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Investigating modified gravity signatures during inflation

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Abstract

Inflation is a period during which the Universe expansion accelerated in the very early universe. Originally introduced to solve the fine tuning problems of the cosmological Hot Big Bang model, it has been a great success in explaining the origin of the small temperature anisotropies of the Cosmic Microwave Background (CMB). Actually the most accepted models of inflation are the so-called standard single-field models of slow-roll inflation. The quantum field theory (QFT) description of such models consists in the presence during inflation of one scalar field, the inflaton, which slowly rolls down an almost flat potential and interacts with Einstein gravity. At the beginning of inflation both the inflaton and the metric tensor have linear oscillations around their background. During inflation these primordial perturbations are stretched by the accelerated expansion on very large (superhorizon) scales, where they get frozen. They form the seeds for the formation of primordial perturbations in the scalar curvature of comoving hypersurfaces, which can explain the temperature anisotropies of the CMB, and perturbations of the metric tensor corresponding to primordial gravitational waves. The statistics of the primordial perturbations predicted by the standard slow-roll models of inflation is almost Gaussian. If we try to develop a non-linear extension of the slow-roll theories we find that there is no possibility to observe the non-Gaussianities predicted given the sensitivity of the actual measurements. In the last years the WMAP and Planck satellite has constrained with increasing precision the level of primordial non-Gaussianity. The best constraints at present are those from the Planck measurements of the temperature (and polarization) CMB anisotropies. Such constraints are compatible with a zero level of primordial non-Gaussianity as predicted by the slow-roll models, but there is still a window of almost two orders of magnitude unexplored. For this reason it is interesting to think about modifications of slow-roll models of inflation in order to achieve signals of non-Gaussianity. Modified gravity theories are an example of such a modification. Being-open minded about a modification of Einstein gravity during inflation is a well motivated question. In fact the high energies of the early universe are not accessible today in the colliders and we do not know if in these conditions gravity follows exactly the Einstein description. In this Thesis we have focused on the analysis of Chern-Simons gravity during inflation, which is parity breaking and polarizes the primordial gravitational waves into circular polarizations. In this case a difference between the large-scale power spectrum of the two different circular polarizations of the primordial gravitational waves arises. For the approximations made to develop the theory we argue that this difference is small but maybe observable with future experiments. As an original contribution we have focused on primordial non-Gaussian signatures. We have computed a non trivial parity breaking pattern into the non-Gaussianity of the primordial perturbations (specifically for the 3-point function correlating primordial gravitational waves and the scalar curvature perturbation). In the slow-roll limit this signature can be large even if there is a small parity breaking in the power spectra. For this reason making a more detailed investigation about non-Gaussianities provided by the Chern-Simons gravity term during inflation can be an interesting issue for the future.

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Introduction

Inflation is a period during which the Universe expansion accelerated in the very early universe. Its role is crucial for explaining the well known horizon, flatness and magnetic monopole problems of the Hot Big Bang cosmological model, the most accepted paradigm of the universe history.

But inflation is considered mainly for another important feature: primordial perturbations production. Actually the slow-roll models of inflation are the most accepted models. They have a quantum field theory description. In according to them the primordial universe is dominated by a scalar field ϕ , the inflaton, which autointeracts with a potential $V(\phi)$ and interacts with standard Einstein gravity. The potential is assumed to be approximately flat. During inflation we can decompose the inflaton into a background value, which is isotropic and homogeneous, and a small fluctuation around the background. We can do the same for the components of the metric tensor g . The background dynamics of the inflaton is responsible to create an accelerated expansion of the primordial universe. Then the initial small fluctuations are stretched on superhorizon cosmological scales by this accelerated expansion and so that their amplitude get "frozen". This happens because at a certain time the wavelength of a certain oscillatory mode λ exits the Hubble horizon $r_H(t)$, which provides a measure of the dimensions of the cosmological regions causally connected in the universe at a fixed time t . At the end of inflation, the inflaton decays into radiation through a process called *reheating* of the universe. The calculations show that two are the relevant types of dynamical perturbations during inflation: scalar perturbations associated to the perturbations of the curvature of comoving hypersurfaces and primordial gravitational waves. The first one, after the reheating, remains frozen until the corresponding wavelength reenters into the Hubble horizon, creating perturbations in the energy density of the radiation fluid. Then through this mechanism we explain essentially how in the universe small perturbations that we observe in the Cosmic Microwave Background (CMB) formed. Such anisotropies represent a photography of the primordial universe at the hydrogen recombination epoch. The same mechanism explains the first seeds from which, via gravitational instability, the Large Scale Structure of the Universe formed during the matter dominated epoch. Up to now, we have not revealed the primordial gravitational waves yet and we have some upper limits on their amplitude.

The slow-roll models of inflation predict an almost Gaussian and almost scale invariant primordial perturbations. This is confirmed by observations of the CMB temperature anisotropies. In the last years the Planck satellite has also put strong constraints on deviations from a pure Gaussian distribution of the primordial perturbations [1]. Such constraints are compatible with a zero level of primordial non-Gaussianity as predicted by the slow-roll models, but there is still a window of almost two orders of magnitude unexplored. The future prospective is to reduce further these errors in order to find out if some signal of non-Gaussianity may arise from the CMB.

In fact non-Gaussianities are important to discriminate between different models of inflation, such as multi-field models that are models in which we add one or more fields besides the inflaton, or modified gravity models in which models of inflation are studied with a modification of the Einstein gravity. In this Thesis we have decided to search for effects provided by a modification of

the Einstein gravity in the slow-roll models of inflation. Being open-minded about this fact is an important question: in fact the high energy of the primordial universe is not accessible today in the colliders, and we do not know if in the primordial cosmological fluid gravity followed precisely the Einstein description or not.

In particular we have concentrated into the Chern-Simons gravity during inflation. In the Lagrangian formalism, the action of this type of gravity is composed of two terms: the first one is the standard Hilbert-Einstein term which describes standard gravity, and the second one is the Chern-Simons term which violates parity and arises from an effective field theory approach in which we admit in the action all the possible covariant terms with at maximum four derivatives of the metric tensor. For reasons of parity invariance this term does not modify the theoretical Gaussian statistics of the primordial scalar perturbations, but it changes the Gaussian statistics of the primordial gravitational waves. Our aim is to investigate if this term can produce or not signatures of non-Gaussianities into the statistics of the primordial perturbations.

The Thesis is organized as follows.

In Chapter 1 we introduce inflation as a powerful mechanism to solve the classical problems of the Hot Big Bang model of cosmology and we describe qualitatively the primordial perturbation production.

In Chapter 2 we analyse quantitatively the standard slow-roll models of inflation, focusing on predictions about the power spectra of the gauge invariant primordial perturbations.

In Chapter 3 we define statistical correlators that describe effects of primordial non-Gaussianity. Then we define a theoretical formalism to search for non-Gaussianities of the primordial perturbations in the slow-roll models of inflation. Finally, we introduce the role of Modified Gravity models of inflation which can be relevant in producing some signatures of primordial non-Gaussianity.

In Chapter 4 we analyze the effects of the addition of a modified gravity term in the context of the slow-roll models of inflation. This term is the Chern-Simons term, which breaks parity symmetry. We describe the effects of such a modified gravity term on the power spectra of the primordial perturbations. We then investigate possible signatures of non-Gaussianity of the primordial perturbations. In particular, as original contribution, we have computed the parity contribution in the 3-point function correlating primordial gravitational waves and the scalar curvature perturbation, showing that we can have a large parity breaking even if there is a small parity breaking in the power spectra.

Chapter 1

Introduction to inflation

In this chapter we review the formalism necessary to give a description of the dynamics of the universe. We start from recalling some basics of the standard Hot Big Bang model. We briefly describe the flatness and the horizon problems of the model, together with the so called cosmic relics problem. We introduce inflation as a powerful paradigm to solve these problems. Finally, we discuss how an inflationary epoch in the primordial universe can produce primordial density perturbations, that can be the "seeds" from which all the actual large scale structures in the universe originated. We follow the Refs. [2, 3, 4, 5] and we work with standard natural units $c = \hbar = 1$.

1.1 The Robertson-Walker metric

The basis of modern cosmology is the assumption that the universe is homogeneous and isotropic on large scales. Today the homogeneity is verified through the observation of the distribution of galaxies and from the Cosmic Microwave Background (CMB), which represents a photography of the primordial universe at the time of hydrogen recombination. It consist of cosmic photons at the same average temperature equal to $T \simeq 2.7K$. These photons come from the last surface scattering with electrons. Isotropy is checked observing in the CMB the low level of relative temperature anisotropies $\frac{\Delta T}{T} \simeq 10^{-5}$. But we must specify the class of observers with respect to which this is valid. They are the so-called *comoving* observers. A comoving observer can be thought of as an observer that moves following the cosmic fluid. For this reason he is not sensible to the energy flows and the universe for such observer can be considered static.

Homogeneity and isotropy fix, univocally, the Robertson-Walker (RW) metric for the universe [2]. In cartesian coordinates it is expressed as :

$$ds^2 = -dt^2 + a^2(t) \left[d\mathbf{x}^2 + k \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 - k\mathbf{x}^2} \right], \quad (1.1)$$

where (t, \mathbf{x}) are a set of coordinates of a comoving observer.

In polar coordinates it is expressed as:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (1.2)$$

In such expressions $a(t)$ represents the *scale factor* of the universe. If we foliate the spacetime in spacelike hypersurfaces Σ_t ¹, $a(t)$ characterizes the time evolution of the physical dimensions of

¹This procedure is called "slicing" .

such surfaces. It is usually normalized as $a_o = 1$, where the suffix o denotes that a is evaluated today. In the standard Hot Big Bang cosmology $a(t)$ turns out to be a monotonic function increasing with time. The parameter k , called *spatial curvature parameter*, represents the curvature of the spatial hypersurfaces. We have three different spatial geometries of the universe depending on the value of k :

- $k < 0$, the hypersurfaces Σ_t have negative curvature, it follows an *open* universe;
- $k = 0$, the hypersurfaces Σ_t are euclidean, it follows a *flat* universe;
- $k > 0$, the hypersurfaces Σ_t have positive curvature, it follows a *closed* universe.

If we use the metric (1.2), we can define for a fixed time t the comoving distance between an object in the universe and the origin of the coordinate system² as:

$$\chi(t) = \int_0^{r(t)} \frac{dr'}{\sqrt{1 - kr'^2}}. \quad (1.3)$$

Instead, the corresponding proper distance $d(t)$ is obtained multiplying (1.3) for the factor scale $a(t)$. We obtain:

$$d(t) = a(t) \int_0^{r(t)} \frac{dr'}{\sqrt{1 - kr'^2}}. \quad (1.4)$$

An important physical quantity for characterizing the universe is the *Hubble parameter* H , defined as:

$$H(t) = \frac{\dot{a}}{a}. \quad (1.5)$$

It gives information about the rate expansion of the universe at a certain cosmological time t . Actually its experimental value is [6]:

$$H_o = (67.8 + / - 0.9) \text{ km Mpc}^{-1} \text{ s}^{-1}. \quad (1.6)$$

Another important quantity to describe the cosmological evolution of the universe is the cosmological redshift z of the cosmic fluid. This redshift is the result of the cosmological expansion of the universe. It is defined as [2]:

$$z = \frac{\lambda_{obs} - \lambda_{emis}}{\lambda_{obs}}, \quad (1.7)$$

where λ_{emis} is the wavelength emitted by an electromagnetic source at a certain time t_{emis} and λ_{obs} is instead the wavelength observed today, which results stretched by the cosmological expansion. We can link this observable with the scale factor $a(t)$ as [2]:

$$1 + z(t_{emis}) = \frac{a_o}{a(t_{emis})}. \quad (1.8)$$

Then, using Eq. (1.8), we can adopt the variable $z(t)$ instead of $a(t)$ to refer to a particular cosmological epoch in the universe history. Moreover a measure of z permits to go back to the time in which the corresponding electromagnetic wave was emitted.

²We should remark that the origin of the coordinate system is totally arbitrary for invariance under traslations of the universe.

1.2 Friedmann equations

The equations that describe the dynamics of the universe are obtained by the solution of the Einstein equation [5]:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} , \quad (1.9)$$

where in the case of a homogeneous and isotropic Universe, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R are obtained by the metric (1.1) and $T_{\mu\nu}$ is the energy-momentum tensor associated to the cosmological fluid. From now on we fix $8\pi G = 1$ for simplicity of notation.

We suppose that the cosmological fluid is a perfect fluid and its energy momentum tensor has the form:

$$T_{00} = \rho(t), \quad T_{ij} = -g_{ij}p(t), \quad T_{0i} = 0 , \quad (1.10)$$

where $p(t)$ is the isotropic pressure density of the fluid and $\rho(t)$ is its energy density.

Then, substituting Eqns. (1.1) and (1.10) into Eq. (1.9), we find two independent equations, the so-called Friedmann equations [3]:

$$H^2 = \frac{\rho}{3} - \frac{k}{a^2} , \quad (1.11)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) . \quad (1.12)$$

These equations link together the scale factor $a(t)$ with the pressure density $p(t)$ and energy density $\rho(t)$ of the cosmological fluid. Assuming to have an expanding universe, so with $\dot{a} > 0$, we can have $\ddot{a} < 0$ only if $\rho + 3p \geq 0$ (from Eq. (1.12)). This last condition³ defines the ordinary matter, which is defined as matter that interacts mainly with gravitational interactions. Here gravity predominates over the electromagnetic force decelerating the expansion rate of the universe. Instead the case $\rho + 3p < 0$ refers to non-ordinary matter, leading in particular to an accelerated expansion of the universe with $\ddot{a} > 0$.

We can define the *critical energy density* as:

$$\rho_c = 3H^2 , \quad (1.13)$$

and the *density parameter*:

$$\Omega = \frac{\rho}{\rho_c} . \quad (1.14)$$

Then, we can rewrite Eq. (1.12) as:

$$\Omega - 1 = \frac{k}{a^2 H^2} . \quad (1.15)$$

From Eq. (1.15) it follows:

³This condition is also equivalent to the so called *strong energy condition* in a Friedmann-Robertson-Walker universe.

- if $\Omega < 1$, then $k < 0$ and we have an open universe;
- if $\Omega = 1$, then $k = 0$ and we have a flat universe;
- if $\Omega > 1$, then $k > 0$ and we have a closed universe.

For this reason the critical density is the value which the universe would have if it was perfectly flat.

1.3 Solutions of the Friedmann equations

In order to find solutions to the Friedmann equations, we combine Eqs. (1.11) and (1.12) to find the *continuity equation*:

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 . \quad (1.16)$$

In order to solve Eq. (1.16), we suppose that the cosmological fluid is barotropic, in the sense that the pressure depends only on the energy density through the linear relation:

$$p(t) = w\rho(t) . \quad (1.17)$$

If we take w to be a constant and substitute Eq. (1.17) into Eq. (1.16), we find the general solution:

$$\rho \propto a^{-3(1+w)} , \quad (1.18)$$

where w is the equation of state of the fluid.

We distinguish three main kinds of fluid depending on the value of w [2]:

Matter fluid: it can consist of *non relativistic matter* and/or *dark matter* (DM). Baryonic matter with an energy mass m greater than the thermal energy $E_T \sim T$ of the universe ($m > T$) is by definition non-relativistic. The dark matter component represents matter that interacts only gravitationally in the universe and does not have any electromagnetic interaction and for this reason we cannot observe it directly. Moreover we know from observations that in order to be a dominant component today in the matter fluid the DM component must be non-relativistic today. For such fluids we can take $p \simeq 0$, $w \simeq 0$ and we have:

$$\rho \propto a^{-3} , \quad (1.19)$$

$$\rho \propto (1+z)^3 . \quad (1.20)$$

Radiation fluid: it consists of *relativistic matter* and *radiation*. The relativistic matter is baryonic matter with an energy mass much smaller than the thermal energy of the universe ($m \ll T$). Radiation is associated to photons. For such fluid $w = -\frac{1}{3}$ and then

$$\rho \propto a^{-4} , \quad (1.21)$$

$$\rho \propto (1+z)^4 . \quad (1.22)$$

Vacuum energy fluid: this kind of fluid is associated to *regions of vacuum* in the universe. For it $w = -1$ and we have:

$$\rho = \text{constant} . \quad (1.23)$$

In this last case the energy momentum tensor becomes of the kind $T_{\mu\nu} = \Lambda g_{\mu\nu}$, with Λ a constant. Then in the literature this fluid is called *cosmological constant*, because it appears in the Einstein equations (1.9) as the cosmological constant firstly introduced by Einstein to find a solution for a static universe [5].

Our universe is composed by all of these three kinds of fluid at the same time. Then the dynamics of the universe is driven by the fluid which is dominant in term of the energy density at a given epoch. The total density parameter (1.14) is obtained by the sum over all the three species:

$$\Omega = \Omega_m + \Omega_r + \Omega_\Lambda . \quad (1.24)$$

The experimental values of the parameters Ω_i was measured by the Planck satellite and tabulated in Ref. [6]. They are exposed in the table 1.1 at the end of this chapter.

Observations confirm that today the predominant fluid is an energy vacuum like fluid with a relative abundance $\Omega_\Lambda \simeq 0.69$. This fluid is also called *Dark Energy*, because it consists of an unknown energy that seems to contrast the attractive gravitational force. The observations indicate also that the universe is fully consistent with a spatially flat universe. If we assume that $k \simeq 0$ during all the universe history, then we can solve exactly equation (1.11), finding

$$a(t) \sim \begin{cases} t^{\frac{2}{3(1+w)}}, & \text{if } w \neq -1, \\ e^{Ht}, & \text{if } w = -1. \end{cases} \quad (1.25)$$

The Hot Big Bang model

The Hot Big Bang model is the most accepted paradigm of the history of the universe given the multiple observational probes on which it is based on. According to it, the early universe had very high temperatures and so it was dominated by radiations and ultrarelativistic particles. We presume also that the universe has expanded for all its history (then $\dot{a} > 0$). In such a scenario from Eqs. (1.15) and (1.25) we find that the spatial curvature parameter k tends to become smaller and smaller in the past. In addition the scale factor a and the energy density ρ evolved⁴ as:

$$a \propto t, \quad \rho(t) \propto t^{-4} . \quad (1.26)$$

We observe that at the time $t = 0$ there is a singularity where the energy density goes to $+\infty$, which is called *Big Bang*.

This ultrarelativistic primordial fluid initially was in a thermodynamic equilibrium, in the sense that the rate of interactions between the particles was much greater than the rate of expansion of the universe, H (Consider, e. g., the electromagnetic interactions between photons γ and electrons e^-). The radiation fluid at equilibrium follows the Planck statistics. By a matching between the energy density imposed by the statistics and the one imposed by the Friedmann equations, we find for such a primordial universe the *time-temperature relation*:

$$T(t) \propto t^{-\frac{1}{2}} . \quad (1.27)$$

⁴We take the solutions of Eqs. (1.25) and (1.18) with $w = \frac{1}{3}$.

This equation tells us that the universe becomes colder as time passes. For this reason at a certain time some baryonic (or dark) matter particles can become non-relativistic. At thermodynamic equilibrium the numerical density $n(t)$ of non-relativistic matter is suppressed by the Boltzmann factor $\sim e^{-\frac{m}{T}}$, where m is the mass of the particle under consideration. Then until the baryonic matter remained in equilibrium with radiation, the energy density of the matter fluid was suppressed. This epoch is called *radiation dominated era*. This continued until the epoch of recombination where hydrogen atoms formed, reducing the number of electrons. This determined the decoupling of the photons from the baryons.

From now, matching Eqs. (1.19) and (1.21), it follows that the energy density of the radiation fluid became soon smaller than the energy density of the matter fluid. When this happened in the universe, it started a new epoch, the *matter dominated era*. During this epoch all the large scale structures that we observe in the universe, such as galaxies and cluster of galaxies, started to form through gravitational instability.

After this epoch, at very recent times, the vacuum energy fluid becomes dominant with respect to both matter and radiation. This can be a consequence of the growing of vacuum regions in the universe.

This is a brief description of the Hot Big Bang model. We remand to Ref. [2] for more details.

1.4 The Problem of the initial conditions

The standard Big Bang cosmology, briefly recalled above, presents in fact some issues. The universe we observe today is the result of very unlikely initial conditions. We could say that the standard Big Bang model has a *fine tuning* problem. In order to understand this problem, we define an important concept, the one of *horizon*, as it is usually used in cosmology.

- **Particle horizon:** In terms of the comoving distance $\chi(t)$, the particle horizon is defined as follows. Given a point A of the universe and a time t , the particle horizon $\chi_p(t)$ is the maximum distance from which a point B may have sent a light signal to A at a certain time in the past t' . If $\chi_p(t)$ is finite, a point C that has reached A by the time t cannot have never sent information to A in the past; at the same time C cannot have never received physical signals from A. The two points have been physically disconnected until the time t . We can define quantitatively $\chi_p(t)$ using the fact that the infinitesimal space-time interval ds of a light ray is zero. From the RW metric (1.2) it follows that the infinitesimal radial distance covered by the light in an infinitesimal time dt at fixed angles is:

$$dr = \frac{\sqrt{1 - kr^2}}{a(t)} dt. \quad (1.28)$$

So, in order to find the quantitative expression of the particle horizon, it is enough to substitute Eq. (1.28) into the general expression of comoving distance $d(t)$ (1.3). We find:

$$\chi_p(t) = \int_0^t \frac{dt'}{a(t')}. \quad (1.29)$$

The corresponding proper particle horizon d_p is given by:

$$d_p = a \int_0^t \frac{dt'}{a(t')}. \quad (1.30)$$

Another horizon that can be defined in a cosmological context is the Hubble horizon.

- **The Hubble horizon:** the *Hubble radius* $R_H(t)$ is defined as the distance covered by the light during the characteristic universe time $\tau = H^{-1}$. So in natural units:

$$R_H(t) = H^{-1}(t) . \quad (1.31)$$

The corresponding *comoving Hubble radius* is:

$$r_H(t) = (aH)^{-1} . \quad (1.32)$$

The comoving Hubble radius represents an estimation of the distance under which two points in the comoving universe are causally connected at a certain time t . For each generic point P of the comoving universe we can define in the space a causally connected spherical region which has radius $r_H(t)$ and is centered in P . This region contain all the points with which the point P is causally connected at the time t . For this reason the comoving Hubble radius is also called *comoving Hubble horizon*.

With a change of variable in (1.29) we can relate the comoving particle horizon to the comoving Hubble horizon as:

$$\chi_p(t) = \int_0^t \frac{dt'}{a(t')} = \int_{a(0)}^{a(t)} \frac{da'}{Ha'^2} = \int_{a(0)}^{a(t)} \frac{d \log(a')}{Ha'} = \int_{a(0)}^{a(t)} d \log(a') r_H(t') , \quad (1.33)$$

where in the last passage we have used Eq. (1.32).

Now, we have all the elements to understand the so called horizon and flatness problems of the standard Hot Big Bang model.

1.4.1 The Horizon problem

We can summarize the horizon problem in the following way. For $w \neq 0$ the comoving Hubble horizon can be written, using Eq. (1.25), as:

$$r_H(t) = \frac{1}{H(t)a} = \frac{1}{H(t_0)} a(t)^{\frac{(1+3w)}{2}} , \quad t_0 < t . \quad (1.34)$$

where t_0 is a certain initial time.

In our model the scale factor a is a monotonus function growing with time. The primordial universe is supposed to be dominated by ordinary ultrarelativistic matter, then with $w \geq -\frac{1}{3}$. In this case the comoving Hubble radius is a monotonus function growing with time too. The same happens also in the matter dominated era in which $w = 0$. Then during almost the entire universe history, the comoving Hubble horizon has grown up. Only recently the universe has begun a vacuum fluid dominated era, in which the comoving Hubble horizon is decreasing.

This means that comoving scales, that were entering inside the comoving Hubble horizon during the matter dominated era, at the time of the baryon-photon decoupling were larger than the comoving Hubble horizon and so causally disconnected. In particular, passing to the comoving particle horizon $\chi_p(t)$, it is possible to compute the ratio between the comoving particle horizon of the CMB today and the one at the time of baryon-photon decoupling. We find [3]:

$$\frac{\chi_{p_o}}{\chi_p(t_{dec})} \sim \frac{10^{28} cm}{10^{23} cm} \sim 10^5 . \quad (1.35)$$

Thus the observed CMB can be divided into 10^5 circular patches that were physically disconnected until the time of recombination. It should be clear from the Eq. (1.35) that going in the past the number of patches physically disconnected only increases. How could all these physically disconnected patches have acquired the same temperature in the past? It does not explain the extreme similarity between regions that may not have been in causal contact between them in the past. In conclusion, the standard Big Bang theory is forced to assume, as initial conditions, an extremely homogeneous and isotropic universe.

1.4.2 The flatness problem

In order to formulate the problem we take the Friedmann equation (1.15) and we write it in terms of the comoving Hubble radius:

$$\Omega - 1 = \frac{k}{a^2 H^2} = k r_H^2(t) . \quad (1.36)$$

We see that if $\Omega = 1$ at a given time t then $k = 0$ for all the times. But realistically speaking there is zero probability that a similar scenario can happen. Observations confirm that $|\Omega_o - 1| < 0.005$ [6]. From the fact that the Hubble radius increases with time, it follows that Ω tends to deviate from the value 1. But this means that in the primordial universe the value of Ω was near to 1 much more than it is today. We take as reference time for the primordial universe the Planck time $t_{pl} \sim 10^{-44} s$ after the Big Bang. This is the time below which the modern quantum field theory description of the nature's laws is incomplete. By a direct estimation we find [3]:

$$\frac{|\Omega(t_{pl}) - 1|}{|\Omega_o - 1|} \simeq 10^{-60} , \quad (1.37)$$

$$|\Omega(t_{pl}) - 1| \simeq 10^{-60} |\Omega_o - 1| \lesssim 10^{-62} . \quad (1.38)$$

We see that the standard Hot Big Bang theory requires a universe to be originated in a very particular state of small spatial curvature.

1.4.3 The problem of cosmological relics

According to several extensions of the Standard Model of particles (e.g. Grand Unification Theories GUT or string theories), if the primordial universe had very high energies, in the early universe at very high energies various cosmological defects, such as magnetic monopoles, could have been produced, which would still be present in the universe, with an abundance that would overclose the universe by many orders of magnitudes. These are called *cosmological relicts*. The magnetic monopoles are an example of these cosmological relics. Historically the problem of the magnetic monopoles was the first theoretical evidence of the necessity to introduce inflation in the primordial universe. We refer to Refs. [4, 7] for more details about other types of cosmological relics.

1.5 Inflation as a powerful solution

1.5.1 General definition of inflation

The solution to both problems is *inflation*. It is defined as a period of accelerated expansion taking place in the early universe, before the radiation dominated era. This allows the universe to start from general initial conditions without the need to impose a very unlikely initial scenario. Quantitatively, it is a period during which:

$$\frac{d^2 a(t)}{dt^2} > 0. \quad (1.39)$$

Starting from Eq. (1.39) we can define also other important features of such a period. First of all, we can rewrite Eq. (1.12) as:

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p). \quad (1.40)$$

From this equation, and requiring the condition (1.39), it follows:

$$\rho + 3p < 0. \quad (1.41)$$

Since ρ is positive, it then follows that the cosmic fluid driving the inflation is characterized by a negative pressure with $p < -\frac{1}{3}\rho$. This means that the cosmic fluid dominating the inflationary epoch is not composed by ordinary matter. The negative pressure is necessary to win against gravity and to produce an accelerated expansion.

In addition we can define the adimensional parameter:

$$\epsilon = -\frac{\dot{H}}{H^2}. \quad (1.42)$$

This parameter describes how much the Hubble parameter changes during the inflationary epoch. Using the definition of the Hubble parameter, (1.5), we can write:

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2(1 - \epsilon). \quad (1.43)$$

Then, requiring $\ddot{a} > 0$, forces $\epsilon < 1$. In particular there can be three possible ranges for the values of ϵ :

- $0 < \epsilon < 1$: the Hubble parameter decreases slowly in time during inflation. In particular in the extreme case $\epsilon \ll 1$ we have a *quasi-de Sitter* inflation. Here approximately $a \sim e^{Ht}$ and $H \sim \text{constant}$;
- $\epsilon = 0$: the Hubble parameter is perfectly constant and we have a *de Sitter inflation*;
- $\epsilon < 0$: the Hubble parameter increases in time during inflation and in the limit $\epsilon \ll -1$ we can have a "super" expansion period, named *super inflation*.

We can also compute the time derivative of the comoving Hubble radius in order to understand how it behaves during inflation:

$$\frac{dr_H(t)}{dt} = \frac{d}{dt} \left(\frac{1}{H(t)a} \right) = -\frac{\dot{a}}{a^2}. \quad (1.44)$$

If $\ddot{a} > 0$, from Eq. (1.44) it follows that the comoving Hubble radius decreases in time during inflationary epoch.

After analyzing the general properties of an inflationary epoch, let us explain how this period solves the fine tuning problems on the initial conditions.

1.5.2 Solution to the Horizon problem

Let us briefly describe in a qualitative way how an inflationary period solves the horizon problem. We have explained in Sect. 4.1 that a certain comoving length scale λ which today⁵ enters the comoving Hubble horizon, was outside this horizon at the time of the matter-radiation decoupling. Then two different points of the universe A and B, separated by a comoving distance λ , were never been in contact until today. But the observations of the CMB proves that also these scales are isotropic to a high degree (1 point to 10^5). To solve the problem we need a time in the past in which the points A and B were causally connected because the comoving Hubble horizon was at least as large as today. But in this case we need also an epoch in which the comoving Hubble horizon was decreasing. We have seen, Eq. (1.44), that this is possible during an inflationary period. Then we can solve the problem considering an inflationary epoch in the primordial universe, before the matter-radiation decoupling (see, e. g., Figure 1.1).

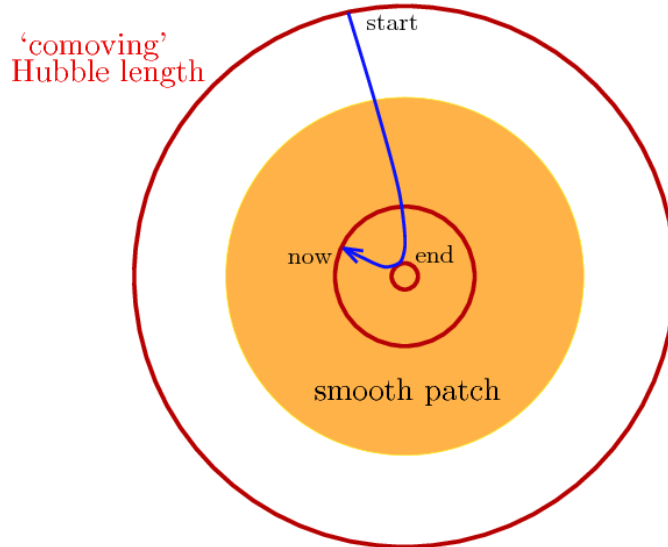


Figure 1.1: This figure illustrates qualitatively the behaviour of the comoving Hubble radius, indicated with a red circle, during inflation. At the beginning of inflation it is larger than the actual comoving particle horizon of the CMB (the purple region); at the end of inflation it is smaller than the comoving particle horizon of the CMB and from that moment it starts back to grow (Figure taken from Ref. [3]).

1.5.3 Solution to the flatness problem

The flatness problem has an immediate solution if one takes into account that during inflation the comoving Hubble radius decreases. In fact, rewriting the Friedmann equation (1.36), we have:

⁵Here with "today" we refers to a time after the decoupling between matter and radiation.

$$\Omega - 1 = kr_H^2(t). \quad (1.45)$$

We see that Ω tends to 1 even if at the beginning of inflation was very different from 1. Thus, requiring an inflationary epoch in the primordial universe, avoids us to require an initial perfect flat geometry for the universe.

1.5.4 Solution to the problem of cosmological relics

We here very briefly mention the inflationary solution to the problem of the cosmological relics. The basic idea is very simple: the density of the cosmological relics can be strongly diluted by the accelerated expansion taking place during inflation, to such low levels that justify the fact that they are not observed. This is just the description of the qualitative solution to the problem. We refer for more details to the Refs. [4, 7].

