\mathrm{m}}$$

up to terms that are not transversal in the strong sense and therefore project to zero by (3.3.11). This equation is solved by

$$\Delta\left(\Delta-d\right)-1-mL^2=0. \tag{3.6.14}$$

We recognize in (3.6.14) equation (2.1.47) upon the redefinition $\Delta=\delta+1$ and we remind that one has to take $\delta\equiv\delta_+$ to have the correct asymptotics at the boundary. The vector propagator is therefore

$$K_\Delta^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1,\boldsymbol{X}_2)=N_\Delta^1N_\Delta\frac{(\mathcal{X}_1\cdot\boldsymbol{X}_2)\,\eta^{\mathrm{mn}}-\mathcal{X}_1^{\mathrm{n}}\boldsymbol{X}_2^{\mathrm{m}}}{(2\mathcal{X}_1^{\mathrm{m}}\boldsymbol{X}_{2\mathrm{m}})^{\Delta+1}}, \tag{3.6.15}$$

for some normalization coefficient N_Δ^1.

Equation (3.6.13) can also be rewritten by applying a differential operator to the scalar propagator

$$K_\Delta^{\mathrm{n}|\mathrm{m}}(\mathcal{X}_1,\boldsymbol{X}_2)=N_\Delta^1D_\Delta^{\mathrm{n}|\mathrm{m}}K_\Delta(\mathcal{X}_1,\boldsymbol{X}_2) \tag{3.6.16}$$

where

$$D_\Delta^{\mathrm{n}|\mathrm{m}}=\eta^{\mathrm{mn}}+\frac{1}{\Delta}\boldsymbol{X}_2^{\mathrm{m}}\frac{\partial}{\partial\boldsymbol{X}_{2\mathrm{n}}}. \tag{3.6.17}$$

3.6.2.3 Higher spins

Analogously to the vector case, one can show that the spin s propagator is

$$K_{\Delta}^{\mathrm{n}_1\ldots\mathrm{n}_s|\mathrm{m}_1\ldots\mathrm{m}_s}(\mathcal{X}_1,\boldsymbol{X}_2)=N_{\Delta}^{s}N_{\Delta}\frac{S^{(\mathrm{n}_1|\mathrm{m}_1}\ldots S^{\mathrm{n}_s)|\mathrm{m}_s}}{(2\mathcal{X}_1\cdot\boldsymbol{X}_2)^{\Delta}}. \tag{3.6.18}$$

It is possible to rewrite also (3.6.18) in an integral form by the Schwinger parameter method

$$\int_0^{+\infty}\frac{dt}{t}S^{(\mathrm{n}_1|\mathrm{m}_1}\ldots S^{\mathrm{n}_s)|\mathrm{m}_s}t^{\Delta}e^{-2(\mathcal{X}_1\cdot\boldsymbol{X}_2)t} \tag{3.6.19}$$

and express (3.6.19) by a differential operator similar to (3.6.17) applied to the scalar propagator (3.6.7):

$$K_{\Delta}^{\mathrm{n}_1\ldots\mathrm{n}_s|\mathrm{m}_1\ldots\mathrm{m}_s}(\mathcal{X}_1,\boldsymbol{X}_2)=N_{\Delta}^{s}D_{\Delta}^{\mathrm{n}_1\ldots\mathrm{n}_s|\mathrm{m}_1\ldots\mathrm{m}_s}K_{\Delta}(\mathcal{X}_1,\boldsymbol{X}_2). \tag{3.6.20}$$

Indeed, consider the following integral

$$\begin{aligned}
\int_0^{+\infty}\frac{dt}{t}t^{\Delta}e^{-2(\mathcal{X}_1\cdot\boldsymbol{X}_2)t}\prod_{i=1}^{k}\frac{\mathcal{X}_1^{\mathrm{n}_i}\boldsymbol{X}_2^{\mathrm{m}_i}}{\mathcal{X}_1\cdot\boldsymbol{X}_2} &= \int_0^{+\infty}dt\frac{\frac{d^k}{dt^k}\left(t^{\Delta+k-1}\right)}{(\Delta+k-1)_{\Delta-1}}e^{-2(\mathcal{X}_1\cdot\boldsymbol{X}_2)t}\prod_{i=1}^{k}\frac{\mathcal{X}_1^{\mathrm{n}_i}\boldsymbol{X}_2^{\mathrm{m}_i}}{\mathcal{X}_1\cdot\boldsymbol{X}_2}\\
&= \int_0^{+\infty}dt\frac{(-1)^k\,t^{\Delta+k-1}}{(\Delta+k-1)_{\Delta-1}}\frac{d^k}{dt^k}\left(e^{-2(\mathcal{X}_1\cdot\boldsymbol{X}_2)t}\right)\prod_{i=1}^{k}\frac{\mathcal{X}_1^{\mathrm{n}_i}\boldsymbol{X}_2^{\mathrm{m}_i}}{\mathcal{X}_1\cdot\boldsymbol{X}_2} \qquad (3.6.21)\\
&= \prod_{i=1}^{k}\boldsymbol{X}_2^{\mathrm{m}_i}\int_0^{+\infty}dt\frac{2^k t^{\Delta+k-1}}{(\Delta+k-1)_{\Delta-1}}e^{-2(\mathcal{X}_1\cdot\boldsymbol{X}_2)t}\prod_{j=1}^{k}\mathcal{X}_1^{\mathrm{n}_j}\\
&= (-1)^k\left(\prod_{i=1}^{k}\boldsymbol{X}_2^{\mathrm{m}_i}\prod_{j=1}^{k}\frac{\partial}{\partial\boldsymbol{X}_{2\mathrm{n}_j}}\right)\int_0^{+\infty}\frac{dt}{t}\frac{t^{\Delta}e^{-2(\mathcal{X}_1\cdot\boldsymbol{X}_2)t}}{(\Delta+k-1)_{\Delta-1}}. \qquad (3.6.22)
\end{aligned}$$

where we used repeatedly integration by parts in (3.6.21) and the Pochhammer symbol n_m defined in the notation appendix. Applying relation (3.6.22) to (3.6.19), one can derive the operator defined in (3.6.20). The resulting expression is quite convoluted and can be simplified by using the index-free formalism. For this purpose we define the index-free bulk-to-boundary propagator as

$$K_{\Delta}^{s}(\mathcal{X}_1,\boldsymbol{X}_2,U_2,V)\equiv K_{\Delta}^{\mathrm{n}_1\ldots\mathrm{n}_s|\mathrm{m}_1\ldots\mathrm{m}_s}(\mathcal{X}_1,\boldsymbol{X}_2)V_{\mathrm{m}_1}\cdots V_{\mathrm{m}_s}U_{2\mathrm{n}_1}\cdots U_{2\mathrm{n}_s}. \tag{3.6.23}$$

Here we used a vector V in place of U_1 to avoid dependence on $\mathcal{X}_1$. This does not guarantee transversality but only total symmetry. The index-free version of (3.6.20) then reads

$$K_{\Delta}^{s}(\mathcal{X}_1,\boldsymbol{X}_2,U_2,V)\equiv N_{\Delta}^{s}D_{\Delta}^{s}(\boldsymbol{X}_2,V,U_2)K_{\Delta}(\mathcal{X}_1,\boldsymbol{X}_2),$$

where

$$D_{\Delta}^{s}(\boldsymbol{X}_2,V,U_2)\equiv V_{\mathrm{m}_1}\cdots V_{\mathrm{m}_s}U_{2\mathrm{n}_1}\cdots U_{2\mathrm{n}_s}D_{\Delta}^{\mathrm{n}_1\ldots\mathrm{n}_s|\mathrm{m}_1\ldots\mathrm{m}_s}(\mathcal{X}_1,\boldsymbol{X}_2). \tag{3.6.24}$$

Using (3.6.22) and

$$\prod_{i=1}^{s}U_{2\mathrm{n}_i}S^{\mathrm{n}_i|\mathrm{m}_i}V_{\mathrm{m}_i}=\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}(U_2\cdot V)^i\,\frac{(V\cdot\boldsymbol{X}_2)^{s-i}(U_2\cdot\mathcal{X}_1)^{s-i}}{\mathcal{X}_1\cdot\boldsymbol{X}_2}$$

on (3.6.19), we finally get

$$D_{\Delta}^{s}(\boldsymbol{X}_2,V,U_2)\equiv\sum_{i=0}^{s}\frac{\binom{s}{i}}{(\Delta+s-i-1)_{\Delta-1}}(U_2\cdot V)^i(V\cdot\boldsymbol{X}_2)^{s-i}\left(U_2\cdot\frac{\partial}{\partial\boldsymbol{X}_2}\right)^{s-i}. \tag{3.6.25}$$

3.6.3 Cubic vertices

Our aim now is to complete the set of tools that allow us to compute Witten diagrams of the Klebanov-Polyakov correspondence with the ambient formalism by finding the 3-point vertices of the higher spin interacting theory. We do this following [54].