1.5.5 Duration of inflation

In fact an inflationary period, that does not last for an enough long interval of time, cannot solve the above problems. The quantity we use to quantify the amount of inflation is the number of e-foldings $N_{e\text{-folds}}(t)$:

$$N_{e\text{-folds}}(t) = \ln \left(\frac{a(t_f)}{a(t)} \right), \quad (1.46)$$

where t_f denotes the time at the end of inflation and $t < t_f$. Then $N_{e\text{-folds}}(t)$ represents the quantity of inflation which takes place from t until t_f . In order to solve the horizon and the flatness problems it is necessary an inflation with [3]:

$$N_{e\text{-folds}}(t_i) \geq 50 - 60, \quad (1.47)$$

where t_i is the time at the beginning of inflation.

1.6 Primordial density perturbation production

Since the proposal of the inflationary scenario, it has been clear that inflation could provide another crucial feature: a mechanism to generate the first primordial fluctuations. These primordial perturbations can be responsible for the small density perturbations we observe today in the CMB (Figure 1.2), and can be the seeds for the formation, during matter dominated epoch, of the galaxies and clusters of galaxies we observe today in the universe.

In particular inflation has a quantum field theory (QFT) description. The field responsible for such an accelerated expansion is a scalar field ϕ , the inflaton, which autointeracts with an almost flat potential $V(\phi)$. Flat means that it changes very slowly with ϕ . Under this condition the kinetic energy of the field becomes very small and the energy momentum tensor of the field ϕ becomes of the kind $T_{\mu\nu} \sim -Vg_{\mu\nu}$. In this case we can have a period of *quasi-de Sitter* expansion driven by the vacuum potential $V(\phi)$. Such a period has all the features to describe an inflationary epoch. Now we give a brief and qualitative description of how primordial perturbations can arise from a period of inflation driven by the inflaton field ϕ . We remand in chapter 2 for a more detailed description.

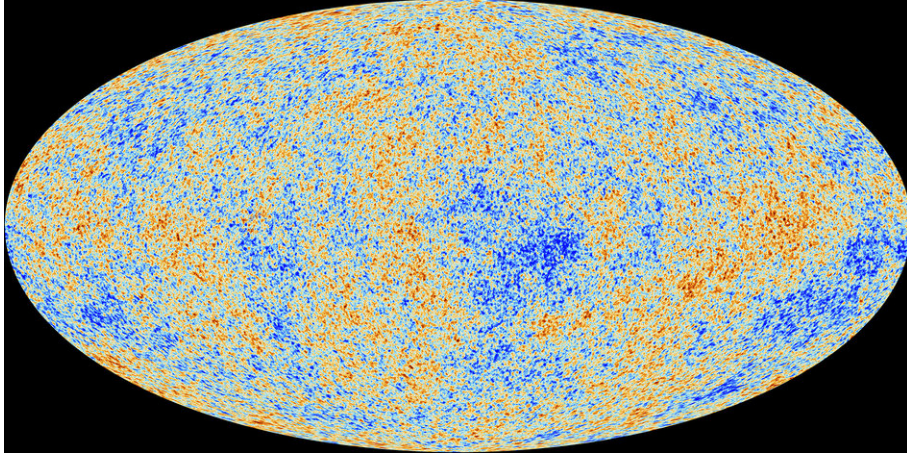


Figure 1.2: An image from the Planck Satellite of the small temperature fluctuations of the CMB. The CMB represents a stretch image of the last scattering surface of electrons with radiation at the epoch of recombination of the hydrogen atom. The blue parts represents points in which the temperature is lower than the average, the red parts represents points in which the temperature is higher than the average. The relative temperature fluctuations are quantified to be $\frac{\Delta T}{T} \simeq 10^{-5}$. The average temperature is $T \simeq 2.7K$. (Figure taken from Ref. [8])

Usually, to study the production and the evolution of cosmological perturbations of a physical quantity, we expand them in Fourier space and we analyze the history of each individual comoving wavenumber k . $\lambda = \frac{2\pi}{k}$ defines a comoving length scale that characterizes a particular mode of the perturbation. Such a procedure turns out to be useful for small perturbations since, at linear regime, each mode evolves independently. Then we split the inflaton field as

$$\phi = \phi_0(t) + \delta\phi(x, t), \quad (1.48)$$

where ϕ_0 is the background value which drives the background evolution of inflation and $\delta\phi$ represents a small perturbation expanded in Fourier space as:

$$\delta\phi(x, \tau) = \frac{1}{(2\pi)^3} \int d^3k \delta\phi(k, \tau) e^{ik \cdot x}, \quad (1.49)$$

where we have passed to the conformal time τ . The coordinate reparametrization which permits to pass from cosmological time to conformal time is the following:

$$dt = a d\tau. \quad (1.50)$$

Inserting Eq. (1.50) into the background metric, which is a spatially flat⁶ RW metric (1.1), it becomes:

$$ds^2 = a^2(\tau)[-d\tau^2 + d\mathbf{x}^2]. \quad (1.51)$$

With this conformal time parametrization, the metric (1.51) differs from the flat Minkowski metric only by a factor equal to the square of the scale factor of the universe. Then in the limit in which we can neglect the cosmic expansion, this metric becomes equal to the Minkowski one. Now we take a single oscillation mode with comoving length λ of the expansion (1.49). We assume that at the beginning of inflation the comoving Hubble horizon $r_H(t)$ was much larger than this scale λ . In this case the oscillation mode does not perceive the cosmic expansion because the microphysics

⁶During inflation we put the curvature parameter $k = 0$ because in section 5.3 we have seen that an inflationary epoch puts rapidly the spatial curvature very near to zero.

is at work. For this reason is taken as initial condition for the mode function the so called Bunch Davies initial state, which corresponds to the solution for the mode function of an oscillation in the Minkowski spacetime. It reads:

$$\delta\phi(k, \tau)_{initial} \propto \frac{1}{\sqrt{2k^3}} e^{-ik\tau} . \quad (1.52)$$

During the accelerated expansion of the universe caused by inflation, the comoving Hubble horizon reduces, unlike the comoving length λ which remains constant. Thus the oscillation is stretched to cosmological scales. At a certain time τ_* the scale λ exits the horizon and thus the microphysics is frozen. The result is that at this time the amplitude of the oscillation is frozen to the value $\delta\phi(\tau_*)$ (The time τ_* of horizon exit is defined as the time in which the length λ becomes approximately equal to the comoving Hubble radius $r_H = \frac{1}{aH}$, then from the condition $\lambda \simeq \frac{1}{aH}$, or $k \simeq aH$). If we quantize the oscillations $\delta\phi$, the frozen amplitude A can be interpreted as a net number density of the quanta of the field ϕ . Such perturbations remain frozen also after the end of inflation during the *re-heating* phase in which inflatons decays into radiation. Then the radiation dominated era starts in which such density perturbations remain frozen until the corresponding wavelength λ reenters into the horizon. This is possible because now the comoving Hubble horizon starts to increase. But if the length λ reenters into the comoving Hubble horizon after the this happens after the baryon-photon decoupling, we expect to observe on this scale an imprint of the inflationary fluctuations. This is what we find experimentally with the temperature anisotropies in the CMB spectra. Not only. Such density perturbations will also be a seed for the formation through gravitational instability of the large scale structures in the present universe.

Parameter	TT+lowP 68 % limits	TT+lowP+lensing 68 % limits	TT+lowP+lensing+ext 68 % limits	TT,TE,EE+lowP 68 % limits	TT,TE,EE+lowP+lensing 68 % limits	TT,TE,EE+lowP+lensing+ext 68 % limits
$\Omega_b h^2$	0.02222 ± 0.00023	0.02226 ± 0.00023	0.02227 ± 0.00020	0.02225 ± 0.00016	0.02226 ± 0.00016	0.02230 ± 0.00014
$\Omega_c h^2$	0.1197 ± 0.0022	0.1186 ± 0.0020	0.1184 ± 0.0012	0.1198 ± 0.0015	0.1193 ± 0.0014	0.1188 ± 0.0010
$100\theta_{MC}$	1.04085 ± 0.00047	1.04103 ± 0.00046	1.04106 ± 0.00041	1.04077 ± 0.00032	1.04087 ± 0.00032	1.04093 ± 0.00030
τ	0.078 ± 0.019	0.066 ± 0.016	0.067 ± 0.013	0.079 ± 0.017	0.063 ± 0.014	0.066 ± 0.012
$\ln(10^{10} A_s)$	3.089 ± 0.036	3.062 ± 0.029	3.064 ± 0.024	3.094 ± 0.034	3.059 ± 0.025	3.064 ± 0.023
n_s	0.9655 ± 0.0062	0.9677 ± 0.0060	0.9681 ± 0.0044	0.9645 ± 0.0049	0.9653 ± 0.0048	0.9667 ± 0.0040
H_0	67.31 ± 0.96	67.81 ± 0.92	67.90 ± 0.55	67.27 ± 0.66	67.51 ± 0.64	67.74 ± 0.46
Ω_Λ	0.685 ± 0.013	0.692 ± 0.012	0.6935 ± 0.0072	0.6844 ± 0.0091	0.6879 ± 0.0087	0.6911 ± 0.0062
Ω_m	0.315 ± 0.013	0.308 ± 0.012	0.3065 ± 0.0072	0.3156 ± 0.0091	0.3121 ± 0.0087	0.3089 ± 0.0062
$\Omega_m h^2$	0.1426 ± 0.0020	0.1415 ± 0.0019	0.1413 ± 0.0011	0.1427 ± 0.0014	0.1422 ± 0.0013	0.14170 ± 0.00097
$\Omega_m h^3$	0.09597 ± 0.00045	0.09591 ± 0.00045	0.09593 ± 0.00045	0.09601 ± 0.00029	0.09596 ± 0.00030	0.09598 ± 0.00029
σ_8	0.829 ± 0.014	0.8149 ± 0.0093	0.8154 ± 0.0090	0.831 ± 0.013	0.8150 ± 0.0087	0.8159 ± 0.0086
$\sigma_8 \Omega_m^{0.5}$	0.466 ± 0.013	0.4521 ± 0.0088	0.4514 ± 0.0066	0.4668 ± 0.0098	0.4553 ± 0.0068	0.4535 ± 0.0059
$\sigma_8 \Omega_m^{0.25}$	0.621 ± 0.013	0.6069 ± 0.0076	0.6066 ± 0.0070	0.623 ± 0.011	0.6091 ± 0.0067	0.6083 ± 0.0066
z_{re}	$9.9^{+1.8}_{-1.6}$	$8.8^{+1.7}_{-1.4}$	$8.9^{+1.3}_{-1.2}$	$10.0^{+1.7}_{-1.5}$	$8.5^{+1.4}_{-1.2}$	$8.8^{+1.2}_{-1.1}$
$10^9 A_s$	$2.198^{+0.076}_{-0.085}$	2.139 ± 0.063	2.143 ± 0.051	2.207 ± 0.074	2.130 ± 0.053	2.142 ± 0.049
$10^9 A_s e^{-2\tau}$	1.880 ± 0.014	1.874 ± 0.013	1.873 ± 0.011	1.882 ± 0.012	1.878 ± 0.011	1.876 ± 0.011
Age/Gyr	13.813 ± 0.038	13.799 ± 0.038	13.796 ± 0.029	13.813 ± 0.026	13.807 ± 0.026	13.799 ± 0.021
z_*	1090.09 ± 0.42	1089.94 ± 0.42	1089.90 ± 0.30	1090.06 ± 0.30	1090.00 ± 0.29	1089.90 ± 0.23
r_*	144.61 ± 0.49	144.89 ± 0.44	144.93 ± 0.30	144.57 ± 0.32	144.71 ± 0.31	144.81 ± 0.24
$100\theta_*$	1.04105 ± 0.00046	1.04122 ± 0.00045	1.04126 ± 0.00041	1.04096 ± 0.00032	1.04106 ± 0.00031	1.04112 ± 0.00029
z_{drag}	1059.57 ± 0.46	1059.57 ± 0.47	1059.60 ± 0.44	1059.65 ± 0.31	1059.62 ± 0.31	1059.68 ± 0.29
r_{drag}	147.33 ± 0.49	147.60 ± 0.43	147.63 ± 0.32	147.27 ± 0.31	147.41 ± 0.30	147.50 ± 0.24
k_D	0.14050 ± 0.00052	0.14024 ± 0.00047	0.14022 ± 0.00042	0.14059 ± 0.00032	0.14044 ± 0.00032	0.14038 ± 0.00029
z_{eq}	3393 ± 49	3365 ± 44	3361 ± 27	3395 ± 33	3382 ± 32	3371 ± 23
k_{eq}	0.01035 ± 0.00015	0.01027 ± 0.00014	0.010258 ± 0.000083	0.01036 ± 0.00010	0.010322 ± 0.000096	0.010288 ± 0.000071
$100\theta_{s,eq}$	0.4502 ± 0.0047	0.4529 ± 0.0044	0.4533 ± 0.0026	0.4499 ± 0.0032	0.4512 ± 0.0031	0.4523 ± 0.0023
f_{2000}^{143}	29.9 ± 2.9	30.4 ± 2.9	30.3 ± 2.8	29.5 ± 2.7	30.2 ± 2.7	30.0 ± 2.7
$f_{2000}^{143 \times 217}$	32.4 ± 2.1	32.8 ± 2.1	32.7 ± 2.0	32.2 ± 1.9	32.8 ± 1.9	32.6 ± 1.9
f_{2000}^{217}	106.0 ± 2.0	106.3 ± 2.0	106.2 ± 2.0	105.8 ± 1.9	106.2 ± 1.9	106.1 ± 1.8

Table 1.1: In this table the experimental values of the principal cosmological parameters are summarized. They are updated to the 2015 Planck results. The table is taken by the Ref. [6].

Chapter 2

Standard slow-roll models of inflation: gaussian profiles of the primordial perturbations

In this chapter we deal with the standard slow-roll models of inflation. We describe in details the primordial perturbations of the inflaton and of the metric tensor. We then introduce some gauge invariant variables, we study their statistic defining some observables to compare the "standard" inflationary predictions to the experimental observations. For the moment we work with linear perturbations, making observational predictions about the gaussian statistics of such perturbations. However we also define a formalism which we will permit us to make also a non-linear analysis in the next chapter. In the calculations we follow Refs [9, 10, 11].

2.1 General introduction to the standard slow-roll models of inflation

The standard model of inflation has a simple field theory description. The ingredients of the model are essentially two: Einstein gravity and one scalar field ϕ , named inflaton, which interacts with gravity through covariant derivatives. The inflaton is characterized by self-interactions described by an almost flat potential $V(\phi)$, which varies very slowly with ϕ . Using these prescriptions, we are able to write the full action of the theory:

$$S = \frac{1}{2} \int d^4x \sqrt{g} \left[M_{pl}^2 R - g_{\mu\nu} D^\mu \phi D^\nu \phi - 2V(\phi) \right]. \quad (2.1)$$

where $g = -\det g_{\mu\nu}$ and D_μ is the covariant derivative.

In the square brackets we recognize two terms: the first one is the Einstein-Hilbert action with the reduced Planck mass $M_{pl}^{-2} = 8\pi G$; the second one is the canonical action of a classical scalar field

with an autointeracting flat potential $V(\phi)$. The inflaton interacts with gravity through the covariant derivative D_μ .

The inflaton and the components of metric are splitted into a background value and a small perturbation:

$$\phi(\vec{x}, t) = \phi_0(t) + \delta\phi(\vec{x}, t) . \quad (2.2)$$

$$g_{\mu\nu}(\vec{x}, t) = g_{\mu\nu}^{(0)}(t) + \delta g_{\mu\nu}(\vec{x}, t) . \quad (2.3)$$

Here the suffix 0 denotes the background values. These are supposed to be invariant under rotations and translations, and so they depend only by the time. There is a phenomenological reason for this choice. In fact inflation is introduced to explain how we can have an isotropic and homogeneous primordial universe also without particular initial conditions. So we need an homogeneous and isotropic background. For this reason, the perturbations take all the spatial dependence.

Technically we can introduce in the full model also other fields, but this is the most simple model that explains the observations [12].

Now, using the action (2.1), we study separately the dynamics of the background and of the perturbations.

2.2 The background dynamics

The background dynamics is important to describe the accelerated expansion taking place during inflation. The background metric is chosen as FRW, (1.1), with zero spatial curvature:

$$d^2s = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j . \quad (2.4)$$

We remember that $a(t)$ is the scale factor of the universe and it is the only dynamical quantity in the background metric.

We can write the equations of motion for the background by putting to zero the 0-th order functional derivatives of the action (2.1) with respect to the inflaton field and the metric $g_{\mu\nu}$. For the inflaton we find the following background equation of motion:

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + \frac{\partial V(\phi_0)}{\partial\phi} = 0 . \quad (2.5)$$

As far as the metric is concerned, we find the Friedmann equations (1.11) and (1.12) with $k = 0$. In order to find the background value of the energy and pressure densities (see, e. g., Ref [10]), we need to evaluate the energy-momentum tensor of the inflaton. Its general expression is [10]:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu} \left[-\frac{1}{2}\partial_\alpha\phi\partial_\beta\phi g^{\alpha\beta} - V(\phi) \right] . \quad (2.6)$$

Then, using the spatially flat FRW metric, we find:

$$T_{00}^{(0)} = \frac{1}{2}\dot{\phi}_0^2 + V(\phi) = \rho_0(t) \quad , \quad (2.7)$$

$$T_{ij}^{(0)} = - \left[\frac{1}{2}\dot{\phi}_0^2 - V(\phi) \right] g_{ij} = - p_0(t) g_{ij} \quad . \quad (2.8)$$

Thus, we read:

$$\rho_0(t) = \frac{1}{2}\dot{\phi}_0^2 + V(\phi) \quad , \quad (2.9)$$

$$p_0(t) = \frac{1}{2}\dot{\phi}_0^2 - V(\phi) \quad . \quad (2.10)$$

Now, since the inflaton slowly rolls under the potential $V(\phi)$, we can neglect the kinetic term $\frac{1}{2}\dot{\phi}_0^2$ with respect to the potential $V(\phi)$ in Eqs (2.9) and (2.10). Thus we have $\rho_0(t) \simeq -p_0(t) = V(\phi)$. Then the background energy-momentum tensor takes the form [10]:

$$T_{\mu\nu}^{(0)} \simeq V(\phi) g_{\mu\nu} \quad . \quad (2.11)$$

The energy-momentum tensor (2.11) is approximately the energy-momentum tensor of a de Sitter space. Thus we have a quasi de-Sitter inflation.

Now, deriving a set of background equations of motion, we derive some adimensional parameters, the so-called *slow-roll parameters*, that we will use to define the slow-roll hypothesis. Inserting the Eqs (2.9) and (2.10) in the Friedmann equation (1.11) , we derive a first background equation of motion¹:

$$3M_{Pl}^2 H^2 = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0) \quad , \quad (2.12)$$

In addition inserting Eqs (2.9), (2.10) and (2.12) into the Friedmann equation (1.12) we derive a second background equation of motion:

$$M_{Pl}^2 \dot{H} = \frac{1}{2}\dot{\phi}_0^2 \quad . \quad (2.13)$$

The fact that we have used (2.12) to derive (2.13) imply that the two equations of motions are not independent.

Now we put the background Eqs (2.5), (2.12) and (2.13) all together:

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + \frac{\partial V(\phi_0)}{\partial \phi} = 0 \quad , \quad (2.14)$$

$$3M_{Pl}^2 H^2 = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0) \quad , \quad (2.15)$$

$$M_{Pl}^2 \dot{H} = -\frac{1}{2}\dot{\phi}_0^2 \quad . \quad (2.16)$$

¹Here we write explicitly the Planck mass.

Since the inflaton slowly rolls under the flat potential $V(\phi)$, on the second member of Eq (2.15) we can neglect the kinetic term $\frac{1}{2}\dot{\phi}_0^2$ in respect to the potential $V(\phi)$. Instead on the first member of the Eq (2.14) we can neglect the term $\ddot{\phi}_0$ in respect to the terms $3H\dot{\phi}_0$ and $\frac{\partial V(\phi_0)}{\partial \phi}$. Thus the previous equations become:

$$3H\dot{\phi}_0 + \frac{\partial V(\phi_0)}{\partial \phi} = 0, \quad (2.17)$$

$$3M_{pl}^2 H^2 = V(\phi_0), \quad (2.18)$$

$$M_{pl}^2 \dot{H} = -\frac{1}{2}\dot{\phi}_0^2. \quad (2.19)$$

Now we define the slow-roll parameters introduced above. They quantify the strenght of the ratio between the first two derivatives of the slow-roll potential $V(\phi)$ with respect to the inflaton and the potential itself. They read:

$$\epsilon_V = \frac{1}{2} \left(\frac{M_{pl} V'}{V} \right)^2 \simeq \frac{1}{2} \frac{\dot{\phi}_0^2}{H^2} M_{pl}^{-2}, \quad (2.20)$$

$$\eta_V = M_{pl}^2 \frac{V''}{V} \simeq -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} + \frac{1}{2} \frac{\dot{\phi}_0^2}{H^2} M_{pl}^{-2}, \quad (2.21)$$

where the ' denotes derivative with respect to the inflaton.

Here the relations with the \simeq are found using Eqns. (2.18), (2.17) and (2.19). An almost flat potential requires:

$$\epsilon_V \ll 1, \quad \eta_V \ll 1. \quad (2.22)$$

Then the slow-roll hypothesis is equivalent to require (2.22).

In particular, using the background equation of motion (2.19), we can relate the time dependence of the Hubble parameter H with the slow-roll parameter ϵ_V . This quantifies how we depart form a de Sitter inflation. We have:

$$\epsilon = -\frac{\dot{H}}{H^2} \simeq \epsilon_V. \quad (2.23)$$

Because under the sow-roll hypothesis $\epsilon_V \ll 1$, this is another confirmation that we have all the correct boundary conditions to produce an inflationary period.

Summarizing briefly: through the background dynamics we can understand the accelerated expansion during inflation in the very early universe, which allows us to solve the classical problems of the Hot Big Bang model, as we have seen in the previous chapter.

2.3 Cosmological perturbations

2.3.1 Perturbations of the inflaton field

In general the perturbations of the inflaton can be expanded as [10, 13, 14, 15, 16]:

$$\delta\phi(\vec{x}, t) = \sum_{n=1}^{\infty} \frac{\delta\phi^{(n)}(\vec{x}, t)}{n!}, \quad (2.24)$$

where each n -th term of this expansion in series corresponds to an n -th perturbation order. In general there is no limit on the orders n , but it is clear that if we consider small fluctuations around the background, the linear part corresponding to $n = 1$ is the dominant one in the expansion series (even if this statement depends on the observable one is interested in). Then we take (for the moment) the linearized expression:

$$\delta\phi(\vec{x}, t) = \delta\phi^{(1)} = \varphi. \quad (2.25)$$

2.3.2 Perturbation of the metric tensor

In general the same expansion is valid also for the metric tensor:

$$\delta g_{\mu\nu}(\vec{x}, t) = \sum_{n=1}^{\infty} \frac{\delta g_{\mu\nu}^{(n)}(\vec{x}, t)}{n!}. \quad (2.26)$$

As done for the inflaton, we take only the linear part of this expansion (for the moment):

$$\delta g_{\mu\nu}(\vec{x}, t) = \delta g_{\mu\nu}^{(1)}(\vec{x}, t). \quad (2.27)$$

Now, we write formally the full metric components adding the perturbations (2.27) to the metric (2.4). We obtain the perturbed metric around the RW space-time:

$$d^2s = -(1 + A)dt^2 + C_i dx^i dt + a^2(t) \left[(1 + B)\delta_{ij} + G_{ij} \right] dx^i dx^j, \quad (2.28)$$

where A , B , C_i and G_{ij} denotes formally the perturbations of the metric.

In cosmology such perturbations are splitted into the so-called scalars, transverse vectors and transverse trace-free tensors [13, 14]. The reason why such splitting are introduced is that, at least in linear theory, the different perturbations are decoupled from each other and evolve independently in the perturbed dynamics equations. For this reason they can be studied separately. Thus in our case we have the following splittings:

$$C_i = \partial_i D_i + E_i, \quad (2.29)$$

$$G_{ij} = D_{ij} F + \partial_i H_j + \partial_j H_i + \gamma_{ij}, \quad (2.30)$$

where D_i and H_i are transverse vectors, $\partial_i D^i = \partial_i H^i = 0$, D_{ij} is a traceless derivative operator, $D_{ij} = \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2$, and γ_{ij} is a transverse and traceless tensor, $\gamma_i^i = 0$, $\partial^i \gamma_{ij} = 0$ (where the latin indices are raised/lowered with δ_{ij}).

2.3.3 3+1 decomposition of the metric

The perturbations just introduced in general are not all dynamical, but there are some redundant degrees of freedom. In order to have an immediate visualization of which perturbations are dynamical and which are not, we rewrite the metric (2.28) using the Arnowitt-Deser-Misner (ADM) formalism of General Relativity [17, 9]. This is a hamiltonian reformulation of the theory in which we foliate the total 4-dimensional spacetime into 3-dimensional spacelike hypersurfaces at fixed time (see, e.g., Figure 2.1).

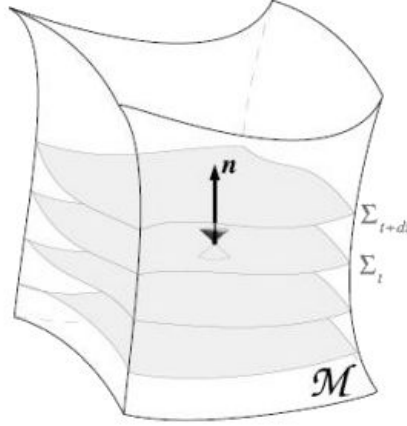


Figure 2.1: Foliation of the spacetime into spacelike slices. The figure is taken from [18].

each hypersurface the 3-metric h_{ij} can be splitted in the same way as in Eq. (2.28). We make the connections between different slices with the remaining independent components of the 4-metric, the so-called lapse function $N = (g^{00})^{-\frac{1}{2}}$ and the shift vector $N_i = g_{0i}$. Then, we can rewrite the metric (2.28) as:

$$d^2s = -(N^2 - N_i N^i) dt^2 + 2N_i dx^i dt + a^2(t) h_{ij} dx^i dx^j, \quad (2.31)$$

where $h_{ij} = (1 + B)\delta_{ij} + G_{ij}$.

The inverse metric components are:

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = -\frac{N^i}{N^2}, \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}, \quad (2.32)$$

where h^{ij} is the inverse of h_{ij} and the latin indices are lowered and raised with the metric h_{ij} and its inverse (i.e. $N^i = h^{ij} N_j$).

We can rewrite in an "ADM form" all the fundamental tensors of general relativity, in a way in which all the contractions are done only with the 3-metric h_{ij} . In Appendix A there are all the formulae we need. Using these formulae, we can rewrite the action (2.1) in an equivalent ADM form:

$$S = \frac{1}{2} \int d^4x \sqrt{h} \left[NR^{(3)} + N(K_{ij}K^{ij} - K^2) + \frac{(\dot{\phi} - \partial_i N^i)^2}{N} - N h^{ij} \partial_i \phi \partial_j \phi - 2NV \right], \quad (2.33)$$

where we have put the reduced mass Planck equal to one for simplicity of notation. It will be easily restored by dimensional analysis. In addition $h = -\det(h_{ij})$ and K_{ij} is the extrinsic curvature tensor defined as:

$$K_{ij} = \frac{1}{2N} [D_i^{(3)} N_j + D_j^{(3)} N_i - \dot{h}_{ij}] \quad K^2 = K_i^i \quad (2.34)$$

and

$$K = K_i^i \quad . \quad (2.35)$$

$D_i^{(3)}$ is the covariant derivative computed with the 3-metric h_{ij} instead of the full metric $g_{\mu\nu}$. $R^{(3)}$ is the scalar curvature computed with the 3-metric h_{ij} . Physically it represents the intrinsic curvature of the spacelike slices at fixed time.

But what are the advantages of this formalism? The metrics (2.28) and (2.31) are equivalent, but the ADM form will permit us to perform more easily a non-linear analysis, in particular as far as the count of the number of the propagating degrees of freedom. In fact, as we will see, the fields N and N_i are not dynamical and can be expanded in power of series of the dynamical degrees of freedom. The zero order value of this expansion is fixed by the background, namely $N_{(0)} = 1$, $N_{(0)}^i = 0$. To find the other orders, we have to derive from the action (2.33) the Eulero-Lagrange equations for N and N_i and then to solve them order by order.

An interesting fact to be pointed out is the following: if we are interested to an expansion of the action (2.33) until cubic order in the dynamical fields, we need to know the expressions of N and N_i only until the first order (we consider the cubic order because in the Chapters 3 and 4 we will study potential non-Gaussian signatures from inflation).

The demonstration of this fact is quite simple. We give an euristic argument: imagine to have in the lagrangian of the theory a term that depends on N :

$$\mathcal{L} = \mathcal{L}(N) \quad . \quad (2.36)$$

We can expand this lagrangian around the first order value $N = N^{(1)}$:

$$\mathcal{L}(N) = \mathcal{L}(N_{(1)}) + \left. \frac{\partial \mathcal{L}}{\partial N} \right|_{N_{(1)}} \cdot \left(\sum_{n=2}^{\infty} N_{(n)} \right) + \left. \frac{\partial^2 \mathcal{L}}{\partial^2 N} \right|_{N_{(1)}} \cdot \left(\sum_{n,n'=2}^{\infty} N_{(n)} N_{(n')} \right) + \dots \quad (2.37)$$

But the second term in this expansion vanishes because it multiplies $\left. \frac{\partial \mathcal{L}}{\partial N} \right|_{N_{(1)}}$, that is zero if we evaluate the Eulero-Lagrange equation for N at first order. Then only the first and the third terms remain. The third term starts with a quartic order term because of the presence of $\left(\sum_{n,n'=2}^{\infty} N_{(n)} N_{(n')} \right)$. Then if we are interested in the contributions until the cubic order, we can take only $\mathcal{L}(N_{(1)})$. An analogous demonstration is valid also for N_i . A more rigorous demonstration is presented in the Ref. [19].

2.3.4 Gauge dependence of the cosmological perturbations

Now we describe the concept of gauge invariance of the cosmological perturbations following the Refs. [15, 13]. In our perturbation theory we assume the existence of a parametric family of solutions of the fields equations of motion to which also the unperturbed physical space belongs (which is the case of the cosmological perturbations). This is a one-parameter family of solutions of the physical perturbed space M_λ (λ is a real parameter) with the condition that $\lambda = 0$ refers to the unperturbed space M_0 . On each of these M_λ we can define the tensor quantities T_λ that are the

physical objects we are interested in. In general we can expand all the tensor quantities in Taylor series of this parameter λ as:

$$M_\lambda = M_0 + \Delta M_\lambda = M_0 + M_1\lambda + M_2\lambda^2 + \dots, \quad (2.38)$$

$$T_\lambda = T_0 + \Delta T_\lambda = T_0 + T_1\lambda + T_2\lambda^2 + \dots, \quad (2.39)$$

where $\Delta_\lambda M$ and $\Delta_\lambda T$ represent the physical perturbations and the parameter λ specifies the order at which such perturbations are taken (i.e. $\lambda = 1$ refers to linear perturbations).

Thus we can define a one-to-one correspondence between the points of the unperturbed space M_0 and the points of the physical perturbed space M_λ (such a definition is defined implicitly in Eq. (2.38)). This correspondence is realized by a map ψ_λ . Then we define a *gauge* as a map which realizes this one-to-one correspondence from the points of the unperturbed space to the points of the physical perturbed space.

Now we take the gauge associated to the map ψ_λ . This maps a generic point P of the unperturbed space to a point Q of the physical space. Then we write $Q = \psi_\lambda(P)$. If we take another gauge φ_λ , different by ψ_λ , it maps the point P in another point S of the physical space different by Q . Then $S = \varphi_\lambda(P)$. Once fixed the gauge ψ_λ , the point Q is the representative of the point P in the physical perturbed space. Instead, in the gauge φ_λ , the point S is the representative in the physical perturbed space of the point P in the unperturbed space. We can pass from the point Q to the point S through the composition between maps:

$$S = \varphi_\lambda(\psi_\lambda^{-1}(Q)). \quad (2.40)$$

This passage defines a gauge transformation, because we have passed from the representative of the point P in the gauge ψ_λ to the representative of the point P in the gauge φ_λ . This way of seeing a gauge transformation is called *active approach*. Notice that the fact that the two different points in the perturbed physical space S and Q have the same representative P in the unperturbed space is crucial for defining the gauge transformation. The *gauge dependence* of the perturbations follows from the fact that we can use arbitrary both gauges ψ_λ and φ_λ to write the perturbations of the tensor quantities $\Delta_\lambda T$ in the physical perturbed space. It follows that there is non-uniqueness in defining the perturbations of the physical tensor quantities that we want to study.

A practical way to consider a gauge transformation is defining it through a coordinate transformation. We explain better this fact. We take the points P , Q and S as defined above. We take also the gauges ψ_λ and φ_λ defined above. Let us call respectively $x^\mu(Q)$ and $y^\mu(S)$ the coordinates of Q and S in the physical perturbed space. Then, using Eq. (2.40), in the coordinate space we can pass from the point $x(P)$ to the point $y(S)$ through the coordinate transformation:

$$y^\mu(S(x)) = y^\mu(\varphi_\lambda(\psi_\lambda^{-1}(x^{-1}x(Q)))) . \quad (2.41)$$

The quantity $y^\mu(\varphi_\lambda(\psi_\lambda^{-1}(x^{-1}x(Q)))) = \theta^\mu(x)$ defines the coordinate transformation associated to the gauge transformation considered. We rewrite better this coordinate transformation as:

$$y^\mu(x) = \theta^\mu(x) . \quad (2.42)$$

This coordinate reparametrization permits to pass from the coordinate system x to the coordinate system y . Then from a practical point of view fixing a gauge is equivalent to fix a coordinate system in the physical space and changing the gauge is equivalent to perform a changing of a coordinate system.

If we work with linear perturbations, then the coordinate transformation to be considered is linear too. In this case we can write $\theta^\mu(x)$ as:

$$\theta^\mu(x) = x^\mu + \xi^\mu, \quad (2.43)$$

where ξ^μ is a vector field which generates a linear infinitesimal gauge transformation.

If we consider a generic tensor quantity in the physical space $T(x)$, the transformed quantity $T'(y)$ through the infinitesimal gauge transformation (2.43) can be written as:

$$T'(y) = T(x) + \mathcal{L}_\xi T(x), \quad (2.44)$$

where \mathcal{L}_ξ is the Lie derivative with respect to the direction ξ^μ .

So we have learned that, at linear level, specifying the time and space components of $\xi^\mu = (\xi^0, \xi^i)$ led to specify a gauge transformation. In particular the spatial component ξ^i can be splitted as done for the cosmological perturbations as $\xi^i = \partial^i k + l^i$, with $\partial_i l^i = 0$. This means that three degrees of freedom are associated to a gauge transformation: two scalars and one transverse vector. Then, if we completely fix a gauge we remove automatically these three degrees of freedom. So a complete gauge fixing permits to remove two scalars and one transverse tensor from the cosmological perturbations seen above. Instead there is no action on tensor perturbations (which at linear level turns out to be gauge invariant). In general we can not fixed completely a gauge, remaining with a residual gauge dependence. In this case pure gauge modes remain in the theory which have no physical meaning. Now we define two useful complete gauge fixings which we use when we are dealing with cosmological perturbations.

2.3.5 Gauge fixing

The two gauges we are going to define will turn to be useful to work with when we use the metric in the form (2.31).

In the first gauge all the scalar perturbations of the 3-metric h_{ij} , F and B , are removed leaving only the scalar perturbation of the inflaton $\delta\phi$, besides the ones in N and N^i . This is called *spatially flat* gauge. Then one is free to remove also the vector perturbation H_i , remaining with a 3-metric h_{ij} of the form [9]:

$$h_{ij} = a^2[\delta_{ij} + \gamma_{ij}], \quad \gamma_i^i = 0, \quad \partial^i \gamma_{ij} = 0, \quad (2.45)$$

together with the scalar inflaton perturbations φ .

In the second gauge the scalar perturbation of the inflaton $\delta\phi$ and F are removed. This is the a-called *comoving gauge*. One is free to remove perturbation H_i also in this case. Then we remain with a 3-metric h_{ij} of the form [9]:

$$h_{ij} = a^2[(1 + B)\delta_{ij} + \gamma_{ij}], \quad \gamma_i^i = 0, \quad \partial^i \gamma_{ij} = 0, \quad \varphi = 0. \quad (2.46)$$

Both gauges (2.45) and (2.46) are completely fixed. But there is a problem which we have left open. Infact the gauge dependence of the cosmological perturbations imply that the value of these perturbations is different on each gauge. At this point we have two methods to avoid such an ambiguity: the first possibility is to identify combinations of perturbations that are gauge-invariant, and so independent of the gauge choice. The second option is to fix one gauge and perform all the calculations in that gauge. In this second possibility pure gauge-modes could appear in the theory if the gauge is not completely fixed. These gauge-modes have not physical meaning and must be eliminated from the full set of the perturbations. In the next section we will start with the second option, working with the complete gauge fixing (2.45), and we will pass to gauge invariant variables only at the end of the calculations.

For the scalar perturbations we will use the gauge invariant quantity ζ . For linear perturbations it is defined as [14, 15, 13, 20]:

$$\zeta = \frac{B}{2} - \frac{H}{\dot{\phi}}\varphi. \quad (2.47)$$

A priori one can define a big number of gauge invariant quantities. The reason for which we decide to use this gauge invariant quantity is that it represents the *curvature perturbation on comoving hypersurfaces* and so it represents the curvature perturbation which feels a comoving observer in the primordial universe.

On the other hand, for what we said above on gauge invariance, the linear tensor (transverse and traceless) perturbations γ_{ij} have not gauge freedom and so are born gauge invariant. Thus we can use γ_{ij} to study tensor perturbations without any ambiguity.

In addition, comparing the definition (2.47) with the gauge fixings (2.45) and (2.46), we have $B = 2\zeta$ and $\varphi = -\frac{\dot{\phi}}{H}\zeta$. So in the gauge (2.46) the gauge invariant quantity ζ appears directly in the metric components and this is the reason for which such gauge is called comoving gauge. Instead in the gauge (2.45) we can connect the perturbation of the inflaton φ to ζ through the linear relation:

$$\varphi = -\frac{\dot{\phi}}{H}\zeta. \quad (2.48)$$

Now we have all the elements to study the evolution of the gauge invariant perturbations ζ and γ_{ij} at linear level.

2.4 Evolution of the perturbations at linear level

We rewrite down the action of the full theory (2.33):

$$S = \frac{1}{2} \int d^4x \sqrt{h} \left[NR^{(3)} + N(K_{ij}K^{ij} - K^2) + \frac{(\dot{\phi} - \partial_i N^i)^2}{N} - Nh^{ij}\partial_i\phi\partial_j\phi - 2NV \right], \quad (2.49)$$

where we have put the reduced Planck mass equal to one for simplicity of the following calculations. It will be easily restored by dimensional analysis. We choose to work with the spatially flat gauge (2.45). The reasoning is that in this gauge the calculations are easier (e.g. we avoid performing a lot of integration by parts). The reader is referred to Ref. [9] for more details of the computations also in the gauge (2.46).

2.4.1 Equations of motion for the fields N and N_i

Following the classic field theory approach, we start deriving the equations of motion for the fields N and N_i by doing the functional derivatives $\frac{\delta S}{\delta N_i}$ and $\frac{\delta S}{\delta N}$ and putting them equal to zero. We find respectively:

$$2D_j^{(3)}K_i^j - 2D_i^{(3)}K_j^j - 2N^{-1}\partial_i\phi(\dot{\phi} - N^j\partial_j\phi) = 0, \quad (2.50)$$

$$R^{(3)} - [K_{ij}K^{ij} - K^2] - N^{-2}(\dot{\phi} - N^i\partial_i\phi)^2 - h^{ij}\partial_i\phi\partial_j\phi - 2V = 0. \quad (2.51)$$

In such equations we see that there is no time derivatives of the fields N and N_i and so they are not propagating degrees of freedom. In order to remove them from the action we have to solve algebraically their equations and substituting the solutions back to the action (2.49). For what said above at the end of the Section 2.3.3, in finding such solutions we can stop to first order in the fields ζ and γ_{ij} . This permits us to develop some arguments that reduce the difficulty of the calculation. In fact at first order the only scalar and vector quantities that we can construct with γ_{ij} and its derivatives are proportional to $\partial_i\partial_j\gamma^{ij}$ or $\partial_i\gamma^{ij}$, that are zero because γ_{ij} is a transverse tensor. So we can put γ_{ij} equal to zero remaining only with ζ .

In addition we can split the field N_i into a scale and a transverse vector part as:

$$N_i = \partial_i\psi + \chi_i, \quad \partial^i\chi_i = 0. \quad (2.52)$$

The only vector we can construct with ζ and derivatives of ζ is something like $\partial_i(a\zeta + b\dot{\zeta} + \dots)$. In the parenthesis there is a combination of all the possibile time derivatives of ζ multiplied for some coefficients that we do not know a priori. Formally we call this quantity C . Then, it must be $\chi_i = \partial_i C$. But from the condition $\partial^i\chi_i = 0$, it follows $\partial^2 C = 0$. If we work with fields that goes to zero to infinity, it follows² that $C = 0$ in all the spacetime and then also $\chi_i = 0$. So we can take only:

$$N_i = \partial_i\psi. \quad (2.53)$$

In the spatially flat gauge $R^{(3)} = 0$, $D_i^{(3)} = \partial_i$ and the extrinsic curvature K_{ij} reads like:

$$K_{ij} = \frac{1}{2N}[\partial_i N_j + \partial_j N_i - 2a^2 H \delta_{ij}]. \quad (2.54)$$

At first order $N = 1 + N^{(1)}$ and then the equations (2.50) and (2.51) become:

² $\partial^2 C = 0$ is the Laplace equation. If we take the boundary condition $C(\infty, t) = 0$ for all times, we know that the solution is $C = 0$ in all the space.

$$2(2H\partial_i N^{(1)} - \dot{\phi}_0(\partial_i\varphi)) = 0 \quad , \quad (2.55)$$

$$-12N^{(1)} - 4Ha^{-2}\partial^2\psi + 2\dot{\phi}_0^2 N^{(1)} - 2\dot{\phi}_0\dot{\varphi} + 2\ddot{\phi}_0\varphi + 6H\dot{\phi}_0\varphi = 0 \quad . \quad (2.56)$$

In deriving (2.55) and (2.56) we have used also the background equations of motion (2.15), (2.14) to remove zero order terms. We find the solutions:

$$N^{(1)} = \frac{\dot{\phi}_0}{2HM_{Pl}^2}\varphi \quad , \quad N_i = \frac{1}{M_{Pl}^2}\partial_i\psi \quad , \quad \psi = -a^2\frac{\dot{\phi}_0^2}{2H^2}\partial^{-2}\left[\frac{d}{dt}\left(\frac{H\varphi}{\dot{\phi}_0}\right)\right] \quad , \quad (2.57)$$

where we have restored the Planck mass by dimensional analysis.

Now we are ready to study the evolution of scalar perturbations.

2.4.2 Evolution of scalar perturbations

Because of the fact that tensor and scalar perturbations evolve independently at linear level (as said in Section 2.3.2), for studying scalar perturbations we can put $\gamma_{ij} = 0$ in the gauge definition (2.45). In addition we observe that the fields N and N_i (Eq. (2.57)) are subdominant with respect to φ in the slow roll limit (Eq. (2.22)). In fact inserting Eq. (2.20) into Eqs. (2.57) we find at leading order in slow-roll:

$$N^{(1)} \propto \sqrt{\epsilon_V}\varphi \quad , \quad N_i \propto \sqrt{\epsilon_V}\left[\partial_i\partial^{-2}\frac{d}{dt}\varphi\right] \quad . \quad (2.58)$$

Then in the action (2.49) the quadratic terms that multiply at least one between N or N_i or their derivatives are subdominant with respect to the terms that multiply two φ or its derivatives. This is true only if we require the slow-roll condition (2.22).

For this reason, when we substitute the solutions (2.57) into the action (2.49), we find the following expression of the action at leading order in slow-roll³:

$$S_{\varphi\varphi} = \frac{1}{2}\int d^4x a^3\left[\dot{\varphi}^2 - \frac{1}{a^2}(\partial_i\varphi)(\partial^i\varphi)\right] \quad , \quad (2.59)$$

where the latin contraction is done with the δ_{ij} .

Now, using the linear relation between ζ and φ (Eq. (2.48)) and the definition of the slow-roll parameter ϵ_V in Eq. (2.20), we find the quadratic action for the gauge invariant variable ζ at leading order in slow-roll:

$$S_{\zeta\zeta} = \frac{(2M_{Pl}^2\epsilon_V)}{2}\int d^4x a^3\left[\dot{\zeta}^2 - \frac{1}{a^2}(\partial_i\zeta)(\partial^i\zeta)\right] \quad . \quad (2.60)$$

We notice that this action is multiplied by the slow-roll parameter ϵ_V . For this reason the requirement $\epsilon_V \neq 0$ is necessary if we want to produce curvature perturbations on comoving hypersurfaces in our model. Now we pass for convenience from the cosmological time t to the conformal time τ . The conformal time is linked to the cosmological time through the following relations:

³In the action the zero-th order terms in the fields vanishes due to the zero-th order background equations of motion (2.15) and (2.14). The first order terms are surface terms and they vanishes too.

$$dt = ad\tau, \quad \frac{d}{dt} = \frac{1}{a} \frac{d}{d\tau}, \quad t = \int_0^\tau a(\tau') d\tau'. \quad (2.61)$$

The action (2.60) becomes:

$$S_{\zeta\zeta} = \frac{1}{2} \int d^4x \mathcal{A}_s^2 [\zeta'^2 - (\partial_i \zeta)(\partial^i \zeta)], \quad (2.62)$$

where now ' denotes derivative with respect to the conformal time and

$$\mathcal{A}_s^2 = (2M_{pl}^2 \epsilon_V) a^2. \quad (2.63)$$

Let us expand ζ in Fourier space:

$$\zeta(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} \zeta(\vec{k}, \tau) e^{i\vec{k}\cdot\vec{x}}. \quad (2.64)$$

Substituting Eq. (2.64) into the action (2.62), we arrive to:

$$S_{\zeta\zeta} = \frac{1}{2} \int d\tau \frac{d^3k}{(2\pi)^3} \mathcal{A}_s^2 [\zeta_{\vec{k}}'^2 - k^2 \zeta_{\vec{k}}^2]. \quad (2.65)$$

Now we make the field redefiniton

$$\zeta = \frac{\Phi}{\mathcal{A}_s} = \frac{\Phi}{\sqrt{2\epsilon} M_{pl} a}, \quad (2.66)$$

and we rewrite the action (2.65) for the field Φ , obtaining:

$$S_{\Phi\Phi} = \frac{1}{2} \int d\tau \frac{d^3k}{(2\pi)^3} \left[\Phi_{\vec{k}}'^2 - k^2 \Phi_{\vec{k}}^2 + \frac{\mathcal{A}_s''}{\mathcal{A}_s} \Phi_{\vec{k}}^2 \right]. \quad (2.67)$$

We derive the equations of motion for the field Φ_k doing the functional derivative of $S_{\Phi\Phi}$ with respect to Φ :

$$\Phi_{\vec{k}}'' + \left(k^2 - \frac{\mathcal{A}_s''}{\mathcal{A}_s} \right) \Phi_{\vec{k}} = 0. \quad (2.68)$$

This is an equation of motion for an harmonic oscillator with an effective mass $\frac{\mathcal{A}_s''}{\mathcal{A}_s}$. Now we make some intermediate steps in order to derive the explicit expression of such potential at first order in the slow-roll parameters:

$$\frac{da}{dt} = aH \longrightarrow \int H d\tau = \int \frac{da}{a^2} \longrightarrow H\tau(1 - \epsilon) = -\frac{1}{a}. \quad (2.69)$$

In the last step we have used (2.61), (2.23) and an integration by parts.

Thus:

$$aH = -\frac{1}{\tau(1 - \epsilon)}. \quad (2.70)$$

We also compute:

$$\frac{\dot{\epsilon}_V}{\epsilon_V H} = 2M_{Pl}^2 \left[-\frac{\partial_{\phi\phi} V}{V} + \frac{(\partial_{\phi} V)^2}{V^2} \right] = 4\epsilon_V - 2\eta_V, \quad (2.71)$$

Here, instead, we have used the definitions (2.20) and (2.21).

Now we start with the computation we are interested in:

$$\frac{\mathcal{A}'_s}{\mathcal{A}_s} = \frac{a(\dot{\mathcal{A}}_s)^2}{2\mathcal{A}_s^2} = aH \left(1 + \frac{\dot{\epsilon}_V}{2\epsilon_V H} \right) = -\frac{1}{\tau} \left(\frac{1 + 2\epsilon_V - \eta_V}{1 - \epsilon_V} \right) \simeq -\frac{1 + 3\epsilon_V - \eta_V}{\tau}, \quad (2.72)$$

where the \simeq means that the result is approximated at first order in the slow-roll parameters.

Finally we obtain:

$$\frac{\mathcal{A}''_s}{\mathcal{A}_s} = \frac{d}{d\tau} \left(\frac{\mathcal{A}'_s}{\mathcal{A}_s} \right) + \left(\frac{\mathcal{A}'_s}{\mathcal{A}_s} \right)^2 \simeq \frac{2 + 9\epsilon_V - 2\eta_V}{\tau^2}. \quad (2.73)$$

Also here the relation with the \simeq are approximated at first order in the slow-roll parameters. We can define the parameter $\nu_s = \frac{3}{2} + 3\epsilon - \eta$ and rewrite the effective mass as:

$$\frac{\mathcal{A}''_s}{\mathcal{A}_s} = \frac{\nu_s^2 - \frac{1}{4}}{\tau^2}. \quad (2.74)$$

Then the equation of motion (2.68) becomes:

$$\Phi''_{\vec{k}} + \left(k^2 - \frac{\nu_s^2 - \frac{1}{4}}{\tau^2} \right) \Phi_{\vec{k}} = 0. \quad (2.75)$$

Now, we are can canonically quantize⁴ the field Φ as a scalar field in a Minkowski space-time. So we can promote it to an operator $\hat{\Phi}$ and expand it into the creation and annihilation operators \hat{a} and \hat{a}^\dagger . This procedure is possible because, after the field redefinition (2.66), the action for the field Φ is becoming equal to an action for a massive scalar field in the Minkowski space-time with an effective mass $\frac{\mathcal{A}''_s}{\mathcal{A}_s}$. So we have:

$$\hat{\Phi}_{\vec{k}} = u(\vec{k}, \tau) \hat{a}(\vec{k}) + u^*(-\vec{k}, \tau) \hat{a}^\dagger(-\vec{k}). \quad (2.76)$$

The annihilation operator is defined as the operator that annihilate the vacuum state of the theory $|0\rangle$ according to the following relations and obeying the following commutation relations:

$$\langle 0 | \hat{a}^\dagger = 0, \quad \hat{a} | 0 \rangle = 0, \quad (2.77)$$

$$[\hat{a}_k, \hat{a}_k^\dagger] = (2\pi)^3 \delta^3(k - k'), \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0. \quad (2.78)$$

Here the $[\cdot, \cdot]$ denotes the commutator operator. It follows that the function $u(\vec{k}, \tau)$ is a classical scalar function obeying the normalization [10]:

$$u^* u' - u u'^* = -i. \quad (2.79)$$

⁴If we quantize this field, we automatically quantize also the field ζ . In fact ζ is linked to the field Φ by the linear relation $\zeta = \frac{\Phi}{\sqrt{2\epsilon M_{Pl} a}}$ (Eq. (2.66)).

Substituting Eq. (2.76) into Eq. (2.75), we find that the function $u(\vec{k}, \tau)$ obeys the same equation as the classical field $\Phi_{\vec{k}}$ before the quantization:

$$u''_{\vec{k}} + \left(k^2 - \frac{v_s^2 - \frac{1}{4}}{\tau^2} \right) u_{\vec{k}} = 0. \quad (2.80)$$

Now, before showing the explicit solution of this equation, we study the limit of the solution for $\tau \rightarrow -\infty$ (initial condition) and $\tau \rightarrow 0$ (superhorizon⁵ scales).

Limit $\tau \rightarrow -\infty$:

In this limit the terms dominant in the parenthesis of Eq. (2.80) is the first one and so the equation becomes:

$$u''_{\vec{k}} + k^2 u_{\vec{k}} = 0. \quad (2.81)$$

But this is the equation of motion of a classical harmonic oscillator with frequency $\omega = k$. Formally this equation is equivalent to the one of a free scalar field in Minkowski spacetime. The solution is like:

$$u_{\vec{k}}^{\tau \rightarrow -\infty} = c_1 e^{-ik\tau} + c_2 e^{+ik\tau}. \quad (2.82)$$

where c_1 and c_2 are some time independent coefficients.

We select the correct initial condition with a physical reasoning. In fact at the beginning of inflation the physical length $\lambda_{ph} = a\lambda$ of a given fluctuation is small and it feels the space around it as a Minkowsky spacetime. Thus the correct solution is:

$$u_{\vec{k}}^{\tau \rightarrow -\infty} = \frac{1}{\sqrt{2k^3}} e^{-ik\tau}, \quad (2.83)$$

which reproduces the correct behaviour in a flat space-time.

The solution (2.83) is known as the *Bunch-Davies initial condition*.

Limit $\tau \rightarrow 0$:

In this limit the term dominant in the potential is the second one in the parenthesis of Eq. (2.80). Then the equation becomes:

$$u''_{\vec{k}} - \left(\frac{v_s^2 - \frac{1}{4}}{\tau^2} \right) u_{\vec{k}} = 0. \quad (2.84)$$

The general solution is a linear combination of two functions (at leading order in slow-roll parameters):

$$u_{\vec{k}} \propto c'_1 \left(\frac{1}{H\tau} \right) + c'_2 (H^2 \tau^2) \propto c'_1 a + c'_2 a^{-2}. \quad (2.85)$$

where c'_1 and c'_2 are some real coefficients. In the last step we have used Eq. (2.70) at leading order in slow-roll.

⁵A fluctuation mode λ will exit the Hubble horizon at a certain conformal time τ_* defined by $-k\tau_* = 1$. Then taking the limit $\tau \rightarrow 0$, formally we take the limit on superhorizon scales.

We know that during inflation the scale factor accelerates. So in the future limit the solution of order a^{-2} in (2.85) tends to 0 and we remain with only:

$$u_{\vec{k}} \propto c'_1 a . \quad (2.86)$$

We have said that the physical quantity we use to refer to scalar perturbations is ζ that is linked with $\Phi_{\vec{k}}$ (and so with $u_{\vec{k}}$) by the linear relation $\Phi_{\vec{k}} \propto a\zeta_{\vec{k}}$ (see Eq. (2.66)). What we understand then is that ζ becomes constant on superhorizon scales. This means that after horizon exit the amplitude of the oscillation is "frozen" at a certain value. This fact is well known (Refs. [15, 13, 14, 16]) and, e.g., in [9] is deduced by using Hamiltonian considerations.

After understanding the asymptotic behaviours of Eq. (2.80), we can now give its exact solution. In fact Eq. (2.80) is a Bessel equation whose general solution for real ν_s is (Refs. [10, 21, 4, 22, 23, 24]):

$$u(\vec{k}, \tau) = \sqrt{-\tau} \left[c_1(k) H_{\nu_s}^{(1)}(-k\tau) + c_2(k) H_{\nu_s}^{(2)}(-k\tau) \right] , \quad (2.87)$$

where $H_{\nu}^{(1)}$ and $H_{\nu}^{(2)}$ are the Henkel functions of first and second kind. If we choose the Bunch-Davies initial condition (2.83), then we find the exact solution:

$$u(\vec{k}, \tau) = \frac{\sqrt{-\pi\tau}}{2} e^{i(\frac{\pi}{4} + \frac{\pi\nu_s}{2})} H_{\nu_s}^{(1)}(-k\tau) . \quad (2.88)$$

We observe that the solution depends only on the modulus of the wave vector \vec{k} and not by its direction, because the classical scalar field u evolves in an isotropic background.

Now, if we return back to the field $\hat{\zeta}$, from $\hat{\zeta} = \frac{\hat{\Phi}}{\mathcal{A}_s}$ and (2.76), we find for $\hat{\zeta}$ the exact solution:

$$\hat{\zeta}(\vec{x}, \tau) = \frac{1}{\mathcal{A}_s} \int \frac{d^3k}{(2\pi)^3} \left[u_{\vec{k}} \hat{a}(\vec{k}) + u_{-\vec{k}}^* \hat{a}^\dagger(-\vec{k}) \right] e^{i\vec{k} \cdot \vec{x}} . \quad (2.89)$$

where $u_{\vec{k}}$ is as in Eq. (2.88).

2.4.3 Evolution of tensor perturbations

In order to study the tensor perturbations we can put to zero all the scalar perturbations, and so we have $\varphi = 0$, $N = 1$ and $N_i = 0$ in the metric defined by Eqns. (2.31), (2.45). The reason is the same of above. At linear level the tensor perturbations are decoupled from the scalar perturbations and then they can be studied separately. Then the action (2.49) becomes at second order in the tensor perturbations γ_{ij} :

$$S_{\gamma\gamma} = \frac{M_{Pl}^2}{4} \int d^4x a^3 \left[\dot{\gamma}_{ij} \dot{\gamma}^{ij} - \frac{1}{a^2} (\partial_i \gamma_{jk}) (\partial^i \gamma^{jk}) \right] , \quad (2.90)$$

where also in this case the contractions between latin indices are done with the δ_{ij} . As done for scalar perturbations, we switch to conformal time τ , obtaining:

$$S_{\gamma\gamma} = \frac{1}{2} \int d^3x d\tau \mathcal{A}_T^2 [\dot{\gamma}'_{ij} \dot{\gamma}'^{ij} - (\partial_i \gamma_{jk}) (\partial^i \gamma^{jk})] . \quad (2.91)$$

where in this case:

$$\mathcal{A}_T^2 = \frac{M_{Pl}^2 a^2}{2}. \quad (2.92)$$

As usual we can expand γ_{ij} in Fourier space as:

$$\gamma_{ij}(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} \epsilon_{ij}(\vec{k}) \gamma(\vec{k}, \tau) e^{i\vec{k}\cdot\vec{x}}, \quad (2.93)$$

where $\gamma(k, \tau)$ is a scalar function and $\epsilon_{ij}(\vec{k})$ is the transverse traceless polarization tensor of the perturbation.

The tensor γ_{ij} has a priori 9 components, but 3 of these can be removed because of the symmetry in the exchange $i \longleftrightarrow j$. Then the traceless and transverse conditions remove other 4 degrees of freedom, remaining with only 2 linear independent components. Then we can split γ_{ij} into 2 linear independent polarizations, named + and \times :

$$\gamma_{ij}(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} [\gamma_{ij}^+(\vec{k}, \tau) + \gamma_{ij}^\times(\vec{k}, \tau)] e^{i\vec{k}\cdot\vec{x}} = \sum_{s=+, \times} \int \frac{d^3k}{(2\pi)^3} \epsilon_{ij}^s(\vec{k}) \gamma_s(k, \tau) e^{i\vec{k}\cdot\vec{x}}. \quad (2.94)$$

The two independent tensor modes $\gamma_{ij}^+(\vec{k}, \tau)$ and $\gamma_{ij}^\times(\vec{k}, \tau)$ just introduced are known as *primordial gravitational waves*. The corresponding polarization tensors $\epsilon_{ij}^+(\vec{k})$ and $\epsilon_{ij}^\times(\vec{k})$ are orthogonal between them. In particular, if the normalized wave vector $\hat{n} = \frac{\vec{k}}{k}$ is written in polar coordinates as $\hat{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, then we have [25]:

$$\epsilon_{ij}^+ = (u_1)_i (u_1)_j + (u_2)_i (u_2)_j, \quad (2.95)$$

$$\epsilon_{ij}^\times = (u_1)_i (u_2)_j + (u_2)_i (u_1)_j, \quad (2.96)$$

where:

$$u_1 = (\sin \varphi, -\cos \varphi, 0), \quad (2.97)$$

$$u_2 = \begin{cases} (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta), & \text{if } \theta < \frac{\pi}{2} \\ (-\cos \theta \cos \varphi, -\cos \theta \sin \varphi, \sin \theta), & \text{if } \theta > \frac{\pi}{2} \end{cases}. \quad (2.98)$$

From these definitions, it follows that $\epsilon_{ij}^s(\vec{k}) = \epsilon_{ij}^s(-\vec{k})$ and the relation:

$$\epsilon_{ij}^s(\vec{k}) \epsilon_{s'j}^{ij}(\vec{k}) = 2\delta_{s'}. \quad (2.99)$$

Then, inserting Eq. (2.94) into the action (2.91) and using Eq. (2.99), the action (2.91) becomes:

$$S_{\gamma\gamma} = \frac{1}{2} \sum_{s=+, \times} \int \frac{d^3k}{(2\pi)^3} d\tau \mathcal{A}_T^2 [\gamma_s'^2 - k^2 \gamma_s^2]. \quad (2.100)$$

From this equation we understand that the dynamical degrees of freedom that describe the evolution of gravitational waves are the mode functions γ_s . We notice in particular the similarity of the action of tensor modes (2.100) with the action (2.65) of the scalar modes. This permits us to use similar techniques to quantize the fields γ_s and solve the corresponding equations of motion.

Before quantizing the fields γ_{\pm} , as done for the scalar perturbations, it is convenient to use the rescaled variable μ_s defined as:

$$\mu_s = \mathcal{A}_T \gamma_s. \quad (2.101)$$

Thus, for the variable μ_s the action (2.100) becomes:

$$S_{\gamma\gamma} = \frac{1}{2} \sum_{s=+,\times} \int \frac{d^3k}{(2\pi)^3} d\tau \left[\mu_s'^2 - k^2 \mu_s^2 + \frac{\mathcal{A}_T''}{\mathcal{A}_T} \right]. \quad (2.102)$$

The equations of motion for such fields are:

$$\mu_s'' + \left(k^2 - \frac{\mathcal{A}_T''}{\mathcal{A}_T} \right) \mu_s = 0. \quad (2.103)$$

We derive also in this case the explicit expression of the effective potential $\frac{\mathcal{A}_T''}{\mathcal{A}_T}$ at leading order in the slow-roll parameters:

$$\frac{\mathcal{A}_T'}{\mathcal{A}_T} = \frac{a}{2} \frac{\dot{\mathcal{A}}_T^2}{\mathcal{A}_T^2} \simeq -\frac{1}{\tau} (1 + \epsilon). \quad (2.104)$$

$$\frac{\mathcal{A}_T''}{\mathcal{A}_T} = \frac{d}{d\tau} \left(\frac{\mathcal{A}_T'}{\mathcal{A}_T} \right) + \left(\frac{\mathcal{A}_T'}{\mathcal{A}_T} \right)^2 \simeq \frac{1}{\tau^2} (1 + \epsilon_V) + \frac{1}{\tau^2} (1 + 2\epsilon_V) = \frac{2 + 3\epsilon_V}{\tau^2}. \quad (2.105)$$

Then, we define in this case the parameter $\nu_T = \frac{3}{2} + \epsilon$ and we rewrite Eq. (2.103) as:

$$\mu_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} \right) \mu_s = 0. \quad (2.106)$$

Now we can canonically quantize⁶ the fields μ_s as done for the case of scalar perturbations:

$$\hat{\mu}_{\vec{k}}^s = z_s(\vec{k}, \tau) \hat{a}_s(\vec{k}) + z_s^*(-\vec{k}, \tau) \hat{a}_s^\dagger(-\vec{k}), \quad (2.107)$$

where the creation and annihilation operators \hat{a}_s^\dagger and \hat{a}_s obeys the same relations of the scalar perturbations (2.77), (2.78).