First of all, we need to define an ambient version of the interacting part of the action. The most natural candidate is an integral over the ambient space, but we need to evaluate it only on the $\mathscr{S}_{d+1}$ sub-manifold. We can do this by inserting a Dirac delta function:

$$\int \frac{d^{d+2}\mathtt{X}}{L}\delta\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right) = \int \sqrt{g}dzd^dx\int_0^{+\infty}\delta(R-1)dR = \int \sqrt{g}dzd^dx,$$

where we used $R \equiv \frac{1}{L}\sqrt{-\mathtt{X}^2}$ and

$$d^{d+2}\mathtt{X} = L\left|\det\left(\frac{\partial \mathcal{X}}{\partial X}\right)\right| d^{d+1}XdR = L\sqrt{\det(g)} \equiv L\sqrt{g}.$$

Then, the vertex is a scalar quantity obtained by the contractions between 3 higher spin fields and a certain number of covariant derivatives (as explained in subsection 1.2.2 one needs to consider higher derivative vertices). We want to write it by means of the index-free formalism as an operator acting on a polynomial $\Phi(\mathcal{X}, U)$ that represents the fields[10]

$$\Phi(\mathcal{X}, U) = \sum_{i=0}^{+\infty} \Phi^{\mathrm{m}_1\ldots\mathrm{m}_i}(\mathcal{X})U_{\mathrm{m}_1}\cdots U_{\mathrm{m}_i}.$$

A simple way to write it will be then

$$S_3 \equiv \int \frac{d^{d+2}\mathtt{X}}{L}\delta\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\mathcal{V}\left(\mathcal{X},\partial_1,\partial_2,\partial_3,\partial_{U_1},\partial_{U_2},\partial_{U_3}\right)\Phi(\mathcal{X}_1,U_1)\Phi(\mathcal{X}_2,U_2)\Phi(\mathcal{X}_3,U_3)\Big|_{\substack{\mathcal{X}_1=\mathcal{X}_2=\mathcal{X}_3=\mathcal{X}\\ U_1=U_2=U_3=0}}, \tag{3.6.26}$$

where[11] $\partial_{i\mathrm{m}} \equiv \frac{\partial}{\partial \mathcal{X}_i^{\mathrm{m}}}$ and $\mathcal{V}$ is a function that can be expanded in a power series of its arguments with $\mathcal{X}$ being always before the derivatives, so that they do not interfere with each other[12]. $\mathcal{V}$ specifies every kind of vertex that one can build with the fields contained in $\Phi(\mathcal{X}, U)$. For example,

$$\mathcal{V} = \partial_1\cdot\partial_{U_2}\partial_2\cdot\partial_{U_3}$$

identifies[13]

$$\partial^{\mathrm{n}}\Phi^{\mathrm{m}}\partial_{\mathrm{m}}\Phi_{\mathrm{n}}\Phi,$$

Notice that we did not employ covariant derivatives inside (3.6.26), contrarily to what one would expect. We made this choice because, as showed by (3.6.4), $\mathcal{D}$ is equivalent to ∂ plus terms proportional to $\mathcal{X}$, but these can be absorbed into the definition of $\mathcal{V}$.

[10] This is analogous to what we did in section 1.1.2 for the ω and C tensors with (1.4.60) and (1.4.61).

[11] In this subsection we employ $i, j, k, l = 1, 2, 3$ to denote the three fields Φ. We do not use Einstein's convention on repeated indices in this case.

[12] As in subsection 3.4 here derivatives ∂_i are not meant to act on U_i, even if, by (3.4.3), it depends on the coordinates $\mathcal{X}_i$. This is just a convention that allows us to have compact expressions as (3.6.26). Since at the end no U_i is present in the result, this ambiguity disappears.

[13] As we saw in the last part of section 3.4, to extract the tensor from its corresponding polynomial we should employ $\mathscr{D}$ instead of ∂_U. In the example this does not matter, since we deal only with vectors and at the end we put $U_i = 0$, so that $\mathscr{D}$ is proportional to ∂_U. However in the general case it is not so. However, $\mathscr{D}$ is a function of ∂_U and so it fits into the definition of $\mathcal{V}$. Moreover, in the following we will need to contract $\mathscr{D}$ with $\mathcal{X}$ and again this is equivalent to using ∂_U. For these reasons we will always employ ∂_U instead of $\mathscr{D}$.

Our goal is then to constrain the form of $\mathcal{V}$ as much as possible. The first property is that it should be compatible with our redundant description of the vertex by 3 different polynomials Φ, namely that it is invariant under

$$\partial_i^{\mathrm{m}} \rightleftarrows \partial_j^{\mathrm{m}}, \quad \partial_{U_i}^{\mathrm{m}} \rightleftarrows \partial_{U_j}^{\mathrm{m}}. \tag{3.6.27}$$

Secondly we require invariance under isometries, that here appear as the elements of the $SO(2,d)$ symmetry algebra of the ambient space. This amounts to requiring that $\mathcal{V}$ is a scalar. Since it is a function of vectors, this is equivalent to saying that it must be a function of all the possible contractions of its elements. Let us analyze them one by one:

1. $\mathcal{X}^2 = L^2$ can be absorbed by the definition of $\mathcal{V}$.

2. $\mathcal{X} \cdot \partial_i$ counts the power of $\mathcal{X}_i$ and therefore is always equivalent to a number and we ignore it.

3. $\mathcal{X} \cdot \partial_{U_i}$ acts generically on a higher spin field that has been differentiated a certain amount of times. By using (3.4.10) we get

$$\mathcal{X} \cdot \partial_{U_i} \partial_j^{\mathrm{m}} \Phi = \partial_j^{\mathrm{m}} \left(\mathcal{X} \cdot \partial_{U_i} \Phi \right) - \partial_{U_i}^{\mathrm{m}} \Phi = -\partial_{U_i}^{\mathrm{m}} \Phi$$

   and similar relations in case of higher derivatives of Φ. Since this term is equivalent to those where ∂_{U_i} appears, we do not consider it.

4. $\partial_i \cdot \partial_j$, if $i = j$ is proportional to the equation of motion (3.6.6) if we choose the proper Δ_i[14] and therefore vanishes. If instead $i \neq j$, for each term in which $\partial_i \cdot \partial_j$ appears, there are always two others where it is replaced by $\partial_i \cdot \partial_k$ and $\partial_j \cdot \partial_k$, for the exchange symmetry (3.6.27). We have then the following schematic equivalence:

$$\begin{aligned} \partial_{\mathrm{m}} \Phi \partial^{\mathrm{m}} \Phi \Phi + \partial_{\mathrm{m}} \Phi \Phi \partial^{\mathrm{m}} \Phi + \Phi \partial_{\mathrm{m}} \Phi \partial^{\mathrm{m}} \Phi &= \partial_{\mathrm{m}} \Phi \partial^{\mathrm{m}} (\Phi \Phi) \\ &\sim -\Phi \Phi \partial^{\mathrm{m}} \partial_{\mathrm{m}} \Phi \end{aligned}$$

   up to total derivative terms (which we will deal later with), so that we fall in the previous case and reach the same conclusion: $\partial_i \cdot \partial_j$ can not appear in $\mathcal{V}$.

5. $\partial_{U_i} \cdot \partial_j$ can be transformed by integration by parts into

$$\partial_{U_i} \cdot \partial_k, \quad \partial_{U_i} \cdot \partial_l, \quad l \neq k \neq j,$$

   up to total derivative terms. However, if we use the combinations

$$\mathtt{D}_1 \equiv \partial_{U_1} \cdot (\partial_2 - \partial_3), \quad \text{cyclic permutations of } 1, 2, 3$$

   we see that integration by parts transforms $\mathtt{D}_1$ into $-\mathtt{D}_1$. These combinations exhaust all the possible cases because of the exchange invariance of $\mathcal{V}$ (3.6.27).

6. $\partial_{U_i} \cdot \partial_{U_j}$ is equivalent to 0 if $i = j$ for tracelessness (3.6.5) expressed by (3.4.9). For $i \neq j$ it establishes a contraction by the double application of (3.4.7). We then define

$$\mathtt{C}_i = \sum_{j,k} |\epsilon_{ijk}| \, \partial_{U_j} \cdot \partial_{U_k}$$

[14] If we made some other choice we would however had an equation of the kind $\partial^2 \Phi^{\cdots} = \alpha \Phi^{\cdots}$ for some α in place of (3.6.6), and therefore this term amounts to a number.

Let us analyze the case of total derivatives that we left behind. When we write the total derivative of the integrand given by $\mathcal{V}$ and the Φs, we can use the definition of the derivative of a distribution with the Dirac delta appearing in (3.6.26): schematically

$$\begin{aligned}\int \frac{d^{d+2}\mathtt{X}}{L}\delta\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\partial_{\mathrm{m}}\left(\mathcal{V}\Phi\Phi\Phi\right)^{\mathrm{m}} &= -\int \frac{d^{d+2}\mathtt{X}}{L}\partial_{\mathrm{m}}\left(\frac{\sqrt{-\mathtt{X}^2}}{L}\right)\frac{d\delta}{dt}\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\left(\mathcal{V}\Phi\Phi\Phi\right)^{\mathrm{m}} \\ &= -\int \frac{d^{d+2}\mathtt{X}}{L}\frac{1}{L}\frac{d\delta}{dt}\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\mathtt{X}_{\mathrm{m}}\left(\mathcal{V}\Phi\Phi\Phi\right)^{\mathrm{m}} \qquad (3.6.28)\end{aligned}$$

We see thus that a total derivative acts as a differentiation on the δ times one of the first 3 terms we listed before, namely the contractions with $\mathcal{X}$, that we already absorbed inside the definition of $\mathcal{V}$. For this reason we rewrite (3.6.26) as

$$\int \frac{d^{d+2}\mathtt{X}}{L}\sum_{n=0}^{+\infty}\delta^{(n)}\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\mathcal{V}^{(n)}\left(\mathtt{C}_1,\mathtt{C}_2,\mathtt{C}_3,\mathtt{D}_1,\mathtt{D}_2,\mathtt{D}_3\right)\Phi(\mathcal{X}_1,U_1)\Phi(\mathcal{X}_2,U_2)\Phi(\mathcal{X}_3,U_3)\Big|_{\substack{\mathcal{X}_1=\mathcal{X}_2=\mathcal{X}_3=\mathcal{X}\\ U_1=U_2=U_3=0}}, \qquad (3.6.29)$$

where $\delta^{(n)}(t)\equiv\frac{d^n}{dt^n}\delta(t)$ and $\mathcal{V}^{(n)}$ are functions that share the same properties of $\mathcal{V}$ and together express its dependence on total derivatives:

$$\delta\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\mathcal{V}\sim\sum_{n=0}^{+\infty}\delta^{(n)}\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\mathcal{V}^{(n)}.$$

Now that we narrowed down the possible terms appearing in (3.6.29), it is time to impose higher spin gauge invariance. By the rule explained at the end of section 3.3 we can write the gauge transformation in AdS (1.4.32) as

$$\delta\Phi_{\mathrm{m}_1\ldots\mathrm{m}_s}=\mathcal{D}_{(\mathrm{m}_1}\epsilon_{\mathrm{m}_2\ldots\mathrm{m}_s)}. \qquad (3.6.30)$$

In the index-free formalism we group the gauge parameters into

$$\epsilon(\mathcal{X},U)=\sum_{k=1}^{+\infty}\epsilon^{\mathrm{m}_1\ldots\mathrm{m}_k}U_{\mathrm{m}_1}\cdots U_{\mathrm{m}_k}. \qquad (3.6.31)$$

When dealing with gauge transformations, our choice to use U such that $U\cdot\mathcal{X}\equiv 0$ (see (3.4.2)) may hide some important terms proportional to $\mathcal{X}$ coming from the covariant derivatives (3.6.30). For this reason, let us ignore this property for a moment and show how to circumvent this difficulty. We express (3.6.4) as

$$\delta\Phi(\mathcal{X},U)=\mathcal{D}\epsilon(\mathcal{X},U)\equiv\left(U\cdot\partial-(U\cdot\mathcal{X})\frac{\mathcal{X}\cdot\partial+U\cdot\partial_U}{L^2}\right)\epsilon(\mathcal{X},U), \qquad (3.6.32)$$

where we defined the index-free covariant derivative operator $\mathcal{D}$ by means of (3.4.8). The operators $\mathcal{X}\cdot\partial$ and $U\cdot\partial_U$ just count the number of U and $\mathcal{X}$ in each term appearing in (3.6.31), namely k and the homogeneity degree $-\Delta_k$. Therefore, if we choose for every $\epsilon^{\mathrm{m}_1\ldots\mathrm{m}_k}$

$$\Delta_k=k,$$

we have

$$(\mathcal{X}\cdot\partial+U\cdot\partial_U)\epsilon(\mathcal{X},U)=(k-k)\epsilon(\mathcal{X},U)=0$$

and transformation (3.6.32) simplifies to

$$\mathcal{D}\epsilon(\mathcal{X},U)=U\cdot\partial\epsilon(\mathcal{X},U),$$

so that it is correct to replace $U\cdot\mathcal{D}$ with $U\cdot\partial$ in our computations as (3.4.2) suggests.

With this choice and using (3.6.27) we can compute the variation of (3.6.29) as

$$\begin{aligned}\delta S_3 &= \int \frac{d^{d+2}\mathtt{X}}{L}\sum_{n=0}^{+\infty}\delta^{(n)}\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\mathcal{V}^{(n)}U_1\cdot\partial_1\epsilon\Phi\Phi\Big|_{\substack{\mathcal{X}_{1,2,3}=\mathcal{X}\\ U_{1,2,3}=0}}\\ &= 3\int \frac{d^{d+2}\mathtt{X}}{L}\sum_{n=0}^{+\infty}\delta^{(n)}\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\left[\mathcal{V}^{(n)},U_1\cdot\partial_1\right]\epsilon\Phi\Phi\Big|_{\substack{\mathcal{X}_{1,2,3}=\mathcal{X}\\ U_{1,2,3}=0}}\end{aligned}$$

since U_1 is set to zero at the end and therefore the terms proportional to $U_1\cdot\partial_1\mathcal{V}^{(n)}$ vanish. Thus, gauge invariance is achieved if

$$\sum_{n=0}^{+\infty}\delta^{(n)}\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right)\left[U_i\cdot\partial_i,\mathcal{V}^{(n)}\right]=0 \tag{3.6.33}$$

up to terms that vanish on shell. As shown in [54], equation (3.6.33) is equivalent to the following condition on $\mathcal{V}^{(n)}$

$$\begin{aligned}&\left(\mathtt{D}_1\frac{\partial}{\partial\mathtt{C}_2}-\mathtt{D}_2\frac{\partial}{\partial\mathtt{C}_1}\right)\mathcal{V}^{(n)}+\frac{1}{L}\left(3\left(\mathtt{D}_1\frac{\partial}{\partial\mathtt{D}_1}-\mathtt{D}_2\frac{\partial}{\partial\mathtt{D}_2}\right)\frac{\partial}{\partial\mathtt{D}_3}-2\mathtt{C}_3\left(\frac{\partial^2}{\partial\mathtt{D}_1\partial\mathtt{C}_1}-\frac{\partial^2}{\partial\mathtt{D}_2\partial\mathtt{C}_2}\right)\right)\mathcal{V}^{(n-1)}\\ &\qquad -\frac{1}{L^2}3\left(\mathtt{C}_1\frac{\partial}{\partial\mathtt{D}_2}-\mathtt{C}_2\frac{\partial}{\partial\mathtt{D}_1}\right)\frac{\partial^2}{\partial\mathtt{D}_3^2}\mathcal{V}^{(n-2)}=0\end{aligned} \tag{3.6.34}$$

For $n=0$, condition (3.6.34) is a homogeneous differential equation solved by any function of the form

$$\mathcal{V}^{(0)}(\mathtt{C}_i,\mathtt{D}_i)\equiv\mathcal{V}^{(0)}(\mathtt{D}_i,\sum_j\mathtt{C}_j\mathtt{D}_j). \tag{3.6.35}$$

This solution suggests an ansatz for $\mathcal{V}$:

$$\mathcal{V}\equiv\mathcal{V}(\mathtt{E}_i,\mathtt{G}),$$

where

$$\mathtt{E}_i\equiv\mathtt{C}_i+\alpha_i\partial\cdot\partial_{U_i},\quad \mathtt{G}\equiv\sum_j\left(\mathtt{C}_j+\beta_j\partial\cdot\partial_{U_i}\right)\mathtt{D}_j$$

take into account also total derivatives, represented by $\partial=\sum_i\partial_i$, and α_i,β_i are constant factors. Equation (3.6.34) fixes all the coefficients α_i,β_i except two, that we call α and β. Expanding in a power series the resulting $\mathcal{V}$ one gets

$$\begin{aligned}S_3 &= \frac{1}{3!}\sum_{s_1,s_2,s_3}^{\infty}\sum_{n=0}^{\min(s_1,s_2,s_3)}g^n_{s_1s_2s_3}\int_{AdS}d\mathcal{X}\left[\partial_{U_1}\cdot(\partial_{23}+\alpha\partial)\right]^{s_1-n}\left[\partial_{U_2}\cdot\left(\partial_1-\frac{\alpha-1}{\alpha+1}\partial\right)\right]^{s_2-n}\\ &\left[\partial_{U_3}\cdot\left(\partial_1-\frac{1+\alpha}{\alpha-1}\partial\right)\right]^{s_3-n}\left[\left(\partial_{U_2}\cdot\partial_{U_3}\right)\left(\partial_{U_1}\cdot(\partial_{23}+\beta\partial)\right)-2(\partial_{U_3}\cdot\partial_{U_1})\left(\partial_{U_2}\cdot\left(\partial_1+\frac{\alpha-\beta}{\alpha+1}\partial\right)\right)\right.\\ &\left.+2\left(\partial_{U_1}\cdot\partial_{U_2}\right)\left(\partial_{U_3}\cdot\left(\partial_{X_1}+\frac{\alpha-\beta}{\alpha-1}\partial\right)\right)\right]^n\Phi(\mathcal{X}_1,U_1)\Phi(\mathcal{X}_2,U_2)\Phi(\mathcal{X}_3,U_3)\Big|_{\substack{\mathcal{X}_1=\mathcal{X}_2=\mathcal{X}_3=\mathcal{X}\\ U_1=U_2=U_3=0}}\end{aligned} \tag{3.6.36}$$

where

$$\int_{AdS}d\mathcal{X}=\int\frac{d^{d+2}\mathtt{X}}{L}\delta\left(\frac{\sqrt{-\mathtt{X}^2}}{L}-1\right).$$

Notice that the coupling constants $g^{s_1 s_2 s_3 n}$ are not determined, since they do not appear in the resulting Vasiliev equations. Moreover, to shorten notation, we left some spurious contributions in (3.6.36) coming from terms like

$$\partial_{U_1} \cdot \partial \sim \partial_{U_1} \cdot (\partial_2 + \partial_3)\,,$$

since $\partial_{U_1} \cdot \partial_1 \sim 0$ due to the gauge constraints (3.6.5).

Chapter 4

The comparison of correlators

In this final chapter we employ the tools that we have developed so far to compute correlation functions by Witten diagrams in the bulk and compare them with those in the boundary. We will focus on the simplest cases, namely the 2-point function and the 3-point correlator between a higher spin field and two scalars with a weight Δ depending on the boundary conditions of the bulk scalar fields (see section (2.2.2)).