The equation of motion for the scalar function z_s is the same of the classical μ_s :

$$z_s'' + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} \right) z_s = 0. \quad (2.108)$$

This equation is exactly the same as Eq. (2.80). Also the discussion about the asymptotic behaviours of the solution is the same. We conclude that also the tensor perturbations $\gamma_{ij}^s(\vec{k}, \tau)$ are constant on superhorizon scales as $\zeta(k, \tau)$. Now we write the exact solution of (2.108) which is the same as Eq. (2.88) with the exchange $\nu_T \longleftrightarrow \nu_s$:

$$z_s = \frac{\sqrt{-\pi\tau}}{2} e^{i(\frac{\pi}{4} + \frac{\pi\nu_T}{2})} H_{\nu_T}^{(1)}(-k\tau). \quad (2.109)$$

Also in this case the solution does not depend on the direction of the wavevector \vec{k} . Then, returning back to the fields $\hat{\gamma}_+$ and $\hat{\gamma}_\times$ using the relation (2.101), we have the final solution:

⁶Also in this case the quantization of the fields μ_s automatically implies the quantization of the fields γ_s which is linked to the fields μ_s by the linear relation (2.101).

$$\hat{\gamma}_{ij}(\vec{x}, \tau) = \sum_{s=+, \times} \int \frac{d^3 k}{(2\pi)^3} \epsilon_{ij}^s(\vec{k}) \hat{\gamma}^s(k, \tau) e^{i\vec{k} \cdot \vec{x}}, \quad (2.110)$$

$$\hat{\gamma}^s(\vec{k}, \tau) = \frac{1}{\mathcal{A}_\tau} \left[z_s \hat{a}_s(\vec{k}) + z_s^* \hat{a}_s^\dagger(-\vec{k}) \right]. \quad (2.111)$$

where z_s is the same as in Eq. (2.109).

2.5 Gaussian statistics of the perturbations

The gaussian statistics of a generic perturbation $\delta(\vec{x}, t)$ is completely determined by the two point function:

$$\langle 0 | \delta^2(\vec{x}, t) \delta^2(\vec{x} + \vec{r}, t) | 0 \rangle, \quad (2.112)$$

where the mean is taken over the statistical ensemble.

The perturbation δ can be expanded in Fourier space as:

$$\delta(x, t) = \frac{1}{(2\pi)^3} \int d^3 k e^{i\vec{k} \cdot \vec{x}} \delta(\vec{k}, t), \quad (2.113)$$

If we substitute the expansion (2.113) in Eq. (2.112), we find for the variance:

$$\langle \delta^2(\vec{x}, t) \rangle = \frac{1}{(2\pi)^6} \int d^3 k \int d^3 k' \left[e^{i(\vec{k} + \vec{k}') \cdot \vec{x}} \langle |\delta(\vec{k}, t) \delta(\vec{k}', t)|^2 \rangle \right]. \quad (2.114)$$

If we recall that the background space is homogeneous and isotropic, then we have:

$$\langle \delta(\vec{k}, t) \delta(\vec{k}', t) \rangle = (2\pi)^3 \mathcal{P}_\delta(k) \delta^3(\vec{k} + \vec{k}'), \quad (2.115)$$

where $\mathcal{P}_\delta(k)$ is defined as the *power spectrum* of the perturbation δ . The Dirac delta $\delta^3(\vec{k} + \vec{k}')$ is due to invariance under translations; invariance under rotations implies that the power spectrum depends only on the wavenumber k and not on the direction of the wavevector \vec{k} .

If we insert Eq. (2.115) into Eq. (2.114), we find:

$$\langle \delta^2(\vec{x}, t) \rangle = \frac{1}{(2\pi)^3} \int d^3 k \mathcal{P}_\delta(k, t) = \frac{1}{2\pi^2} \int_0^\infty dk k^2 \mathcal{P}_\delta(k, t) = \int_0^\infty d \log k \Delta_\delta(k, t), \quad (2.116)$$

where we have also introduced the *dimensionless power spectrum*:

$$\Delta_\delta(k, t) = \frac{k^3}{2\pi^2} \mathcal{P}_\delta(k, t). \quad (2.117)$$

Now we contextualize these general definitions to the quantistical ensemble considering a quantistical scalar perturbation $\hat{\delta}(\vec{x}, t)$. In this ensemble the mean is taken in the vacuum state of the quantistical theory. In general the quantized scalar field $\hat{\delta}(\vec{x}, t)$ can be expanded as:

$$\hat{\delta}(x, t) = \frac{1}{(2\pi)^3} \int d^3 k e^{i\vec{k} \cdot \vec{x}} \hat{\delta}(\vec{k}, t), \quad (2.118)$$

where:

$$\hat{\delta}(\vec{k}, t) = \hat{a}(\vec{k})v(\vec{k}, t) + \hat{a}^\dagger(-\vec{k})v^*(-\vec{k}, t). \quad (2.119)$$

Here $v(\vec{k}, t)$ is a scalar mode function and a, a^\dagger are annihilation and creation operators that obey the usual relations:

$$\langle 0|\hat{a}^\dagger = 0, \quad \hat{a}|0\rangle = 0, \quad (2.120)$$

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = (2\pi)^3 \delta^3(k - k'), \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0. \quad (2.121)$$

In this case the two points function in Fourier space reads:

$$\langle 0|\hat{\delta}(\vec{k}, t)\hat{\delta}(\vec{k}', t)|0\rangle. \quad (2.122)$$

Inserting the expansion (2.119) into Eq. (2.122), Eq. (2.122) becomes:

$$\begin{aligned} \langle 0|\hat{\delta}(\vec{k}, t)\hat{\delta}(\vec{k}', t)|0\rangle &= \langle 0|(\hat{a}(\vec{k})v(\vec{k}, t) + \hat{a}^\dagger(-\vec{k})v^*(-\vec{k}, t))(\hat{a}(\vec{k}')v(\vec{k}', t) + \hat{a}^\dagger(-\vec{k}')v^*(-\vec{k}', t))\hat{\delta}(\vec{k}', t)|0\rangle = \\ &= v(\vec{k}, t)v^*(-\vec{k}', t)\langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(-\vec{k}')|0\rangle = \\ &= v(\vec{k}, t)v^*(-\vec{k}', t)\langle 0|[\hat{a}(\vec{k}), \hat{a}^\dagger(-\vec{k}')]|0\rangle = \\ &= (2\pi)^3 \delta^3(\vec{k} + \vec{k}')|v(\vec{k}, t)|^2, \end{aligned} \quad (2.123)$$

where we have used the relations (2.120) and (2.121).

If we match the result of Eq. (2.123) with the definition (2.115) we find that the power spectrum of the perturbations $\hat{\delta}$ is:

$$\mathcal{P}_\delta(k, t) = |v(\vec{k}, t)|^2. \quad (2.124)$$

So it is the modulus square of the mode function $v(\vec{k}, t)$ which appears in Eq. (2.119).

Now we have all the ingredients to apply this general analysis to the computation of the power spectrum of inflationary scalar and tensor perturbations which we have quantized in the previous section.

2.5.1 Power spectrum of scalar perturbations from inflation

The variable in exam is $\hat{\zeta}$. In order to compare with the observables, we need to compute the power spectrum of $\hat{\zeta}$ on superhorizon scales. In fact, after horizon exit, the amplitude of the perturbation remains frozen in time and also its statistic. Then this is the statistical pattern which we have to compare with the CMB spectra.

For this reason we are interested in the expression of (2.88) in the large scale limit. The time τ_* of horizon crossing occurs when $k = a(\tau_*)H_*$. At leading order in slow-roll parameters $a \simeq -\frac{1}{H\tau}$. Then it follows that horizon crossing arrives at the conformal time $-k\tau_* \simeq 1$. Thus the superhorizon limit correspond to the limit $-k\tau \ll 1$. In this limit, $u(k, \tau)$ has the asymptotic form (see e.g. Ref.[10]):

$$u(k, \tau)_{-k\tau \ll 1} = \sqrt{\frac{-\tau}{2(-k\tau)^3}} e^{i(-\frac{\pi}{4} + \frac{\pi}{2}v_s)} \frac{\Gamma(v_s)}{\Gamma(3/2)} \left(\frac{-k\tau}{2}\right)^{3-2v_s}. \quad (2.125)$$

For what we have said, the large scale power spectrum of scalar perturbations reads

$$\mathcal{P}_s = \frac{|u(k, \tau)_{-k\tau \ll 1}|^2}{\mathcal{A}_s^2} = \frac{k^{-2v_s}}{2M_{Pl}^2 \epsilon_V a^2 \tau^2} \left(\frac{\Gamma(v_s)}{\Gamma(3/2)}\right) \left(\frac{-\tau}{2}\right)^{3-2v_s}. \quad (2.126)$$

At leading order in the slow-roll parameters we find:

$$\mathcal{P}_s \simeq \frac{k^{-2v_s}}{2\epsilon_V} \frac{H_*^2}{M_{Pl}^2} \left(\frac{-\tau}{2}\right)^{3-2v_s}. \quad (2.127)$$

Here H_* refers to the Hubble parameter at the time of the horizon crossing of the mode k . Using Eq. (2.117) the corresponding dimensionless power spectrum is:

$$\Delta_s \simeq \frac{1}{4\pi^2 \epsilon} \frac{H_*^2}{M_{Pl}^2} \left(\frac{-k\tau}{2}\right)^{3-2v_s}. \quad (2.128)$$

2.5.2 Power Spectrum of tensor perturbations from inflation

Here the variable in exam is γ_{ij} . We proceed as in the case of scalar perturbations. The asymptotic mode function in the super-horizon regime has the same expression:

$$z(k, \tau)_{-k\tau \ll 1} = \sqrt{\frac{-\tau}{2(-k\tau)^3}} e^{i(-\frac{\pi}{4} + \frac{\pi}{2}v_T)} \frac{\Gamma(v_T)}{\Gamma(3/2)} \left(\frac{-k\tau}{2}\right)^{3-2v_T}. \quad (2.129)$$

The tensor power spectrum⁷ of each polarization s reads:

$$\mathcal{P}_T^s = 2 \frac{|u^s(k, \tau)_{-k\tau \ll 1}|^2}{\mathcal{A}_T^2} = 4 \frac{k^{-2v_T}}{M_{Pl}^2 a^2 \tau^2} \left(\frac{\Gamma(v_T)}{\Gamma(3/2)}\right) \left(\frac{-\tau}{2}\right)^{3-2v_T}. \quad (2.130)$$

The total power spectrum of tensor perturbations is the sum over the two polarizations $+$ and \times . Then we multiply the previous result for a factor 2.

$$\mathcal{P}_T = 8 \frac{k^{-2v_T}}{M_{Pl}^2 a^2 \tau^2} \left(\frac{\Gamma(v_T)}{\Gamma(3/2)}\right) \left(\frac{-k\tau}{2}\right)^{3-2v_T}. \quad (2.131)$$

At leading order in the slow-roll parameters this expression gives:

$$\mathcal{P}_T \simeq 8k^{-2v_T} \frac{H_*^2}{M_{Pl}^2} \left(\frac{-\tau}{2}\right)^{3-2v_s}. \quad (2.132)$$

Using Eq. (2.117) the corresponding dimensionless power spectrum is:

$$\Delta_T \simeq \frac{4}{\pi^2} \frac{H_*^2}{M_{Pl}^2} \left(\frac{-k\tau}{2}\right)^{3-2v_T}. \quad (2.133)$$

⁷The additional factor 2 comes from the contraction between two polarization tensors $\epsilon_{ij}^s \epsilon_s^{ij} = 2$, see Eq. (2.99).

2.5.3 Tensor-to-scalar perturbation ratio

The *tensor-to-scalar perturbation ratio* r is an important observable that allows to constrain the amplitude of the gravity modes. It is defined as the ratio between the power spectrum of tensor and scalar perturbations. Thus using Eqns. (2.127) and (2.132) we find

$$r = \frac{\Delta_T}{\Delta_s} = 16\epsilon_V . \quad (2.134)$$

In particular we can relate r directly to the potential V as:

$$V^{\frac{1}{4}} = r^{\frac{1}{4}}(4 \times 10^{16} \text{Gev}) . \quad (2.135)$$

In addition, using the background equation of motion (2.18), it follows:

$$V^{\frac{1}{4}} \propto H^{\frac{1}{2}} , \quad (2.136)$$

where H measures the characteristic energy scale of inflation.

So a measure of r can give also a measure of the energy at which inflation occurs. Unfortunately the primordial gravitational waves have not been detected yet in the CMB and so we cannot determine r exactly. But the observations from the Planck satellite have constrained this quantity as [12, 26]:

$$r < 0.12 \text{ (95 \% C.L.)} . \quad (2.137)$$

2.5.4 Spectral index of the perturbations

Another important observable is the spectral index n of the perturbations. It gives information about the scale dependence of the power spectrum Δ , Eq. (2.117). If Δ is scale invariant, then each mode k in the integral (2.116) gives the same contribution to the variance. This type of configuration is called the *Harrison-Zel'dovich spectrum* (for the scalar perturbations it corresponds to $n_s = 1$ exactly). Instead, some dependence of the power spectrum Δ on k means different contributions of different cosmological scales to the variance. In particular, if Δ increases with k we call this type of configuration *blue spectrum*; viceversa, we have a *red spectrum*.

The *scalar spectral index* is defined as:

$$n_s - 1 = \frac{d(\log \Delta_s)}{d \log k} . \quad (2.138)$$

Using Eq. (2.128) we obtain immediately

$$n_s - 1 = 3 - 2\nu_s = -6\epsilon_V + 2\eta_V . \quad (2.139)$$

Instead, the *tensor spectral index* is defined as:

$$n_T = \frac{d(\log \Delta_T)}{d \log k} . \quad (2.140)$$

Using Eq. (2.133) we find:

$$n_T = 3 - 2\nu_T = -2\epsilon_V . \quad (2.141)$$

From the observations of the CMB spectra the Planck satellite has determined [12]:

$$n_s = 0.968 \pm 0.006 \text{ (95 \% C.L.)} . \quad (2.142)$$

The Harrison-Zel'dovich spectrum would correspond to the value $n_s = 1$. This means that scalar perturbations of the CMB are consistent with an approximately scale invariant spectrum and tend to assume the form of a *red spectrum*.

2.5.5 Consistency relation

If we put together Eqns. (2.134) and (2.141), we find the so-called *consistency relation*:

$$r = -8n_T . \quad (2.143)$$

This relation might be a powerful check of the standard slow-roll inflation scenario because it links together two observables in a model independent way (i.e. it holds for every model of slow-roll parameters of inflation). Its verification is an important issue of future experiments (see, e.g., Ref. [27, 28]).

2.5.6 Space of parameters

The observables just introduced allows to infer about the form of the slow-roll potential $V(\phi)$, which for the moment has been left completely general. In fact a priori a large zoology of models exists depending on the explicit expression of $V(\phi)$. In general we can have two main kind of models: *large field models* and *small field models*. Now we see some features of such models following the Refs. [29, 3, 12].

- **Large field models of inflation:** a typically toy model potential for a large field model is of the form:

$$V(\phi) = V_0 \frac{\phi^p}{\mu} , \quad p > 0 , \quad \mu \ll M_{Pl} . \quad (2.144)$$

In particular the potential (2.144) is associated to the so-called *chaotic models* of inflation. In Figure 2.2 there is a visual example of such potential. We can compute the slow-roll parameters ϵ_V and η_V for the potential (2.144). We find:

$$\epsilon_V^{large} = \frac{1}{2} \left(\frac{M_{pl} V'}{V} \right)^2 = \frac{1}{2} p M_{pl}^2 \frac{\phi^{2p-2}}{\phi^{2p}} = \frac{p}{2} \left(\frac{M_{pl}}{\phi} \right)^2 , \quad (2.145)$$

$$\eta_V^{large} = M_{pl}^2 \frac{V''}{V} = p(p-1) M_{pl}^2 \frac{\phi^{p-2}}{\phi^p} = p(p-1) \left(\frac{M_{pl}}{\phi} \right)^2 . \quad (2.146)$$

Thus in the slow-roll limit $\epsilon_V^{large}, \eta_V^{large} \ll 1$, it follows from Eqns. (2.145) and (2.146):

$$\phi \gg M_{pl} . \quad (2.147)$$

This is the reason for which these kind of models are called large fields models, because the background value of the inflaton is forced to be much larger than the Planck mass during chaotic inflation.

We can also compute the field excursion during inflation between the time in which the primordial CMB fluctuations have exited the horizon and the time of the end of inflation. This field excursion is the minimum field excursion which must have had inflation to produce the anisotropies of the CMB. Quantitatively it is defined as:

$$\Delta\phi = \int_{\phi_{CMB}}^{\phi_{end}} \delta\phi . \quad (2.148)$$

With a change of variable we can rewrite (2.148) as:

$$\Delta\phi = \int_{t_{CMB}}^{t_{end}} dt \frac{\dot{\phi}}{H} H \simeq \sqrt{\epsilon} M_{Pl} \int_{t_{CMB}}^{t_{end}} H dt , \quad (2.149)$$

where in the last step we have used the definition of the slow-roll parameter ϵ_V , Eq. (2.20). By using the definition of the number of e-foldings, (1.46), and performing an another change of variable we have:

$$\int_{t_{CMB}}^{t_{end}} H dt = \int_{a_{CMB}}^{a_{end}} \frac{da}{a} = \ln \left(\frac{a(t_{end})}{a(t_{CMB})} \right) = N_{e-folds}^{CMB} , \quad (2.150)$$

where $N_{e-folds}^{CMB}$ are the minimum number of e-foldings that permits to the primordial CMB fluctuations to exit the horizon. Typically we have $N_{e-folds}^{CMB} \simeq 60$.

Thus the field excursion (2.149) becomes:

$$\Delta\phi \simeq 60 \sqrt{\epsilon_V} M_{Pl} . \quad (2.151)$$

And substituting the slow roll parameter ϵ_V^{large} into Eq. (2.148) we find:

$$\Delta\phi \propto \phi . \quad (2.152)$$

Because of the fact that $\phi \gg M_{Pl}$, it follows that a large field model implies also a large field excursion during inflation.

- **Small-field models of inflation:** the toy model potential for a small field model is:

$$V(\phi) \sim V_0 \left[1 - \left(\frac{\phi}{\mu} \right)^p \right] , \quad \phi < \mu \ll M_{Pl} , \quad p > 2 . \quad (2.153)$$

This kind of potential is visualized in the Fig. 2.3. Also in this case we can compute the slow-roll parameters associated to the potential (2.153). We find:

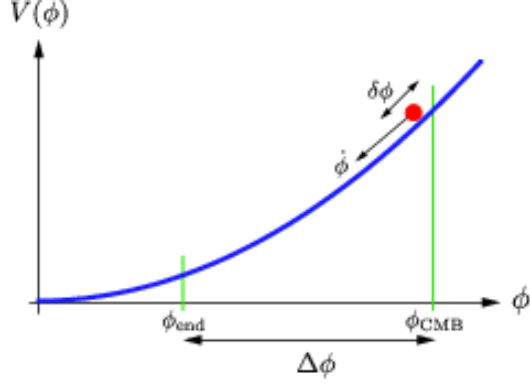


Figure 2.2: This figure illustrates an example of the potential of large field models of inflation. (Figure taken from Ref. [29]).

$$\epsilon_V^{small} = \frac{1}{2} p^2 \frac{M_{pl} \phi^{2p}}{\phi^2 \mu^{2p}} \left[1 - \left(\frac{\phi}{\mu} \right)^p \right]^{-1}, \quad (2.154)$$

$$\eta_V^{small} = -p(p-1) \frac{M_{pl} \phi^p}{\phi^2 \mu^p} \left[1 - \left(\frac{\phi}{\mu} \right)^p \right]^{-1}. \quad (2.155)$$

In this case for p sufficiently larger than 2 the slow-roll limit $\epsilon_V^{small}, \eta_V^{small} \ll 1$ requires $\phi \ll \mu$. In this case the background value of the inflaton can acquire also very small values. This is the reason for which these kind of models are called small field models. We notice from Eq. (2.155) that, differently from the large field models of inflation, in the small field models the slow-parameter η_V acquires a negative sign.

In the same way of the large field models, we can give also in this case an estimation of the excursion of the inflaton field. Substituting the slow-roll parameter ϵ_V^{small} into Eq. (2.151), we find:

$$\Delta\phi \propto \phi \left(\frac{\phi}{\mu} \right). \quad (2.156)$$

Thus, if the background value of the inflaton is small, also the excursion of the inflaton can be small during inflation.

In reality more examples of toy model potentials for large and small field models of inflation exists. There is also a third category of slow-roll inflationary models: the so-called hybrid models. These models are a middle way between large field models and small field models of inflation. We do not analyze all these models in details because it is not the purpose of this work. For more details of other examples we remand to the Refs. [29, 3, 12]. But we remark that the slow roll parameters (2.20) and (2.21) can give informations about which inflationary potential is better in accordance with the experimental data. In particular using Eqns. (2.139) and (2.134), we can extrapolate the value of ϵ_V and η_V through the measure of the tensor-to-scalar ratio r and the scalar spectral index n_s . The experimental bounds on the slow-roll parameters given by the Planck satellite are [12]:

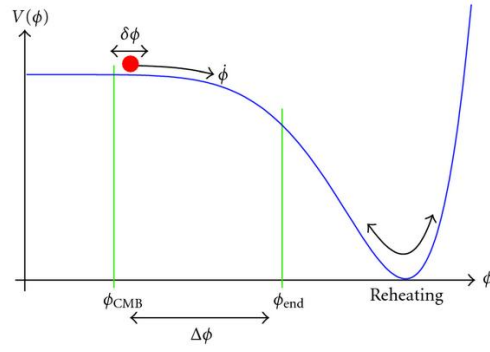


Figure 2.3: This figure illustrates an example of the potential of small field models of inflation. (Figure taken from Ref. [29]).

$$\epsilon_V < 0.012 \text{ (95\% C.L.)} , \quad (2.157)$$

$$\eta_V = -0.0080^{+0.0088}_{-0.0146} \text{ (68\% C.L.)} . \quad (2.158)$$

These constraints tend to favour a small field model w.r.t. a large field one. In fact, as noticed above, a small field model is characterized by $\eta_V < 0$, while for a large field model $\eta_V > 0$.

Chapter 3

Non linear effects: non-Gaussianity as a probe of inflationary models beyond standard slow-roll scenario

In this chapter we perform a non-linear extension of the standard slow-roll models of inflation in order to search for non-Gaussianities of the primordial perturbations. We define the n -th order correlation functions and the bispectrum of a generic perturbation and then we focus to the case of primordial perturbations. We introduce the in-in formalism which allows to compute the n -th order correlation functions from inflation and we perform a computation of the bispectrum of primordial scalar perturbations. We link this computation to experimental results in order to show that non-Gaussianities might arise from scenarios beyond the standard slow-roll theories. In particular at the end we mention some examples of models in which non-Gaussianities can arise by introducing modified gravity terms in the action of slow-roll theories of inflation.

3.1 Why searching for non-Gaussianities?

In the last years there has been an intense investigation about primordial non-Gaussianity signals (performed both by the WMAP and the Planck satellite, see e.g. Refs. [1, 30]). These signatures give contribution to the statistical correlators of higher orders than the two point function considered in the previous chapter. Now we define better these correlators. Given a generic perturbation δ , we define its n -th order correlation function as:

$$C_n(\vec{x}_i, t) = \langle \delta(\vec{x}_1, t) \dots \delta(\vec{x}_i, t) \dots \delta(\vec{x}_n, t) \rangle, \quad (3.1)$$

where the mean is taken over the statistical ensemble.

The first correlator which manifests non-Gaussianity is the 3-point function:

$$\langle \delta(\vec{x}_1, t) \delta(\vec{x}_2, t) \delta(\vec{x}_3, t) \rangle. \quad (3.2)$$

For our purposes we are interested in the Fourier space of the 3-point function, which reads like:

$$\langle \delta(\vec{k}_1, t) \delta(\vec{k}_2, t) \delta(\vec{k}_3, t) \rangle. \quad (3.3)$$

For isotropy and homogeneity of the background space we can parametrize this correlator as:

$$\langle \delta(\vec{k}_1, t) \delta(\vec{k}_2, t) \delta(\vec{k}_3, t) \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B(k_1, k_2, k_3, t). \quad (3.4)$$

The momentum conservation provided by the δ^3 is a consequence of homogeneity. Instead for isotropy $B(k_1, k_2, k_3, t)$ depends only on the wavenumbers k_i and not by the direction of the wavevectors \vec{k}_i . $B(k_i, t)$ is called *bispectrum* of the perturbation δ . In this Thesis we have focused on this correlator because already from its analysis one can learn a lot about inflationary models. The leading contribution of primordial perturbations to the CMB angular bispectrum comes from the scalar three point function [1]:

$$\langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) \zeta(\vec{k}_3) \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_{\zeta\zeta\zeta}(k_1, k_2, k_3), \quad (3.5)$$

where ζ is the comoving gauge invariant curvature perturbation discussed in the previous chapter. In the formula (3.5) is not explicitly exposed the time at which we evaluate the correlator. It is implicitly assumed that we evaluate it in the large scale limit, which correspond to the time after the end of inflation.

In general we can parametrize the bispectrum $B_{\zeta\zeta\zeta}(k_1, k_2, k_3)$ as [31, 19]:

$$B_{\zeta\zeta\zeta}(k_1, k_2, k_3) = S(k_1, k_2, k_3) \frac{\Delta_\zeta^2(k_*)}{k_1^2 k_2^2 k_3^2}, \quad (3.6)$$

where $\Delta_\zeta(k_*)$ is the dimensionless power spectrum of the perturbation ζ (see Eq. (2.128)) evaluated at a fixed momentum scale k_* . The function S is dimensionless and, in the case of scale-invariant bispectra, is invariant under the rescaling of all the three momenta k_i . S permits to define the momentum dependence of the bispectrum. In fact there are two types of the momentum dependence, the *shape* of the bispectrum and the *running* of the bispectrum:

- The *shape* of the bispectrum is the dependence of the function S by the ratios of the momenta $\frac{k_2}{k_1}$ and $\frac{k_3}{k_1}$, while we fix the overall momentum $K = \frac{k_1 + k_2 + k_3}{3}$;
- The *running* of the bispectrum is the dependence of S by the overall momentum K , while we take constant the ratios between the momenta.

In addition we can define the amplitude of non-Gaussianity provided by the bispectrum (3.6), named f_{NL} , as the bispectrum in the equilateral configuration ($k_1 = k_2 = k_3 = k$) normalized for the square of the power spectrum of the perturbation ζ (see Eq. (2.127)) evaluated at the momentum k . In formula it reads

$$f_{NL} = \frac{5}{18} \frac{B(k, k, k)}{\mathcal{P}_\zeta^2(k)}. \quad (3.7)$$

The multiplicative factor $\frac{5}{18}$ is an historical convention. We will see below the reason for this particular normalization. This dimensionless amplitude essentially tells us if a particular shape of non-Gaussianity is detectable or not by the experiment. From this definition we see that it can depend also on the overall momenta K . Moreover, if we substitute Eq. (3.6) into Eq. (3.7), we find the alternative definition:

$$f_{NL} = \frac{5}{18} S(k, k, k). \quad (3.8)$$

So f_{NL} corresponds to the shape function in the equilateral limit less then a factor $\frac{5}{18}$. If the bispectrum is scale invariant in general we can extract the amplitude f_{NL} from the shape function S and parametrize the bispectrum of ζ as:

$$B_{\zeta\zeta\zeta}(k_1, k_2, k_3) = \frac{18}{5} f_{NL} S(k_1, k_2, k_3) \frac{\Delta_\zeta^2(k_*)}{k_1^2 k_2^2 k_3^2}, \quad (3.9)$$

where the shape function $S(k_1, k_2, k_3)$ is normalized as $S(k, k, k) = 1$. Now we briefly describe some different examples of shapes of non-Gaussianities coming from the CMB angular bispectrum. We follow the Refs. [1, 31, 19]:

- **Local shape of non-Gaussianity:** a local non-Gaussian shape arises from a non linear correction to the perturbation ζ_g , where the suffix g denotes that this perturbation coincides with the linear perturbation ζ analyzed in the previous chapter. We can rewrite the new non-linear ζ as:

$$\zeta(\vec{x}) = \zeta_g(\vec{x}) + \frac{3}{5} f_{NL}^{loc} \left[\zeta_g^2(\vec{x}) - \langle \zeta_g^2(\vec{x}) \rangle \right]. \quad (3.10)$$

The reason for the presence of the factor $\frac{3}{5}$ is that, historically, non-Gaussianity was defined firstly for the Bardeen Newtonian potential Φ_g (see Ref. [14]), which at linear level and during the matter era is linked to ζ_g by the linear relation:

$$\Phi_g = \frac{3}{5} \zeta. \quad (3.11)$$

This type of non-Gaussianity is called *local* because the non-linear relation (3.10) is locally defined. In the Eq. (3.10) already appears the amplitude of non-Gaussianity produced. The bispectrum of local non-Gaussianity is:

$$B_{\zeta\zeta\zeta}^{loc.}(k_1, k_2, k_3) = \frac{6}{5} f_{NL}^{local} \times \left[\mathcal{P}_\zeta(k_1) \mathcal{P}_\zeta(k_2) + \mathcal{P}_\zeta(k_1) \mathcal{P}_\zeta(k_3) + \mathcal{P}_\zeta(k_2) \mathcal{P}_\zeta(k_3) \right], \quad (3.12)$$

where $\mathcal{P}_\zeta(k)$ is the power spectrum of the comoving curvature perturbation ζ . If we compute the amplitude of non-Gaussianity by substituting Eq. (3.12) into the definition (3.7) we find perfectly correspondence of our general definition. This is a confirmation that the multiplicative factor in Eq. (3.7) is correct. If we take the expression of $P_\zeta(k)$ (Eq. (2.127)) in the limit in which we neglect the scale dependence (assumption that is justified by slow-roll hypothesis), then Eq. (3.12) becomes:

$$B_{\zeta\zeta\zeta}^{loc.}(k_1, k_2, k_3) = \frac{6}{5} f_{NL}^{loc} \times \frac{\Delta_\zeta^2(k_*)}{k_1^2 k_2^2 k_3^2} \left(\frac{k_1^2}{k_2 k_3} + \frac{k_2^2}{k_1 k_3} + \frac{k_3^2}{k_1 k_2} \right), \quad (3.13)$$

where Δ_ζ is the dimensionless power spectrum of the perturbation ζ in the limit in which we neglect the scale dependence.

Thus, the template for the local shape reads like:

$$S_{local}(k_1, k_2, k_3) = \frac{1}{3} \left(\frac{k_1^2}{k_2 k_3} + 2 \text{ perms.} \right). \quad (3.14)$$

- **Equilateral shape of non-Gaussianity:** This is a shape which peaks in the equilateral configuration $k_1 = k_2 = k_3 = K$. The corresponding shape function has the form:

$$S_{equil.}(k_1, k_2, k_3) = \left(\frac{k_1}{k_3} + 5 \text{ perms.} \right) - \left(\frac{k_1^2}{k_2 k_3} + 2 \text{ perms.} \right) - 2. \quad (3.15)$$

This kind of shape arise from considering higher derivative corrections in the action of several inflationary models of inflation.

- **Orthogonal shape of non-Gaussianity:** this is another shape of non-Gaussianity which arise, as the equilateral shape, from considering higher derivative corrections in inflationary models of inflation. The template associated to this shape is like:

$$S_{ortho.}(k_1, k_2, k_3) = -3.84 \left(\frac{k_1^2}{k_2 k_3} + 2 \text{ perms.} \right) + 3.94 \left(\frac{k_1}{k_3} + 5 \text{ perms.} \right) - 11.10. \quad (3.16)$$

For more details about these shapes and the mechanisms that can create such non-Gaussianities we remand to the Ref. [1]. In the same Ref. the last results of the Planck satellite (updated to year 2015) on primordial non-Gaussianities are exposed. We found the following experimental constraints for the three amplitudes of primordial non-Gaussianities relative to the measures of the CMB temperature anisotropies:

$$f_{NL}^{local} = 2.5 \pm 5.7 \text{ (68\%C.L.)}, \quad (3.17)$$

$$f_{NL}^{equil.} = -16 \pm 70 \text{ (68\%C.L.)}, \quad (3.18)$$

$$f_{NL}^{ortho.} = -34 \pm 33 \text{ (68\%C.L.)}. \quad (3.19)$$

As we can see, these values have high errors, giving compatibility with a zero level of non-Gaussianity. So why is actually so important trying to reduce the errors for better constraining such non-Gaussianities? As we will see in detail in the next section, it is possible to compute at leading order in slow-roll parameters the bispectrum of the gauge invariant perturbation ζ with a non linear extension of the slow-roll models of inflation. We anticipate that the amplitude of such bispectrum is suppressed in the slow-roll limit. Then any signals of non-Gaussianities may come only from an extension or a modification of the slow-roll theories of inflation which, for the moment, are the most accepted paradigms for describing inflation. In particular, such non-Gaussianities can be signatures of possible modifications of the physics at GUT energies (which are the ones of inflation) that are no still achievable in the actual colliders. The reason is that contributions on non-Gaussianities from inflation of the gauge invariant variable ζ arise essentially by autointeraction terms of ζ and by interaction terms between ζ and the primordial gravitational waves γ_{ij}^s . These interactions in the standard slow-roll models of inflation are suppressed. So signals of non-Gaussianity can be signatures of interaction terms between ζ and new fields associated to new degrees of freedom that could appear at high energies in a new physics scenario. But also they can be the signatures of new interactions terms between the primordial perturbations (both ζ and γ_{ij}^s) which arise from a modification of the Einstein theory of gravity at high energies. We will return better to this fact in the last section of this chapter.