4.1 The two point correlator

The easiest CFT correlator that we can compare with the one in the bulk given by the AdS/CFT correspondence is the 2-point function between the currents $\mathcal{J}^{\mu_1\ldots\mu_r}$ and $\mathcal{J}^{\nu_1\ldots\nu_s}$. From the CFT point of view, equation (3.5.20) tells us that $r = s$ and

$$\langle \mathcal{J}^{\mathrm{m}_1\ldots\mathrm{m}_s}(\boldsymbol{X}_1)\mathcal{J}^{\mathrm{n}_1\ldots\mathrm{n}_s}(\boldsymbol{X}_2)\rangle = \mathcal{C}_{12}\frac{\mathcal{S}^{(\mathrm{m}_1|\mathrm{n}_1}\ldots\mathcal{S}^{\mathrm{m}_k)\mathrm{n}_k}}{(\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^{s+1}}. \tag{4.1.1}$$

where we used the proper weight given by (2.2.9). On the other hand, as we learned in subsection 2.1.7, the 2 point function of the operator dual to $\Phi^{\mathrm{m}_1\ldots\mathrm{m}_s}$ is given by the boundary limit of the bulk-to-boundary propagator (3.6.18) (see (2.1.59)):

$$\begin{aligned}\langle \mathcal{J}^{\mathrm{m}_1\ldots\mathrm{m}_s}(\boldsymbol{X}_1)\mathcal{J}^{\mathrm{n}_1\ldots\mathrm{n}_s}(\boldsymbol{X}_2)\rangle &= (s+1)\,L^{d-1}\lim_{\mathcal{X}_1\to\boldsymbol{X}_1}K_{s+1}^{\mathrm{n}_1\ldots\mathrm{n}_s|\mathrm{m}_1\ldots\mathrm{m}_s}(\mathcal{X}_1,\boldsymbol{X}_2)\\ &= (s+1)\,L^{d-1}N_{s+1}N_{s+1}^{s}\frac{\lim\limits_{\mathcal{X}_1\to\boldsymbol{X}_1}S^{(\mathrm{n}_1|\mathrm{m}_1}\ldots S^{\mathrm{n}_s)|\mathrm{m}_s}}{(2\boldsymbol{X}_1\cdot\boldsymbol{X}_2)^{s+1}},\end{aligned}$$

which coincides with (4.1.1) after the identification

$$\mathcal{C}_{12} = \frac{(s+1)L^{d-1}N_{s+1}^{s}N_{s+1}}{2^{s+1}}$$

since

$$\lim_{\mathcal{X}_1\to\boldsymbol{X}_1}S^{\mathrm{n}|\mathrm{m}} = \frac{\mathcal{S}^{\mathrm{m}|\mathrm{n}}}{\boldsymbol{X}_1\cdot\boldsymbol{X}_2}.$$

4.2 The scalar-scalar-higher spin diagram

In this section our goal is to compute the correlator between two scalars and a higher spin field:

$$\mathscr{C}_{00s}^{\mathrm{m}_1\ldots\mathrm{m}_s}(\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3) \equiv \langle\mathcal{J}_\Delta(\boldsymbol{X}_1)\mathcal{J}_\Delta(\boldsymbol{X}_2)\mathcal{J}^{\mathrm{m}_1\ldots\mathrm{m}_s}(\boldsymbol{X}_3)\rangle\,. \tag{4.2.1}$$

Here we are assuming the $N \to \infty$ limit that makes the scalar currents to have weight $\Delta = 1, 2$, depending on the boundary condition chosen, and the higher spin current to have dimension $s+1$, as explained in section 2.2.2. Before proceeding with the computation, we show how to compute the class of integrals that we will be dealing with and prove some basic facts about the operators defined by (3.6.25).

4.2.1 The bulk cubic vertex integral

In our computation of the three point function we will employ the Schwinger parametrization of the bulk-to-boundary propagator (3.6.20) and so we will deal with integrals that can be put in the following form:

$$\mathbb{V}(\boldsymbol{X}_i, l_i) \equiv \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i} \int_{AdS} e^{-2\mathcal{X}\cdot(t_1\boldsymbol{X}_1+t_2\boldsymbol{X}_2+t_3\boldsymbol{X}_3)}. \tag{4.2.2}$$

Our aim is to find a simpler form of (4.2.2). Let $\boldsymbol{T}$ be defined as

$$\boldsymbol{T} \equiv (t_1\boldsymbol{X}_1 + t_2\boldsymbol{X}_2 + t_3\boldsymbol{X}_3)\,.$$

$\boldsymbol{T}$ does not belong to the light-cone[1] and therefore we can perform an $SO(2,3)$ transformation in the ambient space so that

$$\boldsymbol{T} = |\boldsymbol{T}|\left(0, 1, \frac{1}{2}\right), \quad |\boldsymbol{T}| \equiv \sqrt{|\boldsymbol{T}^2|}$$

without changing the measure in (4.2.2). Then, using parametrization (3.2.3), equation (4.2.2) becomes

$$\begin{aligned}
\int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i} \int_{AdS} e^{2\mathcal{X}\cdot\boldsymbol{T}} &= \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i} \int_0^{+\infty} dz \left(\frac{L}{z}\right)^4 \int d^3x e^{-2\mathcal{X}\cdot\boldsymbol{T}} \\
&= \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i} \int_0^{+\infty} dz \left(\frac{L}{z}\right)^4 \int d^3x e^{|\boldsymbol{T}|\frac{L}{z}\left(-1+x^\mu x_\mu - z^2\right)} \\
&= \pi^{\frac{3}{2}} \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i} \int_0^{+\infty} \frac{dz}{|\boldsymbol{T}|^{\frac{3}{2}}} \left(\frac{L}{z}\right)^{\frac{5}{2}} e^{-|\boldsymbol{T}|\frac{L}{z}\left(1+z^2\right)} \\
&= L^4\pi^{\frac{3}{2}} \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i} \int_0^{+\infty} dz \left(\frac{1}{z}\right)^{\frac{5}{2}} e^{-\frac{|\boldsymbol{T}|^2L^2}{z} - z}
\end{aligned}$$

where we performed a gaussian integral[2] in the third step and rescaled $z \to L^{-1}|\boldsymbol{T}|^{-1} z$. Let us also rescale t_i into $t_i\frac{\sqrt{z}}{L}$:

$$\begin{aligned}
\mathbb{V}(\boldsymbol{X}_i, l_i) &= L^{4-\sum_j l_j}\pi^{\frac{3}{2}} \int \prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i} \int_0^{+\infty} dz \left(\frac{1}{z}\right)^{\frac{5}{2}-\frac{1}{2}\sum_j l_j} e^{-z} e^{-|\boldsymbol{T}|^2} \\
&= L^{4-\sum_j l_j}\pi^{\frac{3}{2}}\Gamma\left(\frac{1}{2}\sum_j l_j - \frac{5}{2}\right) \int \left(\prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i}\right) e^{-\sum_{k\neq j} t_k t_j \boldsymbol{X}_j\cdot\boldsymbol{X}_k}
\end{aligned} \tag{4.2.3}$$

Now, consider the following change of variables

$$m_k = \sum_{i,j=1}^{3} \frac{1}{2}|\varepsilon_{ijk}|\, t_i t_j, \iff t_i = \sqrt{\frac{m_{i+1}m_{i+2}}{m_i}}$$

[1] Unless $\boldsymbol{X}_1, \boldsymbol{X}_2$ and $\boldsymbol{X}_3$ are proportional to each other, which means that the three points where the correlator is evaluated coincide and this never happens.

[2] Remember that $\eta_{\mu\nu} = -\delta_{\mu\nu}$ because we are using the Euclidean signature, and therefore $x^\mu x_\mu \leq 0$.

and define

$$\delta_i = \frac{1}{2}\left(\sum_j l_j\right) - l_i.$$

Equation (4.2.3) then becomes

$$\begin{aligned}
\mathbb{V}(\boldsymbol{X}_i, l_i) &= \int \left(\prod_{i=1}^{3} \frac{dt_i}{t_i} t_i^{l_i}\right) e^{-\sum_{k\neq j} t_k t_j \boldsymbol{X}_j \cdot \boldsymbol{X}_k} \\
&= \int \prod_{i=1}^{3} \frac{dm_i}{m_i} m_i^{\delta_i} e^{-\sum_{j,k} |\varepsilon_{ijk}| m_i \boldsymbol{X}_j \cdot \boldsymbol{X}_k} \\
&= \prod_i \frac{\Gamma(\delta_i)}{\left(\sum_{j,k} |\varepsilon_{ijk}|\, \boldsymbol{X}_j \cdot \boldsymbol{X}_k\right)^{\delta_i}}
\end{aligned} \tag{4.2.4}$$

4.2.2 Light-cone projectors and bulk-to-boundary propagators

Let us now give a closer look at the differential operator $D_\Delta^{\mathrm{n}|\mathrm{m}}$ defined by (3.6.17). We originally used it to render the bulk-to-boundary propagator ambient transversal both in the AdS and light-cone sense. In a more general setting, it acts as a projector onto ambient tensors in the light-cone and is analogous to $\mathcal{P}$ defined in (3.3.12). Indeed, consider a generic tensor $\mathcal{T}^{\mathrm{a}_1\ldots\mathrm{a}_k}(\boldsymbol{X})$ with homogeneity degree Δ and its projection $D_\Delta^{\mathrm{a}_1|\mathrm{m}}\mathcal{T}_{\mathrm{m}}^{\mathrm{a}_2\ldots\mathrm{a}_k}$. Then, in view of (3.3.4)

$$\boldsymbol{X}_{\mathrm{a}_1} D_\Delta^{\mathrm{a}_1|\mathrm{m}}\mathcal{T}_{\mathrm{m}}^{\mathrm{a}_2\ldots\mathrm{a}_k} = \boldsymbol{X}^{\mathrm{m}}\mathcal{T}_{\mathrm{m}}^{\mathrm{a}_2\ldots\mathrm{a}_k} + \frac{1}{\Delta}\boldsymbol{X}^{\mathrm{m}}\boldsymbol{X}_{\mathrm{a}_1}\frac{\partial}{\partial \boldsymbol{X}_{\mathrm{a}_1}}\mathcal{T}_{\mathrm{m}}^{\mathrm{a}_2\ldots\mathrm{a}_k} = 0.$$

On the other hand, if $\boldsymbol{X}^{\mathrm{m}}\mathcal{T}_{\mathrm{m}}^{\mathrm{a}_2\ldots\mathrm{a}_k} = 0$ and therefore $\mathcal{T}$ is transversal,

$$D_\Delta^{\mathrm{a}_1|\mathrm{m}}\mathcal{T}_{\mathrm{m}}^{\mathrm{a}_2\ldots\mathrm{a}_k} = \frac{\Delta-1}{\Delta}\mathcal{T}^{\mathrm{a}_1\ldots\mathrm{a}_k} + \frac{1}{\Delta}\frac{\partial}{\partial \boldsymbol{X}_{\mathrm{a}_1}}\boldsymbol{X}^{\mathrm{m}}\mathcal{T}_{\mathrm{m}}^{\mathrm{a}_2\ldots\mathrm{a}_k} = \frac{\Delta-1}{\Delta}\mathcal{T}^{\mathrm{a}_1\ldots\mathrm{a}_k}.$$

So the proper definition for the projector is

$$\boldsymbol{P}_\Delta^{\mathrm{n}|\mathrm{m}} = \frac{\Delta}{\Delta-1} D_\Delta^{\mathrm{n}|\mathrm{m}}. \tag{4.2.5}$$

However, contrary to $\mathcal{P}$, ambient transversality for every index of $\mathcal{T}$ can not be achieved by concatenations of $\boldsymbol{P}_\Delta^{\mathrm{n}|\mathrm{m}}$, because it is a differential operator and it does not commute with itself. Therefore one needs a projector for each rank, given by $\frac{1}{H_\Delta^s} D_\Delta^{\mathrm{n}_1\ldots\mathrm{n}_k|\mathrm{m}_1\ldots\mathrm{m}_k}$ (see (3.6.16)), where H_Δ^s is some normalization constant, which, for the spin 1 case, reads $H_\Delta^1 \equiv \frac{\Delta-1}{\Delta}$ by (4.2.5). This projector can be extracted from equation (3.6.25) in the index-free formalism. Indeed, D_Δ^s can be converted into a differential operator acting on polynomials in V by replacing[3] V with ∂_V in (3.6.25):

$$\boldsymbol{P}_\Delta^s(\boldsymbol{X}, V, U) \equiv \frac{1}{H_\Delta^s}\sum_{i=0}^{s} \frac{\frac{1}{s!}\binom{s}{i}}{(\Delta+s-i-1)_{\Delta-1}} (U\cdot\partial_V)^i (\boldsymbol{X}\cdot\partial_V)^{s-i}\left(U\cdot\frac{\partial}{\partial \boldsymbol{X}}\right)^{s-i}, \tag{4.2.6}$$

so that

$$U_{\mathrm{n}_1}\cdots U_{\mathrm{n}_k} D_\Delta^{\mathrm{n}_1\ldots\mathrm{n}_k|\mathrm{m}_1\ldots\mathrm{m}_k}\mathcal{T}_{\mathrm{m}_1\ldots\mathrm{m}_k}(\boldsymbol{X}) = H_\Delta^s \boldsymbol{P}_\Delta^s(\boldsymbol{X}, V, U)\mathcal{T}(\boldsymbol{X}, V).$$

[3] Notice that, due to the fact that the n-th ∂_V derivative of V^n gives a factor $n!$, we needed to add a constant factor $\frac{1}{s!}$.

Let us show that indeed $\boldsymbol{P}^s_\Delta(\boldsymbol{X},V,U)$ projects onto transversal tensors. Using (3.4.10) we get

$$\begin{aligned}
\boldsymbol{X}\cdot\partial_U D^s_\Delta(\boldsymbol{X}_2,V,U)\mathcal{T}(\boldsymbol{X},V) &= \frac{1}{H^s_\Delta}\boldsymbol{X}\cdot\partial_U\sum_{i=0}^{s}\frac{\frac{1}{(s-i)!i!}}{(\Delta+s-i-1)_{\Delta-1}}(U\cdot\partial_V)^i(\boldsymbol{X}\cdot\partial_V)^{s-i}\left(U\cdot\frac{\partial}{\partial\boldsymbol{X}}\right)^{s-i}\mathcal{T}(\boldsymbol{X},V)\\
&= \frac{1}{H^s_\Delta}\sum_{i=1}^{s}\frac{\frac{1}{(s-i)!(i-1)!}}{(\Delta+s-i-1)_{\Delta-1}}(U\cdot\partial_V)^{i-1}(\boldsymbol{X}\cdot\partial_V)^{s-i+1}\left(U\cdot\frac{\partial}{\partial\boldsymbol{X}}\right)^{s-i}\mathcal{T}(\boldsymbol{X},V)\\
&\quad+\frac{1}{H^s_\Delta}\sum_{i=0}^{s-1}\frac{\frac{1}{(s-i-1)!i!}}{(\Delta+s-i-1)_{\Delta-1}}(U\cdot\partial_V)^i(\boldsymbol{X}\cdot\partial_V)^{s-i}\\
&\quad\cdot(-\Delta-s+i+1)\left(U\cdot\frac{\partial}{\partial\boldsymbol{X}}\right)^{s-i-1}\mathcal{T}(\boldsymbol{X},V)\\
&=0
\end{aligned}$$

where in the last step we used a redefinition of the index in the second sum:

$$i'=i+1.$$

There is a straight-forward way to compute H^s_Δ. Let us illustrate it first with the spin 1 case. We compute $\boldsymbol{P}^1_\Delta$ on a particular transversal tensor with weight Δ given by[4]

$$\mathcal{T}^{\mathrm{m}}_T(\boldsymbol{X})\equiv U^{\mathrm{m}}\left(T^{\mathrm{a}}\boldsymbol{X}_{\mathrm{a}}\right)^{-\Delta}. \tag{4.2.7}$$

for some ambient vector T. Indeed,

$$D^{\mathrm{a|m}}_\Delta U_{\mathrm{m}}\left(T^{\mathrm{n}}\boldsymbol{X}_{\mathrm{n}}\right)^{-\Delta}=U^{\mathrm{a}}\left(T^{\mathrm{n}}\boldsymbol{X}_{\mathrm{n}}\right)^{-\Delta}+\frac{1}{\Delta}\boldsymbol{X}^{\mathrm{m}}\frac{\partial U_{\mathrm{m}}}{\partial\boldsymbol{X}_{\mathrm{a}}}\left(T^{\mathrm{n}}\boldsymbol{X}_{\mathrm{n}}\right)^{-\Delta} \tag{4.2.8}$$

since all the terms proportional to $\boldsymbol{X}^{\mathrm{m}}U_{\mathrm{m}}=0$ vanish because of the transversality of U (3.4.2) and, if we differentiate that constraint, we obtain

$$\boldsymbol{X}_{\mathrm{m}}\frac{\partial}{\partial\boldsymbol{X}_{\mathrm{a}}}U^{\mathrm{m}}=-U^{\mathrm{a}},$$

so that (4.2.8) reads

$$D^{\mathrm{a|m}}_\Delta\mathcal{T}_{T\mathrm{m}}(\boldsymbol{X})=\frac{\Delta-1}{\Delta}\mathcal{T}^{\mathrm{a}}_T(\boldsymbol{X}).$$

Therefore, if we apply $\boldsymbol{P}^1_\Delta$ to the polynomial given by $\mathcal{T}_T(\boldsymbol{X},V)\equiv\mathcal{T}^{\mathrm{m}}_T(\boldsymbol{X})V_{\mathrm{m}}$ and we forget for a moment that $U^2=0$, we can extract the normalization constant from the coefficient in front of the resulting $\mathcal{T}_T(\boldsymbol{X},U)$. This procedure readily generalizes to higher rank case by the definition of

$$\mathcal{T}^{\mathrm{m}_1\ldots\mathrm{m}_s}_{s,T}(\boldsymbol{X})\equiv U^{\mathrm{m}_1}\cdots U^{\mathrm{m}_s}\left(T^{\mathrm{a}}\boldsymbol{X}_{\mathrm{a}}\right)^{-\Delta}$$

[4]As showed by (3.4.4), U^{m} is the ambient representative of some vector u^M in the AdS space. We choose its weight to be 0 for simplicity.

that we use to compute[5]

$$
\begin{aligned}
\boldsymbol{P}^s_\Delta \mathcal{T}_{s,T}(\boldsymbol{X},V) &= \frac{1}{H^s_\Delta}\sum_{i=0}^{s}\frac{\frac{1}{s!}\binom{s}{i}}{(\Delta+s-i-1)_{\Delta-1}}(U\cdot\partial_V)^i(\boldsymbol{X}\cdot\partial_V)^{s-i}\left(U\cdot\frac{\partial}{\partial\boldsymbol{X}}\right)^{s-i}(U\cdot V)^s(T^{\mathrm{a}}\boldsymbol{X}_{\mathrm{a}})^{-\Delta}\\
&= \frac{1}{H^s_\Delta}\sum_{i=0}^{s}\frac{\frac{1}{i!(s-i)!}\frac{s!}{i!}}{(\Delta+s-i-1)_{\Delta-1}}(U\cdot\partial_V)^i(\boldsymbol{X}\cdot\partial_V)^{s-i}\left(U^{\mathrm{m}}\frac{\partial U^{\mathrm{n}}}{\partial\boldsymbol{X}^{\mathrm{m}}}V_{\mathrm{n}}\right)^{s-i}(U\cdot V)^i(T^{\mathrm{a}}\boldsymbol{X}_{\mathrm{a}})^{-\Delta}\\
&= \frac{1}{H^s_\Delta}\sum_{i=0}^{s}\frac{\frac{s!}{i!}}{(\Delta+s-i-1)_{\Delta-1}}\left(U^{\mathrm{m}}\frac{\partial U^{\mathrm{n}}}{\partial\boldsymbol{X}^{\mathrm{m}}}\boldsymbol{X}_{\mathrm{n}}\right)^{s-i}(U^2)^i(T^{\mathrm{a}}\boldsymbol{X}_{\mathrm{a}})^{-\Delta}\\
&= \frac{1}{H^s_\Delta}\left(\sum_{i=0}^{s}\frac{(-1)^{s-i}\frac{s!}{i!}}{(\Delta+s-i-1)_{\Delta-1}}\right)(U^2)^s(T^{\mathrm{a}}\boldsymbol{X}_{\mathrm{a}})^{-\Delta}
\end{aligned}
$$

so that

$$
H^s_\Delta=\sum_{i=0}^{s}\frac{(-1)^{s-i}\frac{s!}{i!}}{(\Delta+s-i-1)_{\Delta-1}}.
$$