3.2 Non linear extension of slow-roll theories of inflation

The first step in our investigation is to understand if observational signals of primordial non-Gaussianities arise naturally by a non linear extension of the standard slow-roll theories of inflation. In the previous chapter we have studied the action only until the second order in the primordial cosmological perturbations, which is the first term that arise from an expansion in series of such perturbations. Now the purpose is to consider the effects of higher order terms and see how they act on the statistics of the primordial perturbations.

In particular, if we are interested in computing the bispectrum of the comoving curvature perturbation ζ , we can stop the expansion of the action until cubic order terms in ζ . For the discussion about the constraints performed in the chapter 2, we can use the expressions of the lapse and the shift functions N and N_i only until the first order in the perturbation ζ , see Eq. (2.57). On the contrary we have to perform a non-linear generalization of the spatial metric h_{ij} . The most natural non-linear extensions of the spatial metric h_{ij} in the gauges (2.45) and (2.46) are respectively [9, 32]

$$\begin{aligned} h_{ij} &= a^2 [\exp \gamma]_{ij}, \quad \gamma_i^i = 0, \quad \partial^i \gamma_{ij} = 0, \\ [exp \gamma]_{ij} &= \delta_{ij} + \gamma_{ij} + \frac{1}{2!} \gamma_{ik} \gamma_j^k + \dots, \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} h_{ij} &= a^2 e^{2\zeta} [\exp \tilde{\gamma}]_{ij}, \quad \tilde{\gamma}_i^i = 0, \quad \partial^i \tilde{\gamma}_{ij} = 0, \\ [exp \tilde{\gamma}]_{ij} &= \delta_{ij} + \tilde{\gamma}_{ij} + \frac{1}{2!} \tilde{\gamma}_{ik} \tilde{\gamma}_j^k + \dots, \\ e^{2\zeta} &= 1 + 2\zeta + \frac{1}{2!} (4\zeta^2) + \dots. \end{aligned} \quad (3.21)$$

As we see, a priori the two transverse traceless gravitational waves are now different in the two gauges. In fact, at the non-linear level, a gauge transformation acts also on transverse traceless tensor quantities. Also the relation between ζ and φ is no more linear. If we do the transformation which allows to pass from the gauge (3.20) to the gauge (3.21) and we impose the equivalence between the two gauges, we find the relations [9]:

$$\begin{aligned} \zeta &= \zeta_1 + \frac{1}{2} \frac{\ddot{\phi}}{\dot{\phi}H} \zeta_1^2 + \frac{1}{4} \frac{\dot{\phi}^2}{H^2} \zeta_1^2 + \\ &+ \frac{1}{H} \dot{\zeta}_1 \zeta_1 - \frac{1}{4} \frac{a^{-2}}{H^2} \partial^{-2} \partial_i \partial_j (\partial_i \zeta_1 \partial_j \zeta_1) + \frac{1}{2H} \partial_i \psi \partial_i \zeta_1 \\ &- \frac{1}{2H} \partial^{-2} \partial_i \partial_j (\partial_i \psi \partial_j \zeta_1) - \frac{1}{4H} \dot{\gamma}_{ij} \partial_i \partial_j \zeta_1. \end{aligned} \quad (3.22)$$

$$\begin{aligned} \gamma_{ij} &= \tilde{\gamma}_{ij} + \\ &+ \frac{1}{H} \dot{\gamma}_{ij} \zeta_1 - \frac{a^{-2}}{H^2} \partial_i \zeta_1 \partial_j \zeta_1 + \frac{1}{H} (\partial_i \psi \partial_j \zeta_1 + \partial_j \psi \partial_i \zeta_1), \end{aligned} \quad (3.23)$$

where ζ_1 is the first order value of ζ (from Eq. (2.48) $\zeta_1 = -\frac{H}{\dot{\phi}}\varphi$), and ψ is as in Eq. (2.57).

Now, we have to convince ourself that such new variables ζ and γ_{ij} ($\tilde{\gamma}_{ij}$) are still constant on superhorizon scales and so are the correct non-linear generalizations of the gauge invariant variables studied in Chapter 2. We can follow the demonstration provided in Ref. [9] (for some first works related to this point see, e.g, Ref [32]). We need to expand the action (2.33) to all orders in the fields, but only up the first order in the derivatives of the fields. In fact we want to demonstrate that, at all orders in the fields, in each term of the action there are at least two derivatives of the fields. So we want to demonstrate that up to first order in the derivatives of the fields there are no contributions to the action.

We can expand N as $N = 1 + \delta N$, when δN is a term which starts with a first order term in the derivatives of ζ and γ_{ij} . On the contrary $D_i N_j$ starts already with a first order term in the derivatives. Then, the equation of motion for N (2.51) up to first order in the derivatives becomes:

$$2V\delta N = 2H(3\dot{\zeta} - D_i N^i). \quad (3.24)$$

Now we choose the gauge (3.21) and we evaluate the action (2.49) up to first order in the derivatives of the fields ζ and γ_{ij} . We find:

$$S = \int d^3x dt \sqrt{h}(-2V - 2V\delta N) = \int d^3x dt a^3 e^{3\zeta} (-6H^2 + \dot{\phi}^2 - 6H\dot{\zeta}) = -2 \int d^3x dt \partial_t(a^3 H e^{3\zeta}), \quad (3.25)$$

where we have used Eq. (3.24) and background equations of motion (2.15) and (2.14) to do some simplifications. Now the last term in the paranthesis of (3.25) is a surface term and can be neglected. The result is that in the full action there are no terms up to first order in the derivatives. If we consider the time region outside the horizon, the spatial derivatives terms are negligible¹. Then it follows that the action at all orders in the fields starts with terms of second order in time derivatives. For this reason the solutions of the equations of motion at all orders in ζ and γ_{ij} are constant outside the horizon.

Thus outside the horizon only the first lines in Eqns. (3.22) and (3.23) are relevant. Notice also that for this reason we can not distinguish between γ_{ij} and $\tilde{\gamma}_{ij}$.

3.3 Non Gaussianities from slow-roll models of inflation

3.3.1 In-in formalism

Now our aim is to perform an explicit computation of the bispectrum $B_{\zeta\zeta\zeta}$, following essentially Refs. [9, 10, 19].

In order to do such a calculation we choose here to adopt the *in-in formalism* (it is summarized in Refs. [9, 10] and described in details in Refs. [19, 33]). Since it will turn out to be useful in the following we are going to describe briefly the method. For another calculation of the primordial non-Gaussianity in the standard single field models of slow-roll inflation, see Ref. [34], where a computation at second order in the perturbations is performed. For the moment we take a generic perturbation $\hat{\delta}(t, \vec{x})$, that must be quantized (suppose it correspond to a scalar field). In general we are dealing with correlators of the type:

¹In fact a spatial derivative term is like $\partial_i A^i \sim \frac{1}{a} \partial_i A_i$. If we pass to Fourier space it becomes of order $\sim \frac{k_i}{a} A_i$. Proceeding in the same way we can show that a term with a number n of spatial derivatives contains a factor of the order $\left(\frac{k_i}{a}\right)^n$, where k_i is a momentum of a single mode of the perturbations. Then, on large scales, which corresponds to the limit $\frac{k}{aH} \ll 1$, the term is suppressed.

$$\langle \Omega | \hat{\delta}(\vec{x}_1, t) \dots \hat{\delta}(\vec{x}_i, t) \dots \hat{\delta}(\vec{x}_n, t) | \Omega \rangle . \quad (3.26)$$

From Eq. (3.26), it is clear that we work in the *Heisenberg picture*, where only the operators evolve and states do not. In this case we call $|\Omega\rangle$ the vacuum of the full theory for a reason which it will be clear in the following. In order to explain the method we work in hamiltonian formalism. We know in fact that the predictions of lagrangian and hamiltonian formalisms are equal. In general the hamilonian of the theory can be decomposed into a quadratic part H_0 and some interaction terms H_{int} as:

$$H_{tot} = H_0 + H_{int} . \quad (3.27)$$

The quadratic part describes essentially the free evolution of the field $\hat{\delta}$. The trick of the method consists in switching to the *interaction picture*. The operator in the interaction picture $\hat{\delta}^I(t)$ is linked to the corresponding one in the Heisenberg picture at the time t , $\hat{\delta}(t)$, by the relation [33]:

$$\hat{\delta}^I(t) = F(t, t_0) \hat{\delta}(t) F^{-1}(t, t_0) , \quad (3.28)$$

where:

$$F(t, t_0) = T \exp \left[-i \int_{t_0}^t H_{int}^I(t') dt' \right] . \quad (3.29)$$

where T represents the time-ordered operator. H_{int}^I is the interaction hamiltonian in the interaction picture which coincides with the one in the Heisenberg picture. The time t_0 is the time in which we switch on the interactions $H_{int}(t)$.

If we insert Eq. (3.28) into Eq. (3.26), Eq. (3.26) becomes:

$$\langle \Omega | \left[\bar{T} \exp \left(i \int_{t_0}^t H_{int}^I(t') dt' \right) \right] \hat{\delta}^I(\vec{x}_1, t) \dots \hat{\delta}^I(\vec{x}_i, t) \dots \hat{\delta}^I(\vec{x}_n, t) \left[T \exp \left(-i \int_{t_0}^t H_{int}^I(t') dt' \right) \right] | \Omega \rangle , \quad (3.30)$$

where \bar{T} is now the anti-time-ordered operator.

In addition the relation between the hamiltonian and the lagrangian in the interaction picture is:

$$H_{int}^I = -L_{int} . \quad (3.31)$$

In fact the Legendre transform which links hamiltonian formalism to lagrangian formalism reads like $H^I \sim (\delta \pi_\delta - L)$, where π_δ is the conjugate momentum of the field δ . But the term $(\delta \pi_\delta)$ is a quadratic term and, if we consider only the interaction terms, the equality (3.31) follows. We can extend this consideration also for the case where there is more than one field in the theory. Then, if we compute the interaction terms in the lagrangian of the theory, we can compute perturbatively the correlator (3.30) by expanding the time(anti)-ordered exponentials. If we drop the expansion of the exponentials up to first order and we use Eq. (3.31), the formula (3.30) becomes simply:

$$i \int_{t_0}^t dt' \langle \Omega | [\hat{\delta}^I(\vec{x}_1, t) \dots \hat{\delta}^I(\vec{x}_i, t) \dots \hat{\delta}^I(\vec{x}_n, t), L_{int}(t')] | \Omega \rangle , \quad (3.32)$$

where $[\cdot, \cdot]$ is the commutator operator.

The last equation is the "master" equation which we use to compute at first order the expectation values. We can easily go to Fourier space by doing on both members the integrals $\prod_i \left[\int d^3 x_i e^{i\vec{k}_i \cdot \vec{x}_i} \right]$. We obtain

$$i \int_{t_0}^t dt' \langle \Omega | [\hat{\delta}^I(\vec{k}_1, t) \dots \hat{\delta}^I(\vec{k}_i, t) \dots \hat{\delta}^I(\vec{k}_n, t), L_{int}(t')] | \Omega \rangle . \quad (3.33)$$

As a final consideration we should remark that the vacuum $|\Omega\rangle$ is the vacuum of the full theory, including also interaction terms in the theory. If we call $|0\rangle$ the vacuum of the theory whose action is dropped at quadratic order in the fields (which is the free vacuum of the theory), we would like to write $|\Omega\rangle$ in function of $|0\rangle$. The reason is that $|0\rangle$ is the vacuum that we have introduced in Chapter 2 to quantize the primordial cosmological perturbations, and so we know how the creation and annihilation operators act on it.

In studying scattering processes in QFT in general the two vacuum states do not coincide due to vacuum fluctuations caused by the interactions. But in our case we are evaluating expectation values. These processes do not generate any non-trivial vacuum fluctuations through interactions. This is a direct consequence of the identity:

$$F^{-1} F = 1 , \quad (3.34)$$

where F is defined in Eq. (3.29).

For this reason we can replace $|\Omega\rangle$ with $|0\rangle$ in (3.33), [33]. This fact is crucial for doing the computations. In fact the fields in the interaction picture evolve as in the free quadratic case. Thus, if we know the free solutions in terms of annihilation and creation operators a and a^\dagger (which are the ones we have introduced in Chapter 2), we can do easily the contractions with the free vacuum state $|0\rangle$.

3.3.2 Computation of the bispectrum $B_{\zeta\zeta\zeta}$

Using the in-in formalism, now we want to evaluate the 3-point function (3.5). We perform a tree level computation, so using the formula (3.33) we have:

$$\langle \hat{\zeta}(\vec{k}_1, 0) \hat{\zeta}(\vec{k}_2, 0) \hat{\zeta}(\vec{k}_3, 0) \rangle = i \int_{-\infty}^0 d\tau' a \langle 0 | [\hat{\zeta}^I(\vec{k}_1, 0) \hat{\zeta}^I(\vec{k}_2, 0) \hat{\zeta}^I(\vec{k}_3, 0), L_{int}(\tau')] | 0 \rangle , \quad (3.35)$$

where we have switched from the cosmological time t to the conformal time τ . In fact all our free solutions are expressed in terms of the conformal time. Here the time at which we evaluate the correlator is at $\tau = 0$ corresponding to the end of inflation and to super-horizon limit. On the contrary, the interactions are switched on when the fluctuation modes are on very sub-horizon scales corresponding to the limit $\tau \rightarrow -\infty$.

In Eq. (3.35) the bra-ket contractions with the vacuum states are non zero only if the interaction lagrangian has the same functional form of the operator in the left side of the commutator operator. For this reason we need to compute the Lagrangian cubic in the field ζ . To simplify the computations, we express the interaction lagrangian in terms of ζ_1 , which is the first order expression of ζ . We will pass to the field ζ at the end through the relation (3.22). For this reason we compute for the moment only the expectation value:

$$\langle \hat{\zeta}_1(\vec{k}_1, 0) \hat{\zeta}_1(\vec{k}_2, 0) \hat{\zeta}_1(\vec{k}_3, 0) \rangle = i \int_{-\infty}^0 dt' a \langle 0 | [\hat{\zeta}_1^I(\vec{k}_1, 0) \hat{\zeta}_1^I(\vec{k}_2, 0) \hat{\zeta}_1^I(\vec{k}_3, 0), L_{int}(\tau')] | 0 \rangle. \quad (3.36)$$

From now we will not write explicitly the suffix I for the fields, anymore impling the fact that they are evaluated in the interaction picture. We start with evaluating the interaction Lagrangian cubic in ζ_1 . In order to do so, we work in spatially flat gauge (3.20) that allows to avoid different integrations by parts. In such a gauge the cubic terms dominant in the slow-roll parameters come from the third and fourth terms in the square bracket of action (2.33). So, at leading order in the slow parameters, the cubic interaction lagrangian looks like [9]:

$$L_{int}(t) = \int d^3x a^3 \frac{1}{M_{Pl}^2} \left[-\frac{\dot{\phi}}{4H} \varphi \dot{\varphi}^2 - a^{-2} \frac{\dot{\phi}}{4H} \varphi (\partial_i \varphi) (\partial^i \varphi) - a^{-2} \dot{\varphi} \partial_i \psi \partial^i \varphi \right], \quad (3.37)$$

where ψ is defined as in (2.57). The contractions between latin indices are done here with the δ_{ij} .

Now we use the linear relation between φ and ζ_1 (Eq. (2.48)) and Eq. (2.57) to rewrite lagrangian (3.37) as:

$$L_{int}(t) = \int d^3x a^3 \frac{1}{M_{Pl}^2} \left[\frac{\dot{\phi}^4}{4H^4} \zeta_1 \dot{\zeta}_1^2 + a^{-2} \frac{\dot{\phi}^4}{4H^4} \zeta_1 (\partial_i \zeta_1) (\partial^i \zeta_1) - \frac{\dot{\phi}^4}{2H^4} \zeta_1 (\partial_i \partial^{-2} \zeta_1) (\partial^i \zeta_1) \right]. \quad (3.38)$$

We make explicit the slow roll dependence inserting the definition (2.20) into (3.38). We find:

$$L_{int}(t) = \epsilon^2 M_{Pl}^2 \int d^3x \left[a^3 \zeta_1 \dot{\zeta}_1^2 + a \zeta_1 (\partial_i \zeta_1) (\partial^i \zeta_1) - 2a^3 \zeta_1 (\partial_i \partial^{-2} \zeta_1) (\partial^i \zeta_1) \right]. \quad (3.39)$$

As a final step, we express the interaction lagrangian as a function of the conformal time τ .

$$L_{int}(\tau) = \epsilon_V^2 M_{Pl}^2 \int d^3x \left[a \zeta_1 \zeta_1'^2 + a \zeta_1 (\partial_i \zeta_1) (\partial^i \zeta_1) - 2a \zeta_1' (\partial_i \partial^{-2} \zeta_1') (\partial^i \zeta_1) \right]. \quad (3.40)$$

If we insert the Fourier decomposition (2.64) of the field ζ_1 into Eq. (3.40), we find

$$L_{int}(\tau) = \int d^3k d^3p d^3q \frac{1}{(2\pi)^6} \delta^3(\vec{k} + \vec{p} + \vec{q}) \epsilon_V^2 M_{Pl}^2 \left[a \zeta_1(\vec{k}) \zeta_1'(\vec{p}) \zeta_1'(\vec{q}) - a (\vec{p} \cdot \vec{q}) \zeta_1(\vec{k}) \zeta_1(\vec{p}) \zeta_1(\vec{q}) + \right. \\ \left. - 2a \frac{(\vec{p} \cdot \vec{q})}{p^2} \zeta_1'(\vec{k}) \zeta_1'(\vec{p}) \zeta_1(\vec{q}) \right]. \quad (3.41)$$

Here the Dirac delta δ^3 comes from an integration $\int d^3x e^{i(\mathbf{k}+\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}$, where the integral is the one in (3.40) and the exponential comes from the Fourier expansions of the fields $\zeta_1(\vec{x}, \tau)$ in (3.40). In Eq. (3.41) the conformal time dependence of the fields is implicitly understood for simplicity of notation.

Now, inserting Eq. (3.41) into Eq. (3.36), we find an expression of the kind:

$$\langle \hat{\zeta}_1(\vec{k}_1) \hat{\zeta}_1(\vec{k}_2) \hat{\zeta}_1(\vec{k}_3) \rangle = \frac{i}{(2\pi)^6} \epsilon^2 M_{Pl}^2 \int d^3K \delta^3(\vec{k} + \vec{p} + \vec{q}) \int_{-\infty}^0 dt' a [\mathcal{A}_1(\tau') + \mathcal{A}_2(\tau') + \mathcal{A}_3(\tau')], \quad (3.42)$$

where $\int d^3 K = \int d^3 k d^3 p d^3 q$ and the \mathcal{A}_n 's stands for the contractions:

$$\mathcal{A}_1 = a \langle 0 | [\hat{\zeta}_1(\vec{k}_1, 0) \hat{\zeta}_1(\vec{k}_2, 0) \hat{\zeta}_1(\vec{k}_3, 0), \hat{\zeta}_1(\vec{k}, \tau) \hat{\zeta}'_1(\vec{p}, \tau) \hat{\zeta}'_1(\vec{q}, \tau)] | 0 \rangle, \quad (3.43)$$

$$\mathcal{A}_2 = -a (\vec{p} \cdot \vec{q}) \langle 0 | [\hat{\zeta}_1(\vec{k}_1, 0) \hat{\zeta}_1(\vec{k}_2, 0) \hat{\zeta}_1(\vec{k}_3, 0), \hat{\zeta}_1(\vec{k}, \tau) \hat{\zeta}'_1(\vec{p}, \tau) \hat{\zeta}'_1(\vec{q}, \tau)] | 0 \rangle, \quad (3.44)$$

$$\mathcal{A}_3 = -2a \frac{(\vec{p} \cdot \vec{q})}{p^2} \langle 0 | [\hat{\zeta}_1(\vec{k}_1, 0) \hat{\zeta}_1(\vec{k}_2, 0) \hat{\zeta}_1(\vec{k}_3, 0), \hat{\zeta}_1(\vec{k}, \tau) \hat{\zeta}'_1(\vec{p}, \tau) \hat{\zeta}'_1(\vec{q}, \tau)] | 0 \rangle. \quad (3.45)$$

We can compute these contractions using the *Wick theorem* [19, 33]. For each A_n we have to sum over all the terms that we obtain in the following way: each term is obtained by doing all the possible bra-ket contractions with the vacuum states between couples of fields evaluated at different times. The terms in which at least one field remains uncontracted are vanishing. We remember that contractions between fields of different type are zero. From the form of our interaction lagrangian we need to compute the following two contractions:

$$\langle 0 | \hat{\zeta}_1(\vec{k}, \tau) \hat{\zeta}_1(\vec{k}', \tau) | 0 \rangle, \quad (3.46)$$

$$\langle 0 | \hat{\zeta}_1(\vec{k}, \tau) \hat{\zeta}'_1(\vec{k}', \tau) | 0 \rangle. \quad (3.47)$$

The field ζ_1 is the first order value of ζ and in the interaction picture its evolution is described by the quadratic action (2.62). Then the solution is the same found in Chapter 2 (see Eq. (2.89)). Inserting such solution into Eqns. (3.46) and (3.47) we find:

$$\begin{aligned} \langle 0 | \hat{\zeta}_1(\vec{k}, \tau) \hat{\zeta}_1(\vec{k}', \tau) | 0 \rangle &= \langle 0 | [u(k, \tau) \hat{a}(\vec{k}) + u^*(k, \tau) \hat{a}^\dagger(-\vec{k})] [u(k', \tau) \hat{a}(\vec{k}') + u^*(k', \tau) \hat{a}^\dagger(-\vec{k}')] | 0 \rangle \\ &= u(k, \tau) u^*(k', \tau) \langle 0 | a(\vec{k}) a^\dagger(-\vec{k}') | 0 \rangle \\ &= u(k, \tau) u^*(k', \tau) \langle 0 | [a(\vec{k}), a^\dagger(-\vec{k}')] | 0 \rangle \\ &= (2\pi)^3 \delta^3(\vec{k} + \vec{k}') u(k, \tau) u^*(k', \tau). \end{aligned} \quad (3.48)$$

In evaluating Eq. (3.48) we have used Eqns. (2.77) and (2.78); $u(k, \tau)$ is the same as in Eq. (2.88). For simplicity of notation, in the computation we have not included the normalization factor $\frac{1}{\mathcal{A}_s}$ which we see in the solution (2.89). It will be restored later when we will consider the explicit expression of the mode function u . Proceeding in the same way we find also:

$$\begin{aligned} \langle 0 | \hat{\zeta}_1(\vec{k}, \tau) \hat{\zeta}'_1(\vec{k}', \tau) | 0 \rangle &= \langle 0 | [u(k, \tau) \hat{a}(\vec{k}) + u^*(k, \tau) \hat{a}^\dagger(-\vec{k})] [u(k', \tau) \hat{a}(\vec{k}') + u^*(k', \tau) \hat{a}^\dagger(-\vec{k}')] | 0 \rangle \\ &= (2\pi)^3 \delta^3(\vec{k} + \vec{k}') u(k, \tau) \frac{d}{d\tau} u^*(k, \tau). \end{aligned} \quad (3.49)$$

Now we can compute the contractions \mathcal{A}_n using the Wick theorem, (3.48) and (3.49). We find:

$$\begin{aligned} \mathcal{A}_1 &= (2\pi)^9 a \left[u(k_1, 0) u(k_2, 0) u(k_3, 0) \left(\frac{d}{d\tau} u^*(k_1, \tau) \right) \left(\frac{d}{d\tau} u^*(k_2, \tau) \right) u^*(k_3, \tau) - c.c. \right] + \\ &\quad + \text{perm}(k_i). \end{aligned} \quad (3.50)$$

$$\mathcal{A}_2 = -(2\pi)^9 a (\vec{k}_1 \cdot \vec{k}_2) [u(k_1, 0)u(k_2, 0)u(k_3, 0)u^*(k_1, \tau')u^*(k_2, \tau')u^*(k_3, \tau') - c.c.] + \text{perm}(\mathbf{k}_i) . \quad (3.51)$$

$$\mathcal{A}_3 = (2\pi)^9 (-2a) \frac{(\vec{k}_1 \cdot \vec{k}_2)}{k_1^2} \left[u(k_1, 0)u(k_2, 0)u(k_3, 0) \left(\frac{d}{d\tau} u^*(k_1, \tau') \right) u^*(k_2, \tau') \left(\frac{d}{d\tau} u^*(k_3, \tau') \right) - c.c. \right] + \text{perm}(\mathbf{k}_i) . \quad (3.52)$$

The permutations over the k_i 's come from all the different ways of contracting the fields. Instead, the minus complex conjugate ($-c.c.$) comes from the commutator operator between fields in the expressions for the \mathcal{A}_n 's. If we insert Eqns. (3.50), (3.51) and (3.52) into Eq. (3.42), we find

$$\begin{aligned} \langle \hat{\zeta}_1(k_1) \hat{\zeta}_1(k_2) \hat{\zeta}_1(k_3) \rangle &= i(2\pi)^3 \delta^3(k_1 + k_2 + k_3) \epsilon_V^2 M_{Pl}^2 \times \\ &\times \text{Im} \left[I_1 - (\vec{k}_1 \cdot \vec{k}_2) I_2 - 2 \frac{(\vec{k}_1 \cdot \vec{k}_2)}{k_1^2} I_3 - c.c. \right] + \\ &+ \text{perm}(\mathbf{k}_i) , \end{aligned} \quad (3.53)$$

where the I_n 's are the integrals

$$I_1 = u(k_1, 0)u(k_2, 0)u(k_3, 0) \int_{-\infty}^0 d\tau' a^2 \left[\left(\frac{d}{d\tau} u^*(k_1, \tau') \right) \left(\frac{d}{d\tau} u^*(k_2, \tau') \right) u^*(k_3, \tau') \right], \quad (3.54)$$

$$I_2 = u(k_1, 0)u(k_2, 0)u(k_3, 0) \int_{-\infty}^0 d\tau' a^2 [u^*(k_1, \tau')u^*(k_2, \tau')u^*(k_3, \tau')], \quad (3.55)$$

$$I_3 = u(k_1, 0)u(k_2, 0)u(k_3, 0) \int_{-\infty}^0 d\tau' a^2 \left[\left(\frac{d}{d\tau} u^*(k_1, \tau') \right) u^*(k_2, \tau') \left(\frac{d}{d\tau} u^*(k_3, \tau') \right) \right]. \quad (3.56)$$

In order to perform these integrals we need an analytic expression for the mode function $u(k, \tau)$. Its exact value is (2.88), but the Hankel functions in general cannot be integrated analytically. Fortunately, if we are interested of a computation at leading order in the slow-roll parameters, we can take the value of u with $\nu_s = \frac{3}{2}$. This corresponds to put to zero the slow roll parameters ϵ_V and η_V in the explicit expression of u . In fact we have already a factor ϵ_V^2 into the Eq. (3.53), and so any other factor of ϵ_V coming from the integrals I_n give automatically a subdominant contribution in the slow-roll parameters. In this case Eq. (2.88) corresponds to the solution for a free massless scalar field in a de Sitter space. Thus, the mode function $u(k, \tau)$ becomes (see, e.g., [9, 10, 19])

$$u(k, \tau) = \frac{iH}{M_{Pl} \sqrt{2\epsilon_V k^3}} (1 + ik\tau) e^{-ik\tau} , \quad (3.57)$$

where we have also restored the correct normalization factor for the variable ζ . We notice that this function can be integrated in the time domain $(-\infty, 0]$. Its time derivative w.r.t. conformal time is

$$\frac{d}{d\tau} u(k, \tau) = \frac{iH}{M_{Pl} \sqrt{2\epsilon_V k^3}} k^2 \tau e^{-ik\tau} . \quad (3.58)$$

Following the same reasoning, we can expand also all the other functions in the integrals I_n in series of slow-roll parameters and take the expression obtained by putting these slow-roll parameters to zero. Then if we put $\epsilon_V = 0$ in the Eq. (2.70), we find:

$$a(\tau) = -\frac{1}{H\tau}. \quad (3.59)$$

In addition we can expand in series near a fixed time t_* the Hubble parameter H :

$$H(t) = H(t_*) + \dot{H}(t - t_*) + \dots = H(t_*) + \epsilon_V H^2(t - t_*). \quad (3.60)$$

Another time we can put $\epsilon_V = 0$ in this last equation because it gives a subdominant contribution in slow-roll parameters when we evaluate the integrals I_n . Then we can take the value of H evaluated at a fixed time t_* which we will define below.

Now, using these prescriptions, we are going to compute the integrals I_n . We start with I_1 , which becomes:

$$I_1 = -H_*^4 k_1^2 k_2^2 \left(\prod_{i=1}^3 \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \int_{-\infty}^0 d\tau' (1 - ik_3 \tau') e^{iK\tau'}, \quad (3.61)$$

where $K = k_1 + k_2 + k_3$. Here the suffix $*$ indicates that the corresponding quantity is evaluated at horizon crossing time. This seems to create an ambiguity because we have three different modes that exit from the horizon at different conformal times. In order to solve this ambiguity we choose the time of horizon crossing of the momentum $K = k_1 + k_2 + k_3$, that corresponds to a time in which we are sure that all the three momenta have already left the horizon.

In order to perform the integral in Eq. (3.61) we have to correct the oscillatory behaviour at $-\infty$ of the exponential $e^{iK\tau'}$. We achieve this by performing a Wick rotation of the real axis [19, 9]. We then promote the real integration variable to a complex variable and we do the change of variable $\tau'' = i\tau'$. This corresponds to a Wick rotation of the time integration contour. Eq. (3.61) becomes now

$$I_1 = iH_*^4 k_1^2 k_2^2 \left(\prod_{i=1}^3 \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \int_{-\infty}^0 d\tau'' (1 - k_3 \tau'') e^{K\tau''}. \quad (3.62)$$

Now in Eq. (3.62) integrals of the type

$$I(n, K) = \int_{-\infty}^0 dx x^n e^{Kx} \quad (3.63)$$

appear. We can solve them by integrating by parts, finding:

$$I(n, K) = (-1)^n \frac{\Gamma(n+1)}{(iK)^{n+1}} = (-1)^n \frac{n!}{K^{n+1}}, \quad (3.64)$$

where $\Gamma(n)$ is the Euler gamma. Applying this formula we find:

$$\int_{-\infty}^0 d\tau'' (1 - k_3 \tau'') e^{K\tau''} = \left(\frac{1}{K} + \frac{k_3}{K^2} \right). \quad (3.65)$$

Thus eq. (3.62) becomes:

$$I_1 = iH_*^4 \left(\prod_{i=1}^3 \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \left(\frac{k_1^2 k_2^2}{K} + \frac{k_1^2 k_2^2 k_3}{K^2} \right). \quad (3.66)$$

More in general, usually in this kind of computations we are dealing with integrals of the type

$$\tilde{I}(n, K) = \int_{-\infty}^0 dx x^n e^{iKx}. \quad (3.67)$$

Proceeding as done for the integral in (3.61), we can deduce the general formula

$$\tilde{I}(n, K) = (-1)^n \frac{\Gamma(n+1)}{(iK)^{n+1}} = (-1)^n \frac{n!}{(iK)^{n+1}}. \quad (3.68)$$

Now we pass to evaluate I_2 :

$$I_2 = -H_*^4 \left(\prod_{i=1}^3 \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \int_{-\infty}^0 \frac{d\tau'}{\tau'^2} (1 - ik_1\tau')(1 - ik_2\tau')(1 - ik_3\tau') e^{iK\tau'}. \quad (3.69)$$

In computing some integrals in (3.69) we use again the formula (3.68), but this time also another type of integral appears, which reads

$$\begin{aligned} \tilde{I} &= \int_{-\infty}^0 \frac{dx}{x^2} (1 - iKx) e^{iKx} = \int_{-\infty}^0 \frac{dx}{x^2} e^{iKx} - iK \int_{-\infty}^0 \frac{dx}{x} e^{iKx} = \\ &= -\frac{e^{Kx}}{x} \Big|_{-\infty}^0 + iK \int_{-i\infty}^0 \frac{dx}{x} e^{iKx} - iK \int_{-i\infty}^0 \frac{dx}{x} e^{iKx} = \\ &= \lim_{x \rightarrow 0} -\frac{1}{x} e^{iKx}, \end{aligned} \quad (3.70)$$

We have used the name variable x instead of τ' for simplicity of notation.