4.2.3 Computation of the correlator

From (3.6.36) we see that there are two kinds of vertices involved in the computation of (4.2.1): if we choose $\alpha=0$, we get

$$
\mathcal{V}^1_{00s} \equiv \frac{s!}{3!}g^0_{s00}\phi^{\mathrm{m}_1\ldots\mathrm{m}_s}\phi\prod_{i=1}^{s}\left(\overleftarrow{\partial}-\overrightarrow{\partial}\right)_{\mathrm{m}_i}\phi \tag{4.2.9}
$$

$$
\mathcal{V}^2_{00s} \equiv \frac{s!}{3!}\left(g^0_{0s0}+g^0_{00s}\right)\phi^{\mathrm{m}_1\ldots\mathrm{m}_s}\phi\prod_{i=1}^{s}\left(2\overleftarrow{\partial}+\overrightarrow{\partial}\right)_{\mathrm{m}_i}\phi. \tag{4.2.10}
$$

However, by partial integrations, we can rewrite the (4.2.10) in the same form as (4.2.9) up to total derivatives that, as we showed in (3.6.28), are equivalent to differentiating in (3.6.36) the Dirac δ function implicitly contained in $\int_{AdS}d\mathcal{X}$ and contracting the rest of the integrand with $\mathcal{X}$. In our case, since the only indices come from the higher spin field $\phi^{\mathrm{m}_1\ldots\mathrm{m}_s}$, these contributions vanish by ambient transversality (3.3.5). For this reason we can define a unique coupling constant for cubic vertices that involve two scalars:

$$
g_{00s}\equiv\frac{s!}{3!}\left(g^0_{s00}+\frac{g^0_{0s0}+g^0_{00s}}{2}.\right)
$$

The computation that we have to perform is thus the following: by using first (4.2.3)[6] and then (4.2.4) we get

[5] As in the spin 1 example, $U\cdot\frac{\partial}{\partial\boldsymbol{X}}$ must act only on $U\cdot V$, otherwise terms proportional to $\boldsymbol{X}\cdot U$ appear.

[6] Notice that we are using the same formula for terms that actually have different powers of t_1 and t_2. In our case this is allowed by the fact that $\sum_j l_j=2s+2\Delta+1$ and, by (4.2.3), no relative coefficients appear.

$$
\begin{aligned}
\mathscr{C}_{00s}(\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3,U_3) =& N^s_{s+1} g_{00s} H^s_{s+1} \boldsymbol{P}^s_{s+1} \\
& \int_{AdS} d\mathcal{X} \left(K_\Delta(\boldsymbol{X}_1,\mathcal{X}) \left[V\cdot \left(\overleftarrow{\partial} - \overrightarrow{\partial} \right) \right]^s K_\Delta(\boldsymbol{X}_2,\mathcal{X}) \right) K_{s+1}(\boldsymbol{X}_3,\mathcal{X}) + (1\rightleftarrows 2) \\
=& \left(N_\Delta\right)^2 N^s_{s+1} N_{s+1} g_{00s} H^s_{s+1} \boldsymbol{P}^s_{s+1} \int \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} t_3^{s+1} t_1^\Delta t_2^\Delta 2^s \\
& \int_{AdS} d\mathcal{X} \left(t_1 \left(\boldsymbol{X}_1\cdot V\right) - t_2 \left(\boldsymbol{X}_2\cdot V\right)\right)^s e^{-2\mathcal{X}\cdot(t_1\boldsymbol{X}_1+t_2\boldsymbol{X}_2+t_3\boldsymbol{X}_3)} + (1\rightleftarrows 2) \\
=& \left(N_1\right)^2 N^s_{s+1} N_{s+1} g_{00s} H^s_{s+1} L^{4-2\Delta-s} \pi^{\frac{3}{2}} \Gamma\left(s-2\right) 2^s \\
& \boldsymbol{P}^s_{s+1} \int \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} t_3^{s+1} t_1^\Delta t_2^\Delta \left(t_1 \left(\boldsymbol{X}_1\cdot V\right) - t_2 \left(\boldsymbol{X}_2\cdot V\right)\right)^s e^{-\sum_{k\neq j} t_k t_j \boldsymbol{X}_{jk}} + (1\rightleftarrows 2) \\
=& \left(N_1\right)^2 N^s_{s+1} N_{s+1} g_{00s} H^s_{s+1} L^{4-2\Delta-s} \pi^{\frac{3}{2}} \Gamma\left(s-2\right) (-1)^s \\
& \boldsymbol{P}^s_{s+1} \left(\left(\boldsymbol{X}_1\cdot V\right) \frac{\partial}{\partial \boldsymbol{X}_{13}} - \left(\boldsymbol{X}_2\cdot V\right) \frac{\partial}{\partial \boldsymbol{X}_{23}} \right)^s \int \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} t_3 t_1^\Delta t_2^\Delta e^{-\sum_{i\neq j} t_i t_j \boldsymbol{X}_{ij}} + (1\rightleftarrows 2) \\
=& \frac{L^{4-2\Delta-s}}{2^{\Delta+\frac{1}{2}}} \left(N_1^0\right)^2 N^s_{s+1} N_{s+1} g_{00s} H^s_{s+1} \pi^{\frac{3}{2}} \Gamma\left(s-2\right) (-1)^s \\
& \boldsymbol{P}^s_{s+1} \left(\left(\boldsymbol{X}_1\cdot V\right) \frac{\partial}{\partial \boldsymbol{X}_{13}} - \left(\boldsymbol{X}_2\cdot V\right) \frac{\partial}{\partial \boldsymbol{X}_{23}} \right)^s \frac{\Gamma^2\left(\frac{1}{2}\right)\Gamma\left(\Delta-\frac{1}{2}\right)}{\boldsymbol{X}_{23}^{\frac{1}{2}}\boldsymbol{X}_{13}^{\frac{1}{2}}\boldsymbol{X}_{12}^{\Delta-\frac{1}{2}}} + (1\rightleftarrows 2)\,,
\end{aligned}
\tag{4.2.11}
$$

where $(1 \rightleftarrows 2)$ just indicates the similar terms with $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ exchanged and we defined

$$
\boldsymbol{X}_{ij} \equiv \boldsymbol{X}_i \cdot \boldsymbol{X}_j .
$$

We can now compute the projection in (4.2.11) by the following observation:

$$
\begin{aligned}
&\left((\boldsymbol{X}_1\cdot V)\frac{\partial}{\partial \boldsymbol{X}_{13}} - (\boldsymbol{X}_2\cdot V)\frac{\partial}{\partial \boldsymbol{X}_{23}} \right)^s \frac{1}{\boldsymbol{X}_{23}^{\frac{1}{2}}\boldsymbol{X}_{13}^{\frac{1}{2}}\boldsymbol{X}_{12}^{\Delta-\frac{1}{2}}} = \\
&= -\frac{1}{2}\left((\boldsymbol{X}_1\cdot V)\frac{\partial}{\partial \boldsymbol{X}_{13}} - (\boldsymbol{X}_2\cdot V)\frac{\partial}{\partial \boldsymbol{X}_{23}} \right)^{s-1} \frac{\frac{\boldsymbol{X}_1\cdot V}{\boldsymbol{X}_{13}} - \frac{\boldsymbol{X}_2\cdot V}{\boldsymbol{X}_{23}}}{\boldsymbol{X}_{23}^{\frac{1}{2}}\boldsymbol{X}_{13}^{\frac{1}{2}}\boldsymbol{X}_{12}^{\Delta-\frac{1}{2}}} = \\
&= \left((\boldsymbol{X}_1\cdot V)\frac{\partial}{\partial \boldsymbol{X}_{13}} - (\boldsymbol{X}_2\cdot V)\frac{\partial}{\partial \boldsymbol{X}_{23}} \right)^{s-2} \left(\frac{1}{4}\frac{\left(\frac{\boldsymbol{X}_1\cdot V}{\boldsymbol{X}_{13}} - \frac{\boldsymbol{X}_2\cdot V}{\boldsymbol{X}_{23}}\right)^2}{\boldsymbol{X}_{23}^{\frac{1}{2}}\boldsymbol{X}_{13}^{\frac{1}{2}}\boldsymbol{X}_{12}^{\Delta-\frac{1}{2}}} + \frac{\left(\frac{\boldsymbol{X}_1\cdot V}{\boldsymbol{X}_{13}}\right)^2 - \left(\frac{\boldsymbol{X}_2\cdot V}{\boldsymbol{X}_{23}}\right)^2}{\boldsymbol{X}_{23}^{\frac{1}{2}}\boldsymbol{X}_{13}^{\frac{1}{2}}\boldsymbol{X}_{12}^{\Delta-\frac{1}{2}}} \right)
\end{aligned}
\tag{4.2.12}
$$

but the second term in (4.2.12) is not transversal and does not contain any transversal part, as one can verify directly[7]. We can therefore drop it. By induction, we finally arrive at

$$
\mathscr{C}_{00s}(\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3,U_3) = \mathscr{N}_{00s} \frac{\left(\frac{\boldsymbol{X}_1\cdot U_3}{\boldsymbol{X}_{13}} - \frac{\boldsymbol{X}_2\cdot U_3}{\boldsymbol{X}_{23}}\right)^s}{\boldsymbol{X}_{23}^{\frac{1}{2}}\boldsymbol{X}_{13}^{\frac{1}{2}}\boldsymbol{X}_{12}^{\Delta-\frac{1}{2}}}, \tag{4.2.13}
$$

where

$$
\mathscr{N}_{00s} \equiv \frac{L^{4-2\Delta-s}}{2^{\Delta+s+\frac{1}{2}}} \left(N_1^0\right)^2 N^s_{s+1} N_{s+1} g_{00s} \pi^{\frac{5}{2}} \Gamma\left(\Delta-\frac{1}{2}\right) \Gamma\left(s-2\right) H^s_{s+1}.
$$

[7] The simplest way to do this without computations (that otherwise would require the explicit form of D_3^2) is by observing that the second term (4.2.12) is antisymmetric under the exchange $1 \rightleftarrows 2$, and therefore so is its projection, by linearity. However, the only transversal polynomial that one can construct with these ingredients is obviously given by the first term of (4.2.12) (we proved it in subsection 3.5.3), which is symmetric under the aforementioned swap.

Notice that only for even spins (4.2.13) is different from zero, since it does not change sign under the exchange of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. Equation (4.2.13) has the same functional form as the 3-point correlator (3.5.25), in compliance with the Klebanov-Polyakov conjecture.

Conclusions and outlook

In this thesis we presented the interacting higher spin field theory in both the frame-like and the metric-like formalism. The first has the advantage to show through unfolding the underlying higher spin symmetry algebra and indeed the Vasiliev equations rest on this formulation. On the other hand, the presence of infinitely many auxiliary fields and of the additional twistorial coordinates tends to hide the physical meaning of the fields. Moreover, the corresponding AdS/CFT duality cannot be expressed in the standard terms, since no explicit form of partition function is available in this formulation. Indeed, the check of the duality performed in [48] is quite involved. The correspondence in this formalism may be anyway formulated as the twistorial holography introduced in [49], which in turn could explain the origin of the duality.

The metric-like formalism offers a more direct physical interpretation of the fields. We used it together with the ambient formalism. In this way we found the bulk-to-boundary propagators easily. Also the cubic vertices can be recovered by imposing gauge invariance to the first order in the fields as showed in [54] and explained in section 3.6.3. This formalism has proven to be useful also to treat the 4-points function case, in particular to find the bulk-to-bulk propagator [56, 55]. The ambient space provides a natural environment also for the boundary CFT, since it realizes the conformal symmetry in a linear way. This simplifies considerably the task of finding the 3-point correlators by imposing conformal invariance without making explicit computations, thus avoiding the issues caused by the fact that different boundary conditions for the scalar field lead to different dual theories. This way, we determined the most general form of such correlation functions (3.5.33), which can be further constrained because the fields involved are conserved currents.

In this framework we could compare such correlators with those computed with Witten diagrams in a simple case, the one involving two scalars and higher spin field, which was completely determined up to a normalization factor in (3.5.25) on the CFT side. This test, which has already been done in [48], amounts to a computation that is conceptually much simpler in this formalism. We found accordance between the two results.

However, while computing the 3-point correlator we did not perform the projection onto (strongly) transversal ambient tensors represented by $\boldsymbol{P}^s_{s+1}$ by using its explicit form (4.2.6), since the result was simple enough to do it "by hand". This procedure, though, cannot be employed in a more general case, where more than one projector appears and the tensor to be projected is more complicated.

Consider for example the immediate generalization of the computation performed in the last chapter, the 3-point correlator between two fields with spin r and s and a scalar. Using the same techniques that allowed us to arrive at (4.2.11), one can write

$$\begin{aligned}\mathscr{C}_{0rs}(\boldsymbol{X}_1,\boldsymbol{X}_2,U_2,\boldsymbol{X}_3,U_3) \;\propto\; & \sum_h \boldsymbol{P}^r_{r+1}(2)\boldsymbol{P}^s_{s+1}(3)\left(A_h\left(\boldsymbol{X}_1\cdot V_2\right)\frac{\partial}{\partial\boldsymbol{X}_{12}}+B_h\left(\boldsymbol{X}_3\cdot V_2\right)\frac{\partial}{\partial\boldsymbol{X}_{23}}\right)^r \\ & \left(D_h\left(\boldsymbol{X}_1\cdot V_3\right)\frac{\partial}{\partial\boldsymbol{X}_{13}}+E_h\left(\boldsymbol{X}_2\cdot V_3\right)\frac{\partial}{\partial\boldsymbol{X}_{23}}\right)^s\frac{1}{\boldsymbol{X}_{23}^{-\frac{\Delta}{2}+1}\boldsymbol{X}_{13}^{\frac{\Delta}{2}}\boldsymbol{X}_{12}^{\frac{\Delta}{2}}},\end{aligned} \tag{4.2.14}$$

where $\boldsymbol{P}^s_\Delta(i)\equiv\boldsymbol{P}^s_\Delta(U_i,V_i,\boldsymbol{X}_i)$ and the sum in h and the coefficients A_h,B_h,D_h,E_h take into account the different vertices that one can extract from (3.6.36). The dependence of (4.2.14) on $\boldsymbol{X}_{ij}$ is in accordance with our boundary computation (3.5.36) and the presence of the projectors implies a tensor structure like the one given by (3.5.36). However the determination of the coefficients C_{b_3} is much more complicated due to the high number of differential operators present in (4.2.14). Even in the simplest case with only one higher spin field examined so far, the explicit computation by means of (4.2.6) is quite convoluted.

A way to circumvent these difficulties would be to perform the whole calculation by means of a computer program. This should not present particular issues, since the only operations involved are differentiations, easily handled by symbolic processors such as Mathematica. However an analytical way to solve the question would be preferable and is surely a road to explore in future developments.

A more ambitious problem for further study will be the computation of 4-point functions correlation functions of higher spin fields and their comparison with the corresponding CFT correlators. By now, only a 4-point correlator of scalars on the boundary has been computed [55].

Notation

This appendix is devoted to resume some of the conventions that occur throughout the thesis and help the reader to find quickly the meaning of the notation used.

For definitions we use the symbol $\equiv$, while, when we impose some equality yet to be verified, we write $\stackrel{!}{=}$.

The Minkowski metric in generic dimensions is taken to be

$$\eta = \operatorname{diag}\{1, -1...\}$$

unless otherwise specified.

The symmetrization and anti-symmetrization of indices are denoted respectively by

$$T_{(\mu_1...\mu_s)} \equiv \sum_{\sigma\in S_s} T_{\mu_{\sigma(1)}...\mu_{\sigma(s)}}, \quad T_{[\mu_1...\mu_s]} \equiv \sum_{\sigma\in S_s} \operatorname{sgn}(\sigma) T_{\mu_{\sigma(1)}...\mu_{\sigma(s)}},$$

where S_n is the symmetric group of the permutations of n objects and $\operatorname{sgn}(\sigma)$ is the sign of the permutation. Notice that when a certain symmetrization is applied to tensor products, we assume that the least number of addends is used.

Derivatives with respect to vectorial quantities different from the coordinates are denoted by

$$\partial_U^\mu \equiv \frac{\partial}{\partial U^\mu}.$$

Index contractions may also be denoted with the dot product as in

$$\partial^\mu V_\mu \equiv \partial \cdot V$$

or with exponents as in

$$V_\mu V^\mu \equiv V^2.$$

While describing a theory by means of the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ duality, the coordinates of AdS (the bulk) are denoted with X and their indices will be $M, N, P, S, ... = 0, 1, ..., d$, while we will employ x for the coordinates of the boundary and we use the indices $\mu, \nu, \rho, \sigma, ... = 0, 1, ..., d-1$.

In the ambient space formalism we embed AdS in the ambient space $\mathcal{A}_{d+2}$ with $d+2$ dimensions, whose points are denoted by $\boldsymbol{X}^{\mathrm{m}}$ or $\mathcal{X}^{\mathrm{m}}$ depending whether they are in the light-cone $\mathscr{L}_{d+1}$ or in the AdS hyperboloid $\mathscr{S}_{d+1}$, and the indices $\mathrm{m}, \mathrm{n}, \mathrm{r}, \mathrm{s}, ... = 0, 1, ..., d{+}1$ or $\mathrm{m}, \mathrm{n}, \mathrm{r}, \mathrm{s}, ... = 0, 1, ..., d{-}1, +, -$ if the light-cone coordinates

$$\boldsymbol{X}_\pm = \frac{\pm\boldsymbol{X}_d + \boldsymbol{X}_{d+1}}{\sqrt{2}}, \quad \mathcal{X}_\pm = \frac{\pm\mathcal{X}_d + \mathcal{X}_{d+1}}{\sqrt{2}}$$

are in use.

The Pochhammer symbol is defined as follows

$$n_m \equiv \frac{\Gamma(n)}{\Gamma(m)}, \quad n, m \in \mathbb{N},$$

where Γ is the Euler gamma function.

Proper subgroups are denoted by the sign $<$: for example

$$SO(2) < SO(3).$$

Bibliography

[1] S.W. Hawking, “Particle creation by black holes.”, Comm.Math.Phys **43** (1975), 199

[2] J. D. Bekenstein, “Universal upper bound on the entropy-to-energy ratio for bounded systems”, Phys Rev D**49** (1994), 1912

[3] L. Susskind, “The World as a Hologram”, J.Math.Phys. **36** (1995) 6377-6396, arXiv:hep-th/9409089

[4] M. Natsuume, “AdS/CFT Duality User Guide”, arXiv:hep-th/1409.3575

[5] E. Brezin, D.J. Wallace, “Critical Behavior of a Classical Heisenberg Ferromagnet with Many Degrees of Freedom”, Phys. Rev. B **7** (1973) 1976

[6] E. Majorana, Nuovo Cimento **9** (1932) 335

[7] S. Esposito, “Searching for an equation: Dirac, Majorana and the others”, Annals Phys. **327** , 1617 (2012), arXiv:hep-th/1110.6878

[8] R. Casalbuoni, “Majorana and the infinite component wave equa - tions,” PoS E **MC2006** (2006) 004, arXiv:hep-th/0610252

[9] P. A. M. Dirac, “Relativistic wave equations”, Proc. Roy. Soc. Lond. **155A**, 447 (1936)

[10] M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field”, Proc. Roy. Soc. Lond. A **173**, 211 (1939)

[11] S. J. Chang, “Lagrange Formulation for Systems with Higher Spin”, Phys. Rev. **161**, 1308 (1967)

[12] L. P. S. Singh and C. R. Hagen, “Lagrangian formulation for arbitrary spin. 1. The boson case”, Phys. Rev. D **9**, 898 (1974)

[13] G. Velo and D. Zwanziger, “Noncausality and other defects of interaction Lagrangians for particles with spin one and higher,” Phys. Rev. **188**, 2218 (1969)