In doing the limit in (3.70) we expand the exponential using the Euler formula. The final result is:

$$\tilde{I} = \lim_{x \rightarrow 0} \left[-\frac{\cos(Kx)}{x} \right] + \lim_{x \rightarrow 0} \left[-i \frac{\sin(Kx)}{x} \right]. \quad (3.71)$$

Only the second limit in Eq. (3.71) is finite and its value is pure imaginary:

$$\lim_{x \rightarrow 0} \left[-i \frac{\sin(Kx)}{x} \right] = -iK. \quad (3.72)$$

Instead, the first limit in Eq. (3.71) give a real divergent contribution to the integral (3.71). Fortunately it doesn't create any problem. Infact, as we see in Eq. (3.53), at the end we have to take only the imaginary part of the integrals that we compute.

Therefore, using the formulae (3.68) and (3.71) in evaluating the integral in Eq. (3.69), we find:

$$I_2 = iH_*^4 \left(\prod_{i=1}^3 \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \left(K - \frac{k_1 k_2 + k_2 k_3 + k_1 k_3}{K} - \frac{k_1 k_2 k_3}{K^2} \right). \quad (3.73)$$

Finally we pass to compute I_3 :

$$I_3 = -H_*^4(k_1^2 k_3^2) \left(\prod_{i=1}^3 \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \int_{-\infty}^0 d\tau' (1 - ik_2 \tau') e^{iK\tau'} . \quad (3.74)$$

The integrals in (3.74) are of the form (3.68). We find:

$$I_3 = iH_*^4(k_1^2 k_3^2) \left(\prod_{i=1}^3 \frac{1}{M_{Pl}^2 2\epsilon_V k_i^3} \right) \left(\frac{1}{K} + \frac{k_2}{K^2} \right) . \quad (3.75)$$

From the results obtained so far, it follows that, if we put the expressions of the I_n 's into the correlator (3.53), we have at leading order in slow-roll parameters:

$$\begin{aligned} \langle \hat{\zeta}_1(\vec{k}_1) \hat{\zeta}_1(\vec{k}_2) \hat{\zeta}_1(\vec{k}_3) \rangle = & (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) 4 \frac{H_*^4}{M_{Pl}^4 \epsilon_V} \left(\prod_{i=1}^3 \frac{1}{2k_i^3} \right) \left[\frac{k_1^2 k_2^2 + k_2^2 k_3^2 + k_1^2 k_3^2}{K} + \right. \\ & \left. + \frac{k_1^2 k_2^2 k_3 + k_1^2 k_3^2 k_2 + k_2^2 k_3^2 k_1}{K^2} \right] . \end{aligned} \quad (3.76)$$

In this result the contribution of I_2 and I_3 sums up to zero when we take into consideration the permutation over the k_i 's.

Now, we are ready to pass to the 3-points function of ζ , which is the non linear generalization of ζ_1 . Infact, as said above, on superhorizon scales the non linear comoving curvature perturbation is link to the linear part ζ_1 by the relation (see Eq. (3.22))

$$\zeta = \zeta_1 + \alpha \zeta_1^2 , \quad (3.77)$$

where:

$$\alpha = \frac{1}{2} \frac{\ddot{\phi}}{\dot{\phi} H} + \frac{1}{4} \frac{\dot{\phi}^2}{H^2} \zeta_1^2 \simeq -\frac{1}{2} \eta_V . \quad (3.78)$$

Here \simeq means that α is evaluated at first order in the slow-roll parameters.

So, when we pass from the variable ζ_1 to the variable ζ through the field redefinition (3.77), then the 3-points function of the variable ζ has an additional contribution to the bispectrum which comes from this field redefinition. Thus, following the Refs. [9, 19], the bispectrum of ζ becomes equal to:

$$\begin{aligned} \langle \hat{\zeta}(\vec{k}_1) \hat{\zeta}(\vec{k}_2) \hat{\zeta}(\vec{k}_3) \rangle = & \langle \hat{\zeta}_1(\vec{k}_1) \hat{\zeta}_1(\vec{k}_2) \hat{\zeta}_1(\vec{k}_3) \rangle + 2\alpha (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \times \\ & \times \left(\mathcal{P}_\zeta(k_1) \mathcal{P}_\zeta(k_2) + \mathcal{P}_\zeta(k_2) \mathcal{P}_\zeta(k_3) + \mathcal{P}_\zeta(k_1) \mathcal{P}_\zeta(k_3) \right) , \end{aligned} \quad (3.79)$$

where $\mathcal{P}_\zeta(k) = \frac{H_*^2}{M_{Pl}^2 2\epsilon_V k^3}$ denotes the scalar superhorizon power spectrum of the mode k .

Then at the end, substituting Eq. (3.78) into Eq. (3.79), we get the final result:

$$\begin{aligned} \langle \hat{\zeta}(\vec{k}_1) \hat{\zeta}(\vec{k}_2) \hat{\zeta}(\vec{k}_3) \rangle &= (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \left(\sum_{i>j} \mathcal{P}_\zeta(k_i) \mathcal{P}_\zeta(k_j) \right) \times \\ &\times \left[-\eta + 2\epsilon_V \left(\frac{k_1^2 k_2^2 k_3 + k_1^2 k_3^2 k_2 + k_2^2 k_3^2 k_1}{K^2 \sum_i k_i^3} \right) + 2\epsilon_V \frac{\sum_{i>j} k_i^2 k_j^2}{K \sum_i k_i^3} \right]. \end{aligned} \quad (3.80)$$

From the computation just performed, we can derive the scalar bispectrum at leading order in the slow-roll parameters predicted by the standard single field models of slow-roll inflation. It is necessary to match the expressions (3.5) and (3.80) to find out [9]:

$$B_{\zeta\zeta\zeta}^{slow-roll}(k_1, k_2, k_3) = \left(\sum_{i>j} P_\zeta(k_i) P_\zeta(k_j) \right) \times \left[-\eta_V + 2\epsilon_V \left(\frac{k_1^2 k_2^2 k_3 + k_1^2 k_3^2 k_2 + k_2^2 k_3^2 k_1}{K^2 \sum_i k_i^3} \right) + 2\epsilon_V \frac{\sum_{i>j} k_i^2 k_j^2}{K \sum_i k_i^3} \right]. \quad (3.81)$$

The fractions that depend on the k_i 's in the square parenthesis are approximately of order $\mathcal{O}(1)$ due to momentum conservation [9]. Then we can take as a good approximation:

$$B_{\zeta\zeta\zeta}^{slow-roll}(k_1, k_2, k_3) \simeq \left(\sum_{i>j} P_\zeta(k_i) P_\zeta(k_j) \right) \times [4\epsilon_V - \eta_V]. \quad (3.82)$$

Now we want to match this result with the non-Gaussianities constrained by the *Planck satellite* in the CMB anisotropies. We notice that the expression (3.82) corresponds to the bispectrum of the local shape of Non-Gaussianity (see Eq. (3.12)). So matching Eq. (3.12) with Eq. (3.82) we predict²:

$$(f_{NL}^{local})_{slow-roll} \simeq \frac{10}{3} \epsilon_V - \frac{5}{6} \eta_V. \quad (3.83)$$

From the experimental constraints on the slow roll parameters (see Eqns.(2.157) and (2.158)), it follows:

$$(f_{NL}^{local})_{slow-roll} \lesssim 10^{-2}. \quad (3.84)$$

This value is very small and definitely compatible with the best constraint on local non-Gaussianity up to date provided by the Planck satellite, which we have exposed above in Eq. (3.17).

For the moment we have considered non-Gaussianities provided only by scalar perturbations. But we know that primordial gravitational waves γ_{ij}^s are unavoidably generated during inflation. Then, we expect contributions to non-Gaussianities also by them. So a priori it is not trivial to analyze also the correlators between graviton and scalar fluctuations and pure graviton correlators. We define these correlators as [9, 36]:

$$\langle 0 | \hat{\gamma}_{s_1}(\vec{k}_1) \hat{\zeta}(\vec{k}_2) \hat{\zeta}(\vec{k}_3) | 0 \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_{\gamma\zeta\zeta}(k_1, k_2, k_3), \quad (3.85)$$

²In reality this fact is not rigorous, because the shape of non-Gaussianity predicted by slow-roll models in general has a dependence on the momenta k_i 's different from the local shape. It turns out that $S(k_1, k_2, k_3) = (6\epsilon - 2\eta) S^{local}(k_1, k_2, k_3) + \frac{5}{3} \epsilon S^{equil}(k_1, k_2, k_3)$ (see, e.g, Ref. [35]). However the estimates given have aim to stress that f_{NL} is of the order of the slow-roll parameters.

$$\langle 0 | \hat{\gamma}_{s_1}(\vec{k}_1) \hat{\gamma}_{s_2}(\vec{k}_2) \hat{\zeta}(\vec{k}_3) | 0 \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_{\gamma\gamma\zeta}(k_1, k_2, k_3), \quad (3.86)$$

$$\langle 0 | \hat{\gamma}_{s_1}(\vec{k}_1) \hat{\gamma}_{s_2}(\vec{k}_2) \hat{\zeta}(\vec{k}_3) | 0 \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B_{\gamma\gamma\gamma}(k_1, k_2, k_3), \quad (3.87)$$

where $\hat{\gamma}_s = \hat{\gamma}^{ij}(\epsilon_{ij}^s)^*$ corresponds to graviton polarization s [36].

The computation of such correlators in the slow-roll models is similar to the explicit computation for the 3-scalar correlator. So we give directly the order of the final result referring to Ref. [9] for more details:

$$\langle 0 | \hat{\gamma}_{s_1}(\vec{k}_1) \hat{\zeta}(\vec{k}_2) \hat{\zeta}(\vec{k}_3) | 0 \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{H_*}{M_{Pl}^4} \frac{1}{\epsilon} \left(\sum_{i>j} \frac{1}{k_i^3 k_j^3} \right) M_1(k_i), \quad (3.88)$$

$$\langle 0 | \hat{\gamma}_{s_1}(\vec{k}_1) \hat{\gamma}_{s_2}(\vec{k}_2) \hat{\zeta}(\vec{k}_3) | 0 \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{H_*^4}{M_{Pl}^4} \left(\sum_{i>j} \frac{1}{k_i^3 k_j^3} \right) M_2(k_i), \quad (3.89)$$

$$\langle 0 | \hat{\gamma}_{s_1}(\vec{k}_1) \hat{\gamma}_{s_2}(\vec{k}_2) \hat{\gamma}_{s_3}(\vec{k}_3) | 0 \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{H_*^4}{M_{Pl}^4} \left(\sum_{i>j} \frac{1}{k_i^3 k_j^3} \right) M_3(k_i), \quad (3.90)$$

where the $M_n(k_i)$'s are dimensionless functions of the momenta k_i of order $\mathcal{O}(1)$. We refer to the original Ref. [9] for explicit expressions of the functions $M_n(k_i)$.

We see that the correlator (3.88) is of the same order in the slow-roll parameters of the 3-scalar correlator (3.80). Instead, the other two correlators (3.89), (3.90) are subdominant in the slow-roll parameters w.r.t. it. Actually we cannot measure directly the graviton non-Gaussianities because we have not revealed primordial gravitational waves yet. But we think that scalar-graviton correlators may give contributions to the CMB bispectrum. The development of the techniques for measuring these correlators from the CMB is an important aim of future analysis.

3.4 Motivation for searching for Modified gravity signatures during inflation

To summarize, in the previous section we have seen that the theoretical bispectrum predicted by slow-roll models of inflation is of the order of the slow-roll parameters. This is fully consistent with what has been measured by the Planck satellite. However we notice that there is still a window of almost two orders of magnitude unexplored, given the present susceptibility to primordial non-Gaussianities. For this reason, it is not trivial to modificate the theory in order to search for signatures of non-Gaussianities. For example we can assume that at the high energies, that are the ones of the primordial universe, the general relativity has no more the exact Einstein description, which instead has many proofs of correctness at low energies. In particular, following an effective field theory approach [37, 38], a starting point for achieving modified gravity terms in the lagrangian of the theory is admitting all the covariant terms built with the contractions of tensors up to two derivatives of the fields of the theory, that are the inflaton ϕ and the metric tensor $g_{\mu\nu}$. This lagrangian reads:

$$\begin{aligned}
\mathcal{L} = & \sqrt{g} \left[\frac{1}{2} M_{Pl}^2 f_1(\phi)^2 R - \frac{1}{2} f_2(\phi) g^{\mu\nu} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right. \\
& + f_3(\phi) \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right)^2 + f_4(\phi) g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \square \phi \\
& + f_5(\phi) \left(\square \phi \right)^2 + f_7(\phi) R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \\
& + f_8(\phi) R g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + f_9(\phi) R \square \phi + f_{10}(\phi) R^2 \\
& \left. + f_{11}(\phi) R^{\mu\nu} R_{\mu\nu} + f_{12}(\phi) C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \right] \\
& + f_{13}(\psi) \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu}{}^{\kappa\lambda} C_{\rho\sigma\kappa\lambda}, \tag{3.91}
\end{aligned}$$

where $\square = g_{\mu\nu} \partial^\mu \partial^\nu$ denotes the covariant laplacian and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, which is the traceless part of the Riemann tensor. In formula:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2} (g_{\mu\rho} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\rho} - g_{\nu\rho} R_{\mu\sigma} + g_{\nu\sigma} R_{\mu\rho}) + \frac{R}{6} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}). \tag{3.92}$$

In the first line of the lagrangian (3.91) we recognize the standard lagrangian of the slow-roll theories of inflation (2.1) apart from the presence of the functions $f_1(\phi)$ and $f_2(\phi)$. In reality, in the slow-roll limit, we expect that these functions can be treated as constants up to slow-roll corrections which we can neglect. All other terms are corrective terms which come from an expansion in series of the derivative of the fields ϕ and $g_{\mu\nu}$. Infact in the first line we have terms with only two derivatives of the fields. Instead all the other terms have four derivative of the fields. A priori we can consider other terms with six derivatives of the fields and so on, but we stop the expansion up to terms with four derivatives in the fields.

In the fourth line of this lagrangian a term particular interesting for producing non-Gaussianities is $f_{10}(\phi)R^2$, which is part of the so-called $f(R)$ theories of modified gravity. In this theories the part of the lagrangian which describes gravity is built by considering an expansion in series of the scalar curvature R . The first correction term in this expansion corresponds to the theory $f(R) = R + \alpha R^2$. Inflation with $R + \alpha R^2$ gravity is studied for example in the Refs. [39, 37]. In particular in Ref. [37] is showed that the so-called $R + \alpha R^2$ theory is equivalent to add an additional scalar field during inflation which, interacting with the inflaton, produces a quasi-local shape of non-Gaussianity with amplitude $f_{NL} \simeq (-1 \text{ to } -30)$. We will see more explicitly this fact in Chapter 4, because it is an interesting toy model for producing non-Gaussianities in the primordial perturbations during inflation. Another term quite interesting in the lagrangian (3.91) is the last one, that is the so-called Chern-Simons term. This term is parity breaking because of the presence of the Levi-Civita pseudotensor which contracts two Weyl tensors. For this reason it can be the source of parity breaking effects in the statistics of the primordial perturbations. We will see in detail the effect of this term in the primordial perturbations in Chapter 4. We anticipate that without the presence of the function $f_{13}(\phi)$ this term is vanishing because it is a total time derivative. So, if the inflaton field was completely static, then this term would give a zero contribution in the action. In Chapter 4 we will perform a computation of non-Gaussianities provided by this term, in order to search if it is possible to achieve a non-zero parity violating contribution in non-Gaussianities of the primordial perturbations. Examples of other searchings of parity violating effects on non-Gaussianities are provided by Refs. [36, 40, 41], where possible parity violation in graviton non-Gaussianities is studied arising by the Weyl cubic tensor contracted with the Levi-Civita pseudotensor. However

in this case non-Gaussianities predicted are well below the sensitivity of future measurements and need the development of new techniques of investigation.

Chapter 4

Searching for modified gravity signatures: the Chern-Simons gravity and parity breaking on the primordial perturbations

In this chapter we modify the standard action of the slow-roll models (2.1) accounting for some terms that introduce a modification in the gravity sector w.r.t. Einstein gravity. These terms essentially come from an expansion in the derivatives of the metric tensor [38]. In particular we start analyzing some $f(R)$ models of inflation following Ref. [37], which is an example of how a modification of gravity during inflation can be probed via primordial non-Gaussianity. After this, we concentrate on the *Chern-Simons term* which violates the parity symmetry. We study if we are able to see a parity violation in the power spectrum of the gauge invariant primordial perturbations following Refs. [11, 25]. Then, using the in-in formalism, we make a computation of the two gravitons and one scalar correlator $\langle \gamma\gamma\varphi \rangle$ produced by the Chern-Simons term. We investigate about the possibility to have a signature of parity violating effects into the bispectrum of the gauge invariant primordial perturbations.

4.1 Modifying gravity with higher derivative terms

When we consider a modification of the Einstein theory during inflation, we have to insert in the action terms that become negligible in the low-energy limit where we know the Einstein description perfectly works. On the contrary, at the beginning of inflation and during inflation the relevant energies can be the temperatures were very high and so we expect that terms that in the present universe are negligible, during the inflationary epoch are not. But in which way we can construct such terms?

The Einstein-Hilbert action is built by admitting covariant terms with a maximum number of two derivatives w.r.t. the metric tensor. In this way the only term admitted is the scalar curvature R . We can relax this condition and consider a more general theory of gravity in which the action is built with an expansion in series of covariant terms that contain an increasing number of derivatives w.r.t. the metric. In order to recover Einstein description in the low-energy limit, it is necessary to require that to each derivative of the new terms corresponds a factor equal to the inverse of some large mass scale [38]. In the primordial universe this mass is the Planck mass M_{Pl} .

In this work we analyze the first correction to the lagrangian which comes from such an expansion in derivatives of the metric tensor. Then, we have to consider the most general covariant terms with

four¹ derivatives of the metric tensor. We build these terms doing tensorial contractions between two tensors with two derivatives of the metric². So we have to do all the possible contractions between two of the fundamental curvature tensors of General Relativity, that are the Riemann tensor $R_{\mu\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R . In fact they are the only three independent tensors that contain two derivatives of the metric. In addition we can multiply these terms for a scalar function which depends by the inflaton field ϕ , which is the field that drives inflation. Thus the most general expression for the additional lagrangian we want to focus on is:

$$\Delta\mathcal{L} = \sqrt{g}[f_1(\phi)R^2 + f_2(\phi)R_{\mu\nu}R^{\mu\nu} + f_3(\phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}] + f_4(\phi)\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{\kappa\lambda}R_{\rho\sigma\kappa\lambda}. \quad (4.1)$$

where the $f_n(\phi)$ are some dimensionless coefficients depending on the inflaton only, $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita antisymmetric pseudo-tensor with $\epsilon^{0123} = 1$ and \sqrt{g} stands for $\sqrt{-\det(g_{\mu\nu})}$, which is part of the covariant integration measure in the action.

We can write lagrangian (4.1) in terms of the Weyl tensor which we have defined at the end of Chapter 3. We rewrite the definition of the Weyl tensor, which is the traceless part of the Riemann tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}) + \frac{R}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}), \quad (4.2)$$

It is possible to demonstrate the relation (see Ref. [42]):

$$\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu}{}^{\kappa\lambda}R_{\rho\sigma\kappa\lambda} = \epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda}. \quad (4.3)$$

In addition by doing the contraction between two Weyl tensors using the definition (4.2), we find an expression of the kind:

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2, \quad (4.4)$$

where α and β are numerical coefficient that we find out with a direct computation.

For what said, if we define

$$g_1(\phi) = f_1(\phi) - f_3(\phi)\beta, \quad (4.5)$$

$$g_2(\phi) = f_2(\phi) - f_3(\phi)\alpha, \quad (4.6)$$

$$g_3(\phi) = f_3(\phi), \quad (4.7)$$

then lagrangian (4.1) becomes:

$$\Delta\mathcal{L} = \sqrt{g}[g_1(\phi)R^2 + g_2(\phi)R_{\mu\nu}R^{\mu\nu} + g_3(\phi)C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}] + g_4(\phi)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda}. \quad (4.8)$$

By renaming the functions $g_n(\phi)$ as $f_n(\phi)$, we find the expression:

$$\Delta\mathcal{L} = \sqrt{g}[f_1(\phi)R^2 + f_2(\phi)R_{\mu\nu}R^{\mu\nu} + f_3(\phi)C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}] + f_4(\phi)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda}. \quad (4.9)$$

¹We are not able to construct a term with three derivatives with respect to the metric and covariant under diffeomorphisms of the metric.

²In fact if we consider a tensorial contraction between a tensor with one derivative and another one with three derivatives, this contraction is automatically zero for the covariance principle.

At the end, in the lagrangian density (4.9) we recognize four different modified gravity corrective terms. As anticipated in Chapter 3, the first term belongs to the so-called $f(R)$ theories, which are very studied in the literature (see, e.g., Refs. [43, 44, 45]). These are modified gravity theories in which we insert in the lagrangian an expansion in series of the scalar curvature R with the condition that the first term in this expansion is the Einstein-Hilbert action of standard gravity. Thus for these kinds of theories the density lagrangian of slow-roll models of inflation modifies as:

$$\mathcal{L} = \sqrt{g} \left[f(R) - \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi - V(\phi) \right]. \quad (4.10)$$

As said, in Ref. [37] this kind of slow-roll model of inflation is analyzed in details, in particular concerning the case $f(R) = R + \alpha R^2$. In the next section we will recall briefly the main results following the original reference because it is a first useful example to see how a modified gravity term can produce primordial non-Gaussianity.

4.2 $f(R)$ theories: quasi-local non gaussianity from $R + \alpha R^2$ modified gravity during inflation

We rewrite down the lagrangian of the $f(R)$ theories during slow-roll inflation (4.10):

$$\mathcal{L} = \sqrt{g} \left[f(R) - \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi - V(\phi) \right], \quad (4.11)$$

We demonstrate now that the term $f(R)$ corresponds in the theory to one additional scalar degree of freedom. It is enough to expand in Taylor $f(R)$ near a value $R = \chi$ and stop the expansion at first order. This expansion reads:

$$f(R) = f(\chi) + f'(\chi)(R - \chi), \quad (4.12)$$

where the ' denotes the derivative with respect to the argument.

Thus inserting the expansion (4.12) into the lagrangian (4.11), yields:

$$\mathcal{L} = \sqrt{g} \left[f(\chi) + f'(\chi)(R - \chi) - \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi - V(\phi) \right]. \quad (4.13)$$

Now we define the auxiliary field ψ :

$$\psi = \frac{2f'(\chi)}{M_{Pl}^2}. \quad (4.14)$$

Inserting the definition (4.14) into the lagrangian (4.13), the lagrangian becomes:

$$\mathcal{L} = \sqrt{g} \left[\frac{1}{2} M_{Pl}^2 \psi R + \Lambda(\psi) - \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi - V(\phi) \right], \quad (4.15)$$

where $\Lambda(\psi) = f(\chi(\psi)) - M_{Pl}^2 \psi \frac{\chi}{2}$.

Now, in order to isolate an Einstein-Hilbert term into the lagrangian (4.15), we perform a conformal Weyl transformation of the metric tensor $g'_{\mu\nu} = \psi g_{\mu\nu}$. After doing this rescaling the lagrangian (4.15) now becomes of the type:

$$\mathcal{L} = \sqrt{g} \left[\frac{1}{2} M_{Pl}^2 R - \frac{1}{2} g^{\mu\nu} \gamma_{ab} D_\mu \phi^a D_\nu \phi^b - V(\phi_1, \phi_2) \right], \quad (4.16)$$

where ϕ_1 and ϕ_2 are two scalar fields defined as:

$$\phi_1 = \sqrt{6} M_{Pl} \omega, \quad \phi_2 = \phi, \quad (4.17)$$

$V(\phi_1, \phi_2)$ is a two-field potential of the form

$$V(\phi_1, \phi_2) = e^{\frac{-4\phi_1}{\sqrt{6}M_{Pl}}} V(\phi_2) + U(\phi_1), \quad (4.18)$$

where $U(\phi_1) = -e^{\frac{-4\phi_1}{\sqrt{6}M_{Pl}}} \Lambda(\psi(\omega(\phi_1)))$, and finally γ_{ab} is the field metric:

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{-2\phi_1}{\sqrt{6}M_{Pl}}} \end{pmatrix}. \quad (4.19)$$

This shows that an $f(R) + scalar$ theory (where the scalar in our context is the inflaton field) is equivalent to a two scalar field model of inflation with a specific field metric and a two field-potential $V(\phi_1, \phi_2)$. Then we can imagine that some interaction terms between the two scalar fields can produce some observable effects, possibly leading to the production of primordial local non-Gaussianity signatures which we have seen in Chapter 3. In particular both the scalar fields participate to the background dynamics and so a priori we have to impose a slow-roll condition over both the fields. But if we impose that one of the two is subdominant during inflation, then we can relax the slow-roll condition at least for only one of the two scalar fields, leaving the other scalar field a free background dynamics.

In Ref. [37] the case of the theory $f(R) = \frac{1}{2} M_{Pl}^2 R + \frac{1}{12M^2} R^2$ is analyzed in details. In this case the two fields potential (4.18) reduces to:

$$V(\phi_1, \phi_2)_{R^2} = \frac{3}{4} M^2 M_{Pl}^4 e^{\frac{-4\phi_1}{\sqrt{6}M_{Pl}}} V(\phi_2) + U(\phi_2). \quad (4.20)$$

After choosing which of the two fields is associated to the inflaton (ϕ_I) and which one is associated to the "extra" field derived from the modifications of the Einstein gravity (ϕ_G), we can compute the interaction vertices between the perturbations of the two fields, neglecting, for simplicity, the interactions with the perturbations of the metric tensor. This is obtained splitting the two fields as

$$\phi_I = \phi_I^{(0)} + \delta\phi_I, \quad (4.21)$$

$$\phi_G = \phi_G^{(0)} + \delta\phi_G, \quad (4.22)$$

where the the suffix 0 denotes the background values and the δ denotes the perturbations.

Substituting these decompositions into the lagrangian (4.15), we find at second order a leading vertex of the kind [37]:

$$\delta\mathcal{L}_2 = \frac{2}{\sqrt{6}M_{Pl}} e^{\frac{-2\phi_G}{\sqrt{6}M_{Pl}}} \dot{\phi}_I^{(0)} \delta\phi_G \dot{\delta}\phi_I. \quad (4.23)$$

Instead, at third order, we find the leading vertex:

$$\delta\mathcal{L}_3 = -\frac{1}{6}U'''(\phi_I^{(0)})\delta\phi_G^3. \quad (4.24)$$

Taking only the multiplicative coefficients of each of these two vertices, it is possible to give an estimate of a quasi-local³ coefficient of primordial non-Gaussianity:

$$f_{NL}^{quasi-local} \simeq \overline{\delta\mathcal{L}_2} \overline{\delta\mathcal{L}_3} \mathcal{P}_\zeta^{-1/2}, \quad (4.25)$$

where the overline denotes the coefficients of the corresponding vertices.

The results obtained in [37] reveal that the αR^2 theory can produce a nearly scale-invariant shape of quasi-local non-Gaussianity, with a level of non-Gaussianity of the order $f_{NL} \approx -(1 \text{ to } 30)$. Thus this analysis is an example of how modification of gravity during inflation can be probed through the potential non-Gaussian signatures they produce.

4.3 Chern-Simons modified gravity during inflation

Now let us analyze another very interesting term which is present in the lagrangian (4.9). This term is the *Chern-Simons term*:

$$\Delta\mathcal{L} = f(\phi)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda}. \quad (4.26)$$

Before computing in details the modifications provided in the slow-roll models of inflation by this term, we analyze some general features. From arguments of differential geometry [38], if we take a metric conformally flat⁴, then the corresponding Weyl tensor is zero. If we take the flat FRW metric (2.3) and we do a time reparametrization passing from cosmological time t to the conformal time τ we find:

$$ds^2 = a^2[-d\tau^2 + h_{ij}dx^i dx^j]. \quad (4.27)$$

From (4.27) we see that with a conformal transformation $g'_{\mu\nu} = a^{-2}g_{\mu\nu}$, the metric becomes flat in every point of the spacetime. We have demonstrated that the FRW metric is conformally flat. Because of the fact that the FRW metric is the background metric of the universe, then the Weyl tensor is vanishing on the background. Thus, the term $\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda}$ does not act on the background dynamics. In addition this term is parity violating due to the presence of the pseudo Levi-civita tensor $\epsilon^{\mu\nu\rho\sigma}$.

Thus the analysis of the Chern-Simons term could reveal a source of signatures that do not modify the slow-roll background dynamics while leading to parity violating effects at the level of primordial perturbations.

For simplicity of notation we will use the following abbreviation to refer to the Chern-Simons term:

$$f(\phi)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda} = fC\tilde{C}. \quad (4.28)$$

³Quasi-local means that the characteristic shape of this kind of primordial non-Gaussianity is intermediate between the local shape and the equilateral shape seen in Chapter 3.

⁴A conformally flat metric is such that it can be transformed in a flat Minkowski metric by a local conformal transformation for all the points of the spacetime.

We want to develop a general theory and so we will leave the function $f(\phi)$ undefined. We start with a quadratic analysis of the term (4.28), making prediction on how it changes the power spectrum of the primordial gauge invariant perturbations. But the main issue is to perform also an investigation about possible non-Gaussian signatures in the primordial gauge invariant perturbations.

4.4 Quadratic analysis of the Chern-Simons term

The Chern-Simons term does not produce any change in scalar perturbations (in the sense that the contribution to the Chern-Simons term from scalar perturbations vanishes). In fact, if we take a scalar field such as the inflaton, the background FRW metric and the scalar perturbation of the metric, and we try to build parity breaking terms we fail from the beginning. In fact both the background metric and the inflaton are parity invariant and we cannot construct parity breaking terms with only parity invariant ones. On the contrary the gravitational waves γ_{ij} are sensitive to parity transformations and for this reason in general the term (4.28) is not zero for tensorial modes. In order to find if we have contributions we have to compute the term $(fC\tilde{C})|_T^{(2)}$, where the suffix 2 denotes that the quantity is at second order in the perturbations, and T denotes that it is the contribution of tensor perturbations only. In studying the tensor perturbations we can put $N = 1$, $N_i = 0$ because at first order they depend only by the inflaton. Moreover, because of the fact that, as said above, the Weyl tensor is vanishing on the background, thus the only contribution to the term which we want to compute will result of the form $f^{(0)}C^{(1)}|_T\tilde{C}^{(1)}|_T$. So we need to compute the Weyl tensor only at first order in the tensor perturbations.

Now we start to compute the components of the Weyl tensor at first order in the tensor perturbations. We adopt the background metric written in terms of the conformal time τ (4.27) for the semplicity of the computations. We can use the 3-metric h_{ij} defined by both Eqns. (3.20) and (3.21) without affecting the final results because, as said, tensor perturbations on large scales are gauge invariant. Thus, using the ADM relations in the Appendix A and Eq. (4.2), we find:

$$C_{0i0j}^{(1)}|_T = -\frac{a^2}{4}[\gamma''_{ij} + \partial^2\gamma_{ij}], \quad (4.29)$$

$$C_{0ijk}^{(1)}|_T = -\frac{a^2}{2}[\partial_j\gamma'_{ik} + \partial_k\gamma'_{ij}], \quad (4.30)$$

$$C_{ijkl}^{(1)}|_T = \frac{a^2}{2}[-\partial_i\partial_k\gamma_{jl} + \partial_i\partial_l\gamma_{jk} + \partial_j\partial_k\gamma_{il} - \partial_j\partial_l\gamma_{ik}] + \frac{1}{4}[\delta_{ik}\square\gamma_{jl} - \delta_{il}\square\gamma_{jk} - \delta_{jk}\square\gamma_{il} + \delta_{jl}\square\gamma_{ik}], \quad (4.31)$$

where in this case $\square \equiv [-\frac{d^2}{d\tau^2} + \partial^2]$ and $'$ denotes derivative in respect to the conformal time. As above, the suffix in the round bracket denotes the order in the perturbations and the suffix at the base denotes the type of perturbations considered in the computation of the term. We will use this notation also below.