[14] C. Fronsdal, “Massless Fields With Integer Spin”, Phys. Rev. D **18**, 3624 (1978)

[15] J. Fang and C. Fronsdal, “Massless Fields With Half Integral Spin,” Phys. Rev. D 18, 3630 (1978)

[16] R. Rahman , “Higher Spin Theory - Part I”, PoS ModaveVIII (2012), p. 004, arXiv:hep-th/1307.3199

[17] N. Bouatta, G. Compere, and A. Sagnotti, “An Introduction to free higher-spin fields”, arXiv:hep-th/0409068

[18] X. Bekaert, N. Boulanger and P. Sundell, “How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples”, Rev.Mod.Phys. **84** (2012) 987-1009 arXiv:hep-th/1007.0435

[19] S. Weinberg, “Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass”, Phys. Rev. **135**, B1049 (1964)

[20] S. Weinberg, “The Quantum theory of fields. Vol. 1: Foundations”, Cambridge University Press, UK, 1995. Cambridge, UK: Univ. Pr. (1995)

[21] S. Weinberg and E. Witten, “Limits on massless particles ”, Phys. Lett. B **96**, 59 (1980)

[22] Florian Loebbert, “The Weinberg-Witten theorem on massless particles: an essay”, Ann. Phys. **17**, No. 9 – 10, 803 – 829 (2008)

[23] M. Porrati, “Universal Limits on Massless High-Spin Particles”, Phys. Rev. D **78** (2008) 065016

[24] C. Aragone and S. Deser, “Consistency Problems of Hypergravity”, Phys. Lett. B **86** (1979) 161

[25] S. R. Coleman and J. Mandula, “All possible symmetries of the s matrix,” Phys. Rev. **159** (1967) 1251–1256

[26] R. Haag, J. T. Lopuszanski and M. Sohnius, “All Possible Generators of Supersymmetries of the s Matrix,” Nucl. Phys. B **88** (1975) 257

[27] F. A. Berends, G. J. H. Burgers and H. van Dam, “On the theoretical problems in constructing interactions involving higher-spin massless particles”, Nucl. Phys. B **260** (1985) 295

[28] Xavier Bekaert, Nicolas Boulanger and Serge Leclercq, “Strong obstruction of the Berends-Burgers-van Dam spin-3 vertex.” J.Phys. A **43** 1854 01,2010. arXiv:hep-th/1002.0289

[29] R. R. Metsaev, “Cubic interaction vertices for massive and massless higher spin fields,” Nucl. Phys. B **759** (2006) 147–201, arXiv:hep-th/0512342

[30] A. Fotopoulos and M. Tsulaia, “Gauge Invariant Lagrangians for Free and Interacting Higher Spin Fields. A Review of the BRST formulation”, Int. J. Mod. Phys. A **24** (2009) 1–60, arXiv:hep-th/0805.1346

[31] E. S. Fradkin and M. A. Vasiliev, “Cubic Interaction in Extended Theories of Massless Higher Spin Fields,” Nucl. Phys. B **291** (1987) 141

[32] Yu. M. Zinoviev, “On spin 2 electromagnetic interactions”, Mod. Phys. Lett. A **24** (2009) 17, arXiv:hep-th/0806.4030

[33] É. Cartan, “Sur les variétés à connexion affine et la théorie de la relativité généralisée”, Part I: Ann. Éc. Norm. **40**, 325–412 (1923) and ibid. **41**, 1–25 (1924); Part II: ibid. **42**, 17–88 (1925)

[34] M. A. Vasiliev, “Gauge form of description of massless fiel ds with arbitrary spin”, Sov. J. Nucl. Phys. **32** (1980) 43

[35] E. S. Fradkin and M. A. Vasiliev, Candidate to the role of higher spin symmetry, Ann. Phys. 177 (1987) 63

[36] V.E. Didenko and E.D. Skvortsov, arXiv:hep-th/1401.2975

[37] M. A. Vasiliev, “Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions” Phys.Lett. B **243** (1990) 378

[38] M. A. Vasiliev, “Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)”, Phys. Lett. B **567** (2003) 139-151, arXiv:hep-th/0304049

[39] J. Erdmenger, “Introduction to gauge/gravity duality”, Lect. Notes Phys. **851** (2012) 99.

[40] G. ’t Hooft, “A Planar Diagram Theory For Strong Interactions”, Nucl. Phys. B **72** (1974) 461.

[41] J. Maldacena, “The Large N Limit of Superconformal Theories and Supergravity”, Adv. Theor. Math. Phys. **2** (1998) 231-252, arXiv:hep–th/9711200.

[42] E. Witten, “Anti–de Sitter Space and Holography”, Adv. Theor. Math. Phys. **2** (1998) 253-291, arXiv:hep–th/9802150.

[43] I. Klebanov and A. Polyakov, “AdS dual of the critical O(N) vector model,” Phys.Lett. B **550** (2002) 213–219, arXiv:hep-th/0210114

[44] E. Witten, “Multi-Trace Operators, Boundary Conditions, And AdS/CFT Correspondence”, arXiv:hep-th/0112258

[45] E. Sezgin and P. Sundell, “Holography in 4D (super) higher spin theories and a test via cubic scalar couplings”, JHEP **07** (2005) 044, arXiv:hep-th/0305040

[46] I.R. Klebanov and E. Witten, “AdS/CFT correspondence and Symmetry Breaking,” Nucl. Phys. B556 (1999) 89, hep-th/9905104

[47] S. Giombi and X. Yin, “The Higher Spin/Vector Model Duality”, J.Phys. A **46** (2013) 214003 arXiv:hep-th/1208.4036

[48] S. Giombi and X. Yin, “Higher Spin Gauge Theory and Holography: The Three-Point Functions”, JHEP **1009** (2010) 115, arXiv:hep-th0912.3462

[49] M. A. Vasiliev, “Holography, Unfolding and Higher-Spin Theory”, J.Phys. **A** 46 (2013) 214013 arXiv:hep-th1203.5554

[50] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, “Spinning Conformal Correlators”, JHEP **1111** (2011) 071, arXiv:hep-th/1107.3554

[51] X. Bekaert and E. Meunier, “Higher spin interactions with scalar matter on constant curvature space-times: conserved current and cubic coupling generating functions”, JHEP **11** (2010) 116, arXiv:hep-th/1007.4384

[52] T. Levi-Civita, “Nozione di parallelismo in una varietà qualunque e consequente specificazione geometrica della curvatura Riemanniana”, Rend. Circ. Mat. Palermo **42** (1917) 73–205

[53] M. F. Paulos, “Towards Feynman rules for Mellin amplitudes”, JHEP **1110** (2011) 074, arXiv:hep-th/1107.1504

[54] E. Joung, M. Taronna, “Cubic interactions of massless higher spins in (A)dS: metric-like approach”, Nucl.Phys. B **861** (2012) 145-174 arXiv:hep-th/1110.5918

[55] X. Bekaert, J. Erdmenger, D. Ponomarev and C. Sleight, “Towards holographic higher-spin interactions: Four-point functions and higher spin exchange”, arXiv:hep-th/1412.0016

[56] M. S. Costa, V. Goncalves, and J. Penedones, “Spinning AdS Propagators”, arXiv:hep-th/1404.5625

Ringraziamenti

Bene, ora dovrei scrivere qualcosa di nostalgico. Fate conto che l'abbia fatto e, con il terrore di aver dimenticato qualcuno, lasciatemi esprimere un sentito grazie:

- A Johanna, che mi ha introdotto a quella cosa meravigliosa e stupefacente che è la corrispondenza AdS/CFT, e a Charlotte, che in pratica ha ideato questa tesi.
- A Kurt, che è stato determinante sia in senso umano che pratico per il completamento di questa tesi (e per molto altro in effetti).
- A Dmitri, il relatore che credo tutti vorrebbero, costantemente disponibile, comprensivo, paziente e capace di farmi capire con poche parole quello che non mi tornava.
- Ai miei genitori, perché scrivendo questa tesi mi sono reso conto quanto profondamente sono figlio loro.
- A mia zia Paola, perché vede sempre in me più di quello che veda io (e non perché ho gli occhiali perennemente lerci).
- A Elena, lei sa i molti perché.
- Ai miei amici di Arezzo Piero, Iacopo, Marco, il Dona (che non si chiama Marco), il Duchi, perché capisci che una persona è importante quando la incontri di nuovo dopo mesi ed è come essersi salutati ieri, con una valigia piena zeppa di Gocciole perché non si sa mai con cosa facciano colazione in Germania.
- (sì, anche a Silvia, che ho messo per ultima solo per darle noia)
- Ai ragazzi di Padova, Luca, il sig. Giacon, Davide, SIS, Riccardo, il Dondi, Lorenzo, Paolo, Agnese (ah, senza l'Agnese sarei stato perduto!), Picco, Ale, StefanoZebbellin, Cosimo e la gran mole di Galileiani che non mi va di scrivere perché sono pigro. Ognuno di loro è veramente speciale, in molti sensi. Anche se sono sparsi per il mondo, spero di incontrarli spesso girovagando qua e là.
- Ai miei coinquilini Ilaria, Emanuele e Caterina. Loro forse non l'hanno capito, ma sono stati importanti davvero, perché mi hanno restituito la casa, nel senso meno architettonico del termine, che ho perso andando in Germania.
- Alla Scuola Galileiana, che mi ha saputo dare molto più che un tetto sopra la testa, un pasto caldo (ancorché alquanto ESU) e un mucchio di conoscenze aggiuntive.
- A Rossella per avermi tolto al momento giusto il prosciutto dagli occhi.
- Al maestro Paolo, che con le sue arringhe contro gli astrologi e il suo senso di scienza ha determinato il mio trovarmi qui, oggi, a laurearmi in Fisica (e a odiare gli astrologi).
- All'Uomo delle Gocciole, a Olivo e a tutto il personale della Forcellini, che in questi anni mi hanno nutrito nel corpo e nello spirito.
- Alla c., che mi ha sempre strappato un sorriso al momento giusto.