If we substitute these expressions in the term $fC\tilde{C}$, we find at quadratic level:

$$\Delta\mathcal{L}|_T = f(\phi_0)\epsilon^{ijk}\frac{\partial}{\partial\tau}\left[(\gamma_{il})'(\partial_j\gamma_k^l)' - (\partial_r\gamma_{il})(\partial_j\partial^r\gamma_k^l)\right]. \quad (4.32)$$

So, at quadratic level, the Chern-Simons term provides the following modification to the action of the tensor modes:

$$\Delta S|_T = \int d^4x f(\phi_0) \epsilon^{ijk} \frac{\partial}{\partial \tau} \left[(\gamma_{il})' (\partial_j \gamma_k^l)' - (\partial_r \gamma_{il}) (\partial_j \partial^r \gamma_k^l) \right], \quad (4.33)$$

where the latin contractions are made with the δ_{ij} . From this last expression we see a fact that we have anticipated also in Chapter 3. If the function $f(\phi)$ was time independent, the Chern-Simons term would be a total time derivative in the action and so it would be vanishing. So the background slow-roll dynamics of the inflaton field is crucial to have a non-vanishing contribution.

The action (4.33) becomes, after integrating by parts the conformal time derivative:

$$\Delta S|_T = - \int d^4x \epsilon^{ijk} f'(\phi_0) [(\gamma_{il})' (\partial_j \gamma_k^l)' - (\partial_r \gamma_{il}) (\partial_j \partial^r \gamma_k^l)]. \quad (4.34)$$

Now we go to Fourier space through the expansion:

$$\gamma_{ij}(\vec{x}, \tau) = \frac{1}{(2\pi)^3} \int d^3k d\tau \sum_{s=s_1, s_2} \epsilon_{ij}^s(\vec{k}) \gamma^s(\tau, k) e^{i\vec{k}\cdot\vec{x}}. \quad (4.35)$$

In order to investigate parity violating effects it is convenient to work with the *left and right circular polarizations* of the gravitational waves. Their respective polarization tensors are a complex superposition of the linear polarizations introduced in (2.95). We define them as:

$$\epsilon_{ij}^R = \frac{1}{\sqrt{2}} (\epsilon_{ij}^+ + i\epsilon_{ij}^\times), \quad (4.36)$$

$$\epsilon_{ij}^L = \frac{1}{\sqrt{2}} (\epsilon_{ij}^+ - i\epsilon_{ij}^\times). \quad (4.37)$$

From these definitions it follows $\epsilon_{ij}^R = (\epsilon_{ij}^L)^*$. Then the two polarizations are complex conjugates. In addition, using the explicit expressions of the linear polarization tensors (2.95), we can show the relations [25]:

$$\epsilon_{ij}^L(\vec{k}) \epsilon_L^{ij}(\vec{k}) = \epsilon_{ij}^L(\vec{k}) \epsilon_L^{ij}(\vec{k}) = 0, \quad (4.38)$$

$$\epsilon_{ij}^L(\vec{k}) \epsilon_R^{ij}(\vec{k}) = 2, \quad (4.39)$$

$$\epsilon_{ij}^L(-\vec{k}) = \epsilon_{ij}^R(\vec{k}) \quad (4.40)$$

$$\frac{k^l}{k} \epsilon_{ml}^j \epsilon_{ij}^s(\vec{k}) = i \alpha^s \epsilon_{im}^s(\vec{k}), \quad (4.41)$$

where k_l is the l -th component of the momentum \vec{k} . Here we have not to make confusion between the Levi-Civita pseudotensor ϵ_{ijk} which has three indices and the polarization tensors ϵ_{ij} that have only two indices. In addition we have $\alpha^R = +1$ and $\alpha^L = -1$.

Now, we substitute the Fourier decomposition (4.35) in (4.34) to find

$$\Delta S|_T = - \sum_{s_1, s_2=L, R} \int d\tau \frac{d^3k}{(2\pi)^3} \epsilon_{ijl} f'(\phi) (-ik_j) \left[(\gamma^{s_1}_{in}(\vec{k}))' (\gamma^{s_2}_{in}(\vec{k}))' + k^2 \gamma^{s_1}_{in}(\vec{k}) \gamma^{s_2}_{in}(\vec{k}) \right]. \quad (4.42)$$

However using Eqns. (4.38) (4.39) and (4.41), we can rewrite this last equation in the more convenient form:

$$\Delta S|_T = - \sum_{s=L,R} 2 \int d\tau \frac{d^3k}{(2\pi)^3} f'(\phi) \alpha_s k \left[|\gamma'_s(\tau, k)|^2 - k^2 |\gamma_s(\tau, k)|^2 \right]. \quad (4.43)$$

If we express also the action of the standard theory (2.100) in function of left and right polarizations we find:

$$S|_T = \sum_{s=L,R} \frac{M_{pl}^2}{4} \int d\tau \frac{d^3k}{(2\pi)^3} a^2 \left[|\gamma'_s(\tau, k)|^2 - k^2 |\gamma_s(\tau, k)|^2 \right]. \quad (4.44)$$

Then, if we put Eqns. (4.43) and (4.44) together, we find the new quadratic action for the tensor perturbations, which reads:

$$S|_T^{tot} = \sum_{s=L,R} \frac{M_{pl}^2}{4} \int d\tau \frac{d^3k}{(2\pi)^3} A_{T,s}^2 \left[|\gamma'_s(\tau, k)|^2 - k^2 |\gamma_s(\tau, k)|^2 \right], \quad (4.45)$$

where now

$$A_{T,s}^2 = a^2 \left(1 - 8\alpha^s \frac{k \dot{f}(\phi)}{a M_{pl}^2} \right). \quad (4.46)$$

For right modes $\alpha^R = +1$ and then there are some values of the physical wave number $k_{phys} = \frac{k}{a}$ in which $A_{T,s}^2$ becomes negative. In particular, from (4.46), this happens for $k_{phys} > \frac{M_{pl}^2}{8\dot{f}(\phi)}$. We define then the Chern-Simons mass scale $M_{C-S} = \frac{M_{pl}^2}{8\dot{f}(\phi)}$. The right modes with physical wave numbers larger than the M_{C-S} acquire a negative kinetic energy and become automatically ghost fields. In a classical field theory this could be not a problem, but when we quantize the fields, the ghost fields states have negative norm and so they create inconsistencies in defining the Hilbert space of the theory. In order to avoid this problem, we have to require to work with modes that are far from the formation of ghosts.

So, we set in the theory an UV cut-off at an energy $\Lambda < M_{C-S}$. This cut-off automatically regularizes the theory.

After introducing the cut-off, we rewrite $A_{T,s}^2$ in a more compact way as:

$$A_{T,s}^2 = a^2 \left(1 - \alpha^s \frac{k}{a \Lambda} \Omega \dot{f}(\phi) \right), \quad (4.47)$$

where

$$\Omega = \frac{\Lambda}{M_{C-S}}. \quad (4.48)$$

From its definition $|\Omega| < 1$.

We expect that M_{C-S} may be almost high but finite. In addition it can be considered also constant. Infact from definition:

$$\dot{f}(\phi_0) = \left[\frac{\partial}{\partial \phi} f(\phi_0) \right] \dot{\phi}_0, \quad (4.49)$$

and its time derivative is

$$\ddot{f}(\phi_0) = \left[\frac{\delta^2}{\delta^2 \phi} f(\phi_0) \right] \dot{\phi}_0^2 + \left[\frac{\delta}{\delta \phi} f(\phi_0) \right] \ddot{\phi}_0 . \quad (4.50)$$

Because of the fact that M_{C-S} depends by the time only through the function $f(\phi)$, it follows from Eqns. (4.49) and (4.50):

$$\frac{\ddot{M}_{C-S}}{H\dot{M}_{C-S}} = \frac{\ddot{f}}{H\dot{f}} = \epsilon_V - \eta_V + \sqrt{2\epsilon_V} M_{Pl} \frac{\frac{\partial^2}{\partial^2 \phi} f(\phi)}{\frac{\partial}{\partial \phi} f(\phi)} , \quad (4.51)$$

where we have used the definition of the slow-roll parameters (Eqns. (2.20) and (2.21)).

Essentially the dimensionless quantity $\frac{\ddot{M}_{C-S}}{H\dot{M}_{C-S}}$ in Eq. (4.51) tells us how much the Chern-Simons mass changes during the characteristic time of inflation. In the slow-roll models of inflation the slow-roll parameters are much smaller than 1. Thus, if we impose the condition

$$M_{Pl}^2 \frac{\partial^2}{\partial^2 \phi} f(\phi) < M_{Pl} \sqrt{2\epsilon_V} \frac{\partial}{\partial \phi} f(\phi) , \quad (4.52)$$

it follows:

$$\frac{\ddot{M}_{C-S}}{H\dot{M}_{C-S}} < 1 . \quad (4.53)$$

So we can neglect in first approximation the time dependence of the Chern-Simons mass, considering it as a constant during inflation.

Now, we derive the equations of motion for the fields γ_s . As done in Chapter 2, before doing the functional derivatives, it is convenient to make the field redefinition

$$\mu_s = A_{T,s} \gamma_s . \quad (4.54)$$

Then the action for the new fields μ_s becomes:

$$S_{TOT}^{\gamma\gamma} = \frac{M_{pl}^2}{4} \int d\tau d^3k \left[|\mu'_s(\tau, k)|^2 - k^2 |\mu_s(\tau, k)|^2 + \frac{A''_{T,s}}{A_{T,s}} |\mu_s(\tau, k)|^2 \right] . \quad (4.55)$$

Again we find an equation of motion similar to that of scalar fields with an effective mass $\frac{A''_{T,s}}{A_{T,s}}$ which is different by the one found in the standard slow-roll case. We derive the equations of motion finding

$$\mu''_s + \left(k^2 - \frac{A''_{T,s}}{A_{T,s}} \right) \mu_s = 0 . \quad (4.56)$$

We see that the equations of motion are different for the two polarizations L and R , because the effective mass depends by the polarization (see Eq. (4.47)). We expect then a different dynamical evolution of the two polarization modes, which signals parity violation.

The differential equations (4.56) in general are not solvable for the explicit value of $\frac{A''}{A_T}$, but we can simplify a bit the effective mass using some facts of our theory.

We start the computation using the slow-roll hypothesis and assuming Ω constant. Infact from Eq. (4.48), Ω depends by the time only by M_{C-s} , which we assume almost constant during inflation. So at first order in slow-roll parameters we have:

$$\frac{A'_{T,s}}{A_{T,s}} = \frac{a \left(\dot{A}_{T,s}^2 \right)}{2 A_{T,s}^2} = aH + \alpha_s \frac{k\Omega}{2M_C} H \simeq -\frac{1+\epsilon}{\tau} + \alpha_s \frac{k\Omega H}{2\Lambda}. \quad (4.57)$$

Now, we notice that the dimensionless quantity $\frac{\Omega}{\Lambda}H$ is smaller than 1. In fact $\Omega < 1$ by definition and $\frac{H}{\Lambda} < 1$; H represents also the charateristic energy of the universe and once we impose a cut off at some energy in the full theory, H cannot overtake it for the autoconsistency of the theory. So in the following calculations we can perform an expansion in $\frac{\Omega H}{\Lambda}$ up to first order.

$$\frac{A''_{T,s}}{A_{T,s}} = \frac{d}{d\tau} \left(\frac{A'_{T,s}}{A_{T,s}} \right) + \left(\frac{A'_{T,s}}{A_{T,s}} \right)^2 \simeq \frac{2+3\epsilon}{\tau^2} - \frac{1}{\tau} \alpha_s \frac{k\Omega H}{2\Lambda} + \mathcal{O}(\epsilon^2, \Omega^2, \epsilon \cdot \Omega). \quad (4.58)$$

Then, calling $\xi = \frac{3}{2} + \epsilon$, and inserting Eq. (4.58) into Eq. (4.56), we have up to first order in the slow roll parameters and $\frac{\Omega H}{\Lambda}$

$$\mu''_s + \left(k^2 - \frac{\xi^2 - \frac{1}{4}}{\tau^2} + \alpha_s \frac{k\Omega H}{\tau\Lambda} \right) \mu_s = 0. \quad (4.59)$$

We notice that this equation differs from the one in Chapter 2 (see Eq. (2.106)) by the additional term $\alpha_s \frac{k\Omega H}{\tau\Lambda}$ which is the correction to the equation of motion for the tensor modes provided by the Chern-Simons term.

Now, as done in Chapter 2, we canonically quantize the fields $\mu_{L,R}$ as:

$$\hat{\mu}_s(k, \tau) = \tilde{z}_r(k, \tau) \hat{a}_r(\vec{k}) + \tilde{z}_r^*(k, \tau) \hat{a}_r^\dagger(-\vec{k}), \quad (4.60)$$

where the creation and annihilation operators obey the usual relations:

$$\langle 0 | \hat{a}_s^\dagger = 0, \quad \hat{a}_s | 0 \rangle = 0, \quad (4.61)$$

$$[\hat{a}_s(k), \hat{a}_{s'}^\dagger(k')] = (2\pi)^3 \delta^3(k - k') \delta_{ss'}, \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0. \quad (4.62)$$

Because of the fact that the field μ_s is linked to the field γ_s by the linear relation (4.54), thus quantizing μ_s is equivalent to quantize also γ_s . From the point of view of the quantum gravity the right-handed field $\hat{\gamma}_R(\vec{k})$ is associated to a graviton with helicity +2. On the contrary the left-handed field $\hat{\gamma}_L(\vec{k})$ is associated to a graviton with helicity -2. The helicity is the component of the spin of a particle in the direction of its momentum. At this point we should emphasize that a parity trasformation changes the sign of the momentum of a particle and so the left-handed gravitons trasform into right-handed gravitons and viceversa. Then, if we insert in the theory a parity violating term such as the Chern-Simons one, a difference between the dynamical evolution of the two different helicity states is expected. This is a confirmation of our computation.

After this small digression, we write the equation of motion for the mode functions $\tilde{z}_s(k, \tau)$ which is equal to the one of the classical μ_s :

$$\tilde{z}_s'' + \left(k^2 - \frac{\xi^2 - \frac{1}{4}}{\tau^2} + \alpha_s \frac{k \Omega H}{\tau \Lambda} \right) \tilde{z}_s = 0. \quad (4.63)$$

This is the Whittaker equation [46]. The exact solution of this equation is given in terms of the Whittaker functions W_1 and W_2 . If we choose the Bunch-Davies initial condition

$$\tilde{z}_s(k, \tau)_{\tau \rightarrow -\infty} \propto \frac{1}{\sqrt{2k}} e^{-ik\tau}, \quad (4.64)$$

then the explicit solution of (4.63) is [11]:

$$\tilde{z}_s(k, \tau) = \sqrt{-\tau} (-2k\tau) e^{-i(\frac{\pi}{4} - \pi\frac{\xi}{2})} e^{-ik\tau} U\left(\frac{1}{2} + \xi - \alpha_s \frac{i \Omega H}{2 \Lambda}, 1 + 2\xi, 2ik\tau\right) e^{\alpha_s \frac{\pi}{4} \frac{\Omega H}{\Lambda}}, \quad (4.65)$$

where U is the confluent hypergeometric function [46].

We observe from (4.65) that the two different circular polarizations have different solutions for the mode function \tilde{z}_s as we should expect. Now we will quantify how large the differences in the power spectrum are.

4.4.1 Power spectrum of circular polarizations

On superhorizon scales the solution (4.65) simplifies, becoming [11]:

$$\tilde{z}_s(k, \tau)_{-k\tau < 1} = \sqrt{\frac{-\tau}{2(-k\tau)^3}} e^{i(-\frac{\pi}{4} + \frac{\pi}{2}\nu_T)} \frac{\Gamma(\xi)}{\Gamma(3/2)} \left(\frac{-k\tau}{2}\right)^{3-2\xi} e^{\alpha_s \frac{\pi}{4} \frac{\Omega H}{\Lambda}}. \quad (4.66)$$

Thus we can compute the dimensionless power spectrum of each polarization mode, defined as:

$$\Delta_T^L = \langle 0 | \hat{\gamma}_{ij}^L(k) \hat{\gamma}_L^{ij}(k') | 0 \rangle = \frac{k^3}{(2\pi)^2} 2 \frac{|\tilde{z}_L(k, \tau)_{-k\tau < 1}|^2}{A_{T,L}^2}, \quad (4.67)$$

$$\Delta_T^R = \langle 0 | \hat{\gamma}_{ij}^R(k) \hat{\gamma}_R^{ij}(k') | 0 \rangle = \frac{k^3}{(2\pi)^2} 2 \frac{|\tilde{z}_R(k, \tau)_{-k\tau < 1}|^2}{A_{T,R}^2}. \quad (4.68)$$

At leading order in the slow-roll parameters we find

$$\Delta_T^L = \frac{\Delta_T}{2} e^{-\frac{\pi}{4} \frac{\Omega H}{\Lambda}}, \quad (4.69)$$

$$\Delta_T^R = \frac{\Delta_T}{2} e^{\frac{\pi}{4} \frac{\Omega H}{\Lambda}}, \quad (4.70)$$

where

$$\Delta_T = \frac{4}{\pi^2} \frac{H_*^2}{M_{Pl}^2} \left(\frac{-k\tau}{2}\right)^{3-2\nu_T}. \quad (4.71)$$

Δ_T is the total dimensionless power spectrum of tensorial perturbations in the standard slow-roll model without the Chern-Simons correction (see also Eq. (2.133)).

The dimensionless coefficient $\frac{H\Omega}{\Lambda}$ is almost time independent and smaller than 1. For this reason we can expand in series the exponentials. Dropping the expansion at third order it we find

$$\Delta_T^L = \frac{\Delta_T}{2} \left(1 - \frac{\pi \Omega H}{4 \Lambda} + \frac{\pi^2 \Omega^2 H^2}{16 \Lambda^2} \right), \quad (4.72)$$

$$\Delta_T^R = \frac{\Delta_T}{2} \left(1 + \frac{\pi \Omega H}{4 \Lambda} + \frac{\pi^2 \Omega^2 H^2}{16 \Lambda^2} \right). \quad (4.73)$$

Now we can define the relative difference between the power spectrum of right (R) and left (L) helicity modes as

$$\Theta_{R-L} = \frac{\Delta_T^R - \Delta_T^L}{\Delta_T^R + \Delta_T^L} = \frac{\pi \Omega H}{2 \Lambda} = \frac{\pi}{2} \frac{H}{M_{C-S}}. \quad (4.74)$$

This observable quantifies the differences between the power spectrum of the helicity polarizations L and R of the gravitational waves. We expect that its value is small for the consistency of the approximations made. We expect to constrain this value with future experiments involving the direct detection of polarized primordial gravitational waves (see, e.g., Refs [47, 48]). We can constrain the value Θ_{R-L} also indirectly through the CMB. In particular in Ref. [49] it is shown that Θ_{R-L} can be measured down to:

$$|\Theta_{R-L}| \gtrsim 0.35 \left(\frac{r}{0.05} \right)^{-0.6}. \quad (4.75)$$

If we take the maximum value of r from the constraint (2.137), then it follows:

$$|\Theta_{R-L}|_{r=0.12} \gtrsim 0.21, \quad (4.76)$$

which is an estimate of the minimum value of Θ_{R-L} that is detectable from the CMB given the sensitivity of the actual experimental instruments.

We can use (4.72) and (4.73) also to compute the modifications to the tensor-to-scalar-ratio. Infact the new total dimensionless power spectrum of tensor perturbations reads

$$\Delta_T^{C-S} = \Delta_T^R + \Delta_T^L = \Delta_T \left(1 + \frac{\pi^2 \Omega^2 H^2}{16 \Lambda^2} \right) = \Delta_T (1 + \Theta_{R-L}^2). \quad (4.77)$$

Instead, the dimensionless scalar power spectrum does not receive any contribution due to parity simmetry of the scalar perturbations. Thus we find

$$r_{C-S} = \frac{\Delta_T^{C-S}}{\Delta_S} = \frac{\Delta_T}{\Delta_S} (1 + \Theta_{R-L}^2) = r (1 + \Theta_{R-L}^2), \quad (4.78)$$

where r is the tensor-to-scalar-ratio of the slow-roll model without the Chern-Simons term. In addition, because of the fact that the parameter Θ_{R-L} does not depend on the comoving wavenumber of the perturbation in exam, then the spectral index of tensorial perturbations n_T remains the same of the standard slow-roll model. For this reason we have a modification to the consistency relation of the kind:

$$r_{C-S} = -8n_T \left(1 + \Theta_{R-L}^2 \right), \quad (4.79)$$

where we have used Eqns. (2.143) and (4.78).

We remind that r is not sensitive to the polarizations of the gravitational waves, because it refers to the total power spectra. Thus a priori, using (4.79), we can measure the effects of the Chern-Simons term also searching for unpolarized primordial gravitational waves. But in this case the corrections are of order Θ_{R-L}^2 and probably this effect generated is more difficult to measure.

4.5 Computation of non-Gaussianities generated by the Chern-Simons term

Now we perform an investigation about the effects of the Chern-Simons term on the bispectrum of the primordial perturbations. We use the same in-in formalism described in Chapter 3. Our aim is to investigate the possibility to observe parity violating effects also in the non-Gaussian part of the statistics of the primordial perturbations. For the moment we work in the spatially flat gauge in which the scalar perturbations are labelled by the inflaton perturbation φ (3.20). This choice is made for simplicity of the computations. We will switch to the gauge invariant quantity ζ only in a second step through the transformation (3.22).

Because of parity conservation in the case of scalar perturbations, there are no changes in the pure scalar bispectrum, which is the one discuss in details in Chapter 3. Then we have to investigate the contributions from the correlators between the inflaton and the left and right circular polarized gravitons. A cubic simple vertex to analyze is the cubic interaction $\varphi\gamma\gamma$. In fact, starting from the lagrangian (4.26), when we evaluate the corresponding action the contributions to this vertex are:

$$\Delta S_{\varphi\gamma\gamma} = \int d^4x \epsilon^{\mu\nu\rho\sigma} \left[\left(\frac{\partial}{\partial\phi} f(\phi) \right) \varphi C_{\mu\nu}^{(1)\kappa\lambda}|_T C_{\rho\sigma\kappa\lambda}|_T + f(\phi) C_{\mu\nu}^{(1)\kappa\lambda}|_S C_{\rho\sigma\kappa\lambda}|_T + C_{\mu\nu}^{(2)\kappa\lambda}|_T C_{\rho\sigma\kappa\lambda}|_S \right]. \quad (4.80)$$

Here the first contribution comes from the expansion in series of the function $f(\phi)$ around the background value of the inflaton multiplied by the contraction of two Weyl tensors at first order in tensor perturbations; the other terms instead come from the contraction between the Weyl tensor at second order in tensor perturbations and the Weyl tensor at first order in the constraints N and N_i . These constraints do not change w.r.t. the ones of the standard slow-roll case (2.57), because at quadratic level the Chern-Simons term gives contribution only to tensor perturbations. We remember that N and N_i are subdominant in the slow-roll hypothesis in comparison with the inflaton perturbation φ . For this reason in the slow-roll limit the term dominant in Eq. (4.80) is the first one. In this case the tensor contractions that we have to do are the same of the quadratic case and we obtain the cubic action in Fourier space:

$$\Delta S_{\varphi\gamma\gamma}(\tau) = \sum_{s=L,R} \alpha_s \int d\tau d^3k d^3p d^3q \frac{\delta^3(\vec{k} + \vec{p} + \vec{q})}{(2\pi)^6} \left(\frac{\partial}{\partial\phi} f(\phi) \right) \varphi(\vec{k}) \frac{\partial}{\partial\tau} \left[p \gamma_{ij}^s(\vec{p}) \gamma_s^{ij}(\vec{q}) + p (\vec{p} \cdot \vec{q}) \gamma_{ij}^s(\vec{p}) \gamma_s^{ij}(\vec{q}) \right]. \quad (4.81)$$

where we remember that the contractions of latin indicies are done with the δ_{ij} .

Thus, if we integrate by parts the action (4.81) with respect to the conformal time, we find the following interaction lagrangian for the vertex $\varphi\gamma\gamma$:

$$\begin{aligned}
L_{int}^{\varphi\gamma\gamma}(\tau) = & - \sum_{s=L,R} \alpha_s \int d^3k d^3p d^3q \frac{\delta^3(\vec{k} + \vec{p} + \vec{q})}{(2\pi)^6} \left[\left(\frac{\partial}{\partial\phi} f(\phi) \right) p\varphi'(\vec{k})\gamma'_{ij}{}^s(\vec{p})\gamma'_{ij}{}^s(\vec{q}) + \right. \\
& + \left(\frac{\partial}{\partial\phi} f(\phi) \right) p(\vec{p} \cdot \vec{q}) \varphi' \gamma_{ij}{}^s(\vec{p}) \gamma_{ij}{}^s(\vec{q}) + a \left(\frac{\partial}{\partial\phi} \dot{f}(\phi) \right) p\varphi(\vec{k})\gamma'_{ij}{}^s(\vec{p})\gamma'_{ij}{}^s(\vec{q}) + \\
& \left. + a \left(\frac{\partial}{\partial\phi} \dot{f}(\phi) \right) p(\vec{p} \cdot \vec{q}) \varphi \gamma_{ij}{}^s(\vec{p}) \gamma_{ij}{}^s(\vec{q}) \right], \tag{4.82}
\end{aligned}$$

where we have express $\left(\frac{\partial}{\partial\phi} f'(\phi) \right) = a \left(\frac{\partial}{\partial\phi} \dot{f}(\phi) \right)$, using the relation $dt = a d\tau$ which links cosmological time to conformal time.

Now, following Ref. [9], we compute the quantum correlator:

$$\langle 0 | \hat{\gamma}^{s_1}(\vec{k}_1, 0) \hat{\gamma}^{s_2}(\vec{k}_2, 0) \hat{\varphi}(\vec{k}_3, 0) | 0 \rangle, \tag{4.83}$$

where $\hat{\gamma}_{s_1}$ and $\hat{\gamma}_{s_2}$ label the circular polarization modes of the primordial gravitational waves.

In the next steps we will omit the time argument $\tau = 0$ for semplicity of notation. By the form of the interaction lagrangian (4.82), it follows that the only non-vanishing correlators of the type (4.83) are:

$$\langle 0 | \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\varphi}(\vec{k}_3) | 0 \rangle, \quad \langle 0 | \hat{\gamma}^L(\vec{k}_1) \hat{\gamma}^L(\vec{k}_2) \hat{\varphi}(\vec{k}_3) | 0 \rangle. \tag{4.84}$$

Instead we have:

$$\langle 0 | \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^L(\vec{k}_2) \hat{\varphi}(\vec{k}_3) | 0 \rangle = \langle 0 | \hat{\gamma}^L(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\varphi}(\vec{k}_3) | 0 \rangle = 0. \tag{4.85}$$

This fact is explained also by a simmetry argument. In fact the mixed interaction vertex $\gamma_L\gamma_R\varphi$ is invariant under parity transformation. The reason is that a parity transformation leaves φ invariant and maps γ_R in γ_L and viceversa.

We start now a computation of the correlator $\langle 0 | \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\varphi}(\vec{k}_3) | 0 \rangle$. The computations for the other correlator will be analogous. Going to the interaction picture, we can use the master formula (3.33), obtaining:

$$\begin{aligned}
\langle \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\varphi}(\vec{k}_3) \rangle = & - \frac{i}{(2\pi)^6} \int d^3K \delta^3(\vec{k} + \vec{p} + \vec{q}) \int_{-\infty}^0 d\tau' \left[\left(\frac{\partial}{\partial\phi} f(\phi) \right) (\mathcal{B}_1(\tau') + \mathcal{B}_2(\tau')) + \right. \\
& \left. + a \left(\frac{\partial}{\partial\phi} \dot{f}(\phi) \right) (\mathcal{B}_3(\tau') + \mathcal{B}_4(\tau')) \right], \tag{4.86}
\end{aligned}$$

where $\int d^3K = \int d^3k d^3p d^3q$ and

$$\mathcal{B}_1 = p \langle 0 | [\hat{\varphi}(\vec{k}_1, 0) \hat{\gamma}^R(\vec{k}_2, 0) \hat{\gamma}^R(\vec{k}_3, 0), \hat{\varphi}'(\vec{k}, \tau') \hat{\gamma}_{ij}^R(\vec{p}, \tau') \hat{\gamma}_R^{ij}(\vec{q}, \tau')] | 0 \rangle, \quad (4.87)$$

$$\mathcal{B}_2 = p (\vec{p} \cdot \vec{q}) \langle 0 | [\hat{\varphi}(\vec{k}_1, 0) \hat{\gamma}^R(\vec{k}_2, 0) \hat{\gamma}^R(\vec{k}_3, 0), \hat{\varphi}'(\vec{k}, \tau') \hat{\gamma}_{ij}^R(\vec{p}, \tau') \hat{\gamma}_R^{ij}(\vec{q}, \tau')] | 0 \rangle, \quad (4.88)$$

$$\mathcal{B}_3 = p \langle 0 | [\hat{\varphi}(\vec{k}_1, 0) \hat{\gamma}^R(\vec{k}_2, 0) \hat{\gamma}^R(\vec{k}_3, 0), \hat{\varphi}(\vec{k}, \tau') \hat{\gamma}_{ij}^R(\vec{p}, \tau') \hat{\gamma}_R^{ij}(\vec{q}, \tau')] | 0 \rangle, \quad (4.89)$$

$$\mathcal{B}_4 = p (\vec{p} \cdot \vec{q}) \langle 0 | [\hat{\varphi}(\vec{k}_1, 0) \hat{\gamma}^R(\vec{k}_2, 0) \hat{\gamma}^R(\vec{k}_3, 0), \hat{\varphi}(\vec{k}, \tau') \hat{\gamma}_{ij}^R(\vec{p}, \tau') \hat{\gamma}_R^{ij}(\vec{q}, \tau')] | 0 \rangle. \quad (4.90)$$

Here the $[\cdot, \cdot]$ denotes the commutator operator.

As done in Chapter 3, we can compute these expressions by using the Wick theorem. We need to compute preliminarily the following contractions between fields:

$$\langle 0 | \hat{\varphi}(\vec{k}, \tau) \hat{\varphi}(\vec{k}', \tau') | 0 \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}') u(k, \tau) u^*(k, \tau'), \quad (4.91)$$

$$\langle 0 | \hat{\varphi}(\vec{k}, \tau) \hat{\varphi}'(\vec{k}', \tau') | 0 \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}') u(k, \tau) \frac{d}{d\tau} u^*(k, \tau'), \quad (4.92)$$

$$\langle 0 | \hat{\gamma}_{ij}^R(\vec{k}, \tau) \hat{\gamma}^R(\vec{k}', \tau') | 0 \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}') \tilde{z}_R(k, \tau) \tilde{z}_R^*(k, \tau') \epsilon_{ij}^R(\vec{k}), \quad (4.93)$$

$$\langle 0 | \hat{\gamma}^R(\vec{k}, \tau) \hat{\gamma}_{ij}^R(\vec{k}', \tau') | 0 \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}') \tilde{z}_R(k, \tau) \tilde{z}_R^*(k, \tau') \epsilon_{ij}^{R*}(\vec{k}), \quad (4.94)$$

$$\langle 0 | \hat{\gamma}_{ij}^R(\vec{k}, \tau) \hat{\gamma}'^R(\vec{k}', \tau') | 0 \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}') \tilde{z}_R(k, \tau) \left(\frac{d}{d\tau} \tilde{z}_R^*(k, \tau') \right) \epsilon_{ij}^R(\vec{k}), \quad (4.95)$$

$$\langle 0 | \hat{\gamma}'^R(\vec{k}, \tau) \hat{\gamma}_{ij}^R(\vec{k}', \tau') | 0 \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}') \left(\frac{d}{d\tau} \tilde{z}_R(k, \tau) \right) \tilde{z}_R^*(k, \tau') \epsilon_{ij}^{R*}(\vec{k}), \quad (4.96)$$

where $u(k, \tau)$ is as in Eq. (2.88) and $\tilde{z}_s(k, \tau)$ is as in Eq. (4.65).

Thus, performing all the contractions, (4.86) becomes:

$$\begin{aligned} \langle \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\varphi}(\vec{k}_3) \rangle &= -i(2\pi)^3 \delta^3(k_1 + k_2 + k_3) \text{Im}\{[k_1(\tilde{I}_1 + \tilde{I}_2) + k_1(\vec{k}_1 \cdot \vec{k}_2)(\tilde{I}_3 + \tilde{I}_4)] \times \\ &\times \epsilon_{ij}^{R*}(\vec{k}_1) \epsilon_{ij}^R(\vec{k}_2) - c.c.\} + (\vec{k}_1 \longleftrightarrow \vec{k}_2), \end{aligned} \quad (4.97)$$

where the \tilde{I}_n are the integrals:

$$\tilde{I}_1 = u(k_1, 0) z_R(k_1, 0) z_R(k_2, 0) \int_{-\infty}^0 d\tau' \left(\frac{\partial}{\partial \phi} f(\phi) \right) \left[\frac{d}{d\tau} u^*(k_1, \tau') \frac{d}{d\tau} \tilde{z}_R^*(k_1, \tau') \frac{d}{d\tau} \tilde{z}_R^*(k_2, \tau') \right], \quad (4.98)$$

$$\tilde{I}_2 = u(k_1, 0) z_R(k_2, 0) z_R(k_3, 0) \int_{-\infty}^0 d\tau' a \left(\frac{\partial}{\partial \phi} f(\phi) \right) \left[u^*(k_1, \tau') \frac{d}{d\tau} \tilde{z}_R^*(k_2, \tau') \frac{d}{d\tau} \tilde{z}_R^*(k_3, \tau') \right], \quad (4.99)$$

$$\tilde{I}_3 = u(k_1, 0) z_R(k_2, 0) z_R(k_3, 0) \int_{-\infty}^0 d\tau' \left(\frac{\partial}{\partial \phi} f(\phi) \right) \left[\frac{d}{d\tau} u^*(k_1, \tau') \tilde{z}_R^*(k_2, \tau') \tilde{z}_R^*(k_3, \tau') \right], \quad (4.100)$$

$$\tilde{I}_4 = u(k_1, 0) z_R(k_2, 0) z_R(k_3, 0) \int_{-\infty}^0 d\tau' a \left(\frac{\partial}{\partial \phi} f(\phi) \right) \left[u^*(k_1, \tau') \tilde{z}_R^*(k_2, \tau') \tilde{z}_R^*(k_3, \tau') \right]. \quad (4.101)$$

We can try to compute analytically these integrals with some approximations that are similar to the ones adopted to compute the bispectrum $B_{\zeta\zeta\zeta}$ in Chapter 3. The first one is to evaluate the Hubble parameter H and the function $f(\phi)$ and its derivatives at the time of horizon crossing of the momentum $K = \sum_i k_i$ and bring them out of the integrals. This approximation is justified by the slow-roll hypothesis in the background dynamics of the inflaton. The second approximation is about the cosmological evolution of the scale factor a . At leading order in slow-roll infact $a \simeq -\frac{1}{H\tau}$. Also this approximation is justified by the slow-roll hypothesis. The last approximation is to expand in series the functions u and \tilde{z}_s around zero values of the slow-roll parameters ϵ_V and η_V and of the parameter $\frac{\Omega H}{\Lambda} \simeq \Theta_{R-L}$. This approximation is justified by the fact that all these parameters are small. Thus, instead of using the functions u and z_R , we use again the mode function of a scalar field in a De Sitter space (3.57):

$$\tilde{z}_s(k, \tau)_{\epsilon_V=\eta_V=\Theta_{R-L}=0} = \frac{iH_*}{M_{Pl} \sqrt{k^3}} (1 + ik\tau) e^{-ik\tau}, \quad (4.102)$$

$$u(k, \tau)_{\epsilon_V=\eta_V=0} = \frac{iH_*}{\sqrt{2k^3}} (1 + ik\tau) e^{-ik\tau}. \quad (4.103)$$

These expressions seem different form (3.57), because they are now normalized respectively for the variables γ and φ .

With the prescriptions just explained we start now the explicit computation of the first integral \tilde{I}_1 . It reads:

$$\tilde{I}_1 = -4M_{Pl}^2 \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k^3} \right) \left(\frac{\partial}{\partial \phi} f(\phi) \right)^* k_1^2 k_2^2 k_3^2 \int_{-\infty}^0 d\tau' \tau'^3 e^{-iK\tau'}, \quad (4.104)$$

where $K = k_1 + k_2 + k_3$.

This integral can be performed using the general formula (3.68), obtaining

$$\tilde{I}_1 = -4M_{Pl}^2 \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k^3} \right) \left(\frac{\partial}{\partial \phi} f(\phi) \right)^* k_1^2 k_2^2 k_3^2 \left(-\frac{3!}{K^4} \right). \quad (4.105)$$

We see that the final result is real and so its imaginary part is zero. Then it does not give any contributions to the correlator (4.97). For the same reason also the integrals \tilde{I}_2 and \tilde{I}_3 do not give contributions. The only integral which is not trivial is \tilde{I}_4 . Let us see its computation in details:

$$\tilde{I}_4 = 4 \frac{M_{Pl}^2}{H_*} \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k^3} \right) \left(\frac{\partial}{\partial \phi} f(\phi) \right)^* \int_{-\infty}^0 \frac{d\tau'}{\tau'} (1 + ik_1\tau')(1 + ik_2\tau')(1 + ik_3\tau') e^{-iK\tau'}, \quad (4.106)$$

We rewrite the integral which appears in (4.106) as:

$$\int_{-\infty}^0 \frac{d\tau'}{\tau'} (1 + ik_1\tau')(1 + ik_2\tau')(1 + ik_3\tau') e^{-iK\tau'}. \quad (4.107)$$

It can be decomposed into the sum of four integrals:

$$\int_{-\infty}^0 \frac{d\tau'}{\tau'} e^{-iK\tau'} + iK \int_{-\infty}^0 d\tau' e^{-iK\tau'} - \prod_{i \neq j} k_i k_j \int_{-\infty}^0 d\tau' \tau' e^{-iK\tau'} - ik_1 k_2 k_3 \int_{-\infty}^0 d\tau' \tau'^2 e^{-iK\tau'} . \quad (4.108)$$

All the integrals apart the first one can be computed using the formula (3.68) and give a real contribution. For this reason they do not give any contribution to the correlator (4.97). Instead, the first integral can be traced back to the exponential integral $Ei(z)$ by promoting the real variable τ' to a complex variable and performing a Wick rotation of the integration contour with the change of variable $\tau' = -i\tau''$. It becomes:

$$\lim_{\tau \rightarrow 0^-} \int_{-i\infty}^{i\tau} \frac{d\tau''}{\tau''} e^{-K\tau''} . \quad (4.109)$$

The *complex exponential integral* is defined as [46]:

$$Ei(z) = \int_{-\infty}^z \frac{dz'}{z'} e^{-z'} \quad |Arg(z)| < \pi . \quad (4.110)$$

It is well defined for all complex numbers z that are off the real negative axis. A good characteristic of this integral is that it is independent by the integration contour but it depends only by z . In particular it can be expressed in terms of the following series representation [46]:

$$Ei(z) = -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-z)^k}{kk!} , \quad (4.111)$$

where γ is the Euler-Mascheroni constant and $\ln z$ is the principal complex logarithm of the complex number z . This series converges for all z that are not in the real axis.

Applying the formula (4.111) to compute the integral (4.109), it becomes:

$$\lim_{K\tau \rightarrow 0^-} \left[-\gamma - \ln(iK\tau) - \sum_{k=1}^{\infty} \frac{(-iK\tau)^k}{kk!} \right] = -\gamma + \left(\lim_{K\tau \rightarrow 0} \ln|K\tau| \right) + i\frac{\pi}{2} , \quad (4.112)$$

where the $\ln|K\tau|$ in this case represents a real logarithm. The divergence provided by the logarithm in (4.112) is not important because at the end we take only the imaginary part of the integral. Thus, we have:

$$\text{Im}(\tilde{I}_4) = 4 \frac{M_{Pl}^2}{H_*} \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k_i^3} \right) \left(\frac{\partial}{\partial \phi} f(\phi) \right)^* \times \left(i\frac{\pi}{2} \right) . \quad (4.113)$$

If we substitute this result into Eq. (4.97) and we consider also the contributions of the permutations, we find the final result:

$$\begin{aligned} \langle \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\phi}(\vec{k}_3) \rangle &= (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) 4 \frac{M_{Pl}^2}{H_*} \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k_i^3} \right) \left(\frac{\partial}{\partial \phi} f(\phi) \right)^* \\ &\times (k_1 + k_2) (\vec{k}_1 \cdot \vec{k}_2) \epsilon_{ij}^R(\vec{k}_1) \epsilon_{ij}^R(\vec{k}_2) . \end{aligned} \quad (4.114)$$

Following the same steps, we are able to compute also the correlator $\langle \hat{\gamma}^L(\vec{k}_1) \hat{\gamma}^L(\vec{k}_2) \hat{\phi}(\vec{k}_3) \rangle$. It is sufficient to substitute in the previous passages R with L and take a relative factor -1 due to the $\alpha_L = -\alpha_R$ in the interaction lagrangian (4.82). Thus, we have:

$$\begin{aligned} \langle \hat{\gamma}^L(\vec{k}_1) \hat{\gamma}^L(\vec{k}_2) \hat{\phi}(\vec{k}_3) \rangle &= - (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) 4 \frac{M_{Pl}^2}{H_*} \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k_i^3} \right) \left(\frac{\partial}{\partial \phi} f(\phi) \right)^* \times \\ &\times (k_1 + k_2) (\vec{k}_1 \cdot \vec{k}_2) \epsilon_{ij}^L(\vec{k}_1) \epsilon_{ij}^L(\vec{k}_2). \end{aligned} \quad (4.115)$$

We can try to express the final result in a way in which we write explicitly the dependence over the wavenumbers k_i . Because of momentum conservation $(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) = 0$, the three momenta form a triangle. For invariance under translations we can put this triangle in the (x, y) -plane without losing any generality. It follows that a triangle can be constructed by:

$$\vec{k}_1 = k_1(1, 0, 0), \quad \vec{k}_2 = k_2(\cos \theta, \sin \theta, 0), \quad \vec{k}_3 = k_3(\cos \Phi, \sin \Phi, 0), \quad (4.116)$$

where $0 \leq \theta \leq \pi$, $\pi \leq \Phi \leq 2\pi$ are the angles in polar coordinates.

With this choices of the momenta, we can write the polarization tensors for L and R using the basis definitions (4.36). We have [41]:

$$\epsilon_{ij}^s(\vec{k}_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i\alpha_s \\ 0 & i\alpha_s & -1 \end{pmatrix}, \quad (4.117)$$

$$\epsilon_{ij}^s(\vec{k}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin^2 \theta & -\sin \theta \cos \theta & -i\alpha_s \sin \theta \\ -\sin \theta \cos \theta & \cos^2 \theta & i\alpha_s \cos \theta \\ -i\alpha_s \sin \theta & i\alpha_s \cos \theta & -1 \end{pmatrix}. \quad (4.118)$$

Thus through an explicit calculation we find:

$$\vec{k}_1 \cdot \vec{k}_2 = k_1 k_2 \cos \theta, \quad \epsilon_{ij}^s(\vec{k}_1) \epsilon_{ij}^s(\vec{k}_2) = \frac{1}{2} (1 - \cos \theta)^2, \quad (4.119)$$

where θ is essentially the angle between the two momenta \vec{k}_1 and \vec{k}_2 of the gravitational modes. By the cosine theorem we can express this angle in function of the three wavenumbers k_i as:

$$\cos \theta = \frac{k_3^2 - k_2^2 - k_1^2}{2k_1 k_2}. \quad (4.120)$$

We can also write:

$$\left(\frac{\partial}{\partial \phi} f(\phi) \right)^* = \left(\frac{\partial^2}{\partial^2 \phi} f(\phi) \right)^* \dot{\phi}_*. \quad (4.121)$$

In the end the correlators (4.114) and (4.115) become

$$\begin{aligned} \langle \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\phi}(\vec{k}_3) \rangle &= (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) 4 \frac{\dot{\phi}_*}{H_*} \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k_i^3} \right) \left(M_{Pl}^2 \frac{\partial^2}{\partial^2 \phi} f(\phi) \right)^* \times \\ &\times (k_1 + k_2) k_1 k_2 \frac{\cos \theta (1 - \cos \theta)^2}{2}, \end{aligned} \quad (4.122)$$

with $\cos \theta = \frac{k_3^2 - k_2^2 - k_1^2}{2k_1 k_2}$ and

$$\langle \hat{\gamma}^L(\vec{k}_1) \hat{\gamma}^L(\vec{k}_2) \hat{\phi}(\vec{k}_3) \rangle = -\langle \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\phi}(\vec{k}_3) \rangle. \quad (4.123)$$

From the angular dependence of (4.122), we see that the correlator (4.122) is maximum when $\cos \theta = -1$. This corresponds essentially to the "squeezed" limit, which is the limit in which the momenta of the gravitational waves k_1, k_2 are much larger than the momentum k_3 of the inflaton perturbation. In fact in this configuration the triangle of the momenta k_i 's appears very squeezed.

Now we pass to gauge invariant variables in order to make predictions about the strength of these correlators in comparison with the one predicted by the standard slow-roll model (3.89). On superhorizon scales and in the slow-roll limit the local relation between the inflaton φ and the gauge invariant curvature perturbation ζ is (see Eqns. (3.77) and (3.78)):

$$\zeta = \zeta_1 - \frac{\eta_V}{2} \zeta_1^2, \quad (4.124)$$

where η is the slow-roll parameter (2.21) and $\zeta_1 = -\frac{H}{\dot{\phi}} \varphi$ is the first order value of ζ .

Then in the coordinate space we have:

$$\langle \hat{\gamma}^R(\vec{x}_1) \hat{\gamma}^R(\vec{x}_2) \hat{\zeta}(\vec{x}_3) \rangle_{C-S} \simeq -\frac{H^*}{\dot{\phi}_*} \langle \hat{\gamma}^R(\vec{x}_1) \hat{\gamma}^R(\vec{x}_2) \hat{\phi}(\vec{x}_3) \rangle, \quad (4.125)$$

where we have not considered the contribution of field redefinition coming from the non-linear part of the relation between ζ and ζ_1 , Eq. (4.124). In fact this contribution represents a disconnected term. The \simeq means that we are evaluating the correlator at first order in the slow-roll parameters.

Thus passing in Fourier space and substituting Eq. (4.122) into Eq. (4.125) we have:

$$\begin{aligned} \langle \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\zeta}(\vec{k}_3) \rangle_{C-S} &\simeq -\frac{H^*}{\dot{\phi}_*} \langle \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\phi}(\vec{k}_3) \rangle = \\ &= -(2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k_i^3} \right) \left(M_{Pl}^2 \frac{\partial^2}{\partial^2 \phi} f(\phi) \right)^* \times \\ &\times 2(k_1 + k_2) k_1 k_2 \cos \theta (1 - \cos \theta)^2. \end{aligned} \quad (4.126)$$

Proceeding with the same reasoning for computing the vertex $\langle \hat{\gamma}^L(\vec{k}_1) \hat{\gamma}^L(\vec{k}_2) \hat{\zeta}(\vec{k}_3) \rangle_{C-S}$, we find:

$$\langle \hat{\gamma}^L(\vec{k}_1) \hat{\gamma}^L(\vec{k}_2) \hat{\zeta}(\vec{k}_3) \rangle_{C-S} = -\langle \hat{\gamma}^R(\vec{k}_1) \hat{\gamma}^R(\vec{k}_2) \hat{\zeta}(\vec{k}_3) \rangle_{C-S} \quad (4.127)$$

A physical quantity in which the parity violating effects in the vertex $\langle \gamma\gamma\zeta \rangle$ are encoded can be the normalized relative difference between the correlators (4.126) and (4.127). In evaluating this difference we have to take into account the contribution of the standard slow-roll model computed by Maldacena in [9], which we have seen in Eq. (3.89) is given by:

$$\langle \hat{\gamma}^S(\vec{k}_1) \hat{\gamma}^S(\vec{k}_2) \hat{\zeta}(\vec{k}_3) \rangle = \frac{H_*^4}{M_{Pl}^4} \left(\sum_{i>j} \frac{1}{k_i^3 k_j^3} \right) F(k_i), \quad (4.128)$$

where the $F(k_i)$ is a particular function of the momenta k_i which is typically of order $\mathcal{O}(1)$ due to momentum conservation. In the expression (4.128) we have not explicited the $(2\pi)^3\delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$ for simplicity of notation. We see that (4.128) is equal for both L and R polarizations.

Similary we see that the correlators just computed in Eq. (4.126) and (4.127) are given by:

$$\langle \hat{\gamma}^R(\vec{k}_1)\hat{\gamma}^R(\vec{k}_2)\hat{\zeta}(\vec{k}_3) \rangle_{C-S} = -\langle \hat{\gamma}^L(\vec{k}_1)\hat{\gamma}^L(\vec{k}_2)\hat{\zeta}(\vec{k}_3) \rangle_{C-S} \simeq -\frac{H_*^6}{M_{Pl}^6} \left(M_{Pl}^2 \frac{\partial^2}{\partial^2\phi} f(\phi) \right)^* \left(\sum_{i>j} \frac{1}{k_i^3 k_j^3} \right) F'(k_i), \quad (4.129)$$

where also in this case we have not explicited the $(2\pi)^3\delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$ for simplicity of notation. $F'(k_i)$ is a function of the momenta k_i of order $\mathcal{O}(1)$ due to momentum conservation. Its explicit expression is:

$$F'(k_i) = \frac{1}{8} \frac{(k_1 + k_2)(k_3^2 - k_2^2 - k_1^2) \left(1 - \frac{k_3^2 - k_2^2 - k_1^2}{2k_1 k_2} \right)^2}{\left(\sum_i k_i^3 \right)}. \quad (4.130)$$

Thus the relative difference between the correlators is given by:

$$B_{R-L}^{\gamma\gamma\zeta} = \frac{\langle \hat{\gamma}^R(\vec{k}_1)\hat{\gamma}^R(\vec{k}_2)\hat{\zeta}(\vec{k}_3) \rangle_{TOT} - \langle \hat{\gamma}^L(\vec{k}_1)\hat{\gamma}^L(\vec{k}_2)\hat{\zeta}(\vec{k}_3) \rangle_{TOT}}{\langle \hat{\gamma}^R(\vec{k}_1)\hat{\gamma}^R(\vec{k}_2)\hat{\zeta}(\vec{k}_3) \rangle_{TOT} + \langle \hat{\gamma}^L(\vec{k}_1)\hat{\gamma}^L(\vec{k}_2)\hat{\zeta}(\vec{k}_3) \rangle_{TOT}} \sim -2 \frac{H_*^2}{M_{Pl}^2} \left(M_{Pl}^2 \frac{\partial^2}{\partial^2\phi} f(\phi) \right)^*, \quad (4.131)$$

where the suffix TOT denotes the total contribution summing Eqns. (4.129) and (4.128).

In this result we are not considering the shape dependence of the bispectrum. We see that this quantity depends essentially from the strength of the second derivative of the function $f(\phi)$ with respect to the inflaton. We notice that we have not specified in the computations the form of $f(\phi)$ because we want to keep the theory as general as possible. In any case we have some constraints on the value of the derivatives of $f(\phi)$ that come from the approximations made to develop the computations themselves. We see now in detail these constraints.

First of all we write the first derivative of f with respect to the inflaton as a function of the other parameters of the theory:

$$M_{Pl} \frac{\partial}{\partial\phi} f(\phi) = M_{Pl} \frac{\dot{f}}{\dot{\phi}} \simeq M_{Pl}^3 \frac{\Omega}{\Lambda \dot{\phi}} \simeq \frac{\Theta_{R-L} M_{Pl}^2}{\sqrt{\epsilon_V} H^2}. \quad (4.132)$$

Then, in our theory we have imposed that the Chern-Simons mass M_{C-S} does not change much in time during inflation. We have said that this is equivalent to require the condition (4.52), which we rewrite:

$$M_{Pl}^2 \frac{\partial^2}{\partial^2\phi} f(\phi) < \frac{M_{Pl}}{\sqrt{\epsilon_V}} \frac{\partial}{\partial\phi} f(\phi). \quad (4.133)$$

Thus, inserting Eqns. (4.132) and (4.133) into Eq. (4.131), it results the constraint:

$$|B_{R-L}^{\gamma\gamma\zeta}| < \mathcal{O} \left(\frac{\Theta_{R-L}}{\epsilon_V} \right). \quad (4.134)$$

The presence of the slow-roll parameter ϵ_V in the denominator in this last equation tells us that we can have a priori a large parity breaking in the vertex $\langle\gamma\gamma\zeta\rangle$ also with a small parity breaking in the power spectrum of the tensor perturbations. This fact is quite interesting, because it guides us to search for signals of the Chern-Simons modified gravity through the analysis of parity violating effects in the CMB bispectrum.

Conclusions

In the Thesis we have investigated through various aspects of inflation. First of all we have introduced the inflationary scenario as a solution of the classical problems of the hot Big Bang model: the horizon, the flatness and the magnetic monopole "problems" .

Then we have introduced the slow-roll models of inflation to study the production and evolution of the primordial perturbations during inflation. These models predict the presence of both scalar and tensor perturbations in the primordial universe. The primordial scalar perturbations can be linked to the temperature anisotropies of the CMB through the gauge invariant quantity ζ , which is the curvature perturbation of comoving hypersurfaces; instead the tensor perturbations are associated to primordial gravitational waves γ_{ij} that are not observed directly yet. We have recalled how to link the large-scale power spectrum of the primordial perturbations to observational constraints from the CMB in order to constrain parameter space of the theory. These parameters are the slow-roll parameters ϵ_V and η_V that depends on the slow-roll potential $V(\phi)$ and its derivatives. Different potentials $V(\phi)$ bring to a large zoology of slow-roll models. We have seen a toy model example of the so-called large and small field models.

Then, always remaining in the context of the slow-roll models, we have made a computation of primordial non-Gaussianities provided by the scalar bispectrum of the primordial perturbations. In order to do the computations we have defined and used the so-called in-in formalism. The scalar bispectrum is suppressed when the slow-roll parameters are small and therefore the non-Gaussianities predicted by the slow-roll models of inflation are very small and presently not detectable given the sensitivity of the actual experimental instruments. This is fully consistent with the observational constraints on non-Gaussianity provided by the Planck satellite. But the errors of these measurements do not exclude a priori the possibility to find out profiles of non-Gaussianity in the next future, being two orders of magnitude larger than the prediction of the standard single field slow-roll models of inflation.

This has motivated us to introduce in the action of the slow-roll theories a modified gravity term which comes naturally by an expansion in the derivatives of the metric tensor. This term is the Chern-Simons term, in formula $f(\phi)\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu}{}^{\kappa\lambda}C_{\rho\sigma\kappa\lambda}$, where $f(\phi)$ is a general function of the inflaton only, $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita pseudotensor and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. The term is parity breaking and does not produce any change in scalar perturbations (in the sense that the contribution to the Chern-Simons term from scalar perturbations vanishes). So we have studied the quadratic modification of the action for the tensor perturbations only. We have seen that at a certain energy scale equal to the so-called Chern-Simons mass M_{C-S} a ghost degree of freedom appears. In order to regularize the theory and avoid the presence of the ghost, we have inserted an UV cut off Λ smaller than the Chern-Simons mass M_{C-S} . Then, assuming M_{C-S} almost constant during inflation, we have derived explicitly the equations of motion for the inflationary gravitational waves. These equations appear different for left(L) and right(R) circular polarizations of the gravitational waves. Therefore there is the possibility to measure a different large-scale power spectra for the two polarizations states. The observable in which this difference can be encoded is:

$$\Theta_{R-L} = \frac{\Delta_T^R - \Delta_T^L}{\Delta_T^R + \Delta_T^L} \simeq \frac{H}{M_{C-S}},$$

where both the Hubble parameter H and M_{C-S} are assumed constant parameters. We expect this quantity to be small for the consistency of the approximations made in the various computations. Given the actual constraint of the tensor-to-scalar ratio r , we hope to constrain the value of Θ_{R-L} from the CMB only if it is larger than approximately 0.21 [49].

A modification on the usual consistency relation of slow-roll models of inflation arises:

$$r_{C-S} = -8n_T \left(1 + \Theta_{R-L}^2\right).$$

The fact that the correction term is of order Θ_{R-L}^2 probably makes this correction difficult to observe with future experiments.

As an original contribution, we have computed at leading order in slow-roll parameters the contribution of the Chern-Simons term to the cubic interaction vertex between two gravitons and one inflaton ($\gamma\gamma\phi$). This interaction vertex gives quantum contributions to the correlator $\langle\gamma\gamma\phi\rangle$. We have found the following results:

$$\begin{aligned} \langle\gamma^R(\vec{k}_1)\gamma^R(\vec{k}_2)\phi(\vec{k}_3)\rangle &= (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) 4 \frac{\dot{\phi}_*}{H_*} \left(\prod_{i=1,2,3} \frac{H_*^2}{M_{Pl}^2 2k_i^3} \right) \left(M_{Pl}^2 \frac{\partial^2}{\partial^2\phi} f(\phi) \right)^* \times \\ &\times (k_1 + k_2) k_1 k_2 \frac{\cos\theta(1 - \cos\theta)^2}{2}, \end{aligned}$$

where $\cos\theta = \frac{k_3^2 - k_2^2 - k_1^2}{2k_1 k_2}$, and

$$\langle\gamma^L(\vec{k}_1)\gamma^L(\vec{k}_2)\phi(\vec{k}_3)\rangle = -\langle\gamma^R(\vec{k}_1)\gamma^R(\vec{k}_2)\phi(\vec{k}_3)\rangle,$$

$$\langle\gamma^L(\vec{k}_1)\gamma^R(\vec{k}_2)\phi(\vec{k}_3)\rangle = \langle\gamma^R(\vec{k}_1)\gamma^L(\vec{k}_2)\phi(\vec{k}_3)\rangle = 0.$$

There the * indicates the epoch of horizon crossing for the mode $K = k_1 + k_2 + k_3$.

Only the "non-L-R-mixed" correlators are non trivial and depend by the second derivative of the function $f(\phi)$ with respect to the inflaton field.

Considering the gauge invariant scalar perturbation ζ , we have computed the order of the relative difference of the correlators $\langle\gamma_R\gamma_R\zeta\rangle$ and $\langle\gamma_L\gamma_L\zeta\rangle$. In fact this adimensional quantity measures the level of parity breaking in the correlator considered. We have found a result proportional to $\left(M_{Pl}^2 \frac{\partial^2}{\partial^2\phi} f(\phi)\right)^*$. In addition accounting for the approximations made to develop the theory, we have found the theoretical constraint:

$$\left|B_{R-L}^{\gamma\gamma\zeta}\right| = \frac{\langle\gamma^R(\vec{k}_1)\gamma^R(\vec{k}_2)\zeta(\vec{k}_3)\rangle_{TOT} - \langle\gamma^L(\vec{k}_1)\gamma^L(\vec{k}_2)\zeta(\vec{k}_3)\rangle_{TOT}}{\langle\gamma^R(\vec{k}_1)\gamma^R(\vec{k}_2)\zeta(\vec{k}_3)\rangle_{TOT} + \langle\gamma^L(\vec{k}_1)\gamma^L(\vec{k}_2)\zeta(\vec{k}_3)\rangle_{TOT}} < \mathcal{O}\left(\frac{\Theta_{R-L}}{\epsilon_V}\right).$$

The presence of the slow-roll parameter ϵ_V at the denominator might allow $B_{R-L}^{\gamma\zeta}$ to be large also in the case of small parity breaking in the power spectrum of the gravitational waves. This result forces us to remain open-minded about the possibilities of measuring the effects of the Chern-Simons modified gravity term through a dedicated analysis of the CMB bispectrum.

In any case our theoretical computation is been just explorative. We are not able actually to link directly our result with the non-Gaussianities of the CMB. It is necessary to develop a template to search for this parity breaking signature in the CMB bispectrum. In addition our computation of the correlator $\langle\gamma\gamma\varphi\rangle$ can be usefull also in other theories of inflation different by the slow-roll theories. The possible future extension of our work is the computation of the contributions of the Chern-Simons term to the correlators $\langle\gamma\zeta\zeta\rangle$ and $\langle\gamma\gamma\gamma\rangle$, in order to search if we obtain also in this case a non trivial parity breaking signature.

Appendix A

ADM expressions of the curvature tensors

The 3+1 decomposition of the metric and inverse metric reads like:

$$g_{00} = -(N^2 - N_i N^i), \quad g_{0i} = N_i, \quad g_{ij} = h_{ij}. \quad (\text{A.1})$$

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = -\frac{N^i}{N^2}, \quad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}, \quad (\text{A.2})$$

where N is called laps function and N_i is called shift function. They represent the linkings between different slices in which we foliate the spacetime. The 3-metric on each slice at fixed time is h_{ij} . If we move on a slice, we can define an intrinsic Riemann curvature $R_{ijkl}^{(3)}$ and an intrinsic covariant derivative $D_i^{(3)}$. Their expressions are obtained replacing in their general definitions the 4-metric $g_{\mu\nu}$ with the 3-metric h_{ij} . In the same way, contracting the 3-Riemann tensor with the metric h_{ij} we can define the intrinsic Ricci tensor $R_{\mu\nu}^{(3)}$ and the intrinsic scalar curvature $R^{(3)}$. The contribution to the total curvature of the spacetime arising from the time component is labelled by the extrinsic curvature tensor:

$$K_{ij} = \frac{1}{2N} [D_i^{(3)} N_j + D_j^{(3)} N_i - \dot{h}_{ij}]. \quad (\text{A.3})$$

This tensor together with the functions N and N_i and the 3-tensors permits us to rewrite the fundamental curvature tensors of the general relativity in a form in which all the contractions can be done only by the spatial 3-metric h_{ij} . By doing some algebra we find the following compact relations:

Riemann tensor components:

$$R_{ijkl} = R_{ijkl}^{(3)} + K_{ik} K_{jl} - K_{il} K_{jk}, \quad (\text{A.4})$$

$$R_{0ijk} = N [D_j^{(3)} K_{ik} - D_k^{(3)} K_{ij}] + N^l [R_{lijk}^{(3)} + K_{lj} K_{ik} - K_{lk} K_{ij}], \quad (\text{A.5})$$

$$R_{0i0j} = N [\dot{K}_{ij} + D_i^{(3)} D_j^{(3)} N + N K_i^k K_{kj} - (D_j^{(3)} K_{ik}) N^k - (D_i^{(3)} N^k) K_{kj} - (D_i^{(3)} K_{kj}) N^k + (D_k^{(3)} K_{ij}) N^k] + N^l N^k [-R_{lijk}^{(3)} - K_{il} K_{jk} + K_{lk} K_{ij}]. \quad (\text{A.6})$$

Ricci tensor components:

$$R_{ij} = R_{ij}^{(3)} + K_{lk}K_{ij}h^{lk} - 2K_{ik}K_{jl}h^{lk} - \frac{1}{N}[\dot{K}_{ij} + D_i^{(3)}D_j^{(3)}N - N^l\partial_l K_{ij} - K_{il}\partial_j N^l - K_{jl}\partial_i N^l], \quad (\text{A.7})$$

$$R_{0i} = -\frac{N^j}{N}\dot{K}_{ij} - \frac{N^j}{N}(D_i^{(3)}D_j^{(3)}N) - N^j K_i^k K_{kj} + \frac{N^j}{N}[(D_j^{(3)}K_{ik})N^k + (D_i^{(3)}N^k)K_{kj} + (D_i^{(3)}K_{kj})N^k - (D_k^{(3)}K_{ij})N^k] + N^l h^{jk} R_{ljk}^{(3)} + N^l h^{jk} K_{il} K_{jk} - N^l h^{jk} K_{lk} K_{ij}, \quad (\text{A.8})$$

$$R_{00} = R_{0i0j} \left(h^{ij} - \frac{N^i N^j}{N^2} \right). \quad (\text{A.9})$$

Scalar curvature:

$$R = R^{(3)} + K_{ij}K^{ij} + (K_i^i)^2 - \frac{2}{N}(\dot{K}^i_i) + \frac{2N^j}{N}(D_j^{(3)}K_i^i) - \frac{2}{N}\Delta^{(3)}N, \quad (\text{A.10})$$

where $\Delta^{(3)}$ is the covariant laplacian built with the 3-metric h_{ij} .

From this last equation in particular we see:

$$R = R^{(3)} + K_{ij}K^{ij} + (K_i^i)^2 + (\text{surface terms}). \quad (\text{A.11})$$

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